PHILOSOPHICAL

TRANSACTIONS

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MDCCCLIII.

ADVERTISEMENT.

THE Committee appointed by the *Royal Society* to direct the publication of the *Philosophical Transactions*, take this opportunity to acquaint the Public, that it fully appears, as well from the Council-books and Journals of the Society, as from repeated declarations which have been made in several former *Transactions*, that the printing of them was always, from time to time, the single act of the respective Secretaries till the Forty-seventh Volume; the Society, as a Body, never interesting themselves any further in their publication, than by occasionally recommending the revival of them to some of their Secretaries, when, from the particular circumstances of their affairs, the *Transactions* had happened for any length of time to be intermitted. And this seems principally to have been done with a view to satisfy the Public, that their usual meetings were then continued, for the improvement of knowledge, and benefit of mankind, the great ends of their first institution by the Royal Charters, and which they have ever since steadily pursued.

But the Society being of late years greatly enlarged, and their communications more numerous, it was thought advisable that a Committee of their members should be appointed, to reconsider the papers read before them, and select out of them such as they should judge most proper for publication in the future *Transactions*; which was accordingly done upon the 26th of March 1752. And the grounds of their choice are, and will continue to be, the importance and singularity of the subjects, or the advantageous manner of treating them; without pretending to answer for the certainty of the facts, or propriety of the reasonings, contained in the several papers so published, which must still rest on the credit or judgement of their respective authors.

It is likewise necessary on this occasion to remark, that it is an established rule of the Society, to which they will always adhere, never to give their opinion, as a Body, upon any subject, either of Nature or Art, that comes before them. And therefore the thanks, which are frequently proposed from the Chair, to be given to the authors of such papers as are read at their accustomed meetings, or to the persons through whose hands they received them, are to be considered in no other light than as a matter of civility, in return for the respect shown to the Society by those communications. The like also is to be said with regard to the several projects, inventions, and curiosities of various kinds, which are often exhibited to the Society; the authors whereof, or those who exhibit them, frequently take the liberty to report and even to certify in the public newspapers, that they have met with the highest applause and approbation. And therefore it is hoped that no regard will hereafter be paid to such reports and public notices; which in some instances have been too lightly credited, to the dishonour of the Society.

The Meteorological Journal hitherto kept by the Assistant Secretary at the Apartments of the Royal Society, by order of the President and Council, and published in the Philosophical Transactions, has been discontinued. The Government, on the recommendation of the President and Council, has established at the Royal Observatory at Greenwich, under the superintendence of the Astronomer Royal, a Magnetical and Meteorological Observatory, where observations are made on an extended scale, which are regularly published. These, which correspond with the grand scheme of observations now carrying out in different parts of the globe, supersede the necessity of a continuance of the observations made at the Apartments of the Royal Society, which could not be rendered so perfect as was desirable, on account of the imperfections of the locality and the multiplied duties of the observer.

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The *fifty* Foreign Members of the Royal Society.

A List of Public Institutions and Individuals, entitled to receive a copy of the Astronomical Observations (including Magnetism and Meteorology) made at the Royal Observatory at Greenwich, on making application for the same directly or through their respective agents, within two years of the date of publication.

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Institutions.

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Falmouth	Royal Cornwall Polytechnic Society.
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Sabine, Colonel, R.A.	Woolwich.
Senftenberg, Baron von	Prague.
Wartmann, Professor Elie	Geneva.
Younghusband, Capt., R.A.	Woolwich.

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PHILOSOPHICAL TRANSACTIONS.

I. *Further Inquiries as to the Structure, Development, and Function of the Liver.* *By* C. HANDFIELD JONES, *M.D. Cantab., F.R.S. Assistant Physician to St. Mary's Hospital.*

Received November 19, 1851,—Read January 17, 1852.

SINCE I presented to the Royal Society in 1847 the description which I have given of the structure of the liver, I have become acquainted with the researches of Professor RETZIUS, and of Dr. LEIDY, and have been favoured by M. NATALIS GUILLOT with a view of the preparations on which he grounds the opinions I noticed on that occasion.

Professor RETZIUS describes the hepatic ducts as forming close networks in the sheaths of GLISSON'S capsule, perilobular or alveolar networks; from which are given off minute lobular networks interwoven with the portal-hepatic plexuses, and constituting with them the substance of the lobules. These plexuses are described as consisting of anastomosing tubes which are formed of a basement or limitary membrane, like those of other glands, and in these tubes I presume Professor RETZIUS considers the hepatic cells to be lodged.

Dr. LEIDY'S account is very similar, at least as far as relates to the structure of the lobules; he figures the cells as lying within tubes, which have walls of basement membrane, and are two or two and a half times the diameter of the secreting cells.

In M. NATALIS GUILLOT'S preparations, the injection thrown in by the hepatic duct is seen lying in the interspaces of the capillary blood-plexus, and occupying the whole extent of the lobules. The view of these anatomists is in great measure accepted and confirmed by Dr. CARPENTER*: he has satisfied himself that a system of canals prolonged from the bile ducts exists in each lobule, but is unable apparently to discover the basement membrane, which RETZIUS and LEIDY agree in describing. The testimony of MULLER, WEBER, and KRONENBERG is also to the same effect; they describe the hepatic ducts as commencing in very fine networks interlaced with the capillary networks between the portal and hepatic veins. And lastly, all the now

* *Vide* Art. Secretion, Cyclop. of Anat. and Physiol.

mentioned authorities are but corroborators of the original description given by the founder of the anatomy of the liver, Mr. KIERNAN, so that one can scarce withhold assent from the doctrine, that a "lobular biliary plexus" really exists*.

To advance a contrary opinion in the face of the testimony of so many and so eminent authorities, can hardly be judged other than presumptuous; yet if it appear that the ascertaining of truth is my sole object I shall not fear this censure, as well knowing that candid minds are glad to confront their own views with all possible objections, which serve as tests to detect the alloy of error, and to bring forth into clearer show the lustre of the real gold of truth.

I will mention first two results I have obtained by the method of injection, which go to confirm the views of those who maintain the existence of a lobular biliary plexus, and are, therefore, opposed to that which I have supported. In reading the description given by chemists of the mode of obtaining the acids of the bile, the cholic, and the choleic, it occurred to me, that the same reagent, the acetate of lead, might be employed to produce a precipitate with the bile while still contained in the ducts.

I tried this first with a Guinea Pig, injecting the ductus com. choled., immediately after the animal had been killed, with a saturated solution of the salt above mentioned. This produced a most abundant precipitate in the gall-bladder, and along the ducts, even to those of extreme minuteness, rendering them visible to the naked eye as ramifying white lines or streaks. In several parts there was seen springing from the sides of ducts running in fissures or small portal canals, a minute plexus consisting apparently of short straight vessels, uniting at nearly right angles with each other. When very thin slices of parts presenting this appearance were minutely examined, it was manifest that these vessels exactly occupied the intercellular spaces; they had no membranous walls, and seemed to consist solely of injection forced in between the cells. Their diameter was very uniform $\frac{1}{10,000}$th of an inch, and the side of the meshes exactly equalled that of the cells, about $\frac{1}{1000}$th of an inch. They originated directly from the ducts of fissures or canals, and not from any such network as Professor RETZIUS describes. This appearance of a plexus was very partial, it occurred only in certain spots, and not, as it appeared to me, in those parts where the ducts were best injected; but was seen as a border of varying width along those portal canals where extravasation had manifestly occurred. I repeated this experiment with the liver of the Sheep, and obtained a result which appeared very striking. The lobules, where the injection had penetrated, presented in their whole extent the appearance of a plexus. This plexus was very different in its aspect to the partial one obtained by injection of the Guinea Pig's liver; it consisted not so much of distinct vessels, but of spots, patches, and very fine streaks, coalescing irregularly together. I am inclined to consider the result in both these cases as produced by the action of the acetate of lead upon the albuminous plasma lying between the cells;

* KÖLLIKER, however, denies the existence of a biliary plexus, and describes the ducts to terminate nearly in the same way that I do.

this agent, when added directly to a portion of parenchyma, produced no marked effect on the cells, rendering them only somewhat more granular, but coagulated a considerable quantity of the free interposed material.

Having described these two results, which, though quite capable, as I conceive, of being otherwise interpreted, appear to support the opinion of there being a lobular biliary plexus, either containing and enclosing the cells in its passages, or interposing its tubes between them, I now proceed to detail those observations which I have recently made, and which after candid consideration do still confirm me in the opinions I formerly expressed, and also throw some light on the mode in which the efferent mechanism may be conceived to perform its function.

The class of Fishes I have found, as before, most easy of examination, and most productive of results, and I would ask inquirers into the subject to direct their attention carefully to these examples. My later dissections have confirmed entirely the description given in my last paper; they show the ducts running a long course with comparatively little branching through the parenchyma, encrusted by it on every side, and terminating without forming any such connection with it as to envelope and surround it. Often the ultimate branches appear as tracts of finely granulous or amorphous matter, in which, when it is broken up, delicate nuclei are discerned: this is perhaps the most common condition, but there are two others which I have most unquestionably observed, and which *quoad* the physiological action of the parts are of great interest. In one of these, the least common, the nuclei are not obscured by the granulous matter, but are distinctly visible, as I figured them in my last paper from the liver of the Perch; they seem to constitute the chief part of the wall of the ductule, and lie close together in a scanty amount of granulous basis substance. This condition, unusual in Fishes, is ordinary in Mammalia; of its import we shall subsequently attempt to give some interpretation. In the other condition, the ducts, not only the terminal, but even the smaller trunks, are *filled* with a pellucid opaline material, in which are imbedded vesicles containing a fluid of the same aspect; often the vesicles are exceedingly numerous, and seem to replace the opaline material; it may indeed be said that the two exist in inverse ratio, and that the one is developed from the other. It seems most probable that the fluid in the interior of the vesicles is bile, and that by their dissolution it is set free in the efferent passage.

A very frequent appearance is, that in the interior of ducts the outlines of vesicles are discerned more or less distinctly in various parts amid translucent free material. The size of the vesicles varies a good deal, from $\frac{1}{3333}$rd of an inch to $\frac{1}{1000}$th of an inch; they seem of exceeding delicacy, but resist sufficiently to alter each other's shape by mutual pressure, and to float about in the water for a short time after they have escaped from the duct. I have called them vesicles, as they do not commonly appear to contain nuclei; but in one instance I found that nuclei did exist in their interior, so that they may sometimes no doubt attain the rank of cells.

An important point to be noticed with regard to the ultimate and penultimate ducts is, that they appear to be *filled* with their epithelium; the central passage must be extremely small, and cannot be actually discerned: this surely is not the condition of a mere channel of passage, a mere outlet for an already elaborated fluid; it surely betokens an active process of attraction, and probably chemical transformation to be going on.

The truth of this remark will be more apparent if we compare the condition of the epithelium of the ultimate or elaborating ducts with that of the large trunks: in the former it has the characters we have just described; in the latter it consists of a simple pavement of short columnar particles; the one is in the condition of the ureter or any simple mucous surface, the other in that of the deeper parts of the gastric or renal tubule.

I subjoin here the details of some examinations I have made in this class of animals, believing that I cannot in any better way convey an idea of the various appearances observed. In a Pike (*Esox lucius*), to whom I had given a dose of calomel, there was observed a peculiar condition of the ducts which seems worth noticing, although there was no reason to regard it as in any way produced by the drug. Many of the larger ducts were obstructed by a concrete deposit in their interior, which, viewed by direct light, was of a glistening white, and probably consisted of cholesterine or some bile-derived matter. At the obstructed parts the duct was considerably narrowed, while beyond it in the direction of ramification it was greatly dilated, sometimes to more than twice its natural magnitude; the distended part forming a pouch of somewhat globular form, into which small absorbing ducts opened; these must doubtless have existed, though they seemed in some cases to be more or less obliterated. The wall of the dilated part was greatly thickened, chiefly by an increase of its fibrous investment; within this a homogeneous tunic was distinctly discernible, supporting a layer of epithelium, which was, I believe, of secreting function, and not a mere lining. The fluid contained in the dilated part was in some instances of a decided bilious colour, which probably depended on its having become concentrated by being retained in the canal longer than usual. The obstructions in the ducts had occasioned no noticeable change in the parenchyma; it was tested for sugar by TROMMER's method, and gave distinct evidence that this was present.

About the middle of February I examined a Roach (*Leuciscus rutilus*) very full of ova, the parenchyma of the liver was of a pale red tint and semitranslucent; it consisted of nuclei, granular matter, and multitudes of delicate semitransparent vesicles packed closely together; one or two tablets of cholesterine, and some biliary deposits were observed in it.

One part of this liver formed a long thin tongue, and this viewed without being cut exhibited a beautiful capillary network perfectly injected with blood; the vessels formed long narrow meshes; and it was especially observable that, while there was scarcely any oil to be seen in other parts of the parenchyma, there were little groups of small oil-drops seated at intervals all along the sides of the vessels, as if it had just exuded from them. The ducts, when dissected out, were seen most perfectly; all the elaborating ones were crowded with delicate pellucid vesicles, varying a good deal in size, and lying together with nuclei imbedded in amorphous matter; similar vesicles and more distinct nuclei were seen in the larger ducts up to a diameter of $\frac{1}{384}$th of an inch; some biliary deposits were seen adhering to the exterior of the ducts, which in this instance were not dilated.

About the same time I dissected another Roach whose abdomen contained but little roe, and who had been dosed, fifteen hours before death, with two grains of blue pill.

The spleen contained several groups of yellow corpuscles; some of the larger were surrounded by a homogeneous envelope; they consisted of various-sized corpuscles of about the diameter of blood-globules, but more spherical and more opake, as well as more brightly and differently coloured. Coloured corpuscles were sometimes seen quite alone, sometimes a very few together; such were not surrounded by envelopes.

The liver was of the buff-yellow colour, which indicates a fatty condition; its parenchyma was loaded with oil, which was in the state of small drops that became detached and floated freely about in the field of view; I satisfied myself of the existence of the pellucid vesicles that were so numerous in the former, but they were not present in nearly so great numbers; the parenchymal substance presented the appearance of imperfectly formed celloid particles coalescing together in a plexiform manner; they were surrounded and enveloped in oily matter. In some parts small biliary deposits were also observed.

On dissecting the ducts out, I found some of the trunks extraordinarily dilated into great pouches: the smaller branches, and especially those nearly terminal, were not enlarged, or not much; but they and all the ducts, even the trunks, were stuffed with their epithelium, which was at first scarcely discernible, the tubes appearing as if filled with a pellucid matter; but when one was ruptured and the epithelial contents escaped, the vesicles, nuclei and amorphous matter came clearly into view.

On examining closely the large ducts, especially after steeping them in solution of bichloride of mercury, it was well seen how their epithelium had encroached on their cavity, so as in some parts to have closed the canal completely; this obstruction had doubtless occasioned the dilatation of the ducts into pouches, which may also have been promoted by the inordinate growth of the epithelium in those parts. The dilatation of the ducts was not confined to those now described; many of the smaller were also affected, so that branches traced in the direction of ramification actually enlarged instead of diminishing, and this manifestly depended on their own stuffing, and the obstruction towards the outlet. In some ducts the vesicles were very apparent, crowded together in groups; in others there was scarce anything to be seen but an amorphous and fluid matter. Neither this liver nor that of the preceding yielded me any evidence of the presence of sugar.

A third Roach was dosed with 1½ grain of blue pill twice, an interval of seventeen hours being allowed to elapse between the doses; it was killed sixteen hours after the last. It contained large masses of ova. The liver was pale, semitranslucent, and contained a good deal of blood in its vessels; its parenchyma consisted of nuclei, granulo-amorphous matter, and very delicate granular globules; there was very little oil, and not any biliary deposits were seen. The gall-bladder was very full of bile. The ducts dissected out appeared occasionally distended, but to a much less extent than in the preceding instance; the epithelium in the larger ones consisted of columnar particles, which viewed *in situ* gave a mosaic or pavement-like appearance; in those of next size the epithelium was in great part vesicular; and in those of smallest size it chiefly consisted of nuclei and amorphous matter, with a few incipient vesicles. In several of the ducts the canal was very much narrowed by the epithelial growth.

These details, though rather minute, are not, I think, devoid of interest, especially as illustrating the actively growing nature of the epithelium of the ducts. With regard to the condition of the parenchyma generally I have nothing to add to my former observations, except that I have usually noticed that when the ova are considerably developed, and form large masses occupying a great space in the abdomen, the quantity of oil in the liver is greatly diminished: it seems to be diverted from its usual place of deposit, and appropriated to the formation of the growing structures.

Development in Fishes.

I now proceed to describe the development of the liver in this class, in which it is observed under very favourable circumstances, as the stages succeed each other slowly, and the transparency of the tissues enables the observer to conduct his examination with little injury to the specimen. In a young Perch (*Perca fluviatilis*), which had recently quitted the ovum, and in which the circulation was distinctly visible, I found the vitelline sac attached to the body, but not included by abdominal walls; there was no intestine yet formed, nor any trace of liver. In the vitelline sac there was seen a smaller one lying near the anterior extremity just behind the heart; this contained a more highly refracting matter than that in the larger vitelline cavity. The next day I observed that the heart was covered in by a strong membrane which passed over the yolk-sac, and was continuous with the inferior vertical fin, and doubtless with the sides of the body, thus closing in the abdominal cavity.

An intestinal tube was now formed, and an oral cavity, but the intestine did not communicate with it, nor had it any anal outlet. By the morning of the next day the intestine had extended quite to the throat, and was enlarged into a fusiform dilatation corresponding to the stomach, while a narrow space intervened between it and the vertebral column. By the evening of the same day the continuity of the intestine as œsophagus into the throat, and the cavity of the mouth was most distinct. I observed a short blind offset passing backwards and upwards towards the vertebral column from the stomachal dilatation; this was evidently the rudiment of the air-bladder. A day later the plicæ of the intestine had begun to appear; its walls were very thick, there was no trace to be seen of liver, or of renal glands. Peristaltic movements were observed in the intestine twenty-four hours later; the duct of the air-bladder was now of pretty good length, but scarcely appeared to be hollow.

Two days from this time the intestine was sufficiently consistent to admit of being detached; its surface was still more plicated; its posterior fourth was separated from the anterior portion by a kind of annular constriction, thus marking off the future large intestine: the walls presented some appearance of developing muscular fibre.

Three days later the cardiac orifice of the stomach presented a more marked constriction, separating it from the œsophagus; no liver nor kidneys were yet formed. I observed at this time that the black pigment was deposited directly as such, constituting globules and streaks of various size, and not previously elaborated and gradually formed in cells. The next morning, that is ten days from the date of the first observation, I observed a mass at the anterior part of the abdomen, extending on a level with the anterior part of the yolk-sac to some distance behind it; this was doubtless the rudiment of the liver; it consisted of granulo-amorphous matter, and was traversed by ramifications of a vessel which spread itself over the yolk-sac; this vessel was a branch given off from the cardinal vein near the anus; it ran along the lower border of the intestine, and after ramifying on the yolk-sac and on the liver, joined a vein entering the heart. A rudiment of the gall-bladder, in the form of a small transparent vesicle, was also observed; the lower end of this appeared to dip down towards the intestine.

On the succeeding day I noted that the parenchymatous mass of the liver was quite distinct, and that the transparent vesicle, the future gall-bladder, adhering to it, had enlarged, but without having yet formed any communication with the intestine. I dissected the intestine out, and saw clearly that there was no protruding of its parietes as if the liver were a development from it. The constriction separating the small and the large intestine was very marked and the appearance of the two very different, the former being thrown into many folds, the latter being more smooth. On the evening of this day I observed the liver having the same shape as in the adult fish, the anterior border being notched; it was traversed by a pretty close plexus of capillaries, and contained the remains of the small yolk; it consisted of granulous and oily matter imbedding delicate vesicular cells. Two days later the parenchyma of the liver had nearly the same aspect; its cells were probably more numerous; they varied in diameter $\frac{1}{3000-6000}$th of an inch, and were imbedded in oily and granulous matter. The gall-bladder was still a transparent vesicle, in diameter $=\frac{1}{143}$rd of an inch, formed by a homogeneous tunic and having some traces of a lining epithelium; its shape was somewhat pyriform, the small end being directed upwards or anteriorly. It was distended by a transparent fluid, which as yet had no outlet. Two days after I noted that the yolk-sac imbedded in the parenchyma of the liver had become very small, and that the gall-bladder was elongated and tended downwards and backwards towards the intestine. The succeeding day the narrow extremity of the gall-bladder was still more elongated; it tended still more downward, and adhered by its apex to the stomach.

After having carried my observations thus far my stock of young fish died, and I was unable in consequence to follow the further progress of development regularly, but after the lapse of about six weeks I procured some minute fish very similar to those I had before been observing. The result of my examination of these was as follows. The liver appeared as a small grayish mass, consisting of delicate vesicular transparent cells, mingled with granulous and oily matter, not in great abundance. The gall-bladder was large, distinct, transparent, full of a colourless fluid, and lined by a distinct epithelium; it had a short duct communicating with the intestinal canal, from which an offset turned up to the mass of the liver; this proceeded but a little way, and terminated by forming a small tract of nuclear particles. The resemblance between the structures observed in these young fish and those whose development I have traced above was so close, that I feel quite justified in concluding that the former presented the ultimate stage of the developmental process, and therefore I venture to offer the following summary as descriptive of the formation of the liver and its excretory apparatus in Fishes.

(1.) The intestine is formed before there is any trace of liver.

(2.) The parenchyma of the liver and a vesicle, the rudiment of the gall-bladder, are first formed; the small yolk mass lies in the substance of the liver.

(3.) The gall-bladder assumes a pyriform shape; its narrow end approaches the intestine and opens into it, while from a part of its extent hepatic ducts are developed that extend into the liver.

(4.) The contents of the gall-bladder at first are not bile, but probably a weakly albuminous fluid; this continues to be the case for some time after the parenchyma of the liver is completely formed.

In my examination of the livers of Reptiles I have been more successful than formerly, and entertain no doubt of their being constructed exactly after the same plan as those of fish. The parenchyma in the Common Frog (*Rana temporaria*) presents no appearance of any lobular division: it is often of very dark colour, owing to the presence of a black pigmentary matter; it consists of delicate, large, feebly-formed nucleated cells, which have no well-marked envelope, and almost appear like masses of granulo-amorphous substance; they are infiltrated with oil-drops, but present no bilious tint. There are also numerous free nuclei, and much diffused granular and oily matter as well as black pigmentary, which is in the molecular condition, and exhibits the peculiar to and fro motion. A considerable quantity of the black matter is in the form of spherical or oval corpuscles, which I described in my previous paper, and which have often a red or deep yellow tint. I supposed them at that time to be biliary concretions, but further inquiry has convinced me that such is not their nature, and that they are what I have termed them above, pigmentary formations. They are scarcely affected by the strongest reagents; liquor ammoniæ, liquor potassæ, sulphuric, nitric and muriatic acids leave them almost unaltered. On comparing them with the coloured corpuscles found in the spleen they appear quite identical, and I have little doubt are truly so; both I am inclined to think are pigmentary deposits, not necessarily derived from decomposing blood-globules. They do not occur in the kidneys, and are quite distinct from the ordinary ramifying pigment cells seen in other parts. I believe them to be a peculiar form of organic matter, rich in carbon, which not being eliminated from the blood in consequence of the feeble grade of respiratory action, settles down as a deposit in these two parts, where accumulation of the blood and retardation of its current, one or both, frequently and almost naturally occur. In a Common Snake (*Coluber natrix*) I found the parenchyma consisting of abundant soft granulous and oily matter, with granular nuclei, faintly formed granular globules and cells; the latter were small, not numerous, and might easily have been overlooked. There were no pigmentary globules in this liver, but in one part of the parenchyma there was a deposit of bile tinging the granular matter, so that that part contrasted manifestly with the surrounding: there was also everywhere a considerable quantity of concrete solid fatty matter, sometimes of coarse crystalline aspect. The ducts in this instance were well seen ramifying on the coats of the vessels; many of them exhibited their tube of limitary membrane filled, and even apparently bulged by a soft granulous substance, which obscured the nuclei imbedded in it, and was mingled with a few minute oil-drops diffused through its mass. The most minute branches appeared to cease gradually, the basement membrane becoming at last indiscernible, and the structure then resembling a minute cylinder of granulous substance. Some of the larger ducts had more transparent contents, in which here and there small vesicles were seen.

In the Frog I have been able to examine the ducts very thoroughly; the smallest ones are I think for the most part like those in the Snake, small cylinders of granulous substance imbedding nuclei, and not usually containing vesicles; those of a somewhat larger size (but having I doubt not the same elaborating function) are usually full of pellucid contents, in which the outlines of exceedingly delicate vesicles, varying from $\frac{1}{1000}$th to $\frac{1}{2500}$th of an inch, are more or less plainly discernible. These vesicles should probably be termed cells, as on addition of nitric acid a number of nuclei are brought into view, which appear to be situate in their interior. It is remarkable how the duct tubes are filled with the vesicular and granulous epithelium; they have exactly the same appearance as their homologues in Fishes, and suggest the same ideas as to their function. In one frog the description now given exactly applied to the ducts of one lobe of the liver, while those of the other were in a very different state; instead of having pellucid contents, they presented a thick epithelial lining of rather coarse granular matter containing nuclei, and stained deeply in various parts with biliary matter. The aspect of this epithelium conveyed to the mind the idea of arrested action; it contrasted most forcibly with the delicate soft-looking granulous or pellucid epithelium of the ducts in the other part; this had the aspect of material rapidly changing into the fluid secretion, that of a stiff inert mass, whose vital changes were well nigh suspended. Connecting with this altered condition of the epithelium the manifest biliary accumulation and retardation in the ducts, the above interpretation will not appear I trust too fanciful. With the alteration in the condition of the ducts, there coexisted also a notable difference in that of the parenchyma; its cells were much more strongly formed, their nuclei and envelopes more apparent, their contents coarsely granular, and containing less oil than in the unaffected part. The ducts were clearly seen in this instance to pursue a long course with very little ramification; one branch I traced for $\frac{1}{67}$th of an inch without observing any branch given off from it, or scarce any diminution in its diameter.

Development in Reptiles.

In REICHERT's description of the development of the embryo of the Frog (von MULLER's Physiology), it is stated that the anterior part of the mass of yolk contained in the cavity of the abdomen isolates itself from the rest of that substance, and becomes an independent body; which constitutes the rudimentary mass from which the liver and pancreas are subsequently formed. This I think I have also observed, or at least that as soon as the yolk mass begins to take the form of an intestinal tube a portion of it remains separate, constituting two masses, a larger and a smaller, which lie one on each side of the nascent intestine. When development had advanced a little further, and the intestine was manifestly formed, I found the parenchyma of the liver lying close in contact with it, and also a small vesicle adhering to the mass of the liver, which was certainly the rudiment of the gall-bladder. The future liver consisted of multitudes of large vesicular nuclei lying imbedded in oily matter, and the whole mass was surrounded by an investing homogeneous tunic. The gall-bladder in its earliest appearance was simply a spherical mass of similar constitution to that of the liver; subsequently it presented a distinct membranous wall lined inside by a close layer of large vesicular nuclei, the interspaces between which were occupied by oil-drops. It assumed very soon more or less of a pyriform shape, and became connected with a narrow tract stretching down to the intestine which clearly represented the cystic duct; this tract consisted of similar formative substance to that composing the liver and the gall-bladder; from it in a more advanced stage offsets were developed having the same constitution, and extending into the substance of the liver; these were beyond doubt rudimentary hepatic ducts. In one instance, where the gall-bladder existed, but was certainly not connected by a duct with the intestine, I saw lying by its side a short tract consisting of similar material, which was partially divided at its peripheral extremity; this seemed to be an early rudiment of an hepatic duct. The rudimentary ducts appeared to me to be solid tracts; I could not at this period detect a cavity in the interior; subsequently they become invested with a homogeneous tunic on their exterior, and the vesicular nuclei arrange themselves in the form of an epithelial lining, the interspaces between the nuclei being occupied with oily matter. Both in the gall-bladder and in the ducts this oily matter gradually disappears; the epithelium of the former comes to present the ordinary form of a pavement, while that of the common duct *developes cilia* which are seen in vigorous action, exciting a current downward from the liver to the intestine. I do not know exactly how long this ciliary action continues to be present in the hepatic duct; I have observed it for about a month from the date of its first appearance. As development advances the oily matter in the parenchyma of the liver diminishes, and is replaced by granulous; pigment granules at the same time are deposited upon and in the vesicular nuclei, and thus those yellow and brown-coloured masses are formed which are so apparent in the liver of the adult animal. The gall-bladder for some time contains only a transparent fluid, which is coagulated in a marked manner by nitric acid, and is therefore probably only a solution of albumen; after a time the fluid in this receptacle has a biliary tinge, and then is not coagulated by the action of nitric acid. This was at least the case in two instances, though in another at a still later period the acid produced a precipitate which however may have been only mucus.

The general conclusions respecting the development of the liver in the Frog may be stated as follows:—

(1.) That a portion of the common formative yolk substance contained in the abdomen is set apart for the development of the liver.

(2.) That this occurs at about the same time when the intestine begins to be formed, so that the liver is not in any way derived from the intestine.

(3.) That the first rudiment of the efferent apparatus is the gall-bladder, and that the ducts are subsequently formed.

(4.) That a cystic duct is first formed. connecting the gall-bladder with the intestine, and that hepatic ducts are subsequently developed.

(5.) That these ducts are at first solid tracts, and as well as the gall-bladder primarily consist of similar formative substance to that which composes the liver.

I may add here that I have seen very distinctly the ciliary motion on the outer tegumentary surface of tadpoles as described by Dr. GAIRDNER; my observations were made several months before I had the pleasure of reading his description of it.

I may remark that I have obtained some very satisfactory views of the minute biliary ducts in the liver of a Rook (*Corvus*) by stripping out branches of the portal vein and washing away the parenchyma. The ducts were then seen here and there lying upon the homogeneous membrane of the vein-branches. Their condition was essentially and closely similar to that described in my former paper.

Development in Birds.

Since the publication of the account I gave in my previous paper of the development of the liver in the Chick, I have twice repeated my observations, and have found the views I advanced in the main correct. I think, however, it may be worth while to review some stages of the progress which it is important to investigate thoroughly.

The formation of the intestine takes place by the constriction of the central transparent portion of the germinal membrane; this seems to me to be a homogeneous membranous expanse, not composed of cells, and covered only with oil-drops, while the rest of the germinal membrane with which it is continuous is covered by adhering yolk-cells, and overspread with ramifications of the omphalo-meseraic vessels. When the constriction of the germinal membrane takes place, two tracts of oily matter appear, one passing forwards and the other backwards from the constricted part, where they are continuous with each other. The margins of these tracts are quite even, and soon become invested with a homogeneous membrane, but at first I doubt if this is actually present; in a chick at the seventy-seventh hour of incubation, the transparent membrane of the vitelline duct appeared to me to lose itself upon the tracts of oily matter. The upper margin of the vitelline duct and of the oily tracts forms at first nearly a straight line, which is separated by a considerable space from the vertebral column, one-half to one-third of this space being occupied by the Wolffian bodies. The posterior tract ceases near the caudal extremity; it is, I think, rather thicker but shorter than the anterior. This, in the earliest periods after passing forward with a slight curve, is lost just as it reaches a quantity of blastema situated close behind the heart. This blastema is the rudiment of the parenchyma of the liver; it does not consist of oily matter like that of the intestine, but of a less opake and more granular substance. As development advances the anterior tract continues to extend; it passes forwards and upwards above and behind the liver, and was traced on one occasion, though becoming more faint, as far as the last branchial fissure; a slight dilatation at one part of its course marks the future stomach, and from near the same part two offsets are given off, one of which runs upon the liver, the other towards the vertebral column. Both oily tracts and the offsets just mentioned are solid; they are to be regarded, I think, as a peculiar deposit of formative matter intended for the formation of particular structures; their appearance and mode of development seem to me by no means consonant with the view that the intestinal cavity is simply a part shut off from the general yolk cavity. In fact, the intestine, as has been said, is not at first a cavity but a solid tract of formative matter, and in this respect is analogous to the embryonic condition of other organs (so far as I can determine), which always seem to arise from a shapeless blastema deposited in one spot. The offset to the liver is more distinct sometimes than at others; it is most marked about the sixth day, but always may be distinguished as a small eminence of opake oily matter in the side of the intestine, until the cystic and hepatic ducts are fully formed and have united with the intestine *at that spot*. The existence of this eminence is certainly very remarkable; it may be well to give it a name and call it the *colliculus*; I do not see that its import at present can be at all explained; nor could we indeed expect this, unless we could obtain some clue to the comprehension of the mysterious principle by which the process of development is governed. I cannot see any ground for the statement that the liver is developed by the protrusion of the walls of the intestinal canal; it certainly appears at a very early period as a parenchymatous mass of blastema; and even if one could grant that this blastema originated in any way from the anterior oily tract which passes close to it, still it remains, according to my observation, perfectly unquestionable, that after the intestine and stomach are formed up to about the ninth or tenth day, the mass of the liver has no connection by ducts with the intestine.

In one instance I obtained a very good view of the developing hepatic duct as it approached the liver; it was formed in a tract of cell substance, which served as a blastema, and was invested on its margin by some fibrous tissue; in this the channel of the duct was excavated, which was lined by a distinct homogeneous membrane, and invested by a delicate transparent epithelium. Towards the hepatic extremity the cavity became less marked, its margin indistinct, and the homogeneous tunic less perceptible; but I satisfied myself, by very careful examination, that the duct-channel terminated by a cul-de-sac, while the blastemous tract, as yet unexcavated, stretched on much further.

In the parenchyma of the liver, just at the beginning of the fifth day, the formation of vessels and blood was very well seen; several groups of blood-globules lay in short channels provided with a fine homogeneous lining; these were oval or more elongated; they did not appear to coalesce, but were manifestly tending to such an arrangement, which a few hours later had actually occurred.

I saw no cells throwing out filamentary processes, such as Kölliker has described; the channels containing the blood-globules were spaces excavated in the blastematous mass; their homogeneous walls I believe to be produced secondarily, and their coalescence to be effected by a simple extension of their cavity.

About the eleventh day I found the parenchyma of the liver consisting of nuclei, cells, amorphous and abundant oily matter; here and there in the midst of the substance there were seen small bright yellow particles, which were doubtless biliary matter; the gall-bladder also was full of green bile, the presence of which, as there was no trace of cysto-hepatic ducts to be seen, I cannot satisfactorily explain; it does not appear to me, however, an idea to be altogether rejected, that before the hepatic ducts are fully developed, and have begun to eliminate bile in the parenchyma, their function should for a time be performed by the epithelium of the gall-bladder, which lies in close contact almost imbedded in the mass of the liver: the fact, at any rate, is, that the gall-bladder contains bile, when there is not a trace of it in the ducts, and when their development is yet incomplete.

The development of the pancreas seems to be very similar in its mode to that of the liver, and the same plan will, I believe, be found to be followed in the formation of all glands. The organ consists at first of blastematous matter, imbedding nuclei and myriads of granular globules, which are clustered over with oily molecules. The sides of the mass are even, and it presents no trace of the ultimate glandular vesicles, which are so characteristic of its adult condition. About the same time that the ducts of the liver appear, those of the pancreas begin to be developed; they are not formed by a protrusion from the intestine, but rather seem to develope themselves on the spot where they are to exist. This at least is the case with the main trunks; the smaller branches are no doubt formed by an extension of the cavities of the larger ones

It must be sufficiently apparent how essentially similar the development of the liver and its apparatus of efferent ducts is found to be in the three lower vertebrate classes. Hitherto I have not been able to pursue the examination in Mammalia; and though I am aware that BISCHOFF's account of it in this class is in accordance with the view of its being a protrusion from the intestine, yet I must remark that the process, as being conducted more slowly in the lower classes, is more favourable for correct observation, and that it seems unlike the known unity of scheme which characterizes the works of Nature, to find an essential difference in one class from that which prevails in the three others.

I have little to add to the anatomical account I previously gave of the condition of the liver in Mammalia; I have studied it recently more particularly with a view to the elucidation of its function, and will now only touch on those points which bear upon this matter. One most important question to determine positively is where the bile is in the healthy state necessarily formed; whether in the cells of the parenchyma, or in the ultimate hepatic ducts. The prevalent opinion among physiologists certainly is that the bile is actually and necessarily formed in the hepatic cells; they are considered homologous to those of the renal tubuli, and on account of this homology and a certain not very manifest yellowish tint, it is usually held that the bile is primarily formed in them, and liberated by their dehiscence in the cavity of the duct. From this opinion, which I long held unquestioningly, and to which I have often referred, observation alone inclines me in some measure to dissent. I have scrutinized numerous specimens of the livers of our ordinary domestic animals with a view to determine this point, and still cannot convince myself that bile is present in the parenchymal cells as *a normal* and *necessary* condition. If present, it ought not to be very difficult of detection, seeing that it is a coloured fluid, and that the cell which contains it is in this class generally provided with a distinct envelope: it ought, in fact, to be as apparent, or nearly so, as it is in the cells of the follicular livers of Mollusca. But this is not the case; according to my observation the cells commonly appear as pale granulous nucleated particles, with a more or less distinct limiting envelope, and presenting a few or perhaps numerous and large oil-drops amid the soft albuminous mass. Not unfrequently a distinct yellow fluid is seen infiltrating the granulous contents, or several yellow highly refracting drops are observed in the same situation; this indicates of course that bile is present in such particles, but they are seldom seen except when the liver is in a state of so-called biliary congestion. This condition is consequently most often observed in human livers. The cells, when massed together and viewed by transmitted light, very commonly, perhaps always, produce a kind of reddish yellow tint; but this seems to me to differ decidedly from the yellow tint of bile, and to be probably dependent on some diffused hematin tinging the cells, or on some modification which the light has undergone by unequal refraction in its passage through the mass; at any rate I could not, were I advocating the opposite opinion, rely on it for a moment as sufficient evidence of the presence of bile in the cells. In some instances of healthy livers in animals, as in the Mole (*Talpa europæa*), the Hare (*Lepus timidus*), and the Squirrel (*Sciurus*), and also in others, I have noted the presence of biliary matter in the cells; nor do I doubt in the least the possibility of its being frequently so found, only it seems to me that evidence is wanting to show that this condition is constant and essential. I have often tried the effect of adding nitric acid to the cells, but I do not think that I have thereby obtained any more decisive evidence regarding the point in question; this reagent makes the cells more opake, contracts them, renders their envelopes more marked, and their contents more granular, but does not impart a more positive biliary tint. I think as soon as a real yellow tint exists the eye will judge of it much more satisfactorily when the cells are in their natural state, than when they are shrunk and entangled amid granular films of coagulated plasma.

Not wishing to rely solely on my own observations, I wrote to three of the most eminent physiologists in London, requesting they would inform me of their views on the matter; they kindly replied to my inquiries, and permit me to subjoin their replies. Mr. BOWMAN's communication is as follows:—"My notion of the matter you refer to is shortly this, that the bile does exist in those hepatic cells that lie nearest to the surface of the lobules, *i. e.* to the portal surface where the ducts commence; not that those cells contain nothing but bile, but that bile comes out of them, either by their rupture and deliquescence, or while they lie *in situ*, undergoing only such slow degeneration as is common to the elements of most or all tissues. That the cells lie in series extending from the hepatic venous to the portal venous surface, and that there is an onward march of the cells from the former to the latter: those near the former being immature, and those near the latter mature; that the bile is gradually elaborated by the agency of all, but probably chiefly by those situated nearest the portal surface. The yellow and brown hepatic substances are the immature and mature hepatic cells respectively. The congestions of KIERNAN seem to me to depend on the bulk of these cells respectively, allowing the capillaries which flow through them to hold more or less blood after death. Many of the morbid appearances of the liver are, in my opinion, dependent on diseased conditions of one end of the series of the hepatic cells (the hepatic venous or the portal), to the exclusion of, or in a different way or degree than that other." This view of Mr. BOWMAN's is nearly the same as that I proposed in my first paper on the Secretory Apparatus of the Liver, published in the Philosophical Transactions; only that I believed the cells not to progress ("march") outwards from the centre, but that the bile was transmitted from cell to cell in order throughout a series. What led me especially to doubt of the truth of this opinion, was simply that in the majority of livers I saw oil only, and not bile in the cells. If oil be regarded as the secretion of the hepatic cells instead of bile, the view of Mr. BOWMAN would receive very strong support, for oil does accumulate in many instances, especially in Sheep's and in certain Human livers*, in a very remarkable manner in the marginal cells of the lobules, so as to form a complete opake border to them; and these same oil-laden cells are seen at times dehiscing, and apparently discharging their contents into the fissure.

But oil is not bile, and therefore this very case becomes an objection to the view of Mr. BOWMAN; for if bile is perfected in the marginal cells, it ought to be manifest and preponderating there, which in these instances, and indeed generally, is not, as far as I have seen, the case. Another condition, which I have several times observed in Human livers not seriously diseased, also appears to speak counter to the view of bile being perfected in the marginal cells of the lobules. In these, for instance, when a transverse section had been made, the central part of the lobules was seen of a distinct yellow tint, the cells manifestly containing bile; this colour shaded off towards the margin, and ceased some distance from thence, while the marginal cells themselves, for a depth of two or three in the central direction, were the seat of considerable *oily* accumulation. This could hardly have occurred if the perfection of bile by the marginal cells were a necessary step in the process†.

Though not at all essential, though I believe the ducts often form bile when none exists in the marginal cells, still there does appear to be a peculiar tendency in the cells in this situation to fill themselves with biliary matter. I have seen bile in the marginal zone of cells when all the rest of the lobules was in a state of complete degeneration; I have observed in the liver of an animal, to whom calomel had been given, bile, though present elsewhere, yet most abundant on the margins of the lobules; I have seen a similar state, accumulation of bile in the marginal cells, occasionally in Human livers; and lastly, I have often found biliary deposits adhering to the ducts in the oily livers of fish, occupying thus an analogous situation to that of the bile in the preceding instances.

The circumstance that oil in many cases accumulates particularly in the marginal zone of cells, seems to me most probably accounted for by the relation of this part to the circulating blood. These marginal cells are, in fact, the first which are exposed to the stream of portal blood as it enters the lobular capillary plexus; they consequently obtain most abundantly the oily matter therein contained, and store it up in their cavities; this, I think, is a more probable view, according to our present knowledge of the habits of cells, than to suppose that the oil is transmitted from the

* *Vide* Report of the Pathological Society, 1846, 1847, p. 105.

† When a liver presents the "nutmeg appearance" in the most marked manner, the following is commonly found to be the condition of the structure. There is fatty degeneration of the marginal zone of the lobules, which is thereby rendered quite pale; while the central part, occupying perhaps rather more than a half, is uniformly saturated by a deep venous congestion. The cells in this part are in great measure destroyed, reduced to a granular detritus, and the remaining ones are converted into spherical masses of orange pigment. The sanguine congestion and the pigmentary accumulation are precisely coterminous. Now unless the ducts themselves are the effective agents, how can such a structure secrete bile? How is the central yellow pigment to be conveyed through the marginal zone of fatty degeneration?

central cells in the direction outward. How to account for the occasional presence of bile sometimes in the central, sometimes in the marginal cells, I do not know; but the variability of its seat, as well as its frequent absence, testify, I think, against the view of the secretion of bile being an *essential* part of the function of the hepatic cells. I may remark that the seat of oily accumulation varies also as that of biliary; in the livers of Dogs I have often found it in the central parts of the lobules around the intralobular vein; this is, I think, less common than the other, and I can offer no explanation of it.

Mr. SIMON states to me, "that in perfectly normal conditions among vertebrate animals, bile, to the best of my belief, *cannot* be demonstrated to exist in the hepatic cells, and I have for a long while strongly inclined to the belief that normally it never lies within them."

Professor PAGET's testimony is to the same effect: "I quite agree with you that in the healthy livers of men the cells do not contain any coloured material, but I think I have seen their contents partially coloured yellow in other cases than those of congestion."

My belief therefore is, that *in perfectly healthy states* of the mammalian liver bile does not exist in its cells, that it is not in any case *necessarily formed* there, but that this may be always effected by the ducts. It remains then to inquire what circumstances produce biliary impletion of the cells, so that they manifestly contain bile. Congestion of the portal-hepatic capillary plexus with blood, and this chiefly of a passive kind, I am inclined to regard as one principal cause. I have frequently observed in cases of heart-disease impeding the onward current of the blood, that the cells of the liver, especially those in the centres of the lobules, were gorged with bile. It is in the central part that congestion of blood is most frequently manifest; here it commences as hepatic venous congestion of the first degree, and from hence it extends outwards towards the interlobular veins. This seems to indicate that a connection obtains between passive congestion of the blood-vessels and biliary impletion of the cells. Another, and probably not less frequent cause, especially of the slighter states, is the cessation, more or less complete, of the excretory action of the ultimate ducts, the spasm of the gall-ducts of former authors; when this occurs, the biliary matter, not being eliminated from the blood by the ultimate ducts, is carried on in the blood current, and deposited in the parenchymal cells, thus producing jaundice of the liver, which, as VIRCHOW remarks, must precede jaundice of the body when this is of hepatic origin. Biliary matter may also collect in the cells no doubt, when a larger quantity of it than natural is present in the mass of circulating blood, whether this depend on insufficient oxygenation in consequence of a sedentary life and rich feeding, or on certain atmospheric influences; in this case the excess which the excretory ducts are unable to carry off passes on, and is deposited in the parenchyma. I think that the accumulation of a large amount of oil in the hepatic cells does not generally coincide with the existence of bile in them, and *vice versâ*; thus in the fatty liver of phthisis and of fish there is scarce any trace of bile to be discerned, while in the follicular liver of Mollusca the dark biliary matter very greatly preponderates over the oily; sometimes, as has been mentioned, we see different parts of the lobules the seat of oil and bile respectively; and this is most usually the case, though there is no doubt that the two products may coexist in the same cell*.

The exact relation which the smaller and terminal ducts bear to the parenchyma is the present "vexata quæstio," and any observations that tend to elucidate this matter, and render it more easily determined, are worth our attention. I have made many examinations since the publication of my last paper relative to this point, and have been by them decidedly confirmed in the views there expressed. I stated that I believed the ducts did not ramify extensively, did not extend far beyond the "spaces," and that many fissures seemed quite destitute of them. This is, I think, generally the case, but in some instances the ducts seem to extend farther than others; with regard, however, to the main point, that the cells are a *parenchyma*, and not an *epithelium*, I entertain no doubt whatever. In many, if not in all London dogs, and in some rabbits, there is found a peculiar condition of the epithelium of the whole excretory apparatus of the liver, which renders it comparatively easy to trace and follow the ducts to their terminations. If thin sections be made of such livers, the ducts are seen as opake tubes or tracts, appearing whitish by direct light, which come up from below at the "spaces," where three lobules adjoin, divide irregularly, and run for a varying distance along the fissures. These tubes run often a long distance without branching; they are most often single, one only traversing each portal canal or fissure, and what is especially remarkable, not maintaining any close relation to the wall of parenchyma, between which and them a distinct interval is often to be seen.

Their mode of termination varies; sometimes they break up into a few delicate branches, which may give the appearance of a plexus; sometimes they terminate by apparently losing themselves, that is to say, they gradually cease, their structure being indiscernible beyond a certain point, or breaking up into a mere streak of oily molecules. Generally I think the ducts, as seen in these specimens, are confined to the canals and fissures, but minute branches are often seen which just enter their substance for a little way: I have never observed anything that in the least indicated that the ducts formed a plexus containing the cells in anastomosing tubes as described by Dr. Leidy. The opacity which renders the excretory ducts thus unusually distinct, is occasioned by a deposit of minute oil-molecules between the nuclei which intercept the light; this oily matter replacing in part the ordinary granular basis substance, and representing the oily matter which occurs so abundantly in the cortical renal tubuli of all London cats.

* I request the reader to bear in mind that Mr. Lindsay Blyth's experiments subjoined to this paper have fully shown that yellow pigment in the cells is not, at least commonly, any proof of the presence of true bile. The above was written before his experiments were performed.

It is perfectly clear that in neither of these cases it interferes in the least with the healthy discharge of the function, and we may learn therefore that excess alone of oil in any part does not constitute "fatty degeneration." The epithelial particles of the gall-bladder in these cases are loaded with oily molecules; they easily break up, and give exit to it very soon after death. No such oily accumulation occurs, at least necessarily, in the parenchymal cells; a further confirmation of the opinion above stated, that these cells are not, *quoad* position, a continuation of the epithelium of the ducts.

In some Human livers I have recently observed the ultimate ducts in a very satisfactory manner, and believe, on the whole, that the mode I am about to mention is the best for obtaining a view of their course and relations. A very thin section of the liver must be made at one or two lines depth from the surface; this is to be treated with acetic acid, and moderately compressed; if the liver be tolerably free from oily matter it will now become much more transparent, and on careful examination of the portal canals, spaces and fissures, some ducts will be found, and their course may be traced. It is manifest that in this way of proceeding the ducts are examined *in situ*; and if the observer have previously acquainted himself with their appearance by dissecting them out as they lie in the Glissonian sheaths, he will have no difficulty in recognizing them and following them, though he may have much difficulty in convincing himself of their actual mode of termination. Their termination by distinctly closed extremities I have certainly observed, as well as the slight amount of ramification they present, and the little intimate relation they appear to hold to the walls of parenchyma between which they run. In many fissures not a trace of them can be seen; and in those where they exist it is manifestly impossible that they can come into relation with any more than a very small portion of the parenchyma. How then, if all the cells secrete bile, could this make its way into the ducts? The ducts as they run in the fissures and canals are in the closest relation to the portal vein branch, and surrounded by the terminal plexus of the hepatic artery; so delicate, so truly homogeneous is the tunic of the vein, that it seems to me scarce possible but that the ducts in the smallest portal canal and fissures should be bathed in plasma exuding from it; from the plexus of slender capillaries, in which the hepatic arterioles terminate, they must also receive part of their supply; and lastly, when the secretion of the marginal cells of the lobules is abundant, or when the liver, as in fish, is a mass of oil, they must be bathed in this fluid.

In considering what is probably the function of the parenchyma, *i. e.* of the chief mass of the liver, we may with advantage enumerate *seriatim*, the chief points which may be regarded as pretty well ascertained respecting it. (1.) It consists, at least in Mammalia, of cells, which as long as they retain their active power exhibit nuclei; these nuclei are imbedded in a soft homogeneous granulous substance which very commonly contains oil-drops, and sometimes biliary molecules, or is tinted with bilious fluid; an envelope much more distinct than in most other glandular cells surrounds the whole particle, and seems to denote a certain amount of permanency in its structure. (2.) These cells are very often arranged in tolerably perfect rows radiating from the centres of the lobules, but this disposition is often exchanged for a plexiform or quite irregular one. (3.) The marginal cells of the lobules, *i. e.* the ends of the rows, are especially prone to be the seat of oily accumulation. (4.) There is no arrangement of the cells comparable to those of the renal tubuli; they form rows, but not in any way the parietes of a tubular passage. (5.) There is no basement membrane to be seen in the lobules, the cells therefore lie in the closest relation to the blood in the capillaries. (6.) Young cells are frequently seen forming in the parenchyma, by collection of granulous matter round free nuclei; these do not seem to form at any particular part more than at another; they are not found solely in the centres of the lobules, *i. e.* at the commencements of the series. (7.) The cells* have in many animals a great affinity for oil, and appropriate it so as to gorge themselves when it exists in any quantity in the blood. (8.) Sugar must doubtless also exist in the cells, as they constitute a far greater part of the parenchyma than the diffused granulous plasma; according to some observations I have made its amount is in inverse ratio to that of the oil present, for in fatty livers of fœtal animals, fish, and diseased persons, I have failed to detect by TROMMER's test the presence of sugar, which in healthy ones is always abundantly manifest. (9.) Bile, or at least the colouring principle of bile, often occurs in the cells, especially in various diseased conditions, but is not proved to exist in perfectly normal states. It thus appears that the hepatic cells differ in several particulars from the cells of other glands; they are more perfectly formed, of more permanent aspect; they are not disposed as a lining to tubes of homogeneous membrane, but in series which tend more or less to plexiformity, and are apparently distant from any free surface open to the exterior. I consider them, therefore, to form a *parenchyma* and not an *epithelium*. Their peculiarly intimate relation to the blood capillaries seems to indicate, as I suggested in my former paper, that they serve as repositories for certain matters absorbed by the blood from the chyme as it passes over the intestinal surface; these matters are however, doubtless, altered by the recipient cells and converted into sugar, perhaps also into oil or biliary matter.

It is clearly proved that sugar is *made* in the liver, that it is not found in the blood entering the liver by the vena porta, but that it exists in very large quantity in the blood passing out by the hepatic vein; the substance also of the liver, the parenchyma, contains abundance of sugar, as I have repeatedly observed. These facts show that one, and probably the chief function, of the hepatic cells is to elaborate sugar from the materials intended to be employed in respiration; and that having done so, they allow this product to return into the circulating fluid, where it perhaps undergoes further changes before it terminates in carbonic acid. The parenchyma of the liver thus resembles closely a ductless gland, such as the suprarenal capsule, allowing its

* To this the liver of a fatted pig presents a remarkable exception; its cells seem inapt to receive the oil which so accumulates in the subcutaneous tissue.

product, like it, to return to the blood from which it had been elaborated: it may also be inferred, as not improbable, that as the elaboration of sugar is certainly one purpose fulfilled by it, so it is more likely that the bile-secreting function properly belongs to another apparatus associated with it, that of the excretory biliary ducts.

Some evidence as to this question which I have sought to obtain from chemical investigation, on the whole corroborates, I think, the view above expressed. I made an alcoholic extract of portions of the liver of several animals, evaporated this on the water-bath to dryness, and employed PETTENKOFER's test for bile. I obtained no decisive characteristic reaction, such as is described to ensue if bile or its constituents are present; a very feeble crimson tint was developed in most cases at the moment of adding the sulphuric acid, but it very quickly disappeared, and was so faint during the brief time it lasted that I cannot think it proves the presence of biliary matter; moreover, as it is quite impossible to obtain the parenchyma quite free from the minute ducts, some of them may have imparted a slight admixture of bile which was not derived from the parenchyma; or, which is perhaps unavoidable, some oleic acid may have been dissolved by the menstruum, and given, as it is said to do, a similar reaction with PETTENKOFER's test as bile itself. Before I was aware that albuminous matter produced with sugar and sulphuric acid the same reaction as bile, I conceived that I had found certain proof of the presence of biliary matter in the cells by applying this test to a thin slice of liver and watching the changes under the microscope; under the action of the acid the tissue became much more translucent, and developed along its margin a beautiful permanent crimson tint. The very same however, even more intense, was produced by treating a section of kidney in a similar way, so that it was manifest the reaction depended on nothing special to the liver.

One other fact we may receive from chemistry, which is also corroborative of the view that the sugar and bile are not produced in the same parts, viz. that the bile does not contain the hepatic sugar. M. BERNARD distinctly states this, and on repeating the experiment I obtained the same result. Sugar therefore seems to be the normal product of the cells, bile of the ultimate biliary ducts.

Mr. NOAD has been kind enough to execute some analyses for me, which go to show that the kind of food influences decidedly the amount of sugar contained or formed in the liver. The subject of the first was a young dog, who was fed for six days on bread and meat; the quantity of sugar contained in his liver, as determined by fermentation, was 20·13 grs. per 1000. The second, a young kitten, was fed for six days on food as far as possible of the saccharine kind, rice, sugar, arrow-root and potatoes; this disordered her bowels after some days, but by leaving off the rice and confining her to potatoes chiefly she recovered her health. Her liver contained 31·66 grs. of sugar per 1000; it also was seen under the microscope to contain much oil. Another kitten of the same age was fed for about a week on bread and butter, and milk, so as to approach nearly to a diet of oily food; her liver was quite opake from the presence of abundance of oil, much more manifestly being present than in the two others; it yielded by fermentation 21·13 grs. of sugar per 1000.

The results stand thus:—

Albuminous food per 1000 grs. of liver	20·13 grs. of sugar.
Saccharine food per 1000 grs. of liver	31·66 grs. of sugar.
Oily food per 1000 grs. of liver.	21·13 grs. of sugar.

Hence the saccharine diet produced fully one-third more sugar than the albuminous or oily. M. BERNARD however has distinctly proved that the liver forms sugar from the blood when only albuminous food is taken, so that it seems certain that a converting change must go on in the organ, which is exerted not only upon saccharine, but also upon albuminous and probably upon oily matters. One of the results of this converting change may sometimes be biliary matter, which would then appear in the cells, as we have seen that it occasionally does; but in the majority of cases observation seems to testify that sugar and oil are the products or results of the energy of the hepatic cells.

I would for a moment refer again to a point noticed in my previous paper, viz. *that at the moment when the liver traced in the ascending animal series assumes a solid parenchymal form, it receives a portal vein which is distributed exclusively to the parenchyma*: this at once seems to imply a special relation between the two, that the one is developed for the sake of the other, and that their functions are coordinate. A portal vein is not needed for the secretion of bile; this product is most copiously formed in the follicular livers of Mollusca, which are supplied by an hepatic artery only; and it therefore is extremely probable that the parenchyma is added for some further and different purpose; this purpose being, as I have suggested, the temporary abstraction from the blood of such matters as being absorbed in considerable quantity from the intestinal surface could not, without disturbing too far the due crasis of the blood, be immediately poured into it. It is impossible to conceive a structure better fitted to fulfil this purpose than the lobules of the liver, which are percolated from their circumference to their centre by capillary blood streams.

The function of the parenchyma I believe, then, to consist chiefly in exerting a modifying action on the portal blood and preparing a product which enters the circulation, and is probably consumed in the service of respiration. The presence of much oil in the cells seems chiefly to depend on the existence of an unusual amount in the blood, which may be occasioned by various causes; thus the fatty liver of fish and of the fœtus seems connected with a low degree of respiration; that of phthisis partly with increased absorption of fat, partly with diminished respiration; that of sheep fed on oil-cake with the oily nature of the food. To three of these cases certainly, if not to all, the term fatty degeneration is quite inapplicable; the condition is that simply of oily accumulation.

The function of the excretory ducts, if my description of their relations is correct, is not merely that of conveying forth out of the organ a product already formed; they must act upon certain matters supplied to them with an energy of their own, and out of these elaborate their secretion themselves. It may be said generally that the ducts which present the structural characters that appear indicative of an active function, *i. e.* those which approach their termination, are in contact with parenchymal cells and celloid particles containing varying quantities of sugar, albumen and oil; they are also surrounded by the rather scanty plexus of the terminal twigs of the hepatic artery, and moreover they run in such close relation with the delicate walled portal vein branches that they may possibly receive plasma exuded from them. In fishes' livers, where the quantity of fibrous tissue is small, it is very remarkable to observe how the ducts are encrusted with the oily parenchyma, as it were bathed in its fluid; in this instance it is impossible to avoid the idea that this abundant oil serves to the ducts as a plasma, *out of which*, by means of their epithelium, they elaborate bile. The same must hold in cases of fatty liver in higher animals; and generally it seems probable that the parenchyma furnishes a suitable fluid (which may often be in part already of biliary nature) which the ultimate ducts employ in their eliminative action.

Now have we any evidence tending to show that an active change is taking place in the ultimate ducts,—that they are more than mere efferent channels? Such I think has been already detailed, and I will refer to it here again.

The presence of abundant granulous matter in the ultimate ducts of fishes' livers speaks in favour of an active change proceeding in these parts, as also do the pellucid vesicles, which often crowd together so closely in them; such surely would not be found in a mere efferent canal. Compare the condition of the epithelium of the cortical renal tubuli of those in the medullary cones as an illustrative instance; in the first, the nuclei, surrounded with abundant granulous matter, constitute celloid particles and sometimes cells; in the other, the nuclei lie applied against the limitary wall almost naked and alone. Manifestly the one is a secreting part, the other a mere channel of exit. In Reptiles and Birds the case is the same; the ultimate ducts are not lined by small epithelial particles, but well nigh filled with vesicles or granulous matter, which must be supposed to be continually breaking up into the fluid secretion and some complementary matter. In Mammalia we meet ordinarily with a seemingly different condition of the ultimate ducts, which, though rare, also occurs in Fishes and Reptiles. The nuclei are not buried in granulous matter, but are set as it were in a thin layer of it, and leave a distinct central channel between the opposite walls; yet they are not formed into distinct particles; there are no columnar cells like those of the larger ducts; the nuclei abide in all their separate entireness. Have we knowledge of any secretory structure similar to this, with which we may compare it, and which may aid us to comprehend it better? Such a one is found in the cavities of the thyroid, where the epithelial layer, precisely similar in constitution to the walls of the ultimate biliary ducts, rests externally upon the limitary membrane, and internally is in contact with the secretion it has elaborated. Can we interpret the import of this arrangement? Can we perceive its peculiar significance? It seems scarcely doubtful that the more perfect a cell is the greater is its permanence; a celloid particle may rapidly decay, but a cell with a well-marked envelope is destined to a longer life; its contents are as it were locked up. In such cells too, when the secretory act is ended, the nucleus disappears, and the cell is henceforth only a repository of the product. This type of secretory action is, compared to the other, gradual and slow. But if a nucleus remain as such, if it exert an unceasing energy, or at least retain the power to exert that energy at intervals, if instead of developing a single cell, filling it with secretion, and then perishing, it continue to attract, to change and to repel charged matter, then we have a structure which is adapted for continuous, and, it may be, rapid action. This we have much reason to believe is the case in the instances of the thyroid and of the biliary ducts.

I might also refer to the case of the intestinal villi during absorption, which is a rapid and temporary process; the nuclei seem to attract the chyle into the villus; they do not develop cells wherein to enclose it, but allow it straightway to flow off by the lacteals, while themselves unchanged continue to attract fresh matter.

I believe then the two conditions of the ultimate ducts to be essentially similar, differing only in the more or less quantity of granulous matter, by whose disintegration in each case the material of the secretion is furnished; the presence of the vesicles in the ducts so often mentioned, seems to me strongly indicative of active growth, though less rapid, and by consequence of change and decay; and thus corroborates decidedly the view I have expressed.

Once I have seen the nuclei of the ultimate ducts deeply bile-tinged; this was in the liver of a Squirrel (*Sciurus*), where many of the parenchymal cells also contained bile; this also seems to indicate an attractive power resident in them; it was remarkable that some ducts were not thus coloured, and in these the nuclei were less elongated, were plumper, more healthy and vigorous-looking, to use a metaphorical expression. The gall-bladder was nearly empty, and the condition, I suppose, was one of sluggish action of the ducts; hence impletion of some of the parenchymal cells with bile, and staining of the nuclei of the ducts with secretion not thrown off. Those unstained were doubtless efficient and active.

As it appeared to be matter of great importance to ascertain whether the parenchyma of the liver really contained bile, that is, not only the colouring matter of the secretion, but its organic salts also, I applied to my friend Mr. Lindsey Blyth, who most kindly undertook at my suggestion a series of experiments to endeavour to decide the question. I may remark, that in the case of the liver there is far more reason to expect that its secretion should be capable of being detected in its substance, than there is in that of the kidney; urea cannot be found in the renal parenchyma, probably because it is so rapidly and completely carried out of the tubes of which it consists, and which present only an extremely thin layer of epithelium to be traversed in its transit from the blood into their cavity. But in the liver, the bile, if it exist in the cells, has no ready means of escape; it must traverse a long series before it can be received into the ducts, and those who affirm that it exists in the cells, point to the yellow matter contained visibly in their interior as proof of its presence. It is to be supposed therefore that if this yellow matter be bile it will give the characteristic reaction of this fluid.

Experiments.

(1.) Healthy Sheep's liver weighing 2 lbs. Separated the mass of the parenchyma as much as possible from the larger vessels, made a watery extract from which the albumen was separated. TROMMER's test showed the presence of sugar in abundance; PETTENKOFER's gave no indications of bile; evaporatedt he extract to perfect dryness, and then treated it with cold absolute alcohol, and then with boiling spirit, neither of which gave any indications of bile. The solutions tested were very concentrated; the whole extract obtained from the mass of 2 lbs. weight was contained in about 1 oz. of fluid.

(2.) A portion of a Pig's liver, treated in a similar manner. Abundance of sugar was obtained, no trace of bile.

(3.) A healthy Human liver, weight 2 lbs.; the whole parenchyma was treated in the same way as that in experiment (1). The watery extract gave abundance of sugar, no bile. The cold alcoholic solution precipitated by ether gave no indication of the presence of a salt of cholic acid.

(4.) A small piece of Human liver. The cells engorged with yellow matter; first treated with alcohol, which on testing gave no trace of bile, but took up a beautiful deep yellow pigment and oily matter. The watery extract gave no trace either of bile or sugar, but the addition of sulphate of copper and liquor potassæ produced a beautiful azure solution, from which nothing was precipitable by heat.

The addition of a very minute portion of bile to the alcoholic solutions caused the characteristic reaction to take place with PETTENKOFER's test.

(5.) A Human liver; the subject a strong healthy man, who died from a fracture of the thigh; it presented a healthy appearance. No trace of sugar or bile was found in it. The patient had partaken of no food for two days previous to death, excepting a small quantity of beef-tea from time to time.

(6.) A portion of Human liver, of decided yellow tint, the cells generally gorged with oil, some containing yellow matter. The man died from the shock of an accident, which caused fracture of the pelvis. The ethereal extract gave a liquid oily product, on which nitric acid produced a play of colours, but no trace of cholic acid was found by PETTENKOFER's test. The alcoholic extract gave a solid fat, but not of so deep a colour as the ethereal, and a very slight reaction with nitric acid. No trace of cholic acid was found in it. The watery extract gave no trace of sugar or bile. Potash and copper being added gave a dirty brown solution, from which nothing was thrown down by heat or was deposited after standing.

(7.) A piece of Human liver of healthy appearance, but the cells around the intralobular veins contain a great deal of deep reddish yellow matter. Four extracts were made. 1st. Chloroform gave a beautiful crystalline residue of solid fat, partly coloured with yellow spots. PETTENKOFER's test gave no trace of cholic acid; nitric acid gave a very slight purple tinge, which changed into green. 2nd. Ether gave an abundant deposit of oils and fats which produced a slight reaction with nitric acid, but no trace of cholic acid with sugar and sulphuric acid. 3rd. Alcohol gave a coloured extract containing fatty matters, more solid than those from ether, but showing on being tested no trace of cholic acid. 4th. Water; the extract gave no trace of cholic acid, but on the addition of liquor potassæ and sulphate of copper, a portion of the copper was reduced by boiling to the red oxide, and the supernatant liquor remained of a beautiful azure tint, in which no change was effected by heat on long standing. This azure colour, as in (4), probably indicates the presence of glycocol, which is united with cholic to form the glycocholic acid of the bile. The fatty matter saponified by potash and extracted with alcohol gave no trace of cholic acid.

(8.) A cat's liver; the animal had been treated with calomel, and had taken twelve grains in twenty hours. The aqueous extract gave some trace of sugar and of fat, but none of cholic acid.

(9.) Liver of a man who died with suppuration in all the divisions of the portal vein, the smaller branches being blocked up. The interior cells of the lobules were filled with yellow matter. The aqueous solution of the parenchyma gave no indication of cholic acid or sugar, but with TROMMER's test a solution of fine blue colour was obtained, from which nothing fell down on boiling or after long standing. This seems to indicate the presence of glycocol. The gall-bladder was inflamed; its contents gave no reaction of cholic acid or sugar.

(10.) Liver of a female extremely fat; died with tubercular cavities in lungs. The presence of sugar and probably of glycocol was ascertained by TROMMER's test. The contents of the gall-bladder were thick and of a dark saffron colour; contained very little cholic acid, no sugar, a great deal of colouring matter.

(11.) Liver of a patient who died from inanition ten days after having cut his throat; it gave no trace of sugar nor of glycocol.

(12.) Liver of a distinct yellow colour, extremely fatty, the cells gorged with oil, and containing much yellow matter also. The aqueous solution gave a deposit of suboxide of copper with TROMMER's test, and the supernatant liquor remained of a purple colour, thus indicating the presence of sugar and glycine. The alcoholic solution gave a beautiful reaction with nitric acid (first green and then purple), characteristic of the colouring matter of the bile, but no trace of cholic acid; it contained also a small quantity of fatty matter. The female from whom this liver was taken, was jaundiced at the time of her death.

The conclusions which may be drawn from the foregoing observations are—

(1.) That sugar, bile-pigment, and probably glycocol, can be detected in the hepatic cells. (2.) That cholic acid cannot be detected in them, and that therefore it is not necessarily united with the bile-pigment. (3.) That after abstinence from food sugar disappears from the liver. If these conclusions be admitted, it is clear that the hepatic cells do not form perfect bile; they form sugar, and two of the biliary principles, but not the organic acid, which is probably the most important.

Explanation of the Plate.

PLATE I.

Fig. 1. Ducts containing vesicles imbedded in pellucid and amorphous matter; they are such as I believe to have an *active* function. Diameter $=\frac{1}{1000}$th of an inch.

Fig. 2. Gall-bladder, cystic, hepatic and common ducts in a very young fish, together with the intestine: the hepatic ducts appear as rudimentary developments from the cystic.

Fig. 3. (*a*) An ultimate duct, with closed extremity, consisting of nuclei set in a sub-granular basis substance from liver of Frog.

(*b*) (*b*) Two ducts containing the vesicular epithelium from a different part of same liver. The diameter of these two was, that of the larger $\frac{1}{714}$th of an inch, that of the smaller $\frac{1}{833}$rd of an inch. The duct (*a*) had a diameter of $\frac{1}{1250}$th of an inch.

(*c*) Duct from a young rabbit, dosed with blue pill; it contains very numerous delicate vesicles with interposed oily molecules. Its diameter $=\frac{1}{1000}$th of an inch, that of the vesicles $=\frac{1}{2500}$th of an inch.

Fig. 4. Alimentary canal and liver of Tadpole, with gall-bladder and rudiment of duct. *s*, indicates the stomach; *t*, the throat; *i*, the commencement of the intestine; *l l*, the two lobes of the liver; *g*, the gall-bladder, lined by a vesicular epithelium; *d*, the rudiment of the cystic duct. The liver consists of mere formative yolk matter; the duct seemed to consist of similar substance but more opake; it was not hollow; its margin was very defined. The gall-bladder measured $\frac{1}{63}$rd of an inch in diameter; its cavity was filled by a clear fluid.

Fig. 5. (A.) A terminal duct from the liver of a Guinea Pig; it had a most distinct closed extremity; its diameter$=\frac{1}{1000}$th of an inch, that of its canal $=\frac{1}{4000}$th of an inch. It was seen lying along the wall of a tolerably sized portal canal. Some of the parenchymal cells are represented.

(B.) Ducts from Squirrel's liver; one is seen lying on the homogeneous tunic of a blood-vessel (a portal vein branch); its extremity was even, and was probably terminal; its nuclei were bile-tinged. The other duct has very distinct nuclei, which appear plump, and are not bile-tinged; its amorphous substance also is paler. A parenchymal cell, about $\frac{1}{1000}$th of an inch in diameter, is figured. The two ducts had each a diameter of $\frac{1}{2500}$th of an inch.

Fig. 6. View of a portal canal and portions of the two parietal lobules, with an ultimate duct running in the space nearer one side than the other. The duct is single; no other existed in this part; it gave off no branches, and had no communication (visible at least) with the lobules. The diameter of the portal canal was $\frac{1}{333}$rd of an inch, that of the duct $\frac{1}{1100}$th of an inch. The specimen was taken from a dog, in whom the epithelium of the gall-bladder and the ducts was in the oily condition described in the text. The larger ducts had a distinct homogeneous tunic, the smaller no perceptible one, but consisted of groups of minute oily molecules, which formed at last mere streaks; a distinct closed rounded extremity was not observed.

Fig. 7. View of one of the glands of the large hepatic duct-trunks in the liver of a Dog; it belongs to the conglomerate type: its cavities were *filled* with an opake granulous matter containing numerous oil-drops, as well as many nuclei. Some of the contents are figured on the side.

II. *On the Morphology of the Cephalous Mollusca, as illustrated by the Anatomy of certain Heteropoda and Pteropoda collected during the Voyage of H.M.S. "Rattlesnake" in* 1846–50.

By THOMAS HENRY HUXLEY, *Esq., F.R.S., Assistant Surgeon R.N.*

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"THE mere description of appearances even of the interior structure, still less of the exterior surface of an animal, without the deductions which they legitimately yield, is of comparatively small value to the philosophic naturalist; for of what value are facts until they have been made subservient to establishing general conclusions and laws of correlation, by which the judgment may be safely guided in regard to future glimpses at new phenomena in nature*?"

If I prefix this admirable exposition of the true aims of anatomical investigation to the present essay, it is that I may justify, by the highest authority, the course which I have taken in considering what of new facts it contains, as of subordinate importance to the reasonings which may be based upon those facts; in making its scope more morphological than zoological.

The morphology of the Cephalous Mollusca is a subject which has been greatly neglected. No SAVIGNY has determined the homologies of their different organs, and so furnished the only scientific basis for anatomical and zoological nomenclature.

It is not settled whether the back of a cuttle-fish answers to the dorsal or to the ventral surface of a Gasteropod. It is not decided whether the arms and funnel of the one have or have not their homologues in the other. The dorsal integument of a *Doris* and the cloak of a Whelk are both called "mantle," without any evidence to show that they are really homologous.

Nor do very much more definite notions seem to have prevailed with regard to the archetypal molluscous form, and the mode in which (if such an archetype exist) it becomes modified in the different secondary types. So far as our knowledge goes among other forms of animal life, we invariably find that, whatever the subsequent variations and aberrations, the primordial embryonic form has its parts arranged symmetrically about a given axis.

No one imagines the Pleuronectidæ belong to an asymmetrical type because they are asymmetrical in their adult shape, and yet there is no stronger evidence for the very common assertion that the typical form of the Mollusca is spiral or asymmetrical†.

This unsatisfactory state of our knowledge appears to me to result from two causes;

* OWEN, Anatomy of *Spirula*, Voyage of Samarang, Zoology, p. 12.

† See Von BAER, Nova Acta Acad. Nat. Cur. vol. xviii. p. 753.

—first, from the want of a clear and definite conception of the fundamental varieties of molluscous structure, and of the nature of the changes in the relations of parts which constitute those varieties; and secondly, from the want of a due regard to the facts presented by the development of the different families, which must stand in the relation of cause to the varieties of form.

Now in order to the former end (the obtaining of a definite conception of the varieties of molluscous form), I propose to set forth the structure of certain Heteropoda and Pteropoda; pelagic animals so transparent, that a perfect knowledge of the arrangement of their parts may be arrived at by simple inspection, without so much as interrupting a beat of their heart.

Afterwards, I shall inquire how far the known laws of development account for these forms, and thence of what archetypal form they may be supposed to be modifications.

PART I.

I. *Anatomy of Firoloïdes* (Plate II. fig. 1).

The species of *Firola* which I examined appears to be identical with the *Firoloïdes Desmarestii* of MM. Eydoux and Souleyet*.

The animal may be described as a transparent cylinder about an inch long, and so generally colourless as to be hardly distinguishable in the water, except by the incessant flapping of its flattened ventral appendage (*pp*).

The only parts which present any colour are the buccal mass, which is brownish; the eyes, almost black, and the mass of the liver, which is brownish green; further, the anus has a pinkish tint.

Attached by a narrow root or pedicle to the ventral surface of the cylindrical body is the broad cheese-cutter-shaped foot, or as I propose to call it, *propodium* (*pp*). Its posterior edge is quite sharp, and carries no sucker-like expansion. The anterior fifth, or thereabouts, of the animal is thinner than the rest of the body, and it narrows again towards its extremity, which is truncated, and forms a circular lip round the aperture of the mouth. Just behind the narrowed fifth, and towards the dorsal surface, we observe the eyes, and immediately below, and as if proceeding from them, are the tentacles, which are short and conical.

The posterior extremity also is abruptly truncated; its uppermost angle slightly projects, and viewed from above appears like a subspiral, richly ciliated band (*d*). Some little distance from this is the aperture of the anus (*a*).

From the two inferior angles, two tubular processes pass in the male (fig. 1). The right process ends in a globular body and is the penis (*p*), while the left (which I shall call the *metapodium*) is long and somewhat pointed (*mt*).

In the female there is only one process (the metapodium) (fig. 2, 3 *mt*) answering

* Figured by them in their beautiful plates illustrative of the Zoology of the Voyage of the Bonite.

to the left of the male, but a long cylindrical egg-tube frequently trails from the aperture of the oviduct (fig. 2).

Alimentary System.—This consists of,—1, the buccal mass; 2, the œsophagus and stomach; 3, the intestine and its termination, the rectum; 4, the liver; and 5, the salivary glands, which are very small and placed above the buccal mass, contrasting singularly with the very large salivary glands of *Atlanta.*

The œsophagus, stomach, and intestine form a straight tube running through the axis of the animal, and suspended by a ligament to the dorsal parietes (fig. 1); having reached the "nucleus," that is, the mass of the liver and ovary, the intestine bends up at a right angle, and so becomes the rectum, which terminates, as has been seen, upon the dorsal surface.

The Buccal Mass, or Tongue (fig. 1 *b*).—This is an oval brownish body, placed below the commencement of the œsophagus, and forming the floor of the cavity of the mouth. The following parts may be distinguished in this mass:—

1. Two ovoid compressed masses of thick-walled clear cells, which somewhat resemble cartilage. I shall call these the "lingual cartilages."

2. These give attachment to muscular fibres by their outer surface, and are enveloped in them. One portion of these fibres is inserted anteriorly into the parietes of the body, and acts therefore as a protractor. The rest are inserted into the edges of a thin plate, the "tongue-plate," which is closely applied to the whole of the upper and a part of the anterior lower surface of the cartilages, being as it were bent over their anterior extremities. The applied surfaces of the plate and of the cartilages are perfectly smooth, so that the former can play readily over the latter, like a rope upon its pulley. The upper surface of the tongue-plate carries in the middle a single row of tridentate teeth; outside these is a row of conical spines and broad flat-edged plates, and most externally there are one or more rows of recurved hooks, which, when the organ is at rest, lie over and nearly meet one another in the middle line.

When this organ is in action, it is commonly more or less protruded from the mouth by the protractor muscles; the large lateral spines of the tongue are divaricated (giving rise to that resemblance to the oral armature of *Sagitta* which has been remarked), so as to get all the teeth to bear; and then by the alternate action of the upper and lower sets of muscles inserted into the tongue-plate, a chain saw-like movement is communicated thereto, in consequence of which the teeth act as a rasp or saw upon any body with which they are brought in contact.

The buccal cartilages take no part in the movement of the tongue-plate, but simply act as its pulley.

The *Œsophagus* widens so gradually into the stomach, that no distinct line of demarcation can be drawn between the two, but the latter narrows suddenly into the intestine at a short distance in front of the "nucleus."

The *Liver* (*l*) is composed of several foliaceous masses containing many oil-globules; it opens by a wide duct into the angle of junction of the intestine and rectum.

The Circulatory System (figs. 1, 2, 3, 6).—This consists of a very perfect heart with an aorta and its branches, but there is no trace of any venous system.

The heart lies anterior to and parallel with the rectum; its axis therefore is nearly at right angles with that of the body. It possesses two chambers, an auricle (*u*) and a ventricle (*v*). The auricle is large and somewhat elongated, extending above into the elevation which carries the subspiral ciliated band. Its wall is formed by branched and interlacing muscular fibres, which are attached partly to the parietes, partly to the walls of a contractile sac (*c*), to be mentioned presently.

The ventricle is almost globular, and has thicker walls, in which the separate muscular fibres are not distinguishable. The lips of the auriculo-ventricular aperture are prolonged on the ventricular side into two valvular folds. Below, the ventricle terminates in a wide aorta, which immediately gives off a large branch backwards to the hepatic and generative organs; then becoming much convoluted, it runs forwards along the intestine and stomach, passing between the latter and the pedal ganglia (fig. 6), and finally terminating, without much alteration in its diameter, in the buccal mass.

As it passes over the pedal ganglia it gives off a considerable branch, "the pedal artery," downwards to the foot; and this pedal artery, just before it enters the foot, gives off a long and delicate "metapodal" branch (*w'*), which passes backwards, parallel to the aorta, and finally terminates in the metapodium, figs. 2 and 3.

The mode of ending of the pedal artery is very remarkable, and physiologically speaking, almost unique (fig. 6). Having entered the foot, it ends suddenly, without narrowing, in a truncated open extremity (*y'*). In the living animal this open end possesses the power of contracting to a very great extent, so as almost to become closed; and its condition must necessarily exercise a very considerable influence upon the direction and rapidity of the animal's circulation.

*Firoloïdes** then affords the most complete ocular demonstration of the truth of M.-MILNE EDWARDS's views with regard to the nature of the circulation in the Mollusca, that can possibly be desired. The perfect transparency of the creature allows the corpuscles of its blood to be seen floating in the visceral cavity between the intestine and the parietes, and drifting more or less rapidly backwards to the heart. Having reached the wall of the auricle, they make their way through its meshes as they best may, sometimes getting entangled therein, if the force of the heart has become feeble. From the auricle they may be followed to the ventricle, and from the ventricle into the aorta, whence they pass, some forwards, to the buccal mass, in which the aorta ends, and through whose tissues it pours them; some downwards, to pour out of the widely open end of the pedal artery, flooding the tissues of the propodium; and a small proportion passes directly backwards to the visceral mass and to the metapodium.

* A similar condition of the circulatory system has been observed by NORDMANN, QUATREFAGES, VAN BENEDEN and ALLMANN, in various Nudibranchiata, though perhaps not quite so distinctly.

Respiratory System.—So delicate a creature would hardly seem to need any special system of this kind, and I found no trace of such organs in any, even the freshest and most uninjured specimens.

In the nearly allied species *Firola Keraudrenii*, however, the gills appear as a row of conical processes, extending along the posterior edge of the body behind the anus; and in other species such processes develope accessory folds, until in *Carinaria* we find fully-formed branchiæ (see EYDOUX and SOULEYET). The ciliated subspiral band (*d*), which will be found to have its homologue in *Atlanta*, is the only structure which appears to be capable of assisting the respiratory function; but its small size must render it of very little importance.

Mantle.—There is no distinct mantle in *Firoloïdes Desmarestii.*

Carinaroides (EYDOUX and SOULEYET) evidently forms the transition from the Heteropods without a mantle to those with one. It is in this genus placed at the lower posterior angle of the body, and carries a minute shell; the branchiæ are developed between it and the anus.

In *Carinaria* there is a proportionably small mantle and shell, but it occupies a position more resembling that of ordinary Mollusks; and in *Atlanta*, as will be seen, the relative proportions of the mantle and body are pretty nearly those found in ordinary Gasteropods.

Contractile Sac or Urinary Organ (*c*, figs. 2, 3).—Between the rectum and the heart, and therefore bathed by the returning venous blood, there lies an elongated, flattened, delicate and transparent sac, whose walls are usually very much wrinkled and sacculated.

This sac opens by a rounded aperture in its upper part upon the right side of the animal, and is of course continually filled by the surrounding water. As the sac is incessantly contracting, however, this water must be continually renewed, and hence the organ, simple as its structure appears and small as its size may be, is probably a very efficient depurating agent*.

Considerations to be stated hereafter, lead me to the belief that it is in fact the urinary organ, at once kidney and urinary bladder.

Reproductive System.—*Firoloïdes* is diœcious. The male may be distinguished at once from the female, by the peculiar penis (*p*) attached to the right posterior inferior angle of the body (fig. 4). It consists of two portions; the larger is cylindrical, but enlarged at its extremity into a globular head, from one side of which a small pointed process projects. The globular body contains many large cells as a sort of lining, and within these there is a cavity which communicates with the exterior through the pointed process. A vast number of small oval fatty-looking particles may be made by pressure to pass out from the cavity.

* In the 'Explication des Planches,' EYDOUX and SOULEYET call a similar organ in *Firola* "organe de la dépuration urinaire ?."

The smaller portion is like a trifid leaf; it is placed at the base of the former portion, and almost reminds one of the sculptor's vine-leaf*.

The *Testis* is not very easily to be distinguished from the liver, behind which it is placed, until the contained spermatozoa are recognized. It occupies about the posterior half of the nucleus. There is hardly any proper vas deferens, but the testis opens by a very short canal at the base of the penis.

The *Ovarium* (*o*) occupies a position similar to that of the testis (figs. 2 and 3). A wide oviduct arises from it about its centre, on the right side, and after making one bend passes downwards, and opens close to the base of the metapodium.

The ovarian ova were oval, about $\frac{1}{300}$dth of an inch in diameter, with a clear, delicate, germinal vesicle about $\frac{1}{600}$dth of an inch in diameter, and a pale, circular, clear, but thick-walled vesicular germinal spot about $\frac{1}{1000}$dth of an inch in diameter. In the oviduct the ova possessed either an entire granular yelk, in which I could not detect any germinal vesicle, or the yelk was broken up into two or more (but not many) spherical or subspherical masses, containing a clear vesicle, the embryo-cell.

In some individuals a long tube (figs. 2 and 5), half as long as the body or more, depended from the orifice of the oviduct. It was colourless and transparent, and appeared as if articulated from its membrane being thrown into regular annular folds.

This egg-tube contained a double series of ova, in which the yelks had undergone division into 8 to 15 masses of very variable size. The ova seemed to have become more divided the nearer they were to the distal extremity of the tube, but I could find none containing a distinct embryo.

The Nervous System (figs. 6 and 7).—Two three-lobed ganglia, closely applied together by their inner edges (fig. 7), are placed between the eyes. Each gives off several branches to the parietes and the following important trunks:—

1. A long branch forwards, which terminates in a small ganglion (*β*) placed in the angle of union of the œsophagus with the buccal mass, and joined to its fellow by a very short commissure beneath the œsophagus. From these "buccal ganglia" various small nerves are given off forwards to the buccal mass and parts about the mouth, and backwards, to the œsophagus.

2. A large optic nerve (*i*).

3. A nerve to the tentacle.

4. A small nerve from the under surface, which terminates in the auditory sac (*j*).

5. A large and long commissural branch which runs backwards and downwards past the stomach, to unite with the pedal ganglion of its side.

The *Pedal Ganglia* (*y*) are two large ovoid masses in contact by their inner edges.

* According to MILNE-EDWARDS (Sur divers Mollusques, Annales des Sciences, 1842), a perfectly similar penis is found in *Carinaria*. He states, however, that the vas deferens traverses one portion of this organ, which is certainly not the case in either *Firoloïdes* or *Atlanta*.

They give off—

1. Two branches downwards to the propodium.
2. Two branches upwards to the dorsal parietes.
3. Two large branches, which run at first separately below the stomach, and then unite to form a single trunk. This runs along the stomach and intestine, sometimes twisting round them and giving off branches to them and to the parietes; and on the intestine it separates again into two chords, which join two small ganglia (3) placed between the aorta and the intestine; one lies on the aorta, the other between this vessel and the intestine, and they are connected by a commissure. From the former of these ganglia, which is the smaller, two nerves pass upwards and join a flattish mass placed immediately beneath the 'subspiral ciliated band.' There was an obscure appearance of branches radiating from this mass, and it is probably ganglionic.

Organs of Sense. The Eyes (fig. 8).—The eyes are very perfectly organized; each eye is contained within a chamber, excavated in a papilla, whose convex wall forms a sort of supplementary cornea, answering to the cornea of Cephalopoda, or to the corresponding cutaneous cornea of the Gasteropoda.

Their eye-proper is suspended within this chamber by a number of irregular muscular bands, which stretch from it to the walls of the chamber, and perform the function of *oculi-motores*. The optic nerve penetrates the inner wall of the chamber and enters the eye from that side.

The eye-proper is elongated and somewhat hour-glass-shaped, being contracted just behind the crystalline lens. The constriction divides the eye into two portions, an internal and an external. The latter is almost spherical, and is formed by the true cornea, which is much thicker in the middle than elsewhere, so as to present a meniscus section. Behind, the cornea is continuous with the sclerotic coat, which is thick, and seems to be continuous with the neurilemma of the optic nerve.

The crystalline lens is spherical, and is separated by a very small interval from the cornea, so that there is hardly any anterior chamber of the eye. There is no iris, but the inner surface of the posterior chamber is coated by a layer of dark chocolate-coloured pigment*.

The Auditory Vesicles (*j*), fig. 7.—These lie behind and a little below the cephalic ganglia. Each is a spherical vesicle, about $\frac{1}{300}$dth of an inch in diameter. Its walls are irregularly thickened here and there, and it contains a spherical otolithe of about half its diameter. I was unable to perceive any motion in the otolithe. The auditory nerve is a delicate thread arising from the under surface of the supra-œsophageal

* I do not see that the eyes of Heteropoda are so "peculiarly formed" as KROHN has it (Ferner Beitrag zur Kenntniss des Schneckenauges. MÜLLER's Archiv, 1839). What I have described as the true cornea, is by KROHN considered as an anterior portion of the vitreous humour; such an arrangement would of course be very peculiar, but I think that my account is correct. The eye, it would appear, projects much less in *Carinaria* and *Pterotrachea*; and in the latter, according to KROHN, there is even a rudimentary eyelid.

KROHN does not say anything about the muscles of the eye in these two genera.

ganglion, just in front of the commissural cord. It appears to terminate suddenly on entering the vesicle.

This origin of the auditory nerves from the cephalic ganglia, *when the pedal ganglia are well-marked and placed below the œsophagus*, is a circumstance common to all the Heteropoda*, and, so far as I am aware, altogether peculiar to them among the Mollusca. The only writers who appear to have been struck by it are MM. Frey and Leuckart: they say that the auditory organs are united with the supra-œsophageal ganglia "only when the lower œsophageal ganglia are wanting (except in *Carinaria*, in which the great length of the lateral commissures of the œsophageal ring appears to have made such a position necessary)." (Beiträge, p. 55.) I must confess I do not see the force of this explanation; and the lower œsophageal ganglia are never "wanting," though they may be united with the upper ones.

II. *Anatomy of Atlanta.* (Plate III.)

This is a very small and very beautiful pelagic mollusk, with a shell not more than one-fourth of an inch in diameter. It appears to be identical with the *Atlanta Lesuerii* of Eydoux and Souleyet. Its structure resembles that of *Firoloïdes* in all its essential points, and the transition between the latter and *Atlanta* is complete through such forms as *Firola, Carinaroides* and *Carinaria.*

The shell is flattened and spiral, none of the whorls projecting beyond the plane of the outermost. The aperture is notched on its dorsal edge, and a deep thin crest surmounts the outer whorl, and is generally broken in several places. The surface of the shell is marked by delicate transverse striations.

The outer fourth of the outer whorl of the shell is occupied by the mantle, the rest of the spire containing the viscera.

When protruded, the body of the animal is as large as the shell and appears trifid; the large head forming the anterior division, the "fin" the middle, and the "tail," with its operculum, the posterior division.

The head is large and subcylindrical. Its anterior extremity is formed by a circular lip surrounding the mouth. The eyes are placed far back, and the longish conical tentacles proceed from the anterior part of their base.

The fin or "propodium" (*pp*) is flattened and fan-shaped; its edge is provided with many long and delicate hairs, and its surface is covered with little asperities. Just below its point of attachment, the posterior edge of the propodium carries a cup-shaped disc (*ms*), also fringed with long hairs. This is commonly called the "sucker," and has no representative in *Firoloïdes*. It may be called the *mesopodium.*

The "tail" or *metapodium*† (*mt*) is subcylindrical at its base, but becomes flattened

* Compare Milne-Edwards, Sur divers Mollusques, Annales des Sciences, 1842, and the figures of Eydoux and Souleyet, so often referred to.

† That this is the same organ as the metapodium of *Firoloïdes*, will be obvious upon comparing the different forms which it assumes in *Carinaria, Carinaroïdes* and *Firola.*

and acuminated inferiorly. The elongated lanceolate transparent operculum is fixed upon its posterior surface.

The animal moves by the vigorous flapping of its fin. When it withdraws within its shell the head is just retracted, then the fin is folded in, and finally, the tail with its operculum covers up the whole.

The male is distinguished from the female by the presence of a peculiar leaf-like penis (*p*), which is attached upon the right side of the body just above where it divides into the three portions of the foot, fig. 1.

The *Alimentary Canal* commences above an oval buccal mass, widens gradually into the stomach, then narrows again and opens into a quadrangular sac, which communicates with the liver. From the anterior and upper part of this sac the rectum is continued and runs forward to the upper and dorsal part of the branchial cavity, in which it terminates by a tubular anus.

The mechanism of the tongue exactly resembles that of *Firoloïdes.* Two long cylindrical salivary glands (*f*) open into the anterior part of the œsophagus. They are simple cæca lined by a thick epithelium.

The *Liver* (*l*, fig. 3) is a wide conical sac with sacculated and glandular walls; its communication with the quadrangular dilatation of the intestine is so wide that it may almost be considered as a diverticulum thereof, and it extends back as far into the spire of the shell as any of the viscera.

The rectum is of a pinkish colour, and is richly ciliated internally.

Circulatory System (figs. 2, 3, 4).—The heart resembles that of *Firoloïdes*, and consists of an auricle (*u*) and a ventricle (*v*). It lies parallel to the rectum, with the auricle forwards at the base of the mantle-cavity, and the animal is therefore prosobranchiate. The aorta proceeds from the apex of the ventricle, and immediately after its origin divides into two branches, one of which runs backwards to the visceral mass, while the other passes forwards close beneath the stomach, until it terminates in the buccal mass. After passing over the subœsophageal ganglia, it gives off a downward branch to the fin, but I did not observe the peculiar termination of this artery which obtains in *Firoloïdes*; this perhaps may be accounted for by the greater muscularity of the fin in *Atlanta,* rendering it less transparent. The venous blood has no distinct channel, but returns to the heart by the cavity of the body. The returning current of blood-corpuscles is very obvious; they seem to pass quite freely in all directions round the intestine, aorta and nervous centres, the general tendency being always backwards towards the heart.

Respiratory System.—Very distinct gills are figured by Eydoux and Souleyet in most Atlantæ, but their presence in this species was decidedly exceptional, the majority of specimens presenting no trace of them. Once I noticed a bundle of long branchial filaments depending from the wall of the mantle-chamber; and in another case rudimentary and undeveloped short processes of the same kind were to be seen; they contained canals, through which a small portion of the returning venous blood was diverted, fig. 4.

The *Mantle* is very well developed; a peculiar thickened and ciliated band crosses it transversely, and seems to be the homologue of the subspiral ciliated band in *Firoloïdes* (*d*).

Contractile Sac.—This resembles the corresponding organ in *Firoloïdes*; it lies between the rectum and the heart, and opens into the bottom of the chamber of the mantle by a well-defined oval aperture, figs. 3 and 4 *c*.

Generative Organs.—The ovary or testis is an elongated mass corresponding to the liver, and occupying the inner and right half of the visceral mass, fig. 3 *t*.

The *Ovary* consists entirely of a mass of ova in course of development, with their characteristic germinal vesicle and spot. The oviduct commences at its anterior larger extremity; it is very wide and passes forwards, forming many convolutions, and terminates in the mantle-cavity by the side of the anus.

The *Testis* contains a mass of small cells and spermatozoa in various stages of development. The vas deferens leads from its anterior extremity, and before terminating in the cavity of the mantle, dilates, occasionally forming thus a dark spherical vesicula seminalis, *s*, fig. 3.

There is a kind of penis attached to the right side of the neck of the animal just above the foot (*p*, fig. 1): it is composed of two portions arising from a common base. The anterior and inner is like a three-pointed leaf; a cæcal ciliated canal runs along its centre. The posterior portion is a tubular cylindrical process, with a kind of ciliated cup at its extremity (fig. 5): a conical body projects into this below. The tube is nearly filled with small oval granules. The resemblance between this and the penis, which has been described in *Firoloïdes*, cannot be misunderstood, and the position of the organ in *Atlanta* corresponds with that in *Firoloïdes*, if we consider the left filamentous process in the latter to be the metapodium. That it is so, is demonstrated not only by the distribution of the metapodal arteries, but by an examination of the intermediate genera above mentioned.

Nervous System (figs. 2 and 6).—This consists of two trilobed supraœsophageal ganglia (fig. 6), which correspond to those of *Firoloïdes*, and give off similar nerves; but in addition their posterior lobes give off each a long nerve*, which runs back upon the stomach, and below its posterior narrowing, between this and the aorta, joins with its fellow in a small ganglion (not figured by EYDOUX and SOULEYET); some branches pass from this to the viscera, and two or three run in the substance of the mantle to the "ciliated band," fig. 4.

The commissural cords which unite the supraœsophageal with the subœsophageal or pedal ganglia, are at first double (fig. 6), but afterwards unite into one. They are connected with the subœsophageal or pedal ganglia (*y*), two large oval masses from which several branches are given to the different parts of the foot; and each gives off one long cord which runs along the lower surface of the intestine, and probably joins the ganglion upon the aorta before mentioned.

* In *Carinaria* a similar commissural nerve, between the cephalic and the parieto-splanchnic system of ganglia, has been shown to exist by MILNE-EDWARDS (*loc. cit.*), but I could find none in *Firoloïdes*.

The *Eyes* (*i*, fig. 6) resemble those of *Firoloïdes*, except that their pigment is black.

The *Auditory Vesicles* are spherical and about $\frac{1}{380}$th of an inch in diameter. Each contains a single strongly refracting globular otolithe of about $\frac{1}{800}$dth of an inch in diameter. In some cases this had a slow movement of rotation upon its axis, fig. 6, *j*.

Now in regarding *Firoloïdes* and *Atlanta*, whose structure has just been described, as illustrations of a typical form, the following circumstances appear to me to be of importance:—

1. The intestine is bent dorsad, or towards the side on which the heart is placed. The visceral mass is situated below and behind the posterior portion of the alimentary canal; it may be called a post-abdomen.

2. *Atlanta* is prosobranchiate; *Firoloïdes* is neither opisthobranchiate nor prosobranchiate.

3. The foot consists of three parts, the propodium, mesopodium and metapodium, in *Atlanta*; but of these the mesopodium disappears in *Firolöides*, and the metapodium becomes very rudimentary.

4. The auditory organs appear to be connected with the cephalic ganglia.

5. The animals are unisexual.

III. *Anatomy of Pteropoda*. (Plate IV.)

The variation of form undergone by the members of this group is perhaps greater than that which takes place in any other, except it may be the Nudibranchiata, and hence their structure is particularly instructive.

Three very distinct modifications of the type present themselves at first sight. The first is characterized by the non-development of the mantle and the full development of the foot, ex. *Pneumodermon*, *Clio*, fig. 1.

The second, by the great development of the mantle, by its cavity opening upon the ventral surface, and by the minuteness or absence of the mesial portion of the foot, ex. *Cleodora*, fig. 4.

The third resembles the second, but the mantle-cavity is placed upon, or at any rate opens upon the dorsal surface, ex. *Spirialis*, *Limacina*.

1. It is very remarkable that Cuvier should not have recognized in the "espèce de menton," and the "deux petits lèvres*" of *Pneumodermon*, nor in the "deux tentacules triangulaires" of *Clio*†, the homologues of the foot of the Gasteropoda. In fact it was on the strength of their having no such appendage that he founded his new order of Pteropoda‡, and yet the resemblance of the inter-alar appendages in these two genera§ to the foot of Gasteropods is so striking as at once to point to their real nature.

* Mém. sur le Pneumoderme, p. 7. † Mém. sur le Clio, p. 6.

‡ Mém. sur le Clio, p. 9; sur l'Hyale et le Pneumoderme, p. 10.

§ This is fully recognised by Leuckart, "Ueber die Morphologie," &c. p. 149.

In truth, the foot, though very small in these genera, is exceedingly well-marked, and shows a clear division into mesopodium and metapodium. It may be a matter of doubt whether the propodium is developed or not, and this question can be settled by embryology alone; but for the present, I think, it may be fairly presumed that it is represented by the tentaculigerous hood of *Pneumodermon* and by the tentaculigerous lobes of *Clio*; following in this case a very common tendency (exemplified in all the Cephalopods and in many Gasteropods) to become developed over and in front of the mouth*.

As CUVIER demonstrated, there is no "mantle" in *Pneumodermon* and *Clio*†; the body of these mollusks answering precisely to that of a *Firoloïdes*. The relation of the gill-laminæ and of the small anomalous shell in *Pneumodermon* sufficiently corroborates CUVIER's view. Gills are never placed upon the outer surface of a mantle; and if anything answers to such an organ it must be the small space covered by the rudimentary shell, so that the relations of the parts are, in fact, similar in *Pneumodermon* to what we find in *Firola* and *Carinaroïdes*.

Finally, new parts, the "alæ," make their appearance in these genera and give its character to the order (figs. 1, 2, 3, 4, 7, *ep*).

Considering the position and relation of these organs as distinct developments from the upper part of the sides of the foot, and the fact that their nerves arise, like those of the foot, from the pedal ganglia, I propose to consider them as parts of the foot and to call them the "epipodia." It has been long since shown by VAN BENEDEN and others that they have nothing to do with the respiratory function.

In this subtype the intestine opens on the right ventral side of the neck, and dissection shows its first bend to be ventral, that is, towards the side of the pedal ganglia.

2. The genus *Psyche*‡ or *Euribia* offers a very interesting transition from the foregoing to the second subtype. This genus is commonly said to have a cartilaginous shell, but this so-called shell appears, upon careful examination, to be only the thickened integument of the body; it is not secreted by a true mantle, like that of *Cleodora*, &c. The notion of a shell has arisen seemingly from the fact that a sort of cleft exists anteriorly from which the locomotive organs of the animal can be protruded and into which they can be retracted, but at the margin of this cleft the softer

* The origin of the nerves of the acetabuliferous tentacula may probably throw some light upon this matter; I have not had the opportunity of dissecting sufficiently large specimens of either *Pneumodermon* or *Clio*, carefully, with regard to this especial point. From the figures of EYDOUX and SOULEYET, one would be led to believe that the nerves of the acetabuliferous tentacles arise from the cephalic ganglia, which would be a very great objection to the view advocated above, since all the other parts of the foot, the mesopodium, metapodium and epipodium are supplied by the pedal ganglia. (See EYDOUX and SOULEYET, plate 15. fig. 30, and plate 15 bis, fig. 8.)

† EYDOUX and SOULEYET, however, call it "manteau" in the description of their figures. LEUCKART also opposes CUVIER's view, but I think without reason. (*Op. cit.* p. 146 note.)

‡ My species appears to be the *Euribia Gaudichaudii* of EYDOUX and SOULEYET.

parts of the body are continuous with the harder, just as the body of a Polyzoon is connected with its cell.

In some individuals I have observed the posterior extremity of the body to be surrounded by two circlets of cilia*.

The head of the animal is provided with two very large tentacles†, which carry a large process upon the inner side of their base, and rudiments of eyes upon the outer. Between the tentacles on the ventral side are two projecting lips with the aperture of the mouth between them, fig. 3.

Behind the mouth there are two lobes, separated by a deep notch; these are the two portions of the mesopodium which had begun to be separated in *Clio*. Behind these, again, there is a single tongue-shaped lobe, the metapodium, which is continuous on each side with two elongated and expanded epipodia.

There are no gills, and the anus opens ventrally upon the left side.

From this genus, to that called *Criseis* by RANG, but which EYDOUX and SOULEYET unite with *Cleodora*, the transition is very easy, figs. 6, 7.

In these forms there is an elongated conical shell, narrow and straight, or wider and slightly curved at its extremity. The body puts one in mind of that of a Cephalopod, being enveloped in a wide mantle, which is united to the body on the dorsal side only (fig. 6). The wall of the mantle is very thick, so that it presents a wide aperture always open upon the ventral side. Its free edge is, as it were, cut down upon its dorsal side, so that ventrally it is considerably longer. On the right side this prolonged portion has a rectangular edge, but upon the left it forms a sort of ram's-horn process‡. The lower part of the inner surface of the mantle is richly ciliated, and is raised into a number of transverse ridges, which must probably be considered to be rudimentary gills.

The head and wings are united with the part of the body covered by the mantle, by a narrow neck; compared with *Euribia*, the change in form is such as would be produced by a lateral expansion of the foot. Behind the mouth is the wide metapodium, and on each side of it are the broad epipodia continuous with the metapodium. About midway between the mouth and their margin the epipodia carry a small triangular lobe (*ms*), which evidently represents one-half of the mesopodium; and nearly at the same level, on their anterior edges, they present two small curved and pointed processes, the representatives of the large tentacles of *Euribia*. Two minute papillæ, the rudiments of the eyes, are placed upon the dorsal surface just behind the anterior edge of the alæ (*i*).

* It is a very interesting fact that Professor MÜLLER has found the larvæ of *Pneumodermon* to be provided with similar ciliated bands, Ueber die Entwickelungs-formen, &c. Monatsbericht d. k. Akad. d. Wiss. zu Berlin, October 1852.

† These are called "branchies" by EYDOUX and SOULEYET (pl. 15.), but why I cannot divine, since these organs are certainly homologous with the tentacles of *Cleodora*.

‡ Is this to be compared with the small posterior curved process of the edge of the mantle in Gasteropteron?

The œsophagus takes a straight course backwards from the mouth, which contains a minute lingual prominence, and widens gradually into a pyriform muscular gizzard, which is provided with two strong curved and conical teeth. The intestine passes from the narrow pylorus, and preserving the same width throughout, bends downwards towards the ventral side, and ultimately terminates in the cavity of the mantle a little to the left of the mesial line, fig. 6.

Just behind the pylorus a very long straight cæcum is given off, and sometimes there is a short one in addition by the side of it. The parietes of these sacs are glandular.

A long "columellar" muscle attaches the animal near to the apex of its shell, and then passes down into the foot, where it spreads out.

The position of the heart varies remarkably in this genus, and this variation is still more remarkable, if, with Eydoux and Souleyet, we consider it to be one with *Cleodora*.

In *C. aciculata* (figs. 6, 7) the mantle-cavity extends considerably beyond the transversely-barred portion of the mantle, and the base of the auricle abuts upon its lower posterior portion. The apex of the heart points backwards and a little to the left side, and in M. Milne-Edwards's arrangement the animal would be prosobranchiate. In *C. virgulata* (E. and S.) the base of the auricle lies behind the right posterior portion of the mantle-cavity, and the apex of the ventricle points directly to the left; it is therefore neither prosobranchiate nor opisthobranchiate.

In *Cleodora curvata*, on the other hand, as will be seen immediately, the base of the auricle is posterior and the ventricle points forwards; it is opisthobranchiate, figs. 4 and 5.

In this one genus, then, we have every transition from the prosobranchiate to the opisthobranchiate type of organization.

The aorta is too delicate to be readily traced in *C. virgulata* and *C. aciculata*. It may however be seen passing through the nervous ring and bifurcating for the epipodia, fig. 7 *w*. There is no rudiment even of a venous system.

The *Cleodora curvata* (figs. 4 and 5) (one of the true *Cleodoræ* as formerly defined) forms a transition from the preceding species to *Hyalæa*. It has the general organization of the former, with the flattened shell more or less fissured laterally, and the filiform appendages to the mantle, of the latter.

The alary expansion forms a more rounded disc than in *C. aciculata* and *virgulata*, the metapodium having become widened out and almost undistinguishable from the epipodia. The triangular lobes, rudiments of the mesopodium, have disappeared.

The intestinal canal resembles that of the two former species; its flexure is ventral, and the anus opens into the cavity of the mantle on that side. The long cæcum has orange-coloured glandular parietes.

The position of the heart has been described. It is comparatively large, and the aorta may be readily traced from it, passing forwards over the stomach and through the nervous ring, and eventually dividing into two branches, one for each epipodium.

There is a more distinct rudiment of a venous system in this mollusk than in *Firoloïdes* or *Atlanta*. A wide canal traverses the mantle towards its upper part; it is crossed by various muscular bands. Another distinct canal can be traced from the auricle towards the right side, skirting the lower border of the branchial chamber. Whether it becomes continuons with the right extremity of the previous canal or not, I could not certainly determine. The blood flows from both ends of the first-mentioned canal towards the auricle, but on the left side there does not appear to be so distinct a venous canal as on the right.

In all species of *Cleodora* there is an elongated sac, in its structure, contractions, and position relatively to the heart, exactly resembling that of the Heteropoda. It communicates by a small aperture with the cavity of the mantle.

The *nervous system* in all these species consists of three ganglia on each side of the œsophagus. Four of these form a mass, placed entirely below the œsophagus, and the other two, placed in contact with and immediately above them at the side of the œsophagus, are united above by a broad flattened supraœsophageal commissure.

The upper (*cephalic*) ganglia give off two principal branches to the rudimentary eyes and tentacles. The anterior pair of the lower mass (the *pedal* ganglia) give off branches to the epipodia and expansion of the foot generally; posteriorly they carry two small auditory vesicles, with many otolithes.

The posterior (*parieto-splanchnic**) ganglia give off their principal branches to the mantle.

The aorta passes between the pedal and parieto-splanchnic ganglia.

