ALGEBRA

FOR

SECONDARY SCHOOLS

BY

CHARLES DAVISON, Sc.D.

VOLUME II

FROM QUADRATIC EQUATIONS

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PREFACE TO VOLUME II

THE subjects included in this volume are those which are usually read after simple quadratic equations as far as the exponential and logarithmic series. A few changes in order are made, the remainder theorem and simple partial fractions being introduced at an earlier stage than usual, while the chapter on the theory and practice of logarithms follows that on indices.

The complete proof of the logarithmic series is beyond the scope of this book; but, with this exception, the theory of the subject is, I hope, sufficiently discussed as regards real and commensurable quantities. Several difficult theorems, and especially those on permutations and combinations, are preceded and illustrated by corresponding numerical examples; and, if desired, the latter may be used temporarily instead of the formal proofs.

The application of graphs is confined to the illustrations which they afford of the methods of solving simultaneous equations and the theory of quadratic equations and expressions. In the solution of simultaneous quadratic equations, they are especially useful, as they serve to show the geometrical meaning of every step in the process employed.

CHARLES DAVISON.

BIRMINGHAM,
November, 1908.

NOTE

*** Sections, articles and exercises preceded by an asterisk may be omitted on the first reading.

CHAPTER XVII.

EQUATIONS OF A HIGHER DEGREE THAN THE SECOND.

170. In this chapter will be considered certain equations of a higher degree than the second, which may be solved like quadratics, or the solution of which depends upon the solution of quadratic equations.

Example 1. Solve the equation $x^4 + 2x^3 - 15x^2 = 0$.

$$x^4 + 2x^3 - 15x^2 = 0,$$
$$\therefore\ x^2(x^2 + 2x - 15) = 0,$$
$$\therefore\ x^2(x - 3)(x + 5) = 0,$$
$$\therefore \text{ the roots are } 0,\ 0,\ 3 \text{ and } -5.$$

Example 2. Solve the equation $x^3 - x^2 - 4x + 4 = 0$.

$$x^3 - x^2 - 4x + 4 = 0,$$
$$\therefore\ x^2(x - 1) - 4(x - 1) = 0,$$
$$\therefore\ (x - 1)(x^2 - 4) = 0,$$
$$\therefore\ (x - 1)(x - 2)(x + 2) = 0,$$
$$\therefore \text{ the roots are } 1,\ 2 \text{ and } -2.$$

Example 3. Solve the equation $x^4 - 10x^2 + 9 = 0$.

$$x^4 - 10x^2 + 9 = 0,$$
$$\therefore\ (x^2 - 9)(x^2 - 1) = 0,$$
$$\therefore\ (x - 3)(x + 3)(x - 1)(x + 1) = 0,$$
$$\therefore \text{ the roots are } 3,\ -3,\ 1 \text{ and } -1.$$

Example 4. Solve the equation $x^2 + \frac{18}{x^2} = 11$.

$$x^2 + \frac{18}{x^2} = 11,$$

$$\therefore\ x^4 - 11x^2 + 18 = 0,$$

$$\therefore\ (x^2 - 9)(x^2 - 2) = 0,$$

$$\therefore\ (x - 3)(x + 3)(x - \sqrt{2})(x + \sqrt{2}) = 0,$$

$\therefore$ the roots are 3, -3, $\sqrt{2}$ and $-\sqrt{2}$.

Example 5. Solve the equation $(x^2 - 4x)^2 - 17(x^2 - 4x) - 84 = 0$.

$$(x^2 - 4x)^2 - 17(x^2 - 4x) - 84 = 0,$$

$$\therefore\ (x^2 - 4x - 21)(x^2 - 4x + 4) = 0,$$

$$\therefore\ (x - 7)(x + 3)(x - 2)^2 = 0,$$

$\therefore$ the roots are 7, -3, 2 and 2

Example 6. Solve the equation $\left(x + \frac{6}{x}\right)^2 - 2\left(x + \frac{6}{x}\right) = 35$.

$$\left(x + \frac{6}{x}\right)^2 - 2\left(x + \frac{6}{x}\right) - 35 = 0,$$

$$\therefore\ \left(x + \frac{6}{x} - 7\right)\left(x + \frac{6}{x} + 5\right) = 0,$$

$$\therefore\ (x^2 - 7x + 6)(x^2 + 5x + 6) = 0,$$

$$\therefore\ (x - 6)(x - 1)(x + 2)(x + 3) = 0,$$

$\therefore$ the roots are 6, 1, -2 and -3.

171. The solution of equations like the preceding may sometimes be simplified by making use of a subsidiary unknown quantity, as in the following examples.

Example 7. Solve the equation

$$\left(x + \frac{6}{x}\right)^2 - 2\left(x + \frac{6}{x}\right) = 35.$$

Put $x + \frac{6}{x} = y$, so that the equation becomes

$$y^2 - 2y - 35 = 0,$$

or $$(y-7)(y+5) = 0,$$

the roots of which are 7 and -5.

(i) $$x + \frac{6}{x} = 7,$$

$$\therefore\ x^2 - 7x + 6 = 0,$$

$$\therefore\ (x-6)(x-1) = 0,$$

the roots of which are 6 and 1.

(ii) $$x + \frac{6}{x} = -5,$$

$$\therefore\ x^2 + 5x + 6 = 0,$$

$$\therefore\ (x+2)(x+3) = 0,$$

the roots of which are -2 and -3.

$\therefore$ the roots of the given equation are 6, 1, -2 and -3.

Example 8. Solve the equation

$$\frac{x^2}{6x-9} + \frac{6x-9}{x^2} = 2.$$

Put $$\frac{x^2}{6x-9} = y,$$

so that $$\frac{6x-9}{x^2} = \frac{1}{y},$$

and the equation becomes

$$y + \frac{1}{y} - 2 = 0,$$

or $$y^2 - 2y + 1 = 0,$$

or $$(y-1)^2 = 0,$$

the roots of which are 1 and 1.

(i) $$\frac{x^2}{6x-9} = 1,$$

$$\therefore\ x^2 = 6x - 9,$$

$$\therefore\ x^2 - 6x + 9 = 0,$$

$$\therefore\ (x-3)^2 = 0,$$

the roots of which are 3 and 3.

(ii) the second root of y leads to the same results,

$\therefore$ the roots of the given equation are 3, 3, 3 and 3.

Example 9. Solve the equation

$$x^2 + \frac{1}{x^2} - x + \frac{1}{x} = 4.$$

Put $$x + \frac{1}{x} = y,$$

so that $$x^2 + 2 + \frac{1}{x^2} = y^2,$$

and therefore $$x^2 + \frac{1}{x^2} = y^2 - 2,$$

and the equation becomes

$$y^2 - 2 - y - 4 = 0,$$

or $$y^2 - y - 6 = 0,$$

or $$(y-3)(y+2) = 0,$$

the roots of which are 3 and -2.

(i) $$x + \frac{1}{x} = 3,$$

$$\therefore\ x^2 - 3x + 1 = 0,$$

the roots of which are

$$\frac{3 \pm \sqrt{9-4}}{2} \text{ or } \frac{3 \pm \sqrt{5}}{2}.$$

(ii) $$x + \frac{1}{x} = -2,$$

$$\therefore\ x^2 + 2x + 1 = 0,$$

$$\therefore\ (x + 1)^2 = 0,$$

the roots of which are -1 and -1.

$\therefore$ the roots of the given equation are

$$-1, -1,\ \tfrac{1}{2}(3 + \sqrt{5}) \text{ and } \tfrac{1}{2}(3 - \sqrt{5}).$$

Exercises 137.

Solve the equations:

1. $x^3 - 5x^2 + 4x = 0.$
2. $x^3 - 6x^2 + 9x = 0.$
3. $x^3 + 7x^2 + 12x = 0.$
4. $x^4 = 4x^2.$
5. $3x^3 + 5x^2 = 0.$
6. $x^4 - 10x^3 + 25x^2 = 0.$
7. $x^3 - x^2 - x + 1 = 0.$
8. $x^3 - 2 = x - 2x^2.$
9. $4x^3 + 8x^2 - x - 2 = 0.$
10. $2x^3 + 3x^2 - 8x - 12 = 0.$
11. $x^4 - 5x^2 + 4 = 0.$
12. $x^4 - 17x^2 + 16 = 0.$
13. $x^4 - 6x^2 + 8 = 0.$
14. $x^4 - 6x^2 + 5 = 0.$
15. $x^2 + \dfrac{36}{x^2} = 13.$
16. $x^2 + \dfrac{2}{x^2} = 3.$
17. $\dfrac{x^2 - 3}{x} = \dfrac{4x}{x^2 - 3}.$
18. $\dfrac{x^2 - 1}{3} + \dfrac{1}{x^2} = 1.$
19. $(x^2 + 2x)^2 - 15(x^2 + 2x) = 0.$
20. $(x^2 + 2x)^2 - 8(x^2 + 2x) = 0.$
21. $(x^2 - x)^2 - 8(x^2 - x) + 12 = 0.$
22. $(x^2 - 2x)^2 - 14(x^2 - 2x) - 15 = 0.$
23. $(x^2 + x + 1)^2 - 4(x^2 + x - 1) - 5 = 0.$
24. $(x^2 - x)(x^2 - x - 14) + 24 = 0.$
25. $(x^2 + 3x)(x^2 + 3x - 14) + 40 = 0.$

26. $\left(x-\frac{2}{x}\right)^2=1.$

27. $\left(x+\frac{4}{x}\right)^2-9\left(x+\frac{4}{x}\right)+20=0.$

28. $\left(x-\frac{6}{x}\right)^2-4\left(x-\frac{6}{x}\right)=5.$

29. $x^4-a^2x^2-x^2+a^2=0.$

30. $x^4+a^2b^2=a^2x^2+b^2x^2.$

31. $x^2+\frac{1}{x^2}=a^2+\frac{1}{a^2}.$

32. $x^3-5x^2+5x-1=0.$

33. $(x+3)(x-4)(x-5)=60.$

34. $6x^3+(5-x)^3=5(5+x)(5+2x).$

35. $\frac{x^2}{2x-1}+\frac{2x-1}{x^2}=2.$

36. $\frac{x^2+3x+1}{x}+\frac{5x}{x^2+3x+1}=6.$

37. $\frac{2x-1}{(x-1)^2}-\frac{2x+1}{(x+1)^2}=4.$

38. $x^2+\frac{1}{x^2}-4\left(x+\frac{1}{x}\right)+6=0.$

39. $x^2+\frac{1}{x^2}-2\left(x-\frac{1}{x}\right)-2=0.$

40. $x^2+\frac{1}{x^2}-5\left(x+\frac{1}{x}\right)+8=0.$

41. $x(x+3)(x+4)(x+7)+36=0.$

42. $(x+1)(x+2)(x+4)(x+5)=4.$

43. $(2x-1)(x-1)(2x-7)(x-3)=9.$

44. $16(x+1)(x+2)(x+3)(x+4)=9.$

CHAPTER XVIII.

QUADRATIC EQUATIONS: PROBLEMS.

172. The problems worked below involve the solution of quadratic equations, each of which has two roots. The method of translating the wording of the problem into an equation is the same as in the problems which lead to simple equations, but the meaning of the double solution, or the inapplicability of one solution, should be noticed in every case.

Example 1. Divide 12 into two parts so that their product may be 32.

Let x be one part and $12-x$ the other,

then
$$x(12-x)=32,$$
$$\therefore\ x^2-12x+32=0,$$
$$\therefore\ (x-4)(x-8)=0,$$
the roots of which are 4 and 8.

If $x=4$, then $12-x=8$, and if $x=8$, then $12-x=4$.

$\therefore$ both values of x lead to the same solution of the problem, namely, that the parts are 4 and 8.

Example 2. One number exceeds another by 4 and their product is 32; find the numbers.

Let x and $x+4$ be the numbers,

$$\therefore\ x(x+4)=32,$$

$$\therefore\ x^2+4x-32=0,$$

$$\therefore\ (x-4)(x+8)=0,$$

the roots of which are 4 and -8.

If $x=4$, then $x+4=8$, and if $x=-8$, then $x+4=-4$.

$\therefore$ if negative solutions be permitted, the problem admits of two solutions, the numbers being either 4 and 8, or -8 and -4.

Example 3. One side of a rectangle is 3 ins. longer than the other, and the area is 270 sq. ins.; find the dimensions of the rectangle.

Let $x+3$ ins. be the length, and x ins. the width, of the rectangle, then

$$(x+3)x=270,$$

$$\therefore\ x^2+3x-270=0,$$

$$\therefore\ (x-15)(x+18)=0,$$

the roots of which are 15 and -18.

If $x=15$, then $x+3=18$, and if $x=-18$, then $x+3=-15$.

The latter pair of values satisfy the equation which is the statement of the problem, but they are inapplicable to the problem itself. The reason of the appearance of this extraneous solution is that the equation is more general than the problem, the equation stating that two numbers differing by 3 have their product equal to 270.

The required dimensions of the rectangle are therefore 18 ins. and 15 ins.

Example 4. A man, riding at a uniform rate between two places 60 miles apart, finds that, if he had ridden two miles an hour faster, he would have accomplished the journey in one hour less; at what rate does he ride?

Let the man's speed be x miles an hour, then the time he takes to ride 60 miles is $\frac{60}{x}$ hrs. If his speed were $x + 2$ miles an hour, the time would be $\frac{60}{x+2}$ hrs. But, this being one hour less than the other, the equation corresponding to the problem is

$$\frac{60}{x} = \frac{60}{x+2} + 1,$$

$$\therefore\ 60x + 120 = 60x + x^2 + 2x,$$

$$\therefore\ x^2 + 2x - 120 = 0,$$

$$\therefore\ (x-10)(x+12) = 0,$$

the roots of which are 10 and -12.

∴ he rides at 10 miles an hour, the negative solution being inapplicable.

Example 5. A poulterer bought a certain number of turkeys for £5; four of them having died, he sold each of the remaining turkeys for 1*s.* 6*d.* more than he gave for it, thereby gaining 4*s.* on the whole; how many turkeys did he buy?

Let x be the number of turkeys, so that the cost price of each was $\frac{100}{x}$ shillings.

He sold $x - 4$ turkeys for 104 shillings.

∴ the selling price of each was $\frac{104}{x-4}$ shillings, which is 1*s.* 6*d.* more than the cost price.

$$\therefore\ \frac{104}{x-4} = \frac{100}{x} + \frac{3}{2},$$

$$\therefore\ 208x = 200x - 800 + 3x(x-4),$$

$$\therefore\ 3x^2 - 20x - 800 = 0,$$

$$\therefore\ (x-20)(3x+40) = 0,$$

the roots of which are 20 and $-13\frac{1}{3}$.

∴ the number of turkeys bought was 20, the negative and fractional root being inapplicable.

Example 6. Find the price of photographs a dozen when, if two more be given for £1, the price is lowered 6*s*. a dozen.

Let x*s*. be the original price of 12 photographs and $x-6$*s*. the price when lowered.

∴ the cost of 1 photograph is $\frac{x}{12}$*s*. in the first case and $\frac{x-6}{12}$*s*. in the second case.

∴ the number that can be bought for 20*s*. is $\frac{20}{\frac{x}{12}}$ or $\frac{240}{x}$ in the first case and $\frac{20}{\frac{x-6}{12}}$ or $\frac{240}{x-6}$ in the second case.

But the latter number is 2 more than the former,

$$\therefore \frac{240}{x-6} = \frac{240}{x} + 2,$$

$$\therefore 240x = 240x - 1440 + 2x(x-6),$$

$$\therefore x^2 - 6x - 720 = 0,$$

$$\therefore (x-30)(x+24) = 0,$$

the roots of which are 30 and -24.

∴ the original price of the photographs was 30*s*. a dozen, the negative root being inapplicable.

Example 7. Find the number which exceeds its square root by 42.

Let x be the number, then

$$x - \sqrt{x} = 42,$$

$$\therefore x - \sqrt{x} - 42 = 0,$$

$$\therefore (\sqrt{x} - 7)(\sqrt{x} + 6) = 0,$$

$$\therefore \sqrt{x} = 7 \text{ or } -6,$$

and $$x = 49 \text{ or } 36.$$

∴ the number is either 49 or 36, the former of which exceeds its positive root by 42, and the latter its negative root by 42.

Exercises 138.

1. Two numbers are such that one is twice the other and their product is 288; find the numbers.

2. Find the number which is equal to 16 times its reciprocal.

3. What number is $\frac{9}{16}$ of its reciprocal?

4. The sum of a number and of 6 times its reciprocal is equal to 5; find the number.

5. The sum of the squares of two consecutive numbers is 41; find the numbers.

6. Find two numbers, one of which is three-fifths of the other, so that the difference of their squares may be 16.

7. There are three numbers, the second of which is twice the first and the third twice the second; if the sum of their squares be 525, find the numbers.

8. Find two numbers differing by 6, the product of which is 160.

9. The difference between the cubes of two consecutive even numbers is 488; find the numbers.

10. Divide 60 into two parts so that their product may be 864.

11. A number consisting of three digits, each digit being the same, is 37 times the square of any digit; find the number.

12. The sum of the squares of three consecutive odd numbers is 5555; find the numbers.

13. The length of a room exceeds its width by 2 ft.; if the area of the floor be 168 sq. ft., find the dimensions of the room.

14. The sides of a rectangle are 20 and 12 feet; what is the width of the border which must be added all round so that the whole area may be 384 sq. ft.?

15. The glass of a picture-frame is 18 ins. long and 12 ins. wide; if the area of the frame be equal to that of the glass, find the width of the frame.

16. The hypotenuse of a right-angled triangle is 91 ft. long, and the longer of the other two sides exceeds the shorter by 49 ft.; find the sides.

17. The number of eggs which can be bought for 1*s.* is equal to the number of pence which 27 eggs cost; how many eggs can be bought for 1*s.*?

18. The number of fives-balls which can be bought for £1 is equal to the number of shillings in the cost of 125 balls; how many can be bought for £1?

19. A certain number consists of two digits; the tens' digit is double of the units' digit, and if the digits be inverted, the product of the number thus formed and the original number is 2268; find the number.

20. "The square root of half the number of a swarm of bees is gone to a shrub of jasmin, and so are eight-ninths of the whole swarm; a female is buzzing to one remaining male, that is humming within a lotus, in which he is confined, having been allured to it by its fragrance at night. Say, lovely woman, the number of bees."

21. A man travelled 105 miles, and then found that, if he had not travelled so fast by two miles an hour, he would have been six hours longer in performing the same journey; find his rate of travelling.

22. The forewheel of a carriage makes six revolutions more than the hind-wheel in 120 yards, and the circumference of the one is a yard less than that of the other; find the circumference of each.

23. If the price of coals were to fall 3*s.* per ton, £17 would purchase 3 tons more than at present; what is the present price per ton?

24. A walks faster than B by one mile an hour; they start together for a town distant 28 miles; if A arrive 2 hrs. 20 mins. before B, at what rates do they walk?

25. Two men, A and B, travel along a road 180 miles long in opposite directions, starting simultaneously from the ends of the road; A travels 6 miles a day more than B, and the number of miles travelled each day by B is equal to double the number of days before they meet; find the number of miles which each travels in one day.

26. A person rents some land for £48; he cultivates 8 acres himself, and subletting the rest for 15*s.* per acre more than he pays, receives in rent £54 a year; find the number of acres.

27. A man buys a certain number of acres of land for £1600; by selling it at £92 an acre, he gains as much as he gave for 3 acres; how many acres did he buy?

28. The price of photographs is raised 3*s.* per dozen, and customers consequently receive seven less than before for a guinea; what are the prices charged?

29. What is the price of eggs a dozen when two less in a shilling's worth raises the price 1*d.* a dozen?

30. Find the price of eggs per score when ten more in half-a-crown's worth lowers the price 1*s.* 3*d.* per hundred.

31. Find two consecutive numbers such that the difference of their squares is 9 more than six times the square root of the smaller.

32. The sum of the square and the cube of a certain number is equal to nine times the next higher number; what is the number?

33. A manufacturer paid £640 to 36 employés, some men and some women; each man received as many pounds as there were women, and each woman as many pounds as there were men; how many men and how many women were there?

34. A traveller starts from A towards B at 12 o'clock, and another starts at the same time from B towards A; they meet at 2 o'clock, at 24 miles from A, and the one arrives at A while the other is still 20 miles from B; what is the distance between A and B?

MISCELLANEOUS EXERCISES.

Exercises 139.

1. Divide the product of $x^2 - 5xy + 6y^2$ and $x - 4y$ by $x - 2y$.

2. Two men receive the same sum; if one were to receive 15*s*. more and the other 9*s*. less, the former would receive three times as much as the other; what sum did each receive?

3. Find the H.C.F. of $9x^5 + 5x - 2$ and $27x^5 - 45x^4 - 16$.

4. Simplify

$$\frac{3a}{2a+6b} + \frac{2a}{3a-9b} - \frac{2a^2-3ab}{a^2-9b^2}.$$

5. Simplify

$$\frac{a^2(b+c) + b^2(c+a) + c^2(a+b) + 2abc}{a+b}.$$

6. Solve the equations

$$\frac{3}{x} - \frac{1}{y} = 2, \quad \frac{4}{x} + \frac{6}{y} = 10.$$

7. Find a fraction such that, when its numerator is increased by 8, the value of the fraction becomes 2, and if the denominator be doubled its value becomes $\frac{3}{7}$.

8. Find the square root of

$$x^6 + \frac{1}{x^6} - 2\left(x^3 - \frac{1}{x^3}\right) - 1.$$

9. Solve the equation

$$\frac{3}{x-5} + \frac{2x}{x-3} = 5.$$

10. Solve the equation

$$3a + 2x^2 = 3ax + 2.$$

11. Solve the equation

$$x^4 - 29x^2 + 100 = 0.$$

12. If the price of wine be raised 3*s.* per dozen, three bottles less are bought for £5. 17*s.* than before; what was the original price per dozen?

Exercises 140.

1. A is three times as old as B; in 12 years he will be only twice as old; when was B born?

2. Find the factors of

$$a^2 - 3a - b^2 + 3b.$$

3. Find the H.C.F. of

$$x^5 - 3x^4 - 2x^3 - 6x^2 - 3x - 3 \text{ and } 3x^4 + x^2 - 2.$$

4. Simplify

$$\frac{9}{x^2 - x - 20} - \frac{7}{x^2 + x - 12} - \frac{2}{x^2 - 8x + 15}.$$

5. Simplify

$$(yz + zx + xy)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) - xyz\left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right).$$

6. Solve the equations

$$\frac{4}{x} + \frac{9}{y} = 5, \quad \frac{5}{x} + \frac{15}{y} = \frac{15}{2}$$

7. Solve the equations

$$\frac{x}{a} + \frac{y}{b} = c, \quad \frac{x}{b} + \frac{y}{a} = -c.$$

8. Find the square root of

$$x^6 + \frac{1}{x^6} - 6\left(x^4 + \frac{1}{x^4}\right) + 15\left(x^2 + \frac{1}{x^2}\right) - 20.$$

9. Solve the equation

$$\frac{1}{x}+\frac{1}{x+4}=\frac{1}{2}.$$

10. Solve the equation

$$\frac{a+1}{x}-\frac{x+1}{a}+a+1=0.$$

11. Solve the equation $x^4-2x^3-15x^2=0$.

12. A bicyclist, having ridden 72 miles and stopped an hour on the way, finds that, if he had ridden faster by one mile an hour, and stopped two hours on the way, he would have accomplished the journey in the same time. At what rate did he ride?

Exercises 141.

Solve the equations:

1. $$\frac{5x-7}{3}-\frac{4x-9}{5}+3x=18\tfrac{4}{5}.$$

2. $$\frac{x+y}{3}-2y=2,\quad \frac{2x-4y}{5}+y=\frac{23}{5}.$$

3. $3x+y-z=0,\quad 4x+3y+z=15,\quad 2x+4y+3z=25.$

4. $$x+y=\frac{b^2+c^2}{abc},\quad bx+cy=\frac{b^3+c^3}{abc}.$$

5. $$\frac{x+10}{x-5}-\frac{10}{x}=\frac{11}{6}.$$

6. $$(b+1)x^2-(b^2+b+1)x+b=0.$$

7. $$\frac{15}{x+1}-\frac{8}{x-2}=\frac{3}{x-5}.$$

8. $$x^3+4x^2-32x=0.$$

9. $$x^4-12x^2+27=0.$$

10. $$(x^2+4x)^2-17(x^2+4x)+60=0.$$

11. $$b^2x^2+\frac{a^2}{x^2}=a^2b^2+1.$$

12. $$(x-4)^3+(x-5)^3=31\{(x-4)^2-(x-5)^2\}.$$

CHAPTER XIX.

GRAPHS OF QUADRATIC EQUATIONS.

173. The method of drawing the graphs of simple equations has been explained in Chapter VIII. In the more complicated graphs which correspond to quadratic equations the same general method is followed, i.e. a number of points are found the co-ordinates of which satisfy the given equation. In particular cases, however, the work may be shortened, as shown in the following examples.

Example 1. Draw the graph of the equation

$$x^2 = 4y.$$

The term containing y being of the first degree, the equation is written in the form

$$y = \frac{1}{4}x^2.$$

It will be noticed that:

(i) Whether x be positive or negative, $\frac{1}{4}x^2$ is positive, and therefore y is always positive, i.e. the curve lies entirely above the axis of x;

(ii) If x be given values which are equal in magnitude but opposite in sign, the value of y is the same; e.g. $y = 4$ when $x = +4$ and when $x = -4$;

(iii) To an infinitely great value of x (positive or negative) there corresponds an infinitely great value of y, i.e. the graph is unlimited in two directions.

Giving different integral values to x from 0 to 7 (say), we have the following pairs of values of x and y:

x	0	1	2	3	4	5	6	7 ...
y	0	·2	1	2·2	4	6·2	9	12·2 ...

and the same values of y for corresponding negative values of x.

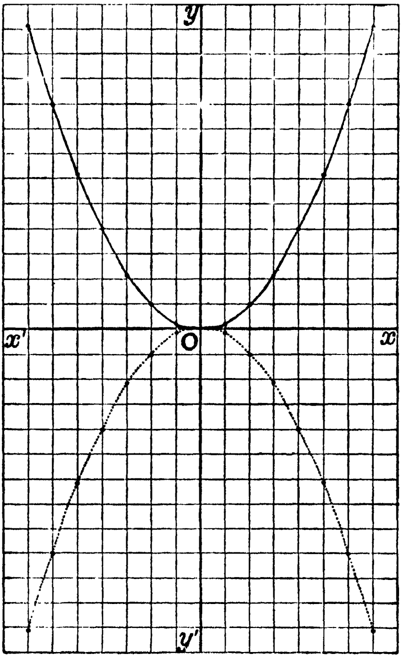


Fig. 10.

From these values of the coordinates we obtain the graph represented by the continuous curve in fig. 10, a curve which is called a *parabola*.

Example 2. Draw the graph of the equation $x^2 = -4y$.

The equation being written in the form $y = -\frac{1}{4}x^2$, it will be noticed that:

(i) Whether x be positive or negative, $-\frac{1}{4}x^2$ or y is always negative, i.e. the curve lies entirely below the axis of x;

(ii) If x be given values which are equal in magnitude but opposite in sign, the value of y is the same;

(iii) The graph is unlimited in two directions. The graph is represented by the dotted curve in fig. 10.

Exercises 142.

Draw the graphs of the following equations:

1. $x^2 = 3y$ and $x^2 = -3y$.
2. $y^2 = 5x$ and $y^2 = -5x$.
3. $y = x^2 - 8x + 12$, $y = x^2 - 8x + 16$, and $y = x^2 - 8x + 20$.
4. $y = x^2 - 4x + 8$ and $y = 6x - x^2 - 4$.
5. $5y = x^2 - 8x + 12$.

174. In the following examples the graphs are confined within finite limits in all directions, i.e. they are closed curves, such as the circle (Example 3) or ellipse (Example 4).

Example 3. Draw the graph of the equation $x^2 + y^2 = 100$.

The equation may be written

$$y^2 = 100 - x^2 \text{ or } x^2 = 100 - y^2,$$

or

$$y = \pm\sqrt{(100 - x^2)} \text{ or } x = \pm\sqrt{(100 - y^2)}.$$

It will be noticed that:

(i) Since $100 - x^2$ must be positive, x cannot be less than -10 or greater than $+10$, and similarly y cannot be less than -10 or greater than $+10$;

(ii) If x be given values, which are equal in magnitude but opposite in sign, the values of y are the same; e.g. $y = \pm 6$ when $x = 8$ and when $x = -8$.

Giving different integral values to x from $x = 0$ to $x = 10$, we have the following pairs of values of x and y:

x	0	1	2	3	4	5
y	± 10	$\pm 9{\cdot}9$	$\pm 9{\cdot}8$	$\pm 9{\cdot}5$	$\pm 9{\cdot}2$	$\pm 8{\cdot}7$
x	6	7	8	9	10	
y	$\pm 8{\cdot}0$	$\pm 7{\cdot}1$	$\pm 6{\cdot}0$	$\pm 4{\cdot}4$	0	

and the same values of y for corresponding negative values of x.

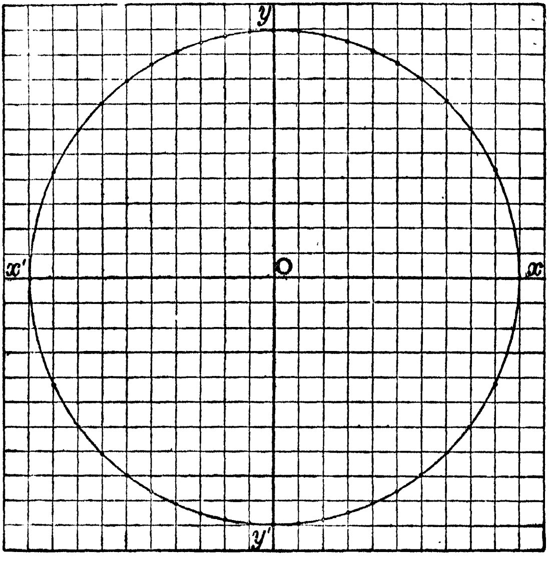


Fig. 11.

From these values of the coordinates we obtain the graph represented in fig. 11, which is obviously a circle.

Example 4. Draw the graph of the equation

$$9x^2 + 16y^2 = 576.$$

The equation may be written

$$y^2 = \frac{576 - 9x^2}{16} \quad \text{or} \quad x^2 = \frac{576 - 16y^2}{9},$$

or $$y = \pm \tfrac{3}{4}\sqrt{(64 - x^2)} \quad \text{or} \quad x = \pm \tfrac{4}{3}\sqrt{(36 - y^2)}.$$

It will be noticed that:

(i) Since $64 - x^2$ and $36 - y^2$ must be positive, x is not less than -8 and not greater than $+8$, while y is not less than -6 and not greater than $+6$.

(ii) If x be given values, which are equal in magnitude but opposite in sign, the values of y are the same; e.g. $y = \pm \frac{3}{4}\sqrt{48}$ when $x = +4$ and when $x = -4$.

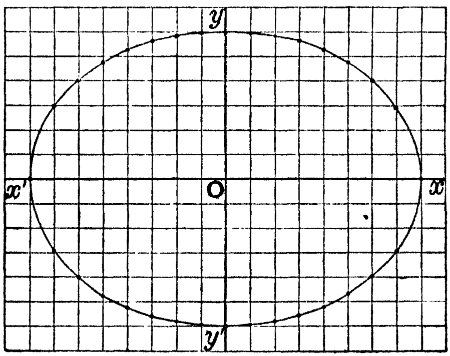


Fig. 12.

Giving different integral values to x from $x = 0$ to $x = 8$, we have the following pairs of values of x and y:

x	0	1	2	3	4	5	6	7	8
y	± 6	$\pm 5{\cdot}9$	$\pm 5{\cdot}8$	$\pm 5{\cdot}6$	$\pm 5{\cdot}2$	$\pm 4{\cdot}7$	$\pm 4{\cdot}0$	$\pm 2{\cdot}9$	0

and the same values of y for corresponding negative values of x.

From these values of the coordinates we obtain the graph represented in fig. 12, a curve which is called an *ellipse*.

Example 5. Draw the graph of the equation

$$x^2 + xy + y^2 = 48.$$

The equation may be written

$$y^2 + xy + x^2 - 48 = 0 \quad \text{or} \quad x^2 + xy + y^2 - 48 = 0,$$

$$\text{or} \quad y = \frac{-x \pm \sqrt{\{x^2 - 4(x^2 - 48)\}}}{2} \quad \text{or} \quad x = \frac{-y \pm \sqrt{\{y^2 - 4(y^2 - 48)\}}}{2},$$

$$\text{or} \quad y = \frac{-x \pm \sqrt{\{3(64 - x^2)\}}}{2} \quad \text{or} \quad x = \frac{-y \pm \sqrt{\{3(64 - y^2)\}}}{2}.$$

It will be noticed that:

Since $64 - x^2$ and $64 - y^2$ must be positive, x is not less than -8 and not greater than $+8$, while y is not less than -8 and not greater than $+8$.

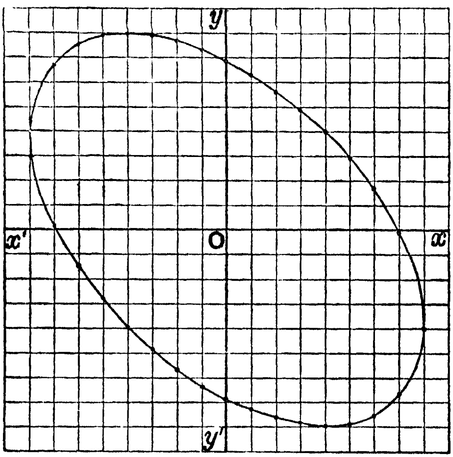


Fig. 13.

Giving different integral values to x from $x = 0$ to $x = 8$, we have the following pairs of values of x and $\sqrt{\{3(64 - x^2)\}}$:

x	0	1	2	3	4	5	6	7	8
$\sqrt{\{3(64 - x^2)\}}$	13·9	13·7	13·4	12·9	12·0	10·8	9·2	6·5	0

and therefore the following pairs of values of x and y:

x	0	1	2	3	4	5	6	7	8
y	$\pm 6{\cdot}9$	$+6{\cdot}3$	$+5{\cdot}7$	$+4{\cdot}9$	$+4{\cdot}0$	$+2{\cdot}9$	$+1{\cdot}6$	$-0{\cdot}2$	-4
		$-7{\cdot}3$	$-7{\cdot}7$	$-7{\cdot}9$	$-8{\cdot}0$	$-7{\cdot}9$	$-7{\cdot}6$	$-6{\cdot}7$	-4

and also

x	-1	-2	-3	-4	-5	-6	-7	-8
y	$+7{\cdot}3$	$+7{\cdot}7$	$+7{\cdot}9$	$+8{\cdot}0$	$+7{\cdot}9$	$+7{\cdot}6$	$+6{\cdot}7$	$+4$
	$-6{\cdot}3$	$-5{\cdot}7$	$-4{\cdot}9$	$-4{\cdot}0$	$-2{\cdot}9$	$-1{\cdot}6$	$+0{\cdot}2$	$+4$

From these values of the coordinates, we obtain the graph represented in fig. 13, an ellipse with its greatest length inclined to the axis of x.

Exercises 143.

Draw the graphs of the following equations:

1. $x^2 + y^2 = 64.$
2. $x^2 + y^2 = 36.$
3. $x^2 + y^2 - 14x = 0.$
4. $16x^2 + 25y^2 = 1600.$
5. $36x^2 + 25y^2 = 900.$
6. $x^2 + xy + y^2 = 7.$
7. $x^2 - xy + y^2 = 3.$

175. *Example* 6. Draw the graph of the equation $xy = 5$.

It will be noticed that:

(i) Since the product of x and y is positive, x and y must be of the same sign; i.e. the graph is confined to the right angles in which x and y are both positive and both negative;

(ii) Writing the equation in the form

$$y = \frac{5}{x} \text{ or } x = \frac{5}{y},$$

as x increases numerically, y decreases numerically, and, when x is indefinitely great, y is indefinitely small; also, when y increases numerically, x decreases numerically, and, when y is indefinitely great, x is indefinitely small; i.e. in both directions the graph approaches indefinitely closely to the axes of x and y.

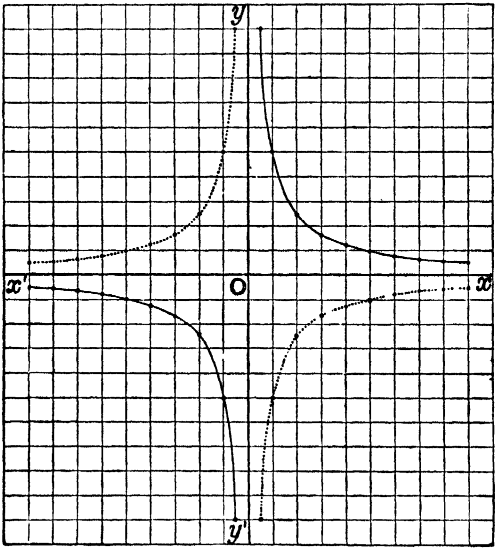


Fig. 14.

Giving x the values ·5, 1, 2, 3, ..., 10, we have the following pairs of values of x and y:

x	·5	1	2	3	4	5	6	7	8	9	10
y	10	5	2·5	1·7	1·2	1·0	0·8	0·7	0·6	0·6	0·5

and the same values of y with their signs changed for corresponding negative values of x.

From these values of the coordinates, we obtain the graph represented by the continuous curve in fig. 14, a curve which is called a *hyperbola*.

Example 7. Draw the graph of the equation $xy = -5$.

It will be noticed that ·

(i) Since the product of x and y is negative, x and y must be of different signs, i.e. the graph is confined to the right angles in which x is positive and y negative, and in which x is negative and y positive;

(ii) In both directions, the curve approaches indefinitely closely to the axes of x and y.

Giving x the values ·5, 1, 2, 3, ..., 10, we have the following pairs of values of x and y:

x	·5	1	2	3	4	5
y	−10	−5	−2·5	−1·7	−1·2	−1·0
x	6	7	8	9	10	
y	−0·8	−0·7	−0·6	−0·6	−0·5	

and the same values of y with their signs changed for corresponding negative values of x.

From these values of the coordinates, we obtain the graph represented by the dotted curve in fig. 14.

176. Def. 25. If, as either coordinate increases indefinitely in numerical value, the graph approach indefinitely closely to a certain line, the line is called an *asymptote* of the graph.

Thus, in Examples 6 and 7 (fig. 14), the axes of x and y are asymptotes.

177. *Example* 8. Draw the graph of the equation

$$x^2 - y^2 = 5.$$

Writing the equation in the form

$$y^2 = x^2 - 5 \quad \text{or} \quad x^2 = y^2 + 5,$$

or $$y = \pm\sqrt{(x^2 - 5)} \quad \text{or} \quad x = \pm\sqrt{(y^2 + 5)},$$

it will be noticed that:

(i) Since $x^2 - 5$ must be positive, x cannot have any value lying between $-\sqrt{5}$ and $+\sqrt{5}$, i.e. between $-2\cdot2$ and $+2\cdot2$; and, since $y^2 + 5$ must be positive, y can have any value whatever, either positive or negative;

(ii) If x be given values, which are equal in magnitude but opposite in sign, the values of y are the same; e.g. $y = \pm 2$, when $x = +3$ or when $x = -3$;

(iii) When x is indefinitely great, y, and therefore also $x + y$, is indefinitely great, and therefore, since

$$(x - y)(x + y) = 5,$$

$x - y$ is indefinitely small.

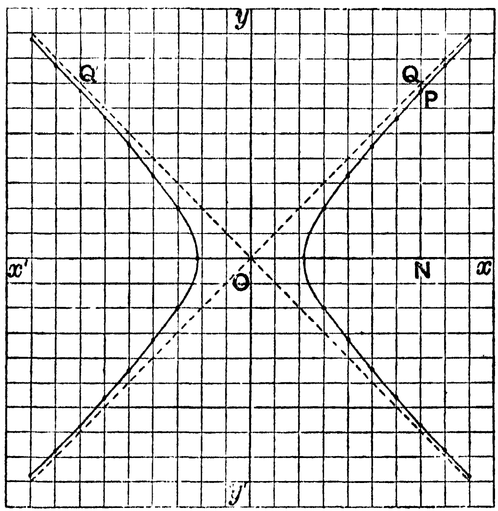


Fig. 15.

Giving different integral values to x from $x = 3$ to $x = 9$ in the equation $y = \pm\sqrt{(x^2 - 5)}$, we have the following pairs of values of x and y:

x	3	4	5	6	7	8	9 ...
y	± 2	$\pm 3\cdot3$	$\pm 4\cdot5$	$\pm 5\cdot6$	$\pm 6\cdot6$	$\pm 7\cdot7$	$\pm 8\cdot8$ ...

and the same values of y for corresponding negative values of x.

From these values of the coordinates, we obtain the graph represented in fig. 15, a curve which is again a hyperbola.

In fig. 15, let the broken lines OQ, OQ′ represent the graphs of $y = x$ and $y = -x$ respectively. From any point N in Ox draw NPQ perpendicular to Ox, cutting the graph of the equation $x^2 - y^2 = 5$ in P and that of the equation $y = x$ in Q. Then, if $\text{ON} = x$, we have $\text{PN} = y$, $\text{QN} = x$, and therefore $\text{PQ} = x - y$. Thus, when x increases indefinitely, PQ decreases indefinitely, i.e. the graph of the equation $x^2 - y^2 = 5$ approaches indefinitely closely to the graph of the equation $y = x$. Similarly, it may be shown that it approaches indefinitely closely to the graph of the equation $y = -x$. Thus, the lines $y = x$ and $y = -x$ are asymptotes to the graph of the equation $x^2 - y^2 = 5$.

Exercises 144.

Draw the graphs of the following equations:

1. $xy = 12$.
2. $xy = 3$ and $xy = -3$.
3. $x^2 - y^2 = 6$ and $y^2 - x^2 = 6$.
4. $x^2 - xy = 2$.
5. $xy - y^2 = 1$.
6. $xy - 2y^2 = 2$.
7. $2x^2 - 3xy = 20$.

CHAPTER XX.

THEORY OF QUADRATIC EQUATIONS AND QUADRATIC EXPRESSIONS.

178. *The roots of the equation* $ax^2 + bx + c = 0$ *are real and different, real and equal, or imaginary and different, according as* b^2 *is greater than, equal to, or less than* $4ac$*

The roots of the equation $ax^2 + bx + c = 0$ are

$$\frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$$

If $b^2 > 4ac$, then $b^2 - 4ac$ is positive, and the roots are real and different.

If $b^2 = 4ac$, then $b^2 - 4ac$ is zero, and the roots are real and equal.

If $b^2 < 4ac$, then $b^2 - 4ac$ is negative, and the roots are imaginary and different.

179. DEF. 26. The expression $b^2 - 4ac$ is called the *discriminant* of the equation $ax^2 + bx + c = 0$.

Thus, the roots of the equation are real and different, real and equal, or imaginary and different, according as the discriminant is positive, zero, or negative.

* Before beginning this theorem, the reader should refer to the examples worked out in Art. 168.

180. *Example* 1. What is the nature of the roots of the equations

(i) $2x^2 + 9x + 3 = 0$, (ii) $4x^2 - 20x + 25 = 0$,

(iii) $3x^2 + 2x + 5 = 0$, (iv) $x^2 - 8x + m = 0$?

(i) The discriminant is $9^2 - 4 . 2 . 3$ or $81 - 24$ or 57,
∴ the roots are real and different.

(ii) The discriminant is $20^2 - 4 . 4 . 25$ or $400 - 400$ or 0,
∴ the roots are real and equal.

(iii) The discriminant is $2^2 - 4 . 3 . 5$ or $4 - 60$ or -56,
∴ the roots are imaginary and different.

(iv) The discriminant is $64 - 4m$ or $4(16 - m)$, which is positive if m be < 16, zero if $m = 16$, and negative if m be > 16: ∴ the roots are real and different, real and equal, or imaginary and different, according as m is less than, equal to, or greater than, 16.

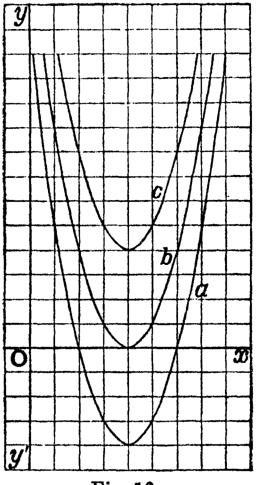


Fig. 16.

These cases are illustrated by the accompanying curves (fig. 16), which represent respectively the expressions (*a*) $x^2 - 8x + 12$, (*b*) $x^2 - 8x + 16$,

(*c*) $x^2 - 8x + 20$.

The first expression is equal to zero for the values 2 and 6, or the roots of the equation $x^2 - 8x + 12 = 0$ are 2 and 6, i.e. are real and different. The second is equal to zero for the values 4 and 4 of x, or the roots of the equation $x^2 - 8x + 16 = 0$ are 4 and 4, i.e. are real and equal. The third expression is never equal to zero, i.e. the roots of the equation $x^2 - 8x + 20 = 0$ are imaginary and different.

Example 2. For what values of m are the roots of the following equations equal?

(i) $2x^2 + 3x + m = 0$,

(ii) $4x^2 + (1 + m)x + 9 = 0$.

(i) The roots of the equation $2x^2 + 3x + m = 0$ are equal if

$$3^2 - 4 \cdot 2 \cdot m = 0,$$

i.e. if $$m = 1\tfrac{1}{8}.$$

(ii) The roots of the equation $4x^2 + (1 + m)x + 9 = 0$ are equal if

$$(1 + m)^2 - 4 \cdot 4 \cdot 9 = 0,$$

i.e. if $$m^2 + 2m - 143 = 0,$$

i.e. if $$(m - 11)(m + 13) = 0,$$

i.e. if $$m = 11 \text{ or } -13.$$

Another method of working these examples should be noticed. If the roots of the equation $2x^2 + 3x + m = 0$, or $x^2 + \frac{3x}{2} + \frac{m}{2} = 0$, be equal, the factors of $x^2 + \frac{3x}{2} + \frac{m}{2}$ must be equal, i.e. $x^2 + \frac{3x}{2} + \frac{m}{2}$ must be a square, and therefore $\frac{m}{2} = \left(\frac{3}{4}\right)^2$, or $\frac{m}{2} = \frac{9}{4}$, or $8m = 9$, or $m = 1\frac{1}{8}$.

Exercises 145.

Find the discriminants of the following equations:

1. $2x^2 + 5x + 1 = 0.$
2. $4x^2 - 12x + 9 = 0.$
3. $4x^2 - 3x + 2 = 0.$
4. $25x^2 + 30x + 9 = 0.$
5. $3x^2 + 2x - 4 = 0.$
6. $6x^2 - x + 5 = 0.$

Determine the nature of the roots of the following equations:

7. $x^2 + 6x + 4 = 0.$
8. $x^2 - 7x + 10 = 0.$
9. $x^2 + 8x + 16 = 0.$
10. $x^2 - 10x + 25 = 0.$
11. $x^2 - 5x + 7 = 0.$
12. $x^2 + 3x + 6 = 0.$
13. $3x^2 - 7x - 2 = 0.$
14. $4x^2 + 2x + 11 = 0.$
15. $2x^2 - 3x + 5 = 0.$
16. $4x^2 - 28x + 49 = 0.$
17. $9x^2 - 30x + 25 = 0.$
18. $2x^2 + x + 5 = 0.$
19. $3x^2 - 8x = 0.$
20. $6x^2 - 5x - 25 = 0.$

For what values of m are the roots of the following equations equal?

21. $x^2 + 6x + m = 0.$
22. $x^2 - 8x + m = 0.$
23. $x^2 + mx + 9 = 0.$
24. $x^3 + mx + 25 = 0.$
25. $mx^2 + 10x + 25 = 0.$
26. $mx^2 + 40x + 16 = 0.$
27. $3x^2 + 2x + m = 0.$
28. $4x^2 - x - 3m = 0.$
29. $mx^2 + 5x + 3 = 0.$
30. $2mx^2 + 3x - 1 = 0.$
31. $3mx^2 + 4x - 1 = 0.$
32. $x^2 + (2 - m)x + 25 = 0.$
33. $x^2 + (3 + m)x + 16 = 0.$
34. $4x^2 + (1 - m)x + 25 = 0.$
35. $9x^2 + (3 + 2m)x + 49 = 0.$

181. *The sum of the roots of the equation* $ax^2 + bx + c = 0$ *is equal to* $-\frac{b}{a}$, *and the product of the roots to* $\frac{c}{a}$.

The roots of the equation $ax^2 + bx + c = 0$ are

$$\frac{-b + \sqrt{(b^2 - 4ac)}}{2a} \text{ and } \frac{-b - \sqrt{(b^2 - 4ac)}}{2a}.$$

$$\therefore \text{ sum of roots} = \frac{-b + \sqrt{(b^2 - 4ac)} - b - \sqrt{(b^2 - 4ac)}}{2a}$$

$$= \frac{-2b}{2a}$$

$$= -\frac{b}{a};$$

$$\text{and product of roots} = \frac{\{-b + \sqrt{(b^2 - 4ac)}\}\{-b - \sqrt{(b^2 - 4ac)}\}}{4a^2}$$

$$= \frac{(-b)^2 - (b^2 - 4ac)}{4a^2}$$

$$= \frac{4ac}{4a^2}$$

$$= \frac{c}{a}.$$

Exercises 146. (*Vivâ Voce.*)

State the sum and product of the roots of the following equations:

1. $x^2 - 7x + 4 = 0$.
2. $x^2 - 5x + 6 = 0$.
3. $x^2 + 11x + 5 = 0$.
4. $x^2 + 8x + 7 = 0$.
5. $x^2 + 5x - 6 = 0$.
6. $x^2 + 4x - 8 = 0$.
7. $x^2 - 11x^2 - 7 = 0$.
8. $x^2 - 10x - 10 = 0$.
9. $2x^2 - 3x + 5 = 0$.
10. $3x^2 - 4x + 3 = 0$.
11. $2x^2 + 7x + 6 = 0$.
12. $3x^2 + 11x + 5 = 0$.
13. $7x^2 - 6x - 4 = 0$.
14. $4x^2 - 5x - 3 = 0$.
15. $5x^2 + x - 1 = 0$.
16. $2x^2 + 2x - 3 = 0$.
17. $6x^2 + 2x + 3 = 0$.
18. $4x^2 - 5x + 7 = 0$.
19. $8x^2 + 2x - 3 = 0$.
20. $10x^2 - 5x - 2 = 0$.

Given one of the roots of the following equations, state the other root:

21. $x^2 + mx + 20 = 0$, one root 4.
22. $x^2 + mx + 12 = 0$, ,, 3.
23. $x^2 + mx - 15 = 0$, ,, 5.
24. $x^2 + mx - 24 = 0$, ,, 2.
25. $x^2 + mx + 16 = 0$, ,, -8.
26. $x^2 + mx + 18 = 0$, ,, -3.
27. $x^2 + mx - 32 = 0$, ,, -8.
28. $x^2 + mx - 40 = 0$, ,, -5.
29. $x^2 - 8x + m = 0$, ,, 5.
30. $x^2 - 11x + m = 0$, ,, 8.
31. $x^2 + 7x + m = 0$, ,, 2.
32. $x^2 + 10x + m = 0$, ,, 4.
33. $x^2 - 12x + m = 0$, ,, -2.
34. $x^2 - 4x + m = 0$, ,, -1.
35. $x^2 + 5x + m = 0$, ,, -3.
36. $x^2 + 10x + m = 0$, ,, -7.

182. *Example* 3. Find the sum of the squares of the roots of the equation

$$2x^2 - 3x + 1 = 0.$$

Let α and β denote the roots of the equation $2x^2 - 3x + 1 = 0$, then

$$\alpha + \beta = \tfrac{3}{2} \text{ and } \alpha\beta = \tfrac{1}{2}.$$

Now
$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$
$$= \tfrac{9}{4} - 1$$
$$= 1\tfrac{1}{4}.$$

Example 4. Find the square of the difference of the roots of the equation

$$3x^2 + 4x - 2 = 0.$$

We have $\alpha + \beta = -\frac{4}{3}, \quad \alpha\beta = -\frac{2}{3}.$

Now
$$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$$
$$= \frac{16}{9} + \frac{8}{3}$$
$$= \frac{40}{9}$$
$$= 4\tfrac{4}{9}.$$

Example 5. Find the sum of the cubes of the roots of the equation
$$4x^2 - 2x - 5 = 0.$$

We have $\alpha + \beta = \frac{1}{2}, \quad \alpha\beta = -\frac{5}{4}.$

Now
$$\alpha^3 + \beta^3 = (\alpha + \beta)(\alpha^2 - \alpha\beta + \beta^2)$$
$$= (\alpha + \beta)\{(\alpha + \beta)^2 - 3\alpha\beta\}$$
$$= \tfrac{1}{2}(\tfrac{1}{4} + \tfrac{15}{4})$$
$$= 2.$$

Example 6. If α and β denote the roots of the equation
$$4x^2 - x + 2 = 0,$$
find the value of $(\alpha + 2\beta)(2\alpha + \beta).$

We have $\alpha + \beta = \frac{1}{4}, \quad \alpha\beta = \frac{1}{2}.$

Now,
$$(\alpha + 2\beta)(2\alpha + \beta) = 2\alpha^2 + 5\alpha\beta + 2\beta^2$$
$$= 2(\alpha^2 + 2\alpha\beta + \beta^2) + \alpha\beta$$
$$= 2(\alpha + \beta)^2 + \alpha\beta$$
$$= 2 \cdot \tfrac{1}{16} + \tfrac{1}{2}$$
$$= \tfrac{5}{8}.$$

Example 7. Show that 1 is a root of the equation
$$(b - c)x^2 + (c - a)x + a - b = 0,$$
and find the other root.

If $x = 1$, the expression
$$(b - c)x^2 + (c - a)x + a - b$$
is equal to $b - c + c - a + a - b$, or zero.

$\therefore$ 1 is a root of the equation
$$(b - c)x^2 + (c - a)x + a - b = 0.$$

Now, the product of the roots is equal to $\frac{a-b}{b-c}$.

$$\therefore \text{ the other root is } \frac{a-b}{b-c} \div 1, \text{ or } \frac{a-b}{b-c}.$$

Exercises 147.

1. If one root of the equation $x^2 + mx + 48 = 0$ be 6, find the other root and also m.

2. If one root of the equation $x^2 - 7x + m = 0$ be 5, find the other root and also m.

3. If one root of the equation $2x^2 - mx + 56 = 0$ be 8, find the other root and also m.

4. If one root of the equation $3x^2 + 5x - 2m = 0$ be $\frac{3}{4}$, find the other root and also m.

Find the sum of the squares of the roots of the following equations, and test the results of the first two by solving the equations and squaring the roots so found:

5. $x^2 - 4x + 4 = 0.$	6. $x^2 + 2x - 8 = 0.$
7. $2x^2 - 4x + 3 = 0.$	8. $2x^2 + 4x - 3 = 0.$
9. $2x^2 - 3x - 1 = 0.$	10. $3x^2 - 6x + 2 = 0.$
11. $3x^2 - 2x + 5 = 0.$	12. $4x^2 - 3x - 1 = 0.$
13. $5x^2 + 3x + 2 = 0.$	14. $5x^2 - 4x - 3 = 0.$

Find the square of the difference of the roots of the following equations, and test the results of the first two:

15. $x^2 - 3x + 2 = 0.$	16. $x^2 + 4x + 3 = 0.$
17. $2x^2 - 3x + 1 = 0.$	18. $2x^2 + 5x + 3 = 0.$
19. $2x^2 - 4x - 5 = 0.$	20. $3x^2 - 5x - 2 = 0.$
21. $3x^2 - 6x + 2 = 0.$	22. $4x^2 + 5x - 2 = 0.$
23. $5x^2 - 4x - 2 = 0.$	24. $7x^2 + x + 1 = 0.$

Find the sum of the cubes of the roots of the following equations, and test the results of the first two:

25. $x^2 - 4x + 3 = 0.$ 26. $x^2 + 7x + 6 = 0.$

27. $2x^2 - 4x + 3 = 0.$ 28. $2x^2 + 5x + 2 = 0.$

29. $2x^2 + 7x - 1 = 0.$ 30. $3x^2 - 7x + 2 = 0.$

31. $3x^2 - 5x - 1 = 0.$ 32. $5x^2 - x - 2 = 0.$

33. $4x^2 + 2x - 3 = 0.$ 34. $6x^2 - 2x - 5 = 0.$

If α and β be the roots of the following equations, find the value of:

35. $\frac{1}{\alpha} + \frac{1}{\beta}$, $x^2 + 5x + 4 = 0.$

36. $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$, $x^2 + 2x - 5 = 0.$

37. $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$, $3x^2 + 2x - 1 = 0.$

38. $\frac{1}{\alpha^3} + \frac{1}{\beta^3}$, $x^2 - 5x + 6 = 0.$

39. $(2\alpha + \beta)(\alpha + 2\beta)$, $2x^2 - x + 2 = 0.$

40. $(2\alpha + \beta)(\alpha + 2\beta)$, $2x^2 - 5x + 1 = 0.$

41. $(2\alpha - \beta)(\alpha - 2\beta)$, $3x^2 + 2x - 1 = 0.$

42. $(2\alpha - \beta)(\alpha - 2\beta)$, $6x^2 - 3x - 2 = 0.$

43. $(3\alpha + \beta)(\alpha + 3\beta)$, $3x^2 - 4x + 5 = 0.$

44. $(3\alpha - \beta)(\alpha - 3\beta)$, $2x^2 + 7x - 3 = 0.$

183. *Example* 8. Find the equations whose roots are

(i) 5 and 3, (ii) $\frac{2}{3}$ and $-\frac{4}{5}$, (iii) $3 \pm \sqrt{2}$.

(i) The equation whose roots are 5 and 3 is

$$(x - 5)(x - 3) = 0,$$

or

$$x^2 - 8x + 15 = 0.$$

(ii) The equation whose roots are $\frac{2}{3}$ and $-\frac{4}{5}$ is

$$(x-\tfrac{2}{3})(x+\tfrac{4}{5})=0,$$

or
$$(3x-2)(5x+4)=0,$$

or
$$15x^2+2x-8=0.$$

(iii) The equation whose roots are $3+\sqrt{2}$ and $3-\sqrt{2}$ is

$$(x-3-\sqrt{2})(x-3+\sqrt{2})=0,$$

or
$$(x-3)^2-2=0,$$

or
$$x^2-6x+7=0.$$

Example 9. Find the equation whose roots are the squares of the sum and difference of the roots of the equation

$$2x^2-x-3=0.$$

We have
$$\alpha+\beta=\tfrac{1}{2},\quad \alpha\beta=-\tfrac{3}{2}.$$

$$\therefore\ (\alpha+\beta)^2=\tfrac{1}{4},\quad (\alpha-\beta)^2=\tfrac{1}{4}+6=\tfrac{25}{4},$$

the equation whose roots are $(\alpha+\beta)^2$ and $(\alpha-\beta)^2$ is

$$(x-\tfrac{1}{4})(x-\tfrac{25}{4})=0,$$

or
$$(4x-1)(4x-25)=0,$$

or
$$16x^2-104x+25=0.$$

Example 10. Find the equation whose roots are the squares of the roots of the equation

$$3x^2+2x-5=0.$$

We have
$$\alpha+\beta=-\tfrac{2}{3},\quad \alpha\beta=-\tfrac{5}{3}.$$

The equation whose roots are α^2 and β^2 is

$$(x-\alpha^2)(x-\beta^2)=0,$$

or
$$x^2-x(\alpha^2+\beta^2)+\alpha^2\beta^2=0.$$

Now
$$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=\tfrac{4}{9}+\tfrac{10}{3}=\tfrac{34}{9},$$

and
$$\alpha^2\beta^2=\tfrac{25}{9}.$$

$\therefore$ the required equation is

$$x^2-\tfrac{34}{9}x+\tfrac{25}{9}=0,$$

or
$$9x^2-34x+25=0.$$

It will be noticed that in Example 9 the roots of the required equation can be expressed in terms of $\alpha+\beta$ and $\alpha\beta$. In Example 10 this is not the case, but the coefficient of x and the term without x can be so expressed.

Exercises 148.

Find the equations whose roots are

1. $4, 2.$	2. $3, 3.$	3. $5, -6.$
4. $-2, 7.$	5. $-3, -5.$	6. $-4, -7.$
7. $0, 5.$	8. $0, -8.$	9. $\frac{1}{2}, \frac{1}{3}.$
10. $\frac{1}{4}, -\frac{1}{5}.$	11. $-\frac{4}{7}, -\frac{2}{3}.$	12. $1\frac{1}{3}, -1\frac{1}{4}.$
13. $2\frac{1}{2}, -2\frac{1}{3}.$	14. $-\frac{2}{3}, -1\frac{1}{2},$	15. $1 \pm \sqrt{2}.$
16. $1 \pm \sqrt{3}.$	17. $3 \pm \sqrt{7}.$	18. $6 \pm \sqrt{3}.$
19. $\frac{5 \pm \sqrt{5}}{2}.$	20. $\frac{3 \pm \sqrt{2}}{5}.$	

If α, β be the roots of the following equations, find the equations whose roots are

21. $\alpha+\beta,\ \alpha\beta;\ 2x^2+3x+1=0.$

22. $\frac{1}{\alpha+\beta},\ \frac{1}{\alpha\beta};\ x^2-4x+7=0.$

23. $\alpha+\beta,\ \frac{\alpha\beta}{\alpha+\beta};\ 3x^2-4x-2=0.$

24. $\frac{1}{\alpha}+\frac{1}{\beta},\ \frac{1}{\alpha\beta};\ 3x^2+8x-4=0.$

25. $\frac{\alpha+\beta}{2},\ \frac{2\alpha\beta}{\alpha+\beta};\ 3x^2+9x-4=0.$

26. $\alpha^2+\beta^2,\ \alpha^2\beta^2;\ x^2+5x+3=0.$

27. $(\alpha+\beta)^2,\ (\alpha-\beta)^2;\ 2x^2-x-5=0.$

28. $(\alpha+\beta)^2,\ (\alpha-\beta)^2;\ 4x^2+2x-3=0.$

29. $\frac{1}{\alpha^2}+\frac{1}{\beta^2}$, $\frac{1}{\alpha^2\beta^2}$; $4x^2-2x+3=0$.

30. $\alpha^3+\beta^3$, $\alpha^3\beta^3$; $x^2-3x+1=0$.

31. $\frac{1}{\alpha^3}+\frac{1}{\beta^3}$, $\frac{1}{\alpha^3\beta^3}$; $x^2+2x+2=0$.

32. $2\alpha+\beta$, $\alpha+2\beta$; $2x^2+7x+4=0$.

33. $2\alpha+\beta$, $\alpha+2\beta$; $5x^2-2x-4=0$.

34. $3\alpha+\beta$, $\alpha+3\beta$; $2x^2-4x-1=0$.

35. $3\alpha+2\beta$, $2\alpha+3\beta$; $x^2-7x+3=0$.

Find the equations whose roots are the squares of the roots of the following equations, and test the results of the first two:

36. $x^2-6x+8=0$.

37. $x^2+7x+12=0$.

38. $2x^2+4x-5=0$.

39. $2x^2-3x-4=0$.

40. $3x^2-7x+1=0$.

41. $3x^2-4x-5=0$.

42. $5x^2+3x+3=0$.

43. $4x^2+x-1=0$.

44. $10x^2-5x-2=0$.

45. $5x^2-10x-2=0$.

46. Find the equation whose roots are the sum and product of the reciprocals of the roots of the equation

$$2x^2-3x+4=0.$$

47. Find the equation whose roots are the square of the sum, and the sum of the squares, of the roots of the equation

$$x^2+7x+7=0.$$

48. Find the equation whose roots are the squares of the sum and difference of the roots of the equation

$$3x^2-7x+2=0.$$

49. Find the equation whose roots are the reciprocals of the roots of the equation

$$4x^2-2x+5=0.$$

50. Find the equation whose roots are the squares of the reciprocals of the roots of the equation

$$x^2+7x-1=0.$$

184. *Example* 11. Find the condition that one root of the equation $ax^2 + bx + c = 0$ may be double the other.

Let the roots be 2α and α, then

$$2\alpha + \alpha = -\frac{b}{a} \text{ and } 2\alpha \cdot \alpha = \frac{c}{a},$$

$$\therefore \ \alpha = -\frac{b}{3a} \text{ and } \alpha^2 = \frac{c}{2a},$$

$$\therefore \ \left(-\frac{b}{3a}\right)^2 = \frac{c}{2a},$$

$$\therefore \ 2b^2 = 9ac.$$

Exercises 149.

1. Find the condition that one root of the equation

$$ax^2 + bx + c = 0$$

may be three times the other.

2. Find the condition that one root of the equation

$$x^2 + px + q = 0$$

may be five times the other.

Find the value of m if one root of the equation:

3. $x^2 + 6x + m = 0$ be double the other.

4. $x^2 + mx + 20 = 0$ be five times the other.

5. $mx^2 - 16x + 3 = 0$ be three times the other, and also find the roots.

6. $2x^2 - mx + 9 = 0$ may be double the other, and also find the roots.

185. *The expression* $ax^2 + bx + c$ *is equal to* $a(x - \alpha)(x - \beta)$,

where
$$\alpha = \frac{-b + \sqrt{(b^2 - 4ac)}}{2a},$$

and $$\beta = \frac{-b - \surd(b^2 - 4ac)}{2a}$$

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$
$$= a\{x^2 - (\alpha + \beta)x + \alpha\beta\}$$
$$= a(x - \alpha)(x - \beta).$$

186. *A quadratic equation has two, and only two, roots.*

The expression $ax^2 + bx + c$ can be resolved into the factors

$$a(x - \alpha)(x - \beta),$$

where $$\alpha = \frac{-b + \surd(b^2 - 4ac)}{2a}, \quad \beta = \frac{-b - \surd(b^2 - 4ac)}{2a}.$$

Now, a is constant, and therefore the expression $ax^2 + bx + c$ can only vanish when one of the factors $x - \alpha$, $x - \beta$ vanishes; i.e. the roots of the equation $ax^2 + bx + c = 0$ are the roots, and the only roots, of the equations

$$x - \alpha = 0 \text{ and } x - \beta = 0.$$

But, an equation of the first degree has one, and only one, root;

∴ a quadratic equation has two, and only two, roots.

187. *Example* 12. If x be real, show that (i) the least value of the expression $x^2 - 4x + 8$ is 4, and (ii) the greatest value of the expression $6x - x^2 - 4$ is 5.

$$\text{(i)} \quad x^2 - 4x + 8 = x^2 - 4x + 4 + 4$$
$$= (x - 2)^2 + 4.$$

Now, if x be real, $(x - 2)^2$ is always positive, whether x be positive or negative, and its least value is zero, when $x = 2$;

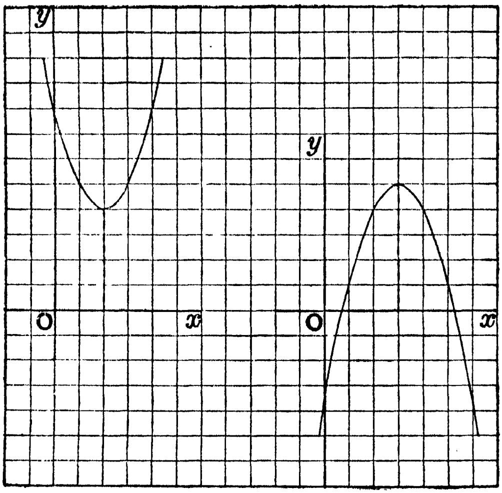
∴ the least value of the expression $x^2 - 4x + 8$ is 4.

$$\text{(ii)} \quad 6x - x^2 - 4 = -(x^2 - 6x + 4)$$
$$= -(x^2 - 6x + 9 - 5)$$
$$= 5 - (x - 3)^2.$$

Now, if x be real, $(x-3)^2$ is always positive, whether x be positive or negative, and its least value is 0, when $x=3$;

$\therefore$ the greatest value of the expression $6x-x^2-4$ is 5.

These results are illustrated by the graphs of the given expressions (fig. 17).



(i) Fig. 17. (ii)

Another method of working the last two examples should be noticed.

Let $$x^2-4x+8=y,$$

$$\therefore\ x^2-4x+8-y=0.$$

Since x is real, $4^2-4(8-y)$ is positive or zero,

$\therefore\ 4-8+y$ is positive or zero,

$\therefore\ y-4$ is positive or zero.

$\therefore\ y \nless 4$, i.e. the least value of x^2-4x+8 is 4.

Again, let $$6x - x^2 - 4 = y,$$

$$\therefore\ x^2 - 6x + 4 + y = 0.$$

Since x is real, $6^2 - 4(4 + y)$ is positive or zero,

$\therefore\ 9 - 4 - y$ is positive or zero,

$\therefore\ 5 - y$ is positive or zero.

$\therefore\ y \not> 5$, i.e. the greatest value of $6x - x^2 - 4$ is 5.

***188.** *Example* 13. If the following expressions be positive, (i) $x^2 - 5x + 6$ cannot lie between 2 and 3, and (ii) $6x - x^2 - 8$ is not less than 2 and not greater than 4.

(i) $$x^2 - 5x + 6 = (x - 2)(x - 3).$$

If $x < 2$, $x - 2$ is negative and $x - 3$ is negative, and their product is positive.

If $x > 3$, $x - 2$ is positive and $x - 3$ is positive, and their product is positive.

If $x > 2$ and < 3, $x - 2$ is positive and $x - 3$ negative, and their product is negative.

$\therefore$ if $x^2 - 5x + 6$ be positive, it can have no value between 2 and 3.

(ii) $$6x - x^2 - 8 = (x - 2)(4 - x).$$

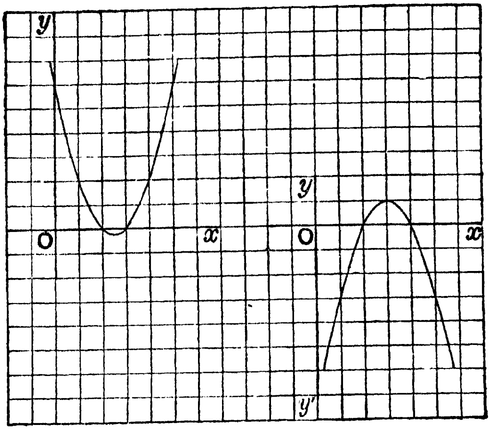
If $x < 2$, $x - 2$ is negative and $4 - x$ positive, and their product is negative.

If $x > 4$, $x - 2$ is positive and $4 - x$ negative, and their product is negative.

If $x > 2$ and < 4, $x - 2$ is positive and $4 - x$ positive, and their product is positive.

$\therefore$ if $6x - x^2 - 8$ be positive, its least value is 2 and its greatest is 4.

These results are illustrated by the following graphs (fig. 18).



(i) Fig. 18. (ii)

*189. *Example* 14. If x be real, show that $\frac{x^2-4x-20}{x-7}$ cannot lie between 8 and 12.

Let $$\frac{x^2-4x-20}{x-7}=y.$$

$$\therefore\ x^2-4x-20=xy-7y,$$

$$\therefore\ x^2-x(4+y)+7y-20=0.$$

Since x is real,

$(4+y)^2-4(7y-20)$ is positive or zero,

$\therefore\ y^2-20y+96$ is positive or zero,

$\therefore\ (y-8)(y-12)$ is positive or zero.

$\therefore$ y cannot lie between 8 and 12.

The graph of the expression is shown in fig. 19.

Example 15. If x be real, show that $\frac{x^2-x+1}{x^2+x+1}$ is not less than $\frac{1}{3}$ and not greater than 3.

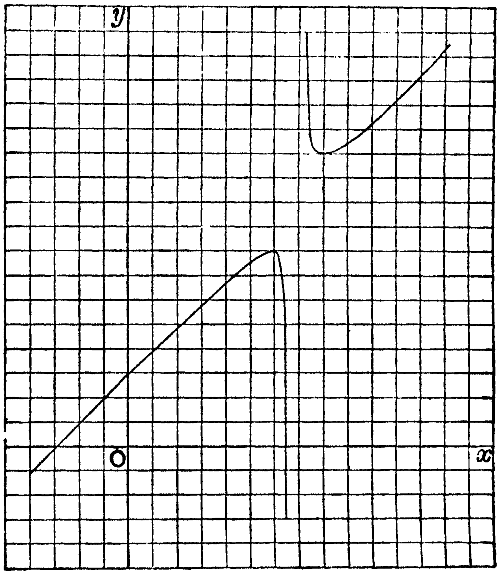


Fig. 19.

Let $$\frac{x^2 - x + 1}{x^2 + x + 1} = y,$$

$$\therefore \quad x^2 - x + 1 = x^2y + xy + y;$$

$$\therefore \quad x^2(1 - y) - x(1 + y) + 1 - y = 0.$$

Since x is real,

$\therefore$ $(1 + y)^2 - 4(1 - y)^2$ is positive or zero;

$\therefore$ $1 + 2y + y^2 - 4 + 8y - 4y^2$ is positive or zero;

$\therefore$ $10y - 3 - 3y^2$ is positive or zero;

$\therefore$ $3y^2 - 10y + 3$ is negative or zero,

$\therefore$ $(3y - 1)(y - 3)$ is negative or zero;

$\therefore$ y is not less than $\frac{1}{3}$ and not greater than 3.

The graph of the expression is shown in fig. 20.

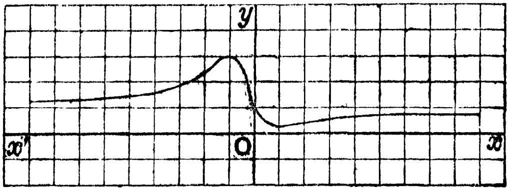


Fig. 20.

Exercises 150.

Find the least value of

1. $x^2 - 6x + 13$.
2. $x^2 + 10x + 32$.
3. $2x^2 + 6x + 5$.
4. $3x^2 + 2x + 2$.

Find the greatest value of

5. $2 + 2x - x^2$.
6. $-8 - 6x - x^2$.
7. $3 + 6x - 2x^2$.
8. $1 - 6x - 5x^2$.

9. The roots of the equation $x^2 + 8x + a = 0$ are real if $a \ngtr 16$.

10. The roots of the equation $x^2 + 4x + 6 - a = 0$ are real if $a > 2$.

11. The roots of the equation $ax^2 + 6x - a = 0$ are real whatever be the value of a.

12. If the roots of the equation $ax^2 + 2x + a = 0$ be real, then $a \ngtr 1$ and $\nless -1$.

13. If the roots of the equation $x^2 + (a - 4)x + 9 = 0$ be real, a cannot have any value between -2 and 10.

14. If the roots of the equation $x^2 + a^2 = 8x + 6a$ be real, show that $a \ngtr 8$ and $\nless -2$.

15. If x be real, show that $x + \frac{1}{x}$ can have no value between -2 and 2.

16. If x be real, show that $x - \frac{1}{x}$ can have any value whatever.

17. If x be real, show that $\dfrac{1}{1+x+x^2}$ cannot be greater than $\frac{4}{3}$.

18. If x be real, prove that $\dfrac{x^2-3x+3}{x-1}$ cannot have any value between -3 and 1.

19. If x be real, show that $\dfrac{2x-7}{2x^2-2x-5}$ can have no real value between $\frac{1}{11}$ and 1.

20. Prove that the greatest and least values of

$$\frac{6x^2-22x+21}{5x^2-18x+17}$$

are $1\frac{1}{4}$ and 1, corresponding to the values 1 and 2 of x.

21. If x be real, show that $\dfrac{x^2-50x+625}{x-50}$ can have no positive value less than 100.

***190.** *If the equations*

$$ax^2+bx+c=0, \quad a'x^2+2b'x+c'=0$$

have a common root, then

$$(ca'-c'a)^2=(bc'-b'c)(ab'-a'b).$$

Let α be the common root, so that

$$a\alpha^2+b\alpha+c=0,$$

$$a'\alpha^2+b'\alpha+c'=0,$$

then

$$\frac{\alpha^2}{bc'-b'c}=\frac{\alpha}{ca'-c'a}=\frac{1}{ab'-a'b},$$

$$\therefore \ \alpha^2=\frac{bc'-b'c}{ab'-a'b}, \text{ and } \alpha=\frac{ca'-c'a}{ab'-a'b};$$

$$\therefore \ \left(\frac{ca'-c'a}{ab'-a'b}\right)^2=\frac{bc'-b'c}{ab'-a'b};$$

$$\therefore \ (ca'-c'a)^2=(bc'-b'c)(ab'-a'b).$$

191. *Example* 16. If α and β denote the roots of the equation $ax^2 + bx + c = 0$, find the equation whose roots are $\alpha + 2$ and $\beta + 2$.

We have $$\alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}.$$

The equation whose roots are $\alpha + 2$ and $\beta + 2$ is

$$(x - \overline{\alpha + 2})(x - \overline{\beta + 2}) = 0,$$

or $$x^2 - (\alpha + \beta + 4)x + \alpha\beta + 2(\alpha + \beta) + 4 = 0,$$

or $$x^2 - \left(-\frac{b}{a} + 4\right)x + \frac{c}{a} - \frac{2b}{a} + 4 = 0,$$

or $$ax^2 + (b - 4a)x + c - 2b + 4a = 0.$$

Example 17. If one of the equations

$$x^2 + x(2a - b) + ab = 0,$$
$$x^2 + x(4a - b) + 3a^2 = 0$$

have equal roots, so has the other.

The condition that the roots of the first equation may be equal is

$$(2a - b)^2 = 4ab,$$

or $$4a^2 - 8ab + b^2 = 0.$$

The condition that the roots of the second equation may be equal is

$$(4a - b)^2 = 12a^2,$$

or $$4a^2 - 8ab + b^2 = 0.$$

Hence, if one equation have equal roots, so has the other.

*Exercises 151.

1. If the roots of the equation $2x^2 - 3x + m = 0$ be equal, what are they?

2. If the roots of the equation $4x^2 + mx + 25 = 0$ be equal, what are they?

3. If α, β be the roots of the equation $x^2 - 8x + m = 0$, what value must m have so that $\alpha^2 + \beta^2$ may be equal to 40?

4. If $3x - 2y$ be a factor of $mx^2 - 11xy + 6y^2$, what is the value of m?

5. Find the condition that the roots of the equation

$$ax^2 + bx + c = 0$$

may be such that m times one root is equal to n times the other.

6 If α, β be the roots of the equation

$$x^2 - (1 + a^2)x + \tfrac{1}{2}(1 + a^2 + a^4) = 0,$$

then $\alpha^2 + \beta^2 = a^2$.

7. One root of the equation

$$(c + a - 2b)x^2 + (a + b - 2c)x + b + c - 2a = 0$$

is 1, what is the other?

8. For what values of m are the roots of the equation $(2m - 1)x^2 + (1 + m)x + 1 = 0$ equal?

9. For what values of m are the roots of the equation $12(m + 2)x^2 - 12(2m - 1)x - 38m - 11 = 0$ equal?

10. Show that the roots of the equation

$$(x - a)(x - b) - h^2 = 0$$

are real if a, b and h be real.

11. Show that the roots of the equation

$$(x - m)(x - n) = mnx^2$$

are real if m and n be real.

12. If a and b be real, the roots of

$$\frac{1}{x} + \frac{1}{x + a} + \frac{1}{x + b} = 0$$

are real.

13. If the roots of the equation

$$(b - c)x^2 + (c - a)x + (a - b) = 0$$

be equal, prove that $2b = a + c$.

14. Find the values of m if the roots of the equation

$$x^2 + (m^2 - 4)x - (m + 1)^2 = 0$$

be equal in magnitude and opposite in sign, and find the roots.

15. Prove that the roots of the equation

$$x^2 + bx + c = 0$$

are rational if $$b = k + \frac{c}{k},$$

where b, c, k are rational quantities.