3. I have mentioned that a third subtype appears to be formed by *Spirialis* and *Limacina*, which are the only Pteropods with spiral shells. *Spirialis*, again, is the only Pteropod with an operculum. But a more important difference for my present purpose consists in the fact, that in these genera the mantle-cavity (and with it the anus) opens on the dorsal side of the animal. I have not myself been fortunate enough to obtain specimens of these genera, and as the attention of those anatomists who have examined them does not appear to have been specially directed to this point, it is impossible to make out with certainty whether the first flexure of the intestine is also dorsal, or whether, as in all other Pteropods, it is ventral. I cannot think that any real variation will be found to occur among closely allied forms, in a matter so fundamentally connected with their whole structure and mode of development; and I would suggest that here also the bend of the intestine is truly ventral, but that by a continuation of the process by which the anus is thrown to the left side in *Cleodora* and to the right in *Pneumodermon*, it (with the branchial cavity) is thrown to the dorsal side in *Limacina* and *Spirialis*. Such a change would be completely paralleled by the arrangement of the parts in the Ascidians, where the first bend of the intestine is dorsal; but the cloaca, which corresponds to the mantle cavity, opens on

* See below, p. 54.

the ventral side, carrying the anus with it; and even in other Pteropoda we find changes in the arrangement of the mantle-chamber, which to a great extent modify, without however essentially altering, the normal arrangement, *e. g.* in *Hyalæa* and *Cymbulia*, where the posterior extremity of the mantle-chamber extends up to the dorsal surface.

Furthermore, the position of the heart, which remains on the ventral side in *Spirialis* (E. and S., plate 11. p. 13, &c.), greatly strengthens this view of the case.

Leaving this question in abeyance until further light is thrown upon it, we may, I think, enunciate the following propositions with regard to the Pteropoda corresponding to those in which the organization of the Heteropoda was summed up:—

1. The intestine is bent towards the ventral side; the visceral mass is placed above, and in front of the anus; it may be called an abdomen.

2. Some Pteropoda are prosobranchiate, others intermediate, others opisthobranchiate.

3. The foot consists of four parts:—three, the propodium, mesopodium, and metapodium, such as are found in the Heteropoda; and a fourth, the epipodium, not found in the Heteropoda. All of these parts (propodium?) may be distinguished in *Pneumodermon* and *Euribia*, while all but the epipodium and metapodium have disappeared in *Cleodora*.

4. The auditory organs are connected with the pedal ganglia.

5. The Pteropoda are hermaphrodite*.

PART II.

The Heteropoda and Pteropoda, whose anatomy I have just endeavoured in a very general way to sketch and illustrate, may be regarded, in some respects, as opposite poles of the development of the archetype of the Cephalous Mollusca. We have now to consider what that archetype is, and by what process it has become modified into the actual forms which have been described; and with the solution of these questions is connected the meaning and justification of certain new terms of which I have made use.

The most proper way of proceeding in this matter would of course be, to trace the *development* of the Heteropoda and Pteropoda. Unfortunately, however, I have had no opportunity of doing this myself; and so far as I am aware, there is no account of the embryogenesis of Mollusks belonging to either of these classes extant†.

* This, it will be observed, is here stated for the first time. In the Heteropoda the nature of the generative system has been a matter of controversy, and I therefore gave an account of it at length in *Firoloïdes* and *Atlanta*. The hermaphrodism of the Pteropoda, on the other hand, is well-known, and a description of their generative organs would only have led to details without any morphological bearing.

† Since the above paragraph was written, this hiatus has been filled, so far as the Pteropoda are concerned, by Vogt (Bilder aus dem Thierleben, p. 289) and by Johannes Müller (Ueber die Entwickelungs-formen einiger niedern Thiere. Monatsbericht d. k. Akad. zu Berlin, October 1852).

But in any natural group of animals the grand laws of development and growth are so uniform (the uniformity in fact constituting the true bond of union of its members), that this want may be supplied by the very full information we possess with regard to other Mollusca. If from these data certain general propositions can be established, it will, I think, be perfectly fair to make these propositions the basis whence deductively to explain and account for facts of organization whose absolute genesis is not known.

The development of the Cephalopoda, Pulmonata, Nudibranchiata, and Tectibranchiata, has been very carefully made out by KÖLLIKER, VAN BENEDEN and WINDISCHMANN, SCHMIDT, GEGENBAUR, SARS, NORDMANN, VOGT, REID, and others. From their observations the following generalizations may be very safely made.

1. The development of a Mollusk commences on the hæmal* side, and spreads round to the neural side, thus reversing the process in Articulata and Vertebrata.

2. In all Mollusks the axis of the body is at first straight, and its parts are arranged symmetrically with regard to a longitudinal vertical plane, just as in a vertebrate or an articulate embryo†. Plate V. fig. 1.

3. The subsequent bent, spiral, or otherwise unsymmetrical arrangement of the parts of the body in Mollusca, depends upon the development of one part at the expense of, or disproportionately to, another; and this asymmetrical over-development never affects the head or the foot of a Mollusk, but only a portion, or the whole of the hæmal surface. Plate V. figs. 2–8.

4. It is to this portion, and its often free projecting edges, that we can alone properly apply the term "*mantle*." When this outgrowth takes place before the anus, I propose to call it an *abdomen*; when it takes place behind the anus, a *post-abdomen*.

5. All embryological evidence goes to show that the Cephalopoda and Pulmonata develope an *abdomen*. The intestine becoming drawn into the abdominal sac becomes in consequence bent towards the neural side. Plate V. figs. 2–5.

6. On the other hand, all the evidence hitherto obtained with regard to the de-

* This very remarkable law has not, it appears to me, received its due importance at the hands of those distinguished anatomists, KÖLLIKER (for the Cephalopoda), VAN BENEDEN and WINDISCHMANN and GEGENBAUR (for the Pulmonata), VOGT (for the Nudibranchiata), and LEYDIG (for the Pectinibranchiata), from whose observations I deduce it. VOGT, however, observes that the order of appearance of organs in the Mollusca is the inverse of that in the Vertebrata; and with regard to the point from whence development commences, he says, "Ce point est facile de trouver, il est situé en arrière des roues a peu près sur la ligne de jonction entre la partie céphalique et la partie ventrale, et même un peu en arrière de cet dernière sur la partie abdominale même," p. 39.

I use the terms *hæmal* and *neural* here to avoid the ambiguity of dorsal and ventral, which have opposite meanings when applied to the Vertebrata and the Invertebrata. The hæmal side is that upon which the vascular centre is developed, it is the dorsal side of Articulata, the ventral of Vertebrata. The neural side is that upon which the nervous centres are developed; it is the dorsal side of Vertebrata, the ventral of Invertebrata.

† "Instead of the radial type of development we meet quite unmistakeably with a lateral symmetrical type; instead of the extended form of the body we find a short compressed body without repetition of segments or lateral appendages."—R. LEUCKART, Morphol. p. 125.

velopment of the Nudibranchiata, Tectibranchiata and Pectinibranchiata, tends to the conclusion, that in them the visceral mass is thrust out *behind* the anus; is in fact a *post-abdomen**. Plate V. figs. 6–8.

A little consideration will show that the intestine drawn into this must become bent towards the *hæmal side*, as in fact it is in the embryos of all three groups†.

Upon embryological grounds, then, we should establish two great primary modifications of the molluscous archetype; the one characterized by the development of an *abdomen*, and a consequent *neural* flexure of the intestine; the other marked by the development of a *post-abdomen*, and the consequent *hæmal* flexure of the intestine.

But these modifications of anatomical structure exactly correspond with those which I have already demonstrated, upon anatomical grounds, to occur in the Pteropoda and Heteropoda; and I trust I am not overstepping the bounds of legitimate analogy in assuming that the anatomical fact of a neural flexure indicates the embryological development of an abdomen; that of a hæmal flexure, the development of a post-abdomen; and that therefore the Pteropoda fall into the same category with the Cephalopoda and Pulmonata; the Heteropoda into that of the Pectinibranchiata, Tectibranchiata and Nudibranchiata.

It is remarkable, that, as regards the flexure of the intestine, similar contrasted modifications of the archetype take place in those animals which are the nearest allies of the Mollusca; I mean the Ascidians and Polyzoa, the Molluscoïdes of Milne-Edwards. In each of these groups the intestine is always bent upon itself; but while in the Ascidian the bend is always *hæmal*, in the Polyzoon it is *neural*. The latter fact is evident to any one who will examine a Polyzoon; the former may seem at first sight to be contradicted by the circumstance, that the ganglion in the Ascidians is placed between the cloacal and branchial apertures. However, as I have endeavoured to show elsewhere‡, whatever be the position of the anus in the Ascidians, the first bend of the intestine is always hæmal. I have already referred to their probable analogy with *Spirialis* in this respect.

Having now determined the changes which take place in the axis of symmetry of the Mollusca, let us examine into those undergone by their principal external organs.

Whether we have to do with a Cephalopod or with an ordinary Mollusk, the first step in development is the separation of the blastoderm into a central elevation, the

* See particularly Leydig, *Ueber Paludina vivipara*, Siebold and Kölliker's Zeitschrift, 1850, where the thrusting forwards of the anus by the development of the mantle is particularly shown, p. 142.

"A considerable change in the position of the anus takes place when the fold of the mantle becomes formed and moves forward, because thereby the intestine and anus are also thrust forward, and to the right side."

† The development of the Pectinibranchiata cannot be said to have been carefully worked out yet, with the exception of that of *Paludina*, but what has been done tends to the conclusions above stated.

‡ Upon the Anatomy and Physiology of *Salpa* and *Pyrosoma*, Philosophical Transactions, 1851, and in a "Report upon the Structure of the Ascidians," read before the British Association, September 1852. The law as regards the Polyzoa was first enunciated by Professor Allman, "On the Homology of the Organs of the Tunicata and Polyzoa," Trans. Royal Irish Acad. vol. xxii.

mantle, and certain lateral portions. Now these portions become in the Gasteropoda, the head and foot; in the Cephalopoda, the head and arms. It follows therefore that the arms of the Cephalopod are homologous with the foot of the Gasteropod.

Again, in the Cephalopod an eminence becomes developed along two lines, which run on each side of the upper part of the "lateral expansions" and meet behind the head; along the anterior portion of this line the eminence remains as a slight ridge, which afterwards becomes one of the muscles of the funnel; along the posterior portion of the line a considerable process is developed, and, uniting with its fellow, becomes the funnel.

In the Gasteropod, it is along the *anterior* halves of two corresponding lines that processes are developed, which become the ciliated alæ or vela of the embryo. The line in question I propose to call the *epipodial line*, and whatever is developed along that line I consider to be the epipodium, or a portion of it. I do not venture upon such a refinement at present, but I think it *probable*, that as we have distinguished three portions in the foot, so it will be necessary to distinguish three portions in the epipodium; anterior, middle and posterior. For instance, in the Cephalopoda the posterior portion only is developed as the funnel; in the Gasteropod larvæ the ciliated vela are the homologues of its anterior portion. The palmated lobes of the Turbinidæ, the "lobes of the mantle" of *Aplysia*, appear to be developed from the whole epipodial line, while it is apparently the middle epipodium alone which is developed into the "wings" of the Pteropoda.

All traces of the epipodium appear to have vanished in the majority of the Pectinibranchiata*.

Of all mollusks *Atlanta* possesses the best developed *foot*-proper, and has its parts

* Leuckart and Lovén have enunciated very different views with regard to the homologies of the external organs of the Mollusca, to which it seems proper I should refer.

Leuckart, for instance (*op. cit.* pp. 155–59), considers that the anterior cephalic lobes of the embryo Cephalopod answer to the cephalic velum of Gasteropoda; the posterior cephalic lobes to the alæ of Pteropoda, while the funnel corresponds with the middle lobe of the foot. The arms he considers to be peculiar structures, mere appendages to the cephalic lobes.

If the halves of the funnel, however, answer to the middle lobes of the foot, how is it that they unite upon the dorsal surface of the neck? If the anterior cephalic lobes answer to the vela of Gasteropoda, how is it that the latter disappear, and do not contribute to the formation of the head in Gasteropoda? Finally, it must be remembered that the arms of the Cephalapoda arise quite independently of the cephalic lobes, the first developed arms being those most distant from the head.

Leuckart considers that the oral lobes of the pulmonate embryo are the homologues of the ciliated vela of Gasteropoda. But their position and number are against this view. It seems to me that these oral lobes correspond with the cephalic lobes of the embryo Cephalopod, and it has been well shown by Gegenbaur (*op. cit.*) that the whole so-called "yelk-sac" of the Pulmonata is the true homologue of the vela in Pectinibranchiata; the "ciliated bands" of Van Beneden and Windischmann turn out to be Wolffian bodies, and to be *internal*, not external organs.

The common view, that the alæ of the Pteropoda are the persistent vela of the embryo, is, so far as I am aware, unsupported by any evidence. Embryology teaches us hitherto that the anterior part of the epipodium best specialized and separated. In the special description of *Atlanta* names have been given to these parts, whose appropriateness is I hope obvious.

From this highly developed condition of the foot, to its diminished state in *Glaucus* and its total absence in *Phyllirrhoë*, we have every gradation. The various portions are still to be distinguished in *Pteroceras* and the *Strombidæ*, but lose their distinctness for the most part in other Gasteropoda. However, the propodium is still marked off by a transverse fissure in *Oliva* and *Ancillaria*, and attains a great development in size; a peculiarity which is still more remarkable in *Natica* and *Sigaretus*. In both these genera it shows a tendency to invest the head. In the Cephalopoda, the anterior arms, which must be considered as the propodium, fairly unite in front of the mouth, and it seems very possible that the cephalic hood of *Gasteropteron*, the "oral" tentacles of *Aplysia*, the hood of *Tethys*, the "lips" of some Pteropoda, and the hood of *Pneumodermon* may be the result of a similar change. But all attempts to settle these points, save by development, must be more or less hypothetical*.

To this same test of development we must refer everything which claims to be called "*mantle*," a word which has perhaps been more vaguely and loosely used than any other term in zoology. Surely if a term is to have any value in either zoological or anatomical nomenclature it must be applied to only a defined thing. The "mantles" of a *Sepia*, a *Cleodora*, or a *Buccinum* may be homologous with one another, but they certainly are not so with what is called the "mantle" in *Doris* or any other Nudibranch†. The simple fact that the cephalic tentacles arise in the midst of the

is never permanently developed, and the position of the alæ would lead to the belief that they correspond to its lateral portions[1].

So far as I can judge from the Latin table affixed to his Swedish essay on the development of the Acephala, Lovén considers the arms of the Cephalopoda, the three pairs of cephalic tentacles of *Clio*, and the cephalic lobes of *Tethys*, to be homologous with the velum of the Gasteropod embryo, while the funnel of Cephalopoda and the alæ of Pteropoda are homologous with the foot of Gasteropoda.

The considerations above cited appear to me to furnish a sufficient refutation of these views, which seem to be the offspring of an idea first propounded by their learned author in his "Contributions to the Embryology of the Mollusks" (Oken's Isis, 1842), that the hood of *Tethys* and the cephalic processes of *Tergipes* are modifications of the cephalic vela of the embryo. This ingenious hypothesis has not however been confirmed by observation so far as regards *Tethys*, and has been directly *dis*proved with respect to *Tergipes* (see Schulze, Ueber die Entwickelung d. *Tergipes lacinulatus*, Wiegmann's Arch. 1849).

* In the absence of any knowledge of development perhaps the source of the nervous supply is one of the best tests of the real homology of a part. Mr. Hancock, in his valuable paper "Upon the Olfactory Apparatus in the *Bullidæ*" (Annals of Nat. Hist. March 1852), has, I observe, applied this test to the cephalic expansion of the *Bullidæ*, to the hood of *Gasteropteron*, &c.; and since it clearly appears that these parts are supplied by nerves from the *cephalic* ganglia, which never give branches to any portion of the foot, the suggestion in the text must be given up.

† Leuckart, believing the hæmal tegument of the Nudibranchiata to represent a mantle, suggests that there

[1] The researches of Vogt, already referred to, have fully confirmed this conclusion. The embryo Pteropod has vela, which eventually disappear, while the epipodia are developed quite distinctly from the upper part of the sides of the "foot."

so-called "mantle" of the latter, is sufficient evidence to show that it cannot be homologous with the "mantle" of the former. The so-called "margins of the cloak" in these genera appear to me to have much more relation to the epipodium.

Cuvier allows a mantle to *Doris*, but denies it to *Glaucus* and *Eolidia*; why, is not obvious.

Leuckart defines a mantle to be "a scutellate (schildförmig) duplicature of the outer integument extending from the neck for a varying distance backwards." By this definition however the upper surface of the anterior division of a *Bulla* would be a "mantle," which it is not, since the true mantle is obviously behind separated from this by a deep cleft, and how would the mantle of *Firola* or *Carinaroides* answer the definition?

If the definition which I have given of the true "mantle" be correct, we must, I think, hesitate for the present in conferring that name upon the dorsal shell-bearing integument of *Chiton*. May this not be homologous with the thick-edged dorsal surface of a *Doris*, in which the calcareous particles instead of being scattered are united into distinct plates*?

With regard to the shells, again, at the risk of being blamed for over-refinement, I would suggest, that it is, to say the least, an open question, whether the shell of *Buccinum* is (as it is commonly supposed to be) homologous with that of *Helix*; that of *Sepia* with that of *Nautilus* and *Ammonites*; that of the embryo *Aplysia* with that of the adult *Aplysia*. Grave differences of development occur in some if not in all of these cases†.

is a difference between their "gills" and those of other Mollusks, which, as he justly observes, are never processes of the mantle, *loc. cit.* p. 130. The argument in the text tends to show, that in this respect there is in reality no difference between the "gills" of the Nudibranchiata and those of other Mollusks. On other grounds however I am inclined to think that Leuckart's distinction is a just one. The organs called gills in the Nudibranchiata appear to me to be in all cases what they undoubtedly are in *Eolis*, viz. gastro-hepatic appendages. Even in *Doris*, where they are gill-like, they are supplied with hepatic blood only. See Hancock and Embleton's admirable memoir "On the Anatomy of *Doris*," Philosophical Transactions, 1852.

* In D'Orbigny's genus *Villiersia*, allied to *Doris*, the calcareous tegumentary particles of the *Doridæ* have united into a flat shell, hidden in the "mantle," which is pierced by the apertures for the tentacles and gills. The disposition of the calcareous particles in the *Doridæ* is very regular, though it seems too much to assume with Lovén that they imitate a subspiral shell (see Lovén, Oken's Isis, 1842).

† The memoir by Dr. Gegenbaur, "Beiträge zur Entwickelungsgeschichte der Land Gasteropoden," which has just appeared in Siebold and Kölliker's "Zeitschrift für Wissenschaftliche Zoologie," furnishes very powerful support to the doubts above suggested, since it demonstrates that the shell of *Clausilia*, and gives good reason for believing that that of *Helix*, are developed within the substance of the mantle, following exactly the type of *Limax*.

"The land Gasteropoda are distinguished by the peculiar mode of development of their shell, if we may draw conclusions for the whole group from *Helix* and *Clausilia*. The original deposition of the shell in the interior of the mantle (as in the *Loligidæ* among the Cephalopoda) is as yet an isolated fact among the land Gasteropoda, of which we find no indication in other Gasteropods."

There seems to be a very curious relation between the internal or external nature of a shell, the curvature of its whorls as regards a vertical plane, and the hæmal or dorsal flexure of the intestine.

Take, first, the case of a true external shell, as that of *Nautilus* or *Argonauta*, or *Atlanta*. Here the direction

From all that has been stated, I think that it is now possible to form a notion of the archetype of the Cephalous Mollusca, and I beg it to be understood that in using this term, I make no reference to any real or imaginary "ideas" upon which animal forms are modelled. All that I mean is the conception of a form embodying the most general propositions that can be affirmed respecting the Cephalous Mollusca, standing in the same relation to them as the diagram to a geometrical theorem, and like it at once imaginary and true.

The archetype of the Cephalous Mollusca, then, it may be said, has a bilaterally symmetrical head and body. The latter possesses on its neural surface a peculiar locomotive appendage, the foot; which consists of three portions from before backwards, viz. the propodium, the mesopodium and the metapodium, and bears upon its lateral surface a peculiar expansion, the epipodium (Plate V. fig. 1).

The hæmal surface of the archetype may or may not secrete a shell upon its surface, or in its interior.

If we compare this unmodified form with the vertebrate or articulate archetype, we find that the three essentially correspond. The appendicular system of the Vertebrata and Articulata is represented by the epipodium in the Cephalous Mollusca (Plate V. figs. 9, 10, 11).

Nevertheless the differences between the three archetypes are so sharp and marked, as to allow of no real transition between them.

In the Cephalous Mollusca *it is the hæmal side of the body which is first developed.* In the Articulata and Vertebrata it is the *neural side which first makes its appearance.*

The archetype of the Cephalous Mollusca further differs from that of the Vertebrata (and resembles that of the Articulata), in the circumstance, that while in the latter the nervous and intestinal axes are parallel, in the former they decussate; that is, the œsophagus opens on the neural side, passing between the great nervous commissures.

The molluscous archetype again differs from that of both Vertebrata and Articulata in its appendicular system (*ep*), which, when it exists, never presents articulations nor anything that can be called an external or internal skeleton (unless indeed the funnel-cartilages of Cephalopoda be such), and which is generally altogether suppressed in the adult state, its place being supplied by the foot, which, as a development of the central neural region into a locomotive organ, is, so far as I am aware, paralleled throughout the Vertebrata and Articulata by nothing but the dorsal fin of a fish.

In the actual forms, the symmetry of the archetype is almost always disturbed by

in which the shell is wound is the same as that in which the intestine is bent. While the aperture of the shell therefore is "hæmad" in *Atlanta* with regard to the axis of columella, it is "neurad" in *Nautilus* and *Argonauta*.

With an internal shell the reverse appears to be the law. Hence the curvature of the shell of *Spirula* is the opposite of that of the shell of *Nautilus*, and that of a pulmonated Gasteropod is the same as that of a Pectinibranch.

the excessive development of a peculiar region of the hæmal surface, into what has been termed the abdomen or post-abdomen, according as it is placed before or behind the anus.

The integument of this outgrowth, commonly modified in structure and having frequently a prolonged anterior or posterior margin, is the "mantle." It may or may not secrete a shell.

The development of an abdomen (Plate V. figs. 2, 3, 4, 5) produces a corresponding *neural* flexure of the intestine, as in Cephalopoda, Pteropoda and Pulmonata; that of a post-abdomen produces a *hæmal* flexure, as in Heteropoda, Pectinibranchiata, Tectibranchiata and Nudibranchiata (Plate V. figs. 6, 7, 8).

From combinations of these primary changes with subsequent greater or less developments of the various parts of the foot, all the varieties of form in the Cephalous Mollusca are produced*.

The formation of an abdomen with a peculiar development of the margins of the foot into elongated processes, and with cohesion of the posterior epipodial lobes, gives us the Cephalopodan subtype.

The formation of an abdomen with an excessive development of the epipodium, at the expense of the foot-proper, characterizes the Pteropoda.

The formation of an abdomen with a moderate development of the foot-proper, and hardly any of the epipodium, marks the Pulmonate subtype.

The Heteropoda combine a great development of the foot-proper with the formation of a post-abdomen (and only a transitory development of the epipodium ?). The Pectinibranchiata seem to differ from them only in degree.

The development of a post-abdomen, coexistent with that of the epipodium, characterizes the Tectibranchiata.

The Nudibranchiata have a post-abdomen and an epipodium in their embryonic condition, but lose both (epipodium ?) more or less completely as they attain maturity. The foot-proper is very moderately developed, or even disappears (*Phyllirrhoë*).

If the "mantle" is to have an analogue anywhere among the Articulata or Vertebrata, it may probably be with the carapace of the Crustacea, inasmuch as this is developed from a corresponding region and has similar functions, *i. e.* to protect the respiratory organs.

Hitherto what has been said has referred to the morphology of the external organs. It remains to show on what plan the internal organs are arranged, and how the archetypal arrangement is modified among the different families. To enter upon this subject fully would belong rather to a formal treatise upon the Mollusca. For the present my object is merely to point out the fundamental unity which obtains

* For clearness' sake I have referred to the "hæmal" and "neural" flexures as if they always took place in a vertical plane, whereas, as every one knows, the anal aperture is almost always either to the right or the left in the Gasteropoda. The only modification of the theory required to meet this fact, is to suppose that the hæmal outgrowth takes place more rapidly on one side than on the other.

among certain of the most important systems of organs, and to bring into prominence some facts in the anatomy of the Mollusca which have hitherto been unknown or neglected.

With these views I propose to treat—1, of the nervous system; 2, of the vascular system; 3, of certain portions of the alimentary system; and 4, of the renal system.

1. *Nervous System.*—The nervous system of every Mollusk consists of two great systems;—A. an excito-motor, or sensory and volitional system; B. a visceral or sympathetic system. The former consists of three pairs of primary ganglia, which always exist, and of a variable number of accessory local ganglia, which may or may not exist*.

* The first record I can find of the distinct enunciation of this very important anatomical fact, is in M. SOULEYET's essay on the Pteropoda (Observations Anatomiques, Physiologiques et Zoologiques sur les Mollusques Ptéropodes), of which an abstract is given in the Comptes Rendus for 1843: he says,—"The central nervous system of the Mollusca is essentially composed of the three orders of ganglia which I have just pointed out (orders answering exactly to those mentioned in the text), and it is in fact reduced to these ganglia in a certain number of animals of this type. But in others the nerves which are given off present numerous enlargements in their course, and this tendency to a ganglionic disposition is so decided among the highest Mollusks, that all the nerves emanating from the central medullary masses produce new ganglia in the parts to which they are distributed" (p. 667).

Again:—"From the facts which have just been stated summarily, I believe I may conclude,—

"1. That the exclusive analogy which many naturalists have wished to establish between the nervous system of the Mollusca, and one of the portions of the same system in the animals of higher classes, is not only contrary to physiological principles, but also to anatomical facts.

"2. That the nervous system of Mollusks corresponds, in fact, in its distribution to the same parts as those which constitute it in the superior animals, the whole difference consisting in the degree of development and disposition of the parts which is in relation with the rank that Mollusks occupy in the series, and the plan which nature has followed in their zoological type.

"3. That the definition very commonly given of this system in Mollusks, that it is *composed of ganglia scattered in different parts of the body*, is not exact, since the parts which by their fixity ought to be considered as those which essentially constitute it, are always grouped round the œsophagus. The others, in fact, are to be regarded only as different degrees of development of these central portions, which is proved by their degradation or disappearance in proportion as we descend in animals of this series.

"4. That the central nervous system of Mollusca is always double, and consequently symmetrical, in opposition to what some anatomists have advanced; that it hardly differs in this respect from the nervous system of the Articulata, except by the centralization of the locomotive ganglia, a centralization which may be observed in many animals of the latter type.

"5. Lastly, that it has been wrongly asserted as a general rule, that the ganglia of which the nervous circle of the Mollusca is composed, tend to approximate the higher the organization of the animal, the position of these ganglia being essentially subordinate to that of the organs which they have to innervate" (p. 669).

The very just and admirable views here set forth seem to have met with strange neglect; foreigners, however, might be pardoned for this, since M. SOULEYET's own countrymen contrive (see BLANCHARD, Sur l'organization des Opisthobranchies, Annales des Sc. Nat. 1848) five years afterwards not to know anything about them.

See also the memoir of HANCOCK and EMBLETON, before cited, in which the first complete demonstration of the true sympathetic system in the Gasteropoda is given; and ALDER and HANCOCK's British Nudibranchiata, in which are contained the most beautiful descriptions and figures of the anatomy of the Mollusca extant.

These three pairs of primary ganglia are the Cephalic, the Pedal, and the Parieto-splanchnic.

I. *The Cephalic Ganglia.*—These are always either in apposition, or are united by a commissure above the œsophagus. They give off either immediately or from the connecting commissure, the following nerves:—

1. Labial, to the lips and anterior parts of the head.
2. Olfactory, to the tentacles.
3. Optic.
4. Buccal, to the buccal mass, tongue, and jaws.

Accessory ganglia may be developed upon all these nerves. They are found upon the labial nerves in *Gasteropteron*, three upon each side (Souleyet and Blanchard); upon the olfactory nerves in many Nudibranchiata (Alder and Hancock, Souleyet, &c.); upon the optic nerves in Cephalopoda and Heteropoda.

The presence of ganglia upon the buccal nerves is almost constant. There seems to be only one inferior buccal ganglion in some Cephalopoda, while in others there are two, one above and one below. In the Heteropoda, Pteropoda, and most Gasteropoda, there is a pair of ganglia placed laterally at the re-entering angle of the œsophagus and buccal mass. In *Patella, Haliotis,* and *Fissurella,* I have found four, two in the latter position, and two anteriorly, just where the buccal nerves come off.

The buccal ganglia are always united by a commissure, so that when the cerebral ganglia are above the œsophagus, an anterior nervous ring is formed; when they are at the side or below, as in Pteropoda, there is no anterior nervous ring.

II. *The Pedal Ganglia.*—These are either in contact or are united by a commissure; *below* the œsophagus they give off—

1. Auditory nerves.
2. Pedal nerves.

The auditory nerves are not commonly present, as their organs are generally sessile; however, they exist in Cephalopoda, and in *Strombus* and *Pteroceras.* As has been stated above, the Heteropoda make an extraordinary exception to the usual position of the auditory organs, since in them these nerves appear to be given off from the cephalic ganglia. Considering, however, that the auditory nerves are invariably attached to the pedal ganglia in all other Mollusks, and that in *Pteroceras* and *Strombus,* genera which so nearly approach the Heteropoda, the auditory nerves are very long, I do not think it very bazardous to suppose, that in the Heteropoda the auditory nerves really proceed from the pedal ganglia, but have become united to the cephalic ganglia.

In any other case their position is quite exceptional, for the supraœsophageal position of the auditory sacs in Nudibranchiatata merely arises from the pedal ganglia being thrust upwards, and united with the cephalic ganglia.

The accessory ganglia of the pedal ganglion appear to be only what may be called *digital ganglia,* developed to meet the wants of certain elongations or expansions of the foot.

Such are the ganglia at the bases of the arms of the Cephalopoda, and such appear to me to be the ganglia which supply the "labial" processes of *Nautilus.*

III. Under the name of *Parieto-splanchnic system of ganglia,* I include the branchial and visceral ganglia of most authors, and the cervical, branchio-cardiac, and angeial ganglia of M. Blanchard. This system consists of two *primary* ganglia, which are always to be found at the side of the œsophagus, connected with both the pedal and cephalic ganglia, and for which I reserve specially the term *parieto-splanchnic* ganglia; from these nerves are given off—

1. *Parietal,* to the sides of the body, as distinct from the foot.

2. *Columellar,* to the shell-muscle or muscles, of which there are two in *Octopus, Nautilus,* and *Cymbulia*; one in the shelled Gasteropoda.

3. *Branchial,* to the branchiæ.

4. *Angeial,* to the heart and great vessels and generative organs.

Separate ganglia, answering to the three latter sets of nerves, may be found in the dibranchiate Cephalopoda; to the two last in the Heteropoda; a single ganglion corresponding to all of them is found in *Aplysia, Buccinum, Turbo, Paludina,* &c. Two such exist in *Strombus* and *Pterocerus.*

The angeial ganglia, wherever they exist separately, are placed above the aorta and united by a commissure.

The visceral or sympathetic nerves ramify extensively over the intestinal canal (see Hancock and Embleton upon *Doris*). They are connected anteriorly with the buccal ganglia, posteriorly with the parieto-splanchnic system.

To sum up, the typical number of ganglia in the Cephalous Mollusca is three pair, with which accessory ganglia and visceral ganglia may be connected in variable number. The primary ganglia are united by commissures which form—1, two *greater* nervous rings, the *cephalo-pedal,* connecting the cephalic and the pedal ganglia, and the *cephalo-splanchnic,* connecting the cephalic and parieto-splanchnic ganglia: these rings surround both the œsophagus and the aorta. 2, Two *lesser* nervous rings, the cephalo-buccal, uniting the cephalic and buccal ganglia, and encircling the œsophagus, and the parieto-angeial, uniting the parieto-splanchnic and angeial ganglia, and sometimes surrounding the aorta alone: this ring does not seem to be invariably present.

The homology of these ganglia with those of other animals does not, I think, present any very great difficulty.

It is needless to point out their identity with those of the Acephala Lamellibranchiata.

In the Articulata we have corresponding cerebral ganglia, while the subœsophageal ganglionic chain answers to the pedal and parieto-splanchnic ganglia united. The nerves of the latter system appear in a distinct form as the *transverse* nerves of Insects.

It seems possible that the series of lateral ganglia in certain Annelida (*Amphinome*) may correspond with the parieto-splanchnic ganglia of Mollusks.

On the other hand, the stomato-gastric nerves with their ganglia in Articulata appear to correspond with the visceral nerves of Mollusca.

To what portions of the nervous system of the Vertebrata do these various ganglia answer? This is a problem which has been variously solved. Unless, with Von Baer, we deny the homology of the centres of the nervous system in the Invertebrata with those of the Vertebrata, an argument whose worth can only be decided by a careful and laborious study of development, it would seem clear that the cerebral ganglia are homologous with the corpora striata and thalami of Vertebrata. Their accessories, the buccal ganglia, answer to the trigeminal ganglia, and supply similar parts.

The cephalo-pedal and cephalo-splanchnic commissures correspond with the crura cerebri; the pedal and parieto-splanchnic ganglia answering to the spinal cord and medulla oblongata. The origin of the auditory nerves would then correspond with that of the seventh pair in Vertebrata, and the pedal nerves with the spinal nerves in function and position.

Again, if the parieto-splanchnic ganglia represent the medulla oblongata, their branches should, as I think they do, represent a pneumogastric nerve*.

Finally, the visceral nerves answer to the sympathetic.

The next question to be considered is, in what manner these typical ganglia are arranged and combined to form the almost infinite varieties in the actual nervous systems of the Cephalous Mollusca.

We find—

1. The ganglia concentrated into a mass above the œsophagus, e. g. *Doris*, *Phyllirrhoë*, the majority of the Nudibranchiata.

2. The ganglia concentrated into a mass below the œsophagus, e. g. Pteropoda.

3. The ganglia concentrated around the œsophagus, some above and some below, e. g. *Buccinum*, *Helix*, *Onchidium*, Cephalopoda.

4. The ganglia scattered and separated in pairs, e. g. Heteropoda, Tectibranchiata, and many Pectinibranchiata.

Among these the parieto-splanchnic ganglia may either be united by apposition with the pedal ganglia, and with the cerebral by commissure, as occurs most commonly, e. g. *Octopus*, *Nautilus*, *Haliotis*; or they may be united with the cerebral ganglia by apposition, and with the pedal by commissure, as in *Strombus* and *Pterocеras*.

Patella, *Aplysia* and *Bullœa*, form a gradual transition from one of these conditions to the other.

It will be seen at once from this enumeration that the concentration of the nervous

* Von Siebold compares the nerves which arise between the nerves to the ganglion stellatum in Cephalopoda to a par vagum. Vergl. Anat. p. 379.

system is by no means a test of high organization in the Mollusca, but rather the reverse.

A peculiarity of the Mollusks belonging to the second and third categories, viz. that their infraœsophageal nervous mass is often perforated by the aorta, may be accounted for by the narrowness of the angeial ring, consequent upon the concentration of the elements of the parieto-splanchnic system, so that they unite directly above the aorta.

The singular variation in the arrangement of the different portions of the nervous system, whereby the Mollusca as a class differ so widely from the other great classes of Vertebrata and Articulata, may, I think, find an explanation in the well-known law, that the constancy of a given arrangement of organs greatly depends upon the period at which they appear in embryonic life. If certain organs are formed early, those which come later must obviously accommodate themselves to their predecessors; and any variations which have taken place in the latter will perturb the normal disposition of the former.

Now in the Mollusca, as has been already stated, the neural side of the embryo is the last to be developed, and the nervous system does not make its appearance until the animal has taken its characteristic form.

Contrast this with the Vertebrata; in them the nervous system is the first to be developed, and it is, of consequence, the most fixed and unchanging feature in the whole of their organization.

On the other hand, the separation of the abdomen or post-abdomen from the body is one of the earliest facts in molluscous development, and it has a corresponding influence over their whole organization.

The Archetypal Vascular System and its modifications.—It may be questioned whether the "archetypal" heart has a single or a double auricle, but it is certain that in proportion as the symmetry of the branchial apparatus and of the whole body is preserved, we approach to the form of heart with a double auricle. Thus we have a double auricle in *Chiton* and *Haliotis*, and a close approach to it in *Tethys*, *Janus*, and the *Eolidæ*.

In the Cephalopoda the contraction of the branchio-cardiac veins has been observed by MILNE-EDWARDS and KÖLLIKER, so that they may be considered to be auricles. This is another curious illustration of the fact, that what is commonly considered the most concentrated and highest organization does not occur in the reputed highest forms of Mollusca.

The heart lies above the intestine, and gives off the aorta anteriorly. This runs forwards through the cephalo-pedal ring with the œsophagus, and terminates eventually in the buccal mass. Its main branches may be classed as visceral and pedal.

It is needless to enter here upon the beautiful discoveries of M. MILNE-EDWARDS, with respect to the incompleteness of the circulation in the Mollusca. The facts I have detailed add ocular proof to his already convincing demonstrations. But it is to be observed, that in this respect, again, the "highest" Cephalopod, *Octopus*, possesses no "higher" organization than the Slug or Snail*

The consideration of the archetypal vascular system leads naturally to that of the value of the distinction made by M. MILNE-EDWARDS between *opisthobranchiate* and *prosobranchiate* Mollusca. If my views be well-founded, it is clear that opisthobranchism is the typical condition of all Mollusks, and that prosobranchism is one result of that asymmetrical development which I have endeavoured to show to be the principal agent in modifying the form of these animals. A little consideration will render it evident, that neither the *neural* nor the *hæmal* flexure will have any effect in altering the position of the heart, so long as the flexure occurs *behind* it, while either flexure will produce prosobranchism, if it take place *before* the heart.

Prosobranchism then indicates that a flexure has taken place, but not in what direction; opisthobranchism indicates only that no flexure has taken place in front of the heart.

As derived characters, therefore, we may expect them to fail in certain cases; and those Mollusks which I have chosen to illustrate this paper are instances of their failure. *Firoloïdes* is nearly opisthobranchiate, while *Atlanta* is very decidedly prosobranchiate; and similar variations, as I have shown, occur among the Pteropoda.

The Archetypal Alimentary Canal consists of—1, lips; 2, jaws; 3, buccal mass and tongue; 4, œsophagus; 5, crop; 6, stomach or gizzard; 7, pyloric appendage; 8, intestine; 9, glandular appendages. I wish here merely to draw attention to some peculiarities of the third and the seventh organs in this list, which have not, I think, been hitherto sufficiently noted.

Of the Structure of the Buccal mass and Tongue (Plate V. figs. 12, 13, 14, 15).—Although the structure of the "tongue" of the Mollusca has been very elaborately investigated, its mechanism appears to me to have been hardly at all understood.

CUVIER, who first described this structure in *Buccinum*, thought that the buccal cartilages were the chief agents in moving the tongue. He considered the 'tongue-plate' to be passive, and that its movements depended upon the protraction, retraction, divergence or approximation of the cartilages†.

This idea is still further carried out in the Leçons d'Anatomie Comparée (2nd ed. t. v. p. 15–25), where the cartilages are compared to rudimentary jaws, though a little consideration would have shown the jaws to be represented by other organs in some of the instances quoted.

Subsequent writers either coincide in CUVIER's view, or substitute for it some vague notion of licking or rasping; so OSLER‡ and TROSCHEL§.

* The "vena cava superior" of Cephalopoda answers to the very short trunk formed by the union of the two afferent branchial trunks in *Aplysia*, &c., which, as receiving the veins of the foot, correspond with the venous circlet at the base of the arms.

† Mém. sur les Mollusques, Grand Buccin, p. 9.

‡ Philosophical Transactions, 1832.

§ WIEG. Arch. 1836.

MIDDENDORF, in his elaborate Monograph upon *Chiton* (Malacozoologia Rossica), gives a very careful and detailed description of the buccal apparatus in that Mollusk, but equally fails in rendering its action clear.

He gives the name of tongue exclusively to a "bifid papillose organ, surrounded by circular folds, which consists mostly of vascular branches, between which masses of muscle are interwoven:" this is placed in the floor of the buccal cavity in front of, and below the buccal mass.

To what is commonly known as the tongue, he gives the name of "radula," "reibplatte." The dentigerous plate is the "lamina radulæ," its expanded portion the "orbis radulæ." What I have called the buccal cartilages are his "folliculi motores."

It is difficult to come at any clear understanding of MIDDENDORF's views, but so far as I can comprehend them, he appears to consider that the "lamina radulæ" acts as a sort of elastic file pushed from behind by a special muscle, the "curvator radulæ," and supported and steadied by the "folliculi motores."

VON SIEBOLD says*, "this organ (the tongue), by its protrusion and retraction, is made use of by the Cephalophora as an ingestive apparatus." He says nothing about the buccal cartilages or the minute structure of the organ.

When I first examined this apparatus carefully six or seven years ago in *Buccinum*, I was convinced that CUVIER had mistaken its mode of operation, and further observation has only strengthened that conviction.

I have already described the manner in which the apparatus may be *seen* working in *Firoloïdes* and *Atlanta*, and I propose now to demonstrate that from the anatomical arrangements the "tongue" has the same chain saw-like mode of operation throughout the Cephalopoda and Gasteropoda. Perhaps *Patella* may be taken as the most convenient illustration, since the organ is here very large, and its parts are distinct and well-developed.

In *Patella* (Plate V. figs, 12, 13) it is an oblong mass, reddish, except where the tongue-plate shines with a somewhat greenish hue. It is bifid posteriorly, and has a sulcus along two-thirds of its upper surface. In this the tongue lies before it enters the cavity of the mouth. The opening of the œsophagus corresponds with about the anterior fourth of the upper surface of the buccal mass.

To the postero-lateral angles of the mass its extrinsic protractor muscles are attached, two on each side. They go to be inserted into the cephalic parietes, two in front of and above, and two behind the supraœsophageal ganglia. The lower ones are united so as to form a broad muscular plate. Two small muscular bands are also sent from the anterior angles of the buccal mass to the skin of the head.

When the muscular expansion formed by the lower protractors is removed, four or five muscular bands (fig. 12 μ) are perceived inserted by their posterior extremities into the posterior and lower part of the "buccal cartilages," and converging anteriorly to be inserted into the lower edge of an "elastic plate."

* Vergleichende Anatomie, p. 320.

From the same point of origin a thick bundle of reddish fibres passes up over the posterior extremity of the cartilages, and is inserted into the upper edge and sides of the "elastic plate." These may be called the intrinsic muscles (fig. 13 μ).

This elastic plate (η) is an elastic transparent membrane, broad posteriorly, and narrower anteriorly, so as to be somewhat heart-shaped. By its superior surface it gives attachment to the "dentigerous plate" (lamina radulæ of MIDD.), on which the teeth are set; inferiorly it is very smooth, and plays over the equally smooth pulley-like surface afforded by the larger *buccal cartilages* (fig. 14). These are four in number, two large and two small accessory ones (δ). The larger are elongated, white, cartilaginous-looking plates, excavated internally, and thick and convex behind; their inner edges are kept together by strong transverse muscular fibres. Their upper edges are in contact, forming the smooth surface mentioned above; the smaller seem to be in a manner sesamoid cartilages; they are connected anteriorly with the tongue-plate and posteriorly with muscular fibres, which are inserted into the larger cartilages.

It is clear that the action of the intrinsic muscular bands (having the insertions described) must be to cause the "elastic plate," and with it the "dentigerous plate," to traverse over the ends of the cartilages, just like a band over its pulley, the cartilages themselves being entirely passive in the matter. The extrinsic bands, again, must serve to protract the whole mass and thrust it more or less firmly against the object to be acted upon.

I have examined *Buccinum*, *Fissurella*, *Doris**, *Aplysia*, *Bullæa*, *Helix*, *Onchidium*, *Cypræa*, *Pteroceras*, *Sigaretus* and *Vermetus*, and in all I have found a structure essentially similar to that here described; the difference consisting in the greater or less length of the dentigerous plate, and the more or less complete development and isolation of the buccal cartilages. These are the less distinct the more the tongue becomes a mere organ of deglutition. *Aplysia* and *Bullæa*, for instance, have the cartilages united and much softer than in most genera. The structure of the Cephalopod tongue closely resembles that of *Aplysia*; and it has the further peculiarity, that the portion of the floor of the buccal cavity in front of the tongue (true tongue of MIDDENDORF), which is plicated and distinct in most Gasteropods†, is in the Cephalopods raised up into a laminated caruncle (or several) larger than the tongue itself‡.

This pulley-like structure of the tongue appears to me to be very characteristic of

* See also the description of the "tongue" in this genus by Messrs. HANCOCK and EMBLETON, *loc. cit.* p. 210.

† OSLER, *loc. cit.*, p. 505, describes a soft striated papilla arising from the floor of the mouth in front of the tongue in *Patella*, which, he says, "is probably the organ of taste."

‡ See OWEN, Article 'Cephalopoda,' Cyc. Anat. and Phys. Is the "horny striated substance" supporting the lingual teeth, "which appears to represent the body of an os hyoides" in *Nautilus*, the representative of the buccal cartilages?

the portion of the molluscous type here considered*, and indeed to be peculiar to it. Its occurrence in *Chiton*, therefore, would effectually determine the molluscous nature of that genus, even if there were no other grounds for the conclusion; while the structure of the buccal armature of *Sagitta*, which has been compared to the protruded tongue of a Heteropod, is in fact so totally different as at once to remove it from the Mollusca†.

I may further remark, that the structure of the tongue in the Cephalopoda adds one more link to the very strong chain of affinity between them and the ordinary Mollusca.

Of the Pyloric Sac.—This appears in various forms in a great number of the Mollusca, and seems to be always in special relation with the liver. In *Atlanta*, it has been seen that its glandular parietes form the liver. In the Cephalopoda the hepatic ducts enter its representative, the spiral sac of *Octopoda*, the elongated sac of *Loligo*. The extreme length of the pyloric sac in *Cleodora*, and the occurrence of a second smaller one, appear to be leading the way to the ramified prolongations of the intestinal canal found in the *Eolidæ*.

In *Pterocera*s a very remarkable structure exists, which, so far as I am aware, has not yet been noticed‡. The existence of a "crystalline style" in connexion with the alimentary canal, has long been known in the Lamellibranchiata, but it has hitherto been supposed to be confined to them. However, in *Pteroceras*, the pyloric sac contains a very complete style, Plate V. figs. 16, 17.

The stomach is a wide somewhat quadrangular cavity. The œsophagus opens into

* It has been noticed by Troschel (Anatomie von *Ampullaria Urceus*, Wieg. Arch. 1845), that the structure of the tongue is the same throughout those Mollusks which have a head: "Under the jaw lies the anterior part of the so-called tongue, a membrane which is present in all Pteropoda, Cephalopoda, Gasteropoda, in short, in all those Mollusks which possess a head. It is wanting in all the so-called Acephala, in the Bivalves and in the Tunicata. It rests, as in all cases where it is present, upon two portions of cartilage of whitish colour, combined by a membrane and moved by many muscles." It is clear, however, from this passage, that Troschel has not recognized the true mechanism of the organ.

† There is a curious similarity between the "tongue" of the Mollusca and the arrangement of the dental apparatus in the Plagiostome fishes, which may be viewed perhaps as another illustration of Von Baer's law, that while the exterior of a vertebrate animal is Articulate in its construction, the interior is Molluscous.

‡ Since writing the above, I find that so far back as 1829, the existence of this organ was distinctly pointed out, though strangely enough the fact has been quite overlooked by every one save Von Siebold; he, however, merely refers to the statement in a note, and says he "does not know what to make of it." (Vergleichende Anatomie, p. 312.) This statement is contained in a valuable paper upon the anatomy of the Mollusca, entitled "General Observations on Univalves," by Mr. Charles Collier, Staff-Surgeon at Ceylon; printed in the Edinburgh New Philosophical Journal for 1829, p. 231. "There is an organ, the crystalline stiletto, confined erroneously by a celebrated naturalist (Cuvier) to bivalves, which is found in every species of *Strombus*, in *Trochus turritus*, and a species (*Vertagus*) of *Murex*. It is enclosed in a sheath, that passes parallel to and by the side of the œsophagus, to the stomach, into which the stiletto enters, leaving its covering; that end which lies within the stomach is obtuse, laminated, and fixed by a hook of similar substance to its situation. The upper portion is circular, homogeneous, slightly tapering, transparent, of gelatinous consistence, and resembling somewhat a pistil with its stigma."

its left anterior angle, while its pyloric orifice, very close to this, is at the right anterior angle. Behind the pyloric orifice the rounded head of the crystalline style projects from the aperture of the pyloric sac, γ, fig. 16.

Two wide apertures communicate with the liver, and act as hepatic ducts.

Several considerable ridges of the gastric membrane rise from the floor of the stomach; the principal one is next to the cardia, and there is a smaller between the cardia and pylorus. The aperture of the pyloric sac is surrounded by an elevated circular ridge, which is slit towards the pylorus, the left edge of the slit overlapping the right. The end of the style projecting from this orifice is opposed by one or two cartilaginous plates upon the principal elevation. It is only the end of the style which is free; for the rest of its length (2 or 3 inches) it lies in the pyloric sac (λ, fig. 17), which runs back over the intestine, in the thickness of the left side of the mantle, and terminates by a rounded extremity.

It seems probable that the "crystalline style" is secreted by the pyloric sac, and that it acts as a gastric plate, assisting in the comminution of the food, although its transparent and delicate texture would not seem to fit it for the performance of any very important office of this kind.

Its resemblance in position and structure to the crystalline style of *Solen* is sufficiently remarkable.

Renal System.—It has been shown that in the Heteropoda and Pteropoda a "contractile sac" exists, placed so as to be bathed by the blood entering the auricle. It has been hinted that this is a renal organ, and I now proceed to give the reasons for my belief that it is so.

A hollow sacculated organ, with yellowish glandular parietes, surrounds the base of the pulmonary sac in the Pulmonata, and opens by the side of the rectum. The secretion of this organ has been shown to contain uric acid*. No contractions have been observed in it.

In *Fusus*, *Cypræa* and other Pectinibranchiata, an aperture, frequently seated upon a kind of papilla placed at the posterior and upper part of the branchial chamber, leads into a wide cavity, which is in relation above with the pericardium, and on the sides with the rectum and generative duct. In its anterior wall a yellow gland is frequently attached, which consists of large vascular laminæ. I observed no contractions of either the sac or the yellow gland, but my attention was not at the time particularly directed to this point.

Now I think that this sac, with its vascular gland, is exactly comparable in position to the "contractile sac" and to the renal organ of *Pulmonata*, while, on the other hand, it closely resembles the serous chambers with their contained venous appendages, which open into the mantle-chamber of the Cephalopoda.

The venous appendages of the Cephalopoda, however, have been demonstrated to be renal organs by containing secreted uric acid, and they possess the faculty of rhythmical contraction†.

* H. Meckel, Müller's Archiv, 1846.

† Kölliker, Entwickelungsgeschichte d. Cephalopoden.

The chambers of the venous appendages, then, in the Cephalopoda answer to a "contractile sac," in which the secreting power and the contractile faculty have become restricted and localized in a portion of the organ*.

I have here touched mainly upon the less commonly understood portions of the internal anatomy of the Cephalopoda and Gasteropoda, but they clearly tend to strengthen the conclusion to be derived from embryology and the more generally known anatomical facts, viz. that the Cephalopoda and Gasteropoda are morphologically one, are modifications of the same archetypal molluscous form.

On the other hand, I have made no reference to the Acephala, nor is it my intention to go into that part of the subject; but, for the sake of the zoological bearings of the question, I may shortly express my belief, that of the two families of the Acephala, there is abundant evidence, both anatomical and embryological, to show that the one, the Lamellibranchiata†, is modelled upon the archetype of the Cephalous Mollusca.

Such evidence as we possess with regard to the Brachiopoda, however, is purely anatomical, and (though I am aware that a great weight of authority lies upon the other side) yet Mr. HANCOCK's opinion, that they are rather to be considered as allied to the Polyzoa than to the Cephalous Mollusca, seems to be quite as plausible as the more general notion.

Should this highly ingenious suggestion be found by embryology to be correct, the Brachiopoda will have the same relation to the Polyzoa as the simple Ascidians to the compound Ascidians, and will form a parallel group to the former in M. MILNE-EDWARDS's section of "*Molluscoïdes*."

In conclusion, I would observe, that the archetypal Cephalous Mollusk (as thus defined) is, in all its modifications, sharply separated from other archetypes, whatever apparent resemblances or transitions may exist. In all cases these will, I believe, on close examination, be found to be mere cases of analogy, not of affinity.

As CUVIER long ago remarked of the Cephalopoda and Fishes, so we may say of the Cephalous Mollusca in general and other types:—"Whatever BONNET and his followers may say, Nature here leaves a manifest hiatus among her productions." For instance, great as are the apparent resemblances between a Lamellibranch and an Ascidian, they all vanish upon closer examination‡. Neither in its anatomical nor in its embryo-

* A renal organ of similar character has been long since demonstrated in the Lamellibranchiata. (See VON SIEBOLD, Vergl. Anat.)

† The Lamellibranchiata are as truly cephalous as many Pteropoda, and the possession of a distinct head is so much a question of degree as to be a very unfit classificatory character.

‡ I beg that I may not be misunderstood here. While I consider that there is no transition between the Cephalous Mollusca as such, and the Ascidians or Polyzoa, I also fully believe (and so far as the Ascidians are concerned I have endeavoured to demonstrate, Report on the Structure of the Ascidians, already referred to) that the archetype of the Cephalous Mollusca, that of the Ascidians and that of the Polyzoa, are all referable to a common archetype, the archetype of the Mollusca generally. It is one thing to believe that certain natural

logical relations does the branchial sac of an Ascidian correspond with the mantle-cavity of a Lamellibranch.

The nervous system is totally different. The three pairs of ganglia, which exist in all Lamellibranchiata (even the apodal genera), are replaced by one in the Ascidians, which is not homologous (as is commonly asserted) with the branchial ganglion, or intersiphonic ganglion of Lamellibranchiata, but with their pedal ganglion.

The organization of the circulatory system is wholly different. The Ascidians have a cellulose test, not a calcareous shell. The larval conditions are totally distinct.

If, however, all Cephalous Mollusks, *i. e.* all Cephalopoda, Gasteropoda and Lamellibranchiata, be only modifications by excess or defect of the parts of a definite archetype, then, I think, it follows as a necessary consequence, that no anamorphism takes place in this group. There is no progression from a lower to a higher type, but merely a more or less complete evolution of one type.

It may indeed be a matter of very grave consideration whether true anamorphosis ever occurs in the whole animal kingdom. If it do, then the doctrine that every natural group is organized after a definite archetype, a doctrine which seems to me as important for zoology as the theory of definite proportions for chemistry, must be given up.

Description of the Plates.

In all the Plates the same letters refer to the same parts.

a. Anus.
b. Buccal mass.
c. Contractile sac.
d. Subspiral ciliated band.
ep. Epipodium.
f. Salivary gland.
g. Tentacles.
h. Head.
i. Eye or optic nerve.
j. Auditory vesicle or nerve.
k. Stomach.
k'. Pyloric cæcum.
l. Liver.
m. Mantle.
mt. Metapodium.
ms. Mesopodium.
n. Branchiæ.
o. Ovary.
p. Penis.
pp. Propodium.
q. Abdomen.
r. Post-abdomen.
s. Vesicula seminalis.
t. Testis.
u. Auricle.
u'. Venous canal.
v. Ventricle.
w. Aorta.
w'. Recurrent artery.
x. Cerebral ganglia.
y. Pedal ganglia.
y'. Pedal artery.
z. Parieto-splanchnic ganglia.

groups have one definite archetype or primitive form upon which they are all modelled; another, to imagine that there exist any transitional forms between them.

Every one knows that Birds and Fishes are modifications of the one vertebrate archetype; no one believes that there are any transitional forms between Birds and Fishes.

β. Buccal ganglia or nerves.
γ. Crystalline style.
λ. Its sheath.
δ. Accessory cartilages of the buccal mass.
θ. Tongue plate.
η. Elastic plate.
μ. Muscles.

PLATE II.

Fig. 1. *Firoloïdes Desmarestii.* (Magnified.) Male.
Fig. 2. *Firoloïdes Desmarestii.* Female. Posterior extremity from the left side.
Fig. 3. *Firoloïdes Desmarestii.* Female. Posterior extremity from the right side.
Fig. 4. Penis of the male.
Fig. 5. Egg-tube.
Fig. 6. Nervous system, with the termination of the pedal artery at *y'*.
Fig. 7. Cerebral ganglia from below.
Fig. 8. Right eye and tentacle.

PLATE III. *Atlanta Lesuerii.*

Fig. 1. Animal in its shell (much magnified), from the right side. Male.
Fig. 2. The same from the left side, to show the arrangement of the nervous and vascular systems.
Fig. 3. Post-abdomen, without the shell; more enlarged, from the right side.
Fig. 4. Portion of the mantle-cavity and post-abdomen from the left side, to show the arrangement of the heart and branchiæ.
Fig. 5. Portion of the penis.
Fig. 6. Cerebral ganglia, from the left side.

PLATE IV.

Fig. 1. *Pneumodermon* ——? A young specimen, to show the form of the foot and its relations.
Fig. 2. *Euribia Gaudichaudii* (Eydoux and Souleyet), from behind, placed, not as it swims, but so as to leave its parts in their normal position. This has been done with each of the figures in this Plate, except it be otherwise expressly mentioned.
Fig. 3. The head and foot of *Euribia*, seen from below.
Fig. 4. *Cleodora curvata* (Eydoux and Souleyet), from the neural side.
Fig. 5. The same, from the hæmal side.
Fig. 6. *Cleodora aciculata*, from the right side, without the shell.
Fig. 7. The same. The head and alæ from the neural side.

PLATE V.

The first eleven figures are to be regarded as mere diagrams, illustrative of the archetypal form of the Mollusca and its more important modifications.

The shaded portion is the hæmal surface, the unshaded the neural surface.

Figs. 2 and 3 are supposed to represent the development of an abdomen, and the changes of position thence undergone by the intestine and heart.

Fig. 4 is a diagram of a Pteropod, corresponding with fig. 2.

Fig. 5 is a diagram of a Cephalopod, corresponding with fig. 3; but in these, changes in the different parts of the foot have been also effected.

Figs. 6 and 7, similarly are supposed to represent the development of a post-abdomen.

Fig. 8. A diagram of *Aplysia*, corresponding with fig. 6. *Atlanta* corresponds exactly with fig. 7.

Figs. 9, 10, 11, are imaginary sections of a Mollusk, a Fish and an articulate animal, respectively, to show the relations of the nervous, alimentary, vascular and appendicular systems.

The Mollusk and articulate animal are in their normal position; the fish is turned upon its back to correspond with them. * The pectoral fins. † The legs of the articulate animal.

Figs. 12–15. The buccal apparatus or tongue of *Patella*.

Fig. 12. From the right side.

Fig. 13. From above.

Fig. 14. The supporting cartilages.

Fig. 15. The elastic plate which plays over them.

Figs. 16, 17. The stomach of *Pteroceras*.

III. *On Rubian and its Products of Decomposition.* By EDWARD SCHUNCK, *F.R.S.*

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PART II.

Action of Alkalies and Alkaline Earths on Rubian.

IN the first part of this paper I have described, in general terms, the effect produced on rubian by alkalies. It is only the fixed alkalies that are capable of effecting the decomposition of rubian. Ammonia produces no perceptible change in a watery solution of this substance, except that of altering its colour from yellow to blood-red. This blood-red colour remains unchanged even after long-continued boiling, and the solution still contains rubian, for on supersaturating the ammonia with acid, the solution again becomes yellow, and not the slightest precipitate is produced. The fixed alkalies, on the other hand, act very differently. On adding caustic soda to a solution of rubian, the colour of the solution first changes to blood-red. On boiling the liquid, however, its colour soon changes again from blood-red to purple. This alteration in colour indicates the formation of alizarine. If the boiling be continued, there is deposited, as the liquid becomes more concentrated, a dark purple powder, which consists mainly of a compound of alizarine and soda, and separates in consequence of its insolubility in caustic lye. After the liquid has been boiled for some time, then, provided the quantity of soda employed be sufficiently large, the rubian is entirely decomposed. On now adding sulphuric acid in excess, a quantity of orange-coloured flocks, exactly like those produced by the action of acids on rubian, is precipitated, while the liquid becomes almost colourless. These flocks are separated by filtration and washed with cold water, until the sulphate of soda and sulphuric acid are entirely removed. They now consist mainly of four substances, viz. 1st, *Alizarine*; 2ndly, *Rubiretine*; 3rdly, *Verantine*; and 4thly, a substance which has not hitherto been observed, and to which I shall give the name of *Rubiadine*.

In order to separate these substances from one another, I adopt almost the same method as that employed for the separation of the products of decomposition with acids. The mixture is first treated with boiling alcohol, which dissolves the greatest part, leaving undissolved however a considerable quantity of a dark brown flocculent substance. This substance invariably accompanies the other products of the action. It is, however, as its properties and composition show, a secondary product of decomposition, the formation of which I shall explain presently. To the dark yellow alcoholic solution, after filtration from this flocculent substance, there is added acetate of alumina, which produces a dark red precipitate. This precipitate, which is a compound of alizarine and verantine with alumina, is decomposed with muriatic acid, and the alizarine and verantine are separated from one another by means of acetate of copper, and purified in the manner which I have before described, when treating of the products of decomposition with acids. The liquid filtered from this alumina compound is still yellow. On adding to it sulphuric acid and a large quantity of water, the substances dissolved in it are precipitated in the shape of yellow flocks, which after filtration and edulcoration are redissolved in boiling alcohol. On adding acetate of lead to this solution, there is produced a dark purplish-brown precipitate, which consists of rubiretine and verantine in combination with oxide of lead. The liquid still remains yellow, and is filtered from the precipitate. The latter, on being decomposed with boiling muriatic acid, gives a brown powder, which, after filtration and edulcoration, is treated with cold alcohol. This dissolves the rubiretine, leaving behind the greatest part of the verantine. The alcoholic liquid, after filtration and evaporation, leaves a residue of rubiretine with its usual appearance and properties. Should it not melt easily when thrown into boiling water, it must again be treated with a small quantity of cold alcohol, when an additional portion of verantine remains undissolved, and the alcohol on evaporation usually gives pure rubiretine. Should it, however, still contain verantine, the process of treating with cold alcohol must be repeated, until it is free from that substance. The alcoholic liquid filtered from the lead precipitate contains the substance which I call rubiadine. In order to obtain it in a state of purity, water is added to the solution. This produces a yellow precipitate, which after filtration is treated with just sufficient boiling alcohol to dissolve it. To the solution while boiling, there is added hydrated oxide of lead, which takes up the impurity, consisting chiefly of rubiretine, and renders the solution of a lighter yellow. The liquid, after being filtered boiling hot, deposits the rubiadine on cooling in small yellow needles. An additional quantity may be procured by evaporating the mother-liquor, but this portion is not sufficiently pure to assume a crystalline form, and has the appearance of an amorphous yellow powder. By exposing it to heat, however, a crystalline sublimate of rubiadine may be obtained.

The acid liquid filtered from the mixture of the four preceding substances still contains in solution another product of decomposition. After neutralizing the sulphuric acid with carbonate of lead, and evaporating the filtered liquid to dryness, a residue is left, which is treated with alcohol. This leaves undissolved the sulphate of soda, and after filtration and evaporation gives a brown glutinous mass, which has all the properties of the sugar produced by the action of acids on rubian. It is this sugar, which, by the continued action on it of the caustic alkali, gives rise to the formation of the dark brown flocculent substance, insoluble in alcohol, which is found mixed with the four preceding products of decomposition. This fact is sufficiently proved by the composition of the latter substance, which is identical with that of the brown substance produced by the action of strong acids on ordinary sugar.

The products formed by the action of caustic baryta on rubian do not differ from those resulting from the action of caustic soda. On adding caustic baryta to a solution of rubian and boiling for some time, the rubian is entirely decomposed; the decomposition, as might be supposed, requiring a rather longer time for its completion than in the case of caustic soda. After the process is finished, the products of decomposition are found for the most part in combination with baryta in the shape of a purple powder, while the liquor still retains a red colour. The purple powder is placed on a filter and slightly washed with cold water. On passing carbonic acid through the filtered liquid, the latter loses its red colour, and a yellow flocculent substance is precipitated together with carbonate of baryta. The liquid, after being boiled in order to decompose the bicarbonate of baryta, and then filtered, is evaporated over sulphuric acid at the ordinary temperature, when it leaves a substance of a light brown colour, which resembles the sugar produced by the action of acids on rubian, but is not quite so deliquescent. This substance, as I shall presently show, is in fact a baryta compound of sugar. The mixture of yellow flocks and carbonate of baryta is now added to the purple powder before mentioned, and the whole is treated with muriatic acid in order to extract the baryta. The acid leaves undissolved a quantity of orange-coloured flocks, which consist, as before, of alizarine, rubiretine, verantine, and rubiadine, as well as of some of the dark brown flocculent substance, insoluble in boiling alcohol, and are separated from one another in the manner just described. If sulphuric acid be employed instead of carbonic acid for separating the baryta from the red liquid, and the excess of acid be neutralized with carbonate of baryta, the liquid on evaporation gives a substance exactly similar to that obtained in the other case, being also a compound of sugar and baryta, but differing from the latter in containing only half the quantity of baryta. The identity of the alizarine and rubiretine obtained by the action of alkalies, with those produced by the action of acids on rubian, is proved by the following analyses.

0·2255 grm. crystallized alizarine obtained by means of caustic soda, lost on being heated in the water-bath, 0·0405 water = 17·96 per cent.

0·1820 grm. of the dry substance, burnt with chromate of lead*, gave 0·4610 carbonic acid and 0·0770 water.

These numbers correspond in 100 parts to—

Carbon	69·07
Hydrogen	4·70
Oxygen	26·23

I. 0·2610 grm. rubiretine, prepared by means of caustic soda, and dried at 100° C., gave 0·6565 carbonic acid and 0·1250 water.

* In order to avoid repetition, I may state that all the organic analyses given in this paper, in which the material used for combustion is not especially mentioned, were performed with chromate of lead.

II. 0·3630 grm. rubiretine, prepared by means of caustic baryta, gave 0·9130 carbonic acid and 0·1715 water.

These numbers correspond in 100 parts to—

	I.	II.
Carbon	68·60	68·59
Hydrogen	5·32	5·24
Oxygen	26·08	26·17

I did not obtain a sufficient quantity of verantine in a pure state for the purpose of analysis, but the properties of the substance formed by the action of alkalies on rubian corresponded so exactly with those of the verantine produced by the action of acids, as to leave no doubt of their identity.

The compound of sugar and baryta, obtained by the action of baryta on rubian, is similar in appearance to the sugar itself. Its colour is light brown, and when dry it is brittle and transparent like dried gum or varnish. It exhibits no trace of crystalline form. It is more easily reduced to a state of dryness than the sugar itself, and deliquesces much less when exposed to a moist atmosphere than the latter. When its solution is mixed with acetate of lead, the whole of the sugar seems to be precipitated in combination with oxide of lead. On analysing a specimen of this substance prepared by precipitating the excess of baryta with carbonic acid, I obtained the following results:—

I. 0·6375 grm. heated for several hours in the water-bath, gave 0·5245 carbonic acid and 0·1945 water.

II. 0·7575 grm. gave 0·6320 carbonic acid and 0·2235 water.

0·4860 grm. gave 0·3480 sulphate of baryta, equivalent to 0·22838 baryta.

From these numbers may be deduced the following composition:—

	Eqs.		Calculated.	I.	II.
Carbon	12	72	22·20	22·43	22·69
Hydrogen . . .	11	11	3·39	3·39	3·26
Oxygen	11	88	27·14		27·06
Baryta	2	153·2	47·27		46·99
		324·2	100·00		100·00

If the baryta contained in the solution of sugar be precipitated with sulphuric acid, and the latter be again neutralized with carbonate of baryta, there is obtained, as I mentioned above, a compound of sugar and baryta containing less baryta than the preceding. Its analysis yielded the following results:—

I. 0·4330 grm. gave 0·4670 carbonic acid and 0·1825 water.

II. 0·3770 grm. heated for several hours longer at 100° C., gave 0·4150 carbonic acid and 0·1570 water.

0·3890 grm. gave 0·1890 sulphate of baryta, equivalent to 0·1240 baryta.

Hence may be deduced the following composition:—

	Eqs.		Calculated.	I.	II.
Carbon	12	72	30·17	29·41	30·01
Hydrogen . . .	10	10	4·19	4·68	4·62
Oxygen	10	80	33·54		33·49
Baryta	1	76·6	32·10		31·88
		238·6	100·00		100·00

This compound, it will be seen, is identical in composition with the baryta compound of cane sugar. If the formulæ of the two compounds be compared together, it is evident that if both be correct, the one containing most baryta must be written thus,—$C_{12}\,H_{10}\,O_{10}+BaO+BaO\,HO$. In this case, however, it would be difficult to conceive how it is possible for the second atom of baryta to escape the action of the carbonic acid used in its preparation. It is therefore more probable that the true formula is $C_{12}\,H_{10}\,O_{10}+2BaO$, which requires in 100 parts—

Carbon	22·84
Hydrogen	3·17
Oxygen	25·39
Baryta	48·60

In fact the amount of carbon and hydrogen found by experiment agree better with these numbers than with those of the other formula.

These analyses lead to the conclusion, that the sugar formed by the decomposition of rubian contains 12, not 14 atoms of carbon. Nevertheless it is doubtful whether in these baryta compounds the sugar exists in the same state as it does when formed by the action of acids on rubian. In the latter state it seems incapable of entering into combination with bases such as baryta or oxide of lead. It is therefore probable that by the continued action on it of alkalies or alkaline earths it undergoes some change, possibly in consequence of the loss of the elements of water.

Among the products of the action of alkalies on rubian, I have mentioned a dark brown substance insoluble in boiling alcohol. This substance is soluble in alkalies with a brown colour, and is reprecipitated by acids. It resembles in all respects the black substances bearing the general name of humus, which are formed by the action of strong acids and alkalies on ordinary sugar. It was analysed with the following result:—

0·3215 grm. gave 0·8040 carbonic acid and 0·1260 water.

In 100 parts therefore it contains—

Carbon	68·20
Hydrogen	4·35
Oxygen	27·45

The ulmic acid of Mulder, obtained by the action of muriatic acid on cane-sugar, contains, according to that chemist (C=6·11),—

Carbon	68·95
Hydrogen	4·23
Oxygen	26·82

The formula given by Mulder for ulmic acid, viz. $C_{40}\ H_{14}\ O_{12}$, requires in 100 parts—

Carbon	68·57
Hydrogen	4·00
Oxygen	27·43

The identity in composition and properties of this acid, and the substance formed by the action of alkalies on rubian, leave no doubt of their being the same.

Rubiadine.—This substance, which has not hitherto been observed among the products derived from madder, bears a close resemblance in its appearance and many of its properties to rubianine, the place of which it in fact occupies in the series of substances produced by the action of alkalies on rubian. Besides its composition, however, there are several properties belonging to it so characteristic, that it cannot be confounded with rubianine or any of the substances previously described. When crystallized from alcohol, it is obtained in the shape of small yellow or orange-coloured needles. A very minute degree of impurity, however, seems to prevent its assuming a crystalline form, in which case it is obtained in small granular masses, or as a yellow amorphous powder. It may be purified by dissolving it in a small quantity of boiling alcohol, and adding to the boiling solution either hydrated oxide of lead, or protoxide of tin. On filtering boiling hot and allowing to cool, it crystallizes out. When heated on platinum foil, it melts and burns with flame. When cautiously heated between two watch-glasses, it may be almost entirely volatilized. On the lower glass a very slight carbonaceous residue is left, while the upper glass is covered with a quantity of partly yellow, partly orange-coloured micaceous scales, endowed with considerable lustre. These scales possess all the properties of rubiadine itself. Rubiadine is insoluble in water. It communicates hardly any colour to boiling water, and the filtered liquid deposits nothing on cooling. It is more soluble in alcohol than rubianine. It dissolves in concentrated sulphuric acid with a dark yellow colour, and is reprecipitated by water in yellow flocks. If the solution in sulphuric acid be boiled, the colour changes to a dark yellowish-brown, a little sulphurous acid is disengaged, and the addition of water now causes a yellowish-brown precipitate. On treating rubiadine with boiling nitric acid, it dissolves, nitrous acid is disengaged, and the liquid on cooling deposits nothing, so that the substance seems to be decomposed by the acid. Towards alkalies rubiadine behaves in a similar manner to rubianine. Ammonia and carbonate of soda change its colour very little in the cold. It is only on boiling the alkaline liquids that it dissolves with a blood-red colour. It is precipitated from its alkaline solutions by acids in thick yellow flocks. The ammoniacal solution on exposure to the air loses its ammonia and deposits the substance in the shape of a yellow uncrystalline pellicle. On adding chloride of barium to the ammoniacal solution, no effect is produced at first, but after some time a slight dark red precipitate falls. Chloride of calcium, added to the ammoniacal solution, produces almost immediately a copious light red precipitate. Acetate of lead gives no precipitate in an alcoholic solution of rubiadine, and water throws down rubiadine in an uncombined state. On adding acetate of copper to the alcoholic solution, no effect ensues at first beyond a darkening of the solution, but after a few moments a dark brownish-red precipitate falls; the supernatant liquid remains yellow, but contains very little rubiadine in solution. When treated with a boiling solution of perchloride of iron, rubiadine does not dissolve. The liquid changes very slightly in colour, and gives after filtration only a very slight precipitate on the addition of muriatic acid. The residue left undissolved by the perchloride of iron is unchanged rubiadine. By its volatility, when exposed to heat, and by its insolubility in boiling water, rubiadine may be easily distinguished from rubianine, which cannot be strongly heated without decomposition, and is soluble in boiling water.

I only obtained sufficient rubiadine in a state of purity for one analysis, which gave the following results:—

0·2575 grm. gave 0·6725 carbonic acid and 0·1120 water.

These numbers correspond in 100 parts to—

Carbon	71·22
Hydrogen	4·83
Oxygen	23·95

There are several formulæ with which this composition agrees, but only two which at the same time satisfactorily explain the formation of this substance, viz. $C_{32}H_{12}O_8$ and $C_{44}H_{15}O_{11}$. The great similarity in properties between rubiadine and rubianine makes it probable that both have a similar composition, and as it is almost certain from the experiments above detailed that the formula of the sugar resulting from the decomposition of rubian is $C_{12}H_{12}O_{12}$, it follows that the true formula of rubiadine must be one of the two just given.

These formulæ require respectively in 100 parts—

	$C_{32}H_{12}O_8$.	$C_{44}H_{15}O_{11}$.
Carbon	71·64	71·93
Hydrogen	4·47	4·08
Oxygen	23·89	23·99

If the first formula be the correct one, then the formation of rubiadine is due to the splitting up of 1 equiv. of rubian, after the assimilation of 2 equivs. of water, into 1 equiv. of rubiadine and 2 equivs. of sugar, as will be seen by the following equation:—

$$\left.\begin{array}{ll}\text{2 equivs. of Sugar} & =C_{24}H_{24}O_{24} \\ \text{1 equiv. of Rubiadine} & =C_{32}H_{12}O_{8}\end{array}\right\}=\left\{\begin{array}{l}C_{56}H_{34}O_{30}=\text{1 equiv. of Rubian.} \\ H_2\,O_2=\text{2 equivs. of Water.}\end{array}\right.$$

$$C_{56}H_{36}O_{32} \qquad C_{56}H_{36}O_{32}$$

If the formula of rubianine be $C_{32} H_{19} O_{15}$, rubiadine will differ from it by containing the elements of 7 equivs. of water less. If the formula of rubiadine be $C_{44} H_{15} O_{11}$, then its formation would presuppose the splitting up of 1 equiv. of rubian into 1 equiv. of rubiadine, 1 equiv. of sugar and 7 equivs. of water, for

$$C_{44} H_{15} O_{11} + C_{12} H_{12} O_{12} + 7HO = C_{56} H_{34} O_{30}.$$

It appears therefore that the action of alkalies on rubian differs in its results from the action of acids only in one respect, that alkalies as well as acids give rise to the formation of five distinct products of decomposition, that four of these are in both cases perfectly identical, and that the fifth substance produced by the action of acids is replaced in the case of alkalies by another, which, though perfectly distinct, is so similar both in properties and composition, that it may be considered as in every respect its equivalent.

Action of Ferments on Rubian.—It has long been suspected by chemists that the colouring matter of madder owes its formation to some process of fermentation, but the exact nature of the process has hitherto remained unknown. Mr. Higgin first pointed out the fact, that a formation of colouring matter takes place even during the short period occupied in the process of dyeing, and he attributes it to the action of some albuminous substance contained in madder on xanthine. That some process of decomposition takes place on extracting madder with cold or tepid water and exposing the extract to a moderate temperature, is proved by the fact that the extract, if concentrated, becomes after some time thick and gelatinous; and that the process of decomposition takes effect chiefly on the rubian is apparent, since the extract, after it has become gelatinous, is found to have lost its bitter taste and the greater part of its yellow colour.

In order to prepare the peculiar substance contained in madder which has the property of effecting the decomposition of rubian, I proceed in the following manner. A quantity of madder having been placed on a piece of calico or fine canvas, there is poured on it for every pound of madder taken about four quarts of distilled water, which may be either cold or of a temperature of about 38° C. The latter temperature is preferable, as the water then percolates more rapidly. To the extract there is added, without any delay, about an equal volume of alcohol, which causes the separation of a quantity of dark reddish-brown flocks. These flocks are allowed to settle, and the supernatant yellowish-brown liquid is decanted. Fresh alcohol is added to the flocks, which are then placed on a paper filter, and washed on the filter with additional quantities of alcohol, until the percolating liquid, which is at first strongly coloured, becomes almost colourless. This process of washing, which often lasts a long time, may be assisted by removing the mass from the filter and shaking it up with moderately warm alcohol. The substance on the filter has now the appearance of a dark reddish-brown mass, which when rubbed between the fingers produces the same granular feeling as coagulated caseine, the sliminess which characterized it on its first precipitation with alcohol having disappeared. Now this substance possesses in an eminent degree the power of effecting the decomposition of rubian. If a quantity of it be added to a solution of rubian, and the mixture be left to stand at the ordinary temperature, a complete change is found to have taken place in the course of a few hours. The liquid is converted into a trembling jelly of a light brown colour, which is perfectly tasteless, insoluble in cold water, and so consistent, that if the solution of rubian was tolerably concentrated, the vessel may be reversed without its falling out. It is manifest from this experiment that it is this substance which constitutes the active fermenting principle of madder, and that for the purpose of understanding the nature of the process of fermentation peculiar to madder, it is necessary to examine the action of this substance on rubian. I shall therefore in the first place describe the products resulting from this action, after which I shall return to the consideration of its other properties.

On adding to a watery solution of rubian a quantity of the ferment, prepared in the manner just described, the latter loses its granular nature and diffuses itself in the solution without actually dissolving, forming a muddy, mucilaginous liquid. On allowing this mixture to stand in a moderately warm place, the rubian is gradually decomposed and converted into a brown jelly, similar in appearance and consistency to coagulated blood. Here and there interspersed in the mass yellow streaks and spots, consisting of long hair-like crystals, are sometimes discernible. If, after twenty-four hours, the jelly on being mixed with cold water still yields to the latter any yellow colour, it is a sign that the rubian is not entirely decomposed, and more ferment must be added, the mixture being again allowed to stand until the liquid has become tasteless and colourless. During this process none of the usual signs of fermentation are manifested. The liquid remains perfectly neutral, and no gas of any kind is disengaged. In order however to be perfectly sure of the latter point, I took a solution of rubian in water, mixed it with ferment, poured the mixture into a graduated glass tube, and inverted the latter over mercury. After standing in this way for several days, not a particle of gas had collected at the top of the tube, notwithstanding which the rubian was entirely decomposed, and on examination the products of decomposition were found to be precisely the same as those formed in the usual manner. This experiment proves not only that no gas is evolved during the fermentation of rubian, but also that, as HIGGIN has observed, the access of atmospheric air is not necessary during the process.

The decomposition of the rubian being completed, a little water is added to the mass, and the gelatinous substance left undissolved by the water is placed on a filter and slightly washed with cold water. The percolating liquid is almost colourless. The gelatinous mass on the filter now consists partly of the ferment employed, and partly of substances formed by the decomposition of the rubian. These substances are six in number, of which three are bodies previously described, and three are new. They are,—1st, *Alizarine*; 2ndly, *Verantine*; 3rdly, *Rubiretine*; 4thly, a substance closely resembling rubiacine, which I shall call *Rubiafine*; 5thly, a substance very similar to rubianine and rubiadine, to which I shall give the name of *Rubiagine*; and, 6thly, a peculiar fatty substance which I shall denominate *Rubiadipine*. These substances are separated in the following manner. The mixture is treated with boiling alcohol. The alcohol, which assumes a dark reddish-yellow colour, is filtered, and the substance on the filter is treated again with boiling alcohol, until the latter acquires thereby only a light yellow colour. The substance left on the filter consists of the ferment, which has undergone no apparent change either in quantity or properties. To the alcoholic liquid there is now added acetate of alumina, which produces a dull yellowish-red precipitate. This precipitate, which consists of alizarine, verantine and rubiafine in combination with alumina, is separated by filtration and decomposed with boiling muriatic acid, which leaves undissolved a quantity of orange-coloured flocks. The liquid filtered from the alumina precipitate has a dark brownish-red colour. By adding to it sulphuric acid and a large quantity of water, the substances contained in it are precipitated in the shape of a yellow powder. This powder yields nothing to boiling water, which proves the absence of rubianine among the products of decomposition. After being separated by filtration, and washed with water to remove the acid and aluminous salts, it is again dissolved in boiling alcohol, and to the boiling solution there is added an excess of sugar of lead, which causes a dark purple precipitate, the liquid becoming dark yellow. The latter is filtered boiling hot and then mixed with a large quantity of water, which produces a dull orange-coloured precipitate, consisting of rubiagine and rubiadipine in combination with oxide of lead. This precipitate is boiled with sulphuric acid, which turns it yellow; and after the excess of acid has been removed with water, it is treated with boiling alcohol, which leaves undissolved a quantity of sulphate of lead, and acquires a yellow colour. The alcohol on evaporation leaves a residue consisting of rubiagine and rubiadipine. This residue is treated with cold alcohol, in which the rubiadipine dissolves easily, and is obtained on again evaporating as a dark brown, soft, fatty mass. The rubiagine left undissolved is treated with a small quantity of warm alcohol, which removes an additional quantity of rubiadipine, together with a portion of the substance itself, and on redissolving it in boiling alcohol and evaporating spontaneously, it appears in the form of a lemon-yellow mass consisting of small crystalline grains. The dark purple precipitate produced by sugar of lead consists of oxide of lead in combination with rubiretine, alizarine, verantine and rubiafine. It is treated with boiling muriatic acid, which dissolves the oxide of lead. The yellow flocks left by the muriatic acid are, after washing with water, treated with cold alcohol which leaves a part undissolved, and on evaporation gives a residue consisting for the most part of rubiretine. By treating this residue again with cold alcohol, an additional quantity of substance is left undissolved, and the rubiretine is now obtained in a state of greater purity. Its appearance and properties are the same as when obtained by the action of acids or alkalies on rubian. The substance left undissolved by the cold alcohol is added to the orange-coloured flocks proceeding from the decomposition of the alumina precipitate, and the mixture is dissolved in boiling alcohol. To the alcoholic solution is added acetate of copper, which gives a dirty purple precipitate, consisting of verantine, rubiafine and a portion of the alizarine in combination with oxide of copper, while the liquid, which has a fine purple colour, contains the greatest part of the alizarine. The latter is obtained from this liquid by precipitating with muriatic acid and water, and purified by crystallization. The precipitate produced by acetate of copper is decomposed with muriatic acid, which leaves undissolved a quantity of red flocks. These, after filtration and washing with water, are dissolved in boiling alcohol. Into the boiling solution is introduced a quantity of hydrated protoxide of tin, which thereby acquires a light brown colour, while the liquid becomes light yellow, and on being filtered boiling hot and allowed to cool, deposits the rubiafine in yellow shining plates and needles, which are purified by recrystallization from alcohol. The oxide of tin-lake left on the filter is treated with cold muriatic acid, which dissolves the greatest part of the oxide of tin, and leaves undissolved a dark reddish-brown powder. This is placed on a filter, washed first with muriatic acid, then with water, and then treated with boiling alcohol. The alcohol leaves a great part of it undissolved, and on being filtered boiling hot deposits on cooling a brown powder consisting of verantine. The liquid on evaporation gives a quantity of alizarine, mixed with verantine. The substance left undissolved by the boiling alcohol is dark brown. It is a compound of verantine and oxide of tin, from which I have not been able to extract the oxide of tin by acids or any other means. It is soluble in ammonia and carbonate of soda, and is precipitated unchanged by acids.

The same products as those just mentioned may be obtained from madder by extracting it with cold or tepid water, allowing the extract to stand until a coagulum has been formed in it, placing the coagulum on a calico strainer and then treating it with boiling alcohol. The alcohol leaves undissolved a quantity of ferment, and is found to contain the same substances as those originating from the direct action of the ferment on rubian.

The liquid filtered from the gelatinous mixture of substances insoluble in water, formed by the action of the ferment, still contains in solution a considerable quantity of sugar. On adding to it a small quantity of caustic baryta, a pinkish-white precipitate falls, consisting probably of pectic acid in combination with baryta. The liquid being filtered, the baryta is precipitated with sulphuric acid, the excess of the latter is removed with carbonate of lead, and a small quantity of lead which dissolves is thrown down with sulphuretted hydrogen. The liquid now leaves, on evaporation at the ordinary temperature over sulphuric acid, a brownish-yellow syrup, having the same appearance and properties as the sugar produced by the action of acids on rubian.

The following analyses prove the identity of the alizarine formed by the fermentation of rubian with that derived from other sources:—

I. 0·5210 grm. alizarine, produced by the action of the ferment on rubian, lost, on being heated in the water-bath, 0·0920 water=17·65 per cent.

0·2500 grm. of the dry substance gave 0·6380 carbonic acid and 0·0960 water.

II. 0·4220 grm., obtained by allowing an extract of madder with tepid water to ferment, lost, on being heated, 0·0750 water=17·77 per cent.

0·3225 grm. of the dry substance gave 0·8230 carbonic acid and 0·1170 water.

These numbers correspond in 100 parts to—

	I.	II.
Carbon	69·59	69·59
Hydrogen	4·26	4·03
Oxygen	26·15	26·38

The quantity of rubiretine and verantine formed by the fermentation of rubian does not seem to be so large as it is, when acids or alkalies are employed for its decomposition, and as the process of separating and purifying the products of fermentation is, on account of the number of products formed, rather more complicated, the quantity of each substance obtained in a pure state is but small. The following analyses of these two substances were made with specimens obtained by the fermentation of an extract of madder; and even by preparing them in this manner I had some difficulty in procuring a sufficient quantity for examination, because the rubiretine formed by fermentation is always found mixed with rubiadipine, from which it is with difficulty freed; and the verantine is for the most part obtained in combination with oxide of tin, from which I have found it impossible to separate it.

0·3445 grm. rubiretine gave 0·8580 carbonic acid and 0·1695 water.

This gives in 100 parts—

Carbon	67·92
Hydrogen	5·46
Oxygen	26·62

0·3705 grm. verantine gave 0·9010 carbonic acid and 0·1420 water.

In 100 parts—

Carbon	66·32
Hydrogen	4·25
Oxygen	29·43

The great excess of carbon in this analysis indicates an admixture of alizarine. This proceeds from the circumstance that the method of separating verantine and alizarine, by means of acetate of copper, is not absolute. When these two substances are present together in solution, acetate of copper precipitates a portion of the alizarine together with the verantine, and the latter can afterwards only be purified by repeated solution in boiling alcohol, and deposition from the boiling solution.

The compound of verantine and oxide of tin, obtained in the process employed for purifying the rubiafine, cannot, as I mentioned before, be separated into its constituents. It dissolves in ammonia and carbonate of soda with a dark-brown colour, and is reprecipitated unchanged by acids. Even if it be dissolved in caustic soda, and an excess of sulphuretted hydrogen be passed through the solution, the precipitate afterwards produced by acids yields nothing to boiling alcohol, which proves that the verantine has not thereby been set at liberty. By treating it with boiling alcohol until all the matter soluble in alcohol is removed, then dissolving it in carbonate of soda, filtering from a small quantity of oxide of tin, then reprecipitating with acid, it is obtained in dark brown flocks, which when dry cohere into black, brittle, shining masses. Its analysis now gave the following results:—

I. 0·5930 grm., dried at 100° C., gave 0·6500 carbonic acid and 0·1800 water.

0·3410 grm., heated in a crucible until all the organic matter was destroyed, gave 0·1580 grm. peroxide of tin, equivalent to 0·1411 protoxide of tin.

II. 0·8390 grm. gave 0·9240 carbonic acid and 0·2455 water.

It contains, therefore, in 100 parts—

	I.	II.
Carbon	29·89	30·03
Hydrogen	3·37	3·25
Oxygen	25·36	
Protoxide of tin	41·38	

The formula $C_{56}H_{36}O_{36}+7SnO=4C_{14}H_5O_5+7SnO+16HO$ requires in 100 parts—

Carbon	29·76
Hydrogen	3·18
Oxygen	25·52
Protoxide of tin	41·54

The sugar obtained by the fermentation of rubian does not differ in its properties, as I mentioned before, from that derived from the action of acids on rubian. In composition too it does not materially differ from the latter. I succeeded, however, by exposing it for a considerable length of time to a temperature of 100° C., in depriving it of two atoms more of water, as will be seen by the following analysis:—

0·5540 grm. gave 0·8745 carbonic acid and 0·3055 water.

These numbers correspond with the formula $C_{12}H_{10}O_{10}$, as will be seen by the following calculation:—

	Eqs.		Calculated.	Found.
Carbon	12	72	44·44	43·84
Hydrogen	10	10	6·17	6·23
Oxygen	10	80	49·39	49·93
		162	100·00	100·00

On subjecting it, however, for some time longer to the same temperature, it became very brown, and its analysis now showed that it had absorbed oxygen.

0·5820 grm. now gave 0·8745 carbonic acid and 0·3000 water.

In 100 parts it contained therefore—

Carbon	40·97
Hydrogen	5·72
Oxygen	53·31

On redissolving the remainder in water, it gave a brown solution, but on adding to the solution hydrated oxide of lead, the brown portion was removed, while the solution became almost colourless, and on evaporation over sulphuric acid left a yellow syrup, the composition of which again corresponded with the formula $C_{12}H_{12}O_{12}$, as will be seen by the following analysis:—

0·3710 grm. gave 0·5535 carbonic acid and 0·2135 water.

In 100 parts:—

Carbon	40·68
Hydrogen	6·39
Oxygen	52·93

Rubiafine.—In my former papers on madder I have described a substance which I called rubiacine, and which I prepared partly from madder itself, and partly by the reduction of rubiacic acid with sulphuretted hydrogen. Now rubiafine cannot be distinguished by any of its properties from rubiacine. It crystallizes from its alcoholic solution in yellow glittering plates and needles, which are sometimes arranged in star-shaped or fan-shaped masses. By carefully heating it, it may be volatilized without leaving much residue, forming a yellow sublimate of small shining needles. It is but slightly soluble in boiling water. It is not decomposed by boiling nitric acid or by concentrated sulphuric acid, but merely dissolved by them. It dissolves in caustic alkalies with a reddish-purple, and in carbonated alkalies with a red colour. Its alcoholic solution gives with sugar of lead a fine crimson precipitate, with acetate of copper an orange-coloured precipitate. It dissolves in a solution of pernitrate of iron with a dark brownish-purple colour. The solution, after being boiled for some time, gives, on the addition of muriatic acid, a yellow precipitate, which is rubiacic acid. These properties, it will be seen, belong also to rubiacine. Nevertheless the composition of rubiafine is different, and as I have succeeded in again preparing a substance of the same composition as the rubiacine formerly obtained, they must be considered as distinct bodies.

I am unable to give more than one analysis of rubiafine, which yielded the following numbers:—

0·1765 grm., dried at 100° C., gave 0·4485 carbonic acid and 0·0725 water.

The relation in which rubiafine stands to rubiacic acid proves that the former contains 32 atoms of carbon. Its composition must then be as follows:—

	Eqs.		Calculated.	Found.
Carbon	32	192	69·31	69·30
Hydrogen	13	13	4·69	4·56
Oxygen	9	72	26·00	26·14
		277	100·00	100·00

If this be the true composition of rubiafine, then the latter stands in the same relation to rubian as rubianine and rubiadine, from which, if the latter also contain 32 atoms of carbon, it only differs by the elements of water. If to rubian be added 3 equivs. of water, there are given the elements of 1 equiv. of rubiafine, and 2 equivs. of sugar, for

$$\left.\begin{array}{ll}\text{2 equivs. of Sugar} \ldots & =C_{24}H_{24}O_{24} \\ \text{1 equiv. of Rubiafine} \ldots & =C_{32}H_{13}O_{9}\end{array}\right\} = \left\{\begin{array}{l} C_{56}H_{34}O_{30} = \text{1 equiv. of Rubian.} \\ H_{3}O_{3} = \text{3 equivs. of Water.}\end{array}\right.$$

$$\overline{C_{56}H_{37}O_{33}} \qquad \overline{C_{56}H_{37}O_{33}}$$

Rubiacine and Rubiacic Acid.—In my former papers on madder, I have given for rubiacic acid the formula $C_{31}H_{8}O_{16}$, and for rubiacine $C_{31}H_{9}O_{10}$. In order to ascertain the correct formula, and also to determine the relation in which they stand to rubian and rubiafine, of which they must be derivatives, I prepared and submitted to analysis a fresh quantity of these substances. For the purpose of preparing rubiacic acid, it is not necessary to take madder itself. If any dyework is at hand in which madder-dyeing is carried on, the liquor in which the goods have been immersed, together with the madder, and which is generally allowed to run to waste, may be employed. This liquid contains a considerable quantity of some substance, which by the action of pernitrate of iron may be converted into rubiacic acid. The mode of operation is as follows. To the liquid, which is brown and muddy, there is added, after the separation of the woody particles of the madder, a quantity of muriatic acid, which gives a brown flocculent precipitate, while the liquid becomes clear and colourless. This precipitate, after being collected on a calico strainer, is treated with pernitrate of iron until nothing more is dissolved. The resulting dark reddish-brown liquid is strained through calico, and acid is added to it, which produces a yellow precipitate. This is filtered, washed and dissolved in boiling carbonate of potash. The solution on cooling deposits crystals of rubiacate of potash, which are placed on a filter and washed with cold water. From a solution of the potash salt, the acid may be precipitated by muriatic or any other strong acid. Some rubiacic acid prepared in this manner was analysed with the following result:—

I. 0·3160 grm. gave 0·7090 carbonic acid and 0·1010 water.

In 100 parts it contained, therefore,—

Carbon	61·19
Hydrogen	3·55
Oxygen[1]	35·26

In my former experiments I obtained as a mean of three determinations the following numbers:—

Carbon	57·28
Hydrogen	2·47
Oxygen	40·25

I was at first unable to explain the great discrepancy here seen, as the new preparation both of the acid and the potash salt had the same appearance as usual. Before proceeding to explain how this discrepancy arose, I may state that the analysis just given corresponds with the formula $C_{64} H_{20} O_{27}$, which requires in 100 parts—

Carbon	61·93
Hydrogen	3·22
Oxygen	34·85

With the same specimen of potash salt as that employed for the preparation of the acid, I made a quantity of silver salt by dissolving the potash salt in water, and adding nitrate of silver. A brilliant cinnabar-red precipitate fell, which was filtered, washed and dried. It was not decomposed either by light or by the heat of boiling water.

0·5800 grm. of this salt, dried in the water-bath, gave 0·9800 carbonic acid and 0·1150 water.

0·3310 grm., treated in a crucible with muriatic acid, and heated until all the organic matter was destroyed, gave 0·1120 chloride of silver.

These numbers lead to the following composition:—

	Eqs.		Calculated.	Found.
Carbon	64	384	46·04	46·08
Hydrogen	18	18	2·15	2·20
Oxygen	25	200	24·00	24·37
Oxide of silver	2	232	27·81	27·35
		834	100·00	100·00

I now took a fresh quantity of waste dye liquor, and divided it into two parts. To the first part I added acid. The precipitate produced by the acid was treated with boiling alcohol, which dissolved a small part with a deep yellow colour, and after being filtered boiling hot and allowed to cool, deposited a quantity of orange-coloured powder. This powder I dissolved again in boiling alcohol, and to the boiling solution I added hydrated protoxide of tin, and filtered boiling hot. The liquid, which had become light yellow, deposited on cooling a quantity of light yellow needles. These needles had the appearance and properties of rubiafine. Their composition, however, proved them to be a distinct substance.

0·1515 grm. gave 0·3730 carbonic acid and 0·0540 water.

In 100 parts, therefore, it consisted of—

Carbon	67·14
Hydrogen	3·96
Oxygen	28·90

To the other part of the liquor I also added acid, and the precipitate, after straining the liquor through calico, and washing with water, I treated with a boiling solution of pernitrate of iron, which dissolved a portion with a dark purplish-brown colour. Muriatic acid produced in the filtered liquid a yellow precipitate, which after filtering and washing with water, I treated with boiling alcohol. The alcohol being filtered boiling hot, deposited on cooling a yellow powder, which on being redissolved in fresh boiling alcohol, yielded a quantity of yellow shining plates and needles. These corresponded in all their properties with rubiafine and rubiaçine. By treatment with pernitrate of iron they were convertible into rubiacic acid.

0·2355 grm. gave 0·5790 carbonic acid and 0·0860 water.

In 100 parts it contained, therefore,—

Carbon	67·05
Hydrogen	4·05
Oxygen	28·90

This analysis proves that the substance contained in the liquor existed in the same state before and after the treatment with pernitrate of iron, and that the latter, in the first instance at least, merely dissolved without changing it. A comparison of the composition of this substance with that of the body which I formerly called rubiacine, proves that they are identical. A specimen of rubiacine obtained on a former occasion from rubiacate of potash by means of sulphuretted hydrogen, I found to contain in 100 parts—

Carbon	67·01
Hydrogen	3·28
Oxygen	29·71

The most probable formula for this substance is $C_{32}\,H_{11}\,O_{10}$, which requires in 100 parts

Carbon	67·84
Hydrogen	3·88
Oxygen	28·28

The remainder of the substance I dissolved in boiling alcohol, and added acetate of lead. This gave a dark red precipitate, which after filtering, washing with alcohol, and drying, was found to have the following composition :—

0·5470 grm., dried at 100° C., gave 0·6630 carbonic acid and 0·0890 water.

0·4060 grm. gave 0·2800 sulphate of lead, containing 0·20602 oxide of lead.

These numbers correspond with the formula $3C_{32}\,H_{11}\,O_{10}+8PbO$, as the following calculation shows :—

	Eqs.		Calculated.	Found.
Carbon	96	576	33·05	33·05
Hydrogen . . .	33	33	1·89	1·80
Oxygen	30	240	13·78	14·41
Oxide of lead . .	8	893·6	51·28	50·74
		1742·6	100·00	100·00

The precipitate produced by muriatic acid in the nitrate of iron solution, was not entirely soluble in boiling alcohol. That part left undissolved by the latter I dissolved again in nitrate of iron; I kept the solution boiling for some time, and then precipitated again with muriatic acid. The precipitate, after filtering and washing, I treated with a boiling solution of carbonate of potash, in which it dissolved. On cooling, a considerable quantity of rubiacate of potash crystallized out with its usual appearance. Some of this salt was decomposed with nitric acid, and the rubiacic acid so obtained was analysed.

II. 0·5250 grm. gave 1·1595 carbonic acid and 0·1465 water.

In 100 parts—

Carbon	60·23
Hydrogen	3·10
Oxygen	36·67

These numbers do not differ very widely from those found in the first analysis. On recrystallizing, however, the remainder of the salt from water, and analysing some acid obtained from the recrystallized salt, I obtained the following numbers:—

III. 0·4300 grm. gave 0·9135 carbonic acid and 0·1100 water.

In 100 parts—

Carbon	57·93
Hydrogen	2·84
Oxygen	39·23

With another portion of the recrystallized potash salt I prepared some silver salt, as before. 0·6260 grm. of the silver salt gave 1·0340 carbonic acid and 0·1070 water, corresponding in 100 parts to—

Carbon	45·04
Hydrogen	1·89

The mother-liquor of the salt from which No. III. was made I precipitated with muriatic acid. Part of the precipitate I dissolved again in carbonate of potash, and the acid from the crystallized potash salt showed the following composition:—

IV. 0·4010 grm. gave 0·8690 carbonic acid and 0·1160 water.

In 100 parts—

Carbon	59·10
Hydrogen	3·21
Oxygen	37·69

The remainder of the precipitate I now dissolved again in pernitrate of iron, and after keeping the solution boiling for several hours I precipitated again with muriatic acid, filtered, washed, dissolved the precipitate in carbonate of potash, filtered and precipitated again with muriatic acid. The precipitated rubiacic acid now gave on analysis the following results:—

V. 0·3925 grm. gave 0·8240 carbonic acid and 0·1030 water.

These numbers correspond with the formula $C_{32} H_9 O_{17}$, as will be seen by the following calculation:—

	Eqs.		Calculated.	Found.
Carbon	32	192	56·97	57·25
Hydrogen . . .	9	9	2·67	2·91
Oxygen	17	136	40·36	39·84
		337	100·00	100·00

This composition differs, as will be seen, very little from that found in my former experiments.

Some fresh rubiacic acid made in the same manner as that of the last analysis, was dissolved in carbonate of potash, the solution was evaporated to crystallization, the crystallized potash salt was again dissolved in boiling water, and nitrate of silver was added to the solution. The precipitate was now no longer red, but of a dull orange colour.

0·1700 grm. of this precipitate gave 0·2720 carbonic acid and 0·0370 water.

In 100 parts—

Carbon	43·63
Hydrogen	2·40

The formula $C_{32} H_8 O_{16} + AgO$ requires in 100 parts—

Carbon	43·24
Hydrogen	1·80

If the formula of the potash salt be similar to that of the silver salt, viz. $C_{32} H_8 O_{16} + KO$, it must contain in 100 parts—

Carbon	51·17
Hydrogen	2·13
Oxygen	34·12
Potash	12·58

In my former experiments I obtained as an average of three determinations—

Carbon	51·37
Hydrogen	2·41
Oxygen	33·18
Potash	13·04

It appears therefore that the four first analyses given above were made with impure acid. The analysis No. I. corresponded, as I have shown, with the formula $C_{64} H_{20} O_{27}$. Now if from this formula be deducted that of the pure acid $C_{32} H_9 O_{17}$, the difference will be $C_{32} H_{11} O_{10}$, which is the formula given above for rubiacine. It is therefore almost certain that it was an admixture of the latter substance with the acid which raised the amount of carbon and hydrogen in the four first analyses. Whether this impure acid is to be considered as a chemical compound of acid and rubiacine, or whether it contains them in a state of mechanical mixture, is a point not easily determined. That the acid of the analysis No. I. contained both substances in atomic proportions may be accidental; and the ease with which rubiacate of potash made from the impure acid yields, by mere recrystallization, a salt containing an almost pure acid, tends to prove that the two substances are merely mechanically mingled. Nevertheless it is difficult to detect the presence of rubiacine in the impure acid, which behaves towards almost all reagents in the same manner as the pure acid. The acid of the analysis No. IV., for instance, could not be distinguished from pure acid by its appearance. When treated with boiling alcohol the latter acquired a yellow colour, but on being filtered boiling hot, no rubiacine crystallized out, as would probably have been the case had the latter only been mixed with the acid. Nevertheless, on heating it cautiously between two watch-glasses, a considerable quantity of yellow shining crystals, doubtless of rubiacine, were formed on the upper glass, while the rubiacic acid of analysis No. V. gave, on being heated in the same manner, only a trace of yellow sublimate and an abundant carbonaceous residue. The potash salt of the impure acid has a more granular and less silky appearance than the salt made from pure acid, and is also of a darker red. When heated it does not detonate so strongly as the pure salt. Its solutions give the same reactions as the pure salt with all reagents except nitrate of silver, which, as I mentioned above, gives with the pure salt an orange-coloured, with the impure salt a bright cinnabar-red precipitate.

In order to ascertain whether rubiacic acid is reconvertible not only into rubiacine but also into rubiafine, I took some rubiacate of potash of the same preparation as that employed for the rubiacic acid No. I., dissolved it in boiling water, added caustic soda, and passed sulphuretted hydrogen through the solution for several hours; I then precipitated with chloride of barium, filtered, washed the precipitate, decomposed it with muriatic acid, and crystallized the residue twice from alcohol. The crystals had the appearance of rubiacine or rubiafine, and possessed considerable lustre. On analysis I obtained the following numbers:—

0·1685 grm. gave 0·4340 carbonic acid and 0·0705 water.

In 100 parts it contained therefore—

Carbon	70·24
Hydrogen	4·64
Oxygen	25·12

Though this is not exactly the composition of rubiafine as given above, still it proves that the hydrogen and oxygen are contained in it in the same proportion as in that substance; for if the formula $C_{32} H_{12\frac{1}{2}} O_{8\frac{1}{2}}$, which differs from that of rubiafine by containing $\frac{1}{2}$HO less, be calculated for 100 parts, it gives—

Carbon	70·45
Hydrogen	4·58
Oxygen	24·97

Though I have not, from want of material, been able to trace the steps of the process with the requisite accuracy, I think I am justified in inferring from these experiments, that by the oxidizing agency of persalts of iron rubiafine is changed first into rubiacine and then into rubiacic acid, and that the latter is reconverted by the action of reducing agents, such as sulphuretted hydrogen, first into rubiacine and then into rubiafine. The presence of rubiacine, however, in the liquor which has been used for dyeing with madder, seems to prove that its direct formation from rubian is possible.

Rubiagine.—This substance belongs to the same group of bodies which includes rubianine, rubiadine and rubiafine, and bears a strong resemblance to these substances in properties and composition. It scarcely ever appears in well-defined crystals. When its alcoholic solution is evaporated spontaneously, it is obtained in the shape of small lemon-yellow spherical grains, which, when crushed and examined under a lens, are found to consist of small crystalline needles grouped round a centre. Occasionally it has an orange tinge, but this is probably due to some impurity. When heated on platinum foil, it melts to a brownish-red liquid and then burns with flame, leaving a large quantity of carbonaceous residue which burns away with difficulty. When heated in a tube, it gives a small quantity of crystalline sublimate mixed with oily drops. When slowly heated between two watch-glasses, it melts to a brownish-red mass, but gives no sublimate. It is quite insoluble in boiling water, to which it hardly communicates a tinge of colour. It is more easily soluble in boiling alcohol than rubianine or even rubiadine, and does not crystallize out on the solution cooling, but is left, on evaporation of the alcohol, in crystalline masses as just described. It is soluble in concentrated sulphuric acid with a dark reddish-brown colour: the solution, on being heated, disengages sulphurous acid and becomes black. Boiling nitric acid dissolves it with a disengagement of nitrous acid to a yellow liquid, while some oily drops rise to the surface. On the solution cooling, a quantity of light yellow crystals, possessed of much lustre, are deposited. Whether these crystals are a product of decomposition, or whether they are the substance itself in a state of purity, the impurities having been destroyed by the nitric acid, I am unable to state. The latter is the more probable view. Rubiagine is soluble in boiling acetic acid with a yellow colour, and crystallizes out again, on the solution cooling, in small needles. Ammonia turns it red, and on boiling dissolves it with some difficulty, forming a blood-red solution, which on evaporation loses its ammonia and leaves the substance behind in the shape of small yellow crystals. It dissolves more easily in caustic soda, with the same colour. It is precipitated from its alkaline solution by acids in lemon-yellow flocks. The ammoniacal solution gives very slight precipitates with the chlorides of barium and calcium, the solution remaining red with chloride of barium, and becoming crimson with chloride of calcium. It is soluble in baryta and lime-water with a blood-red colour, and is reprecipitated by a current of carbonic acid. The alcoholic solution gives, on the addition of acetate of lead, at first no precipitate, but the colour of the solution becomes dark yellow, and after some time, provided the solution be not too dilute, an orange-coloured granular precipitate falls, which is the lead compound of rubiagine. If no deposit is formed, then the addition of water causes an orange-coloured flocculent precipitate, which, after being separated by filtration and washed with water in order to remove the acetate of lead, is found to be very little soluble in boiling alcohol, but is easily soluble in a boiling alcoholic solution of acetate of lead with a dark yellow or orange colour. Acetate of copper changes the colour of the alcoholic solution from light yellow to brownish-yellow, and after some time an orange-coloured precipitate is formed. When rubiagine is treated with a boiling solution of perchloride of iron, the solution acquires a darker colour, but does not assume the deep brownish-purple characteristic of solutions of rubiafine and rubiacine in that menstruum. The liquid being filtered boiling hot, deposits on cooling a quantity of yellow shining scales, but the addition of muriatic acid produces no further precipitate. These scales dissolve easily in boiling alcohol, and the solution on cooling and standing deposits a number of small yellow grains and nodules consisting of crystalline needles, which are apparently nothing but rubiagine itself, for they are not capable of sublimation, and their alcoholic solution is not precipitated by acetate of lead. The greatest part of the rubiagine is left undissolved by the perchloride of iron in the shape of a yellowish-brown powder, which does not dissolve on treating it with an additional quantity of the iron salt. Boiling muriatic acid changes the colour of this powder to yellow, and it has then all the properties of rubiagine. Rubiagine is therefore not changed into rubiacic acid by the action of persalts of iron. Notwithstanding the great resemblance which rubiagine bears to the other bodies belonging to the same series, its reactions prove it to be a distinct substance. It is distinguished from rubianine by its insolubility in water; from rubiadine, for which it might most easily be mistaken, by its being incapable of sublimation; and from rubiafine by its not being convertible into rubiacic acid. Its behaviour towards acetate of lead, which is different from that of all the other three substances, also serves to characterize it.

The analysis of rubiagine gave the following results:—

0·3800 grm., prepared directly from madder, gave 0·9490 carbonic acid and 0·1760 water.

In 100 parts it contained therefore—

Carbon	68·10
Hydrogen	5·14
Oxygen	26·76

There are two formulæ with which this analysis corresponds, and both of which explain the formation of the substance equally well, viz. $C_{32}H_{14}O_{10}$ and $C_{44}H_{17}O_{13}$. These formulæ require respectively in 100 parts—

	$C_{32}H_{14}O_{10}$.	$C_{44}H_{17}O_{13}$.
Carbon	67·13	68·57
Hydrogen	4·89	4·41
Oxygen	27·98	27·02

I am unwilling to draw any inference from the greater or less correspondence of either of these calculations with the experimental result, because I am not convinced of the absolute purity of the specimen employed for analysis.

The lead compound of rubiagine was prepared by adding a small quantity of an alcoholic solution of sugar of lead to a concentrated alcoholic solution of the substance, taking care not to employ an excess of the precipitant. The orange-coloured precipitate was collected on a filter, washed with alcohol, dried and submitted to analysis.

0·4610 grm., dried in the water-bath, gave 0·5290 carbonic acid and 0·1110 water.

0·2010 grm. gave 0·1460 sulphate of lead, containing 0·10742 oxide of lead.

In 100 parts it contained therefore—

Carbon	31·29
Hydrogen	2·67
Oxygen	12·60
Oxide of lead	53·44

These numbers do not decide between the two formulæ, since they correspond equally well with $C_{32}H_{14}O_{10}+3PbO$ and $C_{44}H_{17}O_{13}+4PbO$, as the following calculation shows:—

	Eqs.			Eqs.		
Carbon	32	192	30·91	44	264	31·73
Hydrogen	14	14	2·25	17	17	2·04
Oxygen	10	80	12·89	13	104	12·53
Oxide of lead	3	335·1	53·95	4	446·8	53·70
		621·1	100·00		831·8	100·00

If $C_{32}H_{14}O_{10}$ be the true formula of rubiagine, then it is formed by rubian taking up 4 equivs. of water and splitting up into 1 equiv. of rubiagine and 2 equivs. of sugar, as seen by the following equation:—

$$\left.\begin{array}{ll}\text{2 equivs. of Sugar} & =C_{24}H_{24}O_{24}\\ \text{1 equiv. of Rubiagine} & =C_{32}H_{14}O_{10}\end{array}\right\}=\left\{\begin{array}{l}C_{56}H_{34}O_{30}=\text{1 equiv. of Rubian.}\\ H_4O_4=\text{4 equivs. of Water.}\end{array}\right.$$

$$C_{56}H_{38}O_{34} \qquad C_{56}H_{38}O_{34}$$

If, on the other hand, $C_{44}H_{17}O_{13}$ be the correct formula, then 1 equiv. of rubian loses 5 equivs. of water, and splits up into 1 equiv. of rubiagine and 1 equiv. of sugar, for

$$C_{44}H_{17}O_{13}+C_{12}H_{12}O_{12}+5HO=C_{56}H_{34}O_{30}.$$

Rubiadipine.—This substance is a characteristic product of the fermentation of rubian. I have never subjected rubian to the action of ferment, under ordinary circumstances, without being able to detect it among the bodies formed. In its appearance and general properties it resembles rubiretine. It differs from the latter in always remaining soft and viscid, and never becoming hard and brittle, however long it may be heated. It is similar in appearance to a semifluid fat tinged with colouring matter. Its colour is yellowish-brown. When heated on platinum foil it melts to a brown liquid and then burns with a bright flame, leaving a carbonaceous residue. When heated in a tube it evolves acrid fumes, similar to those produced by fat when exposed to destructive distillation. It is not much affected by boiling nitric acid, but concentrated sulphuric acid chars it when heated. When thrown into boiling water rubiadipine melts, forming oily drops, which rise to the surface. It is soluble in caustic alkalies with a blood-red colour, but the solutions do not froth when boiled like solutions of soap. The ammoniacal solution gives only a slight precipitate with chloride of barium. On adding to the alcoholic solution a small quantity of acetate of lead, a pale reddish-brown precipitate is formed, which is the lead compound. This precipitate is insoluble in boiling alcohol, but dissolves entirely when an excess of acetate of lead is added to the boiling liquid, forming a dark brownish-red solution. From this solution it is again precipitated by water, and after filtering and washing is found to be again insoluble in boiling alcohol. In its behaviour to sugar of lead it therefore resembles rubiagine. The alcoholic solution gives no precipitate on the addition of acetate of copper. The substance itself cannot be obtained in a state fit for analysis, I therefore confined myself to the examination of the lead compound formed in the manner just described. The quantity of the substance obtained was, nevertheless, so small that I had only sufficient for one analysis at my disposal.

0·2020 grm., dried at 100° C., gave 0·3770 carbonic acid and 0·1260 water.

0·1150 grm. gave 0·0490 sulphate of lead, containing 0·03605 oxide of lead.

These numbers lead to the formula $C_{30} H_{24} O_5 + PbO$, as the following calculation shows:—

	Eqs.		Calculated.	Found.
Carbon	30	180	50·60	50·89
Hydrogen . . .	24	24	6·74	6·93
Oxygen	5	40	11·26	10·83
Oxide of lead . .	1	111·7	31·40	31·35
		355·7	100·00	100·00

If this formula represents the true composition of rubiadipine, I confess I am unable to explain its formation from rubian. The great excess of hydrogen contained in it, shows that some substance must be formed simultaneously containing a large proportion of oxygen, but which has hitherto escaped detection.

The experiments above detailed prove that the decomposition which rubian undergoes by fermentation, does not differ in its general nature from that which results from the action of acids or alkalies. The decomposing effect of the ferment extends like that of these agents to three portions of rubian. The first portion of rubian loses water and gives rise to the formation of alizarine. The second loses water and produces rubiretine and verantine in equivalent proportions. The third takes up water and gives sugar and rubiafine, or it takes up 1 equiv. more of water, and yields sugar and rubiagine. The rubianine produced by acids, and the rubiadine formed by alkalies, give place, when ferment is the decomposing agent, to rubiafine and rubiagine. The rubiadipine, in consequence of the uncertainty of its nature and origin, and the minute quantity in which it is formed, I leave out of consideration. The relative proportion in which the various products of decomposition are formed, seems to be about the same in the case of ferment as when acids or alkalies are employed. Alizarine is formed in the smallest quantity; of rubiretine and verantine a little more is produced, but the sugar and the rubiafine and rubiagine exceed all the others in amount.

It appears further from these experiments, that the ordinary process of dyeing with madder is, in fact, mainly a process of fermentation, that it consists essentially of a re-arrangement of the elements of rubian induced by the action of ferment, whereby a formation of colouring matter takes place, and that the precautions necessary to be taken in regard to the regulation of the temperature in madder dyeing have reference principally to the progress of the fermentation. The extraordinary rapidity with which this process of fermentation is completed, is the only circumstance which distinguishes it from other processes of a similar nature.

A very interesting question now arises, suggested by the analogy which must be presumed to exist between this and other processes of fermentation, viz. By what means is the action of the ferment on rubian either destroyed, retarded or promoted; and do not any means exist of so modifying the action as to lead to the formation of particular substances in preference to others? With the view of throwing some light on this question, I instituted a series of experiments, which I shall now proceed to describe.

Having taken a solution of rubian and mixed it with a quantity of ferment sufficiently large to effect its decomposition under ordinary circumstances, I raised the temperature of the mixture to the boiling point and kept it boiling for a short time. After the mixture had stood for some days exposed to a moderate temperature, I found that the rubian had not undergone the least change. The liquid remained yellow and bitter, and after filtration and evaporation left a residue of rubian with its usual appearance and properties.

I took in the next place a quantity of ferment, dried it slowly at a moderate temperature, reduced it to a fine powder and mixed it with a solution of rubian. After the mixture had stood for some days, the rubian was entirely decomposed. The filtered liquid left on evaporation a quantity of sugar, and the residual mass on the filter, which was interspersed with numerous small yellow crystals, was treated with boiling alcohol, which, after being filtered boiling hot and allowed to cool, deposited a large quantity of rubiafine in crystals. The alcoholic liquid, on being examined in the usual manner, was found to contain besides rubiafine, alizarine, rubiagine, rubiretine and verantine, the two latter being rather less in amount than usual.

After drying the ferment at a moderate temperature, then heating it for some hours in a water-bath, before adding it to a solution of rubian, no apparent change took place for some time, but gradually the rubian began to disappear, and after two months was entirely decomposed. The filtered liquid left on evaporation a quantity of sugar, and the residue on the filter, on being treated with alcohol, yielded a solution which contained rubiafine, and a pretty large quantity of rubiretine and verantine, but no alizarine.

To a solution of rubian I now added a sufficient quantity of ferment to effect its decomposition when used alone, and then a small quantity of sulphuric acid. The mixture remained apparently unchanged for some time. The liquid was filtered, the acid was neutralized with carbonate of lead, the liquid was then again filtered, and left on evaporation a yellowish-red, bitter, glutinous substance, which was apparently a mixture of rubian and sugar. The mass left on the filter was treated with boiling alcohol, which was then found to contain a trace of alizarine, a little rubiagine, and a large quantity of rubiretine and verantine.

The same experiment was performed, substituting for the sulphuric acid a minute quantity of carbonate of soda. This had the effect of completely dissolving the ferment, forming a solution to which the rubian imparted a red colour. After standing for some time, the rubian was entirely decomposed, for on neutralizing the soda with sulphuric acid, a yellow precipitate fell, while the liquid became colourless. The latter being filtered, the acid was neutralized with carbonate of lead, and the liquid being again filtered left on evaporation a quantity of sugar mixed with sulphate of soda. The mixture of ferment with the products of decomposition yielded to boiling alcohol a large amount of rubiafine, rather more than the average quantity of alizarine, and a moderate quantity of rubiretine and verantine, but no rubiagine. I repeated this experiment, using caustic soda instead of carbonate of soda; but having employed rather too large a quantity of alkali, I found that the action of the ferment was very much retarded, so much so that after standing for some days there was still a quantity of undecomposed rubian present; and among the products of decomposition formed there was little or no alizarine, rubiafine or rubiagine, but a large quantity of rubiretine and verantine. The addition of lime water produced a similar effect to that of carbonate of soda, though of not quite so marked a character.

Having mixed a solution of rubian with ferment in the usual manner, I added a small quantity of sugar of lead. The ferment, which had been previously diffused through the liquid, was immediately precipitated by the lead salt in brown flocks, leaving a clear yellow solution floating above it. After standing however for some days the colour of the solution became gradually paler, and at length almost disappeared. After filtration and evaporation it left a small quantity of sugar. The mass left on the filter was purple, and contained oxide of lead. It was treated with cold sulphuric acid, which turned it red. The acid liquid after filtration was neutralized with carbonate of lead, and again filtered, when it appeared colourless, and produced no precipitate with basic acetate of lead, a proof that it contained no rubian. The mass which had been treated with cold sulphuric acid, was then treated with boiling alcohol, which on being filtered and allowed to cool deposited some crystallized rubiafine, and was found to contain a trace of alizarine and a large quantity of rubirètine and verantine. It appears therefore that even sugar of lead, which produces an insoluble compound with the ferment, is incapable of arresting the action of the latter on rubian.

The same experiment was made, using corrosive sublimate instead of sugar of lead. The action of the ferment was in this case considerably retarded, for the yellow colour of the liquid did not entirely disappear, even after several days. The filtered liquid left on evaporation some corrosive sublimate with a little rubian, but no sugar. The mass on the filter was treated with boiling alcohol, which was then found to contain no alizarine, rubiafine or rubiagine, but a considerable quantity of rubiretine and verantine.

Arsenious acid had a very similar effect to sugar of lead and corrosive sublimate, that is to say, it retarded the action of the ferment, prevented the formation of alizarine, and promoted that of rubiretine and verantine.

The addition of a large quantity of alcohol to a mixture of rubian and ferment had likewise the effect of retarding, though not of altogether preventing the fermentation. A great part of the rubian remained undecomposed, while the remainder had given rise to the formation of a small quantity of alizarine and a large quantity of rubiretine and verantine.

A solution of rubian having been mixed with ferment, I poured the mixture into a bottle, and then added a quantity of oil of turpentine, after which the whole was well shaken up and left to stand. The decomposition of the rubian was much retarded, and took several days for its completion. The oil of turpentine floating on the surface was removed. The liquid underneath was colourless. After filtration and evaporation it left a quantity of sugar. The remaining products of decomposition were rubiafine, a trace of alizarine, and a large quantity of rubiretine and verantine, but no rubiagine.

From these experiments, which seem to me sufficiently decisive to require no further confirmation, I draw the following inferences:—

1. There exist no means, short of the complete destruction of the ferment, capable of arresting its action on rubian, except exposing it while in a moist state to the temperature of boiling water. Even when exposed to that temperature, after having been previously dried, its fermenting power is not entirely lost, but merely weakened.

2. By the addition of various substances, usually classed as antiseptic, such as sulphuric acid, arsenious acid, sugar of lead, corrosive sublimate, alcohol and oil of turpentine, during the process of fermentation the action of the ferment is not destroyed; it is merely retarded and modified.

3. The more the action of the ferment on rubian is retarded, the more rubiretine and verantine, and the less alizarine are formed, so much so that in some cases the

alizarine disappears entirely from among the products of decomposition, which then consist almost solely of rubiretine and verantine. The formation of rubiafine and rubiagine is promoted when the action of the ferment is moderately retarded, but diminishes again, or entirely ceases when the retardation is very great. Of the two, the rubiagine is the first to disappear when any retardation takes place.

4. By the addition of small quantities of alkalies during the process of fermentation, the action is, as regards its duration, if not promoted, at all events not retarded; and as regards the relative quantities of the various substances produced, the amount of alizarine is thereby decidedly increased, while that of the rubiretine and verantine is diminished.

These experiments also confirm the view derived from analytical results, that there is a mutual relation and interdependence on the one hand between rubiretine and verantine, and on the other between rubiafine and rubiagine and the sugar, whereas alizarine occupies an independent position; for rubiretine is never found among the products of decomposition by fermentation without an accompaniment of verantine, though both may be present to the exclusion of all other products; and the formation of sugar always indicates that of rubiafine or rubiagine, whereas all these substances may be present without a trace of alizarine being at the same time produced.

The deleterious effect resulting from the presence of acids in the dye-bath during the process of madder dyeing hereby finds an additional explanation. Not only do acids act injuriously by seizing hold of the mordants with which the colouring matter ought to combine, not only do they allow the constituents of the root which are injurious in the process to have full scope by depriving them of the alkalies or alkaline earths with which they would otherwise unite, but they also retard the peculiar process by which the alizarine is formed, and even lead to the formation of deleterious substances at the expense of the colouring matter itself.

In order to place in a more striking light the influence exerted respectively by acids and alkalies during the process of the fermentation of rubian, I will here give the results of an experiment to ascertain quantitatively the amount of the various products of decomposition formed on the addition of either during the process. For this purpose I took 5·59 grms. rubian, dissolved it in water, added to the solution a quantity of ferment, and mixed both together very well. I then divided the mixture into two equal parts, and added to one half a small quantity of carbonate of soda, to the other a little sulphuric acid. Both were allowed to ferment for a length of time, and the portion insoluble in water was treated in both cases with boiling alcohol. The alcohol deposited on being filtered boiling hot a quantity of rubiafine, which was collected on a filter and weighed. To the alcoholic liquid I added acetate of alumina. The precipitate thereby occasioned was separated by filtration, washed with alcohol and decomposed with boiling muriatic acid. The red flocks thus obtained were collected on a filter, washed, dried and weighed. They consisted of alizarine, verantine and rubiafine. After being weighed, they were treated with boiling dilute nitric acid, which destroyed the alizarine, leaving the two other substances behind in the shape of a yellow powder, which was filtered, washed, dried and weighed. By deducting its weight from the total weight of the three, the weight of the alizarine was ascertained. The liquid filtered from the alumina precipitate deposited on the addition of sulphuric acid and water a yellow powder, which was collected on a filter, washed, dried and weighed. It was then treated with a little cold alcohol. This left undissolved a quantity of rubiagine, which was again collected on a filter, dried and weighed. The alcohol left on evaporation a dark brown mass, the fatty nature of which showed that it contained rubiadipine in addition to rubiretine. Now I obtained from that half of the solution to which carbonate of soda had been added, the following quantities of these various substances:—0·323 grm. alizarine, 0·220 grm. rubiafine, 0·167 grm. of the mixture of rubiafine and verantine, 0·250 grm. rubiagine, and 0·170 grm. of the mixture of rubiretine and rubiadipine. From the other half, to which sulphuric acid had been added, I obtained the following quantities of the same bodies:—0·2030 grm. alizarine, 0·340 grm. rubiafine, 0·157 grm. of the mixture of rubiafine and verantine, 0·310 grm. rubiagine, and 0·230 grm. of the mixture of rubiretine and rubiadipine. In both cases a quantity of rubian remained undecomposed, but the quantity actually decomposed was capable of being approximately determined from the quantities of products of decomposition formed. Assuming the formula of rubiafine to be $C_{32} H_{13} O_9$, that of rubiagine $C_{32} H_{14} O_{10}$, assuming further that the mixture of rubiafine and verantine consisted entirely of the latter substance, by which no great error is committed (since the quantities of rubiafine and verantine which one atom of rubian is capable of yielding are to one another as 277 to 258), and leaving out of consideration the trifling quantity of rubiadipine formed (the rubiretine being given by the amount of verantine), then the quantities of the various substances obtained correspond respectively to the following quantities of rubian:—

		Rubian.		Rubian.
Alizarine .	0·323 corresponds to	0·407	0·203 corresponds to	0·255
Rubiafine .	0·220 corresponds to	0·484	0·340 corresponds to	0·748
Rubiagine .	0·250 corresponds to	0·533	0·310 corresponds to	0·661
Verantine .	0·167 corresponds to	0·394	0·157 corresponds to	0·371
		1·818		2·135

Hence it appears that though the amount of rubian which actually underwent decomposition was less when carbonate of soda was added than when sulphuric acid was employed in addition to the ferment, still the amount of the alizarine produced was greater in the former than the latter case, while the amount of almost all the other substances was proportionally less. In using carbonate of soda, the quantity of alizarine formed amounted to 17·7 per cent., in using sulphuric acid it amounted to 9·5 per cent. of the rubian decomposed. The quantity which should be formed, if the rubian were entirely converted into alizarine, is 79·3 per cent.

In the first part of this paper I have pointed out the possibility which exists in theory of converting rubian entirely into alizarine, to the exclusion of every other product of decomposition; and though I cannot say that I have approached much nearer to its practical realization, still if, as the experiments just described prove, the fermentative process whereby alizarine is formed is of so plastic a nature as to be capable of yielding under different circumstances, sometimes more, sometimes less, occasionally none, of the latter substance, then the prospect of our being able so to modify it as to lead to the formation of a much larger quantity of alizarine from a given quantity of rubian than is at present possible, becomes less distant.

The experiments which I shall now describe were made in order to ascertain the effect produced on rubian by other fermentative substances besides that contained in madder itself.

I first took a solution of rubian and mixed it well with a solution of albumen made from white of egg. The mixture was allowed to stand in a moderately warm place for two months, but though the albumen had entered into a state of putrefaction, the rubian was still for the most part undecomposed, as was evident from the taste and colour of the liquid. At the bottom of the vessel there was found a small quantity of a dark reddish-brown deposit, which was collected on a filter. On treating this deposit with boiling alcohol, the latter acquired a red colour, and left on evaporation a small quantity of a yellow bitter substance, soluble in water, which only differed from rubian in being somewhat viscid. The remainder of the dark brown deposit I treated with a mixture of muriatic acid and alcohol, when it immediately became almost white, while the substance which had coloured it dissolved in the alcohol with a yellow colour. The alcohol was filtered from the insoluble part, which was probably merely modified albumen, and evaporated to dryness, when it left a reddish-yellow, uncrystalline residue. This residue was impure alizarine, for it gave on being heated in a tube a crystalline sublimate, it dissolved in caustic alkali with a violet colour, and its alcoholic solution became purple on the addition of acetate of copper. In the reddish-brown deposit it was doubtless contained in combination with some base, perhaps lime. The greatest part of the rubian remained in solution mixed with the albumen. The solution was slowly evaporated to dryness, when it left a red transparent substance like gum or varnish, which was treated with boiling alcohol. The alcohol acquired a reddish-yellow colour, and left on evaporation a yellowish-red, deliquescent, bitter residue, consisting apparently of rubian mixed with some nitrogenous substance. It was entirely soluble in water with a reddish colour. This solution gave with acetate of lead a copious brown precipitate, the supernatant liquid being almost colourless, but with acids and alkalies it gave the usual reactions of a solution of rubian. The greater part of the gum-like substance was of course insoluble in alcohol. On being treated with a mixture of sulphuric acid and alcohol in the cold, an additional quantity of yellow liquid was obtained, which after filtration, neutralization with carbonate of lead and evaporation, left a reddish, deliquescent residue similar to the last. It appears, therefore, that the action of albumen on rubian is but slight.

In order to ascertain what effect caseine has on rubian, I mixed a solution of the latter with a quantity of milk, and allowed the mixture to stand in a warm place for several days, until the smell indicated that the caseine had entered into a state of putrefaction. The liquid, which was still yellow and bitter, was filtered from the caseine, which had separated in yellow flocculent masses, and basic acetate of lead was added, which produced a red precipitate. This precipitate, after filtering and washing, was decomposed with cold sulphuric acid; the excess of sulphuric acid was removed with carbonate of lead; and after passing sulphuretted hydrogen through the filtered liquid to precipitate a little lead contained in the solution, it was again filtered and evaporated, when it left a quantity of rubian possessing the usual properties of that substance, with the exception of its being somewhat viscid. The caseine on the filter after being washed with water was treated with boiling alcohol, which deprived it of its yellow colour, and rendered it white. The filtered liquid, which was yellow, gave no precipitate either with acetate of alumina or sugar of lead, proving the absence of alizarine and the usual products of decomposition of rubian. On evaporation it left a residue of rubian mixed with fat, the whole, with the exception of the fat, being soluble in water.

A similar negative result was obtained with gelatine and yeast. A solution of rubian being mixed with a solution of glue, the mixture after standing for several weeks began to emit a putrid smell, and acquired a reddish tinge, but remained bitter, and deposited nothing insoluble. On adding yeast to a solution of rubian, and allowing to stand until putrefaction commenced, no apparent change took place in the solution, which retained its yellow colour and bitter taste.

The action of emulsine on rubian was very different. The emulsine employed was prepared in the usual manner, by treating sweet almonds, after being pounded and pressed between paper to deprive them of their oil, with water, separating the milky liquid from the residue by straining, allowing it to stand until the fat had collected on the surface, and then precipitating the emulsine from the clear liquid underneath, by means of alcohol. Some emulsine prepared in this manner was placed in a solution of rubian, and well mixed up with it. The mixture was allowed to stand in a warm place for some days. By degrees the liquid became almost colourless, while the emulsine acquired a yellow colour. It was filtered and evaporated, when it left a quantity of emulsine. The yellow mass on the filter was treated with boiling alcohol, which became yellow, while the residue lost almost the whole of its colour. The alcoholic liquid was filtered and spontaneously evaporated, when it left a yellowish-red mass, among which crystals of alizarine were discernible. This mass was treated with cold water, in which a part dissolved with a yellow colour. The liquid after filtration and evaporation left a quantity of unchanged rubian. The part insoluble in water was dissolved again in alcohol, and to the solution was added acetate of alumina, which produced a red precipitate. This precipitate, after filtration and washing with alcohol, was decomposed with muriatic acid, and the yellow flocks left by the acid were dissolved in alcohol, on the spontaneous evaporation of which there was left a mass of pure alizarine in well-defined yellow crystals. The liquid filtered from the alumina precipitate was evaporated to dryness, and the residue left was treated with water and muriatic acid, when a brown mass was obtained, which after filtering and washing was treated with cold alcohol. The alcohol left undissolved a brown powder, consisting probably of verantine, and after filtration and evaporation left a brown resinous substance, easily fusible when thrown into boiling water, and consisting doubtless of rubiretine. The quantity of alizarine formed seemed to be much larger in proportion to the rubian employed, and to the quantity of other substances formed, than in the case of the ferment of madder. Nevertheless, I must state, that on repeating the experiment I was unable to attain the same result, the alizarine being formed in much smaller proportion and giving place to other products of decomposition.

Lastly, I resolved to examine what effect, if any, would be produced on rubian by seme fermentative substance derived like that of madder from a vegetable root. For this purpose I chose the albuminous substance described by BRACONNOT as existing in the root of the *Helianthus tuberosus**. I selected this substance, because it has, like the ferment of madder, the property of causing a solution of sugar to undergo an acetous fermentation. It was prepared in the following manner. The tubers were cut into slices, then pounded in a mortar with a little water until the lumps had disappeared. The mass was then placed on a piece of calico, and the liquor expressed with the hands. The muddy liquid which ran through was then mixed with a considerable quantity of alcohol. This produced a gray precipitate, which was collected on a filter and well washed with cold alcohol. Some of this precipitate was then added to a solution of rubian, and the mixture was allowed to stand in a moderately warm place until it began to disengage a putrid smell. The liquid was then filtered, and evaporated to dryness. The residue was treated with boiling alcohol, which dissolved a part with a yellow colour, leaving undissolved a quantity of brown flocks, consisting probably of some of the ferment. The alcohol on evaporation left a residue, consisting of rubian with its usual appearance and properties, and apparently free from sugar. The substance left on the filter had acquired a brownish-yellow colour, which was not removable by washing with water. On being treated with boiling alcohol, it lost its yellow colour, which was now transferred to the alcohol. The alcohol after filtration and evaporation left a dark brown substance in the shape of a pellicle, which yielded nothing to boiling water. On being treated with cold alcohol, a brown powder resembling verantine was left undissolved, while the filtered liquid gave on evaporation a residue of a resinous nature, which melted when thrown into boiling water and became brittle again when cold, and consisted doubtless of rubire-

* Ann. de Chem. et de Phys. xxv. 358.

tine. The effect produced by this substance on rubian, therefore, though it does not equal in energy that of the madder ferment itself, or even of emulsine, exceeds that of albumen or caseine. As regards the substances produced by it, its action resembles that of madder ferment when retarded by the addition of antiseptic substances, and it confirms the law which I have deduced from previous experiments, viz. that the more slowly rubian is decomposed, the more rubiretine and verantine are produced.

It appears from these experiments, that none of the common and well-known fermentative substances, with the exception of emulsine, are capable of effecting in any considerable degree the decomposition of rubian, and that none of them, with that single exception, can be employed as a substitute for the ferment contained in madder itself, which produces an effect on rubian altogether *sui generis*. This circumstance alone, apart from all other considerations, would entitle the ferment of madder to be considered as an entirely distinct and peculiar substance, on which it will therefore be necessary to bestow a distinct name. For this purpose I venture to suggest the name of *Erythrozym* (from ἐρυθρὸς *red*, and ζύμη *ferment*) as most appropriately indicating its chief characteristic, and I shall now proceed to give a short account of its properties and composition.

When prepared in the manner above described by precipitation with alcohol, erythrozym is obtained as a chocolate-coloured granular mass. When dried it coheres into hard lumps, which are almost black, and are with difficulty reduced to powder. When the dry substance is heated on platinum foil it emits a smell somewhere between that of burning peat and burning horn, and then burns without much flame, leaving a considerable quantity of residue, which, on being further heated, is soon converted into a grayish-white ash, consisting almost entirely of carbonate of lime. If erythrozym be well mixed while in a moist state with water, a reddish-brown muddy liquid is formed, having all the appearance of a solution. It is, however, no solution; the erythrozym is merely suspended in the liquid, for on filtering through paper a clear liquid passes through, while a mucilaginous substance remains on the filter. The latter, on being mixed with a solution of rubian, exerts the usual decomposing effect on that substance, while the liquid, when tried in the same way, is found to be entirely without effect. Hence it follows that erythrozym, after having once been precipitated from its watery solution, even by alcohol, cannot again be dissolved in water. The liquid obtained by treating erythrozym with water and filtering, contains a small quantity of a substance, which, from its reactions, I conclude to be pectic acid, or some body nearly allied to it. In fact, the method of preparing erythrozym implies that all substances contained in the watery extract of madder, insoluble in alcohol, must be found mixed with it; but since the erythrozym itself by precipitation with alcohol becomes insoluble in water, these substances may afterwards be easily removed by treating with water. If the watery liquid in which the erythrozym is contained in a state of suspension be boiled, a sort of coagulation takes place, and the erythrozym separates in the shape of dirty red flocks, while the liquid retains a reddish colour. The same effect is produced by adding alcohol or salts, such as common salt or sal-ammoniac, the substance separating in dark reddish-brown flocks, with a clear yellowish liquid floating above them. That erythrozym is not an uncombined substance, but a compound of an organic substance with lime, is proved by its behaviour towards acids. If it be treated with any acid, even acetic acid, its colour changes from reddish-brown to yellowish-brown, and the filtered liquid is found to contain a considerable quantity of lime. The yellowish-brown flocks left on the filter, after all the excess of acid has been removed, do not again form with water a mucilaginous liquid like the original substance; and even an addition of lime water, though it restores the original chocolate colour, does not reproduce that peculiar condition of suspensibility in water characteristic of it in its original state. The brown substance into which erythrozym is changed by the action of acids is soluble in caustic alkalies, forming pale purple turbid liquids. The ammoniacal solution gives reddish-brown flocculent precipitates with most earthy and metallic salts. When the solution in caustic soda is boiled, a disengagement of ammonia takes place. If erythrozym be treated with boiling nitric acid, it is dissolved and decomposed with a disengagement of nitrous acid. A small quantity of a white flocculent substance remains behind. Concentrated sulphuric acid chars it on heating.

If erythrozym be mixed with water, and the mixture be allowed to stand for a length of time in a warm place, signs of a more active process of fermentation begin to show themselves, especially in summer weather; bubbles of gas are given off, and a peculiar smell is emitted, which, though disagreeable, cannot exactly be called putrid. During this process, which is evidently one of putrefaction in the stricter sense, the erythrozym loses its sliminess, and is converted into a red flocculent mass, which may easily be separated by filtration from the liquid. The latter is clear, colourless and quite neutral. After erythrozym has passed through this second stage of decomposition, its power of decomposing rubian is found to have lost much of its intensity. It is during the first period of its decomposition, when no apparent change is taking place, that this power is most energetically exerted. During the second, or more strictly putrefactive stage, it acquires, however, the property of decomposing sugar. If erythrozym be mixed with a solution of cane sugar, and the mixture be allowed to stand for a considerable time until gas begins to be disengaged, the solution acquires by degrees a decided acid reaction. What the acid is which is thereby formed, I have not yet ascertained.

The erythrozym which I submitted to analysis, was prepared by precipitating it from an extract of madder with tepid water, by means of alcohol, collecting it on a filter, then treating it repeatedly with boiling alcohol until all matter soluble therein was removed, and then washing it on a filter with cold water until the percolating liquid no longer gave a precipitate with sugar of lead, after which it was rapidly dried in the water-bath. It follows from the analyses which I have made, that the amount of lime which it contains is tolerably constant, and that it must consequently be considered as a definite compound of an organic substance with lime.

I. 0·6220 grm., dried at 100° C. and burnt with oxide of copper and chlorate of potash, gave 0·8670 carbonic acid and 0·2365 water.

0·7770 grm. gave 0·4035 chloride of platinum and ammonium.

0·9630 grm. gave on being incinerated 0·2350 carbonate of lime.

II. 0·4885 grm. of another preparation gave 0·6865 carbonic acid and 0·2060 water.

0·1540 grm. gave on being incinerated 0·0370 carbonate of lime, containing 0·02076 lime =13·48 per cent. After being treated with sulphuric acid and again heated, it weighed 0·0500 grm., which, estimated as sulphate of lime, is equivalent to 0·02062 lime =13·39 per cent.

From these numbers may be deduced the formula $C_{56}\,H_{34}\,N_2\,O_{40}+4CaO$, as the following calculation shows:—

	Eqs.		Calculated.	I.	II.
Carbon	56	336	40·48	40·93	41·20
Hydrogen . . .	34	34	4·09	4·22	4·68
Nitrogen . . .	2	28	3·37	3·26	
Oxygen	40	320	38·57	37·90	
Lime	4	112	13·49	13·69	13·48
		830	100·00	100·00	

If this be the true composition of erythrozym, it stands in a very interesting relation to that of rubian. If to 1 equiv. of rubian be added the elements of 2 equivs. of nitric acid, the sum will represent the composition of the organic substance contained in the erythrozym in combination with lime, $C_{56}\,H_{34}\,O_{30}+2NO_5=C_{56}\,H_{34}\,N_2O_{40}$.

I shall now give the results of an analysis of erythrozym which had been employed for the decomposition of a quantity of rubian, and then treated successively with cold water and boiling alcohol until all the products of decomposition were entirely removed.

0·5330 grm., dried at 100° C. and burnt with oxide of copper and chlorate of potash, gave 0·8325 carbonic acid and 0·2220 water.

0·2780 grm. gave 0·1835 chloride of platinum and ammonium.

0·0645 grm. gave on being incinerated 0·0130 carbonate of lime, containing 0·00729 lime.

These numbers lead to the following composition:—

	Eqs.		Calculated.	Found.
Carbon	52	312	44·82	44·99
Hydrogen . . .	32	32	4·59	4·62
Nitrogen . . .	2	28	4·02	4·14
Oxygen	30	240	34·51	34·95
Lime	3	84	12·06	11·30
		696	100·00	100·00

If this composition be compared with that of the substance in its undecomposed state, the exact nature of the change which it undergoes during the process of fermentation will be apparent. It appears that the change consisted in this case in the loss of 4 atoms of carbonic acid, 2 atoms of water and 1 of lime, for

$$C_{52}\,H_{32}\,N_2\,O_{30}+3CaO+4CO_2+2HO+CaO=C_{56}\,H_{34}\,N_2\,O_{40}+4CaO.$$

It appears also that no loss had been sustained in the nitrogen. The atom of lime lost was probably abstracted by the carbonic acid disengaged at the same time.

In order to obtain, if possible, the organic substance, which is contained in the erythrozym in combination with lime, and to which the name of erythrozym ought perhaps to be restricted, in a free state, I prepared an extract of madder with tepid water, and added to it tartaric acid. A brown precipitate was produced, which was collected on a paper filter, washed with water in order to remove the excess of acid, then treated with boiling alcohol, until everything soluble in that menstruum was removed, and lastly, washed with cold water, in order to separate any pectic acid which might be present; after which it was dried. Its analysis showed, however, that the acid had removed only half of the lime, and that the substance itself must, during the process of purification, have undergone decomposition.

0·5695 grm., burnt with oxide of copper and chlorate of potash, gave 0·9585 carbonic acid and 0·2600 water.

0·5940 grm. gave on being incinerated 0·0900 carbonate of lime, containing 0·0505 lime =8·50 per cent.

0·2835 grm. gave on being incinerated, and then treated with sulphuric acid, 0·0560 sulphate of lime, containing 0·02310 lime =8·14 per cent. The average per-centage of lime was therefore 8·32.

The nitrogen was not determined; but presuming no loss to have taken place in that constituent, then these numbers correspond to the following composition:—

	Eqs.		Calculated.	Found.
Carbon	52	312	48·00	47·68
Hydrogen . . .	30	30	4·61	5·07
Nitrogen . . .	2	28	4·30	
Oxygen	28	224	34·48	
Lime	2	56	8·61	8·32
		650	100·00	

It appears, therefore, that the decomposition of the substance had advanced in this case still further even than in the last, as it contained 2 equivs. of water less.

The last specimen I examined was one which was prepared in the usual manner by precipitation with alcohol, then mixed with water and allowed to decompose until bubbles of gas began to be disengaged, and the smell peculiar to this substance when in a state of putrefaction began to be emitted, then treated with boiling alcohol until nothing more was dissolved, and dried.

0·4630 grm., obtained in this manner and burnt with oxide of copper and chlorate of potash, gave 0·7260 carbonic acid and 0·1760 water.

0·5270 grm. gave 0·2705 chloride of platinum and ammonium.

0·2990 grm. gave on being incinerated 0·0720 carbonate of lime, containing 0·04040 lime =13·51 per cent. After treatment with sulphuric acid, this yielded 0·0980 sulphate of lime, containing 0·04042 lime =13·52 per cent.

In 100 parts it therefore consisted of—

Carbon	45·65
Hydrogen	4·22
Nitrogen	3·22
Oxygen	33·40
Lime	13·51

The only formula with which this composition agrees, and which is at the same time in accordance with the preceding formulæ, is $C_{52}\,H_{28\frac{1}{2}}\,N_{1\frac{1}{2}}\,O_{28}+3CaO$, which requires in 100 parts—

Carbon	46·60
Hydrogen	4·25
Nitrogen	3·13
Oxygen	33·48
Lime	12·54

From this formula it may be inferred, that during the further progress of decomposition erythrozym loses ammonia in addition to carbonic acid and water, for

$$C_{52}\,H_{28\frac{1}{2}}\,N_{1\frac{1}{2}}\,O_{28}+4CO_2+4HO+\tfrac{1}{2}NH_3=C_{56}\,H_{34}N_2\,O_{40}.$$

I shall conclude this part of my paper with a few remarks on the *vexata quæstio* of the existence or non-existence of more than one colouring matter in the madder root. ROBIQUET, the discoverer of alizarine, attributed the tinctorial power of madder partly to that substance and partly to another, on which he bestowed the name of purpurine. GAULTIER DE CLAUBRY and PERSOZ were of opinion that there are two colouring matters in madder. RUNGE has described three substances obtained by him from madder, all of which he considers as colouring matters playing a part in the process of madder dyeing. SCHIEL, DEBUS and WOLFF and STRECKER all assume the existence of two distinct colouring matters. Notwithstanding, however, the array of authority in favour of this view, I have never been able to convince myself that the entire tinctorial power of madder was not due to alizarine alone, and that consequently all substances derived from madder, if endowed with any such power, owe it to their containing alizarine; and after having isolated and examined the primitive substance, from which all the bodies in madder endowed with a red or yellow colour, or capable of producing these colours, are derived, and after having exposed this substance to the influence of all the reagents with which madder itself under ordinary circumstances is brought into contact, I find myself entirely confirmed in my opinion. I should indeed think it unnecessary to add anything confirmatory of the conclusions which must suggest themselves at once from the perusal of the preceding pages, but as Messrs. Wolff and Strecker have made it one of the special objects of their investigation to prove the existence of more than one colouring matter in madder, I think it may not be out of place here to add a few remarks to show how, in my opinion, these chemists, as well as their predecessors, have been misled, and also to give an account of some experiments still further confirmatory of the opinion which I have always held.

The second colouring matter, which, according to the chemists just mentioned, exists in madder in addition to alizarine, and which has received at various times the names of purpurine, madder-purple, and oxylizaric acid, possesses, according to those observers, the property of being easily soluble with a red colour in boiling alum liquor. Wolff and Strecker assert that this substance is peculiarly a product of fermentation, and in order to prepare it, they mix madder with water and yeast, allow the mixture to stand in a warm place until the effervescence produced by fermentation has ceased, and the liquid has acquired a strong acid reaction and contains alcohol, after which they strain the liquid through a cloth, wash the mass on the cloth with water, and then treat it with boiling alum liquor. From the bright red solution a substance separates on cooling in red flocks, which, as well as the orange-coloured flocks produced by adding sulphuric acid to the liquid, consist, according to them, entirely of purpurine without any trace of alizarine. They purify it by crystallization from alcohol. Now I have shown above that the fermentation of madder, which is in fact synonymous with the fermentation of rubian, is due to the action of a peculiar substance, which I have called erythrozym, on rubian; that the action of this substance is very rapid; that it is not accompanied by any disengagement of gas; that it is terminated long before any effervescence or any acid reaction of the liquid begins to appear; that the products of the action do not differ essentially from those due to the action of acids and alkalies; that the formation of alizarine in about the same proportion as when acids or alkalies are employed is one of the results of the process, and that yeast exerts no decomposing power on rubian whatever. Hence it necessarily follows, that if the purpurine of Wolff and Strecker be not found as such among the products of the fermentation of rubian, it must consist of a mixture of two or more of those products.

Now I have mentioned in the first part of this paper, that though alizarine and verantine are both perfectly insoluble in boiling alum liquor when acted on separately, yet that when a mixture of both is employed, the mixture is found to be soluble in alum liquor with the colour characteristic of purpurine. Hence I concluded that purpurine is in fact a mixture of those two substances, a view with which all that is mentioned regarding purpurine completely coincides. I may mention incidentally, that in making this experiment it is necessary to treat the verantine with a little dilute nitric acid, in order to destroy the alizarine which usually accompanies it, and then to remove the acid by washing with water before employing it, and that it generally succeeds best when a large excess of alizarine is used.

To this synthetical proof of the opinion here advocated, I will now add a few analytical ones. I will show, in the first place, that purpurine prepared in the manner mentioned by Wolff and Strecker is a substance of very variable composition, but that the variations in its composition may be easily explained by supposing it to consist of alizarine and verantine in different proportions; and secondly, that by treatment with nitric acid purpurine yields unchanged verantine and an acid, which is identical with that formed by the action of nitric acid on alizarine.

In the course of my investigation I obtained at the termination of the process for separating the products of the fermentation of rubian, an alcoholic liquid, from which the verantine had been deposited, but still containing a substance, which, from its solubility in alum liquor, would by most chemists be called purpurine. This substance was precipitated from the solution with water. Its colour was brownish-yellow. It was treated with boiling alum liquor, to which it communicated a bright red colour. The liquid was filtered boiling hot, and deposited on cooling a quantity of red flocks. The residue was treated with fresh quantities of alum liquor, until on cooling very few flocks separated. A great proportion of the substance employed remained undissolved. The flocks deposited from the alum liquor were collected on a filter, and washed with water in order to remove all the alum. After drying they formed a dark reddish-brown powder, which was almost entirely soluble in alcohol. The alcoholic solution left on evaporation a bright red mass, in which no trace of anything crystalline was discernible. Its analysis gave the following results:—

0·4820 grm. gave 1·1240 carbonic acid and 0·1800 water.

0·8060 grm. left on being incinerated 0·0310 alumina=3·84 per cent.

After making the proper correction for the alumina, the quantity of which, in relation to that of the other constituents, seems to be indefinite, these numbers correspond in 100 parts to—

Carbon	66·13
Hydrogen	4·31
Oxygen	29·56

This composition does not differ very widely from that given for oxylizaric acid by Debus.

Now the formula $C_{56}H_{20}O_{19}=C_{14}H_5O_4+3C_{14}H_5O_5$ requires in 100 parts—

Carbon	66·14
Hydrogen	3·93
Oxygen	29·93

A quantity of material similar to the last, obtained on a different occasion, was treated in the same way with boiling alum liquor, and the deposit formed on the liquor cooling was submitted to analysis. It had a much lighter colour than the preceding, and when dissolved in alcohol, the latter left on evaporation crystals, apparently of alizarine, mingled with red crystalline masses resembling impure alizarine.

0·4000 grm. gave 0·9870 carbonic acid and 0·1390 water.

0·3840 grm. left on being incinerated 0·0060 alumina=1·56 per cent.

After making the necessary correction for the alumina, these numbers correspond in 100 parts to—

Carbon	68·36
Hydrogen	3·92
Oxygen	27·72

The formula $C_{56}H_{20}O_{17}=3C_{14}H_5O_4+C_{14}H_5O_5$ requires in 100 parts—

Carbon	68·29
Hydrogen	4·06
Oxygen	27·65

These two specimens therefore of a substance prepared in the same way, both of which would, according to the definition of WOLFF and STRECKER, pass for purpurine, possessed a composition, which in the one case corresponded to a mixture of 1 equiv. of alizarine and 3 equivs. of verantine, in the other case to a mixture of 3 equivs. of alizarine and 1 equiv. of verantine. A still more manifest proof of the fact of purpurine not being a substance of uniform composition, is derived from an examination of the liquid from which these specimens were deposited. To the bright red liquid from which the last was deposited, I added after filtration muriatic acid and boiled. A yellow precipitate was produced, which was collected on a filter and washed. A small quantity of it being dissolved in alcohol, the alcohol left on evaporation crystals of apparently pure alizarine. Its analysis also showed that it consisted of alizarine almost in a state of purity.

0·2390 grm. gave 0·6040 carbonic acid and 0·0980 water.

In 100 parts it contained therefore—

		Alizarine.
Carbon	68·92	69·42
Hydrogen	4·55	4·13
Oxygen	26·53	26·45

I now prepared some so-called purpurine from garancine. The garancine was treated with boiling alum liquor, and the liquor was strained boiling hot through calico. On cooling there was formed a copious deposit, which was redissolved in fresh alum liquor. The deposit formed this time, which was very trifling, was separated, and the substance contained in the liquid was precipitated with muriatic acid. The precipitate was yellow. Its analysis proved it to be almost pure verantine.

0·3685 grm. gave 0·8760 carbonic acid and 0·1400 water.

In 100 parts it contained therefore—

		Verantine.
Carbon	64·83	65·11
Hydrogen . . .	4·22	3·87
Oxygen	30·95	31·02

It appears therefore that the substance called purpurine cannot even be called a compound of alizarine and verantine, for it consists sometimes of one alone, sometimes of the other, sometimes of a variable mixture of both.

I now treated some purpurine, made in the same manner as the last, with boiling dilute nitric acid. Nitrous acid was disengaged, the bulk of the substance diminished very much, and its colour became lighter. After the action was completed, I allowed the liquid to cool, added water, collected the yellow flocks on a filter, and washed them with water to remove all the acid. After drying their colour was brownish-yellow, similar to that of pure verantine.

0·3820 grm. gave 0·9070 carbonic acid and 0·1240 water.

In 100 parts—

Carbon	64·72
Hydrogen	3·60
Oxygen	31·68

Its composition therefore, with the exception of a slight deficiency in the amount of hydrogen, was that of verantine.

The acid liquid filtered from the substance was evaporated almost to dryness, when it yielded a quantity of yellow crystals. These were washed with cold water, and then redissolved in a little boiling water. The boiling solution was decolorized with animal charcoal, and after being filtered boiling hot deposited on cooling a quantity of colourless crystals, having the appearance, and as their analysis showed, the composition of Laurent's naphthalic acid.

0·5230 grm. gave 1·1020 carbonic acid and 0·1770 water.

In 100 parts—

		Napthalic acid.
Carbon	57·46	57·83
Hydrogen . . .	3·76	3·61
Oxygen	38·78	38·56

Wolff and Strecker mention, that purpurine yields this acid when subjected to the action of nitric acid, but the verantine which is found in an undecomposed state, after the action of the nitric acid has ceased, seems to have eluded their observation.

From these experiments I infer that purpurine, madder-purple, and the various similar bodies derived from madder, owe their property as colouring matters to an admixture of alizarine, and that they simply consist of the latter substance in a state of impurity.

IV. *Observations on the Structure and Development of Bone. By* JOHN TOMES, *F.R.S., Surgeon-Dentist to the Middlesex Hospital, and* CAMPBELL DE MORGAN, *Surgeon to the Middlesex Hospital.*

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THE structure and development of bone is a subject which has occupied the attention of physiologists both of present and past times, and holds a prominent place in their writings. From the time of CLOPTON HAVERS, who was the first to point out the vascular canals in bone, and whose name they have since borne, to the year 1850, when M. KÖLLIKER's elaborate work on Structural Anatomy appeared, physiological writers have dwelt at considerable length on this branch of their subject. All have, however, concluded with the admission that more remained to be learned before this important structure could be regarded as fully understood*.

It is the purpose of the authors in this communication, to lay before the Royal Society the results of an extended series of observations on the structure and development of bone. Such of these results as are novel might, perhaps, be described without reference to those points which are already well known. It will, however, be seen, in the subsequent pages, that they are in themselves, and in their relations to osseous tissue and its development, as well as to structural anatomy generally, of such a nature, as would hardly admit of an intelligible description apart from some consideration of the whole subject. Hence it will be necessary to point out the relations between that which is already known, and that which it is the special purpose of this paper to communicate, and it is proposed to consider the details of the subject as they present themselves in the progress of investigation with but little reference to authorities, adding in the form of notes, those statements which require acknowledgement, either as agreeing with, or as being opposed to, the views advanced in this paper.

In treating of the structure of bone, it is convenient to commence by describing the appearances which a thin transverse section from the shaft of an adult long bone presents when placed in the field of a microscope having a magnifying power of from 200 to 300 linear. In addition to the large central space which corresponds to the medullary cavity of the bone, the section is seen to be perforated by numerous small apertures, each of which is surrounded by a series of laminæ of osseous tissue concentrically arranged. Lying amongst the laminæ are many small cavities, from which minute tubes radiate. These were called bone-corpuscles by PURKINJE, who

* Mikroskopische Anatomie, von Dr. A. KÖLLIKER. Leipzig, 1850.

first described them; but latterly, with a more correct appreciation of their nature, the name of lacunæ has been substituted at the suggestion of Dr. TODD and Mr. BOWMAN*, who have also described the Haversian canals, their surrounding laminæ, and their lacunæ, under the inclusive term of Haversian systems. The whole substance of the bone is made up of Haversian systems, connected by irregularly-shaped patches of short laminæ which occupy the spaces between the systems (Plate VI. fig. 1 *b*). These have received the name of interstitial laminæ from KÖLLIKER. In addition to the Haversian and interstitial laminæ, we have in most instances others that surround, or partly surround the whole, and form the outer surface of the bone. These, from the position they occupy, are called the circumferential laminæ. A similar series more or less perfect is found to form the wall of the medullary cavity.

If the section be taken from near the end instead of from the middle of the shaft of a long bone, the Haversian canals will appear much larger, and in the place of one great central opening we shall see numerous lesser ones, corresponding to the cancellous part of the bone. From whatever bone, or part of a bone, the section be taken, the arrangement of laminæ and lacunæ is substantially the same; the differences between the compact and cancellous portions consisting in variations in the relative quantities of the bone tissue and the size and number of the canals, and not in any real difference in the structure.

The elongated bones of some of the vertebrata, when cut transversely through their centres, are seen to be composed, at that part, of a single Haversian system. The rib of a Boa may be cited as an example.

Having drawn attention to the several parts which are recognised as entering into the composition of bone, the authors will now proceed to describe the results of their investigations under the subjoined heads:—

1st. The Haversian canals and other spaces.
2nd. The laminæ.
3rd. The lacunæ and canaliculi.
4th. The Haversian systems.
5th. Ossified cartilage of joints.
6th. Ossified cells.
7th. Bone tissue.
8th. Development of bone.
9th. Growth of bone.

Haversian and other Canals of Bone.—The Haversian canals have been described by all recent writers on the subject of bone; their variable size and not unfrequent anastomoses have been pointed out; and the fact of the larger apertures in the spongy parts of bone being similar in the structure of their parietes to the smaller ones in the denser parts of the osseous tissue has not failed to be recognized. It may be remarked, that whether the canals are large or small, their section presents an

* The Physiological Anatomy and Physiology of Man, by Dr. TODD and Mr. BOWMAN, vol. i. p. 109.

oval or circular outline, and that the walls of such canals have a tolerably uniform surface. When examining sections of bone it is necessary to bear this fact in mind, for on a close inspection it will be seen that there are other spaces which possess a different character, although up to the present time, so far as the authors know, this difference has not been recognized.

These spaces are irregular in shape, and have an irregular, festooned and often jagged outline, similar to that found on the surface of bone which has been removed by exfoliation, similar also to the surface left upon the fang of a tooth after a part has been absorbed. Such spaces correspond in shape to the peripheral outline of one or more of the Haversian systems, and are in fact subservient to the development of those systems, and necessarily precede their formation (Plate VI. figs. 2 *a*, 3 *a*, 4 *a*). It is proposed to call these spaces which are produced by absorption, *Haversian spaces*, as this name is indicative of their ultimate connection with the Haversian systems. In the Haversian canals we have surrounding laminæ, but in the spaces the boundaries are formed by portions of several systems, each of which has been in part removed in the formation of the spaces, as shown in figures 2 and 3, and Plate V., which, on being compared with figure 1, will illustrate the difference between the spaces and canals; allowance must, however, be made for the difference of size, as in figure 1 the parts are less magnified than in the other illustrations. In newly-formed bone situated near ossifying cartilage, the Haversian spaces are exceedingly numerous and large, frequently affording room for the development of two or more systems (fig. 5, Plate VI. fig. 24, Plate VIII.), while in older bone they are far less numerous, and generally less in size. In no case, however, are they wholly absent. They are found even in the bones of old subjects, and in connection with them newly developed systems.

This is a very important fact, as it demonstrates that the old tissue is removed in masses, and a new one developed in its place. It has long been taught that the older particles of an internal tissue are removed by absorption and new ones substituted, and this throughout the life of the individual. But the authors believe that they have demonstrated it for the first time.

On examining with care a series of sections, the Haversian spaces will be found in various conditions. In one place the space will have attained a large size, while in another part of the same section its commencement will be seen extending from one side of an Haversian canal. In a third instance, its occupation will have commenced, and its festooned margin will be lined by the peripheral lamina of an Haversian system. Then, again, examples may be found where one side of a space is becoming the seat of a new system, while the opposite one is undergoing further enlargement. In no case is there any difficulty in distinguishing between an enlarging Haversian space and a developing Haversian system. The authors have preparations which show that these conditions of absorption and reproduction take place in old as well as in young or middle-aged subjects; but it would appear that the frequency diminishes with the increasing age of the individual. It may however be observed in the bones of those who have passed the sixtieth year. The more recently formed systems differ from those of older date in the somewhat larger size of their lacunæ and canaliculi, and in the greater abundance of the latter, as shown in fig. 6, Plate VI.

Laminæ of Bone.—Lamination is a tolerably constant character of mammalian bone, although the degree of its distinctness is subject to considerable variation. In the bones of young subjects it is less pronounced than in those of mature animals (fig. 4, Plate VI.); it is also much less observable in the cancellous than in the compact tissue. When lamination reaches its highest degree of development, each lamina is seen to be made up of two parts, an outer, which is highly granular, composed oftentimes of a single line of large granules or cells*, as shown in Plate VI. fig. 8; and an inner portion, which is singularly clear and transparent, and to all appearance without granulation or any recognizable structure.

Kölliker has described and figured a somewhat similar appearance from a specimen treated with turpentine, but he does not seem to regard it as very readily discoverable in ordinary preparations†; there is, however, no difficulty in exhibiting this arrangement of structure in well-made sections, either simply polished or (which is better) put up in hard Canada balsam.

Very favourable views of lamination may be seen in transverse sections from the shafts of long bones; some of the most striking illustrations in the possession of the authors were obtained from the tibia of a subject the whole of whose bones were slightly hypertrophied, and from the femur of a limb which had from infancy been atrophied and useless.

Strongly marked laminæ, with their transparent and granular portions, are however not always met with; yet in their absence the observer will seldom fail to find laminæ of more uniform granularity surrounding the Haversian canals in well-marked concentric rings.

On examining the Haversian systems, it will be seen that the peripheral lamina has its internal surface marked by an even line, while the external is subject to slight bulgings, which correspond with and accurately fit into the indentations that form the margin of a pre-existing Haversian space (Plate VI. fig. 1). Hence the outer lamina of a system will present on its external surface a certain number of alternating dilatations and contractions; a peculiarity possessed by none of the more internal of the series. We may, however, find the whole of the laminæ of a system presenting different degrees of thickness on the opposite sides of a canal, from the latter being a little eccentric in its position. But when the eccentricity is considerable, as in the case illustrated in Plate VI. fig. 7, the laminæ are not all continued round the canal‡. Hence,

* The nature of these bodies will be more fully considered under the head of development and growth of bone.

† Kölliker, *oper. cit.* page 284. In 'The Microscopic Anatomy of the Human Body,' by A. H. Hassall, this appearance is indistinctly figured, but no description of it can be found in the letterpress.

‡ Kölliker, *oper. cit.* at page 283, states that in the eccentric systems the whole of the laminæ generally

in such a system, there are more laminæ on one side than on the other, and those on the side occupied by the greatest number are thicker in their centres than at their extremities, where they become gradually thinner, and are ultimately lost. The inner lamina, whether the systems are eccentric or not, forms, when the development is completed, a perfect ring, and not unfrequently presents a second peculiarity. The tissue of which it is composed is transparent, and affords, with our present means of investigation, little evidence of structure (Plate VI. figs. 6 and 7). It is transparent and glass-like, resembling that tissue which lines the Haversian canals in the antlers of the Cervidæ before they are cast (Plate VII. fig. 9).

The interstitial laminæ are abundant, and strongly marked in transverse sections of long bones, or in any sections of bone in which the Haversian systems are cut transversely. The laminæ vary both as to their number and length, and are usually more or less curved. They are arranged throughout the bone in well-defined isolated groups, in each of which the constituent laminæ are parallel (Plate VI. fig. 1 *b*). Professor Kölliker*, when treating of this subject, observes that it is difficult to account for the presence of the interstitial laminæ, and suggests that the appearance may perhaps be produced by the Haversian systems being cut obliquely, but at the same time admits that this explanation is not satisfactory. M. Kölliker had not recognized the presence of the Haversian spaces and the manner of their production and subsequent occupation, otherwise he would have seen that the interstitial laminæ are the remaining parts of Haversian systems, the larger portions of which had been removed by absorption. But that the term *interstitial laminæ* is already in use and has a definite meaning, the authors would have proposed the substitution of *partial Haversian systems*, as being more descriptive of these parts.

In addition to the laminæ of the perfect and imperfect Haversian systems, others that surround several linearly arranged systems are not uncommon, producing the appearance illustrated in Plate VI. fig. 5. In these instances a large oval system of laminæ has become occupied by several lesser ones. The occurrence of one system within another is by no means rare. We may then see the characteristic emarginated outline of the older system, and within this a second similar outline marking the periphery of the younger system, as shown in Plate VI. fig. 6, in which the lamination of each is delineated†.

The circumferential laminæ are the most external, and where they exist form the surface of the bone. Their occurrence, however, is by no means so constant as is generally described, and even when present they seldom extend perfectly round the bone. The authors have scarcely seen an example in which the laminæ have been continued without interruption around the shaft of a long bone, and in flat bones

encircle the canal, but that they are individually thicker on the one than on the other side. Although this may be the case in some instances it has been but rarely seen by the authors.

* Kölliker, page 289, *oper. cit.*

† This appearance is figured by M. Kölliker, but no reference is made to the peculiarity in the letterpress.

they are even less constant. When present they appear to indicate that the bone is nearly stationary in its growth as regards thickness, for in the fast-growing bones of young animals these laminæ are entirely absent, while in the bones of adults they are usually well-developed in some parts. In number they seem subject to considerable variation, and not unfrequently are arranged in different concentric series with three or four laminæ in each division, separated from the adjacent series by a broad and transparent lamina, suggesting the idea of periods of greater activity of growth, alternating with others of less activity. Like those of the Haversian systems, the circumferential laminæ are perforated by Haversian spaces and systems (Plate VII. fig. 10), and in many examples to such an extent that they are recognized only as interstitial laminæ; but their uniform direction parallel with the surface of the bone will satisfactorily indicate their previous existence as continuous circumferential laminæ. In this part of a bone we may sometimes discover a peculiarity of character, in the presence of Haversian systems, consisting of two or three laminæ, or of even a single lamina only, so small has been the preceding Haversian space.

In the progress of development canals are sometimes formed which proceed from the surface of the bone into its substance, and around these the circumferential laminæ are inflected, thereby forming systems, the laminæ of which are continuous with those of the surface of the bone. On the other hand, Haversian spaces not unfrequently occur which are continued from the surface towards the centre of the bone, cutting the circumferential laminæ abruptly across; systems are subsequently developed in these spaces; but in such instances the laminæ of the systems are not continuous with those of the surface (the latter appearance is shown in Plate VII. fig. 10).

In young rapidly growing bone the circumferential laminæ are replaced by a series, which, from the character of their course, might be termed the undulating laminæ. The bone in its growth sends off outrunning processes between those vessels of the periosteum which lie nearest the surface of the bone. These processes are formed of reduplicated laminæ which are continued from process to process, following the undulating surface of the bone for a considerable distance before they are lost. The growing processes as they increase in size and length arch over towards each other by the dilatation or bifurcation of the outer extremities. The salient points so formed come into contact and unite, thereby enclosing the vessels. The spaces enclosed in the manner described become the seat of Haversian systems. Hence we have at this period of growth, bone composed of Haversian systems with intervening undulating laminæ, a character by which the observer cannot fail to recognise young growing bone as differing from that of adult animals. Over the points where the processes have met and united, laminæ are extended which are continued outwards to form similar and more external processes; these in their turn become united and inclose spaces. This mode of increase by the development of outgrowing processes is repeated again and again until the rate of growth becomes arrested, when the formation of ordinary circumferential laminæ is assumed in some cases to be again replaced by the development of undulating laminæ. It must however be remembered that in the rapidly growing bone of young subjects lamination is much less strongly pronounced than in the bones of the adult (Plate VI. fig. 4), and that in many cases it is indicated more distinctly by the arrangement of the lacunæ than by the presence of well-marked outlines in the laminæ themselves.

In addition to laminated tissue we sometimes find portions of superficial bone run in, as it were, between two or three Haversian systems, and without any appearance of lamination, but with a general transparency, and with lacunæ scattered through it of indefinite form, and wanting in that arrangement which prevails in laminated bone*.

Lacunæ of Bone.—In a transverse section of the Haversian systems the lacunæ are seen as elongated oval spaces, lying amongst, and with their long axes in the direction of, the laminæ; numerous canaliculi proceed from them, the bulk of which in this view pass off towards either the Haversian canals or the outer surface of the systems. The lacunæ are however somewhat different in appearance at different ages of the systems and of the subject. In youth they are large, numerous, and richly supplied with canaliculi; on the other hand, in an old subject they are less numerous and have fewer canaliculi. Moreover, they vary a little in character in different parts of the same preparation, according to the age of the particular spot in which they are examined (Plate VI. fig. 6). Then, again, they frequently exist without canaliculi; while in other cases they may be found, together with the canaliculi, in great part filled up with solid matter, leaving only a small round open space in their centre. This state of consolidation has not, so far as the authors are aware, been previously observed. It is however present to a considerable extent in the long bones of adults who have in infancy suffered from rickets, and whose bones have remained permanently distorted (Plate VII. fig. 12 *a* and *b*). More rarely a similar condition may be seen in the bones of those who have passed the middle period of life without having suffered from rickets in childhood. Hence consolidation must be regarded as a normal condition of the lacunæ, of occasional occurrence, rather than as the result of disease.

In favourable sections it may be shown that both the lacunæ and canaliculi have parietes, which are manifested by appearances similar to those observed in the dentinal tubes†. In lacunæ which have no canaliculi the walls are well-marked, and in

* Kölliker, *oper. cit.* p. 288, objects to the opinion of Arnold that the interstitial laminæ have a different origin from the external and internal circumferential laminæ, the former of which he believes to originate in the cartilage, the latter in the external and internal periosteum.

Kölliker's views are inserted in the text, but it is seen above that the observations of the authors do not coincide with those of either Arnold or Kölliker.

† This point will receive additional confirmation in a subsequent part of the paper, when the development of the lacunæ and canaliculi is described.

a transverse section of the latter the parietes are readily recognized, as shown in figure 11, Plate VII.

In a longitudinal section of Haversian systems we have the laminæ on either side of the canals cut at right angles, or nearly so, while immediately over the canals they are exposed in their breadth; in the one instance we look upon the cut edges, in the other upon the surface of the laminæ. The lacunæ in the two parts present differences in outline. In the former part they resemble those seen in the Haversian systems divided transversely, in the latter they approach a circular, or slightly oval figure, and send out canaliculi indifferently from all parts of their circumference. After examining the lacunæ from these three different points of view, it will be manifest to the observer that the normal form is that of a greatly compressed hollow sphere or oval, from which numerous minute tubes are continued. To this prevailing form we find numerous exceptions, which, although of not unfrequent occurrence, form but a small part of the whole number that may be seen in an ordinary preparation. The outlines assumed by the exceptional lacunæ are too varied and too irregular to admit of such a description as would embrace every departure from the ordinary form. It is however worthy of mention, that lacunæ may sometimes be met with, normal in form, but of three times the usual dimensions.

What has been said of the lacunæ of the Haversian systems will of course apply also to those occupying parts of systems known under the name of interstitial laminæ. In the circumferential laminæ the lacunæ are for the most part similar to those already described. But we have in this situation what may perhaps be regarded as modified lacunæ, presenting themselves in the form of elongated tubes arranged in bundles, or occurring singly, and passing more or less obliquely from the surface towards the interior of the bone (Plate VII. fig. 13). When these tubes attain a considerable length they are commonly bent at a sharp angle once or twice in their course, and at each flexure change their direction, presenting sometimes a loose resemblance to the letter *z*. When these tubules are divided transversely they exhibit parietes. On tracing them in parts abounding with lacunæ, it will be seen that they form lateral connections with the canaliculi, at the same time that they cross these as well as the lacunæ and laminæ obliquely. The presence of the oblique tubes is not however constant. In one part of the circumferential laminæ they will be abundant, while in another they will be sparingly present or altogether absent. Then, again, they may be seen in one series of laminæ, while in the contiguous set they are absent, but may reappear in a third and more deeply-seated series. They are, however, we believe, strictly confined to the limits of the circumferential laminæ, and are seldom present unless these are tolerably abundant.

The lacunæ situated in the transparent non-laminated bone, which is sometimes found on or near the surface, and to which we have before alluded, are less compressed in form than those hitherto described. They are subject to great irregularities of outline, and send off canaliculi equally from all parts of their circumference, thereby imparting a peculiarity to the tissue which cannot fail to be remarked. The presence of persistent nuclei in the lacunæ has been questioned by recent authors; some state that they may be detected at times, while others regard their presence as very problematical. Mr. GOODSIR* describes the lacunæ as containing a little mass of nucleated cells.

M. KÖLLIKER†, after citing SCHWANN, KRAUSE, KOHLRAUSCH, HEISCHMANN, GÜNTHER and DONDERS, all of whom believe in the persistence of the nuclei in the lacunæ, goes on to say that he has paid great attention to the subject. In the bones of an individual eighteen years old he found them after the sections had been treated with acid. The authors have had no difficulty in finding the nuclei in recent bone without the aid of chemical treatment. If a small fragment be taken from the spongy portion of a fresh bone and freed from adherent fat, the nuclei may be seen as small rounded bodies attached to the walls of the lacunæ; their appearance is shown in Plate IX. figs. 27 and 29. In dry sections of bone the nuclei are less readily recognized, although we may find them here and there. The authors have sections of a fossil bone from the Wealden (supposed to be Pterodactyle) in which all the lacunæ have well-defined nuclei (Plate VII. fig. 14).

Haversian Systems.—Having described the several parts entering into the formation of Haversian systems, it now remains to say a few words on the relations of these to each other, and to the other parts of bone which are not included under that term. The union of the Haversian and other systems of laminæ is seen, in a transverse section of a long bone, to be effected by the interposition of a layer of transparent tissue. In a recently-formed system the canaliculi do not appear to pass through this layer, but in older ones they are seen to pass across from the external lacunæ and to anastomose with those from the interstitial or circumferential laminæ, or with those of adjoining Haversian systems (Plate VII. fig. 10). After these intercommunications have been fully established, the connecting layer loses something of its original transparency, but not the whole, for in the bones of aged subjects it may still be distinguished.

In Dr. TODD and Mr. BOWMAN's 'Physiological Anatomy,' it is stated, on the authority of Mr. TOMES, that the canaliculi of adjoining systems anastomose but scantily. A more extended examination into the subject has shown this to be true of the more recently developed systems, but not of those of greater age. In addition to the connection of systems of laminæ by the juxtaposition of their external members, instances not unfrequently occur where several Haversian systems, linearly arranged, are encircled by a series of laminæ common to the whole. This is shown in Plate VI. fig. 5. But although we may find groups of systems thus bound together by enclosing laminæ, in addition to the general or partial investment by circumferential laminæ, such binding together is clearly not necessary to ensure the strength of the bone, as we see many

* Anatomical and Pathological Observations, by JOHN GOODSIR and HENRY D. S. GOODSIR, 1845.

† KÖLLIKER, *oper. cit.* p. 296.

bones altogether destitute of circumferential laminæ, and very few with these laminæ complete; while the encircling of several Haversian systems with laminæ common to the whole is the exception rather than the rule.

In the progress of their investigations the authors have failed to discover any instances in which pre-existing laminæ, partially absorbed, have become continuous with newly-developed ones, end to end; indeed there seems good reason for supposing that such seldom, if ever occurs, otherwise the Haversian systems would not be so strongly marked, neither should we have the festooned outline to the systems lying against the cut ends of the laminæ, in the manner shown in Plates VI. and VII. figs. 1 and 10.

In examining Haversian systems divided transversely, one will be occasionally found in which the canal has been completely obliterated by the development of a lacunal cell in its centre. This condition is illustrated in fig. 1 *c*.

Ossified Cartilage of Joints.—Dr. Sharpey has shown that beneath the articular cartilage lies a layer of bone produced by the gradual ossification of this tissue*. Professor Kölliker† has observed this form of bone to be present in all the articulations, excepting that of the jaw and the hyoid bone, in which situations he has failed to detect it. The authors have numerous sections from the articular process of the lower jaw, in all of which it is present, and although in smaller amount than is usually found in corresponding parts of other bones, still it is sufficiently distinct. The peculiarity of the articular bone consists in the maintenance of the same arrangement of parts as existed in the cartilage prior to its conversion into bone. The cells for the most part are arranged in groups or in lines parallel with the long axis of the bone, each cell presenting a roundish mass surrounded by granules, and rendered rather indefinite in outline. Not uncommonly several cells become connected end to end, thus producing an elongated form. It is when in this condition that the cells and basement tissue may be restored to the appearance of cartilage by the use of hydrochloric acid. Occasionally, however, a cell becomes converted into a well-developed lacuna, and near the ordinary bone the cells may be seen in more or less advanced stages towards their conversion (Plate VII. fig. 17).

In the ossified cartilage, when the tissue attains a considerable thickness, broad and ill-defined lines or bands of opacity may generally be found running parallel with the articular surface, and giving a stratified character to the part in which they are situated, as shown in fig. 17 *b*. When seen by transmitted light, these lines appear of

* Dr. Quain's Anatomy, fifth edition, by Mr. Quain and Dr. Sharpey, part ii. page 158. "In the slow growth of bone which encroaches on the attached surface of articular cartilage, the ossification would almost seem to be produced merely by the impregnation of the cartilaginous matrix with earthy matter (corresponding with the first step of the ordinary process); and in this case, the cells and clusters of cells being surrounded by calcified matrix, may remain as little vacuities or lacunæ in the bone; but this as well as the formation of lacunæ in the crusta petrosa of the teeth, and the production of adventitious bony deposits in different textures, requires further investigation."

† Kölliker, page 318, *oper. cit.*

a deep rich brown colour, and as if possessed of a much higher degree of granularity than the surrounding parts. The cells of the tissue are not uncommonly surrounded by a similar granular condition of the matrix. This state of granularity is most conspicuous near the bone, and less so near the cartilage, where the tissue becomes clear and transparent. The ossified cartilage is generally separated by a well-marked line from the subjacent bone. Exceptions to this, however, are not wanting, where the two pass, the one into the other, by insensible gradations. The bone at the line of junction is usually advanced into the ossified cartilage by rounded projections of variable size; some of which reach nearly to the surface, while others are situated at some little depth. From this arrangement the articular bone is necessarily irregular in quantity. In one part it will dip to a considerable depth into the bone, while at another and perhaps contiguous part it will form but a thin layer. The external or articular surface is but slightly affected by the irregularities of the surface attached to the bone; on the contrary, it presents an even surface, on which the articular cartilage is placed in a uniform layer.

The absence of Haversian canals, and the unfrequency of lacunæ with canaliculi in the articular bone, renders this tissue much less porous than ordinary bone, a condition which probably contributes to its strength, and to the strength of the part in which it occurs.

In the presence of this peculiar form of tissue at the articular extremities of bones we may perceive a striking instance of design.

It would appear that the articular cartilage must rest upon a firm and uniform surface, otherwise it would, under pressure, yield unequally, and the joint would probably be but imperfect in its action. Circumferential laminæ are developed only on free surfaces, and do not present there a very level outer surface. A surface formed by Haversian systems would be irregularly fluted or nodulated, like that beneath the calcified cartilage. Neither of those forms of development, then, would effect a condition of surface similar to that attained by the ossified cartilage. A careful consideration of the foregoing facts will, we think, lead to the conviction, that the presence of ossified cartilage as a basis for articular cartilage must not be regarded simply as the result of a slowly progressing ossification of a tissue prone to ossify; neither must it be regarded as an imperfectly developed tissue in which the formative process was arrested before the part had been perfected*; but it must be regarded as a constant element in the osseous system, having its special use, and necessary to the perfect organization of the skeleton in the higher forms of Vertebrata.

Ossified Lacunal Cells.—In addition to the parts already described, the authors have found a condition of bone which has hitherto escaped observation, or if seen, has not been recognized as a form of osseous tissue. The bones of aged people not unfrequently become extremely light and spongy, readily break, and from the diminished amount of compact tissue, may in the case of the flat bones, such as

* Kölliker, *oper. cit.* page 318, speaks of this tissue as imperfectly formed bone-substance.

those of the pelvis, be indented by firm pressure with the finger. Such bones, after maceration, contain within the spaces enclosed by the cancelli, a white powder, which readily falls out if the bone be broken. If a little of this powder be mounted in Canada balsam, and placed in the field of the microscope, the observer will see that it is mainly composed of large nucleated cells, some of which are detached, others united into masses.

The isolated cells have a spherical or oval form, and appear to consist of a granular nucleus surrounded by a very thick cell-wall, external to which we commonly find an aggregation of granular matter, indefinite in amount and presenting a ragged outline (Plate VII. fig. 15). The cells which are united into masses are connected by this granular matter, or ossified blastema, as it might perhaps be called. The cells themselves are similar to the loose ones, excepting that in some cases their character is rendered more apparent by the nucleus assuming the form of a lacuna. If a section of the bone be examined, similar cells will be found adherent to the walls of the Haversian canals, with the canaliculi of contiguous lacunæ advancing into them, while the nuclei of the adherent cells are assuming the form of lacunæ and throwing out canaliculi.

Again, in the substance of the bone, the outline of the formative cells, similar in size and shape to the loose ones, may be in places recognized, having lacunæ and canaliculi as their centres. The recognition of these ossified cells in their isolated state will be followed by the observation of similar cells occupying the Haversian canals and cancellar spaces in the majority of microscopic preparations of adult bone (Plate VI. fig. 5 *a*); and the same may sometimes be seen on the outer surface of bone (Plate VII. fig. 18). The appearance of the cells is, however, liable to be modified by the various circumstances which operate upon growing bone, such as the presence of a tendon, or of a large pulsating vessel, or in fact, by pressure from any cause. Thus if situated in the immediate neighbourhood of a tendon, they assume an elongated form; while on the other hand they may be flattened by pressure.

It has been thought desirable to describe the ossified lacunal cells at this place, on account of their frequent occurrence in adult bone (Plate VI. fig. 5 *a*, and Plate VII. figs. 16 and 18).

Bone Tissue.—Hitherto we have confined our description to the forms in which the bone substance is arranged, as into laminæ, Haversian systems, &c. We have now to consider the ultimate structure of the tissue itself. But little has been said upon this point until within the last ten years. In the 'Physiological Anatomy' of Dr. TODD and Mr. BOWMAN*, it is stated that bone is made up of an aggregation of granules in a scarcely distinguishable matrix. Latterly, other views have been proposed. Dr. SHARPEY has advanced the hypothesis, that although the bone developed in cartilage is at one period granular†, a reticulate fibrous arrangement of the elements

* Dr. TODD and Mr. BOWMAN, Physiological Anatomy, page 108.

† Dr. QUAIN's Anatomy, edited by Mr. QUAIN and Dr. SHARPEY, page 157. "The primary osseous matter forming the original thin walls of the areolæ is, as Mr. TOMES observes, decidedly granular, and has a

of the tissue takes its place, and must be regarded as the general condition of bone*. Dr. SHARPEY, when he formed this opinion, had directed his attention principally to thin fragments torn from the surface of bone macerated in hydrochloric acid. The authors have seen the appearance both in decalcified bone, and in sections which have not been chemically treated; but they believe the fibrous appearance to be an optical effect produced by the canaliculi and granules when the light is unequally distributed, or the object-glass in bad adjustment. It will be shown in the subsequent details, that the appearance can be produced and dispelled at pleasure by alterations in the direction of the transmitted light. If thin sections of bone are seen by transmitted light (passing through in the axis of the microscope,) the tissue will appear either granular or structureless, and in the laminæ alternations of granular and structureless parts will be seen. But if the light be allowed to pass in one direction only, and with considerable obliquity through the specimen, then the appearance of extremely minute fibres will present itself; and this not only in adult bone, but in the isolated cells of old and in developing young bone. They are most strongly marked over the lacunæ; to which, and to the canaliculi, a confused and broken outline is given. The fibres appear as though arranged in series of short and broken lengths, parallel, and without plaiting. If the specimen be a little out of focus there will be an appearance of reticulated fibres marked by

dark appearance; the subsequent or secondary deposit, on the other hand, is quite transparent, and of a uniform and homogeneous aspect, without obvious granules. This begins to cover the granular bone a very short distance (about the $\frac{1}{80}$th of an inch) below the surface of ossification, and, as already stated, increases further down. The lacunæ first appear in this deposit; there are none in the primary granular bone. In what further regards the nature and formation of the secondary deposit, my own observations lead me to differ considerably from the views of Mr. TOMES. He supposes that it is formed of cells which become impregnated with earthy matter,—the cartilage cells in the first instance, and afterwards the cells newly formed in the blastema. Now although certain appearances render it not improbable that there may be a layer of flattened and calcified cells next to the surface of the granular bone, I am nevertheless disposed to think that the subsequent and chief part of the deposit results from the calcification of successive layers of fibres generated in the blastema, and possibly derived from the granular cells, some cells being perhaps also involved along with the fibres, as in the ossification of the flat bones of the cranium: in short, it appears to me that the deposit in question is formed after the manner of intramembranous ossification already described (page 150). I infer that such is the process from the structure of the layers, for they are made up of fine reticulated fibres like the lamellæ of perfect bone shown at page 143."

* On this point KÖLLIKER, *oper. cit.* p. 289, observes, "Die Grundsubstanz der Knochen besteht nach SHARPEY und HASSALL aus einem Netzwerk feiner Fasern mit rhombischen Maschen, von welchen Fasern ich weder an frischen, noch an mit Reagentien behandelten Knochen eine Spur habe entdecken können. Was ARNOLD neulich Primitiv-fasern der Knochen nennt, sind, wie seine Abbildungen lehren, nur die Granula der Grundsubstanz, an denen er eine lineare Aneinanderreihung zu erkennen glaubt, und seine Querstreifen auf Flächenschnitten, und radiären Linien auf Querschliffen, wenigstens die letzteren sicher, nichts als die Auslaüfer der Knochenhöhlen. Sollten durch irgend ein Reagens in dem Knochengewebe deutliche Fibrillen nachgewiesen werden können, was mir aber noch nicht gelungen ist, so könnten dieselben wohl keine anderen sein, als diejenigen, welche in ossificirenden Knorpel das streifige Ansehen bewirken und die von SHARPEY und mir bei der Knochenbildung aus dem Periost beschriebenen."

nebulous outlines. On the other hand, if we have light passing from all sides obliquely through the preparation, that is, if the object be illuminated by a hollow cone of light, in the manner obtainable by Mr. GILLETT's arrangement of achromatic condenser, or by rendering opake the centre of the lower lens of the ordinary achromatic condenser, the fibres disappear, and we see in their place a granular tissue, with a tendency here and there to a linear arrangement of the granules, and the lacunæ and canaliculi clear and distinct, when well in focus, but with a fibre-like nebulosity when slightly out of focus, especially in those parts where the latter run obliquely through the specimen. This is more especially the case where the canaliculi are filled with fluid, or where they have become solid, a condition very common in the bones of old subjects. Indeed, in many specimens where the bone is not highly developed, a fibrous appearance at once strikes the eye, and it is only on careful examination of the surface of the section that the illusion is dispelled by the orifices of the canaliculi appearing at the termination of the apparent fibres. Amongst a great many examples the authors have not seen an instance in which the fibrous appearance could not be traced to canaliculi, this effect being often increased by the presence of granules. If parallel dentinal tubes be viewed by a side light, in the manner we have described, each tube will throw two or three strong nebulous lines, which might be mistaken for fibres; and if tubes cut transversly be inspected in a similar manner, each will appear to have several concentric lines extending half round it on the side opposite to that from which the light proceeds.

That the presence of granules imbedded in a less refractive tissue may, when viewed by a side light, give an appearance of fibres, is shown on an examination of pus-globules under similar circumstances, when each globule will seem to have a striated surface; and that slightly rounded prominences will produce a like effect is shown in the case of some species of *Navicula*. The surface of these latter objects is marked by rows of slight prominences, which, when the light is oblique and passes through the specimen from one point only, appear as lines or fibres, the direction of which will change with alteration in the direction of the light; that is to say, shadows will extend from one eminence to another in the opposite direction to that from which the light enters. If we examine the projecting spiculæ of bone forming in temporary cartilage under corresponding conditions as regards light, a similar fibrous appearance will be developed.

Under these circumstances the authors are forced to the conclusion that bone substance, instead of being made up of minute reticulate fibres, is composed of granules or granular cells, imbedded in a more or less clear homogeneous or subgranular matrix. This subject will receive further elucidation under the head of development. These views are supported by the appearances presented not only in developing bone of young subjects, but also in the bone of adults.

Thus as regards the basement homogeneous tissue, it will be found that where lamination is highly developed, the laminæ have a transparent and structureless, and a more opake and granular part, to which the former appears to be the matrix. The peripheral lamina of the Haversian systems is generally clear and free from granularity, and the internal lamina sometimes presents a similar structureless appearance. The matter which fills up the Haversian systems in the full-grown antlers of the Cervidæ, affords another and a very striking example of transparent structureless osseous tissue, which in this instance is the more distinct from the absence of canaliculi in its substance. Then, again, we have another instance in the clear tissue which is sometimes found between the superficial Haversian systems of ordinary bone. It has already been described as a non-laminated element found on the *surface* of certain bones. In the instances already cited, and no doubt in many others which may be found in the skeletons of the lower vertebrata, we have bone tissue without obvious granularity and without obvious structure, and although it forms but a small part of the general mass, yet from its constant presence at all ages and in all subjects, it must be regarded as an integral and normal part of mammalian bone. The granular condition of bone tissue is tolerably obvious in all preparations, though it is much more strongly marked in some specimens than in others. The amount of the component granules varies in different parts of the same specimen, and in specimens taken from different parts of the skeleton. Thus in one situation we may see laminæ with a highly granular part gradually merging into a transparent tissue, while in another the laminæ may be granular throughout. Again in young bone developed in cartilage, the part between the cells becomes highly granular; fragments of which may be found in certain adult bones, as in the petrous portion of the temporal bone (Plate VII. fig. 16 *b*). Bone near the articular surface frequently presents a well-marked granularity. Whether the granular and the transparent parts of bone contain different relative amounts of phosphate of lime is a question yet to be decided.

Development of Bone.—Temporary cartilage, when it first appears in the embryo, consists of an aggregation of closely packed nucleated cells, which in the process of growth become separated by the development of a tissue external to them, usually designated the hyaline tissue of cartilage. At this stage of growth we have a granular cell occupying a cavity in the hyaline tissue; at a previous period the granular cell was inclosed by an outer cell-wall, to which it formed the nucleus, but now the cell-wall has merged into the hyaline tissue, from which it can no longer be distinguished, while the granular cell contains itself one or more nuclei. In making sections of cartilage many of these granular cells escape from their cavities in the hyaline tissue, and may be seen detached and floating about in the field of the microscope. Some little confusion of terms has arisen in connexion with this part of the subject. The granular cells are not unfrequently described under the name of cartilage-nuclei containing nucleoli, while in fact they are cells capable of separation from the matrix, and are at one period the only recognisable cartilage cells. They occur singly or in groups, more frequently however two or three are extended in a line. It is proposed in the present communication to call these bodies *granular cartilage cells*, as distinguished from a form which the cells assume previous to ossification. Cartilage previous to its conversion into bone undergoes a rapid growth, which takes place principally in the direction of the long axis of the future bone. Each granular cell becomes divided into two by segmentation transverse to the line of ossific advance. These are again divided, and the process repeated from time to time, till in the place of a single granular cell we have a long line of cells extending from the unchanged cartilage to the point where ossification has taken place. Contemporaneously with this development of lines of cells, other changes are going on in the individual cells composing them. If we examine those situated near the advancing bone, it will be observed that the granular cells have enlarged, have become separated from each other by wide intervals, and that each has become invested with a thick pellucid cell-wall (Plate VIII. figs. 19 and 20). The increase in the size of the cells has occurred at the expense of the hyaline tissue, which, at those points where the rounded cells approach each other, is reduced to a thin film, although in the intervals left by the packing of the uncompressed cells it exists to a considerable amount. Here then we have a matrix containing cartilage cells composed of an outer pellucid coat, within which is a granular cell containing central nuclei, a cell consisting of three distinct parts. The nucleus existed previous to the segmentation of the granular cell, but the outer wall is the product of a subsequent development. It will be observed that growth is effected by two modes; first, by the increase of the number of the cells; and, secondly, by the increase of their size individually.

While these changes are going on for lengthening the shaft of a long bone, cells are being added by a somewhat similar process of division to the surface, so as to increase the diameter of the cartilage. For the development of the epiphysis the cells become multiplied on a similar principle, but instead of occurring in lines they are accumulated in oval or rounded groups*.

From this general view of the subject we will now proceed to consider in detail those conditions of cartilage which precede and prepare for its conversion into bone. These phenomena may be conveniently examined in the cartilage which connects the diaphysis and epiphysis of long bones, and we may take those of a lamb or calf, as they present the same conditions as are found in the human subject, and have the advantage over those of other domestic animals of being readily obtainable at all times without the necessity of destroying the animal for the sole purpose of physiological investigation.

At that point in the cartilage where the linear arrangement of cells commences, several of these bodies lie side by side in the same column, but by following the line downwards towards the advancing bone, it will be seen that the column usually divides, forming two or three distinct lines, in each of which the cells are arranged

* Previous to the ossification of the epiphysis the cartilage at the heads of the bone is increased in its bulk from additions to the surface, by a process somewhat similar to that through which the growth of bone is effected. This process will be described when the latter subject is considered.

in single file. It does not appear that the number of cells entering into the formation of a series is at all limited; hence the lines or columns vary in length. Neither does it appear that the consecutive lines are placed, as a rule, end to end, so as to form one long continuous straight line by the junction of several shorter ones. On the contrary, each series is complete in itself, and is generally placed a little to the one or other side of those above and below it. We have in fact interrupted lines of cells, each line the offspring of a single cell, the outer walls of which have merged into the intercellular or hyaline tissue.

When the destined number of cells forming a linear series has been developed, each cell becomes itself a centre of growth. The granular cell enlarges, together with its nucleus, and becomes invested with a cell-wall. In examining a line of these bodies extending from the forming bone of the diaphysis, we shall see them in various degrees of forwardness. Thus, if attention be directed to the end of the line furthest from the bone, the cells will be found small in size, granular, and with a perceptible nucleus, but without an outer wall, distinguishable from the hyaline substance, which is abundant between the contiguous lines, but small in quantity between the cells composing the lines. But if the other end of the line be examined, very different conditions will be observed. The granular cells will be seen to have become rounded in form, to have increased to three times their original bulk, and to possess well-marked circular nuclei; in addition to which each granular cell will have acquired a thick pellucid outer wall, while the hyaline tissue between contiguous lines of cells will have dwindled down to a thin film, excepting in those parts where spaces are necessarily left in the approximation of spherical bodies (Plate VI. figs. 19, 20). The abundance of the hyaline tissue in the earlier condition of cartilage, affords space for the development of the cells in the breadth of the bone, but in the direction of the length of the bone we find but little of that tissue, yet each component cell of the almost innumerable lines of cells that exist, even in a bone of small size, is itself a centre of growth. At the osseous end of a line, a single cell occupies more space than four or five at the opposite extremity; and as in each situation the cells are imbedded in hyaline tissue, it is evident that there is a concurrent growth of the latter throughout each line of cells, and also of the cells and hyaline tissue in the breadth of the cartilage. But for this provision of nature, cells would grow at the cost of those in their neighbourhood, a condition which obtains wherever space for a secondary development of bone is required, a description of which will follow in the subsequent pages of this paper. Then, again, had we concurrent growth in each cell of a series without a similar growth in all the other series, throughout the breadth of the bone, the growing series would encroach upon the cartilage in which the linear arrangement had not commenced, and ultimately unite by osseous tissue the epiphysis and diaphysis before the bone had acquired its normal length. And supposing concurrent growth to take place over the one-half of the breadth of a bone and not in the other, the bone would become curved and the limb distorted.

We have traced in temporary cartilage those changes which occur preparatory to its conversion into bone: it now remains for the authors to describe the results of their researches on the mode in which that conversion is effected, and concerning which there hitherto has been a great diversity of opinion.

Preceding ossific deposit, the intercolumnar tissue, which may not only be seen between the columns of cells, but also passing more or less perfectly between the individual cells, becomes in some cases slightly fibrous in appearance, and of a light brown colour; this condition speedily gives way to a highly granular state; in fact it has become bone, and encloses, in osseous cavities or crypts, the cartilage cells. For reasons which will hereafter be sufficiently obvious, we shall in future call these lacunal cells. If thin sections be made with a very sharp knife through ossifying cartilage, and placed in water or albumen under the quarter or eighth object-glass, we shall occasionally find lacunal cells which have escaped from their osseous crypts, floating loose in the fluid, and offering a favourable opportunity for the examination of their characters. It may then be seen that the thick pellucid outer investment has become granular, and that the enclosed granular cell has lost its regular outline, and become angular or ragged, while the nucleus has become obscured by the changes in the more external parts of the cell, an early stage of which is seen in Plate VIII. fig. 19 and 22 *a*. Such are the appearances presented in some of the detached lacunal cells, but they may be seen in both earlier and later stages of development than that described. Thus instances may be found where the outer coat has only a few granules on the surface, in which case the granular cell may be seen more distinctly, and minute elongated processes traced, extending from its surface (Plate VIII. figs. 19 and 20). In a favourable section we sometimes find such a cell with its outer coat torn, exposing to full view the granular cell, with numerous and well-marked processes extending from it. Dr. SHARPEY kindly placed at the disposal of the authors the rickety bone of a child who died during the active condition of the disease. In this specimen granular cells with radiating processes can with great readiness be detached by tearing a thin section, for in this case cartilage assumed the arrangement of bone without the impregnation of earthy ingredients (Plate VIII. fig. 23)*.

The third condition, in which we may find isolated lacunal cells detached from the osseous crypts of the intercellular tissue, differs from those already described in their greater solidity. The whole mass has become highly granular, and the granular cell with its processes has united to the outer coat, so that the two can no longer be

* KÖLLIKER, *oper. cit.* at p. 360, describes accurately the formation of lacunæ from cartilage cells in a rickety bone, and although he has not observed the changes with equal distinctness in normal bone, appears to have satisfied himself that in the ossification of cartilage the cells of the latter become converted into lacunæ. M. KÖLLIKER does not however seem to have recognized the relation of the lacunal cells as such to the intercellular tissue, neither does he mention the fact that the granular cells with their long stellated processes can be separated from the investing cell-wall in the manner described in the text.

separated the one from the other, and are no longer to be recognized as distinct parts. If accidentally broken across, we see that they have a hollow centre, in fact, a lacuna; but when loose and entire, they appear as rounded dense masses, projecting from the surface of which we may not unfrequently detect a few short needle-like processes. Lacunal cells of the latter kind are not, however, very easily detached, for when they have advanced thus far in development, their union with ossified intercolumnar tissue is too strong to admit of ready separation; indeed, after this period their separation is impossible. It has been stated that the granular cells of cartilage previous to the formation of lacunal cells send out processes. There are, however, a certain number of granular cells that do not appear to undergo this change, but preserve their state of granulation and their external figure, and show the round nucleus even after they are surrounded by and imbedded in ossified tissue. Indeed, we may find perfectly ossified crypts of intercolumnar tissue at some little depth within the forming bone, containing several of these granular cells in an unaltered condition, and this without the cavities having been enlarged by absorption of their walls (Plate VIII. fig. 19 *d*). It is important to bear this point in mind, as it will subsequently be shown that similar granular cells lie loose in the large cavities formed by absorption, and that they are concerned in the formation of bone when that tissue is not preceded by ordinary cartilage.

If a section of developing bone, including a little of the cartilage, be dried, and mounted in Canada balsam, we shall be able to recognize, first the cartilage with the cells greatly contracted, then the ossified intercolumnar tissue full of little crypts, from which, in the process of drying, the lacunal cells have disappeared (Plate VIII. fig. 24 *a*); and lastly, we shall see bone in which ossified lacunal cells run in lines corresponding to the preceding lines of cartilage cells (fig. 24 *b*). The latter appearance is, however, interrupted by the presence of numerous large spaces produced by the absorption of both the intercolumnar tissue and the lacunal cells (fig. 24 *c*), the process for the removal of which commences concurrently with the impregnation of the lacunal cells with the earthy salt.

These conditions can be also very favourably observed in sections of the long bones of a fœtal lamb, taken previous to the commencement of ossification of the epiphysis, and examined in the liquor amnii; water will not answer as a substitute for this fluid, as the cartilage cells are speedily altered in character by its presence. And we may here observe that the examination of developing bone should be conducted on perfectly fresh subjects, and that care should be taken to keep the knives, glass, &c., which are used in preparing the sections, perfectly free from extraneous matter; otherwise the results will be unsatisfactory. Thus a small quantity of spirits of wine or of acid will produce changes of character in the specimen, and thereby lead the observer into an erroneous conception of the normal characters of the tissue.

Hitherto the authors of this communication have spoken principally of those appearances which are presented in sections made parallel with the advancing bone;

attention must now be directed to sections made across the bone, so as to cut transversely the lines of cartilage cells.

If, then, we take a thin section through the cartilage where ossification of the intercellular tissue is advancing, the following appearances will be seen. The intercellular tissue will present itself in the form of more or less perfect septa lying between and enclosing the lacunal cells, in such a manner as to produce a uniform pattern over the surface of the section (Plate VIII. fig. 20). On close inspection it will be seen that the intercellular tissue is granular, and that in each enclosure it encircles one or two lacunal cells, which in the latter case lie in contact, with the contiguous sides flattened, while the peripheral surface of the two form together a tolerably perfect circle in close contact with the intercellular tissue.

The lacunal cells are beautifully shown in the kind of section under consideration. Their circumference is becoming granular, and frequently may be seen as a perfect ring of granules in close contact with the intercellular tissue, while in the centre we see the granular cell with a well-defined nucleus, and with numerous rudimentary canaliculi, extending from the circumference of the inner cell towards the surface of the outer cell-wall. Where two cells lie in contact, the rudimentary canaliculi may sometimes be found running across from the one cell to the other. If after examining such sections as may be made with a knife, we take one through the bone immediately below the partially ossified cartilage, rendered thin by grinding, and mount it in Canada balsam, the various points described as existing in ossifying cartilage will be seen in that which has been converted into bone. Thus we shall see the intercellular tissue preserving its original form, and highly granular, and that while the outer walls of the lacunal cells have become calcified, the granular cells will have assumed the form of perfect lacunæ and canaliculi, the latter freely intercommunicating where the surface of the lacunal cells is in contact, but seldom extending into the ossified intercellular tissue. These conditions are shown in Plate VII. fig. 25.

The ossified lacunal cells, and the intercellular tissue which have existed in the temporary cartilage, and have there served their part in forming the skeleton of the fœtus, are destined to have but a short existence, for we find that no sooner is the bone formed than large spaces are produced by absorption. Whole lines of ossified lacunal cells with the enclosing intercellular tissue disappear, to be replaced by more or less perfectly developed Haversian systems, which also last but for a time, when they by a similar process are removed to give place to others, or to contribute to the formation of permanent cavities. It has been well stated by Dr. SHARPEY, that the long bones of the fœtus are not equal in size to the medullary cavities of the corresponding bones of the adult. Hence as growth takes place, gradually comes the necessity for the removal of parts to make way for the development of others upon an enlarged scale.

It has been shown that primary bone, if we except two or three of the cranial bones, is merely calcified cartilage which has previously assumed the structural arrangement of bone. It is proposed to restrict the term primary bone to tissue so formed, as it will be subsequently shown that the development of Haversian systems in the spaces formed in primary bone, is effected by processes similar to those which are concerned in the growth of bone generally, and in the extension of the flat cranial bones. Age and situation no doubt modify the conditions of growth, or development of secondary bone, as it may be called, but the processes in either case are alike.

Hence, in describing the manner in which a long bone increases in diameter, we shall in fact be describing the process by which a flat cranial bone is extended. Before however going to this point the authors will state the results of their observations on the subject of the absorption of bone, on that interesting and important process by which the Haversian spaces are formed out of the solid tissue of bone.

During the present winter it became necessary to remove a portion of the femur which protruded from a stump six weeks after the removal of the limb. From the medullary cavity a granulating mass projected and covered the surface of the bone left by the saw, and as the bone was rapidly wasting from the inner or medullary surface, we had in this specimen a favourable opportunity of examining the tissue which lay in immediate contact with the surface of the wasting bone. On cutting through this piece of femur in its length with a very fine jeweller's saw, it was found that a dense pale pink tissue lay in contact with the inner surface of the bone, which was hollowed with numerous minute cavities, into which the soft tissue accurately fitted, but from which it could be detached without tearing. The outer surface of the bone had been deprived of membrane many days before its removal from the limb.

The examination of the tissue thus closely applied to the fast wasting bone, offered as favourable an opportunity for learning something of the means by which absorption is effected as we could reasonably expect to obtain, the more so since the outer surface having been for some time exposed and covered only by dried periosteum, the actions had been confined to the inner surface of the bone. A careful examination showed that the surface of this tissue was composed of minutely granular nucleated cells, which lay in close and immediate contact with the bone, and increased in an exact ratio with its diminution. What the bone lost in bulk the cells gained, the cellular mass presenting a perfect cast of the surface of the bone, suggesting to the mind that the soft was growing at the cost of the hard tissue, or at all events that the former was instrumental in the removal of the latter. The cellular mass was tolerably vascular, but the vessels did not reach the surface in contact with the bone; hence they could not be regarded as having any immediate action in the process of absorption. Section of the bone showed that the medullary cavity had been greatly enlarged by absorption, and no doubt had sufficient time been allowed the femur at that part would have been reduced to a thin scale. A transverse section showed that in many, though not in all instances, the Haversian canals had been enlarged and rendered irregular in shape, but it was evident that the process of removal had been less active in this situation than on the medullary surface of the bone.

If we examine the fangs of temporary teeth when they are undergoing removal, similar states to those described as existing in the portion of femur will be found to obtain. A similar cellular mass will be seen to be closely applied to that surface of the tooth which is in process of removal, and the surface itself will present the characteristic emargination observed in the bone. When we connect these conditions with the fact that the nucleated cells which form the embryo have the power of appropriating the material which lies about them to the purpose of their own growth and their conversion into the various animal tissues, it is difficult to resist the belief that the cells which lie in contact with wasting bone and dentine, take up those tissues and use all or part of their element for the purposes of their own increase or multiplication, or else form a medium through which they are passed into the circulation. But as the process of absorption with concurrent development of cells is most active in primary bone where but few vessels exist, the former hypothesis seems the more probable. An objection may be raised to the supposition, that the bone is absorbed by the cells, on the ground of the density of the former; but it must be borne in mind, that as the density is gradually imparted to the bone through the agency of the adjoining soft parts, there seems no good reason for disbelieving that they may also be instrumental in its removal.

Growth of Bone.—Under this head will be described the extension of flat bones, the increase in the diameter of cylindrical bones, and the development of Haversian systems.

If the advancing edge of a parietal bone be taken either from a human fœtus or a fœtal lamb, and the pericranium and dura mater be carefully removed from their respective surfaces, we shall find the growing bone still invested with soft tissue both on the outer and inner surface, which is prolonged from the free edge. When examined under a favourable light this tissue will show differences of character in different parts, varying with the distance from the bone at which the observations are made. Thus, if attention be directed to the part furthest removed from the bone, it will be seen that the membrane-like mass is composed of oval cells with slight prolongations from the extremities, which are frequently arranged in the form of bands of fibrous tissue (Plate VIII. fig. 26). Dr. SHARPEY has observed that the membrane into which the bone extends is like fibrous tissue in an early stage of development*, and this observation is strictly true when confined to the part indicated, but the analogy

* Mr. QUAIN and Dr. SHARPEY, *oper. cit.* cxlix. and page clix. Dr. SHARPEY says, "When further examined with a higher magnifying power, the tissue or membrane in which ossification is proceeding, appears to be made up of fibres and granular corpuscles, with a soft amorphous or faintly granular uniting matter. The fibres have the character of the white fibres, or rather fasciculi of the cellular or fibrous tissue, and are similarly affected by acetic acid. The corpuscles are for the most part true cells with an envelope and granular contents; some about the size of blood-particles, but many of them two or three times larger. In certain parts the fibres, but in most the corpuscles predominate, and on the whole the structure may be said to be not unlike that of fibrous tissue in an early stage of development." KÖLLIKER, *oper. cit.* p. 289, confirms Dr. SHARPEY's statement, but denies that a fibrous arrangement can be traced in formed bone.

ceases as we extend our examination towards the bone. Here in the place of cells with elongated processes, or cells arranged in fibre-like lines, we find cells aggregated into a mass, and so closely packed as to leave little room for intermediate tissue. The cells appear to have increased in size at the cost of the processes which existed at an earlier stage of development, and formed a bond of union between them. Everywhere about growing bone a careful examination will reveal cells attached to its surface, while the surface of the bone itself will présent a series of similar bodies ossified (Plate IX. figs. 27, 28 and 29). To these we propose to give the name of *osteal cells*, as distinguished from lacunal and other cells.

In microscopic characters the osteal cells closely resemble the granular cells of temporary cartilage, so closely indeed, that the latter when detached from the cartilage could not well be distinguished from them. They are for the most part spherical or oval in form, and lie on the surface of the growing bone in a crowded mass, held together by an intervening and apparently structureless matrix (Plate IX. fig. 27 *b*). Here and there we find a cell which has accumulated about itself an outer investment of transparent tissue, and has in fact become developed into a lacunal cell destined to become a lacuna*. These points are illustrated in fig. 29 *b*.

The process of growth may be thus described. In the meshes of the fibrous tissue

* The various views which have been entertained regarding the formation of the lacunæ and canaliculi have been concisely stated by Dr. Sharpey, *oper. cit.* p. 158. He observes, that "they are generally supposed to be derived from the cells of the soft tissue involved in the ossification by some sort of metamorphosis which has been variously conceived. Some suppose that the cells become the lacuna and send out branches (like the pigment cells) to form the canaliculi (Schwann[1]). Others think that it is not the cell but its nucleus that undergoes this change, and that the substance of the nucleus is afterwards absorbed, leaving the lacuna (Todd and Bowman[2])." The nucleus described by Todd and Bowman is identical with that which in this communication is called the granular cell, and from which the authors have shown the lacuna is formed. "Henle[3] thinks that the lacuna is a cavity left in the centre of a cell which has been partially filled up by calcification, and that the canaliculi are branched passages, also left in consequence of the unequal deposition of the hard matter, as in the instance of the pore cells of plants." "It rather appears to me as if the lacunæ and canaliculi were little varieties left in the tissue during the deposition of the reticular fibres, as open figures are left out in the weaving of some artificial fabrics (but not within a cell, as Henle imagined), and that thus the apposition of the minute apertures existing between the reticulations of the lamellæ gives rise to the canaliculi." "At the same time it seems not unlikely that a cell or a cell-nucleus may originally lie in the lacuna or central cavity, and may perhaps determine the place of its formation." Hassall[4] agrees with Schwann, while Gerber[5] and Bruns[5] appear to hold the views of Todd and Bowman. With the exception of Dr. Sharpey, the abovenamed authorities may perhaps differ more in the use of terms than in matter of fact. The appearances represented in figs. 14 and 27 would at first view seem to justify the opinion expressed by Dr. Sharpey, but a careful examination of the tissue during its development, the unquestionable fact that in the development from cartilage the granular cell becomes converted into a lacuna, together with the circumstance that lacunal cells are frequently found in the Haversian canals and cancellated structure, especially in the bones of old subjects, and at times imbedded in the structure of the bone, have left no room for doubt in the authors' minds that the lacunæ are formed from special nucleated cells, in the manner described in the text.

[1] Mikroscopische Untersuchungen. [2] Physiological Anatomy. [3] Anatomie Générale.
[4] Microscopic Anatomy of the Human Body, p. 310. [5] Allgemeine Anatomie.

on the surface of the bone *osteal cells* are developed and gradually take its place; a few cells become developed into lacunal cells; the earthy salts are added, and concurrently lacunæ and canaliculi are formed; we then have bone presenting the usual characters of that tissue*. In bone developed in the foregoing manner, we find the canaliculi not merely extending to the surface of the cell-wall, or anastomosing with the canaliculi of lacunal cells lying in contact with it, but extending freely in all directions and passing through or amongst the ossified cells, and establishing rich plexuses of anastomosis. Indeed we see the boundary of the original lacunal cells only in those cases where the lacunæ have but few, or are entirely devoid of canaliculi. It would appear to be a law, to which there are few if any exceptions, that when anastomosis is established between adjoining lacunæ, the lacunal cells blend with the contiguous parts, and are no longer recognisable as distinct bodies. The process by which the cylindrical bones are increased in diameter is in all essential points similar to that described as pertaining to the growth of flat bones. Similar osteal and lacunal cells are present, but the relative amount of the matrix is greater; moreover the osteal cells have a disposition to assume a linear arrangement corresponding to the direction of the laminæ of the contiguous bone. In these lines the cells are placed so close to each other as to leave but little room for intervening tissue, but between the lines an appreciable amount may be recognized (Plate IX. fig. 31). This appearance however varies in different specimens. In one the cells predominate, in another the transparent tissue is the more abundant. Generally the younger the animal the greater will be the amount of the intervening transparent tissue, and the smaller the number of the osteal cells. But in all cases, whatever the age of the subject, or from whatever part of the skeleton the specimen be taken, the cells and the intermediate tissue become blended in the process of ossification, and the whole presents a uniform granular appearance, excepting in the instances in which lamination is strongly developed, or in those which have been noticed in the previous part of the paper. We frequently find portions of bone where the osteal cells, lacunal cells, and intermediate tissue are so perfectly fused together that neither can be recognized, but in their place we have a minutely granular mass, divisible only into lacunæ and canaliculi and the tissue in which they lie imbedded. In Plate IX. fig. 32, taken from the tibia of a fœtal lamb, the osteal and lacunal cells have become blended to a considerable extent, but without the

* When speaking of the growth of cartilage, it was stated that the bulk of that tissue at the epiphysis increased laterally by the division of the cells, but the fact that it also increased on the free surface of the greatly enlarged cartilaginous epiphysis of the fœtus by a process similar to that by which the diameter of bones is increased, was reserved to be described in connection with the latter subject. If a longitudinal section be taken from the epiphysis of a fœtal long bone, including some portion of the perichondrium, it will be seen that the cartilage passes gradually into a more or less fibrous tissue, which forms the exterior of the part. Amongst the fibres will be found numerous elongated cells, which on tracing the specimen inwards will be seen to be similar to the cartilage cells at and near the surface of the tissue, while the fibres will be seen to give way to or become converted into hyaline tissue, as shown in Plate IX. fig. 30, in which the cartilage is indicated at *a*, and the cells included in the fibrous tissue at *b*.

occurrence of lamination. The lamina which is represented at the margin is composed of still uncalcified cells. It shows the mode in which the undulating laminæ are formed. Fig. 33 also, taken from the fibula of a calf, shows ossified lacunal and osteal cells without any tendency to lamination.