16. If α, β be the roots of the equation

$$ax^2 + bx + c = 0,$$

find the equation whose roots are

$$\frac{\alpha}{\beta} \text{ and } \frac{\beta}{\alpha}.$$

17. If α, β be the roots of the equation

$$ax^2 + bx + c = 0,$$

find the equation whose roots are

$$\frac{\alpha}{\alpha + \beta} \text{ and } \frac{\beta}{\alpha + \beta}.$$

18. If α, β be the roots of the equation

$$ax^2 + bx + c = 0,$$

find the value of

$$\left(\alpha + \frac{\beta^2}{\alpha}\right)\left(\beta + \frac{\alpha^2}{\beta}\right).$$

19. Find the condition that the roots of the equation

$$a^2x^2 + b^2x + c^2 = 0$$

may be the squares of the roots of the equation

$$ax^2 + bx + c = 0.$$

20. Find the value of m if the sum of the roots of the equation

$$2x^2 + (1 + m)x + 3 = 0$$

be equal to the sum of the roots of the equation

$$3x^2 - 5x + 4 = 0$$

21. Find the values of m if the sum of the roots of the equation

$$x^2 - (m+4)x + 3m^2 = 0$$

be equal to the product of the roots of the equation

$$2x^2 + 7mx + m^2 = 0.$$

22. If α, β be the roots of the equation

$$ax^2 + 2bx + c = 0,$$

find the equation whose roots are

$$1 + \frac{\alpha}{\beta}, \quad 1 + \frac{\beta}{\alpha}.$$

23. If α, β be the roots of the equation

$$ax^2 + bx + c = 0,$$

find the equation whose roots are

$$2\beta - \alpha, \quad 2\alpha - \beta.$$

24. If α, β be the roots of the equation

$$2x^2 + 3x - 6 = 0,$$

find the equation whose roots are

$$\alpha + 1, \quad \beta + 1.$$

25. If α, β be the roots of the equation

$$3x^2 - 4x - 3 = 0,$$

find the equation whose roots are

$$3\alpha - 2, \quad 3\beta - 2.$$

26. If α, β be the roots of the equation

$$ax^2 + 2bx + c = 0,$$

find the equation whose roots are

$$\alpha^2 + 1, \quad \beta^2 + 1.$$

27. If α, β be the roots of the equation

$$x^2 + ax + b = 0,$$

express the roots of the equation

$$x^2 - a^2x + a^2b = 0$$

in terms of α and β.

28. If the roots of the equation $x^2 + \text{P}x + q = 0$ be α, β, and the roots of the equation $x^2 + px + \text{Q} = 0$ be γ, δ, find the roots of the equation $x^2 + px + q = 0$ in terms of α, β, γ, δ.

29. If one of the equations

$$x^2 + (a + 2b)\,x - 3b^2 = 0, \quad x^2 + (a + 4b)\,x + ab = 0$$

have equal roots, so has the other.

30. If one of the equations

$$x^2 + (2a - b)\,x + a^2 = 0, \quad 4x^2 + 4\,(a - 2b)\,x + a^2 + 3b^2 = 0$$

have equal roots, so has the other.

31. Show that the roots of one of the equations

$$16a^2x^2 - 8a^2x + b^2 = 0, \quad 4a^2x^2 + 4b^2x + b^2 = 0$$

must be imaginary.

32. Show that the roots of one of the equations

$$x^2 + 2ax + a\,(a + 2b) = 0, \quad 4x^2 + 4\,(a + 2b)\,x + a^2 + 4b^2 = 0$$

must be imaginary.

33. A and B attempt the same quadratic equation; A, after reducing and dividing by the coefficient of x^2, gets the coefficient of x wrong, and finds the roots to be 2 and 6; B gets the term without x wrong and finds the roots to be 2 and -9; find the correct roots.

34. A and B attempt the same quadratic equation; A, after reducing and dividing by the coefficient of x^2, gets the term without x wrong and finds the roots to be 2 and 6; B gets the coefficient of x wrong and finds the roots to be 1 and 15; find the correct roots.

35. Find whether the roots of the equation

$$x^2 + 2\,(p + q)\,x + 2\,(p^2 + q^2) = 0$$

are real or imaginary.

36. If the roots of the equation

$$q^2x^2 + (p^2 - 2q)\,x + q + 1 - \frac{p^2}{4} = 0$$

be equal, then $p^2 = 4q$.

37. If $2 > a > 0$, prove that the equation

$$(x+3)(1-ax)-(1+a)(x+3)=2$$

has no real roots.

38. Form the quadratic equation the sum of whose roots is 3, and the sum of the cubes of whose roots is 7.

39. Find the condition that the roots of the equation

$$ax^2+2bx+c=0$$

should differ by unity.

40. If ax^2+bx+c vanish for three different values of x, show that a, b, c must all vanish.

41. In a certain quadratic equation, the coefficients of x^2 and x are 1 and 2 respectively, and the addition of 8 to each of the roots changes the sign but not the magnitude of the third term; find the original equation, and the coefficient of x in the transformed equation.

42. If a be a positive quantity and k a real quantity, show that a real value of x can be found which will make $x-\frac{a}{x-k}$ equal to any real quantity.

43. For what values of λ is

$$4x^2-10xy+10y^2+\lambda(3x^2-10xy+3y^2)$$

a complete square?

44. If $(x+b)(x+c)+(x+c)(x+a)+(x+a)(x+b)$ be a complete square, show that $a=b=c$.

45. If the roots of the equation

$$\left(1-q+\frac{p^2}{2}\right)x^2+p(1+q)x+q(q-1)+\frac{p^2}{2}=0$$

be equal, then $p^2=4q$.

46. Show that $\frac{x^2-bc}{2x-b-c}$ has no real value between b and c.

47. Find within what limits the expression $x^2+2px+q$ must lie for real values of x.

48. If the equations $ax^2 + bx + c = 0$, $a^2x^2 + b^2x + c^2 = 0$ have a common root, find that root.

49. Solve the equations

$$28x^2 + 71x + 45 = 0 \text{ and } 32x^2 + 20x - 25 = 0,$$

it being given that they have a common root.

50. Solve the equations

$$66x^2 - 157x + 85 = 0, \ 126x^2 - 243x + 115 = 0,$$

it being given that they have a common root.

51. The equations $x^2 + ax + b = 0$, $x^2 + ax + c = 0$ cannot have a common root if b and c be unequal.

52. If the equations $x^2 + ax + b = 0$, $x^2 + bx + a = 0$ have a root in common, then either $a = b$ or $a + b + 1 = 0$.

53. If $x^2 + px + q$ and $x^2 - qx - p$ have a common factor, then $1 - p + q = 0$: what is their L.C.M.?

54. Find the value of m when the equations $x^2 + x + m = 0$, $x^2 + mx + 1 = 0$ have one, and only one, root in common; and find the roots of both equations.

55. If one root of the equation $x^2 + 2x + m = 0$ be double one root of the equation $x^2 + x + 1 = 0$, find the value of m.

56. The equations $x^2 - 2ax + b = 0$, $x^2 - 2bx + a = 0$ have a common root if $4a + 4b + 1 = 0$.

57. Show that one of the roots of the equation $ax^2 + bx + c = 0$ will be double one of the roots of the equation $cx^2 + bx + a = 0$ if either $c = 2a$ or $2a + c = \pm b\sqrt{2}$.

58. If $ax^2 + bx + c = 0$, $bcx^2 + cax + ab = 0$ have a common root, and if $a + b + c = 0$, prove that $b^4(a - c)^2 = a^2c^2(a - b)(b - c)$.

59. If $4x + 3y = 120$, find the greatest value which xy can have.

60. If a, b, c be given real quantities, prove that the least possible value of

$$x^2 + \frac{b^2c^2}{4x^2} - \tfrac{1}{2}(b^2 + c^2 - a^2) \text{ is } \tfrac{1}{2}(a + b - c)(a + c - b).$$

CHAPTER XXI.

SIMULTANEOUS EQUATIONS, OF A HIGHER DEGREE THAN THE FIRST.

192. A complete quadratic equation in x and y contains terms in x^2, xy, y^2, x and y, and a term without x or y. Two complete equations of this form do not as a rule admit of solution by elementary methods, and we shall therefore consider only the following types of equations, which may be solved by methods within the range of this book:

I. $ax + by = c,\ lx^2 + mxy + ny^2 + px + qy + r = 0.$

II. $ax^2 + bxy + cy^2 = d,\ lx^2 + mxy + ny^2 = p.$

III. $x^3 \pm y^3 = a,\ x \pm y = b.$

IV. $ax + by = c,\ lx^2y^2 + mxy + n = 0.$

V. $x^4 + x^2y^2 + y^4 = a,\ x^2 \pm xy + y^2 = b.$

As a general rule, the "method of substitution" is used in solving simultaneous quadratics. The mode of solving each type will be explained by means of one or two examples, and, as far as possible, every step in the solution will be illustrated by means of appropriate graphs.

TYPE I.

$$ax + by = c,\ lx^2 + mxy + ny^2 + px + qy + r = 0.$$

193. In this type, one equation is of the first degree, and the other of the second degree, in x and y. The method consists in finding one of the unknown quantities, say y, in terms of the other x, and substituting this value in the second equation, which will then become a quadratic in x. Since, for each root of this quadratic, there is a corresponding value of y, it follows that every equation of this type must have two solutions.

194. *Example* 1. Solve the equations

$$y - x = 2, \ x^2 + y^2 = 100.$$

$$y - x = 2, \qquad (1)$$

$$x^2 + y^2 = 100. \qquad (2)$$

From (1), we have $\quad y = x + 2.$

Substituting in (2), we get

$$x^2 + (x + 2)^2 = 100,$$

$$\therefore \ 2x^2 + 4x - 96 = 0,$$

$$\therefore \ x^2 + 2x - 48 = 0,$$

$$\therefore \ (x - 6)(x + 8) = 0,$$

the roots of which are 6 and -8.

From (1), we have

$$\text{if } x = 6, \text{ then } y = 6 + 2 = 8,$$

$$\text{if } x = -8, \text{ then } y = -8 + 2 = -6;$$

$\therefore$ the solutions are

$$\left.\begin{matrix} x = 6 \\ y = 8 \end{matrix}\right\} \text{ and } \left.\begin{matrix} x = -8 \\ y = -6 \end{matrix}\right\}.$$

195. Drawing the graphs of the equations, as in fig. 21, we have the straight line corresponding to equation (1) and the circle PQR to equation (2). The points P, Q, in which the straight line cuts the circle are those whose coordinates 6, 8 and -8, -6, satisfy the given equations simultaneously.

A difficulty is sometimes experienced in understanding why the values of x (6 and 8) are substituted in equation (1) and not in equation (2). The reason will be seen from the diagram. The line $x = 6$, for instance, is PR, and this cuts the circle in two points P and R, of which only the former lies on the straight line PQ represented by equation (1), as well as on the circle represented by equation (2).

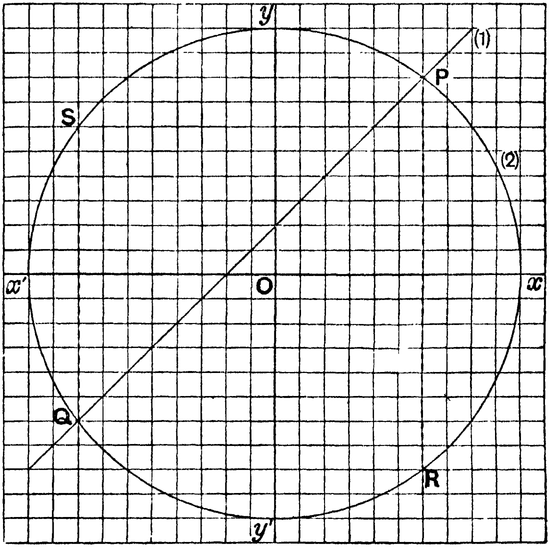


Fig. 21.

196. *Example* 2. Solve the equations

$$4x + 3y = 50, \ x^2 + y^2 = 100.$$

$$4x + 3y = 50, \qquad (1)$$

$$x^2 + y^2 = 100. \qquad (2)$$

From (1) we have $y = \dfrac{50 - 4x}{3}$.

Substituting in (2), we get

$$x^2 + \frac{(50 - 4x)^2}{9} = 100,$$

$$\therefore \ 9x^2 + 2500 - 400x + 16x^2 - 900 = 0,$$

$$\therefore\ 25x^2 - 400x + 1600 = 0,$$

$$\therefore\ x^2 - 16x + 64 = 0,$$

$$\therefore\ (x-8)^2 = 0,$$

the roots of which are 8 and 8.

From (1) we have,

$$\text{if } x = 8, \text{ then } y = \frac{50-32}{3} = 6,$$

$\therefore$ the solutions are

$$\left.\begin{array}{l} x = 8 \\ y = 6 \end{array}\right\} \text{ and } \left.\begin{array}{l} x = 8 \\ y = 6 \end{array}\right\}.$$

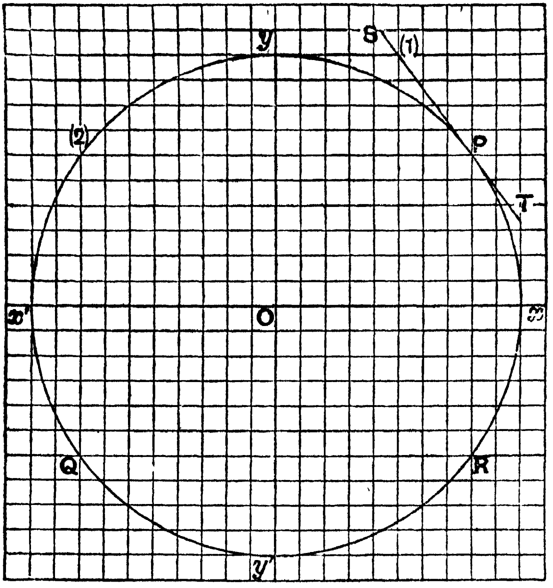


Fig. 22.

197. Drawing the graphs, as in fig. 22, we have the straight line SPT corresponding to equation (1) and the circle PQR to equation (2). The points in which the straight line cuts the circle are coincident, and the meaning of the equal pairs of roots is that the straight line represented by equation (1) touches the circle represented by equation (2).

198. *Example* 3. Solve the equations

$$x - y = 1,\ x^2 - y^2 = 5.$$

$$x - y = 1, \qquad (1)$$

$$x^2 - y^2 = 5. \qquad (2)$$

We have

$$x^2 - y^2 = 5 = 5 \cdot 1 = 5(x - y),$$

$$\therefore\ (x - y)(x + y - 5) = 0, \qquad (3)$$

$$\therefore\ x = y, \qquad (4)$$

or

$$x = 5 - y. \qquad (5)$$

(i) $x = y$.

Substituting in (2), we get

$$y^2 - y^2 = 5,$$

which leads to a pair of infinite roots.

(ii) $x = 5 - y$.

Substituting in (2), we get

$$25 - 10y + y^2 - y^2 = 5,$$

$$\therefore\ -10y = -20,$$

$$\therefore\ y = 2,$$

$$\therefore\ x = 5 - 2 = 3.$$

∴ the solutions are

$$\left.\begin{array}{l} x = 3 \\ y = 2 \end{array}\right\} \text{ and a pair of infinite roots.}$$

199. The meaning of the pair of infinite roots will be seen from the graphs in fig. 23. Here, the graphs of (1) and (2), represented by continuous lines, intersect in the point (3, 2), where they are cut by the line

$$x = 5 - y. \qquad (4)$$

The line

$$x = y \qquad (5)$$

is an asymptote to the curve (2), and therefore the graph of (1), which is parallel to this asymptote, does not meet the curve at a finite distance from the origin.

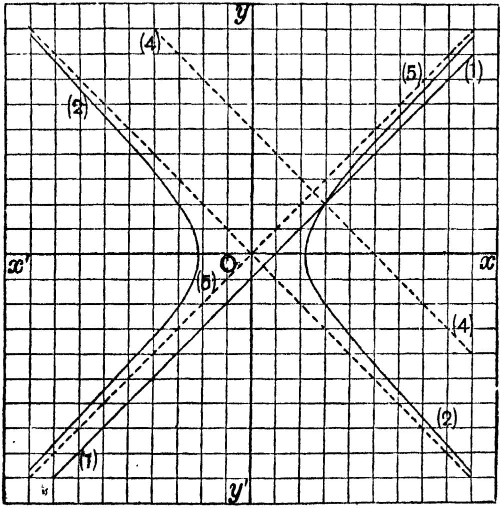


Fig. 23.

Since the line $x = y$ does not intersect the curve $x^2 - y^2 = 5$ in any points at a finite distance from the origin, it follows that the equations

$$x - y = 0, \quad x^2 - y^2 = 5$$

are satisfied by no finite values of x and y.

It will be noticed that the method used in Example 3 is different from that used in Examples 1 and 2. If the latter were used, we should have

$$x = y + 1,$$

and $$\therefore\ y^2 + 2y + 1 - y^2 = 5,$$

or $$2y = 4 \text{ or } y = 2,$$

and thus we should lose sight of the pair of infinite roots.

The same objection applies to the following method.

By division, we get

$$x + y = 5,$$

which, with the equation

$$x - y = 1,$$

gives $$x = 3,\ y = 2.$$

Exercises 152.

Solve the equations:

1. $x + y = 8,\ xy = 15.$
2. $x + y = 13,\ xy = 36.$
3. $x + y = 7,\ xy + 18 = 0.$
4. $x - y = 8,\ x^2 + y^2 = 40.$
5. $x + y = 1,\ \dfrac{1}{x} + \dfrac{1}{y} = -\dfrac{1}{2}.$
6. $x^2 + y^2 = x + y = 2.$
7. $y - 2x = 6,\ 4x^2 - 3xy = 16.$
8. $x - 2y = 1,\ 3xy - y^2 = 8.$
9. $3x + 4y = 2,\ 2xy + 5y^2 = 1.$
10. $y + 3x = 1,\ 3x^2 - xy - 2x - y = 5.$
11. $x^2 - xy + y^2 = 13,\ 2x - y = 5.$
12. $2x - 3y = 7,\ 3xy + 2y^2 + 4 = 0.$
13. $2x - 7y = -1,\ 3x^2 - 5xy = 28.$
14. $5x - 3y = -22,\ \dfrac{6}{x} + \dfrac{10}{y} = -\dfrac{1}{2}.$
15. $9y + x = x(x + y),\ x - y = 2.$
16. $2x + y = x^2 + 2xy = 3.$
17. $4x - 5y = 1 = 3xy - 5(x + y).$
18. $\dfrac{1}{x} - \dfrac{1}{y} = 10,\ \dfrac{1}{x^2} + \dfrac{1}{y^2} = 58.$

19. $x^2-y^2=21,\ x+y=7.$

20. $4x^2-9y^2=160,\ 2x-3y=20.$

21. $\frac{16x^2}{9}-\frac{36y^2}{25}=3,\ \frac{4x}{3}-\frac{6y}{5}=1.$

22. $x^2+xy-2y^2+4x+3y=5=x+2y.$

23. $ax+by=a+b,\ xy=1$ 24. $2ax-by=a^2,\ bxy=a^3$

25. $x(y+1)=y(a+1)=a(x+1).$

TYPE II.

$$ax^2+bxy+cy^2=d,\ lx^2+mxy+ny^2=p.$$

200. In this type, the equations are both homogeneous and of the second degree in x and y. The method consists in deducing from the given equations a single equation in x^2, xy, and y^2 only, from which we obtain two values of y in terms of x, and substituting each value in turn in one of the given equations. Since each of the quadratic equations thus obtained gives two values of x, it follows that every equation of this type must have four solutions. Occasionally, as in Example 5, two of these solutions are infinite. The reason for this will be evident from the corresponding graphs.

201. *Example* 4. Solve the equations

$$2x^2-3xy=20,\ xy-2y^2=2.$$

$$2x^2-3xy=20, \quad (1)$$

$$xy-2y^2=2, \quad (2)$$

$$\therefore\ 2(2x^2-3xy)=20(xy-2y^2),$$

$$\therefore\ 2x^2-3xy=10xy-20y^2,$$

$$\therefore\ 2x^2-13xy+20y^2=0, \quad (3)$$

$$\therefore\ (x-4y)(2x-5y)=0,$$

$$\therefore\ x=4y, \quad (4)$$

or

$$x=\tfrac{5}{2}y. \quad (5)$$

(i) $x = 4y$.

Substituting in (2)

$$4y^2 - 2y^2 = 2,$$
$$\therefore\ y^2 = 1.$$
$$\therefore\ y = \pm 1.$$

Substituting in (4), we have

when $y = 1$, then $x = 4$,

when $y = -1$, then $x = -4$.

(ii) $x = \frac{5}{2}y$.

Substituting in (2),

$$\tfrac{5}{2}y^2 - 2y^2 = 2,$$
$$\therefore\ y^2 = 4,$$
$$\therefore\ y = \pm 2.$$

Substituting in (5), we have

when $y = 2$, then $x = 5$,

when $y = -2$, then $x = -5$.

$\therefore$ the solutions are

$$\left.\begin{matrix} x = 4 \\ y = 1 \end{matrix}\right\},\quad \left.\begin{matrix} x = -4 \\ y = -1 \end{matrix}\right\},\quad \left.\begin{matrix} x = 5 \\ y = 2 \end{matrix}\right\},\ \text{and}\ \left.\begin{matrix} x = -5 \\ y = -2 \end{matrix}\right\}.$$

202. It is clear that all the values of x and y which satisfy equations (1) and (2) simultaneously, and no others, must also satisfy

$$2\,(2x^2 - 3xy) = 2\,.\,20,$$

or

$$2\,(2x^2 - 3xy) = (xy - 2y^2)\,20,$$

since $2 = xy - 2y^2$ by (2), i.e. must satisfy (3). In other words, all the points of intersection of the graphs of (1) and (2) must also lie on the graph of (3). But this latter graph consists of the two straight lines (4) and (5), since all values of x and y which make $x - 4y$ or $2x - 5y$ equal to zero and no others satisfy (3).

Hence, the process of solution corresponds to finding the coordinates of the points which are common to the line (4) and either of the curves (1) and (2), and to the line (5) and either of the curves (1) and (2). In fig. 24, the continuous lines (1) and (2) are the graphs of the given equations, and the broken lines (4) and (5) are the graphs of the corresponding equations.

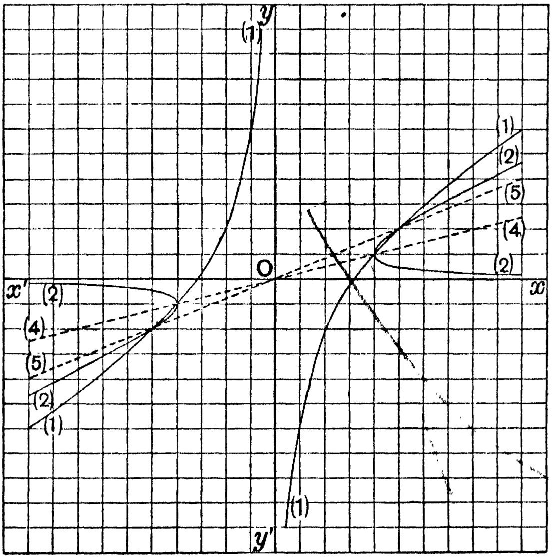


Fig. 24.

203. *Example* 5. Solve the equations

$$x^2 - xy = 2, \ xy - y^2 = 1.$$

$$x^2 - xy = 2, \quad (1)$$

$$xy - y^2 = 1. \quad (2)$$

We have, as in the preceding example,

$$x^2 - xy = 2(xy - y^2),$$

$$\therefore \ x^2 - 3xy + 2y^2 = 0, \quad (3)$$

$$\therefore \ (x - 2y)(x - y) = 0,$$

$$\therefore \ x = 2y, \quad (4)$$

or

$$x = y. \quad (5)$$

(i) $x = 2y$.

Substituting in (2),

$$2y^2 - y^2 = 1,$$
$$\therefore\ y^2 = 1,$$
$$\therefore\ y = \pm 1.$$

Substituting in (4),

when $y = 1$, then $x = 2$,

when $y = -1$, then $x = -2$.

(ii) $x = y$.

Substituting in (2),

$$y^2 - y^2 = 1,$$

which leads to two pairs of infinite roots.

$\therefore$ the solutions are

$\left.\begin{matrix} x = 2 \\ y = 1 \end{matrix}\right\}$, $\left.\begin{matrix} x = -2 \\ y = -1 \end{matrix}\right\}$ and two pairs of infinite roots.

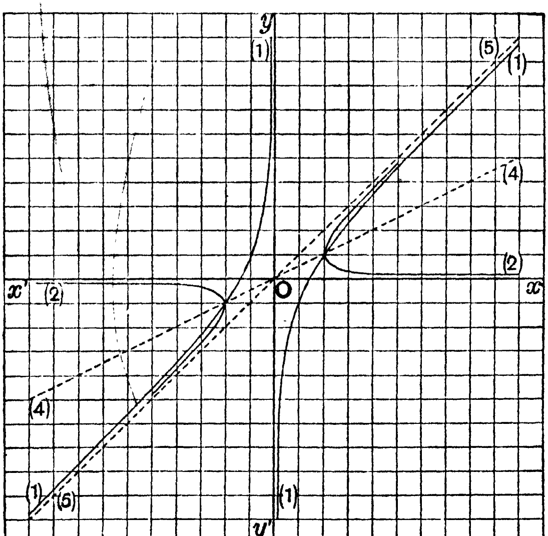


Fig. 25.

204. The meaning of the two pairs of infinite roots will be evident from the graphs in fig. 25. Here, the graphs of equations (1) and (2), represented by the continuous lines, intersect in the points (2, 1) and (– 2, – 1), where they are cut by the line

$$x = 2y. \qquad (4)$$

The line

$$x = y \qquad (5)$$

does not meet either of the curves (1) and (2) at a finite distance from the origin.

205. *Example* 6. Solve the equations

$$x^2 + y^2 = 25, \ xy = 12.$$

$$x^2 + y^2 = 25, \qquad (1)$$

$$xy = 12. \qquad (2)$$

These equations may be solved as in Example 4, but it is usual to adopt the following method.

We have

$$2xy = 24,$$

$$\therefore \ x^2 + 2xy + y^2 = 49,$$

and

$$x^2 - 2xy + y^2 = 1,$$

$$\therefore \ x + y = \pm 7,$$

and

$$x - y = \pm 1.$$

$$\therefore \left.\begin{matrix} x + y = 7 \\ x - y = 1 \end{matrix}\right\}, \left.\begin{matrix} x + y = \ \ 7 \\ x - y = -1 \end{matrix}\right\}, \left.\begin{matrix} x + y = -7 \\ x - y = \ \ 1 \end{matrix}\right\} \text{ and } \left.\begin{matrix} x + y = -7 \\ x - y = -1 \end{matrix}\right\},$$

$$\therefore \left.\begin{matrix} 2x = 8 \\ 2y = 6 \end{matrix}\right\}, \left.\begin{matrix} 2x = 6 \\ 2y = 8 \end{matrix}\right\}, \left.\begin{matrix} 2x = -6 \\ 2y = -8 \end{matrix}\right\} \text{ and } \left.\begin{matrix} 2x = -8 \\ 2y = -6 \end{matrix}\right\}.$$

$\therefore$ the solutions are

$$\left.\begin{matrix} x = 4 \\ y = 3 \end{matrix}\right\}, \left.\begin{matrix} x = 3 \\ y = 4 \end{matrix}\right\}, \left.\begin{matrix} x = -3 \\ y = -4 \end{matrix}\right\} \text{ and } \left.\begin{matrix} x = -4 \\ y = -3 \end{matrix}\right\}.$$

206. The graphs of the given equations are represented by the continuous lines in fig. 26. Instead of finding directly the coordinates of the points in which the curves cut one another, the process corresponds to finding the points of intersection of either of the lines (3) and (4) with either of the lines (5) and (6).

$$x + y = 7, \qquad (3)$$

$$x + y = -7, \qquad (4)$$

$$x - y = 1, \qquad (5)$$

$$x - y = -1. \qquad (6)$$

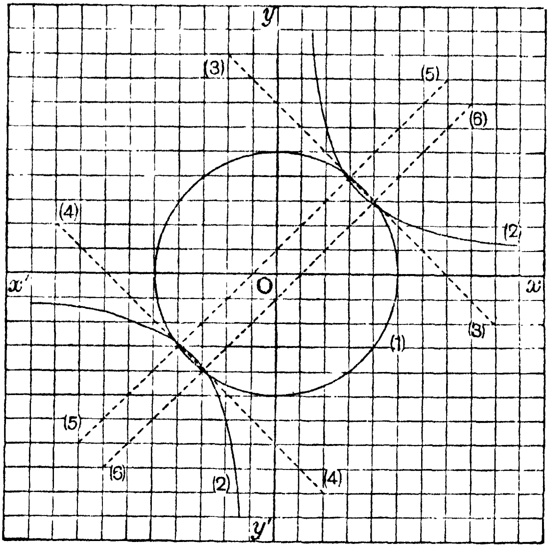


Fig. 26.

Exercises 153.

Solve the equations:

1. $x^2 + 2xy + y^2 = 36$, $x^2 - 2xy + y^2 = 4$.

2. $x^2 + y^2 = 20$, $xy = 8$.

3. $x^2 + y^2 = 34$, $xy = -15$

4. $x^2 + 2xy + y^2 = 100$, $xy = 21$.

5. $y^2 - 4xy = 0,\ 4x^2 + 15xy = 64.$
6. $2x^2 - 3xy = 0,\ xy + y^2 = 40.$
7. $x^2 - 2xy = 0,\ x^2 + 5y^2 = 45.$
8. $x^2 - 5xy = 0,\ x^2 - 3xy + 2y^2 = 18.$
9. $x^2 + y^2 = 5,\ x^2 - y^2 = \frac{3}{2}xy.$
10. $x^2 + 3xy = 45,\ y^2 - xy = 4.$
11. $x^2 + 2xy = 3,\ y^2 - xy = 4.$
12. $x^2 + 2xy = 8,\ y^2 - xy = -1.$
13. $x^2 + xy = 15,\ xy - y^2 = 2.$
14. $x^2 + xy = 10,\ 2xy - y^2 = 3.$
15. $x^2 + 3xy = 18,\ xy - 2y^2 = 1.$
16. $x^2 + 4xy = 35,\ 2xy - 16y^2 = 1.$
17. $x^2 + xy = 15,\ x^2 - xy = 3.$
18. $x^2 + xy = 90,\ xy + y^2 = 10.$
19. $2x^2 + 3xy = -10,\ 3y^2 + 2xy = 8.$
20. $6x^2 + 3xy - 18y^2 = 20,\ 3x^2 + 6xy = 8.$
21. $x^2 - 2xy = 3,\ xy - 2y^2 = 1.$
22. $\frac{1}{x^2} + \frac{1}{y^2} = 5,\ \frac{1}{x^2} + \frac{2}{xy} + \frac{1}{y^2} = 9.$
23. $\frac{1}{x^2} + \frac{2}{y^2} = 3,\ \frac{3}{xy} - \frac{1}{y^2} = 2.$
24. $x^2 + 3xy + 2y^2 = 3a - b,\ x^2 - y^2 = 2b.$
25. $x^2 + y^2 = c^2,\ (ax - by)^2 = c^2(a^2 + b^2).$

TYPE III.

$$x^3 \pm y^3 = a,\ x \pm y = b.$$

207. *Example* 7. Solve the equations $x^3 - y^3 = 7,\ x - y = 1.$

$$x^3 - y^3 = 7, \qquad (1)$$

$$x - y = 1. \qquad (2)$$

$$\therefore \quad x^3 - y^3 = 7(x - y),$$

$$\therefore \quad (x - y)(x^2 + xy + y^2) - 7(x - y) = 0,$$

$$\therefore \quad (x - y)(x^2 + xy + y^2 - 7) = 0, \qquad (3)$$

$$\therefore \quad x = y, \qquad (4)$$

or

$$x^2 + xy + y^2 = 7. \qquad (5)$$

(i) $x = y$.

Substituting in (1),

$$\therefore \quad y^3 - y^3 = 7,$$

which leads to a pair of infinite roots.

(ii) $x^2 + xy + y^2 = 7$.

From (2), we have

$$x = y + 1,$$

$$\therefore \quad (y + 1)^2 + y(y + 1) + y^2 = 7,$$

$$\therefore \quad 3y^2 + 3y - 6 = 0,$$

$$\therefore \quad y^2 + y - 2 = 0,$$

$$\therefore \quad (y - 1)(y + 2) = 0,$$

$$\therefore \quad y = 1 \text{ or } -2.$$

Substituting in (2),

when $y = 1$, then $x = 2$,

when $y = -2$, then $x = -1$.

$\therefore$ the solutions are

$$\left.\begin{matrix} x = 2 \\ y = 1 \end{matrix}\right\}, \quad \left.\begin{matrix} x = -2 \\ y = -1 \end{matrix}\right\} \text{ and a pair of infinite roots.}$$

208. In fig. 27, the graph of (1) is the continuous curve, and the graph of (2) is the continuous straight line. Hence, instead of finding directly the coordinates of the points in which these lines cut one another, we find the coordinates of the points in which the straight line (2) cuts the curve (5) and the straight line (4), the curves (4) and (5) being indicated by broken lines. Since the lines (2) and (4) are parallel, we thus get a pair of infinite roots.

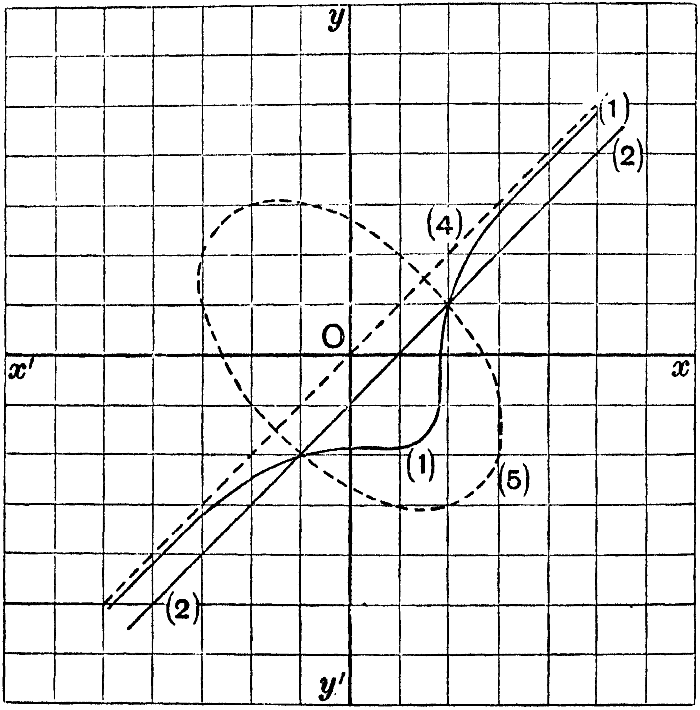


Fig. 27.

Exercises 154.

Solve the equations:

1. $x^3 + y^3 = 35,\ x + y = 5.$
2. $x^3 - y^3 = 7,\ x - y = 1.$
3. $x + y = 7,\ x^3 + y^3 = 91.$
4. $x^3 - y^3 = 386,\ x - y = 2.$
5. $x^3 + y^3 = 28,\ x^2 - xy + y^2 = 7.$

6. $x^3 + y^3 = 126,\ x^2 - xy + y^2 = 21.$

7. $x^3 - y^3 = 56,\ x^2 + xy + y^2 = 28.$

8. $x^2 + xy + y^2 = 93,\ x^3 - y^3 = 279.$

9. $2x - y = 5,\ 8x^3 - y^3 = 65.$

10. $27x^3 + y^3 = 341,\ 3x + y = 11.$

Type IV.

$$ax + by = c,\ lx^2y^2 + mxy + n = 0.$$

209. In this type, one equation is of the first degree, and the other a quadratic in xy. The method consists in replacing the latter by two simple equations in xy, and then solving each of these with the equation of the first degree in x and y, as in examples belonging to Type I. Since each pair of equations has two solutions, it follows that the given equations have four solutions.

210. *Example* 8. Solve the equations

$$x + y = 4,\ x^2y^2 + 2xy - 15 = 0.$$

$$x + y = 4, \qquad (1)$$

$$x^2y^2 + 2xy - 15 = 0. \qquad (2)$$

From (2), we have

$$(xy - 3)(xy + 5) = 0,$$

$$\therefore\ xy = 3, \qquad (3)$$

or

$$xy = -5. \qquad (4)$$

(i) $xy = 3$.

From (1) we have $\quad y = 4 - x.$

Substituting in (3),

$$4x - x^2 = 3,$$

$$\therefore\ x^2 - 4x + 3 = 0,$$

$$\therefore\ (x - 3)(x - 1) = 0,$$

the roots of which are 3 and 1.

Substituting in (1), we have

$$\text{when } x=3, \text{ then } y=1,$$
$$\text{when } x=1, \text{ then } y=3.$$

(ii) $xy=-5.$

From (1) we have

$$y=4-x.$$

Substituting in (4),

$$4x-x^2=-5,$$
$$\therefore\ x^2-4x-5=0,$$
$$\therefore\ (x-5)(x+1)=0,$$

the roots of which are 5 and -1.

Substituting in (1), we have

$$\text{when } x=5, \text{ then } y=-1,$$
$$\text{when } x=-1, \text{ then } y=5.$$

$\therefore$ the solutions are

$$\left.\begin{array}{l}x=3\\y=1\end{array}\right\}, \quad \left.\begin{array}{l}x=1\\y=3\end{array}\right\}, \quad \left.\begin{array}{l}x=\ \ \ 5\\y=-1\end{array}\right\} \text{ and } \left.\begin{array}{l}x=-1\\y=\ \ \ 5\end{array}\right\}.$$

211. In fig. 28, the continuous straight line is the graph of equation (1), and the continuous curves (3) and (4) the graph of equation (2). Hence, the method of solution is to find the coordinates of the points of intersection of the line (1) with the curves (3) and (4) separately.

212. *Example* 9. Solve the equations

$$x+y=0, \ x^2y^2+2xy-15=0.$$

$$x+y=0, \qquad (1)$$

$$x^2y^2+2xy-15=0. \qquad (2)$$

As in the preceding example, we have

$$xy=3, \qquad (3)$$

or

$$xy=-5. \qquad (4)$$

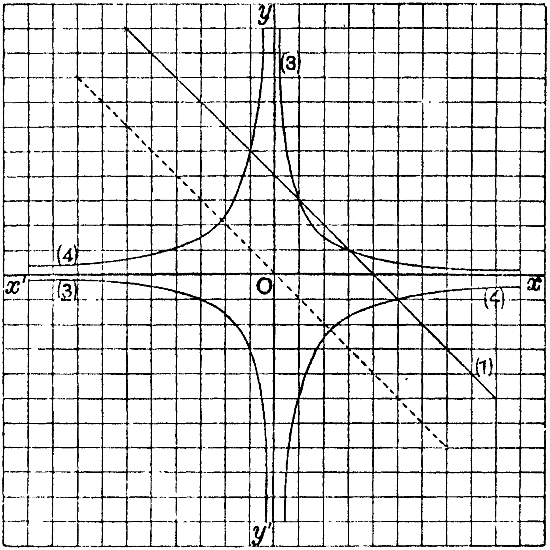


Fig. 28.

(i) $xy = 3$.

From (1) we have

$$y = -x,$$

$$\therefore\ -x^2 = 3,$$

$$\therefore\ x^2 + 3 = 0,$$

the roots of which are imaginary.

(ii) $xy = -5$.

From (1) we have

$$y = -x,$$

$$\therefore\ -x^2 = -5,$$

$$\therefore\ x^2 - 5 = 0,$$

$$\therefore\ (x - \sqrt{5})(x + \sqrt{5}) = 0,$$

the roots of which are $\sqrt{5}$ and $-\sqrt{5}$.

Substituting in (1) or (4), we have

$$\text{when } x = \sqrt{5}, \text{ then } y = -\sqrt{5},$$
$$\text{when } x = -\sqrt{5}, \text{ then } y = \sqrt{5}.$$

∴ the solutions are

$$\left.\begin{array}{l} x = \sqrt{5} \\ y = -\sqrt{5} \end{array}\right\}, \quad \left.\begin{array}{l} x = -\sqrt{5} \\ y = \sqrt{5} \end{array}\right\}, \text{ and two pairs of imaginary roots.}$$

213. In fig. 28, the broken straight line is the graph of $x+y=0$, and the continuous curves as before the graph of equation (2). The broken line cuts the graph of $xy=-5$ in two points $(\sqrt{5}, -\sqrt{5})$ and $(-\sqrt{5}, \sqrt{5})$, while it does not cut the graph of $xy=3$, or it may be said to cut the latter in two imaginary points.

Type V.

$$x^4 + x^2y^2 + y^4 = a, \quad x^2 \pm xy + y^2 = b.$$

214. Dividing

$$x^4 + x^2y^2 + y^4 \text{ by } x^2 \pm xy + y^2,$$

we obtain

$$x^2 \mp xy + y^2,$$

and therefore

$$x^2 \mp xy + y^2 = \frac{a}{b}.$$

This equation with the second of the given equations may be solved as in Type II.

215. *Example* 10. Solve the equations

$$x^4 + x^2y^2 + y^4 = 21, \quad x^2 + xy + y^2 = 7.$$

$$x^4 + x^2y^2 + y^4 = 21, \qquad (1)$$

$$x^2 + xy + y^2 = 7. \qquad (2)$$

Dividing, we have

$$x^2 - xy + y^2 = 3. \qquad (3)$$

Adding and subtracting (2) and (3), we get

$$2x^2 + 2y^2 = 10,$$

or

$$x^2 + y^2 = 5,$$

and

$$2xy = 4,$$

$$\therefore\ x^2 + 2xy + y^2 = 9,$$

and

$$x^2 - 2xy + y^2 = 1,$$

$$\therefore\ x + y = \pm 3,$$

and

$$x - y = \pm 1.$$

$$\therefore\ \left.\begin{matrix} x+y=3 \\ x-y=1 \end{matrix}\right\},\ \left.\begin{matrix} x+y=\ \ 3 \\ x-y=-1 \end{matrix}\right\},\ \left.\begin{matrix} x+y=-3 \\ x-y=\ \ 1 \end{matrix}\right\} \text{ and } \left.\begin{matrix} x+y=-3 \\ x-y=-1 \end{matrix}\right\},$$

$$\left.\begin{matrix} 2x=4 \\ 2y=2 \end{matrix}\right\},\ \left.\begin{matrix} 2x=2 \\ 2y=4 \end{matrix}\right\},\ \left.\begin{matrix} 2x=-2 \\ 2y=-4 \end{matrix}\right\} \text{ and } \left.\begin{matrix} 2x=-4 \\ 2y=-2 \end{matrix}\right\}.$$

$\therefore$ the solutions are

$$\left.\begin{matrix} x=2 \\ y=1 \end{matrix}\right\},\ \left.\begin{matrix} x=1 \\ y=2 \end{matrix}\right\},\ \left.\begin{matrix} x=-1 \\ y=-2 \end{matrix}\right\} \text{ and } \left.\begin{matrix} x=-2 \\ y=-1 \end{matrix}\right\}.$$

216. Thus, as in Example 6, instead of finding directly the coordinates of the points in which the given curves cut one another, the process corresponds to finding the points of intersection of either of the lines

$$x + y = 3 \text{ and } x + y = -3$$

with either of the lines

$$x - y = 1 \text{ and } x - y = -1.$$

Exercises 155.

Solve the equations:

1. $x^2y^2 - 14xy + 45 = 0,\ x + y = 6.$
2. $x + y = 8,\ x^2y^2 + 192 = 28xy.$
3. $x + y = 7,\ x^2y^2 + 24xy - 180 = 0.$

4. $x + y = 10, \ x^2y^2 + 3xy = 504.$

5. $x^2y^2 + 9xy + 18 = 0, \ x + y = 2.$

6. $x^2y^2 + 5xy = 84, \ x + y = 8.$

7. $x^4 + x^2y^2 + y^4 = 273, \ x^2 - xy + y^2 = 13.$

8. $x^4 + x^2y^2 + y^4 = 3, \ x^2 - xy + y^2 = 1.$

9. $x^2 + xy + y^2 = 19, \ x^4 + x^2y^2 + y^4 = 133.$

10. $x^4 + x^2y^2 + y^4 = 16, \ x^2 + xy + y^2 = 4.$

11. $x^4 + x^2y^2 + y^4 = 91, \ x^2 - xy + y^2 = 13.$

12. $x^4 + x^2y^2 + y^4 = 741, \ x^2 + xy + y^2 = 39.$

* Miscellaneous Equations.

217. *Example* 11. Solve the equations

$$x^2 + y^2 + x - y = 22, \ x + y = 6.$$

$$x^2 + y^2 + x - y = 22, \qquad (1)$$

$$x + y = 6. \qquad (2)$$

Put $x + y = u$, $x - y = v$, so that

$$u^2 + v^2 = 2(x^2 + y^2),$$

and the equations become

$$\tfrac{1}{2}(u^2 + v^2) + v = 22, \qquad (3)$$

$$u = 6. \qquad (4)$$

Substituting for u in (3), this equation becomes

$$36 + v^2 + 2v = 44,$$

$$\therefore \ v^2 + 2v - 8 = 0,$$

$$\therefore \ (v - 2)(v + 4) = 0,$$

the roots of which are 2 and -4.

$$\therefore \left.\begin{array}{l} x + y = 6 \\ x - y = 2 \end{array}\right\} \text{ and } \left.\begin{array}{l} x + y = 6 \\ x - y = -4 \end{array}\right\};$$

$$\therefore \left.\begin{array}{l} 2x = 8 \\ 2y = 4 \end{array}\right\} \text{ and } \left.\begin{array}{l} 2x = 2 \\ 2y = 10 \end{array}\right\}.$$

$\therefore$ the solutions are

$$\left.\begin{array}{l} x=4 \\ y=2 \end{array}\right\} \text{ and } \left.\begin{array}{l} x=1 \\ y=5 \end{array}\right\}.$$

Example 12. Solve the equations $x^2+y^2=10$, $x-xy+y=1$.

$$x^2+y^2=10, \tag{1}$$

$$x-xy+y=1. \tag{2}$$

Putting $x+y=u$, $xy=v$, the equations become

$$u^2-2v=10, \tag{3}$$

and

$$u-v=1. \tag{4}$$

Substituting $u-1$ for v in (3), we get

$$u^2-2(u-1)=10,$$

or

$$u^2-2u-8=0,$$

or

$$(u-4)(u+2)=0,$$

the roots of which are 4 and -2.

(i) $x+y=4$, giving by (4)

$$v=xy=3.$$

(ii) $x+y=-2$, giving by (4)

$$v=xy=-3.$$

$\therefore$ the solutions are

$$\left.\begin{array}{l} x=3 \\ y=1 \end{array}\right\}, \left.\begin{array}{l} x=1 \\ y=3 \end{array}\right\}, \left.\begin{array}{l} x=-3 \\ y=1 \end{array}\right\} \text{ and } \left.\begin{array}{l} x=1 \\ y=-3 \end{array}\right\}.$$

Exercises 156.

Solve the equations:

1. $x^2+y^2+x-y=36$, $x+y=8$.
2. $x^2+y^2+x+y=18$, $xy=6$.
3. $x^2+y^2-x-y=8$, $xy-x-y=1$.
4. $\dfrac{x}{y}+\dfrac{y}{x}=\dfrac{5}{2}$, $xy=8$.

5. $x^3y + xy^3 = 290,\ x^2 + y^2 = 29.$

6. $x^2 - 4y = y^2 + 4x = 21.$

7. $2x^2 + 3xy - y^2 - 8(x + y) = 0,\ 3x^2 - 2xy + y^2 + 11(x + y) = 0.$

8. $x + y = 3,\ \dfrac{x^3 + y^3}{x - y} = 9.$

9. $x^3 - xy + y^3 = 4,\ x + y = 1.$

10. $\dfrac{x^2}{y} + \dfrac{y^2}{x} = 27,\ \dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{4}.$

11. $x^2 + y^2 = 13,\ 2x - xy + 2y = 4.$

12. $x^2 + xy + y^2 = 7,\ xy(x + y) = 6.$

13. $x^2 - xy - 2y^2 - x - y = 5,\ 3x^2 - 7xy + 2y^2 - 3x + y = 11.$

14. $x^2 + y^2 + x - y = 6,\ xy + x - y = 3.$

15. $x^2 + 4y^2 + 7x + 14y = 12,\ xy + 6 = 0.$

16. $x^2 + 9y^2 + 3x - 9y = 28,\ xy = 4.$

17. $(x + 2y)^2 + (2x - y)^2 = 85,\ xy = 4.$

18. $(x + y)^2 - (x - y)^2 = 352,\ x(y + 5) = 143.$

19. $x^2 + y = 3,\ x + y^2 = 3.$

20. $x^3 + y^3 + xy(x + y) = 20,\ (x + 5)^2 + (y + 5)^2 = 80.$

2. Problems.

218. *Example* 13. Two lawns, one square and the other rectangular, each contain 400 sq. yds.; the perimeter of the rectangular lawn is one-fourth greater than that of the other; find its dimensions.

Let x yds. and y yds. be the lengths of the sides of the rectangular lawn.

Now, the area of the square lawn being 400 sq. yds., the length of one side is 20 yds., and therefore the perimeter is 80 yds., so that the perimeter of the rectangle is

$$80 + \tfrac{1}{4} \cdot 80, \text{ or } 100, \text{ yds.},$$

$$\therefore\ 2x + 2y = 100,$$

or $$x + y = 50,$$

and $$xy = 400,$$

$$\therefore\ x(50 - x) = 400,$$

$$\therefore\ x^2 - 50x + 400 = 0,$$

$$\therefore\ (x - 40)(x - 10) = 0,$$

the roots of which are 40 and 10.

$$\text{If } x = 40, \text{ then } y = 10;$$

$$\text{if } x = 10, \text{ then } y = 40.$$

$\therefore$ the rectangular lawn is 40 yds. long and 10 yds. wide.

Example 14. A farmer sold 7 oxen and 12 cows for £250, and he sold 3 more oxen for £50 than he did cows for £30; find the price of each.

Let £x be the price of an ox, and £y that of a cow; then

$$7x + 12y = 250.$$

Again, number of oxen sold for £50 is $\frac{50}{x}$, and number of cows sold for £30 is $\frac{30}{y}$; but the former number is greater than the latter by 3,

$$\therefore\ \frac{50}{x} = \frac{30}{y} + 3,$$

$$\therefore\ 50y = 30x + 3xy.$$

But $$x = \frac{250 - 12y}{7},$$

$$\therefore\ 50y = 30 \times \frac{250 - 12y}{7} + 3y \times \frac{250 - 12y}{7},$$

$$\therefore\ 350y = 7500 - 360y + 750y - 36y^2,$$

$$\therefore\ 36y^2 - 40y - 7500 = 0,$$

$$\therefore\ 9y^2 - 10y - 1875 = 0,$$

$$\therefore\ (y - 15)(9y + 125) = 0,$$

the roots of which are 15 and $-13\frac{8}{9}$, the latter being inapplicable as it is both negative and fractional.

Now, when $y = 15$, $x = \dfrac{250 - 180}{7} = 10$,

$\therefore$ he sold each ox for £10 and each cow for £15.

Exercises 157.

1. What two numbers have their sum 23 and the difference of their squares 69?

2. The difference of the reciprocals of two numbers is 2 and the difference of the squares of the reciprocals is 6; find the numbers.

3. The area of an oblong room is 221 sq. ft. and its perimeter is 60 ft.; find its length and breadth.

4. The hypotenuse of a right-angled triangle is 4 ft. less than the sum of the sides, and the area is 30 sq. ft.; find the sides.

5. The diagonal of a rectangle is 130 ft. and its area is 4032 sq. ft.; find the lengths of its sides.

6. Divide 1 into two fractions such that the sum of their cubes is $\frac{1}{3}$.

7. The sum of the cubes of two numbers is 18 times their product, and the sum of the numbers is 12; find the numbers.

8. Find a number of two digits such that, if it be divided by the product of its digits, the quotient is 2, and if 27 be added to the number the order of the digits is reversed.

9. Find two numbers such that the sum of their squares exceeds twice their product by 9, and the difference of their squares is 51.

10. A number consisting of two digits has one decimal place; the difference of the squares of the digits is 20, and if the digits be reversed the sum of the number thus obtained and the original number is 11; find the number.

11. A rectangular plot of ground measures 42 acres and its diagonal is 1243 yds. long; what are its sides?

12. The perimeter of a rectangular field of 36 acres is 2498 yds.; find its length and width.

13. The sum of two numbers multiplied by the greater is 84; their difference multiplied by the less is 10; find the numbers.

14. The product of two numbers is 75, and the quotient of the sum divided by the difference is four times the quotient of the difference divided by the sum; find the numbers.

15. A number of two digits is less than the sum of the squares of the digits by 14, but exceeds twice the product of the digits by 2; find the number.

16. A, B, C bicycle between two places, B rides 2 miles an hour faster than A, and C rides at the rate of 15 miles an hour; B starts 1 hour after A, and C 1 hour after B, and they all reach their destination together; find the distance between the two places, and the rates of riding of A and B.

17. A wine merchant sold 8 dozen of claret and 6 dozen of sherry for £31; he sold of sherry 2 dozen more for £10 than he did of claret for £4; find the price per dozen of each.

18. A man, being asked his age, answered "If you multiply my two digits together, the number formed will be my age 22 years ago, and if you add all the digits of the two ages you will have one-third of my present age"; how old is he?

19. 100 yds. of palisading surround a space in the shape of a rectangle with semi-circular ends and containing 546 sq. yds; find the width and extreme length of the enclosure. ($\pi = 3\frac{1}{7}$.)

20. The area of a rectangular piece of carpet is 168 sq. ft.; that of another, the dimensions of which are 3 ft. and 2 ft. longer than those of the former, is 238 sq. ft.; find the length and breadth of the carpets.

21. A rectangular piece of carpet has an area of 315 sq. ft.; if a piece be taken off so as to reduce its width by 3 ft., the remaining area is $\frac{14}{15}$ of what it would have been had its length been reduced by 3 ft.; what are its dimensions?

22. The sides of three cubes have equal differences, and their sum is 15 ins.; the total volume of the three is 495 c. ins.; find the dimensions and the volumes.

23. A merchant gained as many sovereigns on a certain quantity of coal as there were shillings in the cost price of a ton, or pence in the retail price of a cwt.; how many tons did he sell?

*3. SIMULTANEOUS QUADRATIC EQUATIONS, ETC., WITH THREE UNKNOWNS.

219. The following examples illustrate some of the methods of obtaining solutions of these equations.

Example 15. Solve the equations $yz = 6$, $zx = 3$, $xy = 2$.

$$yz = 6, \qquad (1)$$

$$zx = 3, \qquad (2)$$

$$xy = 2. \qquad (3)$$

Multiply (1) and (2),

$$\therefore\ xyz^2 = 18.$$

Dividing by (3),

$$\therefore\ z^2 = 9,$$

$$\therefore\ z = \pm 3.$$

$$\text{If } z = 3, \quad \text{then } y = 2 \quad \text{and } x = 1,$$

$$\text{if } z = -3, \text{ then } y = -2 \text{ and } x = -1.$$

$\therefore$ the solutions are

$$\left.\begin{array}{l}x=1\\y=2\\z=3\end{array}\right\} \text{ and } \left.\begin{array}{l}x=-1\\y=-2\\z=-3\end{array}\right\}.$$

Example 16. Solve the equations

$$xy=15,\ xz=5,\ y^2+z^2=10.$$

$$xy=15, \tag{1}$$

$$xz=\ 5, \tag{2}$$

$$y^2+z^2=10. \tag{3}$$

Dividing (1) by (2), we have

$$\frac{y}{z}=3 \text{ or } y=3z.$$

Substituting in (3),

$$9z^2+z^2=10,$$

$$\therefore\ z^2=1 \text{ or } z=\pm 1.$$

If $z=1$, then $y=3$ and $x=5$,

if $z=-1$, then $y=-3$ and $x=-5$.

the solutions are

$$\left.\begin{array}{l}x=5\\y=3\\z=1\end{array}\right\} \text{ and } \left.\begin{array}{l}x=-5\\y=-3\\z=-1\end{array}\right\}.$$

***Example* 17.** Solve the equations

$$x(y+z)=12,\ y(z+x)=6,\ z(x+y)=10.$$

The equations may be written.

$$xy+zx=12, \tag{1}$$

$$yz+xy=\ 6, \tag{2}$$

$$zx+yz=10. \tag{3}$$

Subtracting (2) from (3),

$$\therefore\ zx-xy=4,$$

$$\therefore\ 2zx=16,\ 2xy=8,$$

$$\therefore\ zx=8,\ xy=4.$$

Similarly, it may be shown that $yz = 2$,

$$\therefore \frac{x}{4} = \frac{y}{1} = \frac{z}{2} = m, \text{ say.}$$

Substituting in (1),

$$\therefore 4m(m + 2m) = 12 \text{ or } m^2 = 1 \text{ or } m = \pm 1.$$

$\therefore$ the solutions are

$$\left.\begin{array}{l} x = 4 \\ y = 1 \\ z = 2 \end{array}\right\} \text{ and } \left.\begin{array}{l} x = -4 \\ y = -1 \\ z = -2 \end{array}\right\}.$$

Example 18. Solve the equations

$$y^2 + zx = 3, \; z^2 + xy = 3, \; yz = 1.$$

$$y^2 + zx = 3, \qquad (1)$$

$$z^2 + xy = 3, \qquad (2)$$

$$yz = 1. \qquad (3)$$

Subtracting (2) from (1), we have

$$y^2 - z^2 - x(y - z) = 0,$$

$$\therefore (y - z)(y + z - x) = 0,$$

$$\therefore y = z \text{ or } x = y + z.$$

(i) $y = z$, then (3) becomes

$$y^2 = 1, \text{ or } y = \pm 1.$$

If $y = 1$, then $z = 1$, and $1 + x = 3$ or $x = 2$,

if $y = -1$, then $z = -1$ and $1 - x = 3$ or $x = -2$.

(ii) $x = y + z$. Adding (1) and (2),

$$\therefore y^2 + z^2 + x(y + z) = 6,$$

$$\therefore y^2 + z^2 + (y + z)^2 = 6,$$

$$\therefore y^2 + yz + z^2 = 3.$$

But $yz = 1$ by (3),

$$\therefore (y + z)^2 = 4 \text{ and } (y - z)^2 = 0,$$

$$\therefore y = z = 1, \; x = 2, \text{ and } y = z = -1, \; x = -2,$$

$\therefore$ the solutions are

$$\left.\begin{array}{r}x=2\\y=1\\z=1\end{array}\right\},\quad\left.\begin{array}{r}x=-2\\y=-1\\z=-1\end{array}\right\},\quad\left.\begin{array}{r}x=2\\y=1\\z=1\end{array}\right\},\text{ and }\left.\begin{array}{r}x=-2\\y=-1\\z=-1\end{array}\right\}.$$

Example 19. Solve the equations

$$x+y+z=7,\ x^2+y^2+z^2=21,\ yz=8.$$

$$x+y+z=7, \qquad (1)$$

$$x^2+y^2+z^2=21, \qquad (2)$$

$$yz=8. \qquad (3)$$

From (2),

$$y^2+z^2=21-x^2.$$

Squaring (1)

$$y^2+2yz+z^2=49-14x+x^2,$$

$\therefore$ by (3)

$$y^2+z^2+16=49-14x+x^2,$$

or

$$y^2+z^2=33-14x+x^2,$$

$$\therefore\ x^2-14x+33=21-x^2,$$

$$\therefore\ 2x^2-14x+12=0,$$

$$\therefore\ x^2-7x+6=0,$$

$$\therefore\ (x-1)(x-6)=0,$$

the roots of which are 1 and 6.

(i) If $x=1$,

$$\therefore\ y+z=6 \text{ and } yz=8,$$

$$\therefore\ y=4,\ z=2, \text{ and } y=2,\ z=4.$$

(ii) If $x=6$, then $y+z=1$ and $yz=8$, which leads to imaginary roots,

$\therefore$ the real solutions are

$$\left.\begin{array}{r}x=1\\y=4\\z=2\end{array}\right\}\text{ and }\left.\begin{array}{r}x=1\\y=2\\z=4\end{array}\right\}.$$

Example 20. Solve the equations

$$x + y + z = 4,\ \ x^2 + y^2 + z^2 = 14,\ \ x^3 + y^3 + z^3 = 34.$$

$$(x + y + z)^2 = x^2 + y^2 + z^2 + 2\,(yz + zx + xy),$$

$$\therefore\ \ 16 = 14 + 2\,(yz + zx + xy),$$

$$\therefore\ \ yz + zx + xy = 1,$$

$$(x + y + z)^3 = x^3 + y^3 + z^3 + 3\,(y^2z + yz^2 + z^2x + zx^2 + x^2y + xy^2) + 6xyz,$$

$$\therefore\ \ 64 = 34 + 3\,(x + y + z)\,(yz + zx + xy) - 3xyz,$$

$$\therefore\ \ 30 = 12\ \ 1 - 3xyz,$$

$$\therefore\ \ xyz = -6.$$

Now, x, y, z are roots of the equation

$$(\lambda - x)\,(\lambda - y)\,(\lambda - z) = 0,$$

or $$\lambda^3 - \lambda^2\,(x + y + z) + \lambda\,(yz + zx + xy) - xyz = 0,$$

or $$\lambda^3 - 4\lambda^2 + \lambda + 6 = 0,$$

or $$(\lambda + 1)\,(\lambda^2 - 5\lambda + 6) = 0,$$

or $$(\lambda + 1)\,(\lambda - 2)\,(\lambda - 3) = 0,$$

the roots of which are -1, 2 and 3,

$\therefore$ the solutions are

$$\left.\begin{array}{l} x = -1 \\ y = 2 \\ z = 3 \end{array}\right\},\ \left.\begin{array}{l} x = -1 \\ y = 3 \\ z = 2 \end{array}\right\},\ \left.\begin{array}{l} x = 2 \\ y = -1 \\ z = 3 \end{array}\right\},\ \left.\begin{array}{l} x = 2 \\ y = 3 \\ z = -1 \end{array}\right\},\ \left.\begin{array}{l} x = 3 \\ y = -1 \\ z = 2 \end{array}\right\} \text{ and } \left.\begin{array}{l} x = 3 \\ y = 2 \\ z = -1 \end{array}\right\}.$$

Exercises 158.

Solve the equations :

1. $yz = 4,\ zx = 2,\ xy = 2.$
2. $(x + y)\,(x + z) = 30,\ (y + z)\,(y + x) = 18,\ (z + x)\,(z + y) = 15.$
3. $xy = 8,\ xz = 4,\ y^2 + z^2 = 5.$
4. $xy = 2,\ yz = 8,\ x^2 + xz + z^2 = 21.$
5. $y^2z = 36,\ z^2x = 32,\ x^2y = 12.$

6. $yz^2 = 75,\ zx^2 = 20,\ xy^2 = 18.$

7. $xyz = 231,\ xyw = 420,\ yzw = 1540,\ xzw = 660.$

8. $xy = 8 - y,\ xz = 24 - z,\ yz = 12.$

9. $x(y + z) = 3,\ y(z + x) = 4,\ z(x + y) = 3.$

10. $xy^2z^3 = 108,\ yz^2 = 18x,\ 2z = 3yx^2.$

11. $x(y + z) = 14,\ y(x - z) = 50,\ x + y - z = 15.$

12. $x + y = z = 2xy,\ \frac{1}{x} - \frac{1}{y} = 1.$

13. $x^2(y + z) = 64,\ y^2(z + x) = 7,\ xyz = 12.$

14. $x + y + z = 4,\ xy = 1,\ x^2 + y^2 + z^2 = 6.$

15. $x + y - z = 2,\ x^2 + y^2 + z^2 = 6,\ 2xy = 4.$

16. $x + y + z = 6,\ x^2 + y^2 + z^2 = 14,\ yz = 2.$

17. $xy = z^2,\ x + y + z = 19,\ x^2 + y^2 + z^2 = 133.$

18. $x - y - z = 4,\ x^2 + y^2 - z^2 = 88,\ xy = 20.$

19. $x^2 + yz = 5,\ y^2 + zx = 3,\ z^2 + xy = 3.$

20. $2(x^2 + y^2 + z^2) = 3(x + y + z) = 12,\ z^2 = 4xy.$

21. $x^2 + y^2 + z^2 = 14,\ yz + zx + xy = 11,\ yz + zx - xy = -1.$

22. $y + z + yz = 11,\ z + x + zx = 7,\ x + y + xy = 5.$

23. $x + y + z = 5,\ x^2 + y^2 + z^2 = 11,\ x^3 + y^3 + z^3 = 29.$

24. $yz = a^2,\ zx = b^2,\ xy = c^2.$

25. $x(x + y + z) = a^2,\ y(x + y + z) = b^2,\ z(x + y + z) = c^2.$

*CHAPTER XXII.

THE REMAINDER THEOREM AND INDETERMINATE COEFFICIENTS.

1. Remainder Theorem.

220. *If the expression*

$$ax^n + bx^{n-1} + cx^{n-2} + \dots + kx + l$$

be divided by $x - a$, *the remainder is*

$$aa^n + ba^{n-1} + ca^{n-2} + \dots + ka + l.$$

Let the quotient be denoted by Q and the remainder by R, so that

$$ax^n + bx^{n-1} + cx^{n-2} + \dots + kx + l = Q(x - a) + R.$$

Now, since the divisor is of the first degree in x, R does not contain x and therefore does not change when x is changed. Put a for x in this equation, then, if Q become Q′,

$$aa^n + ba^{n-1} + ca^{n-2} + \dots + ka + l = Q'(a - a) + R$$
$$= R,$$

∴ the remainder is obtained by substituting a for x in the given expression.

This theorem is known as the *Remainder Theorem.*

221. (i) *If* a *be a root of the equation*

$$ax^n + bx^{n-1} + cx^{n-2} + \dots + kx + l = 0,$$

then $x - a$ *is a factor of the expression*

$$ax^n + bx^{n-1} + cx^{n-2} + \dots + kx + l,$$

and (ii) *conversely, if $x - a$ be a factor of this expression, then a is a root of the corresponding equation.*

(i) Since a is a root of the given equation,

$$\therefore\ aa^n + ba^{n-1} + ca^{n-2} + \dots + ka + l = 0,$$

$$\therefore\ ax^n + bx^{n-1} + cx^{n-2} + \dots + kx + l = Q(x - a),$$

or $x - a$ is a factor of this expression.

(ii) Since $x - a$ is a factor of this expression,

$$\therefore\ ax^n + bx^{n-1} + cx^{n-2} + \dots + kx + l = Q(x - a),$$

$\therefore$ the remainder, or

$$aa^n + ba^{n-1} + ca^{n-2} + \dots + ka + l,$$

is zero,

i.e. a is a root of the equation

$$ax^n + bx^{n-1} + cx^{n-2} + \dots + kx + l = 0.$$

222. *If n be integral,* (i) *$x^n - a^n$ is divisible by $x - a$ whether n be odd or even, and by $x + a$ if n be even;* (ii) *$x^n + a^n$ is divisible by $x + a$ if n be odd, and is never divisible by $x - a$.*

(i) Since $a^n - a^n$ is zero,

$\therefore$ $x^n - a^n$ is divisible by $x - a$, whether n be odd or even.

Since $(-a)^n - a^n$ is zero when n is even,

$\therefore$ $x^n - a^n$ is divisible by $x - (-a)$ or $x + a$, when n is even.

(ii) Since $(-a)^n + a^n$ is zero when n is odd,

$\therefore$ $x^n + a^n$ is divisible by $x - (-a)$ or $x + a$, when n is odd.

Since $a^n + a^n$ is never zero,

$$\therefore\ x^n + a^n \text{ is never divisible by } x - a.$$

223. *Example* 1. Find the remainder when

$$x^3 - 3x^2 + 5x - 7 \text{ is divided by } x - 2.$$

The remainder is

$$2^3 - 3 \,.\, 2^2 + 5 \,.\, 2 - 7 \text{ or } 8 - 12 + 10 - 7 \text{ or } -1.$$

Example 2. Find the remainder when

$$x^3 + 6x^2 + 5x - 12 \text{ is divided by } x + 3.$$

The divisor may be written $x - (-3)$, thus the remainder is $(-3)^3 + 6(-3)^2 + 5(-3) - 12$, or $-27 + 54 - 15 - 12$, or 0.

Example 3. Find the value of m if

$$x^4 + mx^2 - 4x - 6$$

be divisible by $x - 3$.

Since $x^4 + mx^2 - 4x - 6$ is divisible by $x - 3$, the remainder is zero,

$$\therefore\ 3^4 + m \cdot 3^2 - 4 \cdot 3 - 6 = 0,$$

$$\therefore\ 9m = -81 + 12 + 6 = -63,$$

$$\therefore\ m = -7.$$

Exercises 159.

Find the remainder when:

1. $x^2 + 4x + 5$ is divided by $x - 1$.
2. $5x^2 - 7x + 3$ is divided by $x - 5$.
3. $x^3 - 3x^2 - 18x + 40$ is divided by $x - 2$.
4. $2x^2 + 4x + 3$ is divided by $x + 2$.
5. $x^3 + 2x^2 - 4x - 3$ is divided by $x + 5$.
6. $x^2 - 5x + 2$ is divided by $3x - 1$.
7. $2x^3 + x^2 + 4x + 4$ is divided by $2x - 3$.
8. $x^3 + x^2 - 2x + 1$ is divided by $2x + 3$.
9. Prove that the remainder left when $x^4 + 7x + 3$ is divided by $x - 1$ is the same as that left when $x^4 - 6x + 7$ is divided by $x - 2$.
10. Is $x^3 + 2x^2 - 7x - 6$ divisible by $x - 2$?
11. Is $2x^4 - 5x^2 + 7x - 1$ divisible by $x - 1$?

12. If $x^2 + ax - 4$ be divisible by $x - 1$, what is the value of a?

13. If $x + 2$ divide $x^3 + 4x^2 + 9x + a$ exactly, what is the value of a?

14. What value of a will make $4x^4 - 12x^3 + ax^2 - 6x + 1$ divisible by $2x - 1$?

224. *Example* 4. Find the factors of

$$x^3 + 2x^2 - 7x + 4.$$

By trial, it is seen that

$$1^3 + 2 \cdot 1^2 - 7 \cdot 1 + 4$$

is equal to zero,

$$\therefore\ x - 1 \text{ is a factor of } x^3 + 2x^2 - 7x + 4.$$

The other factor is found by division to be $x^2 + 3x - 4$,

$$\therefore\ x^3 + 2x^2 - 7x + 4 = (x - 1)^2 (x + 4).$$

Example 5. Solve the equation

$$x^3 - 9x^2 + 26x - 24 = 0.$$

By trial, it is seen that

$$2^3 - 9 \cdot 2^2 + 26 \cdot 2 - 24 = 8 - 36 + 52 - 24 = 0,$$

$$\therefore\ x - 2 \text{ is a factor of } x^3 - 9x^2 + 26x - 24.$$

The other factor is found by division to be

$$x^2 - 7x + 12.$$

$\therefore$ the given equation becomes

$$(x - 2)(x - 3)(x - 4) = 0,$$

$\therefore$ the roots are 2, 3 and 4.

Exercises 160.

Find the factors of

1. $x^3 - 6x^2 + 11x - 6.$

2. $x^3 + 2x^2 - 11x - 12.$

3. $x^3 + 9x^2 + 26x + 24.$

4. $2x^3 + 3x^2 - 1.$

5. $x^4 + 2x^3 - 13x^2 - 14x + 24.$

6. $x^4 + 5x^3 + 5x^2 - 5x - 6.$

Solve the equations:

7. $x^3 - 3x^2 - 6x + 8 = 0$, given one root 1.

8. $x^3 + 4x^2 - 7x - 10 = 0$, given one root -1.

9. $x^3 - 2x^2 - x + 2 = 0$.

10. $x^3 - 3x + 2 = 0$.

11. $x^3 - 4x^2 + x + 6 = 0$.

12. $x^3 + x^2 - 8x - 12 = 0$.

225. DEF. 27. An expression which is unchanged when any two of its letters are interchanged is called a *symmetrical* expression.

Thus, $a + b$, $a + b + c$, $a^2 + b^2 + c^2$, $bc + ca + ab$, etc., are symmetrical expressions. The complete symmetrical expression of the first degree in a, b, c is $\mathsf{L}(a + b + c)$, that of the second degree is $\mathsf{L}(a^2 + b^2 + c^2) + \mathsf{M}(bc + ca + ab)$, and that of the third degree is

$$\mathsf{L}(a^3 + b^3 + c^3) + \mathsf{M}(b^2c + bc^2 + c^2a + ca^2 + a^2b + ab^2) + \mathsf{N}abc,$$

where L, M, N are numerical factors.

DEF. 28. An expression which is changed in sign, but is otherwise unchanged, when any two of its letters are interchanged is called an *alternating* expression.

Thus, $b - c$, $(b - c)(c - a)(a - b)$, etc., are alternating expressions.

226. If one symmetrical expression be divided by another, it is evident that the quotient is also symmetrical; and, if one alternating expression be divided by another, that the quotient is symmetrical.

Thus, $a^3 + b^3$ is a symmetrical expression of the third degree, and $a + b$ one of the first degree, and the quotient must be a symmetrical expression of the second degree, for it is unchanged if a and b be interchanged. In order to find it, put

$$a^3 + b^3 = (a + b)\{\mathsf{L}(a^2 + b^2) + \mathsf{M}ab\}.$$

Since this is an identity, it is true for all values of a and b, and therefore when $a = 1$, $b = 0$, in which case

$$1 = 1 . (\mathsf{L} . 1) \text{ or } \mathsf{L} = 1.$$

Next, put $a = 1$, $b = 1$, then

$$1 + 1 = 2\{\mathrm{L}(1+1) + \mathrm{M}\,.\,1\},$$

$$\therefore\ 2\mathrm{L} + \mathrm{M} = 1 \text{ or } \mathrm{M} = -1.$$

$$\therefore\ a^3 + b^3 = (a+b)(a^2 - ab + b^2).$$

Again, $a^3 - b^3$ is an alternating expression of the third degree, and $a - b$ one of the first degree, and, if a and b be interchanged, the quotient is unchanged in sign and is therefore a symmetrical expression of the second degree. In order to find it, put

$$a^3 - b^3 = (a-b)\{\mathrm{L}(a^2 + b^2) + \mathrm{M}ab\}.$$

Put $a = 1$, $b = 0$, then $1 = 1\,.\,(\mathrm{L}\,.\,1)$ or $\mathrm{L} = 1$.

Next, put $a = 2$, $b = 1$,

$$\therefore\ 2^3 - 1^3 = (2-1)\{\mathrm{L}(4+1) + \mathrm{M}\,.\,2\},$$

$$\therefore\ 5\mathrm{L} + 2\mathrm{M} = 7 \text{ or } \mathrm{M} = 1,$$

$$\therefore\ a^3 - b^3 = (a-b)(a^2 + ab + b^2).$$

In the latter case, it should be noticed that different values must be given to a and b, for if both were put equal to unity, both sides of the equation would vanish.

227. Though not a consequence of the Remainder Theorem, the following example illustrates the methods used in this chapter.

Example 6. Expand $(a + b)^4$.

Since $(a+b)^4$ is the product of four symmetrical and homogeneous expressions each of the first degree, its expansion is a symmetrical and homogeneous expression of the fourth degree. Thus, we may write

$$(a+b)^4 = \mathrm{L}(a^4 + b^4) + \mathrm{M}(a^3b + ab^3) + \mathrm{N}a^2b^2.$$

Put $a = 1$, $b = 0$, then $1 = \mathrm{L}$.

Put $a = b = 1$, then

$$16 = 2\mathrm{L} + 2\mathrm{M} + \mathrm{N},$$

or

$$2\mathrm{M} + \mathrm{N} = 14.$$

Put $a=1$, $b=-1$, then

$$0 = 2L - 2M + N,$$

$$\therefore\ 2M - N = 2,$$

$$\therefore\ 4M = 16,\ 2N = 12,$$

$$\therefore\ M = 4,\ N = 6,$$

$$\therefore\ (a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

228. *Example* 7. Find the factors of

$$a^2(b-c) + b^2(c-a) + c^2(a-b).$$

Put $b=c$, then the expression becomes

$$c^2(c-a) + c^2(a-c),$$

which is equal to zero.

$\therefore\ b-c$ is a factor of the expression.

Similarly, $c-a$ and $a-b$ are factors,

$$\therefore\ (b-c)(c-a)(a-b) \text{ is a factor.}$$

This expression is of the third degree in a, b, c, and the given expression is also of the third degree, so that the only other factor is numerical. Let

$$a^2(b-c) + b^2(c-a) + c^2(a-b) = L(b-c)(c-a)(a-b).$$

To find L, put $a=2$, $b=1$, $c=0$, then

$$4 + (-2) = L \,.\, 1 \,.\, (-2) \,.\, 1,$$

$$\therefore\ 2 = -2L \text{ or } L = -1,$$

$$\therefore\ a^2(b-c) + b^2(c-a) + c^2(a-b) = -(b-c)(c-a)(a-b).$$

The value of L may also be determined as follows: The coefficient of a^2b in the given expression is $+1$, and that of a^2b in $(b-c)(c-a)(a-b)$ is -1, for the only term containing a^2b is formed by the product $b(-a)(a)$;

$$\therefore\ L = -1.$$

Example 8. Find the factors of

$$a^3(b-c) + b^3(c-a) + c^3(a-b).$$

It may be shown, as in the previous example, that

$$(b-c)(c-a)(a-b)$$

is a factor. Now, this is an alternating expression of the third degree in a, b, c, and the given expression is one of the fourth degree; thus, the remaining factor must be a symmetrical expression of the first degree in a, b, c, say $\text{L}(a+b+c)$; or

$$a^3(b-c)+b^3(c-a)+c^3(a-b)=\text{L}(a+b+c)(b-c)(c-a)(a-b).$$

Put $$a=2,\ b=1,\ c=0,$$

$$\therefore\ 8+(-2)=\text{L}\,.\,3\,.\,1\,.\,(-2)\,.\,1,\ \text{ or }\ \text{L}=-1,$$

$$\therefore\ a^3(b-c)+b^3(c-a)+c^3(a-b)$$
$$=-(b-c)(c-a)(a-b)(a+b+c).$$

Example 9. Find the factors of

$$a^4(b-c)+b^4(c-a)+c^4(a-b).$$

As before, $(b-c)(c-a)(a-b)$ is a factor, and since this is an alternating expression of the third degree, and the given expression one of the fifth degree, the remaining factor must be a symmetrical expression of the second degree, say

$$\text{L}(a^2+b^2+c^2)+\text{M}(bc+ca+ab),$$

so that

$$a^4(b-c)+b^4(c-a)+c^4(a-b)$$
$$=(b-c)(c-a)(a-b)\{\text{L}(a^2+b^2+c^2)+\text{M}(bc+ca+ab)\}.$$

Put $a=2$, $b=1$, $c=0$, then

$$16+(-2)=1\,.\,(-2)\,.\,1\,(5\text{L}+2\text{M}),$$
$$\therefore\ 5\text{L}+2\text{M}=-7.$$

Next, put $a=1$, $b=-1$, $c=0$, then

$$-1-1=(-1)(-1)\,.\,2\,.\,(2\text{L}-\text{M}),$$
$$\therefore\ 2\text{L}-\text{M}=-1,$$
$$\therefore\ 9\text{L}=-9\ \text{ or }\ \text{L}=-1\ \text{ and }\ \text{M}=-1,$$

$$\therefore\ a^4(b-c)+b^4(c-a)+c^4(a-b)$$
$$=-(b-c)(c-a)(a-b)(a^2+b^2+c^2+bc+ca+ab).$$

Exercises 161.

1. Prove that

$$a(b+c-a)^2 + b(c+a-b)^2 + c(a+b-c)^2 + (b+c-a)(c+a-b)(a+b-c) = 4abc.$$

2. Prove that

$$(b-c)^3 + (c-a)^3 + (a-b)^3 = 3(b-c)(c-a)(a-b).$$

3. Prove that

$$x(y+z)^2 + y(z+x)^2 + z(x+y)^2 - 4xyz = (y+z)(z+x)(x+y).$$

4. Prove that $b+c$ is a factor of

$$(bc+ca+ab)(a+b+c) - abc,$$

and find the other factors.