The manner in which the Haversian spaces become gradually occupied by Haversian systems is peculiarly interesting. To obtain a good view of the process, it is necessary to make a transverse section of the developing systems. It may then be seen that osteal cells arrange themselves in single file within the Haversian space with intermediate lines of transparent tissue, and here and there a lacunal cell; the process commencing at the surface of the Haversian space, and extending gradually inwards till the system is completed. In fact the soft tissue takes the permanent form previous to the addition of the salts of bone, much in the same manner and to the same degree as occurs in temporary cartilage before the earthy ingredients are deposited. Lamination is nothing more than a definite linear arrangement of the osteal cells with their outlines permanently retained in the perfected bone; a character much more strongly marked in the bones of adult than in those of young animals.

In pursuing their inquiries into the growth of bone, the authors found it necessary to take sections of perfectly fresh bone and to examine them in albuminous fluid; spirits of wine, whether diluted or not, obscures the normal appearances, and water is not more favourable as a medium.

Sections permanently mounted in dilute spirit lose a good deal of their character; still the appearances are preserved with more or less of their original distinctness. Sections mounted in Canada balsam show some of the points remarkably well. For instance, the partially ossified osteal and lacunal cells are tolerably well preserved, but unfortunately those cells which have not received any of the indurating salts are represented only by a transparent mass, in which but little structure can be recognized; thus, the part *b* in fig. 32 would have appeared but as a transparent line had the specimen been mounted in Canada balsam. In addition to the necessity for care in the examination of osseous structures, under the most favourable circumstances, as regards the selection of specimens and the fluid used in their preparation, it is equally necessary to have the more recent appliances for the illumination of the objects.

Mr. Gillett's achromatic condenser, with what is called the white cloud illuminator, renders it very easy to demonstrate points which with the ordinary miscroscopic apparatus are shown with difficulty. It need not however be urged, that it is desirable in pursuing structural anatomy to avail ourselves of the most perfect instruments that can be obtained. It is proper that the authors should, before leaving this part of the subject, draw some comparison between the development of bone in cartilage and in the softer tissues.

Temporary cartilage, previous to the development of bone, affords a mechanical support and protection to the surrounding or enclosed softer tissues. These offices could not be rendered by a mere aggregation of soft cells, but are efficiently performed by the dense intercellular substance, which forms so characteristic a feature of the cartilage. Hence in the mechanical function of cartilage the intercellular element of the tissue must be regarded as the most important.

A second, and scarcely less important purpose effected by temporary cartilage, is that of affording a medium for which a more solid tissue may be substituted, without the mechanical support being withdrawn from the adjoining parts during the process of change. It affords also a means by which the long bones are gradually increased in length, without any interference with the functions of the limb. These changes are brought about by the gradual increase in the number of the cartilage cells, at those parts only where ossification is about to commence, and by the conversion of the cells into lacunal cells, at the cost of the intercellular tissue, which, whilst its bulk is diminishing, becomes impregnated with the earthy salts; so that although the quantity is lessened, the strength of that which remains is increased. Now, in examining the process by which growing bones are enlarged, it must be borne in mind that the mechanical function of the cartilage is performed by the bone which has replaced it, and that no such office is required of the new tissue which is gradually adding to the bulk of the bone. The necessity for a dense intercellular tissue no longer exists. Hence we have an aggregation of osteal cells with so much only of intercellular tissue as will serve to connect them into a mass. A certain number of these cells, by additions to their exterior, become lacunal cells. We have then in the place of lacunal cells and dense intercolumnar tissue, lacunal and osteal cells united by a small and almost imperceptible amount of a semifluid intercellular medium; the relative proportions of these several parts varying a little with differences of age. Thus in aged subjects aggregations of lacunal cells may here and there be found with but few osteal cells and but little intercellular tissue, while in a young subject osteal cells will preponderate.

In all cases a direction may be given to the cells of a growing part, by their being in relation with the insertion of a tendon.

A consideration of the foregoing facts will, we think, lead to a conviction that the two different forms in which bone is developed, are designed to meet the requirements of the animal at different periods of life, and that bone developed by means of osteal and lacunal cells is the higher form of the tissue. In early embryonic life a soft cartilaginous skeleton is laid down, which is hard as compared with the surrounding parts, and capable of affording them the required amount of mechanical support. As the various tissues advance in development the cartilage increases in density by the elaboration of its hyaline tissue, and after awhile those changes occur which precede its conversion into bone. But in the bone so formed, it has been shown that the canaliculi anastomose only at those points where the lacunal cells come into contact (Plate VIII. fig. 25). In bone formed from osteal and lacunal cells, the anastomosis of the canaliculi prevails throughout the tissue, while the outline of the formative cells is almost entirely lost in the general blending of the whole into a subgranular mass subject to lamination, in the manner already described.

It will therefore be in accordance with the preceding views, to regard the intercolumnar tissue present in primary bone, rather as an accessory element calculated to give support during the ossification of the lacunal cells, than as an integral and necessary element of osseous tissue; for it has been shown that it offers an obstruction to the anastomosis of the canaliculi, and that the lacunal cells preserve their outline so long as they are enclosed within it, instead of becoming lost in the surrounding tissue, as occurs in secondary bone. It must not, however, be forgotten that small patches of ossified intercolumnar tissue may here and there be found between the Haversian systems, even in the bone of old subjects (Plate VII. fig. 16); so that its presence in a small amount, (as accidentally left when the Haversian spaces are forming), is not incompatible with the normal condition of perfect bone.

It may also be stated in this place, that the undulating laminæ formed on the surface of growing bone, although for the most part removed to make way for Haversian systems, are also found here and there in small patches in the bones of adults.

The subject of absorption, as it relates to the removal of bone, has already been partly discussed, but something more remains to be said. Although the process of absorption has not, and probably cannot be seen in actual operation, yet a consideration of the relative position of the increasing and wasting parts, and of their conditions, will, in the authors' opinion, justify the conclusion that the bone is removed through the agency of cells.

In seeking to find the circumstances under which the absorption of bone takes place most rapidly, and to the greatest extent, the authors have been led to examine bones which had been placed in various conditions, both of health and disease. It has been already stated that bone developed in cartilage is speedily removed; but it does not seem that bone formed by osteal and lacunal cells is absorbed with equal rapidity, although in the course of a short time it not less surely disappears. This difference in the rate of absorption is probably the result of the vascularity and higher state of development of the latter tissue. In bone developed in cartilage Haversian spaces have to be formed before it can be permeated by vessels; hence in this tissue absorption proceeds rapidly, the process being established as soon as the bone is formed. In adult bone, when in a normal condition, we find here and there an Haversian space; but if a bone around which the soft parts are in a state of inflammation be examined, numerous Haversian spaces will be seen. If we examine the same bone in the neighbourhood of that part where the inflamed is merging into the healthy investing texture, we shall find new osseous matter in the process of deposition on the surface of the pre-existing bone. Examples of this may frequently be found in the vicinity of diseased joints.

We have here an instance of the development of new tissue, and the removal of a pre-existing one being set in operation, by what would appear to be a different degree only of the same action. Hence a more particular examination of the cellular mass, by which bone appears to be absorbed in the one case and deposited in the other, naturally suggests itself, and attention has been directed to this point as frequently as favourable opportunities for observation have presented themselves. The differences, however, in microscopic characters between the two tissues, have not been so strongly marked, as to admit of any definite description being given, by which the one may, in all instances, be distinguished from the other, when removed from their natural positions, as is almost necessarily the case with regard to the absorbent cells, from the readiness with which they separate from the wasting bone (Plate IX. figs. 27 and 34).

Lacunal cells moreover will not be found in the absorbing tissue; but it is not easy to recognize them, even in the tissue from which bone is developing, as they are frequently obscured by the osteal cells; so that their presence, although a good ground of distinction, cannot in all instances be demonstrated. When, however, developmental cells are seen *in situ*, their character is readily distinguished. They are closely applied to the surface of the increasing bone, which loses the festooned and assumes an even outline, to which the osteal cells adhere.

The source from which the cells destined for effecting absorption arise is at present unknown to the authors, any further than that they arise in connection with soft tissues, the three situations in which they occur being beneath the periosteum or medullary membrane, or within an Haversian canal. These are no doubt the points from which the development of the cells starts, but the law which regulates their occurrence at one part rather than another, and gives to them their peculiar function, is at present unexplained. Neither is it more easy to understand why, at a particular period, the process of absorption is arrested and development of new tissue commences; the new bone may, for anything we have seen to the contrary, be formed from the same cells that have been concerned in the removal of the old. The organ by which the fangs of teeth are absorbed offers a very favourable object for examination.

This we find to commence within or beneath the periosteum which covers the fang of the tooth, and increases with the wasting of the tooth, until it comes in contact with the pulp, which then assumes a similar function and becomes an absorbing organ, increasing gradually in size, till, if the tooth be left undisturbed, but little of the crown remains excepting the enamel. If, when in this condition, the crown of the tooth be carefully removed, we shall see the absorbent papillæ projecting a little above the gum, firm in substance, and not disposed to bleed unless rudely handled. Within two or three days it becomes covered with epithelium, either by the extension of the epithelial membrane from the surrounding parts, or from a change in character in the superficial cells of the part itself*. Gradually the papilla loses its

* In the 26th volume of the Medico-Chirurgical Transactions, there is a short paper by Mr. Dalrymple, in which he describes a small tumour from the eyelid, consisting essentially of epithelial cells, in which the ordinary contents had been replaced by ossific matter.

distinctness of outline and merges into the surrounding gum, from which it can no longer be distinguished. Here then we have a second instance in which an absorbent tissue has its function finally suspended, and a new one substituted. In the place of increasing at the cost of pre-existing tissue, growth is suspended, and it unites with and assumes a similar office to that of the surrounding gum.

In conclusion, the authors may allude to an interesting fact, with reference to the subject of absorption, which has been recently observed by Mr. STANLEY and subsequently by Mr. BOWMAN. In the treatment of ununited fracture, the practice has of late been successfully adopted of drilling cylindrical holes into the bone in the neighbourhood of the injury, and of driving in pegs of ivory, accurately fitted to the perforations, for the purpose of setting up action in their vicinity. These pegs, after remaining in the bone several weeks, have been removed, and have been found on inspection to present erosions similar to those which are seen on absorbing bone or tooth. In fact, they closely resemble the fangs of temporary teeth recently attacked by absorption. Mr. BOWMAN kindly placed at the disposal of the authors one of the pegs which had been used for this purpose, a section of which presents, at the parts eroded, in its microscopic characters, the usual emarginated outline so frequently alluded to in the foregoing pages. They have not however had the opportunity of investigating the condition of the tissues around the pegs; but it is exceedingly probable that absorption has in these cases been effected by processes similar to those which occur in the formation of Haversian spaces.

EXPLANATION OF THE PLATES.

PLATE VI.

Fig. 1. Transverse section of compact bone, showing the ordinary appearances. *a.* Haversian system. *b.* Interstitial laminæ. *c.* A new Haversian system within an older one, the Haversian canal obliterated by the development of a lacunal cell.

Fig. 2. Transverse section of compact bone, showing an Haversian space, with its characteristic emarginated outline.

Fig. 3. The same, from a less compact part of the bone, from a man aged fifty-six.

Fig. 4. Transverse section from the fibula of a child two years old, showing the general characters of bone in young subjects. *a.* Haversian space.

Fig. 5. Section of compact bone, showing several Haversian systems enclosed within a single series of laminæ. *a.* A lacunal cell attached to the surface of an Haversian canal.

Fig. 6. Section showing the characters of a new Haversian system, developed within an older one.

Fig. 7. An excentric Haversian system, illustrating the arrangement of the laminæ.

Fig. 8. *a.* Lamination of bone, as it appears under a low power. *b.* The same highly magnified, showing the lines of osteal cells and the intermediate transparent tissue.

PLATE VII.

Fig. 9. Transverse section of a stag's antler, after shedding, showing the transparent tissue which lines the Haversian canals.

Fig. 10. Section showing the circumferential laminæ, perforated by an Haversian system.

Fig. 11. Section showing the orifices of the canaliculi, with their parietes.

Fig. 12. Sections from rickety bone. *a.* Consolidated lacunæ. *b.* Lacunal cells, from which no canaliculi have proceeded.

Fig. 13. Section showing circumferential laminæ, with elongated canals passing amongst the lacunæ and canaliculi.

Fig. 14. Section of fossil bone, from the Wealden, supposed to be Pterodactyle, in which the nuclei have been preserved by fossilization.

Fig. 15. Ossified lacunal cells, from the spongy portion of a bone of an old subject.

Fig. 16. Section from the petrous portion of a temporal bone of an adult. *a.* Lacunal cells, containing lacunæ and canaliculi. *b.* Intercellular granular tissue, similar to that found in bone developed in cartilage.

Fig. 17. Section showing ossified articular cartilage. *a.* Lacuna developed from a cartilage cell. *b.* Lines of granules.

Fig. 18. Section from an incus in a subject aged seventy-five, showing osteal and lacunal cells on the surface.

PLATE VIII.

Fig. 19. Longitudinal section of ossifying temporary cartilage, showing—*a.* Ossified intercolumnar tissue. *b.* Lacunal cells. *c.* Granular cells in process of conversion into lacunæ. *d.* Lacunal cells imbedded in ossified intercolumnar tissue, without conversion into lacunæ.

Fig. 20. Transverse section of ossifying cartilage from femur of a child, fourteen days old. *a.* Intercolumnar tissue. *b.* Lacunal cell. *c.* Similar cell, with granular cell in process of conversion into a lacuna.

Fig. 21. Section from femur of same subject, taken a little way below the ossifying cartilage.

Fig. 22. A longitudinal section of new bone, a little further advanced than in fig. 19. *a.* A detached lacunal cell, showing the three parts of which these bodies are composed.

Fig. 23. From rickety bone. *a.* Lacunal cells. *b.* A detached granular cell in process of conversion into a lacuna.

Fig. 24. Longitudinal section of developing bone from the tibia of a calf, mounted in Canada balsam. *a.* Intercolumnar tissue, from which the lacunal cells have disappeared. *b.* Ossified intercolumnar tissue and lacunal cells. *c.* Space produced by absorption. *d.* Haversian system in process of formation.

Fig. 25. Transverse section of same, taken at level of *b* in fig. 24. *a.* Intercolumnar tissue. *b.* Ossified lacunal cell. *c.* Secondary bone.

Fig. 26. Arrangement of cells and fibres in the neighbourhood of developing parietal bone.

PLATE IX.

Fig. 27. Advancing spicula of bone, from parietal bone of fœtal lamb. *a.* Developed bone, showing its cellular character. *b.* Osteal cells arranged around the bone previous to their calcification.

Fig. 28. From the epiphysis of the long bone of a calf, showing osteal and lacunal cells.

Fig. 29. From the parietal of a human fœtus, showing osteal and lacunal cells, occupying the space betwixt two advancing spiculæ of bone.

Fig. 30. Longitudinal section of cartilage with epichondrium attached. *a.* The developed cartilage. *b.* Fibres of perichondrium, with cells of a similar character to those of the cartilage.

Fig. 31. Circumferential laminæ of human tibia, showing the manner in which the laminæ are developed from osteal cells. *b.* A compound cell, the origin? of the osteal cells.

Fig. 32. From the tibia of a fœtal lamb. *a.* Developed bone, showing the cellular character. *b.* Undulating lamina in process of development, and containing lacunal cells. *c.* Osteal cells, not yet arranged. At *d* is shown an Haversian space, enclosed in the advancing ossification, while at *e* the laminæ are approaching towards one another to enclose a space.

Fig. 33. From the fibula of a calf, showing growth on the external surface.

Fig. 34. From an inflamed femur, showing absorption, with the cells contained in the Haversian space.

V. *On the Periodic and Non-periodic Variations of the Temperature at Toronto in Canada. from* 1841 *to* 1852 *inclusive. By Colonel* Edward Sabine, *of the Royal Artillery, Treas. and V.P.R.S.*

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THE interest with which the comparison of the contemporaneous non-periodic variations of the temperature in Europe and America is regarded by meteorologists of great reputation, leads me to hope that the present communication may not be unacceptable to the Royal Society. The geographical situation of Toronto, both as respects latitude and inland position, renders it a very suitable locality for such comparison with stations in the middle parts of Europe, where similar observations have been made and where the conclusions to be derived from them are the subjects of active investigation. The years over which the observations at Toronto extended, 1841 to 1852 inclusive, were years of unusual meteorological activity in Europe; and the period of twelve years embraced by them is one of sufficient duration to supply a fair basis for generalisation on many of the problems which are now receiving the attention of meteorologists.

The observations at Toronto were made at the Magnetical and Meteorological Observatory at that station, by the detachment of the Royal Artillery by whom the duties of the observatory were carried on, and were under the direction of Captain Riddell, R.A., at the commencement of the observations, of Captain Younghusband, R.A., from February 1841 to the end of 1844, and of Captain Lefroy, R.A., from 1845 to 1852 inclusive. The latitude of the observatory is 43° 39′·6 N., and its longitude 79° 21′·5 west of Greenwich; the height above the surface of Lake Ontario is 108 feet, and above the Ocean about 342 feet.

From 1841 to 1844 inclusive, the thermometer was stationed beneath a roof on the outside of the north wall of the observatory. The height of the roof above the ground was about 7 feet, and three of the sides of the space which it covered (the observatory wall being the fourth side) were closed in by venetian-blind shutters descending from the roof to within 4 feet of the ground. The slope of the blinds of the shutters was such as to admit a free current of air whilst it completely screened the thermometer from rain. The shutters and the wall were coloured green, and the ground beneath was a grass-plat. The thermometer was fastened to two narrow strips of wood running horizontally from side to side of the thermometer shed, the ball being perfectly free, and about 6 inches higher than the bottom of the venetian blinds. The thermometer was read from the inside of the observatory through a double window; its distance from the outer window being about 8 or 9 inches. From January 1845 to the end of 1852, the thermometer shed had the additional protection of a second roof and a second enclosure of venetian blinds on three of the sides, similar to those already described, and exterior to them at an interval of space from one to one and a half foot. The length of the exterior shutter on the northern side was 7 feet, and on the east and west sides 5 feet 4½ inches. The thermometer was suspended as before with the ball perfectly free, and was read from the inside of the observatory through a window furnished with a sliding wooden shutter, which was kept closed, except at the times required for observation.

The same thermometer was employed throughout the whole period, excepting from March to December 1845, and in some few other occasional instances, in all of which corrections carefully ascertained were applied to give the values which would have been read by the thermometer usually employed. A comparison of this thermometer with a standard thermometer by Fastré of Paris, was made at all natural temperatures between April 1851 and February 1852, partly with the assistance of a telescope, and partly with the eye alone. The differences between the two thermometers at the several parts of the scale have been applied to all the observed temperatures before they were combined in the Table given in this paper. The standard thermometer by Fastré is divided *à l'échelle arbitraire* according to the method of Regnault. The zero-points were determined as follows:—

	0° Cent.	100° Cent.	For 1° Cent.
1851. Data furnished by M. Fastré	115·7	617·70	5·012
1851. By M. Izard and Capt. Lefroy at Paris	115·5	617·60	5·026

Re-examination at Toronto in 1852.

Freezing-point.		Boiling-point.		Barom. mm.	
January 13.	116·12	January 5.	614·60	743·055	617·76
14.	116·00	12.	616·55	755·136	617·45
22.	116·30	13.	614·00	740·866	617·57
28.	116·22*	28.	614·10	738·530	618·12
28.	116·10†				
	116·14=32° Fahr.				212° Fahr.=617·72

whence 1° Cent.=5·0158; and 1° Fahr.=2·7865 divisions.

This last value has been employed in the comparison in preference to the original determination at Paris, from which however the difference is extremely slight.

During six years, *i. e.* from July 1, 1842 to June 30, 1848, the observations were made hourly,—Sundays, Christmas Days, and Good Fridays excepted, on which days no observations whatsoever were made. In the remaining six years, *i. e.* from January 1, 1841 to June 30, 1842, and from July 1, 1848 to December 31, 1852, the

* Before boiling. † After boiling.

same system was preserved in respect to *days*, but the *hours* of observation were less frequent. The series may consequently be divided into two portions, of which the first, or that containing the six years of hourly observation, will be first discussed.

The following Table exhibits the mean temperature at every hour in the different months on the average of the six years.

Table I.

Mean Temperature of the Air at Toronto, obtained by hourly observations from July 1, 1842 to June 30, 1848.

Toronto astronomical time.	Months.												Hourly means.
	Jan.	Feb.	March.	April.	May.	June	July.	August.	Sept.	October.	Nov.	Dec.	
Hours.													
0.	27·83	27·07	34·00	47·53	58·80	66·55	72·85	72·30	63·52	49·50	39·57	29·93	49·12
1.	28·33	27·93	34·65	48·47	59·72	67·28	73·77	73·07	64·12	49·93	39·97	30·65	49·82
2.	28·60	28·33	35·22	48·85	60·07	67·70	74·62	73·65	64·52	50·28	40·05	30·80	50·22
3.	28·57	28·32	35·02	48·92	60·13	68·08	74·82	74·00	64·55	50·05	39·88	30·55	50·24
4.	28·05	27·77	34·55	48·53	60·08	68·32	74·83	73·85	64·33	49·32	38·98	29·90	49·88
5.	27·05	26·57	33·80	47·80	59·70	67·72	74·37	73·30	63·37	47·57	37·77	28·95	49·00
6.	26·23	25·12	32·12	46·00	57·95	66·42	72·93	71·40	60·70	45·52	36·95	28·25	47·47
7.	25·70	24·13	30·65	43·47	55·08	63·68	69·45	67·42	58·00	44·42	36·38	27·92	45·52
8.	25·38	23·28	29·68	41·88	52·37	60·38	65·25	64·50	56·72	43·68	36·07	27·53	43·89
9.	25·18	22·63	28·68	40·80	50·62	58·22	62·88	62·92	55·68	42·92	35·78	27·28	42·80
10.	24·80	22·08	28·03	40·03	49·65	56·88	61·65	61·90	54·62	42·17	35·43	26·98	42·02
11.	24·48	21·57	27·38	39·53	48·73	55·92	60·47	61·10	53·98	41·50	35·08	26·87	41·38
12.	23·80	21·45	27·33	39·37	47·88	55·37	59·45	60·30	53·63	40·95	34·42	26·53	40·87
13.	23·33	21·07	26·85	38·62	47·02	54·68	58·58	59·65	53·02	40·35	34·13	25·95	40·27
14.	23·25	20·73	26·47	37·95	46·18	53·98	58·02	58·97	52·43	40·03	33·85	25·58	39·79
15.	23·10	20·30	26·18	37·75	45·47	53·20	57·30	58·30	51·97	39·87	33·53	25·45	39·37
16.	23·00	20·00	25·80	37·32	45·00	52·63	56·67	57·92	51·38	39·57	33·37	25·42	39·01
17.	22·82	19·65	25·28	36·95	45·05	52·82	56·62	57·73	50·75	39·40	33·48	25·40	38·83
18.	23·55	19·08	25·00	37·08	47·50	55·47	59·83	59·18	51·43	39·62	33·75	24·98	39·71
19.	23·45	18·95	25·87	39·37	50·48	58·28	63·50	62·15	53·98	40·37	33·75	24·82	41·25
20.	23·68	19·97	27·85	41·62	52·70	60·62	66·10	65·42	56·73	42·62	34·80	25·25	43·11
21.	24·65	22·27	30·02	43·60	55·02	62·50	68·30	67·92	59·15	45·30	36·33	26·43	45·12
22.	25·88	24·28	31·75	45·12	56·72	64·17	70·00	69·90	61·12	47·23	37·77	27·88	46·82
23.	27·05	25·87	32·98	46·50	57·85	65·45	71·55	71·35	62·55	48·60	38·78	29·12	48·14
Monthly means.	25·32	23·27	29·80	42·63	52·91	60·68	65·99	65·76	57·59	44·20	36·24	27·44	44·32

The hourly observations from which this Table is formed will be found in full in the volumes of the Magnetical and Meteorological Observations at the British Colonial Obervatories, of which Toronto was one.

The temperature at any hour, and on any day of the year, corresponding to the mean values obtained in the six years and shown in Table I., may be computed from them by means of Bessel's well-known interpolation-formula*, which may be thus written for the purpose:—

$$\begin{aligned} t_{r} = A_0 &+ A_1 \cos a + B_1 \sin a + A_2 \cos 2a + B_2 \sin 2a \\ &+ A_3 \cos 3a + B_3 \sin 3a + A_4 \cos 4a + B_4 \sin 4a \\ &+ A_5 \cos 5a + B_5 \sin 5a + A_6 \cos 6a, \end{aligned}$$

* Astron. Nach. No. 136.

in which t_x is the temperature on x the required day, A_0 the mean temperature of the year at the hour required, $a = n \times 30°$, n denoting the number of months and parts of a month between January 15 and x, and $A_1, A_2, \ldots. A_6, B_1, B_2 \ldots. B_5$, constants derived from the values in Table I. by the method of least squares.

The constants for each of the hours, and for the mean temperature, derived from the six years of hourly observation, are as follows:—

Table II.

Constants for the calculation of the Temperature on the several Days and Hours from the Mean Monthly Values in six years.

Toronto astronomical time.	A_0.	A_1.	B_1.	A_2.	B_2.	A_3.	B_3.	A_4.	B_4.	A_5.	B_5.	A_6.
Hours.												
0.	49°·12	−23°·38	−2°·53	+0°·60	+0°·29	+0°·62	−1°·11	+0°·31	+0°·54	+0°·25	+0°·44	+0°·31
1.	49·82	−23·48	−2·35	+0·65	+0·31	+0·63	−1·17	+0·30	+0·57	+0·13	+0·45	+0·27
2.	50·22	−23·65	−2·28	+0·73	+0·45	+0·55	−1·16	+0·36	+0·56	+0·09	+0·40	+0·29
3.	50·24	−23·87	−2·21	+0·85	+0·47	+0·59	−1·17	+0·35	+0·60	+0·15	+0·47	+0·25
4.	49·88	−24·23	−2·02	+1·00	+0·46	+0·68	−1·14	+0·31	+0·52	+0·16	+0·48	+0·26
5.	49·00	−24·52	−1·73	−1·17	+0·42	+0·70	−1·36	+0·20	+0·50	+0·16	+0·48	+0·35
6.	47·47	−24·10	−1·69	+1·56	−0·03	+0·48	−1·43	+0·21	+0·57	+0·27	+0·50	+0·35
7.	45·52	−22·54	−2·03	+1·46	−0·41	+0·38	−1·10	+0·23	+0·40	+0·28	+0·46	+0·35
8.	43·89	−20·95	−2·55	+0·91	−0·31	+0·58	−1·09	+0·15	+0·27	+0·43	+0·55	+0·35
9.	42·80	−20·05	−2·89	+0·75	−0·29	+0·69	−1·20	+0·15	+0·30	+0·51	+0·62	+0·34
10.	42·02	−19·61	−2·97	+0·72	−0·33	+0·66	−1·30	+0·14	+0·37	+0·52	+0·60	+0·34
11.	41·38	−19·25	−3·07	+0·68	−0·37	+0·71	−1·42	+0·11	+0·34	+0·55	+0·67	+0·30
12.	40·87	−19·02	−2·93	+0·52	−0·21	+0·68	−1·40	+0·02	+0·17	+0·51	+0·76	+0·21
13.	40·27	−18·84	−3·03	+0·52	−0·17	+0·63	−1·35	−0·05	+0·20	+0·58	+0·81	+0·22
14.	39 79	−18·60	−3·13	+0·58	−0·14	+0·59	−1·29	+0·02	+0·18	+0·63	+0·80	+0·25
15.	39·37	−18·38	−3·20	+0·47	−0·13	+0·59	−1·35	+0·14	+0·11	+0·65	+0·79	+0·22
16.	39·01	−18·11	−3·28	+0·50	−0·19	+0·59	−1·41	+0·13	+0·15	+0·69	+0·75	+0·20
17.	38·83	−18·17	−3·30	+0·60	−0·48	+0·54	−1·37	+0·12	+0·24	+0·73	+0·70	+0·17
18.	39·71	−19·58	−3·05	+1·20	−1·01	+0·65	−1·18	+0·31	+0·38	+0·79	+0·61	+0·47
19.	41·25	−21·48	−2·62	+1·21	−0·92	+0·80	−1·46	+0·43	+0·34	+0·65	+0·66	+0·59
20.	43·11	−22·63	−2·60	+0·85	−0·49	+0·73	−1·51	+0·39	+0·35	+0·70	+0·58	+0·53
21.	45·12	−23·05	−2·59	+0·56	−0·13	+0·69	−1·31	+0·34	+0·50	+0·53	+0·42	+0·45
22.	46·82	−23·20	−2·63	+0·50	+0·07	+0·70	−1·20	+0·24	+0·54	+0·44	+0·37	+0·39
23.	48·14	−23·28	−2·63	+0·55	+0·22	+0·69	−1·17	+0·29	+0·54	+0·34	+0·40	+0·32
Mean of the 24 hours.	44·32	−21·41	−2·64	+0·80	−0·12	+0·63	−1·28	+0·22	+0·38	+0·45	+0·58	+0·32

From the temperatures computed by the formula with the constants in Table II., corrections have been obtained to the mean temperature of the day for each of the hours of observation throughout the year, to be employed whenever a less onerous system than that of hourly or two-hourly (equidistant) observation is adopted, as was the case in the second period (also of six years) at Toronto. Table III. exhibits these corrections for every hour on every fifth day throughout the year; the corrections on the intermediate days admitting of easy interpolation. The corrections are additive when in the larger character, and subtractive when in the smaller.

Table III.

Corrections for every Fifth Day of the Year, to be applied to the Temperature observed at Toronto at any of the hours of Mean Astronomical Time, in order to give the Mean Temperature of the Day.

First Part, January to June. The corrections in the smaller type are subtractive; in the larger type additive.

Month	Days of the month.	0h.	1h.	2h.	3h.	4h.	5h.	6h.	7h.	8h.	9h.	10h.	11h.	12h.	13h.	14h.	15h.	16h.	17h.	18h.	19h.	20h.	21h.	22h.	23h.
		Subtractive.									Additive.													Subtractive.	
January.	5	2·3	2·8	3·1	3·0	2·4	1·5	0·7	0·4	0·0	0·1	0·5	0·7	1·4	1·9	2·1	2·1	2·2	2·3	1·8	1·9	1·6	0·7	0·5	1·6
	10	2·4	2·9	3·1	3·1	2·5	1·6	0·8	0·2	0·0	0·1	0·5	0·8	1·5	2·0	2·1	2·2	2·3	2·4	1·7	1·8	1·6	0·7	0·5	1·7
	15	2·5	3·0	3·3	3·2	2·7	1·7	0·9	0·4	0·1	0·1	0·5	0·8	1·5	2·0	2·1	2·2	2·3	2·5	1·8	1·9	1·6	0·7	0·6	1·7
	20	2·6	3·1	3·4	3·4	2·9	1·9	0·9	0·4	0·0	0·2	0·6	1·0	1·6	2·0	2·1	2·3	2·4	2·7	2·0	2·1	1·8	0·7	0·6	1·8
	25	2·8	3·4	3·7	3·8	3·3	2·1	1·2	0·4	0·0	0·2	0·7	1·1	1·6	2·0	2·1	2·4	2·6	2·8	2·3	2·5	2·1	0·8	0·6	1·9
	30	3·0	3·7	4·0	4·1	3·6	2·4	1·3	0·5	0·0	0·4	0·8	1·2	1·7	2·0	2·2	2·5	2·7	3·0	2·8	3·0	2·5	0·9	0·6	2·0
February.	4	3·3	4·1	4·4	4·5	3·9	2·7	1·5	0·6	0·0	0·5	0·9	1·4	1·6	2·0	2·3	2·6	3·0	3·2	3·3	3·5	2·8	1·0	0·7	2·2
	9	3·5	4·3	4·7	4·8	4·2	3·0	1·6	0·7	0·0	0·6	1·1	1·6	1·8	2·1	2·5	2·7	3·1	3·4	3·8	4·0	3·2	1·0	0·8	2·4
	14	3·8	4·7	5·1	5·0	4·5	3·3	1·8	0·9	0·0	0·6	1·2	1·7	1·8	2·2	2·5	3·0	3·3	3·7	4·2	4·3	3·3	1·0	1·0	2·6
	19	3·9	4·8	5·3	5·2	4·7	3·5	2·0	0·9	0·0	0·7	1·3	1·9	1·9	2·3	2·7	3·1	3·5	3·8	4·5	4·6	3·4	0·9	1·2	2·7
	24	4·1	5·0	5·5	5·4	4·8	3·8	2·1	1·1	0·1	0·8	1·4	2·0	2·0	2·4	2·8	3·3	3·6	4·0	4·7	4·6	3·2	0·6	1·4	2·9
March.	1	4·2	4·9	5·5	5·3	4·8	3·9	2·2	1·0	0·1	0·9	1·5	2·1	2·1	2·6	2·9	3·5	3·8	4·2	4·8	4·5	3·0	0·4	1·6	3·0
	6	4·2	5·0	5·5	5·3	4·8	3·9	2·3	1·0	0·0	0·9	1·6	2·2	2·3	2·7	3·1	3·5	3·8	4·3	4·8	4·4	2·7	0·2	1·7	3·1
	11	4·2	4·9	5·5	5·2	4·8	3·9	2·3	0·9	0·0	1·1	1·7	2·4	2·4	2·9	3·2	3·6	3·9	4·4	4·8	4·2	2·3	0·0	1·8	3·1
	16	4·2	4·8	5·4	5·2	4·7	4·0	2·3	0·8	0·1	1·1	1·8	2·4	2·5	2·9	3·3	3·6	4·0	4·5	4·8	3·9	1·9	0·2	1·9	3·2
	21	4·2	4·8	5·4	5·2	4·8	4·0	2·1	0·7	0·2	1·2	1·9	2·5	2·5	3·0	3·4	3·6	4·0	4·6	4·8	3·6	1·7	0·4	2·0	3·2
	26	4·2	4·9	5·4	5·3	4·9	4·1	2·3	0·7	0·3	1·3	2·0	2·6	2·6	3·0	3·6	3·8	4·1	4·7	4·9	3·6	1·4	0·5	2·0	3·3
	31	4·3	5·1	5·5	5·4	5·0	4·3	2·4	0·6	0·5	1·4	2·1	2·7	2·7	3·3	3·8	3·9	4·3	4·8	5·0	3·5	1·2	0·6	2·1	3·3
April.	5	4·5	5·3	[illegible]	5·7	5·3	4·5	2·7	0·6	0·6	1·6	2·3	2·8	2·8	3·5	4·0	4·1	4·5	5·0	5·2	3·4	1·2	0·7	2·2	3·5
	10	4·6	5·5	5·9	6·0	5·6	4·8	3·0	0·7	0·7	1·7	2·5	3·0	3·0	3·7	4·3	4·5	4·9	5·3	5·4	3·4	1·1	0·8	2·3	3·6
	15	4·9	5·8	6·2	6·3	5·9	5·2	3·4	0·8	0·7	1·8	2·6	3·1	3·3	4·0	4·7	4·9	5·3	5·7	5·5	3·3	1·0	1·0	2·5	3·9
	20	5·1	6·1	6·5	6·6	6·2	5·6	3·7	1·0	0·8	2·0	2·7	3·3	3·5	4·4	5·1	5·3	5·8	6·1	5·7	3·1	0·9	1·1	2·7	4·1
	25	5·4	6·4	6·8	6·8	6·5	5·9	4·0	1·3	0·8	2·1	2·9	3·5	3·9	4·8	5·5	5·9	6·4	6·0	5·7	3·0	0·8	1·3	3·0	4·3
	30	5·6	6·6	7·0	7·1	6·8	6·2	4·4	1·5	0·8	2·1	3·0	3·7	4·2	5·1	5·9	6·4	6·9	6·8	5·7	2·9	0·7	1·6	3·2	4·6
May.	5	5·7	6·7	7·1	7·2	7·0	6·5	4·6	1·7	0·7	2·2	3·1	3·9	4·6	5·5	6·3	6·9	7·3	6·4	5·7	2·7	0·5	1·8	3·5	4·7
	10	5·9	6·8	7·2	7·3	7·1	6·7	4·9	2·0	0·6	2·2	3·1	4·0	4·8	5·7	6·5	7·2	7·6	7·7	5·5	2·5	0·3	2·0	3·7	4·9
	15	5·9	6·8	7·2	7·2	7·2	6·8	5·0	2·2	0·5	2·3	3·3	4·2	5·0	5·9	6·7	7·4	7·9	7·8	5·4	2·4	0·2	2·1	3·8	4·9
	20	5·9	6·8	7·1	7·2	7·3	6·9	5·4	2·4	0·4	2·3	3·3	4·3	5·1	5·9	6·7	7·5	8·0	7·9	5·2	2·3	0·1	2·0	3·9	5·0
	25	5·9	6·7	7·0	7·2	7·3	6·9	5·5	2·5	0·4	2·3	3·4	4·4	5·2	6·0	6·8	7·5	8·0	7·9	5·1	2·3	0·1	2·2	3·9	5·0
	30	5·8	6·6	6·9	7·2	7·3	6·8	5·5	2·6	0·3	2·3	3·5	4·4	5·2	5·9	6·7	7·5	8·0	7·8	5·1	2·3	0·1	2·1	3·8	4·9
June.	5	5·8	6·6	6·9	7·2	7·4	6·9	5·6	2·7	0·3	2·3	3·6	4·5	5·2	5·9	6·6	7·5	8·0	7·8	5·1	2·4	0·1	2·0	3·7	4·8
	10	5·8	6·6	6·9	7·3	7·5	7·0	5·7	2·9	0·2	2·3	3·6	4·6	5·2	5·9	6·6	7·4	7·9	7·9	5·1	2·4	0·0	1·9	3·6	4·8
	15	5·9	6·6	7·0	7·4	7·6	7·0	5·7	3·0	0·3	2·5	3·8	4·8	5·3	6·0	6·7	7·5	8·0	7·9	5·2	2·4	0·1	1·8	3·5	4·8
	20	6·0	6·7	7·2	7·6	7·9	7·2	5·9	3·1	0·3	2·5	3·9	4·9	5·5	6·2	6·8	7·7	8·2	8·1	5·3	2·4	0·1	1·8	3·5	4·8
	25	6·1	6·9	7·5	7·8	8·0	7·3	6·2	3·3	0·4	2·7	4·0	5·1	5·7	6·4	7·0	7·8	8·4	8·3	5·5	2·3	0·0	1·8	3·5	4·9
	30	6·3	7·2	7·8	8·1	8·3	7·7	6·4	3·4	0·4	2·8	4·1	5·2	5·9	6·7	7·3	8·0	8·7	8·6	5·6	2·3	0·0	1·9	3·6	5·0

Table III. (Continued.)

Second Part, July to December. The corrections in the smaller type are subtractive; in the larger type additive.

Month	Days of the month.	0h.	1h.	2h.	3h.	4h.	5h.	6h.	7h.	8h.	9h.	10h.	11h.	12h.	13h.	14h.	15h.	16h.	17h.	18h.	19h.	20h.	21h.	22h.	23h.
		Subtractive.								Additive.												Subtractive.			
July.	5	6·5	7·4	8·1	8·4	8·5	8·0	6·7	3·5	0·5	2·9	4·2	5·3	6·2	6·9	7·6	8·3	8·9	9·0	5·8	2·3	0·1	2·0	3·7	5·2
	10	6·7	7·6	8·4	8·6	8·7	8·2	6·8	3·5	0·6	3·0	4·3	5·5	6·4	7·2	7·8	8·5	9·2	9·2	6·0	2·4	0·1	2·2	3·9	5·4
	15	6·9	7·8	8·6	8·8	8·8	8·4	6·9	3·5	0·7	3·1	4·3	5·5	6·5	7·4	8·0	8·7	9·3	9·4	6·2	2·5	0·1	2·3	4·0	5·6
	20	7·0	7·8	8·7	8·9	8·9	8·4	6·8	3·3	0·9	3·2	4·3	5·5	6·6	7·5	8·0	8·8	9·3	9·4	6·4	2·7	0·1	2·4	4·1	5·7
	25	7·0	7·9	8·8	8·9	8·9	8·4	6·7	3·1	1·0	3·2	4·3	5·4	6·6	7·4	8·0	8·7	9·2	9·3	6·4	2·9	0·0	2·4	4·3	5·8
	30	7·0	7·9	8·7	8·9	8·8	8·3	6·5	2·8	1·1	3·1	4·2	5·3	6·4	7·2	7·8	8·5	9·0	9·0	6·5	3·1	0·1	2·4	4·3	5·8
August.	5	6·9	7·7	8·5	8·8	8·6	8·1	6·3	2·4	1·2	3·1	4·2	5·1	6·1	6·9	7·5	8·2	8·6	8·7	6·6	3·3	0·2	2·4	4·3	5·8
	10	6·6	7·4	8·1	8·4	8·3	7·8	5·9	2·0	1·3	3·1	4·0	5·0	5·9	6·6	7·3	7·9	8·3	8·4	6·7	3·6	0·3	2·2	4·2	5·6
	15	6·5	7·3	7·9	8·2	8·1	7·5	5·6	1·7	1·3	2·8	3·9	4·7	5·5	6·1	6·8	7·5	7·8	8·0	6·6	3·6	0·3	2·2	4·1	5·6
	20	6·3	7·0	7·6	7·9	7·8	7·2	5·2	1·3	1·3	2·7	3·7	4·4	5·1	5·7	6·4	7·1	7·5	7·8	6·5	3·7	0·4	2·0	4·0	5·4
	25	6·2	6·8	7·3	7·6	7·5	7·0	4·7	1·0	1·2	2·5	3·5	4·2	4·8	5·4	6·1	6·7	7·1	7·5	6·5	3·6	0·4	1·9	3·9	5·3
	30	6·1	6·7	7·1	7·4	7·2	6·6	4·3	0·7	1·2	2·3	3·4	4·0	4·5	5·1	5·8	6·4	6·8	7·3	6·4	3·6	0·5	1·8	3·8	5·2
September.	5	6·0	6·6	7·0	7·2	7·0	6·3	3·9	0·6	1·0	2·1	3·2	3·9	4·3	4·8	5·5	6·1	6·6	6·7	6·3	3·6	0·6	1·7	3·6	5·1
	10	5·9	6·5	6·9	7·0	6·9	6·1	3·4	0·4	1·0	2·0	3·1	3·7	4·1	4·7	5·3	5·8	6·4	7·0	6·2	3·6	0·7	1·6	3·6	5·0
	15	5·9	6·5	6·9	7·0	6·7	5·8	3·1	0·4	0·9	1·9	3·0	3·6	4·0	4·6	5·2	5·6	6·2	6·8	6·2	3·6	0·9	1·6	3·5	5·0
	20	5·9	6·5	6·9	6·9	6·5	5·5	3·0	0·3	0·8	1·8	2·9	3·5	3·9	4·5	5·0	5·4	6·0	6·7	6·1	3·7	1·0	1·4	3·4	4·9
	25	5·9	6·5	6·9	6·7	6·4	5·1	2·6	0·4	0·7	1·7	2·7	3·4	3·8	4·4	4·9	5·3	5·8	6·4	6·0	3·8	1·2	1·4	3·4	4·9
	30	5·8	6·4	6·8	6·6	6·2	4·7	2·3	0·3	0·6	1·6	2·6	3·3	3·7	4·3	4·8	5·0	5·5	6·1	5·7	3·9	1·4	1·3	3·3	4·8
October.	5	5·7	6·2	6·6	6·4	5·9	4·3	1·9	0·3	0·6	1·5	2·4	3·2	3·6	4·2	4·6	4·8	5·3	5·7	5·4	4·0	1·5	1·2	3·2	4·7
	10	5·6	6·0	6·4	6·2	5·6	3·9	1·7	0·3	0·5	1·4	2·2	2·9	3·4	4·0	4·4	4·6	4·9	5·2	5·0	3·9	1·5	1·2	3·2	4·6
	15	5·3	5·7	6·1	5·8	5·1	3·4	1·3	0·2	0·5	1·3	2·0	2·7	3·2	3·8	4·2	4·3	4·6	4·8	4·6	3·8	1·6	1·1	3·0	4·4
	20	5·0	5·4	5·7	5·5	4·7	2·9	1·1	0·2	0·5	1·1	1·8	2·4	3·1	3·6	3·9	4·0	4·4	4·4	4·1	3·6	1·5	1·0	2·9	4·1
	25	4·7	5·0	5·3	5·1	4·2	2·5	1·0	0·1	0·4	1·0	1·6	2·2	2·9	3·3	3·6	3·7	4·0	4·0	3·6	3·3	1·4	0·9	2·6	3·9
	30	4·3	4·6	4·8	4·6	3·7	2·1	0·8	0·1	0·4	0·9	1·4	1·9	2·7	3·0	3·3	3·5	3·7	3·6	3·2	3·1	1·4	0·7	2·4	3·5
November.	5	3·9	4·3	4·4	4·2	3·3	1·9	0·8	0·1	0·3	0·7	1·2	1·6	2·4	2·7	2·9	3·2	3·4	3·3	2·9	2·8	1·3	0·6	2·1	3·2
	10	3·6	3·9	4·1	3·9	3·0	1·6	0·7	0·1	0·3	0·6	1·0	1·4	2·1	2·4	2·7	3·0	3·1	3·0	2·6	2·6	1·4	0·3	1·8	2·8
	15	3·3	3·7	3·8	3·6	2·7	1·5	0·7	0·1	0·2	0·5	0·8	1·2	1·8	2·1	2·4	2·7	2·9	2·8	2·5	2·5	1·4	0·1	1·5	2·5
	20	3·1	3·5	3·6	3·4	2·6	1·5	0·5	0·2	0·1	0·4	0·7	1·0	1·6	1·9	2·2	2·5	2·6	2·4	2·5	2·5	1·6	0·2	1·2	2·3
	25	2·9	3·4	3·5	3·3	2·5	1·4	0·5	0·2	0·1	0·3	0·6	0·8	1·3	1·6	2·0	2·3	2·5	2·4	2·5	2·5	1·8	0·5	1·0	2·0
	30	2·8	3·4	3·5	3·2	2·5	1·5	0·6	0·3	0·0	0·2	0·5	0·7	1·1	1·5	1·9	2·2	2·3	2·2	2·5	2·6	2·0	0·7	0·8	1·9
December.	5	2·7	3·3	3·4	3·2	2·5	1·5	0·7	0·4	0·0	0·2	0·5	0·6	1·0	1·4	1·8	2·1	2·1	2·1	2·5	2·7	2·1	0·9	0·6	1·8
	10	2·5	3·2	3·4	3·1	2·4	1·5	0·7	0·4	0·0	0·2	0·5	0·6	0·9	1·5	1·9	2·1	2·1	1·9	2·5	2·7	2·2	1·0	0·5	1·8
	15	2·5	3·2	3·4	3·1	2·5	1·5	0·8	0·5	0·1	0·2	0·5	0·6	0·9	1·5	1·9	2·0	2·0	2·0	2·5	2·6	2·2	1·0	0·4	1·7
	20	2·4	3·1	3·3	3·0	2·4	1·5	0·8	0·5	0·1	0·1	0·4	0·6	1·0	1·6	1·9	2·1	2·0	2·0	2·3	2·5	2·1	1·0	0·4	1·6
	25	2·4	3·0	3·2	3·0	2·4	1·6	0·8	0·4	0·1	0·1	0·4	0·6	1·1	1·7	1·9	2·0	2·0	2·1	2·2	2·2	1·9	0·9	0·4	1·6
	30	2·3	2·9	3·1	3·0	2·2	1·5	0·8	0·4	0·1	0·1	0·5	0·7	1·2	1·8	2·0	2·0	2·1	2·1	1·9	2·0	1·8	0·8	0·5	1·6

The temperatures at the several hours of the day and on the several days of the year, computed in the manner which has been described, furnish data for the delineation of the Chrono-Isothermal Lines in Plate X. figs. 1 and 2. The representation in this Plate of the phenomena of the temperature at Toronto is on the same principle which has long been used in maps of the magnetic isogonic, isoclinal, and isodynamic lines, in which three variables are comprehended, of which one, the magnetical element, is dependent upon the other two, viz. the geographical latitude and longitude. The principle is also the same as that adopted in ordinary isothermal maps; but, whereas in those maps the two variables upon which the meteorological variation is dependent are the geographical latitude and longitude, in the present case the two variables are the hour of the day and the day of the year. The variation of the temperature is here referred to *time* and not to *space*. This difference is intended to be conveyed by the term Chrono-Isothermals as applicable to lines of the present description, whilst those of the more customary form might not inappropriately be termed, for contradistinction, as suggested to me by Mr. Wheatstone, Topo-Isothermals. Representations of the Chrono-Isothermals in different localities might materially facilitate the comparison of the phenomena of the temperature in different parts of the globe; and if similar plates were constructed for each of the meteorological elements at the same locality, they might be useful for intercomparison, and for illustrating the mutual dependence of the elements on each other, to those who prefer a representation to the eye to the instruction conveyed by tabular abstracts. In fig. 1 the isothermals are represented which are comprised between the mean temperature of the year and the *highest* isothermal; in fig. 2 between the mean temperature and the *lowest* isothermal. In fig. 1, consequently, the isothermals increase in value from the circumference to the centre; whilst in fig. 2 they decrease from the circumference to the centre. To avoid too great a multiplicity of lines only every alternate degree (the even degrees of Fahrenheit's scale) are drawn. Figs. 1 and 2 are so arranged on the Plate that the summer and winter solstices (marked by a dotted line) are in the same vertical. The abnormal character of the winter temperature of the North American Continent is strikingly manifested in this Plate.

From the temperatures computed from the six years of observation and represented in the plate, we learn many facts regarding the temperature at Toronto, which are interesting in themselves, and may become particularly so in their comparison with the phenomena in other parts of the globe. Amongst these may be noticed the following:—The mean annual range, or the difference between the mean temperatures of the coldest and hottest months (February and July), is 42°·7. The warmest day of the year is July 28, being thirty-seven days after the summer solstice. The coldest day is February 14, being fifty-five days after the winter solstice. The mean temperature of the year is passed through on April 19 and October 15. The warmest and coldest days, and the days on which the mean temperature is passed through, deduced by a similar process at Königsberg by Bessel, at Paris, Turin and Padua by Kämtz, at Berlin by Mädler, and at Prague by Fritsch and Jelinek, are collected by the last-named meteorologist in his memoir "On the daily march of the principal meteorological elements deduced from hourly observations at the Prague Observatory," published in the Transactions of the Imperial Academy of Sciences at Vienna in 1850, and are as follows:—

	Maximum.	Minimum.	Days on which the mean temp. of the year is passed through.	
Königsberg	August 1	January 9	April 21	October 20
Berlin (18 years) . .	July 18	January 19	April 19	October 21
Berlin (92 years) . .	July 22	January 12	April 17	October 16
Prague (8 to 9 years) .	July 24	January 26	April 16	October 20
Prague (76 years) . .	July 23	January 19	April 15	October 18
Paris	July 28	January 15	April 18	October 19
Turin	July 27	January 3	April 18	October 26
Padua	July 26	January 15	April 20	October 15

These may be compared with the corresponding epochs at Toronto, as derived respectively from the six-years and the twelve-years series discussed in this paper:

Toronto (1842, 5 to 1848, 5)	July 28	February 14	April 19	October 15
Toronto (1841 to 1852)	July 28	February 12	April 25	October 17

The anomalous character of the North American winter, so visible in the Chrono-Isothermal Plate, is also marked by the very late occurrence of the epoch of the *minimum* temperature, and the great dissimilarity in that respect from all the other stations. The systematic character of this anomaly is further shown by the fact, that every hour in the twenty-four has its minimum temperature between the 7th and 17th of February; the minimum occurs earliest, viz. on the 7th of February, at the hour of 2 P.M.; the minima of the hours of the night, or from 9 P.M. to 7 A.M. inclusive, fall the latest, viz. on the 15th, 16th and 17th of February; those of the intermediate hours on the intermediate days and in regular progression. The hours from 6 A.M. to 9 P.M. inclusive, or those of the *day*, have their *maximum* temperature between the 20th and 30th of July; those of the *night*, or from 11 P.M. to 5 A.M. inclusive, from the 3rd to the 12th of August. The portion of the twenty-four hours which is warmer than the mean temperature of the day varies considerably at different seasons; in part of November there are fourteen of the observation hours colder, and only ten warmer than the mean temperature of the day; in the greater part of July twelve of the observation hours are colder and twelve warmer; and in all the rest of the year thirteen hours colder and eleven warmer. This is seen in detail in Table III.; the dark lines which separate the additive from the subtractive corrections pass between the hours which are above and those which are below the mean temperature on every fifth day of the year. On the average of the whole year the mean temperature is passed through about 8^h 31^m A.M., and 7^h 44^m P.M., making intervals of 11^h 13^m and 12^h 47^m. The hours from 9 P.M. to 7 A.M. inclusive are throughout the year colder than the mean temperature of the day; those from 10 A.M. to 7 P.M. are throughout the year warmer than the mean temperature of the day; 8 and 9 A.M., and 8 P.M., are sometimes warmer and sometimes colder than the mean temperature; 8 A.M. is colder except for about three weeks in July, and 9 A.M. is warmer except from November 20 to March 11; 8 P.M. is colder from the middle of March till late in November, and either coincides with the mean temperature, or is slightly warmer during the remainder of the year.

The hours of highest and lowest temperature on every fifth day of the year, and the amount by which the temperature at those hours exceeds or falls short of the mean temperature of the day, may be examined in detail in Table III. From the third week of September until April, 2 P.M. is the warmest hour, with the exception of some days in January and February, when 3 P.M. is warmer; from April to the middle of May, and again from the end of July to the middle of September, 3 P.M. is the warmest hour; and from the middle of May to the middle of July, 4 P.M. The coldest hour from the latter part of April to the end of June, and again from the end of October to late in November, is 4 A.M.; from the middle of July to the middle of October, in January, and for a short time in the middle of April, it is 5 A.M.; from the latter end of February to early in April it is 6 A.M.; and generally in December and February 7 A.M. The range from the minimum to the maximum in the day is greatest in July (18°·2), and least at the end of December (5°·2). The daily range has but one maximum in the year, which is in July; not as at Prague, where June and July have a less range than the months immediately preceding and following them, and where consequently there are two maxima; a phenomenon attributed to the greater prevalence of clouds in June and July.

It may be desirable to add a few words on the assistance to observers of tables which furnish corrections to the mean temperature of the day for every hour of every day in the year, such as Table III. Besides their direct use at the station itself, they have a useful bearing, within a reasonable distance from the station, on the selection of observation hours in the many cases in which it may not be possible to observe at hourly or two-hourly intervals, by affording a ready means of estimating the amount of error to which a deduction from any limited combination of hours is subject. If we desire for example to seek the observation hours within the command of a single observer, which may give the best approximation to the mean temperature of the day, and to that of the month, and of the year, as well as to the *climatic* difference (*i. e.* the difference between the hottest and the coldest months), we find, that of homonymous hours, the best pairs at Toronto are 9^h—9^h and 10^h—10^h, 10^h—10^h being the better of the two; but that 8^h—8^h, which is a combination frequently adopted by observers, does not suit so well at Toronto as either 9^h—9^h or 10^h—10^h. The average errors in the different months, when the temperature is inferred from observations at 9^h—9^h or 10^h—10^h instead of from hourly observations, are shown on the average of six years to be as follows:—the sign + denoting that the mean daily temperature in the month is given too high, and − that it is given too low, by the mean of the two compared with the mean of the twenty-four observations.

9^h—9^h				10^h—10^h			
January	−0°·4	July	−0°·4	January	0°·0	July	−0°·2
February	−0·8	August	−0·3	February	−0·1	August	+0·1
March	−0·5	September	−0·2	March	+0·1	September	+0·3
April	−0·4	October	−0·1	April	−0·1	October	+0·5
May	−0·1	November	−0·2	May	+0·3	November	+0·4
June	−0·3	December	−0·6	June	−0·2	December	0·0
Mean of the year −0°·36				Mean of the year +0°·09			

Difference of the hottest and coldest months:

Too great by 0°·4 — Too great by 0°·15

For the purpose of combining with an approximate mean temperature of the day, an approximation also to the hottest and coldest hours of the day, and to the hours of maximum and minimum of other meteorological elements, *three* equidistant observations are frequently adopted in preference to a binary system, and the hours of 6 A.M., 2 P.M. and 10 P.M. appear to be usually preferred. These hours are still within the command of a single observer, though we often find substituted for them the non-equidistant hours of 7 A.M., 2 P.M. and 9 P.M., doubtless because they suit better the convenience of observers. On comparing the mean temperatures in the different months derived from 6^h, 2^h, 10^h, or 7^h, 2^h, 9^h, with the full complement of twenty-four hours, we find that the approximation to the mean temperature obtained by 7^h, 2^h, 9^h is not quite so good as by 6^h, 2^h, 10^h; and that either of the triplets give a less correct mean temperature than 10^h—10^h: $6\frac{1}{2}$, 2, $9\frac{1}{2}$ would appear a more suitable combination as far as regards approximation to the mean temperature.

6 A.M., 2 P.M., 10 P.M.				7 A.M., 2 P.M., 9 P.M.			
January	+0°·3	July	−0°·6	January	+0°·4	July	+1°·0
February	−0·1	August	−0·9	February	0·0	August	+0·5
March	−0·4	September	−0·7	March	+0·1	September	+0·5
April	−0·6	October	−0·2	April	+0·4	October	+0·3
May	−0·5	November	+0·2	May	+0·8	November	+0·3
June	−0·7	December	+0·1	June	+0·7	December	+0·2
Mean of the year −0°·34				Mean of the year +0·43			

Difference of the hottest and the coldest months:

Too small by 0°·4 — Too great by 1°·0

Three equidistant observations in the twenty-four hours are the utmost that can be perseveringly maintained by a single observer. When there are two or more observers, there is no difficulty in multiplying the times of observation so as to comprehend all the objects that may be desired, each in the manner and by the means which are most suitable to it and will be most satisfactory. But as the work of observation at by far the greater number of meteorological stations is usually carried out by a single observer, and as this is likely to be always the case, it should be a primary object with meteorologists who are furnished with sufficient means, to form tables of corrections to the mean daily temperature for every hour of the day, upon the basis of a sufficient number of years of observation, to be used at the respective localities, or within the distances to which such tables may be severally applicable, by persons whose means or convenience may restrict them in respect to the number and choice of hours of observation. With such a table, similar to Table III. of this paper, the choice is disembarrassed of its chief difficulty, that of selecting hours which by their combination will give an approximate mean temperature for the several months and for the year; and the observer is left free to give a preference, independent of such consideration, either to the hours when the phenomena change least rapidly, and when consequently small irregularities in the times of observation will be least injurious, or to the hours which will furnish the best approximation to the daily maxima and minima of the meteorological elements generally, viz. of the temperature, the tension of vapour, the pressure of the gaseous atmosphere, and the force of the wind; or to the hours which will have the most effective bearing upon other points of meteorological or climatic interest, to which the observer's attention may be directed.

The equation of the mean annual variation of the temperature, in the form most convenient for use, deduced from the twelve monthly means in the lowest horizontal line of Table I., is as follows:—

$$t_x = +44^\circ{\cdot}32 - 21^\circ{\cdot}57 \sin(a + 82^\circ\ 58') + 0^\circ{\cdot}81 \sin(2a + 278^\circ\ 32')$$
$$- \ 1^\circ{\cdot}43 \sin(3a + 333^\circ\ 48') + 0^\circ{\cdot}44 \sin(4a + 30^\circ\ 04')$$
$$+ \ 0^\circ{\cdot}73 \sin(5a + 37^\circ\ 48') + 0{\cdot}32 \cos 6a,$$

in which $a = n \times 30^\circ$, n denoting the number of months and parts of a month between January 15th and x.

The equation of the mean diurnal variation of the temperature deduced from the twenty-four hourly means in the last vertical column of Table I. is—

$$t_x = +44^\circ{\cdot}32 + 5^\circ{\cdot}513 \sin(a + 53^\circ\ 40') + 0^\circ{\cdot}82 \sin(2a + 59^\circ\ 08')$$
$$- 0^\circ{\cdot}48 \sin(3a + 41^\circ\ 41') - 0^\circ{\cdot}06 \sin(4a + 51^\circ\ 23')$$
$$+ 0^\circ{\cdot}04 \sin(5a + 20^\circ\ 35'),$$

in which $a = n \times 15^\circ$, n being the number of hours and parts of an hour between x, and 0 hours or astronomical mean noon.

I now proceed to the second portion of the series, or to the six years in which the observations were for the most part made on a less perfect system than that of twelve or twenty-four equidistant observations in the twenty-four hours. The hours of observation during this portion of the series were as follows:—

January 1841 to June 1842 inclusive; two-hourly equidistant.
July 1848 to December 1848 inclusive; 2, 3, 4, 9, 10, 11, 17, 19, 21, 22.
1849. January to December inclusive; 1, 2, 3, 5, 6, 9, 10, 11, 18, 19, 20, 22.
1850. January to April inclusive; hourly.
1850. May to June inclusive; 2, 3, 10, 11, 18, 19.
1850. July and August; hourly.
1850. September to December inclusive; two-hourly equidistant.
1851. January to April inclusive; two-hourly equidistant.
1851. May to December inclusive; 2, 4, 10, 12, 18, 20.
1852. January to December inclusive; 2, 4, 10, 12, 18, 20.

Whenever the observations during these six years were either hourly, or two-hourly and equidistant, a mean of the observations simply, without corrections applied to any of them, has been taken as the mean temperature of the day. In all the other cases corrections taken from Table III. have been applied individually to the observations at the hours to which they refer, and the mean temperature of the day has been computed from the mean of the observations so corrected.

Having thus obtained the mean daily temperatures during the second period of six years, we have the mean monthly temperatures derived from the twelve years as follows:—

January .	24°·97	April .	41°·14	July . .	66°·41	October .	44°·93
February .	23·40	May .	51·18	August . .	66·16	November .	36·51
March . .	30·23	June .	61·05	September	58·02	December .	26·75

Mean of the whole 44°·23.

Employing these values, we obtain the formula representing the mean annual variation of the temperature at Toronto as follows:—

$$t_x = +44^\circ{\cdot}23 - 21^\circ{\cdot}81 \sin(a+81^\circ\ 27') + 1^\circ{\cdot}06 \sin(2a+71^\circ\ 32')$$
$$- 0^\circ{\cdot}80 \sin(3a+347^\circ\ 42') + 0^\circ{\cdot}22 \sin(4a+37^\circ\ 27')$$
$$+ 0^\circ{\cdot}88 \sin(5a+50^\circ\ 41') + 0^\circ{\cdot}325 \cos 6a,$$

$a = n \times 30^\circ$; n being reckoned as before from January 15th.

Table IV. (pages 154 to 159 inclusive) exhibits in column 1 the mean temperature of every day of the year, computed by the preceding formula; and in columns 2 to 13, under the respective years 1841 to 1852, the differences of the mean temperatures actually observed on each day from the mean of the twelve years computed as described and shown in column 1. These "differences" are the *non-periodic variations* of each day, on the assumption that the monthly means of twelve years furnish a sufficient basis for the deduction of approximate normal values; which is certainly true within the limits which are at present required for the comparison of the non-periodic variations in Europe and America. The observed temperatures themselves may be obtained, if they are required, by adding or subtracting (as the case may be) from the mean daily temperature in column 1, the difference which stands on the same horizontal line with the *day*, and in the same vertical line with the *year*. The final column (14) shows for each day the *average* non-periodic variation in twelve years. We may learn, consequently, from this column, the average non-periodic variation in twelve years of any particular day of the year which may be surmised to be subject to some special physical peculiarity, causing it to be warmer or colder than the general progression of the temperature in the part of the year to which it belongs. An example of its application may be given by the reply which the values in this column furnish to the question*, whether the three days of May (the 11th, 12th, and 13th), which MÄDLER has stated to be characterized, on the average of eighty-six years of observation at Berlin, by a depression exceeding 2° FAHR. when compared with the general march of the temperature at that season, undergo a similar depression in North America. On a reference to the month of May in Table IV., it is seen in column 14 that on the average of the twelve years from 1841 to 1852, the 11th of May was 0°·1 *below*, and on the 12th and 13th of May respectively 3°·1 and 2°·4 *above* the general mean of the temperature in those years. It may be seen also that the average non-periodic variation in the five days from the 8th to the 12th of May inclusive, is in the same twelve years 1°·1 *above*, and in the five days from the 13th to the 17th inclusive 1°·0 *above*, the general mean of the temperature. The meteorological observations at Toronto during these twelve years do not therefore support the supposition that the depression of temperature on the 11th, 12th and 13th of May, observed at Berlin, is a general and periodically recurring phenomenon over the whole globe; such as would be occasioned by a partial obscuration of the sun's disc by the intervention of a periodical stream of aërolites; but they tend rather to indicate that the depression observed in Europe may have been a partial phenomenon, having a local cause.

The not unfrequent occurrence of differences of large amount on single days, shown in columns 2 to 13, is an indication of the great variability of the climate of Toronto in regard to temperature; and the still remaining occasional magnitude of the daily averages in column 14, shows that the influence of non-periodic variations is by no means extinguished in the means of twelve years.

* Kosmos, Bd. i. S. 407. Anm. 56.

TABLE IV.—Non-Periodic Variations of the Temperature at Toronto, from January 1841 to December 1852 inclusive.

Month	Day	Daily temperature derived from the monthly mean of the 12 years.	Difference on each day of the twelve years from the temperature derived from the monthly means.												Mean daily difference.
			1841.	1842.	1843.	1844.	1845.	1846.	1847.	1848.	1849.	1850.	1851.	1852.	
		(1.)	(2.)	(3.)	(4.)	(5.)	(6.)	(7.)	(8.)	(9.)	(10.)	(11.)	(12.)	(13.)	(14.)
January.	1	25°·2	− 9°·6	+ 8°·2	Sunday.	− 0°·1	+ 9°·9	+ 8°·5	+ 9°·8	+19°·0	−11°·2	− 3°·7	− 2°·0	+ 0°·4	+2°·7
	2	25·2	− 9·6	Sunday.	− 1·3	+ 8·1	+ 5·4	+ 9·0	+10·2	Sunday.	−26·0	− 5·1	− 3·8	− 2·3	−0·5
	3	25·1	Sunday.	− 9·6	−14·7	+ 7·4	+12·4	+ 7·8	Sunday.	+11·5	−17·0	− 2·7	− 9·5	− 2·1	−1·7
	4	25·1	−18·4	− 3·7	− 7·2	− 2·6	+10·2	Sunday.	+11·7	− 1·1	− 6·4	− 3·9	−11·0	Sunday.	−3·2
	5	25·1	− 4·8	−11·5	+ 3·3	− 4·8	Sunday.	+ 4·8	+10·2	− 0·1	− 4·1	− 4·2	Sunday.	− 1·4	−1·3
	6	25·1	+11·5	+ 7·6	+11·8	+ 1·6	− 7·1	+ 8·1	+ 6·4	−13·4	− 6·5	Sunday.	+ 2·5	− 4·7	+1·6
	7	25·1	+10·6	+ 0·2	+13·4	Sunday.	− 1·4	+10·3	− 4·6	− 3·3	Sunday.	+ 4·5	− 3·1	−15·8	+1·1
	8	25·1	− 3·4	+ 2·5	Sunday.	−10·0	+ 4·4	+ 6·9	−15·4	+ 4·6	− 1·7	+ 3·1	+ 1·4	− 3·3	−1·0
	9	25·1	− 2·2	Sunday.	+ 2·9	− 1·7	+ 7·2	+ 3·9	− 9·4	Sunday.	−15·4	+ 8·0	+11·8	+ 2·2	+0·7
	10	25·1	Sunday.	− 0·9	+ 7·8	− 4·4	+ 3·6	+ 4·7	Sunday.	−21·3	−27·8	+ 0·7	+ 9·1	− 3·8	−3·2
	11	25·0	+ 5·3	+ 5·7	+ 5·6	− 5·9	+ 1·0	Sunday.	−14·4	− 6·1	−26·6	+12·0	+ 7·1	Sunday.	−1·6
	12	25·0	+ 4·4	− 1·7	+ 7·4	+10·3	Sunday.	− 4·0	− 8·1	− 1·1	− 7·3	+ 5·1	Sunday.	−10·0	−0·5
	13	25·0	+ 0·1	− 6·6	+ 2·0	+ 5·5	− 8·2	− 3·8	+ 7·6	+11·3	+10·1	Sunday.	+ 5·7	−14·7	+0·8
	14	25·0	+ 3·7	+ 9·7	− 1·3	Sunday.	− 7·5	+ 3·1	+11·8	+13·7	Sunday.	− 7·3	+12·3	− 1·0	+3·7
	15	25·0	+ 6·5	0·0	Sunday.	+ 7·6	+ 4·7	+10·3	+15·0	+15·3	+ 2·8	+ 6·0	+10·9	−21·5	+5·2
	16	24·9	+11·2	Sunday.	− 1·6	+12·5	− 0·7	− 0·7	− 1·0	Sunday.	− 5·8	+ 9·7	+11·8	− 8·5	+2·7
	17	24·9	Sunday.	+ 9·1	+ 1·4	+ 2·9	− 5·3	−16·4	Sunday.	+ 2·3	− 4·8	+10·2	− 7·2	−18·4	−2·6
	18	24·9	−22·5	+10·1	+11·1	− 0·8	−10·2	Sunday.	+ 3·7	− 8·0	−14·2	+ 4·9	−15·2	Sunday.	−4·1
	19	24·8	−15·4	+12·8	+14·6	− 9·4	Sunday.	−14·7	−16·2	− 4·9	−10·7	+ 2·3	Sunday.	−23·6	−6·5
	20	24·8	− 0·1	+10·4	+13·3	−14·7	+ 2·1	− 5·2	− 9·1	+12·1	− 0·4	Sunday.	+ 7·2	−24·3	−0·8
	21	24·7	+ 4·7	+ 1·3	+18·8	Sunday.	+ 6·3	− 7·3	−16·8	+ 6·6	Sunday.	+11·1	+ 5·8	−14·5	+1·6
	22	24·7	+ 0·8	−12·3	Sunday.	− 1·9	+ 1·5	−13·6	− 9·9	− 2·4	− 5·5	+ 5·2	+10·3	−17·0	−4·1
	23	24·6	+ 4·6	Sunday.	+ 9·1	+12·7	+ 8·2	− 9·0	+ 2·5	Sunday.	+ 0·4	+ 6·5	+ 8·3	− 7·1	+3·6
	24	24·5	Sunday.	− 3·2	+ 7·4	−10·5	+10·7	+ 4·1	Sunday.	+ 3·0	+ 7·7	+11·6	+ 8·8	+ 1·4	+4·1
	25	24·5	+ 4·8	+ 9·1	−10·1	−25·8	− 0·4	Sunday.	+ 0·5	+11·6	+11·5	+12·4	+ 6·6	Sunday.	+2·0
	26	24·4	+ 5·8	+ 5·6	− 7·8	−25·5	Sunday.	+ 8·1	+ 1·1	+14·1	+ 3·3	+ 9·0	Sunday.	− 6·4	+0·7
	27	24·3	+10·5	− 3·0	+ 3·1	−24·7	+ 9·1	− 0·3	−14·5	+13·0	− 2·4	Sunday.	− 3·7	− 1·3	−0·4
	28	24·3	+ 4·0	+12·0	− 0·3	Sunday.	+11·4	+ 9·8	− 7·5	+ 7·1	Sunday.	+ 2·8	+ 8·9	+ 8·7	+5·7
	29	24·2	+ 8·4	+16·2	Sunday.	−20·9	+ 0·5	+11·7	+ 2·3	+ 5·6	+ 9·6	+ 0·6	−22·5	+10·4	+2·0
	30	24·1	+ 4·5	Sunday.	+ 6·0	−15·4	− 8·3	+12·3	− 9·0	Sunday.	−10·2	+ 2·4	−25·6	+ 1·0	−4·2
	31	24·0	Sunday.	+ 8·2	+ 6·4	−18·9	−18·1	− 9·1	Sunday.	+ 8·2	−14·7	+ 9·2	−10·3	− 0·2	−3·9
February.	1	23·9	−10·0	+ 7·1	−13·5	− 5·3	−20·6	Sunday.	+ 6·3	+ 0·9	+ 4·4	+ 3·5	− 1·9	Sunday.	−2·9
	2	23·9	− 0·7	+14·5	− 9·3	− 2·3	Sunday.	+ 4·7	+14·2	+ 8·9	+ 6·2	+ 5·6	Sunday.	+ 0·2	+4·2
	3	23·8	− 6·6	+18·3	− 0·8	− 5·3	− 3·2	+11·7	+ 2·3	+ 8·9	− 4·5	Sunday.	+ 5·5	+ 5·6	+2·9
	4	23·7	− 0·8	+10·7	+ 1·1	Sunday.	− 7·3	+11·3	− 7·9	+10·9	Sunday.	−16·3	+ 8·1	+13·6	+2·3
	5	23·6	+ 4·3	+ 7·6	Sunday.	+ 9·7	−11·3	+10·3	− 7·3	+ 2·3	− 8·7	−14·1	+ 1·6	+ 9·3	+0·3
	6	23·6	+ 5·8	Sunday.	−14·9	+ 5·8	−12·4	+ 8·8	− 3·7	Sunday.	+ 2·2	− 2·1	−12·9	+ 8·1	−1·5
	7	23·5	Sunday.	+ 6·3	−12·1	− 1·4	− 6·1	+ 7·9	Sunday.	− 1·9	−12·8	+ 8·3	−10·9	+ 5·6	−1·7
	8	23·5	+ 2·8	−14·9	−12·7	− 1·2	−11·7	Sunday.	+ 9·1	+ 0·1	+ 1·2	+12·9	−18·7	Sunday.	−3·3
	9	23·4	+ 4·3	− 0·5	− 9·7	−12·6	Sunday.	−13·3	+ 5·5	+ 6·8	− 7·9	+11·9	Sunday.	+10·0	−0·5
	10	23·4	−10·4	+ 7·4	+ 4·8	− 2·3	+ 6·2	−13·6	+ 1·2	−11·4	− 1·4	Sunday.	+ 3·2	+11·5	−0·4
	11	23·4	−18·9	+12·4	− 4·2	Sunday.	+ 7·4	− 8·6	− 6·0	−13·0	Sunday.	+ 5·7	− 8·7	+10·8	−2·3
	12	23·4	−17·1	+ 5·7	Sunday.	+ 5·0	− 7·6	− 7·6	− 6·0	− 5·5	−17·5	+ 3·7	+ 1·2	− 3·8	−4·5
	13	23·4	− 9·2	Sunday.	− 9·0	+ 9·4	−16·7	+ 3·7	− 2·5	Sunday.	−10·2	+ 4·2	+ 9·6	− 5·1	−2·6
	14	23·4	Sunday.	− 8·0	−15·5	− 1·6	+ 3·4	+ 4·5	Sunday.	+ 6·5	−19·1	+ 4·2	+14·4	− 8·6	−2·0
	15	23·4	− 5·4	+ 0·1	−13·7	+ 6·6	+12·8	Sunday.	− 1·8	+ 1·7	−21·5	− 9·3	+15·0	Sunday.	−1·5
	16	23·5	+ 4·7	− 4·0	−20·0	+ 6·2	Sunday.	− 0·8	− 4·8	+ 5·0	−20·6	+ 1·6	Sunday.	− 6·4	−3·9
	17	23·5	+ 0·5	−11·3	−20·0	− 7·8	+11·4	− 1·3	+ 0·5	+ 4·7	−16·5	Sunday.	+ 6·3	−11·4	−4·1
	18	23·6	+ 4·0	+ 7·1	−18·4	Sunday.	+ 9·0	− 8·1	+ 5·5	+ 7·8	Sunday.	+ 8·8	+ 4·0	−21·0	−0·1
	19	23·7	− 1·1	− 5·6	Sunday.	+ 9·7	+11·4	+ 0·4	+ 6·1	+12·1	−19·8	− 8·0	+10·4	−17·4	−0·2
	20	23·8	+ 7·2	Sunday.	− 9·4	+14·4	+14·9	+ 2·5	− 2·3	Sunday.	− 6·2	+ 9·8	+14·5	−13·7	+3·2
	21	23·9	Sunday.	+ 0·1	− 3·2	+11·5	+15·4	+ 1·9	Sunday.	+14·8	+ 3·7	+ 3·9	+11·3	+ 1·6	+6·1
	22	24·0	+13·6	− 0·9	−11·8	+11·8	+13·7	Sunday.	− 9·9	+10·5	+ 6·5	−12·0	+12·6	Sunday.	+3·4
	23	24·1	−11·4	+ 4·6	−14·8	+ 1·5	Sunday.	− 5·6	−17·5	+ 5·8	+ 8·2	− 3·5	Sunday.	+ 7·2	−2·5
	24	24·3	− 3·4	+ 5·8	− 3·3	− 9·4	+13·7	−12·8	− 6·7	− 1·2	+ 9·3	Sunday.	+11·1	+ 9·7	+1·2
	25	24·5	+ 5·9	+ 1·7	+ 3·6	Sunday.	+18·0	−17·4	− 7·5	− 7·4	Sunday.	+10·8	+ 6·0	+ 3·4	+1·7
	26	24·7	+ 7·4	+ 9·1	Sunday.	+ 9·4	+ 9·1	−25·4	+ 0·2	+ 2·5	+10·0	+13·9	+11·0	− 7·2	+3·6
	27	24·9	+ 9·6	Sunday.	− 2·5	+ 4·9	+ 5·9	−16·4	+ 4·2	Sunday.	+ 9·8	+ 7·4	+ 9 2	− 5·8	+2·6
	28	25·1	Sunday.	+10·0	− 4·7	+ 6·6	+ 5·1	− 9·8	Sunday.	+ 5·4	+11·7	+ 9·4	− 3·0	+ 2·5	+3·3

Table IV. (Continued.)

Month	Day	Daily temperature derived from the monthly mean of the 12 years.	Difference on each day of the twelve years from the temperature derived from the monthly means.												Mean daily difference.
			1841.	1842.	1843.	1844.	1845.	1846.	1847.	1848.	1849.	1850.	1851.	1852.	
		(1.)	(2.)	(3.)	(4.)	(5.)	(6.)	(7.)	(8.)	(9.)	(10.)	(11.)	(12.)	(13.)	(14.)
March.	1	25°·4	+ 2°·7	+11°·3	−10°·7	+15°·3	+11°·3	Sunday.	− 0°·7	−13°·2	+ 0°·8	+ 5°·9	+ 2°·1	− 8°·1	+1°·5
	2	25·6	− 7·4	+15·1	−14·4	+ 9·2	Sunday.	− 8·4	− 1·7	− 9·0	− 2·5	+ 4·1	Sunday.	−12·7	−2·8
	3	25·9	− 2·4	+17·5	−10·8	Sunday.	+ 4·0	− 3·8	+ 1·1	− 5·2	− 5·2	Sunday.	− 4·0	−13·0	−2·2
	4	26·2	− 8·9	+12·7	−11·2	− 8·7	+ 9·6	+ 8·1	− 2·3	− 7·3	Sunday.	− 3·0	+ 4·9	− 5·3	−1·0
	5	26·4	− 6·6	+ 3·9	Sunday.	+ 3·7	+ 9·2	+ 8·5	− 1·0	Sunday.	+ 5·3	+ 6·3	+ 3·1	+ 1·0	+3·3
	6	26·7	− 2·5	Sunday.	−14·1	+ 5·9	+ 7·1	− 5·2	+ 5·6	− 3·0	+ 7·7	+ 7·7	− 0·6	− 7·3	+0·1
	7	27·0	Sunday.	+ 3·3	−13·1	+ 8·2	+11·8	− 1·4	Sunday.	+ 9·3	+10·0	+ 6·1	− 2·6	Sunday.	+3·5
	8	27·4	− 0·5	+ 7·3	− 2·8	+10·8	+18·5	Sunday.	− 1·5	+ 9·6	+ 2·8	− 1·3	− 3·5	+ 5·0	+4·0
	9	27·7	− 0·3	+13·0	− 3·6	+ 3·1	Sunday.	+ 5·2	− 5·5	− 2·8	+ 3·3	+ 2·7	Sunday.	+10·0	+2·5
	10	28·0	− 8·9	+ 8·3	+ 4·4	Sunday.	+ 4·5	+ 4·9	+ 0·4	− 5·8	+ 2·8	Sunday.	+ 3·8	+ 1·0	+1·5
	11	28·4	− 9·7	− 3·6	− 2·0	+10·4	+ 4·6	+ 6·5	−13·1	− 0·4	Sunday.	− 3·2	+ 6·7	+ 4·8	+0·1
	12	28·7	− 1·7	− 4·1	Sunday.	+11·4	+ 7·3	+11·1	−12·8	Sunday.	+ 2·4	+ 0·1	− 4·8	+10·2	+1·9
	13	29·1	− 1·5	Sunday.	− 3·3	+ 9·2	+ 7·7	+10·1	−10·4	− 9·2	+ 7·3	+ 7·7	+ 3·0	+11·0	+2·9
	14	29·5	Sunday.	+ 2·3	− 5·4	+ 1·2	+ 1·3	+ 5·2	Sunday.	−20·2	+ 3·5	+ 9·3	− 0·6	Sunday.	−0·4
	15	29·9	−15·3	+ 4·1	− 9·6	+ 3·6	−11·7	Sunday.	−12·8	−18·8	+ 8·3	+ 3·7	+10·0	+ 4·1	−3·1
	16	30·2	−13·5	+ 9·3	− 9·3	+ 3·4	Sunday.	− 2·0	−13·0	− 5·8	+ 7·4	+ 1·8	Sunday.	+ 2·4	−1·9
	17	30·6	−15·0	+14·7	− 5·8	Sunday.	− 1·5	− 3·1	− 1·4	− 7·8	+ 0·1	Sunday.	+ 3·1	+ 7·5	−0·9
	18	31·0	− 3·2	+ 5·7	− 8·3	−15·3	− 7·5	+ 3·3	+ 2·5	− 1·7	Sunday.	+ 0·4	+ 2·3	− 6·6	−2·6
	19	31·4	+ 7·6	+16·9	Sunday.	− 4·6	− 9·5	+ 6·2	+ 0·6	Sunday.	+ 1·2	− 7·6	− 2·8	−10·4	−0·2
	20	31·8	+13·4	Sunday.	− 9·3	− 5·6	− 3·2	+ 7·4	+ 3·6	+ 3·9	+ 7·6	−13·1	+ 1·9	−15·1	−0·8
	21	32·2	Sunday.	− 1·6	− 9·3	− 8·2	Good Friday	+ 0·6	Sunday.	+ 7·3	+ 4·9	− 6·3	+ 1·9	Sunday.	−1·3
	22	32·6	+ 3·8	+ 0·1	−11·5	− 5·4	+ 2·5	Sunday.	− 0·8	+ 1·9	0·0	+ 0·5	+ 3·6	− 8·0	−1·2
	23	32·9	+ 4·9	+ 0·2	−20·3	− 6·0	Sunday.	+ 6·6	+ 3·1	+ 1·4	+ 9·1	− 1·6	Sunday.	− 6·0	−0·9
	24	33·3	+ 7·3	+ 2·0	−15·0	Sunday.	− 0·1	+ 6·5	+ 0·3	+ 3·1	+ 1·7	Sunday.	+ 6·4	− 5·3	+0·7
	25	33·7	+ 4·9	Good Friday	−12·7	+ 6·0	+ 4·3	+ 5·1	+ 4·2	+ 6·1	Sunday.	−10·1	− 2·8	− 7·7	−0·3
	26	34·1	+ 8·2	+ 6·0	Sunday.	+ 4·0	+10·6	+ 4·7	− 3·0	Sunday.	− 4·6	− 8·2	+ 8·3	− 1·3	+2·5
	27	34·5	+ 3·7	Sunday.	− 7·0	− 1·5	+10·3	+ 4·0	−12·8	− 0·2	− 4·6	− 4·9	+16·4	− 3·1	0·0
	28	34·8	Sunday.	− 0·8	− 4·0	+ 3·6	+ 8·7	+ 1·0	Sunday.	+ 4·6	+ 5·5	− 2·1	− 0·8	Sunday.	+1·7
	29	35·2	− 5·1	+ 2·8	− 7·4	− 7·7	+14·6	Sunday.	− 1·8	+ 8·2	+ 5·5	Good Friday	+ 4·0	− 1·9	+1·1
	30	35·6	−13·8	+10·2	−10·1	−14·8	Sunday.	− 0·9	−12·3	+ 4·2	+ 8·4	− 1·7	Sunday.	− 7·3	−3·8
	31	36·0	− 3·6	− 8·8	−10·4	Sunday.	+18·5	− 0·7	−17·4	+10·9	+ 3·5	Sunday.	+ 1·7	− 2·0	−0·8
April.	1	36·3	+ 2·7	− 0·6	− 9·3	− 3·6	+ 0·2	− 1·7	− 9·5	− 3·0	Sunday.	+ 1·3	+ 3·5	− 4·3	−2·2
	2	36·7	− 1·7	+17·5	Sunday.	+ 2·9	+ 1·9	− 1·6	Good Friday	Sunday.	+ 1·4	+ 4·5	+ 6·6	− 8·1	+2·6
	3	37·1	+ 1·3	Sunday.	− 7·2	+14·6	− 3·5	+ 1·1	− 1·5	+ 7·4	+ 8·6	+ 8·0	+ 1·6	− 9·5	+1·9
	4	37·4	Sunday.	0·0	− 4·6	+11·8	− 1·5	+ 5·4	Sunday.	+ 3·4	+ 6·9	+ 3·1	− 4·2	Sunday.	+2·3
	5	37·8	− 2·1	+ 2·6	− 3·0	Good Friday	−10·7	Sunday.	+ 0·5	− 2·0	− 0·9	− 1·6	− 1·9	− 6·2	−2·5
	6	38·1	− 2·1	+ 3·5	− 4·0	− 0·4	Sunday.	+ 8·6	+ 5·4	+ 3·0	Good Friday	− 1·4	Sunday.	− 5·8	+0·8
	7	38·5	− 3·0	+ 2·6	− 2·4	Sunday.	−13·6	+ 9·5	+ 6·0	+ 0·9	+11·5	Sunday.	+ 1·0	− 6·4	+0·6
	8	38·8	− 1·1	− 3·2	+ 3·0	+12·5	−13·6	− 3·6	+ 6·5	+ 6·0	Sunday.	−13·1	+ 3·2	− 4·3	−0·7
	9	39·1	Good Friday	− 0·2	Sunday.	+ 9·2	− 3·8	− 0·4	+ 0·9	Sunday.	+ 2·7	−10·4	+ 0·7	− 3·4	−0·5
	10	39·5	−12·7	Sunday.	− 3·0	+11·4	+ 1·4	Good Friday	+ 2·0	+15·3	+ 6·0	− 7·4	− 3·2	− 3·4	+0·6
	11	39·8	Sunday.	+ 9·8	− 0·3	+10·6	− 4·9	+ 2·4	Sunday.	+ 6·5	− 3·8	−10·3	− 8·2	Sunday.	+0·2
	12	40·2	− 5·1	0·0	+ 0·9	+14·2	+ 0·3	Sunday.	+ 0·1	+ 7·2	− 3·9	− 4·1	− 8·0	− 0·3	+0·1
	13	40·5	− 2·9	− 3·2	− 1·2	+17·0	Sunday.	− 8·9	− 0·7	− 2·1	− 2·6	− 9·3	Sunday.	− 2·8	−1·7
	14	40·8	− 8·5	− 1·3	Good Friday	Sunday.	+ 6·5	− 2·9	− 1·7	+ 0·2	−16·7	Sunday.	− 1·6	+ 0·3	−2·9
	15	41·1	− 7·0	− 3·2	+ 5·1	+11·1	+ 9·4	− 9·5	− 8·6	+ 8·0	Sunday.	− 9·5	− 1·9	+ 2·5	−0·4
	16	41·5	− 0·2	− 2·8	Sunday.	+ 9·4	+ 1·3	− 3·1	− 6·5	Sunday.	− 9·9	−14·3	+ 3·4	+ 2·0	−2·1
	17	41·8	+ 6·8	Sunday.	+ 0·2	− 1·8	+ 0·1	+ 6·7	−11·8	− 2·2	− 3·3	−14·9	+ 0·8	+ 2·7	−1·5
	18	42·1	Sunday.	− 2·4	− 6·5	− 3·2	+ 2·0	+ 8·9	Sunday.	−11·8	− 5·8	− 9·6	Good Friday	Sunday.	−3·6
	19	42·4	− 3·8	+ 1·6	− 2·9	+ 1·5	0·0	Sunday.	− 5·1	− 9·5	− 4·1	− 2·6	− 3·4	+ 4·6	−2·2
	20	42·8	−11·4	+ 5·2	− 1·2	+ 5·6	Sunday.	+ 8·3	− 0·1	− 1·6	− 8·3	− 5·1	Sunday.	+ 1·3	−0·7
	21	43·1	− 4·7	+11·9	+ 4·9	Sunday.	− 1·3	+14·3	+12·4	Good Friday	− 7·5	Sunday.	− 2·0	+ 4·5	+3·6
	22	43·4	− 2·4	+18·3	+ 4·8	+ 6·5	+ 4·2	+ 3·1	− 0·6	+ 0·2	Sunday.	+ 6·3	+ 3·7	− 2·7	+3·8
	23	43·7	+ 2·4	− 1·7	Sunday.	+11·4	+11·6	+ 9·4	− 7·1	Sunday.	− 2·9	− 8·9	+ 4·8	−10·0	+0·9
	24	44·0	+ 7·3	Sunday.	+ 5·8	+11·3	+12·0	+ 7·3	− 5·2	− 6·9	− 6·3	− 1·6	+ 5·7	− 5·5	+2·2
	25	44·4	Sunday.	+ 3·1	+ 2·3	+ 5·7	− 1·2	− 2·4	Sunday.	− 4·5	− 0·7	+ 2·9	+ 5·5	Sunday.	+1·2
	26	44·7	+ 4·4	+ 1·1	+ 6·2	− 2·4	+ 3·9	Sunday.	+ 6·1	− 6·2	− 8·6	+ 4·4	+ 0·5	− 5·9	+0·3
	27	45·0	− 4·2	− 4·2	+ 2·8	− 5·2	Sunday.	+ 4·0	− 8·2	− 1·7	− 7·1	+ 8·1	Sunday.	− 6·4	−2·2
	28	45·4	− 0·5	− 4·2	+ 9·7	Sunday.	+ 7·5	+ 7·1	− 8·7	− 1·6	+ 2·5	Sunday.	− 1·5	− 6·3	+0·4
	29	45·7	− 2·5	− 1·7	− 8·0	+ 2·3	+ 7·3	+ 2·0	− 7·5	− 2·8	Sunday.	− 2·5	− 0·4	− 2·3	−1·5
	30	46·0	+ 0·6	+ 2·6	Sunday.	+ 6·3	+ 7·3	+ 6·0	− 7·7	Sunday.	+ 2·1	+ 2·7	− 2·1	− 1·5	+1·6

Table IV. (Continued.)

Month	Day	Daily temperature derived from the monthly mean of the 12 years.	Difference on each day of the twelve years from the temperature derived from the monthly means. 1841.	1842.	1843.	1844.	1845.	1846.	1847.	1848.	1849.	1850.	1851.	1852.	Mean daily difference.
		(1.)	(2.)	(3.)	(4.)	(5.)	(6.)	(7.)	(8.)	(9.)	(10.)	(11.)	(12.)	(13.)	(14.)
	1	46·4	− 3·2	Sunday.	− 7·9	+11·6	+ 9·3	+ 7·5	Not obs.	− 3·7	− 3·9	− 7·9	− 6·6	− 0·9	−0·6
	2	46·7	Sunday.	− 1·1	− 6·0	+12·7	+ 1·3	+ 7·3	Sunday.	+ 0·6	− 7·9	− 4·2	− 9·4	Sunday.	−0·7
	3	47·0	−14·6	+ 0·6	− 2·1	+ 8·5	+ 6·9	Sunday.	Not obs.	+ 5·4	− 7·2	− 0·2	− 8·7	− 0·6	−1·2
	4	47·4	− 8·7	− 2·5	− 6·7	+ 2·3	Sunday.	+ 7·4	− 1·0	+ 6·0	− 0·6	− 1·9	Sunday.	− 4·6	−1·0
	5	47·7	− 7·1	+ 1·5	− 7·3	Sunday.	− 6·9	+ 7·8	− 0·3	+12·8	−10·2	Sunday.	− 5·8	+ 4·2	−1·1
	6	48·0	− 5·8	+ 4·2	− 1·5	+ 2·6	− 3·2	+ 2·3	+ 1·3	+14·7	Sunday.	− 5·6	− 5·4	+ 6·1	+0·9
	7	48·4	− 6·4	− 4·8	Sunday.	+ 7·3	− 8·3	+ 4·4	+ 7·7	Sunday.	− 5·9	− 2·1	− 1·1	+ 8·0	−0·1
	8	48·7	− 2·7	Sunday.	− 2·4	+ 6·3	− 5·9	+ 8·6	+ 6·4	+ 2·0	− 3·6	+ 0·3	+ 3·7	+10·1	+2·1
	9	49·1	Sunday.	− 6·8	+ 0·7	+ 1·0	− 1·2	+ 5·2	Sunday.	+ 3·1	− 2·9	− 7·6	+10·5	Sunday.	+0·2
	10	49·4	− 1·6	+ 6·1	+ 1·3	− 2·5	+ 1·9	Sunday.	+ 6·8	+ 1·1	− 4·3	− 8·6	+ 2·2	− 2·4	0·0
	11	49·8	+ 4·8	− 3·7	+ 3·4	+ 9·0	Sunday.	− 8·1	+ 7·5	− 2·0	− 5·1	− 7·4	Sunday.	+ 0·5	−0·1
	12	50·1	− 2·0	− 2·7	+ 6·3	Sunday.	+14·7	− 0·2	+10·0	− 5·3	− 3·4	Sunday.	+ 9·9	+ 4·1	+3·1
	13	50·5	− 5·1	− 6·9	+ 8·9	− 7·1	+12·1	+ 8·9	+ 6·7	−10·8	Sunday.	+ 7·0	+10·3	+ 2·8	+2·4
	14	50·8	− 6·8	− 6·7	Sunday.	− 0·8	+ 7·9	+11·2	+ 2·8	Sunday.	− 4·0	− 0·3	+ 0·6	+ 2·5	+0·6
	15	51·2	− 4·1	Sunday.	+11·5	+ 3·6	−10·7	+ 1·1	+ 7·5	− 2·0	− 3·6	+ 5·3	− 0·8	− 1·3	+0·6
	16	51·5	Sunday.	+ 1·8	+ 1·3	+ 3·0	−10·3	+ 2·9	Sunday.	− 0·3	+ 1·0	+ 4·7	+ 5·7	Sunday.	+1·1
	17	51·9	+ 3·1	+ 0·3	− 6·4	+ 1·7	− 0·3	Sunday.	+ 5·8	+ 2·4	+ 1·4	− 7·3	+ 5·2	− 5·0	+0·1
	18	52·2	+ 1·1	+ 8·2	− 3·6	− 1·6	Sunday.	− 5·1	+ 6·3	+ 8·6	+ 1·9	− 7·5	Sunday.	− 6·9	+0·1
	19	52·5	− 4·3	− 6·6	− 3·3	Sunday.	+ 3·9	− 6·5	+ 6·8	+ 9·9	+ 2·0	Sunday.	− 0·6	− 6·9	−0·6
	20	52·9	+ 1·4	− 9·3	− 0·2	− 5·3	− 5·9	+ 1·2	+ 1·2	+ 9·7	Sunday.	−15·8	+ 7·3	−11·8	−2·5
	21	53·2	+ 5·2	− 2·3	Sunday.	−12·5	− 2·7	−10·2	+ 1·7	Sunday.	+ 5·7	− 9·6	+ 1·5	− 0·4	−2·4
	22	53·6	+ 3·6	Sunday.	− 1·5	− 6·3	−10·3	− 1·8	+ 6·1	+ 3·5	− 1·0	− 8·0	+ 0·3	+ 2·9	−1·1
	23	53·9	Sunday.	− 1·6	− 3·5	+ 1·6	− 3·1	+ 5·2	Sunday.	+ 2·7	− 0·6	− 2·9	− 8·7	Sunday.	−1·2
	24	54·2	+ 8·3	− 3·2	− 0·1	+ 5·5	−10·8	Sunday.	+ 2·7	+ 6·7	−11·7	+ 3·4	− 7·1	+ 7·5	+0·1
	25	54·6	+ 6·0	− 2·6	− 1·3	+11·4	Sunday.	+11·3	+ 0·1	+ 7·0	− 9·2	− 0·7	Sunday.	− 1·5	+2·1
	26	54·9	+ 4·3	− 1·7	− 2·0	Sunday.	+ 7·3	+11·8	− 8·3	+ 1·6	− 4·8	Sunday.	+ 2·7	− 4·3	+0·7
	27	55·2	+ 3·6	− 5·8	− 3·9	+ 7·2	+ 5·4	+12·5	− 3·0	+ 2·5	Sunday.	−17·0	+ 0·4	− 3·7	−0·2
	28	55·6	+ 8·6	− 3·9	Sunday.	+ 4·2	− 2·7	+ 7·3	+ 7·2	Sunday.	− 1·2	+ 8·0	+ 2·2	− 0·5	+2·9
	29	55·9	+ 0·6	Sunday.	− 2·7	+ 1·6	−18·3	+ 8·5	+ 1·2	+ 7·4	− 2·4	+ 3·1	− 6·6	+ 5·4	−0·2
	30	56·2	Sunday.	− 6·2	−13·6	+ 1·3	−12·4	+ 8·7	Sunday.	− 1·6	− 4·7	− 8·6	− 4·0	Sunday.	−4·6
	31	56·5	+ 0·7	− 2·2	−16·2	− 1·9	− 3·1	Sunday.	−10·4	− 7·9	− 3·0	− 8·2	+ 0·9	− 2·9	−4·9
June.	1	56·9	+ 3·7	− 3·4	−16·8	+ 2·9	Sunday.	+ 8·0	− 2·6	− 4·2	− 2·2	− 4·1	Sunday.	− 2·0	−2·1
	2	57·2	+ 6·9	+ 1·2	−13·8	Sunday.	+ 5·1	+ 2·7	− 2·8	+ 5·6	− 0·3	Sunday.	− 2·6	+10·3	+1·3
	3	57·5	− 1·0	− 1·3	− 9·6	− 4·4	+ 5·4	+ 5·4	− 1·7	+ 6·6	Sunday.	+ 4·2	− 7·1	+ 3·5	0·0
	4	57·8	+ 9·5	+ 0·8	Sunday.	− 2·4	+ 6·5	+ 3·5	− 6·6	Sunday.	+ 1·6	+ 6·4	− 3·2	− 7·0	+0·9
	5	58·1	+12·7	Sunday.	−11·1	+ 1·8	− 1·6	− 5·3	− 5·4	− 1·5	− 4·2	+ 5·6	− 5·4	− 5·0	−2·2
	6	58·4	Sunday.	−12·6	−11·3	+ 3·4	− 6·6	− 7·9	Sunday.	− 8·4	− 1·7	+12·9	− 5·4	Sunday.	−4·2
	7	58·7	+11·6	− 8·8	− 7·1	− 4·6	− 5·1	Sunday.	+ 1·2	− 3·6	− 1·3	+13·0	− 1·8	− 1·9	−0·8
	8	59·0	+13·8	−10·7	− 5·5	− 6·9	Sunday.	− 2·9	− 1·8	− 0·8	− 5·5	+ 7·3	Sunday.	− 1·7	−1·5
	9	59·4	+ 6·8	− 4·0	+ 3·1	Sunday.	+10·2	− 1·0	+ 7·4	+ 6·2	− 3·6	Sunday.	− 5·9	− 1·9	+1·7
	10	59·7	+10·8	−17·9	−12·1	− 10·4	+ 6·7	+ 4·0	+ 2·1	+ 4·7	Sunday.	− 4·3	− 2·7	−11·7	−2·8
	11	59·9	+ 6·0	−23·2	Sunday.	− 7·8	+ 3·6	+ 0·2	− 3·5	Sunday.	− 2·6	− 3·7	− 2·1	− 5·9	−3·9
	12	60·2	− 6·4	Sunday.	− 3·2	− 4·7	+ 4·9	− 3·0	− 2·5	−10·6	− 7·0	+ 6·3	− 3·0	− 4·5	−3·1
	13	60·5	Sunday.	− 7·5	− 2·2	− 3·0	+ 2·6	− 1·2	Sunday.	− 8·0	− 0·8	+ 4·0	− 4·1	Sunday.	−2·2
	14	60·8	− 0·7	− 6·4	− 3·3	− 0·2	+ 3·9	Sunday.	−14·8	− 5·0	+ 0·1	+ 4·2	− 1·9	+ 9·5	−1·3
	15	61·1	− 4·2	− 2·9	− 8·6	+ 0·3	Sunday.	+ 7·7	−12·0	+16·8	+ 3·8	− 1·2	Sunday.	+13·2	+1·3
	16	61·3	− 1·5	− 0·7	− 4·9	Sunday.	− 8·7	+ 0·9	−10·6	+14·1	+ 3·0	Sunday.	− 3·9	+12·9	+0·1
	17	61·6	− 2·7	− 5·2	− 3·5	+ 2·1	− 7·4	+ 4·8	− 6·3	+10·7	Sunday.	+ 1·3	− 4·8	+ 6·4	−0·4
	18	61·9	− 2·8	+ 0·3	Sunday.	+ 8·2	− 4·3	+ 7·1	− 2·4	Sunday.	+ 4·8	+ 9·9	− 3·8	+ 5·1	+2·2
	19	62·1	+ 0·4	Sunday.	− 3·7	+ 5·0	− 2·6	+ 7·3	− 5·1	+ 6·1	+ 9·2	+11·0	+ 1·2	+ 0·4	+2·7
	20	62·4	Sunday.	− 6·7	+ 3·1	− 0·8	+ 1·6	− 6·2	Sunday.	+ 1·5	+12·2	+ 8·4	+ 2·8	Sunday.	+1·8
	21	62·6	+ 2·3	− 2·0	+ 6·2	− 2·4	+ 0·4	Sunday.	− 3·2	− 0·7	+12·0	+ 4·8	+ 0·2	+ 0·5	+1·6
	22	62·9	+ 7·7	− 1·3	+ 6·4	− 3·0	Sunday.	− 2·9	− 4·0	+ 1·1	+ 8·1	+ 0·3	Sunday.	− 2·5	+1·0
	23	63·1	+ 7·2	− 6·5	+ 4·2	Sunday.	+ 5·5	+ 3·6	− 0·7	+ 1·0	+ 5·0	Sunday.	+ 0·2	− 4·6	+1·5
	24	63·3	+ 3·5	− 5·0	+ 2·5	+ 3·9	+ 5·6	+ 6·5	+ 1·6	+ 0·3	Sunday.	− 0·6	− 1·7	− 8·7	+0·7
	25	63·5	+ 3·0	− 7·2	Sunday.	+ 6·2	− 6·0	+ 8·8	+ 1·9	Sunday.	+ 3·3	− 4·1	+ 0·5	− 5·7	+0·1
	26	63·8	+ 5·9	Sunday.	+ 3·5	− 0·9	− 3·7	+ 6·1	+ 5·6	+ 5·9	+ 4·9	− 0·3	+ 4·0	+ 0·1	+2·8
	27	64·0	Sunday.	− 2·5	+ 7·2	− 3·4	− 0·2	+ 1·4	Sunday.	+ 6·3	+ 6·8	− 3·4	+ 0·3	Sunday.	+1·4
	28	64·2	+ 7·5	− 3·4	+ 3·9	− 4·2	− 7·5	Sunday.	+ 1·6	+ 1·5	+ 2·1	+ 0·3	+ 1·3	− 1·1	+0·2
	29	64·4	+14·3	− 3·3	+ 3·0	− 3·3	Sunday.	+ 5·1	− 1·0	+ 0·2	+ 2·9	+ 4·2	Sunday.	− 1·1	+2·1
	30	64·5	+ 6·2	+ 1·3	+ 5·8	Sunday.	− 9·8	+ 8·1	− 1·7	+ 4·2	+ 5·0	Sunday.	+ 2·8	− 0·8	+2·1

TABLE IV. (Continued.)

Month	Day	Daily temperature derived from the monthly means of the 12 years.	Difference on each day of the twelve years from the temperature derived from the monthly means. 1841.	1842.	1843.	1844.	1845.	1846.	1847.	1848.	1849.	1850.	1851.	1852.	Mean daily difference.
		(1.)	(2.)	(3.)	(4.)	(5.)	(6.)	(7.)	(8.)	(9.)	(10.)	(11.)	(12.)	(13.)	(14.)
July.	1	64°·7	+3°·0	− 5°·6	+10°·4	+4°·1	− 8°·3	+ 5°·5	+ 1°·3	− 8°·5	Sunday.	+3°·1	−4°·0	− 7°·7	−0°·6
	2	64·9	−5·1	− 5·6	Sunday.	+1·4	− 7·2	+ 1·8	+ 3·1	Sunday.	− 3·7	+3·5	−7·3	− 6·1	−2·5
	3	65·1	−5·4	Sunday.	− 7·5	−6·4	− 7·8	− 1·9	+ 5·9	− 5·6	− 2·7	+5·0	−8·0	− 6·0	−3·7
	4	65·2	Sunday.	− 3·3	− 2·7	−9·6	− 8·2	+ 2·9	Sunday.	− 4·4	− 0·7	+4·9	−2·8	Sunday.	−2·6
	5	65·3	+2·3	− 7·9	− 5·2	−5·4	− 2·5	Sunday.	+ 6·8	− 2·7	− 1·8	+4·7	−0·6	− 2·0	−1·3
	6	65·5	−4·9	−10·1	− 0·9	+4·1	Sunday.	+ 4·1	+ 3·6	− 1·1	+ 0·1	+1·3	Sunday.	+ 2·0	−0·2
	7	65·6	−4·6	− 5·6	− 0·3	Sunday.	+ 5·5	+ 1·1	+ 5·9	− 4·1	+ 4·0	Sunday.	−4·0	+ 7·3	+0·5
	8	65·7	−1·3	− 5·5	+ 4·6	−3·1	+ 4·8	+ 2·9	+ 7·9	− 8·4	Sunday.	−2·5	−5·2	+10·2	+0·4
	9	65·9	−1·4	− 6·2	Sunday.	+2·0	− 3·2	+ 6·7	+ 7·6	Sunday.	+ 6·6	−2·2	+0·5	+ 3·6	+1·4
	10	66·0	−5·1	Sunday.	− 4·0	+2·8	− 0·5	+12·5	+ 6·4	− 2·6	+10·8	−0·5	−2·0	+ 6·0	+2·2
	11	66·1	Sunday.	− 3·5	−10·7	+1·1	+ 7·1	+ 8·9	Sunday.	+ 0·9	+10·7	+3·0	−3·9	Sunday.	+1·5
	12	66·2	−0·9	+ 2·4	− 6·4	+1·7	+15·8	Sunday.	+ 8·1	+ 2·1	+12·6	+6·6	−2·0	− 3·8	+3·3
	13	66·3	+3·2	+ 4·7	− 3·2	+1·1	Sunday.	− 3·0	+ 4·0	+ 5·2	+12·1	+1·2	Sunday.	− 2·8	+2·2
	14	66·3	+6·3	+ 2·0	+ 2·7	Sunday.	+ 9·7	−10·2	− 3·6	+ 4·2	− 4·6	Sunday.	+2·3	+ 0·8	+1·0
	15	66·4	−2·0	− 1·4	− 1·2	−3·1	+ 8·2	− 9·2	+ 1·2	− 0·5	Sunday.	+8·4	−0·6	− 0·7	−0·1
	16	66·5	−5·9	+ 0·1	Sunday.	−3·5	+10·7	− 8·3	+ 7·0	Sunday.	− 1·6	+6·8	+6·1	− 1·5	+1·0
	17	66·6	−4·4	Sunday.	+ 1·1	−2·6	+ 7·0	− 5·2	+ 8·0	− 2·6	+ 0·8	+5·2	+6·2	+ 2·6	+1·5
	18	66·6	Sunday.	+ 5·4	+ 5·2	+1·1	− 0·5	− 1·8	Sunday.	+ 2·9	+ 3·6	−2·2	−1·5	Sunday.	+1·4
	19	66·7	+1·7	+ 4·8	− 6·9	+0·2	− 2·5	Sunday.	+10·6	+ 3·0	+ 7·2	−0·9	−3·3	− 5·6	+0·8
	20	66·7	+0·6	− 6·0	− 9·4	−1·7	Sunday.	+ 5·9	+ 7·0	+ 6·5	+ 2·5	+3·0	Sunday.	+ 2·4	+1·1
	21	66·7	+4·4	− 6·9	− 4·3	Sunday.	+ 6·4	+ 2 6	+ 6·0	+ 3·9	− 0·4	Sunday.	−2·2	+10·0	+1·9
	22	66·8	+8·7	+ 0·3	+ 0·3	+3·9	+ 0·8	− 1·1	+ 0·2	− 3·6	Sunday.	+3·2	−0·2	+ 8·0	+1·9
	23	66·8	+7·1	+ 4·5	Sunday.	+1·2	− 4·7	+ 4·8	− 3·0	Sunday.	− 1·7	+2·6	+2·9	+ 2·3	+1·6
	24	66·8	+4·3	Sunday.	+ 0·1	−1·9	− 6·8	− 0·8	− 2·9	+ 0·3	+ 2·3	+8·2	−0·9	− 1·2	+0·1
	25	66·9	Sunday.	− 6·9	− 2·5	−3·9	− 2·1	+ 4·8	Sunday.	− 3·1	0·0	−1·1	+2·8	Sunday.	−1·3
	26	66·9	−3·8	+ 4·8	+ 0·7	−5·0	+ 0·8	Sunday.	−12·5	− 0·2	+ 3·1	−7·5	+4·9	+ 0·8	−1·4
	27	66·9	−4·0	+ 4·0	− 1·8	−2·8	Sunday.	− 0·7	−10·8	− 1·3	− 4·6	−7·1	Sunday.	− 3·9	−3·3
	28	66·9	−9·5	− 2·6	+ 3·9	Sunday.	− 3·8	+ 2·1	− 8·5	− 1·6	− 2·7	Sunday.	−0·2	− 0·1	−2·3
	29	66·9	−4·7	+ 3·9	− 7·1	+3·6	− 5·2	+ 8·5	− 2·6	− 3·2	Sunday.	+4·6	−5·2	+ 6·0	−0·1
	30	66·9	−8·8	− 5·7	Sunday.	+2·4	−11·4	+ 8·0	− 7·5	Sunday.	+ 4·6	+1·1	−5·3	− 0·6	−2·3
	31	66·9	−8·4	Sunday.	− 5·6	+5·3	− 7·9	+ 1·0	− 5·9	0·0	− 6·0	+1·9	−4·2	−10·5	−3·7
August.	1	66·9	Sunday.	−11·0	− 8·2	+1·2	− 7·8	+ 2·0	Sunday.	− 5·9	− 3·9	+ 3·0	−5·1	Sunday.	−4·0
	2	66·8	−4·2	− 7·7	− 6·5	−1·3	− 5·4	Sunday.	− 4·0	− 1·3	− 0·4	+ 3·1	−2·5	− 9·4	−3·6
	3	66·8	+1·2	− 7·7	− 2·1	−3·7	Sunday.	+ 3·2	− 3·6	+ 1·2	+ 3·0	+ 1·8	Sunday.	− 5·8	−1·3
	4	66·8	+0·3	− 4·5	+ 1·4	Sunday.	− 0·2	+ 8·4	− 0·6	+ 0·2	− 1·8	Sunday.	−4·4	− 2·9	−0·4
	5	66·8	−1·6	+ 0·2	+ 2·0	−3·0	+ 1·4	+11·6	+ 0·5	− 3·0	Sunday.	+ 6·2	−2·9	− 4·0	+0·7
	6	66·8	−3·8	− 3·8	Sunday.	−4·2	+ 1·5	+ 6·1	+ 2·2	Sunday.	+ 0·3	+ 5·2	−0·2	− 4·5	−0·1
	7	66·7	−3·5	Sunday.	+ 2·5	−0·4	+ 5·2	+ 5·9	− 2·3	− 1·9	− 1·8	+ 6·6	+2·4	+ 0·2	+1·2
	8	66·7	Sunday.	− 1·0	− 0·7	+3·2	+ 2·4	+ 0·4	Sunday.	− 0·1	− 0·7	+ 3·3	−2·4	Sunday.	+0·5
	9	66·6	−1·8	− 3·7	− 2·4	+3·1	+ 5·8	Sunday.	+11·4	+ 5·1	+ 0·8	+ 4·1	−3·7	− 1·0	+1·6
	10	66·6	−3·8	− 2·6	+ 0·8	−3·3	Sunday.	+ 0·5	+ 3·6	+ 7·4	− 7·2	− 0·4	Sunday.	− 2·8	−0·8
	11	66·5	−8·1	− 1·5	+ 1·1	Sunday.	+ 2·1	− 2·6	− 1·0	+ 6·9	− 4·6	Sunday.	−1·3	− 3·3	−1·2
	12	66·4	−5·9	+ 1·7	+ 3·9	−8·1	− 0·5	+ 5·8	+ 1·9	+ 6·3	Sunday.	+ 6·7	+2·1	− 1·3	+1·1
	13	66·3	−1·7	+ 0·3	Sunday.	−4·3	− 3·3	+ 6·6	+ 4·2	Sunday.	− 0·5	− 2·6	+2·6	+ 0·2	+0·2
	14	66·3	−5·2	Sunday.	+ 2·2	−2·1	− 2·1	+ 3·2	+ 4·9	+11·3	− 3·3	− 2·2	−7·3	+ 2·7	+0·2
	15	66·2	Sunday.	+ 1·6	− 2·2	+0·5	+ 0·1	+ 6·6	Sunday.	+ 6·7	− 4·3	− 5·5	−8·8	Sunday.	−0·6
	16	66·1	+1·9	+ 3·5	+ 3·6	+4·7	+ 2·9	Sunday.	+ 7·0	+ 4·8	− 1·9	− 5·9	−6·7	− 8·6	+0·5
	17	66·0	+4·0	+ 3·9	+ 2·0	+1·0	Sunday.	− 2·7	+ 0·7	− 2·6	+ 0·6	− 7·1	Sunday.	− 4·4	−0·5
	18	65·9	+4·3	+ 2·3	− 3·2	Sunday.	+ 6·3	− 8·7	− 9·1	+ 0·1	+ 3·9	Sunday.	−6·0	0·0	−1·0
	19	65·8	−1·4	− 1·5	− 4·5	+4·8	+ 1·1	− 2·6	− 8·1	− 0·3	Sunday.	− 2·1	−6·3	+ 5·3	−1·4
	20	65·6	−0·8	− 5·3	Sunday.	−2·6	+ 4·4	− 2·2	− 4·7	Sunday.	+ 2·9	− 3·6	−4·3	+ 2·5	−1·2
	21	65·5	+1·3	Sunday.	− 4·3	−4·7	+ 6·9	+ 2·0	− 2·3	− 2·0	+ 3·8	− 2·8	+0·5	+ 3·3	+0·2
	22	65·4	Sunday.	− 2·7	− 3·0	+0·2	+ 3·1	− 0·5	Sunday.	+ 1·2	+ 0·9	− 1·6	+2·7	Sunday.	0·0
	23	65·2	−6·5	− 0·2	− 2·2	−3·0	+ 7·2	Sunday.	− 5·7	+ 4·6	+ 2·7	+ 1·2	+2·1	+ 7·8	+0·7
	24	65·0	−5·6	+ 1·8	− 3·1	−6·5	Sunday.	− 4·6	− 3·6	+ 3·3	+ 2·6	+ 1·9	Sunday.	+ 7·2	−0·7
	25	64·8	−2·6	+ 6·6	− 0·5	Sunday.	+ 4·0	+ 0·4	− 2·9	+ 3·1	+ 4·8	Sunday.	−0·8	+ 6·4	+1·9
	26	64·6	−2·7	+ 6·8	+ 4·8	−6·7	+ 3·0	+ 1·3	+ 1·6	+ 2·7	Sunday.	− 3·9	−8·7	+ 6·6	+0·4
	27	64·4	−0·9	+ 6·0	Sunday.	−7·4	− 3·6	+ 3·2	− 6·3	Sunday.	+ 4·2	− 9·8	−9·3	− 1·0	−2·5
	28	64·2	+1·4	Sunday.	+ 3·7	−5·7	− 1·6	+ 3·7	− 3·8	+ 5·7	− 2·5	− 2·7	−5·5	− 3·1	−0·9
	29	63·9	Sunday.	+ 1·7	+ 3·1	−4·7	+ 6·2	+ 2·4	Sunday.	+ 3·8	+ 4·2	+ 0·1	−1·3	Sunday.	+1·7
	30	63·7	−0·6	− 1·1	+ 6·9	−1·6	+ 3·3	Sunday.	− 0·9	+ 6·0	+ 6·7	+ 4 1	+5·3	− 1·0	+2·5
	31	63·4	−0·2	+ 4·3	+10·1	+1·5	Sunday.	+ 8·0	− 8·9	+ 6·4	− 5·8	+ 7·3	Sunday.	+ 2·9	+2·6

Table IV. (Continued.)

	Daily temperature derived from the monthly mean of the 12 years.	Difference on each day of the twelve years from the temperature derived from the monthly means.												Mean daily difference.
		1841.	1842.	1843.	1844.	1845.	1846.	1847.	1848.	1849.	1850.	1851.	1852.	
	(1.)	(2.)	(3.)	(4.)	(5.)	(6.)	(7.)	(8.)	(9.)	(10.)	(11.)	(12.)	(13.)	(14.)
September. 1	63·1	+ 4·1	+10·7	+ 8·5	Sunday.	+ 0·7	+10·8	+ 5·2	+ 1·8	− 8·1	Sunday.	+ 0·8	+ 7·8	+4·2
2	62·8	+ 7·1	+ 0·6	+ 9·7	+ 4·2	+ 2·0	+11·6	+ 0·2	+ 1·0	Sunday.	− 3·6	+ 1·7	+ 8·8	+3·9
3	62·5	+ 5·7	− 6·0	Sunday.	+ 0·4	+ 4·6	+ 8·3	+ 0·5	Sunday.	− 2·2	− 0·5	0·0	+ 1·8	+1·3
4	62·2	+ 4·6	Sunday.	+ 7·5	− 1·5	+ 2·4	+10·8	− 5·8	+ 7·9	+ 3·7	+ 2·7	+ 1·4	− 2·5	+2·8
5	61·9	Sunday.	− 2·7	+ 1·3	− 3·2	− 3·8	+ 9·7	Sunday.	+ 1·0	+ 6·9	− 2·4	+ 5·4	Sunday.	+1·4
6	61·5	+ 6·9	− 1·0	+ 4·6	+ 0·1	− 2·3	Sunday.	− 2·9	− 4·0	− 5·4	− 3·7	+ 6·3	+ 4·9	+0·3
7	61·2	+ 8·1	+ 2·0	+ 4·9	+ 0·2	Sunday.	+12·5	+ 1·7	− 0·9	−11·0	− 0·7	Sunday.	+ 4·5	+2·1
8	60·8	+ 8·4	− 6·3	+ 1·1	Sunday.	− 8·9	+ 1·4	+ 3·9	+ 2·0	−10·0	Sunday.	+10·1	+ 3·8	+0·6
9	60·4	+ 9·4	− 8·1	− 7·7	+ 3·7	− 3·6	+ 0·3	− 7·7	− 4·7	Sunday.	− 2·6	+ 9·2	+ 7·1	−0·4
10	60·1	+ 8·2	− 3·7	Sunday.	+ 4·5	− 6·7	+ 3·8	− 5·2	Sunday.	− 3·5	− 2·8	+13·7	+ 9·0	+1·7
11	59·7	− 3·5	Sunday.	− 8·6	+ 4·5	− 8·2	+10·7	− 4·6	− 1·4	− 0·5	− 5·7	+13·3	+ 4·7	+0·1
12	59·3	Sunday.	− 0·9	− 7·9	+ 2·7	− 7·4	+ 9·3	Sunday.	− 7·1	+ 2·3	−11·6	+15·6	Sunday.	−0·6
13	58·9	− 3·2	− 1·4	− 1·7	+ 0·7	− 3·7	Sunday.	− 9·6	− 5·9	+ 3·1	− 8·2	− 3·0	−11·5	−4·0
14	58·4	− 3·9	− 0·4	− 2·6	+ 7·0	Sunday.	+14·7	−12·7	− 3·3	+ 4·1	− 8·7	Sunday.	− 9·0	−1·5
15	58·0	+ 3·0	+ 1·8	+ 3·0	Sunday.	− 1·0	− 1·2	− 9·3	− 8·6	+ 5·0	Sunday.	−10·5	− 9·4	−2·7
16	57·6	+ 5·7	− 2·5	+ 3·5	+ 8·1	− 9·2	+ 1·5	− 5·4	− 6·9	Sunday.	− 2·1	− 7·0	− 8·9	−2·1
17	57·1	+ 3·8	− 3·5	Sunday.	+10·9	+ 1·4	+ 1·2	+ 1·2	Sunday.	+ 4·7	+ 2·8	− 5·2	− 5·9	+1·1
18	56·7	− 6·7	Sunday.	+ 5·8	+ 1·9	+ 3·5	+ 2·0	+ 1·4	− 0·1	− 1·0	+ 2·9	+ 1·4	− 0·1	+1·0
19	56·2	Sunday	−10·0	+ 2·9	+ 8·7	− 1·3	+ 6·9	Sunday.	+ 3·1	− 4·0	− 3·0	+ 5·9	Sunday.	+1·0
20	55·7	+ 4·3	− 2·2	+ 4·9	+13·9	− 3·6	Sunday.	− 0·8	− 4·1	+ 0·7	− 2·2	+ 9·7	− 2·8	+1·6
21	55·2	+ 8·0	−11·4	+12·2	+ 2·8	Sunday.	− 3·6	− 1·9	− 9·5	+ 4·6	+ 1·7	Sunday.	+ 5·5	+0·8
22	54·7	+11·1	−17·6	− 0·1	Sunday.	− 6·0	− 0·9	− 2·3	−14·3	+ 7·6	Sunday.	− 2·7	− 1·7	−2·7
23	54·3	+ 7·4	−10·5	+11·4	− 5·9	− 5·1	+10·2	+ 5·7	− 4·9	Sunday.	+12·5	+ 4·4	− 4·3	+1·9
24	53·8	+ 6·3	− 7·0	Sunday.	− 5·4	− 6·1	+ 4·9	+ 0·1	Sunday.	+ 0·3	+ 8·3	−10·5	+ 0·9	−0·8
25	53·3	+ 4·3	Sunday.	− 0·1	− 7·2	− 3·4	− 1·5	− 0·6	− 3·7	+ 6·9	− 2·2	− 4·6	+ 2·0	−0·9
26	52·8	Sunday.	− 3·6	− 9·0	− 9·8	− 3·1	+ 0·3	Sunday.	−14·2	− 3·7	− 2·0	+ 3·8	Sunday.	−4·6
27	52·3	− 2·6	+ 6·3	−11·6	−11·9	− 3·5	Sunday.	+ 7·2	− 5·2	− 4·8	0·0	− 7·7	− 5·7	−3·6
28	51·9	− 2·0	+ 5·2	− 7·2	− 8·6	Sunday.	+ 1·2	− 2·8	− 8·0	+ 5·9	− 1·1	Sunday.	− 4·0	−2·1
29	51·4	− 4·1	+ 6·1	+ 0·1	Sunday.	+12·1	+ 9·3	− 4·9	− 0·7	+ 2·8	Sunday.	+ 1·2	− 7·3	+1·4
30	50·9	− 5·3	+ 5·3	+ 3·1	− 5·4	+ 6·7	+10·9	− 2·1	− 8·0	Sunday.	− 5·1	− 2·2	− 0·5	−0·2
October. 1	50·5	− 9·1	+ 5·3	Sunday.	− 6·2	− 0·2	− 0·5	− 4·7	Sunday.	− 1·9	+ 1·1	− 3·2	+ 3·0	−1·6
2	50·0	− 6·7	Sunday.	+ 2·0	+ 4·4	+ 4·9	− 4·1	− 2·0	− 1·9	− 5·0	− 6·6	+ 3·4	+ 9·6	−0·2
3	49·6	Sunday.	− 1·8	− 3·0	+ 1·3	+ 3·2	− 2·1	Sunday.	+ 5·0	− 0·5	− 3·4	+ 2·7	Sunday.	+0·2
4	49·1	− 4·6	− 5·1	− 3·7	+ 1·2	+ 2·5	Sunday.	+ 6·0	+ 4·7	+ 1·6	+ 2·5	− 1·2	+ 5·9	+0·9
5	48·7	− 2·7	− 4·8	+ 0·8	− 0·9	Sunday.	+ 3·7	+10·7	+ 4·0	− 1·3	+ 2·2	Sunday.	+ 2·6	+1·4
6	48·3	− 2·2	− 3·7	+ 8·0	Sunday.	− 6·9	+ 7·4	+ 8·8	+ 0·3	− 0·9	Sunday.	+ 2·0	+ 9·0	+2·2
7	47·9	0·0	+ 6·7	+ 5·1	−10·1	+ 1·9	+15·3	+ 7·8	+ 3·3	Sunday.	−10·2	+ 5·9	+ 6·7	+2·9
8	47·5	− 2·2	+ 6·3	Sunday.	+ 4·8	+ 7·5	+11·5	+ 5·8	Sunday.	− 1·4	+ 0·9	+ 8·0	+ 9·2	+5·0
9	47·1	− 1·6	Sunday.	− 3·5	+ 7·7	+ 8·4	+ 5·3	+ 4·0	+ 5·1	+ 3·6	+ 7·4	+ 8·3	+ 6·1	+4·6
10	46·7	Sunday.	+ 1·0	− 0·7	+ 1·5	+10·4	− 6·0	Sunday.	− 1·8	− 2·2	+ 7·8	+ 9·1	Sunday.	+2·1
11	46·3	+ 5·1	+ 9·3	+ 3·1	− 4·3	+ 6·1	Sunday.	− 2·8	− 6·1	+ 0·1	+ 7·7	+11·0	+ 3·6	+3·0
12	46·0	− 2·4	+ 0·8	− 3·1	− 2·0	Sunday.	+10·3	− 2·9	− 3·9	+ 2·0	− 1·7	Sunday.	+ 1·5	−0·1
13	45·6	− 4·1	0·0	− 7·4	Sunday.	+ 4·5	− 0·1	− 6·1	− 1·7	− 4·1	Sunday.	+ 9·7	− 4·4	−1·4
14	45·3	+ 2·3	+ 1·0	− 8·5	+ 2·5	− 3·6	− 0·9	−10·8	+ 5·2	Sunday.	− 6·3	+ 2·3	− 1·7	−1·7
15	44·9	+ 2·2	− 1·7	Sunday.	+ 0·2	− 9·9	− 0·8	− 4·5	Sunday.	+ 4·8	− 2·0	− 4·2	− 6·0	−2·2
16	44·6	− 6·8	Sunday.	− 3·5	− 1·6	− 6·3	+ 5·0	+ 5·3	+10·6	+ 8·7	+ 8·7	− 8·0	− 8·0	+0·4
17	44·3	Sunday.	− 0·1	− 5·0	− 2·0	− 3·0	−11·0	Sunday.	− 8·7	+ 5·1	+10·7	− 1·3	Sunday.	−1·7
18	44·1	− 8·5	− 5·8	− 3·4	+ 0·1	+ 3·8	Sunday.	+ 8·9	− 9·2	− 0·4	− 1·2	+ 8·6	+11·2	+0·4
19	43·8	− 3·6	− 5·1	− 3·6	− 6·1	Sunday.	− 1·1	− 0·9	− 3·3	− 3·7	− 2·4	Sunday.	− 0·4	−3·0
20	43·6	− 4·2	− 6·2	+ 8·9	Sunday.	− 9·9	− 3·4	− 2·3	− 1·4	− 2·7	Sunday.	+ 9·7	− 1·7	−1·3
21	43·3	− 4·5	− 0·8	− 1·1	− 1·9	−14·7	− 5·0	− 2·2	− 3·3	Sunday.	+ 0·2	+ 1·6	+ 5·1	−2·4
22	43·0	− 7·6	+ 3·5	Sunday.	+ 0·5	−12·0	−12·0	− 4·4	Sunday.	+ 6·0	+ 5·6	− 2·7	+ 2·1	−2·1
23	42·8	− 9·0	Sunday.	−10·4	+ 0·1	− 1·2	− 6·4	− 0·8	− 0·7	+ 2·8	+ 2·0	− 2·2	− 2·6	−2·6
24	42·5	Sunday.	+ 8·4	− 6·6	+ 5·4	+ 1·9	− 3·4	Sunday.	+ 5·3	− 2·3	− 1·0	+ 3·1	Sunday.	+1·2
25	42·3	−11·2	− 1·4	− 5·5	+ 7·2	+ 2·6	Sunday.	− 3·5	+ 2·2	+ 3·3	− 3·1	− 6·4	+ 2·8	−1·2
26	42·0	− 3·7	− 1·4	− 7·3	+ 0·6	Sunday.	+ 7·1	−15·4	− 0·4	+ 3·5	− 2·7	Sunday.	− 2·9	−2·3
27	41·8	− 9·9	− 4·6	−10·8	Sunday.	+ 6·0	+ 1·0	−11·6	+ 3·0	+ 2·4	Sunday.	− 9·8	+ 5·6	−2·9
28	41·5	− 5·2	+ 2·2	− 6·6	−13·5	+ 7·4	− 8·1	− 7·2	+ 7·4	Sunday.	+ 2·9	− 1·6	+11·1	−1·0
29	41·3	+ 3·0	+ 2·4	Sunday.	−12·5	+12·8	− 1·0	− 1·5	Sunday.	+ 7·1	− 8·2	+ 8·1	+ 6·2	+1·6
30	41·0	+ 8·5	Sunday.	− 9·1	−10·7	+11·1	− 5·4	+ 1·4	+ 9·7	− 3·9	− 4·1	+10·4	+ 5·4	+1·2
31	40·8	Sunday.	− 1·4	− 7·7	−10·9	+11·9	− 5·1	Sunday.	+ 2·1	−10·9	+ 3·1	+ 1·4	Sunday.	−1·9

Table IV. (Continued.)

		Daily temperature derived from the monthly mean of the 13 years.	Difference on each day of the twelve years from the temperature derived from the monthly means. 1841.	1842.	1843.	1844.	1845.	1846.	1847.	1848.	1849.	1850.	1851.	1852.	Mean daily difference.
		(1.)	(2.)	(3.)	(4.)	(5.)	(6.)	(7.)	(8.)	(9.)	(10.)	(11.)	(12.)	(13.)	(14.)
November.	1	40·5°	+13·2°	+3·9°	−4·4°	−2·3°	+7·8°	Sunday.	+10·1°	−1·7°	−4·0°	+6·8°	+3·5°	+3·5°	+3·3°
	2	40·3	+6·6	−0·5	−5·3	−1·5	Sunday.	+9·8	+11·6	−4·7	+1·2	+5·4	Sunday.	+3·1	+2·6
	3	40·0	+2·4	−4·9	−9·0	Sunday.	+1·5	+7·7	+10·8	−0·3	+6·9	Sunday.	−4·3	+0·5	+1·1
	4	39·8	−3·2	−4·6	−12·8	+0·5	+1·5	+4·9	+3·6	+3·3	Sunday.	+10·4	−8·1	+1·2	−0·3
	5	39·5	−2·2	+0·3	Sunday.	+0·4	+2·0	0·0	−0·9	Sunday.	+13·8	+14·8	−11·0	−0·3	+1·7
	6	39·2	−6·2	Sunday.	−9·4	−0·2	+2·4	+2·0	−6·2	−2·8	+12·6	+4·2	−7·9	+1·8	−0·9
	7	39·0	Sunday.	+5·4	−5·2	+3·9	+0·7	+7·6	Sunday.	−8·7	+10·1	−4·7	−1·9	Sunday.	+0·8
	8	38·7	−0·7	+0·4	−5·6	−4·4	−5·1	Sunday.	+9·4	−8·6	+5·2	−1·7	+0·5	−0·9	−1·0
	9	38·4	−1·1	−2·1	−3·2	−4·5	Sunday.	+13·6	+7·9	−16·3	+6·2	−3·1	Sunday.	−3·0	−0·6
	10	38·1	+0·8	−1·8	−1·3	Sunday.	+1·1	+14·2	−0·5	−12·6	+9·2	Sunday.	−6·5	−3·6	−0·1
	11	37·8	+4·3	+3·1	−3·1	+3·2	+0·2	+12·2	−1·8	−9·0	Sunday.	−0·5	−14·0	+1·1	−0·4
	12	37·5	+3·7	+1·3	Sunday.	+4·8	−1·5	+9·2	−0·7	Sunday.	+5·1	+1·4	−6·8	+0·4	+1·7
	13	37·2	+0·9	Sunday.	−9·5	−3·7	+7·0	+8·7	+0·8	−4·1	+9·0	−0·8	−5·6	−4·3	−0·1
	14	36·9	Sunday.	+0·7	−11·8	−2·9	+4·7	+7·5	Sunday.	−0·2	+6·4	+3·1	−5·0	Sunday.	+0·3
	15	36·5	−6·4	−3·1	−0·9	−0·3	+2·3	Sunday.	+0·6	+0·9	+0·8	+6·1	−3·8	−4·2	−0·7
	16	36·2	−1·8	−1·2	+2·7	+3·2	Sunday.	+8·4	+6·6	+0·6	+4·8	−2·1	Sunday.	−4·5	+1·7
	17	35·8	−4·5	+2·2	+4·0	Sunday.	+9·6	+10·3	+11·7	−2·1	+4·8	Sunday.	−4·0	−4·5	+2·8
	18	35·5	−7·1	−14·0	+4·7	−5·9	+14·3	+10·5	+7·9	−4·9	Sunday.	+0·4	−5·1	−1·2	0·0
	19	35·1	−4·9	−6·8	Sunday.	+0·2	+4·6	+4·1	−8·4	Sunday.	+10·5	+1·4	−3·1	−1·6	−0·4
	20	34·8	−1·6	Sunday.	+3·6	+0·7	+8·9	+3·5	−8·1	−2·5	+9·9	+1·9	+1·4	+0·3	+1·6
	21	34·4	Sunday.	−8·5	+4·4	+2·4	−1·1	+6·1	Sunday.	−0·8	+7·8	−6·0	+3·8	Sunday.	+0·9
	22	34·1	+8·6	−4·1	−2·7	+7·3	+0·3	Sunday.	+7·4	−0·2	+11·6	−6·8	−3·1	−0·7	+1·6
	23	33·7	+1·4	0·0	+5·6	+4·3	Sunday.	+1·2	+15·2	+1·0	+7·2	−1·8	Sunday.	−5·8	+2·8
	24	33·3	+0·6	−4·3	+6·5	Sunday.	−8·4	+0·7	+11·7	+9·0	+8·5	Sunday.	−4·9	−7·6	+1·2
	25	33·0	−5·2	−5·1	+0·9	−12·3	+1·9	−12·8	+3·8	+1·3	Sunday.	+3·8	−3·5	+3·5	−2·2
	26	32·6	−10·7	−7·9	Sunday.	−5·9	−1·6	−7·2	−8·7	Sunday.	+7·7	+3·8	−6·3	+7·4	−2·9
	27	32·2	−8·9	Sunday.	−12·0	−15·7	−14·5	−0·2	−3·9	−2·2	+2·7	+11·5	+1·8	+2·7	−3·5
	28	31·9	Sunday.	−12·9	−6·0	−11·3	−18·1	+6·2	Sunday.	+7·0	−1·3	+4·3	+3·3	Sunday.	−3·2
	29	31·5	−13·4	−11·4	−1·5	−0·9	−11·9	Sunday.	−17·9	+9·3	+0·3	+5·1	+3·4	+1·4	−3·4
	30	31·1	−1·5	−5·2	−4·4	+3·6	Sunday.	−7·4	−5·9	−2·0	+3·7	+5·6	Sunday.	+3·9	−1·0
December.	1	36·8	+3·1	−5·7	−0·2	Sunday.	−12·1	−3·5	+6·1	−3·6	−9·4	Sunday.	−9·2	+4·7	−3·0
	2	30·5	+4·7	+1·5	−0·3	−1·3	−22·3	+10·6	+4·4	+6·4	Sunday.	+1·2	−7·9	+4·3	+0·1
	3	30·1	+9·5	+6·1	Sunday.	+2·7	−9·8	+0·3	+0·5	Sunday.	−0·1	+9·7	−5·6	+10·5	+2·4
	4	29·8	+5·5	Sunday.	+5·4	+5·1	−3·6	+2·4	−0·3	+4·8	+4·1	+5·2	−3·8	+11·3	+3·3
	5	29·4	Sunday.	−4·1	−9·5	+4·3	−6·5	+2·0	Sunday.	+0·2	+5·3	+2·4	−6·0	Sunday.	−1·3
	6	29·1	+0·4	−10·2	−0·5	+6·6	−6·1	Sunday.	+2·0	−1·3	−3·0	+0·6	+0·3	+14·5	+0·3
	7	28·7	+2·2	−3·1	+1·1	+9·5	Sunday.	+9·7	+6·2	+4·1	−4·4	−6·5	Sunday.	+19·0	+3·8
	8	28·5	+10·7	+4·0	+1·5	Sunday.	+0·7	+6·5	+3·1	+12·2	−4·1	Sunday.	+9·3	+10·5	+5·4
	9	28·2	+12·5	−0·1	−0·3	+0·1	+0·6	+3·3	+13·7	+4·5	Sunday.	−2·9	+1·2	+8·4	+3·7
	10	27·9	+13·1	−8·6	Sunday.	−1·1	−13·8	+1·9	+14·5	Sunday.	+0·6	−10·2	+0·6	+5·7	+0·3
	11	27·7	+9·7	Sunday.	+4·9	+1·9	−22·5	−1·7	+5·0	+1·8	−4·5	0·0	−6·4	+5·7	−0·6
	12	27·4	Sunday.	+3·0	−16·0	+7·3	−12·0	−9·0	Sunday.	+0·7	−3·5	−10·5	+3·3	Sunday.	−4·1
	13	27·2	+11·6	−3·9	−3·1	+7·0	+0·1	Sunday.	+6·3	+4·8	+1·1	−16·7	−16·3	+1·2	−0·7
	14	27·0	+13·2	−1·6	+7·9	+5·6	Sunday.	−10·7	+6·0	+10·6	+5·1	−4·1	Sunday.	+0·5	+3·3
	15	26·8	+7·5	−1·7	+7·6	Sunday.	+2·5	−11·3	−1·2	+6·8	+7·0	Sunday.	−12·3	+0·7	+0·6
	16	26·6	−4·6	−5·3	+5·9	−7·7	+1·7	−3·7	−8·8	+12·8	Sunday.	+2·7	−21·8	+9·5	−1·8
	17	26·4	−12·9	−0·9	Sunday.	−9·3	+8·4	−5·1	−5·4	Sunday.	+3·3	−9·8	−21·5	+1·1	−5·2
	18	26·2	−10·5	Sunday.	+4·3	−7·8	+2·2	−0·2	+0·6	+11·2	−1·1	−8·4	−10·1	−0·2	−1·8
	19	26·1	Sunday.	+1·1	+5·2	−9·0	−16·1	+4·0	Sunday.	+11·4	+10·2	−11·1	−5·2	Sunday.	−1·1
	20	25·9	−11·2	−1·8	+9·5	−7·7	−14·4	Sunday.	−14·0	−0·5	+11·9	−3·9	−10·5	−0·3	−3·9
	21	25·8	−17·9	+6·8	+8·7	+1·9	Sunday.	−4·2	−6·3	−10·7	+2·0	+4·8	Sunday.	−9·1	−2·4
	22	25·7	−8·5	−15·2	+8·8	Sunday.	−11·5	−5·3	−4·9	−17·0	+0·4	Sunday.	−12·2	−1·5	−6·7
	23	25·6	+8·0	−12·4	+8·0	+0·4	−4·9	−9·8	−2·7	−6·6	Sunday.	−14·9	−10·5	+7·4	−3·5
	24	25·5	−0·4	−7·6	Sunday.	+6·0	−1·8	+7·4	−5·3	Sunday.	−4·8	−18·5	−2·1	+13·5	−1·4
	25	25·4	Xmas day	Xmas day.	Xmas day.	Xmas day	Xmas day.	Xmas day.	Xmas day.	Xmas day.	Xmas day.	Xmas day.	Xmas day.	Xmas day.	
	26	25·3	Sunday.	+5·3	+12·1	+10·4	−10·4	+2·6	Sunday.	−4·2	−12·8	+3·2	−24·4	Sunday.	−2·0
	27	25·3	+1·4	+2·5	+11·2	−5·1	−1·0	Sunday.	−5·1	+3·2	−0·2	+1·8	−0·1	+5·5	+1·3
	28	25·2	+5·5	+1·1	+5·9	−2·4	Sunday.	+4·0	+9·5	+1·5	−1·3	−9·5	Sunday.	+6·8	+2·1
	29	25·2	+3·4	+7·4	+1·7	Sunday.	+8·5	+3·9	+15·5	+1·7	−0·5	Sunday.	+11·4	+2·1	+5·5
	30	25·2	+6·9	−3 6	+1·1	+9·3	−2·0	+11·4	+17·5	+1·6	Sunday.	−20·0	+14·0	+3·5	+3·6
	31	25·2	−1·7	−9·7	Sunday.	+8·3	−3·4	+12·5	+18·6	Sunday.	−10·3	−11·0	+7·6	−1·8	+0·9

With a view of assisting the inquiry whether non-periodic variations of a uniform character manifest a tendency to recur at the same epochs in different years, the daily averages in column 14 of Table IV. have been collected in Table V. into "five-day means," each of the five-day means thus represented being the average non-periodic variation in those five-days in twelve years. And for the purpose of supplying a more satisfactory indication of the degree of variability to which the temperature of any single day is subject at different seasons, than can be obtained by the mere inspection of the differences in columns 2 to 13 of Table IV., the "probable daily non-periodic variation" in column 3 of Table V. has been computed for each of the five-day periods, upon the same principle and by the same process that the "probable error" of each of several partial results is derived from their mutual differences; whilst in column 2 is shown the probable non-periodic variation corresponding to the averages of five days in twelve years contained in column 1, bearing a similar analogy to the probable error of the arithmetical mean of a series of partial results.

TABLE V.

Five-day periods.	Non-periodic variations in twelve years averaged in five-day means.	Probable non-periodic variation in the averages of five days in Column 1.	Probable daily non-periodic variation.	Five-day periods.	Non-periodic variations in twelve years averaged in five-day means.	Probable non-periodic variation in the averages of five days in Column 1.	Probable daily non-periodic variation.
	(1.)	(2.)	(3.)		(1.)	(2.)	(3.)
Jan. 3 to 7 inclusive.	−0°·7	±0°·8	±5°·8	July 3 to 7 inclusive.	−1°·5	±0°·5	±3°·4
8 to 12.	−1·1	±0·9	±6·2	8 to 12.	+1·8	±0·6	±4·2
13 to 17.	+2·0	±0·8	±6·1	13 to 17.	+1·1	±0·5	±3·5
18 to 22.	−2·8	±1·1	±7·7	18 to 22.	+1·4	±0·4	±3·1
23 to 27.	+2·0	±0·9	±6·7	23 to 27.	−0·9	±0·4	±3·0
28 to Feb. 1.	−0·7	±1·0	±7·3	28 to Aug. 2.	−2·7	±0·5	±3·7
Feb. 2 to 6.	+1·6	±0·8	±6·0	Aug. 3 to 7.	0·0	±0·4	±2·6
7 to 11.	−1·6	±0·9	±6·3	8 to 12.	+0·2	±0·4	±2·8
12 to 16.	−2·9	±0·9	±6·7	13 to 17.	0·0	±0·4	±3·0
17 to 21.	+1·0	±1·0	±7·0	18 to 22.	−0·7	±0·4	±2·7
22 to 26.	+1·5	±0·9	±6·9	23 to 27.	0·0	±0·4	±3·3
27 to March 3.	−1·4	±0·8	±5·8	28 to Sept. 2.	+2·3	±0·5	±3·6
Mar. 4 to 8.	+2·0	±0·7	±5·0	Sept. 3 to 7.	+1·6	±0·5	±3·4
9 to 13.	+1·8	±0·6	±4·6	8 to 12.	+0·3	±0·7	±4·7
14 to 18.	−1·8	±0·8	±5·6	13 to 17.	−1·8	±0·6	±4·3
19 to 23.	−0·9	±0·7	±5·0	18 to 22.	+0·3	±0·6	±4·2
24 to 28.	+0·9	±0·6	±4·5	23 to 27.	−1·6	±0·6	±4·4
29 to April 2.	−0·6	±0·7	±5·3	28 to Oct. 2.	−0·5	±0·5	±3·7
Apr. 3 to 7.	+0·6	±0·6	±4·0	Oct. 3 to 7.	+1·5	±0·5	±3·6
8 to 12.	−0·1	±0·7	±4·7	8 to 12.	+2·9	±0·5	±3·9
13 to 17.	−1·7	±0·6	±4·6	13 to 17.	−1·3	±0·5	±3·9
18 to 22.	+0·2	±0·6	±4·4	18 to 22.	−1·7	±0·5	±3·9
23 to 27.	+0·5	±0·6	±4·2	23 to 27.	−1·6	±0·5	±3·9
28 to May 2.	−0·2	±0·5	±4·0	28 to Nov. 2.	+1·0	±0·6	±4·9
May 3 to 7.	−0·5	±0·6	±4·3	Nov. 3 to 7.	+0·5	±0·6	±4·3
8 to 12.	+1·1	±0·5	±3·9	8 to 12.	−0·1	±0·6	±4·1
13 to 17.	+1·0	±0·6	±4·0	13 to 17.	+0·8	±0·5	±3·6
18 to 22.	−1·3	±0·6	±4·3	18 to 22.	+0·7	±0·6	±4·2
23 to 27.	+0·3	±0·6	±4·3	23 to 27.	−0·9	±0·7	±4·9
28 to June 2.	−1·2	±0·6	±4·6	28 to Dec. 2.	−2·1	±0·7	±5·3
June 3 to 7.	−0·8	±0·6	±4·4	Dec. 3 to 7.	+1·7	±0·6	±4·4
8 to 12.	−1·9	±0·7	±5·0	8 to 12.	+0·9	±0·8	±5·5
13 to 17.	−0·5	±0·6	±4·6	13 to 17.	−0·8	±0·8	±5·8
18 to 22.	+1·9	±0·5	±3·6	18 to 22.	−3·2	±0·8	±6·0
23 to 27.	+1·3	±0·4	±3·1	23 to 28.	−0·7	±0·9	±5·8
28 to July 2.	+0·3	±0·4	±3·5	29 to Jan. 2.	+2·4	±0·8	±6·1

If we collect the values in column 3 of Table V. into monthly averages, and arrange the months in meteorological seasons, we obtain thereby the following comparison of the probable variability of the daily temperature at different seasons :—

Winter.		Spring.		Summer.		Autumn.	
December .	$\overset{\circ}{5}$·6	March . . .	$\overset{\circ}{5}$·1	June	$\overset{\circ}{4}$·0	September .	$\overset{\circ}{4}$·2
January . .	6·6	April	4·6	July.	3·5	October. . .	4·0
February . .	6·6	May.	4·2	August . . .	3·0	November .	4·4
Means . . .	6·3		4·6		3·5		4·2

Whence we learn that during the winter months the temperature of a day is as likely to differ from its mean or normal state 6°·3 or *more*, as it is to differ *less* than that amount. Regarding 6°·3 therefore as the *index of variability* in the winter months, the index in the spring months is 4°·6, in the summer months 3°·5, and in autumn 4°·2. The probable variability is a maximum in the last half of January and first half of February, and a minimum in August; it is progressive from the maximum to the minimum and from the minimum to the maximum, and its amount is about twice as great at the maximum as at the minimum.

The principal variations from the general progression shown in the five-day averages of the twelve years, accompanied in each case by the "probable variability," are as follows :—

January 18 to 22 inclusive	$-\overset{\circ}{2}$·8	$\pm\overset{\circ}{1}$·1
February 12 to 16 inclusive	−2·9	±0·9
August 28 to September 2 inclusive . .	+2·3	±0·5
October 8 to 12 inclusive	+2·9	±0·5
November 28 to December 2 inclusive .	−2·1	±0·7
December 18 to 22 inclusive	−3·2	±0·8
December 29 to January 2 inclusive . .	+2·4	±0·8

Of these, more than half, four, occur in the winter season, two in the autumn, and one in the summer. Their connexion with particular winds and with other meteorological circumstances will be more properly examined in the general discussion of the meteorological observations at Toronto in the volumes of that observatory; but it has appeared desirable to bring into immediate notice the principal instances, furnished by the five-day means in twelve years, of such more considerable departures from the regular progression of the temperature, as may be supposed to indicate in some degree a tendency to periodical recurrence.

For the convenience of those who study *monthly* averages, the mean temperatures of every month in each of the twelve years have been collected in Table VI., together with the average temperature of each month in the whole period. By the comparison of these numbers, the amount by which the temperature of any particular month during the twelve years exceeded or fell short of the average may at once be seen.

February and March appear to be months most liable to extreme variations; July and August the least so; the probable variability of the several months is as follows:—

January	±2°·7	April	±1°·9	July . .	±1°·1	October .	±1°·4
February	±2·6	May .	+1·8	August .	±1·2	November .	±2·1
March .	±2·8	June .	±2·0	September	±1·8	December .	±2·5

The mean *annual* temperature derived from the whole body of the observations in twelve years is 44°·23; and on the supposition that no constant errors, instrumental or observational, or occasioned by insufficient protection or defective exposure of the thermometer, are involved, and that the variations of the temperature in different years may be regarded strictly as accidental oscillations around a mean value, and of equally probable occurrence in every year, the probable error of this result is ±0°·18. The probable variability of a *single year* is ±0°·63; showing that there is an equal probability that the mean temperature of any one year will fall within the limits of 43°·60 and 44°·86, as that it will exceed those limits; a conclusion which perhaps would scarcely have been anticipated considering the great range of the thermometer in the course of the year, and the magnitude of the non-periodic variations in short intervals. The climate of Toronto presents a remarkable combination of great regularity in the annual temperature with great variability occurring in the course of the year. The mean temperatures of the several years differed from the average mean temperature as follows:—

1841.	−0°·31	1844.	+0°·25	1847.	−0°·53	1850.	+0°·22
1842.	−0·27	1845.	+0·35	1848.	+0·85	1851.	−0·25
1843.	−1·88	1846.	+2·13	1849.	−0·14	1852.	−0·39

The excess of cold in 1843 (1°·88) was due chiefly to the occurrence of very low temperatures in February and March of that year; the excess of heat in 1846 (2°·13) was more generally diffused throughout the year, all the months excepting February and October being above their average.

Table VI. exhibits also the normal temperatures of the different months in the geographical latitude, 43° 40′ N., in which Toronto is situated, taken from Dove's 'Verbreitung der Wärme,' Berlin, 1852; as well as the 'Thermic Anomaly,' or differences between these normal values and the temperatures observed at Toronto. It appears from this comparison that every month of the year at Toronto is colder than the normal temperature of the parallel in which it is situated, the mean annual temperature being nearly 7° below the normal. The thermic anomaly is least in July and August (between 2° and 3°) and greatest in February, when it exceeds 11°. Its sudden increase in October and decrease in November are deserving of notice. In viewing the bearing of the thermic anomaly at Toronto on the more general question of the thermic anomaly in the part of North America in which it is situated, it is necessary to bear in mind that the thermometer at Toronto was about 342 feet above the sea-level, equivalent, as usually estimated, to a diminution of rather more than 1° of Fahr. on account of vertical elevation. Dove's normal temperatures are all reduced to the sea-level, and when the monthly temperatures at Toronto have undergone the same reduction, the thermic anomaly indicated by them is diminished to about 1° in July and August, but in February still reaches the large amount of 10°; in both respects therefore confirming Dove's conclusion, that the summers of North America are *not* warmer than is due to their latitude, whilst the winters are *much* colder.

Table VI.

Showing the mean monthly and annual temperatures in each of the twelve years from 1841 to 1852 inclusive, and the average monthly and annual temperatures in the whole period. Also the mean or normal temperature of the several months in the latitude of 43° 40′ N., taken from Dove's 'Verbreitung der Wärme,' and the 'Thermic Anomaly,' shown by the observations at Toronto.

Years.	January.	Feb.	March.	April.	May.	June.	July.	August.	Sept.	Oct.	Nov.	Dec.	Mean annual temperature.
1841.	25°·6	22°·4	27°·7	39°·2	50°·5	65°·6	65°·0	64°·4	61°·3	41°·6	35°·0	28°·7	43°·92
1842.	27·9	26·9	35·8	43·1	49·1	55·6	64·7	65·7	55·7	45·1	33·3	24·7	43·96
1843.	28·7	14·5	21·3	40·9	49·1	58·4	64·5	66·4	59·1	41·8	33·5	30·0	42·35
1844.	20·2	26·0	31·3	47·5	53·6	59·9	66·0	64·3	58·6	43·3	34·9	28·2	44·48
1845.	26·5	26·0	35·4	42·1	49·6	61·0	66·2	67·9	56·0	46·4	36·8	21·1	44·58
1846.	26·7	20·4	33·1	44·0	55·5	63·3	68·0	68·4	63·6	44·6	41·3	27·5	46·36
1847.	23·3	21·5	26·2	39·2	54·4	58·4	68·0	65·1	55·6	44·0	38·6	30·1	43·70
1848.	28·7	26·6	28·6	41·3	54·1	62·9	65·5	69·2	54·2	46·3	34·5	29·1	45·08
1849.	18·5	19·5	33·5	39·0	48·0	63·2	68·4	66·3	58·2	45·3	42·6	26·5	44·09
1850.	29·7	26·0	29·8	37·9	47·6	64·3	68·9	66·8	56·5	45·4	38·8	21·7	44·45
1851.	25·5	27·6	32·4	41·3	51·3	59·2	65·0	63·6	60·0	47·4	32·9	21·5	43·98
1852.	18·4	23·4	27·7	38·2	51·4	60·8	66·8	65·9	57·5	48·0	36·0	31·9	43·84
Means in 12 years.	24·97	23·40	30·23	41·14	51·18	61·05	66·41	66·16	58·02	44·93	36·51	26·75	44·23
Normal temperatures in latitude 43° 40′ N.	32·8	34·7	40·1	50·2	58·1	64·6	68·7	68·5	61·5	53·8	43·2	36·0	51·0
'Thermic Anomaly' at Toronto.	−7·8	−11·3	−9·9	−9·1	−6·9	−3·5	−2·3	−2·3	−3·5	−8·9	−6·7	−9·2	−6·8

Plate XI. represents the five-day means in the twelve years, both as computed from the monthly means, and as given by the actual averages of every five days; the darker line shows the computed and the fainter line the observed five-day means. The darker line consequently shows the mean march of the temperature, the fainter line its actual march, and their difference the non-periodic variations for intervals of five days. The monthly means are distinguished by a small circle drawn around the point which marks the temperature; they are connected by faintly dotted vertical lines with the monthly means (similarly distinguished) of the normal temperatures of the latitude of Toronto, derived by DOVE from the mean monthly temperatures of thirty-six equidistant points in each of the parallels of 40° and 50° as shown by his isothermal maps. The normal temperature corresponds to the level of the sea; the temperatures at Toronto are those given directly by the observations, uncorrected for the elevation, which, as already stated, is about 342 feet above the sea.

VI. *On Periodical Laws in the larger Magnetic Disturbances.*

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HAVING submitted the hourly observations of the magnetic declination made at the St. Helena and the Cape of Good Hope Observatories to a course of examination similar to that undertaken by Colonel Sabine for Toronto and Hobarton, and published by him in the Philosophical Transactions, Part I. for 1851, and Part I. for 1852, also in Vol. II. Magnetical and Meteorological Observations at Hobarton, and Vol. II. Magnetical and Meteorological Observations at Toronto (now in the press), with the object of investigating some of the laws which govern the occurrence of the larger magnetic disturbances, I have found that at these two stations, as well as at the two others, the evidence is complete of the existence of laws of a periodical character: the facts appear to be important materials towards elucidating the general laws of disturbances, and I therefore venture to communicate them to the Royal Society.

The observations which have been examined are comprised between September 1842 and September 1847 at St. Helena, and between July 1841 and July 1846 at the Cape of Good Hope; these periods include all the hourly observations that could be made available for discussion.

The observations were made simultaneously at all the magnetic stations, viz. at the commencement of each Göttingen hour; we have, therefore, the best opportunity of judging of the degree in which the disturbances may be referred to one general cause, while by arranging them with regard to local time, the means are afforded of showing the operation of laws which have respect to the sun's position towards the place, both in relation to the hour of the day and the season of the year.

At St. Helena, every observation that differed by 2·5 scale divisions from its normal place (meaning by that, the mean place at the hour, and in the month in which the observation occurs) has been regarded as a disturbed observation, and separated from the others; the number of observations thus separated amounts to 2659; the whole number of hourly observations made in the period amounted to 36,054, wherefore one observation in 13·6 has been regarded as disturbed. At the Cape, in like manner, every observation that differed by more than 2·5 scale divisions has been regarded as disturbed and separated from the others; the number of separated observations is 3038, the number of hourly observations made in the period is 36,571, wherefore one observation in every 12·0 is disturbed.

The limit beyond which an observation should be regarded as disturbed, was, of course, arbitrarily fixed; and as the principle on which the disturbances were separated was the same as that which formed a guide in the previous investigation, it is only necessary here to repeat, that the absolute numbers which express the frequency and amount of the disturbances cannot be compared at any two stations, but a comparison may with propriety, and has been made, between the ratios of each year, or season or hour, to the respective mean quantities.

In a paper by Colonel Sabine already alluded to, a comparison has been instituted of the disturbances at Toronto and Hobarton, stations almost as far from each other, geographically, as it is possible for two places to be situated on the surface of the earth, but possessing considerable analogy in their magnetical relations. The result of the comparison was to show a striking connection between the two places in the laws of the occurrence of disturbances, whether as regards the hour of the day, the season of the year, or the year itself. Now St. Helena and the Cape of Good Hope, situated near together, are unlike magnetically, as well as in geographical position, to either Toronto or Hobarton; and I am desirous, in addition to presenting an investigation into the periodical character of the laws of disturbance operating at the new stations, to point out in what respect the character agrees, and in what it differs, from that which has been found at Toronto and Hobarton.

We may expect, from facts already known, some agreement in the amount of disturbance in the same years at different places. Table I. exhibits the ratios of the numbers and aggregate values of the disturbed observations in the different years to the average annual number and aggregate value, also the average value of one disturbance in each year.

Table I.

Years.	St. Helena.			Cape of Good Hope.		
	Numbers.	Aggregate values.	Average value of one disturbance.	Numbers.	Aggregate values.	Average value of one disturbance.
			′			′
1841	....	....	....	0·97 *	1·07 *	3·10 *
1842	0·98 †	0·96 †	2·43 †	1·25	1·26	2·86
1843	0·82	0·76	2·31	0·57 ‡	0·53 ‡	2·62 ‡
1844	0·90 ‖	0·90 ‖	2·47 ‖	0·90	0·88	2·76
1845	1·00	0·97	2·41	0·93	0·91	2·76
1846	1·06	1·03	2·42	1·38 §	1·36 §	2·78 §
1847	1·25 ¶	1·37 ¶	2·72 ¶	....	....	....

The observations are here too incomplete, and their duration too limited to derive more than a general indication of the action of the same law, which has been shown to exist at Toronto and Hobarton, of the periodical character of the number and magnitude of the disturbances during a cycle of years; both at Toronto and Hobarton

* From six months' observations. † From four months' observations. ‡ From ten months' observations. ‖ From ten months' observations. § From six months' observations. ¶ From eight months' observations.

the year 1843 has been shown to be that of least disturbance, and 1848 of greatest; and we have here some additional evidence of a general cause operating over the whole globe to produce a minimum magnetic effect in 1843 progressing towards a maximum in 1848. In judging of the fulfilment of the general law and of the degree of accordance of the ratios at the two stations, it is to be borne in mind that three years only are complete years of observation at each station, and that in the other three years at each station the ratios are necessarily computed from the portions of those years to which the observations are limited; also, that although the observations were everywhere made simultaneously, as stated above, yet that the cessation on Sundays was regulated by local time; and as the Saturdays' observations were closed at midnight, and the Mondays' observations commenced at 1 A.M. civil time, there are a certain number at each station which have none corresponding at any other.

Table II. contains the ratios of the numbers and aggregate values of the disturbances, divided into their westerly and easterly components, to the average annual number and aggregate value.

Table II.

Years.	St. Helena.				Cape of Good Hope.			
	Westerly.		Easterly.		Westerly.		Easterly.	
	Numbers.	Aggregate values.	Numbers.	Aggregate values.	Numbers.	Aggregate values.	Numbers.	Aggregate values.
1841	......	......	......	......	0·95	1·07	1·00	1·08
1842	0·91	0·92	1·02	0·96	1·14	1·16	1·39	1·40
1843	0·75	0·70	0·89	0·86	0·56	0·52	0·59	0·54
1844	0·89	0·91	0·90	0·92	0·97	0·95	0·80	0·77
1845	0·96	0·94	1·03	1·04	0·96	0·96	0·88	0·84
1846	1·07	1·04	1·02	1·05	1·42	1·36	1·36	1·36
1847	1·43	1·49	1·14	1·17	......	......	......	......

If we separate the whole number of observations at St. Helena and the Cape into their easterly and westerly components, we find that each of the two divisions singly, indicate the same periodical law which they did when combined, viz. a fall to a minimum in 1843, from whence an increase towards a maximum in 1848 takes place; and this is apparent both in the number of observations disturbed and the amount of the disturbance. By an easterly disturbance is to be understood, that the end of the needle which points north is disturbed from its normal place towards the east, and *vice versâ*, and this whether the mean declination has an easterly or westerly value, and whether the station be in the northern or southern hemisphere.

Table III. contains the ratios of the westerly to the easterly disturbed observations in each year, in numbers and values.

TABLE III.

Years.	St. Helena.		Cape of Good Hope.	
	Numbers. W. to E.	Values. W. to E.	Numbers. W. to E.	Values. W. to E.
1841	....	....	1·36 to 1	1·42 to 1
1842	1·12 to 1	1·28 to 1	1·17 to 1	1·19 to 1
1843	1·07 to 1	1·09 to 1	1·35 to 1	1·38 to 1
1844	1·26 to 1	1·45 to 1	1·73 to 1	1·77 to 1
1845	1·18 to 1	1·20 to 1	1·57 to 1	1·66 to 1
1846	1·33 to 1	1·34 to 1	1·51 to 1	1·40 to 1
1847	1·60 to 1	1·70 to 1	......	......

The westerly disturbances therefore decidedly preponderate at both stations over the easterly in the aggregate of each year, both in frequency and amount. At Hobarton a similar preponderance of westerly over easterly was found in every year; but at Toronto, on the contrary, the easterly disturbance exceeded the westerly in every year (though they were nearly equal in 1841 and 1847); yet on the average of seven and a half years' observation the easterly disturbances were in proportion to the westerly, as 1·17 to 1 in number, and as 1·16 to 1 in value.

Dividing the disturbances into the several months of their occurrence, we have in Table IV. the ratios in each month of the number of disturbances, of their values, and of the average value of one disturbance to the mean monthly values.

TABLE IV.

Months.	St. Helena.			Cape of Good Hope.		
	Numbers.	Values.	Average value of one disturbance.	Numbers.	Values.	Average value of one disturbance.
January	1·60	1·76	1·12	1·71	1·77	1·04
February ...	1·39	1·41	1·03	1·49	1·60	1·08
March	0·99	1·00	1·02	0·95	0·90	0·95
April	1·36	1·41	1·05	1·36	1·29	0·96
May	0·83	0·80	0·97	0·51	0·48	0·94
June	0·36	0·34	0·97	0·42	0·37	0·88
July	0·64	0·59	0·94	0·49	0·58	1·18
August	0·75	0·72	0·97	0·48	0·47	0·96
September ...	1·21	1·20	1·01	0·91	0·91	1·00
October	0·76	0·71	0·95	1·14	1·09	0·96
November ...	0·96	0·91	0·96	1·32	1·32	1·00
December ...	1·13	1·14	1·02	1·21	1·23	1·01

From this table we find that January is the month of greatest disturbance in frequency and value, and June of least disturbance, both at St. Helena and the Cape. On referring to the Hobarton observations, the very same months are found to be those of greatest and least disturbance respectively, but at Toronto *both* January and June are months of minimum disturbance; the maximum disturbances in the year occurring at Toronto in April and September.

In comparing the observations at Toronto and Hobarton alone, it was found that although a minimum of disturbance occurred at Toronto in midsummer, still the disturbances in the four summer months had a considerable preponderance over the disturbances in the four winter months; at Hobarton also the disturbances in the four summer months greatly exceeded those in the four winter months, and it seemed reasonable to refer the maxima and minima periods of disturbance in their *mean effects* to extreme seasons; St. Helena however offers a remarkable objection. The sun is vertical at St. Helena early in November, and again early in February, but there is no indication of the occurrence of the disturbances being connected with the sun's position relative to the place. The temperature reaches its maximum in the middle of March, and its minimum early in September; and if we consider the periods of extreme temperature to be those of extreme seasons at St. Helena, we shall find that at *both* those times the disturbances have their mean value, proving that insofar as St. Helena has extreme seasons, they are undoubtedly not coincident with extreme disturbances. While, therefore, we can class St. Helena, Cape of Good Hope and Hobarton together, places differing so widely in climate and geographical position, as exhibiting their maximum and minimum disturbance in the same months of the year contrasted with Toronto as presenting a different law, we may conclude that the principal causes which operate in producing the annual variation of the disturbances are not dependent upon local seasons.

There is, however, a striking point of similarity at all the stations about the times of the sun's passing the equator. At Toronto, April and September are both months of maxima disturbance in the year, September being that of primary maximum. At St. Helena a distinct tendency towards maxima is shown in the same two months, rather more strongly marked in April than in September. At the Cape of Good Hope the tendency towards maxima in the same two months is also quite apparent, but still more strongly marked in April over September than at St. Helena. At Hobarton the tendency to maxima in April and September is likewise evident, though not so marked, and here, as at Toronto, September's maximum preponderates over that of April.

The average value of a single disturbance scarcely differs in any of the months, either at St. Helena or the Cape; at Toronto and Hobarton the average value of one disturbance appears to be least in their summer months, about a mean in winter, and greatest in the equinoctial months.

Table V. contains the ratios of the numbers and values of the westerly to the easterly disturbances in the different months. The westerly disturbances exceed the easterly in every case; the preponderance has its greatest excess in the month of May, and its least in the month of February at both stations. It also appears that the ratio of westerly *values* is greater than the ratio of westerly *numbers*. At Hobarton the westerly tendencies were also found to exceed the easterly in every month, while at Toronto the easterly exceeded the westerly in every month; and it is worthy of remark, that at these two stations, as well as at the Cape and St. Helena, the ratios of *values* likewise exceeded the ratios of *numbers*; the amount in every instance is small, but decided.

Table V.

Ratios of Westerly to Easterly disturbed observations in every month in the year in numbers and values, at St. Helena and the Cape of Good Hope, the numbers and values of the easterly disturbances being represented in each case by unity.

Months.	St. Helena.		Cape of Good Hope.	
	Numbers.	Values.	Numbers.	Values.
January	1·23	1·18	1·16	1·14
February......	0·96	1·03	1·21	1·22
March.........	1·21	1·20	1·90	2·03
April	1·41	1·56	1·39	1·50
May	1·88	2·07	3·13	3·20
June	1·13	1·07	2·09	1·99
July	1·70	1·63	1·70	1·55
August	1·11	1·17	1·20	1·44
September ...	1·24	1·34	1·54	1·55
October	1·15	1·28	1·28	1·26
November ...	1·50	1·55	1·43	1·49
December ...	1·18	1·28	1·31	1·43
Means.........	1·31	1·37	1·61	1·65

Table VI. exhibits the average value of an easterly compared with that of a westerly disturbed observation in the several months, found by dividing the aggregate easterly or westerly amount of disturbance by the corresponding number of occurrences.

Table VI.

Average values of an Easterly and of a Westerly disturbance in the several months.

Months.	St. Helena.		Cape of Good Hope.	
	Easterly.	Westerly.	Easterly.	Westerly.
January	2·67	2·89	2·93	2·88
February ...	2·35	2·54	3·01	3·03
March	2·43	2·43	2·55	2·70
April	2·38	2·62	2·55	2·75
May	2·18	2·40	2·57	2·64
June	2·37	2·25	2·55	2·43
July	2·31	2·22	3·48	3·18
August	2·24	2·36	2·42	2·91
September ...	2·30	2·50	2·78	2·79
October	2·15	2·39	2·69	2·66
November ...	2·24	2·34	2·73	2·85
December ...	2·32	2·52	2·69	2·94
Means.........	2·33	2·45	2·75	2·81

It appears from this table that an easterly disturbance is very generally less in amount than a westerly disturbance. At St. Helena there is but one month in the year in which the westerly disturbance falls below the easterly, and but three at the

Cape of Good Hope. On the average of the year the easterly is decidedly a smaller disturbance than the westerly. The same fact is shown from the Hobarton observations. At that station the easterly disturbance is smaller than the westerly in eleven months of the year, the average annual easterly disturbance being considerably less than the westerly. At Toronto also we find the same fact occurring; the value of a disturbed easterly observation is less than a westerly on the average of the year; the difference is however in this case inconsiderable, and there are several discrepancies in the different months.

Next, distributing the disturbances into the several hours of their occurrence, the following table is obtained. The facts, besides being very remarkable when viewed alone, acquire so much additional interest when exhibited in contrast with the law which has already been shown to prevail at Toronto and Hobarton, that I have placed the corresponding tables of the ratios at Toronto and Hobarton beside those of St. Helena and the Cape for the sake of more convenient comparison.

Table VII.

Ratios of the numbers and of the aggregate values at the several hours, to the mean hourly numbers and values at St. Helena, Cape of Good Hope, Hobarton and Toronto.

Mean time at station, Astronomical reckoning.	St. Helena.		Cape of Good Hope.		Hobarton.		Toronto.	
	Numbers.	Values.	Numbers.	Values.	Numbers.	Values.	Numbers.	Values.
Day.								
18 hours.	0·23	0·24	0·98	1·00	0·73	0·68	1·04	1·08
19	0·44	0·38	1·30	1·29	1·01	0·96	1·02	1·28
20	0·95	0·88	1·97	1·86	1·12	1·07	1·06	1·14
21	1·58	1·49	2·01	1·97	1·05	0·91	1·08	1·09
22	1·92	1·94	2·25	2·28	1·02	0·85	1·06	0·92
23	2·49	2·55	2·12	2·17	1·04	0·88	1·01	0·83
Noon.	2·70	2·77	1·97	1·93	1·04	0·85	0·86	0·67
1	2·82	2·92	1·57	1·51	1·11	0·90	0·72	0·55
2	2·65	2·72	1·31	1·22	1·14	0·97	0·59	0·49
3	2·29	2·27	0·83	0·76	1·04	0·91	0·66	0·52
4	1·86	1·78	0·77	0·69	1·00	0·91	0·69	0·58
5	1·18	1·10	0·52	0·48	0·90	0·86	0·69	0·60
Sum of ratios at day-hours.	21·11	21·04	17·60	17·16	12·20	10·75	10·48	9·75
Night.								
6 hours.	0·60	0·56	0·63	0·60	1·04	0·90	0·77	0·78
7	0·29	0·29	0·67	0·66	0·81	0·86	0·92	0·89
8	0·32	0·32	0·71	0·81	0·88	0·93	1·11	1·17
9	0·22	0·26	0·62	0·65	0·98	1·17	1·26	1·48
10	0·21	0·24	0·59	0·69	1·11	1·26	1·27	1·33
11	0·20	0·24	0·44	0·53	1·35	1·66	1·15	1·15
Midnight.	0·22	0·26	0·45	0·52	1·28	1·45	1·21	1·24
13	0·15	0·16	0·52	0·52	1·10	1·12	1·24	1·35
14	0·17	0·18	0·49	0·50	0·99	1·15	1·21	1·21
15	0·16	0·15	0·44	0·42	1·01	1·08	1·15	1·12
16	0·15	0·17	0·51	0·49	0·95	0·94	1·22	1·25
17	0·17	0·18	0·44	0·45	0·60	0·64	1·16	1·26
Sum of ratios at night-hours.	2·86	3·01	6·51	6·84	12·10	13·16	13·67	14·23

The above table shows in a very striking manner that at St. Helena the disturbances are almost, without exception, confined to the hours of the day. At the Cape the hours of the day are likewise those of greatest disturbance in a very marked degree, as for instance, at eleven o'clock in the forenoon they are between four and five times as frequent, and as great in amount, as at the corresponding hour at night twelve hours later. Hobarton and Toronto present however very different features: there the hours of the *night* are of greatest disturbance; but it will be seen that the sum of the ratios at the twelve hours of the night does not exceed the sum of the ratios in the twelve hours of the day in the same degree that the sum of the day ratios exceeds the sum of the night ratios at St. Helena and the Cape. At Hobarton the night ratios only *slightly* exceed those of the day. At Toronto the preponderance is greater; but at the Cape we find the day-ratios between two and three times as great as those of the night, and at St. Helena about seven times as great. We must proceed further to gain a more perfect knowledge of the laws of the daily and nightly occurrences, by separating the disturbances classed according to the hours of their occurrence into their easterly and westerly components.

Table VIII.

Ratios of the Easterly and Westerly numbers and values at the several hours, to the mean easterly and westerly numbers and values.

Mean time at the station, Astronomical reckoning.	St. Helena.				Cape of Good Hope.			
	Easterly.		Westerly.		Easterly.		Westerly.	
	Numbers.	Values.	Numbers.	Values.	Numbers.	Values.	Numbers.	Values.
Day. 18 hours	0·08	0·08	0·34	0·36	1·10	1·19	0·91	0·86
19	0·39	0·32	0·48	0·43	1·56	1·56	1·12	1·10
20	1·08	0·99	0·85	0·80	2·29	2·28	1·74	1·57
21	1·65	1·62	1·52	1·40	2·25	2·32	1·84	1·73
22	2·12	2·22	1·76	1·73	2·60	2·77	2·00	1·94
23	2·80	2·96	2·24	2·24	2·56	2·67	1·81	1·82
Noon.	3·00	3·12	2·47	2·50	2·27	2·29	1·76	1·68
1	3·02	3·17	2·66	2·73	1·79	1·82	1·42	1·29
2	3·06	3·19	2·32	2·37	1·54	1·49	1·15	1·03
3	2·59	2·61	2·05	2·02	0·92	0·89	0·76	0·66
4	2·16	2·07	1·61	1·56	0·63	0·59	0·86	0·76
5	1·08	0·98	1·26	1·18	0·42	0·34	0·60	0·58
Sum of ratios at day-hours.	23·03	23·33	19·56	19·32	19·93	20·21	15·97	15·02
Night. 6 hours	0·41	0·35	0·76	0·71	0·42	0·38	0·77	0·75
7	0·14	0·12	0·40	0·43	0·31	0·24	0·92	0·96
8	0·12	0·10	0·47	0·49	0·29	0·22	1·01	1·21
9	0·06	0·05	0·34	0·41	0·23	0·19	0·89	0·97
10	0·04	0·03	0·34	0·39	0·23	0·20	0·84	1·03
11	0·00	0·00	0·35	0·41	0·17	0·16	0·64	0·78
Midnight.	0·00	0·00	0·39	0·45	0·23	0·24	0·61	0·71
13	0·00	0·00	0·27	0·28	0·40	0·41	0·60	0·60
14	0·02	0·02	0·29	0·31	0·44	0·37	0·53	0·59
15	0·04	0·04	0·26	0·24	0·44	0·40	0·45	0·44
16	0·00	0·00	0·27	0·29	0·54	0·48	0·49	0·50
17	0·02	0·02	0·29	0·31	0·46	0·50	0·43	0·42
Sum of ratios at night-hours.	0·85	0·73	4·43	4·72	4·16	3·79	8·18	8·96

Separating the whole body of disturbances at St. Helena and the Cape of Good Hope into their easterly and westerly components, it is thus found that the law of the hourly disturbance has the same general characteristic feature when viewed separately in either direction as when together, viz. the preponderance of disturbances during the day over those during the night. This would be anticipated from the large excess of the total daily disturbances; but the fact is interesting as evidencing a law different from that by which the disturbances both at Toronto and Hobarton are regulated. When the disturbances at these places were separated into their easterly and westerly components, it was found that for the purpose of comparison they should be arranged into two classes, viz. the easterly disturbances at Toronto and the westerly at Hobarton into one class, and the westerly disturbances at Toronto and the easterly at Hobarton into another; the results are elsewhere published in detail*, but it may be convenient here to repeat the general conclusion, viz. as respects the first class (easterly at Toronto and westerly at Hobarton), it was found that, at Toronto, the nightly disturbances were to the daily as 2·0 to 1 in number and as 2·8 to 1 in value. At Hobarton the nightly disturbances were to the daily as 1·6 to 1 in number and as 2·4 to 1 in value, the disturbance being greater at both places at *any* hour of the night than at *any* hour of the day. Respecting the second class (westerly at Toronto and easterly at Hobarton), the period of twenty-four hours required to be otherwise divided. At Toronto (westerly) the numbers and values were uniformly less at every hour from noon to midnight than at *any* hour from midnight to noon. At Hobarton (easterly) the numbers and values were greater at *any* hour from 6 A.M. to 5 P.M. than at *any* hour from 6 P.M. to 5 A.M., or the easterly *day* disturbance greater than the easterly *night* disturbance. In this last we recognize a similarity in the law to that in both directions combined, or either separately, at the Cape and St. Helena; and although we found that the *total* disturbance at Hobarton showed a preponderance during the night to that during the day, it is evident this was caused by the westerly maxima having overridden and masked the easterly maxima.

It has already been shown, as the result of the separation of the disturbances at St. Helena and the Cape into easterly and westerly movements, that in each separate instance, as in both combined, there is the same general law, viz. the maxima of frequency and amount occurring in the day hours and the minima in the night hours. When closely followed out, however, the law of easterly and westerly disturbances presents some differences. Table IX. contains the total amounts of easterly and of westerly disturbances observed during all the years in which the observations under discussion were made, in frequency and value, at the several hours of their occurrence at St. Helena and the Cape of Good Hope.

* Philosophical Transactions for 1852, p. 103.

TABLE IX.

Mean time at Station, Astronomical reckoning.	St. Helena.				Cape of Good Hope.			
	Numbers.		Values.		Numbers.		Values.	
	Easterly.	Westerly.	Easterly.	Westerly.	Easterly.	Westerly.	Easterly.	Westerly.
Day.								
19 hours.	19	30	52·6	91·7	81	83	301·4	308·1
20	53	53	160·7	171·9	119	129	440·0	440·4
21	81	94	261·2	301·6	117	136	447·9	484·2
22	104	109	357·5	372·3	135	148	533·7	544·6
23	137	139	479·0	481·7	133	134	515·1	510·8
Noon.	147	153	503·2	539·2	118	130	441·5	470·8
1	148	165	511·0	589·7	93	105	350·3	361·3
2	150	144	515·1	510·8	80	85	287·9	289·6
3	127	127	420·4	436·3	48	56	171·2	185·9
4	106	100	333·4	336·2	33	64	114·1	212·9
5	53	78	158·3	255·1	22	44	66·4	161·9
6	20	47	55·7	153·9	22	57	72·7	211·1
Night.								
7 hours.	7	25	19·2	91·8	16	68	46·2	267·9
8	6	29	16·4	105·8	15	75	42·4	338·6
9	3	21	7·8	89·4	12	66	35·9	271·1
10	2	21	5·1	83·7	12	62	37·6	289·3
11	0	22	0·0	89·9	9	47	31·0	218·7
Midnight.	0	24	0·0	96·3	12	45	47·2	199·6
13	0	17	0·0	61·0	21	44	78·7	167·8
14	1	18	2·8	66·6	23	39	70·6	166·5
15	2	16	6·2	50·7	23	33	76·0	123·2
16	0	17	0·0	62·6	28	36	92·0	140·0
17	1	18	2·5	65·9	24	32	96·8	115·8
18	4	21	12·4	77·0	57	67	229·5	241·4

From this table it appears that at St. Helena, although few disturbances occurred at night in comparison with those during the day, almost all the nightly disturbances were in the westerly direction. During five years of observation 183 disturbances, in all, occurred between the hours of 9 P.M. and 5 A.M. inclusive, having a value of 690·5 scale divisions; of these but nine were in the easterly direction with a value of only 24·4 scale divisions, the remaining 174 were in the westerly direction with a value of 666·1 scale divisions. During the same number of hours of maximum disturbance, from 9 A.M. to 5 P.M. inclusive, there were 2162 disturbances having a value of 7362·0 scale divisions; of these 1053 were to the east and 1109 to the west, the values being 3539·1 and 3822·9 scale divisions respectively; westerly are therefore slightly the greater. At the Cape of Good Hope the preponderance of westerly over easterly disturbance is greater during the day hours than at St. Helena, and not so great during the night hours; and the analogy with St. Helena is still so far carried out that the excess during the night hours is considerably greater than the excess in the day hours.

St. Helena is situated in that part of the globe in which the diurnal movement of the declination needle has peculiar interest. Unlike those places in middle latitudes, where the north end of the needle attains its extreme easterly and westerly positions at the same hours in all seasons, at St. Helena the curve of diurnal movement is precisely reversed at opposite periods of the year; in general terms, it may be said to correspond to that of the diurnal variation in the northern hemisphere, (at Toronto for example), when the sun is north of the equator, and to the diurnal variation in the southern hemisphere (or as at Hobarton), when the sun is south of the equator. It appeared therefore desirable to examine whether the law of the occurrence of the disturbance, as respects the hour of the day, remained the same in every month in the year, whether in fact there was any analogy between the law of hourly disturbance at one period of the year compared with northern latitudes, or at Toronto, and of the opposite period compared with southern latitudes, or at Hobarton.

Table X. will show that the law is the same in all the months of the year, the hours of the day being in every month periods of greatest, and the night hours of least disturbance.

Table X.

Monthly Statement of the number of Disturbed Observations in Five Years, at St. Helena, distributed into the several months and hours of their occurrence.

Hours.	Jan.	Feb.	March.	April.	May.	June.	July.	August.	Sept.	Oct.	Nov.	Dec.	Sums.
Noon.	34	30	28	42	16	11	32	19	25	13	20	30	300
1	26	24	29	42	25	12	32	19	34	19	23	28	313
2	29	31	20	35	20	12	21	22	37	12	27	28	294
3	28	30	18	23	23	10	25	13	29	12	17	26	254
4	28	22	18	18	16	6	21	12	21	12	14	18	206
5	16	10	14	12	8	5	9	9	16	10	11	11	131
6	7	5	8	7	5	0	6	6	3	3	6	11	67
7	3	2	4	8	1	1	3	1	2	0	4	3	32
8	3	1	5	6	3	2	3	3	1	2	3	3	35
9	4	1	3	6	2	1	1	2	0	0	2	2	24
10	4	2	2	6	2	0	1	3	0	0	1	2	23
11	3	1	3	6	2	0	2	1	0	0	2	2	22
Midnt.	3	2	2	6	3	0	2	0	0	0	2	4	24
13	1	2	2	4	0	0	2	0	1	1	2	2	17
14	1	2	2	2	1	0	3	0	1	2	3	2	19
15	2	1	1	2	1	0	2	3	1	1	3	1	18
16	3	2	0	2	1	1	1	2	1	1	3	0	17
17	3	1	0	2	0	0	4	1	1	2	5	0	19
18	3	1	0	2	2	1	2	3	3	2	5	1	25
19	5	2	1	2	1	2	5	3	11	10	5	2	49
20	15	8	8	9	4	1	5	6	26	15	4	5	106
21	22	15	12	16	18	4	9	10	26	14	13	16	175
22	27	24	17	18	17	5	17	11	15	19	18	25	213
23	27	38	25	28	16	7	25	20	17	20	22	31	276
Sums.	297	257	222	304	187	81	233	169	271	170	215	253	2659

It now only remains to notice the influence of the disturbances on the mean position of the magnet at the different hours; this we shall determine by obtaining the aggregate excess of disturbance in one direction over that in the opposite direction, and dividing by the whole number of days over which the period of observation extends.

Table XI. contains the excess of easterly or westerly values of the disturbances at the several hours, and the mean effect on the position of the magnet.

Table XI.

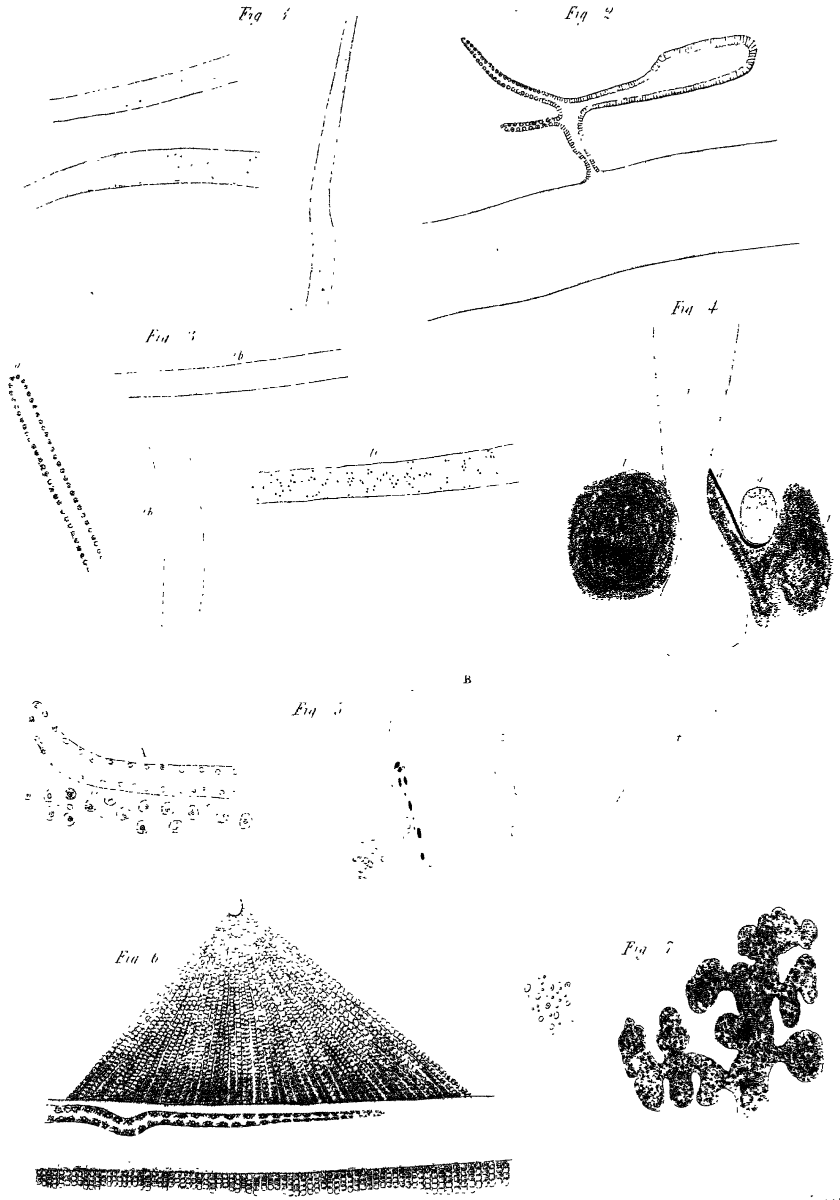
Mean time at Station, Astronomical reckoning	St. Helena.	Cape of Good Hope.
	Mean diurnal variation occasioned by the disturbed observations.	
18 hours	′ 0·31 W.	′ 0·06 W.
19	0·18 W.	0·03 W.
20	0·05 W.	0·02 W.
21	0·19 W.	0·18 W.
22	0·07 W.	0·05 W.
23	0·01 W.	0·02 E.
Noon.	0·17 W.	0·14 W.
1	0·37 W.	0·05 W.
2	0·02 E.	0·01 W.
3	0·08 W.	0·08 W.
4	0·01 W.	0·49 W.
5	0·46 W.	0·49 W.
6	0·46 W.	0·68 W.
7	0·34 W.	1·09 W.
8	0·43 W.	1·46 W.
9	0·38 W.	1·16 W.
10	0·37 W.	1·24 W.
11	0·43 W.	0·92 W.
13	0·46 W.	0·75 W.
Midnight.	0·29 W.	0·44 W.
14	0·30 W.	0·47 W.
15	0·21 W.	0·23 W.
16	0·30 W.	0·23 W.
17	0·30 W.	0·09 W.

The effect of the disturbances in its mean quantity is therefore *constantly* to draw the north end of the magnet to the west both at St. Helena and the Cape; there is but one exceptional hour at St. Helena and one at the Cape, and these from the smallness of the amount may perhaps be considered as accidental; the influence on the night hours is decidedly greater than in the day, at both places, and is more strongly marked at the Cape than at St. Helena. Once more comparing this law with that at Toronto and Hobarton, it is interesting to observe, that whereas we have just seen the mean effect of the disturbance is constant in drawing the north end of the magnet to the west in every hour of the twenty-four at the Cape and St. Helena, and more energetically in the night than in the day; at Toronto the effect is west during the day and east during the night; and at Hobarton east during the day and west during the night.

In the foregoing paper exact numerical results have not been aimed at. It would be difficult, even if instrumental means were more perfect, to arrive at a very precise knowledge of the normal positions of the magnet, and therefore to separate *all* the disturbances, including those of smallest amount; nor would any commensurate advantage be gained by the attempt.

By the method pursued a sufficient body of disturbances has been dealt with (and those of the larger amount) to make it probable that the numerical ratios would suffer little alteration by a more refined process, which it is doubtful the observations themselves would sustain. The nature of the laws has in every instance been indicated; and their regular and systematic character, evidenced in the classification adopted, bear abundant testimony that the method (first employed by Colonel Sabine) of separating observations that differ by a certain prescribed amount from the mean, and dealing with them as disturbances, is the true method of investigating the laws by which the abnormal fluctuations are regulated.

Woolwich, February 15th, 1853.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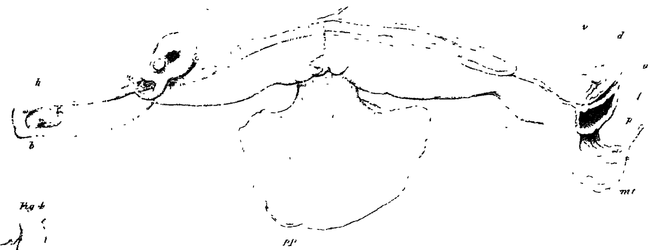

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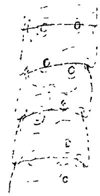

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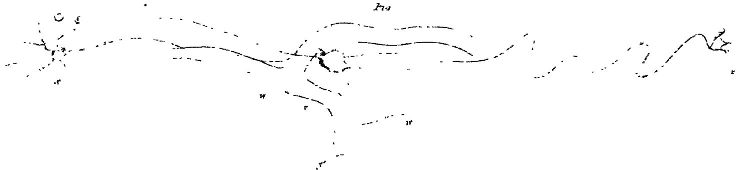

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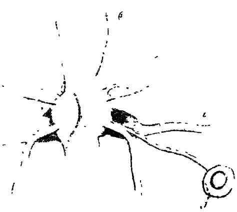

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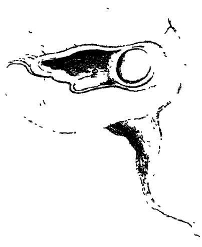

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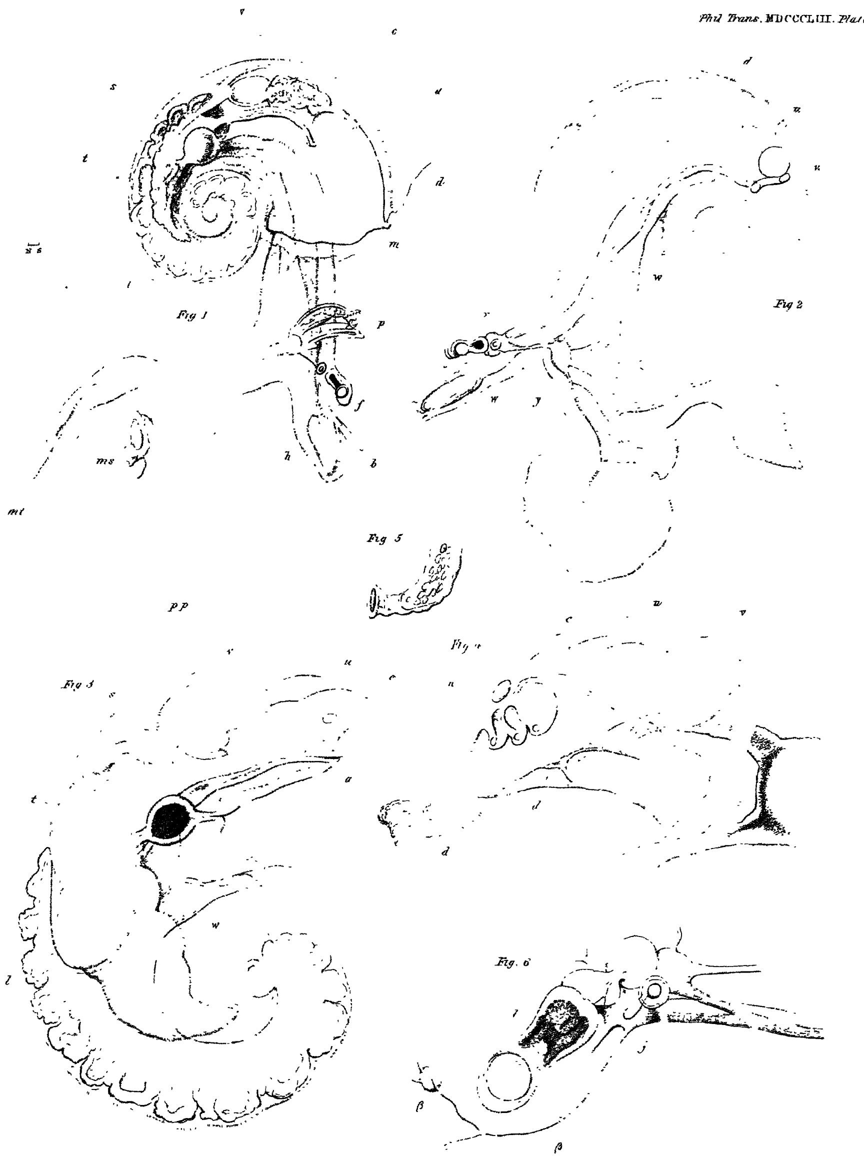
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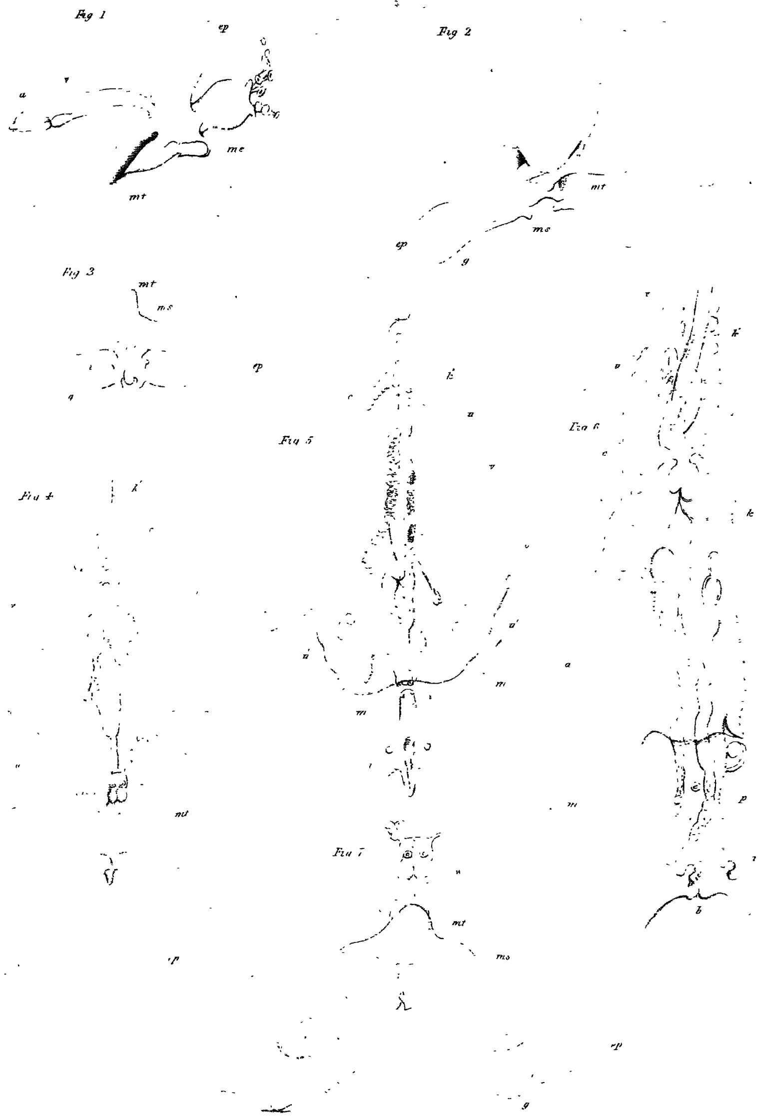
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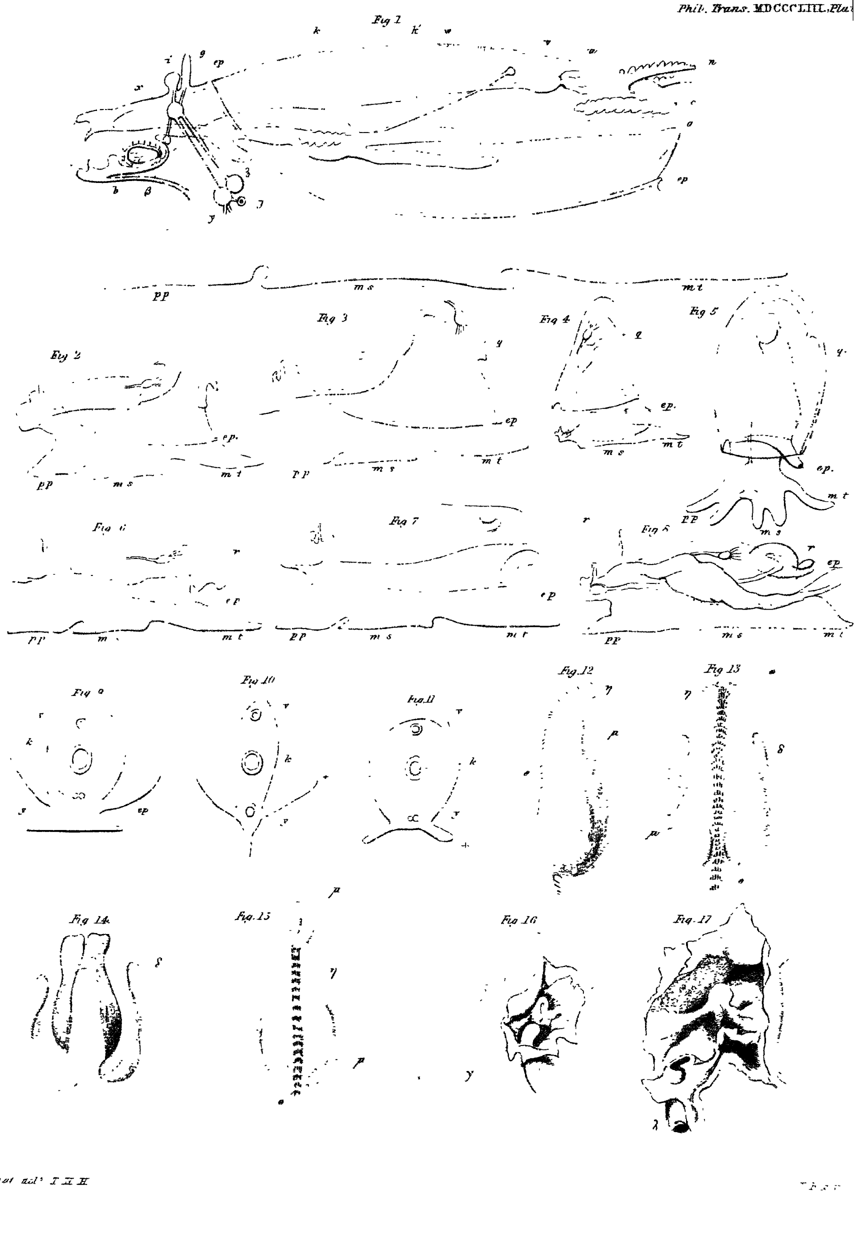

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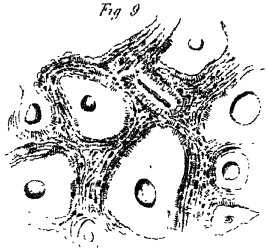

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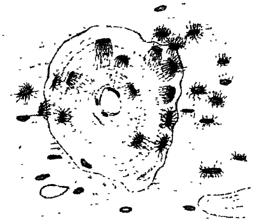

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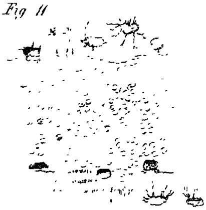

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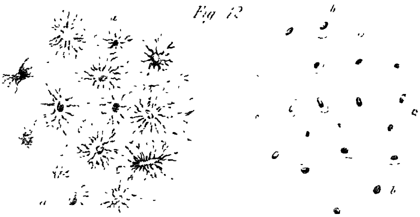

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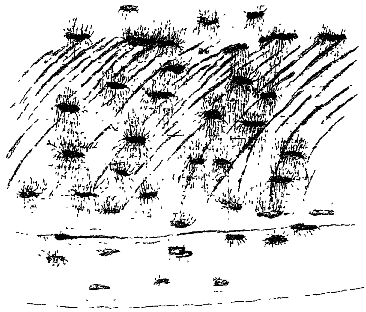

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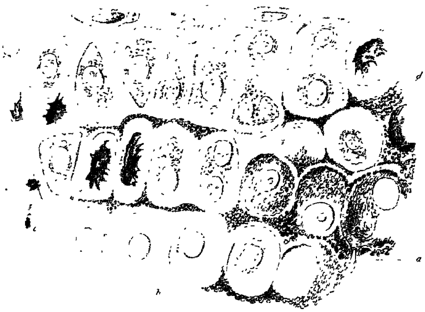
Fig. 19



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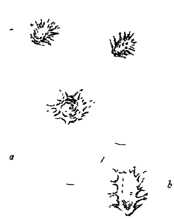

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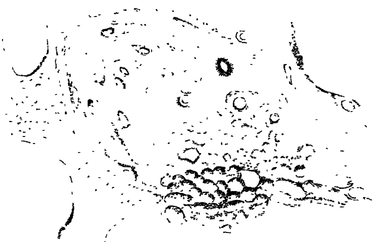

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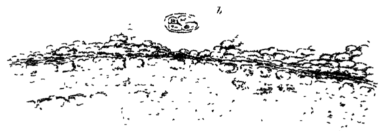

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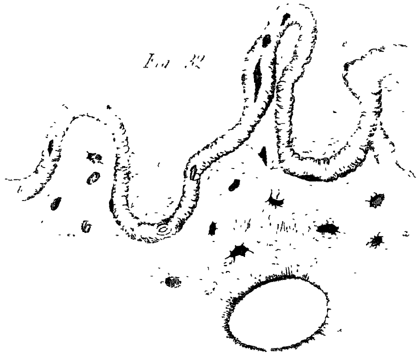

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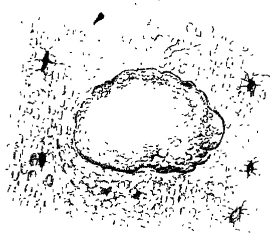
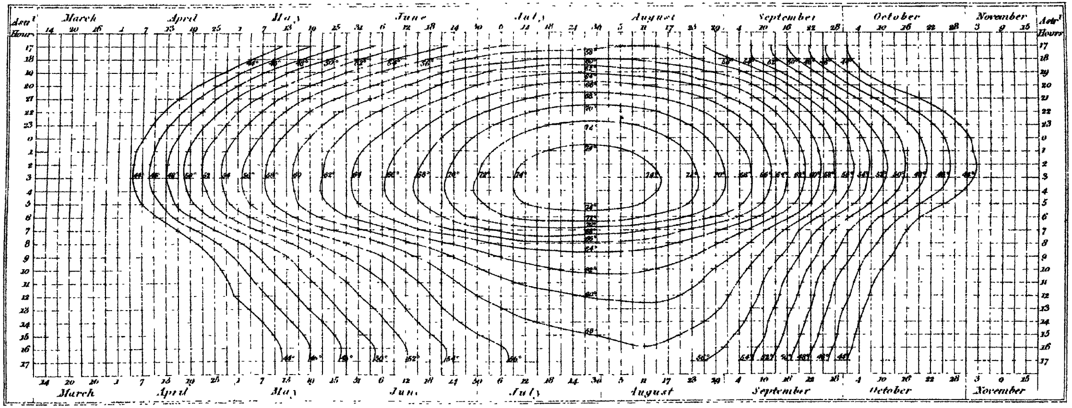
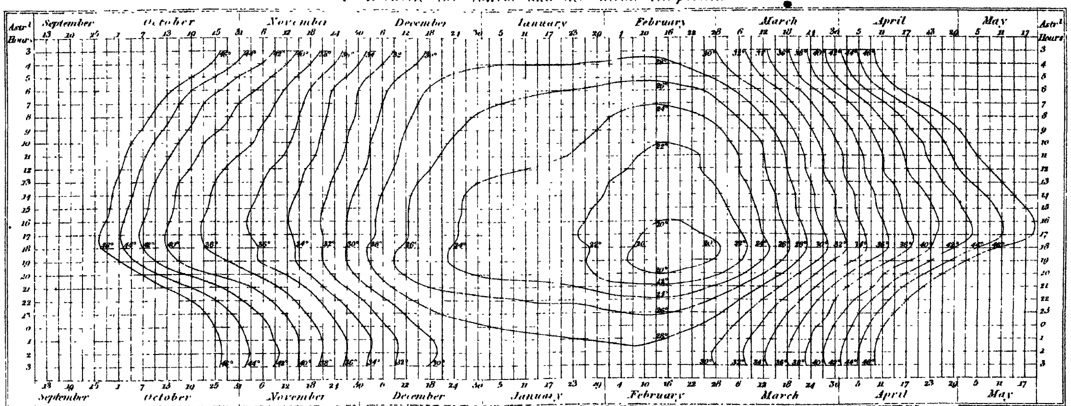


Chrono Isothermals at Toronto; derived from six years of hourly observations

1° Between the highest and the mean Temperature.



2° Between the lowest and the mean Temperature.



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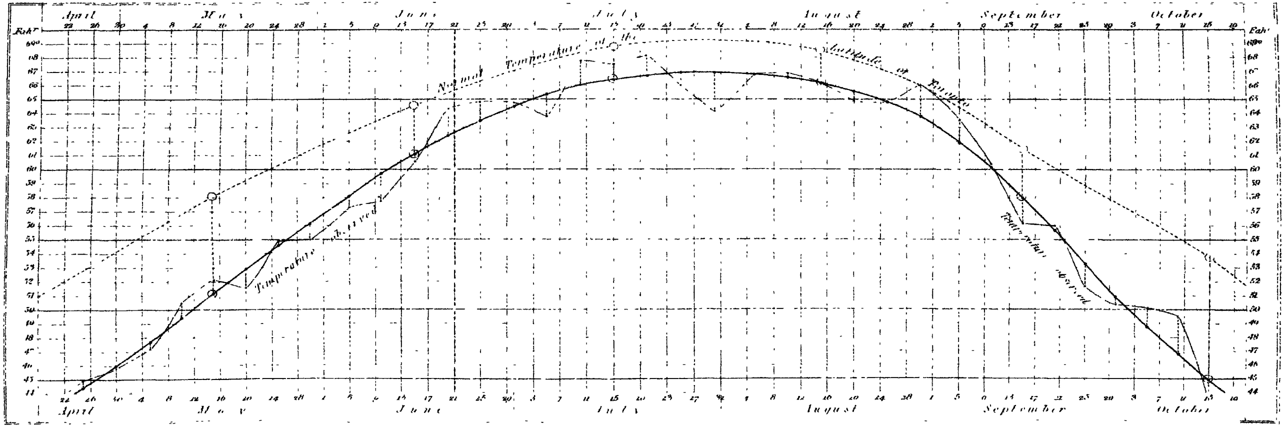
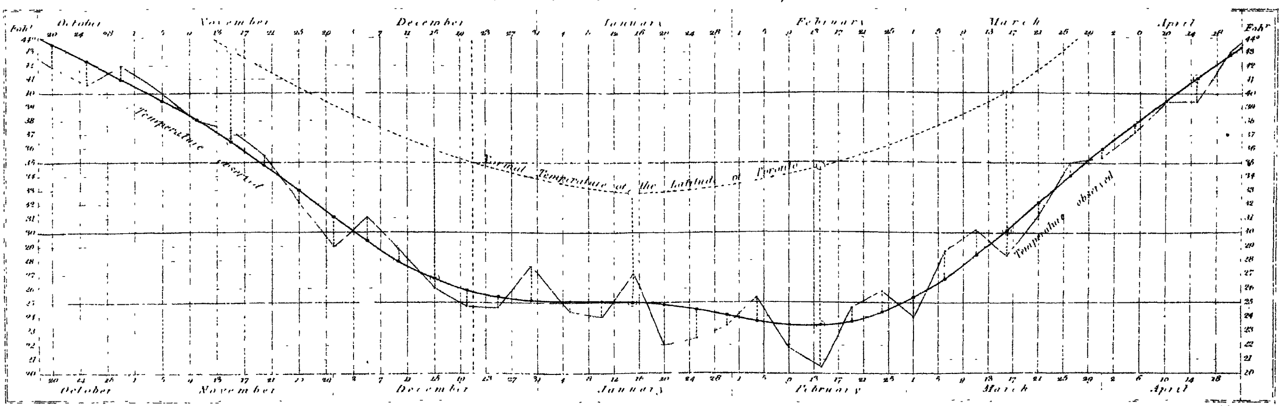
Five-day Means of the Temperature at Toronto from 12 years of Observation

1° Between the highest and the mean Temperature



2° Between the lowest and the mean Temperature



CONTENTS.

PHILOSOPHICAL TRANSACTIONS.

VII. *On the Eclipses of* AGATHOCLES, THALES, *and* XERXES.

By G. B. AIRY, *Esq., Astronomer Royal.*

Received December 15, 1852,—Read February 3, 1853.

SECTION I. *Preliminary.—On the Elements of the Lunar Tables.*

1. TILL the beginning of the present century, neither the mechanical theory of the moon's motion, nor the numerical determination of her principal elements, nor the lunar tables founded on these, were sufficiently accurate for the computation of a distant eclipse. And (perhaps in consequence of the evident imperfection of these essential grounds of calculation) the mode of treating chronological eclipses was, in most instances, extremely lax. The general result of these deficiencies is, that in any point of the slightest delicacy, the calculations made before 1810 are absolutely worthless.

2. The extension and general improvement of the lunar theory by LAPLACE, and in particular the determination of the secular equations depending on the square of the time, very greatly altered the state of lunar and chronological science. Partly by the stimulation of foreign academies, partly by individual enterprise, lunar tables were soon produced which embodied the principal results of the new theory, and which were founded on more numerous and more carefully reduced observations than had been used before. The extensive tables by BÜRG, printed by the Bureau des Longitudes in 1806, and the smaller tables by OLTMANNS from the same elements, printed in the fourth supplementary volume of the Berliner Jahrbuch in 1808, will long be remarked as important steps in lunar calculation.

3. The first valuable deduction which was drawn from these, in reference to chronological computation, was the series of calculations in the paper by our late Fellow, Mr. FRANCIS BAILY, "On the Solar Eclipse which is said to have been predicted by THALES," communicated to the Royal Society on 1811, March 14, and printed in the Philosophical Transactions for 1811. Although there can now be no doubt that the eclipse on which Mr. BAILY fixed was a wrong one, yet this paper (the first, I believe,

of Mr. Baily's astronomical publications) must be considered as possessing the highest value; for its due appreciation of the characteristic circumstances of a total eclipse, for its accuracy of computation, and for the caution and good faith with which the results are announced. Mr. Baily, in the first place, pointed out that only a *total* eclipse could satisfy the account of Herodotus, and that a total eclipse *would* suffice. He lived to witness the total eclipse of 1842, but he observed it from a room of a house, where probably he could scarcely remark the general effect of the eclipse. I have myself seen two total eclipses (those of 1842 and of 1851), being on both occasions in the open country; and I can fully testify to the sudden and awful effect of a total eclipse. I have seen many large partial eclipses, and one annular eclipse concealed by clouds; and I believe that a body of men, intent on military movements, would scarcely have remarked on these occasions anything unusual. Mr. Baily then, adopting Bürg's tables, exhibited in detail the results of computation of the eclipses of B.C. 585, May 28; 583, October 1; 607, July 30; 597, July 9; 601, September 20; 603, May 18; 626, February 3; and 610, September 30; and stated that he had computed all the eclipses which seemed likely to have been visible in Asia Minor from B.C. 650 to B.C. 580. He found that only the eclipse of 610, September 30, passed over Asia Minor; that the centre of its shadow crossed the river Halys at its mouth, passing a short distance south of the Caspian Sea; and he adopted it as the true eclipse of Thales. But he then subjoined a computation which threw doubt on the whole. Upon applying the elements of the same tables to the computation of the eclipse of Agathocles, B.C. 310, August 15, he found that they would not give a total eclipse for any place in which it was possible to locate Agathocles at the time when, according to the historical record, the shadow passed over him. Mr. Baily inferred from this that the tabular motion of the moon's node must be altered; and he admitted, as a probable effect of such an alteration, that the eclipse of B.C. 610 might be found insufficient for the history of Herodotus. But he confidently believed that no other eclipse between B.C. 650 and 580 would be made to pass centrally over any part of Asia Minor.

4. Since I first read this paper, I have always attached greater importance to the last portion of it than to any other part. It has always appeared to me that not the eclipse of Thales but that of Agathocles ought to be considered as the true cardinal eclipse for chronology and lunar astronomy. And I have long since contemplated the recalculation of the latter eclipse with the view of obtaining correct elements for the computation of the former.

5. About the same time in which Mr. Baily was employed in his calculations, Mr. Oltmanns was also engaged (apparently without any knowledge of Mr. Baily's labours) on the same subject. His paper was presented to the Berlin Academy on 1812, November 26, and is printed in the Berlin Memoirs 1812–1813. He points out (as Mr. Baily had done) that a total eclipse is required: he insists also that it must have been visible as a large eclipse in Ionia; he then, using his own tables

(equivalent to BÜRG's), exhibits the detailed elements of several eclipses, and finds that the eclipse of B.C. 610, September 30, was total on the Halys and at Erzeroum (a result agreeing precisely with Mr. BAILY's), and, like Mr. BAILY, he adopts that as the true eclipse of THALES. He then subjoins a very valuable table, exhibiting roughly the track of the central shadow in every eclipse, whether total or not (113 in number), as computed by means of the same tables, from B.C. 631 to 585.

6. The successive alterations in the adopted motions of the moon's node have been nearly as follows. I must premise that there is much confusion among the original writers, from the circumstance that some in speaking of secular motions refer that term to a Julian century, while others use an ordinary Gregorian century, which differs from the Julian by one day, and produces a difference of 3′ 16″·4 in the motion of the node. I have endeavoured to remove this obscurity by giving the motion in all cases for a Julian century.