5. Given that $a+b+c$ is a factor of $a^3+b^3+c^3-3abc$, find the other factor.

Find the factors of:

6. $(a+b+c)^2 - (a+b-c)^2$.

7. $bc(b-c) + ca(c-a) + ab(a-b)$.

8. $a(b^2-c^2) + b(c^2-a^2) + c(a^2-b^2)$.

9. $(y+z)yz + (z+x)zx + (x+y)xy + 2xyz$.

10. $(x+y+z)^3 - x^3 - y^3 - z^3$.

11. $bc(b^2-c^2) + ca(c^2-a^2) + ab(a^2-b^2)$.

12. Prove that

$$a(b-c)^3 + b(c-a)^3 + c(a-b)^3 = (b-c)(c-a)(a-b)(a+b+c).$$

13. Prove that

$$a(b^3-c^3) + b(c^3-a^3) + c(a^3-b^3) = (b-c)(c-a)(a-b)(a+b+c).$$

14. Find the factors of

$$(a+b+c)^4 - (b+c)^4 - (c+a)^4 - (a+b)^4 + a^4 + b^4 + c^4.$$

15. Prove that $(b-c)^5 + (c-a)^5 + (a-b)^5$ is divisible by $(b-c)(c-a)(a-b)$, and find the other factor.

16. Prove that $(x+y+z)^5 - x^5 - y^5 - z^5$ is divisible by

$$(y+z)(z+x)(x+y),$$

and find the other factor.

17. Show that $(y-z)(z-x)(x-y)$ is a factor of

$$x^3y^2 + y^3z^2 + z^3x^2 - x^2y^3 - y^2z^3 - z^2x^3,$$

and find the other factor.

Find the factors of:

18. $a^3(b^2-c^2) + b^3(c^2-a^2) + c^3(a^2-b^2)$.

19. $a(b^4-c^4) + b(c^4-a^4) + c(a^4-b^4)$.

20. $a^2(b^3-c^3) + b^2(c^3-a^3) + c^2(a^3-b^3)$.

21. $b^2c^2(b^2-c^2) + c^2a^2(c^2-a^2) + a^2b^2(a^2-b^2)$.

22. $a(b^5-c^5) + b(c^5-a^5) + c(a^5-b^5)$.

23. Prove that the difference of the expressions

$$ax^3 + bx^2 + cx + d \text{ and } am^3 + bm^2 + cm + d$$

is divisible by $x - m$, and write down the quotient.

24. Prove that $(x-1)^2$ is a common factor of

$$nx^{n+1} - (n+1)x^n + 1 \text{ and } x^n - nx + n - 1.$$

Simplify:

25. $$\frac{bc}{(c-a)(a-b)} + \frac{ca}{(a-b)(b-c)} + \frac{ab}{(b-c)(c-a)}.$$

26. $$\frac{a^2}{(a-b)(a-c)} + \frac{b^2}{(b-c)(b-a)} + \frac{c^2}{(c-a)(c-b)}.$$

27. $$\frac{1}{a(a-b)(a-c)} + \frac{1}{b(b-c)(b-a)} + \frac{1}{c(c-a)(c-b)}.$$

28. $$\frac{y^2z^2(y^2-z^2) + z^2x^2(z^2-x^2) + x^2y^2(x^2-y^2)}{yz(y-z) + zx(z-x) + xy(x-y)}.$$

29. $$\frac{(y-z)^5 + (z-x)^5 + (x-y)^5}{(y-z)^3 + (z-x)^3 + (x-y)^3}.$$

30. If $f(x)$ denote any expression involving x, and be divided by $(x-a)(x-b)$, the remainder being $\lambda x+\mu$, show that

$$\lambda=\frac{f(a)-f(b)}{a-b}, \quad \mu=\frac{bf(a)-af(b)}{b-a}.$$

2. Indeterminate Coefficients.

229. *If two expressions involving x and each containing a finite number of terms be equal for all values of x, the coefficients of the same powers of x in the two expressions are equal.*

Let $ax^n+bx^{n-1}+\ldots+hx^2+kx+l$ be equal to

$$a'x^n+b'x^{n-1}+\ldots+h'x^2+k'x+l'$$

for all values of x; to show that

$$a=a', \; b=b', \ldots, \; h=h', \; k=k', \text{ and } l=l'.$$

Since the expressions are equal,

$$(a-a')x^n+(b-b')x^{n-1}+\ldots+(h-h')x^2+(k-k')x+l-l'$$

is equal to zero for all values of x.

Put $x=0$, then $l-l'=0$, or $l=l'$.

Dividing by x, the expression reduces to

$$(a-a')x^{n-1}+(b-b')x^{n-2}+\ldots+(h-h')x+k-k'$$

which is equal to zero for all values of x.

Put $x=0$, then $k-k'=0$ or $k=k'$.

Similarly, $\qquad h=h', \ldots, b=b', a=a'$.

Cor. If one expression be of lower dimensions than the other, then the coefficients of all higher powers of x in the other are equal to zero.

230. *Example* **10.** Find the values of A and B when

$$A(x-2)+B(x+3)$$

is equal to $10x$ for all values of x.

Since $\qquad A(x-2)+B(x+3)=10x,$

$$\therefore \quad A+B=10 \text{ and } -2A+3B=0,$$

$$\therefore \quad 5B=20 \text{ or } B=4 \text{ and } A=6.$$

Another, and generally briefer, method is to put $x=-3$, then

$$A(-5)=-30 \text{ or } A=6;$$

again, put $x=2$, then

$$B.5=20 \text{ or } B=4.$$

Example 11. If x^2-2x+3 be a factor of x^4-4x^2+Ax+B, find A and B.

Let $x^4-4x^2+Ax+B=(x^2-2x+3)\left(x^2+\lambda x+\frac{B}{3}\right)$

$$=x^4+x^3(\lambda-2)+x^2\left(3-2\lambda+\frac{B}{3}\right)+x\left(3\lambda-\frac{2B}{3}\right)+B.$$

Equating the coefficients of x^3, x^2 and x in the two expressions, we have

$$\lambda-2=0, \quad 3-2\lambda+\frac{B}{3}=-4 \text{ and } 3\lambda-\frac{2B}{3}=A,$$

$$\therefore \quad \lambda=2, \quad 3-4+\frac{B}{3}=-4, \quad 6-\frac{2B}{3}=A,$$

$$\therefore \quad B=-9, \quad A=6+6=12.$$

The values of A and B may also be found by division as follows:

$$\begin{array}{r|l}
x^2-2x+3\,)\,x^4 \qquad -4x^2+Ax+B & (\,x^2+2x-3 \\
\underline{x^4-2x^3+3x^2} \qquad\qquad & \\
2x^3-7x^2+Ax+B & \\
\underline{2x^3-4x^2+6x} \qquad & \\
-3x^2+(A-6)x+B & \\
\underline{-3x^2 \qquad +6x-9} & \\
x(A-12)+B+9 &
\end{array}$$

Since the remainder is zero for all values of x, we have

$$A=12, \quad B=-9.$$

Example 12. If $ax^3+3bx^2+3cx+d$ be divisible by

$$ax^2+2bx+c,$$

the former expression is a perfect cube and the latter a perfect square.

Let $$ax^3+3bx^2+3cx+d=(ax^2+2bx+c)\left(x+\frac{d}{c}\right),$$

then $$3b=2b+\frac{ad}{c} \text{ and } 3c=c+\frac{2bd}{c},$$

$$\therefore\ b=\frac{ad}{c} \text{ and } c=\frac{bd}{c},$$

$$\therefore\ bc=ad \text{ and } c^2=bd.$$

Equating values of d from these equations, we have

$$\frac{bc}{a}=\frac{c^2}{b} \text{ or } b^2=ac,$$

$\therefore$ $ax^2+2bx+c$ is a perfect square.

Again, since $$c=\frac{b^2}{a} \text{ and } d=\frac{bc}{a}=\frac{b^3}{a^2},$$

$$\therefore\ ax^3+3bx^2+3cx+d=a\left(x^3+\frac{3b}{a}x^2+\frac{3c}{a}x+\frac{d}{a}\right),$$

$$=a\left(x^3+3\frac{b}{a}x^2+3\frac{b^2}{a^2}x+\frac{b^3}{a^3}\right),$$

$$=a\left(x+\frac{b}{a}\right)^3,$$

which is a perfect cube.

Exercises 162.

1. If $A(x-2)+B(x-3)=1$ for all values of x, find A and B.

2. If $(x+1)(x+2)+A(x+3)+B=x^2$ for all values of x, find A and B.

3. If $A(x+a)+B(x+b)=x$ for all values of x, find A and B.

4 If $A(x-2)^2+B(x-2)(x+3)+C(x+3)=50$ for all values of x, find A, B and C.

5. If

$$A(x-2)(x-3)+B(x-3)(x-1)+C(x-1)(x-2)=2x+6$$

for all values of x, find A, B and C.

6. If $(x+a)^3=x^3+Ax^2+Bx+C$ for all values of x, find A, B and C

7 Find A, B, C, so that $A(n-1)^2+B(n-2)^2+C(n-3)^2=n^2$ for all values of n.

8. Determine A, B and C so that

$$(x-a)(x-b)(x-c)+A(x-a)(x-b)+B(x-a)+C$$

may be identically equal to x^3.

9. Find A and B so that the coefficients of x^4 and x^3 in the product of x^2+x+1 and x^3+Ax^2+Bx+C may be zero.

10. Show that $5x^2+19x+18$ can be put in the form

$$l(x-2)(x-3)+m(x-3)(x-1)+n(x-1)(x-2),$$

and find l, m and n.

11. What values of a will make $6x^4-5x^3-29x^2+2x+a$ exactly divisible by $3x^2+2x-4$?

12. Find the values of a and b for which x^4+ax^2+bx+3 is exactly divisible by x^2-7x+1.

13. Find a value of x for which the expression

$$x^5+10x^4+30x^3+32x^2+12x-4$$

is divisible by x^2+3x+1.

14. Find b and c in order that x^4+1 may be divisible by

$$x^2+bx+c.$$

15. If x^2+px+q be a factor of x^3+ax^2+bx+c, prove that $pq+c=aq$ and $q^2+cp=bq$.

16. Find the condition that

$$adx^3 - x^2(bd + ah) - x(dc - bh) + f$$

may be exactly divisible by $ax^2 - bx - c$.

17. Find the value of c in order that

$$x^2 + 3x + c \text{ and } x^2 - 5x + 2c$$

may have a common factor.

18. If a be not zero, what value must a have in order that $x^3 - x - a$ and $x^2 + x - a$ may have a common factor? What is the H.C.F.?

19. The H.C.F. of $ax^3 + bx^2 + c$ and $ax^2 + bx + d$ is an expression of the first degree in x, prove that $ac^2 + bcd + d^3 = 0$.

20. If $x^4 + 8x^3y + bx^2y^2 + 24xy^3 + 9y^4$ be the square of

$$ax^2 + 4xy + 3y^2,$$

find the values of a and b.

21. Find the relations between a, b, c, d in order that

$$x^4 + ax^3 + bx^2 + cx + d$$

may be a perfect square.

22. If $ax^3 + bx^2 + cx + d$ be a perfect cube, then

$$b^2 = 3ac \text{ and } c^2 = 3bd.$$

23. Show that there is only one value of x which makes $x^3 + 3cx^2 + 2c^2x + 5c^3$ equal to the cube of $x + c$ and find it.

24. Assuming

$$\frac{1 + c}{1 + c + x} = 1 + a_1x + a_2x^2 + a_3x^3 + \dots,$$

find, without dividing, the values of a_1, a_2, a_3 ....

3. Partial Fractions.

231. If a and b be unequal, then

$$\frac{\mathrm{A}}{x+a}+\frac{\mathrm{B}}{x+b}=\frac{\mathrm{A}(x+b)+\mathrm{B}(x+a)}{(x+a)(x+b)}$$

$$=\frac{(\mathrm{A}+\mathrm{B})x+\mathrm{A}b+\mathrm{B}a}{(x+a)(x+b)}$$

$$=\frac{\mathrm{P}x+\mathrm{Q}}{(x+a)(x+b)},$$

where $\mathrm{A}+\mathrm{B}=\mathrm{P}$ and $\mathrm{A}b+\mathrm{B}a=\mathrm{Q}$.

These two equations give A and B in terms of P, Q, a and b; hence it is always possible to resolve a fraction of the form

$$\frac{\mathrm{P}x+\mathrm{Q}}{(x+a)(x+b)}$$

into the sum of two partial fractions of the form

$$\frac{\mathrm{A}}{x+a} \text{ and } \frac{\mathrm{B}}{x+b}\,*.$$

Similarly, it may be shown that, if a, b and c be all different, a fraction of the form

$$\frac{\mathrm{P}x^2+\mathrm{Q}x+\mathrm{R}}{(x+a)(x+b)(x+c)},$$

can always be resolved into the sum of three partial fractions of the form

$$\frac{\mathrm{A}}{x+a},\ \frac{\mathrm{B}}{x+b} \text{ and } \frac{\mathrm{C}}{x+c}.$$

---

* To find A and B from the equations

$$\mathrm{A}+\mathrm{B}=\mathrm{P} \text{ and } \mathrm{A}b+\mathrm{B}a=\mathrm{Q},$$

we have $$\mathrm{A}a+\mathrm{B}a=\mathrm{P}a \text{ and } \mathrm{A}b+\mathrm{B}b=\mathrm{P}b,$$

$$\therefore\ \mathrm{A}(a-b)=\mathrm{P}a-\mathrm{Q} \text{ and } \mathrm{B}(a-b)=\mathrm{Q}-\mathrm{P}b,$$

or $$\mathrm{A}=\frac{\mathrm{P}a-\mathrm{Q}}{a-b} \text{ and } \mathrm{B}=\frac{\mathrm{Q}-\mathrm{P}b}{a-b}.$$

Now, if $\mathrm{P}a-\mathrm{Q}$ were equal to zero, we should have

$$\mathrm{A}a+\mathrm{B}a=\mathrm{A}b+\mathrm{B}a \text{ or } \mathrm{A}a=\mathrm{A}b \text{ or } a=b.$$

Similarly if $\mathrm{Q}-\mathrm{P}b$ were equal to zero, we should have $a=b$.

Hence, since a and b are unequal, the values of A and B are always determinate.

If the numerator be of the same dimensions in x as the denominator, or of higher dimensions, the fraction can be reduced by division into an integral part and a fractional part of the above form.

In the following examples, the factors of the denominator are all of the first degree in x and are all different. Fractions in which the factors of the denominator are of the second or a higher degree in x, or in which one or more of the factors are repeated, are beyond the scope of this book.

232. *Example* 13. Resolve $\dfrac{5x-4}{(x-2)(x+1)}$ into partial fractions.

Let $$\frac{5x-4}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1},$$

$$\therefore\ 5x-4 = A(x+1) + B(x-2), \qquad (1)$$

$$\therefore\ A + B = 5 \text{ and } A - 2B = -4,$$

$$\therefore\ 3B = 9 \text{ or } B = 3 \text{ and } A = 2.$$

$\therefore$ the given fraction is equal to

$$\frac{2}{x-2} + \frac{3}{x+1}.$$

Another, and generally simpler, method of determining A and B is to put, first, $x = 2$ in equation (1),

$$\therefore\ 10 - 4 = A\,.\,3 \text{ or } A = 2,$$

and next $x = -1$,

$$\therefore\ -5 - 4 = B(-1-2) \text{ or } B = 3.$$

Example 14. Resolve $\dfrac{x^3+x^2+3x-13}{(x-1)(x-2)(x+3)}$ into partial fractions.

Dividing the numerator by the denominator, the fraction reduces to

$$1 + \frac{x^2+10x-19}{(x-1)(x-2)(x+3)}.$$

Let $$\frac{x^2+10x-19}{(x-1)(x-2)(x+3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x+3},$$

$$\therefore\ x^2+10x-19 = A(x-2)(x+3) + B(x+3)(x-1) + C(x-1)(x-2).$$

Putting x equal successively to 1, 2 and -3 in this equation, we have

$$1 + 10 - 19 = A(-1) . 4 \quad \text{or } A = 2,$$
$$4 + 20 - 19 = B . 5 . 1 \quad \text{or } B = 1,$$
$$9 - 30 - 19 = C(-4)(-5) \quad \text{or } C = -2.$$

$\therefore$ the given fraction is equal to

$$1 + \frac{2}{x-1} + \frac{1}{x-2} - \frac{2}{x+3}.$$

Exercises 163.

Resolve into partial fractions:

1. $\dfrac{1}{(x+1)(x+2)}$.

2. $\dfrac{2x-3}{(x-1)(x-2)}$.

3. $\dfrac{3x}{x^2-x-2}$.

4. $\dfrac{1}{(x+1)(2x+1)}$.

5. $\dfrac{13x-1}{(2x+1)(x-2)}$.

6. $\dfrac{x^2+2x-5}{x^2-3x+2}$.

7. $\dfrac{3(x+2)}{4(x+4)(x+3)}$.

8. $\dfrac{3x^2+12x+11}{(x+1)(x+2)(x+3)}$.

9. $\dfrac{x^2+1}{(x-1)(x-2)(x-3)}$.

10. $\dfrac{6x^2-22x+18}{(x-1)(x-2)(x-3)}$.

11. $\dfrac{x^2-17x}{12(x^2-1)(x-2)}$.

12. $\dfrac{3x^2+x+1}{(x-1)(x^2-4)}$.

13. $\dfrac{x^3+4x^2+7x+5}{(x+1)(x+2)}$.

14. $\dfrac{x^4}{(x-1)(x-2)(x-3)}$.

CHAPTER XXIII.

INDICES.

233. A *power* has been defined (art. 7) as a product consisting of the same factor repeated, and an *index* as a figure or letter which indicates the number of times the factor occurs in the power. Using this definition, it has been shown (art. 30) that, if m and n be positive integers,

$$a^m \times a^n = a^{m+n},$$

and consequently that

$$a^m \div a^n = a^{m-n} \text{ or } \frac{1}{a^{n-m}} \text{ according as } m \text{ is } > \text{ or } < n.$$

234. The above definition of an index applies only to the case in which the index is a positive integer. A fractional index and a negative index have not yet been defined, and we are at liberty to define them in any way we please, provided the new definition include the other as a particular case, or provided both satisfy a common law. We shall assume then that the Index Law

$$a^m \times a^n = a^{m+n}$$

holds whether m and n be positive or negative, integral or fractional, and deduce from this relation the meaning: (i) of a^m, when m is a positive integer, (ii) of $a^{\frac{m}{n}}$, when m and n are positive integers, i.e. when the index is a commensurable fraction, (iii) of a^0, and (iv) of a^{-m} when m is either an integer or a commensurable fraction.

235. *To find the meaning of a^m, when m is a positive integer.*

By the Index Law, we have

$$a \times a \times a \times \dots \text{ to } m \text{ factors} = a^{1+1+1+\dots \text{ to } m \text{ terms}}$$
$$= a^m.$$

Hence, if m be a positive integer, a^m is the product of m factors each equal to a.

236. *Example* 1. Find the meaning of $a^{\frac{1}{2}}$.

By the Index Law, we have

$$a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1,$$
$$\therefore\ a^{\frac{1}{2}} = \sqrt{a}.$$

Example 2. Find the meaning of $a^{\frac{2}{3}}$.

By the Index Law, we have

$$a^{\frac{2}{3}} \times a^{\frac{2}{3}} \times a^{\frac{2}{3}} = a^{\frac{2}{3}+\frac{2}{3}+\frac{2}{3}} = a^2.$$
$$\therefore\ a^{\frac{2}{3}} = \sqrt[3]{a^2}.$$

Again, $$a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^{\frac{2}{3}},$$
$$\therefore\ a^{\frac{2}{3}} = (a^{\frac{1}{3}})^2.$$

Exercises 164.

Deduce from the Index Law the meaning of:

1. $a^{\frac{1}{3}}$. 2. $a^{\frac{1}{4}}$ 3. $a^{\frac{1}{5}}$. 4. $a^{\frac{3}{2}}$. 5. $a^{\frac{3}{4}}$.
6. $a^{\frac{4}{3}}$. 7. $a^{\frac{2}{5}}$. 8. $a^{\frac{7}{4}}$. 9. $a^{\frac{5}{6}}$. 10. $a^{\frac{3}{5}}$.

237. *To find the meaning of $a^{\frac{1}{n}}$.*

By the Index Law, we have

$$a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times \dots \text{ to } n \text{ factors} = a^{\frac{1}{n}+\frac{1}{n}+\frac{1}{n}+\dots \text{ to } n \text{ terms}}$$
$$= a^1.$$
$$\therefore\ a^{\frac{1}{n}} = \sqrt[n]{a}.$$

Thus, $a^{\frac{1}{3}} = \sqrt[3]{a}$, $a^{\frac{1}{10}} = \sqrt[10]{a}$, etc.

238. *To find the meaning of* $a^{\frac{m}{n}}$.

By the Index Law, we have

$$a^{\frac{m}{n}} \times a^{\frac{m}{n}} \times a^{\frac{m}{n}} \times \dots \text{ to } n \text{ factors} = a^{\frac{m}{n}+\frac{m}{n}+\frac{m}{n}+\dots \text{ to } n \text{ terms}}$$

$$= a^m.$$

$$\therefore\ a^{\frac{m}{n}} = \sqrt[n]{a^m}.$$

Again,

$$a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times \dots \text{ to } m \text{ factors} = a^{\frac{1}{n}+\frac{1}{n}+\frac{1}{n}+\dots \text{ to } m \text{ terms}}$$

$$= a^{\frac{m}{n}},$$

$$\therefore\ a^{\frac{m}{n}} = (a^{\frac{1}{n}})^m.$$

Thus, $a^{\frac{2}{3}} = \sqrt[3]{a^2} = (a^{\frac{1}{3}})^2$, $a^{\frac{4}{5}} = \sqrt[5]{a^4} = (a^{\frac{1}{5}})^4$, etc.

239. *Example* 3. Simplify $\sqrt{a} \times \sqrt[3]{a^2} \div \sqrt[6]{a^5}$.

$$\sqrt{a} \times \sqrt[3]{a^2} \div \sqrt[6]{a^5} = a^{\frac{1}{2}} \times a^{\frac{2}{3}} \div a^{\frac{5}{6}}$$

$$= a^{\frac{1}{2}+\frac{2}{3}-\frac{5}{6}}$$

$$= a^{\frac{3+4-5}{6}}$$

$$= a^{\frac{1}{3}}$$

$$= \sqrt[3]{a}.$$

Exercises 165.

Express with radical signs:

1. $a^{\frac{1}{2}}$. 2. $a^{\frac{1}{3}}$. 3. $a^{\frac{2}{3}}$. 4. $a^{\frac{3}{4}}$. 5. $a^{\frac{3}{2}}$.

6. $a^{\frac{4}{3}}$. 7. $a^{\frac{5}{3}}$. 8. $\frac{1}{a^{\frac{1}{4}}}$. 9. $\frac{1}{a^{\frac{5}{2}}}$. 10. $\frac{1}{a^{\frac{4}{3}}}$.

Express with fractional indices:

11. $\sqrt{a}$. 12. $\sqrt[3]{a}$. 13. $\sqrt[3]{a^2}$. 14. $\sqrt{a^3}$. 15. $\sqrt[3]{a^4}$.

16. $\sqrt{a^5}$. 17. $\sqrt[4]{a^7}$. 18. $\dfrac{1}{\sqrt[3]{a}}$. 19. $\dfrac{1}{\sqrt[4]{a^5}}$. 20. $\dfrac{1}{\sqrt[6]{a^2}}$.

Find the values of:

21. $4^{\frac{1}{2}}$. 22. $9^{\frac{1}{2}}$. 23. $25^{\frac{1}{2}}$. 24. $8^{\frac{1}{3}}$. 25. $27^{\frac{1}{3}}$.

26. $16^{\frac{1}{4}}$. 27. $8^{\frac{2}{3}}$. 28. $27^{\frac{2}{3}}$. 29. $16^{\frac{3}{4}}$. 30. $32^{\frac{3}{5}}$.

Simplify, expressing the results with fractional indices:

31. $a^{\frac{1}{2}} \times a^{\frac{1}{4}}$. 32. $a^{\frac{1}{3}} \times a^{\frac{1}{6}}$. 33. $a^{\frac{2}{5}} \times a^{\frac{1}{5}}$. 34. $a^{\frac{2}{5}} \times a^{\frac{1}{10}}$.

35. $a^{\frac{3}{8}} \times a^{\frac{3}{8}}$. 36. $a^{\frac{1}{2}} \times a^{\frac{1}{2}} \times a^{\frac{1}{2}}$. 37. $a^{\frac{1}{3}} \times a^{\frac{1}{4}}$. 38. $a^{\frac{2}{3}} \times a^{\frac{5}{6}}$.

39. $a \times a^{\frac{2}{3}}$. 40. $a \times a^{\frac{1}{2}} \times a^{\frac{1}{4}}$. 41. $a^{\frac{1}{2}} \div a^{\frac{1}{4}}$. 42. $a^{\frac{2}{3}} \div a^{\frac{1}{6}}$.

43. $a^{\frac{5}{2}} \div a^{\frac{1}{2}}$. 44. $a^{\frac{5}{4}} \div a$. 45. $a^{\frac{7}{12}} \div a^{\frac{1}{4}}$. 46. $a^{\frac{1}{2}} \div a^{\frac{2}{5}}$.

47. $a^{\frac{7}{8}} \div a^{\frac{3}{8}}$. 48. $a^{\frac{11}{12}} \div a^{\frac{1}{4}}$. 49. $a^{\frac{4}{3}} \div a^{\frac{1}{4}}$. 50. $a^{\frac{3}{2}} \div a^{\frac{4}{5}}$.

Simplify and express with fractional indices:

51. $\sqrt{a} \times \sqrt[3]{a} \times \sqrt[5]{a}$. 52. $\sqrt[4]{a^5} \times \sqrt[5]{a^3}$.

53. $\sqrt[3]{a^4} \times \sqrt{a^5} \times \sqrt[6]{a^7}$. 54. $\sqrt{a^3} \times \sqrt[3]{a^5} \div a^{\frac{1}{2}} \div a^{\frac{4}{3}}$.

55. $\sqrt[4]{a^7} \times \sqrt[6]{a^5} \div a^{\frac{1}{3}} \div a^{\frac{5}{12}}$.

240. *To find the meaning of a^0.*

By the Index Law, we have

$$a^m \times a^0 = a^{m+0} = a^m,$$

$$\therefore \ a^0 = a^m \div a^m = 1.$$

241. *To find the meaning of a^{-m}, when m is an integer or a commensurable fraction.*

By the Index Law, we have

$$a^m \times a^{-m} = a^{m-m} = a^0 = 1,$$

$$\therefore \ a^{-m} = \frac{1}{a^m}.$$

Thus, $\quad a^{-2} = \dfrac{1}{a^2}, \quad a^{-\frac{3}{4}} = \dfrac{1}{a^{\frac{3}{4}}} = \dfrac{1}{\sqrt[4]{a^3}}$, etc.

242. *Example* 4. Express with positive indices

$$\frac{3a^4x^{-3}}{5y^2b^{-1}}.$$

$$\frac{3a^4x^{-3}}{5y^2b^{-1}} = 3a^4 \cdot \frac{1}{x^3} \div 5y^2 \cdot \frac{1}{b}$$

$$= \frac{3a^4}{x^3} \times \frac{b}{5y^2}$$

$$= \frac{3a^4b}{5x^3y^2}.$$

Example 5. Find the value of $\dfrac{27^{\frac{2}{3}}}{16^{-\frac{3}{4}}}$.

$$27^{\frac{2}{3}} \div 16^{-\frac{3}{4}} = 27^{\frac{2}{3}} \div \frac{1}{16^{\frac{3}{4}}}$$

$$= 27^{\frac{2}{3}} \times 16^{\frac{3}{4}}$$

$$= 9 \times 8$$

$$= 72.$$

Example 6. Simplify $4a^{\frac{1}{2}}b^{\frac{2}{3}} \times 3a^{-2}b^{-1} \div 6a^{-\frac{5}{2}}b^{-\frac{1}{3}}$.

$$4a^{\frac{1}{2}}b^{\frac{2}{3}} \times 3a^{-2}b^{-1} \div 6a^{-\frac{5}{2}}b^{-\frac{1}{3}} = 2\,(a^{\frac{1}{2}} \times a^{-2} \div a^{-\frac{5}{2}})\,(b^{\frac{2}{3}} \times b^{-1} \div b^{-\frac{1}{3}})$$

$$= 2a^{\frac{1}{2}-2+\frac{5}{2}}b^{\frac{2}{3}-1+\frac{1}{3}}$$

$$= 2a^1b^0$$

$$= 2a.$$

Exercises 166.

Express with positive indices:

1. a^{-2}. 2. a^{-3}. 3. $a^{-\frac{1}{2}}$. 4. $a^{-\frac{2}{3}}$.

5. $a^{-\frac{3}{4}}$. 6. $2a^{-1}$. 7. $3a^{-\frac{1}{4}}$. 8. $\dfrac{4}{a^{-4}}$.

9. $\dfrac{1}{2a^{-\frac{3}{2}}}$. 10. $\dfrac{a^{-1}b}{xy^{-1}}$. 11. $\dfrac{x^3y^{-2}}{a^{-1}b^2}$. 12. $\dfrac{2a^2x^2}{3by^{-3}}$.

Find the values of:

13. $4^{-\frac{1}{2}}$. 14. $8^{-\frac{1}{3}}$. 15. $4^{-\frac{3}{2}}$. 16. $25^{-\frac{1}{2}}$.

17. $27^{-\frac{4}{3}}$. 18. $\dfrac{1}{8^{-\frac{2}{3}}}$. 19. $\dfrac{1}{25^{-\frac{3}{2}}}$. 20. $\dfrac{1}{16^{-\frac{1}{2}}}$.

21. $\dfrac{2}{64^{-\frac{2}{3}}}$. 22. $\dfrac{5}{36^{-\frac{1}{2}}}$. 23. $\dfrac{4^{\frac{1}{2}}}{32^{-\frac{1}{5}}}$. 24. $\dfrac{9^{-\frac{1}{2}}}{27^{-\frac{1}{3}}}$.

Simplify, expressing the results with positive indices:

25. $a^{\frac{3}{4}} \times a^{-\frac{1}{4}}$. 26. $a \times a^{-\frac{1}{2}}$. 27. $a^5 \times a^{-8}$.

28. $a^2 \times a^{-\frac{6}{5}}$. 29. $a \times a^{-\frac{3}{2}} \times a^{\frac{1}{2}}$. 30. $2a^{\frac{2}{3}} \times 3a^{-\frac{1}{6}}$.

31. $5a^{\frac{5}{4}} \times 2a^{-\frac{3}{8}}$. 32. $ab^{-\frac{1}{2}} \times a^{-1}b$. 33. $a^{\frac{2}{3}}x^{\frac{1}{4}} \times a^{-\frac{1}{3}}x^{-\frac{3}{4}}$.

34. $2a^{\frac{3}{5}} \times 4a^{-\frac{7}{5}}$. 35. $a^{\frac{3}{2}} \div a^{-\frac{1}{2}}$. 36. $a^{-1} \div a^{-2}$.

37. $a^{-\frac{1}{2}} \div a^{\frac{3}{4}}$. 38. $4a^{\frac{5}{3}} \div 2a^{-\frac{1}{3}}$. 39. $x^2y^{-1} \div x^{-1}y^2$.

40. $3a^{\frac{2}{3}}b^{-\frac{1}{2}} \div 6a^{-\frac{1}{3}}b^{-\frac{3}{2}}$.

243. *Example* 7. Multiply

$$a^{\frac{1}{2}} + 2 - 3a^{-\frac{1}{2}} \text{ by } a^{\frac{1}{2}} - 2 + 3a^{-\frac{1}{2}}.$$

$$\begin{array}{llll}
a^{\frac{1}{2}} + 2 - 3a^{-\frac{1}{2}} & & & \\
a^{\frac{1}{2}} - 2 + 3a^{-\frac{1}{2}} & & & \\
\hline
a & + 2a^{\frac{1}{2}} - 3 & & \\
 & - 2a^{\frac{1}{2}} - 4 + & 6a^{-\frac{1}{2}} & \\
 & \quad + 3 + & 6a^{-\frac{1}{2}} & - 9a^{-1} \\
\hline
a & \quad - 4 + & 12a^{-\frac{1}{2}} & - 9a^{-1}
\end{array}$$

Example 8. Divide

$$a + 2a^{\frac{2}{3}}b^{\frac{1}{3}} - 11a^{\frac{1}{3}}b^{\frac{2}{3}} + 6b \text{ by } a^{\frac{1}{3}} - 2b^{\frac{1}{3}}.$$

$$\begin{array}{l}
a^{\frac{1}{3}} - 2b^{\frac{1}{3}}\ )\ a + 2a^{\frac{2}{3}}b^{\frac{1}{3}} - 11a^{\frac{1}{3}}b^{\frac{2}{3}} + 6b\ (\ a^{\frac{2}{3}} + 4a^{\frac{1}{3}}b^{\frac{1}{3}} - 3b^{\frac{2}{3}} \\
\qquad\qquad\quad a - 2a^{\frac{2}{3}}b^{\frac{1}{3}} \\
\hline
\qquad\qquad\qquad 4a^{\frac{2}{3}}b^{\frac{1}{3}} - 11a^{\frac{1}{3}}b^{\frac{2}{3}} + 6b \\
\qquad\qquad\qquad 4a^{\frac{2}{3}}b^{\frac{1}{3}} - \ \ 8a^{\frac{1}{3}}b^{\frac{2}{3}} \\
\hline
\qquad\qquad\qquad\qquad\quad - \ 3a^{\frac{1}{3}}b^{\frac{2}{3}} + 6b \\
\qquad\qquad\qquad\qquad\quad - \ 3a^{\frac{1}{3}}b^{\frac{2}{3}} + 6b \\
\hline
\end{array}$$

Exercises 167.

Multiply

1. $a^{\frac{1}{2}}+b^{\frac{1}{2}}$ by $a^{\frac{1}{2}}-b^{\frac{1}{2}}$.
2. $x^{\frac{2}{3}}-x^{\frac{1}{3}}a^{\frac{1}{3}}+a^{\frac{2}{3}}$ by $x^{\frac{1}{3}}+a^{\frac{1}{3}}$.
3. $x+1+x^{-1}$ by $x-1+x^{-1}$.
4. $x^{\frac{1}{2}}+x^{\frac{1}{4}}-1$ by $x^{\frac{1}{2}}-x^{\frac{1}{4}}-1$.
5. $a+2a^{\frac{1}{2}}+1$ by $a^{\frac{1}{2}}-2+a^{-\frac{1}{2}}$.
6. $a-3a^{\frac{2}{3}}+3a^{\frac{1}{3}}-1$ by $a^{\frac{2}{3}}-2a^{\frac{1}{3}}+1$.

Divide

7. $x-y$ by $x^{\frac{1}{2}}+y^{\frac{1}{2}}$.
8. $a^{\frac{3}{2}}-b^{-\frac{3}{2}}$ by $a^{\frac{1}{2}}-b^{-\frac{1}{2}}$.
9. x^2+1+x^{-2} by $x-1+x^{-1}$.
10. x^2-y^{-2} by $x^{\frac{1}{2}}+y^{-\frac{1}{2}}$.
11. $a+3a^{\frac{2}{3}}b^{\frac{1}{3}}+3a^{\frac{1}{3}}b^{\frac{2}{3}}+b$ by $a^{\frac{1}{3}}+b^{\frac{1}{3}}$.
12. $x^{\frac{4}{3}}+x^{\frac{2}{3}}y^{-\frac{2}{3}}+y^{-\frac{4}{3}}$ by $x^{\frac{2}{3}}-x^{\frac{1}{3}}y^{-\frac{1}{3}}+y^{-\frac{2}{3}}$.

244. *To prove that $(a^m)^n = a^{mn}$, when m and n are either positive or negative integers or commensurable fractions.*

Let m be either a positive or a negative integer or commensurable fraction.

(i) Let n be a positive integer.

Then $$(a^m)^n = a^m \times a^m \times a^m \times \dots \text{ to } n \text{ factors}$$
$$= a^{m+m+m+\dots\text{ to } n \text{ terms}}$$
$$= a^{mn}.$$

(ii) Let n be a positive commensurable fraction, say $\frac{p}{q}$, where p and q are positive integers.

Then $$(a^m)^n = (a^m)^{\frac{p}{q}} = \sqrt[q]{(a^m)^p}$$
$$= \sqrt[q]{(a^{mp})}, \text{ by case (i)},$$
$$= a^{\frac{mp}{q}}$$
$$= a^{mn}.$$

(iii) Let n be negative and either an integer or commensurable fraction, say $-r$.

Then $$(a^m)^n = (a^m)^{-r} = \frac{1}{(a^m)^r}$$

$$= \frac{1}{a^{mr}}, \text{ by case (i) or (ii),}$$

$$= a^{-mr} = a^{m(-r)} = a^{mn}.$$

245. *Example* 9. Simplify $[(a^{-3})^{\frac{1}{2}}]^{-\frac{4}{3}}$.

$$[(a^{-3})^{\frac{1}{2}}]^{-\frac{4}{3}} = [a^{-\frac{3}{2}}]^{-\frac{4}{3}}$$

$$= a^2.$$

Example 10. Simplify $(a^{-3})^2 \times (a^{\frac{3}{4}})^{-4} \div (a^{\frac{2}{3}})^{-6}$.

$$(a^{-3})^2 \times (a^{\frac{3}{4}})^{-4} \div (a^{\frac{2}{3}})^{-6} = a^{-6} \times a^{-3} \div a^{-4}$$

$$= a^{-6-3+4}$$

$$= a^{-5}$$

$$= \frac{1}{a^5}.$$

Exercises 168.

Simplify, expressing with positive indices:

1. $(a^4)^{\frac{1}{2}}$.
2. $(a^{\frac{1}{2}})^{-2}$.
3. $(x^{-3})^{-\frac{2}{3}}$.
4. $[(x^{\frac{1}{2}})^{\frac{1}{2}}]^{\frac{1}{2}}$.
5. $[(a^2)^{\frac{1}{3}}]^{\frac{3}{4}}$.
6. $[(x^{\frac{1}{3}})^{\frac{2}{5}}]^{-30}$.
7. $\sqrt[4]{[x^3\sqrt[3]{\{x^2\sqrt{(x^5)}\}}]}$.
8. $(x^6)^{\frac{1}{3}} \times (x^{\frac{1}{2}})^4$.
9. $(x^4)^{\frac{1}{2}} \times (x^3)^{\frac{2}{3}}$.
10. $(x^3)^{\frac{1}{2}} \div (x^2)^{\frac{1}{3}}$.
11. $(a^{-4})^{-\frac{1}{2}} \div (a^{-6})^{-\frac{2}{3}}$.
12. $(x^2)^3 \times (x^4)^{\frac{1}{2}} \times (x^3)^{\frac{2}{3}}$.
13. $(a^2)^{\frac{1}{2}} \times (a^{-3})^{-\frac{1}{3}} \times (a^4)^{-\frac{2}{3}}$.
14. $(a^5)^{\frac{1}{2}} \times (a^{-3})^{\frac{2}{3}} \div (a^{-1})^{-1}$.
15. $(a^x)^{y-z} \times (a^y)^{z-x} \times (a^z)^{x-y}$.
16. $a^{2^3} \div (a^2)^3$.

246. *To prove that $(ab)^n = a^n b^n$, when n is positive or negative, an integer or a commensurable fraction.*

(i) Let n be a positive integer.

Then $(ab)^n = (ab) \times (ab) \times (ab) \times \ldots$ to n factors

$$= a \times b \times a \times b \times a \times b \times \ldots$$

$$= (a \times a \times a \times \ldots \text{ to } n \text{ factors}) \times (b \times b \times b \times \ldots \text{ to } n \text{ factors}),$$

$$= a^n b^n.$$

(ii) Let n be a positive commensurable fraction, say $\frac{p}{q}$, where p and q are positive integers.

Then $[(ab)^n]^q = [(ab)^{\frac{p}{q}}]^q = (ab)^p$, by art. 244,

$$= a^p b^p, \text{ by case (i)}.$$

Also, $[a^n b^n]^q = [a^{\frac{p}{q}} b^{\frac{p}{q}}]^q = (a^{\frac{p}{q}})^q \times (b^{\frac{p}{q}})^q$, by case (i),

$$= a^p b^p,$$

$$\therefore\ (ab)^n = a^n b^n.$$

(iii) Let n be negative, and either an integer or a commensurable fraction, say $-r$.

Then
$$(ab)^n = (ab)^{-r} = \frac{1}{(ab)^r}$$

$$= \frac{1}{a^r b^r}, \text{ by case (i) or (ii)},$$

$$= \frac{1}{a^r} \times \frac{1}{b^r} = a^{-r} b^{-r}$$

$$= a^n b^n.$$

COR.
$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

247. *Example* 11. Find the value of $\left(\frac{64}{125}\right)^{\frac{2}{3}}$.

$$\left(\frac{64}{125}\right)^{\frac{2}{3}} = \frac{64^{\frac{2}{3}}}{125^{\frac{2}{3}}} = \frac{(\sqrt[3]{64})^2}{(\sqrt[3]{125})^2}$$

$$= \frac{4^2}{5^2}$$

$$= \frac{16}{25}.$$

Example 12. Simplify $(a^{\frac{2}{3}} b^{-\frac{3}{4}})^{-12}$.

$$(a^{\frac{2}{3}} b^{-\frac{3}{4}})^{-12} = (a^{\frac{2}{3}})^{-12} \times (b^{-\frac{3}{4}})^{-12}$$

$$= a^{-8} \times b^9$$

$$= \frac{b^9}{a^8}.$$

Example 13. Simplify $\sqrt[4]{[a^2 b^2 \sqrt{(abc^5)}]}$.

$$\sqrt[4]{[a^2 b^2 \sqrt{(abc^5)}]} = [a^2 b^2 (abc^5)^{\frac{1}{2}}]^{\frac{1}{4}},$$

$$= [a^2 b^2 a^{\frac{1}{2}} b^{\frac{1}{2}} c^{\frac{5}{2}}]^{\frac{1}{4}},$$

$$= [a^{\frac{5}{2}} b^{\frac{5}{2}} c^{\frac{5}{2}}]^{\frac{1}{4}},$$

$$= a^{\frac{5}{8}} b^{\frac{5}{8}} c^{\frac{5}{8}}.$$

Exercises 169.

Find the value of :

1. $(\frac{1}{16})^{-\frac{1}{2}}$.
2. $(\frac{8}{27})^{-\frac{2}{3}}$.
3. $(\frac{1}{1000})^{-\frac{2}{3}}$.
4. $(\frac{9}{25})^{-\frac{3}{2}}$.
5. $(\frac{8}{125})^{-\frac{4}{3}}$.

Simplify :

6. $(x^{\frac{1}{3}} y)^3$.
7. $\left(\frac{a^4}{b^6}\right)^{\frac{1}{2}}$.
8. $(a^{\frac{2}{3}} b^{-\frac{1}{3}})^3$.
9. $(x^{\frac{5}{4}} y^{-\frac{1}{4}})^4$.
10. $(x^3 y^{-\frac{3}{5}})^{\frac{5}{3}}$.
11. $\left(\frac{a^{-4}}{b^{-6}}\right)^{\frac{2}{3}}$.
12. $(a^{-\frac{3}{4}} b^{\frac{1}{2}})^{\frac{4}{3}}$.

13. $\left(\dfrac{b^{-3}}{16a^2}\right)^{\frac{1}{4}}$.

14. $(16x^{-4}y^8)^{-\frac{3}{4}}$.

15. $(a^{\frac{1}{2}}b^{\frac{2}{3}}c^{\frac{3}{4}})^{-12}$.

16. $(x^2)^3 \div (x^3)^{-2}$.

17. $(x^2y^3)^{\frac{1}{2}} \times (x^{\frac{1}{3}}y^{-\frac{1}{2}})^3$.

18. $\sqrt[3]{(a^2b^3)} \times \sqrt{(a^2b)}$.

19. $\sqrt[3]{(a^2b^5c^8)} \div \sqrt[4]{(a^3b^6c^9)}$.

20. $\sqrt[6]{[a^3b^4\sqrt[5]{(a^3bc)}]}$.

21. $(a^{\frac{1}{3}}b^{-2} \div ab)^{-3}$.

22. $(ab^{-2}c^3)^{\frac{1}{2}} \times (a^3b^2c^{-3})^{\frac{1}{3}}$

23. $\{(x^{\frac{1}{2}}y^{\frac{1}{2}})^3x^{-1}y^{-\frac{3}{2}}z^{\frac{1}{2}}\}^4$.

24. $[a^{\frac{1}{3}}b^{\frac{1}{2}}(a^{-\frac{1}{6}}b^{-\frac{1}{4}})^{-\frac{1}{2}}]^{-\frac{4}{5}}$.

25. $\sqrt[3]{(a^6b^{\frac{2}{3}}c^{\frac{1}{2}})} \times b^{\frac{7}{9}} \times (c^{\frac{1}{3}}a^2)^{-\frac{1}{2}}$.

248. *Example* 14. Simplify

$$\frac{a^{-\frac{1}{2}}b^{\frac{4}{3}}}{\sqrt{(a^{-5}b^6)}} \div \frac{ab^{-\frac{7}{3}}}{\sqrt[3]{(a^{-3}b^{-5})}}.$$

$$\begin{aligned}\frac{a^{-\frac{1}{2}}b^{\frac{4}{3}}}{\sqrt{(a^{-5}b^6)}} \div \frac{ab^{-\frac{7}{3}}}{\sqrt[3]{(a^{-3}b^{-5})}} &= \frac{a^{-\frac{1}{2}}b^{\frac{4}{3}}}{a^{-\frac{5}{2}}b^3} \times \frac{a^{-1}b^{-\frac{5}{3}}}{ab^{-\frac{7}{3}}}\\ &= a^{-\frac{1}{2}-1+\frac{5}{2}-1} \cdot b^{\frac{4}{3}-\frac{5}{3}-3+\frac{7}{3}}\\ &= a^0b^{-1}\\ &= \frac{1}{b}.\end{aligned}$$

Example **15.** Find the square root of

$$4a^{\frac{2}{3}} - 4a^{\frac{1}{3}} + 13 - 6a^{-\frac{1}{3}} + 9a^{-\frac{2}{3}}.$$

$$\begin{array}{l|l}
2a^{\frac{1}{3}} & 4a^{\frac{2}{3}} - 4a^{\frac{1}{3}} + 13 - 6a^{-\frac{1}{3}} + 9a^{-\frac{2}{3}} \;(\,2a^{\frac{1}{3}} - 1 + 3a^{-\frac{1}{3}} \\
 & 4a^{\frac{2}{3}} \\ \hline
4a^{\frac{1}{3}} - 1 & -4a^{\frac{1}{3}} + 13 - 6a^{-\frac{1}{3}} + 9a^{-\frac{2}{3}} \\
 & -4a^{\frac{1}{3}} + \;\;1 \\ \hline
4a^{\frac{1}{3}} - 2 + 3a^{-\frac{1}{3}} & 12 - 6a^{-\frac{1}{3}} + 9a^{-\frac{2}{3}} \\
 & 12 - 6a^{-\frac{1}{3}} + 9a^{-\frac{2}{3}} \\ \hline
\end{array}$$

$\therefore$ square root is $2a^{\frac{1}{3}} - 1 + 3a^{-\frac{1}{3}}$.

Example 16. Find the value of $x^3 - 6x$, when $x = 2^{\frac{2}{3}} + 2^{\frac{1}{3}}$.

$$x^3 - 6x = (2^{\frac{2}{3}} + 2^{\frac{1}{3}})^3 - 6\,(2^{\frac{2}{3}} + 2^{\frac{1}{3}})$$
$$= (2^{\frac{2}{3}})^3 + 3\,(2^{\frac{2}{3}})^2\,(2^{\frac{1}{3}}) + 3\,(2^{\frac{2}{3}})\,(2^{\frac{1}{3}})^2 + (2^{\frac{1}{3}})^3 - 6\,(2^{\frac{2}{3}} + 2^{\frac{1}{3}})$$
$$= 2^2 + 3\,.\,2^{\frac{4}{3}+\frac{1}{3}} + 3\,.\,2^{\frac{2}{3}+\frac{2}{3}} + 2 - 6\,(2^{\frac{2}{3}} + 2^{\frac{1}{3}})$$
$$= 4 + 3\,.\,2^{\frac{5}{3}} + 3\,.\,2^{\frac{4}{3}} + 2 - 6\,(2^{\frac{2}{3}} + 2^{\frac{1}{3}})$$
$$= 4 + 6\,.\,2^{\frac{2}{3}} + 6\,.\,2^{\frac{1}{3}} + 2 - 6\,(2^{\frac{2}{3}} + 2^{\frac{1}{3}})$$
$$= 6.$$

Exercises 170.

Simplify:

1. $\dfrac{x^{a+b}\,.\,x^{a-b}\,.\,x^{c-2a}}{x^{c-a}}$.

2. $x^{\frac{1}{2}}\,.\,x^{\frac{1}{3}}\,.\,x^{\frac{1}{4}}\,.\,x^{\frac{1}{5}} \div x^{\frac{17}{15}}$.

3. $\sqrt[3]{a^7} \times \sqrt[5]{a^9} \times a^{-\frac{1}{3}} \div a^{\frac{4}{5}}$.

4. $\left(\dfrac{16^{-2}a^{\frac{1}{2}}b^{-3}}{81a^{-\frac{1}{2}}b^3}\right)^{-\frac{1}{4}}$.

5. $\left(\dfrac{x^{p+q}}{x^q}\right)^p \div \left(\dfrac{x^q}{x^{q-p}}\right)^{p-q}$.

6. $\left\{(a^m)^{m-\frac{1}{m}}\right\}^{\frac{1}{m+1}}$.

7. $\sqrt[m]{[a^{m^2} \div (a^m)^2]}$.

8. $\dfrac{\{x^{-nr}\,(x^{n+1}y)^r\,x\}^n}{x^n\,(xy)^{nr}}$.

9. $\{ab^2(ab^3)^{\frac{1}{2}}(a^2b^3)^{\frac{1}{3}}\}^{\frac{1}{2}}$.

10. $\{(a^{\frac{4}{3}}b^{\frac{3}{4}})^{-\frac{1}{2}} \times (a^{-\frac{3}{4}}b^{-\frac{4}{3}})^{\frac{1}{2}}\}^{-24}$

11. $\left(\dfrac{a^{\frac{1}{m}}\,x^m}{a^{\frac{1}{n}}\,x^n}\right)^{\frac{1}{m-n}} \times \sqrt[mn]{a}$.

12. $\left\{\sqrt[3]{\left(\dfrac{a^{-\frac{1}{2}}b^2}{b^{\frac{1}{2}}a}\right)} \times \sqrt{\left(\dfrac{b\sqrt{a^{-2}}}{\sqrt{(ab^2)}}\right)}\right\}^2$.

Multiply

13. $a^{\frac{3}{2}}b^{\frac{1}{2}} - ab + a^{\frac{1}{2}}b^{\frac{3}{2}}$ by $a^{\frac{1}{2}}b^{-\frac{1}{2}} + a^{-\frac{1}{2}}b^{\frac{1}{2}}$.

14. $a^{\frac{4}{3}}b^{\frac{4}{3}} - a^{\frac{1}{3}}bc^{\frac{1}{3}} + b^{\frac{2}{3}}c^{\frac{2}{3}}$ by $a^{\frac{2}{3}}b^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}} + c^{\frac{2}{3}}$.

15. $a^{\frac{2}{3}} + b^{\frac{4}{3}} + c^2 - cb^{\frac{2}{3}} - ca^{\frac{1}{3}} - a^{\frac{1}{3}}b^{\frac{2}{3}}$ by $a^{\frac{1}{3}} + b^{\frac{2}{3}} + c$.

16. $\frac{1}{2}a^{\frac{1}{2}} - \frac{1}{3}a^{\frac{1}{3}} + \frac{1}{4}a^{\frac{1}{4}}$ by $\frac{2}{3}a^{\frac{1}{2}} + \frac{3}{4}a^{\frac{1}{3}} - \frac{4}{5}a^{\frac{1}{4}}$.

17. Find the continued product of

$$2a^{\frac{3}{2}} - 3ax^{\frac{1}{2}},\ 3a^{-\frac{1}{2}} + 2x^{-\frac{1}{2}},\ 4a^{\frac{1}{2}}x^{\frac{1}{2}} + 9a^{-\frac{1}{2}}x^{\frac{3}{2}}.$$

18. Simplify:

$$(x^{\frac{1}{2}} + y^{\frac{1}{2}} + z^{\frac{1}{2}})(-x^{\frac{1}{2}} + y^{\frac{1}{2}} + z^{\frac{1}{2}})(x^{\frac{1}{2}} - y^{\frac{1}{2}} + z^{\frac{1}{2}})(x^{\frac{1}{2}} + y^{\frac{1}{2}} - z^{\frac{1}{2}}).$$

Divide

19. $x^{\frac{2}{3}} + 4y$ by $x^{\frac{1}{6}}y^{-\frac{1}{4}} + 2 + 2x^{-\frac{1}{6}}y^{\frac{1}{4}}$.

20. $a - x + 4a^{\frac{1}{4}}x^{\frac{3}{4}} - 4a^{\frac{1}{2}}x^{\frac{1}{2}}$ by $a^{\frac{1}{2}} + 2a^{\frac{1}{4}}x^{\frac{1}{4}} - x^{\frac{1}{2}}$.

21. $a^{\frac{5}{2}} - a^{\frac{3}{2}}b + ab^{\frac{3}{2}} - 2a^{\frac{1}{2}}b^2 + b^{\frac{5}{2}}$ by $a^{\frac{3}{2}} - ab^{\frac{1}{2}} + a^{\frac{1}{2}}b - b^{\frac{3}{2}}$.

22. $x^{\frac{5}{2}} - 5x^2y^{\frac{1}{3}} + 10x^{\frac{3}{2}}y^{\frac{2}{3}} - 10xy + 5x^{\frac{1}{2}}y^{\frac{4}{3}} - y^{\frac{5}{3}}$
by $x - 2x^{\frac{1}{2}}y^{\frac{1}{3}} + y^{\frac{2}{3}}$.

Find the square root of:

23. $x^{\frac{2}{3}} + 9x^{-\frac{2}{3}} + 6$.

24. $x^2 + 2x + 3 + 2x^{-1} + x^{-2}$.

25. $a^2 + 4a^{\frac{3}{2}} - 2a - 12a^{\frac{1}{2}} + 9$.

26. $4x^2a^{-2} - 12xa^{-1} + 25 - 24x^{-1}a + 16x^{-2}a^2$.

27. $a^4 - 4a^{\frac{10}{3}}b^{\frac{2}{3}} + 4a^{\frac{8}{3}}b^{\frac{4}{3}} + 2a^{\frac{7}{3}}b^{\frac{5}{3}} - 4a^{\frac{5}{3}}b^{\frac{7}{3}} + a^{\frac{2}{3}}b^{\frac{10}{3}}$.

28. $a^{-2} + 2a^{-1}(2 - b^{-2}) + b^{-4} + 4(1 - b^{-2})$.

Simplify:

29. $$\frac{3^{\frac{2}{3}} + 3^{\frac{1}{3}} + 1}{3^{\frac{1}{3}} + 1} + \frac{3^{\frac{2}{3}} - 3^{\frac{1}{3}} + 1}{3^{\frac{1}{3}} - 1}.$$

30. $\dfrac{(\frac{3}{2})^{\frac{1}{2}} - (\frac{2}{3})^{\frac{3}{2}}}{6^{\frac{1}{2}} + (\frac{2}{3})^{\frac{1}{2}}}$.

31. $\dfrac{xy^{\frac{1}{2}} - 2x^{\frac{3}{4}}y^{\frac{3}{4}} + x^{\frac{1}{2}}y}{xy^{\frac{1}{2}} - x^{\frac{1}{2}}y}$.

32. $\dfrac{2y^{-\frac{2}{3}} - 2y^{-\frac{1}{3}} + 1 - y^{\frac{1}{3}}}{3 - 3y^{\frac{1}{3}} + 4y^{\frac{2}{3}} - 4y}$.

33. $\dfrac{x - 7x^{\frac{2}{3}} + 16x^{\frac{1}{3}} - 12}{3x - 14x^{\frac{2}{3}} + 16x^{\frac{1}{3}}}$.

34. $\dfrac{a^2 - 3a^{\frac{3}{2}}b^{\frac{1}{2}} + 2ab}{a - b} \times \dfrac{a^2b^{\frac{1}{2}} + a^{\frac{1}{2}}b^2}{a^{\frac{3}{2}} - 8b^{\frac{3}{2}}}$.

35. $\dfrac{(x - x^{-1})(x^2 + x^{-2}) + (x + x^{-1})(x^2 - x^{-2})}{(x + x^{-1})^2 + (x^2 + x^{-2})}$.

36. Express the square root of $\dfrac{1 + 4x^{\frac{1}{2}} - 2x - 12x^{\frac{3}{2}} + 9x^2}{1 - 4x^{\frac{1}{2}} + 6x - 4x^{\frac{3}{2}} + x^2}$ in its simplest form.

37. Divide the product of $a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} - 2b^{\frac{2}{3}}$ and $2a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} - b^{\frac{2}{3}}$ by $2a^{\frac{2}{3}} - 3a^{\frac{1}{3}}b^{\frac{1}{3}} - 2b^{\frac{2}{3}}$.

38. Find the coefficients of x and x^3 in the product of $x^{\frac{5}{2}} + p^{\frac{1}{2}}x^2 + 2px^{\frac{3}{2}} - 3p^{\frac{3}{2}}x + 2p^2x^{\frac{1}{2}} + p^{\frac{5}{2}}$ and $(x^{\frac{1}{2}} - p^{\frac{1}{2}})^3$.

39. If $x = 2^{\frac{1}{3}} - 2^{-\frac{1}{3}}$, prove that $2x^3 + 6x = 3$.

40. If $m = a^x$, $n = a^y$ and $a^2 = (m^y n^x)^z$, show that $xyz = 1$.

41. If $a = xy^{p-1}$, $b = xy^{q-1}$, $c = xy^{r-1}$, prove that
$$a^{q-r}b^{r-p}c^{p-q} = 1.$$

42. If $a^x = b^y = c^z$, and $b^2 = ac$, prove that $\dfrac{1}{x} + \dfrac{1}{z} = \dfrac{2}{y}$.

43. If $x^y = y^x$, prove that $\left(\dfrac{x}{y}\right)^{\frac{x}{y}} = x^{\frac{x}{y} - 1}$.

44. Simplify $[1 - \{1 - (1 - x^2)^{-1}\}^{-1}]^{-\frac{1}{2}}$.

45. Solve the equation $2\sqrt[3]{x} + 2x^{-\frac{1}{3}} = 5$.

CHAPTER XXIV.

LOGARITHMS.

249. Def. 29. If $a^x = m$, then x is called the *logarithm* of m to the base a.

Thus, since

$2^4 = 16$,	4 is the logarithm	of	16	to the base	2.
$4^3 = 64$,	3 ,,	,,	64	,, ,,	4.
$10^2 = 100$,	2 ,,	,,	100	,, ,,	10.
$10^5 = 100000$,	5 ,,	,,	100000	,, ,,	10.
$10^{-1} = \frac{1}{10} = \cdot 1$,	-1 ,,	,,	$\cdot 1$	,, ,,	10.
$10^{-3} = \frac{1}{1000} = \cdot 001$,	-3 ,,	,,	$\cdot 001$	,, ,,	10.

Exercises 171. (*Vivâ Voce.*)

State:

1. $\log_2 4$. 2. $\log_2 8$. 3. $\log_3 9$.
4. $\log_5 25$. 5. $\log_{10} 10$. 6. $\log_2 32$.
7. $\log_4 16$. 8. $\log_6 216$. 9. $\log_{10} 1000$.
10. $\log_6 36$. 11. $\log_2 2$. 12. $\log_3 27$.
13. $\log_7 49$. 14. $\log_7 7$. 15. $\log_5 125$.
16. $\log_3 81$. 17. $\log_2 64$. 18. $\log_4 256$.
19. $\log_7 343$. 20. $\log_9 81$. 21. $\log_{10} 100$.

22. $\log_2 \frac{1}{2}$. 23. $\log_{10} 10000$. 24. $\log_9 729$.
25. $\log_5 5$. 26. $\log_8 \frac{1}{8}$. 27. $\log_8 64$.
28. $\log_9 9$. 29. $\log_9 \frac{1}{81}$. 30. $\log_2 \frac{1}{4}$.
31. $\log_4 \frac{1}{4}$. 32. $\log_8 512$. 33. $\log_6 \frac{1}{36}$.
34. $\log_{10} \cdot 01$. 35. $\log_3 \frac{1}{9}$. 36. $\log_5 \frac{1}{125}$.
37. $\log_{10} \cdot 0001$. 38. $\log_7 \frac{1}{343}$. 39. $\log_{10} \cdot 000001$.
40. $\log_2 \frac{1}{32}$.

250. The logarithm of m to the base a is written $\log_a m$. Thus, if $a^x = m$, then $\log_a m = x$.

From the results of the previous article, we have

$$\log_2 16 = 4,$$
$$\log_4 64 = 3,$$
$$\log_{10} 100 = 2,$$
$$\log_{10} 100000 = 5,$$
$$\log_{10} \cdot 1 = -1,$$
$$\log_{10} \cdot 001 = -3.$$

If the base be omitted in the present chapter, it may always be assumed to be 10. Thus, log 100 is assumed to be $\log_{10} 100$.

Exercises 172. (*Vivâ Voce.*)

Transform the following into equations of the form $a^x = m$:

1. $\log_2 8 = 3$. 2. $\log_3 9 = 2$. 3. $\log_4 64 = 3$.
4. $\log_8 64 = 2$. 5. $\log_2 64 = 6$. 6. $\log_{10} 10 = 1$.
7. $\log_{10} 1000 = 3$. 8. $\log_5 125 = 3$. 9. $\log_2 32 = 5$.
10. $\log_3 81 = 4$. 11. $\log_{10} \cdot 01 = -2$. 12. $\log_2 \frac{1}{2} = -1$.
13. $\log_{10} 10000 = 4$. 14. $\log_2 \frac{1}{4} = -2$. 15. $\log_4 \frac{1}{4} = -1$.
16. $\log_3 \frac{1}{27} = -3$. 17. $\log_2 \frac{1}{16} = -4$. 18. $\log_{10} \cdot 001 = -3$.
19. $\log_8 \frac{1}{64} = -2$. 20. $\log_{10} \cdot 1 = -1$.

251. *Example* 1. Find (i) $\log_4 8$, (ii) $\log_{27} 81$, (iii) $\log_{100} \cdot 001$.

(i) Let $\log_4 8 = x$,

then $4^x = 8$,

$\therefore (2^2)^x = 2^3$,

$\therefore 2^{2x} = 2^3$,

$\therefore 2x = 3$ or $x = 1 \cdot 5$.

(ii) Let $\log_{27} 81 = x$,

then $27^x = 81$,

$\therefore (3^3)^x = 3^4$,

$\therefore 3^{3x} = 3^4$,

$\therefore 3x = 4$ or $x = 1\frac{1}{3} = 1 \cdot 3333....$

(iii) Let $\log_{100} \cdot 001 = x$,

then $100^x = \cdot 001 = \frac{1}{1000}$,

$\therefore (10^2)^x = 10^{-3}$,

$\therefore 10^{2x} = 10^{-3}$,

$\therefore 2x = -3$ or $x = -1 \cdot 5$.

Exercises 173.

Find:

1. $\log_8 16$. 2. $\log_9 27$. 3. $\log_{16} 64$. 4. $\log_{100} 10$.
5. $\log_{16} 8$. 6. $\log_4 \frac{1}{2}$. 7. $\log_9 \frac{1}{3}$. 8. $\log_8 \frac{1}{16}$.
9. $\log_{16} \frac{1}{2}$. 10. $\log_{100} \frac{1}{10}$.

252. *Construction of a simple table of logarithms.* The general method of calculating logarithms to any base is explained in Chapter XXXIV. The logarithms of a few numbers may, however, be found easily by arithmetic. In this article, logarithms to the base 10 will alone be considered.

For instance, the square root of 10 may be shown to be 3·16227766..., or, to four places of decimals, 3·1623;

i.e. $10^{\frac{1}{2}} = 3{\cdot}1623$ or $\log_{10} 3{\cdot}1623$ is $\frac{1}{2}$ or ·5000.

Again, the fourth root of 10, or the square root of 3·16227766, is 1·7782794, and therefore

$$\log 1{\cdot}7783 \text{ is } \tfrac{1}{4} \text{ or } {\cdot}2500.$$

In the same way, from the 8th and 16th roots of 10, we find that

$$\log 1{\cdot}3335 \text{ is } \tfrac{1}{8} \text{ or } {\cdot}1250,$$

$$\log 1{\cdot}1548 \text{ is } \tfrac{1}{16} \text{ or } {\cdot}0625.$$

Now,

$$10^{\frac{3}{16}} = 10^{\frac{1}{8}} \times 10^{\frac{1}{16}} = 1{\cdot}3335 \times 1{\cdot}1548 = 1{\cdot}5399,$$

$$10^{\frac{5}{16}} = 10^{\frac{1}{4}} \times 10^{\frac{1}{16}} = 1{\cdot}7783 \times 1{\cdot}1548 = 2{\cdot}0535,$$

and so on; i.e. $\log 1{\cdot}5399 = \frac{3}{16} = {\cdot}1875,$

$\log 2{\cdot}0535 = \frac{5}{16} = {\cdot}3125,$

and so on.

Proceeding in this way as far as $10^{\frac{15}{16}}$, we obtain the following table of logarithms to the base 10 (a table which the reader should calculate for himself).

Number	Logarithm
1·1548	·0625
1·3335	·1250
1·5399	·1875
1·7783	·2500
2·0535	·3125
2·3714	·3750
2·7384	·4375
3·1623	·5000
3·6517	·5625
4·2170	·6250
4·8697	·6875
5·6234	·7500
6·4938	·8125
7·4989	·8750
8·6596	·9375
10·0000	1·0000

253. (i) *The logarithm of the base itself is unity, and* (ii) *the logarithm of unity to any base is zero.*

Let a be the base.

(i) Since $$a^1 = a,$$
$$\therefore \log_a a = 1.$$

(ii) Since $$a^0 = 1,$$
$$\therefore \log_a 1 = 0.$$

254. *The logarithm of the product of two or more numbers is equal to the sum of the logarithms of the factors.*

Let the numbers be m and n, and let x, y be their logarithms to the base a, so that

$$m = a^x, \quad n = a^y,$$
$$\therefore \quad mn = a^x \times a^y = a^{x+y},$$
$$\therefore \quad \log_a mn = x + y = \log_a m + \log_a n.$$

Again, $$\log_a mnp = \log_a mn + \log_a p$$
$$= \log_a m + \log_a n + \log_a p,$$

and similarly, for any number of factors.

255. *Example* 2. Multiply 1·7783 by 2·3714 to 4 places of decimals (see table on p. 375).

$$\log(1{\cdot}7783 \times 2{\cdot}3714) = \log 1{\cdot}7783 + \log 2{\cdot}3714$$
$$= {\cdot}2500$$
$$+ {\cdot}3750$$
$$= {\cdot}6250$$
$$= \log 4{\cdot}2170,$$
$$\therefore \quad 1{\cdot}7783 \times 2{\cdot}3714 = 4{\cdot}2170.$$

Example 3. Find the square of 1·5399 to 4 places of decimals.

$$\log(1{\cdot}5399 \times 1{\cdot}5399) = \log 1{\cdot}5399 + \log 1{\cdot}5399$$
$$= {\cdot}1875 \times 2$$
$$= {\cdot}3750$$
$$= \log 2{\cdot}3714,$$

$\therefore$ the square of 1·5399 is 2·3714.

256. *The logarithm of the quotient of one number divided by another is equal to the logarithm of the former diminished by the logarithm of the latter.*

Let the dividend be m and the divisor n, and let x, y be their logarithms to the base a, so that

$$m = a^x, \quad n = a^y,$$

$$\therefore \frac{m}{n} = \frac{a^x}{a^y} = a^{x-y},$$

$$\therefore \log_a \frac{m}{n} = x - y = \log_a m - \log_a n.$$

257. *Example* 4. Divide 6·4938 by 3·6517 to 4 places of decimals.

$$\begin{aligned} \log(6{\cdot}4938 \div 3{\cdot}6517) &= \log 6{\cdot}4938 - \log 3{\cdot}6517 \\ &= {\cdot}8125 \\ &\quad - {\cdot}5625 \\ &= {\cdot}2500 \\ &= \log 1{\cdot}7783, \\ \therefore \; 6{\cdot}4938 \div 3{\cdot}6517 &= 1{\cdot}7783. \end{aligned}$$

258. *The logarithm of any power of a number is equal to the logarithm of the number multiplied by the index.*

Let m be the number, r the index, and let x be the logarithm of m to the base a, so that

$$m = a^x,$$

$$\therefore m^r = (a^x)^r = a^{xr},$$

$$\therefore \log_a m^r = rx = r \,.\, \log_a m.$$

259. *Example* 5. Find the 5th power of 1·5399 to 4 places of decimals.

$$\begin{aligned} \log(1{\cdot}5399)^5 &= 5 \,.\, \log 1{\cdot}5399 \\ &= {\cdot}1875 \times 5 \\ &= {\cdot}9375 \\ &= \log 8{\cdot}6596, \\ \therefore \; (1{\cdot}5399)^5 &= 8{\cdot}6596. \end{aligned}$$

Example 6. Find the 7th root of 7·4989 to 4 places of decimals.

$$\log (7\cdot 4989)^{\frac{1}{7}} = \tfrac{1}{7} \times \log 7\cdot 4989$$
$$= \cdot 8750 \div 7$$
$$= \cdot 1250$$
$$= \log 1\cdot 3335,$$
$$\therefore \sqrt[7]{7\cdot 4989} = 1\cdot 3334$$

Exercises 174.

Find, with the aid of the table of logarithms on p. 123, the values of:

1. $1\cdot 1548 \times 1\cdot 7783$.
2. $3\cdot 1623 \times 2\cdot 0535$.
3. $4\cdot 2170 \times 2\cdot 3714$.
4. $(1\cdot 3335)^2$.
5. $(2\cdot 0535)^3$.
6. $(1\cdot 7783)^4$.
7. $7\cdot 4989 \div 4\cdot 8697$.
8. $10 \div 3\cdot 6517$.
9. $6\cdot 4398 \div 1\cdot 7783$.
10. The square root of 4·2170.
11. The cube root of 5·6234.
12. The fifth root of 8·6596.
13. $(5\cdot 6234)^{\frac{2}{3}}$.

260. DEF. 30. The integral part of the logarithm of a number is called its *characteristic*, and the decimal part its *mantissa*.

261. It has been shown that

$$\log 1\cdot 5399 = \cdot 1875.$$

Now, $$15\cdot 399 = 1\cdot 5399 \times 10,$$
$$\therefore \log 15\cdot 399 = \log 1\cdot 5399 + \log 10$$
$$= \cdot 1875 + 1$$
$$= 1\cdot 1875.$$

Again, $1539 \cdot 9 = 1 \cdot 5399 \times 1000,$

$$\therefore \log 1539 \cdot 9 = \log 1 \cdot 5399 + \log 10^3$$
$$= \cdot 1875 + 3 . \log 10$$
$$= 3 \cdot 1875.$$

Also, $\cdot 015399 = 1 \cdot 5399 \div 100,$

$$\therefore \log \cdot 015399 = \log 1 \cdot 5399 - \log 10^2$$
$$= \cdot 1875 - 2 . \log 10$$
$$= \bar{2} \cdot 1875,$$

where $\bar{2} \cdot 1875$ stands for $-2 + \cdot 1875$.

Lastly, $\cdot 00015399 = 1 \cdot 5399 \div 10000,$

$$\therefore \log \cdot 00015399 = \log 1 \cdot 5399 - \log 10^4$$
$$= \cdot 1875 - 4 . \log 10$$
$$= \bar{4} \cdot 1875.$$

Thus, if $1 \cdot 5399$ be multiplied or divided by some power of 10 (i.e. if the number so obtained consist of the same succession of significant figures), the logarithms of such numbers have the same mantissa as the logarithm of $1 \cdot 5399$.

Again, the characteristics of $\log 1 \cdot 5399$, $\log 15 \cdot 399$ and $\log 1539 \cdot 9$ are respectively 0, 1 and 3; i.e. in each case the characteristic is one less than the number of digits in the integral part of the number. Also, the characteristics of $\log \cdot 015399$ and $\log \cdot 00015399$ are respectively $\bar{2}$ and $\bar{4}$; i.e. in each case the characteristic is negative and numerically one greater than the number of ciphers before the first significant figure.

262. *To prove that* (i) *in the logarithm of a number containing $n + 1$ digits, the characteristic is n, and* (ii) *in the logarithm of a decimal containing n ciphers before the first significant figure, the characteristic is $-(n + 1)$.*

Since $\log 1 = 0$ and $\log 10 = 1$, it follows that the logarithm of any number between 1 and 10 must lie between 0 and 1, i.e. its characteristic is zero.

Let N be any number between 1 and 10, and let m be its logarithm, so that m is a decimal. Then, if n be any integer, positive or negative,

$$\log(N \times 10^n) = \log N + \log 10^n$$
$$= m + n \,.\, \log 10$$
$$= m + n,$$

i.e. the characteristic of $\log(N \times 10^{n})$ is n, and, whatever be the value of n, the mantissa is the same, namely, m.

(i) Now, N is a number consisting of 1 digit, $N \times 10$ of 2 digits, $N \times 10^2$ of 3 digits, ..., $N \times 10^n$ of $n+1$ digits.

Hence, if the integral part of a number contain $n+1$ digits, the characteristic of its logarithm is n.

(ii) Again, in $N \times 10^{-1}$ there is no cipher before the first significant figure, in $N \times 10^{-2}$ there is one such cipher, in $N \times 10^{-3}$ there are 2 ciphers, ..., and in $N \times 10^{-(n+1)}$ there are n ciphers.

Hence, if a decimal contain n ciphers before the first significant figure, the characteristic of its logarithm is $-(n+1)$.

Exercises 175. (*Vivâ Voce.*)

What is the characteristic of the logarithm of:

1.	27.	2.	352.	3.	9·6.	4.	85·27.
5.	673·4.	6.	2748·1.	7.	3·2.	8.	52730.
9.	·2761.	10.	·00452.	11.	·079.	12.	·00062.
13.	·8324.	14.	·00635.	15.	371·4.	16.	·06295.
17.	7015·2.	18.	572000.	19.	·2918.	20.	·000853.

Given log 3·6517 = ·5625, what are the logarithms of:

21.	36·517.	22.	365·17.	23.	·0036517.
24.	36517.	25.	·36517.		

Given $\log 8{\cdot}6596 = {\cdot}9375$, what are the logarithms of:

26. ·086596. 27. 8659·6. 28. 86·596.

29. ·00086596. 30. 8659600.

Given $\log 5{\cdot}6234 = {\cdot}7500$, what are the logarithms of:

31. 5623·4. 32. ·56234. 33. 56·234.

34. ·0056234. 35. 562340.

Given $\log 2{\cdot}0535 = {\cdot}3125$, what are the logarithms of:

36. 20·535. 37. ·020535. 38. 20535.

39. ·20535. 40. ·000020535.

263. *To prove that* $\log_b m = \log_a m \div \log_a b$.

Let $\log_a m = x$ and $\log_b m = y$, so that

$$m = a^x = b^y,$$

$$\therefore\ a^{\frac{x}{y}} = b,$$

$$\therefore\ \frac{x}{y} = \log_a b,$$

$$\therefore\ y = x \div \log b,$$

$$\therefore\ \log_b m = \log_a m \div \log_a b.$$

Thus, if the logarithms of m and b to the base a be known, the logarithm of m to the base b can be determined by multiplying the logarithm of m to the base a by the reciprocal of the logarithm of b to the base a.

264. *To prove that* $\log_b a \times \log_a b = 1$.

This may be proved by putting $m = a$ in the result of the previous article, or as follows:

Let $\log_b a = x$ and $\log_a b = y$,

so that $a = b^x$ and $b = a^y$,

and therefore $a = (a^y)^x = a^{xy}$,

$$\therefore\ xy = 1,$$

i.e. $\log_b a \times \log_a b = 1.$

Applications of Logarithms to Arithmetic.

265. The present section contains a few illustrations of the application of logarithms to problems of arithmetic and mensuration. They are adapted for working with tables of logarithms calculated to 4 decimal places.

Example 7. Find the value of $471 \times 3{\cdot}7^2 \div 8{\cdot}76^3$ to 3 places of decimals.

Let $u = 471 \times 3{\cdot}7^2 \div 8{\cdot}76^3$,

then
$$\begin{aligned}\log u &= 2{\cdot}6730 + {\cdot}5682 \times 2 - {\cdot}9425 \times 3\\ &= 2{\cdot}6730\\ &\quad + 1{\cdot}1364 - 2{\cdot}8275\\ &= 3{\cdot}8094\\ &\quad - 2{\cdot}8275\\ &= {\cdot}9819,\end{aligned}$$
$$\therefore\ u = 9{\cdot}591.$$

Example 8. Find the amount of £475 in 6 years at 3 per cent. compound interest (to the nearest pound).

In 1 year, £1 amounts to $(1 + \frac{3}{100})$ £,

" " £$(1 + \frac{3}{100})$ " " $(1 + \frac{3}{100}) + (1 + \frac{3}{100})\frac{3}{100}$ £
$= (1 + \frac{3}{100})^2$ £,

∴ in 2 years £1 " " $(1 + \frac{3}{100})^2$ £,

∴ in 3 " " " " $(1 + \frac{3}{100})^3$ £,

∴ in 6 " " " " $(1 + \frac{3}{100})^6$ £,

∴ " " £475 " " $475 \times (\frac{103}{100})^6$ £.

Let A be this amount,
$$\begin{aligned}\log A &= \log 475 + 6(\log 103 - \log 100)\\ &= 2{\cdot}6767 + (2{\cdot}0128 - 2) \times 6\\ &= 2{\cdot}6767\\ &\quad + {\cdot}0768\\ &= 2{\cdot}7535,\end{aligned}$$
$$\therefore\ A = £567.$$

Example 9. The volume of a cube is 437·2 c. ins., find the length of an edge.

Let u be the length of an edge in inches,

$$\therefore \ u = (437 \cdot 2)^{\frac{1}{3}},$$

$$\therefore \ \log u = 2 \cdot 6407 \times \tfrac{1}{3}$$

$$= \cdot 8802,$$

$$\therefore \ u = 7 \cdot 59 \text{ ins.}$$

Exercises 176.

Find:

1. 53×64.
2. $15 \times 27 \times \cdot 32$ (1 place).
3. $87 \times 22 \div 29$.
4. $124 \times 27 \div 53$ (2 places).
5. $\cdot 258 \times \cdot 719 \div \cdot 526$ (4 places).
6. $2413 \div 57 \div 132$ (4 places).
7. $17 \times \cdot 25 \times 27 \times \cdot 018$ (3 places).
8. $2 \cdot 421 \times 1 \cdot 907 \div \cdot 263$ (2 places).
9. Square of 1·573 (3 places).
10. Cube of 7·58 (1 place).
11. 5th power of 2·17 (2 places).
12. 9th power of 1·027 (3 places).
13. Cube root of 357·2 (3 places).
14. 4th root of 2719 (3 places).
15. 10th root of 83·4 (3 places).
16. $3 \cdot 5^2 \times 2 \cdot 1^3 \times 1 \cdot 2^4$ (1 place).
17. $26^3 \times 84^4 \div 91^5$ (1 place).
18. $2 \cdot 531^2 \div 1 \cdot 297 \div 9 \cdot 26^3$ (6 places).
19. $11^{\frac{2}{3}} \times 21^{\frac{3}{4}}$ (2 places).
20. $12 \cdot 4^{\frac{1}{3}} \times 8 \cdot 27^{\frac{2}{3}} \div 6 \cdot 19^{\frac{3}{4}}$ (3 places).