	°	′	″
LALANDE, in the third edition of his Tables, 1792, had made the secular regression of the moon's node	134	11	15
LAPLACE, in the Additions to the Connaissance des Temps, An. VIII., from ALBATENIUS' discussion of PTOLEMY, found that the motion was to be diminished to	134	8	25
BÜRG and OLTMANNS, 1806 and 1808, adopted	134	11	42
BOUVARD, Monatliche Correspondenz, 1811, May, found from the discussion of numerous eclipses	134	9	42
BURCKHARDT's Tables, 1812, adopt	134	10	12
OLTMANNS, Berliner Jahrbuch, 1817 (printed about three years earlier), found from eclipses in 1239 and 1241	134	10	28
WURM, Zeitschrift für Astronomie, 1817, January and February, found from twelve ancient lunar eclipses	134	8	23·8
and from eight total or annular solar eclipses	134	8	25·6
DAMOISEAU's Tables, 1824, adopt	134	9	57·5

7. In the Berliner Jahrbuch, 1824, is a paper by OLTMANNS, dated 1821, May 15, in which for the first time Mr. BAILY's researches are mentioned. M. OLTMANNS remarks that the close agreement between Mr. BAILY's results and his own on the eclipse of THALES proves the correctness of their calculations; and then he proceeds to say that the researches of BOUVARD, BURCKHARDT, and WURM, as well as his own, show that the secular regression of the node must be diminished 2′ (or reduced to 134° 9′ 42″); that with this the eclipse of AGATHOCLES was possible, supposing AGATHOCLES near Cape Passaro; that to make the eclipse central there the regression must be further diminished (as I understand him) by 18″, and to make it barely possible it must be increased by 9″. But after insisting on the certainty of this correction of the node, and after having called attention to his former calculations on the eclipse of THALES, he never so much as hints that his former conclusions must now necessarily be erroneous. I am wholly unable to account for this extraordinary silence.

8. In the Ergänzungs-Heft to the Astronomische Nachrichten, published in 1849, Professor HANSTEEN has given a most interesting discussion of the total eclipse which occurred at the battle of Stiklastad, A.D. 1030, August 31. The certainty as to the exact spot on which this battle was fought, and the attainable precision of the astronomical determination (for the eclipse was annular when it commenced upon the earth, and the dark shadow was therefore extremely small at Stiklastad), will render this eclipse unusually valuable. Professor HANSTEEN infers from it the value of the secular regression of the node 134° 8′ 28″·5; but as it appears that in the calculation (which was avowedly made rather for the verification of history and chronology than for the correction of the lunar elements) BURCKHARDT's tables were used without alteration, and as we shall hereafter give reason for supposing that, in one important point, these tables are incorrect, this result is not entirely free from suspicion. It is much to be wished that Professor HANSTEEN would calculate this eclipse with special reference to the correction of the elements of the moon's motion.

9. In the year 1846, the reduction of the Greenwich Lunar Observations from 1750 to 1830 was nearly completed under my direction, and I was able to exhibit the apparent errors of the moon's epoch of mean longitude to Professor HANSEN. The immediate result of this was, that Professor HANSEN discovered two inequalities of long period in the moon's motion, produced by the attraction of Venus. Their formulæ are given in the Astronomische Nachrichten, No. 597. The values of the coefficients, as I understand, are not entirely free from doubt.

10. In a paper communicated to the Royal Astronomical Society on 1848, June 9, and printed in vol. xvii. of their Memoirs, I gave the principal results of the Lunar Reductions above cited. It is unnecessary to recapitulate here the corrections to the various elements of the moon's orbit for different years; I shall only remark, that though I do not doubt that the mean motion of the node may be found from them far more accurately than it was ever before found from meridional observations, yet I conceive it to be still open to correction from observations of distant eclipses. Indeed, the uncertainty, as to the part of the field of view at which the Greenwich observers in BRADLEY's time were accustomed to observe the moon's declination, leaves considerable uncertainty as to the place of the node.

11. In alluding to these corrections, it is proper to advert to the changes which their apparent values must receive from HANSEN's new inequalities. And first, as to the change in the correction to the moon's motion of mean longitude. HANSEN's inequalities affect the apparent error of epoch very differently in the first part and in the last part of the interval through which the reductions extend, the numerical amount of their influence going through a gradual change without repeated reversion of sign. For the time, therefore, through which the reductions extend, these inequalities produce an apparent alteration in the moon's mean motion; and therefore, when, in the comparison of observations with theory, they are duly taken into account, the resulting value of the moon's secular motion in longitude will be sensibly different

from that which would have been found if they had not been considered. This change is fully recognized in the paper in volume xvii. of the Royal Astronomical Society's Memoirs.

12. Secondly, as to the correction to the motion of the moon's node. The comparison, between an observed latitude of the moon (supposed near the node) and the corresponding tabular latitude of the moon as calculated from DAMOISEAU's tables, gives in fact a comparison between the true argument of latitude and DAMOISEAU's argument of latitude; and therefore it gives the sum of two corrections to two different elements of DAMOISEAU's tables; namely, the correction to DAMOISEAU's longitude of the moon, and the correction to DAMOISEAU's supplement of longitude of moon's node. And when, from the groups of results, we infer the secular correction to the motion of the argument of latitude, we have found the sum of two corrections; first, the secular correction to DAMOISEAU's mean motion of the moon, such as will best reconcile DAMOISEAU's longitude of the moon, unaffected by new inequalities. with the observed longitude of the moon, during the period to which the reductions apply; and, secondly, the secular correction to DAMOISEAU's regression of the node.

13. Now for the difference between DAMOISEAU's longitude unaffected by new inequalities, and the observed longitude, we must refer to the second column of the table, Royal Astronomical Society's Memoirs, vol. xvii. p. 35; and, comparing the first four numbers with the last four, we find the correction to DAMOISEAU's motion of the moon during 38·1 years $=-0''\cdot 30$. But in page 54, the correction to DAMOISEAU's motion of argument of latitude during the same time is $-24''\cdot 26$. Hence the correction to DAMOISEAU's regression of the node during that time is $-24''\cdot 26+0''\cdot 30=-23''\cdot 96$, or the secular correction is $-62''\cdot 9$. Applying this to DAMOISEAU's secular regression in a Julian century (namely, 134° 9′ 57″·5), we find the corrected regression 134° 8′ 54″·6, which is probably the most accurate that can yet be deduced from meridional observations.

14. But, in correcting DAMOISEAU's tables for further use, we must not apply the quantity $-62''\cdot 9$ to his secular motion of argument of latitude. The moon's tabular longitude at any time, putting H for the value of HANSEN's inequalities at that time, will require (see page 36 of the same Memoir) the correction $+39''\cdot 3\times$ number of centuries $+$H; and as the tabular supplement of node requires the correction $-62''\cdot 9\times$ number of centuries, the tabular argument of latitude (which is the sum of moon's longitude and supplement of node) will require a correction equal to the sum of these two corrections, or $-23''\cdot 6\times$ number of centuries $+$H. For a distant eclipse, the quantity H may be safely neglected.

15. Thirdly, as to the correction to the motion of the moon's perigee. The same considerations, in every respect, which have been used for determining the correction to the motion of node and to the motion of argument of latitude, are to be used for determining the correction to the motion of perigee and to the motion of mean anomaly. Thus, in p. 40 of the Memoir above cited, the apparent correction of motion

of anomaly in 38·1 years is $-12''\cdot03$: by the second column of the table on p. 35, the apparent correction of mean motion in the same time (the tabular values being unaffected by HANSEN's inequalities) is $-0''\cdot30$; hence the true correction to the motion of perigee (estimated as a regression) in 38·1 years is $-12''\cdot03+0''\cdot30=-11''\cdot73$. And the true correction to the moon's motion of mean longitude in 38·1 years is $+14''\cdot99$. Therefore the true correction to the motion of the mean anomaly in 38·1 years is $+14''\cdot99-11''\cdot73=+3''\cdot26$; and the true secular correction is $+8''\cdot56$. This supposes that HANSEN's inequalities in longitude are not accompanied with sensible inequalities in the place of perigee.

16. The position of the moon at any time, as affecting the circumstances of an eclipse, will depend on the moon's mean longitude, the longitude of perigee, and the longitude of node. The values of these three elements for any late year are known with very great accuracy (their values for certain years are given in the Memoir repeatedly cited); and the annual motions of mean longitude and longitude of perigee for a Julian century at the present time are very accurately known; in that of the longitude of node there is a very minute uncertainty. But the secular motion of each of these elements changes from century to century; and terms are thus introduced into the expression for each of these elements depending on the square and higher powers of the time. LAPLACE was the first who computed (from theory) the coefficients of these terms; and his numbers were adopted, with insignificant alterations, in BÜRG's and BURCKHARDT's tables. DAMOISEAU, on repeating the investigation, obtained different values for the coefficients; in particular, he introduced in the coefficient which relates to the place of perigee a change of such magnitude as very greatly to modify the circumstances of any calculated distant eclipse. PLANA and HANSEN, by independent investigations conducted in different ways, have in general confirmed these alterations; the result, however, of HANSEN's last investigation differs somewhat from that of his former investigation, though by a very much smaller quantity than the difference of each from LAPLACE's values. I shall give here the coefficients of the square of the number of centuries obtained by these writers; the signs of those numbers which relate to perigee and node being applicable to progression of perigee and regression of node. The reader must remark that a change of 1″ in the coefficient for mean longitude, of 9″ in that for longitude of perigee, or of 11″ in that for longitude of node, produces, in the moon's place for a perigeal eclipse at the time of THALES, an effect of about 10′, and that this will alter the place of the eclipse-shadow at a given time not less than 10° on the earth's surface.

LAPLACE, Mécanique Céleste, vol. iii. pages 237, 273, 274.

Coefficient for mean longitude	$+10''\cdot18$
Coefficient for longitude of perigee	$-30\cdot55$
Coefficient for suppl. longitude of node	$-\ 7\cdot49$

BÜRG's and BURCKHARDT's Tables.

Coefficient for mean longitude	$+10''\cdot00$
Coefficient for longitude of perigee	$-29\cdot98$
Coefficient for suppl. longitude of node	$-\ 7\cdot35$

DAMOISEAU, Mém. Savans Etrangers, Third Series, tome i. p. 544, and Tables.

Coefficient for mean longitude	$+10''\cdot72$
Coefficient for longitude of perigee	$-39\cdot70$
Coefficient for suppl. longitude of node	$-\ 6\cdot56$

PLANA, Théorie de la Lune, vol. i. p. 724.

Coefficient for mean longitude	$+10''\cdot58$
Coefficient for longitude of perigee	$-40\cdot23$
Coefficient for suppl. longitude of node	$-\ 6\cdot79$

HANSEN, Astronomische Nachrichten, No. 443.

Coefficient for mean longitude	$+11''\cdot93$
Coefficient for longitude of perigee	$-39\cdot18$
Coefficient for suppl. longitude of node	$-\ 6\cdot49$

HANSEN, Astronomische Nachrichten, No. 597.

Coefficient for mean longitude	$+11''\cdot47$
Coefficient for longitude of perigee	$-36\cdot31$

(The coefficient for suppl. longitude of node is not computed).

17. I believe that I have now stated, without important omission, the progress of Lunar Theory, as bearing on distant eclipses, to the present time; and I shall now proceed with the special calculations of this paper.

SECTION II. *Methods of Computation adopted in this Paper.*

18. The tables used in these computations, for the sun's longitude and the obliquity of the ecliptic, are CARLINI's tables attached to the Effemeridi di Milano, 1833. The precepts of the tables have been strictly followed, with these exceptions; that from Table V. the number has been taken which corresponds to lunar syzygies, and the mean values of Table VI. and Table XIII. have been taken, and the sum of these numbers or $7''\cdot5$ has been used as a constant. The sun's semidiameter has been slightly altered for the change of excentricity of the earth's orbit. The sun's longitude is found for two adjacent hours, and is then changed into right ascension and north polar distance. The following error has been remarked in the printed tables: Table XXX., *for* $16'\ 44''\cdot10$ *read* $16'\ 34''\cdot10$.

19. The lunar tables employed here are the same (except in the epochs) as those used in the Reduction of the Greenwich Lunar Observations. With the exception

of some small inequalities and some small changes of coefficients, entirely insignificant in the computation of an ancient eclipse, they are the same as DAMOISEAU's tables.

20. The arguments which DAMOISEAU distinguishes by the letters u, x, t, z, y, are formed in the following manner. To DAMOISEAU's epochs, for the year in the nineteenth century which differs by a whole number of centuries from the year for which the calculation is to be made, are added the numbers in his Table II. for the whole number of centuries taken backwards, and the corrections for u and z given at the foot of that table, and the corrections proportional to the square of the time in Table III., and the motions for $12^h\ 9^m\ 21^s{\cdot}5$ (to reduce Paris civil time to Greenwich astronomical time), and the numbers for the month, day, and hour of Greenwich mean solar time. The numbers thus formed are called DAMOISEAU's Elements. It will be seen from this statement that I have adopted DAMOISEAU's coefficients of the terms depending on the square of the time. Then the following corrections are added. For u (the mean longitude), the secular motion is increased by $+1^{\backprime}\ 21^{\backprime\backprime}{\cdot}4$, reckoning the years from 1814; the same correction is applied to t (the moon's elongation from the sun). For x (the mean anomaly), the secular motion is increased by $+26^{\backprime\backprime}{\cdot}4$, reckoned from 1788. For y (the argument of latitude), the secular motion is increased by $-72^{\backprime\backprime}{\cdot}8$, reckoned from 1782. These are called, in the subsequent articles, the Greenwich Corrections. By the addition of these, values of u, x, t, z, y are formed which are called the Unvaried Greenwich Elements, and these are the fundamental arguments used for the calculation of the moon's places. The longitude and ecliptic polar distance thus found for a certain hour are altered by the horary motions found from the tables, and longitude and ecliptic polar distance are thus obtained for a second hour. These are converted into right ascension and north polar distance with the obliquity found in the solar calculations. The following error has been remarked in DAMOISEAU's tables: Table II., -2200 years, *for* $113^g{\cdot}6634$ *read* $113^g{\cdot}8634$.

21. With these right ascensions and declinations of the sun and moon, the circumstances of the eclipse have been computed by the use of WOOLHOUSE's methods in the Appendix to the Nautical Almanac, 1836.

22. The next step is, to examine the effect of a possible change of elements. And here it may be remarked, that when the track of an eclipse is not highly inclined to the parallel upon the earth (which is true with regard to the eclipses here under consideration), a small change in the moon's longitude produces little effect in the track of the eclipse. Partly for this reason, and partly because the place of node appears liable to the greatest uncertainty, I have recognized no error in the moon's place except as depending on a possible error in the argument of latitude. In order to take account of this, I have in each calculation increased the argument of latitude by $20^{\backprime}$ centesimal; and this changed element, taken in combination with the other elements unchanged, constitute the system which I call "Elements with Variation." With this new system of elements, the moon's place is computed and the track of the central part of the shadow is computed, exactly as with the "Unvaried Greenwich Elements." The breadth of the shadow, as laid down on an ordinary chart, is

assumed to be the same as with the "Unvaried Greenwich Elements." The further inferences, as to the multiple, positive or negative, of "variation" (always using that word in the technical and precise sense of "increase of argument of latitude by 20‵‵") which is best adapted to the circumstances of the problem, are deduced by graphical process, as will be seen hereafter.

23. The whole of the calculations have been made by Mr. H. BREEN, Assistant at the Royal Observatory. I have every reason to trust in their accuracy.

SECTION III. *Eclipse of* AGATHOCLES.

24. The account given by DIODORUS, lib. xx., and supported in all important particulars by that of JUSTIN, lib. xxii., is as follows. AGATHOCLES, being blockaded in Syracuse by the Carthaginian fleet, secretly formed the design of invading the Carthaginian territories, and placed men on board ships in the harbour, but was unable for several days to pass the enemy's fleet. At length a convoy of provision-ships appeared; the blockading ships left their station to attack the convoy; AGATHOCLES took the opportunity of leaving the harbour; the Carthaginians immediately left the convoy and followed him; he escaped with difficulty under cover of night; and "the next day there was such an eclipse of the sun that the day wholly put on the appearance of night, stars being seen everywhere." After he had sailed "six days and the same number of nights" he made the African shore, and again barely escaped a Carthaginian fleet (it does not appear whether it was the same as that which had blockaded Syracuse; it was probably a different fleet); and landed at a place called "The Quarries." He shortly took two cities, of which the second (White Tunis) was 2000 stadia from Carthage; it does not appear however whether the distance from Carthage was measured in the most direct line or in reference to the route afterwards pursued by AGATHOCLES; and there is no mention of the distance or direction of the city from the landing-place. It is stated by DIODORUS that the troops, before sailing, supposed that they were to make an attack either on Italy or on the Carthaginian part of Sicily; and by JUSTIN, that, while on the voyage, they supposed that they were going on a marauding expedition either to Italy or to Sardinia.

25. The eclipse was evidently total; and the principal task which remains for us, in order to render this eclipse available for the correction of the lunar tables, is, to investigate from these materials the probable place of AGATHOCLES when the shadow passed over him. The first thing is, to discover the position of his landing-place. Mr. BAILY supposed this to be in the Gulf of Khabes. By the assistance chiefly of Captain WILLIAM HENRY SMYTH, R.N., I am enabled to indicate, and (as I conceive) with perfect certainty, a very different locality. On the west of Cape Bon, at a place called Alhowareah, are quarries of immense extent, proceeding from the sea cliffs and worked into the solid rock, and lighted by holes from above. They are undoubtedly the quarries from which Utica and Carthage were built. "Alhowareah" appears to be a corruption of the Roman name "Aquilaria;" the place at which CURIO

landed B.C. 49. It would appear to have been a usual place of landing, at least in coming from Sicily. It is said to be a well-sheltered harbour (which indeed is implied by its use as a shipping port for such large quantities of stone); and the great height of Cape Bon renders it an admirable point to approach from the sea. There can be no doubt that Alhowareah is the place where AGATHOCLES landed. In some maps there is marked in this district a valley called "Wady Abiad," "the white valley;" it is possible that White Tunis may have been situate here.

26. The adoption of Alhowareah as the landing-place of AGATHOCLES leaves the question open whether AGATHOCLES sailed on the north side of Sicily or on the south side. I am entirely indebted to J. W. BOSANQUET, Esq. for the suggestion that AGATHOCLES may possibly have passed the straits of Messina and sailed on the north side of Sicily; and I am also indebted to that gentleman for the heads of the following reasons for supposing that AGATHOCLES really did take the northern course.

(A.) The distance from Syracuse to Alhowareah by the southern route is about 200 nautical miles; that by the northern route about 330 miles. Either of these distances is considerably less than we should expect a fleet to traverse in six days and nights (according to the usual rate of sailing of ancient ships); and, so far, the northern route, as being the longer, is the more probable of the two.

(B.) Selinus, Himera, and other towns in the extreme west of Sicily, had always been Carthaginian. Agrigentum had been maintained in the Carthaginian interest even before the battle of Himera. After that battle, all the Greek cities on the northern coast, and all north of Syracuse on the eastern coast, and even Camarina on the south, submitted to the Carthaginians. Gela alone was firm in the interest of Syracuse. The predominant party there was supported by a Syracusan garrison; and the town was so strongly fortified as to defy the attack of HAMILCAR. The expression "campi Geloi" of VIRGIL, contrasted as it is with "Acragas magnanimûm generator equorum," seems to imply a great breadth of corn-lands; and we know from DIODORUS that the harvest was just gathered in. It cannot, I think, be doubted that the provision-ships, whose approach drew off the attention of the blockading ships, were coming from Gela; in that case, they approached from the south; the blockading ships therefore started towards the south to attack them; and AGATHOCLES, as soon as he passed out from the mouth of the harbour, necessarily went towards the north.

(C.) The belief of the troops that they were on their way to Italy or Sardinia implies that they were on the northern route.

27. I have no doubt that AGATHOCLES did really take the northern course. But as the usual opinion is that he took the southern course, I think it proper to exhibit the results of calculations made on both suppositions. As we do not know the hour of day at which AGATHOCLES sailed out of harbour, and as we have no information on the comparative rate of sailing on the different days, we cannot judge very precisely on the place of AGATHOCLES at the time of the eclipse ($7\frac{1}{2}$ A.M. on the next morning). But it seems likely that the following may be near enough to the truth:

Possible northern position of AGATHOCLES at the time of the total eclipse . . . . } Latitude 37° 55′. Longitude E. 15° 30′.
Possible southern position . . . . Latitude 36° 35′. Longitude E. 15° 0′.

28. The calculations of the places of the sun and moon are made in the manner described above, for the times −309, August 14, 19ʰ and 20ʰ, Greenwich Mean Solar Time (or, in civil reckoning, B.C. 310, August 15, 7ʰ and 8ʰ A.M.). The following are the fundamental arguments of the moon's place for −309, August 14, 20ʰ, G.M.S.T.

	u.	x.	t.	z.	y.
	g	g	g	g	g
DAMOISEAU's Elements	149·64068	26·1319	395·6437	283·4513	1·6953
Greenwich Corrections	−·25737	−·0554	−·2574	....	+ ·1522
Greenwich Unvaried Elements	149·38331	26·0765	395·3863	283·4513	1·8475
Elements with Variation	149·38331	26·0765	395·3863	283·4513	2·0475

With the Greenwich Unvaried Elements, the following places are obtained:

	19ʰ.	20ʰ.
	° ′ ″	° ′ ″
Obliquity of Ecliptic	23 43 42·0	
Sun's Longitude	136 33 57·6	136 36 24·1
Sun's Right Ascension	139 4 59·5	139 7 24·7
Sun's North Declination	16 3 39·0	16 2 54·4
Sun's Semidiameter	15 56·7	
Moon's Longitude	136 1 19·8	136 38 52·6
Moon's North Latitude	16 17·3	19 45·6
Moon's Right Ascension	138 37 44·4	139 16 8·1
Moon's North Declination	16 29 3·3	16 20 58·4
Moon's Equatorial Horizontal Parallax	61 1·4	
Moon's Geocentric Semidiameter	16 40·1	
Sun's True Right Ascension, in time	....	9ʰ 16ᵐ 29ˢ·65
Sun's Mean Right Ascension, in time	....	9ʰ 14ᵐ 10ˢ·76

From these, the following numbers are deduced:—

Greenwich Mean Solar Time of conjunction in R.A. −309, Aug. 14ᵈ 19ʰ 45ᵐ 27ˢ.
R.A. of Sun and Moon at conjunction . . . . 139° 6′ 49″·5.
Sun's North Declination at conjunction . . . . 16° 3′ 5″·2.
Moon's North Declination at conjunction . . . . 16° 22′ 56″·1.
Greenwich Mean Solar Time of Middle of General Eclipse . . . . 19ʰ 52ᵐ 28ˢ.

Using $\frac{1}{300}$ for the earth's compression, the following coordinates of points on the earth's surface are obtained; which it must be remembered are deduced from Greenwich Unvaried Elements.

Points on the central path of the shadow.

Greenwich Mean Solar Time.	East Longitude.	North Latitude.
h m s	° ′	° ′
18 20 0	7 2	34 37
18 22 30	10 56	35 29
18 25 0	14 15	36 9
18 27 30	17 10	36 41
18 30 0	19 50	37 8

Points on the limits of the total shadow.

Northern Limit.		Southern Limit.	
East Longitude.	North Latitude.	East Longitude.	North Latitude.
° ′	° ′	° ′	° ′
5 53	35 14	8 5	34 0
9 57	36 9	11 49	34 49
13 24	36 51	15 2	35 27
16 25	37 24	17 52	35 58
19 9	37 52	20 27	36 23

29. From the Elements with Variation the following numbers are obtained:—

	19h.	20h.
	° ′ ″	° ′ ″
Moon's Longitude	136 1 17·2	136 38 50·0
Moon's North Latitude	17 13·8	20 42·2
Moon's Right Ascension	138 37 59·7	139 16 23·5
Moon's North Declination	16 29 58·3	16 21 53·2

Greenwich Mean Solar Time of conjunction in R.A. −309, Aug. 14d 19h 45m 2s.
R.A. of Sun and Moon at conjunction . . . . 139° 6′ 48″·5.
Sun's North Declination at conjunction . . . . 16° 3′ 5″·5.
Moon's North Declination at conjunction . . . . 16° 23′ 54″·3.
Greenwich Mean Solar Time of Middle of General Eclipse . . . . 19h 52m 24s.

Coordinates of Points on the central path of the shadow.

Greenwich Mean Solar Time.	East Longitude.	North Latitude.
h m s	° ′	° ′
18 20 0	5 50	35 21
18 22 30	9 55	36 16
18 25 0	13 22	36 58
18 27 30	16 23	37 32
18 30 0	19 7	38 0

The breadth of the shadow, as marked on an ordinary map, for the Elements with Variation, will not sensibly differ from that for Greenwich Unvaried Elements.

30. The results of these calculations are laid down on the first map accompanying this paper, Plate XII. The large dots show the two possible positions of Agathocles. The strong line shows the central path of shadow, and the faint dotted lines show the limits of the dark shadow, from Greenwich Unvaried Elements. The interrupted line shows the central path of shadow from Elements with Variation (the term "Variation" signifying an increase of the Mean Argument of Latitude by 0s·20). The other lines show the positions of the central path of shadow necessary for satisfying respectively the four following conditions:—1. that the Northern boundary of the shadow will touch the Southern position of Agathocles; 2. that the Northern boundary will touch the Northern position; 3. that the Southern boundary will touch

the Southern position; 4. that the Southern boundary will touch the Northern position. The shaded irregular ovals are intended merely to show the breadth of the dark shadow in the direction transverse to the path of shadow, without any relation to the extent of shadow at any one instant in the direction of the path.

31. From an inspection of this map, with some geographical measures, the following conclusions will be easily deduced:—

(*a.*) With Unvaried Elements, the dark shadow passes over the Southern position of AGATHOCLES, but not over the Northern position.

(*b.*) In order that the eclipse may be total at the Southern position, the change of Mean Argument of Latitude must be included between those corresponding to Condition 1 and Condition 3, or between

$$-0{\cdot}55 \times \text{Variation and} +1{\cdot}14 \times \text{Variation.}$$

(*c.*) In order that the eclipse may be total at the Northern position, the change must be included between those corresponding to Condition 2 and Condition 4, or between

$$+0{\cdot}68 \times \text{Variation and} +2{\cdot}38 \times \text{Variation.}$$

The latter is the system of limitations which, for reasons already explained, I am disposed to adopt.

32. Assuming the uncertainty in the circumstances of an eclipse to be due entirely to uncertainty in the place of the moon's node, these statements supply us with an easy method of correcting (within certain limits) the elements for any other eclipse, according as we adopt one or the other position of AGATHOCLES. It is only to be remarked that if, in making the application to another eclipse, we adopt in the calculations for that eclipse (as will be convenient) a Variation of $0^{g}{\cdot}20$, then the factors corresponding to these Conditions must be increased in the same proportion as the interval of time backwards from the present age (when the place of the node is well known) to the age of the eclipse in question. Thus in applying them to the eclipse of THALES we must increase the factors by about $\frac{1}{7}$th part.

SECTION IV. *Eclipse of* THALES, *as recorded by* HERODOTUS.

33. The account of this eclipse given in the first book of HERODOTUS is in substance as follows. "Upon the refusal of ALYATTES (king of the Lydians) to give up some Scythian fugitives to CYAXARES (king of the Medes), the Lydians and the Medes were at war for five years; during which the Medes often defeated the Lydians and the Lydians often defeated the Medes; they had also in this war a sort of night-battle; and while they were still carrying on the war with equal success, and met for battle in the sixth year, it happened that on the battle being joined the day suddenly became night. THALES the Milesian predicted to the Ionians that this change would happen, fixing beforehand this very year, in which the change did occur. The Lydians and the Medes, when they saw that it was night instead of day, ceased from fighting, and on both sides endeavoured more anxiously to obtain peace.

The persons who brought them together were SYENNESIS the Cilician and LABYNETUS the Babylonian."

34. It is to be remarked that this war was one of a different character from that which was subsequently undertaken by CRŒSUS and which ended in his ruin. The war of ALYATTES was a struggle between two nations, in which the Medes apparently made the first movement; and though it is impossible to say how far in so long a time the places of conflict may have been shifted, yet it is likely that they would always have reference to the great lines of military communication between the two warring countries. The war of CRŒSUS, on the other hand, was undertaken to obtain possession of the province of Cappadocia. It must be remembered that the limits of this province, in the geography of HERODOTUS, were very different from those assigned to it in later times, and which are generally traced in our maps. With him, the province (then an independent kingdom) of Cilicia included both banks of the upper part of the Halys (and therefore extended very much further north than in later times); then the Matieni occupied the right bank of the river, and the Phrygians the left; then, from their boundary to the mouth of the Halys, the Cappadocians (who he says were called Syrians by the Greeks) occupied the right bank of the Halys (thus including what was afterwards the kingdom of Pontus) and the Paphlagonians the left. Thus, in the Cappadocian enterprise of CRŒSUS (which, as the attack on Pteria shows, was principally directed against the inhabitants of the coast), it was necessary to pass the Halys near its mouth, and with the difficulties described by HERODOTUS; but in the Median war of ALYATTES there was not necessarily any movement so far north. The circumstance that the armies in this eclipse-battle were accompanied by the forces of their principal allies, and that the kings were present in person ready to make a treaty, shows that it was no skirmish of detachments, but a meeting of the main armies. It will be well therefore to consider in what part of the country such armies were likely to meet. I am indebted to M. PIERRE DE TCHIHATCHEFF and W. J. HAMILTON, Esq., for much of the information on which the following remarks are founded.

35. Asia Minor is bounded on its eastern side by a wide-spreading cluster of mountains, which, apparently, presents to the west an unbroken front, extending from the Euxine Sea to the Gulf of Issus; and on its southern side by a narrow range of mountains joining the former near Issus. The difficulties of passing the eastern mountains appear to be great. There is one road leading from Erzeroum by Sebaste or Sivas towards Cæsarea, and another road nearly parallel to this, thirty or forty miles S.E. of it; but both are rough and pass through very extensive tracts which provide little food. A rough road leads in the S.E. direction from Cæsarea by El Bostan. The best road appears to be that which leads from Sivas to Guroun, and then accompanies one of the feeders of the Euphrates by Melitene or Malatieh. In the southern mountains, the best pass towards the shore of the Mediterranean is that of Tarsus, leading thence by Issus to Antioch or Aleppo. Mesopotamia has only

once (I believe) been invaded from the Euxine Sea; namely, by an army directed by the Byzantine emperor HERACLIUS (A.D. 623), which landed at Trebizond and made its way through the mountains; returning however by the way of Issus. There is only one instance of an army marching along the north coast of Asia Minor, namely, that of the ten thousand Greeks in their return from the Anabasis; but this route was not adopted from choice; and the difficulties which they experienced show that it is not likely that a large army would willingly take that line. It would appear therefore that there are but two routes really practicable for armies; that of Melitene and that of Issus. At Melitene was fought the important battle, A.D. 572, between the Emperor TIBERIUS and CHOSROES NUSHIRVAN. One great battle, that between TIMUR and BAJAZET, A.D. 1402, was fought as far north as Ancyra; it was perhaps preceded by movements on the Melitene road. It is probable also that other marches have been made on the same line. But the far greater number of marches in both directions have been by the pass of Tarsus and the coast line to Issus. This was the route of the younger CYRUS; of ALEXANDER, although he marched from Ancyra; of VALERIAN and JULIAN; of SAPOR in marching from Armenia to the Cappadocian Cæsarea (for which the pass of Melitene would have appeared more direct); of the Crusaders in the first and second crusades; and of many other armies. When, in marching eastward, the valley of Antioch or the more open plains of the Euphrates are gained, it is difficult to define with the same strictness the probable march of a military force. The account of HERODOTUS however conveys the impression that the eclipse-battle took place in or very near to Asia Minor.

36. I conceive therefore that we are limited, as to the battle field, to the country within no great distance of the line from Sardes to Melitene; that it may have been anywhere south of that line, especially near Issus, but that it cannot have been far north of it; and that it cannot have been far east of Issus.

37. The approximate examination of the eclipses which could pass near this tract is very greatly facilitated by the tables in the *Art de vérifier les dates*, but still more by the calculations of Mr. BAILY and Mr. OLTMANNS. It is only necessary to observe that the correction of the moon's elements increases the argument of latitude (by which the track of the shadow at every eclipse in ascending node will be thrown three or four degrees northward, and that at every eclipse in descending node will be thrown as much southward); and that it increases the secular equation of anomaly, and thus increases the moon's longitude at every perigeal eclipse (by which the track of the shadow in every eclipse will be thrown several degrees to the east). Thus I have examined every total eclipse in Mr. OLTMANNS' table, extending from B.C. 631 to B.C. 585; and find only one (namely, that of B.C. 585, May 28*) which can have passed near to Asia Minor; that of B.C. 610, Sept. 30, which was adopted by Messrs.

* The first publication of results relating to the eclipse of B.C. 585, derived from careful calculations on good elements, so far as I know, was that by J. R. HIND, Esq., in the Athenæum for 1852, August 28, during the preparation of the present memoir.

BAILY and OLTMANNS, is now thrown north even of the sea of Azov. I have likewise formed the first approximate elements of the eclipses from B.C. 630 to B.C. 576, by the use of M. LARGETEAU's very convenient tables inserted in the Additions to the Connaissance des Temps, 1846, and am led to the same conclusion.

38. I shall now proceed with the computation of the eclipse of B.C. 585, May 28. The elements are formed precisely in the same manner as those for the eclipse of AGATHOCLES. The following are the primary numbers for the moon, for −584 (or B.C. 585), May 28, 2h, Greenwich Mean Solar Time.

	u.	x.	t.	z.	y.
	g	g	g	g	g
DAMOISEAU's Elements	67·37639	388·0970	1·0941	200·9173	4·6101
Greenwich Corrections	—·29112	—·0626	—·2911	....	+·1723
Greenwich Unvaried Elements	67·08527	388·0344	0·8030	200·9173	4·7824
Elements with Variation	67·08527	388·0344	0·8030	200·9173	4·9824

With the Greenwich Unvaried Elements, the following places are obtained:—

	2h.	3h.
Obliquity of Ecliptic	23° 45′ 55″·6	
Sun's Longitude	59 34 37·0	59° 37′ 0″·0
Sun's Right Ascension	57 18 49·9	57 21 18·7
Sun's North Declination	20 20 5·3	20 20 36·4
Sun's Semidiameter	15 44·4	
Moon's Longitude	59 23 41·1	60 1 34·9
Moon's North Latitude	17 4·5	20 34·8
Moon's Right Ascension	57 3 27·9	57 42 11·2
Moon's North Declination	20 34 21·8	20 46 1·1
Moon's Equatorial Horizontal Parallax	61 18·8	
Moon's Geocentric Semidiameter	16 44·8	
Sun's True Right Ascension, in time	3h 49m 15s·32	
Sun's Mean Right Ascension, in time	3h 58m 23s·04	

From these, the following numbers are deduced:—

Greenwich Mean Solar Time of conjunction in R.A. −584, May 28d 2h 25m 27s.
R.A. of Sun and Moon at conjunction . . . . 57° 19′ 53″·0.
Sun's North Declination at conjunction . . . . 20° 20′ 18″·5.
Moon's North Declination at conjunction . . . . 20° 39′ 18″·3.
Greenwich Mean Solar Time of Middle of General Eclipse . . . . . 2h 15m 29s.

From which the following coordinates of points are obtained:—

Points on the central path of the shadow.

Greenwich Mean Solar Time.	East Longitude.	North Latitude.	Sun's Zenith Distance.
h m s	° ′	° ′	° ′
3 48 0	23 1	39 46	71 27
3 49 0	24 56	39 14	73 11
3 50 0	27 3	38 37	75 8
3 51 0	The formulæ fail.		77 22
3 52 0	32 26	36 54	80 5
3 53 0	36 34	35 29	83 54

Points on the limits of the total shadow.

Northern Limit.		Southern Limit.	
East Longitude.	North Latitude.	East Longitude.	North Latitude.
° ′	° ′	° ′	° ′
24 35	40 29	21 37	39 2
26 38	39 53	23 25	38 32
28 56	39 12	25 24	37 58
31 39	38 20	27 37	37 18
35 9	37 9	30 14	36 29
42 14	34 36	33 33	35 22

39. From the Elements with Variation, the following numbers are obtained:—

	2^h.	3^h.
	° ′ ″	° ′ ″
Moon's Longitude..................	59 23 38·6	60 1 32·3
Moon's North Latitude............	18 1·1	21 31·4
Moon's Right Ascension	57 3 12·0	57 41 55·5
Moon's North Declination	20 35 16·4	20 46 55·7

Greenwich Mean Solar Time of conjunction in R.A. −584, May 28^d, 2^h 25^m 53^s.
R.A. of Sun and Moon at conjunction . . . . $57°$ $19'$ $54''\cdot1$.
Sun's North Declination at conjunction . . . . $20°$ $20'$ $18''\cdot7$.
Moon's North Declination at conjunction . . . . $20°$ $40'$ $18''\cdot1$.
Greenwich Mean Solar Time of Middle of General Eclipse . . . . 2^h 15^m 24^s.

Coordinates of Points on the central path of the shadow.

Greenwich Mean Solar Time.	East Longitude.	North Latitude.
h m s	° ′	° ′
3 48 0	24 36	40 22
3 49 0	26 39	39 46
3 50 0	28 57	39 5
3 51 0	31 41	38 12
3 52 0	35 13	37 1

It will be remembered that the term "Variation" here has the technical sense of "increase of the mean argument of latitude by 20′." And it must be remarked that, to find the positions of the path of shadow which correspond respectively to Conditions 1, 2, 3, 4 in the eclipse of Agathocles, the effects of "Variation" must be multiplied by −0·63, +0·78, +1·31, +2·72.

40. In the second map which accompanies this paper (Plate XIII.), I have shown by a strong line the central path of the shadow corresponding to Unvaried Elements, and by an interrupted line the path corresponding to Elements with Variation of 20′ in the Argument of Latitude; and also, by finer continuous lines, the paths corresponding to Conditions 1, 2, 3, 4 in the eclipse of Agathocles; together with those that correspond to Central eclipse on the Southern position of Agathocles and

Central eclipse on the Northern position of AGATHOCLES. The oval whose centre is on the line of Unvaried Elements is intended only to show the breadth of the shadow, which will be sensibly the same for each of the positions of central path. On examining these, the following remarks will at once suggest themselves.

41. If the centre of the shadow followed the line of Condition 1, the shadow would obscure no open country except a very small distance north-east of the Pisidian Mountains; and even there the obscurity would be short. The western towns however from Halicarnassus to Pergamum would be shaded.

If the centre followed the line corresponding to a central eclipse for the Southern position of AGATHOCLES, the shadow would be almost central at what was afterwards Antioch, Celænæ and Pergamum; and there would be a great total eclipse at Issus and Tarsus, and on all the western towns from Ephesus to Troy; and the southern plain of Iconium would be in the shade. But Cæsarea, Melitene, Ancyra, would be in light.

If the centre followed the line of Condition 3, Issus, Tarsus, and a great length of the southern road, would be covered; but the shadow would not extend to Cæsarea, Ancyra, or Melitene. Sardes would now be out of the shadow.

If the centre followed the line corresponding to central eclipse for the Northern position of AGATHOCLES, Issus, Tarsus, Cæsarea, Iconium, would be within the shadow; Melitene, Pergamum, Ancyra, would be in light. Generally, the centre of the peninsula would be in shade.

If the central point followed the line of Condition 4, Melitene would be shaded; Cæsarea and Ancyra would be nearly in the middle of the shadow; and the northern plains would be covered; but the plain of Iconium and the whole western coast south of Lampsacus would be free from shade.

42. Any one of these tracks of the shadow (perhaps excepting that of Condition 1) is compatible with a conceivable place of engagement. In balancing the probabilities, we must in some measure be guided by the extent of ground proper for large military operations which the total shadow would cover. Judging thus, I should fix on a course between that of Condition 3 and that corresponding to central eclipse at the North position of AGATHOCLES as most probable. This selection, it will be remarked, excludes the possibility of AGATHOCLES being at the South position; and therefore, if adopted, would decide absolutely that AGATHOCLES sailed on the north side of Sicily.

43. If in conformity with this selection we suppose that the Unvaried Argument of Latitude ought to be increased by $1{\cdot}53 \times$ variation of 20ʹ, or by 30ʹ 60ʺ, and if we remark that the interval of time from 1782, when DAMOISEAU's epoch is nearly correct, is $-23{\cdot}66$ centuries, it will appear that the secular motion of argument of latitude ought to be diminished by 1ʹ 29ʺ or 42ʺ nearly. Assuming our motion of mean longitude to be correct, the same correction ought to be applied to the regression of the node. This makes the secular regression of the node for a Julian century equal to 134° 8ʹ 13ʺ, which is rather less than the smallest of those found by other inves-

tigators. If we had supposed the shadow to follow the line of Condition 3, the number would have agreed very closely with those smallest numbers.

44. In terminating this section, I may remark on the causes of uncertainty which yet remain in the theoretical calculations. The mean motion of the moon is determined by observations which are more completely free from constant error than almost any other observations applicable to one element; and I have no doubt of its extreme correctness. The great theorists of the present age, however, do not agree very closely in the value which they ascribe to the coefficient of the term in the moon's longitude depending on the square of the time. The present motion of the moon's perigee is determined with a certainty only inferior to that of mean motion. But Professor HANSEN in his last published investigations has proposed to alter the coefficient of the squares of centuries in the place of perigee by 3″, which will affect the moon's longitude in these eclipses by more than 3′, and will produce an effect opposite to that of regression of node in the eclipse of AGATHOCLES, but combining with it in the eclipse of THALES. The determination of the movement of the node from observation is liable to uncertainty only from the negligence of the Greenwich observers in the last century, who did not carefully observe the zenith distance of the moon's limb at the precise instant when it passed the meridian; and the effect of this error may be considerable. The theoretical term depending on the square of the time appears (if we may so infer from the consent of the investigators) to be well determined.

45. I conclude therefore that the terms to which at present it is most desirable that the attention of theorists should be directed, are those in the mean longitude and in the longitude of perigee depending on the square of the time. A careful discussion of eclipses will then supply, what meridional observations at present are hardly able to supply with the requisite accuracy, the motion of the node.

SECTION V. *Eclipse recorded by the Persian Historians.*

46. In Sir JOHN MALCOLM's History of Persia, Chapter VII., is a comparison of the Persian history, as recorded in Persian poetry (founded undoubtedly on authentic history, though with many changes and very great omissions), with that recorded by Greek writers. It appears that the KAI KAOOS of the Persians is the same as the ASTYAGES of the Greeks, or that the events of his reign are those of both ASTYAGES and CYAXARES; and Sir JOHN MALCOLM adds, "the most remarkable agreement is in the expedition of KAI KAOOS to Mazanderam. We are told by the Persian poet that in a battle which was fought in that province, the prince and his army were struck with a sudden blindness, which had been foretold by a magician."

47. In the range of years through which my examination has extended, there was no total eclipse in Mazanderam, and only two which could be visible in the eastern dominions of Persia. One was the eclipse of B.C. 610, Sept. 30; of which the central path, as computed by J. R. HIND, Esq., from elements not very unlike mine (which,

as well as every part of the results, Mr. HIND has most obligingly communicated to me), passed over Kœnigsberg, Astrakhan, and Khiva. The other is that of B.C. 603, May 17, which crossed the Persian Gulf in a north-easterly direction.

48. I imagine therefore that if this be a record of a total eclipse (which I see no sufficient reason for doubting), it relates to the same eclipse as that recorded by HERODOTUS. It appears, from Sir JOHN MALCOLM's remarks on the Persian historical traditions in general, that the names of provinces are in many instances given erroneously.

SECTION VI. *Eclipse of* XERXES.

49. In the spring of the same year in which the battle of Salamis was fought (to which the date B.C. 480 is usually assigned), there occurred a phenomenon which is thus described by HERODOTUS, Book vii. "With spring, the army [of the Persians], being ready, set out from Sardes on its march to Abydos; and as it was setting out, the sun leaving his seat in heaven was invisible, when there were no clouds but a perfectly clear sky; and instead of day it became night. XERXES, who saw this and heard about it, felt some anxiety, and inquired of the Magi what the appearance portended; they replied that the deity prognosticated to the Greeks the desertion of their cities; saying that the sun was the prognosticator for the Greeks, the moon for the Persians. When XERXES heard this he was very joyful, and proceeded on his march."

50. This account, interpreted as a record of a total eclipse of the sun, has given great trouble to chronologers, and not without reason. The only solar eclipse which it is worth while even to examine is that in the morning of the 19th of April, B.C. 481. The numbers computed from Greenwich Unvaried Elements are as follows:—

For —480, April 18, 16^h, Greenwich Mean Solar Time.

	u.	*x.*	*t.*	*z.*	*y.*
	g	g	g	g	g
DAMOISEAU's Elements	21·90139	44·9528	397·9207	156·6612	191·9616
Greenwich Corrections	—·27849	—·0599	—·2785	....	+·1648
Greenwich Unvaried Elements......	21·62290	44·8929	397·6422	156·6612	192·1264

	16^h.	17^h.
Obliquity of Ecliptic..............................	23° 44′ 56″·7	
Sun's Longitude	22 47 45·1	22° 50′ 9″·1
Sun's Right Ascension	21 2 27·8	21 4 42·9
Sun's North Declination	8 58 37·3	8 59 31·4
Sun's Semidiameter	15 48·2	
Moon's Longitude..................................	22 37 22·9	23 14 5·5
Moon's North Latitude............................	19 34·8	16 12·0
Moon's Right Ascension	20 45 16·4	21 21 1·9
Moon's North Declination	9 12 51·7	9 23 31·5
Moon's Equatorial Horizontal Parallax	60 19·3	
Moon's Geocentric Semidiameter..................	16 28·6	
Sun's True Right Ascension, in time	1^h 24^m 9^s·85	
Sun's Mean Right Ascension, in time...........	1^h 26^m 7^s·57	

Greenwich Mean Solar Time of conjunction in R.A.	−480, April 18^d, 16^h 30^m 47^s.
R.A. of Sun and Moon at conjunction	21° 3′ 37″·4.
Sun's North Declination at conjunction	8° 59′ 5″·0.
Moon's North Declination at conjunction	9° 18′ 20″·0.
Greenwich Mean Solar Time of Middle of General Eclipse	16^h 21^m 18^s.

And from these,

Beginning of Central Eclipse on the Earth	14^h 41^m 47^s
in Longitude 49° 17′ East, Latitude 1° 25′ North.	
Central Eclipse at Noon	16^h 30^m 47^s
in Longitude 111° 49′ East, Latitude 27° 49′ North.	
End of Central Eclipse on the Earth	18^h 0^m 51^s
in Longitude 173° 13′ East, Latitude 34° 2′ North.	

51. If a diagram is constructed to exhibit the path of the shadow in this eclipse over the earth, and if it is remarked that the longitude of Sardes is about 28° East, it will be found that there could not be even a partial eclipse for Sardes, the whole penumbra having entered completely upon the earth before sunrise at Sardes. Nor, if the calculations above are correct (as I have great reason to believe), does it appear possible by alteration of secular movements to make an eclipse visible at Sardes. For if the moon's longitude were diminished, to make this eclipse possible, it must also be diminished in B.C. 585, and that would make the eclipse of THALES impossible, as the moon would not then have entered upon the sun's disk before sunset.

52. Abandoning then the idea of explaining this account by a solar eclipse, I have examined into the possibility of referring it to some other phenomenon. First, I cannot doubt that there was something unusual and alarming, as the solemn consultation of the Magi by XERXES seems to have been a matter of notoriety. Secondly, HERODOTUS repeatedly expresses himself doubtful on matters of detail which occurred during the movements of XERXES on the eastern side of the Ægean sea. Thirdly, the notion that the Sun was the peculiar divinity of the Greeks and the Moon that of the Persians, is entirely opposed to all that we know of the religious ideas of the Persians generally, or of XERXES in particular. For instance, when XERXES was preparing to cross the Hellespont, he waited for the rising of the Sun, and then addressed to the Sun his prayers for success. The Greeks however appear to have attached great importance to the appearance of the Moon, as is evident from their terror, and its calamitous consequences, at the lunar eclipse in the Syracusan war (THUCYDIDES, book vii.). The reply of the Magi therefore, which (as given by HERODOTUS) is, on the face of it, absurd, would seem to be much more plausible if we suppose that the information received by HERODOTUS was wrong in one particular, and that the observation in question was an eclipse of the moon, instead of the sun.

53. Now there was an eclipse of the moon on the morning of the 14th of March, B.C. 479, which answers well to the conditions of the history. The elements of computation are, for −478, March 13, 15^h, Greenwich Mean Solar Time.

	u.	x.	t.	z.	y.
	g	g	g	g	g
DAMOISEAU's Elements	181·75704	118·9084	197·7783	116·6227	392·6488
Greenwich Corrections	—·27849	—·0599	—·2785	....	+·1648
Greenwich Elements Unvaried......	181·47855	118·8485	197·4998	116·6227	392·8136

	14h.	15h.
Sun's Longitude	347° 26′ 10″·7	347° 28′ 37″·2
Moon's Longitude..................	167 23 0·0	167 55 2·2
Moon's South Latitude	12 10·4	9 12·8

The opposition in longitude occurred therefore at 14^h 6^m 28^s Greenwich Mean Solar Time, or about 15^h 59^m Sardes Mean Solar Time; and the Moon's South Latitude was 11′ 51″·1; which would be reduced to about 10′ 35″ by the corrections at which I arrived in Section IV. It was therefore a total eclipse, nearly central (the moon's limb being at least 16′ within the inner boundary of the penumbra), and it is probable that the moon disappeared completely, and was lost for nearly two hours.

54. I think it extremely probable that this really was the eclipse to which the account of HERODOTUS refers. But for its adoption it is necessary to bring down the date of the battle of Salamis one year later than in the chronology generally received.

G. B. AIRY.

Royal Observatory, Greenwich,
1852, *December* 10.

VIII. *On the Dissolution of Urinary Calculi in dilute Saline Fluids, at the Temperature of the Body, by the aid of Electricity.*

By Henry Bence Jones, *M.D., F.R.S., Physician to St. George's Hospital.*

Received October 12,—Read December 16, 1852.

In the year 1842 I published a statement of some of the attempts which had been made to dissolve urinary calculi. I did not then know that MM. Prevost and Dumas had made any researches on this subject. Some time afterwards I was told that they had used electricity, but I could not obtain the reference to their original paper; as no practical use was made of their experiments, I did not search for the record of them until after my own experiments were completed.

In 1845 I first devised the investigation which from circumstances I did not commence until 1848, when I made some experiments on the solution of uric acid calculi.

It appeared to me not unlikely that when a solution of nitrate of potassa was made to divide into potassa and nitric acid by means of electricity, if a piece of uric acid were held between the electrodes of the galvanic battery, then the potassa at the negative electrode and the nitric acid at the positive electrode would both act on the calculus.

Thus I expected that the fluid about the negative electrode would dissolve the uric acid, whilst the fluid about the positive electrode would decompose the uric acid. I considered that no great excess of acid or alkali would result, because after passing round the calculus I expected they would recombine, re-forming a neutral solution.

The battery used consisted of six pairs of Grove's plates. The action was continued about four hours. There was an effect produced on the uric acid at the negative pole, but no very decided result was obtained.

A second experiment gave so little encouragement that for the time the experiments were discontinued.

In the year 1851 I proposed again trying, under the same circumstances, the effect of nitre on uric acid, of sulphate of potash on oxalate of lime, and of lactate of soda on phosphatic calculi. I was unable to make any experiments until July 1852, when, through the kindness of Dr. Faraday, I was allowed to have the use of the batteries at the Royal Institution, and I had the valuable assistance of Mr. Anderson, by which the perfect action of these batteries was ensured.

Section I. *On the Solution of Uric Acid Calculi.*

After a few preliminary experiments, on the 3rd of July the first experiment was made; a piece of very compact uric acid calculus was put into a saturated solution of nitrate of potassa, between the electrodes of a battery consisting of ten pairs of Grove's plates. The fluid soon boiled; the action was continued between three and four hours, when the calculus was removed. It was found to be so deeply corroded and worn on both sides, that apparently nearly half the piece of uric acid calculus was gone (see Plate XIV. fig. (*a*) and (*b*). The solution of nitre was boiling the whole time. On the removal of the stone a few drops of the liquid were evaporated, and the evidence of uric acid was most distinct. The fluid was evaporated to dryness, treated with a drop or two of nitric acid, again evaporated to dryness, and exposed to the fumes of ammonia. The characteristic reaction of uric acid was obtained.

Hence, by this qualitative experiment, it was certain that at a boiling temperature uric acid between the electrodes could be dissolved in a saturated solution of nitre.

The question immediately arose, what effect would be produced when the solution of nitre was diluted, and when the temperature was about that of the human body.

Experiment 2.—The number of plates in the battery was reduced to one-half in order that the temperature might rise less rapidly. Only five pairs of plates were used. Ten cubic inches of saturated solution of nitre were mixed with thirty cubic inches of distilled water; the specific gravity of the mixture was 1029. The temperature was kept low by changing the solution in which the calculus and the electrodes were placed. Each glass contained nearly a pint of the solution of nitre. The piece of calculus weighed 48 grains, and it consisted of very compact uric acid.

The action began at 9^h 45^m A.M. and was continued to 1 P.M., the temperature of the solution being between 65° and 108°.

The action recommenced at 2^h 10^m P.M. and was continued to 5 P.M., the temperature being between 65° and 112°.

Total time 6^h 5^m.

The calculus was then removed and perfectly dried in a water-bath. It was not washed with water, so that some nitre probably remained on the calculus; when perfectly dry the uric acid weighed 37 grains. Hence, in six hours at least, 11 grains of uric acid were dissolved. The solution appeared to take place chiefly at the negative pole. The nitre solution became very slightly alkaline.

Experiment 3.—I was then anxious to know whether a larger piece of calculus and increased power of the battery would give me a satisfactory result; the piece of calculus, weighing 113 grains, consisted chiefly of uric acid, but the nucleus was oxalate of lime. The battery consisted of ten pairs of plates, and the electrodes and solution were the same as in the last experiment.

The action began at 10^h 10^m A.M. and was continued to 1 P.M., the temperature of the solution being between 67° and 104°.

The action was recommenced at 2^h 10^m P.M. and was continued to 5^h 30^m P.M., the temperature being between 70° and 108°.

Total time 6^h 10^m.

The calculus was then removed, dried in a water-bath, after which it weighed 99 grs., so that in six hours about 14 grains of uric acid were dissolved. The external surface of the calculus which had been in contact with the negative electrode was corroded, whilst the internal surface, which consisted of oxalate of lime and which had been in contact with the positive electrode, was apparently very slightly acted on. The solution of nitre was much discoloured, and on evaporating, even a few drops, uric acid was immediately detectable.

Experiment 4.—A whole uric acid calculus, weighing 111 grains, was put in a solution of nitre of specific gravity 1024·5. Ten pairs of plates were employed.

The action began at 9^h 35^m A.M. and was continued to 1^h 5^m P.M., the temperature of the solution being between 67° and 100°.

The action recommenced at 2^h 10^m P.M. and was continued until 5 P.M., the temperature being from 70° to 102°.

Total time 6^h 20^m.

The calculus when dried weighed 95 grains, so that in little more than six hours 16 grains of uric acid were dissolved. The solution took place almost entirely at the negative or alkaline electrode. The solution of nitre became slightly alkaline.

I was now desirous of trying whether the form of the electrodes at that part which was in contact with the calculus made any perceptible difference. Instead of a piece of thick platinum wire flattened, of the exact size and form of fig. 2, the wire was not flattened but only bent back. The breadth and length of the electrodes was nearly the same as in the previous experiments, but the surface in the former case was continuous, in the latter interrupted by a small space between the bent portions of the wire.

Experiment 5.—A whole calculus, consisting chiefly of uric acid, with a thin external coating of oxalate of lime with a little phosphate of lime, was employed. It weighed 57 grains. At first the lower electrode was made positive, and the upper electrode negative. The power of the battery and the strength of the solution were the same as in the last experiment.

The action began at 9^h 30^m A.M. and was continued to 1 P.M., the temperature of the solution being between 64° and 110°.

The action recommenced at 2 P.M., the electrodes were then reversed, the negative being below, and the action was continued to 5 P.M.; the temperature was from 70° to 120°.

Total time 6^h 45^m.

When dry the stone weighed 45 grains, so that the loss was 12 grains. Very little action was apparent on the calculus until the alkali was evolved from below. At five o'clock a voltameter interposed in the circuit showed that 2 cubic inches of mixed gases were produced in three minutes, but at this time the action of the battery was going down.

No deduction could be made as to the influence of the different forms of the electrodes.

Being anxious to know whether increased power of the battery and a better arrangement for keeping the temperature about that of the human body would be attended with advantage, another uric acid calculus was taken. It was very flat, and had been divided. The half used weighed $107\frac{1}{2}$ grains. The electrodes were those used in the first experiments. The portions not in contact with the calculus were covered with glass. The negative electrode was on the external surface of the stone, and was below. The solution consisted of the same strength of nitre as before, but to three pints of the solution eight ounces of urine were added; this being taken as the largest quantity likely to be secreted during the time that the experiment was likely to be continued in the human body. The battery consisted of twenty pairs of GROVE's plates. By means of a siphon leading from the vessel in which the calculus was placed, and by means of a reservoir with a stopcock, a continuous current of the solution was allowed to flow around the stone; thus the temperature could be most easily regulated.

Experiment 6.—The action began at 9^h 28^m A.M., the temperature of the solution then being 64°.

At 9^h 55^m A.M., the voltameter showed that 1 cubic inch of mixed gases was evolved in thirty-five seconds.

The action was stopped at 12^h 45^m.

Total time 3^h 17^m.

The calculus when dry weighed 80 grains, so that the loss was $27\frac{1}{2}$ grains in three hours and a quarter. There was a deep red stain on the calculus when it was removed from the solution; this increased on drying, but it was more marked at the negative than at the positive electrode. The calculus was partially cut in halves. (See fig. 3.) On the negative electrode there was a deposit, which, when scraped off and examined, proved to be earthy phosphate, probably a deposit from the urine. The calculus itself on the surface, near the negative electrode, was made white from a slight deposit of earthy phosphate.

The current of the solution of nitre was continuous throughout, and the temperature was kept between 90° and 106°; when the action was stopped, the fluid was still slightly acid from the urine which was originally added.

No other experiments were made with uric acid calculi, so that no deductions have to be made for unsuccessful experiments.

The results may be thus arranged:—

Exp. 1 lasted 4^h in saturated solution of nitre, temp. 212° with 10 pairs. Half dissolved.
Exp. 2 lasted 6^h 5^m, $\frac{1}{4}$ nitre, $\frac{3}{4}$ water, average 109° with 5 pairs, 11 grs. dissolved.
Exp. 3 lasted 6^h 10^m, $\frac{1}{4}$ nitre, $\frac{3}{4}$ water, average 101° with 10 pairs, 14 grs. dissolved.
Exp. 4 lasted 6^h 20^m, $\frac{1}{4}$ nitre, $\frac{3}{4}$ water, average 100° with 10 pairs, 16 grs. dissolved.
Exp. 5 lasted 6^h 45^m, $\frac{1}{4}$ nitre, $\frac{3}{4}$ water, average 106° with 10 pairs, 12 grs. dissolved.
Exp. 6 lasted 3^h 17^m, $\frac{1}{4}$ nitre, $\frac{3}{4}$ water, average 98° with 20 pairs, $27\frac{1}{2}$ grs. dissolved.

So that uric acid calculi can be dissolved in a moderately dilute solution of nitrate of potassa at the temperature of the human body, by the aid of electricity, at the rate of from two to nine grains an hour.

SECTION II. *On the Solution of Oxalate of Lime Calculi.*

My original conjecture was, that sulphate of soda would prove the best solvent of oxalate of lime. Having however obtained so good an action on uric acid by a solution of nitre, I determined first to see what the effect of the same solution would be on oxalate of lime.

Experiment 1.—Half a small oxalate of lime calculus, weighing 42·5 grains, was taken. The battery consisted of only five pair of GROVE's plates.

The solution of nitre was of the specific gravity 1028.

The action began at 9^h 25^m A.M., the temperature of the solution then was 65°.

The action was stopped at 4^h 35^m P.M., the temperature never rose above 90°.

Total time 7^h 10^m.

The calculus was then removed and perfectly dried; it then weighed 42 grains, so that in upwards of seven hours only half a grain was dissolved.

Experiment 2.—The same calculus, weighing 42 grains, was again placed in the same solution. The strength of the battery was doubled, being ten pairs.

The action began at 9^h 35^m A.M. and was continued to 1^h 15^m P.M., the temperature of the solution being between 70° and 115°.

The action recommenced at 2^h 30^m P.M. and continued to 5^h 50^m P.M., the temperature being between 75° and 104°.

Total time 7^h.

For the last two hours there was a great diminution in the power of the battery. The calculus when dry weighed 40 grains, so that in seven hours only 2 grains had been dissolved. It appeared that the divided surface was the one attacked; to this the positive electrode had been applied.

Experiment 3.—As the nitre solution had so little action, I proceeded to try the effect of a solution of sulphate of soda.

A solution of sulphate of soda was made, having the specific gravity 1034. The battery was of the same strength as in the previous experiment, and the same calculus was used, so that the comparative action of the solution of nitre and the solution of sulphate of soda might be determined. The weight of the calculus then was 40 grains.

The action commenced at 9^h 10^m A.M. and was continued to 1 P.M., the temperature of the solution being between 64° and 104°.

The action recommenced at 2^h 5^m P.M. and was continued to 4^h 30^m P.M., the temperature being between 70° and 102°.

Total time 6^h 15^m.

The battery was in good condition throughout the day. When the calculus was dry it weighed 38 grains, so that in six hours and a quarter only 2 grains had been dissolved.

Comparing this with the previous experiment, there is no advantage in using the sulphate of soda instead of the nitrate of potash.

I proceeded to try whether a solution of chloride of sodium would be more efficacious than the sulphate of soda or nitrate of potash.

Experiment 4.—The calculus was taken which had already been used in the previous experiments, in order that the comparative result might be as conclusive as possible. The chloride of sodium was dissolved in undistilled water, and the specific gravity of the solution was 1053·5. The battery was of the same strength as before.

The action began at 9^h 30^m A.M. and was continued to 1 P.M., the temperature of the solution being between 68° and 108°.

The action recommenced at 2^h 15^m P.M. and was continued to 4^h 30^m P.M., the temperature being between 70° and 100°.

Total time 5^h 45^m.

During the whole action much chlorine gas was evolved. The stone was perfectly bleached in places. When dry it weighed 37 grains. In five hours and three quarters only 1 grain was dissolved; thus it was evident that a solution of common salt was inferior to a solution of Glauber-salt or of nitre. In consequence of these experiments, it appeared most desirable to try whether the solution of nitre could be made still more active, either by increasing the strength of the battery, by adding other substances to the solution of nitre, or by increasing the strength of the solution of nitre. For determining these questions the following experiments were made.

Experiment 5.—The battery was increased to double the number of plates; that is, to twenty pairs. The calculus was the same as before, and the solution of nitre had the specific gravity 1026.

The action began at 9^h 35 A.M. and was continued to 1 P.M., the temperature of the solution being between 67° and 130°.

The action recommenced at 2 P.M. and was continued to 4^h 45^m P.M., the temperature being between 70° and 100°.

Total time 6^h 10^m.

The action of the battery diminished at one o'clock, so that the full action was only 3^h 25^m.

The calculus when dry weighed 32 grains, so that the loss was 6 grains in 6^h 10^m. The solution was chiefly on the cut surface of the calculus, which was in contact with the positive electrode.

Thus it appeared that by increasing the intensity of the battery the solution of nitre acted more energetically. The calculus however was a very small one; it originally weighed 42½ grains only. One quarter of this however was dissolved.

Experiment 6.—The addition of other substances to the nitre solution was then tried; and first, phosphate of soda was employed. To each pint of the nitre solution two ounces of a saturated solution of phosphate of soda were added. A divided oxalate of lime calculus was taken which weighed 173 grains. The battery consisted of twenty pairs of plates.

The action commenced at 9^h 30^m A.M. and was continued to 12^h 49^m A.M., the temperature of the solution being between 66° and 110°.

The total time was 3^h 19^m.

The calculus when dry weighed 172 grains, so that in 3^h 19^m the loss was only 1 grain.

The calculus was exceedingly compact. There was no action whatever apparent at the negative or alkaline electrode which was on the upper and cut surface; the solution was of course alkaline throughout the whole time. The battery was in excellent action.

Experiment 7.—Two ounces of a strong solution of bichromate of potash were then added to each pint of the nitre solution. The same calculus was taken; it weighed 172 grains. Twenty pair of plates were again used.

The action began at 9^h 30^m A.M. and was continued to 12^h 45^m A.M., the temperature being between 65° and 120°.

The total time was 3^h 15^m.

When the calculus was dry it weighed 170 grains, so that the loss in three hours and a quarter was only 2 grains. Apparently this was taken from the cut surface of the calculus. The external surface, which was exceedingly hard, was not attacked, though it was in contact with the positive electrode. Finding thus no success I returned again to the nitre solution.

Experiment 8.—The solution of nitre was taken of double the strength of that which had previously been used, that is, one-half saturated solution and one-half distilled water. I was desirous of knowing whether the oxalate of lime would be more soluble in this than in a weaker solution; the battery and the calculus were the same as before. The calculus weighed 170 grains.

The action began at 9^h 30^m A.M. and was continued to 12^h 47^m A.M.; the temperature of the solution was between 64° and 110°.

Total time was 3^h 17^m.

When dry the calculus weighed 167½ grains, so that the total loss in three hours and seventeen minutes was only 2½ grains. There was very slight action on the external surface of the stone which was in contact with the positive electrode. The cut surface around the nucleus appeared most acted on. The very frequent changes of the solution consequent on the resistance being lessened, led to the adoption of a continuous stream of solution.

Experiment 9.—It appeared from the last experiment, that by doubling the strength of the solution no corresponding increase in the action on the calculus occurred. To be still further convinced, the same calculus with the same battery was taken, and a solution was used of only half the strength of the previous one; that is, it was one-fourth saturated solution of nitre and three-fourths water, by means of a small tube siphon which carried off an ounce of fluid in twenty seconds, and a vessel with a stopcock which supplied fresh fluid. The temperature of the solution in which the calculus was kept remained about 92°, never rising higher than 96°.

The action began at 10 A.M., the temperature then was 64°.

The action was stopped at 12^h 50^m A.M., the temperature then was 92°.

Total time 2^h 50^m.

The calculus when dry weighed 165 grains, so that in two hours and fifty minutes of continuous action 2½ grains only were dissolved. By comparing this experiment with the previous one, it is so far certain, that, by increasing the strength of the solution of nitre, no more action on the calculus was obtained. The next question was whether, by increasing the strength of the battery, the solution of the oxalate of lime would more rapidly take place.

Experiment 10.—The same calculus was used as before; it weighed now 165 grains. The solution of nitre was the same strength as in the last experiment. The battery was increased to forty pairs of GROVE's plates. The temperature was kept down by about six pints of nitre solution, and as the liquid passed out of the decomposing glass it was cooled by ice, and then returned to the reservoir. The temperature was thus kept throughout the whole experiment from rising above 102°; one ounce ran through in five seconds.

The action began at 9^h 25^m A.M., the temperature then was 66°.

The action began at 9^h 47^m A.M. A voltameter placed in the circuit showed that 1 cubic inch of mixed gases was evolved in twenty seconds.

The action was stopped at 12^h 25^m A.M., the temperature throughout was between 98° and 102°.

The total time 3^h.

The calculus when dried weighed 160 grains, so that the total loss in three hours was only 5 grains. There were very slight marks of action on the external surface. The hardness of this surface was between that of fluor and calc spar; Mr. TENNANT placed it for me at 3·5.

The current was continuous for three hours; the battery was in perfect order, though diminishing in action towards the end of the experiment.

The result of these experiments with oxalate of lime may be thus arranged:—

	h	m		Average temp.	Pairs of plates.	Grs. dissolved.
Exp. 1 lasted	7	10,	in solution $\frac{1}{4}$ nitre, $\frac{3}{4}$ water . . .	90°	5	$\frac{1}{2}$
Exp. 2 lasted	7	10,	in solution $\frac{1}{4}$ nitre, $\frac{3}{4}$ water . . .	104	10	2
Exp. 3 lasted	6	15,	in sulphate of soda	101	10	2
Exp. 4 lasted	5	45,	in common salt	102	10	1
Exp. 5 lasted	6	10,	in nitre $\frac{1}{4}$, water $\frac{3}{4}$	108	20	6
Exp. 6 lasted	3	19,	in nitre $\frac{1}{4}$ with phosphate of soda .	110	20	1
Exp. 7 lasted	3	15,	in nitre $\frac{1}{4}$ with bichrom. of potassa	111	20	2
Exp. 8 lasted	3	17,	in nitre $\frac{1}{2}$, water $\frac{3}{4}$	110	20	$2\frac{1}{2}$
Exp. 9 lasted	2	50,	in nitre $\frac{1}{4}$, water $\frac{3}{4}$	92	20	$2\frac{1}{2}$
Exp. 10 lasted	3,		in nitre $\frac{1}{4}$, water $\frac{3}{4}$	100	40	5

Hence oxalate of lime calculi can be only very slowly dissolved in the solution of nitrate of potassa, which acts most energetically on uric acid calculi. The solution of the oxalate of lime is from $\frac{1}{2}$ a grain to 2 grains nearly an hour, so that the action is certainly four times as slow as in the case of uric acid calculi.

Such being the result when the calculus consisted of oxalate of lime only, it appeared desirable to determine what would be the effect when the calculus consisted of oxalate of lime with uric acid, or of oxalate of lime with the earthy phosphates.

Experiment 11.—Half an oxalate of lime calculus, which contained uric acid also on the external surface, weighed 258 grains. The solution of nitre, consisting of one-fourth nitre and three-fourths distilled water, with twenty pairs of plates, was employed.

The action began at 9^h 35^m A.M. and was continued to 1 P.M.; the temperature of the solution was between 64° and 120°.

Total time 3^h 25^m.

When dry the calculus weighed 254 grains, so that in three hours twenty-five minutes the loss was 4 grains. There was very little action on the calculus at the positive or acid electrode. The negative electrode was in contact with the external surface, and there the action had taken place.

Experiment 12.—The experiment with the same stone was repeated with the same solution and the same strength of battery. The solution was changed thirty-three times; the temperature was 110°; the time of action was three hours twenty-three minutes, in which the calculus lost 5 grains. There was very little action at the positive electrode, but very distinct at the negative electrode, which was on the upper and external surface.

Experiment 13.—Another small half calculus, consisting of oxalate of lime and uric acid, weighing 64 grains, was submitted to the action of twenty pairs of plates, whilst in a continuous stream of nitre of the same strength as before.

The action began at 9^h 32^m A.M., the temperature throughout was from 90° to 98°. and was stopped at 12^h 48^m A.M.

The total time 3^h 16^m.

When dry the stone weighed 60 grains, so that in three hours and a quarter 4 grains were dissolved.

The loss was chiefly on the cut surface, which was in contact with the negative or alkaline electrode. The surface was not regularly attacked, but some ridges were left, one very distinct, consisting of oxalate of lime. The external surface was below, and the acid appeared to have little or no effect on it.

It seemed from these experiments that uric acid and oxalate of lime together were more easily dissolved than oxalate of lime alone.

To determine whether oxalate of lime with the earthy phosphates was more soluble than oxalate of lime alone, the following experiments were made.

Experiment 14.—A whole calculus, consisting externally of oxalate of lime with phosphate of lime, weighing 186 grains, was acted on by twenty pairs of plates in the solution of nitre of the same strength as before.

The action began at $9^h\ 30^m$ A.M. and was continued to 1 P.M., the temperature of the solution being between 68° and 120°.

The action recommenced at $2^h\ 5^m$ P.M. and was continued to $3^h\ 35^m$ P.M., the temperature being between 70° and 100°.

The total time was 5^h. The last hour there was very little action of the battery.

The calculus when dry weighed 159 grains, so that in five hours 27 grains were dissolved; the result in this experiment was so much beyond my expectation that the same calculus was again submitted to the same reagents.

Experiment 15.—One grain having been removed for analysis the calculus weighed 158 grains.

The action began at $9^h\ 40^m$ A.M. and was continued to $12^h\ 45^m$ A.M., the temperature being between 65° and 130°.

The total time was $3^h\ 5^m$.

When the calculus was dry it weighed $143\frac{1}{2}$ grains, so that in three hours and five minutes the calculus lost $14\frac{1}{2}$ grains.

The result of these experiments with oxalate of lime and uric acid, and with oxalate of lime and phosphates, may be thus arranged:—

	Average temp. (°)	Pairs of plates.	Grs. dissolved.
Exp. 11 lasted 3h 25m, in solution of nitre $\frac{1}{4}$, water $\frac{3}{4}$	109	20	4
Exp. 12 lasted 3h 23m, in solution of nitre $\frac{1}{4}$, water $\frac{3}{4}$	110	20	5
Exp. 13 lasted 3h 16m, in solution of nitre $\frac{1}{4}$, water $\frac{3}{4}$	94	20	4
Exp. 14 lasted 5h, in solution of nitre $\frac{1}{4}$, water $\frac{3}{4}$	103	20	27
Exp. 15 lasted 3h 5m, in solution of nitre $\frac{1}{4}$, water $\frac{3}{4}$	113	20	$14\frac{1}{2}$

Hence the oxalate and urate together can be slowly dissolved in the nitre solution, whilst the oxalate and phosphate together are very rapidly dissolved, namely, at the rate of from $4\frac{1}{2}$ to $5\frac{1}{2}$ grains an hour.

It appears then from these experiments that calculi consisting of oxalates with phosphates are very easily dissolved in a dilute solution of nitre at the temperature of the body by the aid of electricity; but that calculi consisting of oxalates with urates are dissolved with difficulty, whilst calculi of oxalate of lime alone are acted on very slowly.

SECTION III. *On the Solution of Phosphatic Calculi and of Carbonate of Lime.*

Having found the solution of nitre so efficacious in its action on uric acid, and as it was at least as good if not better than any other solution in its action on oxalate of lime, it was most desirable that it should be found efficacious in its action on phosphatic calculi.

Experiment 1.—A piece of very hard phosphate of lime calculus, weighing 64 grains, was put into the nitre solution (one-fourth saturated solution and three-fourths distilled water). A battery of ten pairs of plates was first tried.

The action began at 9^h 15^m A.M. and was continued until 1^h 30^m P.M., the temperature being between 68° and 106°.

The action recommenced at 2 P.M. and was continued until 5 P.M., the temperature being between 70° and 106°.

The total time was 7^h 15^m.

The calculus when dry weighed 49 grains. Hence in seven hours and a quarter 15 grains were dissolved. The solution took place chiefly at the part in contact with the positive electrode.

Experiment 2.—A divided fusible calculus, weighing 85 grains, was subjected to the action of a battery of twenty pair. The solution of nitre was the same as before, but to three pints of the solution eight ounces of urine were added, being the quantity, or more than the quantity which would be secreted during the time of the experiment.

The action began at 9^h 27^m A.M.; the temperature of the solution was 70°.

The action was stopped at 10^h 40^m A.M.; the temperature was kept between 90° and 102°.

The total time was 1^h 13^m.

The calculus when dry weighed only 54 grains, so that the loss in one hour and thirteen minutes was 31 grains. The current was continuous throughout.

When the stone had been some time in the liquid, it was taken out with the electrodes still in contact. It then appeared that the stone had imbibed sufficient water to allow of the action going on through the wet stone without any liquid surrounding it. Hence probably the very rapid action which took place.

The negative electrode was found to be coated with earthy phosphates. The solution remained feebly acid throughout.

The positive electrode was below in contact with the external surface of the calculus which was cut through. See fig. 4.

On the Solution of Carbonate of Lime.

Experiment 3.—A piece of marble, weighing 170 grains, was put into the solution of nitre of specific gravity 1026. A galvanic battery of ten pairs of plates was employed.

The action began at 9^h 30^m A.M. and was continued until 1 P.M., the temperature being between 66° and 106°.

The action recommenced at 2 P.M. and was continued until 4 P.M., the temperature being between 70° and 108°.

The total time was $5\frac{1}{2}^h$.

The piece of marble when dry weighed 142½ grains, so that in five and a half hours 27½ grains were dissolved. The solution became slightly alkaline; the marble was dissolved at the part in contact with the positive electrode. See fig. 5.

Experiment 4.—Another mass of marble, with the same battery, was tried in a solution of sulphate of soda, specific gravity about 1034.

The action began at 9^h 30^m A.M. and was continued until 1 P.M., the temperature being between 87° and 100°.

The action recommenced at 2 P.M. and was continued until 5 P.M., the temperature being between 100° and 106°.

The total time was $6\frac{1}{2}^h$.

When the marble was dry it weighed 156½ grains. Hence in six hours and a half only 4½ grains of marble were dissolved. See fig. 6.

Without doubt the difficult solubility of the sulphate of lime was the cause that hindered the solution of the marble.

The contrast between the solution of nitre and the solution of Glauber-salt is very remarkable; the former being seven and a quarter times more efficacious than the latter.

The result of these experiments on the earthy phosphates and carbonate of lime may be thus arranged:—

	Average temp.	Pairs of plates.	Grs. dissolved.
Exp. 1 lasted 7 h 15 m in nitre solution	102°	10	15
Exp. 2 lasted 1 13 in nitre solution	96	20	31
Exp. 3 lasted 5 30 in nitre solution	104	10	27½
Exp. 4 lasted 6 30 in Glauber-salt	101	10	4½

Hence it is evident that calculi consisting of the earthy phosphates or of carbonate of lime can be most rapidly dissolved in a nearly neutral dilute solution of nitre, at the temperature of the human body, by the aid of the galvanic battery.

Conclusions.

Thus, then, by the aid of electricity, from 2 to 9 grains of uric acid calculi can be dissolved in an hour, whilst in the same time from 2 to 25 grains of phosphatic calculi can be dissolved in a neutral dilute solution of nitre at the temperature of the body. However, only from half a grain to 2 grains of oxalate of lime can be removed by the same means in the same time. Still, if the stone consists of oxalate with urate, from 1 to 2 grains may be dissolved in an hour; and if it consist of oxalate with phosphate, from 4½ to 5½ grains can be taken away in an hour.

Such at least are the results which can be obtained with calculi which have been long removed from the bladder and have been dried at 212°. The harder the calculi the greater the difficulty in dissolving them; and as previous to their removal from the bladder they are far less hard than they are after they have become dry, and as calculi which are wet may perhaps allow of the passage of the electricity through their substance, instead of only through the surrounding solution, there is good reason to expect that there will be less chemical difficulty in dissolving them in the bladder than out of it.

The skill which has been acquired in catching calculi in the bladder, the art of plating with platinum, the manufacture of vulcanized India rubber, will, with Mr. Weiss's help, I hope, enable me to state at some future time what can be done in the body as well as out of it. Looking back now from the results that have been obtained, it appears as if they might have been foreseen.

The uric acid and the phosphatic calculi are produced by the opposite states of over acidity and alkalescence occurring in the urine for a considerable time. By means of the galvanic battery acting on a saline solution, acid and alkali can be made to appear at any desired spot, in any quantity and for any time. Thus these opposite conditions may be produced on the surface of the stone at the spot, and only at the spot, where the action is wanted to correct the consequence of too much acidity or alkalescence; whilst around the place of action an almost neutral solution will remain in contact with the bladder without causing irritation.

I cannot conclude without stating what is due to MM. Prevost and Dumas. Their note is printed in the Annales de Chemie, vol. xxiii. p. 202, 1823, Sur l'Emploi de la Pile dans le traitment des Calculs de la Vessie*.

In this note the *mechanical action* of the mixed gases (evolved from the decomposition of water) on a fusible calculus is stated to have been 12 grains in 12 hours; twenty-five pairs of plates being used and recharged each hour.

The possibility of dissolving uric acid calculi is rejected, and no mention is made of calculi consisting of oxalate of lime.

In a postscript it is said that the addition of a "certain quantity" of nitre gave, with phosphatic calculi, a better result than when water alone was used.

M. Dumas appears afterwards to have occupied himself with experiments on the possibility of employing currents of electricity within the bladder of living animals, and not with the determination of the *chemical action* of substances evolved by the galvanic battery on the surface of different kinds of calculi.

I am indebted to Dr. Du Bois Reymond for a reference to a paper, vol. i. p. 154, Oesterrichische Medicinischen Jahrbucher 1848, On the Effect of Galvanism on Urinary Calculi, by Dr. Ludwig Melicher of Vienna. He has employed the electric current for the electrolysis of the calculus itself, not for setting substances free which can act chemically upon it. He has made experiments on the living human body in two cases, it is said, with success.

* See also Dr. Paris's Pharmacologia, vol. i. p. 231. ed. 6.

Appendix to a paper on the Solution of Urinary Calculi in dilute Neutral Solutions, by the aid of Electricity.

By HENRY BENCE JONES, *M.D., F.R.S., Physician to St. George's Hospital.*

Received December 18, 1852.

SINCE the original paper was communicated to the Royal Society, it has appeared to me desirable to try whether more dilute solutions of nitrate of potash would have sufficient effect, when the strength of the battery and all other circumstances were the same as in the previous experiments.

The solution of nitre was therefore made to contain only 10 grains of nitre to an ounce of water; it had a specific gravity of 1015. Two ounces of urine were added to three pints of the nitre solution.

A fusible calculus, weighing when dry 352 grains, was placed in this solution between the electrodes of twenty pairs of GROVE's plates.

The action commenced at 9^h 20^m A.M., the temperature being 50°.

The action was stopped at 12^h 45^m A.M., the temperature being 98°, so that the total time was 3^h 23^m. The calculus when dry weighed 324 grains, so that the total loss was 67 grains. See fig. 7.

Hence a solution of nitre of one-half the strength of that previously used is sufficient when decomposed to dissolve a fusible calculus at the rate of 19 grains an hour.

In order to observe the effect of reducing the strength of the battery when a dilute solution of nitre was employed, another calculus, consisting externally of phosphates, was placed in a solution of nitre, specific gravity 1015, and this was decomposed by ten pairs of GROVE's plates.

The weight of the calculus when dry was 217 grains.

The action began at 9 A.M., the temperature then being 50°; at 12^h 45^m A.M. the action was stopped, the temperature being 98°.

The action was recommenced at 2 P.M., temperature 60°.

It was finally stopped at 4^h 45^m P.M., temperature 98°.

The total time was 6^h 30^m. The solution of nitre did not require to be changed during the whole time, the temperature remaining constant. After being acted on, the calculus when dry weighed 197 grains, so that the total loss was 20 grains.

Hence in twice the time not one-third of the effect shown in the previous experiment was obtained, though this was chiefly owing to the diminished power of the battery, yet the greater hardness of the calculus used in this last experiment no doubt influenced the result.

I then proceeded to try the effect of the dilute solution of nitre on a uric acid calculus.

A uric acid calculus, weighing 403 grains, was placed in a solution of nitre, specific gravity 1015, which was decomposed by twenty pairs of GROVE's plates.

The action commenced at 8^h 56^m A.M., temperature being 50°. The action was stopped at 12^h 45^m A.M., temperature being 98°. The action recommenced at 2 P.M., temperature 52°. Finally stopped at 4^h 41^m P.M., temperature being 97°.

Hence the total time was 6^h 30^m. The calculus when dried weighed 381 grains, so that the total loss was 22 grains, or about 3½ grains each hour. See fig. 8.

It appears then from this experiment that the effect on a uric acid calculus, when a dilute solution of nitre is employed, is less than one-half of the effect produced when a solution of nitre of double the strength is used.

From these experiments, it is evident that phosphatic and uric acid calculi can be dissolved in a very dilute solution of nitrate of potash, but the weaker the solution the longer the time required to obtain equal results.

Lastly, an experiment made on an undried phosphatic calculus, fresh from the bladder, showed that the action on the wet calculus was more rapid than when it was previously dried; but the nature of the experiment does not admit of a numerical statement of the result. See fig. 9. The electrodes in this experiment were much longer than those previously used.

The mechanical difficulties in making an instrument (litholyte) to be used in the body have not yet been completely overcome.

It is essentially requisite in such an instrument,—

First. That the insulation should be perfect, in order that the chemical action may be transferred to the surface of the stone.

Secondly. That no chemical action should be set up on the surface of the bladder or the urethra, whereby the mucous membrane might be injured.

Thirdly. It may be necessary that a double passage for the injection of the solution of nitre should exist, in order to keep down the temperature and to admit of the escape of the gas evolved in the bladder.

Already many instruments have been made for me by Mr. WEISS, yet hitherto a perfectly safe one has not enabled me to make experiments on the solution of calculi within the body.

Explanation of the Plate.

PLATE XIV.

Representing the exact size of the calculi and the extent of action.

Fig. 1 (*a*). Uric acid before it was acted on.
(*b*). The same after action.
Fig. 2. The electrodes of the exact size and shape used.
Fig. 3. Uric acid calculus acted on in the nitre solution.
Fig. 4. Phosphatic calculus in the same solution.
Fig. 5. Piece of marble in solution of nitre.
Fig. 6. Piece of marble in solution of Glauber-salt.
Fig. 7. Phosphatic calculus in a more dilute solution of nitre.
Fig. 8. Uric acid calculus in a more dilute solution of nitre.
Fig. 9. A calculus as moist as when taken from the bladder, acted upon in nitre with larger electrodes.

IX. *On Molecular Influences.*—PART I. *Transmission of Heat through Organic Structures. By* JOHN TYNDALL, *F.R.S.*

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THE various solid substances which are met with in nature allow themselves to be classed under three general heads:—Amorphous, Crystalline and Organized. In amorphous bodies the component particles are confusedly mingled, without any regard to symmetry of arrangement. In crystalline bodies, on the contrary, the particles are symmetrically arranged; the mass appears as if built up according to certain architectural rules, and the result is an exterior form whose angular dimensions are perfectly constant for all crystals of the same class. Organized bodies, as the name implies, are bodies endowed with, or composed of, organs formed with reference to the special functions they are intended to discharge, and in the construction of which a molecular architecture of a very composite order comes into play. The granules, cells, glands, tubes, &c. of animal and vegetable tissues are all, of course, the visible products of this architecture. Crystalline bodies appear to bridge the chasm which separates the amorphous from the organized. Like the former, they are devoid of the powers of assimilation and reproduction—like the latter, their particles are arranged according to rule; as if nature, in the case of crystals, had made her first structural effort. The student of nature has ever looked upon these molecular combinations with an inquiring eye, and, perhaps, at no age of the world more than at present. The molecular peculiarities of any substance declare themselves by the manner in which a force is modified in its passage through the substance. The polarization and bifurcation of a luminous ray in doubly refracting media is an old example of molecular action; and the rotation of the plane of polarization, observed by Professor FARADAY, may be the result of a mechanical change of the medium, effected by the current or the magnet. SENARMONT's* and KNOBLAUCH's† experiments demonstrate the influence of crystalline structure upon the transmission of heat; and the magnecrystallic discoveries of PLÜCKER and FARADAY receive, I believe, their true explanation by reference, simply, to the modification of the magnetic and diamagnetic forces which peculiarity of aggregation induces. Matter, in this aspect, may be regarded as a kind of organ through which force addresses our senses; if the organ be changed, it is reasonable to infer that the utterance will be correspondingly modified,—an inference which is abundantly corroborated by experiment. Thus, mechanical pressure will polarize a ray, and the same may be

* Annales de Chimie et de Physique, vols. xxi. xxii. xxiii.

† POGGENDORFF's Annalen, vol. lxxxv. p. 169.

applied with success to produce all the phenomena of magnecrystallic action. The anomalies which owe their origin to peculiarities of aggregation are indeed manifold, and constitute one of the most important subjects of study which can engage the attention of the natural philosopher.

Organic structures furnish an ample field for inquiries into molecular action. For here, as before remarked, nature, to attain her special ends, has arranged her materials in a particular manner. To ascertain what effect the molecular structure of wood has upon the transmission of heat through it, constitutes the object of the first part of this investigation.

Upwards of twenty years ago MM. De la Rive and DeCandolle instituted an inquiry into the conductive power of wood*, and in the case of five specimens examined established the fact of the feeble conductivity of the substance, and also that the velocity of transmission was greater along the fibre than across it. The manner of experiment was that usually adopted in inquiries of this nature, and applied to metals by M. Depretz†. A bar of the substance was taken, one end of which was brought into contact with a source of heat and allowed to remain so until a stationary temperature was assumed. The temperatures attained by the bar, at various distances from its heated end, were ascertained by means of thermometers fitting into cavities made to receive them; from these data, with the aid of a well-known formula, the conductivity of the wood was determined. Since the publication of their results by the distinguished men above mentioned, nothing, so far as I am aware of, has been done in connexion with this subject.

The mode of experiment here indicated is, however, by no means sufficiently delicate for an inquiry like the present. Some other mode must therefore be devised. I will not trouble the reader with a rehearsal of the long series of trials which have led to the construction of the instrument employed in these researches, but will proceed at once to the description of it.

QQ′ RR′, fig. 1, is an oblong piece of mahogany 3·4 inches long, 1·8 inch wide and 0·5 of an inch deep. A is a bar of antimony, B is a bar of bismuth, each measuring 1·5 inch in length, 0·07 of an inch in breadth and 0·3 of an inch in depth. The ends of the two bars are kept in close contact by the ivory jaws I, I′, and the other ends are let into a second piece of ivory *i*, in which they are firmly fixed. Soldered to these ends are two pieces of platinum wire which proceed to the little ivory cups MM′, enter through the sides of the cups and communicate with a drop of mercury placed in the interior. The wood is cut away, so that the bars A and B are sunk to a depth which places their upper surfaces a little below the general level of the slab of mahogany. The ivory jaws I, I′ are sunk similarly. Two small projections are observed in the figure jutting from I, I′; across from one projection to the other a fine membrane is drawn, thus enclosing a little chamber *m*, in front of the wedge-like end of the bismuth and antimony junction; the chamber has an ivory bottom. S is a wooden

* Mém. de la Soc. de Genève, vol. iv. p. 70.

† Annales de Chim. et de Phys. December 1827.

slider, which can be moved smoothly back and forward along a bevelled groove, by means of the lever L. This lever turns on a pivot near Q, and fits into a horizontal slit in the slider, to which it is attached by the pin *p'* passing through both; in the lever an oblong aperture is cut through which *p'* passes, and in which it has a certain amount of lateral play, so as to enable it to push the slider forward in a straight line. A small chamber, *m'*, is cut out at the end of the slider, and across, from projection

Fig. 1

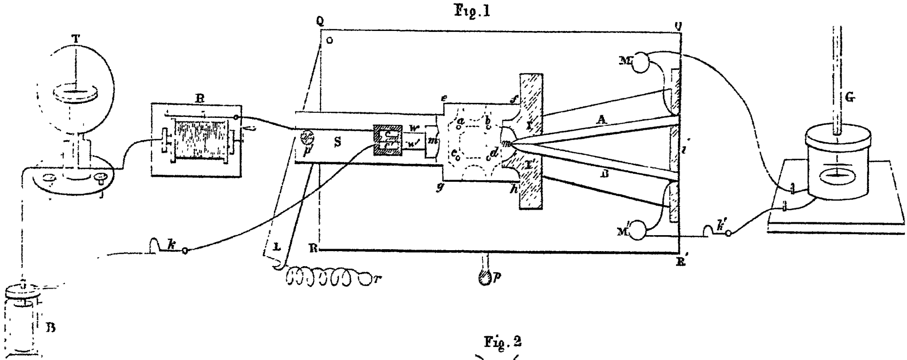
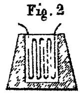


Fig. 2



to projection, a thin membrane is stretched; a chamber is thus formed bounded on three sides and the bottom by wood, and in front by the membrane. A thin platinum wire, bent up and down several times, so as to form a kind of micrometer grating, is laid against the back of the chamber and imbedded in the end of the slider by the stroke of a hammer; the end in which the wire is imbedded is then filed down until about half the latter is removed, and the whole is reduced to a uniform flat surface. Against the common surface of the slider and wire an extremely thin plate of mica is glued, sufficient, simply, to interrupt all contact between the bent wire and a quantity of mercury which the chamber *m'* is destined to contain: the ends *w w'* of the bent wire proceed to two small cisterns, *c c'*, hollowed out in a slab of ivory; they enter through the substance into the cisterns, and come thus into contact with mercury which fills the latter. The end of the slider and its bent wire are shown in fig. 2. The rectangular space *efgh*, fig. 1, is cut quite through the slab of mahogany, and a brass plate is screwed to the latter underneath; from this plate (which, for reasons to be explained presently, is cut away as shown by the dotted lines in the figure) four conical ivory points, *abcd*, project upwards; though appearing to be upon the same plane as the upper surfaces of the bismuth and antimony bars, the points are in reality 0·3 of an inch below the said surfaces.

The body to be examined is reduced to the shape of a cube, and is placed, by means of a pair of pliers, upon the four supports *abcd*; the slider S is then drawn up against the cube, and the latter becomes firmly clasped between the projections of the piece of ivory II′, on the one side, and those of the slider S, on the other. The chambers *m m′* being filled with mercury, the membrane in front of each is pressed gently against the cube by the interior fluid mass, and in this way perfect contact is secured. In fact the principle here applied is the same as that made use of by Fourier* in his thermometer of contact, although both instruments have nothing else in common.

The problem which requires solution is the following:—It is required to apply a source of heat of a strictly measurable character, and always readily attainable, to that face of the cube which is in contact with the membrane at the end of the slider, and to determine the quantity of this heat which crosses the cube to the opposite face in a minute of time. For the solution of this problem two things are required: —first, the source of heat to be applied to the left hand of the face of the cube, and secondly, a means of measuring the amount which has made its appearance at the opposite face at the expiration of a minute.

To obtain a source of heat of the nature described the following method was adopted:—B is a small galvanic battery, from the negative pole of which a current proceeds to the galvanometer of tangents T; passes round the ring of the instrument, deflecting, in its passage, the magnetic needle which hangs in the centre of the ring. The strength of the current is, as is known, proportional to the tangent of the angle of permanent deflection. From T the current proceeds to the rheostat R; this instrument consists of a cylinder of serpentine stone, round which a German silver wire is coiled spirally; by turning the handle of the instrument any required quantity of this powerfully resisting wire is thrown into the circuit, the current being thus regulated at pleasure. The sole use of these two last instruments in the present series of experiments is to keep the current perfectly constant from day to day. From the rheostat the current proceeds to the cistern *c*, thence through the bent wire, and back to the cistern *c′*, from which it proceeds to the other pole of the battery.

The bent wire, during the passage of the current, becomes heated; this heat is transmitted through the mercury in the chamber *m′* to the membrane in front of the chamber; this membrane becomes the proximate source of heat which is applied to the left-hand face of the cube. The quantity transmitted from this source, through the mass of the cube, to the opposite face, in any given time, will, of course, depend on the conductivity of the latter, and its amount may be estimated from the deflection which it is able to produce upon the needle of a galvanometer connected with the bismuth and antimony pair. G is a galvanometer used for this purpose; from it proceed wires to the mercury cups M M′, which, as before remarked, are connected by platinum wires with A and B. The galvanometer is a carefully constructed and delicate instrument from the workshop of that skilful mechanic, Kleiner, in Berlin.

* Annales de Chimie et de Physique, March 1828.

The sole use of the mercury in the chambers m and m' is to secure good and equable contact; when the chambers are filled with pure mercury, and this is allowed to remain in them throughout an entire series of experiments, it is certain that the conditions of contact are perfectly constant, and thus the most fruitful source of doubt and error is effectually excluded. In rough experiments the chambers might be dispensed with, and the bent wire itself might be brought into contact with one face of the cube, while the other face might immediately press against the bismuth and antimony. The result however of many hundreds of experiments made with the instrument in this state, has been to prove the impossibility of preserving the conditions of experiment constant, and to compel me to devise some means of avoiding the irregularities which exhibited themselves. The instrument just described meets the requirements of the case; care is necessary in the use of it, but when care is taken, an accuracy is attainable by it which, I believe, has been hitherto unequalled.

The action of mercury upon bismuth, as a solvent, is well known; an amalgam is speedily formed where the two metals come into contact. To preserve our thermo-electric couple from this action, their ends are protected by a sheathing of the same membrane as that used in front of the chambers $m\,m'$.

Previous to the cube being placed between the two membranes, the latter, by virtue of the fluid masses behind them, bulge out a little, thus forming a pair of soft and slightly convex cushions. When the cube is placed upon its supports and the slider is brought up against it, both cushions are pressed flat, and thus perfect contact is secured. The surface of the cube is larger than the surface of the membrane in contact with it*; and thus the former is always firmly caught between the opposed rigid projections, the slider being held fast in this position by means of the spring r, which is then attached to the pin p. The exact manner of experiment is as follows:—Having first seen that the needle of the galvanometer points to zero, when the thermo-circuit is complete, the latter is interrupted by means of the break-circuit key k'. At a certain moment, marked by the seconds-hand of a watch, the voltaic circuit is closed by the key k, and the current is permitted to circulate for sixty seconds; at the sixtieth second the voltaic circuit is broken by the left hand at k, while almost at the same instant the thermo-circuit is closed by the right hand at k'. The needle of the galvanometer is instantly deflected, and the limit of the first impulsion is noted; the amount of this impulsion depends, of course, upon the quantity of heat which has reached the bismuth and antimony junction through the mass of the cube during the time of action, and consequently upon the conductive power of the latter. The limit of the first impulsion being noted, the cube is instantly removed, and the instrument is allowed to cool until the needle of the galvanometer returns to zero. To expedite the cooling, the metallic surfaces of A and B are to a great extent exposed; the wood is cut away all round them, and from the space between them; they do not rest upon the wood, their sole points of support being the ivory i at one end and the

* The edge of each cube measured 0·3 of an inch.

jaws II′ at the other. The cube, as before explained, does not touch the brass plate underneath it, but is supported on its four conical points, and the plate which bears these is itself as much as possible cut away to permit of a free circulation of air through the space *efgh*. Time is a precious commodity to the experimenter, and by the means described the cooling is hastened and the experiments can succeed each other more quickly. To hasten the cooling further I made use of a pair of small bellows during the first minute after the removal of each cube, and, afterwards, a plate of thin glass was placed over the junction, but not in contact with it. On the glass two drops of ether were suffered to fall from a pipette; its evaporation caused a refrigeration of the air underneath, which, in virtue of its increased density, sank and diffused itself around the place of junction. In this way the temperature at the junction was brought a little lower than that of the surrounding air; the needle of the galvanometer being thus brought back, not only to zero, but to a certain point at the other side of it; at this point the glass was removed and a new cube was introduced; the thermo-circuit was permitted to remain closed until the needle descended to zero, which it slowly did, when the cause of local cooling was removed; the thermo-circuit was then broken at *k*′, and things stood as at the commencement of the former experiment. The voltaic circuit was once more closed, the current permitted to circulate sixty seconds, then interrupted by the left hand, the thermo-circuit being closed at the same moment with the right, and the first impulsion measured as before.

When however these artificial means of cooling are adopted great care is necessary. We must not use the bellows in some experiments and neglect the use of it in others; and if the ether be applied once, it must be applied throughout the entire series of experiments. It must continue to act for the same time, and the same quantity should be applied in all cases. Of course such precautions are only necessary when great accuracy is required, but here they are absolutely necessary. Judging from the description, the mode of experiment may appear complicated, but in reality it is not so. A single experimenter has the most complete command over the entire arrangement. The wires from the small galvanic battery (four of Bunsen's cells) remain undisturbed from day to day; all that is to be done is to connect the battery with them, and every thing is ready for experiment.

There are in wood three lines at right angles to each other, which the mere inspection of the substance enables us to fix upon as the necessary resultants of molecular action: the first line is parallel to the fibre; the second is perpendicular to the fibre, and to the ligneous layers which indicate the annual growth of the tree; while the third is perpendicular to the fibre and parallel, or rather tangent to the layers. From each of a number of trees a cube was cut, so that every two opposite faces were parallel to one of the above lines. Thus, two faces were parallel to the ligneous layers, two perpendicular to them, while the remaining two were perpendicular to the fibre. It was proposed to examine the velocity of calorific transmission through the mass in these three directions. It may be remarked that the cubes were fair average specimens of the woods, and were in all cases well-seasoned and dry.

The cube was first placed upon its four supports *abcd*, so that the line of flux from *m'* to *m* was parallel to the fibre, and the deflection produced by the heat transmitted in sixty seconds was observed. The position of the cube was then changed so that its fibre stood vertical, the line of flux from *m'* to *m* being perpendicular to the fibre and parallel to the ligneous layers; the deflection produced by a minute's action in this case was also determined. Finally, the cube was turned 90° round, its fibre being still vertical, so that the line of flux was perpendicular to both fibre and layers, and the consequent deflection was observed. In the comparison of these two latter directions the chief delicacy of manipulation is necessary. It requires but a rough experiment to demonstrate the superior velocity of propagation along the fibre, but the velocities in all directions perpendicular to the fibre are so nearly equal that it is only by the greatest care and, in the majority of cases, by numerous experiments, that a differential action can be securely established.

The following table contains the results of the inquiry; it will explain itself.

Table I.—Strength of Current used to heat the bent Wire, as measured by the tangent galvanometer:—constantly 35°. Deflections, the line of flux being of:—

Description of wood.	I. Parallel to fibre of wood.	II. Perpendicular to fibre and parallel to ligneous layers.	III. Perpendicular to fibre and to ligneous layers.
American Birch......	35°	9°	11°
Oak....................	33	9·5	11
Beech..................	33	8·0 9·0 9·7 9·5 Mean 8·8	11·0 10·0 11·2 11·0 Mean 10·8
Coromandel-wood. An exceedingly hard wood from Ceylon	33	10·2 10·2 9·5 9·5 Mean 9·8	12·2 12·0 12·0 13·0 Mean 12·3
Quebec Pine	33	10	11
Beef-wood. A red-coloured wood from New South Wales..............	33	9·8 10·0 10·5 9·8 Mean 10·0	11·0 12·0 11·5 11·0 Mean 11·4

Description of wood.	I. Parallel to fibre of wood.	II. Perpendicular to fibre and parallel to ligneous layers.	III. Perpendicular to fibre and to ligneous layers.
Black Ebony	32	10·5° 9·5 10·5 8·0 Mean 9·5	11·0° 11·0 9·5 10·5 Mean 10·5
Bird's-eye Maple ...	31	11	12
Lance-wood	31	9·8 10·0 11·5 11·0 Mean 10·6	12·0 11·0 12·0 13·5 Mean 12·1
Zebra-wood. The produce of the Brazils	31	8·0 8·0 8·5 Mean 8·2	10·0 10·0 10·0 Mean 10·0
Box-wood	31	9·8 10·0 10·0 10·0 Mean 9·9	12·3 11·7 12·0 12·0 Mean 12·0

Table I. (Continued.)

Description of wood.	I. Parallel to fibre of wood.	II. Perpendicular to fibre and parallel to ligneous layers.	III. Perpendicular to fibre and to ligneous layers.
	°	11·5°	12·5°
		11·5	12·0
		10·5	12·0
		11·0	12·0
Tamarind-wood	31	Mean 11·1	Mean 12·1
		10·0	12·0
		9·5	13·0
		10·0	13·0
		10·0	11·5
Teak-wood	31	Mean 9·9	Mean 12·4
		10·5	12·0
		10·0	13·5
		10·0	13·0
		11·0	12·0
Rose-wood	31	Mean 10·4	Mean 12·6
		10·5	13·0
		10·5	12·0
Mazatlan-wood	30	Mean 10·5	Mean 12·5
		12·5	12·5
		11·0	12·0
		12·0	12·7
		12·0	12·0
Satin-wood. From St. Domingo	30	Mean 11·9	Mean 12·3
		9·0	11·0
		9·0	10·5
		10·0	11·0
		9·0	11·5
Braziletto. From Jamaica	30	Mean 9·2	Mean 11·0
		10·0	10·5
		10·0	11·5
		10·0	11·0
Locust-wood. From North America.	30	Mean 10·0	Mean 11·0
		11·0	11·0
		10·0	11·0
		10·0	11·5
Ruby-wood. From Calcutta	30	Mean 10·3	Mean 11·2
		11·0	12·0
		10·5	12·0
		10·5	11·0
Peruvian-wood	30	Mean 10·7	Mean 11·7

Description of wood.	I. Parallel to fibre of wood.	II. Perpendicular to fibre and parallel to ligneous layers.	III. Perpendicular to fibre and to ligneous layers.
	°	10·0°	12·0°
		10·5	12·0
		10·5	11·0
King-wood. Called also violet-wood, from the Brazils	30	Mean 10·3	Mean 11·7
Bladder-nut-wood	29	10·0	12·0
Larch	29	10·0	11·0
		11·0	14·0
		10·0	12·0
		11·2	14·0
		12·0	12·0
Princes-wood. From Jamaica	29	Mean 11·1	Mean 13·1
		11·5	13·0
		11·5	12·0
		12·5	12·8
		11·2	12·5
Green Heart. From Jamaica	29	Mean 11·4	Mean 12·6
Walnut	28	11·0	13·0
Drooping Ash	28	11·0	12·0
		10·0	13·0
		10·0	12·6
		10·5	12·2
		9·0	12·0
Botany Bay Oak. Does not belong to the same genus as the European	28	Mean 9·9	Mean 12·4
		12·0	13·0
		12·5	14·0
		12·0	13·5
		11·2	14·0
Cocoa-wood. From the West Indies	28	Mean 11·9	Mean 13·6
		11·3	12·0
		11·2	12·5
		10·5	11·8
		10·0	11·0
Madagascar Red-wood	28	Mean 10·7	Mean 11·8
		11·0	12·0
		10·5	12·5
		9·5	11·5
		9·0	11·0
Sandal-wood. From Malabar	28	Mean 10·0	Mean 11·7

Table I. (Continued.)

Description of wood.	I. Parallel to fibre of wood.	II. Perpendicular to fibre and parallel to ligneous layers.	III. Perpendicular to fibre and to ligneous layers.
Tulip-wood. From Brazil	28°	11·0° 11·0 11·0 11·0 Mean 11·0	11·5° 12·5 12·5 12·0 Mean 12·1
Camphor-wood. From China	28	9·0 9·0 8·0 Mean 8·6	10·0 10·0 10·0 Mean 10·0
Olive-wood. From Leghorn	28	10·0 11·0 10·5 Mean 10·5	14·0 12·0 13·0 Mean 13·2
Gaffle Deal	27	10	11
Ash	27	9·5	11·5
Green Ebony. From Jamaica	27	11·0 11·0 9·0 11·0 Mean 10·5	12·5 12·0 13·0 11·5 Mean 12·2
Black Oak	27	8·0 7·0 8·0 8·0 Mean 8·0	9·0 10·0 10·0 9·0 Mean 9·4
Apple-tree	26	10	12·5
Cam-wood. An African dye-wood	26	13·0 13·0 14·0 13·5 Mean 13·4	15·0 15·0 15·0 15·0 Mean 15·0

Description of wood.	I. Parallel to fibre of wood.	II. Perpendicular to fibre and parallel to ligneous layers.	III. Perpendicular to fibre and to ligneous layers.
Iron-wood. Sp. gr. 1·426	26°	9·5° 9·5 10·2 10·5 Mean 10·2	12·5° 12·0 13·0 12·0 Mean 12·4
Chestnut	26	10·2 10·0 Mean 10·1	12·0 11·0 Mean 11·5
Sycamore	26	10·2 11·0 Mean 10·6	12·5 12·0 Mean 12·2
Spruce Fir	25	11·8	12·5
Honduras Mahogany	25	9·0	10·0
Brazil-wood. A red dye-wood, called also Pernambuco	25	12·0 13·0 11·5 11·2 Mean 11·9	13·0 14·5 13·0 14·0 Mean 13·9
Yew	24	11·0	12·0
Elm	24	10·0	11·5
Plane-tree	24	10·0	12·0
Portugal Laurel	24	10·0	11·5
Bullet-wood. From the Virgin Isles, West Indies	24	10·0 10·5 9·5 Mean 10·0	12·0 12·0 11·0 Mean 11·7
Spanish Mahogany	23	11·5	12·5
Scotch Fir	22	10·0	12·0
Laurel	22	12·0	15·0

To enable the eye to detect, at once, the law of action established by the experiments, we will present the results in a more condensed form.

Deflections.

Description of wood.	I. Parallel to fibre.	II. Perpendicular to fibre and parallel to ligneous layers.	III. Perpendicular to fibre and to ligneous layers.	Description of wood.	I. Parallel to fibre.	II. Perpendicular to fibre and parallel to ligneous layers.	III. Perpendicular to fibre and to ligneous layers.
1 American Birch	35°	9°·0	11°·0	28 Botany Bay Oak	28°	9°·9	12°·4
2 Oak	34	9·5	11·0	29 Cocoa-wood	28	11·9	13·6
3 Beech	33	8·8	10·8	30 Madagascar Red-wood	28	10·7	11·3
4 Coromandel-wood......	33	9·8	12·3	31 Sandal-wood	28	10·0	11·7
5 Quebec Pine	33	10·0	11·0	32 Tulip-wood	28	11·0	12·1
6 Beef-wood	33	10·0	11·4	33 Camphor-wood	28	8·6	10·0
7 Black Ebony............	32	9·5	10·5	34 Olive-tree	28	10·5	13·2
8 Bird's-Eye Maple	31	11·0	12·0	35 Gaffle-Deal	27	10·0	11·0
9 Lance-wood	31	10·6	12·1	36 Ash	27	9·5	11·5
10 Zebra-wood	31	8·2	10·0	37 Green Ebony	27	10·5	12·2
11 Box-wood	31	9·9	12·0	38 Black Oak..............	27	8·0	9·4
12 Tamarind-wood.........	31	11·1	12·1	39 Apple-tree.............	26	10·0	12·5
13 Teak-wood	31	9·9	12·4	40 Cam-wood..............	26	13·4	15·0
14 Rose-wood..............	31	10·4	12·6	41 Iron-wood	26	10·2	12·4
15 Mazatlan-wood	30	10·5	12·5	42 Chestnut	26	10·1	11·5
16 Satin-wood	30	11·9	12·3	43 Sycamore	26	10·6	12·2
17 Braziletto	30	9·2	11·0	44 Spruce Fir...............	25	11·8	12·5
18 Locust-wood	30	10·0	11·0	45 Honduras Mahogany	25	9·0	10·0
19 Ruby-wood	30	10·3	11·2	46 Brazil-wood	25	11·9	13·9
20 Peruvian-wood	30	10·7	11·7	47 Yew	24	11·0	12·0
21 King-wood	30	10·3	11·7	48 Elm	24	10·0	11·5
22 Bladder-nut-wood......	29	10·0	12·0	49 Plane-tree	24	10·0	12·0
23 Larch.....................	29	10·0	11·0	50 Portugal Laurel	24	10·0	11·5
24 Princes-wood............	29	11·1	13·1	51 Bullet-wood	24	10·0	11·7
25 Green-heart	29	11·4	12·6	52 Spanish Mahogany ...	23	11·5	12·5
26 Walnut	28	11·0	13·0	53 Scotch Fir...............	22	10·0	12·0
27 Drooping Ash	28	11·0	12·0	54 Laurel	22	12·0	15·0

The above table furnishes us with the fullest corroboration of the result arrived at by De la Rive and DeCandolle, regarding the superior conductivity of the wood in the direction of the fibre. Evidence is also afforded as to how little mere density affects the velocity of transmission. There appears to be neither law nor general rule here. American Birch, a comparatively light wood, possesses undoubtedly a higher transmissive power than any other in the list—a result which has been established by numerous experiments, although but one appears cited in the table. Iron-wood, on the contrary, with a specific gravity of 1·426, stands low. Again, Oak and Coromandel-wood,—the latter so hard and dense that it is used for sharp war-instruments by savage tribes,—stand near the head of the list, while Scotch Fir and other light woods stand low.

We further find that the lateral conductivity bears no definite ratio to the longitudinal conductivity. Indeed the tendency appears to be that those woods which possess the lowest power of transmission, along the fibre, possess the highest power across it. But here the exceptions are so numerous that we have no warranty for a general conclusion.

But the most remarkable result of the experiments remains yet to be stated. If we cast our eyes along the second and third columns of the tabular summary, we shall find that in every instance the velocity of propagation is greatest in a direction perpendicular to the ligneous layers. This result is, of course, wholly independent of the specific heat of the wood, inasmuch as it is two directions through the self-same cube which are here compared with each other. The law of molecular action, as regards the transmission of heat through wood, may therefore be expressed as follows:—

At all the points not situate in the centre of the tree, wood possesses three unequal axes of calorific conduction, which are at right angles to each other. The first, and principal axis, is parallel to the fibre of the wood; the second, and intermediate axis, is perpendicular to the fibre and to the ligneous layers; while the third and least axis is perpendicular to the fibre and parallel to the layers.

The researches of SAVART on the sonorous vibrations of wood naturally suggest themselves here; for, doubtless, the same molecular structure which imparts to this substance the peculiar elastic properties discovered by SAVART, must be regarded as the cause of the differential action established above. SAVART* took bars of equal size, and in different directions, from a mass of wood; determining their resistance to flexure from the number of vibrations carried out by each in a certain time, he found that wood possessed three axes of elasticity. *These axes coincide with the axes of calorific conduction established by the foregoing experiments.* The axis of greatest elasticity coincides with that of highest conductive capacity, and the axis of least elasticity with that of lowest conductive capacity.

A few exact experiments made with a view to ascertain the influence of molecular structure upon the cleavage of wood would have formed an interesting addition to this communication; fortunately, however, the mere sense of touch, to fingers accustomed to seek for the cleavages of crystals, affords sufficient evidence here. If a piece of wood be taken, on which the rings which mark the growth of the tree plainly appear, and a penknife or a chisel be laid across the rings, it will be found that the pressure necessary to cleave the wood is less in this than in any other direction. The cohesion in the direction parallel to the layers and perpendicular to the fibre is therefore a minimum. In the same way, it will be found that of all lines perpendicular to the fibre the line of greatest cohesion is perpendicular to the ligneous layers; while the cohesion in the direction of the fibre is far greater than along either of the lines just referred to. Hence—

Wood possesses three axes of cohesion which coincide with the axes of calorific conduction—the greatest with the greatest, and the least with the least.

It would have also added interest to the inquiry to have examined the permeability

* TAYLOR's Scientific Memoirs, vol. i. p. 139.

of wood to fluids in various directions. Here, again, however, the experimental knowledge already amassed by housewives and cask-makers comes to our aid. It is well known that fluids would ooze with facility through a stave cut perpendicular to the fibre; a wooden plate, for instance, cut perpendicular to the axis of a tree would be totally unfit for the bottom of a vessel destined to hold spirits, water, or brine. Further precautions, however, must be taken in choosing staves for casks. If the surface of the stave be parallel to the ligneous layers, the liquid, though with greater difficulty than in the case just mentioned, will still make its way through. The stave must be cut perpendicular to the layers; for, in crossing such a stave, the resistance offered to the passage of the fluid is a maximum. Hence—

Wood possesses three axes of fluid permeability which coincide with those of calorific conduction,—the greatest with the greatest, and the least with the least.

To sum up:—In this single substance we have pointed out the existence of three new systems of axis; the axes of calorific conduction, of cohesion, and of fluid permeability; all of which coincide with a fourth system of axes of elasticity discovered by SAVART. The experiments also furnish an illustration of the theory of Professor STOKES, who proves that the flux of heat through any body may be referred to three rectangular axes, which he calls the thermic axes of the body*.

MM. DE LA RIVE and DECANDOLLE have remarked upon the influence which its feeble conducting power in a lateral direction must exert in preserving within a tree the warmth which it acquires from the soil. In virtue of this property a tree is able to resist sudden changes of temperature which would probably be prejudicial to it; it resists alike the sudden abstraction of heat from within and the sudden accession of it from without. But nature has gone further, and clothes the tree with a sheathing of worse-conducting material than the wood itself, even in its worst direction. The following are the deflections obtained by submitting a number of cubes of bark of the same size as the cubes of wood to the same conditions of experiment:—

	Deflection.	Corresponding deflection produced by the wood.
	°	
Beech-tree Bark	7	10·8
Oak-tree Bark	7	11·0
Elm-tree Bark	7	11·5
Pine-tree Bark	7	12·0

The direction of transmission, in these cases, was from the interior surface of the bark outwards.

The average deflection produced by a cube of wood, when the flux is lateral, may be taken at

$$12°;$$

* Cambridge and Dublin Mathematical Journal, November 1851.

a cube of rock-crystal (pure silica) of the same size produces a deflection of

90°.

This single experiment is sufficient to show how different must be the meteorological effects of these two substances, when they exist in sufficient quantity to exercise an influence upon climate. Among the more prominent influences here, HUMBOLDT mentions the nature of the soil and of vegetation. The general influence of an arid and exposed soil has been long known, but the part played by this substance, silica, has hitherto had no particular importance attached to it. Were gypsum, however, instead of silica, the prevalent mineral in Sahara, a very different state of things from the present would assuredly exist. A cube of the latter substance examined in the usual manner produces a deflection of

19°

only. It is scarcely superior to wood, while there is the strongest experimental grounds for the belief that silica *possesses a higher conductive power than some of the metals*. These grounds shall be adduced in a future paper.

Let us consider, for a moment, the process which takes place from sunrise to the hour of maximum temperature in a region overspread with forests, and compare it with that which must take place in the African Desert. In the former case, the heat slowly and with difficulty penetrates the masses of wood and leaves on which it falls, and after the point of maximum temperature is passed, the yielding up of the heat acquired is proportionately slow. In the desert, however, the mass of silica exposed to the sun becomes burning hot as the hour of maximum temperature approaches; but, after this is passed, the heat is yielded up with proportionate facility. Hence a maximum and minimum thermometer must, in the latter case, mark a far wider range of temperature during the twenty-four hours than in the former. This agrees with observation. In Sahara, to use the words of Mrs. SOMERVILLE, during "the glare of noon the air quivers with the heat reflected from the red sand, and in the night it is chilled under a clear sky sparkling with its host of stars*." Were gypsum, however, the prevailing mineral, it is *à priori* certain that this could not be the case to anything like its present extent.

The following experiments furnish some notion of the transmissive power of a few other organic structures: cubes of the substances were examined in the usual manner.

Tooth of Walrus	16
Tusk of East Indian Elephant .	17
Whalebone	9
Rhinoceros's-horn	9
Cow's-horn	9

Considering the density and elasticity of ivory, we might be disposed to attribute to it a comparatively high conductive power; but the experiment proves it to be a very

* Physical Geography, vol. i. p. 147.

bad conductor—much inferior, indeed, to wood in the direction of the fibre. Doubtless this conduces to the animal's comfort. Exposed to the rays of a tropical sun, if these huge bony masses were capable of assuming a high temperature during the day and losing it again at night, it must be a source of the greatest inconvenience to the animal, as at present constituted. The horns of the Rhinoceros and Cow, however, still more strikingly exemplify that fitness of parts which is perpetually presented to the student of natural science. In the latter case especially, the mass of horn in close contact with the skull, and therefore capable of transmitting heat directly to the animal's brain, must be attended with very unpleasant consequences, if horn were a good conductor. Given such a constitution, the substance fixed upon by our own enlightened intellect to furnish the animal with such weapons of defence, would be just such as nature has chosen.

As a general rule, sudden changes of temperature are prejudicial to animal and vegetable health; the substances used in the construction of organic tissues are exactly such as are best calculated to resist those changes. Coal enters largely into the composition of such tissues, and it is an exceedingly bad conductor. Here are the deflections obtained with three different descriptions of this substance:—

Sunderland coal	8°
Boghead cannel	8
Lesmahago cannel	8

The following results illustrate the subject in a still more striking manner. It is almost needless to remark that each of the substances mentioned was reduced to the cubical form, and submitted to an examination similar in every respect to that of wood and quartz. While, however, a cube of the latter substance produces a deflection of 90°, a cube of

Sealing-wax produces a deflection of	0°
Sole leather	0
Bees'-wax	0
Glue produces a deflection of	0°
Gutta-percha	0
India-rubber	0
Filbert-kernel	0
Almond-kernel	0
Boiled ham-muscle	0
Raw veal-muscle	0

The substances here named are all of them animal and vegetable productions; and the experiments demonstrate the extreme imperviousness of every one of them. Starting from the principle that sudden accessions or deprivations of heat are prejudicial to animal and vegetable health, we see that the materials chosen are precisely those which are best calculated to avert such changes. It is yet to be estimated what influence the extreme non-conductibility of muscular tissue exerts in producing the remarkable constancy of temperature observed in the human body in different climates. The cuticle is an exceedingly bad conductor, and this explains the insensibility to heat of hands on which the skin has been thickened by exposure. Probably many escapes from the fiery ordeal, which have been hitherto referred to collusion, might be scientifically explained by reference to this fact. While studying at Marburg, I have sometimes heard Professor Bunsen make a good-humoured remark on the tenderness of his pupils' fingers. Accustomed as he was to the manipulation of the glass used in his admirable eudiometrical researches, his fingers had acquired an insensibility to heat sufficient to carry him safely through an ordeal which, in other cases, would undoubtedly invoke the judicial condemnation of the middle ages. The experiments of Chantrey and Blagden are often referred to as illustrations of the surprisingly high temperature to which the human body may for a short time be exposed without injury. These experimenters owed their safety to two things,—to the non-conductibility of their tissues, and the non-conductibility of the air in contact with them. Were either of these materials changed, the experiments could not have been made. If air were a good conductor, and parted with its heat readily, their hands and faces would have shared the fate of the beefsteak and eggs which were cooked in contact with tin in the same oven. Were their bodies good conductors, they would have become heated like the tin, the heat would have been transferred to the deeper tissues and organs, to the probable destruction of the latter. As it was, however, both the causes mentioned contributed to the success of the experiment, and a mere surface irritation was the only inconvenience felt.

X. *On the Impregnation of the Ovum in the Amphibia.* (*Second Series, Revised.*) *And on the Direct Agency of the Spermatozoon.* By George Newport, *F.R.S., F.L.S. &c.*

Received May 10,—Read June 17, 1852.

HAVING shown in a former series of investigations, which has been honoured by a place in the Philosophical Transactions for 1851, that the sole agent of impregnation of the ovum, in all cases of communion of the sexes, is the spermatozoon,—and having then supplied both direct and negative evidence that impregnation is not effected by the *liquor seminis*,—I endeavoured, in a subsequent communication to the Royal Society, in June 1851*, to arrive at some knowledge of the manner in which impregnation is effected, and of the nature of the impregnating influence.

Up to that period, and indeed, until very recently, I had never been able to detect any evidence of the existence of spermatozoa within the envelopes of the fecundated egg, but had constantly found them in great abundance, and easily recognized, in contact with the exterior surface. Experiment also, made by immersion of the egg in coloured fluids, showed that the substance of the envelopes, although permeable by fluids, is uniform in its structure; but no evidence was afforded by it of any natural canal, fissure, or perforation through the envelopes of the egg of the Frog, capable of admitting the spermatozoon to the interior, as has been supposed to exist in the egg of the Mammalia†.

Hence the conclusion which seemed to be fairly led to, was, that some influence was transmitted from the spermatozoon on the surface of the envelopes, through their substance, to the yelk which they inclose,—as the *commencement* of impregnation. But it was especially pointed out in my First Series‡, that "simple contact of the spermatozoon does not appear to be sufficient to determine the transmission of more or less of the material structural characters of the male parent to the offspring;" and that, "possibly, we may hereafter find that the first changes induced by contact of the impregnating body are completed by its diffluence, and by the material constituents into which it is dissolved, being transferred to the yelk by endosmosis;" Further, in my second communication, I remarked,—"I am not yet prepared to assent to the view, that simple contact alone, even of the spermatozoon, is sufficient to complete the changes which result in the formation of the embryo§." Again, in

* Proceedings of Royal Society, vol. vi. p. 82.

† Philosophical Transactions, 1840, p. 533, Plate XXII. figs. 165, 167.

‡ Philosophical Transactions, 1851, p. 242.

§ See "Impregnation of the Ovum (Second Series)," June 19th, 1851, MS. No. 762, p. 49, in the Archives of the Royal Society.

a subsequent part of the same communication, these considerations are further referred to, as being those alone on which the transmission of structural peculiarities seemed likely to be explained*.

Recent observation has now supplied to me a fact which renders these considerations, which heretofore were regarded as hypothetical, much more probable; but, at the same time, it necessitates a revision of the view that an impregnating influence is transmitted from the surface to the interior of the egg; and also some correction of the deductions from experiment by which this view seemed to be supported.

The fact referred to, is, that the spermatozoon of the Frog penetrates bodily into the substance of the thick envelopes of the egg, and comes into direct communication with, at least, the membrane which incloses the yelk in the interior.

It is my duty, therefore, to Science, and to the Royal Society, knowing this to be the case, to revise and to extend my views, and to submit them again to the Society in their amended form; making, in self-correction, the certainty of fact now take the place of negative observation.

In doing this, I shall endeavour to show, that, although still regarding the spermatozoon, for reasons to be adduced, as the organ of a special condition of force, or vitality in the male body,—its influence on the egg in fecundation is direct and immediate, and not operative, as heretofore supposed, merely through the envelopes; but, nevertheless, as I have before shown, that it is exerted only so long as the spermatozoon continues to give evidence of its vitality or force in its power of motion.

Before proceeding to show that the spermatozoon penetrates into the envelopes of the egg it is necessary that I should point out some of the conditions which affect the fulfilment of its fecundatory function, both in regard to the spermatozoon itself and to the ovum. These are, on the one hand, the persistence of vital power in the spermatozoon as influenced by the temperature of the surrounding medium; and on the other, the endosmic property of the gelatinous coverings of the egg, and the susceptibility of the yelk which they inclose, to become impregnated.

1. THE VITALITY OF THE SPERMATOZOON AND OVUM COMPARED.

1. *Of the Spermatozoon.*—It has been elsewhere shown† that when the mature spermatic fluid of the Frog has been more than four hours removed from the living body, and mixed with water, at a temperature of 50° FAHR. or but a few degrees higher, it usually has already ceased to have any power to fecundate the egg; and its efficiency has been found to have been diminished in proportion to the length of time it has been obtained and so mixed. Nearly the whole of the observations I have since made have coincided with these results. Yet there have been two marked instances in which fecundation was effected by fluid, which had been obtained, and

* *Loc. cit.* p. 56.

† Philosophical Transactions, 1851, p. 213.

mixed with a small quantity of water, *twenty-four hours* before it was employed. In the first case, the fluid was obtained when the temperature of the atmosphere was 51° Fahr., and afterwards sunk to 49° Fahr., but had again risen to 51° Fahr. at the time of the experiment. When examined with the microscope, immediately before it was used, the fluid still contained a quantity of very active vibratile spermatozoa, and also an abundance of spermatozoal cells, still in course of development. This fact seemed to afford an explanation of the cause of the efficiency of the fluid for so great a length of time, spermatozoa being liberated from the cells during the whole period. Thus the fact of the prolonged fecundity of this fluid seems to confirm, instead of oppose the conclusion arrived at,—that, the more mature the fluid is before its removal from the body, the more efficacious it is, but only for a given extent of time, since it is then composed almost entirely of very active spermatozoa, with but very few cells of development. Spallanzani, as formerly mentioned*, found that the fluid of the fœtid terrestrial Toad (*Bufo calamita*?) at a high temperature of the season, 70° Fahr. to 73° Fahr., at which this species spawns in Italy, had lost its fecundatory influence at the end of six hours†; but that in the temperature of an ice-house, 40° Fahr. it retained its efficacy for twenty-five hours‡. Further, that when the fluid was preserved in the testes of the dead animal, at the temperature of the season mentioned, it became effete in nine or ten hours; but that it continued to be efficacious, when retained in those organs in the temperature of an ice-house for thirty-four hours. Again, he found that the fluid of the green aquatic Frog (*Rana esculenta*), when mixed with a large quantity of water, and placed in an ice-house, the temperature of which he gives at 3½ Reaum. (39·87° Fahr.) retained its fecundatory property for thirty-five hours§. But it must be borne in mind, with regard to these observations, that the species of Toad (*Bufo*), pair later in the season, and at a higher temperature than those of the Frog; and also that the fluid employed by Spallanzani in all his experiments, was obtained by vivisection, either from the testes or the vesicles; and no doubt contained, as in my experiment just mentioned, many undeveloped cells; and, consequently, not being fully matured, retained its influence longer than it would have done, under the same circumstances, in its perfect condition.

But Prevost and Dumas‖, in some experiments made to ascertain the length of time during which the fluid of the species they experimented on,—the *Rana esculenta*,—retains its fecundatory property, observed that it was efficient at *twenty-four* hours, although removed from the body and preserved during that time, in a temperature that varied from 18 to 22 cents., or from 64′ to 71° Fahr.¶ This result, differing so much, in regard to temperature, from the results formerly arrived at by Spallanzani, in regard to the Frog, and from the majority of those since obtained by myself, may, I think, in great part be accounted for in the fact that, like the Toads, the *Rana*

* *Loc. cit.* p. 212. † Dissertations, &c., vol. ii. p. 169. ‡ *Loc. cit.* p. 169. § *Loc. cit.* p. 194.

‖ Annales des Sc. Naturelles, tom. ii. 1824. ¶ *Loc. cit.* p. 140.

esculenta spawns at a higher temperature, and later in the season than the *R. temporaria*, the subject of my observations, and that the fluid employed by PREVOST and DUMAS, was always obtained, as they state, by vivisection, and was expressed from the testes, and necessarily must have been in great part immature, and contained many spermatozoal cells, from which the fecundatory agents were liberated at different periods during the observations, and thus appeared to show that the fluid continues to be efficacious, at a high temperature of the atmosphere, for a much longer period than is really the case. In my own experiments the fluid employed being obtained, without vivisection, by simple compression of the body from the seminal ducts and vesicles, has usually contained only perfectly mature and very active spermatozoa, with but few cells of development. In this condition, as I have stated, it has retained its fecundatory property only during the first three or four hours. Yet, in the exception above mentioned, and also in the second one alluded to, which occurred more recently, the fluid was efficacious at the end of *twenty-four hours* in the first, and *twenty-six hours* in the second instance, the temperature being similar on the two occasions. In both instances the fluid contained a very large quantity of developmental cells, and had been procured from individuals which were not fully prepared for the exercise of their fecundatory function. Further, I may add, that in each of these instances, after the fluid had been supplied to some eggs to test its effect, I found many undeveloped spermatozoal cells adhering to the surface of the eggs, and detected some spermatic bodies in the act of being liberated from them. These facts, then, are confirmatory of the suggestion respecting the results obtained by PREVOST and DUMAS.

The presence of undeveloped cells in the fluid obtained from the living Frog is explicable in two or three ways: first, in that of the animal having but recently been taken from its natural haunts, either early in the season, or at a later period, after a continuance of unusually cold weather and easterly wind, or after the sexes have but recently united; in either of which cases, as also in certain others, in which the individual itself has been late in the development of its reproductive organism, developmental cells are thrown off from the testes together with already liberated spermatozoa, and are usually abundant in the efferential ducts and vesicles. It may thus be seen that in judging of the length of time during which the fluid preserves its fecundatory influence, and consequently, the spermatozoon its vitality, after removal from the body, it is necessary to take into consideration, not only the temperature of the surrounding medium, but also the precise condition of the fluid itself. In perfectly natural impregnation by the Frog there is reason to believe that the fluid contains but very few undeveloped cells, and that it is not employed until it has acquired its full maturity, and that,—as the time of its actual employment is exceedingly brief,—the oviposition of *Rana temporaria* occupying scarcely a single minute, the retention of the impregnating influence and vital force of the spermatozoon may be even for a much shorter space of time than I have mentioned—from three to four hours. But this time may differ in different species of the *Anoura*, and of the higher vertebrata, and may have some relation to the particular economy of each animal.

The general conclusion which seems to be deducible from a comparison of the observations of SPALLANZANI and of PREVOST and DUMAS, with those by myself, in regard to the tail-less Amphibia, is, that—making allowance for the difference of species and habit of the animals experimented on, and for the degree of maturity of the fluid, which there seems reason to think does not attain its perfect condition until it has passed into the efferential ducts and vesicles—the vitality of the spermatozoon, and the duration of its fecundatory power, are in a ratio *inverse* to that of an increase of temperature in the surrounding medium.

2. EVOLUTION OF VITALITY IN THE SPERMATOZOON AND OVUM.

WAGNER and LEUCKARDT have already pointed out* that the power of motion possessed by the spermatozoon is closely connected with the completion of its structure and composition, and is gradually evolved as the development of this body proceeds. This power of motion I shall presently endeavour to show I regard as the visible exponent of its fecundatory force, or form of vitality, and that a similar power is evolved concurrently with this in the spermatozoon in the contents of the egg, the evolution in both being more or less influenced by temperature, according to the species. These facts are well shown both in the Frog and Toad.

Several pairs of Frogs were collected on the 2nd of March, when the temperature of the season was low, and strong easterly winds prevailed, during which Frogs seldom spawn, except in very sheltered places. As one female had passed some eggs, and believing that the others were equally advanced, I selected a pair, which appeared to be the most mature, for experiment. The fluid from the male was obtained with the greatest ease, in full quantity of a white and somewhat opake colour. When examined with the microscope it was found to consist in chief part of developmental cells, containing each its motionless immature spermatozoon. Besides these cells it also included many others less far advanced, and in which I was unable to distinguish this body. In addition to these there were some also from which the spermatozoa were in the act of being liberated, besides many very active spermatozoa already set free. These constituted nearly one third of the whole mass. It was evident from these circumstances that the fluid was immature, and not fitted for experiment. Five days afterwards, March 9th, I again examined fluid from this male, which had been kept since the previous examination, in a separate vessel, so that there was no mistake in its identity. The mean temperature of the room during the interval was 42°·5 FAHR., and the water in which the Frogs were preserved 40° FAHR., but none of the Frogs had spawned. The fluid was then less dense and of a less white colour, and was composed almost entirely of very active and fully developed spermatozoa, with but very few cells from which these bodies had not

* Article "Semen" in Cyclopædia of Anatomy and Physiology, part xxxvi. (January, 1849) vol. iv. p. 504.

been liberated. There were, however, some of which the contents appeared to be perfectly granular, and these I regarded as spermatozoal cells in the earlier stage of development.

The female with which this male had been paired was killed on the 4th, at the time the experiments were to have been commenced. It was then found that the eggs, like the fluid from the male, were immature. The whole of the eggs had escaped from the ovaries into the cavity of the peritoneum among the viscera, and were crowded together in a mass on each side, at the anterior part of the abdomen, and were without any gelatinous covering. A few only had begun to enter the oviduct on the right side of the body, and acquire their envelopes. In most of these eggs the germinal vesicle had already disappeared.

The fluid and eggs of the common Toad (*Bufo vulgaris*) are developed in precisely the same way as in the Frog. On the 30th of March 1851, I examined the bodies of a male and female Toad for the purpose of experiment. The great efferential ducts and vesicles in the male were then entirely empty, no fluid having yet descended into them. But the testes contained a vast quantity of spermatozoal cells, some of which were distended by the spermatozoon, as if in the act of being liberated. But not a single spermatozoon was free, or showed any sign of motion. The whole were still immature, the caudal portion being as yet short and imperfect.

In the female the ovaries were distended with eggs and occupied the whole of the visceral cavity between the intestines. The eggs were of a sooty black colour, with but a slight trace of the grey surface, and appeared to be nearly mature, but had not yet left the ovary. The germinal vesicle of a large size, circular in its outline, and somewhat flattened, still existed in nearly the whole of them.

It is thus evident that the pairing of the sexes, both of Frogs and Toads, takes place before the semen is fully matured, or the eggs have descended into the oviducts, and that there is a near concurrence in the time of maturity in both.

These facts seem to lead to the conclusion, that when, as in some of the fore-mentioned experiments, the male fluid contains a very large proportion of developmental cells, which occasion its white appearance, it is not fully matured for its function; that the first portions of fluid which descend into the efferential vessels are usually immature; and that it is not absolutely necessary that the spermatozoa should be fully formed before the cells escape from the testes. It seems fair then to infer from these facts that the completion of the development of the spermatozoal bodies takes place in the deferential vessels, or in the vesicles, perhaps in both: further, that the quantity of fluid produced, when too great to be contained in the ducts, may become accumulated in the vesicles, to be furnished at the instant it is required; and that there is a near concurrence in the time of maturity in the fluid in the male, and of the eggs in the female, the former being only slightly in advance, in regard to time, of the latter, at the natural period of their encounter; a condition which perhaps may be necessary to the full and healthy fecundation of the whole brood.

3. OF THE OVUM.

The vitality of the eggs of the Frog appears to be of longer duration than that of the fertilizing agent, the spermatozoon. The egg, as formerly shown*, leaves the ovary quickly after, or at the time when its germinal vesicle disappears; but, as experiment has proved, it is not then entirely fitted for its function. It has yet to be enveloped in a thick covering, which it gains in its transit through the oviduct, before it is susceptible of impregnation. Its condition when escaping from the ovary to the oviduct seems to be analogous to that of the spermatozoon when it leaves the testis to enter the efferential duct. Neither the one nor the other has yet attained its full maturity. When the eggs are collected in the dilated or uterine portions of the oviducts they seem to be in a condition parallel to that of the spermatozoa in the lower portion of the efferential ducts and vesicles. When nearly the whole of the eggs are collected in the uterine cavities their maturity is then almost completed; but a period of retention, even in these structures, seems to be necessary to ensure the fertilization of the whole, as some of them,—as I have found by the persistence of the germinal vesicle for a short time, even after the egg has escaped from the ovary into the cavity of the abdomen, as the undeveloped spermatozoal cell passes from the testis into the efferential duct,—are later in their development than others. Further, it may be remembered†, that so far from the contents of the yelk being in a dormant condition at the time of oviposition, as some inquirers have supposed‡, there are changes still going on within it, and which are perceptible to the eye, in the condition of the white surface, for ten or twelve minutes after oviposition; after which they become less and less marked, and soon entirely cease if the egg be not fecundated.

Thus then from a comparison of the state of the fecundatory agent with the body to be fecundated, we might have expected to have found some close coincidence in the retention of their vitality, and this indeed, in a state of nature, seems to be the fact. Spawning seldom or never takes place by the act of the species until each sex has attained its full maturity of function. The eggs are sometimes voluntarily retained by the female Frog, for several hours or days within the uteri, although arrived at maturity, if the proper evolution of spermatozoa in the male be not completed. I have had evidence of this in the fact, that when the union of the sexes has been of long continuance, the female has sometimes passed a very few ova, and then has continued without further oviposition for many hours, sometimes for a day or two, before the mass was expelled, and the function of the sexes consummated. These first extruded eggs have almost always been found to be unimpregnated.

When the eggs have been deposited, the circumstances which affect their fecundation are precisely similar to those which affect the spermatozoa,—the temperature of the surrounding medium, and the length of time during which they remain im-

* Philosophical Transactions, 1851, p. 180. † *Loc. cit.* p. 185. ‡ WAGNER and LEUCKARDT, *loc. cit.*

mersed in water after expulsion from the body. The length of time during which they are susceptible of impregnation has relation to each of these conditions; and these have now acquired additional interest in the establishment of the fact that the spermatozoon penetrates into the envelopes of the eggs, and that, consequently, the rate of expansion of the envelopes has a further direct relation to its function.

The following experiments will illustrate the foregoing observations:—

It may here be remarked, that although, as in my former paper, the date of the experiment is recorded, it is so only for the convenience of reference, and not with any regard to priority of period at which the experiments were made; the details being given only in illustration of the general statements enunciated.

April 2, 1852. Atmosphere 55° FAHR. The first three of the following experiments were made with the eggs and fluid from a pair of Frogs which had been obtained from their natural haunts only three hours before the female was killed, by division of the spinal cord, for the purpose of the experiments.

No. 1. *Eighty-six eggs* passed from the Frog at *one hour and a half* after death, were supplied with fluid from the male with which this female had been paired,—the fluid having been obtained and mixed with water *one hour and a half* before it was employed.

On the seventh day *fifty-two embryos* had been produced from these eggs, and some of them were then escaping from their envelopes.

No. 2. *Thirty-seven eggs*, obtained from the same female, were supplied with a portion of the same fluid as the above, after it had been mixed with water about *two hours and three-quarters*.

Twenty-eight embryos were produced from these eggs at the same time as the above.

No. 3. *Sixty-eight eggs* from the same Frog at *six hours and three-quarters* after death were immersed in the same portion of mixed fluid and water in which the eggs of No. 2 had been fecundated. In this experiment the fluid had been *six hours and three-quarters* mixed with water.

At the end of three days the eggs still had a healthy appearance, but on the seventh day not one had given any evidence of having been impregnated, nor was any embryo ultimately produced. Thus the fluid had lost its fecundatory property, at a temperature of 55° FAHR. between the end of the third and of the seventh hours.

As it was possible that the failure in this experiment was due as much to the eggs as to the fluid employed, it having already been shown that impregnation is sometimes effected with fluid which has been *twenty-four hours* removed from the body,—the following experiments were then made:—

No. 4. *Sixty-nine eggs* obtained from the same female at *one hour and twenty minutes* after death were bathed with fluid procured from another male, and which had been mixed with water *thirty hours* previous. These eggs preserved their regularity of outline for nearly *six hours*, at the end of which time one egg showed traces of partial impregnation, while the yelks of the others were changing form and becoming irregular. But not a single embryo was produced.

No. 5. *Eighty-one eggs* were obtained from another Frog, a few minutes after it had been killed by division of the spinal cord, and a portion of fluid which had been mixed with water *twenty-four hours and a half*, and kept in a temperature which had sunk from 54° FAHR. to 51° FAHR., but at the time the fluid was employed had again risen to 55° FAHR.—was supplied to them.

This fluid on examination by the microscope was found to contain an abundance of spermatozoa, the whole of which were perfectly motionless, so that I was not able to detect even one which gave any sign of vitality. Yet the eggs in this experiment retained their natural healthy appearance for several hours; but not one was impregnated, nor was even a single embryo produced. On the contrary, after thirteen hours, the yelks of some of them assumed a pyriform shape, while those of others became shrivelled and withered.

No. 6. *Thirty-six eggs* were passed from a Frog, which had been killed about *three-quarters of an hour*, and some fluid which had been obtained and mixed with water *forty-four hours* before was shed over them, and pure water was then added.

The envelopes of these eggs became expanded as under perfectly healthy and favourable conditions; but within two hours of the contact of the eggs with the fluid the yelks of the whole became very irregular, shrivelled, and contracted, and assumed an appearance very similar to that which first results from the application of a strong solution of potass. Not one egg retained its spherical shape, but the whole entirely perished; and it may be needless to add, not an embryo was produced.

These facts then support the statement already made, that only while the spermatozoon continues to give evidence of vitality in its power of motion does it exert any fecundatory influence on the egg. They seem, too, to show that the period during which this vital force is retained by the mature spermatozoon after it is passed from the body of the Frog is perhaps shorter even than what I have stated, and that the limit, in regard to time, depends much on physical influences. In addition to this, they seem to show that after all ocular evidence of power in the spermatozoon is lost, this body is inert, until decomposition has commenced, when its material constituents become injurious and destructive.

The following experiments have more direct reference to the vitality of the egg, and appear to indicate that this is much longer retained than that of the spermatozoon, and is less quickly affected by external causes.

April 2, 1851. Atmosphere 55°.

No. 1. *Thirty-eight eggs* were passed from a Frog killed *twenty-four hours and a half* before, and some fluid which had been obtained and mixed with water about *one hour and a quarter* was then added to them.

Segmentation commenced in some of these eggs in about *four hours and a half*, and on the seventh day *eight embryos* had been produced.

No. 2. *Seventy-three eggs* were taken from a Frog which had been killed about *thirty hours*, and were bathed with fluid which had been *one hour and twenty minutes* mixed with water.

At the end of six hours several of these eggs had become irregular, but *one* had certainly been fecundated, as one *embryo* afterwards came to maturity.

No. 3. *One hundred and sixty eggs* from the Frog which supplied those employed in the experiment No. 2, were bathed with fluid from another male, immediately after it had been obtained; but not a single egg was impregnated.

No. 4. *Thirty-seven eggs* were passed from a Frog which had been killed *forty-four hours* before, the temperature of the atmosphere during this period having risen from 50° Fahr. to 55° Fahr. These eggs were bathed with fluid which had been obtained about *three quarters of an hour*. At the end of six hours I found, to my surprise, that one egg had become segmented, but that the yelks of the others were flattened, shrivelled, and irregular, as if they had been affected by the solution of potass, or by decomposing seminal fluid. On the seventh day the egg which had been segmented had produced *an embryo*.

From these results, then, it appears that while the spermatozoon at a mean temperature of the atmosphere of 55° Fahr. usually loses its vitality, and is inert in less than four hours, and but rarely is efficient at a longer period after immersion in water, the egg retains its reproductive property for a very much longer time, especially if not removed from the body of the dead animal, only losing it at twenty-four hours after the death of the parent, and occasionally retaining it for forty hours.

The experiment No. 4, in which the latter fact is shown, is interesting, from the circumstance that the same results are produced by the living vibratile spermatozoon on the yelks of the dead eggs, as the majority of these were, as that which is occasioned by the application of dead and decomposing spermatozoa to the envelopes of the living one, as shown in No. 6 of the preceding set; and further, that these results so closely resemble in appearance the first effects produced on the yelks of living eggs by the application of strong solution of potass; thereby showing not only that powerful endosmic action takes place rapidly through the coverings of the egg, but also that the influence of the spermatozoon on the yelk, whatever may be its precise nature, is direct and immediate.

4. ENDOSMOSIS OF THE EGG IN RELATION TO ITS VITALITY.

It has before been shown that the endosmic action of the envelopes of the egg, on immersion in water, is closely connected with the act of fecundation; and we have now further proof of this in the fact that the vitality of the egg may be preserved for many hours if the envelopes be not brought into contact with water; so that the insusceptibility of the egg to become impregnated, after a lengthened period of immersion, is due to a diminution of the expansive property of the tissue of the envelopes, and the distension of the tissue with fluid, rather than to the loss of vitality in the contents of the yelk.

This view has led me to endeavour to ascertain more precisely than heretofore,—*first*, at what length of time, after immersion in water, the egg usually becomes insusceptible of impregnation; and *next*, to what extent the act of fecundation is accelerated by a given increase of temperature, when the experiments are made with portions of fluid from the same male, and with eggs from the same female, at, as nearly as possible, the same time, all the conditions being similar except in regard to the temperature of the surrounding medium.

With this object, I have recently made two sets of observations consisting each of nine experiments. One set was conducted in a room, the temperature of which was raised artificially to 61° FAHR., and the water in which the eggs were immersed to 60° FAHR.; while the other was carried on in an adjoining room where the temperature was only 55° FAHR., and the water employed 53° FAHR. In order to place the whole of the experiments under as similar conditions as possible, it was necessary that the corresponding experiments in each set should not only be made at about the same time, but should also be supplied with fluid in exactly the same state, and the eggs be immersed in like quantities of water, for the time specified in each experiment, before the addition of the fluid. When the time of immersion had elapsed the water was quickly withdrawn, and the fluid poured over the eggs, and fresh water was immediately supplied to them. As much time was unavoidably occupied in passing the eggs into their respective vessels, and noting the period of immersion, it became necessary to commence the experiments with those which had been longest in water, in order that the fluid supplied to the whole should be as nearly as possible in the same condition, in each experiment, at the moment of its application, and be furnished to the eggs in similar quantities. To ensure this, the fluid was mixed, as soon as it was obtained, with fifteen times its quantity of water, and was employed at about *thirty-five minutes* afterwards. The quantity of this mixed fluid poured over the eggs of each experiment was ten minims by measure, excepting only to the first and second experiments of each set, to which twenty minims were added; but even with this advantage, as was seen, with no more favourable change in the result. After the end of the first hour, the eggs which had remained in the lower temperature were removed to the higher, and the two sets of experiments were then placed side by side to watch the time of commencement of segmentation, as indicatory of the more or less accelerated impregnation. The number of embryos subsequently produced showed the ratio in which the eggs had been fecundated after different periods of immersion*. Each set consisted of nine separate experiments, the time of immersion of the eggs being forty-five, thirty-five, twenty-five, fifteen, ten, seven, five, three, and one or two minutes in each.

At the highest temperature, 61° FAHR. atmosphere, and 60° FAHR. water, a tem-

* For the details of the experiments, see MS. paper in the Archives of the Society.

perature which is always most favourable to fecundation, there were no signs that any change had even been commenced in those eggs which had been immersed for a longer period than thirty-five minutes. But it must be borne in mind that the fluid employed had been thirty-five minutes mixed with water, and consequently had begun to lose some of its efficacy. The results of the corresponding experiments at the lower temperature—Atmosphere 55°, Water 53°—were similar, but less marked.

On comparing the two sets of investigations, it was found that no evidence of fecundation occurred in either of them, when the eggs had been immersed during forty-five minutes before the impregnating fluid was supplied to them; and that only the very earliest symptoms of any influence having been communicated to the eggs, after thirty-five minutes' immersion, occurred in those which were submitted to the higher temperature, in the irregular contraction of the yelks; while in each set, both in the higher and lower temperature, some of the eggs, after twenty-five minutes' immersion, showed unequivocal signs of *partial fecundation* in the formation of the *respiratory chamber*, the result of the contraction and depression of the upper portion of the yelk previous to segmentation. But in both sets there were some eggs which, after fifteen minutes' immersion, had become fecundated, and produced embryos. The relative number of embryos produced after immersion for this and shorter spaces of time was certainly greatest in the higher temperature; since, when the total number of eggs employed in the last six experiments in each of the two sets, in which only any embryos were produced, were compared with the number of embryos, it was found that in the higher temperature there were seventy-eight embryos from five hundred and thirty-six eggs; while in the lower temperature, from six hundred and two eggs, there were only seventy-six embryos.

The period at which segmentation occurred in the fecundated eggs of the two sets corresponded also with the above results. In most of the eggs fecundated in the higher temperature segmentation commenced about ten minutes earlier than in those which had been fecundated, and remained during the first hour in the lower temperature.

These results seem to show, that while the eggs are more rapidly affected, they are also more certainly impregnated, and the embryos produced in greater numbers at a higher than at a lower temperature. But it must be borne in mind that the number of eggs fecundated in these experiments can only be regarded as comparative, and as mere approximations to what takes place in a state of nature, the experiments having been made with fluid which had been removed from the body, and considerably diluted with water, more than half an hour before it was employed, and consequently when its efficacy had been considerably diminished.

These facts, then, go to confirm the previous conclusions, that the organic force or vitality of the spermatozoon is of shorter duration than that of the egg; since, if the results of these experiments be compared with those obtained from former ones, it will be seen that the insusceptibility of the egg depends very much more on the endosmic action of its envelopes after immersion in water, than on any loss of vitality:

further, that the non-fecundation of the egg may depend much more on diminished vitality and inefficiency of the spermatozoon, than on the insusceptibility of the egg itself.

5. QUANTITY OF SPERMATOZOA ESSENTIAL TO FECUNDATION.

On a former occasion* I endeavoured to arrive at some conclusion as to the minimum quantity of spermatozoa necessary to effect impregnation, and to draw some comparison between the results obtained by myself and certain of the more remarkable ones by SPALLANZANI†. It then seemed to me, that although the general results obtained were in accordance with SPALLANZANI'S, yet that a larger proportion of fluid, and consequently a greater quantity of spermatozoa, was required than was supposed by him. I have, therefore, made further investigation on this subject, and have sought to obtain some knowledge of the quantity really necessary to fecundate the egg. PREVOST and DUMAS formerly made a similar attempt‡, and M. QUATREFAGES has recently done the same§. The results of the observations by PREVOST and DUMAS, QUATREFAGES, and myself, although differing greatly in detail, coincide with the general results obtained by SPALLANZANI,—that only an exceedingly small quantity of seminal influence, and consequently only a very limited number of vibratile spermatozoa is necessary to effect impregnation. They agree, too, in the fact that the number of eggs impregnated is always much fewer than that of the spermatozoa supplied to them; and, consequently, that there is reason to think that fecundation is not the result of a single, isolated spermatozoon, although it has not been ascertained what is the number actually required. The course I have pursued in putting this curious question to the test, has been different from that of either of the distinguished naturalists mentioned. It has not been that of diluting the impregnating fluid in a very large quantity of water, and then applying a minute drop of the water to the Frog's egg, as done by SPALLANZANI;—nor has it been that of reckoning the number of spermatozoa on a given surface of a micrometer plate, and then immersing the plate in water with a quantity of eggs, and afterwards observing how many of them became fecundated,—the course followed by PREVOST and DUMAS;—nor has it been by estimating the number of spermatozoa in a given bulk of fluid, and then ascertaining the number of eggs fecundated by them, as done by QUATREFAGES with those of the *Hermella* and *Teredo*; but it has been by direct application of the spermatozoa on the point of a pin to the surface of a single egg in a glass cell; and then observing beneath the microscope the number of spermatozoa deposited from the point of the pin, applied in a similar way as to the egg, on a plate of glass. Yet only approximative results can be arrived at by this, or by the other modes of proceeding, although it is satisfactory to find that the general results obtained in this way are in agreement with those obtained by the observers mentioned; and further, that the mode adopted has led to some additional results, which seem to be of value.

* Philosophical Transactions, 1851, p. 206.

† Dissertations, &c., vol. ii. p. 189.

‡ Annales des Sc. Naturelles, tom. ii. 1824.

§ Annales des Sc. Nat. 3me série, tom. xiii. p. 129.

6. TESTS OF FECUNDATION.

I have given some account in my former paper* of a space, or *chamber*, which is developed between the vitellary envelope of the egg and the upper surface of the yelk, within the first hour and a half, subsequent to the encounter of the spermatozoon with the egg; and I have referred to it in the details of these experiments, as occurring in some of the eggs which had been immersed during *twenty-five minutes* before the fecundatory agent was supplied to them, and as marking their having been affected by its influence. The formation of this chamber is one of the earliest and most certain indications that the egg has been more or less impregnated. The earliest symptom of impregnation occurs somewhat sooner than the formation of the chamber, or within the first half-hour, and consists in a marked increase in the degree of undulatory heaving and sinking of portions of the surface of the yelk, motions which take place slightly in the unimpregnated egg, but are increased in the impregnated; although, even in that, they are not sufficiently permanent to be easily recognized as proofs of fecundation. The irregular contraction and shrivelled appearance of the yelks, noticed in the preceding observations, may, perhaps, be due in part to this cause. The formation of the chamber is the result of the contraction and depression of the upper surface of the yelk, occasioned, as there is reason to think, by changes which are going on within the substance of the upper hemisphere of the yelk, around the embryo vesicle, or its progeny, and which changes appear to be due to the operation of the spermatozoon; as I have never observed the chamber in any egg which has not been more or less fecundated. The fact of the existence of a chamber, therefore, may always be regarded as a *proof of fecundation*. It is commenced sooner or later according to the temperature of the surrounding medium, and to the more or less complete fecundation of the egg. It is usually perceptible in about *one hour*, or *one hour and a quarter* after the impregnating fluid has been supplied to the egg, and is distinctly marked in less than one hour and a half, when the eggs are preserved in a temperature of from 50° to 55° FAHR., and its size is increased during the first three hours. It is commenced in a slight depression in the centre of the upper surface of the yelk, at a point which corresponds to the entrance to the central canal, and it takes place much sooner in eggs which have been completely fecundated, even at moderate temperatures, than in others which have been more sparingly affected. I have seen it begun in a fully impregnated egg, in a temperature of 52° FAHR., at *fifty-three minutes* after encounter with very mature spermatozoa. When it has attained its greatest superficial extent it appears like a clear watch-glass cavity, filled with a perfectly transparent, limpid fluid above the dark upper surface of the egg. After the third hour it becomes slightly further enlarged in its axis from above downwards by the formation of a shallow, funnel-shaped depression on the surface of the yelk, the centre of which is the minute orifice seen by PREVOST and DUMAS, and BAER, and this is continuous with the canal above alluded to. I have never found this chamber

* Philosophical Transactions, 1851, p. 187.

absent from any egg which has produced an embryo. Its formation is preliminary to the subsequent cleavage of the yelk. It thus affords, within a very short space of time, a certain *test of fecundation*, and as such is exceedingly valuable in all experiments by artificial impregnation.

7. PARTIAL, OR INCOMPLETE FECUNDATION.

But although the formation of the chamber gives certain proof that the egg has been influenced by the spermatozoon, it is only the actual segmentation of the yelk, carried through the first four stages, which shows that its fecundation has been completed; since the egg may be only partially influenced, and consequently may not produce an embryo, although the respiratory chamber may be formed, and the early stages of segmentation of the yelk be commenced. When this is the case the change seldom proceeds beyond the second or third stage of cleavage, and even then there is usually some irregularity in its occurrence. Sometimes only the chamber itself is developed, and the process then stops; at other times the chamber is completed and a little *white spherical body* makes its appearance, and is but *just perceptible within the entrance to the central canal*; but more frequently a white spherical body, not more than one-fourth the size of the original germinal vesicle, appears on the *surface of the yelk* within the chamber; at other times there are *two of these white bodies*, each of which is scarcely more than one-half the size of the single one, and sometimes the bodies are of a dark or grey colour. In other cases the yelk begins to be divided in the margins of the canal, but does not proceed with its change; or it may go on, as just stated, to the several stages. In other cases the yelk divides very unequally. These instances of *partial impregnation* or *fecundation* may occasionally be produced by application of exceedingly minute quantities of seminal influence, as some of the following experiments will show.

8. DEFICIENCY OF FECUNDATORY INFLUENCE.

(*a.*) *Pin-point* application of the fluid.—These experiments were commenced in the season of last year (1851), but have been repeated at different periods during the season of the present year.

March 30, 1851. Atmosphere 56° FAHR.—A *single egg* was passed into each of four glass cells, with a small quantity of water, sufficient only to preserve the eggs moist and promote the expansion of their envelopes. Within two minutes afterwards two of the eggs were each touched once with the point of a common-sized pin, which had been so lightly dipped, into a mixture of one part of fecundatory fluid, obtained about one hour and five minutes before, and four parts of water,—as to cover a portion of its surface equal to about one thirtieth of an inch in extent. The two remaining eggs were touched, the first *once* only, and the second *twice*, with the point of a pin slightly curved, and which thus was made to retain a much greater number of spermatozoa in the fluid which adhered to it than the pin with the straight point.

Each cell was then filled with water, so that the egg which it contained was completely immersed. Within the second minute after the eggs had been touched with the loaded pin, there arose from each, at the point of contact, a microscopic bubble of air, the egg which had been twice touched having two bubbles. These were found by subsequent observations to be due to the circumstance of the eggs having been exposed to the air, for a short time before they were touched, and their surface become slightly dried, so that when the fluid was applied to it a small bubble of air became involved and imprisoned in it, before the eggs were immersed in water. This circumstance and its explanation are mentioned simply to show that the occurrence had no relation to the act of fecundation, as at first I had supposed was probable.

I endeavoured to obtain some idea of the number of spermatozoa deposited from the pin point, during its momentary contact with the egg, in these experiments. But the result seemed to give but little hope that either of the experiments would be successful, or that any fecundation could be effected by the application of so minute a quantity of influence, as it was found that the number of spermatozoa deposited on a plate of glass, beneath the microscope, when the pin point was applied to the glass in the same way as to the egg, did not exceed from six to ten or twelve of these bodies. Yet the results of the experiments were watched. Within the first half-hour the eggs were removed from the temperature of 56° FAHR. to one of 65° FAHR. But at the end of ten hours it was found that only in one instance had any fecundation been effected. The grey central spot on the dark surface of the yelk was still present in three of the eggs, and no respiratory chamber had been formed in either of them. But partial fecundation had been effected in the fourth egg, which had been twice touched with the loaded pin point. In this instance the dark surface of the yelk was depressed, and the respiratory chamber had been formed between it and the envelope, and *two spherical opake white bodies* appeared within it, lying side by side on the middle of the surface of the yelk; but no segmentation of this had taken place, or even had been commenced.

One cause of failure, in these first experiments, appears to have been the diminished efficacy of the fluid, which had been obtained and mixed more than one hour before; as it will presently be seen that even a less proportion of the fluid sometimes is effectual when employed immediately it is procured.

As it was late in the season of last year when these experiments with minimum quantities were commenced, the spawning of the frogs being then nearly at an end, I was fortunate in procuring, on the second of April, a single unspawned pair of frogs from their natural haunts, and thus was enabled to repeat the experiments with the fewest disadvantages. This was done within four hours after the frogs were captured. The fluid in this instance was very mature, was obtained in good abundance, and was mixed with an equal quantity of water, and employed in the experiments a few minutes afterwards.

April 2, 1851. Atmosphere 55°.—*Ten eggs* were passed into a large glass cell, and each egg was touched *once* only with the pin point dipped lightly into the fluid for each egg. This experiment was perfectly successful, as the chamber was formed above the yelk in four or five of these eggs, and at the end of five days three of them had produced embryos. In these instances the pin had been allowed to remain in contact with the egg, and the fluid to drain off it for one or two seconds.

Six eggs were also placed in another cell and each was touched with the loaded pin point when the fluid employed had been obtained about a quarter of an hour. *Two* of these eggs became partially fecundated, as shown in the formation of the respiratory chamber. In each of these the yelk became slightly elongated, a symptom which always precedes the first cleavage; yet they did not undergo further alteration, but with the remaining four were abortive.

Twenty-three eggs were included in a large cell as a further experiment, and each egg was touched once with the pin-point loaded each time, when the fluid employed had been one hour and three quarters mixed with water. *Four* of these also had the chamber formed, and segmentation commenced in them, at the end of four hours and a half; but the majority were only *partially fecundated,* as only one of the twenty-three eggs produced *an embryo.* The fact of this production, however, shows, that an exceedingly small amount of influence, even at a long period after the fecundatory fluid has been removed from the body in which it is generated, is sometimes sufficient to occasion the development of an animated being; and that although a plurality of spermatozoa, supplied to the egg, appears always to be necessary to ensure fecundation, yet that the result seems to depend less upon the abolute given number, or numerical relation of these bodies, to the egg, than on the measure of vitality possessed by those which are brought into actual encounter with it, as a very small number appears to be more efficient when employed immediately after their removal from the body, than a much larger number after a prolonged interval.

(*b.*) *Pin-head* application of fluid.—The preceding experiments having been made with the smallest quantity of fluid, and consequently with the fewest spermatozoa that could be applied directly to the egg, it was desirable to increase the number of these bodies without applying them in excess, and for this purpose the head of a very small pin, the smallest size used by insect-collectors, was employed, instead of the point of a larger sized pin. An endeavour was also made, as in the previous trials, to ascertain the number of spermatozoa which the head of this sized pin, dipped lightly into fluid, was likely to be the means of conveying to the egg. It was then found that the loaded pin-head deposited on a plate of glass, in the minute quantity of fluid which adhered to it, from at least fifty to one hundred and fifty spermatozoa; a number which was not only fully sufficient to fecundate each egg to which the pin-head was applied, but also other eggs which might happen to come into contact with these in the same cell and water in which the experiment was made.

Six eggs, placed in a single cell, were each touched once only with the pin-head, which had been dipped into fluid only once for the whole, the time occupied in touching them being about five seconds. Each of these eggs became fertilized, and segmentation commenced in two of them in exactly *five hours*, and in the remaining four in from three to four minutes later. The period at which segmentation commenced having been very carefully watched, and the whole of the eggs having been submitted, during the interval, to the same physical conditions of heat, light, quantity of water and degree of aëration, the result, in regard to the difference of time at which they began to change, seemed to be accounted for only on the assumption that it was due to a greater quantity of influence applied to the first two than to the others; and this opinion has since been more fully borne out by other experiments. The fluid at the time it was employed had been thirty-five minutes mixed with water. On the *fifth* day each of these eggs was producing an *embryo*.

Three eggs were placed in separate cells, and each egg was touched once with the pin-head loaded only once for the three, the fluid employed having been *twenty-three* minutes mixed with water. After *four hours and forty minutes* two of these eggs were undergoing segmentation, and on the fifth day both were producing *embryos*. The third egg was not impregnated.

Twenty-two eggs, in a large cell, were each touched once with the pin-head loaded for each egg when the fluid had been *one hour and forty-seven minutes* mixed with water. Some of the eggs in this experiment were touched on the white surface, others on the dark, and others at the side, or about midway between the white and dark surfaces.

In the same cell with these eggs, but at a little distance from them, were placed eighteen other eggs, which were not touched with the pin-head, but were simply allowed to remain in the same cells with the twenty-two eggs experimented on, and the cell was then filled with pure water. At the end of *four hours and fifty minutes* the chamber had been formed, and segmentation had commenced in each of the twenty-two eggs experimented on. But in addition to these there were also *two* of the eighteen g which had not been experimented on undergoing segmentation. These, during the expansion of their envelopes, had come into contact with some of the fecundated eggs, and had also become fecundated by some of the spermatozoa which had been supplied to them. *Five days* afterwards the whole of these *twenty-four eggs* were producing embryos! so that not only had the quantity of spermatozoa employed been fully sufficient to fecundate the eggs actually touched, but also others present in the same water with them.

In another experiment, made before the preceding, but with some of the same brood of eggs and same sample of fluid, I placed five eggs in a cell, and touched each once, very lightly, with the pin's head loaded once only for the whole. This was when the fluid had been only *eighteen minutes* mixed with water. The object of this experiment was to learn whether there is any difference in the result when the eggs are touched at different parts of their surface. Accordingly the trial was made by touching some eggs on the *dark surface*, others on the under or *white surface*, and others at the sides between the two. At the end of *five hours and seven minutes* the chamber had been formed and segmentation was taking place in each egg, and on the *fifth day* each had produced *an embryo*, thus *appearing to show* that it is of little consequence, with reference to fecundation, as to which part of the egg the fecundatory agent is applied to. But this conclusion is subject to certain conditions, as I shall proceed to explain.

9. PORTION OF THE YELK MOST SUSCEPTIBLE OF FECUNDATION.

On attentively examining the preceding experiments, it seemed difficult to understand how it happened that when similar quantities of the same fluid were applied by the pin-head to different eggs of the same brood, at the same time, and the whole of them placed in exactly similar conditions in regard to heat, light and aëration, that some should become fecundated and others not, if, what *appeared* to be shown in the last detailed experiment,—that the egg can be fecundated equally well by the application of the fecundatory agent to any part of its surface, be strictly correct. There at first seemed to be no other way of accounting for the differences in the results, except on the hypothesis, either that it was due to some insusceptibility in the entire egg itself,—which thus presupposed great imperfection in each brood of eggs, a conclusion at variance with observation,—or that it resulted from the mode in which fecundation was attempted.

During the past season (1852) I have put the question, thus raised to a more rigid test, in order to learn whether the egg really is as susceptible of fecundation in one part of its structure as in another.

In order to obtain some decisive results I repeated my experiments on three separate occasions, with eggs and fluid from three pairs of frogs, at nearly similar temperatures of the atmosphere. On the first occasion, April 5th, the temperature was 54° FAHR., on the second, the following day, it was 55° FAHR., and on the last occasion, April 9th, it was 53° FAHR.

In all the preceding experiments the eggs had been allowed to remain in exactly the same position, at the time of applying the fluid, as that in which they had passed from the body of the frog; so that sometimes the dark, and sometimes the white surface was uppermost, and at other times inclined more or less to either side, and occasionally the two were perfectly horizontal. When the fluid was applied by means of the pin's-head to the part desired, some portion of it necessarily flowed over other parts, if the surface which received it happened to be inclined to one side or the other.

But in my more recent investigations care was taken to place every egg in a perfectly vertical position, before attempting to fecundate it, so that in one set of eggs the central part of the dark surface of the yelks was uppermost, and in another the centre of the white surface. These precautions led to very definite results. In the first experiment, made on the 5th of April, with fluid which had been mixed with water about *one hour and fifty minutes* before, *twelve eggs* contained in a single cell were each twice or thrice freely touched with the loaded pin's-head over the centre of the *white* surface. *Three* of these eggs became *partially* impregnated, as shown in the formation of the chamber in each, and segmentation commenced in one of them, but *no embryo was produced* from either. *Fourteen eggs* in another cell were touched at the same time, and in the same way as the preceding, with the loaded pin's-head, over the centre of the *black* surface of the yelk. *The whole of these eggs produced healthy and well-formed embryos.*

These experiments were repeated on the following day on the eggs of another frog, with fluid which had been obtained only about *twenty* minutes before it was employed. *Fourteen eggs* were each touched *once* only on the centre of the *white surface*, the pin-head being loaded for each egg. But *not one of these eggs became fecundated*, nor was even one of them partially impregnated. In another cell, at the same time with these, *thirteen eggs* were touched on the centre of the *dark surface*, and *four* of them became partially fecundated, but only *two embryos* were produced. The failure in this experiment certainly appeared to be due to the eggs, rather than to the mode of attempting their fecundation.

At the time of making this experiment I placed two eggs, each in a separate cell, with their white surface uppermost, and then filled the cells, the one with nearly *pure* fluid, and the other with equal parts of fluid and water. Each of these eggs became fecundated, and underwent segmentation, although more slowly than usual, but the development of their embryos was not completed, as it became arrested at a particular period, a circumstance which is presently to be explained. Yet these experiments showed that when the egg is completely immersed in fluid its position at the moment has but little reference to the act of fecundation, provided the fecundatory agents are present in good abundance. A further trial was made on the 9th of April, when *six eggs*, placed in one cell, were each *twice* touched on their *white* surface, the pin's-head being loaded each time it was employed. But neither of these eggs produced an *embryo*. *Six eggs*, placed in another cell, were also *twice* touched, like the preceding, on their *dark* surface. The whole of these eggs became fecundated, but only *five embryos* were sufficiently matured as to leave their envelopes. The fluid employed in these two experiments had been mixed with water about *fifty-five minutes* before it was applied to the eggs, and consequently was becoming slightly deteriorated.

The conclusion which seems to be deducible from a comparison of these experiments is,—that when the egg of the frog is completely immersed in water charged with the fecundatory agents, or to which these are supplied quickly after the immersion of the egg, as in the natural fecundation of the species, the actual position of the egg at the moment of its encounter with the spermatozoon, has but little reference to the act of fecundation, every part of its surface having then an equal chance of becoming affected. In like manner when the fluid is applied directly to any given part of the surface artificially, when the egg is exposed in an empty cell, and the fluid is allowed to gravitate, before the egg is immersed in water, it is of little consequence what part is first touched with it, since it may spread over the surface to the most susceptible portion; there being in reality one part of the egg which is more susceptible than another. The extremes of susceptibility and insusceptibility appear to be, on the one hand the *centre* of the *dark*, or future upper surface of the yelk, and on the other that of the *white* or under surface*.

That these conclusions, deduced from experiment, are correct, seems to be shown in the fact, formerly pointed out†, that the germinal vesicle originally, and its progeny subsequently, at the time of fecundation, occupies the centre, not of the entire egg itself, but of the upper or dark hemisphere of the yelk; and this possibly may be the structure to be fecundated.

These facts deserve consideration in connexion with that of the first changes in the yelk taking place in this hemisphere; first, in the shrinking of the yelk at this part; the formation of the chamber above it, subsequent to fecundation; the *existence of a canal in its centre*, in the margin of which the cleavage of the yelk commences; and, lastly, in this being the portion of the yelk in which the first lineaments of the embryo become apparent.

In connexion with these remarks, it is but just to mention that a former observer, Dr. Martin Barry, has distinctly referred to the changed germinal vesicle as being the structure in which the future embryo, in the Mammalia, originates‡.

10. EXCESS OF FECUNDATORY INFLUENCE.

At the time of commencing the foregoing experiments with small quantities of fluid, I made others of a directly opposite character with an excess of the same. *Four eggs* were placed each in a separate cell, and within one minute afterwards the cells were filled with a mixture composed of equal parts of seminal fluid and water. This was half an hour after the fluid had been obtained. I had expected, from the vast quantity of spermatozoa present in the fluid, that segmentation of the yelk in these eggs would have taken place very quickly; but to my great surprise, although the eggs were preserved in a temperature of 60° Fahr. no segmentation whatever had commenced in them at the expiration of twelve hours! at which time an abundance of spermatozoa adhered to every part of the surface of their envelopes. But no chamber was formed in either of these eggs, as neither of them had been fecundated.

* During the present year I have most fully confirmed these views by numerous experiments, and have proved to my full satisfaction that the *white surface* of the egg is least, and perhaps not at all, susceptible; while of the *dark surface, that the centre, which is occupied by the canal, and its contents*, is the fertilizable part.—G. N., April 18, 1853.

† Philosophical Transactions, 1851, p. 176.

‡ Ibid. 1840.

This result appeared to be far more inexplicable than any I had previously noticed. A somewhat similar one had occurred once before, but I had attributed that to some imperfection in the eggs employed, an explanation which did not apply in this case, as eggs from the same female had all been fecundated in other experiments.

Four eggs were then passed from another frog, immediately after death, each into a separate cell, which was filled with a mixture of two parts of water and one of seminal fluid, obtained only a few minutes before it was employed. Two of the eggs were allowed to remain in this mixture in their cells, but the others, after a few minutes' immersion, were well-washed, and supplied with pure water, and the whole were then placed in a temperature of 60° FAHR. At the end of four hours I found the surface of the first two eggs covered with an abundance of spermatozoa; and a chamber had been commenced, and the upper surface of the yelk was depressed, in each, thus showing that some degree of fecundation had been effected, but no segmentation had taken place in either of these eggs at the end of *five hours and a half*, although the temperature of the room had been raised during the interval from 60° FAHR. to 67° FAHR. The sixth hour was almost completed before either of these yelks gave any indication of the probability of this change. Segmentation then commenced, but in a very irregular manner. In one egg the division was not in the centre, but at the side of the yelk, and proceeded no further than to about one-half the extent across the surface. In the other the surface of the yelk within the chamber merely became sulcated twice in the same direction. Neither of these eggs proceeded further with their changes, and of course no embryos were produced. The two remaining eggs did not undergo any change whatever, and consequently had not been impregnated.

Although these failures were still attributed to some immaturity of the eggs, I began to suspect that they might be occasioned by the immersion, or *smothering of the eggs* in fluid of too great a density; yet this suspicion appeared to be so hypothetical as scarcely to merit consideration, when it was remembered that on a former occasion when I had covered eggs with a solution of gum* and afterwards applied the impregnating fluid to them, some of them became fecundated. I was well aware that SPALLANZANI had found in his experiments that the greatest number of frogs' eggs became fecundated when the fluid employed was very much diluted with water†; but I did not then know that M. QUATREFAGES had made an observation‡ which completely coincided with the result obtained by myself. This acute naturalist found that when he employed the pure semen of the sea-worms, the *Hermella* and the *Teredo*, not a single egg was fecundated, and that when he mixed it with only a small quantity of water, so as to obtain " un liquide très opalin," that then only a very small number of eggs showed any traces of fecundation. He ascribed these results simply

* Philosophical Transactions, 1851, p. 236.

† Dissertations, vol. ii. p. 190.

‡ Annales des Sc. Naturelles, 3me Série, tom. xiii. p. 129.

to the circumstance of the fluid being too thick; but as we shall presently find, this, perhaps, may admit of a different interpretation.

My first object having been to learn how small a number of spermatozoa is sufficient to fecundate an egg, it may well be supposed that the facts just mentioned were exceedingly puzzling, and they were rendered still more so by the circumstance that a different result occurred with three other eggs from the same frog, employed at the same time as those in the preceding experiments. *Two* of these eggs, placed like the others, each in a separate cell, in a mixture of one part of seminal fluid to six parts of water, *two hours and twenty minutes* after the fluid had been obtained, became fecundated. In each of these the chamber was formed above the yelk, segmentation took place, and proceeded regularly, and each egg subsequently produced an embryo. In the third egg the chamber was formed, but the yelk became irregular in outline, and did not undergo segmentation. It had merely three parallel depressions on its upper surface, after which it became strangely altered in form, and was abortive. The cause of this failure seemed to be the same as in the preceding experiments,—an excess of spermatozoa in encounter with the egg, as subsequent experiments tended to show.

Seven eggs were passed into one large cell, and one egg into each of *two* separate cells, and the cells were then filled with a mixture of fluid and water, which was so dense as to appear turbid and opaline. This appearance was in part due to the fluid containing a large proportion of developmental cells. The result in each instance was very marked. The fluid at the time it was employed had been mixed with water about *one hour and five minutes*, and the temperature at the time the experiment was made was 56° FAHR., but the eggs were soon afterwards removed to a temperature of 62° FAHR. At the end of *five hours* the yelks of two of the eggs had become partially segmented, but the changes were continued only in one. The remaining eggs had been more injuriously affected. Their yelks had become *shrivelled and contracted, and resembled withered apples.* In this experiment it was evident that the effect had been occasioned by an excess of impregnating influence, but in what way it was not then easy to understand. The yelks of these eggs looked very like those of eggs which had been in contact with solution of potass, or of those to which stale and decomposing fluid had been supplied.

But it was possible that this change of form might be due to some injury sustained by the eggs in their removal from the body of the frog, or to some accidental cause in their removal to the cells, before fecundation. In order, therefore, to remove all doubt in this respect, another trial was made with two other eggs, which were so carefully passed into cells as to avoid all suspicion of injury to them, and the cells were then filled with mixed fluid like the preceding. The yelks of each of these eggs also changed their form, and one of them became irregularly contracted and pear-shaped. Of the whole of the eggs employed in these experiments only *one* underwent segmentation, and produced an *embryo.* The others were not fecundated. It was thus evident that an excess of fecundatory fluid is as unfavourable to the production of the embryo as a deficiency.

11. DEFICIENCY AND EXCESS OF FECUNDATORY INFLUENCE COMPARED.

The results of these experiments with the fecundatory agent in excess require to be further explained, and also to be compared with those obtained with deficiency of the agent, in order that the nature of both may be properly understood. If this be done it will be seen that fecundation of the frog's egg depends very much on the *physical influence of temperature*; and that this has a close relation with the intensity of the force, or the degree of vitality evolved in the spermatozoon, at the time of its application to the egg; and probably also with the influence which the spermatozoon supplies through a greater or less amount of material substance to the body to be fecundated, at the time of its encounter, as in the following experiments made in March 1852.

The experiments by pin-point application of fluid show that only a very small amount of influence is actually required to set up those changes in the yelk which result in the formation of the embryo; and other observations have convinced me that the result is the more certain at a slight increase than at ever so slight a diminution of temperature. The effect of the application of minimum quantities is most certain when the influence has only very recently been obtained, and consequently while still endowed with its greatest degree of vitality. But even at that time a given amount is required, and if such be not supplied the result is incomplete, and partial fecundation only is effected. First then in regard to minimum quantities.

Six eggs were placed each in a separate cell, and three of these were touched *once* only with the pin-head loaded with fluid obtained about twenty-five minutes before, while the other three were each touched once with the fine pin-point loaded in like manner. Yet although there was only an interval of one minute between the application of fluid to the two sets of eggs, two of the former by pin-head application had the chamber developed, and segmentation commenced at the end of *three hours and forty minutes*; while one egg only of the three supplied by pin-point application began to be segmented in *three hours and forty-eight minutes*, although the whole were placed under precisely similar conditions in regard to light, heat, quantity of water and degree of aëration. The third egg of the first set was not impregnated, neither were the two remaining ones of the second set. Thus there was a difference between the two sets of eggs, not only in the extent to which they were fecundated, but also in the *rate of the accomplishment of fecundation* to the extent of *eight minutes*, in about three hours and three-quarters. There was also a perceptible difference of time in the formation of the respiratory chamber of the two, after the first hour, as was remarked by a distinguished physiologist who was present with me when these comparative trials were made. The two eggs of the first set, and the single egg of the second, which underwent segmentation, afterwards produced embryos; the remaining eggs were abortive.

A second comparative trial was made on the 15th of March. *Three eggs* in separate cells were each touched once with the fine pin-point loaded with fluid which had been obtained only twenty minutes before. The temperature at the time of the experiment was 56° FAHR., but the eggs were removed within a few minutes afterwards to a higher temperature, which was gradually increased to 64°·5 FAHR. Segmentation took place in one of these eggs in *three hours and thirty-nine minutes*, and this egg afterwards produced an *embryo*. But the remaining two were not impregnated. Three other eggs in separate cells, were, at the same time as the preceding, each once touched with the loaded pin-head, and two out of the three underwent segmentation at the end of *three hours and thirty-three minutes*. In a further experiment, made at the same time, *sixteen eggs* included in one cell were touched with the loaded pin's-head when the fluid had been obtained about half an hour before. *Four* of these eggs underwent segmentation in *three hours and fifty-three minutes*, and afterwards produced *embryos*, the remaining eggs being unproductive. Another experiment was made with *nineteen eggs* in a single cell, by pin-head application of the fluid, and eleven of these were fecundated and began to be segmented in *three hours and forty-two minutes*, and ultimately produced embryos, the remaining eight being sterile. Some of these failures were probably due to the circumstance that the fecundatory fluid was not applied to the most susceptible part of the egg, yet the general results sufficiently show not only that fecundation is more surely and quickly effected at a moderately high than at a low temperature, but also that it is so, more or less certainly and quickly, in proportion to the amount of influence supplied.

These facts lead us to some further consideration of the results of the application of definite quantities of the fecundatory agent to the egg, as affecting the development of the embryo, and *possibly also as influencing both the evolution of its physical structure and psychical condition.* I have little hesitation in believing that whatever be the precise nature of the influence communicated by the fecundatory agent to the egg, that it is only after full and complete impregnation that perfectly normal and healthy embryos are formed, and ultimately attain to the maturity of the species. And yet we have already seen that when there is great *excess* of the agent, either that no embryo is produced, or that its development is not completed.

A few days after making the last-mentioned experiments I repeated them with another object in view, and then obtained a result which had occurred on previous occasions, but being less distinctly marked had been almost overlooked.

About *twenty eggs* had been placed in separate cells, and fecundation attempted by pin-head application of fluid which had been obtained from a partially exhausted and debilitated male, at the end of the season, and which had been employed from want of a more healthy individual. The fluid, when examined by the microscope, was found to contain very many perfectly motionless spermatozoa, besides a large quantity of cells. The inefficiency of this male was inferred from the circumstance that the female was prepared to spawn, as was proved by her having passed a few eggs; but these remained unimpregnated, and she still retained the remainder within her for nearly a day, at which time the male was removed for the experiment. The employment of this male was thus accidental, it being the only one I then possessed.

The fluid was applied to the eggs within a quarter of an hour of its being obtained, and the eggs were at once placed in a temperature of 61° FAHR. I remarked that development appeared to go on more slowly than usual, although most of the eggs underwent segmentation, and *fifteen* of them produced embryos, which were placed together in one vessel. A few days afterwards I was surprised to find that two of these embryos were greatly malformed. One had a short falcated tail narrowed at its base, where it was not broader than at its extremity, and the tail of the other was scarcely more than a short narrow stump, so that the embryo had much difficulty of locomotion. A few days later three other of the embryos became distorted, and one of them died much altered in form at the period of absorption of the external branchiæ.

These circumstances are recorded merely as simple occurrences which point to the necessity for further experiment on this curious and important subject, and not as results from which any very positive conclusions can be deduced. Nevertheless, I may add, that I think I have before observed that embryos which have been the result of fecundation with very small quantities of fluid, are usually smaller than others which are produced from full and natural impregnation.

The effect produced by immersion of the egg in pure fluid is as curious as the results above mentioned obtained from minimum quantities, and at first it is less easily to be understood. I can find no more expressive term by which to indicate it than that of *smothering*, and this, perhaps, may be found to be a more literally correct expression of the fact than at first is apparent. It has already been shown that free aëration is most essential to the development of the embryo, and I believe it is equally so, in a minor degree, to fecundation. Thus while the egg may be fecundated in a dense fluid, composed of equal parts of seminal fluid and water, its changes usually take place much more slowly than when the proportion of water is considerably increased; while if it be immersed in pure fluid, and be allowed to remain in that for some hours, it may still be fecundated, and its changes be commenced, but these will not proceed to the full development of the embryo. If the egg when immersed in pure fluid be retained in a moderately low temperature, then the development of the embryo may go on slowly for a time, but with less rapidity than when the egg is freely aërated in pure water. But if, on the other hand, it be retained in seminal fluid, at a high temperature, then it usually happens that although the respiratory chamber be formed, as the result of fecundation, segmentation will proceed slowly, or irregularly, and the embryo, which is begun to be formed, perish. That want of proper aëration is in part the cause of this infertility seems to be shown in the following experiments.

April 5, 1852. Atmosphere 54°.—*Two eggs* were placed in separate cells, which were immediately filled with nearly pure fluid. In about one hour afterwards the respiratory chamber was beginning to be formed in both these eggs, but its progress was slow. One egg was then washed as completely as possible with a powerful jet of water, thrown upon it by a syringe, and the cell was then filled with pure water, while the cell with the other egg was replenished with more of the fecundatory mixture. After the addition of fresh water to the first egg its envelopes began to expand, and it was then seen that the substance of these coverings was *filled with spermatozoa, lying in every direction and so crowded around the envelope which immediately invests the yelk, as to render the whole semi-opake.* The envelopes of the second egg were much more opake than those of the first. Yet the respiratory chamber was developed, after a lengthened interval, in both, but earlier in the egg which had been washed and placed in water than in that which remained in the fluid. At the expiration of twelve hours the second egg was washed and placed in pure water like the first. Both eggs were preserved, from within an hour or two after their first encounter with seminal fluid, for four days in a temperature which ranged from 59° Fahr. to 64° Fahr. At the end of three days and a quarter, or about seventy-eight hours, in this temperature the development of the embryo had advanced so far in the first as to the union of the laminæ dorsales, and the narrowing and elongation of the body, a stage of development at which the function of aëration is becoming more energetic in the evolution of ciliary action. At this stage of formation the development of the embryo became arrested, although the egg was properly supplied with water, and from this period it decayed. The second egg, which had remained twelve hours in the mixed fluid before it was supplied with fresh water, perished much earlier, and before there were any distinct evidences of the formation of an embryo.

On other occasions I have found that the egg does not become fertilized at all, but the yelk contracts irregularly, and resembles the yelk of eggs affected by solution of potass.

These are the usual results of immersion in undiluted fluid, which, contrary to what occurs with minimum quantities, appears to be most prejudicial to the egg, in its pure state, *when the temperature of the surrounding medium is much increased*; so that the cause of failure in these cases *of excess* may reasonably be attributed, in great part, to a smothering of the egg, through impeded aëration; accelerated, perhaps, by some chemical change, through increased temperature, in the remains of the spermatozoa, with which the envelopes of the egg are crowded in their interior.

12. THE MOTION OF THE SPERMATOZOON IN RELATION TO ITS FUNCTION.

The possibility of the fecundatory function being in some way connected with the motion exhibited by the contents of the pollen in plants, and with that of the spermatozoa in animals, has not escaped the consideration of the best observers. Mr. Brown, the most distinguished of botanists, many years ago, made this the subject of his particular investigation *; and M. BRONGNIART †, SCHLEIDEN ‡, NÄGELI §, GRIFFITH ‖, and more recently SUMINSKI ¶ and HENFREY ** have done the same. But although neither of these able inquirers succeeded by direct experiment in proving that the motion of the particles of plants is essential to the *act* of impregnation, M. HOFMEISTER ††, and very recently also Mr. HENFREY ‡‡, have noticed facts in regard to that of the spermatozoid filaments discovered by NÄGELI and SUMINSKI, in the Cryptogamia, which seem to show that, in plants, it is of great importance to the function of these bodies. The motion of the spermatozoon in animals has equally attracted the attention of zoologists. PREVOST and DUMAS §§, as already stated, SIEBOLD, MÜLLER, WAGNER, KÖLLIKER, BISCHOFF, QUATREFAGES, and especially WAGNER and LEUCKARDT, have studied it attentively; but so intricate is the inquiry concerning its nature and import, that the last two authors dismiss the consideration of the question without arriving at any conclusion, and state that they do not venture to decide ‖‖. Heretofore I regarded impregnation as being *commenced* by transmission from the spermatozoon on the surface of the egg, to the contents in the interior, of some influence characterized by motion. But I have regarded this motion as being only the visible indication of a *peculiar force*, or *form of vitality*, in the impregnating agent, the spermatozoon, by which it is destined to arrive at, and is to expend on the object to be fecundated, and the effect of which is to *strengthen*, to *augment*, and possibly also to *modify* the nature of the formative changes, which are going on in the yet unimpregnated egg, *per se*; but which will subside, and soon entirely cease, if not reinforced through the agency of the spermatozoon. Nevertheless, I have not been prepared to assent to the view that simple contact of the spermatozoon, *even with the vitelline membrane*, is sufficient to complete the changes which result in the formation of the embryo (see p. 233). The powerful endosmic action of the envelopes of the ovum, at the time of oviposition, is opposed to this conclusion; since, if simple contact

* A brief account of microscopical observations, made in the months of June, July and August, 1827, on the particles contained in the pollen of plants, and on the general existence of active molecules in organic and inorganic bodies, by ROBERT BROWN, F.R.S., etc., 8vo. July, 1828; also additional remarks on active molecules. (*Id.*) July, 1829.

† Recherches sur la génération et le développement de l'embryon dans les Végétaux Phanérogames (Notes). Annales des Sciences Nat. tom. ix. 1828.

‡ Grundzüge der wissenschaftliche Botanik.

§ SCHLEIDEN und NÄGELI'S Zeitschr. für Wiss.-Botanik; Heft i. 168; Zurich, 1844.

‖ Transactions of the Linnean Society of London, vol. xx.

¶ Zur Entwickelungsgeschichte der Farrnkräuter, 4to. Berlin, 1848.

** On the development of the ovule in *Orchis morio*, Transactions of the Linnean Society of London, vol. xxi., and Proceedings, vol. ii. p. 27, April 8, 1849.

†† Untersuchung des Vorganges bei der Befruchtung der Œnothereen, Botanische Zeitung, v. 785, 1847.

‡‡ Annals and Magazine of Natural History, vol. ix. June, 1852; Transactions of Linnean Society, vol. xxi. Part II.; also Proceedings, June 17, 1852, vol. ii.

§§ Annales des Sciences Naturelles, tom. ii. 1824.

‖‖ Article "Semen," Cyclopædia of Anatomy and Physiology, vol. iv. par. xxxiv. p. 508.

or encounter of the spermatozoon with the ovum were sufficient, it might be expected, as has been well remarked*, that the influence of the spermatozoon of any animal of the same class would be competent to effect the impregnation of any species.

Yet all the phenomena connected with the origin and death of the spermatozoon seem to be in accordance with the view, that its *motion* is essential to its function. Whatever be the relation of this motion to its peculiar faculty, it is evident that motion is intimately *associated with, and dependent on, its material composition, and structural development.* In the Frog, the spermatozoa are usually completed very early in the season, when the animals begin to emerge from their hybernacula, in the beginning or middle of February; but, as we have seen, they only begin to pass into the efferential ducts from the testicle, at the time when the pairing of the sexes is commenced; from which time, to that of spawning, a period of from ten days to a fortnight or three weeks, according to the temperature of the season, they are more fully matured, and acquire greater vibratory power, and are collected in the vesiculæ seminales for expulsion at the instant after oviposition. In the Toad the spermatozoa are developed at a later period of the season, but at a relatively corresponding period in the life of the animal. The male Toad, like that of the Frog, usually emerges from its hiding-place a few days, or a week or two earlier than the female. I have taken the males in the middle and at the end of March, at which time I have not been able to detect any seminal fluid in their vesiculæ seminales; yet they are then exceedingly salacious and disposed to pair, and I have sometimes found the contents of the reproductive organs in a similar state, even after the sexes have been for two or three days in union. The testicles, nevertheless, are then filled with an abundance of spermatozoal cells in the course of development, and also with a great quantity of spermatozoa, each still included in its vesicle of development; but as yet immature, *motionless*, and with only a very short caudal extension from its thick cylindrical body. Besides these there are usually a few spermatozoa, more matured than the rest, *which exhibit movements while still retained within their cells*, which are enlarged, and from which they are soon to be liberated.

Thus motive power in the spermatozoon is coincident with the completion of its structure and composition, and as such, may fairly be regarded as *essential to its function.* In the Frog the motion of the spermatozoon is most intense and persistent at the full period of connubiality. On the other hand, I have constantly noticed in all my experiments on artificial impregnation, that where impregnation has not been effected, all the conditions being favourable to it, or when I have found by trial that the male fluid *has ceased to be efficient*, that then nearly the whole, or perhaps all of the spermatozoa, have been perfectly motionless, and apparently dead. This is also the condition in which I have usually found the few spermatozoa which are retained in the reproductive organs a week or two after pairing, when the male Frog and Toad may be regarded as in the state of aged individuals, the season of reproduction

* Wagner and Leuckardt, *loc. cit.* p. 508.

having passed, and the function being fulfilled. Thus too, I have noticed that, when from accident, but more especially when from reduction of the temperature in the surrounding medium, the season of spawning has been greatly retarded, the impregnating power of the male is much diminished, and perhaps is almost exhausted, through constant shedding of the spermatic fluid, which, I have found, often takes place when the oviposition of the female is delayed, and the individuals are disturbed or interfered with. The female is then forsaken by her partner, and when this occurs it rarely happens that the connubial intercourse of these two individuals is recommenced. When this separation has taken place, there is usually but a small quantity of fluid remaining in the male organs, and even in that, the number of spermatozoa is considerably diminished, and *their power of motion is exceedingly feeble*; while the quantity of molecules and cells is increased. When several days, or a week or two have elapsed, there are not only fewer spermatozoa, but those which remain are much more feeble in action. This is exactly what occurs also in the Toad. On the sixth of June I found that the testes and efferential ducts in a male Toad, which had been kept from pairing during the whole season, were still filled with spermatozoa, together with a very small quantity of liquor seminis with active molecules moving in it; but that, though the spermatozoa were in full abundance, nearly the whole of them were entirely motionless, while the motions of the few which still gave evidences of vitality, were exceedingly feeble, whether the spermatozoa were examined simply in the fluid portion of the semen, or whether they were mixed with water, in which, as is well known, the motions are always at first greatly increased.

A similar reduction in the number of the spermatozoa and diminution of their motive power, appears to exist in animals which have become exhausted through long confinement or want of food; at least, if we may so judge, from a few observations on the Tritons. A *Triton palustris*, which had been captured on the seventh of May, and accidentally confined without food till the sixth of June, was examined immediately after death. The efferential ducts were well-filled with spermatozoa contained in a distinctly perceptible quantity of liquor seminis. When the spermatozoa were examined, without the addition of water, their motions were regular, but apparently very much slower than usual, being uniform and undulating, without that peculiar rapid ciliary action of the spirally twisted tail, which is so constantly referred to as characteristic of the spermatozoa of the Tritons and of some other Amphibia. When water was added, the motions were immediately accelerated, and the tail, which before was merely flexed, and almost longitudinally extended from the body, became folded and entwined around it, and its rapid ciliary movements were commenced. But these gradually subsided within a very few minutes. In another specimen, which had been captured at the same time as the preceding, and confined under similar circumstances, but which, at the time of examination, had been already dead for more than twenty-four hours, scarcely any spermatozoa remained in the testes, or in the efferential ducts. There was a great quantity of cells, with granular nuclei, in the body of each testis, but scarcely a single spermatozoon. The few spermatozoa which remained were most of them dead, and decomposition appeared to have commenced in them, and there was a large proportion of active molecules. Amongst these were one or two spermatozoa still in action, and the molecules were accumulated around them by the attraction of the current in the fluid, induced by the ciliary action of the tail, and were propelled onwards, frequently in a spiral direction, from the base of the tail to the anterior extremity of the body; thus showing that the motions around the spermatozoon are due to the action of the filamentous tail, flexed upon, and twisted spirally round the body; in accordance with the observation of WAGNER and LEUCKARDT.

But it has been remarked, in opposition to the view, that power of motion in the spermatozoon is essential to its function, that motion is not detected, or but very faintly, in many instances in which the spermatozoa are united into simple and uniform cords, as they often are in the deferential vessels of insects; and that then only a slight waving, or trembling of the mass, the consequence of hygroscopic conditions induced by the fluid around, is, under such circumstances, observed. An aggregation of the spermatozoa into cords takes place in the Tritons as well as in insects. When the semen is expelled by the Triton at the time of conjugation, it is composed almost entirely of white cord-like, flocculent masses of spermatozoa, which, at that moment, exhibit only the faint undulatory motion noticed in them while still within the deferential vessels. Yet, immediately after these masses are ejected from the vessels, and are in contact with water, at the moment of being received into the cloaca of the female, as I shall hereafter have occasion to show,—the motions of the spermatozoa are not only greatly increased, but that peculiar vibratory ciliary action of the tail, which, while the spermatozoa are still within the seminal ducts is very indistinct, and perhaps can scarcely be said to occur—is immediately set up, and the motion of the body of the spermatozoon is changed from the undulatory action before observed, to the jerking or watch-spring movement pointed out by observers, and thought to be the motion peculiar to the spermatozoon of this class of animals.

Thus then, the fact of only a slight wavy motion of the spermatozoon being perceived in some cases, ceases to be of importance, as an argument against the view that motion is closely connected with the function, since it is evident that the true spermatic force is not set up until the spermatozoa have become more or less separated from each other, through their residence in some fluid vehicle. The movements of the spermatozoon in insects are very similar to those of the spermatozoon of the Triton, and the circumstances which affect both are in full accordance with the fact of the existence of, and the necessity for, a distinct liquor seminis, as part of the normal composition of the seminal fluid, in the Mammalia; while this portion of the fluid is not needed, and consequently is almost entirely absent in the Amphibia, Fishes and most Invertebrata. A comparison of the facts in the two divisions of animals seems to point to the true nature and use of the liquor seminis, as being that of a mere vehicle for the ready transmission of the spermatozoon to the ova, and as allowing greater freedom of motion to these bodies.

It is true, as Wagner and Leuckardt have stated, that but little motion can be observed in the spermatozoa of Insects when they are united into cords; yet this seems to be due rather to their union into these masses, and to have reference to their mode of ejection from the body of the male, as in the Triton, and is known to take place during their collection in the deferential vessels, than to any real absence of a power of motion in them. This is not their condition when brought into contact with the egg at the time of impregnation. When the fluid of the male insect, during pairing, is passed into the spermatheca of the female, where it may be destined to remain for an indefinite time, the spermatozoa compose the chief portion of it, as in the Tritons; but, during their residence in the spermatheca, the spermatozoa are mixed with a fluid, which is supplied by a gland attached to that organ, and which becomes to them a vehicle like the liquor seminis, and allows of their separation and independent motion, and thus appears to answer the purpose of the true liquor seminis in the Mammalia. At the period when the ova are descending from the ovaries, the movements of the spermatozoa become more distinct; and when these bodies are brought into contact with the egg, as it passes the outlet of the spermatheca, they become more isolated, and their movements more intense, as I have seen in the case of the Orthoptera. From these circumstances I am led to believe that some degree of motion will ultimately be observed to mark the perfect condition of the impregnating agent in all animals. The facts already mentioned coincide with this view, and point to the probability that the degree of impregnating force in each individual may perhaps be indicated relatively by the degree or intensity of motion in the spermatozoon, and the duration of this force by the length of time which the spermatozoon continues in motion.

The established fact, that a difference in the structural conformation of a body is the invariable result of a difference in the relations, proportions, and composition of its material constituents, has always afforded reason for presuming that some material influence may be transferred from the substance of the spermatozoon to the contents of the ovum, at the time of impregnation. The function of impregnation appears to be one of definite relations and proportions. Thus we have seen, when only a few spermatozoa were applied to the ovum on the point of a pin, that full impregnation was but rarely effected. In most instances of such limited application, the yelk underwent only partial segmentation, and its changes were then gradually arrested, and no embryo was produced. If, however, the pin point which had been charged with spermatozoa, instead of being applied to the ovum for an instant only, was allowed to remain in contact with the egg for a second or two, and thus by capillary attraction became drained of the spermatozoa which adhered to it, and, as a consequence, thus made to deposit a greater number of these bodies, which were not afterwards attempted to be removed or destroyed, then impregnation was sometimes completed, and an embryo became developed. But if, instead of the spermatozoa being thus supplied in a minimum quantity from the *point* of a pin, they were supplied in greater quantity from the *head*, then impregnation was almost always effected; and this result was rendered the more certain by allowing the head to be drained of the adhering bodies for a moment or two, as from the point, by which a much larger quantity of the impregnating agents was deposited. In these instances it was rare that impregnation was not effected. These results appear to show, that, whatever may be the precise quantity necessary to effect healthful impregnation, it has some definite relation to the effect to be produced in the contents of the ovum—that a definite quantity of spermatozoa, or spermatic influence is required to fecundate, and that the perfection of fecundation has relation to the degree of impregnating influence. They seem to show, also, that fecundation is not the simple result of the penetration into the egg of *a single isolated spermatozoon*, but probably of some definite number of these bodies, or of a definite amount of influence supplied through their encounter. We are thus led to perceive that the same law of relation between cause and effect,—between a definite amount of influence expended, and definite results,—which has long constituted the basis of our knowledge respecting chemical affinities, and which is now being demonstrated, as that also of the other forces of inorganic nature, may equally pervade and control these material combinations among the organic affinities. Further, the observations, now mentioned, seem to put to rest the last remaining question respecting the independent animality of the spermatic bodies, and to show that these, like the cilia, are mere elementary parts of the adult male organization,—as many physiologists believe*,—as the contents of the ova are of that of the female.

All my experiments on the egg of the Amphibia serve to show, that even with a definite quantity or number of spermatozoa, no impregnation is effected, if, before the spermatozoa are brought into contact with the egg, they have all ceased to exhibit that motion which constitutes their marked characteristic. Thus then, it seems that, independent of the important question as to whether these bodies yield any material substance for combination with the constituents of the egg, through the usual chemical affinities of matter,—the quantity of influence to be supplied to produce the healthful result is definite, and that, whatever be its nature or essence, it is always characterized by a definite degree of *motion* in the spermatozoon. Further, all the observations I have made on the spermatozoa tend to show that their motion is rendered more vivid and intense by an increase of heat, but that, in proportion to such increase, it is so much the sooner exhausted; as, on the other hand, it has long been known that it is diminished by reduction of temperature, but increased in duration. Many of the observations lead to the view, that the power, in a given quantity or number of the spermatozoa, to effect impregnation, is more in proportion to the amount, *quantity, or intensity of the motion* exhibited by these bodies than to their actual

* Kölliker, Siebold, Müller, Wagner, &c.

numbers. Thus the spermatozoa, like the ova, are more efficient at a given temperature, within the first two or three minutes after they are passed from the body, than at a later period. At that time there is not only a greater number of spermatozoa in a given quantity of fluid in a state of activity; but the whole of them exhibit a greater intensity of motion than after they have been for some time mixed with water. In the Toad, which, in this country, as already stated, pairs at a few weeks later in the season than the Frog, and which seems not only to require a higher temperature of the surrounding medium, but also a relatively higher temperature for the development of the ova, the spermatozoa are most active, and most fitted to impregnate at the instant of expulsion; as is shown in the mode of intercourse of the sexes in that animal, and in the fact that, if the fecundatory fluid be not immediately applied to the strings of ova as these are passed, the unsprinkled ova are infertile. The oviposition of the Toad is an exceedingly slow process, and usually lasts, as SPALLANZANI observed, from ten to fifteen hours: in one instance I found it continued during seventeen hours, but it was completed in others in eight or ten. The act of impregnation, therefore, is necessarily also prolonged. But the length of time during which the spermatozoa continue efficient to impregnate after removal from the body, is shorter than in the Frog. SPALLANZANI found that, at a temperature of 81° FAHR., it does not exceed fifteen minutes. I also have noticed that the motions of the spermatozoa cease much earlier than in the spermatozoa of the Frog.

Thus the conclusions to be drawn from the facts of natural impregnation in the Toad, fully agree with those deduced from artificial impregnation in the Frog, and seem to establish the view, that while an increase of temperature is required for the fulfilment of the reproductive function in that animal, and to maintain the efficiency of the spermatozoa, the fecundatory force of the agent is of shorter duration, and corresponds to its more early cessation of motion.

13. PENETRATION BY THE SPERMATOZOON IN EFFECTING FECUNDATION.

The penetration of the spermatozoon into the substance of the egg, or even into its envelopes, has been so much disputed, and so repeatedly denied, that it is only on what may perhaps be regarded as indisputable evidence, that any one ought to reassert it. Up to a very recent period I could not, myself, admit it as probable; because, throughout my investigations, I had not been able to detect any appearance in the fecundated egg or its envelopes, either of the Frog or Newt, which rendered it likely that the spermatozoon penetrates into or through them. Yet the fact of penetration was stated by the older naturalists, not so much, perhaps, from any decided proof of its occurrence, as from a confidence that it was only in this way that the function of the spermatozoon could then be understood. LEEUWENHOEK, and those of his day, not only believed that the spermatozoon penetrates bodily into the substance of the egg, but also that it becomes the future embryo. In later times, M. PREVOST having, with M. DUMAS, seen the spermatozoon within the gelatinous envelope of the egg of the Frog, as they show, conceived that this body becomes,—not the embryo itself, but the foundation of the nervous system of the embryo. Since then, Dr. MARTIN BARRY announced to the Royal Society that he had *seen* spermatozoa within the substance of the egg of the Rabbit; through the envelopes of which, he has stated, that there is a natural perforation or cleft at the time of fecundation*; a view which has met with much opposition, as no physiologist has hitherto (1852) verified his observation with regard to the existence of a perforation or cleft in the egg-envelopes; or has announced that he has *seen* the spermatozoon *within* the ovum of that animal. Indeed so general has been the opinion that there is no penetration by the spermatozoon, even into the envelopes of the egg, notwithstanding WAGNER's† announcement that he had seen spermatozoa within the envelopes in the eggs of Fishes, that one of the most recent investigators, M. QUATREFAGES, in alluding to MM. PREVOST and DUMAS' views, observes, that "it is useless to allude to the question of penetration by the spermatozoon, as he believes that the only living author of the theory has himself renounced it‡." But very recently it has been announced, in a paper on the *Ascaris Mystax*, by Dr. NELSON, communicated to the Royal Society §, that the spermatozoon in *that* animal *does penetrate* into the substance of the yelk; and a somewhat similar account of that of the Earth-worm was formerly given by Dr. ARTHUR FARRE ||. Yet, with all due respect for the observations of these able investigators, I must still have hesitated to admit the fact of any penetration, even into the envelopes of the egg, had I not more recently been convinced of this fact by direct observation, in correction of my former opinions, in so far as relates to the penetration of the spermatozoon into the envelopes of the eggs of the Frog, and its arrival at the vitelline membrane. To them, then, be all honour on this subject, while I subjoin my testimony to a fact which I had heretofore failed to observe.

It was during the month of March last (1852), while making experiments on the Frog's egg by artificial impregnation, before some scientific friends, that certain appearances were noticed beneath the microscope within the expanded jelly of the egg, which led to a suspicion of the probability of penetration by the spermatozoon. Heretofore I had employed, in my observations, either a deep glass cell, in which the egg was completely immersed in water, or the common object-glass of the microscope. In the experiments now referred to, a shallow glass cell was employed, which was capable of containing only a single egg. This, while it admitted light freely around the egg, was too shallow for its complete immersion, and the egg was therefore covered by a drop of water, into which the object-glass of the microscope was passed,

* Philosophical Transactions, Part II. 1840, p. 533, Plate XXII. figs. 164, 165 and 167. Ibid. Part I. 1843, p. 33.

† Elements of Physiology (English Edit. by Dr. WILLIS), note, p. 74, 1841.

‡ Annales des Sciences Nat. 3me Série, tom. xiii. 1850.

§ Proceedings, June 19, 1851, vol. vi. p. 86. Philosophical Transactions, 1852.

|| See Dr. CARPENTER's Principles of Human Physiology, 1st edit. 1842, p. 617.

while viewing the egg, instead of the observation being made through the double medium of air and water. This was the method of examination in all the subsequent experiments.

The appearance which first led me to suspect that spermatozoa do penetrate into the envelope, was produced by an acicular body, which seemed to have its narrowed extremity in near proximity to the vitelline membrane. The direction of the longitudinal axis of this body was in a line with the centre of the yelk. But what appeared to be its larger end was farthest removed from the yelk; and this circumstance seemed to show that the appearance could hardly be due to the presence of a spermatozoon, which, if the observation were correct, must have penetrated in a direction the reverse of that of its usual motion.

It was necessary, therefore, that the suspicion raised by this observation should be settled. I had again and again found, as before shown, that the egg of the frog may be impregnated, under certain conditions, by the direct application of spermatozoa to almost any part of its surface, and this enabled me to put the question of penetration to the test. In the first trial, an egg was placed in a single cell, and immersed in water for *one minute*—the water was then removed, and the egg touched on one point only of its surface with the head of a pin, loaded with the fecundating fluid, which had been obtained and mixed with water about two hours before. I had expected that on watching the egg beneath the microscope, from the instant of contact with the pin's-head and the re-filling of the cell with water, to have been able to detect the spermatozoon during its passage through the envelope. But this I failed to do, in the present instance,—no spermatozoa were detected *in the interior* of the envelope, although many were easily observed on the *surface* at the point to which they had been applied by the pin. Yet, this egg, placed in a temperature of 66° FAHR., underwent segmentation in *three hours* and *thirty-two minutes*, and ultimately produced an embryo. A similar trial was made, at the same time, with an isolated egg after immersion for *two minutes* in water, which was then withdrawn, and the fecundating fluid applied as before. In this instance, several spermatozoa had penetrated for a short distance into the envelope, but had not reached the covering which immediately invests the yelk. No *respiratory chamber* was formed above the yelk in this egg, nor was any embryo afterwards produced. In a third instance, with an egg, which had been immersed for *five minutes*, the experiment was equally unsuccessful. On the 24th of March a further trial was made with an isolated egg, after *one minute's* immersion. The fecundating fluid employed had been obtained only sixteen minutes before it was used, and was applied by the head of a pin, once only, to one point of the egg. In this case, at the expiration of half an hour, I distinctly saw a single spermatozoon sticking by its larger extremity into the vitelline membrane, and a few minutes later there was evidence of the respiratory chamber being about to be formed. This egg underwent segmentation, and afterwards produced a good embryo. In a second egg immersed for *three minutes*, and in a third for *five minutes*, there were no appearances of spermatozoa in the interior of the jelly, although they were in abundance at the point on the surface, to which they had been applied by the pin. Neither of these eggs underwent any change. But in a *fourth trial*, with an egg immersed for *ten minutes*, and to which the spermatozoa were applied in the same way as in the preceding, I saw, at the expiration of fifty minutes, four spermatozoa sticking in the vitellary membrane; soon after which the chamber was commenced, and at a later period segmentation of the yelk took place, and an embryo was produced. This was with fluid which had been obtained nearly an hour before it was employed. In a further trial, with an isolated egg, to which a full quantity of fluid was added, by three or four applications of the loaded pin's-head, after the egg had been *one minute* in the water, I detected several spermatozoa, within half an hour afterwards, sticking like the preceding into the vitelline membrane, and this egg also produced a good embryo.

It was thus evident, that in some cases, spermatozoa certainly penetrated through the envelope; but as there was also one instance, in which the egg had become fecundated, and afterwards produced an embryo, in which I had not seen spermatozoa within the envelopes, it was necessary to pursue the investigation further, to be quite assured of the fact of penetration as connected with the act of fecundation.

On the following day an opportunity occurred to me of examining the egg after impregnation by the natural union of the sexes. I had several pairs of frogs in a basin of water covered by a glass bell jar, when, having my attention directed for a few minutes to some other object, one pair of frogs spawned suddenly, as I knew by the sound of a plunge or splash in the water, which always occurs after the act, at the moment of separation of the sexes. Nearly the whole of the eggs were deposited in a mass, but there were a few which were detached from the rest, and were lying at the bottom of the water. This seemed a good opportunity to examine these eggs, which had been expelled naturally, singly, in search of the spermatozoa within the envelopes. *Fifteen* of these detached eggs were placed, each in separate cells, and carefully examined, during the first hour and a half after their expulsion. In the first eight or nine of these eggs, I was not able to detect even a single spermatozoon within their substance, and I began to look upon the previous observations, made by artificial impregnation, rather as the result of accident. In the tenth egg, however, I most distinctly saw spermatozoa in contact with the vitellary membrane, but in one or two eggs examined after this no spermatozoa were to be seen. Out of the fifteen eggs examined, there were only *two* in which I could detect spermatozoa. As each egg had been carefully marked, and a note made, as to whether or not any spermatic bodies had been detected within it; and as previous investigation had shown, that when any fecundation has been effected the yelk of the fecundated egg becomes depressed, and a chamber begins to be formed, in about one hour, or little more, after the spermatozoon is supplied to the egg, between its upper surface and the investing membrane,—there were ready means of learning, within a short time, whether any, and which of these eggs had been fecundated. At the expiration of *one hour and twenty minutes*, the temperature being 54° FAHR., the chamber was beginning to be formed in the two eggs in which I had seen spermatozoa sticking in the vitellary membrane, and *these bodies were still readily detected in them*, and continued to be so for two or three hours afterwards. These two eggs ultimately produced embryos. But no change took place in any of the thirteen eggs, in which I could not detect spermatozoa,—no respiratory chamber was formed within, nor was any embryo produced. It was evident, therefore, that these had not been fecundated, and it is probable that the whole of the detached eggs were those which had last been ejected from the oviducts, possibly after the act of fecundation by the male had been completed. Having found spermatozoa in two only of these detached eggs, I then examined several of those from the mass. In each of them I found *numerous spermatozoa sticking around the yelk membrane*, and very many, in the clear space between the membrane and the granular middle portion of the envelope, which had not arrived at the membrane, and in *every instance* the eggs had been fecundated, as the chamber was being formed at the time of examination, about an *hour and a quarter* after the eggs had been deposited, so that the act of penetration by the spermatozoon through the envelopes as far as the vitellary membrane, seemed thus to be clearly established as connected with the act of fecundation. This was the case with all the eggs taken from the upper and middle portion of the mass. But in a very few eggs taken from the sides of the mass, I was not able to detect any spermatozoa, within the envelopes, or found only solitary instances of them. In these cases it was remarkable that no chamber had yet been formed above the yelk, although in some, in which spermatozoa were detected, it commenced at a later period, so that these appeared to confirm the deduction from experiment with reference to quantity of influence, or number of spermatozoa required to effect fecundation; while the penetration of the spermatozoa, as far as the vitellary membrane, and the subsequent development of the chamber above the yelk, appeared in the relation of *cause* and *consequence*, direct or indirect. I could not then observe, in the eggs thus examined, any penetration by the spermatozoa completely *through* the vitellary membrane into the substance of the yelk; although numerous spermatozoa were attached, by their larger ends, to every portion of the membrane, sticking out from it at right angles, with what, at first, appeared to be a knob or knot at the distal end, or rather as if that part of the spermatozoon had been shrivelled up or scorched. This appearance, as I subsequently found, was due to a loop, or distortion of the tail of the spermatozoon, consequent, apparently, on the death of this body. It was this looping of the tail which gave the appearance of a thick end to this part of the spermatic body first detected within the envelopes in a previous observation. All the spermatozoa seen in connexion with the vitellary membrane were perfectly motionless; so that when the looping of the tail has taken place the fecundatory influence of these bodies may be held to have been already exercised and exhausted.

At the expiration of three hours from the commencement of these observations, although the number of spermatozoa which had first been noticed within the coverings of the eggs submitted to examination appeared to have been diminished, there was still an abundance attached to the vitellary membrane. These appeared to afford a good opportunity for endeavouring to ascertain whether any part of the body of the spermatozoon is passed through the membrane into the cavity of the yelk. With this object in view, I placed a glass cell, which contained an egg, on one side, and waited for the rotation, or rather the gravitation, of the yelk within its envelopes. By this change of position of the yelk within its coverings—and which always takes place, if fecundation, or any change in the position of the entire egg, has been effected,—the flattened surface of the yelk, and the chamber above it, were brought beneath that part of the vitellary membrane into which spermatozoa appeared to have sunk deepest,—so that if any of these bodies had passed, or were in the act of passing, or had been partially protruded through the membrane into its cavity, which contained the yelk, it was fair to expect that they might thus have been detected. But not the slightest trace of penetration, even by a single spermatozoon, could then be observed*.

I may here remark, in anticipation of a future communication, that, previous to segmentation, the mass of the yelk does not appear to be invested by any distinct envelope within its so-called vitellary membrane, but the whole seems to be kept together by the natural coherence of the granules or cells of which it is composed; so that when the position of the egg is changed the yelk mass rotates within the vitellary envelope as a consequence of this change, owing probably to the excentric position, within its substance, of the progeny of the germinal vesicle†, contained in the

* Since this paper was communicated to the Society, I have succeeded, through the adoption of a different mode of examination, in detecting spermatozoa *within the vitelline cavity in direct communication with, and penetrating into the yelk.* They were first seen by myself, in company with a friend, on the 25th of March of the present year (1853) within the clear chamber above the yelk, at about forty minutes after fecundation, when the chamber begins to be formed. I have since repeatedly observed them within the chamber, and in some instances still in motion, in which state I have had opportunities of showing them to my friend Professor Ellis of University College, and to two other medical friends, so that the presence of active spermatozoa *within the vitelline cavity* in the fecundated egg of the Frog may now be regarded as indisputable. The details of my investigation I reserve for a future communication, and will merely now add, that the spermatozoa do not reach the yelk of the Frog's egg by *any special orifice or canal* in the envelopes, but actually *pierce the substance of the envelopes at any part with which they may happen to come into contact*; as I have constantly observed while watching their entrance: sometime after they have entered the yelk chamber they become disintegrated, and are resolved into elementary granules. The importance of this fact of actual penetration by the spermatozoon into the yelk is indicated by Wagner and Leuckardt, in their late Article, "Semen" (*loc. cit.* p. 507), in the following remark:—"The truth is, the '*how*' of the fecundation is as far from our knowledge to-day as it was thousands of years ago; this process is still enveloped in what we feel inclined to consider 'its sacred mystery.' *It would be different if we could prove that the spermatozoa really yielded the material foundation for the body of the embryo; that they penetrated into the ovum,* and were developed into the animal (which was the assumption of Leeuwenhœk, Andry, Guatier), or else, that they became metamorphosed into the central parts of the nervous system."—G. N., April 18, 1853.