21. Find the simple interest on £325 at $4\frac{1}{2}$ per cent. for $3\frac{1}{2}$ years.

22. Find the simple interest on £85 at 4 per cent. for 7 years.

23. Find the amount at compound interest of £1 at 5 per cent. for 10 years.

24. Find the compound interest on £55 at 4 per cent. for 8 years.

25. In how many years will a sum of money double itself at 5 per cent. compound interest?

26. In how many years will a sum of money double itself at 4 per cent. compound interest?

27. If the death-rate in a town of 8341 inhabitants be 17·2 per 1000 per year, how many persons die in a year?

28. A man invests £1340 in $3\frac{1}{2}$ per cent. stock at $101\frac{1}{2}$; find his interest to the nearest pound.

29. Find the cost of carpeting a room 24 ft. long and 19 ft. wide with carpet $2\frac{1}{2}$ ft. wide at 3*s*. 11*d*. a yard.

30. Find the volume of a rectangular solid 3 ft. 5 ins. long, 2 ft. 11 ins. wide and 1 ft. 7 ins. high.

31. Find the length of the side of a square which contains 42·72 sq. ins.

32. Find the length of the edge of a cube which contains 157·4 c. ins.

33. The area of a triangle whose sides are a, b, c and semi-perimeter s is $\sqrt{[s(s-a)(s-b)(s-c)]}$; find the area of a triangle whose sides are 17, 11 and 9 ins.

34. The area of a rectangle which is twice as long as it is wide is $21\frac{1}{8}$ sq. ft.; find the lengths of the sides.

35. A plate of metal 14·4 ins. long, 4·8 ins. wide and 3·6 ins. thick is melted down and cast into a solid cube; find the edge of the cube.

36. The surface of a sphere of radius r is $4\pi r^2$, where $\pi = 3{\cdot}142$; find that of a sphere of radius 6·45 ins.

37. Find the radius of a sphere, the surface of which contains 321·4 sq. ins.

38. The volume of a sphere of radius r is $\frac{4}{3}\pi r^3$, where $\pi = 3{\cdot}142$; find the volume of a sphere of radius 2·135 ins.

39. Find the radius of a sphere, the volume of which is 1027 c. ins.

40. Find the surface of a cube, the volume of which is 120·4 c. ins.

41. Find the surface of a sphere, the volume of which is 524 c. ins.

42. The radius of the circumcircle of a triangle, the sides of which are a, b, c and the semi-perimeter s, is

$$abc \div 4\sqrt{[s(s-a)(s-b)(s-c)]};$$

find that of a triangle the sides of which are 14, 11 and 9 ins. long.

CHAPTER XXV.

SURDS.

266. DEF. 31. A root of a number which cannot be expressed as a commensurable fraction is called a *surd* or *irrational quantity.*

Thus, $\sqrt{3}$, $\sqrt[3]{4}$ and $\sqrt[4]{10}$ are surds, but $\sqrt{4}$, $\sqrt[3]{27}$ and $\sqrt[4]{16}$ are not surds, for they can be expressed as rational quantities.

DEF. 32. Surds are of the same *order* when the same roots are taken.

Thus $\sqrt{8}$ and $\sqrt{24}$ are surds of the same order.

DEF. 33. A *quadratic* surd or surd of the *second order* is one in which the square root of a number is taken; a *cubic* surd or surd of the *third order* one in which the cube root is taken; and a surd of the *nth order* one in which the *n*th root is taken.

Thus, $\sqrt{5}$ is a quadratic surd, $\sqrt[3]{12}$ a cubic surd, and $\sqrt[4]{10}$ a surd of the fourth order.

DEF. 34. *Similar* or *like* surds are those which contain the same irrational factor.

Thus, $\sqrt{2}$ and $5\sqrt{2}$ are similar surds; so also are $3\sqrt[3]{4}$ and $8\sqrt[3]{4}$.

Every quadratic surd possesses two values, equal numerically, but opposite in sign. Thus, the square root of 5 is either $+\sqrt{5}$ or $-\sqrt{5}$. We shall, however, confine ourselves throughout this chapter and the next to the principal or positive root only, and consider that $\sqrt{2}$, for instance, is $+1{\cdot}414\ldots$.

267. The following properties of surds result from first principles or from the theory of indices:

(i) A rational quantity may be expressed in the form of a surd, e.g.

$$2=\sqrt{4}=\sqrt[3]{8}=\sqrt[4]{16}=\ldots$$

(ii) Sum or difference of surds of the same order. We have

$$3\sqrt{2}+2\sqrt{2}=5\sqrt{2},\quad 4\sqrt{3}-\sqrt{3}=3\sqrt{3},\quad a\sqrt{x}\pm b\sqrt{x}=(a\pm b)\sqrt{x}.$$

(iii) Product or quotient of surds of the same order. Since $a^{\frac{1}{n}}\times b^{\frac{1}{n}}=(ab)^{\frac{1}{n}}$ and $a^{\frac{1}{n}}\div b^{\frac{1}{n}}=\left(\frac{a}{b}\right)^{\frac{1}{n}}$, it follows that

$$\sqrt{3}\times\sqrt{2}=\sqrt{6},\quad \sqrt{8}\div\sqrt{2}=\sqrt{\frac{8}{2}}=\sqrt{4}=2,\quad 4\sqrt{3}\times 3\sqrt{2}=12\sqrt{6},$$

$$2\sqrt{3}\div 4\sqrt{2}=\tfrac{1}{2}\sqrt{\frac{3}{2}},\quad \sqrt[n]{a}\times\sqrt[n]{b}=\sqrt[n]{ab},\quad \sqrt[n]{a}\div\sqrt[n]{b}=\sqrt[n]{\frac{a}{b}}.$$

So also

$$4\sqrt{2}=\sqrt{16}\times\sqrt{2}=\sqrt{32},\quad 3\sqrt[3]{3}=\sqrt[3]{27}\times\sqrt[3]{3}=\sqrt[3]{81}.$$

(iv) Product or quotient of surds of different orders. We have

$$\sqrt{2}\times\sqrt[3]{3}=\sqrt[6]{2^3}\times\sqrt[6]{3^2}=\sqrt[6]{8}\times\sqrt[6]{9}=\sqrt[6]{72},$$

$$\sqrt{5}\div\sqrt[4]{10}=\sqrt[4]{25}\div\sqrt[4]{10}=\sqrt[4]{\frac{25}{10}}=\sqrt[4]{\frac{5}{2}},$$

$$\sqrt[m]{a}\times\sqrt[n]{b}=\sqrt[mn]{a^n}\times\sqrt[mn]{b^m}=\sqrt[mn]{(a^nb^m)}.$$

(v) Simplification of surds. We have

$$\sqrt{32}=\sqrt{16}\times\sqrt{2}=4\sqrt{2},$$

$$\sqrt[3]{81}=\sqrt[3]{27}\times\sqrt[3]{3}=3\sqrt[3]{3},$$

$$\sqrt{(a^5b^3)}=\sqrt{(a^4b^2)}\times\sqrt{(ab)}=a^2b\sqrt{(ab)}.$$

Exercises 177. (*Vivâ Voce.*)

Express in the form of quadratic surds:

1. 4. 2. 7. 3. 9. 4. 12.

Express in the form of cubic surds:

5. 2. 6. 3. 7. 5. 8. 10.

Express as single surds:

9. $2\sqrt{2}+\sqrt{2}$. 10. $3\sqrt{3}-2\sqrt{3}$. 11. $5\sqrt[3]{3}+7\sqrt[3]{3}$.
12. $4\sqrt[4]{5}-2\sqrt[4]{5}$. 13. $\sqrt{5}\times\sqrt{2}$. 14. $\sqrt{7}\times\sqrt{3}$.
15. $\sqrt[3]{5}\times\sqrt[3]{6}$. 16. $\sqrt[4]{5}\times\sqrt[4]{2}$. 17. $\sqrt{6}\div\sqrt{2}$.
18. $\sqrt{10}\div\sqrt{5}$. 19. $\sqrt[3]{6}\div\sqrt[3]{3}$. 20. $\sqrt[5]{12}\div\sqrt[5]{4}$.

Simplify:

21. $\sqrt{8}$. 22. $\sqrt{12}$.
23. $\sqrt{18}$. 24. $\sqrt{32}$.
25. $\sqrt[3]{16}$. 26. $\sqrt[3]{54}$.
27. $\sqrt[3]{24}$. 28. $\sqrt[3]{(a^4b^3)}$.
29. $\sqrt[4]{(x^5y^5)}$. 30. $\sqrt[6]{(a^8b^6)}$.
31. $(\sqrt{2}+1)(\sqrt{2}-1)$. 32. $(2+\sqrt{2})(2-\sqrt{2})$.
33. $(3+\sqrt{3})(3-\sqrt{3})$. 34. $(5-\sqrt{6})(5+\sqrt{6})$.
35. $(7-\sqrt{5})(7+\sqrt{5})$.

268. *Example* 1. Simplify:

(i) $\sqrt{63}-\sqrt{32}+\sqrt{98}-\sqrt{28}$; (ii) $\sqrt{15}\times\sqrt{45}\div\sqrt{12}$;
(iii) $\sqrt{8}\times\sqrt[4]{4}$.

(i) $$\sqrt{63}-\sqrt{32}+\sqrt{98}-\sqrt{28}=3\sqrt{7}-4\sqrt{2}+7\sqrt{2}-2\sqrt{7}$$
$$=\sqrt{7}+3\sqrt{2}.$$

(ii) $$\sqrt{15}\times\sqrt{45}\div\sqrt{12}=\sqrt{5}\times\sqrt{3}\times 3\sqrt{5}\div 2\sqrt{3}$$
$$=5\times 3\times\sqrt{3}\div 2\sqrt{3}$$
$$=7\tfrac{1}{2}.$$

(iii) $\sqrt{8} \times \sqrt[4]{4} = \sqrt[4]{64} \times \sqrt[4]{4} = \sqrt[4]{256}$
$= \sqrt{16}$
$= 4.$

Example 2. Arrange in descending order of magnitude $\sqrt{2}$, $\sqrt[3]{3}$ and $\sqrt[6]{6}$.

$$\sqrt{2} = \sqrt[6]{8}, \qquad \sqrt[3]{3} = \sqrt[6]{9}, \qquad \sqrt[6]{6} = \sqrt[6]{6},$$

$\therefore$ order is $\sqrt[3]{3}$, $\sqrt{2}$, $\sqrt[6]{6}$.

Exercises 178.

Simplify:

1. $\sqrt{1250}$.
2. $2\sqrt{720}$.
3. $\sqrt{8} + \sqrt{18} - \sqrt{50}$.
4. $\sqrt{12} - \sqrt{8} + \sqrt{18} - \sqrt{3}$.
5. $\sqrt{98} + \sqrt{32} - \sqrt{200}$.
6. $\sqrt{24} + \sqrt{54} - \sqrt{96} + \sqrt{6}$.
7. $\sqrt{128} - 2\sqrt{50} + \sqrt{72} - \sqrt{18}$.
8. $\sqrt{75} - \sqrt{147} + \sqrt{50} + 3\sqrt{12} - 3\sqrt{18}$.
9. $\sqrt{(8x)} + \sqrt{(18x)} + \sqrt{(50x)}$.
10. $a\sqrt{(a^2x)} + b\sqrt{(b^2x)} + c\sqrt{(c^2x)}$.
11. $b\sqrt{(a^3b)} + \sqrt{(a^3b^3)} - \dfrac{2b}{a}\sqrt{(a^5b)}$.
12. $\sqrt{6} \times \sqrt{12} \times \sqrt{18}$.
13. $5\sqrt{32} \times \sqrt{48} \times 2\sqrt{54}$.
14. $\sqrt{7} \times \sqrt{14} \times \sqrt{28} \times \sqrt{216}$.
15. $\sqrt{28} \times \sqrt{35} \times \sqrt{45} - \sqrt{147} \times \sqrt{75}$.
16. $\sqrt{12} \times \sqrt{50} \div \sqrt{24}$.
17. $\sqrt{98} \div \sqrt{12} \div \sqrt{54}$.
18. $\sqrt{2} \times \sqrt[3]{4}$.
19. $\sqrt[3]{4} \times \sqrt[4]{8}$.
20. $\sqrt[3]{3} \times \sqrt{3} \times \sqrt[3]{9}$.
21. $\sqrt{2} \times \sqrt[3]{2} \times \sqrt[6]{2}$.
22. Which is the greater $2\sqrt{3}$ or $3\sqrt{2}$?
23. Which is the greater $3\sqrt{5}$ or $4\sqrt{3}$?
24. Which is the greater $3\sqrt[3]{2}$ or $2\sqrt[3]{7}$?

25. Which is the greater $\sqrt{5}$ or $\sqrt[3]{10}$?

26. Arrange in ascending order of magnitude

$$5\sqrt{3},\ 6\sqrt{2},\ 4\sqrt{5}.$$

27. Arrange in descending order of magnitude

$$5,\ 2\sqrt{7},\ 4\sqrt[4]{3}.$$

269. *Example* 3. Express with rational denominators:

(i) $\dfrac{6}{\sqrt{3}}$; (ii) $\dfrac{4}{\sqrt{5}-\sqrt{2}}$; (iii) $\dfrac{1}{\sqrt{3}-\sqrt{2}+1}$.

(i) $\dfrac{6}{\sqrt{3}} = \dfrac{6\sqrt{3}}{3} = 2\sqrt{3}.$

(ii)
$$\frac{4}{\sqrt{5}-\sqrt{2}} = \frac{4(\sqrt{5}+\sqrt{2})}{(\sqrt{5}-\sqrt{2})(\sqrt{5}+\sqrt{2})} = \frac{4(\sqrt{5}+\sqrt{2})}{5-2}$$
$$= \tfrac{4}{3}(\sqrt{5}+\sqrt{2}).$$

(iii)
$$\frac{1}{\sqrt{3}-\sqrt{2}+1} = \frac{\sqrt{3}+1+\sqrt{2}}{(\sqrt{3}+1)^2-(\sqrt{2})^2} = \frac{\sqrt{3}+\sqrt{2}+1}{3+1+2\sqrt{3}-2}$$
$$= \frac{\sqrt{3}+\sqrt{2}+1}{2(\sqrt{3}+1)} = \frac{(\sqrt{3}+\sqrt{2}+1)(\sqrt{3}-1)}{2(3-1)}$$
$$= \frac{3+\sqrt{6}+\sqrt{3}-\sqrt{3}-\sqrt{2}-1}{4}$$
$$= \tfrac{1}{4}(2-\sqrt{2}+\sqrt{6}).$$

Exercises 179.

Express with rational denominators:

1. $\dfrac{1}{\sqrt{2}}, \dfrac{2}{\sqrt{3}}, \dfrac{10}{\sqrt{5}}$. 2. $\dfrac{1}{\sqrt{2}-1}$. 3. $\dfrac{2}{\sqrt{5}-1}$.

4. $\dfrac{6}{\sqrt{7}+2}$. 5. $\dfrac{1}{\sqrt{5}+\sqrt{2}}$. 6. $\dfrac{3+\sqrt{5}}{\sqrt{5}+2}$.

7. $\dfrac{\sqrt{7}+\sqrt{2}}{\sqrt{7}-\sqrt{2}}$. 8. $\dfrac{2\sqrt{3}+3\sqrt{2}}{5+2\sqrt{6}}$. 9. $\dfrac{3\sqrt{5}-4\sqrt{2}}{2\sqrt{5}-3\sqrt{2}}$.

10. $\dfrac{1}{\sqrt{2}-1}-\dfrac{1}{\sqrt{2}+1}$.

11. $\dfrac{5}{4+\sqrt{3}}+\dfrac{5}{4-\sqrt{3}}$.

12. $\dfrac{1}{\sqrt{3}-\sqrt{2}}+\dfrac{\sqrt{3}}{\sqrt{2}+1}-\dfrac{2\sqrt{2}}{\sqrt{3}-1}$.

13. $\dfrac{3\sqrt{2}}{\sqrt{3}+\sqrt{6}}-\dfrac{4\sqrt{3}}{\sqrt{6}+\sqrt{2}}+\dfrac{\sqrt{6}}{\sqrt{2}+\sqrt{3}}$.

14. $\dfrac{5\sqrt{3}-3\sqrt{2}+2\sqrt{6}}{3\sqrt{6}+5\sqrt{2}}$.

15. $\dfrac{1}{1+\sqrt{3}+\sqrt{2}}$.

16. $\dfrac{1}{\sqrt{6}+\sqrt{3}-\sqrt{2}}$.

17. $\dfrac{2}{\sqrt{5}+\sqrt{2}-1}$.

18. $\dfrac{3+2\sqrt{5}}{1-\sqrt{3}+\sqrt{5}}$.

19. $\dfrac{1}{1+\sqrt{2}+\sqrt{3}}+\dfrac{1}{1+\sqrt{2}-\sqrt{3}}+\dfrac{1}{1-\sqrt{2}+\sqrt{3}}+\dfrac{1}{-1+\sqrt{2}+\sqrt{3}}$.

20. $\dfrac{\sqrt{3}}{\sqrt{3}+\sqrt{5}+\sqrt{7}}-\dfrac{\sqrt{3}}{\sqrt{5}+\sqrt{7}-\sqrt{3}}+\dfrac{\sqrt{3}}{\sqrt{7}+\sqrt{3}-\sqrt{5}}+\dfrac{\sqrt{3}}{\sqrt{3}+\sqrt{5}-\sqrt{7}}$.

270. *A quadratic surd cannot be equal to the sum or difference of a rational quantity and a quadratic surd.*

If possible, let $\sqrt{a}=x\pm\sqrt{y}$

where x is rational and $\sqrt{a}$ and $\sqrt{y}$ are surds.

$$\therefore\ a=x^2+y\pm 2x\sqrt{y},$$

$$\therefore\ \pm\sqrt{y}=\frac{a-x^2-y}{2x},$$

i.e. a quadratic surd is equal to a rational quantity, which is impossible;

$\therefore$ a quadratic surd is not equal to the sum or difference of a rational quantity and a quadratic surd.

271. *If* $a + \sqrt{b} = x + \sqrt{y}$, *where* a *and* x *are rational and* $\sqrt{b}$ *and* $\sqrt{y}$ *are surds, then* $a = x$ *and* $\sqrt{b} = \sqrt{y}$.

Let $$a = x + m,$$
then $$x + m + \sqrt{b} = x + \sqrt{y}$$
or $$m + \sqrt{b} = \sqrt{y},$$
which is impossible, unless $m = 0$, in which case $a = x$ and therefore $\sqrt{b} = \sqrt{y}$.

272. *If* $$\sqrt{(a + \sqrt{b})} = \sqrt{x} + \sqrt{y},$$
then $$\sqrt{(a - \sqrt{b})} = \sqrt{x} - \sqrt{y}.$$

Since $$\sqrt{(a + \sqrt{b})} = \sqrt{x} + \sqrt{y},$$
$$\therefore\ a + \sqrt{b} = x + y + 2\sqrt{(xy)},$$
$$\therefore\ a = x + y \text{ and } \sqrt{b} = 2\sqrt{(xy)},$$
$$\therefore\ a - \sqrt{b} = x + y - 2\sqrt{(xy)},$$
$$\therefore\ \sqrt{(a - \sqrt{b})} = \sqrt{x} - \sqrt{y}.$$

273. *Example* 4. Find the square root of

(i) $30 + 10\sqrt{5}$, (ii) $57 - 12\sqrt{15}$.

(i) Let $$\sqrt{(30 + 10\sqrt{5})} = \sqrt{x} + \sqrt{y},$$
$$\therefore\ 30 + 10\sqrt{5} = x + y + 2\sqrt{(xy)},$$
$$\therefore\ x + y = 30 \text{ and } 2\sqrt{(xy)} = 10\sqrt{5},$$
$$\therefore\ (x - y)^2 = (x + y)^2 - 4xy = 900 - 500 = 400,$$
$$\therefore\ x - y = 20,$$
$$\therefore\ 2x = 50 \text{ and } 2y = 10, \text{ or } x = 25 \text{ and } y = 5,$$
$$\therefore\ \sqrt{(30 + 10\sqrt{5})} = 5 + \sqrt{5}.$$

(ii) Let $$\sqrt{(57 - 12\sqrt{15})} = \sqrt{x} - \sqrt{y},$$
$$\therefore\ 57 - 12\sqrt{15} = x + y - 2\sqrt{(xy)},$$
$$\therefore\ x + y = 57 \text{ and } 2\sqrt{(xy)} = 12\sqrt{15},$$
$$\therefore\ (x - y)^2 = (x + y)^2 - 4xy = 3249 - 2160 = 1089,$$
$$\therefore\ x - y = 33,$$
$$\therefore\ 2x = 90 \text{ and } 2y = 24, \text{ or } x = 45 \text{ and } y = 12,$$
$$\therefore\ \sqrt{(57 - 12\sqrt{15})} = \sqrt{45} - \sqrt{12} = 3\sqrt{5} - 2\sqrt{3}.$$

Example 5. Simplify:

(i) $\sqrt{2+\sqrt{3}+\sqrt{7-4\sqrt{3}}}$;

(ii) $1+\sqrt{12}-\sqrt{2}-\sqrt{3}-\sqrt{5-2\sqrt{6}}$.

(i) Let $\sqrt{(7-4\sqrt{3})}=\sqrt{x}-\sqrt{y}$,

so that $x+y=7$, $2\sqrt{(xy)}=4\sqrt{3}$ and $(x-y)^2=49-48=1$,

$$\therefore\ 2x=8,\ 2y=6 \text{ or } x=4,\ y=3,$$

$$\therefore\ \sqrt{2+\sqrt{3}+\sqrt{7-4\sqrt{3}}}=\sqrt{2+\sqrt{3}+2-\sqrt{3}}=\sqrt{4}$$

$$=2.$$

(ii) Let $\sqrt{(5-2\sqrt{6})}=\sqrt{x}-\sqrt{y}$,

so that $x+y=5$, $2\sqrt{(xy)}=2\sqrt{6}$ and $(x-y)^2=25-24=1$,

$$\therefore\ 2x=6,\ 2y=4 \text{ or } x=3,\ y=2,$$

$$\therefore\ 1+\sqrt{12}-\sqrt{2}-\sqrt{3}-\sqrt{(5-2\sqrt{6})}$$

$$=1+2\sqrt{3}-\sqrt{2}-\sqrt{3}-\sqrt{3}+\sqrt{2}$$

$$=1.$$

Exercises 180.

Find the square root of:

1. $4+\sqrt{12}$.
2. $7+2\sqrt{10}$.
3. $28-5\sqrt{12}$.
4. $49-20\sqrt{6}$.
5. $57+28\sqrt{2}$.
6. $69+28\sqrt{5}$.
7. $27-12\sqrt{5}$.
8. $126+72\sqrt{3}$.
9. $103-20\sqrt{21}$.
10. $3-\sqrt{5}$.
11. $9-\sqrt{77}$.

Simplify:

12. $\sqrt{(19+8\sqrt{3})}+\sqrt{(19-8\sqrt{3})}$.
13. $\sqrt{6-\sqrt{17-12\sqrt{2}}}$.
14. $\sqrt{\left(\dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}\right)}$.
15. $1+\sqrt{8}+\sqrt{2}-\sqrt{27}-\sqrt{12}+\sqrt{75}-\sqrt{(19+6\sqrt{2})}$.
16. $\sqrt{(3+\sqrt{5})}+\sqrt{(3-\sqrt{5})}$.

17. Prove that

$$\sqrt{(9\sqrt{6}+6\sqrt{12})}+\sqrt{(9\sqrt{6}-6\sqrt{12})}=2\sqrt[4]{216}.$$

18. Simplify

$$\left(\frac{\sqrt{10+2\sqrt{5}}}{2}+\frac{5\sqrt{10-2\sqrt{5}}}{2}\right)^2+(\sqrt{5}+1)^2.$$

19. Find the square root of $2-x-2\sqrt{1-x}$.

20. Find the square root of $3x-1+2\sqrt{(2x^2+x-6)}$.

274. *If $m+\sqrt{n}$ be a root of the equation $ax^2+bx+c=0$, where a, b, c are rational, then $m-\sqrt{n}$ is also a root of the equation.*

Since $m+\sqrt{n}$ is a root of the equation $ax^2+bx+c=0$,

$$\therefore\ a(m+\sqrt{n})^2+b(m+\sqrt{n})+c=0,$$

$$\therefore\ am^2+an+2am\sqrt{n}+bm+b\sqrt{n}+c=0,$$

$$\therefore\ am^2+an+bm+c+\sqrt{n}(2am+b)=0,$$

$$\therefore\ am^2+an+bm+c=0 \text{ and } \sqrt{n}(2am+b)=0,$$

$$\therefore\ am^2+an+bm+c-\sqrt{n}(2am+b)=0,$$

$$\therefore\ a(m^2-2m\sqrt{n}+n)+b(m-\sqrt{n})+c=0,$$

$$\therefore\ a(m-\sqrt{n})^2+b(m-\sqrt{n})+c=0,$$

$\therefore$ $m-\sqrt{n}$ is also a root of the equation $ax^2+bx+c=0$.

Exercises 181.

1. Simplify

$$3\sqrt{20}-\sqrt{27}+6\sqrt{75}-4\sqrt[3]{3375},$$

given $\sqrt{5}=2\cdot236,\ \sqrt{3}=1\cdot732.$

2. Simplify

$$\left(\frac{\sqrt{x}}{1+\sqrt{x}}+\frac{1-\sqrt{x}}{\sqrt{x}}\right)\div\left(\frac{\sqrt{x}}{1+\sqrt{x}}-\frac{1-\sqrt{x}}{\sqrt{x}}\right).$$

3. If $$a+\frac{1}{a}=\sqrt{3},$$

prove that $$a^3+\frac{1}{a^3}=0.$$

4. Find the value of

$$\left(\frac{x+a}{x-a}\right)^2 - \frac{x}{3a},$$

when $$x = a(1 + 2\sqrt{3}).$$

5. Find the value of $x^2 + 3xy + y^2$

when $$x = \frac{\sqrt{5}+2}{\sqrt{5}-2}, \quad y = \frac{\sqrt{5}-2}{\sqrt{5}+2}.$$

6. Find the value of $(x^2 + xy + y^2) \div (x^2 - xy + y^2)$

when $$x = \frac{\sqrt{3}+1}{\sqrt{3}-1}, \quad y = \frac{\sqrt{3}-1}{\sqrt{3}+1}.$$

7. Find the value of $27x + 48x^3 - 8x^4$,

when $$x = \tfrac{1}{4}(\sqrt{21} - 3).$$

8. If $$z = x\sqrt{(1+y^2)} + y\sqrt{(1+x^2)},$$

prove that $$\sqrt{(1+z^2)} = xy + \sqrt{(1+x^2)}\sqrt{(1+y^2)}.$$

9. Prove that $\sqrt{-\sqrt{3} + \sqrt{3 + 8\sqrt{7 + 4\sqrt{3}}}} = 2.$

10. Simplify $$\frac{\sqrt{3}+\sqrt{2}}{\sqrt{2}+\sqrt{2+\sqrt{3}}} - \frac{\sqrt{3}-\sqrt{2}}{\sqrt{2}-\sqrt{2-\sqrt{3}}}.$$

11. Find the product of

$$\sqrt{2+\sqrt{3}},\ \sqrt{2+\sqrt{2+\sqrt{3}}},\ \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}},\ \sqrt{2-\sqrt{2+\sqrt{2+\sqrt{3}}}}.$$

12. Divide $15(a-b) + 16\sqrt{(ab)}$ by $3\sqrt{a} + 5\sqrt{b}$.

13. Find the square root of $6 + 2\sqrt{2} + 2\sqrt{3} + 2\sqrt{6}$.

14. If $2 + \sqrt{3}$ be a root of $x^2 - 4x + m = 0$, find m.

15. If $\sqrt{(x+a+b)} + \sqrt{(x+c+d)} = \sqrt{(x+a-c)} + \sqrt{(x-b+d)}$

then $$b + c = 0.$$

Solve the equations:

16. $x^3 + 2x^2 - 3x - 6 = 0$, given one root $\sqrt{3}$.

17. $x^3 - 6x^2 + 7x + 4 = 0$, " " " $\sqrt{2} + 1$.

18. $x^4 - 3x^3 - 7x^2 + 7x + 2 = 0$, " " $\sqrt{5} + 2$.

CHAPTER XXVI.

IRRATIONAL EQUATIONS.

275. Def. 35. An *irrational equation* is one which contains one or more irrational terms.

We shall suppose:

(i) That all such irrational terms are real, i.e. that in the term $\sqrt{x}$ only positive values of x are admissible, and, in $\sqrt{(3-x)}$, only negative values of x or positive values $\ngtr 3$;

(ii) That the positive value of the square root is taken, i.e. that $\sqrt{9} = +3$, and that, when $x = 1$, $\sqrt{(x+3)} = 2$.

Exercises 182.

Find the square of:

1. $\sqrt{x} + 5$. 2. $\sqrt{x} - 3$. 3. $6 - \sqrt{x}$.
4. $\sqrt{(x+1)} + 2$. 5. $\sqrt{(x-1)} - 3$. 6. $2\sqrt{x} + 1$.
7. $3 - 2\sqrt{x}$. 8. $1 - 2\sqrt{(x-1)}$.
9. $\sqrt{2(x+1)} + 5$. 10. $7 - \sqrt{3(x-2)}$.
11. $\sqrt{(x+1)} + \sqrt{x}$. 12. $\sqrt{x} - \sqrt{(x-1)}$.
13. $\sqrt{(x+2)} + \sqrt{(x+1)}$. 14. $\sqrt{(2x+3)} - \sqrt{(x-2)}$.
15. $\sqrt{(4x+5)} + \sqrt{(2x-3)}$. 16. $2\sqrt{(x+1)} - \sqrt{(x-1)}$.
17. $3\sqrt{x} - \sqrt{(x-1)}$. 18. $3\sqrt{(2x+1)} + 2\sqrt{(3x-1)}$.

276. *Example* 1. Solve the equation

$$x + \sqrt{x-3} = 9.$$

$$x + \sqrt{x-3} = 9,$$

$$\therefore \sqrt{x-3} = 9 - x.$$

Squaring both sides,

$$\therefore x - 3 = 81 - 18x + x^2,$$

$$\therefore x^2 - 19x + 84 = 0,$$

$$\therefore (x-7)(x-12) = 0,$$

the roots of which are 7 and 12.

If $x = 7$, then $x + \sqrt{x-3} = 7 + 2 = 9$.

If $x = 12$, then $x + \sqrt{x-3} = 12 + 3 = 15$.

$\therefore$ the required root is 7.

The double value of x thus found is due to the squaring of both sides of the equation

$$\sqrt{x-3} = 9 - x.$$

The same result, namely, $x^2 - 19x + 84 = 0$, would also have been obtained if the equation had been

$$-\sqrt{x-3} = 9 - x,$$

or

$$x - \sqrt{x-3} = 9,$$

a root of which is clearly 12.

Example 2. Solve the equation

$$\sqrt{x+5} + \sqrt{x} = 1.$$

$$\sqrt{x+5} + \sqrt{x} = 1,$$

$$\therefore \sqrt{x+5} = 1 - \sqrt{x}.$$

Squaring both sides,

$$\therefore x + 5 = 1 + x - 2\sqrt{x},$$

$$\therefore 4 = -2\sqrt{x},$$

$$\therefore 2 = -\sqrt{x}.$$

Squaring both sides, $4 = x$.

Putting $x = 4$ in the given equation, we have

$$\sqrt{x+5} + \sqrt{x} = \sqrt{9} + \sqrt{4} = 3 + 2 = 5.$$

$\therefore$ the given equation is not satisfied by any value of x, the reason being that x must be positive in order that $\sqrt{x}$ may be real, so that

$$\sqrt{x+5} > \sqrt{5} > 1.$$

Example 3. Solve the equation $\sqrt{x+5} - \sqrt{x} = 1$.

$$\sqrt{x+5} - \sqrt{x} = 1.$$

$$\therefore \sqrt{x+5} = 1 + \sqrt{x}.$$

Squaring both sides,

$$x + 5 = 1 + x + 2\sqrt{x},$$

$$\therefore 4 = 2\sqrt{x},$$

$$\therefore 2 = \sqrt{x}.$$

Squaring both sides, $4 = x$.

Putting $x = 4$ in the given equation, we have

$$\sqrt{x+5} - \sqrt{x} = \sqrt{9} - \sqrt{4} = 3 - 2 = 1.$$

$\therefore$ the required root is 4.

Example 4. Solve the equation $\sqrt{x+1} + \sqrt{2x+3} = 5$.

$$\sqrt{x+1} + \sqrt{2x+3} = 5.$$

Squaring both sides,

$$x + 1 + 2x + 3 + 2\sqrt{(x+1)(2x+3)} = 25,$$

$$\therefore 2\sqrt{(x+1)(2x+3)} = 21 - 3x.$$

Squaring both sides,

$$4(2x^2 + 5x + 3) = 441 - 126x + 9x^2,$$

$$\therefore x^2 - 146x + 429 = 0,$$

$$\therefore (x-3)(x-143) = 0,$$

the roots of which are 3 and 143.

If $x = 3,\ \sqrt{x+1} + \sqrt{2x+3} = \sqrt{4} + \sqrt{9} = 5.$

If $x = 143,\ \sqrt{x+1} + \sqrt{2x+3} = \sqrt{144} + \sqrt{289} = 12 + 17 = 29.$

$\therefore$ the required root is 5.

Example 5. Solve the equation $\sqrt{2(2x-3)} = \sqrt{x+1} + \sqrt{3x-2}.$

$$\sqrt{2(2x-3)} = \sqrt{x+1} + \sqrt{3x-2}.$$

$$\therefore\ 4x - 6 = x + 1 + 3x - 2 + 2\sqrt{(x+1)(3x-2)},$$

$$\therefore\ -5 = 2\sqrt{3x^2 + x - 2},$$

$$\therefore\ 25 = 4(3x^2 + x - 2),$$

$$\therefore\ 12x^2 + 4x - 33 = 0,$$

$$\therefore\ (2x-3)(6x+11) = 0,$$

the roots of which are $1\frac{1}{2}$ and $-1\frac{5}{6}$.

If $x = 1\frac{1}{2}$, then $\sqrt{2(2x-3)} = 0$

and $\sqrt{x+1} + \sqrt{3x-2} = \sqrt{2\frac{1}{2}} + \sqrt{2\frac{1}{2}}.$

If $x = -1\frac{5}{6}$, then $\sqrt{2(2x-3)}$ is imaginary.

$\therefore$ the given equation has no root.

Example 6. Solve the equation $\sqrt{x+5} - \sqrt{2x-4} = \sqrt{2x+1} - \sqrt{x}.$

$$\sqrt{x+5} - \sqrt{2x-4} = \sqrt{2x+1} - \sqrt{x},$$

$$\therefore\ x + 5 + 2x - 4 - 2\sqrt{(x+5)(2x-4)} = 2x + 1 + x - 2\sqrt{(2x+1)x},$$

$$\therefore\ \sqrt{(x+5)(2x-4)} = \sqrt{(2x+1)x},$$

$$\therefore\ 2x^2 + 6x - 20 = 2x^2 + x,$$

$$\therefore\ 5x = 20,$$

$$\therefore\ x = 4.$$

If $x = 4$, then $\sqrt{x+5} - \sqrt{2x-4} = \sqrt{9} - \sqrt{4} = 3 - 2 = 1,$

and $\sqrt{2x+1} - \sqrt{x} = \sqrt{9} - \sqrt{4} = 3 - 2 = 1.$

$\therefore$ the required root is 4.

Exercises 183.

Solve the equations :

1. $x + \sqrt{x} = 6.$ 2. $x - \sqrt{x} = 6.$ 3. $\sqrt{x+5} + x = 7.$
4. $\sqrt{2-x} + x = 0.$ 5. $x - \sqrt{x+11} = 1.$
6. $2x + \sqrt{x+1} = 8.$ 7. $\sqrt{x+7} = \sqrt{x} + 1.$
8. $\sqrt{x-24} = \sqrt{x} - 2.$ 9. $\sqrt{x+9} = 2\sqrt{x} - 3.$
10. $\sqrt{x+7} = 3\sqrt{x} - 5.$ 11. $\sqrt{x+4} + \sqrt{x-4} = 4.$
12. $\sqrt{x-5} + \sqrt{x+7} = 6.$ 13. $\sqrt{2x+6} - \sqrt{x-1} = 2.$
14. $\sqrt{9x-8} = 3\sqrt{x+4} - 2.$ 15. $\sqrt{x+4} + \sqrt{x-3} = \sqrt{7}.$
16. $\sqrt{x+2} + \sqrt{x-3} = \sqrt{3x+4}.$
17. $\sqrt{x-2} + \sqrt{x+8} = \sqrt{2x+30}.$
18. $\sqrt{2x-1} + \sqrt{3x+1} = \sqrt{9x+4}.$
19. $\sqrt{x+7} + \sqrt{2x-9} + \sqrt{5x+4} = 0.$
20. $\sqrt{2(2x-1)} = \sqrt{x+2} + \sqrt{3x+1}.$
21. $\sqrt{3x+4} - \sqrt{2x+3} = \sqrt{10(x+1)}.$
22. $\sqrt{8+x} = 2\sqrt{3+x} - \sqrt{x}.$
23. $\sqrt{x+4} + 2\sqrt{x-4} = \sqrt{4x+5}.$
24. $(x-1)\sqrt{4-x} = 2.$
25. $\sqrt{\dfrac{1+x}{1-x}} - \sqrt{\dfrac{1-x}{1+x}} = \dfrac{3}{2}.$
26. $\sqrt{2x+3} - \sqrt{3x-5} = \sqrt{4x-3} - \sqrt{x+1}.$
27. $\sqrt{x} + \sqrt{x+5} = \sqrt{x+12} + \sqrt{x-3}.$
28. $\sqrt{x+9} + \sqrt{x+1} = \sqrt{x+16} + \sqrt{x}.$
29. $\sqrt{x-1} + \sqrt{x-6} = \sqrt{x+6} + \sqrt{x-9}.$
30. $\sqrt{(x-1)(x-2)} + \sqrt{(x-4)(x-5)} = \sqrt{8}.$
31. $5\sqrt{\dfrac{3}{x}} + 7\sqrt{\dfrac{x}{3}} = 22\tfrac{2}{3}.$
32. $\sqrt{x+a} + \sqrt{x+b} = \sqrt{a-b}.$
33. $\dfrac{\sqrt{a+x} + \sqrt{a-x}}{\sqrt{a+x} - \sqrt{a-x}} = b.$

277. *Example* 7. Solve the equation

$$x^2 - 5x + 16 - 5\sqrt{x^2 - 5x + 10} = 0.$$

Put $$\sqrt{x^2 - 5x + 10} = y,$$

so that $$x^2 - 5x + 16 = y^2 + 6,$$

and the equation becomes

$$y^2 + 6 - 5y = 0$$

or $$(y - 2)(y - 3) = 0,$$

the roots of which are 2 and 3.

(i) $$\sqrt{x^2 - 5x + 10} = 2,$$

$$\therefore\ x^2 - 5x + 10 = 4,$$

$$\therefore\ x^2 - 5x + 6 = 0,$$

$$\therefore\ (x - 2)(x - 3) = 0.$$

(ii) $$\sqrt{x^2 - 5x + 10} = 3,$$

$$\therefore\ x^2 - 5x + 10 = 9,$$

$$\therefore\ x^2 - 5x + 1 = 0,$$

$\therefore$ the roots are $2,\ 3,\ \dfrac{5 + \sqrt{21}}{2}$ and $\dfrac{5 - \sqrt{21}}{2}$.

Example 8. Solve the equation

$$2x^2 + 8x - \sqrt{x^2 + 4x - 4} = 9.$$

Put $$\sqrt{x^2 + 4x - 4} = y,$$

so that $2x^2 + 8x - 8 = 2y^2$ and $2x^2 + 8x - 9 = 2y^2 - 1$,

and the equation becomes

$$2y^2 - 1 - y = 0$$

or $$(y - 1)(2y + 1) = 0,$$

the roots of which are 1 and $-\frac{1}{2}$.

(i) $$\sqrt{x^2 + 4x - 4} = 1,$$

$$\therefore\ x^2 + 4x - 4 = 1,$$

$$\therefore\ x^2 + 4x - 5 = 0,$$

$$\therefore\ (x - 1)(x + 5) = 0.$$

(ii) The roots corresponding to

$$\sqrt{x^2 + 4x - 4} = -\tfrac{1}{2}$$

do not satisfy the given equation, for here the negative value of the square root is taken.

$\therefore$ the roots are 1 and -5 *.

Exercises 184.

Solve the equations:

1. $x^2 - 10x + 31 - 5\sqrt{x^2 - 10x + 25} = 0.$
2. $2x^2 + 5x - 8\sqrt{2x^2 + 5x + 2} + 17 = 0.$
3. $2x(x-1) + 2\sqrt{2x^2 - 7x + 6} = 5x - 6.$
4. $x^2 - 2x + 6\sqrt{x^2 - 2x + 6} = 21.$
5. $x^2 + 3 = 2\sqrt{x^2 - 2x + 2} + 2x.$
6. $x^2 - 5x + 2 + \sqrt{x^2 - 5x + 22} = 0.$
7. $5x^2 + 11x - 12\sqrt{(x+4)(5x-9)} = 36.$
8. $x^2 + 3x + 4\sqrt{x^2 + 3x - 3} = 48.$
9. $x^2 + 4x + 2\sqrt{x^2 + 4x + 11} = 13.$
10. $x^2 + 7x - 3 = \sqrt{2x^2 + 14x + 2}.$
11. $(x+1)^2 = x + 3\sqrt{3x^2 + 3x - 11}.$
12. $6x^2 - 10x + 6 - 5\sqrt{3x^2 - 5x + 2} = 0.$
13. $2x^2 - 3x + \sqrt{x^2 - 2x - 4} = x + 9.$
14. $x + \sqrt{x^2 - ax + b^2} = \frac{x}{a} + b.$

* The values of x arising from the value $-\frac{1}{2}$ of y are $-2 \pm \frac{1}{2}\sqrt{33}$, and it will be found that these satisfy the equation

$$2x^2 + 8x + \sqrt{x^2 + 4x - 4} = 9.$$

MISCELLANEOUS EXERCISES.

Exercises 185.

1. Write down all the numbers between 20 and 50 which are included in the general expression $6n+5$.

2. Divide $(x+y)^4+(x^2-y^2)^2+(x-y)^4$ by $3x^2+y^2$.

3. Find the factors of $x^5-y^5-(x-y)^5$.

4. Find the factors of $x^6-14x^4+49x^2-36$.

5. If $x+\frac{1}{x}=y$, express $x^3+\frac{1}{x^3}$ and $x^5+\frac{1}{x^5}$ in terms of y.

6. Prove that if the square of any odd number be divided by 8, the remainder is 1.

7. If $a+b=x$, $a-b=y$, express a^3-b^3 in terms of x and y. Hence, find the value of 5002^3-4998^3.

8. Find the square root of $(x+1)(x+2)(x+3)(x+4)+1$.

9. What value of x will make the product of $3-8x$ and $3x+4$ equal to the product of $6x+11$ and $3-4x$?

10. If x^4+1-kx^2 vanish when $x=a$, show that it vanishes also when x is equal to $\frac{1}{a}$ or $-a$ or $-\frac{1}{a}$.

11. Divide $x^{\frac{5}{2}}-x^2-4x^{\frac{3}{2}}+6x-2x^{\frac{1}{2}}$ by $x^{\frac{3}{2}}-4x^{\frac{1}{2}}+2$.

12. Simplify $\dfrac{(a^3)^4\,(b^{\frac{1}{2}})^8}{(a^2)^{\frac{1}{2}}\,(b^3)^6}\div\dfrac{(a^{\frac{1}{2}})^8\,(b^{\frac{2}{3}})^3}{(a^{\frac{1}{3}})^6\,(b^4)^{\frac{1}{2}}}$.

Exercises 186.

1. What is the nature of the roots of the equations
(i) $x^2-12x+36=0$, (ii) $2x^2+5x=0$, (iii) $2x^2-3x+4=0$?

2. Prove, without solving, that the positive root of the equation $x^2-8x-8=0$ is greater than 8.

3. One root of the equation $3x^2 - 2x = m$ is 4; what is the other root, and what is the value of m?

4. If the roots of the equation $4x^2 + (2 + m)x + 25 = 0$ be equal, find the value of m.

5. Find the sum of the cubes of the roots of the equation

$$2x^2 - x - 2 = 0.$$

6. If α, β be the roots of the equation $3x^2 + 7x - 1 = 0$, find the equation whose roots are $\alpha + \beta$ and $\dfrac{\alpha\beta}{\alpha + \beta}$.

7. Find the equation whose roots are the squares of the roots of the equation $3x^2 - 10x + 2 = 0$.

8. If one root of the equation $3x^2 + 4x + 2m = 0$ be double the other, find the value of m.

9. Find the equation whose roots are each equal to the sum of the reciprocals of the roots of the equation $ax^2 + bx + c = 0$.

10. If α, β be the roots of the equation $ax^2 + bx + c = 0$, find the equation whose roots are $\dfrac{1 - \alpha}{1 + \alpha}$ and $\dfrac{1 - \beta}{1 + \beta}$.

11. The difference of the roots of an equation is 3 and the product of the roots is 54; find the equation.

12. Find the maximum value of $23 - 4x^2 - 12x$, and the minimum value of $x^2 - 10x + 27$.

Exercises 187.

1. What is the remainder when $x^3 + 3x^2 - 20x + 5$ is divided by (i) $x - 4$, (ii) $x + 4$?

2. If $x^4 - ax + 4$ be divisible by $x - 2$, what is the value of a?

3. Find the factors of $x^3 + 2x^2 - 7x + 4$.

4. Prove that $ax^5 + bx^4 + cx^3 + cx^2 + bx + a$ is divisible by $x + 1$.

5. Show that $(2x - y - z)(2y - z - x) - 9xy$ is divisible by $x + y + z$.

6. Find the factors of

$$(y+z)(y-z)^3+(z+x)(z-x)^3+(x+y)(x-y)^3.$$

7. Simplify $\dfrac{a^3(b-c)+b^3(c-a)+c^3(a-b)}{a^2(b-c)+b^2(c-a)+c^2(a-b)}$.

8. Prove that

$$\frac{a^4(b^2-c^2)+b^4(c^2-a^2)+c^4(a^2-b^2)}{a^2(b-c)+b^2(c-a)+c^2(a-b)}=(b+c)(c+a)(a+b).$$

9. If $\log 108 = 2{\cdot}0334$ and $\log 24 = 1{\cdot}3802$, find $\log 2$ and $\log 3$.

10. Resolve $\dfrac{10-x}{(x^2-4)(x-1)}$ into partial fractions.

11. If x^4+px+q be divisible by x^2+ax+b, find the values of p and q in terms of a and b.

12. If one root of the equation $x^4-3x^3-5x^2+9x-2=0$ be $2+\sqrt{3}$, find all the roots.

Exercises 188.

Solve the equations:

1. $\dfrac{x}{3}-\dfrac{3}{x}=4\frac{4}{5}$.

2. $ax^2-c^2x=a^2x-ac^2$. 3. $x^4-20x^2+64=0$.

4. $(x^2+x)^2-26(x^2+x)+120=0$.

5. $(x+1)(x-2)(x+5)(x-6)=60$.

6. $x^2+4y^2-3x+y=67,\ \ x-2y=1$.

7. $x+y=3,\ \ x^3+y^3=65$.

8. $3x^2+xy=6,\ \ 4xy-y^2=3$.

9. $x^4+x^2y^2+y^4=741,\ \ x^2-xy+y^2=19$.

10. $\dfrac{3}{1+\sqrt{x}}+\dfrac{3}{1-\sqrt{x}}=4$.

11. $\sqrt{2x+9}-\sqrt{x-4}=\sqrt{x+1}$.

12. $x^2-3x+6=3\sqrt{(x^2-3x+4)}$.

CHAPTER XXVII.

RATIO, PROPORTION AND VARIATION.

1. Ratio.

278. Def. 36. The *ratio* of one quantity a to another b is the quotient of a by b.

The ratio of a to b is written $a:b$. The quantities a, b are called the *terms* of the ratio, the former being called the *antecedent*, and the latter the *consequent*, of the ratio.

From the above definition, it follows that the ratio $a:b$ is measured by the fraction $\frac{a}{b}$. Also, since $\frac{a}{b}=\frac{am}{bm}$, we have

$$a:b=am:bm;$$

i.e. if both terms of a ratio be multiplied or divided by the same number, the ratio is unaltered.

Thus, $\frac{2}{3}:\frac{3}{4}=\frac{2}{3}\times 12:\frac{3}{4}\times 12=8:9.$

279. *Example* 1. Find the value or values of the ratio $x:y$, if (i) $3x=5y$, (ii) $2x^2-7xy+3y^2=0$.

(i) $$3x=5y.$$

Divide both sides by $3y$,

then $$\frac{x}{y}=\frac{5}{3} \text{ or } x:y=5:3.$$

(ii)
$$2x^2 - 7xy + 3y^2 = 0.$$
$$\therefore\ (2x - y)(x - 3y) = 0$$
or
$$\left(\frac{x}{y} - \frac{1}{2}\right)\left(\frac{x}{y} - 3\right) = 0,$$
$\therefore$ the values of the ratio $x : y$ are $\frac{1}{2}$ and 3.

Example 2. What number must be added to each term of the ratio 3 : 7 to make the ratio equal to 2 : 3?

Let x be the number;

then
$$\frac{3 + x}{7 + x} = \frac{2}{3},$$
$$\therefore\ 9 + 3x = 14 + 2x,$$
$$\therefore\ x = 5.$$

Example 3. Two numbers are in the ratio of 3 : 5; if 6 be added to each term, the ratio becomes 5 : 7; what are the numbers?

Let the numbers be $3x$ and $5x$;

then
$$\frac{3x + 6}{5x + 6} = \frac{5}{7},$$
$$\therefore\ 21x + 42 = 25x + 30,$$
$$\therefore\ -4x = -12,$$
$$\therefore\ x = 3,$$
$\therefore$ the numbers are 9 and 15.

Example 4. At present A's age is to B's in the ratio of 3 : 2; in 15 years' time they will be in the ratio of 4 : 3; find their ages.

Let A's age be $3x$ years and B's $2x$ years.

Then
$$\frac{3x + 15}{2x + 15} = \frac{4}{3},$$
$$\therefore\ 9x + 45 = 8x + 60,$$
$$\therefore\ x = 15,$$
$\therefore$ A's age is 45 years and B's 30 years.

Exercises 189.

1. If $3x = 5y$, what is the ratio of x to y?

2. If $6x = 9y$, what is the ratio of x to y?

3. If $\frac{2x + 3y}{4x - y} = \frac{2}{3}$, what is the ratio of x to y?

4. If $\frac{3x + 4y}{5x + 2y} = \frac{3}{4}$, what is the ratio of x to y?

5. Find the ratio of x to y from the equation

$$2x^2 - 5xy + 2y^2 = 0.$$

6. Find the ratio of x to y from the equation

$$2x^2 - 9xy + 10y^2 = 0.$$

7. Find the value of the ratio $\frac{3x + 2y}{2x - y}$ if $x : y = 3 : 2$.

8. If $a : b = 5 : 4$, find the value of the ratio $a^2 - b^2 : a^2 + b^2$.

9. Find the value of $\frac{x + a}{x - a} + \frac{x - a}{x + a}$ when $x : a = \sqrt{5}$.

10. What number must be added to each term of the ratio $3 : 5$ to make it equal to $3 : 4$?

11. What number must be taken from each term of the ratio $17 : 19$ to make it equal to $2 : 3$?

12. Find a number which, when subtracted from each term of the ratio $9 : 13$, will make it equal to $13 : 9$.

13. Two numbers are in the ratio $7 : 4$ and their difference is 27; find the numbers.

14. Find two numbers in the ratio $7 : 11$, such that, if 3 be added to each, they may be in the ratio $2 : 3$.

15. Two numbers are in the ratio $2 : 3$, and, if 9 be added to each, they are in the ratio $3 : 4$; find the numbers.

16. Two numbers are in the ratio 3 : 5, but if 10 be taken from the greater and added to the smaller, the ratio is reversed; find the numbers.

17. Find two numbers which differ by 7 whose ratio becomes 1 : 2 when 6 is subtracted from each.

18. Find two numbers whose difference is 7, and which are such that the ratio of their sum to the sum of their squares is 1 : 25.

19. At present the ratio of B's age to A's age is 5 : 2, but, in 30 years' time, the ratio will be 35 : 23; find their ages.

20. The perimeter of a right-angled triangle is six times as long as the shortest side; what is the ratio of the two sides containing it?

280. Def. 37. A ratio is said to be one of *greater inequality*, of *equality*, or of *less inequality*, according as the antecedent is greater than, equal to, or less than, the consequent.

Thus, 7 : 5 is a ratio of greater inequality, 5 : 5 one of equality, and 3 : 5 one of less inequality.

281. *If the same quantity be added to both terms of a ratio, the ratio is increased or diminished according as it is one of less or greater inequality.*

Let $a : b$ be the ratio and x the quantity added to both its terms; then

$$\frac{a+x}{b+x}-\frac{a}{b}=\frac{b(a+x)-a(b+x)}{b(b+x)},$$

$$=\frac{bx-ax}{b(b+x)},$$

$$=\frac{x(b-a)}{b(b+x)}.$$

Hence, $a+x : b+x$ is greater or less than $a : b$ according as $b-a$ is positive or negative, i.e. as a is $<$ or $>$ b, i.e. as the ratio is one of less or greater inequality.

282. *If the same quantity be subtracted from both terms of a ratio, the ratio is increased or diminished according as it is one of greater or less inequality.*

Let $a:b$ be the ratio and x the quantity subtracted from both its terms; then

$$\frac{a-x}{b-x}-\frac{a}{b}=\frac{b(a-x)-a(b-x)}{b(b-x)},$$

$$=\frac{ax-bx}{b(b-x)},$$

$$=\frac{x(a-b)}{b(b-x)}.$$

Hence, if $x<b$, $a-x:b-x$ is greater or less than $a:b$ according as $a-b$ is positive or negative, i.e. according as a is greater or less than b, i.e. as the ratio is one of greater or less inequality.

***283.** *If* $ax+by+cz=0$ *and* $a'x+b'y+c'z=0$, *then*

$$\frac{x}{bc'-b'c}=\frac{y}{ca'-c'a}=\frac{z}{ab'-a'b}.$$

$$ax+by+cz=0,$$

$$a'x+b'y+c'z=0,$$

$$\therefore\ (ac'-a'c)x+(bc'-b'c)y=0,$$

$$\therefore\ x(ca'-c'a)=y(bc'-b'c),$$

$$\therefore\ \frac{x}{bc'-b'c}=\frac{y}{ca'-c'a}.$$

Again, $$(ab'-a'b)x+(b'c-bc')z=0,$$

$$\therefore\ (ab'-a'b)x=(bc'-b'c)z,$$

$$\therefore\ \frac{x}{bc'-b'c}=\frac{z}{ab'-a'b},$$

$$\therefore\ \frac{x}{bc'-b'c}=\frac{y}{ca'-c'a}=\frac{z}{ab'-a'b}.$$

***284.** *Example* 5. Solve the equations

$$x+y+z=0,\quad 2x+y+4z=0,\quad x^2+y^2+z^2=14.$$

$$x+y+z=0,$$

$$2x+y+4z=0,$$

$$\therefore\ \frac{x}{4-1}=\frac{y}{2-4}=\frac{z}{1-2},$$

$$\therefore\ \frac{x}{3}=\frac{y}{-2}=\frac{z}{-1}=m, \text{ say},$$

$$\therefore\ x=3m,\quad y=-2m,\quad z=-m.$$

Substituting in the 3rd equation, we have

$$9m^2+4m^2+m^2=14,$$

$$\therefore\ m^2=1 \text{ or } m=\pm 1.$$

$\therefore$ the solutions are

$$\left.\begin{aligned} x&=3\\ y&=-2\\ z&=-1\end{aligned}\right\} \text{ and } \left.\begin{aligned} x&=-3\\ y&=2\\ z&=1\end{aligned}\right\}.$$

*Exercises 190.

Find the ratios of x, y, z from the equations:

1. $x+y+5z=0,\quad x-y+z=0.$
2. $2x-y-z=0,\quad 5x-y-3z=0.$
3. $x+3y-5z=4x-2y-7z=3x-y-3z.$
4. $\dfrac{x+y}{5}=\dfrac{2x-3y}{7}=\dfrac{x+2y+5z}{9}.$
5. $x+y+z=0,\quad a^2x+b^2y+c^2z=0.$

Solve the equations:

6. $\dfrac{x}{3}=\dfrac{y}{2},\quad x^2+y^2=13.$

7. $\dfrac{x}{4}=\dfrac{y}{5}=\dfrac{z}{6}, \quad 3x^2-2y^2-7x+4z=2.$

8. $\dfrac{x-1}{5}=\dfrac{y-3}{4}=\dfrac{z-5}{3}=y^2-zx.$

9. $3x+y-z=0, \quad x-5y+z=0, \quad x^2+2y^2=3.$

10. $2x+y=2z, \quad 9z-7x=6y, \quad x^3+y^3+z^3=216.$

***285.** *If the ratios* $\dfrac{a}{b}, \dfrac{c}{d}, \dfrac{e}{f}, \ldots$ *be all equal, then each is equal to*

$$\left(\frac{pa^n+qc^n+re^n+\ldots}{pb^n+qd^n+rf^n+\ldots}\right)^{\frac{1}{n}}.$$

Let
$$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\ldots=m,$$

so that
$$a=bm, \quad c=dm, \quad e=fm, \ldots,$$

$$\therefore\ a^n=b^nm^n, \quad c^n=d^nm^n, \quad e^n=f^nm^n, \ldots,$$

$$\therefore\ pa^n=pb^nm^n, \quad qc^n=qd^nm^n, \quad re^n=rf^nm^n, \ldots,$$

$$\therefore\ pa^n+qc^n+re^n+\ldots=(pb^n+qd^n+rf^n+\ldots)m^n,$$

$$\therefore\ m=\left(\frac{pa^n+qc^n+re^n+\ldots}{pb^n+qd^n+rf^n+\ldots}\right)^{\frac{1}{n}}.$$

Second proof. Let

$$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\ldots=m,$$

so that
$$a=bm, \quad c=dm, \quad e=fm, \ldots,$$

then

$$\left(\frac{pa^n+qc^n+re^n+\ldots}{pb^n+qd^n+rf^n+\ldots}\right)^{\frac{1}{n}}=\left(\frac{pb^nm^n+qd^nm^n+rf^nm^n+\ldots}{pb^n+qd^n+rf^n+\ldots}\right)^{\frac{1}{n}}$$

$$=(m^n)^{\frac{1}{n}},$$

$$=m.$$

COR. 1. If $$n = 1,$$

then $$\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \dots = \frac{pa + qc + re + \dots}{pb + qd + rf + \dots}.$$

COR. 2. If also $p = q = r = \dots = 1$,

then $$\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \dots = \frac{a + c + e + \dots}{b + d + f + \dots}.$$

COR. 3. If ratios $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \dots, \frac{k}{l}$ be unequal, $\frac{a}{b}$ being the greatest and $\frac{k}{l}$ the least, then

$$\frac{a + c + e + \dots + k}{b + d + f + \dots + l} \text{ is } < \frac{a}{b} \text{ and } > \frac{k}{l}.$$

Since $$\frac{a}{b} = \frac{a}{b}, \quad \frac{c}{d} < \frac{a}{b}, \quad \frac{e}{f} < \frac{a}{b}, \dots, \quad \frac{k}{l} < \frac{a}{b},$$

$$\therefore \ a = \frac{a}{b} \cdot b, \quad c < \frac{a}{b} \cdot d, \quad e < \frac{a}{b} \cdot f, \dots, \quad k < \frac{a}{b} \cdot l,$$

$$\therefore \ a + c + e + \dots + k < \frac{a}{b} (b + d + f + \dots + l),$$

$$\therefore \ \frac{a + c + e + \dots + k}{b + d + f + \dots + l} < \frac{a}{b}.$$

Similarly, it may be shown that

$$\frac{a + c + e + \dots + k}{b + d + f + \dots + l} > \frac{k}{l}.$$

***286.** *Example* 6. If

$$\frac{3x + 2y}{3} = \frac{x + 2y}{2},$$

then each ratio is equal to $\frac{7x + 6y}{8}$.

$$\frac{3x + 2y}{3} = \frac{x + 2y}{2} = \frac{p(3x + 2y) + q(x + 2y)}{3p + 2q}$$
$$= \frac{(3p + q)x + 2(p + q)y}{3p + 2q}.$$

Let p and q be chosen so that

$$3p + q = 7, \quad p + q = 3,$$

or

$$2p = 4 \text{ or } p = 2, \quad q = 1.$$

$\therefore$ each ratio is equal to $\dfrac{7x + 6y}{8}$.

*Exercises 191.

1. Divide a into two parts in the ratio of b to c.

2. If $\dfrac{a}{b} > \dfrac{c}{d}$, prove that $\dfrac{ma + nc}{mb + nd} >$ or $< \dfrac{na + mc}{nb + md}$, according as m is $>$ or $< n$, all the letters denoting positive quantities.

3. If $x : y = a : a - b$, show that the ratio of $x^2 - xy + y^2$ to $a^2 - ab + b^2$ can be expressed in terms of the ratio of x to a alone.

4. If $x + y + z = 0$, and $\left(\dfrac{1}{b} + \dfrac{1}{c}\right)x + \left(\dfrac{1}{c} + \dfrac{1}{a}\right)y + \left(\dfrac{1}{a} + \dfrac{1}{b}\right)z = 0$, find the ratio $x : y : z$.

5. Show that the ratio $a : b$ is the square of the ratio $a + c : b + c$ if $c^2 = ab$.

6. What value must be given to the integer x so that $\sqrt{2}$ shall lie between $\dfrac{x+3}{x}$ and $\dfrac{x+4}{x+1}$?

7. If $\dfrac{l}{b - c} = \dfrac{m}{c - a} = \dfrac{n}{a - b}$, show that $l + m + n = 0$.

8. If

$$\frac{x}{b + c - a} = \frac{y}{c + a - b} = \frac{z}{a + b - c},$$

then

$$(b - c)x + (c - a)y + (a - b)z = 0.$$

9. If $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$ be unequal, then $\frac{la+mc+ne}{lb+md+nf}$ is intermediate between the greatest and least of these fractions.

10. If
$$\frac{a}{b}=\frac{c}{d}=\frac{e}{f},$$
prove that each of these ratios is equal to
$$\frac{30a+15c+5e}{30b+15d+5f}.$$

11. If $\frac{a^2-b^2-c^2}{a-c}=\frac{a^2-b^2+c^2}{a+c}$, find the value of either fraction in terms of c.

12. If
$$\frac{a+b}{a-b}=\frac{b+c}{b-c}=\frac{c+d}{c-d},$$
prove that each
$$=\frac{a+d}{a-2b+2c-d}.$$

13. If
$$\frac{3x+4y-5z}{3}=\frac{5x+6y-7z}{8},$$
then each
$$=\frac{11x+16y-21z}{5}.$$

14. If $\frac{bx+cy}{b+c-a}=\frac{cx+ay}{c+a-b}=\frac{ax+by}{a+b-c}$, and no numerator or denominator vanishes, show that either $x=y$ or $a=b=c$.

15. If $\frac{bx-ay}{cy-az}=\frac{cx-az}{by-ax}=\frac{z+y}{x+z}$, then will each of these fractions $=\frac{x}{y}$, unless $b+c=0$.

16. Prove that if the three fractions
$$\frac{2x-3y}{3z+y},\ \frac{z-y}{z-x},\ \frac{x+3z}{2y-3x}$$
be all equal, each is also equal to $\frac{x}{y}$, and hence show that either $x=y$ or $z=x+y$.

17. If $$\frac{p}{a^2-bc}=\frac{q}{b^2-ca}=\frac{r}{c^2-ab},$$

prove that $$\frac{a}{p^2-qr}=\frac{b}{q^2-rp}=\frac{c}{r^2-pq}.$$

18. If $$\frac{x}{a+2b+c}=\frac{y}{2a+b-c}=\frac{z}{4a-4b+c},$$

then $$\frac{a}{x+2y+z}=\frac{b}{2x+y-z}=\frac{c}{4x-4y+z}.$$

19. If $$\frac{y\mathrm{C}-z\mathrm{B}}{yc-zb}=\frac{z\mathrm{A}-x\mathrm{C}}{za-xc},$$

show that each is equal to $\dfrac{x\mathrm{B}-y\mathrm{A}}{xb-ya}$.

20. Having given
$$\frac{2a-b-c}{x+a-b-c}=\frac{2b-c-a}{x+b-c-a},$$

show that each fraction is equal to $\frac{3}{2}$ and also to
$$\frac{2c-a-b}{x+c-a-b}.$$

21. If $$\frac{ad-bc}{a-b-c+d}=\frac{ac-bd}{a-b-d+c},$$

then each is equal to $\dfrac{a+b+c+d}{4}$.

22. A woman buys apples at x a penny and as many pears at y a penny; she sells them all at $(x+y)$ for $2d.$ and loses 4 per cent.; find the ratio of x to y.

23. If $ax+by+cz=0$ and $a^2x+b^2y+c^2z=0$, show that $a^3x+b^3y+c^3z$ cannot vanish unless two of the quantities a, b, c be equal.

2. Proportion.

287. Def. 38. Quantities are said to be *in proportion* if the ratios of successive pairs be equal.

Thus, a, b, c, d are in proportion if

$$a : b = c : d, \text{ i.e. if } \frac{a}{b} = \frac{c}{d},$$

and a, b, c, d, e, f are in proportion if

$$a : b = c : d = e : f, \text{ i.e. if } \frac{a}{b} = \frac{c}{d} = \frac{e}{f}.$$

Def. 39. If four quantities a, b, c, d be in proportion, the quantities a and d are called the *extremes*, and b and c the *means*; also the quantity d is called the *fourth proportional* to a, b, c.

Def. 40. Quantities are said to be *in continued proportion* if the ratios of the first to the second, the second to the third, etc., be all equal.

Thus, a, b, c are in continued proportion if

$$a : b = b : c, \text{ i.e. if } \frac{a}{b} = \frac{b}{c}.$$

Def. 41. If three quantities a, b, c be in continued proportion, the quantity b is called the *mean proportional between* a and c; and also the quantity c is called the *third proportional* to a and b.

288. *If four quantities be in proportion, the product of the extremes is equal to the product of the means; and conversely.*

(i) Let a, b, c, d be four quantities in proportion;

then $$\frac{a}{b} = \frac{c}{d}.$$

Multiply both sides by bd,

then $$ad = bc.$$

(ii) Let $$ad = bc.$$

Divide both sides by bd,

then $$\frac{a}{b} = \frac{c}{d},$$

i.e. a, b, c, d are in proportion.

COR. If three quantities a, b, c be in continued proportion, then $b^2 = ac$; also, if $b^2 = ac$, then a, b, c are in continued proportion.

289. *If* $a : b = c : d$, *then*

(i) $a : c = b : d$; (ii) $b : a = d : c$; (iii) $a + b : b = c + d : d$;

(iv) $a - b : b = c - d : d$; (v) $a + b : a - b = c + d : c - d$.

Since $$a : b = c : d,$$

$$\therefore\ ad = bc.$$

(i) Divide both sides by cd,

$$\therefore\ \frac{a}{c} = \frac{b}{d}, \text{ i.e. } a : c = b : d.$$

(ii) Also $$1 \div \frac{a}{b} = 1 \div \frac{c}{d},$$

$$\therefore\ \frac{b}{a} = \frac{d}{c}, \text{ i.e. } b : a = d : c.$$

(iii) $$\frac{a}{b} + 1 = \frac{c}{d} + 1,$$

$$\therefore\ \frac{a+b}{b} = \frac{c+d}{d}, \text{ i.e. } a + b : b = c + d : d.$$

(iv) $$\frac{a}{b} - 1 = \frac{c}{d} - 1,$$

$$\therefore\ \frac{a-b}{b} = \frac{c-d}{d}, \text{ i.e. } a - b : b = c - d : d.$$

(v) $$\frac{a+b}{b} \div \frac{a-b}{b} = \frac{c+d}{d} \div \frac{c-d}{d},$$

$$\therefore \frac{a+b}{a-b} = \frac{c+d}{c-d}, \text{ i.e. } a+b : a-b = c+d : c-d.$$

290. *Example* 7. What number must be subtracted from each of the numbers 8, 10, 13, 17, so that the remainders may be in proportion?

Let x be the number,

then $$8-x : 10-x = 13-x : 17-x,$$

$$\therefore (8-x)(17-x) = (10-x)(13-x),$$

$$\therefore 136 - 25x + x^2 = 130 - 23x + x^2,$$

$$\therefore -2x = -6,$$

$$\therefore x = 3.$$

Example 8. If $a:b = c:d$, then $a^2+b^2 : \frac{a^3}{b} = c^2+d^2 : \frac{c^3}{d}$.

Let $\frac{a}{b} = \frac{c}{d} = m$, so that $a = bm$, $c = dm$,

$$\therefore \frac{a^2+b^2}{\frac{a^3}{b}} = \frac{b^2m^2+b^2}{\frac{b^3m^3}{b}} = \frac{b^2(m^2+1)}{b^2m^3} = \frac{m^2+1}{m^3},$$

and $$\frac{c^2+d^2}{\frac{c^3}{d}} = \frac{d^2m^2+d^2}{\frac{d^3m^3}{d}} = \frac{d^2(m^2+1)}{d^2m^3} = \frac{m^2+1}{m^3};$$

$$\therefore a^2+b^2 : \frac{a^3}{b} = c^2+d^2 : \frac{c^3}{d}.$$

Example 9. If $a:b = b:c$, then $a^2+ab : b^2 = b^2+bc : c^2$.

Let $\frac{a}{b} = \frac{b}{c} = m$, so that $a = bm$, $b = cm$,

$$\therefore \frac{a^2+ab}{b^2} = \frac{b^2m^2+b^2m}{b^2} = m^2+m,$$

and $$\frac{b^2+bc}{c^2} = \frac{c^2m^2+c^2m}{c^2} = m^2+m,$$

$$\therefore a^2+ab : b^2 = b^2+bc : c^2.$$

Example 10. If $2a+b:2a-b=2c+d:2c-d$, then $a:b=c:d$.

Since
$$2a+b:2a-b=2c+d:2c-d,$$
$$\therefore\ (2a+b)(2c-d)=(2a-b)(2c+d),$$
$$\therefore\ 4ac+2bc-2ad-bd=4ac-2bc+2ad-bd,$$
$$\therefore\ 4bc=4ad,$$
$$\therefore\ bc=ad,$$
$$\therefore\ a:b=c:d.$$

Exercises 192.

1. Find a fourth proportional to (i) 3, 4, 9; (ii) 16, 12, 4; (iii) 15, 12, 8; (iv) a^4, a^3b, ab.

2. Find a third proportional to (i) 25, 20; (ii) 6, 10; (iii) $4\frac{1}{2}$, 6; (iv) $(a-b)^2$, a^2-b^2.

3. Find a mean proportional to (i) 3, 12; (ii) 9, 16; (iii) 9, $6\frac{1}{4}$; (iv) a^3b, ab^5.

4. Find x when:

(i) $x:7=4:8$; (ii) $5:x=6:5$; (iii) $12:5=x:2$.

5. If $2x+1:x+4=3:2$, find x.

6. If $x+1$, $x+3$, $x+7$ be in continued proportion, find x.

7. What number added to the numbers 6, 8, 10, 13 will make the sums proportional?

8. What number subtracted from each of the numbers 10, 11, 14, 16 will make the remainders proportionals?

If $a:b=c:d$, prove that

9. $3a+4c:5a+4c=3b+4d:5b+4d$.

10. $a^2+c^2:b^2+d^2=ac:bd$.

11. $a^2+b^2:a^2-b^2=c^2+d^2:c^2-d^2$.

12. $ab:cd=a^2+b^2:c^2+d^2$.

13. $a:c=\sqrt[4]{a^4+b^4}:\sqrt[4]{c^4+d^4}$.

14. $a^2c^3 : b^2d^3 = a^5 + c^5 : b^5 + d^5$.

15. $3a^2 + ab + 2b^2 : 3a^2 - 2b^2 = 3c^2 + cd + 2d^2 : 3c^2 - 2d^2$.

16. $a^2 + c^2 : b^2 + d^2 = \sqrt{a^4 + c^4} : \sqrt{b^4 + d^4}$.

If $a : b = b : c$, prove that

17. $a - b : b - c = b : c$.

18. $a : c = a^2 + b^2 : b^2 + c^2$.

19. $a^2 + ab + b^2 : b^2 + bc + c^2 = a : c$.

If $a : b = c : d = e : f$, prove that

20. $a^2 : b^2 = ce \cdot df$.

21. $a + b \cdot a - b = c + d : c - d = e + f : e - f$.

22. $ace : bdf = a^3 + c^3 + e^3 : b^3 + d^3 + f^3$.

23. If $a : b = b : c = c : d$, prove that

$$a : d = a^3 : b^3 \text{ and } a + b : c + d = a^2 : b^2.$$

24. If $2a + 3b : 2c + 3d = 3a - 4b : 3c - 4d$, prove that

$$a : b = c : d.$$

25. If $10a + b : 10c + d = 12a + b : 12c + d$, show that

$$a : b = c : d.$$

26. If $(a + b + c + d)(a - b - c + d) = (a - b + c - d)(a + b - c - d)$, prove that $a : b = c : d$.

27. If $a + b - 2c : a - b + 2c = 2a + 2b - c : 2a - 2b + c$, then will either $a = 0$ or $c = 0$.

28. If $x + y : x - y = x + z : x - z$, then either $x = 0$ or $y = z$.

29. Find a number such that its excess over 8 bears the same ratio to its excess over 9 as the number itself does to its excess over 2.

30. Find three numbers in continued proportion whose sum is 7, and such that the extremes together exceed by 25 the double of the mean.

291. *Example* 11. If $ma + nc : pa + qc = mb + nd : pb + qd$, then either $a : b = c : d$ or $m : n = p : q$.

Since $$ma + nc : pa + qc = mb + nd : pb + qd,$$

$$\therefore\ (ma + nc)(pb + qd) = (pa + qc)(mb + nd),$$

$$\therefore\ mpab + npbc + mqad + nqcd = mpab + mqbc + npad + nqcd,$$

$$\therefore\ bc(np - mq) = ad(np - mq),$$

$$\therefore\ (bc - ad)(np - mq) = 0,$$

$$\therefore\ \text{either } ad = bc \text{ or } mq = np,$$

$$\therefore\ \text{either } a : b = c : d \text{ or } m : n = p : q.$$

Example 12. A cask, holding 10 gallons, is filled with wine and water in the ratio of 3 : 2; if a certain number of gallons be drawn off and replaced by water, the ratio of wine to water in the new mixture is 2 : 3; how many gallons were drawn off?

Let x gallons of the first mixture be drawn off, then $\frac{3x}{5}$ gallons of wine and $\frac{2x}{5}$ gallons of water are drawn off. These are replaced by x gallons of water; so that number of gallons of wine in the new mixture is $\frac{3}{5} . 10 - \frac{3x}{5}$, and number of gallons of water is $\frac{2}{5} . 10 - \frac{2x}{5} + x$,

$$\therefore\ 6 - \frac{3x}{5} : 4 + \frac{3x}{5} = 2 : 3,$$

$$\therefore\ 30 - 3x : 20 + 3x = 2 : 3,$$

$$\therefore\ 3(30 - 3x) = 2(20 + 3x),$$

$$\therefore\ -9x - 6x = 40 - 90,$$

$$\therefore\ -15x = -50,$$

$$\therefore\ x = 3\tfrac{1}{3}:$$

$\therefore$ $3\frac{1}{3}$ gallons are drawn off from the first mixture.

Exercises 193.

1. If $a : b = (a + x)^2 : (b + x)^2$, prove that $x^2 = ab$.

2. If $a + b : b + c = c + d : d + a$, prove that either $a = c$ or
$$a + b + c + d = 0.$$

3. If $a : b = x - 2y : y + 2x$, prove that $x : y = a + 2b : b - 2a$.

4. If $(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) = (ax + by + cz)^2$, show that
$$x : a = y : b = z : c.$$

5. If $ay - bx : c = cx - az : b = bz - cy : a$, then either x, y, z are proportional to a, b, c or $a^2 + b^2 + c^2 = 0$.

6. Find x if $2^{x^2} : 2^{2x} = 8 : 1$.

7. If $x : 3 = y : 4 = z : 5$, prove that each of these ratios is equal to $3x^3 - x^2y + y^2z : 4(x^2 + y^2) + z^2$.

8. If four quantities be proportional, and the second be a mean proportional between the third and fourth, then the third is a mean proportional between the first and second.

9. If α, β be the roots of the equation $(x + a)(x + b) = c^2$, prove that $a + \alpha : \alpha + b = \beta + b : \beta + a$.

10. Find three numbers in continued proportion, whose sum is 13, and sum of their squares 91.

11. If $z^2 = x^2 + y^2$, prove that
$$y - z + x : z - x + y = x - y + z : x + y + z.$$

12. If a, b, c, d be in proportion, prove that the sum of the extremes is greater than the sum of the means.

13. If $a + b : c + d = a : c$, and $a > b$, then $a - c > b - d$.

14. If a, b, c, d be unequal, show that $a + b$, $b + c$, $c + d$, $d + a$ cannot be in continued proportion.

15. The length and breadth of a room are as $3 : 2$, and if 3 ft. be added to each the new area of the floor is to the old as $35 : 27$; find the dimensions of the room.

16. A cask is filled with wine and water mixed together in the ratio of $5:3$; if 16 gallons of the mixture be drawn off and the cask filled up with water, the ratio becomes $3:5$; how many gallons does the cask hold?

17. Two casks, A and B, contain mixtures of wine and water, A in the ratio of $8:3$, and B in the ratio of $5:1$; in what ratio must liquid be drawn from each cask to give a mixture in the ratio of $4:1$?

18. Two casks, A and B, are filled with two kinds of sherry, mixed, in the cask A in the ratio $2:7$, and in the cask B in the ratio $1:5$; what quantity must be taken from each to form a mixture which shall consist of 2 gallons of the first kind and 9 gallons of the second kind?

3. Variation.

292. Def. 42. A quantity x is said to *vary directly* (or to *vary*) as another quantity y, when the ratio of the measure of x to the measure of y is constant.

Def. 43. A quantity x is said to *vary inversely* as another quantity y, when x varies as the reciprocal of y.

Thus, the area of a field mowed in a given time varies as the number of men employed; while the time required to mow a field of given area varies inversely as the number of men employed.

The symbol $\propto$ is used to denote "varies as."

293. *If x vary as y, then x is equal to y multiplied by a constant quantity; and, conversely, if $x = my$, where m is a constant quantity, then x varies as y.*

(i) Since x varies as y,

$\therefore$ the ratio $x:y$ is constant; let it equal m,

$$\therefore \frac{x}{y} = m,$$

$$\therefore x = my.$$

(ii) Conversely, if $x = my$, where m is constant,

$$\therefore \ \frac{x}{y} = m,$$

i.e. the ratio $x : y$ is constant,

$$\therefore \ x \propto y.$$

294. DEF. 44. A quantity x is said to *vary jointly* as two other quantities y and z when x varies as the product yz.

DEF. 45. A quantity x is said to vary *directly* as y and *inversely* as z when x varies as $\frac{y}{z}$.

If a man walk with uniform velocity of v miles an hour, the number of miles (s) that he walks in t hours is given by the equation $s = vt$, from which we have

$$v = \frac{s}{t}.$$

Thus, the distance he walks varies jointly as his velocity and the time for which he walks; while his velocity varies directly as the distance and inversely as the time.

295. *If x vary as y when z is constant, and vary as z when y is constant, then x varies as yz when both y and z vary.*

Let $x = m \, . \, yz$; then we have to show that m does not change when y and z change.

Since $x \propto y$, when z is constant,

$$\therefore \ mz \text{ is constant with respect to } y.$$

But z does not change when y changes,

$$\therefore \ m \text{ is constant with respect to } y.$$

Again, since $x \propto z$, when y is constant,

$$\therefore \ my \text{ is constant with respect to } z,$$

$$\therefore \ m \text{ is constant with respect to } z \text{ as well as with respect to } y,$$

$$\therefore \ x \propto yz.$$

296. *Example* 13. If $y \propto x^2$, and if $y = 32$ when $x = 4$, find the value of y when $x = 5$.

Since $y \propto x^2$,

$$\therefore\ y = mx^2, \text{ where } m \text{ is constant.}$$

Since $y = 32$, when $x = 4$,

$$\therefore\ 32 = m \,.\, 16, \text{ or } m = 2,$$

$$\therefore\ y = 2x^2.$$

Hence, when $x = 5$,

$$y = 2 \,.\, 25 = 50.$$

Example 14. If $x \propto y^3 z^4$ and $y \propto z^8 x^2$, then x varies as a certain power of y only.

Since $x \propto y^3 z^4$ and $y \propto z^8 x^2$,

$\therefore\ x = my^3 z^4$ and $y = nz^8 x^2$, where m and n are constants.

$$\therefore\ x^2 = m^2 y^6 z^8 = m^2 y^6 \,.\, \frac{y}{nx^2},$$

$$\therefore\ x^4 = \frac{m^2}{n} \,.\, y^7,$$

$$\therefore\ x = \left(\frac{m^2}{n}\right)^{\frac{1}{4}} \,.\, y^{\frac{7}{4}},$$

$$\therefore\ x \propto y^{\frac{7}{4}}.$$

Example 15. If x vary as the sum of two quantities, one of which varies as t and the other as t^2; and if $x = 104$ when $t = 2$, and $x = 500$ when $t = 5$, find x when $t = 10$.

Let $x = mt + nt^2$, where m and n are constants;

then $$104 = 2m + 4n,$$

and $$500 = 5m + 25n,$$

$$\therefore\ m + 2n = 52 \text{ and } m + 5n = 100,$$

$$\therefore\ 3n = 48 \text{ or } n = 16 \text{ and } m = 52 - 32 = 20,$$

$$\therefore\ x = 20t + 16t^2.$$

Hence, if $t = 10$, $x = 200 + 1600 = 1800$.

Example 16. A man spends on charitable objects an amount which varies as the square of his income; if he spend £35 more when his income is £1200 than when it is £900, find his charitable expenditure in each case.

Let his income be £x, and his charitable expenditure with this income £mx^2, where m is constant, then

$$m(1200)^2 = m(900)^2 + 35,$$

$$\therefore\ m = \frac{35}{(1200-900)(1200+900)},$$

$$= \frac{35}{300 \cdot 2100} = \frac{1}{60 \cdot 300} = \frac{1}{18000}.$$

Hence, if his income be £1200, his charitable expenditure is £$\frac{1200 \times 1200}{18000}$ or £80. If his income be £900, his charitable expenditure is £$\frac{900 \times 900}{18000}$ or £45.

Example 17. If $a+b \propto a-b$ when a and b both vary, then $a^2+b^2 \propto ab$.

Let $a+b = m(a-b)$, where m is constant, then

$$a^2 + 2ab + b^2 = m^2(a^2 - 2ab + b^2),$$

$$\therefore\ (a^2+b^2)(m^2-1) = 2ab(m^2+1),$$

$$\therefore\ a^2+b^2 = \frac{2(m^2+1)}{m^2-1}ab,$$

$$\therefore\ a^2+b^2 \propto ab.$$

Exercises 194.

1. If $y \propto x$, and $y=4$ when $x=2$, find the value of y when $x=5$.

2. If y vary inversely as x, and $y=5$ when $x=2$, find the value of y when $x=20$.

3. If $x \propto yz$, and $x=24$ when $y=3$, $z=2$, find the value of x when $y=\frac{1}{2}$, $z=\frac{1}{2}$.

4. If $x \propto yz$, and $x = 12$ when $y = 2$, $z = 3$, find the value of y when $x = 18$, $z = 9$.

5. If x vary directly as y and inversely as z, and if $x = 8$ when $y = 4$, $z = 3$, find the value of x when $y = 5$, $z = 2$.

6. If $x \propto \frac{yz}{t^2}$ and if $x = 10$ when $y = 2$, $z = 25$, $t = 1$, find the value of x when $y = 2$, $z = 12$, $t = 2$.

7. If $t \propto pv^{\frac{3}{2}}$, and $t = 224$ when $p = 28$ and $v = 16$, what is the value of t when $p = 32$, $v = 25$?

8. If $x \propto y + z$, and if 1, 2, 3 be simultaneous values of x, y, z; find the value of z when $x = 1$, $y = 4$.

9. If $z \propto (x + 2)(y + 3)$ and be equal to 4 when $x = \frac{1}{3}$, $y = \frac{3}{7}$; find the value of z when $x = 5$, $y = 3$.

10. If the square of x vary as the cube of y, and $x = 2$ when $y = 3$, find the relation between x and y.

11. If the cube of x vary as the square of $y + 1$, and $x = 4$ when $y = 3$, find the relation between x and y.

12. If $y^2 \propto a^2 - x^2$, and $x = 0$ when $y = b$, express y in terms of x.

13. If x vary inversely as y, and y vary inversely as z, show that $x \propto z$.

14. If $x \propto \frac{w}{y^2}$ and $w \propto \frac{y^3}{z}$, and $z^2 \propto y$, show that $x \propto \sqrt{y}$.

15. If $A \propto B^2$, while $B \propto \sqrt[3]{C}$ and $C \propto D^6$, find the relation between A and D.

16. If $\frac{1}{x} - \frac{1}{y}$ vary inversely as $x - y$, prove that $x^2 + y^2 \propto xy$.

17. The time required to mow a field varies directly as its area and inversely as the number of men employed; if 10 men mow a field of 15 acres in 3 days, how many men will be required to mow a field of 12 acres in 4 days?

18. The volume of a cylinder varies as the area of the base and the height jointly; if the volume of one cylinder be to that of a second as 11 : 8, and the height of the first to that of the second as 3 : 4, and if the base of the first have an area of 16·5 sq. yds., what is the area of the base of the second?

19. The volume of a right circular cone varies jointly as its height and the square of the radius of its base; if the volume of a cone 7 ft. high with a base of radius 3 ft. be 66 c. ft., find that of a cone twice as high standing on a base with radius half as large as the other.

20. The volume of a sphere varies as the cube of its radius; find the radius of a single sphere formed by melting together three spheres of radii 6, 8 and 10 ins.

21. A hollow sphere of metal weighs $\frac{7}{8}$ of a solid sphere of the same substance and radius; compare the inner and outer radii.

22. The space described by a moving point varies as the sum of two quantities, one of which varies as the time and the other as the square of the time; in 5 seconds it describes 500 ft., and in 6 seconds 672 ft.; how far will it go in 10 seconds?

23. If x vary as the sum of two quantities, and x^2 vary as the sum of their squares, then x^2 will also vary as their product.

24. If x, y, z be variable, but their sum constant, and if $(x-y+z)(x+y-z)$ vary as yz, prove that $y+z-x$ varies as yz.

25. If z vary as $(x+a)(y+b)$ and be equal to $\frac{1}{2}(a+b)^2$ when $x=b$, $y=a$, show that $z=2(a+b)^2$ when $x=a+2b$, $y=2a+b$.

26. If x vary as the sum of the cubes of y and z, the sum of which is constant, find the value of x when $y=2$, it being given that $x=3$ when $y=3$, $z=3$.

27. If z be equal to the sum of two quantities, one of which is constant and the other varies jointly as x and y, and if $y=5$ when x and z each equal 2, and $x=9$ when y and z each equal 10, find the value of z when x and y each equal 5.

28. If y be the sum of two numbers, of which the first varies directly, and the second inversely, as x, and if $y=7$ when $x=2$ and $y=-1$ when $x=1$, show that $y=5x-\frac{6}{x}$.

29. The pressure of wind on a plane surface varies jointly as the area of the surface and the square of the wind's velocity; the pressure on a sq. ft. is 1 lb. when the wind is moving at the rate of 15 miles an hour; find the velocity of the wind when the pressure on a sq. yd. is 16 lbs.

30. Determine the resistance of the air to a projectile 16 ins. in diameter at a velocity of 1500 ft. per sec., supposing the resistance of the air to vary as the square of the diameter and the cube of the velocity; given that the resistance of the air to a projectile 1 in. in diameter at a velocity of 1000 ft. per sec. is $2\frac{1}{2}$ lbs. weight.

CHAPTER XXVIII.

SERIES: ARITHMETICAL AND GEOMETRICAL PROGRESSIONS.

297. DEF. 46. An expression in which each term is formed by means of a definite law from one or more of the preceding terms is called a *series*.

In some cases, a series is also called a *progression*.

DEF. 47. An *infinite series* is one in which the number of terms is infinite.

In this chapter, we shall consider two important series known as arithmetical and geometrical progressions; in the next chapter a third series known as harmonical progression and a few simple series formed according to different laws.

1. ARITHMETICAL PROGRESSION.

298. DEF. 48. An *arithmetical progression* is a series in which each term is formed from the preceding by adding to it a constant quantity.

This quantity is called the *common difference*, and it is obtained by subtracting any term of the series from the next succeeding term.

An arithmetical progression is generally denoted, for brevity, by A.P.

The following are examples of series in A.P.:

$$1,\quad 2,\quad 3,\quad 4,\quad 5,\quad \ldots,$$
$$1,\quad 3,\quad 5,\quad 7,\quad 9,\quad \ldots,$$
$$9,\quad 6,\quad 3,\quad 0,\quad -3,\quad \ldots,$$
$$2\tfrac{1}{2},\quad 1\tfrac{1}{4},\quad 0,\quad -1\tfrac{1}{4},\quad -2\tfrac{1}{2},\quad \ldots,$$

the common differences of these series being respectively, 1, 2, -3 and $-1\frac{1}{4}$.

299. *To find the nth term of an* A.P., *of which the first term is a and the common difference d.*

The first term being a, the second term is $a+d$, the third $a+2d$, the fourth $a+3d$, etc.

Now, the coefficient of d in the first term is 0, and in each succeeding term increases by 1, so that the coefficient of d in any term is one less than the number of the term, i.e. the coefficient of d in the nth term is $n-1$,

$$\therefore \text{ the } n\text{th term is } a+(n-1)\,d.$$

300. *Example* 1. Find

(i) the 20th term of the series 1, 3, 5, ...,

(ii) the 100th term of the series 9, 6, 3, ...

(i) Here $a=1$, $d=3-1=2$,

$\therefore$ the 20th term is $1+(20-1)\,.\,2=1+38=39$.

(ii) Here $a=9$, $d=6-9=-3$,

$\therefore$ the 100th term is

$$9+(100-1)(-3)=9-99\,.\,3=9-297=-288.$$

301. If the last of a series of n terms in A.P. be denoted by l, we have

$$l=a+(n-1)\,d.$$

Thus, if any three of the four quantities a, d, l, n be known, the fourth can be found from this equation. In Example 1, a, d and n are given or can be found by inspection, and l is determined from the equation.

The following examples illustrate cases in which a different set of three quantities is given.

Example 2. Which term of the series 3, 6, 9, ... is 999?

Let 999 be the nth term,

then (since $a = 3$, $d = 3$, $l = 999$),

$$999 = 3 + (n-1)\,3 = 3n,$$

$$\therefore\ n = 333.$$

Example 3. If 14 be the first term and -30 the 23rd term of an A.P., what is the common difference?

Let d be the common difference,

then
$$-30 = 14 + (23-1)\,d,$$

$$\therefore\ 22d = -44,$$

$$\therefore\ d = -2.$$

Example 4. If the 8th term of an A.P. be 50 and the 21st term be 115, find the 50th term.

Let a be the first term and d the common difference, then (since the 8th term is $a + 7d$, etc.),

$$a + 7d = 50 \text{ and } a + 20d = 115,$$

$$\therefore\ 13d = 65 \text{ or } d = 5,$$

$$\therefore\ a = 50 - 35 = 15.$$

$$\therefore \text{ the 50th term} = 15 + 49\,.\,5 = 15 + 245$$
$$= 260.$$

Example 5. How many numbers between 1000 and 2000 are divisible by 7?

Let n be the number of such numbers.

The first number above 1000 that is divisible by 7 is 1001, and the last number less than 2000 that is divisible by 7 is 1995.

$$\therefore\ 1995 = 1001 + (n-1)\,7,$$
$$\therefore\ 7n = 1995 - 1001 + 7 = 2002 - 1001$$
$$= 1001,$$
$$\therefore\ n = 143.$$

302. If the last term of an A.P. be l, and the common difference d, it is evident that the second term from the end is $l-d$, the third term from the end $l-2d$, and, as in art. 299, that the nth term from the end is $l-(n-1)\,d$.

Example 6. Find the 14th term from the end of the series 2, 6, 10, ... 86.

The 14th term from the end $= 86 - (14-1).4 = 86 - 13.4$
$= 86 - 52 = 34.$

Exercises 195.

Find the following terms:

1. 10th term of the series 3, 5, 7, ...
2. 21st ,, ,, ,, 10, 14, 18, ...
3. 15th ,, ,, ,, 14, 12, 10, ...
4. 32nd ,, ,, ,, -4, -2, 0, ...
5. 100th ,, ,, ,, 1, 5, 9, ...
6. 48th ,, ,, ,, 2, $2\frac{1}{2}$, 3, ...
7. 18th ,, ,, ,, 3, $1\frac{3}{4}$, $\frac{1}{2}$, ...
8. 20th ,, ,, ,, 38, 36, 34, ...
9. 45th ,, ,, ,, $8\frac{1}{3}$, 9, $9\frac{2}{3}$, ...
10. 50th ,, ,, ,, $7\frac{3}{4}$, $9\frac{1}{2}$, $11\frac{1}{4}$, ...

Find the nth term of the following series:

11. 1, 2, 3, ... 12. 2, 4, 6, ...

13. 1, 3, 5, ... 14. 4, 7, 10, ...

15. 20, 18, 16, ... 16. $3a-2b$, $4a-3b$, $5a-4b$, ...

Which term of the series:

17. 7, 10, 13, ... is 43? 18. 84, 80, 76, ... is 0?

19. 5, 10, 15, ... is 1000000?

20. $19\frac{1}{2}$, 18, $16\frac{1}{2}$, ... is -18?

21. 75, 72, 69, ... is 9, and which is -9?

22. If the first term of an A.P. be 14 and the 20th term 128, what is the common difference?

23. If the first term of an A.P. be 58 and the 26th term -17, what is the common difference?

24. If the second term of an A.P. be -19 and the 50th term 101, what is the common difference?

25. What is the first term of an A.P. in which the 14th term is 51 and the common difference 3?

26. What is the first term of an A.P. in which the 43rd term is 51 and the 44th term 45?

27. The second term of an A.P. is 14 and the 12th term is 44; find the 22nd term.

28. The 11th term of an A.P. is 42 and the 22nd term is -42; find the 35th term.

29. The 15th term of an A.P. is -40 and the 31st term is 40; find the 23rd term.

30. The 12th term of an A.P. is $13\frac{3}{4}$ and the 17th term is $17\frac{1}{2}$; find the 8th term.

31. The 6th term of an A.P. is 10 and the 13th term is -4; which term is -40?

32. The 11th term of an A.P. is -8 and the 29th term is 4; which term is 18?

33. How many odd numbers are there between 80 and 220?

34. How many numbers divisible by 3 are there between 40 and 340?

35. How many numbers divisible by 8 are there between 900 and 1700?

36. The sum of the 10th and 32nd terms of an A.P. is 176, and the 13th term is 48; find the 20th term.

37. The sum of the 10th and 25th terms of an A.P. is 47, and the sum of the 5th and 16th terms is 33; find the first term and the common difference.

38. The sum of the 10th and the 24th terms of an A.P. is 34, and the difference between the 8th and 22nd terms is 21; find the series.

39. Find the 15th term from the end of the series 7, 10, 13, ..., 103.

40. Find the sum of the 12th term from the beginning and the 12th term from the end of the series 48, $46\frac{1}{2}$, 45, ..., 3.

41. If 7 and 43 be the first and last of 13 numbers in A.P., find the middle term.

42. If 16 and 76 be the first and last of 16 numbers in A.P., find the two middle terms.

43. The 15th, 45th and nth terms of an A.P. are 33, 153 and 217 respectively; find n.

44. Show that $(a-b)^2$, a^2+b^2, and $(a+b)^2$ are in A.P.

45. If $3a+2b$, $a+4b$ and $x+6b$ be in A.P., find the value of x.

303. *To find the sum of n terms of an* A.P. *of which the first term is a and the common difference d.*

Let s be the sum of n terms,

then $$s = a + (a+d) + (a+2d) + \ldots + (a+\overline{r-1}\,.\,d) + \ldots$$
$$+ (l-2d) + (l-d) + l,$$

also $s = l + (l - d) + (l - 2d) + \dots + (l - \overline{r-1}\,.\,d) + \dots$
$+ (a + 2d) + (a + d) + a.$

$\therefore\ 2s = (a + l) + (a + l) + (a + l) + \dots$ to n terms $= n(a + l)$;

$$\therefore\ s = \frac{n}{2}(a + l).$$

But $$l = a + (n - 1)\,d,$$

$$\therefore\ s = \frac{n}{2}\{2a + (n - 1)\,d\}.$$

304. *Example* 7. Find the sum of the series:

(i) $14 + 17 + 20 + \dots$ to 40 terms,

(ii) $98 + 94 + 90 + \dots$ to 50 terms.

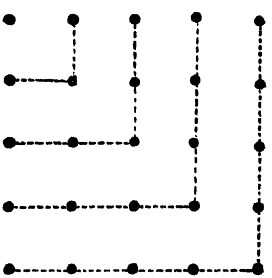
(i) $s = \frac{40}{2}\{2\,.\,14 + (40 - 1)\,.\,3\} = 20\,(28 + 117)$
$= 20\,.\,145 = 2900.$

(ii) $s = \frac{50}{2}\{2\,.\,98 + (50 - 1)(-4)\} = \frac{50}{2}(196 - 196) = 0.$

Example 8. Find the sum: (i) of the first n numbers, (ii) of the first n odd numbers.

(i) $s = \frac{n}{2}\{2\,.\,1 + (n - 1)\,.\,1\} = \frac{1}{2}n\,(n + 1).$

(ii) $s = \frac{n}{2}\{2\,.\,1 + (n - 1)\,.\,2\} = \frac{n}{2}\,.\,2n = n^2.$



The last case is illustrated in the accompanying diagram, which shows that $1 + 3 + 5 + 7 + 9 = 5^2$.

Example 9. How many terms of the series $21 + 18 + 15 + \ldots$ have their sum equal to 66?

Let n be the number of terms;

then $$66 = \frac{n}{2}\{2 . 21 + (n-1)(-3)\},$$

$$\therefore\ 132 = n(42 - 3n + 3),$$

$$\therefore\ 3n^2 - 45n + 132 = 0,$$

$$\therefore\ n^2 - 15n + 44 = 0,$$

$$\therefore\ (n-4)(n-11) = 0,$$

$$\therefore\ n = 4 \text{ or } 11.$$

The first four terms of the series are 21, 18, 15, 12, the sum of which is 66. The next 7 terms are 9, 6, 3, 0, -3, -6, -9, the sum of which is 0. Thus, the sum of 4 terms, and the sum of 11 terms, both amount to 66.

Example 10. Find the sum of all the numbers less than 1000 which are divisible by 3.

Let n be the number of such numbers, the first of which is 3 and the last 999;

then $$999 = 3 + (n-1) . 3 = 3n,$$

$$\therefore\ n = 333.$$

$$\therefore\ s = \tfrac{333}{2}\{2 . 3 + (333-1) . 3\} = \tfrac{333}{2}(6 + 996)$$

$$= 333(3 + 498) = 333 . 501$$

$$= 166833.$$

Exercises 196.

Find the sum of the series whose first and last terms and number of terms are:

1. 14, 50; 10. 2. 31, 75; 40. 3. 2, 100; 50.

4. 5, 71; 36. 5. $48\frac{1}{2}$, 20; 18. 6. $5\frac{3}{4}$, $-13\frac{1}{4}$; 24.

Find the sum of the series:

7. $1 + 2 + 3 + \dots$ to 100 terms.

8. $1 + 3 + 5 + \dots$ to 40 terms.

9. $10 + 14 + 18 + \dots$ to 24 terms.

10. $60 + 57 + 54 + \dots$ to 41 terms.

11. $104 + 109 + 114 + \dots$ to 70 terms.

12. $5 + 11 + 17 + \dots$ to 23 terms.

13. $8 + 4 + 0 + \dots$ to 32 terms.

14. $101 + 99 + 97 + \dots$ to 101 terms.

15. $2 + 3\frac{1}{2} + 5 + \dots$ to 51 terms.

16. $-9 - 7\frac{1}{2} - 6 - \dots$ to 13 terms.

17. $3\frac{1}{2} + 2\frac{1}{2} + 1\frac{1}{2} + \dots$ to 15 terms.

18. $16\frac{1}{2} + 14 + 11\frac{1}{2} + \dots$ to 14 terms.

19. $8 + 7\frac{1}{9} + 6\frac{2}{9} + \dots$ to 19 terms.

20. $-4\frac{1}{2} - 4\frac{5}{6} - 5\frac{1}{6} - \dots$ to 17 terms.

21. $1 + \frac{1}{3} - \frac{1}{3} - \dots$ to 9 terms.

22. $13{\cdot}1 + 11{\cdot}4 + 9{\cdot}7 + \dots$ to 15 terms.

23. Find the sum (i) of 5 terms, (ii) of 16 terms, of the series 20, 18, 16, ...

Sum to n terms the series:

24. $2 - \frac{1}{2} - 3 - \frac{11}{2} - \dots$

25. $1\frac{1}{7} + 1\frac{17}{21} + 2\frac{10}{21} + \dots$

26. $(p + 1) + (p + 3) + (p + 5) + \dots$

27. $(a) + (2b) + (4b - a) + (6b - 2a) + \dots$

28. $\dfrac{n-1}{n} + \dfrac{n-2}{n} + \dfrac{n-3}{n} + \dots$

29. $(a + b)^2 + (a^2 + b^2) + (a - b)^2 + \dots$

30. Show that

$$3 \,.\, (1 + 3 + 5 + \dots + 99) = 101 + 103 + 105 + \dots + 199.$$

31. How many terms of the series $24 + 28 + 32 + \ldots$ amount to 1240?

32. How many terms of the series $32 + 28 + 24 + \ldots$ amount to 120?

33. How many terms of the series $-16 - 14 - 12 - \ldots$ amount to 0?

34. How many terms of the series $23 + 19 + 15 + \ldots$ must be taken in order that the sum may be 12?

35. How many terms of the series $1 + 1\frac{3}{4} + 2\frac{1}{2} + \ldots$ must be taken to make $61\frac{1}{2}$?

36. The first term of an A.P. is $1\frac{1}{2}$ and the 6th term is $-1\frac{1}{2}$; find the sum of 12 terms.

37. The third term of an A.P. is 17 and the 13th term is 87; find the sum of 19 terms.

38. The sum of 12 terms of an A.P. is 522, and the common difference is -1; find the first term.

39. The sum of 7 terms in A.P. is 28 and the common difference is 3; find the series.

40. If the first term of an A.P. be 2 and the sum of 8 terms 100, find the common difference.

41. If the 4th term of an A.P. be 35 and the sum of 12 terms be 570, find the common difference.

42. The sum of 5 terms in A.P. is 10, and the sum of 17 terms is -17; find the series.

43. The sum of 20 terms of an A.P. is 365, and the sum of 30 terms is $772\frac{1}{2}$; find the sum of 15 terms.

44. The first and third terms of an A.P. are 22 and 14; how many terms must be taken so that the sum may be 64?

45. In an A.P., the first term is 3 and the last term 27, also the sum of the terms is 495; how many terms are there?

46. Find the sum of all the odd numbers less than 1000.

47. Find the sum of all the numbers between 100 and 200 that are divisible by 3.

48. Find the sum of all the numbers between 1000 and 3000 that are divisible by 7.

49. Find the sum of all the numbers less than 1000 whose units' digit is 5.

50. Find the sum of all the numbers less than 200 which are odd multiples of 7.

305. DEF. 49. If three terms be in A. P., the middle term is called the *arithmetic mean* between the first and third.

DEF. 50. If any number of terms be in A. P., the intermediate terms are called the *arithmetic means* between the first and last.

306. *If* A *be the arithmetic mean between two quantities* x *and* y, *then* $A = \frac{1}{2}(x + y)$.

Since x, A, y are in A. P.

$$\therefore\ A - x = y - A;$$

$$\therefore\ 2A = x + y \quad \text{or} \quad A = \tfrac{1}{2}(x + y).$$

307. *Example* 11. Insert 12 arithmetic means between 15 and 54.

Let d be the common difference, then, since there are 12 means and two end terms, 54 is the 14th term of the A. P. of which the first is 15 and the common difference d.

$$\therefore\ 54 = 15 + (14 - 1)d,$$

$$\therefore\ 13d = 39 \quad \text{or} \quad d = 3,$$

$\therefore$ the means are 18, 21, 24, ..., 51.

Exercises 197.

Find the arithmetic means of the following quantities:

1. 7, 21.   2. 16, 24.   3. 15, 30.

4. 42, 85.   5. $2a - b$, $4a + 5b$.   6. $7x + 4y$, $3x - 6y$.

7. If O, A, B, C be four points in order on a straight line such that B bisects AC, and if OA be 6 ins., and OC 24 ins., long, find OB.

8. Insert 6 arithmetic means between 1 and 29.

9. Insert 10 arithmetic means between 13 and $29\frac{1}{2}$.

10. Insert 7 arithmetic means between 3 and 18.

11. Insert 3 arithmetic means between $\frac{11}{15}$ and $-\frac{8}{5}$.

12. Insert 9 arithmetic means between $4a+5b$ and $5b-6a$.

13. Insert 24 arithmetic means between 15 and 165, and find their sum.

14. Insert 100 arithmetic means between 48 and -255, and find their sum.

15. a, b, c are three numbers in A. P.; if x be the arithmetic mean of a and b, and y that of b and c, show that a, x, b, y, c are in A. P.

308. If the number of terms in an A. P. be odd, the equations may be simplified by taking a for the middle term. Thus, the sum of 3 terms $a-d$, a, $a+d$ is $3a$, and the sum of their squares $(a-d)^2$, a^2, $(a+d)^2$ is $3a^2+2d^2$, the terms in d and ad disappearing.

Example 12. The sum of 3 numbers in A. P. is 27, and the sum of their squares is 275; find the numbers.

Let the numbers be $a-d$, a, $a+d$,

then $$(a-d)+a+(a+d)=27,$$

and $$(a-d)^2+a^2+(a+d)^2=275;$$

$$\therefore\ 3a=27,$$

and $$3a^2+2d^2=275;$$

$$\therefore\ a=9 \text{ and } 3\,.\,81+2d^2=275,$$

$$\therefore\ 2d^2=275-243=32,$$

$$\therefore\ d=\pm 4,$$

∴ the numbers are 5, 9, 13; the value -4 for d giving the same numbers in the opposite order.

Example 13. The first, middle and last terms of an odd number of terms in A. P. are also in A. P.

Let a be the middle term, d the common difference, and $2m+1$ the number of terms; then, since there are m terms on each side of the middle term, the middle term is the $(m+1)$th.

∴ the first term $= a - md$, and the last term $= a + md$.

Now, $a-(a-md)=md$ and $(a+md)-a=md$,

∴ the first, middle and last terms are in A.P., the common difference being md.

Example 14. Sum to n terms the series whose nth term is $3n-5$.

The first term (which is obtained by putting $n=1$) is $3.1-5$ or -2.

∴ the sum of n terms $=\frac{n}{2}(-2+\overline{3n-5})=\frac{1}{2}n(3n-7)$.

Or thus: Sum to n terms

$$=(3.1-5)+(3.2-5)+(3.3-5)+\ldots+(3.n-5)$$
$$=3(1+2+3+\ldots+n)-5n$$
$$=3.\tfrac{1}{2}.n(n+1)-5n=\tfrac{1}{2}n(3n-7).$$

Example 15. Find the series whose sum to n terms is an^2-bn.

Let x be the first term and y the common difference,

$$\therefore \text{ sum to } n \text{ terms } = \frac{n}{2}\{2x+(n-1)y\}=\tfrac{1}{2}n^2y+\tfrac{1}{2}n(2x-y)$$
$$=an^2-bn,$$
$$\therefore \tfrac{1}{2}y=a \text{ and } x-\tfrac{1}{2}y=-b,$$
$$\therefore y=2a \text{ and } x=a-b,$$

∴ the series is $a-b$, $3a-b$, $5a-b$, ...

Example 16. The series of natural numbers, beginning with unity, is divided into groups of which the first contains 1, the second 2, the third 3, and so on; prove that the sum of the numbers in the nth group is $\frac{1}{2}n(n^2+1)$.

The number of terms in the first $n-1$ groups is

$$1+2+3+\ldots+n-1 \quad \text{or} \quad \tfrac{1}{2}(n-1)n,$$

$\therefore$ the first term in the nth group is $\dfrac{n^2-n}{2}+1$.

$\therefore$ the sum of the numbers in the nth group

$$=\frac{n}{2}\left\{2\left(\frac{n^2-n}{2}+1\right)+(n-1).1\right\}$$

$$=\frac{n}{2}(n^2-n+2+n-1)$$

$$=\tfrac{1}{2}n(n^2+1).$$

Exercises 198.

1. If a, b, c be in A. P., then ma, mb, mc are also in A. P.

2. If a, b, c be in A. P., show that $2a-3b$, $2b-3c$ and $3b-4c$ are also in A. P.

3. If a, b, c be in A. P., then $b+c$, $c+a$, $a+b$ are also in A. P.

4. If a, b, c, d be in A. P., then $a+d=b+c$.

5. If the third term of an A. P. be 5 and the 15th (and last) be -13, find the middle term.

6. The sum of 21 terms in A. P. is 63; find the 11th term.

7. Prove that if any even number of terms of the series 1, 3, 5, ... be taken, the sum of the second half is three times the sum of the first half.

8. The sum of the 5th and 15th terms of an A. P. is equal to the sum of the first and last terms; how many terms are there? If it be further known that the middle term is 16, what is the sum of the A. P.?

9. Show that the sum of the cubes of four numbers in A. P. is always divisible by the sum of the numbers.

10. The sum of 10 terms of an A. P. is 285, and the sum of the next 10 terms is 585; find the sum of the next group of 10 terms.

11. If n, the number of terms of an A. P., be odd, the sum of the terms is n times the middle term.

12. If $a : b = b : c$, then $\log a$, $\log b$, $\log c$ are in A. P.

13. Sum the series whose rth term is $3r + 2$ to n terms.

14. The sum of the first 3 terms of an A. P. is 9, and the sum of the next 3 terms is 27; find the series.

15. The sum of 5 numbers in A. P. is 10, and the sum of their squares is 60; find the numbers.

16. Show that the sum of the first n odd numbers is to the sum of the first n even numbers as n to $n + 1$.

17. Insert 4 arithmetic means between a and b, and show that the sum of their squares is equal to $\frac{2}{5}(3a^2 + 4ab + 3b^2)$.

18. 300 trees are planted regularly in rows in the shape of an isosceles triangle, and the numbers in successive rows diminish by one; how many trees are there in the row which forms the base of the triangle?

19. Find the sum of all the numbers in the first thousand which are not divisible by 3.

20. If a, b, c, d be in A. P. prove that $ad < bc$.

21. The last term of an A. P. is 10 times the first, and the last but one is equal to the sum of the fourth and fifth; find the number of terms, and show that the common difference is equal to the first term.

22. The first and last of 46 terms are -5 and 25 respectively; find the two middle terms.

23. The middle term of an A. P. of 21 terms is 13, and the sum of the terms that follow it is 12 times the sum of the terms that precede it; find the series.

24. A man arranges to pay off a debt of £3600 by 40 annual instalments, which form an arithmetic series; when 30 of the instalments are paid, he dies, leaving a third of the debt unpaid; find the value of the first and second instalments.

25. The $(n+1)$th term of a series is $p-qn$; prove that the sum of the first $2n+1$ terms is $(p-qn)(2n+1)$.

26. If the sum of n_1 terms of an A.P. be equal to the sum of n_2 terms, show that the number n_1+n_2 depends only on the ratio of the first term to the common difference.

27. The nth term of an A.P. is $\frac{1}{6}(3n-1)$; find the first term, the common difference and the sum of n terms.

28. Find the sum of the square roots of all odd numbers between 100 and 10000 which are perfect squares.

29. The sum of n terms of an A.P. is always $n(n+2)$; find the common difference.

30. The sum of n terms of an A.P. is always $n-n^2$; find the nth term.

31. The sum of n terms of an A.P. is $n(a-b)^2+n^2ab$; find the series.

32. If, in an A.P. the pth term be q and the qth term be p, prove that the $(p+q)$th term vanishes.

33. If the mth term of an A.P. be p and the nth term be q, find the sum of $m-n$ terms.

34. If the mth term of an A.P. be n, and the nth term be m, how many terms must be taken so as to give the sum

$$\tfrac{1}{2}(m+n)(m+n-1)?$$

35. Find the condition that in an A.P. the sum of any two terms whatever may form a term of the same series.

36. If $\dfrac{a}{b+c}$, $\dfrac{b}{c+a}$, $\dfrac{c}{a+b}$ be in A.P., then a^2, b^2, c^2 are also in A.P.

37. Show that the sums of successive groups of n terms of an A. P. are in A. P.

38. Find the sum of all the numbers in the first thousand which are not divisible by either 2 or 5.

39. The common difference of 4 numbers in A. P. is 2 and their product is 156009; find them.

40. Two arithmetical progressions, each of n terms, have for their first term and common difference a, b and b, a respectively, where a and b are the roots of the equation $x^2 - px + q = 0$; prove that the sums of the series are the roots of the equation in x

$$4x^2 - 2n(n+1)px + n^2\{2(n-1)p^2 + (n-3)^2 q\} = 0.$$

41. If three positive integers a, b, c be respectively the pth, qth, rth, terms of an A. P., then will p, q, r be the ath, bth, cth terms of another A. P.; find the relation between the common differences of the two series.

42. The sum of the first x integers is a number composed of three equal figures; find x.

43. If S_n be the sum of the first n terms of an A. P. of common difference b, show that $S_n - 2S_{n+1} + S_{n+2} = b$.

44. If S_n, S_{2n}, S_{3n} be the sums of any A. P. to n, $2n$, $3n$ terms, respectively, prove that $S_{3n} = 3(S_{2n} - S_n)$.

45. If s_1, s_2, s_3, ... be the sums of m arithmetic series, each to n terms, the first terms being 1, 2, 3, ... respectively, and the common differences 1, 3, 5, ... respectively, show that

$$s_1 + s_2 + s_3 + \dots + s_m = \tfrac{1}{2}mn(mn+1).$$

46. If the sum of the first p terms of an A. P. be zero, the sum of the next q terms is $\dfrac{-a(p+q)q}{p-1}$.

47. If $\dfrac{a}{b}$, $\dfrac{c}{d}$ be unequal fractions, prove that $\dfrac{a+c}{b+d}$ cannot be their arithmetic mean unless $b = d$.

48. If the roots of the equation $(b-c)x^2 + (c-a)x + (a-b) = 0$ be equal, show that a, b, c are in A. P.

2. Geometrical Progression.

309. Def. 51. A *geometrical progression* is a series in which each term is formed from the preceding by multiplying it by a constant quantity.

This quantity is called the *common ratio* or *common factor*, and it is obtained by dividing any term of the series by the next preceding term.

A geometrical progression is generally denoted, for brevity, by G. P.

The following are examples of series in G. P.:

$$1, 2, 4, 8, 16, \ldots,$$
$$1, 3, 9, 27, 81, \ldots,$$
$$32, 16, 8, 4, 2, \ldots,$$
$$27, -18, 12, -8, 5\tfrac{1}{3}, \ldots;$$

the common ratio of these series being respectively 2, 3, $\frac{1}{2}$ and $-\frac{2}{3}$.

310. *To find the nth term of a* G.P. *of which the first term is a and the common ratio r.*

The first term being a, the second term is ar, the third ar^2, the fourth ar^3, etc.

Now, the index of r in the first term is 0, and in each succeeding term increases by 1;

∴ the index of r in any term is one less than the number of the term, i.e. the index of r in the nth term is $n-1$.

∴ the nth term is ar^{n-1}.

311. *Example* 17. Find (i) the 9th term of the series 1, 2, 4, 8, ...; (ii) the 8th term of the series 27, −18, 12, −8, ...

(i) Here $a = 1,\ r = \frac{2}{1} = 2,$

∴ the 9th term is $1 \cdot 2^8 = 16^2 = 256$.

(ii) Here $a = 27,\ r = \dfrac{-18}{27} = -\frac{2}{3},$

∴ the 8th term is

$$27\left(-\tfrac{2}{3}\right)^7 = -\frac{3^3 \cdot 2^7}{3^7} = -\frac{2^7}{3^4} = -\tfrac{128}{81} = -1\tfrac{47}{81}.$$

312. If we denote the last term of a series of n terms in G.P. by l, we have the equation $l = ar^{n-1}$.

Thus, if any three of the four quantities a, r, l, n be known, the fourth is given by this equation. In Example 17, a, r and n are given or can be found by inspection, and l is determined from the equation.

The following examples illustrate cases in which a different set of three quantities is given.

Example 18. If 18 be the first term and $\frac{2}{3}$ the fourth term of a G.P., find the common ratio.

Let r be the common ratio,

then $$\tfrac{2}{3} = 18 \cdot r^3.$$

$$\therefore\ r^3 = \tfrac{1}{27},$$

$$\therefore\ r = \tfrac{1}{3}.$$

Example 19. If 40 be the third term and 160 the fifth term of a G.P., find the series.

Let a be the first term and r the common ratio,

then $$ar^2 = 40,$$

and $$ar^4 = 160.$$

$$\therefore\ r^2 = 160 \div 40 = 4,$$

$$\therefore\ r = \pm 2,$$

and $$a = 40 \div 4 = 10,$$

∴ the series is 10, ± 20, 40, ± 80, 160, ± ....

Example 20. Which term of the series 5, 10, 20, ... is 320?

Let 320 be the nth term,

then $$320 = 5 \cdot 2^{n-1}.$$

$$\therefore\ 2^{n-1} = 64 = 2^6,$$

$$\therefore\ n - 1 = 6,$$

and $$n = 7.$$

313. If the last term of a G.P. be l, and the common ratio r, it is evident that the second term from the end is $\frac{l}{r}$, the third from the end is $\frac{l}{r^2}$, and, as in art. 310, that the nth term from the end is $\frac{l}{r^{n-1}}$.

Exercises 199.

Find the following terms:

1. 10th term of the series 1, 2, 4, ...
2. 8th term of the series 2, 6, 18, ...
3. 10th term of the series $\frac{1}{4}$, $-\frac{1}{2}$, 1, ...
4. 5th term of the series 36, 12, 4, ...
5. 9th term of the series 40, -20, 10, ...
6. 7th term of the series 9, 6, 4, ...
7. 6th term of the series 1, $\frac{3}{4}$, $\frac{9}{16}$, ...
8. 11th term of the series 4, 6, 9, ...
9. 6th term of the series 40, -30, $22\frac{1}{2}$, ...
10. 7th term of the series 100, 80, 64, ...
11. 12th term of the series $1 + 2x + 4x^2 + 8x^3 + \ldots$.

Find the nth term of the following series:

12. $(a-b)^2$, a^2-b^2, $(a+b)^2$, ...
13. 1, $2x$, $3x^2$, $4x^3$, ...
14. $1+x$, $2+x^2$, $3+x^3$, ...
15. The 6th term of a G.P. is $\frac{5}{8}$, and the 9th term is $\frac{5}{64}$; find the first two terms of the series.
16. The 7th and 8th terms of a G.P. are 4 and -8; find the 3rd and 4th terms.

17. The 4th term of a G.P. is 64, and the 5th term is 128; find the series.

18. The third term of a G.P. is 45, and the 5th term is 405; find the 8th term.

19. Which term of the series 625, 125, 25, ... is $\frac{1}{625}$?

20. Which term of the series 36, 24, 16, ... is $4\frac{20}{27}$?

314. *To find the sum of* n *terms of a* G.P. *of which the first term is* a *and the common ratio* r.

Let s be the sum of n terms,

then $$s = a + ar + ar^2 + \dots + ar^{n-2} + ar^{n-1}$$

and $$sr = \quad ar + ar^2 + \dots \quad + ar^{n-1} + ar^n;$$

$$\therefore \; s - sr = a - ar^n,$$

$$\therefore \; s(1-r) = a(1-r^n),$$

$$\therefore \; s = \frac{a(1-r^n)}{1-r} \text{ or } \frac{a(r^n-1)}{r-1}.$$

315. *Example* 21. Find the sum of the series:

(i) $1 + 2 + 4 + 8 + \dots$ to 8 terms.

(ii) $9 - 6 + 4 - 2\frac{2}{3} + \dots$ to 6 terms.

(i) $$s = \frac{1 \, . \, (2^8 - 1)}{2 - 1} = 2^8 - 1 = 256 - 1 = 255.$$

(ii) $$s = \frac{9\{1 - (-\frac{2}{3})^6\}}{1 - (-\frac{2}{3})} = \frac{27}{5}(1 - \frac{64}{729})$$
$$= \frac{27}{5} \times \frac{665}{729} = \frac{133}{27}$$
$$= 4\frac{25}{27}.$$

Example 22. In the series 1, $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, ..., find the sum of (i) 2 terms, (ii) 5 terms, and (iii) 10 terms.

(i) $$s = \frac{1 - \frac{1}{10^2}}{1 - \frac{1}{10}} = \frac{10}{9}(1 - \cdot 01) = 1 \cdot 1.$$

(ii) $$s = \frac{1 - \frac{1}{10^5}}{1 - \frac{1}{10}} = \frac{10}{9}(1 - \cdot 00001) = 1 \cdot 1111.$$

(iii) $$s = \frac{1 - \frac{1}{10^{10}}}{1 - \frac{1}{10}} = \frac{10}{9}(1 - \cdot 0000000001) = 1 \cdot 111111111.$$

Thus, as the number of terms increases, the sum approaches more and more nearly to $\frac{10}{9}$ or $1 \cdot \dot{1}$.

316. *To find the limit of the sum of an infinite number of terms of a* G.P. *of which the first term is* a *and the common ratio* r, r *being less than unity.*

Let s be the number of terms,

then $$s = \frac{a(1 - r^n)}{1 - r} = \frac{a}{1 - r} - \frac{a}{1 - r} r^n.$$

Now, r being less than 1, it follows that r^n decreases as n increases. Let $$\frac{a}{1 - r} r^n = \epsilon,$$

then $$\log a - \log(1 - r) + n \log r = \log \epsilon,$$

$$\therefore \; n = \frac{\log \epsilon - \log a + \log(1 - r)}{\log r};$$

$\therefore \; \frac{a}{1 - r} r^n$ can be made less than any quantity ϵ, however small, by making n greater than the value given by the above equation, that is, the limit of the sum of an infinite number of terms is $\frac{a}{1 - r}$.

317. *Example* 23. Find the sum of an infinite number of terms of the series:

(i) 8, 4, 2, 1,.... (ii) 81, −27, 9, −3, ...

(i) $s = \frac{8}{1 - \frac{1}{2}} = 16.$ (ii) $s = \frac{81}{1 - (-\frac{1}{3})} = \frac{81 . 3}{4} = \frac{243}{4} = 60\frac{3}{4}.$

Exercises 200.

Find the sum of the series :

1. $1 + 2 + 4 + \dots$ to 8 terms.

2. $2 + 6 + 18 + \dots$ to 7 terms.

3. $8 + 4 + 2 + \dots$ to 10 terms.

4. $81 + 27 + 9 + \dots$ to 9 terms.

5. $2 - 1 + \frac{1}{2} - \dots$ to 8 terms.

6. $1 + \frac{1}{10} + \frac{1}{100} + \dots$ to 6 terms.

7. $\frac{1}{16} - \frac{1}{8} + \frac{1}{4} - \dots$ to 12 terms.

8. $1 - \frac{3}{4} + \frac{9}{16} - \frac{27}{64} + \dots$ to 6 terms.

9. $\frac{4}{9} + \frac{2}{3} + 1 + \dots$ to 8 terms.

10. $1\frac{4}{5} + 1\frac{1}{5} + \frac{4}{5} + \dots$ to 10 terms.

Find the sum of n terms of the series;

11. $1\frac{1}{2} + 4\frac{1}{2} + 13\frac{1}{2} + \dots$.

12. $2 - \frac{1}{2} + \frac{1}{8} - \frac{1}{32} + \dots$.

13. $\frac{2}{3} - \frac{1}{2} + \frac{3}{8} - \dots$.

14. $\frac{2}{3} + \frac{1}{2} + \frac{3}{8} + \dots$.

15. $x + 2x^2 + 4x^3 + 8x^4 + \dots$.

16. Sum the series $(1\cdot05) + (1\cdot05)^2 + (1\cdot05)^3 + \dots + (1\cdot05)^{11}$, it being given that $(1\cdot05)^{12} = 1\cdot79586$ nearly.

17. Sum to $2n$ terms $\frac{a}{3} - \frac{b}{9} + \frac{a}{27} - \frac{b}{81} + \frac{a}{243} - \frac{b}{729} + \dots$.

18. Sum to n terms $(a + b)^2 + (a^2 - b^2) + (a - b)^2 + \dots$.

Find the sum to infinity of the series :

19. $1 + \frac{1}{2} + \frac{1}{4} + \dots$.

20. $1 - \frac{1}{2} + \frac{1}{4} - \dots$.

21. $1 + \frac{1}{3} + \frac{1}{9} + \dots$

22. $3 - 2 + \frac{4}{3} - \dots$.

23. $48 - 36 + 27 - \dots$.

24. $3 - \frac{2}{3} + \frac{4}{27} - \dots$.

25. $9\cdot6 + 7\cdot2 + 5\cdot4 + \dots$.

26. $\frac{\sqrt{2} + 1}{\sqrt{2} - 1} + \frac{1}{2 - \sqrt{2}} + \frac{1}{2} + \dots$.

27. Find the sum of 5 terms in G.P., the second term being 5 and the fifth 625.

28. The first term of a G.P. is a and the second b; find the sum of n terms.

318. DEF. 52. If three terms be in G.P., the middle term is called the *geometric mean* between the first and third.

DEF. 53. If any number of terms be in G.P., the intermediate terms are called the *geometric means* between the first and last.

319. *If* G *be the geometric mean between two quantities* x *and* y, *then* $G = \sqrt{xy}$.

Since x, G, y are in G.P.,

$$\therefore \frac{G}{x} = \frac{y}{G},$$

$$\therefore G^2 = xy,$$

$$\therefore G = \sqrt{xy}.$$

Cor. Thus, the mean proportional between any two numbers is also their geometric mean and vice versâ.

320. *Example* 24. Insert 3 geometric means between 3 and 48.

Let r be the common ratio, then, since there are 3 means and 2 end terms, 48 is the 5th term of the G.P. of which the first term is 3 and the common ratio r;

$$48 = 3 . r^4.$$

$$\therefore r^4 = 16 \text{ or } r^2 = 4,$$

$$\therefore r = \pm 2;$$

$\therefore$ the means are 6, 12 and 24, or -6, 12 and -24.

Example 25. The arithmetic mean between two numbers is $6\frac{1}{2}$, and the geometric mean is 6; find the numbers.

Let x and y be the numbers,

then $$\frac{x+y}{2} = 6\tfrac{1}{2} \text{ and } \sqrt{xy} = 6.$$

$$\therefore\ x + y = 13 \text{ and } xy = 36,$$

the roots of which are

$$\left.\begin{array}{l} x = 4 \\ y = 9 \end{array}\right\} \text{ and } \left.\begin{array}{l} x = 9 \\ y = 4 \end{array}\right\}.$$

$\therefore$ the numbers are 4 and 9.

Exercises 201.

Find the geometric means of:

1. 4, 16. 2. 9, 16. 3. 4, 25.

4. a^3b, ab^3. 5. $6x^2y^5$, $24x^4y^3$.

6. Insert 2 geometric means between 1 and -1.

7. Insert 5 geometric means between 4 and 256.

8. Insert 4 geometric means between 9 and $\frac{1}{27}$.

9. Insert 5 geometric means between $\frac{8}{9}$ and $10\frac{1}{8}$.

10. Insert 3 geometric means between $4\frac{113}{128}$ and 2.

11. Insert 3 geometric means between a^4 and b^4c^4.

12. Find the geometric mean between $a+b$ and $a^3-a^2b-ab^2+b^3$.

13. Find the geometric mean between the pth and qth terms of a G.P.

14. Insert 5 geometric means between 5 and 320, and find their sum.

15. The product of two numbers and their geometric mean is 216, find the geometric mean.

16. Show that the geometric mean of the 1st and 9th terms of the arithmetic series 4, 8, 12, ... is $\frac{3}{4}$ of the geometric mean of the 2nd and 8th terms.

17. The arithmetic mean between two numbers is 5, and the geometric mean is 4; find the numbers.

18. The geometric mean of two numbers is greater than one of them by 8, and less than the other by 24; find the numbers.

321. If the number of terms in a G.P. be odd, the equations may sometimes be simplified by taking a for the middle term. Thus, the product of the three terms $\frac{a}{r}$, a and ar is a^3.

Example 26. The sum of three numbers in G.P. is $43\frac{1}{3}$, and their product is 1000; find the numbers.

Let the numbers be $\frac{a}{r}$, a and ar,

then $$\frac{a}{r} + a + ar = 43\tfrac{1}{3}$$

and $$\frac{a}{r} \cdot a \cdot ar = a^3 = 1000.$$

$$\therefore \; a = 10,$$

$$\therefore \; 10\left(\frac{1}{r} + 1 + r\right) = \frac{130}{3},$$

$$\therefore \; 1 + r + r^2 = \tfrac{13}{3}r,$$

$$\therefore \; 3r^2 - 10r + 3 = 0,$$

$$\therefore \; (r-3)(3r-1) = 0,$$

the roots of which are 3 and $\frac{1}{3}$.

$\therefore$ the numbers are $3\frac{1}{3}$, 10 and 30.

Example 27. The first, middle and last terms of an odd number of terms in G.P. are also in G.P.

Let a be the middle term, r the common ratio, and $2m+1$ the number of terms; then, since there are m terms on each side of the middle term, the middle term is the $(m+1)$th.

$\therefore$ the first term is $\frac{a}{r^m}$ and the last term ar^m.

Now, $$a \div \frac{a}{r^m} = r^m$$

and $$ar^m \div a = r^m.$$

∴ the first, middle and last terms are in G.P., the common ratio being r^m.

Example 28. Find an infinite G.P. in which each term is 10 times the sum of all the terms that follow it.

Let a be any term, and r the common ratio. Then, since the next term after a is ar, the sum of the terms that follow a is $\frac{ar}{1-r}$.

$$a = 10 . \frac{ar}{1-r}.$$

$$\therefore\ 1 - r = 10r,$$

$$\therefore\ r = \tfrac{1}{11}.$$

∴ the series beginning with a is $a, \frac{1}{11}a, \frac{1}{121}a, \ldots$.

Example 29. Sum to n terms the series whose nth term is $n - 1 + \frac{1}{2^{n-1}}$.

Let s be the sum of n terms,

then $s = 1 + 2 + 3 + \ldots + n - (1 + 1 + 1 + \ldots$ to n terms$)$

$$+ \frac{1}{1} + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}}$$

$$= \frac{1}{2} n(n+1) - n + \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}}$$

$$= \frac{1}{2} n(n-1) + 2 - \frac{1}{2^{n-1}}.$$

Example 30. Show that the product of any odd number (n) of consecutive terms of a G.P. is equal to the nth term power of the middle term.

Let a be the middle term and r the common ratio, and let $n = 2m + 1$, where m is zero or a positive integer, so that the middle term is the $(m+1)$th, and therefore the first term is $\frac{a}{r^m}$ and the last ar^m; then, the product of the series of terms

$$= \frac{a}{r^m} \cdot \frac{a}{r^{m-1}} \cdots \frac{a}{r} \cdot a \cdot ar \cdot \ldots ar^{m-1} \cdot ar^m$$

$$= a^{2m+1} = a^n.$$

Exercises 202.

1. Find the sum to n terms of the series whose rth term is 2^{r+2}.

2. Sum to 12 terms

$$(200 + 2) + (205 - 4) + (210 + 8) + (215 - 16) + (220 + 32) + \ldots.$$

3. Sum $x + a$, $x^3 + 3a$, $x^5 + 5a$, ... to n terms.

4. Sum to n terms

$$(1 + 2) + (2 + 2^2) + (3 + 2^3) + (4 + 2^4) + \ldots.$$

5. Sum to n terms the series whose nth term is $1 + 2n + 3^n$.

6. Find the condition that the sum of an infinite number of terms of the series

$$1 - \frac{b}{a-b} + \left(\frac{b}{a-b}\right)^2 - \ldots$$

may be obtained.

7. What is the number which must be added to each of the numbers 20, 50 and 100, so that the results may be in G.P.?

8. If a, b, c, d be four quantities in G.P., then $ad = bc$.

9. Two terms are taken equidistant from the beginning and end of a geometric series; prove that their product is constant.

10. Show that in every G.P. in which the common ratio is 3, the arithmetic mean between the 2nd and 4th terms is 5 times the 2nd term.

11. If a, b, c be positive quantities in G.P., prove that $a^2 + c^2 > 2b^2$.

12. The difference of two numbers is 32, and the arithmetic mean exceeds the geometric mean by 8; find the numbers.

13. Find an infinite G.P. in which each term is n times the sum of all that follow it.

14. The first two terms of an infinite G.P. are together equal to 1, and every term is twice the sum of all the terms that follow it; find the series.

15. The sum of three numbers in G.P. is 126, and their product is 13824; find the numbers.

16. Divide 183 into three parts in G.P., such that the sum of the first and third is $2\frac{1}{20}$ times the second.

17. The continued product of three numbers in G.P. is 512, and the sum of the products of them in pairs is 224; find the numbers.

18. If a, x, y, b be in A.P., and c^3, x, y, d^3 in G.P., show that $a + b = cd(c + d)$.

19. If a, b, c, d be in G.P., prove that $a + d > b + c$, unless $a + b$ be negative.

20. If a, b, c be in A.P. and x, y, z in G.P., then

$$x^b y^c z^a = x^c y^a z^b.$$

21. How many terms of the series $1 + \frac{5}{4} + (\frac{5}{4})^2 + \ldots$ must be taken so that the sum may exceed 396? ($\log 2 = \cdot 30103$).

22. If r be positive, and

$$(1 + r + r^2 + \ldots \text{ to } \infty)(1 + p + p^2 + \ldots \text{ to } \infty) = (1 + rp + r^2p^2 + \ldots \text{ to } \infty),$$

show that p must be negative and that r must be less than $\frac{1}{3}$, or vice versâ.

23. If $x = 1 + a + a^2 + \ldots$ to ∞, $y = 1 + b + b^2 + \ldots$ to ∞, prove that $1 + ab + a^2b^2 + \ldots \text{ to } \infty = \dfrac{xy}{x + y - 1}$.

24. In any G.P. of an odd number of terms, prove that the sum of the extremes is numerically greater than twice the middle term.

25. The sum of an A.P. of 11 terms is equal to the square of the 6th term, and the 4th, 7th and 11th terms are in G.P.; find the series.

26. If A be the arithmetic mean, and G the geometric mean between x and y, find the ratio of x to y, if $A : G = 13 : 5$.

27. Find two numbers whose difference is 12, and of which the arithmetic mean bears to the geometric mean the ratio $5 : 4$.

28. Between each two consecutive terms of a series consisting of the squares of numbers in A.P. a geometric mean is inserted; show that each term in the original series is an arithmetic mean between the inserted terms adjacent to it.

29. The sum of the first and fourth terms of a G.P. is 13, and of the first four terms 25; find the series.

30. Find the least number of terms for which the sum of $1 + 1\frac{1}{4} + 1\frac{9}{16} + 1\frac{61}{64} + \ldots$ exceeds 3996 ($\log 2 = \cdot 30103$).

31. If S_n be the sum of n terms of the series $a, ar, ar^2, \ldots$ find the value of $S_1 + S_2 + S_3 + \ldots + S_n$.

32. If P be the product of n terms in G.P. of which the first is a and the last l, show that $P^2 = (al)^n$.

33. Find m and n in terms of a and b, so that $\dfrac{ma + nb}{m + n}$ may be the arithmetic mean between m and n and the geometric mean between a and b.

34. If s be the sum of an odd number of terms of a series in G.P., and s' the sum of the series in which the signs of alternate terms are changed; prove that the sum of the squares of the terms will be equal to $\pm ss'$, the upper or lower sign being taken according as the signs of the even or odd terms are changed.

35. If l, m, n, r be in G.P., prove that

$$(l + m + n + r)^2 = (l + m)^2 + (n + r)^2 + 2(m + n)^2.$$

*CHAPTER XXIX.

SERIES, CONT.: HARMONICAL PROGRESSION AND MISCELLANEOUS SERIES.

3. Harmonical Progression.

322. Def. 54. A *harmonical progression* is a series in which every three consecutive terms, a, b, c are such that

$$a : c = a - b : b - c.$$

A harmonical progression is generally denoted, for brevity, by H.P.

323. *The reciprocals of quantities in* H.P. *are in* A.P.

Let a, b, c be three quantities in H.P.,

then $$a : c = a - b : b - c,$$

$$\therefore\ a(b - c) = c(a - b).$$

Dividing both sides by abc, we get

$$\frac{1}{c} - \frac{1}{b} = \frac{1}{b} - \frac{1}{a},$$

i.e. $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.

324. No formula can be given for the sum of any number of terms in H.P. If, however, the common difference of the corresponding A.P. be very small compared with the first term of the A.P., it will be seen (art. 391, Example 10) that an expression for the sum can be determined.

325. *Example* 1. The numbers 1, $\frac{4}{5}$, $\frac{2}{3}$ are in H.P., find the 4th and 5th terms of the H.P.

The numbers 1, $\frac{5}{4}$, $\frac{3}{2}$ are in A.P.

$\therefore$ the common difference of the corresponding A.P. is $\frac{1}{4}$,

$\therefore$ the 4th and 5th terms of the A.P. are $\frac{7}{4}$ and $\frac{8}{4}$,

$\therefore$ the 4th and 5th terms of the H.P. are $\frac{4}{7}$ and $\frac{1}{2}$.

Example 2. The first two terms of an H.P. are a and b, find the nth term.

The common difference of the corresponding A.P. is $\frac{1}{b} - \frac{1}{a}$ or $\frac{a-b}{ab}$.

$\therefore$ the nth term of the A.P. is

$$\frac{1}{a} + (n-1)\frac{a-b}{ab} = \frac{b+(n-1)(a-b)}{ab}$$

$$= \frac{(n-1)a-(n-2)b}{ab}.$$

$\therefore$ the nth term of the H.P. is

$$\frac{ab}{(n-1)a-(n-2)b}.$$

Exercises 203.

1. Find the third term of an H.P. of which the first two are 2 and $1\frac{1}{3}$.

2. Find the 5th and 7th terms of the H.P. of which $\frac{1}{4}$ and $\frac{1}{8}$ are the first and third.

3. Find the 5th term of the H.P. 2, 3, ...

4. Continue the H.P. 15, 20, 30, for two terms.

5. Continue the H.P. $2\frac{2}{3}$, $1\frac{7}{9}$, $1\frac{1}{3}$, for two terms on both sides.

6. Which term of the H.P. $\frac{1}{8}, \frac{1}{6}, \frac{1}{4}, \ldots$ is $-\frac{1}{4}$?

7. Which term of the H.P. $2, 1\frac{1}{5}, \frac{6}{7}, \ldots$ is $\frac{2}{5}$?

8. The 1st term of an H.P. is 1 and the 6th term is $\frac{1}{11}$; find the 3rd term.

9. The 4th term of an H.P. is 1 and the 10th term is $\frac{1}{2}$; find the first three terms.

10. The 3rd term of an H.P. is 2 and the 8th term is $\frac{2}{3}$; find the 10th term.

11. The 7th and 8th terms of an H.P. are $\frac{1}{2}$ and $\frac{4}{9}$; find the sum of the first three terms.

12. Find the H.P. of which $17\frac{1}{2}$ is the first and $1\frac{9}{26}$ the 9th term.

326. DEF. 55. If three terms be in H.P., the middle term is called the *harmonic mean* between the first and third.

DEF. 56. If any number of terms be in H.P., the intermediate terms are called the *harmonic means* between the first and last.

327. *If* H *be the harmonic mean between two quantities* x *and* y, *then* $H = \dfrac{2xy}{x+y}$.

Since x, H, y are in H.P.

$\therefore \dfrac{1}{x}, \dfrac{1}{H}, \dfrac{1}{y}$ are in A.P.

$$\therefore \frac{1}{H} = \frac{1}{2}\left(\frac{1}{x} + \frac{1}{y}\right) = \frac{y+x}{2xy},$$

$$\therefore H = \frac{2xy}{x+y}.$$

328. *If* A, G *and* H *be the arithmetic, geometric and harmonic means between two quantities, then* (i) G *is the geometric mean of* A *and* H, *and* (ii) A, G *and* H *are in descending order of magnitude.*

If x and y be the two quantities,

then $$\text{A} = \frac{x+y}{2}, \quad \text{G} = \sqrt{xy}, \quad \text{H} = \frac{2xy}{x+y}.$$

$$\text{(i)} \quad \text{AH} = \frac{x+y}{2} \cdot \frac{2xy}{x+y} = xy = \text{G}^2.$$

$\therefore$ A, G, H are in G.P.

$$\text{(ii)} \quad \text{A} - \text{G} = \frac{x+y}{2} - \sqrt{xy}$$
$$= \tfrac{1}{2}(x + y - 2\sqrt{xy})$$
$$= \tfrac{1}{2}(\sqrt{x} - \sqrt{y})^2, \text{ which is positive.}$$
$$\therefore \text{A} > \text{G}.$$

Also $$\text{AH} = \text{G}^2,$$
$$\therefore \text{H} < \text{G},$$

$\therefore$ A, G, H are in descending order of magnitude.

329. *Example* 3. Insert 3 harmonic means between $\frac{5}{6}$ and $1\frac{1}{2}$.

Let d be the common difference of the corresponding A.P., of which the first term is $1\frac{1}{5}$ and the 5th is $\frac{2}{3}$,

then $$\tfrac{2}{3} = 1\tfrac{1}{5} + 4d,$$
$$\therefore 4d = \tfrac{2}{3} - \tfrac{6}{5} = -\tfrac{8}{15},$$
$$\therefore d = -\tfrac{2}{15}.$$

$\therefore$ the arithmetic means are $\frac{16}{15}, \frac{14}{15}, \frac{12}{15}$,

$\therefore$ the harmonic means are $\frac{15}{16}, \frac{15}{14}, \frac{5}{4}$.

Example 4. If G be the geometric mean between two quantities, x and y, show that the ratio of the arithmetic and harmonic means of x and G is equal to the ratio of the arithmetic and harmonic means of G and y.

Let a, h be the arithmetic and harmonic means of x and G, a', h' those of G and y, then

$$\text{G}^2 = xy,$$

also $$\frac{a}{h} = \frac{x+G}{2} \div \frac{2xG}{x+G} = \frac{(x+G)^2}{4xG} = \frac{x^2+2xG+G^2}{4xG}$$

$$= \frac{x^2+2xG+xy}{4xG} = \frac{x+2G+y}{4G}.$$

Similarly, $$\frac{a'}{h'} = \frac{y+2G+x}{4G}.$$

$$\therefore \; a:h = a':h'.$$

Exercises 204.

Find the harmonic means of:

1. 4, 6. 2. 10, 20. 3. 1, 9.

4. 5, $2\frac{1}{2}$. 5. 14, 10.

6. Insert two harmonic means between $\frac{1}{4}$ and 1.

7. Insert two harmonic means between 2 and 3.

8. Insert three harmonic means between 6 and 11.

9. Insert five harmonic means between 2 and $\frac{2}{13}$.

10. Find the sum of the first and last of five harmonic means inserted between $2\frac{1}{2}$ and 1.

11. Find the first and last of n harmonic means inserted between x and y.

12. The harmonic mean between two numbers, one of which is double the other, is $2\frac{2}{3}$; find the numbers.

13. The arithmetic mean between two numbers is 4 and the harmonic mean is 3; find the numbers.

14. The geometric mean between two numbers is 4 and the harmonic mean is $3\frac{1}{5}$; find the numbers.

15. Find two numbers whose arithmetic mean exceeds their geometric mean by 2, and whose harmonic mean is one-fifth of the larger number.

330. *Example* 5. If the mth term of a harmonic series be n, and the nth term be m, prove that the rth term is $\frac{mn}{r}$.

Let a, d be the first term and common difference of the corresponding A.P.; then

$$\frac{1}{n} = a + (m-1)\,d \quad \text{and} \quad \frac{1}{m} = a + (n-1)\,d.$$

$$\therefore \ (m-n)\,d = \frac{1}{n} - \frac{1}{m} = \frac{m-n}{mn},$$

$$\therefore \ d = \frac{1}{mn},$$

$$\therefore \ a = \frac{1}{n} - \frac{m-1}{mn} = \frac{1}{mn}.$$

$\therefore$ rth term of the A.P. is

$$\frac{1}{mn} + \frac{r-1}{mn} = \frac{r}{mn}.$$

$\therefore$ rth term of the H.P. is $\frac{mn}{r}$.

Example 6. To each of three terms of a geometric series, the second of the three is added; show that the sums are in H.P.

Let a, b, c be any three terms in G.P. so that $b^2 = ac$; then $a+b$, $2b$, $b+c$ are the required sums.

Now,
$$\frac{2\,(a+b)\,(b+c)}{(a+b)+(b+c)} = 2\,.\,\frac{ab+b^2+ac+bc}{a+2b+c}$$

$$= 2\,.\,\frac{ab+2b^2+bc}{a+2b+c}$$

$$= 2b.$$

$\therefore$ $2b$ is the harmonic mean of $a+b$ and $b+c$.

Exercises 205.

1. Continue for two terms each of the series: (i) $\frac{1}{2}, -\frac{3}{4}, -2$; (ii) $\frac{1}{2}, -\frac{3}{4}, \frac{9}{8}$; (iii) $\frac{1}{2}, -\frac{3}{4}, -\frac{3}{14}$.

2. Show that the numbers 1, -2, 4 may be arranged in A.P., G.P. and H.P.

3. Find the equation whose roots are the arithmetic and harmonic means of the roots of the equation $x^2 - px + q = 0$.

4. The arithmetic mean between two numbers is $6\frac{1}{2}$ and the geometric mean is 6; find the harmonic mean.

5. The arithmetic mean between two numbers is 17, and the harmonic mean is $13\frac{4}{17}$; find the geometric mean.

6. If the arithmetic mean between two numbers be 1, show that the harmonic mean is the square of the geometric mean.

7. What number must be taken from each of the numbers 13, 15, 19, so that the remainders may be in H.P.?

8. If $x^a = y^b = z^c$, and x, y, z be in G.P., then will a, b, c be in H.P.

9. If a be the arithmetic mean between b and c, b the geometric mean between a and c, prove that c is the harmonic mean between a and b.

10. If the mth term of an H.P. be n, and the nth term be m, prove that the $(m+n)$th term is $\frac{mn}{m+n}$.

11. If the arithmetic mean of two numbers exceed the geometric mean by $2\frac{1}{2}$ and the geometric mean exceed the harmonic mean by 2, find the numbers.

12. If x, y, z be in H.P., find in its simplest form the geometric mean between $x - \frac{y}{2}$ and $z - \frac{y}{2}$.

13. If a, b, c, d be in H.P., show that $a + d > b + c$.

14. If P, Q, R be the pth, qth, rth terms of an H.P., then
$$QR(q-r)+RP(r-p)+PQ(p-q)=0.$$

15. If a, b, c be in H.P., then $2a-b$, b, $2c-b$ are in G.P.

16. If a, b, c be in H.P., then a, $a-c$, $a-b$ and c, $c-a$, $c-b$, are also in H.P.

17. If a, b, c be in H.P., then $\frac{a}{b+c}$, $\frac{b}{c+a}$, $\frac{c}{a+b}$ are also in H.P.

18. If A be the rth term when $2n-1$ arithmetic means are inserted between a and b, and H the rth term when the same number of harmonic means is inserted, prove that $\frac{A}{a}+\frac{b}{H}$ is independent of the values of n and r.

19. If a, b, c, d be in A.P., a, e, f, d in G.P., and a, g, h, d in H.P., prove that $ad=ef=bh=cg$.

20. If the arithmetic, geometric and harmonic means between two quantities be A, G, H and if X be the arithmetic mean between G and H, Y the geometric mean between H and A, and Z the harmonic mean between A and G, prove that X, Y, Z are in G.P.

21. Between two quantities A and B a harmonic mean H is inserted; between A and H, and between H and B geometric means G_1 and G_2 are inserted, and it is found that G_1, H and G_2 are in A.P.; find the ratio of A to B.

22. If $x-a$, $y-a$, $z-a$ be in G.P., prove that twice $y-a$ is the harmonic mean between $y-x$ and $y-z$.

23. Three numbers in H.P. are such that the product of the first two is equal to the third, and the sum of the squares of the first two is less than three times the third by 5; find the numbers.

24. If $2p$ and $3q$ be the pth and qth terms of an H.P., prove that the $(p+q)$th term is $6(p-q)$.

25. If $\frac{1}{a}+\frac{1}{c}=\frac{1}{b-a}+\frac{1}{b-c}$, then either $b=a+c$ or a, b, c are in H.P.

4. Miscellaneous Series.

1. **331.** *To find the sum of the squares of the first n integers.*

Let $S = 1^2 + 2^2 + 3^2 + \dots + n^2$.

Since $$(n+1)^3 = n^3 + 3n^2 + 3n + 1,$$

we have $$(n+1)^3 - n^3 = 3n^2 + 3n + 1.$$

Also, putting in succession $n-1$ for n, we get

$$n^3 - (n-1)^3 = 3(n-1)^2 + 3(n-1) + 1,$$

$$(n-1)^3 - (n-2)^3 = 3(n-2)^2 + 3(n-2) + 1,$$

$$\dots\dots\dots\dots\dots\dots$$

$$3^3 - 2^3 = 3 \cdot 2^2 + 3 \cdot 2^2 + 1,$$

$$2^3 - 1^3 = 3 \cdot 1^2 + 3 \cdot 1 + 1.$$

∴ by addition,

$$(n+1)^3 - 1^3 = 3S + 3(1 + 2 + 3 + \dots + n) + n.$$

$$\therefore\ n^3 + 3n^2 + 3n = 3S + 3 \cdot \frac{n(n+1)}{2} + n,$$

$$\therefore\ 6S = 2n^3 + 6n^2 + 6n - 3n^2 - 3n - 2n$$

$$= 2n^3 + 3n^2 + n = n(2n^2 + 3n + 1).$$

$$\therefore\ S = \tfrac{1}{6} n(n+1)(2n+1).$$

2. **332.** *To find the sum of the cubes of the first n integers.*

Let $S = 1^3 + 2^3 + 3^3 + \dots + n^3$.

Since $$(n+1)^4 = n^4 + 4n^3 + 6n^2 + 4n + 1,$$

we have $$(n+1)^4 - n^4 = 4n^3 + 6n^2 + 4n + 1.$$

Also, putting in succession $n-1$ for n, we get

$$n^4 - (n-1)^4 = 4(n-1)^3 + 6(n-1)^2 + 4(n-1) + 1,$$

$$(n-1)^4 - (n-2)^4 = 4(n-2)^3 + 6(n-2)^2 + 4(n-2) + 1,$$

$$\dots\dots\dots\dots\dots\dots$$

$$3^4 - 2^4 = 4 \cdot 2^3 + 6 \cdot 2^2 + 4 \cdot 2 + 1,$$

$$2^4 - 1^4 = 4 \cdot 1^3 + 6 \cdot 1^2 + 4 \cdot 1 + 1;$$

by addition,

$$(n+1)^4 - 1^4 = 4S + n(n+1)(2n+1) + 2n(n+1) + n,$$

$$\therefore \quad 4S = n^4 + 4n^3 + 6n^2 + 4n - 2n^3 - 3n^2 - n - 2n^2 - 2n - n$$

$$= n^4 + 2n^3 + n^2$$

$$= n^2(n+1)^2,$$

$$\therefore \quad S = \tfrac{1}{4}n^2(n+1)^2 = \{\tfrac{1}{2}n(n+1)\}^2.$$

Hence, the sum of the cubes of the first n integers is equal to the square of the sum of the first n integers.

333. *Example* 7. Find the sum of the squares of the first n odd numbers.

Let $S = 1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2$.

Then
$$S = 1^2 + 2^2 + 3^2 + 4^2 + \ldots + (2n-1)^2 + (2n)^2 - \{2^2 + 4^2 + 6^2 + \ldots + (2n)^2\}$$

$$= 1^2 + 2^2 + 3^2 + 4^2 + \ldots + (2n-1)^2 + (2n)^2 - 4(1^2 + 2^2 + 3^2 + \ldots + n^2)$$

$$= \tfrac{1}{6} \cdot 2n(2n+1)(4n+1) - \tfrac{4}{6}n(n+1)(2n+1)$$

$$= \tfrac{1}{3}n(2n+1)(4n+1-2\cdot\overline{n+1})$$

$$= \tfrac{1}{3}n(2n+1)(2n-1)$$

$$= \tfrac{1}{3}n(4n^2-1).$$

334. *To find the sum of n terms of the series*

$$a + (a+b)r + (a+2b)r^2 + (a+3b)r^3 + \ldots.$$

Let $S = a + (a+b)r + (a+2b)r^2 + \ldots + (a + \overline{n-1}\cdot b)r^{n-1}$,

then
$$Sr = \qquad ar + (a+b)r^2 + \ldots + (a+\overline{n-2}\cdot b)r^{n-1} + (a+\overline{n-1}\cdot b)r^n.$$

$$\therefore \quad S(1-r) = a + br + br^2 + \ldots + br^{n-1} - (a+\overline{n-1}\cdot b)r^n$$

$$= a + \frac{br(1-r^{n-1})}{1-r} - (a+\overline{n-1}\cdot b)r^n$$

$$= \frac{a(1-r) + br(1-r^{n-1}) - (a+\overline{n-1}\cdot b)r^n + (a+\overline{n-1}\cdot b)r^{n+1}}{1-r}.$$

$$\therefore \quad S = \frac{a - (a-b)r - (a+nb)r^n + (a+\overline{n-1}\cdot b)r^{n+1}}{(1-r)^2}.$$

If r be less than 1, the sum of an infinite number of terms is therefore

$$\frac{a-(a-b)r}{(1-r)^2}.$$

335. *To find the sum of n terms of the series*

$$1.2.3+2.3.4+3.4.5+\ldots.$$

Let

$$S=1.2.3+2.3.4+3.4.5+\ldots+n(n+1)(n+2)$$

and

$$S'=1.2.3.4+2.3.4.5+\ldots+n(n+1)(n+2)(n+3),$$

also

$$S'=\qquad 1.2.3.4+\ldots+(n-1)n(n+1)(n+2)+n(n+1)(n+2)(n+3).$$

$\therefore$ by subtraction,

$$\begin{aligned}0&=1.2.3.4+2.3.4.(5-1)+3.4.5.(6-2)+\ldots\\&\quad+n(n+1)(n+2)(\overline{n+3}-\overline{n-1})\\&\quad-n(n+1)(n+2)(n+3)\\&=4(1.2.3+2.3.4+3.4.5+\ldots+n.\overline{n+1}.\overline{n+2})\\&\quad-n(n+1)(n+2)(n+3)\\&=4S-n(n+1)(n+2)(n+3).\end{aligned}$$

$$\therefore\ S=\tfrac{1}{4}n(n+1)(n+2)(n+3).$$

336. *To find the sum of n terms of the series*

$$\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\frac{1}{4.5.6}+\ldots.$$

Let

$$S=\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\ldots+\frac{1}{n(n+1)(n+2)},$$

and

$$S'=\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots+\frac{1}{n(n+1)}+\frac{1}{(n+1)(n+2)},$$

also

$$S'=\qquad\frac{1}{1.2}+\frac{1}{2.3}+\ldots\qquad+\frac{1}{n(n+1)}+\frac{1}{(n+1)(n+2)}.$$

∴ by subtraction,

$$0 = \frac{1}{2} - \left(\frac{1}{1.2} - \frac{1}{2.3}\right) - \left(\frac{1}{2.3} - \frac{1}{3.4}\right) - \dots$$

$$- \left(\frac{1}{n.\overline{n+1}} - \frac{1}{\overline{n+1}.\overline{n+2}}\right) - \frac{1}{(n+1)(n+2)}$$

$$= \frac{1}{2} - \frac{3-1}{1.2.3} - \frac{4-2}{2.3.4} - \dots$$

$$- \frac{\overline{n+2}-n}{n(n+1)(n+2)} - \frac{1}{(n+1)(n+2)}$$

$$= \frac{1}{2} - 2S - \frac{1}{(n+1)(n+2)}.$$

$$\therefore\ S = \frac{1}{2}\left\{\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right\}.$$

Hence, the sum of an infinite number of terms is $\frac{1}{4}$.

337. *Example* 8. Find the sum of n terms

$$1.2 + 2.3 + 3.4 + \dots + n(n+1).$$

Let $S = 1.2 + 2.3 + 3.4 + \dots + n(n+1).$

Now, since $n(n+1) = n^2 + n$, it follows that

$$1.2 = 1^2 + 1,\quad 2.3 = 2^2 + 2,\quad 3.4 = 3^2 + 3, \text{ etc.}$$

$$\begin{aligned}\therefore\ S &= (1^2 + 1) + (2^2 + 2) + (3^2 + 3) + \dots + (n^2 + n)\\ &= (1^2 + 2^2 + 3^2 + \dots + n^2) + (1 + 2 + 3 + \dots + n)\\ &= \tfrac{1}{6}n(n+1)(2n+1) + \tfrac{1}{2}n(n+1)\\ &= \tfrac{1}{6}n(n+1)(2n+1+3)\\ &= \tfrac{1}{3}n(n+1)(n+2).\end{aligned}$$

Example 9. Find the sum of

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots + \frac{1}{n(n+1)}.$$

Let $S = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots + \frac{1}{n(n+1)}.$

Now, since $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$, it follows that

$$\frac{1}{1.2} = \frac{1}{1} - \frac{1}{2}, \quad \frac{1}{2.3} = \frac{1}{2} - \frac{1}{3}, \quad \frac{1}{3.4} = \frac{1}{3} - \frac{1}{4}, \text{ etc.}$$

$$\therefore\ S = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}.$$

Example 10. Find the series whose sum to n terms is n^3.

The rth term = (sum of r terms) − (sum of $\overline{r-1}$ terms)

$$= r^3 - (r-1)^3 = 3r^2 - 3r + 1.$$

$$\therefore\ \text{1st term} = 3 - 3 + 1 = 1,$$

$$\text{2nd term} = 3.4 - 3.2 + 1 = 7,$$

$$\text{3rd term} = 3.9 - 3.3 + 1 = 19.$$

$\therefore$ the series is 1, 7, 19, ..., $(3r^2 - 3r + 1)$,....

Exercises 206.

Find the sum of:

1. $2^2 + 4^2 + 6^2 + \ldots + (2n)^2$.

2. $2^3 + 4^3 + 6^3 + \ldots + (2n)^3$.

3. $1^3 + 3^3 + 5^3 + \ldots + (2n-1)^3$.

4. 8 terms of the series of which the nth term is $n(2-3n)$.

5. $1 + 2x + 3x^2 + \ldots + nx^{n-1}$, proving that when $x = 1 + \frac{1}{n}$, the sum of the series is n^2.

6. $1 + 2.\frac{1}{2} + 3.\frac{1}{2^2} + 4.\frac{1}{2^3} + \ldots + \frac{n+1}{2^n} + \ldots$ to ∞.

7. $1.3 + 3.5 + 5.7 + \ldots$ to n terms.

8. $1 + 3 + 6 + \ldots + \frac{1}{2}n(n+1)$.

9. $\frac{1}{1.2.3.4} + \frac{1}{2.3.4.5} + \frac{1}{3.4.5.6} + \ldots$ to n terms.

10. $\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+\dots$ to ∞.

11. $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\dots$ to n terms.

12. $\frac{1}{1.4}+\frac{1}{2.5}+\frac{1}{3.6}+\dots$ to ∞.

13. $\frac{1}{1.3}+\frac{1}{3.5}+\dots+\frac{1}{(2n+1)(2n+3)}$.

14. $\frac{1}{1.3}+\frac{1}{2.4}+\frac{1}{3.5}+\dots$ to n terms.

15. $1+3+7+\dots+(n^2-n+1)$.

16. $1.2+3\ 4+5.6+\dots$ to n terms.

17. $1.3+2.4+3.5+4.6+\dots$ to n terms.

18. $1.5+2.6+3.7+\dots$ to n terms.

19. $2.1^2+4.3^2+6.5^2+\dots$ to n terms.

20. $1^2.2+2^2.3+\dots+n^2(n+1)$.

21. $1.2^2+2.3^2+3.4^2+\dots$ to n terms.

22. $2.4+5.9+8.14+\dots$ to n terms.

23. $1.2.3+2.3.5+3.4.7+4.5.9+\dots$ to n terms.

24. The sum of n terms of a series is $n.2^n$, find the rth term.

25. Sum to n terms the series whose rth term is $\frac{2r-1}{2^r}$.

26. Sum to n terms the series whose rth term is $r(r+1)(r+3)$.

27. Prove that the sum of n terms of the two series whose rth terms are $4r^3-3r^2-1$ and $8r^3+2r-1$, respectively, are to one another as $n-1$ to $2n$ (n being >1).

28. Sum to n terms

$$n+2(n-1)+3(n-2)+4(n-3)+\dots+r(n-r+1)+\dots.$$

29. Find the sum of

$$1 \cdot 2n + 2 \cdot 3(n-1) + 3 \cdot 4(n-2) + \dots + n(n+1).$$

30. Find the sum of the 4th powers of the first n integers.

31. Sum to infinity $1 + 3x + 5x^2 + 7x^3 + \dots$ (x being < 1).

32. Find the value of $s_1 + s_2 + \dots + s_n$, where s_r is the sum of r terms of an A.P., whose first term is a and common difference b.

33. If $a_0, a_1, a_2, a_3, \dots$ be in A.P., show that

$$a_0 + a_1 x + \dots + a_n x^n$$

may always be written in the form

$$\frac{A + Bx + Px^{n+1} + Qx^{n+2}}{1 - 2x + x^2},$$

and find the values of A, B, P, Q when a_0, a_1 and n are given.

34. Sum the infinite series

$$1 + r + (1 + a)r^2 + (1 + a + a^2)r^3 + \dots.$$

35. If $y = x + 4x^2 + 7x^3 + 10x^4 + \dots$ to ∞, ($x < 1$), find x in terms of y.

36. Sum to n terms $1^2 + 2^2x + 3^2x^2 + 4^2x^3 + \dots$

CHAPTER XXX.

MATHEMATICAL INDUCTION.

338. Formulae involving a quantity which is known to be a positive integer may often be proved simply by the method of *mathematical induction.* The method, which is illustrated in the following examples, consists in showing: (i) that the formula holds true for the smallest possible value of the integer; and (ii) that, assuming it to hold true for some value n of the integer, it also holds true for the next higher value, $n+1$, of the integer. Thus, knowing that the formula is true when n is, say, 1, it is also true when n is $1+1$ or 2, therefore when n is $2+1$ or 3, therefore when n is 4, and therefore when n is any integer. For instance, in the second of the following examples, it is shown that, if $x^n - a^n$ be divisible by $x-a$, then $x^{n+1} - a^{n+1}$ is also divisible by $x-a$. But $x^n - a^n$ is obviously divisible by $x-a$ when $n=1$, therefore it is divisible when $n=2$, therefore when $n=3$, and therefore when n has any integral value.

In some cases, a formula is only true when n is either odd or even, and in such cases the next higher value of n is $n+2$. Thus, if $x^n + a^n$ be divisible by $x+a$, it can be shown that $x^{n+2} + a^{n+2}$ is divisible by $x+a$, but that $x^{n+1} + a^{n+1}$ is not so divisible. Now, $x^n + a^n$ is obviously divisible by $x+a$ when $n=1$, therefore it is divisible when $n=3$, therefore when $n=5$, and so on; thus $x^n + a^n$ is divisible by $x+a$ when n is an odd number.

339. *Example* 1. Prove, by induction, that the sum of n terms in G.P. is $\dfrac{a(r^n-1)}{r-1}$, where a is the first term and r the common ratio.

Assume that

$$a + ar + ar^2 + \dots + ar^{n-1} = \frac{a(r^n-1)}{r-1},$$

then

$$\begin{aligned} a + ar + ar^2 + \dots + ar^{n-1} + ar^n &= \frac{a(r^n-1)}{r-1} + ar^n \\ &= \frac{a(r^n - 1 + r^{n+1} - r^n)}{r-1} \\ &= \frac{a(r^{n+1}-1)}{r-1}. \end{aligned}$$

Thus, if the formula hold true for n terms, it also holds true for $n+1$ terms.

But it is obviously true when $n=1$, therefore when $n=2$, therefore when $n=3$, and so on; and therefore universally.

Example 2. Prove, by induction, that $x^n - a^n$ is divisible by $x-a$, when n is any integer.

Assume that $x^n - a^n$ is divisible by $x-a$,

then
$$\begin{aligned} x^{n+1} - a^{n+1} &= x(x^n - a^n) + xa^n - a^{n+1} \\ &= x(x^n - a^n) + a^n(x-a). \end{aligned}$$

$\therefore$ $x^{n+1} - a^{n+1}$ is also divisible by $x-a$.

But $x^n - a^n$ is divisible by $x-a$ when $n=1$, therefore when $n=2$, therefore when $n=3$, and so on; and therefore $x^n - a^n$ is divisible by $x-a$, when n is any integer.

Example 3. Prove, by induction, that $9^n - 8n + 1$ is divisible by 64.

Assume that $9^n - 8n - 1 = \text{M}(64)$, where M (64) denotes some multiple of 64,

then $9^{n+1} - 8(n+1) - 1 = 9(9^n - 8n - 1) + 72n + 9 - 8(n+1) - 1$
$$= 9 \,.\, \text{M}(64) + 64n$$
$$= \text{M}(64).$$

Thus, if the result hold true for n, it also holds true for $n + 1$.

But it is obviously true when $n = 2$, therefore when $n = 3$, therefore when $n = 4$, and therefore universally.

Exercises 207.

Prove, by induction, that

1. The sum of the first n integers is $\frac{1}{2}n(n+1)$.

2. The sum of the first n odd numbers is n^2.

3. The sum of n terms in A. P. is $\frac{n}{2}\{2a + (n-1)d\}$.

4. The sum of the cubes of the first n integers is equal to the square of the sum of the integers.

5. $1 \,.\, 2 \,.\, 3 + 2 \,.\, 3 \,.\, 4 + 3 \,.\, 4 \,.\, 5 + \ldots$ to n terms
$$= \tfrac{1}{4}n(n+1)(n+2)(n+3).$$

6. $x^n + a^n$ is divisible by $x + a$, if n be odd.

7. $x^n - a^n$ is divisible by $x + a$, if n be even.

8. $a^n(b-c) + b^n(c-a) + c^n(a-b)$ is divisible by $(b-c)(c-a)(a-b)$.

9. $(1^5 + 1^7) + (2^5 + 2^7) + (3^5 + 3^7) + \ldots$ to n terms $= \frac{1}{8}n^4(n+1)^4$.

10. $(a_1 + a_2 + \ldots + a_n)^2 = a_1^2 + a_2^2 + \ldots + a_n^2$
$$+ 2a_1a_2 + 2a_1a_3 + \ldots + 2a_1a_n + 2a_2a_3 + \ldots + 2a_{n-1}a_n.$$

11. $n(n+1)(2n+1)$ is divisible by 6.

12. $3^{2n+2} - 8n - 9$ is divisible by 64 when n is a positive integer.

13. $(3n+1) \,.\, 7^n - 1$ is divisible by 9 when n is a positive integer.

CHAPTER XXXI.

PERMUTATIONS AND COMBINATIONS.

340. DEF. 57. The different orders in which a group of things can be arranged, either all together or a certain number together, are called their *permutations*.

Thus, the different permutations of the letters a, b, c one at a time are a, b, c; two at a time, ab, ac, ba, bc, ca, cb; all together, abc, acb, bac, bca, cab, cba.

The number of permutations of n things taken r at a time is denoted by ${}_nP_r$. Thus ${}_3P_1 = 3$, ${}_3P_2 = 6$, ${}_3P_3 = 6$.

341. *Example* 1. In how many ways can a chairman and a secretary be elected from a committee of five, all the members being supposed equally eligible for the posts?

Let the different members be represented by the letters a, b, c, d, e.

The chairman may be chosen in 5 different ways.

If a be elected chairman, there are 4 men, b, c, d, e, left from whom the secretary may be chosen, i.e. the secretary may be chosen in 4 ways. Similarly, if b be chosen chairman, the secretary may be chosen in 4 ways. Thus, for each choice of chairman, the secretary may be elected in 4 ways.

$\therefore$ the total number of ways of choosing both chairman and secretary is 5×4 or 20.

Example 2. In how many different ways can a peal of 5 bells be rung?

The first bell may be rung in 5 ways.

With each way of ringing the first bell, the second may be rung in 4 ways.

$\therefore$ number of ways of ringing the first two bells is $5 \cdot 4$.

With each way of ringing the first two bells, the third may be rung in 3 ways.

$\therefore$ number of ways of ringing the first three bells is $5 \cdot 4 \cdot 3$.

With each way of ringing the first three bells, the fourth may be rung in 2 ways.

$\therefore$ number of ways of ringing the first four bells is $5 \cdot 4 \cdot 3 \cdot 2$.

With each way of ringing the first four bells, the fifth or last may be rung in only 1 way.

$\therefore$ number of ways of ringing the 5 bells is $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$.

Exercises 208.

1. In how many ways can a cricket eleven choose a captain and deputy captain?

2. Find the number of permutations of the letters of the word *advise* taken (i) 3 at a time, (ii) all together.

3. How many changes can be rung with 5 out of 8 bells, and how many with the whole peal?

4. In how many ways can the letters a, b, c, d, e be arranged two at a time; and in how many of these will a stand first?

5. In how many ways can the letters a, b, c, d, e, f be arranged 3 at a time; and in how many of these will (i) a, and (ii) ab, stand first?

6. Find the number of permutations taken all together of the letters of the word *chance*.

342. *To find the number of permutations of n different things taken r at a time.*

First method. The first thing may be taken in n ways.

With each way of taking the first thing, the second may be taken in $n-1$ ways;

$\therefore$ the number of ways of taking the first two things is $n(n-1)$.

With each way of taking the first two things, the third may be taken in $n-2$ ways;

$\therefore$ the number of ways of taking the first three things is $n(n-1)(n-2)$.

Now, for each additional thing taken, the number of things that remain is diminished by one. Thus, the second term of the last factor is always one less than the number of things taken; so that, when r things are taken, the second term of the last factor is $r-1$.

$$\therefore \; \mathrm{P} = n(n-1)(n-2)\ldots(n-\overline{r-1})$$
$$= n(n-1)(n-2)\ldots(n-r+1).$$

Second method. Let the n different things be represented by the letters a_1, a_2, a_3, ..., a_n, and let the required number of permutations be ${}_n\mathrm{P}_r$, so that the number of permutations of n things taken $r-1$ at a time is ${}_n\mathrm{P}_{r-1}$.

Take any permutation of $r-1$ letters, say $a_1 a_2 a_3 \ldots a_{r-1}$, and to it add one of the remaining $n-(r-1)$ or $n-r+1$ letters a_r, a_{r+1}, ..., a_n. In this way, we obtain permutations of r letters in each of which $a_1 a_2 \ldots a_{r-1}$ stands first, and the number of such permutations is $n-r+1$. If we do the same to each of the ${}_n\mathrm{P}_{r-1}$ permutations of n things taken $r-1$ at a time, the total number of permutations so formed will be ${}_n\mathrm{P}_{r-1} \times (n-r+1)$. But, as in these different permutations, each letter occupies in its turn every place in the series, the number of permutations so formed must be ${}_n\mathrm{P}_r$.

$$\therefore \; {}_n\mathrm{P}_r = {}_n\mathrm{P}_{r-1} \times (n-r+1).$$

Since this equation is true for all values of r not greater than n, it is true when $r-1$ is substituted for r, i.e.

$$_nP_{r-1} = {_nP_{r-2}} \times (n - \overline{r-2}) \text{ or } {_nP_{r-2}} \times (n - r + 2).$$

Similarly,

$$_nP_{r-2} = {_nP_{r-3}} \times (n - \overline{r-3}) \text{ or } {_nP_{r-3}} \times (n - r + 3),$$

and so on, until finally,

$$_nP_3 = {_nP_2} \times (n-2),$$
$$_nP_2 = {_nP_1} \times (n-1) = n \times (n-1).$$

Hence,

$$_nP_r \cdot {_nP_{r-1}} \cdot {_nP_{r-2}} \cdots {_nP_3} \cdot {_nP_2}$$
$$= {_nP_{r-1}} \cdot {_nP_{r-2}} \cdot {_nP_{r-3}} \cdots {_nP_2} \cdot \times (n-r+1)(n-r+2)\ldots(n-1) \cdot n.$$
$$\therefore\ {_nP_r} = n(n-1)(n-2)\ldots(n-r+1).$$

343. *To find the number of permutations of* n *different things taken all together.*

We have

$$_nP_r = n(n-1)(n-2)\ldots(n-r+1).$$
$$\therefore\ {_nP_n} = n(n-1)(n-2)\ldots(n-n+1)$$
$$= n(n-1)(n-2)\ldots 3 \,.\, 2 \,.\, 1.$$

344. DEF. 58. The continued product of the first n numbers is called *factorial* n, and is denoted by the symbol $\underline{|n}$ or $n!$

Exercises 209.

1. Find $\underline{|3}$, $\underline{|4}$ and $\underline{|5}$.

Prove that

2. $\underline{|8} = 8\,\underline{|7}$ and $\underline{|10} = 90\,\underline{|8}$.

3. $\underline{|8} \div \underline{|5} = 336$.

4. $\underline{|n} = n\,\underline{|n-1}$ and $(n-1) : \underline{|n-1} = \dfrac{1}{\underline{|n-2}}$.

5. $n\,\underline{|n} = \underline{|n+1} - \underline{|n}$.

6. $\dfrac{n^2}{\underline{|n}} = \dfrac{1}{\underline{|n-2}} + \dfrac{1}{\underline{|n-1}}$.

7. If $\underline{|n} = 56\,\underline{|n-2}$, find n.

8. Prove that $\underline{|2n} = 2^n\,\underline{|n} \,.\, 1 \,.\, 3 \,.\, 5 \ldots (2n-1)$.

345. *Example* 4. Find the number of permutations taken all together of the letters of the word *essence*.

Let x be the required number of permutations.

Take any one of these x permutations, say

e s s e n c e,

and in it change the three letters *e* into three new letters different from one another and from all the rest, say a, b, d, so that the permutation becomes

a s s *b* n c *d*.

From this one permutation, the number of permutations that can be formed by interchanging the three letters a, b, d in all possible ways is $\underline{|3}$;

∴ from all the x permutations, the number of permutations that can be formed by interchanging the three letters a, b, d in all possible ways is $x \times \underline{|3}$.

Take any one of these $x \times \underline{|3}$ permutations, say

a s s b n c d,

and in it change the two letters *s* into two new letters different from one another and from all the rest, say f, g, so that the permutation becomes

a *f g* b n c d.

From this one permutation, the number of permutations that can be formed by interchanging the two letters f, g in all possible ways is $\underline{|2}$;

∴ from all the $x \times \underline{|3}$ permutations, the number of permutations that can be formed by interchanging the two letters f, g in all possible ways is $x \times \underline{|3} \times \underline{|2}$.

But the seven letters are now all different, and, as in the different rearrangements each letter occupies in its turn every place in the series, the number of permutations so formed must be $|\underline{7}$.

$$\therefore\ x \times |\underline{3} \times |\underline{2} = |\underline{7}.$$

$$\therefore\ x = \frac{|\underline{7}}{|\underline{3}\,|\underline{2}}.$$

346. *To find the number of permutations of n letters taken all together, there being p letters a, q letters b, r letters c, etc.*

Let x be the required number of permutations.

In any one of these x permutations, say

$$aaa \ldots abbb \ldots bccc \ldots cdef \ldots,$$

let the p letters a be changed into p new letters different from one another and from all the rest, say $a_1, a_2, a_3, \ldots, a_p$, so that the permutation becomes

$$a_1 a_2 a_3 \ldots a_p bbb \ldots bccc \ldots cdef \ldots.$$

From this one permutation, the number of permutations that can be formed by interchanging the p letters $a_1, a_2, a_3, \ldots, a_p$ in all possible ways is $|\underline{p}$;

$\therefore$ from all the x permutations, the number of permutations that can be formed by interchanging the p letters $a_1, a_2, a_3, \ldots, a_p$ in all possible ways is $x \times |\underline{p}$.

In any one of these $x \times |\underline{p}$ permutations, say

$$a_1 a_2 a_3 \ldots a_p bbb \ldots bccc \ldots cdef \ldots,$$

let the q letters b be changed into q new letters different from one another and from all the rest, say $b_1, b_2, b_3, \ldots, b_q$, so that the permutation becomes

$$a_1 a_2 a_3 \ldots a_p b_1 b_2 b_3 \ldots b_q ccc \ldots cdef \ldots.$$

From this one permutation, the number of permutations that can be formed by interchanging the q letters $b_1, b_2, b_3, \ldots, b_q$ in all possible ways is $|\underline{q}$;

$\therefore$ from all the $x \times \underline{|p}$ permutations, the number of permutations that can be formed by interchanging the q letters b_1, b_2, b_3, ..., b_q in all possible ways is $x \times \underline{|p} \times \underline{|q}$.

Similarly, the number of permutations that can be formed by changing the r letters c into r new letters different from one another and from all the rest is $x \times \underline{|p} \times \underline{|q} \times \underline{|r}$; and so on.

But the n letters are now all different and, as in the different rearrangements each letter occupies every place in the series, the number of permutations so formed must be $\underline{|n}$.

$$\therefore\ x \times \underline{|p} \times \underline{|q} \times \underline{|r} \times \ldots = \underline{|n}.$$

$$\therefore\ x = \frac{\underline{|n}}{\underline{|p}\,\underline{|q}\,\underline{|r}\ldots}.$$

347. *Example* 5. How many permutations can be formed of the letters of the word *severe* all together, each of which begins with *s* and ends with *e*?

Excluding these letters, there are 4 others, of which 2 are *e* and the others different.

The number of permutations of these 4 letters is $\underline{|4} \div \underline{|2}$ or 12.

$\therefore$ the number of permutations which begin with *s* and end with *e* is 12.

Example 6. How many different permutations can be formed of the word *severe* all together, in which the letters *v* and *r* do not come together?

Excluding the letters *v* and *r*, there are 4 others, of which 3 are *e*.

Let the letters *v* and *r* be supposed joined together in the order *vr*. Then the number of letters may be regarded as 5, of which 3 are *e* and the others are different. The number of permutations of these 5 letters all together is $\underline{|5} \div \underline{|3}$ or $5 . 4$ or 20.

In any one of these 20 permutations, the letters v and r may be rearranged in 2 ways.

$\therefore$ the total number of ways in which v and r come together is $20 \,.\, 2$ or 40.

But the total number of ways in which the 6 letters may be arranged is $\underline{|6} \div \underline{|3}$ or $6 \,.\, 5 \,.\, 4$ or 120.

$\therefore$ the total number of ways in which v and r do not come together is $120 - 40 = 80$.

Exercises 210.

1. Find the values of ${}_6P_3$, ${}_8P_8$, ${}_{12}P_2$ and ${}_{10}P_4$.

2. Show that ${}_{10}P_4 = {}_8P_4 \times 3$.

3. If ${}_nP_2 = 110$, find n.

4. If ${}_nP_3 = 20n$, find n.

5. If ${}_nP_3 : {}_{n+2}P_3 = 5 : 12$, find n.

Find the number of permutations taken all together of the letters of the words:

6. *Magnitude.*
7. *Manna.*
8. *Humourous.*
9. *Trinitarian.*

10. Find the number of permutations taken all together of the letters of the word *monotone*; how many of these will begin and end with n?

11. Find the number of permutations taken all together of the letters of the word *orchid*; how many of these will begin with o, and how many will begin with o and end with d?

12. Find the number of permutations 4 at a time that can be formed with the letters of the word *portal*; how many of these will begin and end with vowels?

13. How many numbers greater than 1000 can be formed with the digits 1, 2, 3, 4, and how many of these will be greater than 3000?

14. How many numbers greater than 1000000 can be formed with the digits 3, 3, 5, 5, 7, 7, 7?

15. How many numbers greater than 10000 can be formed with the digits 1, 2, 3, 4, 0?

16. In how many different ways can 4 Englishmen and 4 Americans be arranged at a round table so that no two of the same nation sit together?

17. In how many ways can a party of 6 ladies and 6 gentlemen be seated at a round table, so that no two ladies sit together?

18. In how many different ways can the letters a, b, c, d be arranged without letting b and c come together?

19. Find the number of ways in which 8 books can be arranged on one shelf so that two particular books shall never come together.

20. In how many different ways can 4 flags of each of 6 different colours be hoisted in a vertical line?

21. In how many ways can a man distribute 5 votes among 5 candidates, it being allowable to give more than one vote to each?

22. In how many ways may 3 different books be given to 5 persons, (i) if no person may have more than one book, (ii) if he may have more than one?

23. In how many ways can 6 men and 5 women sit in a row, so that men and women are seated alternately?

24. Find the number of ways in which n differently coloured beads can be (i) arranged in a line, (ii) formed into a circle on a table, and (iii) formed into a necklace.

25. If there be n objects, of which r are alike and the rest unlike, prove that the number of ways in which they can be arranged in a row is $(r+1)(r+2)\dots n$.

348. DEF. 59. The different sets which can be formed from a group of things, either all together or a certain number together, are called their *combinations.*

Thus ab and ba, though different permutations, form only one combination. The different combinations of the letters a, b, c, one at a time, are a, b, c; two at a time, are bc, ca, ab; all together, abc.

The number of combinations of n things taken r at a time is denoted by ${}_nC_r$. Thus

$${}_3C_1 = 3, \quad {}_3C_2 = 3, \quad {}_3C_3 = 1.$$

349. *To find the number of combinations of n different things taken r at a time.*

First method. Let ${}_nC_r$ denote the number of combinations of n things taken r at a time.

Take any combination of r things. The number of permutations that can be formed by arranging these r things in all possible orders is $\lfloor r$.

$\therefore$ the number of permutations that can be formed from all the ${}_nC_r$ combinations is

$${}_nC_r \times \lfloor r.$$

But all the groups of r letters arranged in all possible orders are the same as all the permutations of n things taken r at a time.

$$\therefore \; {}_nC_r \times \lfloor r = {}_nP_r.$$

$$\therefore \; {}_nC_r = \frac{{}_nP_r}{\lfloor r} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{\lfloor r}$$

$$= \frac{n(n-1)(n-2)\ldots(n-r+1)\,.\,(n-r)\ldots 3\,.\,2\,.\,1}{\lfloor r\,.\,(n-r)\ldots 3\,.\,2\,.\,1}$$

$$= \frac{\lfloor n}{\lfloor r \, \lfloor n-r}.$$

Second method (without assuming the formula for the number of permutations of n things taken r at a time). Let ${}_nC_{r-1}$ denote the number of combinations of n different things, each containing $r-1$ things. If to each of these combinations be added one of the remaining $n-(r-1)$ things, we obtain $n-r+1$ different combinations each containing r things. But each combination so obtained will be obtained r times over by the addition of each of its r things to a previous group of $r-1$ things*. Thus the total number of combinations of n things taken r at a time is

$$ {}_nC_{r-1} \times \frac{n-r+1}{r}. $$

Thus $$ {}_nC_r = {}_nC_{r-1} \times \frac{n-r+1}{r}. $$

Since this equation is true for all values of r not greater than n, it is true when $r-1$ is substituted for r, i.e.

$$ {}_nC_{r-1} = {}_nC_{r-2} \times \frac{n-r+2}{r-1}. $$

Similarly $$ {}_nC_{r-2} = {}_nC_{r-3} \times \frac{n-r+3}{r-2}, $$

and so on, until finally,

$$ {}_nC_2 = {}_nC_1 \times \frac{n-1}{2}. $$

Hence, by multiplication,

$$ {}_nC_r = {}_nC_1 \times \frac{n-1}{2} \cdot \frac{n-2}{3} \dots \frac{n-r+1}{r} $$

$$ = \frac{n(n-1)(n-2)\dots(n-r+1)}{\underline{|r}}. $$

COR. 1. The number of combinations of n things taken r at a time so as always to include a particular thing is ${}_{n-1}C_{r-1}$, for there are $r-1$ things to be chosen to make up r and $n-1$ things to choose from.

* For example, the combination *abcd* will be obtained by adding *a* to *bcd*, *b* to *acd*, *c* to *abd* and *d* to *abc*.

COR. 2. The number of combinations of n things taken r at a time so as always to exclude a particular thing is ${}_{n-1}C_r$, for there are r things to be chosen and $n-1$ to choose them from.

350. *To prove that* ${}_nC_r = {}_nC_{n-r}$.

First proof. With each way of taking r things out of n, there is a group left of $n-r$ things;

$\therefore$ the number of ways of taking r things out of n is the same as the number of ways of leaving $n-r$ things out of n;

$$\therefore \quad {}_nC_r = {}_nC_{n-r}.$$

Second proof.
$$ {}_nC_r = \frac{\underline{|n}}{\underline{|r}\;\underline{|n-r}},$$

$$ {}_nC_{n-r} = \frac{\underline{|n}}{\underline{|n-r}\;\underline{|n-(n-r)}} = \frac{\underline{|n}}{\underline{|n-r}\;\underline{|r}};$$

$$\therefore \quad {}_nC_r = {}_nC_{n-r}.$$

351. *To prove that* ${}_{n+1}C_r = {}_nC_{r-1} + {}_nC_r$.

The number of combinations of n things taken r at a time is equal to the number which include a particular thing (namely, ${}_nC_{r-1}$) together with the number which exclude the same thing (namely, ${}_nC_r$);

$$\therefore \quad {}_{n+1}C_r = {}_nC_{r-1} + {}_nC_r.$$

352. *Example* 8. Find the number of different sums that can be formed with 2 sovereigns, 3 shillings and 4 pence.

(It should be noted that no one sum can be formed by a set of any others. For instance, if there had been 12 pence instead of 4, the same sum would be formed by taking 1 shilling or 12 pence.)

Either 2 sovereigns, 1 sovereign or 0 sovereigns may be taken;

$\therefore$ 3 sums may be formed with the sovereigns only (counting the case in which no sovereigns are taken).

With each way of choosing the sovereigns, either 3 shillings, 2 shillings, 1 shilling or 0 shillings may be taken;

∴ 3 . 4 sums may be formed with the sovereigns only and shillings only (counting the case in which no sovereigns and no shillings are taken).

With each way of choosing the sovereigns and shillings, either 4 pence, 3 pence, 2 pence, 1 penny or 0 pence may be taken;

∴ 3 . 4 . 5 sums may be formed with the sovereigns, shillings and pence (counting the case in which no sovereigns, no shillings and no pence are taken);

∴ the total number of sums is $3 . 4 . 5 - 1$ or 59.

353. *To find the number of different groups of things of which p things are alike, q things are alike and different from the rest, and r things are different from one another and from all the rest.*

Of the first group of things, either p, $p-1$, $p-2$, ..., 2, 1 or 0 may be taken;

∴ $p+1$ groups may be formed from the first set (counting the case in which none are taken).

With each way of choosing the first set of p things, either q, $q-1$, $q-2$, ..., 2, 1 or 0 of the second set may be taken;

∴ $(p+1)(q+1)$ groups may be formed with the first two sets only (counting the case in which none of either set is taken).

With each way of choosing the first two sets of things, either 1 or 0 of the first of the r things may be taken;

∴ $(p+1)(q+1)\,2$ groups may be formed (counting the case in which none of either set is taken), and so on;

∴ the total number of groups is $(p+1)(q+1)2^r - 1$.

Exercises 211.

1. Find the values of ${}_5C_2$, ${}_8C_3$, ${}_{15}C_4$, ${}_{10}C_7$.

2. Show that ${}_nC_r \div {}_nC_{r-1} = \dfrac{n-r+1}{r}$.

3. If ${}_{n+2}C_4 \div {}_nC_2 = 11$, find n.

4. If ${}_nC_6 : {}_{n-1}C_5 = 5 : 3$, find n.

5. There are 12 points in a plane, no three of which are in a straight line; how many lines can be drawn joining the points?

6. There are 20 stations on a railway; find the number of tickets required so that a person may travel from any one station to any other.

7. There are 15 points in a plane, no three of which are in a straight line; how many triangles can be formed by joining these points?

8. There are 10 points in a plane, no three of which are in a straight line; how many triangles can be formed by joining these points, every such triangle having a vertex at a particular point?

9. How many triangles are in general formed by n straight lines in a plane?

10. How many groups of 4 men can be selected from 12 men, so as always to include a particular man?

11. How many pickets of 5 men can be chosen from a group of 20 men; and in how many of these will 2 particular men be found?

12. Prove that ${}_{n+2}C_{r+1} = {}_nC_{r+1} + 2 \cdot {}_nC_r + {}_nC_{r-1}$.

13. In how many ways can a committee of 3 liberals and 3 conservatives be chosen from 10 liberals and 12 conservatives?

14. In how many ways may 12 be divided into 3 sets of 4?

15. In how many ways may 16 people be divided into 4 groups of 4 persons in each?

16. Of 15 men, 10 can row and cannot steer, and 5 can steer and cannot row; find how many boats' crews of 8 rowers and a coxswain can be formed out of the 15 men.

17. How many combinations can be made of 6 things taken any number at a time?

18. The number of permutations of $m+n$ things taken 2 together is 56, and of $m-n$ things taken 2 together is 12; find the number of combinations of m things n together.

19. Two points are taken on each side of a triangle, and the 6 points thus given are joined in every possible way; how many triangles are there in general in the resulting figure, when all the lines in it are produced indefinitely?

20. A certain council consists of a chairman, two vice-chairmen, and 12 other members; how many different committees consisting of 6 members can be chosen, including always the chairman and one of the vice-chairmen?

21. The number of combinations of n letters taken 5 together, in which a, b, c occur, is 21; find the number of combinations of them taken 6 together in which a, b, c, d occur.

22. In how many ways could two ladies and two gentlemen be chosen to make a set at tennis from a party of 6 ladies and 8 gentlemen?

23. Prove that the total number of ways in which a selection can be made of n articles is 2^n-1.

24. The total number of combinations of n things of which p are alike and the rest unlike is $(p+1)\,2^{n-p}-1$.

25. How many different sums can be formed with 5 sovereigns, 3 half-crowns and 2 shillings?

26. How many different sums of money can be formed with 2 crowns, 1 florin and 2 shillings?

27. How many sums can be paid out of 3 pence, 2 sixpences, 5 shillings, 1 half-crown and 4 sovereigns?

***354.** *Example* 9. Show that ${}_6C_r$ is greatest when $r=3$ and that ${}_7C_r$ is greatest when $r=3$ or 4.

We have the following table of values:

${}_6C_1$	${}_6C_2$	${}_6C_3$	${}_6C_4$	${}_6C_5$	${}_6C_6$	
6	15	30	15	6	1	
${}_7C_1$	${}_7C_2$	${}_7C_3$	${}_7C_4$	${}_7C_5$	${}_7C_6$	${}_7C_7$
7	21	35	35	21	7	1

$\therefore$ ${}_6C_r$ is greatest when $r=3$, i.e. $\frac{6}{2}$, and ${}_7C_r$ is greatest when $r=3$ or 4, i.e. $\frac{7-1}{2}$ or $\frac{7+1}{2}$.

***355.** *To prove* (i) *that* ${}_nC_r$ *at first continually increases and afterwards continually decreases as* r *increases from* 1 *to* n, *and* (ii) *that* ${}_nC_r$ *is greatest when* $r=\frac{n}{2}$ *if* n *be even, and when* $r=\frac{n-1}{2}$ *or* $\frac{n+1}{2}$ *if* n *be odd.*

(i) We have

$${}_nC_r=\frac{n(n-1)\ldots(n-r+2)(n-r+1)}{1\,.\,2\,.\ldots(r-1)\,.\,r}$$

and
$${}_nC_{r-1}=\frac{n(n-1)\ldots(n-r+2)}{1\,.\,2\,.\ldots(r-1)};$$

$$\therefore\ {}_nC_r={}_nC_{r-1}\times\frac{n-r+1}{r}={}_nC_{r-1}\times\left(\frac{n+1}{r}-1\right).$$

Now, as r increases, the factor $\frac{n+1}{r}-1$ decreases; but so long as it is greater than unity ${}_nC_r>{}_nC_{r-1}$, i.e. ${}_nC_r$ continually increases as r increases; and, when $\frac{n+1}{r}-1$ becomes less than unity, then ${}_nC_r<{}_nC_{r-1}$, i.e. ${}_nC_r$ continually decreases. If $\frac{n+1}{r}-1=1$, or $n+1=2r$, then ${}_nC_r={}_nC_{n-r}$; i.e. there will be two maximum values of ${}_nC_r$ if n be odd, but only one if n be even.

(ii) Let ${}_nC_r$ be that value which is not less than ${}_nC_{r-1}$ or ${}_nC_{r+1}$ and is greater than all the other values.

Now
$$\frac{{}_nC_r}{{}_nC_{r-1}}=\frac{n-r+1}{r}\ \text{and}\ \frac{{}_nC_{r+1}}{{}_nC_r}=\frac{n-r}{r+1},$$

and since
$${}_nC_r\nless{}_nC_{r-1}\ \text{or}\ {}_nC_{r+1},$$

$$\therefore\ \frac{n-r+1}{r}\nless 1\ \text{and}\ \frac{r+1}{n-r}\nless 1,$$

$$\therefore\ n-r+1\nless r\ \text{and}\ r+1\nless n-r,$$

$$\therefore 2r \ngtr n+1 \text{ and } \nless n-1,$$

$\therefore$ the only values which $2r$ can have are $n-1$, n and $n+1$.

If n be even, then $r = \dfrac{n}{2}$.

If n be odd, then $r = \dfrac{n-1}{2}$ or $\dfrac{n+1}{2}$.

356. *Example* 10. Twelve cards are dealt to four persons, find (i) how many different hands may be held by any one of them, and (ii) in how many ways the cards may be distributed among the four.

(i) The number of different hands is the number of combinations of 12 things 3 at a time or $\dfrac{12 \,.\, 11 \,.\, 10}{1 \,.\, 2 \,.\, 3}$ or 220.

(ii) The first hand may be dealt in ${}_{12}C_3$ ways, leaving 9 cards to be dealt. With each way of dealing the first hand, the next hand may be dealt in ${}_9C_3$ ways, leaving 6 cards to be dealt;

$\therefore$ the first two hands may be dealt in ${}_{12}C_3 \times {}_9C_3$ ways.

With each way of dealing the first two hands, the third hand may be dealt in ${}_6C_3$ ways, leaving 3 cards to be dealt;

$\therefore$ the first three hands may be dealt in ${}_{12}C_3 \times {}_9C_3 \times {}_6C_3$ ways.

With each way of dealing the first three hands, the fourth hand may be dealt in one way;

$\therefore$ the total number of ways of distributing the cards is

$$\frac{\underline{|12}}{\underline{|3}\,\underline{|9}} \times \frac{\underline{|9}}{\underline{|6}\,\underline{|3}} \times \frac{\underline{|6}}{\underline{|3}\,\underline{|3}} \text{ or } \frac{\underline{|12}}{(\underline{|3})^4}, \text{ or } 369{,}600 \text{ ways.}$$

Example 11. There are n points in a plane, no three of which are in a straight line, except p which are in a straight line; how many straight lines can be drawn joining them?

First solution. If no three of the n points were collinear, the number of such lines would be $\frac{1}{2}n(n-1)$.

If no three of the p points were collinear, the number of such lines would be $\frac{1}{2}p(p-1)$. But one line can be drawn through the p points,

∴ the total number of lines joining the n points is

$$\tfrac{1}{2}n(n-1)-\tfrac{1}{2}p(p-1)+1.$$

Second solution. There are $n-p$ points, no three of which are in a straight line, and p which are in a straight line.

The number of lines that can be drawn joining the $n-p$ points is $\frac{1}{2}(n-p)(n-p-1)$.

Each of the p points can be joined to each one of the $n-p$ points in $n-p$ ways.

∴ all the p points can be joined to all the $n-p$ points in $p(n-p)$ ways.

Also, one line can be drawn through the p points.

∴ the total number of lines joining the n points is

$$\tfrac{1}{2}(n-p)(n-p-1)+p(n-p)+1,$$

or $$\tfrac{1}{2}(n^2-2np+p^2-n+p)+np-p^2+1,$$

or $$\tfrac{1}{2}(n^2-p^2-n+p)+1,$$

or $$\tfrac{1}{2}n(n-1)-\tfrac{1}{2}p(p-1)+1.$$

Example 12. There are 10 white and 6 red balls in a bag; in how many different ways may 6 balls be drawn out so that in each drawing there may be at least 2 red balls?

The number of red balls that may be drawn out is either 2, 3, 4, 5 or 6.

Now, 2 red and 4 white balls may be drawn out in

$$\frac{6.5}{1.2}\times\frac{10.9.8.7}{1.2.3.4} \text{ or } 3150 \text{ ways};$$

3 red and 3 white balls may be drawn out

$$\frac{6.5.4}{1.2.3}\times\frac{10.9.8}{1.2.3} \text{ or } 2400 \text{ ways};$$

4 red and 2 white balls may be drawn out

$$\frac{6.5}{1.2}\times\frac{10.9}{1.2} \text{ or } 675 \text{ ways};$$

5 red and 1 white balls may be drawn out in

$$6 \times 10 \quad \text{or} \quad 60 \text{ ways.}$$

6 red and 0 white balls may be drawn out in 1 way.

∴ total number of ways is

$$3150 + 2400 + 675 + 60 + 1 \quad \text{or} \quad 6286 \text{ ways.}$$

**Example* 13. Find (i) the number of combinations, (ii) the number of permutations, which can be made of the letters of the word *examination* taken 4 together.

The letters are *a a i i n n e m o t x*.

(i) We may take 2 pairs of similar letters, or one pair of similar letters and 2 different letters, or 4 different letters.

2 pairs of similar letters out of 3 pairs may be chosen in 3 ways.

1 pair of similar letters out of 3 pairs and 2 different letters out of 7 may be chosen in $3 \times \frac{7 \cdot 6}{2}$ or 63 ways.

4 different letters out of 8 may be chosen in $\frac{8 \cdot 7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4}$ or 70 ways.

∴ the total number of combinations is $3 + 63 + 70$ or 136.

(ii) From each combination consisting of 2 pairs of similar letters may be formed $\frac{\lfloor 4}{\lfloor 2 \lfloor 2}$ or 6 permutations. From each combination consisting of one pair of similar letters and 2 different letters may be formed $\frac{\lfloor 4}{\lfloor 2}$ or 12 permutations. From each combination consisting of four different letters may be formed $\lfloor 4$ or 24 permutations.

∴ total number of permutations is $3 \times 6 + 63 \times 12 + 70 \times 24$ or $18 + 756 + 1680$ or 2454.

*Exercises 212.

1. Prove that the greatest number of combinations that can be formed with $2n$ things, each combination containing the same number of things, is double the greatest number that can be formed with $2n-1$ things.

2. How many numbers, expressed by five unequal digits, have three of their digits taken from the first six, and the other two from the last three, of the nine digits?

3. In how many ways can 5 ladies and 4 gentlemen sit at a round table, so that no two gentlemen sit together?

4. Eight men are to form a boat's crew, two of them can only row on one side, and two on the other; how many crews are possible?

5. From a company of 10, 5 are to be selected so as either to exclude one particular person but include two others, or else to include the person previously excluded but to exclude the two who were previously included; in how many ways can the selection be made?

6. In how many ways can 10 people be divided into two groups of four and one group of two?

7. There are 8 letters, of which a certain number are alike, and 336 different words can be formed out of them; how many are alike?

8. How many different permutations can be made of the letters of the word *dividend*, and how many of these will both begin and end with vowels?

9. Eight men sit down to two tables at whist; how many different sets of partners can be arranged?

10. A purse contains a sovereign, a half-sovereign, a half-crown, a florin, a shilling, a threepenny-piece and a penny; how many different sums of money can be obtained by taking out four of the coins?

11. How many numbers over 1000 can be made out of the figures 0, 0, 3, 5, 6?

12. 26 persons are to travel by an omnibus which can carry 12 inside and 14 outside; if 8 of them will not ride outside and 6 will not ride inside, in how many different ways can the party travel?

13. In the fully expanded product a^2b^m (m being any integer), find the number of different arrangements in which the 2 a's are (i) together, (ii) separated.

14. A cricket club consists of 13 members, of whom only four can bowl; in how many ways can an eleven be chosen so as to include at least two bowlers?

15. How many different pairs can be made out of $2n$ different things?

16. In how many different ways can an even number ($2n$) of things be arranged in two equal groups?

17. Show that the number of ways in which pn things may be given to n persons is $\frac{\lfloor pn}{(\lfloor n)^p}$ or n^{pn}, according as each person has n things, or any number of the pn things.

18. How many different arrangements are possible of the letters of the word *solace* with the restriction that two vowels must not be together?

19. How many different arrangements are possible of the letters of the word *palace* with the restriction that two vowels must not be together?

20. Prove that, with two dice, the number of throws is 21; how many different throws can be made with three dice?

21. How many words of three letters can be formed out of 5 vowels and 21 consonants, each word to have a vowel between two consonants, which may be the same letter repeated or different letters?

22. In how many ways can a committee of 5 be chosen from 7 liberals and 4 conservatives so as always to give a liberal majority?

23. Find the total number of combinations that can be made with the letters of the word *divide*.

24. Find the number of arrangements of three different letters which can be formed out of the ten letters q to z; q, when it occurs, being always followed by u?

25. Find in how many ways 8 different objects can be arranged in a row, so that two given objects may have one object between them.

26. There are n points in a plane, no three of which are in a straight line; find the number of points of intersection of the lines which join them in all possible ways.

27. There are n points in a plane, no three of which are in a straight line, except p, which are in a straight line; how many triangles can be formed by joining all the points?

28. There are p points in a straight line and q points in a parallel straight line; how many triangles can be formed having three of these points for vertices?

29. If there be n given points, of which p lie in one straight line and q on another, how many triangles can be formed with their vertices at these points?

30. Given 8 straight lines, 3 of which are parallel, and no three are concurrent; how many different triangles do they form?

31. There are $2n$ letters, of which 2 are a, 2 are b, and so on; find how many different algebraical products can be formed, in each of which the sum of the indices is 3.

32. In how many different ways may 10 plus signs and 4 minus signs be placed in a row, so that no two minus signs may come together?

33. Show that, when all the numbers that can be formed of the same four different digits is divided by the sum of the digits, the quotient is 6666.

34. Prove that the number of different arrangements, three together, of the letters of the word *university* is 528.

35. How many permutations are there of the letters of the word *corridor*, (i) all together, (ii) taken 4 at a time, 2 consonants and 2 vowels, with the latter in the even places?

36. How many words, consisting of 2 consonants and 2 vowels, can be formed from 20 consonants and 5 vowels, the vowels not coming together?

37. Find the number of permutations of the letters of the word *infinite*; how many of them begin and end with a consonant?

38. How many different words of two consonants and one vowel can be formed from the letters in the word *Woolwich*?

39. How many permutations can be formed from the letters of the word *terrace* four at a time?

40. Find the number of different combinations of the letters of the word *zoology* taken four at a time.

41. Find how many even numbers can be made up of the nine digits, using them all for each number; how many of these will be multiples of four?

42. How many different quadratic expressions, each containing three terms, can be formed with the symbols 3, 5, 8, x, x^2, +, −, and no others? Of these, how many can be resolved into two real simple factors?

43. How many different combinations and permutations may be formed of the letters of the word *college* taken four at a time?

44. How many combinations and permutations, each of four letters, can be made from the letters of the word *parabola*?

45. Find the number of permutations, 5 together, that can be made of the letters of the word *convolvulus.*

46. Show that eight differently coloured equilateral triangles can be formed into a regular octahedron in 1680 different ways.

CHAPTER XXXII.

BINOMIAL THEOREM: POSITIVE INTEGRAL INDEX.

357. We have seen that

$$(a+b)^2 = a^2 + 2ab + b^2,$$

and
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

The object of the Binomial Theorem is to find the value of $(a+b)^n$, and, in the present chapter, the case in which n is a positive integer will alone be considered.

If n be a small positive integer, the expansion can be obtained by continued multiplication, or, more simply, by the method employed in the next example (see also art. 227).

358. *Example* 1. Expand $(a+b)^3$.

Since $a+b$ is a symmetrical and homogeneous expression of the first degree in a and b, it follows that $(a+b)^3$ is a symmetrical and homogeneous expression of the third degree in a and b, and must therefore consist of terms in a^3, a^2b, ab^2, and b^3. Since it is symmetrical, the coefficients of a^3 and b^3 must be equal, and so also those of a^2b and ab^2.

Now $(a+b)^3$ is the product of three factors each equal to $a+b$.

The term in a^3 is the product of the a's in all the factors. These can be selected in one way only. Thus, the coefficient of a^3, and therefore also of b^3, is 1.

The term in a^2b is the product of the a's in two factors and the b in one. Now, two a's can be chosen out of three, or one b out of three, in 3 ways. Thus, the coefficient of a^2b, and therefore also of ab^2, is 3.

$$\therefore\ (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

359. *To expand* $(a+b)^n$.

Since $a+b$ is a symmetrical and homogeneous expression of the first degree in a and b, it follows that $(a+b)^n$ is a symmetrical and homogeneous expression of the nth degree in a and b. Since its expansion is homogeneous, it must consist of terms such as a^n, $a^{n-1}b$, $a^{n-2}b^2$, ... a^2b^{n-2}, ab^{n-1}, and b^n. Since it is symmetrical, the coefficients of a^n and b^n must be equal, and so also those of $a^{n-1}b$ and ab^{n-1}, of $a^{n-2}b^2$ and a^2b^{n-2}, etc.

Now $(a+b)^n$ is the product of n factors each equal to $a+b$.

The term in a^n is the product of the a's in all the factors. These can be selected in one way only. Thus, the coefficient of a^n, and therefore also of b^n, is 1.

The term in $a^{n-1}b$ is the product of the a's in $n-1$ of the factors and the b in one. Now, $n-1$ a's can be chosen out of n, or one b out of n, in ${}_nC_1$, or n, ways. Thus, the coefficient of $a^{n-1}b$, and therefore also of ab^{n-1}, is n.

The term in $a^{n-r}b^r$ is the product of the a's in $n-r$ of the factors and of the b's in r of the factors. Now, $n-r$ a's can be chosen out of n, or r b's out of n, in ${}_nC_r$ ways. Thus, the coefficient of $a^{n-r}b^r$, and therefore also of a^rb^{n-r}, is ${}_nC_r$.

Thus, $(a+b)^n = a^n + {}_nC_1 a^{n-1}b + {}_nC_2 a^{n-2}b^2 + \dots$

$$+ {}_nC_r a^{n-r}b^r + \dots + {}_nC_2 a^2b^{n-2} + {}_nC_1 ab^{n-1} + b^n,$$

or

$$= a^n + na^{n-1}b + \frac{n(n-1)}{\underline{|2}}a^{n-2}b^2 + \dots$$

$$+ \frac{n(n-1)}{\underline{|2}}a^2b^{n-2} + na\,b^{n-1} + b^n.$$

Cor. 1. Since the indices of b in successive terms are respectively 0, 1, 2, 3, ..., n, it follows that the number of terms in the expansion of $(a+b)^n$ is $n+1$.

Cor. 2. Since the expansion of $(a+b)^n$ is symmetrical with respect to a and b, it follows that the coefficients of $a^{n-r}b^r$ and $a^r b^{n-r}$ are equal, i.e. that the coefficients of terms equidistant from the beginning and end are equal.

360. *Example* 2. Expand (i) $(a+b)^5$, (ii) $(2a-3b)^4$.

(i) $$(a+b)^5 = a^5 + 5a^4b + \frac{5\,.\,4}{1\,.\,2}a^3b^2 + \frac{5\,.\,4}{1\,.\,2}a^2b^3 + 5ab^4 + b^5$$
$$= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

(ii) $$(2a-3b)^4 = (2a)^4 + 4\,(2a)^3(-3b)$$
$$+ \frac{4\,.\,3}{1\,.\,2}(2a)^2(-3b)^2 + 4(2a)(-3b)^3 + (-3b)^4$$
$$= 16a^4 - 96a^3b + 216a^2b^2 - 216ab^3 + 81b^4.$$

361. *To find the* $(r+1)$*th term in the expansion of* $(a+b)^n$.

The index of b in any term is less than the number of the term by unity.

∴ the index of b in the $(r+1)$th term is r. But the sum of the indices of a and b is n.

∴ the index of a is $n-r$.

Now the number of ways of choosing $n-r$ a's out of n, or r b's out of n, is ${}_nC_r$.

$$\therefore (r+1)\text{th term} = {}_nC_r\,.\,a^{n-r}b^r = \frac{\lfloor n}{\lfloor r\,\lfloor n-r}\,a^{n-r}b^r.$$

362. *Example* 3. Find (i) the 8th term of $(a+b)^{10}$, (ii) the middle term of $(a-2b)^8$, and (iii) the term in x^8 in $\left(2x^2-\frac{1}{3x}\right)^8$.

(i) 8th term of $(a+b)^{10} = {}_{10}C_7 a^3b^7 = \frac{10\,.\,9\,.\,8}{1\,.\,2\,.\,3}\,.\,a^3b^7$
$$= 120a^3b^7.$$

(ii) Since there are 9 terms in the expansion of $(a-2b)^8$, there are 4 terms on each side of the middle term, so that the middle term is the 5th;

$$\therefore \text{ middle term of } (a-2b)^8 = \frac{8 \cdot 7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4} a^4 (-2b)^4 = 70a^4 \cdot 16b^4$$

$$= 1120a^4b^4.$$

(iii) Let the term which contains x^3 be the $(r+1)$th, which is

$$= {}_6C_r (2x^2)^{6-r} \left(-\frac{1}{3x}\right)^r = (-1)^r \cdot {}_6C_r \cdot \frac{2^{6-r}}{3^r} \frac{x^{12-2r}}{x^r}$$

$$= (-1)^r \cdot {}_6C_r \cdot \frac{2^{6-r}}{3^r} \cdot x^{12-3r}.$$

Now, $12 - 3r = 3$ or $r = 3$.

$\therefore$ the required term is

$$-{}_6C_3 \cdot \frac{2^3}{3^3} x^3 = -\frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} \cdot \frac{8}{27} \cdot x^3$$

$$= -\tfrac{160}{27} x^3.$$

Exercises 213.

Expand:

1. $(1+x)^4$. 2. $(a+b)^5$. 3. $(1+2x)^5$.

4. $(2a-3b)^4$. 5. $\left(2x+\frac{a}{2}\right)^5$. 6. $(1+x)^5+(1-x)^5$.

7. $(a-b)^6$. 8. $(1-x)^7$. 9. $(1-2x)^6$.

10. $\left(2a-\frac{x}{2}\right)^7$. 11. $(1+x)^{12}$ to 5 terms.

12. $(2a-b)^{10}$ to 4 terms.

Find:

13. $1 \cdot 1^4$. 14. $1 \cdot 01^3$. 15. $\cdot 9^4$. 16. $\cdot 99^3$.

Find:

17. 4th term of $(1+x)^9$.
18. 4th term of $(1-2x)^{10}$.
19. 5th term of $\left(2a+\frac{b}{2}\right)^{12}$.
20. 3rd term of $\left(2x^2-\frac{3}{2x}\right)^8$.
21. 3rd term of $(3a-7b)^7$.
22. 9th term of $\left(\frac{3}{2}-\frac{x}{3}\right)^{12}$.
23. 8th term of $\left(a-\frac{b}{2}\right)^{11}$.
24. Coefficient of x^5 in $(a-x)^9$.

Find the middle term of

25. $(1+x)^6$.
26. $\left(x-\frac{2y}{3}\right)^{10}$.
27. $(2+\frac{1}{2}x)^{20}$.
28. $(1+x)^{2n}$.

Find the two middle terms of

29. $(1-x)^7$.
30. $\left(x+\frac{1}{x}\right)^{2n+1}$.
31. $(a-x)^{29}$.

Find the coefficient of

32. x^3 in $\left(2x-\frac{1}{2x}\right)^{11}$.
33. x^{-1} in $\left(x-\frac{1}{x^2}\right)^8$.
34. x in $\left(x^2+\frac{a^8}{x}\right)^5$.

35. Find the term in the expansion of $\left(x+\frac{1}{x}\right)^{2n}$ which is independent of x.

36. Find $\cdot 99^4$ to 5 places of decimals.

37. Find the value of $\cdot 9^7$ to 4 places of decimals.

38. Find the coefficient of x^3 in the expansion of $(1+x+x^2)^4$.

39. Find the coefficient of x^4 in the expansion of $(1+3x-x^2)^5$.

40. Prove that the coefficient of x^r in the expansion of $(1+x)^{n+1}$ is equal to the sum of the coefficients of x^{r-1} and x^r in the expansion of $(1+x)^n$.

***363.** *To find the greatest coefficient or coefficients in the expansion of* $(1+x)^n$.

The coefficient of the $(r+1)$th term is ${}_nC_r$.

Hence, if n be even, the greatest coefficient is that for which

$$r = \frac{n}{2}.$$

If n be odd, the greatest coefficients are those for which

$$r = \frac{n-1}{2} \text{ or } \frac{n+1}{2}$$

***364.** *To find the greatest term or terms in the expansion of* $(1+x)^n$.

Let the $(r+1)$th term be not less than the rth term or the $(r+2)$th term.

Now, $$\frac{(r+1)\text{th term}}{r\text{th term}} = \frac{{}_nC_r \cdot x^r}{{}_nC_{r-1} \cdot x^{r-1}} = \frac{n-r+1}{r}x,$$

$$\therefore \frac{(r+2)\text{th term}}{(r+1)\text{th term}} = \frac{n-\overline{r+1}+1}{r+1}x = \frac{n-r}{r+1}x,$$

$$\therefore \frac{n-r+1}{r}x \not< 1 \text{ and } \frac{n-r}{r+1}x \not> 1,$$

$$\therefore (n-r+1)x \not< r \text{ and } (n-r)x \not> r+1,$$

$$\therefore (n+1)x \not< (1+x)r \text{ and } nx-1 \not> (1+x)r,$$

$$\therefore r \not> \frac{(n+1)x}{1+x} \text{ and } \not< \frac{nx-1}{1+x}.$$

***365.** *Example* 4. Find which is the greatest term in the expansion of $(1+\frac{2}{3})^{10}$.

Let the greatest term be the $(r+1)$th.

Now

$$\frac{(r+1)\text{th term}}{r\text{th term}} = \frac{{}_{10}C_r \cdot (\frac{2}{3})^r}{{}_{10}C_{r-1}(\frac{2}{3})^{r-1}} = \frac{10-r+1}{r}\cdot\frac{2}{3} = \frac{11-r}{r}\cdot\frac{2}{3},$$

and $$\frac{(r+2)\text{th term}}{(r+1)\text{th term}} = \frac{11-(r+1)}{r+1}\cdot\frac{2}{3} = \frac{10-r}{r+1}\cdot\frac{2}{3},$$

$$\therefore \frac{11-r}{r}\cdot\frac{2}{3} \not< 1 \text{ and } \frac{10-r}{r+1}\cdot\frac{2}{3} \not> 1,$$

$$\therefore 22-2r \not< 3r \text{ and } 20-2r \not> 3r+3,$$

$$\therefore 5r \not> 22 \text{ and } \not< 17,$$

$$\therefore 5r = 20 \text{ or } r = 4;$$

$\therefore$ the 5th term is the greatest.

***366.** We know that ${}_nC_r = {}_nC_{n-r}$, where r has any value from 1 to $n-1$ inclusive. But the number of ways of taking all of n things is equal to the number of ways of leaving none behind, that is, unity, hence ${}_nC_0 = {}_nC_n = 1$.

In the present and following articles, the symbol c_r is used to denote ${}_nC_r$.

Now, since $c_0 = 1$ and $c_r = c_{n-r}$, it follows that we may write

$$(1+x)^n = c_0 + c_1x + c_2x^2 + \dots + c_{n-2}x^{n-2} + c_{n-1}x^{n-1} + c_nx^n,$$

or $$(1+x)^n = c_n + c_{n-1}x + c_{n-2}x^2 + \dots + c_2x^{n-2} + c_1x^{n-1} + c_0x^n.$$

***367.** *To prove that*

(i) $c_0 + c_1 + c_2 + \dots + c_n = 2^n$,

(ii) $c_0 - c_1 + c_2 - \dots + (-1)^n c_n = 0$,

(iii) $c_0^2 + c_1^2 + c_2^2 + \dots + c_n^2 = \frac{\lfloor 2n}{\lfloor n \lfloor n}$.

(i) We have

$$(1+x)^n = c_0 + c_1x + c_2x^2 + \dots + c_nx^n.$$

Put $x = 1$,

then $$2^n = c_0 + c_1 + c_2 + \dots + c_n.$$

(ii) In the same expansion, put $x = -1$,

then $$0 = c_0 - c_1 + c_2 - \dots + (-1)^n c_n.$$

(iii) We have

$$(1+x)^n = c_0 + c_1 x + c_2 x^2 + \dots + c_{n-2}x^{n-2} + c_{n-1}x^{n-1} + c_n x^n,$$

and $$(1+x)^n = c_n + c_{n-1}x + c_{n-2}x^2 + \dots + c_2 x^{n-2} + c_1 x^{n-1} + c_0 x^n.$$

Now, coefficient of x^n in the product of these two series

$= c_0^2 + c_1^2 + c_2^2 + \dots + c_n^2$

= coefficient of x^n in the product of $(1+x)^n$ and $(1+x)^n$, i.e. in the expansion of $(1+x)^{2n}$

$$= \frac{\lfloor 2n}{\lfloor n \lfloor n}.$$

***368.** *Example* 5. Prove that

$$c_0 + c_2 + c_4 + \dots = c_1 + c_3 + c_5 + \dots = 2^{n-1}.$$

We have $$c_0 - c_1 + c_2 - c_3 + \dots = 0,$$

$$\therefore c_0 + c_2 + c_4 + \dots = c_1 + c_3 + c_5 + \dots.$$

But $$(c_0 + c_2 + c_4 + \dots) + (c_1 + c_3 + c_5 + \dots) = 2^n,$$

$$\therefore c_0 + c_2 + c_4 + \dots = c_1 + c_3 + c_5 + \dots = \frac{2^n}{2} = 2^{n-1}.$$

Example 6. Prove that

$$2c_0 + 2^2 . \frac{c_1}{2} + 2^3 . \frac{c_2}{3} + 2^4 . \frac{c_3}{4} + \dots = \frac{3^{n+1}-1}{n+1}$$

$$(n+1)\left(2c_0 + 2^2 . \frac{c_1}{2} + 2^3 . \frac{c_2}{3} + 2^4 . \frac{c_3}{4} + \dots\right)$$

$$= (n+1) . 2 + \frac{(n+1)n}{\lfloor 2} . 2^2 + \frac{(n+1)n(n-1)}{\lfloor 3} 2^3 + \dots$$

$$= 1 + (n+1) . 2 + \frac{(n+1)n}{\lfloor 2} . 2^2 + \frac{(n+1)n(n-1)}{\lfloor 3} 2^3 + \dots - 1$$

$$= (1+2)^{n+1} - 1,$$

$$\therefore 2c_0 + 2^2 . \frac{c_1}{2} + 2^3 . \frac{c_2}{3} + 2^4 . \frac{c_3}{4} + \dots = \frac{3^{n+1}-1}{n+1}.$$

*Exercises 214.

Find the greatest coefficient in the expansion of

1. $(1+x)^8$.
2. $(1+x)^{11}$.

Find the greatest term in the expansion of

3. $(1+\frac{1}{2})^{12}$.
4. $(1+\frac{2}{3})^{10}$.
5. $(1+2\,.\,\frac{1}{2})^{30}$.
6. Prove that $c_r + c_{r-1} : c_{r-1} = n+1 : r$.
7. Show that

$$1 + 2n + \frac{2n(2n-1)}{\lfloor 2} + \frac{2n(2n-1)(2n-2)}{\lfloor 3} + \ldots$$
$$= \left\{1 + n + \frac{n(n-1)}{\lfloor 2} + \frac{n(n-1)(n-2)}{\lfloor 3} + \ldots\right\}^2.$$

Prove that

8. $c_1 + 2c_2 + 3c_3 + \ldots + nc_n = n\,.\,2^{n-1}$.
9. $c_0 + 2c_1 + 3c_2 + \ldots + (n+1)\,c_n = (n+2)\,.\,2^{n-1}$.
10. $c_1 - 2c_2 + 3c_3 - \ldots - (-1)^n\,nc_n = 0$.
11. $2c_0 + c_1 + 2c_2 + c_3 + 2c_4 + c_5 + \ldots = 3\,.\,2^{n-1}$.
12. $c_0 + \frac{1}{2}c_1 + \frac{1}{3}c_2 + \ldots + \frac{1}{n+1}c_n = \frac{2^{n+1}-1}{n+1}$.
13. $(n+1)\,c_0 - nc_1 + (n-1)\,c_2 - \ldots + (-1)^n\,c_n = 0$.
14. $1^2c_1 + 2^2c_2 + 3^2c_3 + \ldots + n^2c_n = n\,(n+1)\,.\,2^{n-2}$.
15. $c_0c_1 + c_1c_2 + c_2c_3 + \ldots + c_{n-1}c_n = \frac{\lfloor 2n}{\lfloor n+1\,\lfloor n-1}$.
16. $c_0c_2 + c_1c_3 + c_2c_4 + \ldots + c_{n-2}c_n = \frac{\lfloor 2n}{\lfloor n+2\,\lfloor n-2}$.
17. $c_0^2 - c_1^2 + c_2^2 - c_3^2 + \ldots + (-1)^n\,c_n^2 = 0$ if n be odd and $(-1)^{\frac{n}{2}}\frac{\lfloor n}{\lfloor \frac{1}{2}n\,\lfloor \frac{1}{2}n}$ if n be even.

18. $c_0^2 + \dfrac{c_1^2}{2} + \dfrac{c_2^2}{3} + \ldots + \dfrac{c_n^2}{n+1} = \dfrac{\underline{|2n+1}}{\underline{|n+1}\,\underline{|n+1}}$.

19. If $y^6 + y^5 - 5y^4 - 4y^3 + 6y^2 + 3y - 1 = 0$, and $y = x + \dfrac{1}{x}$, prove that $\dfrac{x^{13} - 1}{x - 1} = 0$.

20. Find the relation between n and r so that the coefficients of the $(2r+3)$th and $(3r+2)$th terms of $(1+x)^{4n}$ may be equal.

21. If three successive coefficients in the expansion of $(1+x)^n$ be 220, 495 and 792, find n.

22. If a_r denote the coefficient of x^r in the expansion of $(1-x)^{2m-1}$, show that $a_{r-1} + a_{2m-r} = 0$.

23. Show that

$$(1-x^2)^n = (1+x)^{2n} - 2nx(1+x)^{2n-1} + \frac{2n(2n-2)}{\underline{|2}} x^2 (1+x)^{2n-2} - \ldots.$$

CHAPTER XXXIII.

CONVERGENCY AND DIVERGENCY OF SERIES.

369. A series has been defined (art. 297) as an expression in which each term is formed from one or more preceding terms according to some definite law; and an infinite series as one in which the number of terms is infinite.

One of the simplest examples of an infinite series is the ordinary geometric series

$$a,\ ar,\ ar^2,\ ar^3,\ \ldots,\ ar^{n-1},\ \ldots.$$

The sum of such a series has been shown (art. 314) to be

$$\frac{a(1-r^n)}{1-r} \text{ or } \frac{a}{1-r} - \frac{ar^n}{1-r}.$$

Now, if r be equal to or greater than 1, the sum of an infinite number of terms is infinitely great; if r be less than 1, the sum of an infinite number of terms tends to the limit $\frac{a}{1-r}$, since the difference between the sum of n terms and the sum of an infinite number of terms can be made less than any quantity we please by making n large enough; and, lastly, if $r=-1$, the series becomes $a-a+a-a+a\ldots$, and the sum of an infinite number of terms, though finite, does not tend to a definite limit, for the sum of an even number of terms is 0 while the sum of an odd number of terms is a.

370. DEF. 60. An infinite series is said to be *convergent* when the sum of n terms can be made to differ from a finite quantity by less than any quantity we please by making n large enough.

DEF. 61. An infinite series is said to be *divergent* when the sum of n terms can be made greater than any finite quantity we please by making n large enough.

DEF. 62. An infinite series is said to be *oscillating* when the sum of n terms is finite, however great n may be, but does not differ from a fixed quantity by less than any quantity we please by making n large enough.

Thus, a geometric series is convergent if the common ratio be less than 1, divergent if the common ratio be greater than or equal to 1, and oscillating if the common ratio be equal to -1.

Since the terms of an oscillating series must be alternately positive and negative, it follows that, if the sum of n terms of a series, in which the terms are all positive, be less than a given finite quantity (however great n may be), then the series is convergent.

As a general rule, the infinite series will be expressed as follows:

$$u_1 + u_2 + u_3 + \dots + u_n + \dots,$$

and, unless otherwise mentioned, it will be supposed that each term of the series is positive.

The sum of n terms of such a series will be denoted by S_n, and the sum of an infinite number of terms by S or S_∞.

The following are some of the principal tests for determining whether a series is convergent or divergent.

371. *Test I. If every term of an infinite series be greater than some finite quantity, however small, the series is divergent.*

If every term be greater than a, then the sum of n terms is greater than na, which can be made greater than any finite quantity by making n large enough.

Hence, the given series is divergent.

372. *Test II. An infinite series is convergent if, after any term, the ratio of each term to the preceding is less than some fixed quantity which is itself less than unity.*

Let the series, beginning at any term, be

$$u_1 + u_2 + u_3 + \dots + u_n + \dots,$$

and let each of the ratios $u_2 : u_1$, $u_3 : u_2$, $u_4 : u_3$, ... be less than k, where k is itself less than unity.

Then $\quad S = u_1 + u_2 + u_3 + u_4 + \dots$

$$= u_1 \left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_1} + \frac{u_4}{u_1} + \dots\right)$$

$$= u_1 \left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_2} \cdot \frac{u_2}{u_1} + \frac{u_4}{u_3} \cdot \frac{u_3}{u_2} \cdot \frac{u_2}{u_1} + \dots\right)$$

$$< u_1 (1 + k + k^2 + k^3 + \dots);$$

$$\text{i.e.} < u_1 \cdot \frac{1}{1-k}.$$

Hence, the given series is convergent.

373. *Example* 1. Show that the series

$$\frac{1}{\underline{|1}} + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \frac{1}{\underline{|4}} + \dots$$

is convergent.

Here $\qquad u_r = \dfrac{1}{\underline{|r}}$ and $u_{r+1} = \dfrac{1}{\underline{|r+1}}$;

∴ the test-ratio

$$u_{r+1} : u_r = \frac{1}{\underline{|r+1}} \div \frac{1}{\underline{|r}} = \frac{1}{r+1},$$

which is equal to $\frac{1}{2}$ when $r = 1$, and is less than $\frac{1}{2}$ for all other values of r;

∴ the series is convergent.

374. *Test III. An infinite series is divergent if, after any term, the ratio of each term to the preceding be greater than some fixed quantity which is itself greater than unity.*

Let the series beginning at any term be

$$u_1 + u_2 + u_3 + u_4 + \dots,$$

and let each of the ratios $u_2 : u_1$, $u_3 : u_2$, $u_4 : u_3$, ... be greater than a quantity which is itself greater than unity.

Then $$u_2 > u_1, \ u_3 > u_2 > u_1, \ \dots;$$

$$\therefore \ S > nu_1;$$

$\therefore$ the series is divergent.

375. *Example* 2. Show that the series

$$\frac{\underline{|1}}{x} + \frac{\underline{|2}}{x^2} + \frac{\underline{|3}}{x^3} + \frac{\underline{|4}}{x^4} + \dots$$

is divergent.

Here, the test-ratio $u_{r+1} : u_r$ is

$$\frac{\underline{|r+1}}{x^{r+1}} \div \frac{\underline{|r}}{x^r} \text{ or } \frac{r+1}{x}.$$

Hence, for every term after that for which $r+1$ is just greater than x, the test-ratio is greater than $\frac{r+1}{x}$, which is itself greater than unity;

$\therefore$ the series is divergent.

376. *Test IV. If there be two infinite series, and if the ratio of corresponding terms in both series be always finite, then the series are both convergent or both divergent.*

Let the series be

$$u_1 + u_2 + u_3 + u_4 + \dots$$

and $$v_1 + v_2 + v_3 + v_4 + \dots,$$

and let the ratio $u_r : v_r$ be finite for all values of r.

Now, the ratio

$$u_1 + u_2 + u_3 + \dots : v_1 + v_2 + v_3 + \dots$$

lies between the greatest and least of the ratios

$$u_1 : v_1, \;\; u_2 : v_2, \;\; u_3 : v_3, \;\; \dots,$$

and is therefore finite.

Thus, if the sum of one of the given series be finite, the other is so also; if one be infinite, the other is so also;

∴ the series are both convergent or both divergent.

COR. 1. If each term of an infinite series be less than the corresponding term of an infinite series which is known to be convergent, the given series is also convergent.

COR. 2. If each term of an infinite series be greater than the corresponding term of an infinite series which is known to be divergent, the given series is also divergent.

377. *Example* 3. Show that the series

$$\text{(i)} \quad \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \dots$$

is convergent,

$$\text{(ii)} \quad \lfloor 1 + \lfloor 2 + \lfloor 3 + \dots$$

is divergent.

(i) $\lfloor 3 = 2 \,.\, 3$, which is $> 2^2$, $\lfloor 4 > 2^3$, and so on,

$$\therefore \; S < 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots,$$

which is a geometric series, in which the common ratio is less than unity;

∴ the series is convergent.

(ii) $S > 1 + 2 + 2^2 + 2^3 + \dots,$

every term of which after the first is greater than unity;

∴ the series is divergent.

378. *Test V. An infinite series is convergent, if the signs of the terms be alternately positive and negative, if each term be numerically less than the preceding term, and if the nth term be infinitely small when n is infinitely great.*

Let the series be

$$u_1 - u_2 + u_3 - u_4 + u_5 - u_6 + \ldots.$$

Then $$S = (u_1 - u_2) + (u_3 - u_4) + (u_5 - u_6) + \ldots.$$

But each of the quantities $u_1 - u_2$, $u_3 - u_4$, $u_5 - u_6$, ... is positive;

$$\therefore\ S \text{ is positive.}$$

Again $$S = u_1 - (u_2 - u_3) - (u_4 - u_5) - \ldots,$$

and each of the quantities $u_2 - u_3$, $u_4 - u_5$, $u_6 - u_7$, ... is positive;

$$\therefore\ S < u_1.$$

Thus, the series is either convergent or oscillating.

Lastly,

$$S - S_n = \pm (u_{n+1} - u_{n+2} + u_{n+3} - u_{n+4} + \ldots),$$

and, as above, $S - S_n$ is numerically less than u_{n+1}, which is infinitely small when n is infinitely great;

∴ the series is convergent.

379. *Example* **4.** Show that the series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \ldots$$

is convergent.

The terms are alternately positive and negative, each term being numerically less than the preceding, and the numerical value of the nth term is $\frac{1}{n}$, which is infinitely small when n is infinitely great.

∴ the series is convergent.

It will be seen, in art. 381, that the series

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \ldots,$$

which satisfies the above tests, with the exception of the first, is divergent.

380. In the next article, an important series is considered, namely,

$$\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \dots.$$

It should be noticed that the second and third tests are not applicable in this case, for the test-ratio

$$u_{r+1} : u_r = \frac{r^p}{(r+1)^p} = \left(1 - \frac{1}{r+1}\right)^p,$$

which becomes indefinitely nearly equal to unity when r is made very great. Thus, the ratio of the $(r+1)$th term to the rth term is not less than, or greater than, a quantity which is itself less than, or greater than, unity.

381. *To show that the series*

$$\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \dots$$

is convergent if p be greater than unity, and divergent if p be equal to or less than unity.

(i) Let p be greater than unity. Let the terms of the series be divided into groups, so that the first group contains 1 term, the second 2, the third 4, the fourth 8, and so on.

Now, the first term is 1.

The sum of the next two terms is $\frac{1}{2^p} + \frac{1}{3^p}$, which is $< \frac{1}{2^p} + \frac{1}{2^p}$, since $3^p > 2^p$,

$$\text{i.e.} < \frac{2}{2^p} \text{ or } \frac{1}{2^{p-1}}.$$

The sum of the next four terms is

$$\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p},$$

which is

$$< \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p},$$

$$\text{i.e.} < \frac{4}{4^p} \text{ or } \frac{1}{4^{p-1}} \text{ or } \frac{1}{2^{2(p-1)}}.$$

Similarly, the sum of the next eight terms is $< \frac{8}{8^p}$ or $\frac{1}{8^{p-1}}$ or $\frac{1}{2^{3(p-1)}}$ and so on.

$$\therefore \quad S < 1 + \frac{1}{2^{p-1}} + \frac{1}{2^{2(p-1)}} + \frac{1}{2^{3(p-1)}} + \dots,$$

which is a geometrical progression in which the common ratio is less than unity;

$\therefore$ the series is convergent (Test IV.).

(ii) Let p be equal to unity. Let the terms of the series be divided into groups, so that the first group contains 1 term, the second 1, the third 2, the fourth 4, the fifth 8, and so on.

The sum of the third group of terms is $\frac{1}{3}+\frac{1}{4}$, which is $> \frac{1}{4}+\frac{1}{4}$ or $\frac{1}{2}$ and so on.

$$\therefore \quad S > 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \dots;$$

$\therefore$ the series is divergent (Test I.).

(iii) Let p be less than unity.

Then, since $2^p < 2$, $3^p < 3$, and so on,

$$S > 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \dots;$$

$\therefore$ the series is divergent (Test IV.).

382. *If the two infinite series*

$$a + bx + cx^2 + dx^3 + \dots \text{ and } a' + b'x + c'x^2 + d'x^3 + \dots$$

be equal for all values of x for which both series are convergent, then

$$a = a', \quad b = b', \quad c = c', \textit{ etc.}$$

For all the values of x for which the given series are convergent, we have

$$(a-a') + (b-b')x + (c-c')x^2 + (d-d')x^3 + \dots = 0. \qquad (1)$$

Now, both series are convergent when $x = 0$; hence, putting $x = 0$, we have

$$a - a' = 0 \text{ or } a = a'.$$

$$\therefore \quad (b-b')x + (c-c')x^2 + (d-d')x^3 + \dots = 0.$$

$$\therefore \quad x[(b-b') + (c-c')x + (d-d')x^2 + \dots] = 0. \qquad (2)$$

Now, for all values of x for which the series (1) is convergent, it follows that

$$(b-b')\,x+(c-c')\,x^2+(d-d')\,x^3+\ldots$$

is convergent, and therefore that

$$(b-b')+(c-c')\,x+(d-d')\,x^2+\ldots$$

is convergent. Hence, by (2) we have, for all such values of x,

$$(b-b')+(c-c')\,x+(d-d')\,x^2+\ldots=0.$$

Hence, as before, $\qquad b=b'$,

and similarly $\qquad c=c', \quad d=d'$, etc.

Exercises 215.

Determine whether the following series are convergent, oscillating or divergent:

1. $\frac{1}{5!}+\frac{1}{6!}+\frac{1}{7!}+\ldots.$

2. $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots$

3. $\frac{2}{1!}+\frac{2^2}{2!}+\frac{2^3}{3!}+\frac{2^4}{4!}+\ldots.$

4. $\frac{1}{2^0}+\frac{1}{2^1}+\frac{1}{2^2}+\ldots.$

5. $\frac{3}{5}+\frac{3}{5.2}+\frac{3}{5.2^2}+\frac{3}{5.2^3}+\ldots.$

6. $\frac{2}{3}-\frac{3}{4}+\frac{4}{5}-\frac{5}{6}+\ldots.$

7. $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\ldots.$

8. $\frac{3}{5}+\frac{4}{6.2}+\frac{5}{7.2^2}+\frac{6}{8.2^3}+\ldots.$

9. $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots.$

10. $\frac{3}{2}+\frac{3}{2.2}+\frac{3}{2.3}+\frac{3}{2.4}+\ldots.$

11. $\frac{2}{3}+\frac{2^2}{3.4}+\frac{2^3}{3.4.5}+\frac{2^4}{3.4.5.6}+\ldots.$

12. $\frac{4}{3} + \frac{5}{4 \,.\, 2} + \frac{6}{5 \,.\, 3} + \frac{7}{6 \,.\, 4} + \dots$

13. $\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \dots$

14. $\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \dots$

15. $1 + \frac{1}{2\,\underline{|2}} + \frac{2}{3\,\underline{|3}} + \frac{3}{4\,\underline{|4}} + \dots$

16. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \dots$

17. $\frac{1}{\sqrt{7}} + \frac{1}{\sqrt{8}} + \frac{1}{\sqrt{9}} + \frac{1}{\sqrt{10}} + \dots$

18. $1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \dots$

19. $1 - \frac{5}{4} + \frac{6}{5} - \frac{7}{6} + \dots$

20. $\frac{8}{5} + \frac{2}{5} + \frac{4}{15} + \frac{8}{45} + \dots$

21. $\frac{2}{3} + \frac{4}{3 \,.\, 2} + \frac{5}{4 \,.\, 3} + \frac{6}{5 \,.\, 4} + \dots$

22. $\frac{4}{5} + \frac{5}{6\,\underline{|2}} + \frac{6}{7\,\underline{|3}} + \frac{7}{8\,\underline{|4}} + \dots$

23. $\frac{8}{10} + \frac{8}{8} + \frac{15}{32} + \frac{75}{128} + \dots$

24. $\frac{16}{3} - \frac{8}{4} + \frac{4}{5} - \frac{2}{6} + \frac{1}{7} + \dots$

25. $\frac{27}{32} + \frac{9}{16} + \frac{3}{8} + \frac{1}{4} + \dots$

26. $\frac{64}{81} + \frac{32}{27} + \frac{16}{9} + \frac{8}{3} + \dots$

27. $1 - \frac{4}{5} + \frac{5}{6} - \frac{6}{7} + \dots$

28. $1 + \frac{1}{\sqrt{2}} + \frac{1}{2} + \frac{1}{2\sqrt{2}} + \dots$

29. $\frac{1}{1 \,.\, 2 \,.\, 3} + \frac{1}{2 \,.\, 3 \,.\, 4} + \frac{1}{3 \,.\, 4 \,.\, 5} + \dots$

30. $1 + 2x + 3x^2 + 4x^3 + \dots$

31. $\frac{x}{1 \,.\, 2} + \frac{x^2}{2 \,.\, 3} + \frac{x^3}{3 \,.\, 4} + \frac{x^4}{4 \,.\, 5} + \dots$

32. $1 + \frac{1}{2^3} + \frac{2}{3^3} + \frac{3}{4^3} + \dots$

CHAPTER XXXIV.

BINOMIAL THEOREM: ANY INDEX.

383. The object of this chapter is to prove that, if n be a positive or negative integer or commensurable fraction,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{\lfloor 2} x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{\lfloor r} x^r + \ldots.$$

It is sufficient to prove the theorem for the binomial expression $1+x$, for $(a+b)^n$ may be written in the form $a^n\left(1+\frac{b}{a}\right)^n$.

384. *To prove that the series*

$$1 + nx + \frac{n(n-1)}{\lfloor 2} x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{\lfloor r} x^r + \ldots$$

is convergent if x be less than 1.

If u_r denote the rth term of the series, we have

$$\frac{u_{r+1}}{u_r} = \frac{n-r+1}{r} x = \left(\frac{n+1}{r} - 1\right) x.$$

(i) Let n be positive. When $r > n+1$, the factor $\left(\frac{n+1}{r} - 1\right)$ is negative and numerically less than unity, and thus, if $|x| < 1$, the ratio u_{r+1}/u_r is numerically less than a quantity which is itself less than unity. Moreover, if x be positive and $r > n+1$, the ratio u_{r+1}/u_r is negative and the subsequent terms of the series alternate in sign. If x be negative and $r > n+1$, the ratio u_{r+1}/u_r is positive and the subsequent terms are of the same sign. Thus, if $|x| < 1$, the series is convergent, in the former case by Art. 378, in the latter case by Art. 372.

(ii) Let n be negative. If n be numerically less than unity, the factor $\left(\frac{n+1}{r} - 1\right)$ is negative for all integral values of r and numerically less than unity. If $n = -1$, the factor $\left(\frac{n+1}{r} - 1\right)$

is equal to -1. In both cases, if $|x|<1$, the series, as above, is convergent. If n be numerically greater than unity, the factor $\left(\frac{n+1}{r}-1\right)$ is negative for all integral values of r and numerically greater than unity; but, if $|x|<1$, the ratio u_{r+1}/u_r can, by taking r great enough, be made numerically less than a quantity which is itself less than unity, and the series, as above, is convergent.

Thus, if $|x|<1$, the series is convergent.

385. The following abbreviations will be used in proving the binomial theorem.

Any function of x, i.e. any expression containing x, may be denoted by $f(x)$ or some similar form. Thus, if $f(x)$ denote the function $1+2x+3x^2+4x^3$, then $f(y)$ denotes the same function of y, namely $1+2y+3y^2+4y^3$, $f\left(\frac{h}{k}\right)$ the same function of $\frac{h}{k}$, namely

$$1+2\frac{h}{k}+3\frac{h^2}{k^2}+4\frac{h^3}{k^3},$$

$f(-z)$ the same function of $-z$, namely

$$1+2(-z)+3(-z)^2+4(-z)^3 \text{ or } 1-2z+3z^2-4z^3,$$

while $f(0)$ denotes the same function of 0, namely

$$1+2.0+3.0+4.0 \text{ or } 1.$$

Again, if m be positive or negative, integral or fractional, m_r will be used to denote the product of r factors

$$m(m-1)(m-2)\dots(m-r+1);$$

so that if m be a positive integer, $m_r = {}_m\mathrm{P}_r$.

Also,

$$m_r \times (m-r) = m(m-1)\dots(m-r+1)(m-r) = m_{r+1}.$$

***386.** *To prove by induction that, if r be a positive integer, and m and n any quantities, positive or negative, integral or fractional,*

$$(m+n)_r = m_r + r \,.\, m_{r-1}n_1 + \frac{r(r-1)}{\underline{|2}}\, m_{r-2}n_2 + \dots + r \,.\, m_1 n_{r-1} + n_r.$$

Assume the result to be true for the value r. Multiply both sides of the equation by $m+n-r$, so that

$$(m+n)_r \times (m+n-r) = m_r(m+n-r) + r\,.\,m_{r-1}n_1 \times (m+n-r) + \ldots$$
$$\ldots + {}_r\mathrm{C}_s\,.\,m_{r-s}n_s \times (m+n-r)$$
$$+ {}_r\mathrm{C}_{s+1}\,.\,m_{r-s-1}n_{s+1} \times (m+n-r)$$
$$+ \ldots + n_r \times (m+n-r).$$

Now,

$$(m+n)_r \times (m+n-r) = (m+n)_{r+1},$$
$$m_r \times (m+n-r) = m_r \times (m-r) + m_r \times n = m_{r+1} + m_r n_1,$$
$$r\,.\,m_{r-1}n_1 \times (m+n-r) = r\,.\,m_{r-1}(m-r+1) \times n_1 + r\,.\,m_{r-1}n_1 \times (n-1)$$
$$= r\,.\,m_r n_1 + r\,.\,m_{r-1}n_2,$$
$${}_r\mathrm{C}_s\,.\,m_{r-s}n_s(m+n-r) = {}_r\mathrm{C}_s\,.\,m_{r-s} \times (m-r+s)\,n_s + {}_r\mathrm{C}_s\,.\,m_{r-s}n_s \times (n-s)$$
$$= {}_r\mathrm{C}_s\,.\,m_{r-s+1}n_s + {}_r\mathrm{C}_s\,.\,m_{r-s}n_{s+1},$$
$${}_r\mathrm{C}_{s+1}\,.\,m_{r-s-1}n_{s+1}(m+n-r) = {}_r\mathrm{C}_{s+1}\,.\,m_{r-s-1} \times (m-r+s+1)\,.\,n_{s+1}$$
$$+ {}_r\mathrm{C}_{s+1}\,.\,m_{r-s-1}n_{s+1}(n-s-1)$$
$$= {}_r\mathrm{C}_{s+1}\,.\,m_{r-s}n_{s+1} + {}_r\mathrm{C}_{s+1}m_{r-s-1}n_{s+2},$$
$$n_r \times (m+n-r) = n_r \times (n-r) + n_r \times m = n_{r+1} + n_r m_1.$$
$$\therefore\ (m+n)_{r+1} = m_{r+1} + m_r n_1(1+r) + \ldots + m_{r-s}n_{s+1}({}_r\mathrm{C}_s + {}_r\mathrm{C}_{s+1}) + \ldots + n_{r+1}$$
$$= m_{r+1} + (r+1)\,m_r n_1 + \ldots + {}_{r+1}\mathrm{C}_{s+1}\,.\,m_{r-s}n_{s+1} + \ldots + n_{r+1}.$$

Thus, if the theorem be true for r, it is also true for $r+1$. But, it is true when $r=1$, for $(m+n)_1 = m+n = m_1+n_1$; hence it is also true when $r=2$, therefore when $r=3$, and so on; and therefore when r is any positive integer.

387. *If $f(m)$ denote the series*

$$1 + m_1 x + \frac{m_2}{\underline{|2}}x^2 + \ldots + \frac{m_r}{\underline{|r}}x^r + \ldots,$$

where $x<1$, to prove that

$$f(m) \times f(n) \times f(p) \times \ldots = f(m+n+p+\ldots).$$

Since

$$f(m) = 1 + m_1 x + \frac{m_2}{\underline{|2}} x^2 + \dots + \frac{m_r}{\underline{|r}} x^r + \dots,$$

$$\therefore f(n) = 1 + n_1 x + \frac{n_2}{\underline{|2}} x^2 + \dots + \frac{n_r}{\underline{|r}} x^r + \dots,$$

and

$$f(m+n) = 1 + (m+n)_1 x + \frac{(m+n)_2}{\underline{|2}} x^2 + \dots + \frac{(m+n)_r}{\underline{|r}} x^r + \dots;$$

and, since $x < 1$, each of these three series is a convergent series.

Now, coefficient of x^r in the product of the series $f(m)$ and $f(n)$ is

$$\frac{m_r}{\underline{|r}} + \frac{m_{r-1}}{\underline{|r-1}} \cdot \frac{n_1}{\underline{|1}} + \frac{m_{r-2}}{\underline{|r-2}} \cdot \frac{n_2}{\underline{|2}} + \dots + \frac{m_1}{\underline{|1}} \cdot \frac{n_{r-1}}{\underline{|r-1}} + \frac{n_r}{\underline{|r}},$$

which, by art. 386, is $\frac{(m+n)_r}{\underline{|r}}$. Hence, for all values of r, the coefficient of x^r in the product of the series $f(m)$ and $f(n)$ is equal to the coefficient of x^r in the series $f(m+n)$.

$$\therefore f(m) \times f(n) = f(m+n).$$

Similarly,

$$f(m) \times f(n) \times f(p) = f(m+n) \times f(p) = f(m+n+p),$$

$$\therefore f(m) \times f(n) \times f(p) \times \dots = f(m+n+p+\dots).$$

388. *To prove the binomial theorem that*

$$(1+x)^n = 1 + n_1 x + \frac{n_2}{\underline{|2}} x^2 + \frac{n_3}{\underline{|3}} x^3 + \dots + \frac{n_r}{\underline{|r}} x^r + \dots.$$

(i) Let n be a positive commensurable fraction and equal to $\frac{h}{k}$, where h and k are positive integers.

By the preceding theorem,

$$f\left(\frac{h}{k}\right) \times f\left(\frac{h}{k}\right) \times f\left(\frac{h}{k}\right) \times \dots \text{ to } k \text{ factors}$$

$$= f\left(\frac{h}{k} + \frac{h}{k} + \frac{h}{k} + \dots \text{ to } k \text{ terms}\right),$$

$$\therefore \quad \left[f\left(\frac{h}{k}\right)\right]^k = f(h) = (1+x)^h,$$

since h is a positive integer,

$$\therefore \quad (1+x)^{\frac{h}{k}} = f\left(\frac{h}{k}\right) = 1 + \frac{h}{k} + \frac{1}{\underline{|2}}\frac{h^2}{k^2}x^2 + \ldots.$$

$\therefore$ if n be a positive commensurable fraction,

$$(1+x)^n = 1 + n_1 x + \frac{n_2}{\underline{|2}}x^2 + \ldots + \frac{n_r}{\underline{|r}}x^r + \ldots.$$

(ii) Let n be a negative integer or commensurable fraction and equal to $-s$.

Now, $$f(-s) \times f(s) = f(-s+s) = f(0) = 1,$$

$$\therefore \quad f(-s) = \frac{1}{f(s)} = \frac{1}{(1+x)^s},$$

since s is a positive integer or commensurable fraction,

$$\therefore \quad f(-s) = (1+x)^{-s},$$

$$\therefore \quad (1+x)^{-s} = 1 + (-s)\,x + \frac{(-s)(-s-1)}{\underline{|2}}x^2 + \ldots.$$

$\therefore$ if n be a negative integer or commensurable fraction,

$$(1+x)^n = 1 + n_1 x + \frac{n_2}{\underline{|2}}x^2 + \ldots + \frac{n^r}{\underline{|r}}x^r + \ldots.$$

389. *To find the general term in the expansion* of $(1+x)^n$.

The index of x in the $(r+1)$th term is r,

$\therefore$ the $(r+1)$th term is $\dfrac{n_r}{\underline{|r}}x^r$.

390. *Example* 1. Expand to five terms:

(i) $(1+x)^{\frac{5}{2}}$, (ii) $(1-x)^{-3}$, (iii) $(1+2x)^{-\frac{3}{2}}$.

(i) $$(1+x)^{\frac{5}{2}} = 1 + \tfrac{5}{2}x + \frac{\frac{5}{2}\left(\frac{5}{2}-1\right)}{\lfloor 2} x^2 + \frac{\frac{5}{2}\left(\frac{5}{2}-1\right)\left(\frac{5}{2}-2\right)}{\lfloor 3} x^3 + \frac{\frac{5}{2}\left(\frac{5}{2}-1\right)\left(\frac{5}{2}-2\right)\left(\frac{5}{2}-3\right)}{\lfloor 4} x^4 + \ldots$$

$$= 1 + \tfrac{5}{2}x + \frac{5 \,.\, 3}{2^2 \,.\, 2} x^2 + \frac{5 \,.\, 3 \,.\, 1}{2^3 \,.\, 6} x^3 + \frac{5 \,.\, 3 \,.\, 1\,(-1)}{2^4 \,.\, 24} x^4 + \ldots$$

$$= 1 + \tfrac{5}{2}x + \tfrac{15}{8}x^2 + \tfrac{5}{16}x^3 - \tfrac{5}{128}x^4 + \ldots.$$

(ii) $$(1-x)^{-3} = 1 + (-3)(-x) + \frac{(-3)(-3-1)}{\lfloor 2}(-x)^2 + \frac{(-3)(-3-1)(-3-2)}{\lfloor 3}(-x)^3 + \frac{(-3)(-3-1)(-3-2)(-3-3)}{\lfloor 4}(-x)^4 + \ldots$$

$$= 1 + 3x + \frac{3 \,.\, 4}{1 \,.\, 2}x^2 + \frac{3 \,.\, 4 \,.\, 5}{1 \,.\, 2 \,.\, 3}x^3 + \frac{3 \,.\, 4 \,.\, 5 \,.\, 6}{1 \,.\, 2 \,.\, 3 \,.\, 4}x^4 + \ldots$$

$$= 1 + 3x + 6x^2 + 10x^3 + 15x^4 + \ldots.$$

(iii) $$(1+2x)^{-\frac{3}{2}} = 1 + \left(-\tfrac{3}{2}\right)2x + \frac{\left(-\frac{3}{2}\right)\left(-\frac{3}{2}-1\right)}{\lfloor 2}(2x)^2 + \frac{\left(-\frac{3}{2}\right)\left(-\frac{3}{2}-1\right)\left(-\frac{3}{2}-2\right)}{\lfloor 3}(2x)^3 + \frac{\left(-\frac{3}{2}\right)\left(-\frac{3}{2}-1\right)\left(-\frac{3}{2}-2\right)\left(-\frac{3}{2}-3\right)}{\lfloor 4}(2x)^4 + \ldots$$

$$= 1 - 3x + \frac{3 \,.\, 5}{2^2 \lfloor 2} \,.\, 2^2x^2 - \frac{3 \,.\, 5 \,.\, 7}{2^3 \lfloor 3} \,.\, 2^3x^3 + \frac{3 \,.\, 5 \,.\, 7 \,.\, 9}{2^4 \lfloor 4} \,.\, 2^4x^4 - \ldots$$

$$= 1 - 3x + \tfrac{15}{2}x^2 - \tfrac{35}{2}x^3 + \tfrac{315}{8}x^4 - \ldots.$$

Example 2. Find the $(r+1)$th term in the expansion of

(i) $(1+x)^{\frac{5}{2}}$, (ii) $(1-x)^{-3}$, (iii) $(1+2x)^{-\frac{3}{2}}$.

(i) $$(r+1)\text{th term} = \frac{\frac{5}{2}\left(\frac{5}{2}-1\right)\left(\frac{5}{2}-2\right)\ldots\left(\frac{5}{2}-r+1\right)}{\lfloor r} x^r.$$

Now the first three factors of the numerator of the coefficient are positive and the rest negative,

$$\therefore\ (r+1)\text{th term} = (-1)^{r-3} \,.\, \frac{5 \,.\, 3 \,.\, 1 \,.\, 1 \,.\, 3 \,.\, \ldots\, (2r-7)}{2^r \lfloor r} x^r.$$

(ii) $(r+1)$th term

$$= \frac{(-3)(-3-1)(-3-2)\ldots(-3-r+1)}{\lfloor r} (-x)^r.$$

Now, there are r factors equal to -1 in the numerator of the coefficient and the same number in $(-x)^r$.

$$\therefore\ (r+1)\text{th term} = (-1)^{2r}.\ \frac{3.4.5.\ldots r(r+1)(r+2)}{1.2.3.\ldots r} x^r$$

$$= \tfrac{1}{2}(r+1)(r+2)x^r.$$

(iii) $(r+1)$th term

$$= \frac{(-\frac{3}{2})(-\frac{3}{2}-1)(-\frac{3}{2}-2)\ldots(-\frac{3}{2}-r+1)}{\lfloor r} (2x)^r$$

$$= (-1)^r.\ \frac{3.5.7\ldots(2r+1)}{2^r.\lfloor r} 2^r x^r$$

$$= (-1)^r.\ \frac{3.5.7\ldots(2r+1)}{\lfloor r} x^r.$$

Example 3. Find the first negative term in the expansion of $(1+x)^{\frac{7}{3}}$.

The first negative term is

$$\frac{\frac{7}{3}(\frac{7}{3}-1)(\frac{7}{3}-2)(\frac{7}{3}-3)}{\lfloor 4} x^4$$

$$= \frac{7.4.1.(-2)}{3^4\lfloor 4} x^4$$

$$= -\frac{7}{243} x^4.$$

Example 4. Find the value of $99^{\frac{1}{2}}$ to 5 places of decimals.

$$99^{\frac{1}{2}} = (100 - 1)^{\frac{1}{2}} = 100^{\frac{1}{2}} \left(1 - \tfrac{1}{100}\right)^{\frac{1}{2}}$$

$$= 10 \left[1 + \tfrac{1}{2}\left(-\tfrac{1}{100}\right) + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)}{\lfloor 2} \left(-\tfrac{1}{100}\right)^2 \right.$$

$$\left. + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)\left(\frac{1}{2} - 2\right)}{\lfloor 3} \left(-\tfrac{1}{100}\right)^3 + \dots \right]$$

$$= 10\left[1 - \tfrac{1}{2} \times \cdot 01 - \tfrac{1}{8} \times \cdot 0001 - \tfrac{1}{16} \times \cdot 000001 - \dots\right]$$

$$= 10\left[1 - \cdot 005 - \cdot 0000125 - \cdot 00000006\dots\right]$$

$$= 10\left[1 - \cdot 00501256\dots\right]$$

$$= 10 \times \cdot 9949874$$

$$= 9 \cdot 94987.$$

Exercises 216.

Expand to 5 terms:

1. $(1 + x)^{\frac{1}{2}}$.
2. $(1 + x)^{\frac{3}{2}}$.
3. $(1 - x)^{\frac{11}{2}}$.
4. $(1 - x)^{-1}$.
5. $(1 - x)^{-2}$.
6. $(1 + x)^{-3}$.
7. $(1 - 2x)^{-\frac{1}{2}}$.
8. $(1 - 3x)^{-\frac{2}{3}}$.
9. $(1 - x^2)^{-\frac{3}{2}}$.
10. $\dfrac{1}{\sqrt[3]{(1 - x)}}$.
11. $(x + x^2)^{-4}$.
12. $\left(\sqrt{x} - \dfrac{3}{\sqrt{x}}\right)^{\frac{7}{3}}$.

Find the

13. 5th term of $(1 + x)^{\frac{5}{2}}$.
14. 4th term of $(1 + 2x)^{\frac{1}{2}}$.
15. 6th term of $(1 + 2x)^{\frac{3}{2}}$.
16. 6th term of $(1 + x)^{-3}$.
17. 4th term of $(1 - 2x)^{-\frac{1}{2}}$.
18. 6th term of $(1 - x)^{-\frac{4}{3}}$.

Find the $(r + 1)$th term of

19. $(1 + 2x)^{\frac{1}{2}}$.
20. $(1 - x)^{-1}$.
21. $(1 - x)^{-2}$.
22. $(1 + x)^{-3}$.
23. $(1 - x)^{-\frac{1}{2}}$.
24. $(1 - 2x)^{-\frac{3}{2}}$.

25. $\left(1-\frac{x}{2}\right)^{-5}$. 26. $(1-x)^{-n}$. 27. $(1-x)^{\frac{10}{3}}$.

28. $(1-nx)^{\frac{1}{n}}$. 29. $\left(1+\frac{x}{2}\right)^{2n}$. 30. $\left(x^n-\frac{n}{x^n}\right)^{-\frac{1}{n}}$.

31. Find the 5th term in the expansion of $\left(2x-\frac{x^2}{2}\right)^{\frac{3}{2}}$.

32. Find the 8th term in the expansion of $(a-2x)^{-\frac{1}{2}}$.

33. Expand $(1-3x)^{\frac{2}{3}}$ to 5 terms, and write down the general term.

34. Expand $(1-4x)^{-\frac{3}{2}}$ to 5 terms, and show that the general term is $\frac{\lfloor 2r+1}{(\lfloor r)^2}x^r$.

35. Write down the first negative term in the expansion of $(1+x)^{\frac{7}{2}}$.

36. Write down the first negative term in the expansion of $\left(1+\frac{4x}{3}\right)^{\frac{11}{2}}$.

37. Prove that every term in the expansion of $(1-x)^{\frac{1}{2}}$ after the first is negative.

38. Show that only two terms in the expansion of $(1-x)^{\frac{12}{5}}$ are positive.

39. Prove that the coefficient of x^m in the expansion of $(1-x)^{-(n+1)}$ is equal to the coefficient of x^n in the expansion of $(1-x)^{-(m+1)}$.

40. Find the value of x when the 4th term in the expansion of $(1+x)^{\frac{5}{2}}$ is $\frac{5}{2}$.

41. The coefficient of the 3rd term in the expansion of $(1-x)^{-n}$ is $\frac{2}{9}$; find n and the coefficient of the 5th term.

42. Find the value of $101^{\frac{1}{2}}$ to 6 places of decimals.

43. Find the value of $1{\cdot}04^{\frac{1}{4}}$ to 7 places of decimals.

44. Find the value of $\sqrt[5]{3128}$ to 6 places of decimals.

45. Show that $\sqrt[5]{\cdot 875} = \cdot 97365$ nearly.

***391.** *Example* 5. Find which are the greatest terms in the expansion of

$$\text{(i)}\quad (1+\tfrac{6}{7})^{\frac{7}{2}}. \qquad\qquad \text{(ii)}\quad (1-\tfrac{3}{4})^{-8}.$$

(i) Let the $(r+1)$th term be not less than any other.

$$\text{Then}\quad \frac{\frac{7}{2}-r+1}{r}\cdot\frac{6}{7} \not< 1 \quad\text{and}\quad \frac{\frac{7}{2}-r}{r+1}\cdot\frac{6}{7} \not> 1,$$

$$\therefore\ 27-6r \not< 7r \quad\text{and}\quad 21-6r \not> 7r+7,$$

$$\therefore\ 13r \not> 27 \quad\text{and}\quad \not< 14,$$

$$\therefore\ 13r = 26 \quad\text{or}\quad r = 2.$$

$$\therefore\ \text{the 3rd term is the greatest.}$$

(ii) Let the $(r+1)$th term be not less than any other. Then $\dfrac{-3-r+1}{r}\cdot\dfrac{3}{4}$ is numerically $\not< 1$ and $\dfrac{-3-r}{r+1}\cdot\dfrac{3}{4}$ is numerically $\not> 1$,

$$\therefore\ \frac{2+r}{r}\cdot\frac{3}{4} \not< 1 \quad\text{and}\quad \frac{3+r}{r+1}\cdot\frac{3}{4} \not> 1,$$

$$\therefore\ 6+3r \not< 4r \quad\text{and}\quad 9+3r \not> 4r+4,$$

$$\therefore\ r \not> 6 \quad\text{and}\quad \not< 5,$$

$$\therefore\ r = 5 \text{ or } 6.$$

$$\therefore\ \text{the 6th and 7th terms are the greatest.}$$

Example 6. Find the coefficient of x^n in the expansion of $\dfrac{5(1-x)}{1-x-6x^2}$; and write down the coefficients of x^4 and x^5.

$$\text{Let}\qquad \frac{5(1-x)}{1-x-6x^2} = \frac{A}{1-3x} + \frac{B}{1+2x},$$

$$\therefore\ 5-5x = A(1+2x) + B(1-3x).$$

Put $x = \frac{1}{3}$, then $\frac{10}{3} = \text{A} \cdot \frac{5}{3}$, or $\text{A} = 2$.

Put $x = -\frac{1}{2}$, then $\frac{15}{2} = \text{B}\,(1 + \frac{3}{2})$, or $\text{B} = 3$.

$$\therefore \quad \frac{5\,(1-x)}{1-x-6x^2} = \frac{2}{1-3x} + \frac{3}{1+2x}$$

$$= 2\,(1-3x)^{-1} + 3\,(1+2x)^{-1}$$

$$= 2\,(1 + 3x + 3^2x^2 + \ldots + 3^n x^n + \ldots)$$
$$+ 3\,\{1 - 2x + 2^2x^2 + \ldots + (-1)^n 2^n x^n + \ldots\}.$$

$$\therefore \text{ coefficient of } x^n = 2 \cdot 3^n + (-1)^n\, 3 \cdot 2^n.$$

$\therefore$ coefficient of $x^4 = 2 \cdot 3^4 + 3 \cdot 2^4 = 2 \cdot 81 + 3 \cdot 16 = 162 + 48 = 210$;
and coefficient of $x^5 = 2 \cdot 3^5 - 3 \cdot 2^5 = 2 \cdot 243 - 3 \cdot 32 = 486 - 96 = 390$.

Example 7. Find the coefficient of x^n in the expansion of $\frac{(1+x)^2}{(1-x)^3}$, and write down the coefficients of x^5 and x^7.

$$\frac{(1+x)^2}{(1-x)^3} = (1 + 2x + x^2)\,(1-x)^{-3}$$

$$= (1 + 2x + x^2)\,\{1 + 3x + 6x^2 + \ldots + \tfrac{1}{2}(n+1)(n+2)\,x^n + \ldots\}.$$

$$\therefore \text{ coefficient of } x^n = \tfrac{1}{2}n\,(n+1) + 2 \cdot \tfrac{1}{2}(n-1)\,n + \tfrac{1}{2}(n-2)(n-1)$$
$$= \tfrac{1}{2}(n^2 + n + 2n^2 - 2n + n^2 - 3n + 2)$$
$$= \tfrac{1}{2}(4n^2 - 4n + 2) = 2n^2 + 2n + 1.$$

$\therefore$ coefficient of $x^5 = 2 \cdot 5^2 + 2 \cdot 5 + 1 = 50 - 10 + 1 =$ ~~41~~; 61
and coefficient of $x^7 = 2 \cdot 7^7 - 2 \cdot 7 + 1 = 98 - 14 + 1 = 85$.

Example 8. In the expansion of $\frac{5-6x}{(1-x)^2}$, show that the coefficient of x^5 is zero, that the coefficients of all lower powers of x are positive, and of all higher powers of x negative.

$$\frac{5-6x}{(1-x)^2} = (5-6x)\,(1-x)^{-2}$$

$$= (5-6x)\,(1 + 2x + 3x^2 + \ldots + \overline{n+1} \cdot x^n + \ldots).$$

$$\therefore \text{ coefficient of } x^n = 5\,(n+1) - 6n = 5 - n.$$

$\therefore$ coefficient of x^n is zero when $n = 5$, is positive for values of $n < 5$, and negative for values of $n > 5$.

Example 9. If x be a small fraction, show that

$$\frac{(1+x)^{\frac{1}{2}}(1-2x)^{\frac{1}{2}}}{1-x}$$

is equal to $1 + \frac{1}{2}x$ approximately.

$$\begin{aligned}
\frac{(1+x)^{\frac{1}{2}}(1-2x)^{\frac{1}{2}}}{1-x} \\
&= (1+x)^{\frac{1}{2}}(1-2x)^{\frac{1}{2}}(1-x)^{-1} \\
&= (1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \dots)(1 - x - \tfrac{1}{2}x^2 - \dots)(1 + x + x^2 + \dots) \\
&= \{1 + x(\tfrac{1}{2} - 1) + x^2(-\tfrac{1}{8} - \tfrac{1}{2} - \tfrac{1}{2}) + \dots\}(1 + x + x^2 + \dots) \\
&= (1 - \tfrac{1}{2}x - \tfrac{9}{8}x^2 - \dots)(1 + x + x^2 + \dots) \\
&= 1 + \tfrac{1}{2}x + \text{terms in } x^2 \text{ and higher powers of } x \\
&= 1 + \tfrac{1}{2}x, \text{ approximately.}
\end{aligned}$$

Example 10. Find the sum of n terms of the H.P.

$$\frac{1}{a+b} + \frac{1}{a+2b} + \frac{1}{a+3b} + \dots,$$

where b is so small compared with a that terms in b^2, b^3, etc. may be neglected.

$$\begin{aligned}
S &= \frac{1}{a\left(1+\frac{b}{a}\right)} + \frac{1}{a\left(1+\frac{2b}{a}\right)} + \dots + \frac{1}{a\left(1+\frac{nb}{a}\right)} \\
&= \frac{1}{a}\left[\left(1+\frac{b}{a}\right)^{-1} + \left(1+\frac{2b}{a}\right)^{-1} + \dots + \left(1+\frac{nb}{a}\right)^{-1}\right] \\
&= \frac{1}{a}\left[1 - \frac{b}{a} + 1 - \frac{2b}{a} + \dots + 1 - \frac{nb}{a}\right],
\end{aligned}$$

neglecting squares and higher powers of b,

$$\begin{aligned}
&= \frac{1}{a}\left[n - \frac{b}{a}\cdot\frac{n(n+1)}{2}\right] \\
&= \frac{n}{2a^2}\left[2a - (n+1)b\right].
\end{aligned}$$

*Exercises 217.

1. Find the greatest term in the expansion of $(1+\frac{2}{3})^{\frac{11}{2}}$.

2. Find the greatest term in the expansion of $(1+\frac{1}{4})^{\frac{3}{5}}$.

3. Which is the greatest term in the expansion of $(1+\frac{4}{7})^{-7}$?

4. Show that there are two terms in the expansion of $(1-\frac{15}{17})^{-\frac{6}{5}}$ greater than all the others.

5. For what values of x is the series for $(a+2x)^{-\frac{3}{2}}$ convergent?

6. Find the coefficient of x^3 in the expansion of $\frac{a+bx}{p+qx}$ in ascending powers of x.

Find the coefficient of x^r in the expansions of:

7. $(1+x)^3(1-x)^{-2}$. 8. $\frac{1+x+x^2}{(1-x)^3}$. 9. $\frac{(1+x)^2}{(1-x)^4}$.

10. $(a-x)(a+x)^{-3}$. 11. $\left(\frac{1-x}{1-2x}\right)^2$.

12. Expand $\frac{(1-x)^2}{1+x+x^2}$ in a series of ascending powers of x; and, if a_r be the coefficient of x^r in this expansion, prove that

$$a_{3n}=0, \quad a_{3n+2}-a_{3n+1}=6.$$

13. Find the coefficient of x^4 when $\frac{(1+x)^2}{(1-x)^3(1-2x)^{\frac{1}{2}}}$ is expanded in ascending powers of x.

14. Prove that the coefficient of x^n in the expansion of $\frac{2-3x}{1-3x+2x^2}$ is 2^n+1.

15. Expand in powers of x to three terms and find the coefficient of x^n in $\frac{4}{(x-1)(x-3)}$.

16. Find the coefficient of x^r in the expansion of $\frac{1}{1-3x+2x^2}$.

17. Find the $(r+1)$th term in the expansion of

$$\frac{x}{(1-ax)(1-bx)}.$$

18. Find the coefficient of x^n in the expansion of $(1+x-2x^2)^{-1}$.

19. Expand each of the expressions $(1-x+x^2)^{-1}$ and $(1-x-2x^2)^{-1}$; and, if p_n, q_n be the coefficients of x^n in the two series, show that $3q_n - p_{3n} = 2^{n+1}$.

20. If x be a small fraction, show that $\dfrac{\sqrt{(1+x)}+\sqrt[3]{(1-x)^2}}{1+x+\sqrt{(1+x)}}$ is very nearly equal to $1-\frac{5}{6}x$.

21. If x be a small fraction, show that

$$\frac{(1-x)^{-\frac{2}{3}}-(1+x)^{\frac{2}{3}}}{(1-x)^{-1}-(1+x)}=\frac{2}{3}-\frac{2}{9}x$$

very nearly.

22. Show that three consecutive terms of the expansion of $(1+x)^n$ can be in continued proportion only when $n+1=0$.

23. If t_n denote the middle term in the expansion of $(1+x)^{2r}$, then

$$t_0+t_1+t_2+\ldots=(1-4x)^{-\frac{1}{2}}.$$

CHAPTER XXXV.

EXPONENTIAL AND LOGARITHMIC SERIES.

392. In the present chapter, we shall use the letter e to denote the infinite series

$$1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \dots + \frac{1}{\lfloor n} + \dots.$$

Since the test-ratio $u_{r+1} : u_r = \frac{1}{\lfloor r} \div \frac{1}{\lfloor r-1} = \frac{1}{r}$, it follows that the series is convergent. This may also be shown in the following manner:

Since $$\lfloor 3 = 3 \,.\, 2 > 2^2,$$

$$\frac{1}{\lfloor 3} < \frac{1}{2^2}.$$

Similarly, $\frac{1}{\lfloor 4} < \frac{1}{2^3}$, $\frac{1}{\lfloor 5} < \frac{1}{2^4}$, and so on.

$$\therefore\ e < 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^{n-1}} + \dots,$$

i.e. $$< 1 + \frac{1}{1 - \frac{1}{2}} \text{ or } 3.$$

Thus, e lies between 2 and 3.

The value of e may be found to any number of decimal places required by taking a sufficient number of terms.

393. *Example* 1. Find the value of e to 4 places of decimals.

Since $\frac{1}{\lfloor 2} = \cdot 5$, $\frac{1}{\lfloor 3} = \cdot 5 \div 3$, and so on,

$$\begin{aligned}\therefore\ e = &\ 1\\ &+1\\ &+\ \cdot 5\\ &+\ \cdot 166667\\ &+\ \cdot 041667\\ &+\ \cdot 008333\\ &+\ \cdot 001400\\ &+\ \cdot 000200\\ &+\ \cdot 000025\\ &+\ \cdot 000003\\ &\ \ \ \ \cdots\cdots\cdots\end{aligned}$$

$e = 2\cdot 7183$, to four places of decimals.

394. *To prove that e is incommensurable.*

If possible, let e be equal to a commensurable fraction $\frac{h}{k}$, where h and k are positive integers. Then

$$\frac{h}{k} = 1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \ldots + \frac{1}{\lfloor k} + \frac{1}{\lfloor k+1} + \frac{1}{\lfloor k+2} + \ldots.$$

Multiply both sides by $\lfloor k$, then

$$h \lfloor k-1 = \text{an integer} + \frac{1}{k+1} + \frac{1}{(k+1)(k+2)} + \ldots.$$

Now $\frac{1}{k+1} + \frac{1}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)} + \ldots$

$$< \frac{1}{k+1} + \frac{1}{(k+1)^2} + \frac{1}{(k+1)^3} + \ldots$$

$$< \frac{\dfrac{1}{k+1}}{1 - \dfrac{1}{k+1}}$$

or $\frac{1}{k}$, which is a proper fraction.

$\therefore$ e cannot be of the form $\frac{h}{k}$, i.e. it is incommensurable.

395. *To prove that the series*

$$1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \dots + \frac{x^n}{\lfloor n} + \dots$$

is convergent for all values of x.

The test-ratio $u_{r+1} : u_r = \frac{x^r}{\lfloor r} \div \frac{x^{r-1}}{\lfloor r-1} = \frac{x}{r}$.

Hence, for all values of $r > x$, the test-ratio is less than a quantity which is itself less than unity;

$\therefore$ the series is convergent.

396. *If $f(m)$ denote the infinite series*

$$1 + m + \frac{m^2}{\lfloor 2} + \frac{m^3}{\lfloor 3} + \dots + \frac{m^r}{\lfloor r} + \dots$$

to prove that

$$f(m) \times f(n) \times f(p) \times \dots = f(m + n + p + \dots).$$

We have

$$f(m) = 1 + m + \frac{m^2}{\lfloor 2} + \frac{m^3}{\lfloor 3} + \dots + \frac{m^r}{\lfloor r} + \dots \quad (1)$$

$$f(n) = 1 + n + \frac{n^2}{\lfloor 2} + \frac{n^3}{\lfloor 3} + \dots + \frac{n^s}{\lfloor s} + \dots \quad (2)$$

$$f(m+n) = 1 + (m+n) + \frac{(m+n)^2}{\lfloor 2} + \dots + \frac{(m+n)^{r+s}}{\lfloor r+s} + \dots \quad (3)$$

each of these three series being a convergent series.

The coefficient of $m^r n^s$ in the product of the series in (1) and (2) is $\frac{1}{\lfloor r \lfloor s}$. Now, the term in $m^r n^s$ in the series in (3), being of

$r+s$ dimensions, can only occur in the expansion of the term $\frac{(m+n)^{r+s}}{\underline{|r+s}}$, and the coefficient of $m^n n^s$ in this expansion is

$$\frac{1}{\underline{|r+s}} \times \frac{\underline{|r+s}}{\underline{|r}\,\underline{|s}} \text{ or } \frac{1}{\underline{|r}\,\underline{|s}}.$$

Thus, the coefficient of $m^r n^s$ in the product of the series (1) and (2) is equal to the coefficient of $m^r n^s$ in the expansion of the different terms of the series in (3); and this is true for all values of r and s.

$$\therefore\ f(m) \times f(n) = f(m+n).$$

Again,

$$f(m) \times f(n) \times f(p) = f(m+n) \times f(p) = f(m+n+p),$$

and, similarly,

$$f(m) \times f(n) \times f(p) \times \ldots = f(m+n+p+\ldots).$$

397. *To prove the exponential theorem, that*

$$e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \ldots + \frac{x^r}{\underline{|r}} + \ldots.$$

(i) Let x be a positive integer.

In the equation

$$f(m) \times f(n) \times f(p) \times \ldots = f(m+n+p+\ldots)$$

put $m = n = p = \ldots = 1$, and let the number of the quantities $m, n, p, \ldots$ be x; then

$$[f(1)]^x = f(x).$$

Now, $$f(1) = 1 + \frac{1}{\underline{|1}} + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \ldots + \frac{1}{\underline{|r}} + \ldots = e,$$

$$\therefore\ e^x = f(x)$$

$$= 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \ldots + \frac{x^r}{\underline{|r}} + \ldots.$$

(ii) Let x be a positive commensurable fraction, $\frac{h}{k}$, where h and k are positive integers.

Let $m = n = p = \ldots = \dfrac{h}{k}$, and let the number of the quantities $m, n, p, \ldots$ be k; then

$$\left[f\left(\frac{h}{k}\right)\right]^k = f\left(\frac{h}{k} \times k\right) = f(h) = e^h,$$

since h is a positive integer.

$$\therefore\ e^{\frac{h}{k}} = f\left(\frac{h}{k}\right) = 1 + \frac{h}{k} + \frac{1}{\underline{|2}}\left(\frac{h}{k}\right)^2 + \ldots + \frac{1}{\underline{|r}}\left(\frac{h}{k}\right)^r + \ldots,$$

$$\therefore\ e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \ldots + \frac{x^r}{\underline{|r}} + \ldots.$$

(iii) Let x be a negative integer or commensurable fraction and equal to $-y$.

Now $$f(-y) \times f(y) = f(-y + y) = f(0) = 1,$$

$$\therefore\ f(-y) = \frac{1}{f(y)} = \frac{1}{e^y} \text{ by parts (i) and (ii),}$$

$$= e^{-y},$$

$$\therefore\ e^{-y} = 1 + (-y) + \frac{(-y)^2}{\underline{|2}} + \ldots + \frac{(-y^r)}{\underline{|r}} + \ldots,$$

$$\therefore\ e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \ldots + \frac{x^r}{\underline{|r}} + \ldots.$$

398. *To prove that*

$$a^x = 1 + x\log a + \frac{x^2}{\underline{|2}}(\log a)^2 + \ldots + \frac{x^r}{\underline{|r}}(\log a)^r + \ldots.$$

Let $a = e^k$, so that $k = \log a$,

then $$a^x = (e^k)^x = e^{kx} = 1 + kx + \frac{k^2x^2}{\underline{|2}} + \ldots + \frac{k^r x^r}{\underline{|r}} + \ldots$$

$$= 1 + x\log a + \frac{x^2}{\underline{|2}}(\log a)^2 + \ldots + \frac{x^r}{\underline{|r}}(\log a)^r + \ldots.$$

399. *Example* 2. Find the coefficient of x^n in the expansions of

$$\text{(i)}\quad (a+bx)\,e^x, \qquad \text{(ii)}\quad \frac{a+bx}{e^x}.$$

(i) $(a+bx)\,e^x = (a+bx)\left(1 + x + \dfrac{x^2}{2!} + \ldots + \dfrac{x^{n-1}}{(n-1)!} + \dfrac{x^n}{n!} + \ldots\right),$

∴ coefficient of

$$x^n = \frac{a}{n!} + \frac{b}{(n-1)!} = \frac{a+bn}{n!}.$$

(ii) $\dfrac{a+bx}{e^x} = (a+bx)\,e^{-x}$

$$= (a+bx)\left\{1 - x + \frac{x^2}{2!} + \ldots + (-1)^{n-1}\frac{x^{n-1}}{(n-1)!} + (-1)^n\frac{x^n}{n!} + \ldots\right\},$$

∴ the coefficient of

$$x^n = (-1)^n\left(\frac{a}{n!} - \frac{b}{(n-1)!}\right) = (-1)^n\frac{a-bn}{n!}.$$

Example 3. Sum to infinity the series

$$1 + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^2}{4!} + \ldots.$$

The nth term of the series is

$$\frac{n^2}{n!} \text{ or } \frac{n}{(n-1)!} \text{ or } \frac{n-1}{(n-1)!} + \frac{1}{(n-1)!} \text{ or } \frac{1}{(n-2)!} + \frac{1}{(n-1)!}.$$

$$S = 1 + \left(\frac{1}{0!} + \frac{1}{1!}\right) + \left(\frac{1}{1!} + \frac{1}{2!}\right) + \left(\frac{1}{2!} + \frac{1}{3!}\right) + \ldots + \left(\frac{1}{(n-2)!} + \frac{1}{(n-1)!}\right) + \ldots$$

$$= 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{(n-1)!} + \ldots + 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{(n-2)!} + \ldots$$

$$= 2e.$$

Exercises 218.

1. If $x = 2$, show that every term in the expansion of

$$1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \ldots$$

after the 8th is $< \cdot 01$.

2. Find the coefficient of x^6 in the expansion of $(1 + 2x + x^2)e^x$.

3. Find the coefficient of x^n in the expansion of $(n - x)e^x$.

4. Find the coefficient of x^n in the expansion of $\dfrac{a + bx + cx^2}{e^x}$.

5. Find the sum of $\dfrac{1}{\underline{|2}} - \dfrac{1}{\underline{|3}} + \dfrac{1}{\underline{|4}} - \ldots$ to infinity.

6. Expand e^{-x} and show that

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} + \ldots + \frac{x^{2n}}{\underline{|2n}} + \ldots.$$

7. Show that

$$\frac{e^x - e^{-x}}{2} = x + \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} + \ldots + \frac{x^{2n+1}}{\underline{|2n+1}} + \ldots.$$

8. Prove that $\dfrac{1}{2e} = \dfrac{1}{\underline{|3}} + \dfrac{2}{\underline{|5}} + \dfrac{3}{\underline{|7}} + \ldots.$

9. Show that the coefficient of x^n in the series

$$1 + \frac{a + bx}{\underline{|1}} + \frac{(a + bx)^2}{\underline{|2}} + \ldots + \frac{(a + bx)^r}{\underline{|r}} + \ldots \text{ is } e^a \frac{b^n}{\underline{|n}}.$$

10. Show that

$$\frac{1^3}{\underline{|1}} + \frac{2^3}{\underline{|2}} + \ldots + \frac{r^3}{\underline{|r}} + \ldots \text{ to } \infty = 5e.$$

11. Show that

$$\frac{2\frac{1}{2}}{\underline{|1}} - \frac{3\frac{1}{3}}{\underline{|2}} + \frac{4\frac{1}{4}}{\underline{|3}} - \frac{5\frac{1}{5}}{\underline{|4}} + \ldots \text{ to } \infty = 1 + e^{-1}.$$

400. *To prove that the series*

$$x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \ldots - (-1)^n \frac{1}{n}x^n + \ldots$$

is convergent if $x < -1$ *and* $\not> 1$.

The test-ratio $u_{r+1} : u_r = -\frac{x^{r+1}}{r+1} \div \frac{x^r}{r} = -\frac{r}{r+1}x.$

Hence, if x be numerically less than 1, the series is convergent.

If $x = 1$, the series is $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ which is convergent (Art. 379).

If $x = -1$, the series is $-(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots)$ which is divergent (Art. 381).

$\therefore$ the given series is convergent if $x > -1$ and $\not> 1$.

401. *To prove the logarithmic series that*

$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots - (-1)^n \cdot \frac{1}{n}x^n + \ldots.$$

We have $\quad a^y = 1 + y\log a + \frac{1}{\underline{|2}}y^2(\log a)^2 + \ldots.$

Put $a = 1 + x$,

then $\quad (1+x)^y = 1 + y\log(1+x) + \frac{y^2}{\underline{|2}}(\log \overline{1+x})^2 + \ldots$

$$= 1 + yx + \frac{y(y-1)}{\underline{|2}}x^2 + \frac{y(y-1)(y-2)}{\underline{|3}}x^3 + \ldots$$

Equating the coefficients of y on both sides of this equation, we have*

$$\log(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \ldots.$$

* It should be noticed that the above reasoning forms an incomplete *proof* of the theorem, for it is not shown that the second of the above series for $(1+x)^y$ remains convergent when its terms are rearranged according to powers of y; or that, if convergent, the limit towards which it converges is the same as before.

402. *To prove that*

(i) $$\log\frac{1+x}{1-x}=2\left(x+\frac{1}{3}x^3+\frac{1}{5}x^5+\dots\right).$$

(ii) $$\log\frac{m}{n}=2\left[\frac{m-n}{m+n}+\frac{1}{3}\left(\frac{m-n}{m+n}\right)^3+\frac{1}{5}\left(\frac{m-n}{m+n}\right)^5+\dots\right].$$

(i) We have

$$\log(1+x)=\quad x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\tfrac{1}{4}x^4+\tfrac{1}{5}x^5-\dots,$$

$$\log(1-x)=-x-\tfrac{1}{2}x^2-\tfrac{1}{3}x^3-\tfrac{1}{4}x^4-\tfrac{1}{5}x^5-\dots,$$

$$\log\frac{1+x}{1-x}=\log(1+x)-\log(1-x),$$

$$=2\left(x+\tfrac{1}{3}x^3+\tfrac{1}{5}x^5+\dots\right).$$

(ii) Put $\dfrac{1+x}{1-x}=\dfrac{m}{n}$ or $n+nx=m-mx$, or $x=\dfrac{m-n}{m+n}$,

$$\therefore\ \log\frac{m}{n}=2\left[\frac{m-n}{m+n}+\frac{1}{3}\left(\frac{m-n}{m+n}\right)^3+\frac{1}{5}\left(\frac{m-n}{m+n}\right)^5+\dots\right].$$

403. *Example* 4. Find log 2 to 4 places of decimals.

Put $m=2$, $n=1$, so that $\dfrac{m-n}{m+n}=\dfrac{1}{3}$ and

$$\log 2=2\left[\frac{1}{3}+\frac{1}{3}\cdot\frac{1}{3^3}+\frac{1}{5}\cdot\frac{1}{3^5}+\frac{1}{7}\cdot\frac{1}{3^7}+\dots\right]$$

$$=2\left[\cdot333333\dots+\tfrac{1}{3}\times\cdot037037\dots+\tfrac{1}{5}\times\cdot004115\dots+\tfrac{1}{7}\times\cdot000457\dots+\tfrac{1}{9}\times\cdot000051\dots+\tfrac{1}{11}\times\cdot000006+\dots\right]$$

$$=2\left[\begin{array}{r}\cdot333333\\+\ \cdot012346\\+\ \cdot000823\\+\ \cdot000065\\+\ \cdot000006\end{array}\right]$$

$$=\cdot346573\dots\times 2$$

$$=\cdot6931\dots.$$

404. Having found log 2, we can find in succession log 3 (by putting $m = 3$, $n = 2$), log 4, and so on. These logarithms are all calculated to the base e. To obtain the logarithms of the corresponding numbers to the base 10, we have by Art. 263,

$$\log_b m = \log_a m \div \log_a b,$$

and therefore

$$\log_{10} m = \log_e m \div \log_e 10$$

$$= \log_e m \times \frac{1}{\log_e 10}.$$

Now,

$$\frac{1}{\log_e 10} = \cdot 43429\ldots.$$

Thus, the logarithm of any number to the base 10 may be found by multiplying the logarithm of the number to the base e by ·43429.

Exercises 219.

1. Prove that $\log(1 + 3x + 2x^2) = 3x - \frac{5x^2}{2} + \frac{9x^3}{3} - \frac{17x^4}{4} + \ldots.$

2 Find the coefficient of x^n in the expansion of

$$\log(1 + x + x^2 + x^3)$$

in powers of x, distinguishing the various cases which may arise.

3. If $x < 1$, find the sum to infinity of the series of which the nth term is $\frac{x^n}{n(n+1)}$.

4. Show that the difference between $\log n$ and $\log(n+1)$ diminishes as n increases.

5. Prove that

$$2\log n - \log(n+1) - \log(n-1) = \frac{1}{n^2} + \frac{1}{2n^4} + \frac{1}{3n^6} + \ldots.$$

6. Prove that

$$\log\frac{(1+x)^{\frac{1-x}{2}}}{(1-x)^{\frac{1+x}{2}}} = x + \frac{5x^3}{2\,.\,3} + \frac{9x^5}{4\,.\,5} + \frac{13x^7}{6\,.\,7} + \ldots.$$

7. Prove that the coefficient of x^n in the expansion of $\log(1 + x + x^2 + \ldots + x^{m-1})$ is either $-\frac{m-1}{n}$ or $\frac{1}{n}$ according as n is, or is not, a multiple of m.

8. Obtain the first four terms in the expansion of

$$\log\left(1 - \frac{2x^2}{1+x}\right)$$

in ascending powers of x.

9. Find the coefficient of x^n in the expansion of

$$\log\frac{1}{6 + x - x^2}.$$

CHAPTER XXXVI.

MISCELLANEOUS GRAPHS.

405. *Example* 1. Draw the graph of the equation

$$y = \frac{x-1}{x-4}.$$

It will be noticed that

(i) If x be negative, or if x be positive and < 1, then $x - 1$ and $x - 4$ are both negative, and therefore y is positive; if x be > 1 and < 4, then $x - 1$ is positive and $x - 4$ is negative, and therefore y is negative; and, if $x > 4$, then both $x - 1$ and $x - 4$ are positive, and therefore y is positive.

(ii) If x be very nearly equal to, but less than, 4, then y is negative and very great numerically; and, if x be very nearly equal to, but greater than, 4, then y is positive and very great; i.e. the line $x = 4$ is an asymptote, such that, at a great distance from Ox, the curve is below Ox on the left side of the asymptote, and above it on the right.

(iii) If the equation be written in the form

$$y = \frac{x-4+3}{x-4} = \frac{x-4}{x-4} + \frac{3}{x-4}$$

$$= 1 + \frac{3}{x-4},$$

then, as x increases indefinitely, $\frac{3}{x-4}$ diminishes indefinitely, and thus y can be made to differ from 1 by a quantity that is indefinitely small; also, if x be positive and > 4, $\frac{3}{x-4}$ is positive and y is > 1, and if x be negative, $\frac{3}{x-4}$ is negative and y is < 1; i.e. the line $y = 1$ is an asymptote, such that, at a great distance from Oy, the curve is above the asymptote on the right side of Oy and below it on the left.

Giving different values to x in the given equation, we have the following pairs of values of x and y:

x	-6	-2	0	1	2	3	3·5	4·5	5	6	7	10
y	·7	·5	·2	0	$-$·5	-2	-5	7	4	2·5	2	1·5

From these values of the coordinates, and from the above inferences with regard to the asymptotes, we obtain the graph represented in Fig. 29, a curve which is a rectangular hyperbola, since the asymptotes are at right angles.

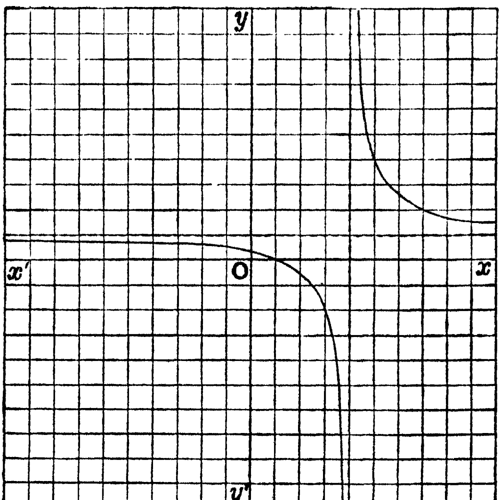


Fig. 29.

406. *Example* 2. Draw the graph of the equation

$$y=\frac{(x+2)(x-5)}{x-3}.$$

It will be noticed that

(i) If x be negative and numerically >2, then $x+2$, $x-3$ and $x-5$ are all negative and therefore y is negative; if $x>-2$ and <3, then $x+2$ is positive and $x-3$ and $x-5$ are negative, and therefore y is positive; if $x>3$ and <5, then $x+2$ and $x-3$ are positive and $x-5$ negative, and therefore y is negative; and if $x>5$, then $x+2$, $x-3$ and $x-5$ are all positive, and therefore y is positive.

(ii) If x be very nearly equal to, but less than, 3, then y is positive and very great; and, if x be very nearly equal to, but greater than, 3, then y is negative and very great numerically; i.e. the line $x=3$ is an asymptote, such that, at a great distance from Ox, the curve is above Ox on the left side of the asymptote, and below it on the right.

(iii) If the equation be written in the form

$$y=\frac{x^2-3x-10}{x-3}=x-\frac{10}{x-3},$$

then, as x increases indefinitely, $\frac{10}{x-3}$ diminishes indefinitely, and thus y can be made to differ from x by a quantity that is indefinitely small; also, if x be positive and >3, $\frac{10}{x-3}$ is positive, and y is $<x$; and, if x be negative, $\frac{10}{x-3}$ is negative and y is $>x$; i.e. the line $y=x$ is an asymptote such that, at a great distance from Oy, the curve is below the asymptote on the right side of Oy and above it on the left.

Giving different values to x in the given equation, we have the following pairs of values of x and y:

x	-8	-6	-4	-2	-1	0	1	2	4	5	6	8	10
y	$-7{\cdot}1$	$-4{\cdot}9$	$-2{\cdot}6$	0	$1{\cdot}5$	$3{\cdot}3$	6	12	-6	0	$2{\cdot}7$	6	$8{\cdot}5$

From these values of the coordinates, and from the above inferences with regard to the asymptotes, we obtain the graph represented in Fig. 30, a curve which is a hyperbola.

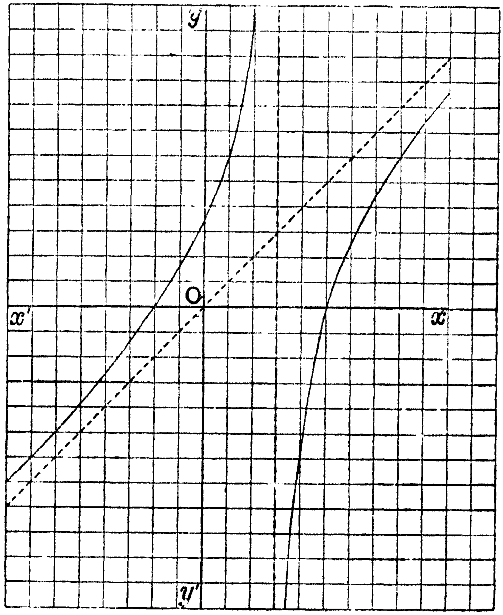


Fig. 30.

407. *Example* 3. Draw the graph of the equation

$$y = \frac{(x-1)(x-5)}{(x-2)(x-4)}.$$

It will be noticed that

(i) If x be negative or positive and < 1, y is positive; if $x > 1$ and < 2, y is negative; if $x > 2$ and < 4, y is positive; if $x > 4$ and < 5, y is negative; and, if $x > 5$, y is positive.

(ii) The lines $x=2$ and $x=4$ are asymptotes, such that, at a great distance from Ox, the curve is below Ox on the left side of the asymptote $x=2$ and on the right side of the asymptote $x=4$, and above Ox on the right side of the asymptote $x=2$ and on the left side of the asymptote $x=4$.

(iii) If the equation be written in the form

$$y=\frac{x^2-6x+5}{x^2-6x+8}=1-\frac{3}{x^2-6x+8},$$

it will be seen that the line $y=1$ is an asymptote such that, at a great distance from Oy, the curve is below the asymptote on the right side of Oy and above it on the left.

It is obvious that the curve cuts Ox at the points for which $x=1$ and $x=5$, and Oy at the point for which $y=\cdot 6$. In order to draw the curve accurately, the values of y corresponding to each of the following values of x should also be found: $-6, -4, -2, 1\cdot 5, 1\cdot 8, 2\cdot 2, 2\cdot 5, 3, 3\cdot 5, 3\cdot 8, 4\cdot 2, 4\cdot 5, 5, 6, 8$.

From these values of the coordinates, and from the above inferences with regard to the asymptotes, we obtain the graph represented in Fig. 31.

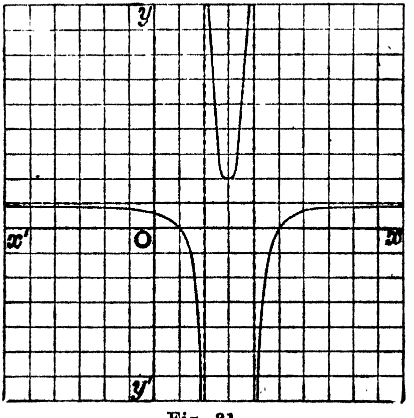


Fig. 31.

408. *Example* 4. Draw the graph of the equation

$$y=\frac{(x-1)(x-4)}{(x-2)(x-5)}.$$

It will be noticed that

(i) If x be negative or positive and <1, y is positive; if $x>1$ and <2, y is negative; if $x>2$ and <4, y is positive; if $x>4$ and <5, y is negative; and, if $x>5$, y is positive.

(ii) The lines $x=2$ and $x=5$ are asymptotes, such that, at a great distance from Ox, the curve is below Ox on the left side of both asymptotes, and above Ox on the right side of both.

(iii) If the equation be written in the form

$$y=\frac{x^2-5x+4}{x^2-7x+10}=\frac{x^2-7x+10+2x-6}{x^2-7x+10}$$

$$=1+\frac{2x-10+4}{x^2-7x+10}=1+\frac{2}{x-2}+\frac{4}{x^2-7x+10},$$

it will be seen that the line $y=1$ is an asymptote, such that, at a great distance from Oy, the curve is above the asymptote on the right side of Oy, and below it on the left.

It is obvious that the curve cuts Ox at the points for which $x=2$ and $x=5$, and Oy at the point for which $y=\cdot 3$. In order to draw the curve, the value of y corresponding to each of the following values of x should be found: $-8, -6, -4, -2, -1, 0, 1, 1\cdot 25, 1\cdot 5, 1\cdot 8, 3, 4, 4\cdot 5, 4\cdot 8, 5\cdot 5, 6, 8, 10$.

From these values of the coordinates and from the above inferences with regard to the asymptotes, we obtain the graph represented in Fig. 32.

409. *Example* 5. Draw the graph of the equation

$$y^2=x(x-1)(x-2).$$

It will be noticed that

(i) If x be negative, then x, $x-1$ and $x-2$ are all negative

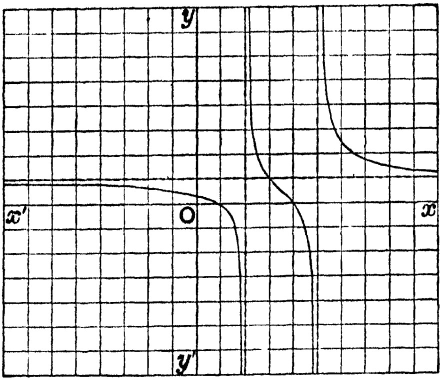


Fig. 32.

and therefore $x(x-1)(x-2)$ is negative; if $x > 0$ and < 1, then x is positive and $x-1$ and $x-2$ are negative, and therefore $x(x-1)(x-2)$ is positive; if $x > 1$ and < 2, then x and $x-1$ are positive and $x-2$ negative, and therefore $x(x-1)(x-2)$ is negative; and, if $x > 2$, then x, $x-1$ and $x-2$ are all positive, and therefore $x(x-1)(x-2)$ is positive; i.e. since y^2 is positive only for values of x between 0 and 1 or greater than 2, that no part of the curve can lie to the left of Oy or in that part to the right for which $x > 1$ and < 2.

(ii) For any value of x between 0 and 1 or greater than 2, there is one value of y^2, giving rise to two values of y that are equal in magnitude and opposite in sign.

It is obvious that the curve cuts Ox at the points for which $x = 0$, $x = 1$ and $x = 2$. In order to draw the curve, the values of y corresponding to the following values of x should be found: ·5, 2·5, 3, 4, 5, 6.

From these values of the coordinates and from the above reasoning, we obtain the graph represented in Fig. 33.

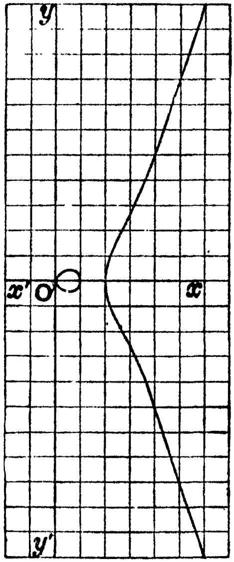


Fig. 33.

410. *Example* 6. By means of the graph of the equation

$$y = \frac{(x+2)(x-5)}{x-3},$$

find approximately the roots of the equation

$$x^2 - 4x - 7 = 0.$$

Since $$x^2 - 4x - 7 = 0,$$

$$\therefore \; x^2 - 3x - 10 = x - 3,$$

$\therefore$ the roots of the equation $x^2 - 4x - 7 = 0$ are the values of x which make

$$\frac{(x+2)(x-5)}{x-3} = 1.$$

Now, the graph of the equation

$$y = \frac{(x+2)(x-5)}{x-3}$$

is given in Fig. 30, and the graph of the equation

$$y = 1$$

is a line parallel to Ox and at unit distance from it on the positive side. Hence, the values of x corresponding to the points in which the two graphs intersect are those which make

$$\frac{(x+2)(x-5)}{x-3} = 1 \quad \text{or} \quad x^2 - 4x - 7 = 0.$$

From Fig. 30, it is evident that these roots are approximately 5·3 and -1·3.

Exercises 220.

Draw the graphs of the following equations:

1. $y = \dfrac{5}{x-2}$.
2. $y = \dfrac{6}{x+2} - 3$.
3. $y = \dfrac{x+5}{x-3}$.
4. $y = \dfrac{x-4}{x+4}$.
5. $y = \dfrac{x}{2x+3}$.
6. $y = \dfrac{x+2}{2-x}$.
7. $y = \dfrac{5-2x}{x}$.
8. $y = \dfrac{(x+3)(x-3)}{x-2}$.
9. $y = \dfrac{(x-2)(x-3)}{x-5}$.
10. $y = \dfrac{(x+1)(x-4)}{x-2}$.
11. $y = \dfrac{4}{(x+2)(x-2)}$.
12. $y = \dfrac{x+4}{(x+2)(x-2)}$.
13. $y = \dfrac{x}{(x-1)(x-5)}$.
14. $y = \dfrac{x-3}{(x-1)(x-5)}$.
15. $y = \dfrac{(x+1)(x+2)}{(x-1)(x-2)}$.
16. $y = \dfrac{(x-5)(x-7)}{(x+1)(x-3)}$.

17. $y=\dfrac{(x+1)(x-4)}{x(x-6)}$.

18. $y=\dfrac{(x-2)(x-5)}{(x-1)(x-3)}$.

19. $y=\dfrac{(2x+1)(3x-7)}{(4x-5)(x+1)}$.

20. $y=\dfrac{x^2-16}{1-x^2}$.

21. $y=x^3$.

22. $y=(x-1)(x-3)(x-5)$.

23. $y=(x-2)^2(x-4)$.

24. $y^2=x^3$.

25. $y^2=(x-1)(x-4)$.

26. $y^2=(x-1)(x-3)(x-5)$.

27. $y^2=(x-2)^2(x-4)$.

28. $y^2=\dfrac{x-2}{x-6}$.

29. Draw the graph of the equation $y=\dfrac{x^2+6x+8}{x+6}$ and thence find approximately the roots of the equation

$$x^2+5x+2=0.$$

30. Draw the graph of the equation $y=\dfrac{(x+1)(x-3)}{4-x}$ and thence find approximately the roots of the equation

$$x^2-x-7=0.$$

31. Draw the graph of the equation $y=\dfrac{(x-1)(x+7)}{2x+3}$ and thence find approximately the roots of the equation

$$x^2+2x-13=0.$$

MISCELLANEOUS PROBLEMS.

1. If $x=a(b-c)$, $y=b(c-a)$, $z=c(a-b)$, prove that $x^3+y^3+z^3=3xyz$.

2. If $a+\dfrac{1}{a}=\sqrt{3}$, prove that $a^3+\dfrac{1}{a^3}=0$.

3. If $a+b+c=0$, prove that $a^2-bc=b^2-ca=c^2-ab$.

4. Express $2(x-y)(x-z)+2(y-z)(y-x)+2(z-y)(z-x)$, as the sum of three squares.

5. Express (i) $(a^2+b^2)(c^2+d^2)$ as the sum of two squares, and (ii) $(x^2-y^2)(a^2-b^2)$ as the difference of two squares.

6. Show that the value of $(a^2-bc)+(b^2-ca)+(c^2-ab)$ is not altered if the quantities a, b, c be each increased or diminished by the same amount.

7. Prove that

$$(a^2+b^2+c^2)(a'^2+b'^2+c'^2)$$
$$=(aa'+bb'+cc')^2+(bc'-b'c)^2+(ca'-c'a)^2+(ab'-a'b)^2.$$

8. Find the value of

$$x^3-8y^3+29z^3+18xyz, \text{ when } 2y=x+3z \text{ and } z=5.$$

9. If $x=a+\frac{1}{a}$, $y=a-\frac{1}{a}$, prove that $\frac{x^2-xy+y^2}{x^2+xy+y^2}=\frac{a^4+3}{3a^4+1}$.

10. Prove that, if

$$x=\frac{b-c}{a},\quad y=\frac{c-a}{b},\quad z=\frac{a-b}{c},$$

then $xyz+x+y+z=0$.

11. If a, b, n be positive integers, arrange in order of magnitude

$$\frac{na}{(n+1)b},\quad \frac{(n+1)a}{nb},\quad \frac{na}{(n-1)b},\quad \frac{(n-1)a}{nb}.$$

12. If

$$(a+b)(b+c)(c+d)(d+a)=(a+b+c+d)(bcd+cda+dab+abc),$$

then $ac=bd$.

13. If $a=\frac{1}{1-b}$, $b=\frac{1}{1-c}$, $c=\frac{1}{1-d}$, prove that $a=d$.

14. Show that the value of $a^3+b^3+c^3-3abc$ is quadrupled if $b+c-a$, $c+a-b$, $a+b-c$ be substituted respectively for a, b, c.

15. Show that the expression

$$(a^2+b^2+c^2)(x^2+y^2+z^2)-(ax+by+cz)^2$$

can never be negative whatever real values be given to the letters.

16. Prove that the expression

$$\frac{(x^4 - 4x^3 + 8x - 4)^2}{(x^2 - 2x + 2)^4}$$

is unaltered by the substitution of $\frac{2}{2-x}$ for x.

17. Show that the fourth power of any odd number when diminished by 1 is divisible by 8.

18. Show that $m(m-1)(m-3)(m-6)$ is divisible by 8, when m is an integer.

19. Show that the difference between the cube of a number of two digits and the cube of the number formed by interchanging the digits is divisible by 27.

20. If $a+b+c=0$, then $a^4+b^4+c^4=2(bc+ca+ab)^2$.

21. Show that the expression $\frac{1}{b-c}+\frac{1}{c-a}+\frac{1}{a-b}$ is positive, if a, b, c or b, c, a or c, a, b be in descending order of magnitude.

22. Find the H.C.F. of $(x-y)^5-x^5+y^5$ and $(x-y)^3-x^3+y^3$.

23. If a and b denote two numbers which have no H.C.F., then a^2+b^2 and a^2-b^2 have no common factor unless it be 2.

24. If $\frac{(a-b)(c-d)}{(a-c)(b-d)}=x$, then the result of interchanging b, d or a, c is $1-x$.

25. Show that the value of the expression

$$\frac{b^2c^2(c^2-b^2)+c^2a^2(a^2-c^2)+a^2b^2(b^2-a^2)}{(b+c)(c+a)(a+b)}$$

is not altered if the quantities be each increased or diminished by the same amount.

26. If $x+\frac{1}{y}=1$ and $y-\frac{1}{z}=1$, prove that $xyz=1$.

27. If $(c-a)^2=2a(b+c)$ and $d^2-b^2=2c(a+b)$, show that $a=b+c\pm d$.

28. If

$$\left(\frac{m}{m+n}\right)^2+\left(\frac{n}{n+l}\right)^2+\left(\frac{l}{l+m}\right)^2=\left(\frac{n}{m+n}\right)^2+\left(\frac{l}{n+l}\right)^2+\left(\frac{m}{l+m}\right)^2,$$

show that two out of the three quantities l, m, n must be equal.

29. If $x+y+z=a$, $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{b}$ and $(y+z)(z+x)(x+y)=c^3$, show that $xyz=\frac{bc^3}{a-b}$.

30. If $z^2=a-\frac{a^2}{y^2}$ and $y^2=a-\frac{a^2}{x^2}$, show that $x^2=a-\frac{a^2}{z^2}$.

31. If x, y, z, a, b, c, be all real and

$$x^2+y^2+z^2+a^2+b^2+c^2=2(ax+by+cz),$$

show that $x=a$, $y=b$, $z=c$.

32. Prove that $bc(b+c)^2+ca(c+a)^2+ab(a+b)^2$ is divisible by $a+b+c$.

33. If $$\left(\frac{1}{x}+\frac{2}{y}+\frac{1}{z}\right)^2=\frac{(x+2y+z)^2}{xy^2z},$$

show that either $x=z$ or $y^2=zx$.

34. If 11 be taken from a certain integer, we get the square of a whole number, and if 24 be added to the same number, we get the square of the next higher whole number; find the integer.

35. If $\frac{x+b}{x+B}=\frac{x+c}{x+C}$ be satisfied by any value of x when $b+C=B+c$, show that it is satisfied by all values of x.

36. Find values of A, B (different from a, b) which will render the equation

$$m(x+A)^2+n(x+B)^2=m(x+a)^2+n(x+b)^2$$

an identity for all values of x.

37. A woman buys eggs at x a penny and an equal number at y a penny; she sells the whole at the rate of $x+y$ for $2d.$; prove that, unless $x=y$, she loses.

38. If

$$\phi(x)=\frac{a^x-a^{-x}}{a^x+a^{-x}} \quad \text{and} \quad F(x)=\frac{2}{a^x+a^{-x}},$$

show that

$$\phi(x+y)=\frac{\phi(x)+\phi(y)}{1+\phi(x)\phi(y)} \quad \text{and} \quad F(x+y)=\frac{F(x)F(y)}{1+\phi(x)\phi(y)}.$$

39. Prove that $ax^2+bxy+cy^2$ and $bx^2-2(a-c)xy-by^2$ cannot have a common factor unless the first be square.

40. If $x^3+y^3+ax^2+bxy+cy^2$ can be expressed as the product of two real factors of the first and second degrees, prove that either $a+c=b$ or $a=c=-b$.

41. Find the condition that a, b, c may be the pth, qth and rth terms of an A.P.

42. If s_n be the sum of the first n terms of a G.P., find the sum of the series $s_1+s_2+s_3+\ldots+s_n$.

43. If a, x, y, b be in A.P., a, p, q, b in G.P., is $x+y$ greater or less than $p+q$?

44. If the sum of n terms in A.P. be denoted by S_n, prove that

$$S_m-S_n=\frac{m-n}{m+n}S_{m+n}.$$

45. If $T_n=1^2+2^2+3^2+\ldots+n^2$, prove that

$$T_n-3T_{n+1}+3T_{n+2}-T_{n+3}=-2.$$

46. Eliminate x, y, z from the equations

$$x+y+z=a, \quad x^2+y^2+z^2=b^2, \quad \text{and} \quad yz+zx+xy=c^2.$$

47. If $\dfrac{x}{y+z}=a$, $\dfrac{y}{z+x}=b$, $\dfrac{z}{x+y}=c$, prove that

$$bc+ca+ab+2abc=1.$$

48. Eliminate x, y from the equations

$$x+y=a, \quad x^2+y^2=b^2, \quad x^3+y^3=c^3.$$

49. Eliminate x from the equations

$$x^3+\frac{1}{x^3}+3\left(x+\frac{1}{x}\right)=m, \quad x^3-\frac{1}{x^3}-3\left(x-\frac{1}{x}\right)=n.$$

50. If $2s=a+b+c$, prove that

(i) $s^2+(s-a)^2+(s-b)^2+(s-c)^2=a^2+b^2+c^2$.

(ii) $s(s-a)+s(s-b)+s(s-c)=s^2$.

(iii) $a^2(s-a)+b^2(s-b)$
$+c^2(s-c)-4(s-a)(s-b)(s-c)=abc$.

(iv) $\dfrac{(s-a)^3+(s-b)^3+(s-c)^3-3(s-a)(s-b)(s-c)}{a^3+b^3+c^3-3abc}=\dfrac{1}{2}$.

(v) $(s-a)^3+(s-b)^3+(s-c)^3+3abc=s^3$.

(vi) $\dfrac{1}{s-a}+\dfrac{1}{s-b}+\dfrac{1}{s-c}-\dfrac{1}{s}=\dfrac{abc}{s(s-a)(s-b)(s-c)}$.

51. Prove that the difference of the squares of any two odd numbers is always divisible by 8; and by 24, if neither of the numbers be divisible by 3.

52. Prove that the difference between the cube of any number whose units digit is 4 and the square of the next higher odd number falls short of a multiple of 40 by 1.

53. If n be an odd number, show that $(n^2+3)(n^2+7)$ is divisible by 32.

54. If n be an odd number, prove that $n^2(n^2-1)(n^4-1)$ is a multiple of 5760.

55. Prove that if the square of any number be divided by 3, it cannot have a remainder 2.

56. If $x+y+z=\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=1$, prove that one of the quantities x, y, or z must be equal to 1.

57. If $\dfrac{x^2}{y^2}+\dfrac{y^2}{x^2}=3$, show that the value of $\dfrac{x^3}{y^3}-\dfrac{y^3}{x^3}$ is 4 or -4.

58. If $x+\dfrac{1}{y}=y+\dfrac{1}{z}=\pm 1$, then each is equal to $z+\dfrac{1}{x}$.

59. If $\frac{bz - cy}{a} = \frac{cx - az}{b} = \frac{ay - bx}{c}$, prove that $x : y : z = a : b : c$.

60. If $(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) = (ax + by + cz)^2$, and a, b, c, x, y, z be all real, prove that $x : a = y : b = z : c$.

61. If $bz - cy = cx - az = ay - bx$, then either $x + y + z = 0$ and $a + b + c = 0$, or $x : a = y : b = z : c$.

62. If $\frac{x}{y + z} = a$, $\frac{y}{z + x} = b$, $\frac{z}{x + y} = c$, show that

$$\frac{x^2}{a - abc} = \frac{y^2}{b - abc} = \frac{z^2}{c - abc}.$$

63. If $ax + by + cz = 0$ and $bcx + cay + abz = 0$, prove that

$$a^2x + b^2y + c^2z + (bc + ca + ab)(x + y + z) = 0.$$

64. If $x = cy + bz$, $y = az + cx$, $z = bx + ay$, then

$$\frac{x^2}{1 - a^2} = \frac{y^2}{1 - b^2} = \frac{z^2}{1 - c^2}.$$

65. If the equation $x^2 + px + q = 0$ have equal roots, prove that one of the roots of the equation $ax^2 + p(a + b)x + q(a + 2b) = 0$ is equal to them, and find the other.

66. If the equations $x^2 + ax + bc = 0$ and $x^2 + bx + ca = 0$ have a common root, show that their other roots satisfy the equation $x^2 + cx + ab = 0$.

67. Find the limits within which x must lie in order that $\frac{5x^2 - 11x + 3}{2x^2 - 9x + 4}$ may be less than unity.

68. A horse can draw one-third of a ton of coal a certain distance in a given time; if his load exceed this, the time he takes is increased by a quantity proportional to the square of the excess; when his load is four-thirds of a ton, he takes four times as long as at first; show that he is most efficient when his load is two-thirds of a ton.

69. If $a(b - c)x^2 + b(c - a)xy + c(a - b)y^2$ is a perfect square, then a, b, c are in H.P.

70. Find the sum of the odd numbers less than 1000 which are not divisible by 7.

71. How many numbers are there less than 1000 which contain the digit 7 at least once?

72. If $m, a_1, a_2, \ldots, a_r, n$ be in A.P. and $m, h_1, h_2, \ldots, h_r, n$ be in H.P., prove that $a_s \,.\, h_{r-s+1} = mn$.

73. If the sum of a pair of numbers be n, find the sum of all the harmonic means which can be inserted between all the pairs of such numbers.

74. How many pieces are there in a complete set of dominoes when the number of spots on each half of a piece ranges from 0 to n?

75. Prove that the sum of all the products in a multiplication table going up to n times n is $\frac{1}{4}n^2(n+1)^2$.

76. Prove that the middle term in the expansion of $\left(x+1+\frac{1}{x}\right)^7$ is 393.

77. Eliminate x, y, z from the equations

$$\frac{x}{y+z}=a, \quad \frac{y}{z+x}=b, \quad \frac{z}{x+y}=c.$$

78. If

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{a+x}=0, \quad \frac{1}{a}+\frac{1}{c}+\frac{1}{a+y}=0, \quad \text{and} \quad \frac{1}{a}+\frac{1}{x}+\frac{1}{y}=0,$$

then $$a+b+c=0.$$

79. Eliminate x, y, z from the equations

$$x^2-yz=a^2, \quad y^2-zx=b^2, \quad z^2-xy=c^2, \quad \text{and} \quad x+y+z=0.$$

80. Eliminate l, m, n from the equations

$$a^2l^3+b^2m^3+c^2n^3=d^2l+e^2m+f^2n, \quad al=bm=cn,$$

and $$l^2+m^2+n^2=1.$$

81. If $a=b+c$, show that

$$a^4+b^4+c^4+2a^2(b-c)^2=2bc(a+b)(a+c).$$

82. Prove that

$$(a^2-bc)^3+(b^2-ca)^3+(c^2-ab)^3-3(a^2-bc)(b^2-ca)(c^2-ab)$$
$$=(a^3+b^3+c^3-3abc)^2.$$

83. If $\dfrac{1}{y-z}+\dfrac{1}{z-x}+\dfrac{1}{x-y}=0$, prove that

$$(y-z)^3=(z-x)^3=(x-y)^3.$$

84. If $x+y+z=0$, show that

$$\left(\frac{y-z}{x}+\frac{z-x}{y}+\frac{x-y}{z}\right)\left(\frac{x}{y-z}+\frac{y}{z-x}+\frac{z}{x-y}\right)-9=0.$$

85. If $x=b-c$, $y=c-a$, $z=a-b$, prove that

$$\frac{x^7+y^7+z^7}{7}=\frac{x^3+y^3+z^3}{3}\times\left(\frac{x^2+y^2+z^2}{2}\right)^2.$$

86. If $a+b+c=0$, prove that

$$\frac{a^4+b^4+c^4}{2}=\left(\frac{a^2+b^2+c^2}{2}\right)^2.$$

87. If $a+b+c=0$, prove that

$$10(a^7+b^7+c^7)=7(a^2+b^2+c^2)(a^5+b^5+c^5).$$

88. The duration of a railway journey varies directly as the distance and inversely as the velocity; the velocity varies directly as the square of the quantity of coal used per mile and inversely as the number of carriages in the train; in a journey of 25 miles in half an hour with 18 carriages, 10 cwt. is required; how much coal will be consumed in a journey of 15 miles in 20 minutes with 20 carriages?

89. A, B, and C run a mile race at uniform speeds; A wins by 160 yards, B comes in second, beating C by $76\frac{12}{23}$ yards in distance and by a quarter of a minute in time; what is the pace of each?

90. A and B run a mile; in the first heat A gives B a start of 20 yards and beats him by 30 seconds; in the second heat, A gives B a start of 32 seconds and beats him by $9\frac{5}{11}$ yards; at what rate does A run?

91. A and B run a mile; when A gives B a start of 44 yards he beats him by 51 seconds; when A gives B a start of 1 m. 15 s., he is beaten by 88 yards; find the time in which each runs a mile.

92. A starts from P to Q half an hour after B and overtakes him midway between P and Q, and arrives at Q at 2 p.m.; after waiting $7\frac{1}{2}$ minutes at Q, he returns and meets B in 10 minutes more; at what time did each start from P?

93. A can beat B by x yards in a 300 yards race, C can beat B by x yards in a 200 yards race, while C can beat A by 9 yards in a race of $594x$ yards; find x.

94. A number consists of three digits, x, y, z, in ascending order of magnitude from left to right; if either 108 or 189 be added, the new number consists of the same digits as the original number, only that no digit occupies its former place; prove that $x + z = 2y$.

95. If a number and its square root contain respectively p and q digits, show that $2p - 4q + 1 = (-1)^p$.

96. If n be a positive integer, show that $3 . 5^{2n+1} + 2^{3n+1}$ is divisible by 17.

97. If n be a positive integer, prove that $n^7 - n$ is divisible by 42.

98. I sent cash to a grocer for a certain number of pounds of sugar, at the rate of 7 lbs. for 1*s.* $1\frac{1}{2}$*d.*, but before the order reached him the price of sugar had risen, and the money was sufficient only to buy a quantity less by $10\frac{1}{2}$ lbs. than that which I had intended; so I sent in addition 5*s.* $7\frac{1}{2}$*d.* and received one-fifth as much again as I had at first ordered; find the number of lbs. ordered at first, the rise in price being less than $\frac{1}{2}$*d.* a lb.

99. A, B and C start at the same time for a town a miles distant; A walks at a uniform rate of u miles an hour, and C and B drive at a uniform rate of v miles an hour; after a certain time B dismounts and walks forward at the same rate as A, while C returns to meet A; A mounts with C, and they enter the town at the same time as B, C driving uniformly all the time; show that the whole time of the journey is $\frac{a}{v}\cdot\frac{3v+u}{3u+v}$ hours.

100. Three bells commenced tolling simultaneously, and tolled at intervals of 25, 29 and 33 seconds respectively; in less than half an hour the first ceased, and the second and third tolled 18 seconds and 21 seconds respectively after the cessation of the first and then ceased; how many times did each bell toll?

ANSWERS.

Exercises 137.

1. $0, 1, 4$.
2. $0, 3, 3$.
3. $0, -3, -4$.
4. $0, 0, 2, -2$.
5. $0, 0, -1\frac{2}{3}$.
6. $0, 0, 5, 5$.
7. $1, 1, -1$.
8. $1, -1, -2$.
9. $\frac{1}{2}, -\frac{1}{2}, -2$.
10. $2, -2, -1\frac{1}{2}$.
11. $1, -1, 2, -2$.
12. $1, -1, 4, -4$.
13. $2, -2, \sqrt{2}, -\sqrt{2}$.
14. $1, -1, \sqrt{5}, -\sqrt{5}$.
15. $2, -2, 3, -3$.
16. $1, -1, \sqrt{2}, -\sqrt{2}$.
17. $1, -1, 3, -3$.
18. $1, -1, \sqrt{3}, -\sqrt{3}$.
19. $0, -2, 3, -5$.
20. $0, -2, 2, -4$.
21. $2, -1, 3, -2$.
22. $1, 1, 5, -3$.
23. $0, -1, 1, -2$.
24. $2, -1, 4, -3$.
25. $1, -4, 2, -5$.
26. $2, -2, 1, -1$.
27. $2, 2, 1, 4$.
28. $6, -1, 2, -3$.
29. $a, -a, 1, -1$.
30. $a, -a, b, -b$.
31. $a, -a, \frac{1}{a}, -\frac{1}{a}$.
32. $1, 2+\sqrt{3}, 2-\sqrt{3}$.
33. $0, 7, -1$.
34. $0, 5, -6$.
35. $1, 1, 1, 1$.
36. $1, 1, -1, -1$.
37. $\sqrt{3}, -\sqrt{3}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$.
38. $1, 1, 1, 1$.
39. $1, -1, 1+\sqrt{2}, 1-\sqrt{2}$.
40. $1, 1, \frac{1}{2}(3+\sqrt{5}), \frac{1}{2}(3-\sqrt{5})$.
41. $-1, -6, -1, -6$.
42. $-3, -3, -3+\sqrt{5}, -3-\sqrt{5}$.
43. $2, 2, \frac{1}{2}(4+\sqrt{13}), \frac{1}{2}(4-\sqrt{13})$.
44. $-2\frac{1}{2}, -2\frac{1}{2}, \frac{1}{2}(-5+\sqrt{10}), \frac{1}{2}(-5-\sqrt{10})$.

Exercises 138.

1.	12, 24.	2.	4.	3.	$\frac{3}{4}$.
4.	2, 3.	5.	4, 5.	6.	5, 3.
7.	5, 10, 20.	8.	16, 10.	9.	8, 10.
10.	24, 36.	11.	333.	12.	41, 43, 45.
13.	14 ft., 12 ft.	14.	2 ft.	15.	3 ins.
16.	35 ft., 84 ft.	17.	18.	18.	50.
19.	63.	20.	72.	21.	7 miles an hour.
22.	4 yds., 5 yds.	23.	£1.	24.	4, 3 miles an hour.
25.	18, 12 miles a day.			26.	32 acres.
27.	20.	28.	9*s*., 12*s*., a dozen.		
29.	8*d*.	30.	1*s*. 3*d*.	31.	16, 17.
32.	3.	33.	16 men, 20 women; 20 men, 16 women		
34.	60 miles.				

Exercises 139.

1.	$x^2-7xy+12y^2$.	2.	21*s*.	3.	$3x^2-3x+2$.
4.	$\frac{a}{6(a-3b)}$.	5.	$(c+a)(b+c)$.	6.	1, 1.
7.	$\frac{6}{7}$.	8.	$x^3-1-\frac{1}{x^3}$.	9.	4, 7.
10.	$1, \frac{1}{2}(3a-2)$.	11.	$2, -2, 5, -5$.	12.	36*s*.

Exercises 140.

1.	12 years ago.	2.	$(a-b)(a+b-3)$.	3.	x^2+1.
4.	0.	5.	$2(x+y+z)$.	6.	2, 3.
7.	$-\frac{abc}{a-b}, \frac{abc}{a-b}$.	8.	$x^3-3x+\frac{3}{x}-\frac{1}{x^3}$.	9.	$\sqrt{8}, -\sqrt{8}$.
10.	$-1, a^2+a$.	11.	$0, 0, 5, -3$.	12.	8 miles an hour.

Exercises 141.

1. 5. 2. 11, 1. 3. 1, 2, 5. 4. $\frac{b}{ac}, \frac{c}{ab}$. 5. 15, −4.
6. $b, \frac{1}{b+1}$. 7. 14, $3\frac{1}{2}$. 8. 0, 4, −8. 9. 3, −3, $\sqrt{3}$, $-\sqrt{3}$.
10. 1, 2, −5, −6. 11. $a, -a, \frac{1}{b}, -\frac{1}{b}$. 12. 10, $4\frac{1}{2}$, −1.

Exercises 145.

1. 17. 2. 0. 3. −23. 4. 0. 5. 52. 6. −119.
7. real and diff. 8. real and diff. 9. real and equal.
10. real and equal. 11. imag. and diff. 12. imag. and diff.
13. real and diff. 14. imag. and diff. 15. imag. and diff.
16. real and equal. 17. real and equal. 18. imag. and diff.
19. real and diff. 20. real and diff. 21. 9. 22. 16.
23. 6, −6. 24. 10, −10. 25. 1. 26. 25. 27. $\frac{1}{8}$.
28. $-\frac{1}{48}$. 29. $2\frac{1}{12}$. 30. $-1\frac{1}{8}$. 31. $-1\frac{1}{3}$. 32. −8, 12.
33. 5, −11. 34. −19, 21. 35. $19\frac{1}{2}$, $-22\frac{1}{2}$.

Exercises 147.

1. 8, −14. 2. 2, 10. 3. $3\frac{1}{2}$, 23. 4. $-2\frac{5}{12}$, $2\frac{23}{32}$. 5. 8.
6. 20. 7. 1. 8. 7. 9. $3\frac{1}{4}$. 10. $2\frac{2}{3}$. 11. $-2\frac{8}{9}$.
12. $1\frac{1}{16}$. 13. $-\frac{11}{25}$. 14. $1\frac{21}{25}$. 15. 1. 16. 4. 17. $\frac{1}{4}$.
18. $\frac{1}{4}$. 19. 14. 20. $5\frac{4}{9}$. 21. $1\frac{1}{3}$. 22. $3\frac{9}{16}$. 23. $2\frac{6}{25}$.
24. $-\frac{27}{49}$. 25. 28. 26. −217. 27. −1. 28. $-8\frac{1}{8}$.
29. $-48\frac{1}{8}$. 30. $8\frac{1}{27}$. 31. $6\frac{8}{27}$. 32. $\frac{31}{125}$. 33. $-1\frac{1}{4}$.
34. $\frac{47}{54}$. 35. $-1\frac{1}{4}$. 36. $\frac{14}{25}$. 37. 10. 38. $\frac{35}{216}$.
39. $1\frac{1}{2}$. 40. 13. 41. $3\frac{8}{9}$. 42. $3\frac{1}{2}$. 43. 12. 44. $60\frac{3}{4}$.

Exercises 148.

1. $x^2-6x+8=0$. 2. $x^2-6x+9=0$. 3. $x^2+x-30=0$.
4. $x^2-5x-14=0$. 5. $x^2+8x+15=0$. 6. $x^2+11x+28=0$.
7. $x^2-5x=0$. 8. $x^2+8x=0$. 9. $6x^2-5x+1=0$.
10. $20x^2-x-1=0$. 11. $21x^2+26x+8=0$. 12. $12x^2-x-20=0$.
13. $6x^2-x-35=0$. 14. $6x^2+13x+6=0$. 15. $x^2-2x-1=0$.
16. $x^2-2x-2=0$. 17. $x^2-6x+2=0$. 18. $x^2-12x+33=0$.

19. $x^2-5x+5=0$. 20. $25x^2-60x+34=0$. 21. $4x^2+4x-3=0$.
22. $28x^2-11x+1=0$. 23. $6x^2-5x-4=0$. 24. $4x^2-5x-6=0$.
25. $18x^2+11x-24=0$. 26. $x^2-28x+171=0$.
27. $16x^2-168x+41=0$. 28 $16x^2-56x+13=0$.
29. $81x^2+36x-320=0$. 30. $x^2-19x+18=0$.
31. $16x^2-10x+1=0$. 32. $2x^2+21x+53=0$.
33. $25x^2-30x-12=0$. 34. $x^2-8x+10=0$.
35. $x^2-35x+297=0$. 36. $x^2-20x+64=0$.
37. $x^2-25x+144=0$. 38. $4x^2-36x+25=0$.
39. $4x^2-25x+16=0$. 40. $9x^2-43x+1=0$.
41. $9x^2-46x+25=0$. 42. $25x^2+21x+9=0$.
43. $16x^2-9x+1=0$. 44. $100x^2-65x+4=0$.
45. $25x^2-120x+4=0$. 46. $8x^2-10x+3=0$.
47. $x^2-84x+1715=0$. 48. $81x^2-666x+1225=0$.
49. $5x^2-2x+4=0$. 50. $x^2-51x+1=0$.

Exercises 149.

1. $3b^2=16ac$. 2. $5p^2=36q$. 3. 8. 4. 12, -12.
5. 16; $\frac{1}{4}$, $\frac{3}{4}$. 6. 9, -9; 3, $1\frac{1}{2}$; -3, $-1\frac{1}{2}$.

Exercises 150.

1. 4. 2. 7. 3. $\frac{1}{2}$. 4. $1\frac{2}{3}$. 5. 3. 6. 1.
7. $7\frac{1}{2}$. 8. $2\frac{4}{5}$.

Exercises 151.

1. $\frac{3}{4}$. 2. $-2\frac{1}{2}$. 3. 12. 4. 3. 5. $mnb^2=(m+n)^2ca$.
7. $\dfrac{b+c-2a}{c+a-2b}$. 8. 1, 5. 9. -1, $-\frac{1}{2}$.
14. 2, -2; 3, -3 if $m=2$; 1, -1 if $m=-2$.
16. $acx^2-(b^2-2ac)x+ac=0$. 17. $b^2x^2-b^2x+ca=0$.
18. $\dfrac{(b^2-2ac)^2}{a^3c}$. 19. $b^2=ac$. 20. $-4\frac{1}{3}$. 21. 4, -2.
22. $cax^2-4b^2x+4b^2=0$. 23. $a^2x^2+abx-2b^2+9ac=0$.
24. $2x^2-x-7=0$. 25. $x^2-13=0$.
26. $a^2x^2-2(a^2+2b^2-ac)x+(a-c)^2+4b^2=0$.
27. $a(a+\beta)$, $\beta(a+\beta)$. 28. $\frac{1}{2}[\gamma+\delta\pm\sqrt{\{(\gamma+\delta)^2-4a\beta\}}]$.
33. -3, -4. 34. 3, 5. 35. Imaginary.

38. $9x^2-27x+20=0$. 39. $a^2=4(b^2-ac)$.
41. $x^2+2x-24=0$, -14. 43. $\frac{3}{4}$, $-1\frac{1}{4}$. 47. $q-p^2$ and infinity.
48. $\frac{c(c-a)}{b(a-b)}$. 49. $-1\frac{1}{4}$, $-1\frac{2}{7}$; $-1\frac{1}{4}$, $\frac{5}{8}$. 50. $\frac{5}{6}$, $1\frac{6}{11}$; $\frac{5}{6}$, $1\frac{2}{21}$.
53. $(x+1)(x^2-x-pq)$. 54. $m=-2$; 1, -2; 1, 1.
55. 4. 59. 300.

Exercises 152.

1. 5, 3; 3, 5. 2. 9, 4; 4, 9. 3. 9, -2; -2, 9.
4. 6, -2; 2, -6. 5. 2, -1; -1, 2. 6. 1, 1; 1, 1.
7. -1, 4; -8, -10. 8. 3, 1; $-2\frac{1}{5}$, $-1\frac{3}{5}$. 9. 2, -1; $\frac{2}{21}$, $\frac{3}{7}$.
10. 1, -2; -1, 4. 11. 1, -3; 4, 3. 12. 2, -1; $2\frac{15}{26}$, $-\frac{8}{13}$.
13. -4, -1; $4\frac{5}{11}$, $1\frac{32}{77}$. 14. -2, 4; $-26\frac{2}{5}$, $-36\frac{2}{3}$.
15. 3, 1; 3, 1. 16. 1, 1; 1, 1. 17. 4, 3; 0, $-\frac{1}{6}$.
18. $\frac{1}{3}$, $-\frac{1}{7}$; $\frac{1}{7}$, $-\frac{1}{3}$. 19. 5, 2; a pair of infinite roots.
20. 7, -2; a pair of infinite roots.
21. $1\frac{1}{2}$, $\frac{5}{6}$; a pair of infinite roots.
22. 1, 2; a pair of infinite roots. 23. 1, 1; $\frac{b}{a}$, $\frac{a}{b}$.
24. a, $\frac{a^2}{b}$; $-\frac{a}{2}$, $-\frac{2a^2}{b}$. 25. a, a; $-\frac{a+1}{a}$, $-\frac{1}{a+1}$.

Exercises 153.

(NOTE. The answers to nos. 6, 16, 23, may be expressed more simply when Chapter XXV. has been read.)

1. 4, 2; 2, 4; -4, -2; -2, -4. 2. 4, 2; 2, 4; -4, -2; -2, -4.
3 5, -3; -3, 5; 3, -5; -5, 3. 4. 7, 3; 3, 7; -7, -3; -3, -7.
5. 4, 0; -4, 0; 1, 4; -1, -4.
6. 0, $\sqrt{40}$; 0, $-\sqrt{40}$; 6, 4; -6, -4.
7. 0, 3; 0, -3; $2\sqrt{5}$, $\sqrt{5}$; $-2\sqrt{5}$, $-\sqrt{5}$.
8. 0, 3; 0, -3; $5\sqrt{\frac{3}{2}}$, $\sqrt{\frac{3}{2}}$; $-5\sqrt{\frac{3}{2}}$, $-\sqrt{\frac{3}{2}}$.
9. 2, 1; -2, -1; 1, -2; -1, 2.
10. $7\frac{1}{2}$, $-\frac{1}{2}$; $-7\frac{1}{2}$, $\frac{1}{2}$; 3, 4; -3, -4.
11. 3, -1; -3, 1; $\frac{1}{\sqrt{3}}$, $\frac{4}{\sqrt{3}}$; $-\frac{1}{\sqrt{3}}$, $-\frac{4}{\sqrt{3}}$.
12. 2, 1; -2, -1; $\frac{4}{\sqrt{3}}$, $\frac{1}{\sqrt{3}}$; $-\frac{4}{\sqrt{3}}$, $-\frac{1}{\sqrt{3}}$.

13. $3, 2; -3, -2; \frac{5}{\sqrt{2}}, \frac{1}{\sqrt{2}}; -\frac{5}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$.

14. $2, 3; -2, -3; \frac{5}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{5}{\sqrt{3}}, -\frac{1}{\sqrt{3}}$.

15. $3, 1; -3, -1; \frac{12}{\sqrt{10}}, \frac{1}{\sqrt{10}}; -\frac{12}{\sqrt{10}}, -\frac{1}{\sqrt{10}}$.

16. $5, \frac{1}{2}; -5, -\frac{1}{2}; \frac{56}{\sqrt{96}}, \frac{1}{\sqrt{96}}; -\frac{56}{\sqrt{96}}, -\frac{1}{\sqrt{96}}$.

17. $3, 2; -3, -2$; two pairs of infinite roots.

18. $9, 1; -9, -1$; two pairs of infinite roots.

19. $5, -4; -5, 4$; two pairs of infinite roots.

20. $2, -\frac{1}{3}; -2, \frac{1}{3}$; two pairs of infinite roots.

21. $3, 1; -3, -1$; two pairs of infinite roots.

22. $1, \frac{1}{2}; -1, -\frac{1}{2}; \frac{1}{2}, 1; -\frac{1}{2}, -1$.

23. $1, 1; -1, -1, \frac{2}{7}\sqrt{\frac{19}{4}}, \sqrt{\frac{19}{4}}; -\frac{2}{7}\sqrt{\frac{19}{4}}, -\sqrt{\frac{19}{4}}$.

24. $\frac{a+b}{\sqrt{(2a)}}, \frac{a-b}{\sqrt{(2a)}}; -\frac{a+b}{\sqrt{(2a)}}, -\frac{a-b}{\sqrt{(2a)}}$; two pairs of infinite roots.

25. $\frac{ac}{\sqrt{(a^2+b^2)}}, -\frac{bc}{\sqrt{(a^2+b^2)}}; -\frac{ac}{\sqrt{(a^2+b^2)}}, \frac{bc}{\sqrt{(a^2+b^2)}};$
$\frac{ac}{\sqrt{(a^2+b^2)}}, -\frac{bc}{\sqrt{(a^2+b^2)}}; -\frac{ac}{\sqrt{(a^2+b^2)}}, \frac{bc}{\sqrt{(a^2+b^2)}}$.

Exercises 154.

1. 3, 2; 2, 3.
2. 2, 1; −1, −2.
3. 4, 3; 3, 4.
4. 9, 7; −7, −9.
5. 3, 1; 1, 3.
6. 5, 1; 1, 5.
7. 4, 2; −2, −4.
8. 7, 4; −4, −7.
9. 2, −1; $\frac{1}{2}$, −4.
10. 2, 5; $1\frac{2}{3}$, 6.

Exercises 155.

1. 5, 1; 1, 5; 3, 3; 3, 3.
2. 6, 2; 2, 6; 4, 4; 4, 4.
3. 6, 1; 1, 6; 10, −3; −3, 10.
4. 7, 3; 3, 7; 12, −2; −2, 12.
5. $3, -1; -1, 3; 1+\sqrt{7}, 1-\sqrt{7}; 1-\sqrt{7}, 1+\sqrt{7}$.
6. $7, 1; 1, 7; 4+\sqrt{28}, 4-\sqrt{28}; 4-\sqrt{28}, 4+\sqrt{28}$.
7. 4, 1; 1, 4; −4, −1; −1, −4.
8. 1, 1; 1, 1; −1, −1; −1, −1.
9. 3, 2; 2, 3; −3, −2; −2, −3.
10. 2, 0; 0, 2; −2, 0; 0, −2.
11. 3, −1; −3, 1; 1, −3; −1, 3.
12. 5, 2; 2, 5; −5, −2; −2, −5.

Exercises 156.

1. 2, 6; 5, 3.
2. 3, 2; 2, 3; $-3+\sqrt{3}$, $-3-\sqrt{3}$; $-3-\sqrt{3}$, $-3+\sqrt{3}$.
3. 3, 2; 2, 3; $-1+\sqrt{2}$, $-1-\sqrt{2}$, $-1-\sqrt{2}$, $-1+\sqrt{2}$.
4. 4, 2; 2, 4; -2, -4; -4, -2.
5. 5, 2; 2, 5; -5, -2; -2, -5.
6. 3, -3; -7, 7; 5, 1; -1, -5.
7. 0, 0; 0, 0; 1, -2; $-2\frac{1}{19}$, $-15\frac{14}{19}$.
8. 3, 0; 2, 1.
9. $1\frac{1}{2}$, $-\frac{1}{2}$; $-\frac{1}{2}$, $1\frac{1}{2}$.
10. 12, 6; 6, 12; $-3+\sqrt{33}$, $-3-\sqrt{33}$; $-3-\sqrt{33}$, $-3+\sqrt{33}$.
11. 3, 2; 2, 3; 2, -3; -3, 2.
12. 2, 1; 1, 2; 1, -3; -3, 1; 2, -3; -3, 2.
13. 4, 1; -2, $-\frac{1}{2}$; two pairs of infinite roots.
14. 2, 1; -1, -2; $\sqrt{3}$, $\sqrt{3}$; $-\sqrt{3}$, $-\sqrt{3}$.
15. 2, -3; -6, 1; $-\frac{1}{2}(-3+\sqrt{57})$, $-\frac{1}{4}(-3-\sqrt{57})$; $-\frac{1}{2}(-3-\sqrt{57})$, $-\frac{1}{4}(-3+\sqrt{57})$
16. 4, 1; -3, $-1\frac{1}{3}$; 2, 2; -6, $-\frac{2}{3}$.
17. 4, 1; 1, 4; -4, -1; -1, -4.
18. 11, 8.
19. 2, -1; -1, 2; $\frac{1}{2}(-1+\sqrt{13})$, $\frac{1}{2}(-1+\sqrt{13})$; $\frac{1}{2}(-1-\sqrt{13})$, $\frac{1}{2}(-1-\sqrt{13})$.
20. 3, -1; -1, 3; $\frac{1}{2}(1+\sqrt{39})$, $\frac{1}{2}(1-\sqrt{39})$; $\frac{1}{2}(1-\sqrt{39})$, $\frac{1}{2}(1+\sqrt{39})$.

Exercises 157.

1. 13, 10.
2. 2, $\frac{4}{5}$.
3. 17 ft., 13 ft.
4. 12 ft., 5 ft.
5. 126 ft., 32 ft.
6. $\frac{2}{3}$, $\frac{1}{3}$.
7. 8, 4.
8. 36.
9. 10, 7.
10. 6·4 or 4·6.
11. 1232 yds., 165 yds.
12. 1089 yds., 160 yds.
13. 7, 5.
14. 15, 5.
15. 26.
16. 60 miles, 10 and 12 miles an hour; or $7\frac{1}{2}$ m., 3 and 5 m. an hour.
17. £2, £2. 10*s.*
18. 78.
19. 14 yds., 42 yds.
20. 14 ft., 12 ft.; 17 ft., 14 ft.; 18 ft., $9\frac{1}{3}$ ft.; 21 ft., $11\frac{1}{3}$ ft.
21. 21 ft., 15 ft.
22. 3 ins., 5 ins., 7 ins.; 27 c. ins., 125 c. ins., 343 c. ins.
23. 30.

Exercises 158.

1. 1, 2, 2; -1, -2, -2.
2. 4, 2, 1; -4, -2, -1.
3. 4, 2, 1; -4, -2, -1.
4. 1, 2, 4; -1, -2, -4.
5. 2, 3, 4.
6. 2, 3, 5.
7. 20, 3, 7, 11.
8. 3, 2, 6; -5, -2, -6.
9. 1, 2, 1; -1, -2, -1.

10. 1, 2, 3; −1, −2, −3.
11. 2, 10, −3; 7, 5, −3; −2, 5, −12; −7, 10, −12.
12. $\frac{2}{3}$, 2, $2\frac{2}{3}$. 13. 4, 1, 3. 14. 1, 1, 2. 15. 2, 1, 1; 1, 2, 1
16. 3, 1, 2; 3, 2, 1; 3, 1, 2; 3, 2, 1. 17. 4, 9, 6; 9, 4, 6.
18. 10, 2, 4; −2, −10, 4.
19. 2, 1, 1; −2, −1, −1; $\frac{1}{\sqrt{2}}, -\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}; -\frac{1}{\sqrt{2}}, \frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}$.
20. 1, 1, 2; 1, 1, 2. 21. 3, 2, 1; 2, 3, 1; −3, −2, −1; −2, −3, −1.
22. 1, 2, 3; −3, −4, −5. 23. 3, 1, 1; 1, 3, 1; 1, 1, 3.
24. $\frac{bc}{a}, \frac{ca}{b}, \frac{ab}{c}; -\frac{bc}{a}, -\frac{ca}{b}, -\frac{ab}{c}$.
25. $\frac{a^2}{\sqrt{(a^2+b^2+c^2)}}, \frac{b^2}{\sqrt{(a^2+b^2+c^2)}}, \frac{c^2}{\sqrt{(a^2+b^2+c^2)}}$.

Exercises 159.

1. 10. 2. 93. 3. 0. 4. 3. 5. −58. 6. $\frac{1}{5}$.
7. 19. 8. $2\frac{1}{3}$. 12. 3. 13. 10. 14. 13.

Exercises 160.

1. $(x-1)(x-2)(x-3)$. 2. $(x+1)(x+4)(x-3)$.
3. $(x+2)(x+3)(x+4)$. 4. $(x+1)^2(2x-1)$.
5. $(x-1)(x-3)(x+2)(x+4)$. 6. $(x-1)(x+1)(x+2)(x+3)$.
7. 1, 4, −2. 8. −1, −5, 2. 9. 1, 2, −1. 10. 1, 1, −2.
11. −1, 2, 3. 12. 3, −2, −2.

Exercises 161.

4. $c+a, a+b$. 5. $a^2+b^2+c^2-bc-ca-ab$. 6. $4c(a+b)$.
7. $-(b-c)(c-a)(a-b)$. 8. $(b-c)(c-a)(a-b)$.
9. $(y+z)(z+x)(x+y)$. 10. $3(y+z)(z+x)(x+y)$.
11. $-(b-c)(c-a)(a-b)(a+b+c)$. 14. $12abc(a+b+c)$.
15. $5(a^2+b^2+c^2-bc-ca-ab)$. 16. $5(x^2+y^2+z^2+yz+zx+xy)$.
17. $-(yz+zx+xy)$. 18. $-(b-c)(c-a)(a-b)(bc+ca+ab)$.
19. $(b-c)(c-a)(a-b)(a^2+b^2+c^2+bc+ca+ab)$.
20. $(b-c)(c-a)(a-b)(bc+ca+ab)$.
21. $-(b-c)(c-a)(a-b)(b+c)(c+a)(a+b)$.
22. $(b-c)(c-a)(a-b)(a^3+b^3+c^3+b^2c+bc^2+c^2a+ca^2+a^2b+ab^2+abc)$.
23. $a(x^2+mx+m^2)+b(x+m)+c$. 25. −1. 26. 1.
27. $\frac{1}{abc}$. 28. $(y+z)(z+x)(x+y)$. 29. $\frac{5}{3}(x^2+y^2+z^2-yz-zx-xy)$.

Exercises 162.

1. 1, -1. 2. -3, 7. 3. $-\frac{b}{a-b}$, $\frac{a}{a-b}$. 4. 2, -2, 10.
5. 4, -10, 6. 6. $3a$, $3a^2$, a^3. 7. 3, -3, 1.
8. $a+b+c$, a^2+ab+b^2, a^3. 9. -1, 0. 10. 21, -76, 60.
11. 20. 12. -45, -14. 13. 5. 14. $\sqrt{2}$, 1; $-\sqrt{2}$, -1.
16. $f=ch$. 17. -88 or 0. 18. 6, $x-2$. 20. 1, 22.
21. $a^3+8c=4ab$, $c^2=a^2d$. 23. $4a$.
24. $-\frac{1}{1+c}$, $\frac{1}{(1+c)^2}$, $-\frac{1}{(1+c)^3}$, $\frac{1}{(1+c)^4}$, ...

Exercises 163.

1. $\frac{1}{x+1}-\frac{1}{x+2}$. 2. $\frac{1}{x-1}+\frac{1}{x-2}$. 3. $\frac{2}{x-2}+\frac{1}{x+1}$.
4. $\frac{2}{2x+1}-\frac{1}{x+1}$. 5. $\frac{3}{2x+1}+\frac{5}{x-2}$. 6. $1+\frac{2}{x-1}+\frac{3}{x-2}$.
7. $\frac{3}{2(x+4)}-\frac{3}{4(x+3)}$. 8. $\frac{1}{x+1}+\frac{1}{x+2}+\frac{1}{x+3}$.
9. $\frac{1}{x-1}-\frac{5}{x-2}+\frac{5}{x-3}$. 10. $\frac{1}{x-1}+\frac{2}{x-2}+\frac{3}{x-3}$.
11. $\frac{2}{3(x-1)}+\frac{1}{4(x+1)}-\frac{5}{6(x-2)}$.
12. $\frac{15}{4(x-2)}+\frac{11}{12(x+2)}-\frac{5}{3(x-1)}$. 13. $x+1+\frac{1}{x+1}+\frac{1}{x+2}$.
14. $x+6+\frac{1}{2(x-1)}-\frac{16}{x-2}+\frac{81}{2(x-3)}$.

Exercises 165.

1. $\sqrt{a}$. 2. $\sqrt[3]{a}$. 3. $\sqrt[3]{a^2}$. 4. $\sqrt[4]{a^3}$. 5. $\sqrt{a^3}$. 6. $\sqrt[5]{a^4}$.
7. $\sqrt[8]{a^5}$. 8. $\frac{1}{\sqrt[4]{a}}$. 9. $\frac{1}{\sqrt{a^5}}$. 10. $\frac{1}{\sqrt[3]{a^4}}$. 11. $a^{\frac{1}{2}}$.
12. $a^{\frac{1}{3}}$. 13. $a^{\frac{2}{3}}$. 14. $a^{\frac{3}{2}}$. 15. $a^{\frac{4}{3}}$. 16. $a^{\frac{5}{2}}$. 17. $a^{\frac{7}{4}}$.
18. $\frac{1}{a^{\frac{1}{3}}}$. 19. $\frac{1}{a^{\frac{5}{4}}}$. 20. $\frac{1}{a^{\frac{2}{5}}}$. 21. 2. 22. 3. 23. 5.
24. 2. 25. 3. 26. 2. 27. 4. 28. 9. 29. 8.

30. 8. 31. $a^{\frac{3}{4}}$. 32. $a^{\frac{1}{2}}$. 33. a. 34. $a^{\frac{1}{2}}$. 35. $a^{\frac{3}{4}}$.
36. $a^{\frac{3}{2}}$. 37. $a^{\frac{7}{12}}$. 38. $a^{\frac{3}{2}}$. 39. $a^{\frac{5}{3}}$. 40. $a^{\frac{7}{4}}$.
41. $a^{\frac{1}{4}}$. 42. $a^{\frac{1}{2}}$. 43. a^2. 44. $a^{\frac{1}{4}}$. 45. $a^{\frac{1}{3}}$.
46. $a^{\frac{1}{10}}$. 47. $a^{\frac{1}{2}}$. 48. $a^{\frac{2}{3}}$. 49. $a^{\frac{13}{12}}$. 50. $a^{\frac{7}{10}}$.
51. $a^{\frac{31}{30}}$. 52. $a^{\frac{57}{20}}$. 53. a^6. 54. $a^{\frac{4}{3}}$. 55. $a^{\frac{11}{6}}$.

Exercises 166.

1. $\frac{1}{a^2}$. 2. $\frac{1}{a^3}$. 3. $\frac{1}{a^{\frac{1}{2}}}$. 4. $\frac{1}{a^{\frac{2}{3}}}$. 5. $\frac{1}{a^{\frac{3}{4}}}$. 6. $\frac{2}{a}$.
7. $\frac{3}{a^{\frac{1}{4}}}$. 8. $4a^4$. 9. $\frac{1}{2}a^{\frac{3}{2}}$. 10. $\frac{by}{ax}$. 11. $\frac{ax^3}{b^2y^2}$.
12. $\frac{2a^2x^2y^3}{3b}$. 13. $\frac{1}{2}$. 14. $\frac{1}{2}$. 15. $\frac{1}{8}$. 16. $\frac{1}{5}$. 17. $\frac{1}{81}$.
18. 4. 19. 125. 20. 4. 21. 32. 22. 30. 23. 4.
24. 1. 25. $a^{\frac{1}{2}}$. 26. $a^{\frac{1}{2}}$. 27. a^2. 28. $a^{\frac{4}{5}}$. 29. 1.
30. $6a^{\frac{1}{2}}$. 31. $10a^{\frac{7}{8}}$. 32. $b^{\frac{1}{2}}$. 33. $\frac{a^{\frac{1}{3}}}{x^{\frac{1}{2}}}$. 34. $\frac{8}{a^{\frac{1}{5}}}$.
35. a^2. 36. a. 37. $\frac{1}{a^{\frac{5}{4}}}$. 38. $2a^2$. 39. $\frac{x^3}{y^3}$. 40. $\frac{1}{2}ab$.

Exercises 167.

1. $a-b$. 2. $x+a$. 3. x^2+1+x^{-2}. 4. $x-3x^{\frac{1}{2}}+1$.
5. $a^{\frac{3}{2}}-2a^{\frac{1}{2}}+a^{-\frac{1}{2}}$. 6. $a^{\frac{5}{3}}-5a^{\frac{4}{3}}+10a-10a^{\frac{2}{3}}+5a^{\frac{1}{3}}-1$.
7. $x^{\frac{1}{2}}-y^{\frac{1}{2}}$. 8. $a+a^{\frac{1}{2}}b^{-\frac{1}{2}}+b^{-1}$. 9. $x+1+x^{-1}$.
10. $x^{\frac{3}{2}}-xy^{-\frac{1}{2}}+x^{\frac{1}{2}}y^{-1}-y^{-\frac{3}{2}}$. 11. $a^{\frac{2}{3}}+2a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}$.
12. $x^{\frac{2}{3}}+x^{\frac{1}{3}}y^{-\frac{1}{3}}+y^{-\frac{2}{3}}$.

Exercises 168.

1. a^2. 2. $\frac{1}{a}$. 3. x^2. 4. $x^{\frac{1}{3}}$. 5. $a^{\frac{1}{2}}$. 6. $\frac{1}{x^4}$.
7. $x^{\frac{2}{3}}$. 8. x^4. 9. x^4. 10. $x^{\frac{5}{6}}$. 11. $\frac{1}{a^2}$. 12. x^{10}.
13. $\frac{1}{a^{\frac{2}{3}}}$. 14. $\frac{1}{a^{\frac{1}{2}}}$. 15. 1. 16. a^2.

Exercises 169.

1. 4. 2. $2\frac{1}{4}$. 3. 100. 4. $4\frac{11}{27}$. 5. $39\frac{1}{16}$. 6. xy^3.

7. $\frac{a^2}{b^3}$. 8. $\frac{a^2}{b}$. 9. $\frac{x^5}{y}$. 10. $\frac{x^5}{y}$. 11. $\frac{b^4}{a^{\frac{8}{3}}}$.

12. $\frac{b^{\frac{2}{3}}}{a}$. 13. $\frac{1}{2a^{\frac{1}{2}}b^{\frac{1}{2}}}$. 14. $\frac{x^3}{8y^6}$. 15. $\frac{1}{a^6b^8c^9}$. 16. x^{12}.

17. x^2. 18. $a^{\frac{5}{3}}b^{\frac{3}{2}}$. 19. $\frac{b^{\frac{1}{6}}c^{\frac{5}{12}}}{a^{\frac{1}{12}}}$. 20. $a^{\frac{3}{5}}b^{\frac{7}{10}}c^{\frac{1}{30}}$. 21. a^2b^9.

22. $\frac{a^{\frac{3}{2}}c^{\frac{1}{2}}}{b^{\frac{1}{3}}}$. 23. x^2z^3. 24. $\frac{1}{a^{\frac{1}{3}}b^{\frac{1}{2}}}$. 25. ab.

Exercises 170.

1. x^a. 2. $x^{\frac{3}{20}}$. 3. a^3. 4. $\frac{12b^{\frac{3}{2}}}{a^{\frac{1}{4}}}$. 5. x^{pq}. 6. a^{m-1}.

7. a^{m-2}. 8. 1. 9. $a^{\frac{11}{12}}b^2$. 10. $a^{25}b^{25}$. 11. x. 12. $\frac{b}{a^{\frac{5}{2}}}$.

13. $a^2-a^{\frac{3}{2}}b^{\frac{1}{2}}+2ab-a^{\frac{1}{2}}b^{\frac{3}{2}}+b^2$. 14. $a^{\frac{4}{3}}b^2+a^{\frac{2}{3}}b^{\frac{4}{3}}c^{\frac{2}{3}}+b^{\frac{2}{3}}c^{\frac{4}{3}}$.

15. $a+b^2+c^3-3a^{\frac{1}{3}}b^{\frac{2}{3}}c$. 16. $\frac{1}{3}a+\frac{11}{12}a^{\frac{5}{6}}-\frac{7}{30}a^{\frac{3}{4}}-\frac{1}{4}a^{\frac{2}{3}}+\frac{109}{240}a^{\frac{7}{12}}-\frac{1}{5}a^{\frac{1}{2}}$.

17. $16a^2-81x^2$. 18. $2yz+2zx+2xy-x^2-y^2-z^2$.

19. $x^{\frac{1}{2}}y^{\frac{1}{4}}-2x^{\frac{1}{3}}y^{\frac{1}{2}}+2x^{\frac{1}{6}}y^{\frac{3}{4}}$. 20. $a^{\frac{1}{2}}-2a^{\frac{1}{4}}x^{\frac{1}{4}}+x^{\frac{1}{2}}$.

21. $a+a^{\frac{1}{2}}b^{\frac{1}{2}}-b$. 22. $x^{\frac{3}{2}}-3xy^{\frac{1}{3}}+3x^{\frac{1}{2}}y^{\frac{2}{3}}-y$. 23. $x^{\frac{1}{3}}+3x^{-\frac{1}{3}}$.

24. $x+1+x^{-1}$. 25. $a+2a^{\frac{1}{2}}-3$. 26. $2xa^{-1}-3+4x^{-1}a$.

27. $a^2-2a^{\frac{4}{3}}b^{\frac{2}{3}}+a^{\frac{4}{3}}b^{\frac{5}{3}}$. 28. $a^{-1}+2-b^{-2}$. 29. $\frac{6}{3^{\frac{2}{3}}-1}$. 30. $\frac{5}{24}$.

31. $\frac{x^{\frac{1}{4}}-y^{\frac{1}{4}}}{x^{\frac{1}{4}}+y^{\frac{1}{4}}}$. 32. $\frac{2+y^{\frac{2}{3}}}{3y^{\frac{2}{3}}+4y^{\frac{4}{3}}}$. 33. $\frac{x^{\frac{2}{3}}-5x^{\frac{1}{3}}+6}{3x^{\frac{2}{3}}-8x^{\frac{1}{3}}}$.

34. $\frac{a^{\frac{3}{2}}b^{\frac{1}{2}}(a-a^{\frac{1}{2}}b^{\frac{1}{2}}+b)}{a+2a^{\frac{1}{2}}b^{\frac{1}{2}}+4b}$. 35. $x-x^{-1}$. 36. $\frac{1+3x^{\frac{1}{2}}}{1-x^{\frac{1}{2}}}$.

37. $a^{\frac{2}{3}}-b^{\frac{2}{3}}$. 38. $6p^3$, $2p$. 44. x. 45. 8, $\frac{1}{8}$.

Exercises 173.

1. 1·3333 .... **2.** 1·5. **3.** 1·5. **4.** ·5. **5.** ·75. **6.** – ·5.
7. – ·5. **8.** – 1·3333.... **9.** – ·25. **10.** – ·5.

Exercises 174.

1. 2·0535. **2.** 6·4938. **3.** 10. **4.** 1·7783. **5.** 8·6596.
6. 10. **7.** 1·5399. **8.** 2·7384. **9.** 3·6517. **10** 2·0535.
11. 1·7783. **12.** 1·5399. **13.** 3·1623.

Exercises 176.

1. 3392. **2.** 129·6. **3.** 66. **4.** 63·17. **5.** ·3526.
6. ·3207. **7.** 2·065. **8.** 17·55. **9.** 2·474. **10.** 435·5.
11. 48·12. **12.** 1·271. **13.** 7·096. **14.** 7·221. **15.** 1·556.
16. 235·2. **17.** 140·2. **18.** ·006220. **19.** 48·52. **20.** 2·412.
21. £51. 3*s.* 10*d.* **22.** £23. 16*s.* **23.** £1. 12*s.* 7*d.*
24. £20. 5*s.* 5*d.* **25.** 14·2 yrs. **26.** 17·6 yrs. **27.** 143.
28. £46. **29.** £11. 18*s.* 2*d.* **30.** 15 c. ft. 1350 c. ins., nearly.
31. 6·54 ins. **32.** 5·40 ins. **33.** 44·5 sq. ins. **34.** $6\frac{1}{2}$ ft., $3\frac{1}{4}$ ft.
35. 6·29 ins. **36.** 523 sq. ins. **37.** 5·06 ins. **38.** 40·77 c. ins.
39. 6·26 ins. **40.** 146·3 sq. ins. **41.** 314·3 sq. ins. **43.** 7·003 ins.

Exercises 178.

1. $25\sqrt{2}$. **2.** $24\sqrt{5}$. **3.** 0. **4.** $\sqrt{3}+\sqrt{2}$. **5.** $\sqrt{2}$.
6. $2\sqrt{6}$. **7.** $\sqrt{2}$. **8.** $4(\sqrt{3}-\sqrt{2})$. **9.** $10\sqrt{(2x)}$.
10. $(a^2+b^2+c^2)\sqrt{x}$. **11.** 0. **12.** 36. **13.** 2880.
14. $168\sqrt{21}$. **15.** 105. **16.** 5. **17.** $\frac{7}{18}$. **18.** $2\sqrt[6]{2}$.
19. $2\sqrt[12]{32}$. **20.** $3\sqrt{3}$. **21.** 2. **22.** $3\sqrt{2}$. **23.** $4\sqrt{3}$.
24. $2\sqrt[3]{7}$. **25.** $\sqrt{5}$. **26.** $6\sqrt{2}, 5\sqrt{3}, 4\sqrt{5}$. **27.** $2\sqrt{7}, 4\sqrt[4]{3}, 5$.

Exercises 179.

1. $\frac{1}{2}\sqrt{2}, \frac{2}{3}\sqrt{3}, 2\sqrt{5}$. **2.** $\sqrt{2}+1$. **3.** $\frac{1}{2}(\sqrt{5}+1)$. **4.** $2(\sqrt{7}-2)$.
5. $\frac{1}{3}(\sqrt{5}-\sqrt{2})$. **6.** $\sqrt{5}-1$. **7.** $\frac{1}{5}(9+2\sqrt{14})$. **8.** $3\sqrt{2}-2\sqrt{3}$.
9. $\frac{1}{2}(6+\sqrt{10})$. **10.** 2. **11.** $3\frac{1}{13}$. **12.** 0. **13.** 0.
14. $\frac{1}{4}(66+45\sqrt{2}-38\sqrt{3}-25\sqrt{6})$. **15.** $\frac{1}{4}(2+\sqrt{2}-\sqrt{6})$.
16. $\frac{1}{23}(12-7\sqrt{2}+5\sqrt{3}-\sqrt{6})$. **17.** $\sqrt{10}-\sqrt{5}+2\sqrt{2}-3$.
18. $1+\sqrt{3}+\sqrt{5}$. **19.** $\frac{1}{2}(2+\sqrt{2}+\sqrt{6})$. **20.** $1\frac{49}{59}$.

Exercises 180.

1. $\sqrt{3}+1$. **2.** $\sqrt{5}+\sqrt{2}$. **3.** $5-\sqrt{3}$. **4.** $5-2\sqrt{6}$.
5. $7+2\sqrt{2}$. **6.** $7+2\sqrt{5}$. **7.** $\sqrt{15}-2\sqrt{3}$. **8.** $6\sqrt{2}+3\sqrt{6}$.
9. $5\sqrt{3}-2\sqrt{7}$. **10.** $\dfrac{\sqrt{5}-1}{\sqrt{2}}$. **11.** $\dfrac{\sqrt{11}-\sqrt{7}}{\sqrt{2}}$. **12.** 8.
13. $\sqrt{2}+1$. **14.** $\sqrt{3}-\sqrt{2}$. **15.** 0. **16.** $\sqrt{10}$. **18.** 71.
19. $1-\sqrt{(1-x)}$. **20.** $\sqrt{(2x-3)}+\sqrt{(x+2)}$.

Exercises 181.

1. ·18. **2.** $\dfrac{1}{2x-1}$. **4.** 1. **5.** 325. **6.** $1\frac{2}{13}$. **7.** 18.
10. $2-\sqrt{2}$. **11.** 1. **12.** $5\sqrt{a}-3\sqrt{b}$. **13.** $1+\sqrt{2}+\sqrt{3}$.
14. 1. **16.** $\sqrt{3},\ -\sqrt{3},\ -2$. **17.** $\sqrt{2}+1,\ -\sqrt{2}+1,\ 4$.
18. $\sqrt{5}+2,\ -\sqrt{5}+2,\ 1,\ -2$.

Exercises 182.

1. $x+10\sqrt{x}+25$. **2.** $x-6\sqrt{x}+9$. **3.** $36-12\sqrt{x}+x$.
4. $x+5+4\sqrt{(x+1)}$. **5.** $x+8-6\sqrt{(x-1)}$. **6.** $4x+4\sqrt{x}+1$.
7. $9-12\sqrt{x}+4x$. **8.** $4x-3-4\sqrt{(x-1)}$.
9. $2x+27+10\sqrt{2(x+1)}$. **10.** $3x+43-14\sqrt{3(x-2)}$.
11. $2x+1+2\sqrt{(x^2+x)}$. **12.** $2x-1-2\sqrt{(x^2-x)}$.
13. $2x+3+2\sqrt{(x^2+3x+2)}$. **14.** $3x+1-2\sqrt{(2x^2-x-6)}$.
15. $6x+2+2\sqrt{(8x^2-2x-15)}$. **16.** $5x+3-4\sqrt{(x^2-1)}$.
17. $10x-1-6\sqrt{(x^2-x)}$. **18.** $30x+5+12\sqrt{(6x^2+x-1)}$.

Exercises 183.

1. 4. **2.** 9. **3.** 4. **4.** -2. **5.** 5. **6.** 3. **7.** 9.
8. 49. **9.** 16. **10.** 9. **11.** 5. **12.** 9. **13.** 5, 5.
14. 12. **15.** 3. **16.** 7. **17.** 10. **18.** 5. **19.** No root.
20. No root. **21.** -1. **22.** 1. **23.** 5. **24.** 3, 3.
25. $\frac{3}{5}$. **26.** 3. **27.** 4. **28.** 0. **29.** 10. **30.** 3, 3.
31. 27, $\frac{25}{147}$. **32.** $-b$. **33.** $\dfrac{2ab}{1+b^2}$.

Exercises 184.

1. 2, 3, 7, 8. **2.** $1,\ -3\frac{1}{2},\ \frac{1}{4}(-5\pm\sqrt{209})$. **3.** $1\frac{1}{2}$, 2.
4. 3, -1. **5.** 1, 1, 1, 1. **6.** 2, 3. **7.** $1\frac{4}{5},\ 5,\ -4,\ -7\frac{1}{5}$.
8. 4, -7. **9.** 1, -5. **10.** $\frac{1}{2}(-7\pm\sqrt{77})$.
11. $4,\ -5,\ \frac{1}{2}(-1\pm\sqrt{21})$. **12.** $2,\ -\frac{1}{3},\ \frac{1}{2},\ 1\frac{1}{6}$. **13.** $\frac{1}{2}(2\pm\sqrt{21})$.
14. $0,\ a,\ \frac{1}{2}(a\pm\sqrt{5a^2-8ab})$.

Exercises 185.

1. 23, 29, 35, 41, 47. **2.** x^3+3y^2. **3.** $5xy(x-y)(x^2-xy+y^2)$.
4. $(x-1)(x+1)(x-2)(x+2)(x-3)(x+3)$. 5. y^3-3y, y^5-5y^3+5y.
7. $\frac{1}{4}y(3x^2+y^2)$, 300000016. **8.** x^2+5x+5. **9.** 7.
11. $x-x^{\frac{1}{2}}$. 12. $\frac{a^9}{b^{14}}$.

Exercises 186.

1. Real and equal, real and different, imaginary and different.
3. $-3\frac{1}{3}$, 40. 4. 18, -22. 5. $1\frac{5}{8}$. 6. $21x^2+46x-7=0$.
7. $9x^2-88x+4=0$. 8. $\frac{16}{27}$. 9. $c^2x^2+2bcx+b^2=0$.
10. $(a-b+c)x^2-2(a-c)x+a+b+c=0$. 11. $x^2\pm 15x+54=0$.
12. 32, 2.

Exercises 187.

1. 37, 69. 2. 10. 3. $(x-1)^2(x+4)$.
6. $2(y-z)(z-x)(x-y)(x+y+z)$. 7. $a+b+c$.
9. ·3010, ·4771. 10. $\frac{2}{x-2}+\frac{1}{x+2}-\frac{3}{x-1}$.
11. a^3-2ab, a^2b-b^3. 12. 1, -2, $2+\sqrt{3}$, $2-\sqrt{3}$.

Exercises 188.

1. 15, $-\frac{3}{5}$. 2. $a, \frac{c^2}{a}$. 3. 2, -2, 4, -4. 4. 2, -3, 4, -5.
5. 0, 1, $\frac{1}{2}(1\pm\sqrt{129})$. 6. 7, 3; $-4\frac{3}{4}$, $-2\frac{1}{3}$. 7. 7, -4; -4, 7.
8. 1, 3; -1, -3; $2\sqrt{\frac{3}{7}}$, $\sqrt{\frac{3}{7}}$; $-2\sqrt{\frac{3}{7}}$, $-\sqrt{\frac{3}{7}}$.
9. 5, 2; 2, 5; -5, -2; -2, -5. 10. No root. 11. 8.
12. 0, 3; and two imaginary roots.

Exercises 189

1. 5 : 3. 2. 3 : 2. 3. 11 : 2. 4. 10 : 3. 5. 2 : 1, 1 : 2.
6. 2 : 1, 5 : 2. 7. 13 : 4. 8. 9 : 41. 9. 3 : 1. 10. 3.
11. 13. 12. 22. 13. 63, 36. 14. 21, 33. 15. 18, 27.
16. 15, 25. 17. 13, 20. 18. 21, 28. 19. 16, 40 yrs.
20. 12 : 5.

Exercises 190.

1. $3:2:-1$. 2. $2:1:3$. 3. $9:5:1$. 4. $110:15:17$.
5. $b^2-c^2:c^2-a^2:a^2-b^2$. 6. $3, 2; -3, -2$.
7. $-4, -5, -6; -4, -5, -6$. 8. $6, 7, 8; 21, 19, 17$.
9. $1, 1, 4; -1, -1, -4$. 10. $3, 4, 5$.

Exercises 191.

1. $\frac{ab}{b+c}, \frac{ac}{b+c}$. 4. $a(b-c):b(c-a):c(a-b)$. 6. 7.
11. a. 22. $3:2, 2:3$.

Exercises 192.

1. $12, 3, 6\frac{2}{5}, b^2$. 2. $16, 16\frac{2}{3}, 8, (a+b)^2$.
3. $\pm 6, \pm 12, \pm 7\frac{1}{2}, \pm a^2b^3$. 4. $3\frac{1}{2}, 4\frac{1}{6}, 4\frac{4}{5}$. 5. 10. 6. 1.
7. 2. 8. 6. 29. 16. 30. $4, -6, 9$.

Exercises 193.

6. $3, -1$. 10. 9, 3, 1. 15. 27 ft., 18 ft. 16. 40.
17. $11:24$. 18. $3:8$.

Exercises 194.

1. 10. 2. $\frac{1}{2}$. 3. 1. 4. 1. 5. 15. 6. $1\frac{1}{5}$.
7. 500. 8. 1. 9. 21. 10. $27x^2=4y^3$. 11. $x^3=4(y+1)^2$.
12. $y^2=\frac{b^2}{a^2}(a^2-x^2)$. 15. $A \propto D^4$. 17. 6. 18. 9 sq. yds.
19. 33 c. ft. 20. 12 ins. 21. $1:2$. 22. 1600 ft. 26. 4.
27. $3\frac{1}{2}$. 29. 20 miles an hour. 30. 2160 lbs. weight.

Exercises 195.

1. 21. 2. 90. 3. -14. 4. 58. 5. 397. 6. $25\frac{1}{2}$.
7. $-18\frac{1}{4}$. 8. 0. 9. $37\frac{2}{3}$. 10. $93\frac{1}{2}$. 11. n. 12. $2n$.
13. $2n-1$. 14. $3n+1$. 15. $22-2n$. 16. $(n+2)a-(n+1)b$.
17. 13th. 18. 22nd. 19. 200000th. 20. 26th.
21. 23rd, 29th. 22. 6. 23. -3. 24. $2\frac{1}{2}$. 25. 12.
26. 303. 27. 74. 28. $-141\frac{3}{11}$. 29. 0. 30. $10\frac{3}{4}$.
31. 31st. 32. 50th. 33. 70. 34. 100. 35. 100.
36. 83. 37. 7, 1. 38. $-7, -5\frac{1}{2}, -4, \ldots$; or $41, 39\frac{1}{2}, 38, \ldots$.
39. 61. 40. 51. 41. 25. 42. 44, 48. 43. 61. 45 $-a$.

Exercises 196.

1. 320. **2.** 2120. **3.** 2550. **4.** 1368. **5.** $616\frac{1}{2}$.
6. -90. **7.** 5050. **8.** 1600. **9.** 1344. **10.** 0.
11. 19355. **12.** 1633. **13.** -1728. **14.** 101.
15. $2014\frac{1}{2}$. **16.** 0. **17.** $-52\frac{1}{2}$. **18.** $3\frac{1}{2}$. **19.** 0.
20. $-121\frac{5}{8}$. **21.** -15. **22.** 18. **23.** 80, 80. **24.** $\frac{1}{4}n(13-5n)$.
25. $\frac{1}{21}n(7n+17)$. **26.** $n(p+n)$. **27.** $\frac{1}{2}n\{2b(n-1)-a(n-3)\}$.
28. $\frac{1}{2}(n-1)$. **29.** $n(a^2+b^2)-n(n-3)ab$. **31.** 20. **32.** 5, 12.
33. 17. **34.** 12. **35.** 12. **36.** $-21\frac{3}{5}$. **37.** 1254.
38. 49. **39.** $-5, -2, 1, \ldots$. **40.** 3. **41.** 5. **42.** $3, 2\frac{1}{2}, 2, \ldots$.
43. $217\frac{1}{2}$. **44.** 4 or 8. **45.** 33. **46.** 250000. **47.** 4950.
48. 571571. **49.** 50000. **50.** 1372.

Exercises 197.

1. 14. **2.** 20. **3.** $22\frac{1}{2}$. **4.** $63\frac{1}{2}$. **5.** $3a+2b$.
6. $5x-y$. **7.** 15 ins. **8.** 5, 9, 13, 17, 21, 25.
9. $14\frac{1}{2}, 16, 17\frac{1}{2}, \ldots, 28$. **10.** $4\frac{7}{8}, 6\frac{3}{4}, 8\frac{5}{8}, \ldots, 16\frac{1}{8}$.
11. $\frac{2}{5}, \frac{1}{15}, -\frac{4}{15}$. **12.** $3a+5b, 2a+5b, \ldots, 5b-5a$.
13. 21, 27, 33, ..., 159; 2160. **14.** 45, 42, 39, ..., -252; -10350.

Exercises 198.

5. $-2\frac{1}{2}$. **6.** 3. **8.** 19, 304. **10.** 885. **13.** $\frac{1}{2}n(3n+7)$.
14. 1, 3, 5, ... **15.** $-2, 0, 2, 4, 6$.
17. $\frac{1}{5}(4a+b), \frac{1}{5}(3a+2b), \frac{1}{5}(2a+3b), \frac{1}{5}(a+4b)$. **18.** 24.
19. 333667. **21.** 10. **22.** $9\frac{2}{3}, 10\frac{1}{3}$. **23.** $-7, -5, -3, \ldots$.
24. £51, £53. **27.** $\frac{1}{3}, \frac{1}{2}, \frac{1}{12}n(3n+1)$. **28.** 2475. **29.** 2.
30. $2(1-n)$. **31.** $a^2-ab+b^2, a^2+ab+b^2, a^2+3ab+b^2, \ldots$.
33. $\frac{1}{2}\{p(m-3n+1)+q(m+n-1)\}$. **34.** $m+n-1$ or $m+n$.
35. $a+(p+q-r-1)d=0$. **38.** 200000. **39.** 17, 19, 21, 23.
41. Reciprocal. **42.** 36.

Exercises 199.

1. 512. **2.** 4374. **3.** -128. **4.** $\frac{4}{9}$. **5.** $\frac{5}{32}$. **6.** $\frac{64}{81}$.
7. $\frac{243}{1024}$. **8.** $230\frac{169}{256}$. **9.** $-9\frac{63}{128}$. **10.** $26\frac{134}{625}$. **11.** $2048x^{11}$.
12. $\frac{(a+b)^{n-1}}{(a-b)^{n-3}}$. **13.** nx^{n-1}. **14.** $n+x^n$. **15.** 20, 10.
16. $\frac{1}{4}, -\frac{1}{2}$. **17.** 8, 16, 32, ... **18.** ± 10935. **19.** 9th. **20.** 6th.

Exercises 200.

1. 255. 2. 2186. 3. $15\frac{63}{64}$. 4. $121\frac{40}{81}$. 5. $1\frac{21}{64}$.
6. $1\frac{111111}{1000000}$. 7. $-85\frac{5}{16}$. 8. $\frac{481}{1024}$. 9. $21\frac{257}{288}$.
10. $5\frac{670}{2187}$. 11. $\frac{3}{4}(3^n-1)$. 12. $\frac{8}{5}\left\{1-(-1)^n\frac{1}{4^n}\right\}$.
13. $\frac{8}{21}\left\{1-(-1)^n\frac{3^n}{4^n}\right\}$. 14. $\frac{8}{3}\left(1-\frac{3^n}{4^n}\right)$. 15. $\frac{x(2^nx^n-1)}{2x-1}$.
16. 14·917 nearly. 17. $\frac{3a-b}{8}\left(1-\frac{1}{3^{2n}}\right)$. 18. $\frac{(a+b)^3}{2b}\left\{1-\left(\frac{a-b}{a+b}\right)^n\right\}$.
19. 2. 20. $\frac{2}{3}$. 21. $1\frac{1}{2}$. 22. $1\frac{4}{5}$. 23. $27\frac{3}{7}$. 24. $2\frac{5}{11}$.
25. $38\frac{2}{5}$. 26. $4+3\sqrt{2}$. 27. 781. 28. $\frac{b^n-a^n}{a^{n-2}(b-a)}$.

Exercises 201.

1. ± 8. 2. ± 12. 3. ± 10. 4. $\pm a^2b^2$. 5. $\pm 12x^3y^4$.
6. $-1, 1$. 7. $\pm 8, 16, \pm 32, 64, \pm 128$. 8. $3, 1, \frac{1}{3}, \frac{1}{9}$.
9. $\pm 1\frac{1}{3}, 2, \pm 3, 4\frac{1}{2}, \pm 6\frac{3}{4}$. 10. $\pm 3\frac{29}{32}, 3\frac{1}{8}, \pm 2\frac{1}{2}$.
11. $\pm a^3bc, a^2b^2c^2, \pm ab^3c^3$. 12. $\pm(a^2-b^2)$. 13. $\pm ar^{\frac{p+q}{2}-1}$.
14. 10, 20, 40, 80, 160, sum 310; -10, 20, -40, 80, -160, sum -110.
15. 6. 17. 8, 2. 18. 4, 36.

Exercises 202.

1. $8(2^n-1)$. 2. 0. 3. $\frac{x(x^{2n}-1)}{x^2-1}+n^2a$.
4. $\frac{1}{2}n(n+1)+2(2^n-1)$. 5. $n(n+2)+\frac{3}{2}(3^n-1)$. 6. $a>2b$.
7. 25. 12. 36, 4. 13. $a, \frac{a}{n+1}, \frac{a}{(n+1)^2}, \ldots$.
14. $\frac{3}{4}, \frac{1}{4}, \frac{1}{12}, \ldots$. 15. 6, 24, 96. 16. 48, 60, 75.
17. 4, 8, 16. 21. 21. 25. 6, 7, 8, ... 26. 25 or $\frac{1}{25}$.
27. 4, 16. 29. $9\frac{1}{7}, 6\frac{6}{7}, 5\frac{1}{7}, \ldots$, or $3\frac{6}{7}, 5\frac{1}{7}, 6\frac{6}{7}, \ldots$. 30. 31.
31. $\frac{a\{r^{n+1}-(n+1)r+n\}}{(r-1)^2}$. 33. $\frac{2b\sqrt{a}}{\sqrt{a}+\sqrt{b}}, \frac{2a\sqrt{b}}{\sqrt{a}+\sqrt{b}}$.

Exercises 203.

1. 1. 2. $\frac{1}{12}, \frac{1}{16}$. 3. -6. 4. 60, ∞.
5. $\infty, 5\frac{1}{3}, \ldots, 1\frac{1}{15}, \frac{8}{9}$. 6. 7th. 7. 7th. 8. $\frac{1}{5}$. 9. $2, 1\frac{1}{2}, 1\frac{1}{5}$.
10. $\frac{10}{19}$. 11. $4\frac{1}{3}$. 12. $17\frac{1}{2}, 7, 4\frac{3}{8}, \ldots$.

Exercises 204.

1. $4\frac{4}{5}$. 2. $13\frac{1}{3}$. 3. $1\frac{4}{5}$. 4. $3\frac{1}{3}$. 5. $11\frac{2}{3}$. 6. $\frac{1}{3}, \frac{1}{2}$.
7. $2\frac{1}{4}, 2\frac{4}{7}$. 8. $6\frac{10}{13}, 7\frac{12}{17}, 9\frac{3}{29}$. 9. $\frac{2}{3}, \frac{2}{5}, \frac{2}{7}, \frac{2}{9}, \frac{2}{11}$. 10. $3\frac{1}{9}$.
11. $\frac{(n+1)xy}{x+ny}, \frac{(n+1)xy}{nx+y}$. 12. 2, 4. 13. 6, 2. 14. 8, 2. 15. 9, 1.

Exercises 205.

1. $-3\frac{1}{4}, -4\frac{1}{2}; -1\frac{11}{16}, 2\frac{17}{32}; -\frac{1}{8}, -\frac{3}{34}$.
3. $2px^2-(p^2+4q)x+2pq=0$. 4. $5\frac{7}{13}$. 5. ± 15. 7. 7.
11. 20, 5. 12. $\frac{1}{2}y$. 21. $7 \pm 4\sqrt{3}$. 23. 2, 3, 6; -3, -7, 21.

Exercises 206.

1. $\frac{2}{3}n(n+1)(2n+1)$. 2. $2n^2(n+1)^2$. 3. $n^2(2n^2-1)$.
4. -540. 5. $\frac{1-(n+1)x^n+nx^{n+1}}{(1-x)^2}$. 6. 4.
7. $\frac{1}{3}n(4n^2+6n-1)$. 8. $\frac{1}{6}n(n+1)(n+2)$.
9. $\frac{1}{18}-\frac{1}{3(n+1)(n+2)(n+3)}$. 10. $\frac{1}{2}$. 11. $\frac{n}{2n+1}$. 12. $\frac{11}{18}$.
13. $\frac{n+1}{2n+3}$. 14. $\frac{n(3n+5)}{4(n+1)(n+2)}$. 15. $\frac{1}{3}n(n^2+2)$.
16. $\frac{1}{3}n(n+1)(4n-1)$. 17. $\frac{1}{6}n(n+1)(2n+7)$.
18. $\frac{1}{6}n(n+1)(2n+13)$. 19. $\frac{1}{3}n(n+1)(6n^2-2n-1)$.
20. $\frac{1}{12}n(n+1)(n+2)(3n+1)$. 21. $\frac{1}{12}n(n+1)(n+2)(3n+5)$.
22. $\frac{1}{2}n(10n^2+7n-1)$. 23. $\frac{1}{2}n(n+1)^2(n+2)$. 24. $(r+1)2^{r-1}$.
25. $3-\frac{3}{2^n}-\frac{n}{2^{n-1}}$. 26. $\frac{1}{12}n(n+1)(n+2)(3n+13)$.
28. $\frac{1}{6}n(n+1)(n+2)$. 29. $\frac{1}{12}n(n+1)(n+2)(n+3)$.
30. $\frac{1}{30}(6n^5+15n^4+10n^3-n)$. 31. $\frac{1+x}{(1-x)^2}$.
32. $\frac{1}{6}n(n+1)(3a+\overline{n-1}\,.\,b)$.
33. a_0, a_1-2a_0, $(n-1)a_0-na_1$, $(n-1)a_1-(n-2)a_0$.
34. $\frac{1-ar+ar^2}{(1-r)(1-ar)}$. 35. $\frac{2y+1\pm\sqrt{12y+1}}{2(y-2)}$.
36. $\frac{1+x-(n^2-2n+3)x^n+(2n^2-2n+1)x^{n+1}-n^2x^{n+2}}{(1-x)^3}$.

Exercises 208.

1. 110. **2.** 120, 720. **3.** 6720, 40320. **4.** 20, 4.
5. 120, 20, 4. **6.** 360.

Exercises 209.

1. 6, 24, 120. **7.** 8.

Exercises 210.

1. 120, 336, 132, 5040. **3.** 11. **4.** 6. **5.** 7. **6.** 362880.
7. 30. **8.** 30240. **9.** 415800. **10.** 3360, 120.
11. 720, 120, 24. **12.** 360, 24. **13.** 24, 12. **14.** 210.
15. 96. **16.** 144. **17.** 86400. **18.** 12. **19.** 30240.
20. 1296. **21.** 3125. **22.** 60, 125. **23.** 86400.
24. $|\underline{n}$, $|\underline{n-1}$, $\frac{1}{2}|\underline{n-1}$.

Exercises 211.

1. 10, 56, 1365, 120. **3.** 10. **4.** 10. **5.** 66. **6.** 380.
7. 455. **8.** 36. **9.** $\frac{1}{6}n(n-1)(n-2)$. **10.** 165.
11. 15504, 816. **13.** 26400. **14.** $|\underline{12}/(|\underline{4})^3$. **15.** $|\underline{16}/(|\underline{4})^4$.
16. 225. **17.** 63. **18.** 15. **19.** 455. **20.** 990.
21. 15. **22.** 420. **25.** 71. **26.** 14. **27.** 717.

Exercises 212.

2. 7200. **3.** 2880. **4.** 6. **5.** 70. **6.** $|\underline{10}/(|\underline{4})^2$.
7. 5. **8.** 3360, 360. **9.** 315. **10.** 35. **11.** 72.
12. 495. **13.** $m+1$, $\frac{1}{2}m(m+1)$. **14.** 78. **15.** $|\underline{2n}/2^n$.
16. $|\underline{2n}/(|\underline{n})^2$. **18.** 72. **19.** 36. **20.** 56. **21.** 2205.
22. 371. **23.** 35. **24.** 520. **25.** 4320.
26. $\frac{1}{8}(n+1)n(n-1)(n-2)$. **27.** $\frac{1}{6}(n-p)(n^2+np+p^2-3n-3p+2)$.
28. $\frac{1}{2}pq(p+q-2)$.
29. $\frac{1}{6}[n(n-1)(n-2)-p(p-1)(p-2)-q(q-1)(q-2)]$. **30.** 40.
31. $\frac{1}{6}n(n-1)(n+4)$. **32.** 330. **35.** 3360, 21. **36.** 22800.
37. 3360, 720. **38.** 78. **39.** 270. **40.** 15. **41.** $4|\underline{8}$, $18|\underline{7}$.
42. $4|\underline{8}$, $16|\underline{7}$. **43.** 18, 270. **44.** 30, 500. **45.** 8220.

Exercises 213.

1. $1+4x+6x^2+4x^3+x^4$. 2. $a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$.
3. $1+10x+40x^2+80x^3+80x^4+32x^5$.
4. $16a^4-96a^3b+216a^2b^2-216ab^3+81b^4$.
5. $32x^5+40x^4a+20x^3a^2+5x^2a^3+\frac{5}{8}xa^4+\frac{1}{32}a^5$.
6. $2(1+10x^2+5x^4)$. 7. $a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6$.
8. $1-7x+21x^2-35x^3+35x^4-21x^5+7x^6-x^7$.
9. $1-12x+60x^2-160x^3+240x^4-192x^5+64x^6$.
10. $128a^7-224a^6x+168a^5x^2-70a^4x^3+\frac{35}{2}a^3x^4-\frac{21}{8}a^2x^5+\frac{7}{32}ax^6-\frac{1}{128}x^7$.
11. $1+12x+66x^2+220x^3+495x^4+\ldots$.
12. $1024a^{10}-5120a^9b+11520a^8b^2-15360a^7b^3+\ldots$.
13. 1·4641. 14. 1·030301. 15. ·6561. 16. ·970299.
17. $84x^3$. 18. $-960x^3$. 19. $7920a^8b^4$. 20. $4032x^{10}$.
21. $250047a^6b^2$. 22. $\frac{55}{144}x^8$. 23. $-\frac{165}{64}a^4b^7$. 24. $-126a^4$.
25. $20x^3$. 26. $-\frac{896}{27}x^5y^5$. 27. $\frac{\underline{|20}}{\underline{|10}\,\underline{|10}}x^{10}$. 28. $\frac{\underline{|2n}}{\underline{|n}\,\underline{|n}}x^n$.
29. $-35x^3$, $35x^4$. 30. $\frac{\underline{|2n+1}}{\underline{|n}\,\underline{|n+1}}x$, $\frac{\underline{|2n+1}}{\underline{|n}\,\underline{|n+1}}\frac{1}{x}$.
31. $\frac{\underline{|29}}{\underline{|14}\,\underline{|15}}a^{15}x^{14}$, $-\frac{\underline{|29}}{\underline{|14}\,\underline{|15}}a^{14}x^{15}$. 32. 2640. 33. -56.
34. $10a^9$. 35. $\underline{|2n}/(\underline{|n})^2$. 36. ·96060. 37. ·4783.
38. 16. 39. 145.

Exercises 214.

1. 70. 2. 462. 3. $30\frac{15}{16}$. 4. $41\frac{13}{27}$. 5. $\underline{|30}/(\underline{|15})^2$.
20. $4n-5r-3=0$. 21. 12.

Exercises 215.

1. conv. 2. conv. 3. conv. 4. div. 5. conv.
6. osc. 7. div. 8. conv. 9. conv. 10. div. 11. conv.
12. div. 13. conv. 14. conv. 15. conv. 16. conv.
17. div. 18. conv. 19. osc. 20. conv. 21. div.
22. conv. 23. div. 24. conv. 25. conv. 26. div.
27. osc. 28. conv. 29. conv.
30. conv. if $x<1$, div. if $x=$ or >1.
31. conv. if $x<$ or $=1$, div. if $x>1$. 32. conv.

Exercises 216.

1. $1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3-\frac{5}{128}x^4+\ldots.$

2. $1+\frac{3}{2}x+\frac{3}{8}x^2-\frac{1}{16}x^3+\frac{3}{128}x^4-\ldots.$

3. $1-\frac{11}{2}x+\frac{99}{8}x^2-\frac{231}{16}x^3+\frac{1155}{128}x^4-\ldots.$

4. $1+x+x^2+x^3+x^4+\ldots.$

5. $1+2x+3x^2+4x^3+5x^4+\ldots.$

6. $1-3x+6x^2-10x^3+15x^4-\ldots.$

7. $1+x+\frac{3}{2}x^2+\frac{5}{2}x^3+\frac{35}{8}x^4+\ldots.$

8. $1+2x+5x^2+\frac{40}{3}x^3+\frac{110}{3}x^4+\ldots.$

9. $1+\frac{3}{2}x^2+\frac{15}{8}x^4+\frac{35}{16}x^6+\frac{315}{128}x^8+\ldots.$

10. $1+\frac{1}{3}x+\frac{2}{9}x^2+\frac{14}{81}x^3+\frac{35}{243}x^4+\ldots.$

11. $\frac{1}{x^4}-\frac{4}{x^3}+\frac{10}{x^2}-\frac{20}{x}+35-\ldots.$

12. $x^{\frac{7}{6}}\left(1-\frac{7}{x}+\frac{14}{x^2}-\frac{14}{3x^3}-\frac{7}{3x^4}-\ldots\right).$

13. $-\frac{5}{128}x^4.$

14. $\frac{1}{2}x^8.$

15. $-\frac{3}{8}x^5.$

16. $-21x^5.$

17. $\frac{5}{2}x^3.$

18. $\frac{1456}{729}x^5.$

19. $(-1)^{r-1}\frac{1\,.\,3\,.\,5\,\ldots\,(2r-3)}{\underline{|r}}x^r.$

20. $x^r.$

21. $(r+1)x^r.$

22. $(-1)^r\frac{(r+1)(r+2)}{2}x^r.$

23. $\frac{1.3.5\ldots(2r-1)}{2^r\,\underline{|r}}x^r.$

24. $\frac{3\,.\,5\,.\,7\,\ldots\,(2r+1)}{\underline{|r}}x^r.$

25. $\frac{(r+1)(r+2)(r+3)(r+4)}{3\,.\,2^{r+3}}x^r.$

26. $\frac{n(n+1)(n+2)\ldots(n+r-1)}{\underline{|r}}x^r.$

27. $\frac{10\,.\,7\,.\,4\,.\,2\,.\,5\,\ldots\,(3r-13)}{3^r\,\underline{|r}}x^r.$

28. $(-1)^{r-1}\frac{(n-1)(2n-1)\ldots(\overline{r-1}\,.\,n-1)}{\underline{|r}}x^r.$

29. $\frac{2n(2n-1)(2n-2)\ldots(2n-r+1)}{\underline{|r}\,.\,2^r}x^r.$

30. $\frac{(1+n)(1+2n)\ldots(1+\overline{r-1}\,.\,n)}{\underline{|r}}\cdot\frac{1}{x^{2rn+1}}$

31. $\frac{3\sqrt{2}}{16384}x^{\frac{11}{2}}.$

32. $\frac{429}{16}\frac{x^7}{a^{\frac{15}{2}}}.$

33. $1-2x-x^2-\frac{4}{3}x^3-\frac{7}{3}x^4-\ldots;\ -\frac{2\,.\,4\,.\,7\,\ldots\,(3r-5)}{\underline{|r}}x^r.$

34. $1+6x+30x^2+140x^3+630x^4+\ldots.$

35. $-\frac{7}{256}x^5.$

36. $-\frac{88}{729}x^7.$

40. 2.

41. $\frac{1}{3},\ \frac{35}{243};\ -\frac{4}{3},\ \frac{5}{243}.$

42. 10·049876.

43. 1·0098534.

44. 5·000960.

Exercises 217.

1. 3rd, $5\frac{1}{2}$. 2. 1st, 1. 3. 8th, 9th. 5. $x<\frac{1}{2}a$.

6. $\dfrac{q^2(bp-aq)}{p^4}$. 7. $4(2r-1)$. 8. $\frac{1}{2}(3r^2+3r+2)$.

9. $\frac{1}{3}(r+1)(2r^2+4r+3)$. 10. $(-1)^r\dfrac{r+1}{a^{r+2}}$. 11. $(r+3)\,2^{r-2}$.

12. $1-3x+3x^2-3x^4+3x^5-\ldots$ 13. $102\frac{3}{8}$.

15. $\frac{4}{3}+\frac{16}{9}x+\frac{52}{27}x^2+\ldots$; $2\left(1-\dfrac{1}{3^{n+1}}\right)$. 16. $2^{r+1}-1$. 17. $\dfrac{a^r-b^r}{a-b}\,x^r$

18. $\frac{1}{3}\{1+(-1)^r\,.\,2^{r+1}\}$. 19. $1+x-x^3-\ldots$, $1+x+3x^2+5x^3+\ldots$

Exercises 218.

2. $\frac{43}{720}$. 3. 0. 4. $(-1)^n(a-nb+n\,.\,\overline{n-1}\,.\,c)/|\underline{n}$. 5. e^{-1}.

Exercises 219.

2. $-\dfrac{3}{n}$ if n be a multiple of 4, $\dfrac{1}{n}$ in other cases.

3. $1+\dfrac{1-x}{x}\log(1-x)$. 8. $-2x^2+2x^3-4x^4+6x^5-\ldots$.

9. $\dfrac{1}{2}\left\{\dfrac{1}{3^n}+(-1)^n\dfrac{1}{2^n}\right\}$.

Miscellaneous Problems.

4. $(y-z)^2+(z-x)^2+(x-y)^2$.
5. $(ac+bd)^2+(ad-bc)^2$ or $(ac-bd)^2+(ad+bc)^2$; $(ax+by)^2-(ay+bx)^2$ or $(ax-by)^2-(ay-bx)^2$.

8. 250. 11. $\dfrac{na}{(n-1)b}$, $\dfrac{(n+1)a}{nb}$, $\dfrac{na}{(n+1)b}$, $\dfrac{(n-1)a}{nb}$.

22. $xy(x-y)$. 34. 300. 36. $\dfrac{(m-n)a+2nb}{m+n}$, $\dfrac{2ma-(m-n)b}{m+n}$.

41. $a(q-r)+b(r-p)+c(p-q)=0.$ 42. $a\frac{r(r^n-1)-n(r-1)}{(r-1)^2}.$

43. Greater. 46. $a^2=b^2+2c^2.$ 48. $a^3+2c^3=3ab^2.$

49. $m^{\frac{2}{3}}-n^{\frac{2}{3}}=4.$ 65. $-\frac{p(a+2b)}{2a}.$

67. $x>-\frac{1}{3}$ and $<\frac{1}{2}$ or $x>1$ and <4. 70. 214713.

71. 271. 73 $\frac{1}{3}(n^2-1).$ 74. $\frac{1}{2}(n+1)(n+2).$

77. $2abc+bc+ca+ab=1.$ 79. $a^2=b^2=c^2.$

80. $a^2b^2c^2(bc+ca+ab)=(b^2c^2+c^2a^2+a^2b^2)(bcd^2+cae^2+abf^2).$

88. 10 cwt. 89. 352, 320, $306\frac{2}{23}$ yds. per min.

90. 352 yds. per min. 91. 5 mins., 6 mins.

92. 10 a.m., 9.30 p.m. 93. 3 yds.

98. 105 lbs. 100. 49, 43, 38.

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