† Philosophical Transactions, Part I. 1851, p. 176.

middle of the dark hemisphere of the yelk, and which, possibly, may be of less specific gravity than the contents of the light-coloured hemisphere.

The facility with which the yelk can thus be made to rotate, and its flattened surface, and the clear space, or chamber above it,—commenced within one hour after the first encounter of spermatozoa with the egg,—can be brought beneath any portion of the vitellary membrane into which spermatozoa have penetrated,—thus enabled me to observe, within a comparatively short period after fecundation, whether any of those bodies are at that time in the act of passing through it. But although no spermatozoa could then be detected within the vitellary cavity, it must be borne in mind that these observations were made after the commencement of the formation of the clear space, or *respiratory chamber*, the consequence of a shrinking and depression of the surface of the dark hemisphere of the yelk as the *result* of impregnation, so that these negative observations did not afford any *proof* that penetration by the spermatozoon had not previously taken place. Indeed, all circumstances considered, there seems reason to believe that impregnation is effected very quickly after the encounter of the spermatozoon with the egg, as appeared to be shown in my former experiments with solutions of potass*. It is probable that it is *commenced*, as then suggested, *within a few seconds, or at most a very few minutes*, after such encounter; as in one instance of artificial impregnation in some observations made subsequently to those above detailed, I detected a spermatozoon in contact with the vitellary membrane, *within one minute* after the impregnating bodies had been supplied to an egg beneath the microscope. Certainly I believe that it is commenced, and probably completed, within *the first half-hour*; because, after that lapse of time, I have not yet been able to detect any change of place or position of the spermatozoa which have passed into the envelopes of the egg; whether they may have already arrived at, and become partially imbedded in the vitellary membrane, or whether they are still contained in the clear substance of the outer coverings, their force of penetration being exhausted before arrival at the membrane. After the lapse of half an hour, and frequently of a much shorter time, the caudal portion of the body of a spermatozoon which has penetrated into the envelopes becomes looped on itself, and this is a certain indication of the death of the object.

As no spermatozoa were seen, in these observations†, within the vitellary cavity at the early period above stated, it was not to be expected that they were likely to be observed at a later; although the eggs which had been the subject of these investigations, and which had been fecundated naturally, were continued to be watched until segmentation of the yelks commenced. This took place in the two eggs, in which alone, out of the fifteen first examined, spermatozoa were detected,—at the expiration of *five hours and twenty minutes*. But this was much earlier than in the great mass of eggs, of which these were a part; and was due to their having been exposed to a higher temperature, owing to heat radiated from my own body during their

* Philosophical Transactions, 1851.

† See the preceding note.

constant observation beneath the microscope,—rather than to that to which the undisturbed mass of eggs was exposed. The temperature of the atmosphere of the room at the time the eggs were deposited was 54° Fahr., but it gradually sunk to 51°·5 Fahr., and the water in which the eggs were contained to 50° Fahr. at the time when segmentation commenced, at the end of *six hours and thirty-five minutes.* In the most submerged it did not take place until *six hours and forty-five minutes* had elapsed.

In nearly the whole of the eggs of this mass which had been fecundated by the natural union of the sexes, I was struck with the fact that the quantity of spermatozoa which had penetrated into every part of their envelopes was very considerable. Very many of these had arrived at, and were sticking by their larger end into the vitellary membrane, from which they projected like spines from the head of a thistle; while there were many others which had not arrived at this part, their power of penetration being exhausted, and their progress inwards being arrested before they had passed more than half-way through the outer gelatinous coverings. Others, again, had penetrated to scarcely more than their own length into them, while a still greater number were simply in contact with, and adhering to the external surface.

When segmentation took place in these fully impregnated eggs, their yelks seemed to contract more powerfully, and the clear space, or respiratory chamber, formed in each became much larger than in others from the same mass in which but a few spermatozoa were detected; so that the inference deducible from the fact seemed to be, that a *plurality* of spermatozoa is necessary for the full impregnation of the egg, and the production of the robust and healthy embryo—although simple fecundation may result from the influence of only a very few spermatozoa, especially when such influence is aided by a considerable increase in the temperature of the surrounding medium.

Mode of Penetration by the Spermatozoon.—The preceding observations on naturally fecundated eggs were so decisive of the fact of penetration by the spermatozoa into at least the envelopes of the egg, and of the arrival at, and partial imbedment of these bodies in the vitellary membrane, that it seemed desirable to make further observations, by the artificial method, with a view to ascertain more directly the mode and circumstances of their entry; and these it was hoped might be learned through the facility with which the egg may be impregnated by direct application of spermatozoa to almost any part of its surface. Accordingly, on the following day, I placed an egg in a glass cell, beneath the microscope, and quickly afterwards applied to one side of it, by means of a pin-head, a quantity of spermatic fluid, obtained from the male only a few minutes before, and immediately filled the cell with water, and commenced the observation. The fluid was applied four times to the same part of the egg, the pin-head being loaded for each application, in order that a full sufficiency of spermatozoa might be furnished, before the water was added. The temperature of the room at the time of the experiment had been intentionally raised to 65° FAHR., to afford a greater chance of success; as all my previous experiments had shown that more eggs became fecundated at a moderately elevated, than, under otherwise similar circumstances, at a low temperature. The instant the object-glass (the half-inch of Ross's microscope) was brought into focus, a vast quantity of spermatozoa were seen adhering to the surface of the egg at the part to which the pin's head had been applied. Many of them, attached laterally to the surface, had already ceased to move; others, only partially attached by their larger end, still vibrated the caudal or ciliated extremity rapidly, but did not appear to penetrate; while others, more centripetally attached by their body portion, vibrated the free extremity rapidly, and were seen in the act of gradually penetrating into the substance of the envelopes. I distinctly recognised one of these bodies which had just entered the envelope to a depth equal to about twice its own length, and when first seen had not reached so far as the middle or granulous layer of the envelope. Its motion was then slightly serpentine, and its course from without inwards was in a perfectly centripetal direction, with its thicker or body portion extended forwards, and in a line with the centre of the yelk, its progress inwards, as seen beneath the microscope, being as continued and as direct as that of an arrow. I kept this object in focus for several seconds, and watched it through the granulous layer, but ultimately lost it in the more dense and, as yet, unexpanded portions of the inner layers of the envelope, after it had been distinctly seen by a friend, who was with me at the time of the observation. A few seconds afterwards, as the envelopes became more expanded, very many of these spermatic bodies were seen to have already arrived at, and be in the act of passing through the inner or laminated portion of the envelopes,—the portion which, when the envelopes have acquired their full distension, by the imbibition of water, is seen to be that which immediately covers the vitellary membrane. A few seconds later a great abundance of them were seen in contact with the vitellary membrane itself; and some were even partially imbedded by their thicker extremity in its substance, and some of these showed an appearance as if they were actually penetrating through it. But in no one instance could I then satisfy myself that they did really pass through; since by alternately elevating and depressing the lens, and carefully noticing when the margin of the vitellary membrane was most distinctly defined, the appearances of perforation which some of them showed, seemed to be due to the spermatic bodies being imbedded in the membrane at some inclination to the plane of observation and to that of their direction being not quite centripetal. Yet there were other circumstances which seemed to show a likelihood that some spermatozoa do *actually pass through the membrane*. Thus, within the first few minutes after the impregnating fluid had been supplied to the egg under observation, and at the time when the spermatozoa, which had penetrated its envelopes, were first seen to have arrived at, and begun to enter the laminated portion, or *zona pellucida*, some of them were noticed to pass gradually onwards for a time, and then suddenly to disappear in an instant; as if, having passed into this tissue, they had also escaped through the vitelline membrane which it covers. This mode of disappearance certainly is that by which the spermatozoa might be supposed to penetrate to the yelk, if in reality they do so*. Thus, presuming the body, or thicker portion of the spermatozoon, to perforate and pass through the vitelline membrane, the more slender or tail portion would follow quickly. But if this occurs with some, then it would be fair to expect that it would happen to the whole which arrive at and become partially imbedded in the membrane; especially to those which have sunk into it to a depth equal to one-half the length of the thicker or body portion; or, further, that the progress of others might be arrested before they had completely passed into the vitelline cavity; and, consequently, that some would occasionally be seen protruding into the interior. But I was not able, in the instance of the observations now detailed, nor in others afterwards made, to prove either of these suggested conditions.

In the observation now referred to, as well as in others since made, there were many spermatozoa which remained distinctly visible for several hours, in the same place, and in almost precisely the same position, sticking into the vitelline membrane, and retaining, at the end of a lengthened period, the same appearance as at first, excepting only that they seemed to have become smaller in diameter and to have their caudal portion more looped. In the present instance, at the high temperature of 65° FAHR. to 66° FAHR., not only were they distinctly seen at the commencement of segmentation of the yelk, which happened at the end of *three hours and twenty-two minutes*, but many of them were present until after the yelk had undergone several of its subsequent divisions. This was the case not only with those which had arrived at the vitelline membrane, but also with others which had never reached it, and had not penetrated further than to about the middle, or granulous portion of the envelopes; as happened with many spermatozoa, both in the eggs which were fecundated naturally, as well as in those which were the subjects of experiment, and were artificially affected. It was those which remained in the substance of the envelopes which usually disappeared earliest, becoming at first gradually fainter, and then more undefined in outline. This change was supposed to be due to a gradual diffluence of the substance of the spermatozoon, through the influence of the water imbibed by the envelopes; but whether this happened as part of the fecundatory process, or whether it was simply the natural process of decay, as other circumstances to be mentioned seemed to intimate, there was no distinct proof. It was remarkable, however, with reference to the *act* of fecundation, that in almost every instance, even of those spermatozoa which never arrived at the vitelline membrane, *the body portion was always directed towards the yelk*, usually peripherally, but sometimes inclined at slight angles to one side or the other; thus showing that it is invariably the body portion which penetrates.

* That this is really the fact, and that the actual penetration by the spermatozoon into the yelk chamber *was observed on this occasion*, is now rendered almost certain by my recent observations stated in the preceding note, p. 271.—G. N., April 18, 1853.

It may be matter of surprise, that, considering the immense quantity of spermatozoa which exist even in a microscopic drop of fluid, and considering also the abundance which come in contact with the surface of the egg, even when but a small quantity of fluid is employed, as in artificial impregnation, a much greater number do not penetrate than are usually observed to do so. There are conditions and circumstances which affect this result. Thus I have found, in repeated observations, that only those spermatozoa which, at the moment of first contact with the egg-envelopes, are in rapid action, and have their body portion directed, either perfectly centripetally towards the yelk, or at angles but slightly inclined to it, do by any possibility enter; while those which happen to be directed horizontally to the surface of the egg at the instant of contact, always adhere to it laterally, and lose their power of motion, but do not penetrate; and the like also is the case with those which become attached by their caudal end, and even with many which adhere by their thicker end, when they come into contact with the egg at very acute angles. Further, I have noticed, that a relatively much greater number of spermatozoa penetrate the envelopes when supplied to the egg immediately after this has been removed from the female into water;—especially when the spermatic fluid also has been recently passed from the male;—and more decidedly so when passed from a male in full season, at which time the movements of the spermatozoa are most energetic. Thus the chances of penetration through the envelopes, and consequently of fecundation of the egg by the spermatozoon, are in direct relation to these circumstances; and inversely to those of an opposite character, being less in proportion to the length of time the egg has been removed from the female, or the fluid from the male; the healthfulness of the parent, and consequent power of motion in the spermatozoon;—the temperature of the season, and the quantity supplied to the egg.

These were the conclusions deduced from the previous observations, and they have been fully borne out by subsequent experiments, some of which have been made in the presence of my friends Professors BELL, BOWMAN, CARPENTER, and Mr. BUSK, Fellows of the Royal Society, and Professor ELLIS, who permit me to mention the circumstance.

Narcotization of the Spermatozoon.—I may now mention some experiments which were made with the view to test the fecundatory influence of recently obtained spermatozoa when narcotized by chloroform. These experiments were suggested by a communication made to me by Mr. BUSK, F.R.S., who, after witnessing my mode of procuring the eggs of the Frog, conceived that a similar result might be attained by narcotizing the gravid animal without killing it, as is necessarily done in my experiment; and on putting this opinion to the test he found that it may be accomplished with ease and success. It then occurred to him to try the effect of exposing the spermatozoa to the vapour of chloroform, by simply covering the spermatic fluid contained in a watch-glass covered with blotting-paper wetted with the liquid, and afterwards to apply the narcotized bodies to some eggs in the way done by myself.

The facility with which the spermatozoa can be narcotized by the mode now mentioned has enabled me to test the relation of their power of motion to that of their fecundatory property, and although my results differ somewhat from those obtained by Mr. Busk, it may be well, perhaps, to relate them.

MM. Prevost and Dumas found, in addition to their many other excellent results, that spermatozoa are rendered motionless by an electric shock, and that then they do not impregnate the egg. They also found that opium and strychnine have a similar paralysing effect on these bodies. But it has since been suggested that the latter agents act on these bodies only in so far as they affect the chemical composition of their substance, and that the operation of electricity on them also is similar. But it yet remains to be shown whether any chemical change is produced in the substance of the spermatozoon, when simply narcotized by the vapour of chloroform, and not mixed with it in the fluid state; and when, although the power of motion is arrested, the vitality of the body is not destroyed.

Three eggs were placed in separate cells, and the spermatic fluid, immediately it had been obtained, was applied, *once* only to each, by means of the pin's head. These trials with the fluid in its natural state, mixed only with a small quantity of water, were made for the purpose of comparing their results, with those of others to be made with portions of the same fluid after it had been narcotized, and applied to eggs in this state at different periods. *One minute* after the application of the fluid I found an abundance of spermatozoa on the surface of each egg at the part touched, and some spermatozoa had not only already penetrated into the envelopes, but had arrived at, and were in contact with the vitelline membrane, or rather the *zona pellucida*. The respiratory chamber was afterwards formed above the yelk in each of these eggs, and two of these subsequently formed embryos; the third was only partially fecundated.

The spermatic fluid was then exposed to the influence of chloroform, in the way mentioned, about fifteen minutes after it had been obtained; and after seven minutes' exposure to it, and when the majority of the spermatozoa it contained had become narcotized, was employed in experiments. The signs of full narcotization are the entire cessation of all motion in the spermatozoon, which lies with its body extended at length, and not looped on itself. In the latter condition it is usually dead.

Three eggs placed in separate cells were then supplied with the narcotized spermatozoa applied to each egg three times by means of the pin's head. On examining the eggs six minutes afterwards, I was unable to detect even a single spermatozoon within the envelopes of either of them, either in contact with the vitelline membrane or in any part of the substance of their envelopes; although there was a great abundance of perfectly motionless spermatozoa on the surface. No chamber was formed above the yelk in either of them, nor did either of them produce an embryo.

Twenty minutes after narcotization, and during which time not a single spermatozoon had been observed in motion, a portion of them were applied in the same way as above to *three* other *eggs*, but not one of these became fecundated, no chamber being formed within them, and no embryo was produced. Many of the spermatozoa at this time were dead, as was shown by the looping of their bodies, but there was still a quantity which had been simply narcotized and lay extended at length.

One hour after narcotization another portion was applied to *two eggs*, one of which became partially fecundated. The chamber appeared in it above the yelk, the yelk underwent segmentation, and afterwards an embryo was begun to be formed, but its development was not completed, and the egg then perished. The second was not fecundated. The circumstance of this egg having been fertilized at so long a period after the fluid had been narcotized, leads to the conclusion that impregnation had been effected by *revived spermatozoa.*

On the following day I repeated these experiments with very mature fluid immediately after it was obtained, and in which the spermatozoa were exceedingly vigorous. *One minute and a half* after it had been exposed to chloroform, I found that the same effect had been produced on the spermatozoa as that which is produced in them by their admixture with very weak solution of potass; their power of motion was greatly increased, and the whole were in a state of intense action. After a few minutes' longer exposure their motion was perceptibly diminished, and at the end of about ten minutes it had almost entirely ceased, as only a few of them were then observed to move.

Three eggs were supplied, by means of the pin's head, with these narcotized spermatozoa. The respiratory chamber was formed in two of these eggs, and one of them underwent segmentation, and at the end of the fifth day an embryo had begun to be produced, but its development did not proceed, and this egg like the other two perished.

Six eggs were then supplied copiously by means of the loaded pin's head, three times applied to each, nearly the whole of the spermatozoa employed being then motionless, and having already remained so for about *ten minutes*, only an occasional one being observed in feeble action. Two of these eggs became partially impregnated, the usual chamber being formed in them, but neither of them underwent segmentation, nor was any embryo formed either in these or the others.

Three eggs were then supplied as above after the spermatozoa had been motionless during *twenty minutes*, not one being observed with the slightest action at the time of the experiment. No chamber was formed in either of these, but the whole remained unimpregnated.

Three eggs were the subject of further trial, when the spermatozoa employed had been motionless during *half an hour*. At this time one or two spermatozoa were again observed to be in feeble action, but no impregnation was effected.

Thirteen eggs were employed when the spermatozoa had remained motionless during *one hour*, and when there was reason to believe that the majority of them had perished, but when there were still a very few among them in motion, having either revived, or but recently escaped from spermatozoal cells contained in the fluid. No impregnation took place in either of these instances.

Thus the results of these experiments appear to show that the spermatozoon does not impregnate *when entirely deprived of its power of motion by narcotization*, and disenabled to penetrate into the envelopes of the egg; and, consequently, that its fecundatory power has a close relation with its motion, or force of vitality. In this, then, they coincide with those of the previously detailed experiments, and go with them to show that the *act* of fecundation of the egg of the Frog, and probably also of all the Vertebrata, is the result of a power in the spermatozoon, which, in its operative condition, is characterized *by motion*; and that this power is totally independent of everything approaching to volitional influence, in the impregnating body, but seems to be in direct relation to physical causes.

MM. Prevost and Dumas appear to have thought that the spermatozoa which they observed within the egg-envelopes had entered through means of the infiltration, or imbibition of water by the envelopes; and the gist of their experiments was to show that particles of solid matter *do enter* during such infiltration, or, as it has since been designated, *endosmosis*. But some experiments, elsewhere detailed*, have led me to believe, that only such solid particles as are very much smaller in diameter than the spermatozoon can so enter with the water by endosmosis; while the circumstances now detailed fully prove that the entrance of the spermatozoon is not the result of simple infiltration, but is that of the operation of direct mechanical or physical power in that body. Thus, if it were mainly dependent on endosmosis, the perfectly motionless spermatozoon would enter the tissues and fecundate the egg equally well with the active, but this, as we have seen, is not the case, and this was shown by the physiologists now referred to. On the other hand, the circumstance that the spermatozoon invariably enters the tissues with its thicker or body portion directed forwards, and sometimes even at an angle slightly inclined to the centre of the yelk, and always impelled by an oscillatory or vibratile motion of its caudal portion, seems to show that its power of penetration is not only a necessary condition of its function, but is inherent as such in its organic composition. It may yet be true, nevertheless, that the passing of the spermatozoon through the substance of the envelopes to the vitellary membrane may be indirectly *aided* by the endosmic action of the envelopes. But this aid does not consist of any imbibing property in the tissues. It appears to be simply of a negative character, and to be the result of a decrease of density in the tissues, which, through their imbibition of water, and consequent expansion, are caused to offer less and less resistance to the propulsive force of the spermatozoon. This I believe is the proper explanation of the fact of penetration by this body.

But although penetration be not caused by the endosmic action of the envelopes,

* Philosophical Transactions, 1851, p. 224.

it may yet be thought to be induced by some attractive power in the substance of the yelk itself; and that, therefore, the entrance of the spermatozoon may be as much due to the egg as to any power of motion in the penetrating body. An accident has enabled me to test the validity of this surmise.

I had promised to show the fact of penetration to a friend, but circumstances prevented me from doing so until the season had nearly passed, and the whole of my frogs had spawned. I determined therefore, as a last resource, to endeavour to obtain some fecundatory fluid from a male which had already paired two or three days previously, and to employ it with the only eggs I had then left, which remained in the body of a frog that had been killed *twenty-six hours* before, and which, as former experiments had shown, it was probable had lost their vitality. The fluid required was obtained with ease, but mixed with a large quantity of spermatozoal cells. This was supplied to some eggs from the dead frog, since, although I did not expect that the eggs would be fecundated, I hoped for an opportunity of again witnessing the penetration by the spermatozoon. The spermatozoa in the fluid obtained were very active, and fully efficient, and were supplied in abundance to several eggs in separate cells under the microscope. The envelopes of the eggs expanded as usual, and endosmosis went on in a perfectly natural way, and an abundance of spermatozoa adhered to their surface. At the expiration of from fourteen to twenty minutes I found that several spermatozoa had penetrated the envelopes and were adhering in the usual way to the vitelline membrane. In one egg there were six, in another five, and in a third four, distinctly visible in the plane of observation that could be brought at once within view with the microscope, besides others recognisable on changing the focus. It was thus evident that spermatozoa, even of the previously paired Frog, still retained their *power of penetrating into dead eggs*, as these ultimately proved to be, after careful preservation to the sixth day in a favourable temperature. The spermatozoa were distinctly visible within the envelopes, without change of position, for several hours, but no fecundation was effected by them; no chamber was formed in either of the eggs, no segmentation took place, nor was any embryo produced. These circumstances seem to show that the eggs were already dead, as was supposed, before contact of the spermatozoon; consequently that the entrance of the spermatozoon into the envelope is due to a power inherent in the penetrating body, and not simply to an attraction on the part of the yelk; although from the fact that the spermatozoa usually enter in a centripetal direction, it is probable that some influence may be exerted by the yelk or its vesicle, although penetration is mainly the result of force in the spermatozoon.

The facts now stated of penetration by the spermatozoon seem to lead us better to understand the nature of some experiments with solutions of caustic potass, which are detailed in my former paper. I have repeated these experiments, during the past season, in the presence of several friends, Professors SHARPEY, ELLIS, and BELL, and Messrs. BUSK, TOMES and WATERHOUSE, with results precisely similar to those which are detailed in that paper; viz. that when diluted spermatic fluid, recently obtained, is applied to a set of eggs, and—as soon afterwards as the experiment can be made, a solution of potass,—of such a strength as is known by previous microscopic observation to have the property of instantly decomposing the spermatic body,—the solution being washed away quickly after its application, by repeated quantities of water,—to prevent its affecting the egg itself—that then—in some instances—even when the interval of time between the application of the spermatic fluid and the subsequent application of the potass does not exceed a few seconds,—impregnation of the egg is effected; as is proved by the formation of the chamber, the segmentation of the yelk, and perhaps the formation of an embryo. Further, that, all circumstances being similar, excepting only that the interval of time between the application of the spermatic fluid, and, subsequently, that of the solution of potass be prolonged,—the production of an embryo is not only more certain to take place, but the number of embryos produced is increased. The now ascertained fact of *almost instantaneous penetration* by the spermatozoon, which, as before shown, sometimes arrives at the vitelline membrane in less than *one minute* after its application to the egg, confirms the principal conclusion deduced from the potass experiments at the time they were made, viz. that impregnation is *commenced at the instant the spermatozoon is in contact with the egg*; while it also seems to afford the true explanation of the nature of those experiments, in which it may now be presumed that some spermatozoa had actually penetrated into the substance of the envelopes before the application of the solution of potass, and thus had already passed out of the reach of its destructive influence, the effect of which on the egg itself was obviated by speedy dilution and ablution with water.

The potass experiments may thus be regarded as confirming by anticipation the results now obtained by direct observation with the microscope, with respect to the rapidity of operation by the spermatozoon; and they seem also to support the view of the essentiality of the motor power of this body to its functional action. A similar view may be taken, as indeed was held at the time, of the nature of the Carmine, the Gum, and the Starch experiments, that the operation of these substances in preventing the fecundation of the egg is *entirely mechanical*, and that they do so simply by offering a mechanical impediment to the spermatozoon, a conclusion which seems to be fully supported in the present inquiry.

14. NATURE OF THE INFLUENCE OF THE SPERMATOZOON.

Having seen in the preceding experiments and observations that fecundation of the egg is effected by the spermatozoon only while this body retains and continues to give evidence of its vitality in its power of motion, and that its vitality either may be destroyed, or its operation be for a time entirely arrested by electricity and by chloroform,—the question naturally arises—in what way, then, is its fecundatory influence to be explained? Is it *simply* by diffluence of the substance of the spermatozoon, and the *chemical fusion* or combination of this with the contents of the egg, after the spermatozoon has penetrated through the envelopes and arrived at the vitelline membrane, or the yelk? Or is it by its expenditure on the yelk, of a *force* or power of vitality, which is inherent in the body of the spermatozoon? Or is it by the joint cooperation of both these conditions—the expenditure of a *force* with diffluence of material substance? I have endeavoured to put the first of these questions to the test of experiment;—although, it must be remarked, that the experiments made seem to be open to some objections, which at present cannot be fully answered. Yet knowing what we do of the disposition and great readiness of all matter to enter into new combinations, and affect, or entirely change the condition of the whole body operated upon,—as,—not to mention the effect of the poison of serpents, or the introduction of dead matter into the living, in dissection,—can be shown in the effect, on the yelk of the Frog's egg, of extremely minute quantities of chemical compounds diffused in water,—there has seemed fair reason to think that the question thus started may be examined by experiment. The rapid endosmic action of the envelopes of the egg at the instant of their contact with any aqueous fluid, and the rapidity with which the yelk itself then becomes affected, seemed to favour the idea that if the bodies of the newly obtained spermatozoa could be quickly reduced to a state of diffluence simply by mechanical means, without the addition of any menstruum, saving only a very small quantity of water, and be applied in this state of diffluence to the egg at the instant after it has left the body of the female,—that then an experiment thus made would be a fair test of the question. Some imbibition of the substance of the spermatozoon with the water might thus be expected to take place, and some changes in the egg to follow, if impregnation be the result of simple chemical combination of the substance of the male with that of the female. The presence in the experiment of a portion of water holding the diffluent spermatic substance in suspension, appeared at first sight to be an objection, but this seemed to be met by the fact that water *must* always be present to ensure the natural fecundation of the Frog's egg; and that it not only permeates the envelopes, and is the means of facilitating the entrance of the spermatozoon, but that it passes even to the substance of the yelk itself, as shown in the experiments with large quantities of potass before alluded to, and also in others made by immersion of eggs in water with given proportions of potass in solution. As the details of these latter experiments are contained in a paper which is now in the Archives of the Royal Society*, I need give but little more than the results of these experiments, in the present communication, before proceeding to relate those now referred to with the diffluent spermatozoa.

Immersion of Ova in Solutions of Potass and Soda.—These experiments were commenced in April 1850, before the publication of some which were made with solutions of caustic potass, by M. QUATREFAGES, on the spermatozoa of one of the marine worms, *Hermella*, and read to the Institute on the 24th of June 1850†.

* *MS.*, No. 762, p. 13 to 26; also Proceedings, vol. vi. p. 83.

† Comptes Rendus, June 24, 1850, and Annales des Sciences Naturelles, 3me série, tom. xiii. in Nos. *marked* "February" and "March 1850."

The object which I had in view was to endeavour to learn what proportion of the *carbonates* of *potass* and *soda* may be present in a given quantity of water without destroying the fertility of the egg. The first trials with potass were made with eggs which had already been fecundated, and were undergoing segmentation. It was then found that when fecundated eggs, arrived at the second stage of segmentation of the yelk, were immersed in a solution, composed of *twenty grains* of the *fused carbonate of potass* to one ounce of water, the yelks became shrivelled, and decomposition was commenced within *three minutes* after immersion; thus proving that even when the envelopes of the egg are almost fully expanded, and, consequently, when their endosmic action may be expected to have become greatly diminished, that endosmosis may still be going on with much energy. It is right to mention, however, that in this and the following experiments, the endosmic action may have been, and probably was, increased by the removal of the eggs—which already retained water in their tissues—to the more dense fluid solution; but as the experiments are comparative, the circumstance is not of much importance.

On the contrary, when some eggs were immersed in a solution of *one-fourth of a grain* of the salt to one ounce of water, and the potass in admixture with the water was thus reduced to $\frac{1}{1920}$th part of the whole, development went on more quickly than in other eggs of the same brood, impregnated at the same time, but which were placed in pure water apparently through the potass occasioning a slight increase of temperature.

These results appear to be explicable only on the assumption that the weak potass solution acts as a chemical stimulus,—perhaps both to the egg and to the spermatozoon,—at the time of fecundation, a view which derives some support, in so far as refers to the spermatozoon, from direct observation with the microscope. Thus, when a drop of the *two-grain* solution of potass is applied to spermatozoa on a plate of glass, covered with talc, beneath the microscope, the movements of these bodies are not instantly arrested, as by the stronger solutions, but gradually become slower and slower, until they entirely cease, apparently, in proportion as their substance becomes affected by the potass. The *one-grain* solution very much accelerates the movements of these bodies during the first few seconds; but in a few seconds longer these also become diminished like the preceding. But the effect of the *half-grain* solution, although exciting the spermatic bodies to activity as soon as it comes into contact with them, continues its effect for a much longer time without acting injuriously upon their substance; and it is not until after the lapse of many minutes that the intensity of the motion excited by it is observed to be diminished, and the movement to slacken and afterwards gradually cease.

With regard to the *carbonates of soda*, I may state that the general effect of these on the impregnated and on the unimpregnated egg are very similar to those of potass, but are far less prejudicial, or marked, so that it is unnecessary to detail the experiments.

The general conclusions deducible from these observations, with reference to the nature of the influence of the spermatozoon in impregnation, seemed to be, that as water which holds in suspension or solution any chemical substance, conveys some portion of that substance at all times, with great rapidity, to the immediate vicinity of the yelk itself, which becomes affected by it, and especially so at the time of its first coming into contact with the envelopes of the newly deposited egg,—the substance of the spermatozoon also, if diffused in water, might be so conveyed, in the experiments proposed. In addition to this conclusion, there are others, which, although less directly connected with the object in view, are not less important. Thus it appears that an alkaline fluid, in large quantity, is as injurious to the vitality and fecundatory influence of the spermatozoon as to the fertility of the egg.

Trituration Experiments.—The endeavour in these experiments was to put to the test the question proposed in the preceding observations,—whether, if the spermatozoon be reduced to a state of diffluence in water quickly after it has left the body of the Frog, and before it can reasonably be supposed that any change in the chemical constituents of its body has taken place, the egg can be fecundated through simple imbibition of the substance of the spermatozoon, conveyed to the vicinity of the yelk with the water during endosmosis, at the time when endosmosis of the envelopes of the egg is most energetic?

Although it had been constantly found, both by the authorities before cited, and since by myself, that when the spermatozoon has ceased to move, and is believed to be dead, no impregnation results from the contact of its motionless body with the egg; yet there still appeared to remain some doubt as to what this want of operation is then due; whether it is to be attributed to the organic death and incipient decomposition of the body, or whether to the loss or suspension of some power which is characterized by *motion*, and through which its function is exercised?

The following experiments, with reference to this question, were commenced before I was aware of the fact that the spermatozoon penetrates into the envelopes of the egg; a fact which, when known, seemed further to necessitate the inquiry, as probably tending to show whether impregnation is merely the result of a union of the substance of the spermatozoon with that of the yelk *or its vesicle*; or whether it be not primarily due to the transmission of some dynamic influence, force, or peculiar vitalizing power from the spermatozoon to the egg at the time of, or previous to any fusion with it of its material substance.

The first experiments were made by breaking down the bodies of the spermatozoa mixed with water, and, after filtration, applying the filtered liquid directly to the egg; the following being the method pursued:—

The bottom and sides of a very small glass mortar, and the glass pestle employed with it were ground together with fine sand and water, for the purpose of slightly roughening their surfaces, and of ensuring their contact at every part, and thus to render the crushing of the spermatozoa more certain.

The spermatic fluid first employed was procured from a Frog early in the season, and consequently contained a large quantity of spermatozoal cells from which the spermatic bodies had not been liberated. A difficulty immediately occurred which had not been anticipated. The fluid was divided into two parts, one of which was triturated gently with the pestle and mortar for two minutes, and was then examined with the microscope. It still contained an abundance of active spermatozoa, and many cells. The trituration was continued for three or four minutes longer, without any marked result, as active spermatozoa were still abundant in the specimens of fluid examined. It was then repeated for several minutes with increased rapidity and force, when the fluid suddenly became partially coagulated, and separated into opake white flocculi, composed almost entirely of cells and granules, and a transparent fluid in which there were scarcely any granules. It was then found that nearly the whole of the spermatozoa had been destroyed, as I was not able to detect even one in motion. The thinner portion only of this fluid, when placed on the filter, passed through, and it did so with great slowness and difficulty.

These circumstances are stated to show the mode of proceeding, and the accidents to be guarded against. The conclusions deduced from these trials were, that the spermatic fluid was not fully matured, and probably contained albumen, which, it is stated*, does not exist in mature spermatozoa; also that the substance of the spermatozoa had undergone some chemical decomposition, perhaps from excess of heat evolved in the act of trituration, through too much force being employed; so that the results of these experiments could not be depended on. A further trial was therefore made a few days afterwards. The fluid obtained from a mature Frog was divided into two parts, one of which was reserved for simple artificial impregnation; while the other was triturated slowly and carefully in the mortar with but little more exertion than was necessary to keep the pestle in constant motion for about fifteen minutes; no water being added to the fluid. When placed on the filter it became necessary to add to it about an equal quantity of water to facilitate its filtration. Portions of the fluid which then passed through, holding the substance of the broken-down spermatozoa in solution, were caught in separate glass cells.

Ten eggs were passed into a cell about half filled with the filtered fluid, and the cell was afterwards filled up with water. No other change than a slight enlargement, after the lapse of some hours, occurred in these eggs; no chamber was formed, nor did any segmentation take place in either of them, consequently no embryos were produced.

Fifteen eggs were placed in a cell which contained about half the quantity of filtered fluid employed in the preceding; but no fecundation took place in either of these eggs.

Eleven eggs in a third cell with filtered fluid were equally unfruitful.

Ten eggs in a fourth cell gave similar results.

* Article "Semen," Cyclop. Anatomy and Physiology, vol. iv. p. 506.

One hundred and forty-seven eggs were then placed on the topmost of the filter papers, which had been employed in separating the fluid used in the preceding trials, and on which were the remains of the broken-down spermatozoa, with, probably, some which had not been injured. This was about four hours and a half after the spermatic fluid had been obtained, so that on this ground but little success could be expected. Not a single embryo was produced.

The portion of reserved fluid which had not been triturated was then applied to *four eggs* in separate cells, by means of the pin's head, when each of these eggs became fecundated, and not only underwent segmentation, but each afterwards produced an embryo. As this experiment was made for the purpose of comparison with the preceding, it is clear that the failure in the case of the mass of eggs on the filter paper was not due to the length of time the fluid had been obtained, but to the effect of trituration; and it seems equally clear that the substance of the broken-down spermatozoa dissolved in water and passed through the filter is as inefficient as the *liquor seminis* has heretofore been proved to be*.

These experiments were afterwards repeated, in the presence of Professors SHARPEY and ELLIS, with similar results. Usually there are still some spermatozoa which escape being crushed during trituration, and, being uninjured, remain on the filter paper capable of effecting impregnation when eggs are passed upon it in water, as was the case on the following occasion.

Seventy eggs were first added to some filtered fluid, which resulted from the trituration of spermatic fluid, and had passed through four filter papers, but not one egg became affected, not one embryo was afterwards produced from them; while of *sixty-two eggs* placed on the topmost filter paper in water, amidst the remains of broken-down spermatozoa, with many which were still active and uninjured, more than one-third of the eggs became fertilized, and *twenty-three* afterwards produced *embryos*.

It was thus evident that no impregnation results from an immersion of the egg in fluid which is derived from the broken-down body of the spermatozoon, mixed with water, and separated from the more solid parts by filtration. But as some objections may fairly be offered as to the validity of this experiment, it seemed desirable that trial should be made without having recourse to filtration, and without removing any portion of the destroyed spermatozoon; but, on the contrary, by applying the whole of the triturated fluid immediately to the egg before it is immersed in water; and at the same time to compare the result obtained with that of an experiment by simple artificial impregnation, with a sample of the same spermatic fluid which had not been triturated; the experiments in both cases being made at the same time, and with eggs from the same female, and placed under precisely similar conditions in regard to light, heat, &c. A comparison of the relative number of embryos which might result from these experiments, it was thought, would show, not merely whether

* Philosophical Transactions, 1851.

impregnation can or cannot be effected by the direct application of the diffluent substance of the spermatozoon to the egg, but also whether the influence of the spermatic body is essentially dynamic in its character.

The quantity of spermatic fluid obtained for these experiments, mixed with a small proportion of water, measured about fifteen minims; and to this a further quantity of water was added, making the whole thirty minims. This was divided into three equal portions. The first portion (*a*) was triturated slowly and continuously for fifteen minutes, after which it was slightly turbid, and on examination with the microscope was found to contain a large quantity of granules, each of which did not exceed in size one half of the diameter of a spermatozoon. It also still contained a good abundance of active spermatozoa, although the quantity of these was greatly reduced, and the movements of those which remained were impeded by the quantity of granules. The second portion of fluid (*b*) was triturated with the addition to it of about one grain weight of well-washed sand, and after three or four minutes presented a turbid appearance. At the end of seven minutes, when examined by the microscope, it was found to contain a great abundance of organic granules, intermixed with the sand, and the number of spermatozoa had been greatly reduced; and three or four minutes later nearly the whole of the spermatozoa were destroyed. The following experiments were then made with eggs obtained from the female with which the male that supplied the fluid had been paired. The temperature of the atmosphere at the commencement of the experiments was 57° FAHR., and that of the water employed 55° FAHR. At the end of the experiment, the commencement of segmentation of the yelks, the atmosphere was 61° FAHR., and the water 59° FAHR.

One hundred and seventy-nine eggs were passed into a glass dish, and the triturated fluid (*a*) was immediately poured over them, and the dish quickly filled with water. The yelks of many of the eggs soon became irregular and contracted, and at the end of *four hours and thirty-one minutes* segmentation had commenced in some of them, and ultimately, *sixty-seven embryos* were produced; so that one hundred and twelve eggs were not fecundated.

One hundred and sixty-seven eggs were passed into a second dish, and the fluid (*b*) which had been triturated with sand was immediately added to them, and afterwards the dish filled with water as in the preceding. But *not a single egg* became fecundated, nor was *a single embryo* produced. It must be stated, however, that much of this result was afterwards found to have been due to the admixture of sand with the fluid acting mechanically, as I shall presently show.

Two hundred and thirty-nine eggs were also passed into a dish of the same dimensions as the preceding, and the third portion of fluid (*c*) reserved for simple artificial impregnation, and which had not been triturated, was supplied to them at, as nearly as possible, the same time as the triturated fluid to the preceding. This was at thirty-five minutes after it was obtained. Segmentation commenced in nearly the whole of these eggs in *four hours and fourteen minutes*, and at the end of the fifth day *two* *hundred and six embryos* were produced, so that only *thirty-three* eggs failed; thus leading to the inference that the failure in the experiments (*a*) and (*b*) was owing to the spermatozoa being destroyed; and, consequently, that the application of the substance of the body of the spermatozoon to the egg is not alone sufficient to effect fecundation; so that fecundation cannot be regarded as the result of simple chemical combination of the substance of the spermatozoon with that of the egg, but, essentially, may be due to some dynamical influence in that body.

At the time of making these experiments I made also two others with a view to this hypothesis; and for the sake of correct comparison employed eggs from the same female, and placed them in a precisely similar condition with regard to light, heat, and the quantity of water employed. The fecundatory fluid, however, was not from a male in full season, as above, but was what may properly be regarded as *senile*, it being purposely obtained from some frogs which had been kept separate from the females, having already paired and spawned from five to ten days previous. It was slightly translucent, and contained, relatively, but few living spermatozoa, the majority of which were languid in their movements; but there were many which appeared perfectly dead and motionless, and there was a good proportion of spermatozoal cells. In the first experiment *one hundred and sixty-seven eggs* were supplied with a portion of this fluid, and segmentation commenced in some of them in *four hours and forty-two minutes*, or not so soon as in the experiment (*c*) by *twenty-eight minutes*, and afterwards only *forty-three* embryos were produced. In the second trial, made at the same time with *one hundred and seventy-three eggs*, segmentation took place in *four hours and forty-one minutes*, and *one hundred and twenty-six embryos* were produced, forty-seven eggs being unimpregnated. If the difference in the length of time which elapsed between the encounter of the spermatozoon with the egg and the commencement of segmentation in these two experiments, as compared with the preceding one, be not fairly referable to a less degree of vitalizing power in the spermatozoon in the latter, it seems difficult to understand to what other cause it can be assigned; seeing that the eggs employed were from the same female in each experiment, and that all the conditions were similar. It seems, as it were, *inversely*, to show, that the most important condition of the impregnating agent is its possession of some dynamical quality, the degree or intensity of which is expressed in that of its power of motion, and which, possibly, it may transmit from itself to the egg, during the act of fecundation.

These results sufficiently mark the injurious effect produced on the fecundatory fluid by trituration, whether with or without admixture of foreign substances; and while these seem to prove that attrition of the spermatic bodies with particles of solid matter, by lacerating and mechanically destroying them, is fatal to their function, other experiments showed that even their simple admixture with any finely comminuted solid materials greatly interferes with their operation, by the mechanical impediment which such materials oppose to their free motion, and to their penetration into the envelopes of the eggs, whether such admixture be made with the bodies themselves, or with the water in which the eggs are immersed before they are supplied to it.

The question proposed in these experiments by trituration we may now look upon as to some extent, if not entirely, answered; as it seems that although the spermatozoon be broken down mechanically before any chemical change can be supposed to have taken place in it,—indeed, at the very time it is giving evidence of its vitality,—and its organic substance so broken down be quickly afterwards brought into contact with the egg in water, at the moment when the egg is most susceptible of its influence, yet that no fecundation is then effected by it. It may be urged, it is true, that there was no direct *proof* that the spermatic substance was actually conveyed to the yelk by, or with, the water in these trials; but there was one circumstance which gave fair reason to believe that it really was so conveyed. The yelks of some of the eggs became much contracted, and, as it were, shrivelled, as when affected by potass solution; of the penetration of which to the yelk there is absolute proof in the decomposition of the egg which quickly succeeds to its introduction. It has also been shown in a preceding experiment (p. 241) that the yelk becomes similarly affected when decomposing spermatic fluid is applied to it.

The conclusion then which seems to me to be deducible from these investigations is, that fecundation is not simply the result of a mere fusion or chemical admixture of the substance of the spermatozoon with that of the egg; although such fusion, probably, is necessary to the production of the organized body of the embryo, in determining its structural and psychical peculiarities, and its definite species; and more or less of which may, possibly, help to determine the sex, and the extent to which the structural and psychical peculiarities of the male parent are transmitted to the offspring. That this fusion of some portion of the spermatic substance with the egg does actually take place, either when the spermatozoon has arrived at, and become imbedded in the vitelline membrane, as in the Frog; or, as stated by a recent observer*, to occur in the *Ascaris Mystax*, when it is in immediate contact with the yelk itself, is probable, from the considerations now adduced. These views are countenanced by the now established fact that the spermatozoon of the Frog, and probably also of other Vertebrata, does not fertilize the egg either when it is perfectly motionless,—whether from actual death, or from suspension of vitality by narcotization; nor even, as we now find, though living, while simply in contact with the surface; nor until after it has actually passed into the envelopes, and arrived at the immediate vicinity of the yelk,—facts which seem, I think, inferentially, to show a probability that, *whatever conjugation of materials may be effected, some vitalizing dynamic influence is also expended by the spermatozoon on the contents of the yelk* in the production of its changes,—phenomena which have not been found to take place

* Dr. Nelson, "On the Reproduction of the *Ascaris Mystax*," Proceedings of the Royal Society, vol. vi. p. 86. Philosophical Transactions, 1852.

on the simple application to the egg of the diffluent spermatic substance. In the absence, then, of all proof that simple fusion only of the substance of the spermatozoon with the contents of the egg is the chief condition of fecundation, we have, in the facts now referred to, much reason to regard fecundation as primarily dependent on the existence of a force or power in the spermatozoon, by which this body is enabled to arrive at the object to be fertilized, and which, visibly expressed in that of its power of motion, is lost quickly after it has penetrated into the yelk-membrane or yelk:—this, probably, is the instant of fecundation. May not the spermatozoon, then, be viewed as *the organ of a special form of force in the male body* for the production of these results, in the same way as the nervous structure is regarded as that of the nervous, and the muscular as that of muscular power? We have already seen that its function is exercised more or less readily and perfectly in relation to two conditions: first, that of its full maturity, in which its power of motion is most intense; and next, in relation to the influence of external physical agencies, which cause this power to be evolved through its organic composition, in a greater or less degree, whether the external influence be that of heat, or the operation of a chemical power, as in the instance of potass in the preceding experiments.

XI. *Description of some Species of the extinct Genus* Nesodon, *with Remarks on the primary Group* (Toxodontia) *of Hoofed Quadrupeds, to which that Genus is referable. By Professor* OWEN, *F.R.S. &c.*

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THE application of part of the Government Grant, allotted to procuring drawings of rare and nondescript fossil remains, has furnished the requisite illustrations of the following Memoir on some extinct Herbivorous animals, which indicate a primary group of the *Ungulata*, distinct from and probably equivalent to one of the great divisions called *Proboscidia, Artiodactyla*, and *Perissodactyla*.

Genus NESODON*, O.

The genus so called, from most of the enamel-folds penetrating the crown of the molars being, when exposed by use, of an insular form, was founded by me in 1846†, on portions of an upper and lower jaw, with teeth, transmitted by Captain SULIVAN, R.N., from the S.W. coast of Patagonia.

I have since received from the same zealous contributor of materials towards the advancement of our knowledge of the ancient zoology of South America, more extensive portions of both upper and lower jaws with teeth of the generic character of *Nesodon*, but belonging to a smaller species than that of which the first described specimens formed part; the second collection also contained detached teeth, indicating much larger species than either of the two above indicated.

I propose to commence with the description of the remains of the smaller species, which from its size may be termed—

Nesodon ovinus, Plates XV. and XVI.

With the upper jaw of this species (figs. 1–5) are preserved the articular surface (figs. 1 and 3, *g*) for the right ramus of the lower jaw, the zygomatic arch (figs. 1 and 2, $_{26}$ and $_{27}$) and part of the nasal bones (figs. 1 and 2, $_{15}$).

The total length of this portion of skull from the post-glenoid process‡ (figs. 1 and 3, *pg*) to the anterior incisor (fig. 1, $d\,i_1$) is 6 inches 4 lines. The breadth of the

* νῆσος insula, ὀδοὺς dens.

† Abstracts of Papers in 'Reports of the British Association' for 1846, vol. xvi. p. 66.

‡ The process, sometimes much developed, as in the Rhinoceros, *e.g.*, is so called in my Catalogue of the Osteology in the Mus. Coll. Chir., which answers to the part described as the "middle root of the zygoma" in Anthropotomy (Bourgery, Traité Complet de l'Anatomie de l'Homme, tom. 1[re], plate 23, fig. 10, *c*.)

glenoid cavity is 1 inch 4 lines. It is slightly concave from side to side, slightly convex from behind forwards, where it forms the 'eminentia articularis' (fig. 3, *g*), which is about 3 lines in antero-posterior extent at its middle, widening to twice that dimension at the outer and the inner ends of the eminence. The narrower concavity behind this part is bounded behind by the thick process (*pg*), extending a little below the level of the outer end of the eminentia articularis, convex anteriorly and below, concave behind, into which deep concavity a venous canal or sinus appears to have opened.

The sutural surface on the inner side of the squamosal (fig. 3, $_{27}$), for junction with the alisphenoid and tympanic, extends from the eminentia articularis (*g*) to the lower end of the post-glenoid process (*pg*), behind which the broken surface of the back part of the squamosal shows part of a smooth cavity.

The zygoma (fig. 1, $_{26}$, $_{27}$) is deep and strong, bent from behind forwards in a gentle sigmoid curve, first convex upwards, then more deeply concave to form the lower part of the orbit, very slightly arched outwards, and formed by the squamosal ($_{27}$) and malar ($_{26}$) in almost equal shares, the major part of the suture between them describing a strong curve, convex forwards. The zygoma is thickest at its two extremities, and of least vertical extent near its anterior one, where it forms the orbit (fig. 1, *o*). There is no trace of post-orbital process from either the malar or squamosal parts of the zygoma, nor of any sutural surface for a descending post-orbital process, as in the Horse. The orbit communicates as extensively with the temporal fossa as in the Tapir and Rhinoceros. The maxillary bone (fig. 1, $_{21}$) sends backwards a plate along the inner side of the fore-part of the zygoma, which unites by a squamous suture with the malar; the malar terminates forwards, gradually contracting as it ascends, at about the middle of the anterior boundary of the orbit, whence it extends but a very short way upon the face. The articular surface for the lacrymal bone may be seen upon the conjoined orbital portions of the malar and maxillary, but the lacrymal itself is lost: its extent upon the face must have been as trifling as that in the Hyrax. The facial part of the maxillary is lofty but not long, gradually decreasing in depth to its anterior border, which is nearly vertical, measuring 1 inch 4 lines in that extent, and wholly united to the pre-maxillary. The outer facial surface of the subquadrate facial plate of the maxillary is concave, and perforated by the antorbital canal (*s*), which opens 6 lines in advance of the orbital border of the maxillary, and 13 lines from the suture with the premaxillary. There is a second small foramen in advance of the large ant-orbital one. The palatal plate (fig. 3, $_{21}$) of the maxillary extends backwards beyond the alveolus of the last molar tooth in place, where its breadth is equal to that at the fore-part of the plate, viz. 5 lines: the plate is 10 lines broad at the middle of its extent, where it forms the anterior end of the oblique rough suture (ib. $_{20}$) for the palatine. The palatines are lost in the specimen; they appear to have formed the middle two-fourths of the produced back part of the bony palate.

The extent of the suture of one maxillary with the opposite maxillary is 2 inches, and behind this the extent of the suture of the maxillary with the palatine is 1 inch: at this part the bone gains much in depth. The palatal plate of the maxillary slopes from the alveolar border towards the median suture, with a curve concave downwards, giving an arched form to the bony roof of the mouth at its anterior two-thirds, behind which the bony palate is less concave transversely, and is convex longitudinally. The post-palatal canal opens obliquely forwards by a foramen situated a little in advance of the middle of the palatal plate of the maxillary a little behind the interval between the third (d_2) and fourth (d_3) teeth in the present specimen. The vacuity behind the maxillo-palatal plates, which was filled up by the palatines, is of a nearly equilateral triangular form.

The premaxillaries (figs. 1, 3 and 4, $_{22}$) are short in proportion to their height, are of a subquadrate form, with the part containing the mid-incisor, *i* 1, bent rather abruptly inwards to join its fellow below the nostril, the face being terminated by an obtuse broad muzzle. The right premaxillary is most entire; it is a little dislocated outwards, and has been separated about 4 lines from its fellow, and to a less degree from the maxillary of its own side. The upper border, which joined the nasal ($_{15}$), is 9 lines in extent; that which joined the maxillary ($_{21}$) is 1 inch 4 lines; that which joined its fellow ($_{22}$) is 10 lines. The palatal plate (fig. 3, $_{22}$) is obliquely grooved by the prepalatal or incisive foramina, the thin inner parietes of which have been broken away.

The portion of the nasal bones preserved (figs. 1 and 2, $_{15}$) show them to have been long and rather broad, meeting above and bent, so as to form an arch: they formed, with the premaxillaries, the external bony aperture of the nostrils, the maxillaries being excluded.

The lower jaw (mandible, Plate XVI. fig. 6) is remarkable for the great proportion which the ascending ramus bears to the horizontal one. The part of the condyle preserved shows it to have been transversely extended and rather convex; the coronoid process rose above its level, and is separated from it by a deep emargination. The entry to the dental canal is 2 inches 2 lines below the condyle. From the condyle to the lower border of the ramus beneath the last molar tooth in place measures 4 inches; from the same part of the lower border to the socket of the outer incisor is 2 inches 9 lines, the depth of the ramus at the same part is 1 inch 4 lines. There is a faint linear impression along the part of the lower border which is preserved. The incisive part of the mandible slightly expands; the outer surface of the horizontal ramus is, lengthwise, convex at its posterior two-thirds, and concave in the rest.

The teeth preserved in the present specimen of *Nesodon ovinus* are,—three incisors, a canine, and five molars on the right side, upper jaw; the canine and five molars on the left side, upper jaw; and the outer incisor, the canine and five molars in the right ramus of the lower jaw.

The upper incisors, unequivocally defined by their position in the premaxillary bone, diminish from the first (innermost) to the third, and vary more in form than size. The first (figs. 1 and 4, i_1) is curved, with the convexity foremost; the fang is slender and contracts as it penetrates the bone; the crown expands and presents the true trenchant incisive form, is flattened transversely at the fore-part, and slightly convex at the back, with the edge worn flat, but with a slightly raised thin enamel ridge. The second incisor (ib. i_2) does not quite touch the first; its crown is curved, with the most prominent convexity at its fore-part, but it is narrower than in the first, and is three-sided, the outer and hinder facets being broader than the front one. The enamel is not continued upon the hinder facet: the contracted fang shows it to be, like the first incisor, a tooth of restricted growth. It is inserted at the angle of the premaxillary, where the front wall bends to form the side wall, and is external and a little posterior to the first incisor. The third incisor (ib. i_3) is double its own breadth's distance from the second, and is implanted behind it, close to the suture with the maxillary; the crown is short, subtrihedral, and so much smaller than the others, as almost to be termed rudimental; it is, however, worn obliquely, like the crown of the second incisor.

The canine (figs. 1, *d c* and 3, *c*) is intermediate in size between the third and second incisors; it is implanted near the suture: the summit of its crown, which was conical and subcompressed, is worn away; its outer side presents a middle convexity bounded by a narrow longitudinal channel near each margin.

The first grinder (ib. d_1) is about the size of the second incisor, and is 1½ line distant from the canine; its crown is thicker as well as broader and longer than that of the canine; the external convexity is produced into a longitudinal ridge near the fore-part of the crown, and its summit forms the most prominent part of the grinding surface, which surface slopes away both forwards and backwards from it, the hinder worn facet being much the largest. The base of the crown contracts and divides into two short nearly parallel fangs. The entire length of the tooth is 7 lines.

The second grinder (ib. d_2), of nearly twice the size of the first, close behind which it is placed, presents two longitudinal ridges dividing three grooves on its outer surface, and a triradiate termination of an inflected fold of enamel at the middle of the grinding surface. The length (vertical diameter) of the crown is 5 lines, and it is implanted by short diverging fangs, two of which are external, and are exposed in Plate XVI. fig. 5, d_2; the total length of this tooth is 7½ lines.

The third grinder (d_3, figs. 1, 3, 5) presents somewhat more than a proportional increase over the second, the hind fourth of which it overlaps, its breadth being 6½ lines. The anterior of the two longitudinal eminences on the outer side of the crown is more prominent and angular than the second convexity which has begun to expand and subside, and the anterior overlapping border of the crown is compressed and produced like a ridge. A fold of enamel penetrates the middle of the inner side of the crown, dips into its substance, contracts as it proceeds outwards and forwards, and bifurcates; behind this there is a small pit or island of enamel:

the postero-external angle of the crown is slightly produced; the increase in the size of the crown is chiefly in its antero-posterior extent and length, which latter is 7½ lines before it divides into fangs, two of which are external, short and divergent. The total length of this tooth is 10 lines.

The fourth grinder (ib. d_4), presents an increase of size in the same directions as the third was enlarged, but to a minor degree in antero-posterior extent than in length, in proportion to which the crown is narrow transversely at its grinding surface. The posterior external longitudinal eminence has much subsided, and has lost all the character of a ridge; the anterior external longitudinal ridge continues to be angular and well-marked; it is near the similarly produced ridge-like anterior angle of the crown, which projects outwards, in advance of the hind part of the third grinder. The bifurcating internal enamel-fold penetrates the anterior half of the grinding surface; a second wide simple fold, parallel with the first, answers, in this less worn tooth, to the enamel island in the third grinder; the postero-external angle is more produced than in that tooth. The length of the crown is 1 inch, following its curvature, which is now well-marked, and one-half of the crown penetrates the substance of the maxillary, contracting before it divides into fangs, two of which are external, short and divergent; the entire length of the tooth is 1 inch 5 lines.

The crown of the fifth molar (ib. m_1) had but recently begun to protrude from its socket, and is only abraded at the anterior half of the grinding surface. The branches of the forked enamel-fold appear as two distinct folds, and there is a slight depression behind the beginning of the main fold. In the hinder lobe of the tooth there is a wide fold on the inner side, and a depression at its end, which would give rise to an island of enamel at a certain stage of abrasion of the crown. The thickness of the exposed and worn surface of this grinder does not exceed that of the preceding tooth; the breadth or antero-posterior extent is a little greater; but the length of the crown, which is imbedded in the jaw-bone, is considerably greater; it measures, following the outer convexity, 1 inch 8 lines, increasing in thickness, and slightly diminishing in breadth, and terminating in a wide pulp-cavity, without trace of fangs. The ridge-like anterior angle is produced external to, and in advance of, the fourth grinder, in the same imbricated manner as the fourth overlaps the third grinder, and this the second.

In the lower jaw the only incisor remaining (Plate XVI. fig. 6, i_3) is that answering to the third in the upper jaw; but it is double the size of that tooth; the crown has the usual trenchant incisive form convex externally, slightly concave on the opposite surface, with a long, straight fang, contracting as it sinks in the socket.

The canine (ib. *c*) is somewhat less, with the crown similarly compressed, but with its worn margin more angular than in i_3.

The first molar (ib. d_1) much resembles the canine, but is somewhat larger, and the longitudinal convexity and depressions of its inner surface are more marked; its root is single and contracted like that of the canine.

The second molar (ib. d_2) has been markedly larger than the first, but is too mutilated for description; it was implanted by two fangs.

The third molar (ib. d_3), of still increasing size, and chiefly in antero-posterior extent, presents a long compressed crown, divided into two lobes, convex externally, by an oblique, unequal-sided longitudinal angular indentation: the inner side of the hinder and larger lobe is penetrated by two folds, extending obliquely outwards and forwards, fig. 8, d_3.

The fourth molar (ib. d_4), with a slight increase of antero-posterior extent, has its anterior lobe indented internally.

The crown of the fifth molar (ib. m_1) resembles that of the fourth in configuration and size, but is narrower, from having been less worn. The grinders are placed close together one behind another, in a straight line and a little overlapping.

All the teeth present a deep brown, in some parts almost black, colour, with a polished shining surface. The enamel is remarkably thin, and is closely blended with the dentine, which renders it very difficult to determine its extent upon the exterior of the crown: it appears to be continued much further upon the outer than the inner side of the tooth.

The incisors and canines being indicated by their position, there remained then to determine the nature of the five grinding teeth in place, on each side of both jaws. That this was not the entire number characterising the dental formula of the animal, was shown by the germ of a molar (Plate XVI. fig. 5, m_2) in the fractured hinder protuberance of the maxillary, behind the last grinder in place, the crown of which (ib. m_1) had very recently emerged from the socket.

To obtain further insight into that formula—so important in the elucidation of the true generic character and affinities of the *Nesodon*—I removed the inner parietes of the sockets of the molars in the right ramus of the lower jaw, and, excavating below the base of the coronoid process, exposed there the germ of a molar (ib. fig. 7, m_2) answering to that forming the sixth (m_2, fig. 5) in the upper jaw. The third (d_3) and fourth (d_4) grinders were implanted by two long tapering and diverging fangs; but the fifth grinder (m_1) presented a marked difference; its crown being continued undivided, with a slight and gradual diminution in antero-posterior extent, and corresponding increase in transverse thickness, to a level, half an inch lower in the deepening ramus of the jaw, than that of the end of the fangs of the antecedent grinder, d_4. The middle indentation on the inner side of the crown of the fifth grinder (m_1) gradually terminates 9 lines below the summit, the rest of the inner side of the crown being almost flat: at its base a wide pulp-cavity is exposed (fig. 9, m_1), into which the obtuse conical end of a deeply inflected enamel-fold projects from above, indicating great variety in the pattern of the grinding surface at different stages of abrasion of the crown.

From the difference between the fourth (d_4) and fifth (m_1) grinders, I conclude them to belong to different series of those teeth; and, from the analogy of the times of their appearance in other *Herbivora*, I regard the fifth tooth in place as the first true molar (m_1 of my dental notation*), the succeeding germ to be the second true molar (m_2), and the preceding four grinders in place as being the deciduous grinders (d_1 to $_4$). At the stage of growth here exemplified, with the crown of the first true molar (m_1) only beginning to emerge from its socket, the germs of the vertical successors (premolars) of the deciduous molars could scarcely be expected to be calcified; but the depth of the jaw beneath those molars, and the space provided by their narrow and diverging fangs, indicate a provision for lodging the matrices of successional teeth, and the small cavity in the substance of the jaw (fig. 7, p_4) beneath the fangs of the fourth molar (d_4) is probably the beginning of the formative nidus of its successor, the fourth premolar (p_4).

The small size of the incisors, in comparison with the premaxillary bones, and their scattered disposition therein, indicate them to belong to the deciduous series, to which, also, we must refer the diminutive canine.

The shape, size, disposition and complex structure of the molar teeth prove the present extinct species to be herbivorous, and the number and nature of the teeth exposed indicate it to be ungulate. No herbivorous or other Rodent has canines and six incisors in the upper jaw; and no Mammal of the Order *Bruta* (*Edentata*, Cuv.) has deciduous teeth properly so called, or molars divisible into two kinds. All the herbivorous Marsupials have a single pair of large procumbent incisors, and no canines, in the lower jaw.

In comparing the fragments of skull and the teeth above-described with the corresponding parts of existing *Ungulata*, we find the most numerous and important correspondences to be with those belonging to the order *Perissodactyla*†.

Both upper and lower true molars in the *Nesodon* have very long crowns, indicating the roots to be comparatively short, and formed late, as in the genus *Equus* The close imbricated disposition of the upper molars and the pattern of the enamel-folds of the crown find their nearest parallel in the *Rhinoceros*.

Both the Tapir and the Horse resemble the *Nesodon* in the number and relative proportions of the incisors and canines. The Tapir, amongst existing *Ungulata*, offers the closest resemblance to the *Nesodon*, in the form and proportions of the zygomatic arch, of the widely communicating orbital and temporal fossæ, and of the articular eminence and cavity for the lower jaw. The premaxillaries join the nasals as in the Horse. The proportions and shape of the post-glenoid process in the *Nesodon* approach more nearly those in the Tapir or Horse, than those in any of the Artiodactyles‡ in which that process is well-developed, as *e.g.* the Peccari and Hippopotamus, in neither of which is the 'eminentia articularis' developed as in the *Nesodon*. The principal osteological character in which the *Nesodon* deviates from the perissodactyle, and seems to approach the artiodactyle, type, is the production of the osseous roof of

* Philosophical Transactions, Part II. 1850, p. 491.

† *Pachydermes à doigts impaires*, Cuvier.

‡ *Pachydermes à doigts paires* et *Ruminans*, Cuvier.

the mouth backwards as far as, and a little beyond, the last molar in place; but this part of the palate is formed for some extent externally by the maxillary bones (Plate XV. fig. 3, 21), whilst in the Artiodactyles it is formed by the palatines, except where a very thin plate of the maxillary completes the socket of the last molar on its inner side.

Neither family nor subordinal correspondence resulting from the comparisons with existing *Ungulata*, they were next extended to the extinct forms, and especially to those that had previously been discovered in South America. But not to prolong the description with needless details, I may confine myself to the remark, that, in this inquiry, the nearest resemblance to the *Nesodon* was found to be offered by that hitherto anomalous genus for which I have proposed the name of *Toxodon**.

In the *Toxodon platensis* the bony palate is continued, as in existing Artiodactyles, beyond the molar series; but the proportion of this back part which is contributed by the maxillaries resembles that in the *Nesodon*, and is greater than, and different in form from, that in the Hog-tribe, the Hippopotamus and the Ruminants. The palate is arched in the *Nesodon*, and also, but with a deeper concavity, in the *Toxodon*: the post-palatal canals in both have the same advanced position of their oblique outlets upon the maxillary; in both Perisso- and Artio-dactyles those outlets are close to the maxillo-palatal suture. In the form of the 'cavitas et eminentia articularis,' and of the post-glenoid process, the *Toxodon* more resembles the *Nesodon* than does the Tapir or the Horse: the convex and protuberant front and lower part of the process and the excavation of its back part are repeated in the *Toxodon*. The strong, deep, sigmoidally bent zygoma, and the vertically extended orbit communicating widely behind with the temporal fossa, are characters common to the *Nesodon* and *Toxodon*; but the orbit is relatively wider, the malar bone is narrower, and its post-orbital process less prominent in the smaller extinct South American Herbivore. In the concavity of the outer surface of the deep facial part of the maxillary, and the anterior expansion of the premaxillaries, the *Toxodon* resembles the *Nesodon*; but the most important evidence of their mutual affinity is afforded by the dental system.

The molar series has a similar arrangement in the *Toxodon*, describing a line gently convex externally, the teeth gradually decreasing as they advance, and having the same close-set, imbricated disposition. The length, curvature and deep implantation of the undivided crown of the upper molars of the *Nesodon*, as exemplified in the fifth molar of the specimen described, fig. 5, *m* 1, are important marks of agreement with the upper molars of the *Toxodon*, in which those characters are present in a greater degree. The same molars of the *Toxodon* also present an obvious similarity to those of the *Nesodon* in structure. The enamel is very thin: a deep oblique fold, penetrating from the inner side, divides the crown into two lobes; and a short fold indents the inner side of the hinder lobe. The anterior external angle of the tooth is produced like a ridge to form the overlapping part in the imbricated molar series:

* Zoology of the Voyage of the Beagle, Fossil Mammalia, p. 16, Plates I. to V. 4to, 1840.

there are, also, two longitudinal risings along the outer side of the tooth. The chief differences presented by the *Toxodon* are seen in the simplicity of the enamel-folds, the low and equal development of the two external longitudinal ribs or risings, the relatively narrower grinding surface, and the more evident interruption of the external coat of enamel at the anterior, posterior, and internal parts of the tooth. Moreover, there is no trace of canines or of outer incisors (i_3) in the adult *Toxodon*, and the anterior upper incisors are proportionally very long and large; but the difference might be less in the permanent upper incisors of the *Nesodon*.

The correspondence between the teeth of the lower jaw of the *Toxodon* and *Nesodon* is of the same instructive character. No quadruped, recent or extinct, presents so close a resemblance to the lower molars of the *Toxodon*, in regard to their narrowness, and the length and deep implantation of the straight undivided crown, as does the *Nesodon*, especially in the first true molar, exposed in the section shown in fig. 7, m_1. The (most probably) deciduous incisors of the same specimen deviate, however, considerably from the peculiar disposition, size, shape and curvature of the six lower incisors in the adult *Toxodon platensis*, and this difference becomes of more importance as illustrative of the generic distinction of the two animals, from the proportions and arrangement of the permanent incisors and canines in the lower jaw of the adult *Nesodon*, first noticed in the 'Reports of the British Association' for 1846 (Transactions of the Sections, p. 66), where the affinity of the *Nesodon* to the *Toxodon* is indicated, and the interval between the latter genus and *Macrauchenia* is stated to be "partly filled by the newly-discovered *Nesodon*."

We may, now, therefore, regard the following as the generic characters of the *Nesodon*:—

Dental formula,—$i\,\frac{3-3}{3-3}$, $c\,\frac{1-1}{1-1}$, $p\,\frac{4-4}{4-4}$, $m\,\frac{3-3}{3-3}=44$. The teeth subequal, and in an unbroken series. *Incisors* trenchant, with long, slightly curved crowns, but not disproportionately large, and of limited growth, being implanted by a tapering fang: *Canines* smaller, and not exceeding in length the contiguous premolars: *Molars*, above, with long, curved, transversely compressed crowns, which contract, as they penetrate the bone, and ultimately develope fangs: the outer side of the crown ridged, the inner side penetrated by two more or less complex folds of enamel, leaving insular patches on the worn crown: the enamel is deposited in a thin layer. The lower molars with long and straight, transversely compressed crowns, divided by an external longitudinal indent into two unequal lobes, both penetrated at the inner side by a fold of enamel, which is complex in the hinder lobe.

The bony palate entire and extending back beyond the molars, the maxillaries and palatines forming the back part in equal proportions. A distinct articular cavity and eminence for the lower jaw, the eminence long and concave transversely, short and convex longitudinally; a protuberant post-glenoid process; a strong and deep zygoma; the orbit and temporal fossa widely intercommunicating; premaxillaries joining the nasals.

Of the genus offering the above osteological and dental characters, I have already received evidence of four species.

The first is that about the size of the Llama, called *Nesodon imbricatus**, founded on the fore-part of the lower jaw and on two molars of the upper jaw. The second species, of the size of the Zebra, called *Nesodon Sulivani* in honour of its discoverer, is indicated by detached molar teeth of the lower jaw. The species to which the portions of skull above described belonged, seems, from the size of the first and second true molar tooth, to have been about the size of the *Vicugna*, or of a large sheep, and I propose to call it *Nesodon ovinus*. A fourth species, as large as a Rhinoceros, is indicated by an upper molar tooth; and I propose to call it *Nesodon magnus*.

That the specimens on which the species *Nesodon ovinus* is founded, did not belong to a young individual of the *Nesodon imbricatus*, is shown by the difference in the extent occupied by the first four (deciduous?) grinders in the lower jaw, Plate XVI. figs. 6, 7, 8, d_{1-4}; and in that occupied by the first four grinders (premolars) in the lower jaw of the *Nesodon imbricatus*, Plate XVII. fig. 11, p_{1-4}; and by the great difference in the size of the teeth, which, on the hypothesis of the two alleged species being one, must hold the relation of temporary and successional teeth. For, assuming, as is most probable, that the first four grinders in the lower jaw of the *Nesodon ovinus* are milk-teeth; yet these are displaced and succeeded vertically by the premolars, which are in most Herbivores smaller and less complex, and in no Mammals larger and more complex, than the teeth they so displace; such premolars never occupying a greater longitudinal extent, although they sometimes, when fewer in number than the deciduous grinders, occupy a less extent, in the jaw of the adult animal. We may therefore safely infer that the four teeth occupying the extent of 3 inches in the lower jaw of the *Nesodon imbricatus*, p_{1-4}, fig. 11, could not have risen into the place of the four molars, which only occupy the space of 1 inch 9 lines in that of the *Nesodon ovinus*, d_{1-4}, fig. 6.

If, next, we compare the crown of the first true molar in the upper jaw of the *Nesodon ovinus* (fig. 3, m_1) with either of those (fig. 10) referred in my former Memoir† to the *Nesodon imbricatus*, we shall obtain equally conclusive evidence that they have not belonged to the same species.

Upper Molar Teeth of NESODON IMBRICATUS.

The antero-posterior extent of the crown of m_1 in the *Nesodon ovinus* is 9 lines; in the *Nesodon imbricatus* it is 15 lines; the greatest transverse diameter of m_1 in *N. ovinus* is 5 lines; in *N. imbricatus* it is 9 lines. The generic conformity of the shape of the crown and the disposition of the enamel is very close. Both the upper molars (Plate XVII. fig. 10) referred to the *Nesodon imbricatus* have been much worn, the anterior one, of course, the most. In this tooth the posterior division of the forked

* Reports of the British Association, 1846, p. 66.

† *Loc. cit.* p. 67.

fold, *a*, is obliterated except at its end, which forms the enamel-island *b*, fig. 10, and the deep termination of the second fold forms the second island, *c*, which is separated from the rest of the fold, which also forms an island, *d*.

In the next molar, the deep beginning of the second fold, *d*, still communicates with the enamel covering the inner side of the molar: the two branches of the forked fold, *a*, retain the continuity with the main stem, and the only island is that (*c*) formed by the sunken end of the second fold. On the implanted side of the crown of the first molar the obtuse blind ends of both folds project into the bottom of the pulp-cavity of the crown, from which the beginnings of short roots are seen to diverge. The bases of the two folds are seen to be more distinct in the wider pulp-cavity of the second and less worn tooth. The crowns of both teeth have had the same curved form as in the *Nesodon ovinus*; but the greater part has been worn away.

Part of the Lower Jaw and some Teeth of the NESODON IMBRICATUS.

The fossil which originally indicated the genus, and was referred to the species *imbricatus**, consists of the symphysis and part of the right ramus of the lower jaw (Plate XVII. figs. 11, 12, 13, 14). The symphysis is obliterated by complete confluence of the rami at that part; it is rounded anteriorly and slopes downwards and backwards at an angle of 120° with the lower border of the rami; these are slightly concave along the outer walls of the sockets of the anterior premolars, as if the rami had been gently pressed together at that part, in advance of which the symphysis slightly expands before forming the convex termination supporting the incisors.

These teeth ($i_{1,2,3}$, figs. 11, 12, 14) incline forwards at the same angle as the symphysis, and increase in size from the first to the third; the summits of the crowns have been worn to an almost flat elongated oval or crescentic surface, from $1\frac{1}{2}$ to 2 lines in the smaller diameter. Those of the first pair (i_1) are in contact, the contracting bases slightly diverging to their insertion, which is by a long and slender fang; the length of the enamelled crown is 8 lines, the breadth at the worn summit $4\frac{1}{2}$ lines; this is thickest at the inner border, when the thickness so increases towards the fang as to give a three-sided figure to the transverse section of the base of the crown: the length of the fang is 1 inch, fig. 12.

The second incisor (i_2) is situated partly behind and to the outer side of the first, so that the mesial angle of the crown is overlapped anteriorly by the outer angle of that of the first incisor. The crown of the second has its anterior surface almost equally divided into two facets, which meet at an angle which marks the extent of the overlapped part of the crown. The posterior surface of the crown is divided unequally, the intervening angle or longitudinal eminence being near the inner rounded border of the crown. This is 9 lines in length, and nearly 6 lines in breadth at the worn surface. The fang contracts as it descends in the socket, and is 1 inch 1 line in length, fig. 12.

* *Loc. cit.* p. 66.

The third incisor (i_3) has the anterior surface turned more outwards, but similarly subequally divided into two facets, the mesial one lying directly behind and upon the second incisor; the posterior surface is more flattened; the inner border is sharper and more produced; the outer angle of the crown is rounded off. The length of the crown is 10 lines, its greatest breadth, which is below the worn end, is $7\frac{1}{2}$ lines; the fang is somewhat shorter and thicker, but contracts, like the others, to its end; its length is 11 lines, fig. 12.

The canine (c) rises and slopes forwards close behind the third incisor; it is of smaller size, with the fore-part of the crown more convex, and unequally divided by a longitudinal eminence nearer its outer border; the posterior surface is concave; the summit is thin, rounded, with the enamel not worn off, although it is on the same level with those of the incisors. The crown is 9 lines long and 6 lines broad, half-way down. The enamel is continued much further down the fore than the back part of the crown in the incisors and canines, and the length of the crown has been taken from the fore-part. The fang of the canine (c, fig. 12) is 7 lines in length.

The first premolar (p_1) rises behind and to the inner side of the canine; the two surfaces of the crown are directed outwards and inwards; the outer surface has its anterior half convex, its posterior half concave; the convex part rests on the concave surface of the canine, and the concave part receives the lower part of the anterior convexity of the second premolar. The working border rises to an obtuse point formed by the summit of the longitudinal prominence dividing the two parts of the outer surface. The inner surface is slightly concave anteriorly. The height or length of the crown is 9 lines, the breadth of the crown is 8 lines. It is implanted by a single, short and thick, moderately tapering, and obtusely ended fang, 6 lines in length (p_1, fig. 12).

The second premolar (p_2) has the crown divided into two equal lobes by a wide and deep longitudinal angular indentation on its outer surface. The anterior lobe is the most prominent one, and has three sides, the antero-external surface being transversely convex: the postero-external one flat, and the internal surface concave. The inner concavity of the hinder lobe sinks at its lower end into the substance of that lobe. The whole crown is convex vertically on the outside, concave on the inside, being slightly bent inwards in the direction of its length; this measures 10 lines, the breadth is 8 lines: the crown gradually contracts to the fangs (p_2, fig. 12), which are two in number, and diverge as they penetrate the sockets; their transverse breadth is considerable; their length 6 lines. The anterior one passes external to the fang of the first premolar, which is wedged, as it were, into the internal interspace between the canine and the second premolar.

The third premolar (p_3, figs. 12 and 14) presents the general complex form and structure characteristic of the true molars. The external entering angle marks off the anterior third of the crown as the anterior lobe, which is convex transversely, and more prominent than the larger and less convex posterior lobe; it is slightly indented on the inner side. The posterior lobe shows the two islands of enamel *a* and *b*, fig. 14, Plate XVII. on its grinding surface and the terminal fossa (*c*) of the internal longitudinal fold almost reduced to the state of an island. The length of the crown is 9 lines, the breadth (now the antero-posterior extent) is 11 lines, the thickness of the worn surface of the anterior lobe 4 lines.

The crown of the fourth premolar (p_4, figs. 11 and 14) is 1 inch (12 lines) in breadth, but sinks much deeper in the substance of the jaw before dividing. The narrow curved enamel-fold, answering to the island *a* in p_3, is seen in the present less-worn tooth to be a fold marking off the hind boundary of the anterior lobe on the inner side of the crown: the circular island (*b*) is repeated; the inner fold (*c*) of the hind lobe is wider. The grinding surface of the fourth premolar does not exceed that of the third in breadth.

Only the anterior lobe of the first true molar (m_1) is preserved; it has been worn down two lines below the level of the last premolar, and the internal depression is wider and deeper than in the premolars. Its length is 1 inch 10 lines, of which more than 1 inch is implanted in the jaw, and it terminates below in a widely open pulp-cavity, fig. 13.

The last three premolars and first true molar are in close apposition, one behind the other in the same line; the anterior surface of one being pressed so close against the back part of the next as to have caused the disappearance of the enamel, if it had been originally laid upon the anterior surface of the crown so worn by pressure.

The part of the jaw here preserved shows it to have been deep and narrow; the hinder half probably resembled in its height that in the *Nesodon ovinus*. Two small foramina on the same horizontal line open upon the outer surface, midway between the upper and lower borders; one of them 1 inch 9 lines behind the anterior end of the symphysis, the other 2 inches 3 lines behind the same part. The antero-posterior extent of the symphysis seems to have been not more than 2 inches.

The breadth of the jaw at the alveoli of the first premolars is 1 inch 8 lines; it becomes wider below, as well as in front and behind those teeth.

The lower molars of the *Toxodon*, in their great length of the undivided crown, their breadth, their thinness, the narrow and prominent anterior lobe, and the broader but thinner and internally penetrated posterior lobe, manifest the same general type of structure, as those above-described, in the *Nesodon*; the generic distinction is seen in the less complex disposition of the enamel.

When the lower jaw and teeth of the *Nesodon imbricatus* are compared with those of the *Nesodon ovinus*, differences present themselves, which can only be referred, in the present state of zoological principles or philosophy, to a distinction of species.

The crown of the outer incisor in *Nesodon ovinus* (Plate XVI. figs. 6 and 8, i_3) is less than half the size of that in *Nesodon imbricatus*, so likewise is the crown of the canine and first grinder; and, supposing these teeth to be the deciduous ones, they are proportionally much smaller than the same milk-teeth are in comparison with their permanent successors in the Horse or Ruminant. In no existing Herbivore is the crown of the third or fourth premolar longer than that of the third or fourth corresponding milk-teeth; but the antero-posterior extent of the crown of the fourth premolar in *Nesodon imbricatus* equals that of both third and fourth (milk?) molars in the *Nesodon ovinus*.

The difference of the size of the fifth molar (*m* 1) in *Nesodon ovinus*, compared with the portion of the first true molar (*m* 1) in *Nesodon imbricatus*, is decisive as to the superior size of that species to the adult of the *Nesodon ovinus*, on the hypothesis that the teeth anterior to m_1 are deciduous in the specimen on which I have founded the latter species.

The summit of the thin but broad crown of m_1, fig. 7, has but just begun to be abraded in the *Nesodon ovinus*; but the entire extent of the inflected fold of enamel is displayed in the imbedded part of the tooth, which has been exposed at m_1, figs. 7 and 9, the vertical extent, from the summit of the tooth, being 1 inch 4 lines.

The vertical extent of the part of the first molar (fig. 13, m_1) preserved in the lower jaw of the *Nesodon imbricatus* is 1 inch 10 lines, and the crown continues to expand to the widely open lower termination, which measures 6 lines in thickness, the same diameter of the bottom of the crown of the first molar (m_1) in the *Nesodon ovinus* being 2 lines.

Here, then, we have two species of the extinct genus restored; and the fossils upon which the *Nesodon imbricatus* is founded supply the characters of the permanent teeth which are so singularly clustered in overlapping arrangement, at the fore-part of the lower jaw, whilst the characters of the first or deciduous dentition of the same part of the jaw are illustrated in the more complete fossils that give the characters of the *Nesodon ovinus*.

Lower Molar Teeth of NESODON SULIVANI.

I next proceed to the description of the remains of the third species of the present remarkable genus.

Nesodon Sulivani.—This species, which appears to have been about the size of a Zebra, was founded, in my original Memoir*, on some detached teeth, of which I now subjoin more detailed descriptions, with figures.

Plate XVIII. fig. 15, represents a lower grinder, with the crown worn down, and the root broken away, an equal extent of both parts of the tooth being preserved. The root has been fractured at the part where its two divisions had not begun to separate; but where their proportions are marked out by a median external and internal longitudinal indentation, connected by a line traversing the consolidated substance of the fractured base, and indicating it to consist of two connate fangs. By the equality of the two lobes of the crown, indicated by the external longitudinal

* *Loc. cit.* p. 67.

notch, *n*, this tooth corresponds best with the second premolar (*p* 2) in the lower jaw of the *Nesodon imbricatus*, fig. 11; and, the anterior lobe being indicated by its greater prominence, it is shown to have come from the right ramus of the jaw. The crown has been worn below the part which bears the internal depressions in the *N. imbricatus*; and a smooth field of dentine is exposed, with the indented plate of enamel on the outer side, and a smooth, slightly convex plate of enamel on the inner side, the two plates being interrupted on the anterior and the posterior sides of the crown, due apparently to the unequal degrees in which the enamel was extended towards the fang along different parts of the crown. The outer coat of enamel is thicker than the inner one; and the former best shows the delicate parallel close-set transverse striæ, indicative of the successive formation of that substance; it is also impressed by a transverse row of minute close-set punctations. The fang is enclosed by a thin layer of cement, which is continued upon the parts of the crown undefended by enamel. The hinder lobe is worn lower than the other, as it is also in the *Nesodon imbricatus*. It differs from the tooth of corresponding form in that species, not only in its greater size, as indicated by the dimensions of breadth and thickness, but also by the thicker coat of enamel and the greater length of the undivided root. There is, also, a worn, smoothly excavated surface on the fore-part of the crown, which indicates close contact with an anterior premolar of a different shape from the *p* 1 in the *Nesodon imbricatus*.

The length of the enamelled crown here remaining is 10 lines, its breadth 9 lines, its thickness 6 lines.

A portion of the corresponding tooth of the left side of the same jaw is preserved, in which the bottom of the enamel-fold is still unobliterated, answering to the island, *a*, p_2, fig. 14, in the second premolar of the *Nesodon imbricatus*. The length of this fragment is 1 inch 9 lines; its anterior surface shows the same abrasion from pressure against the tooth in advance, and the fractured surface exposes a contracted pulp-cavity in the crown, which becomes obliterated in the fang.

The tooth, Plate XVII. fig. 16, apparently the premolar (p_3), succeeding the one last described, is also worn down so near the bottom of the crown, that the inequality of the two lobes appears much less than it would be if the same proportion of the crown had been preserved as in p_3 of the *Nesodon imbricatus*; it is probably, therefore, notwithstanding the actual difference in the proportions of the two lobes, the tallying tooth. Of the inflected fold of enamel only the bottom of one remains, forming an island (*a*) in the substance of the grinding surface of the crown, opposite the angle of the fold, *n*, indenting the outer surface of the crown.

The difference in thickness of the outer and inner plates of enamel, which are disconnected at the fore and back parts of the grinding surface, is well-marked, that of the inner plate being less than that of the insular fold, *a*. The inner plate terminates below by a narrow process continued upon the base of each division of the fang. The base of the crown begins to divide above the lower boundary of the outer plate of enamel, into two almost consolidated fangs, the ends of which are broken off, showing in each the very contracted remnant of the pulp-canal. The enamel is continued 9 lines below the termination of the external groove, is continued a short way down each division of the root, and is characterized by four transverse rows of close-set small circular pits. The length of the enamelled crown is 1 inch 5 lines, its breadth 11 lines, its thickness 8 lines. The tooth above described is from the left side of the jaw; its fellow from the right side is preserved, but in a more mutilated condition.

A tooth, Plate XVII. fig. 17, which, in the unequal size of the two lobes of the crown, tallies with the third and fourth premolars of the *Nesodon imbricatus*, and, by its superior size to fig. 16, answers to the fourth premolar (p_4), has had the complexities of the grinding surface reduced by extensive attrition to the bottom of the deepest of the internal folds, *a*. It consequently resembles the foregoing tooth (fig. 16), except in its superior size, and in the greater extent of the hinder lobe, which is less convex externally. The outer plate of enamel, below the longitudinal indentation, shows the same coarse and punctate transverse markings. The more regular and delicate striæ of growth are beautifully and clearly shown. The posterior of the two roots, which is the largest, has a narrow central pulp-cavity. The external coat of cement is continued a little way upon the enamelled parts of the crown, and very plainly coats the anterior and posterior tracts where the enamel is interrupted. The length of the remainder of the enamelled crown of this tooth is 2 inches 2 lines, its breadth 1 inch 1 line, its thickness 7 lines.

A portion of tooth (Plate XVII. fig. 18), including the base of the outer lobe of the crown, corresponds in colour, structure and markings with the same part in the foregoing tooth, but is somewhat larger, and has had a much larger crown; it is most probably a part of the succeeding grinder in the same jaw, viz. the first true molar, m_1. An extent of 1 inch and 6 lines of the outer enamelled part of the outer lobe is preserved, yet the anterior margin of the enamel plate is straight, vertical, and the cement covering the anterior surface of the same shows no trace of abrasion from pressure against the adjoining tooth, as in p_4, at the same distance above the lower border of the enamel covering the external surface of the anterior lobe. The striæ of growth are as conspicuous upon the enamel of the present tooth (m_1) as in that of p_4; and there are two transverse rows of punctations. The lower pointed termination of the internal plate of enamel is preserved at the back part of the fragment; it is 1 inch 4 lines higher than the termination of the enamel on the fore-part of the same tooth.

The tooth, fig. 18, as the reduced size of the enamel-island *a* demonstrates, has had more of its crown worn away, has been longer in use than the one, fig. 17, which, on the conclusion that it is a premolar (p_4) of the same jaw, must have stood in advance of fig. 18; but, though more worn, what remains of the crown of fig. 18 is longer. The first condition—greater degree of abrasion—accords with the determination of this tooth as the first true molar, m_1; and, if the tallying tooth in the lower jaw of *Nesodon ovinus* be referred to, Plate XVI. fig. 7, m_1, it will be seen that that tooth would most probably have a longer crown than the successor of d_4, and would most certainly show a more worn-down summit when the fourth premolar had come into use in advance of it. These evidences of the nature or homology of the fragment of tooth, fig. 18, strengthen my surmise that the teeth (figs. 15–20, Plate XVIII.) are all from the same under jaw.

Plate XVII. fig. 19, is a part of the inner enamelled portion of the crown of a molar 3 inches and a half long, including part of a widely open pulp-cavity at the base, but with an unknown quantity of the worn or exposed end of the crown broken away; it indicates, therefore, a straight, rootless, molar of considerable length; but the character of the enamel accords so closely with that of the better-defined broken teeth above-described, as to indicate it to have belonged to the same genus, species, and, probably, individual, as the other teeth with which it was associated.

The enamel is thinner at its upper broken end than that on the outer side of the above-described teeth of the lower jaw; and it becomes still thinner as it approaches its lower termination. It forms a nearly flat, long plate, terminating by a free, thinned-off border on each side along its lower half, where its breadth is from 8 to 10 lines. The upper half-inch of enamel shows the transverse wavy striæ of growth; the next inch is roughened by the punctate impressions and coarser transverse lines: then the regular striæ of growth are continued for two-thirds of an inch; and again, the punctate character appears for an extent of 3 lines, the alternation of striated and punctated tracts continuing to the irregular rugged termination of the enamel plate. A thin layer of cement covers the outer surface of the lower part of this fragment, which is not plated by the enamel.

Plate XVII. fig. 20, is a similar but smaller portion of a somewhat larger, long and straight tooth, nearly 2 inches in length, and with the enamel-plate 11 lines in breadth, terminating on each side in a thin free border. Both upper and lower ends of this portion have been broken away, the latter above the pulp-cavity; and the enamel and cement here enclose a solid mass of dentine, of a three-sided shape, one of the sides being much narrower than the other two, indicating a long straight tooth, thus in proportion to its breadth. The outer surface of the enamel shows the same alternation of striated and punctated transverse markings as in the preceding fragment.

As the teeth in the portion of jaw of the *Nesodon imbricatus* (Plate XVII. figs. 11–14) are all of the permanent series, the differences in respect of size which they present in comparison with their homologues above-described, are decisive as to their belonging to a smaller variety, if not, as is more probable, to a smaller species. That the degree of distinction is specific, is further indicated by the punctate markings of the enamel, and the more strongly marked striæ of growth; also, by the greater relative thickness of the teeth of the *Nesodon Sulivani*, and by their relatively as well as absolutely thicker enamel. The outer surface of the anterior lobe of p_2, and of both lobes of p_3, is less convex in *Nesodon Sulivani* than in *Nesodon imbricatus*.

Upper Molar Tooth of NESODON MAGNUS.

The superiority of size, indicated by the portion of tooth in Plate XVII. figs. 21, 22 and 23, of the animal to which it belonged, over the *Nesodon Sulivani*, is too considerable to be interpreted as a mere variety in a wild animal. That the dental fragment in question belongs to a species of the present genus, is to be inferred by the close resemblance of the enamelled exterior surface of the tooth to the same part in the upper molars of the known species of *Nesodon*.

This remarkable fragment, which might pardonably have been mistaken for part of the tooth of a Rhinoceros, is the outer side of the crown of an upper molar, worn low down, showing the natural termination of the enamel upon the base of the fangs which, with the inner part of the crown itself, have been broken away.

The remaining part of the crown presents its grinding surface worn obliquely to the enamel, which forms almost a trenchant edge; the produced and ridge-like anterior angle (fig. 23, *a*) is preserved; the longitudinal obtuse ridge of enamel (fig. 21, *b*) on the outside of the crown, near that border, and the gently convex surface of the broad part of the crown behind the ridge, equally repeat the characters of the outer surface of the upper grinders of *Nesodon*, as exemplified in the smaller species (fig. 5, d_4). A narrow and low ridge of enamel is continued from the base of the longitudinal rising, *b*, along that of the rest of the crown, 3 or 4 lines from the radical border of enamel, and then curves upwards along the hinder border of the crown to the posterior angle of the outer part of the grinding surface, *c*.

The crowns of the upper molars of the *Nesodon ovinus* do not show this marginal ridge.

The breadth of the remains of the crown of the upper molar of the *Nesodon magnus* at the grinding surface is 2 inches 8 lines; the length of the enamelled part is 2 inches 4 lines. The resemblance of the fragment in its shape, and in the disposition of the outer plate of enamel, to the similarly sized upper molar of the *Rhinoceros* is very close; but the characteristic thinness of the crown, as shown by the bend of the enamel at the fore-part of the fragment, fig. 23, the thinness of the enamel, and the uniform oblique abrasion of the outer part of the grinding surface, are all marks of closer resemblance to the upper molars of the genus *Nesodon*.

The surface of the enamel in the present tooth of *Nesodon magnus* is polished, but is minutely wrinkled, especially towards its basal termination; it shows no striæ of growth.

The remarkable portion of molar tooth here described and figured indicates a species of *Nesodon* of a size equal to that of the largest extinct species of *Rhinoceros*, which genus we may suppose the *Nesodon* to have represented in the American Continent during the pliocene and perhaps miocene periods.

Concluding Remarks.

The osteological characters defining the orders of hoofed quadrupeds, called *Proboscidia, Perissodactyla* and *Artiodactyla,* are associated with modifications of the soft parts of such importance, as not only to establish the accuracy of the principles of that ternary division of the great Natural group of *Ungulata**, but to indicate that the known modifications of the skeleton of the extinct *Toxodons* and *Nesodons* of South America, in the degree in which they differ from the osteology of the already defined orders of *Ungulata,* must have been associated with concomitant modifications of other parts of their structure, which would justify, and indeed compel, the consistent classifier to place them in a distinct division of the *Ungulata,* of equal value, if not with the *Perissodactyla* and *Artiodactyla,* at least with the *Proboscidia.* Like the *Proboscidia,* this group, which I propose to call *Toxodontia,* is more nearly allied to the Perisso- than to the Artio-dactyle orders.

This is shown by the large and complex third and fourth premolars (p_3 and p_4), by their close similarity with the true molars, by the unsymmetrical oblique foldings and islands of the enamel, and by the great length of the crowns of the molars, to which the Horse alone offers any near approach amongst existing Ungulates. By the form and proportions of the eminentia articularis, of the glenoid cavity, and of the post-glenoid process,—and by those, also, of the lacrymal bone, of the zygomatic arch, and of the orbit,—the *Toxodontia* are most closely matched by the Tapir and Rhinoceros in the Perissodactyle order.

The dental and osteological characters detailed in the text, whilst they illustrate the closer mutual affinities between the Nesodons and Toxodons, establish their claim to be regarded as types of a distinct order of *Ungulata*; and they also tend to dissipate much of the obscurity supposed to involve the true nature of the genus *Toxodon,* and to reconcile the conflicting opinions as to its proper place in the Mammalian Class.

The fossils above described were discovered on the coast of Patagonia to the south of Port St. Julian, and my friend Mr. CHARLES DARWIN, F.R.S., has kindly communicated to me the following opinion as to the formation in which they were imbedded:—"These beds resemble mineralogically the upper ancient tertiary formation of Patagonia, but EHRENBERG found the included microscopical organisms wholly different from those of the ancient tertiary formation, being of freshwater and brackish origin (p. 117 of my Geological Observations on South America). Hence these beds are of unknown age, probably younger than the old tertiary and older than the superficial beds in which *Macrauchenia* was found."

* Memoir on the Anthracotherioid Animals, in 'Quarterly Journal of the Geological Society,' Nov. 1847, and 'Osteological Catalogue of the Museum of the Royal College of Surgeons,' 4to, p. 629.

Description of the Plates.

PLATE XV.

Nesodon ovinus.

Fig. 1. Side view of a portion of the skull.
Fig. 2. Upper view of the same.
Fig. 3. Under view of the same.
Fig. 4. Front view of the right premaxillary and incisor teeth.

PLATE XVI.

Nesodon ovinus.

Fig. 5. Portion of the left upper maxillary bone, with the roots of the teeth exposed.
Fig. 6. Inside view of the right ramus of the lower jaw.
Fig. 7. Outline of the same, with the implanted parts of the teeth exposed.
Fig. 8. Grinding surface of the teeth.
Fig. 9. Under surface or border of the same ramus.

PLATE XVII.

Nesodon imbricatus.

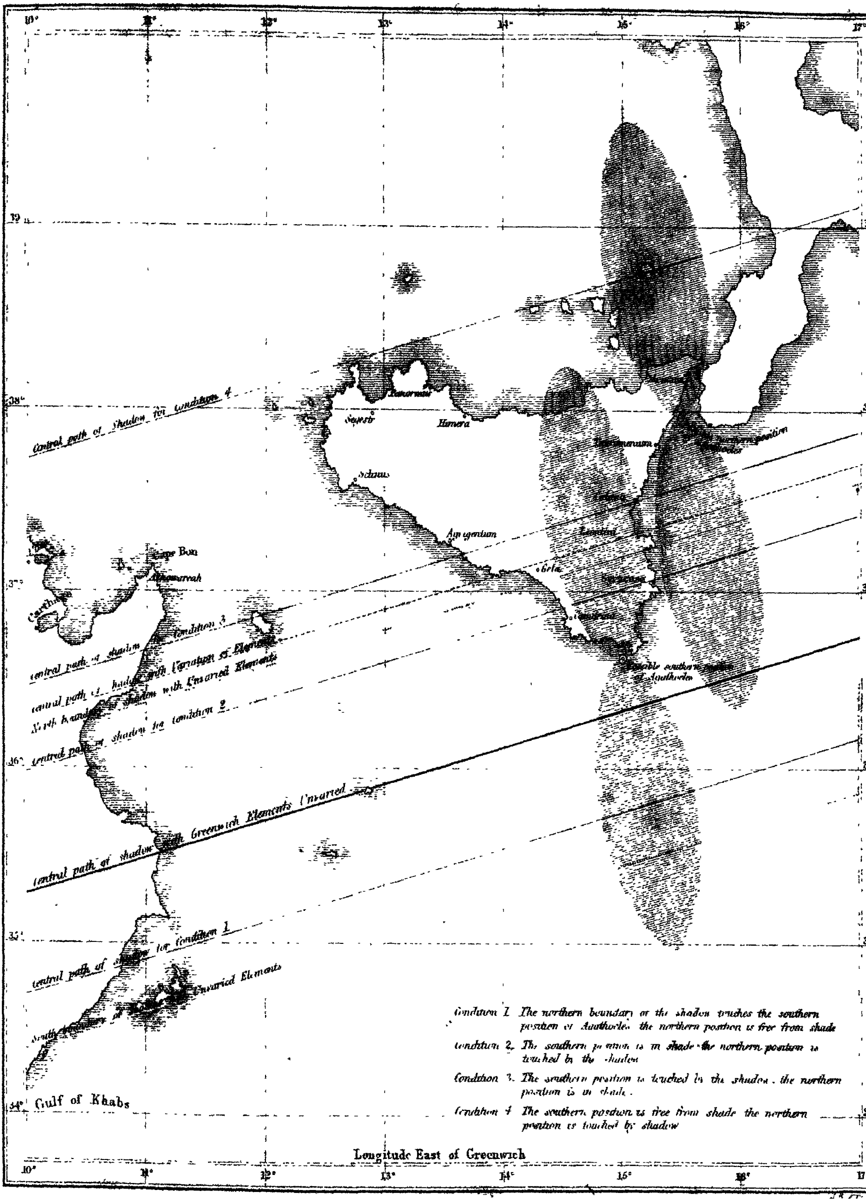
Fig. 10. Grinding surface of two molars, right side upper jaw.
Fig. 11. Part of the right ramus of the lower jaw.
Fig. 12. Part of the left ramus of the same jaw with the roots of the teeth exposed.
Fig. 13. Transverse section of the right ramus of the same jaw, showing the implanted part of the anterior lobe of the first true molar.
Fig. 14. The grinding surface of the teeth in fig. 11.

PLATE XVIII.

Figs. 15–20. Portions of lower molar teeth of the *Nesodon Sulivani.*
Figs. 21–23. Portion of an upper molar tooth of the *Nesodon magnus.*

The figures, letters and symbols, are explained in the text.

MAP TO ILLUSTRATE THE ECLIPSE OF AGATHOCLES.



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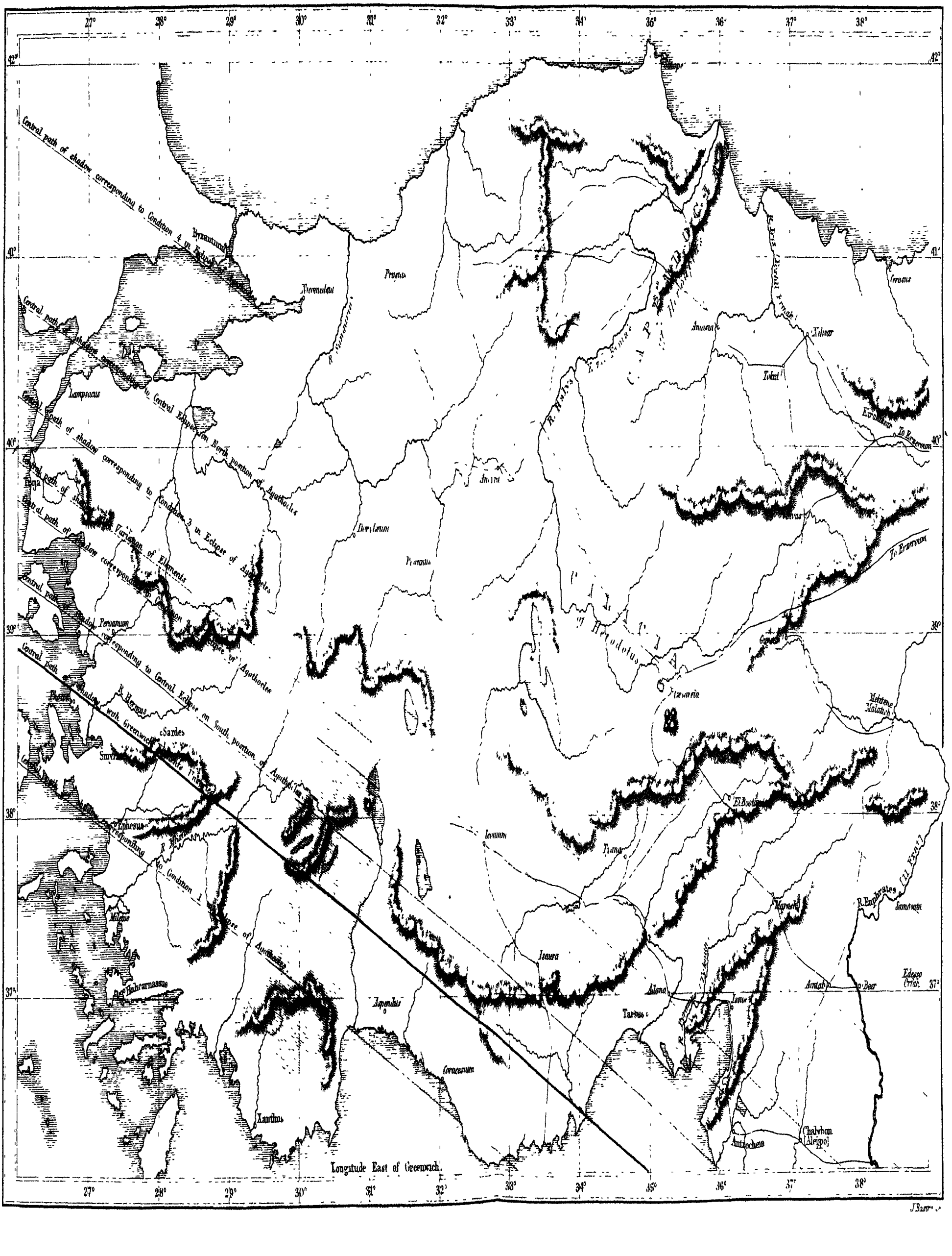

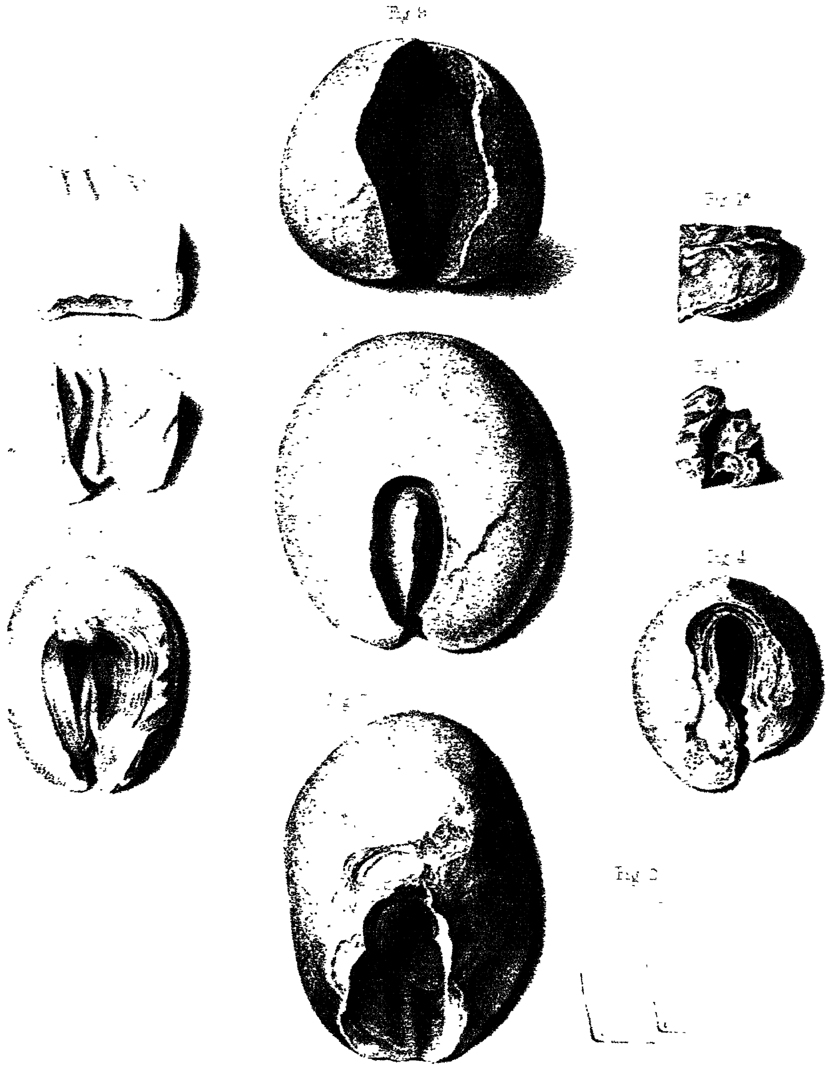
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Fig 4

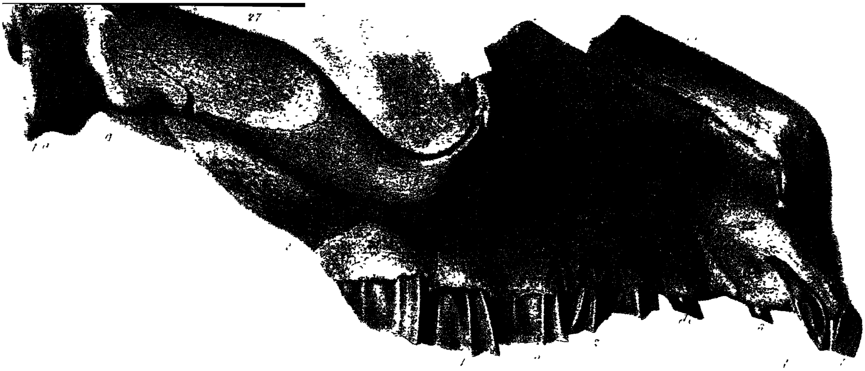
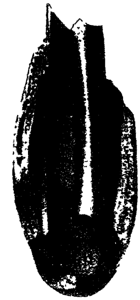
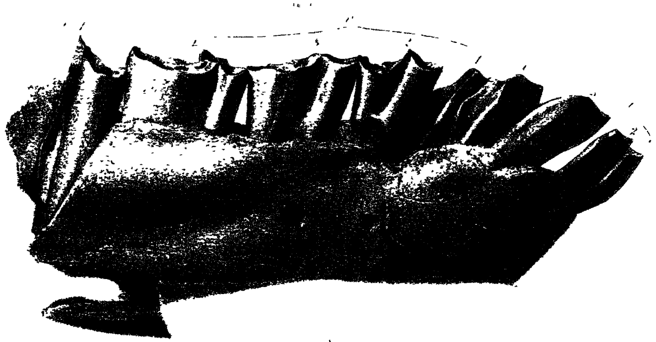
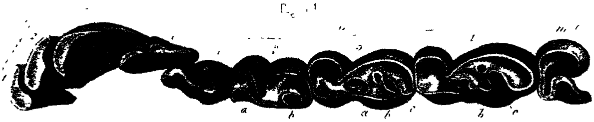


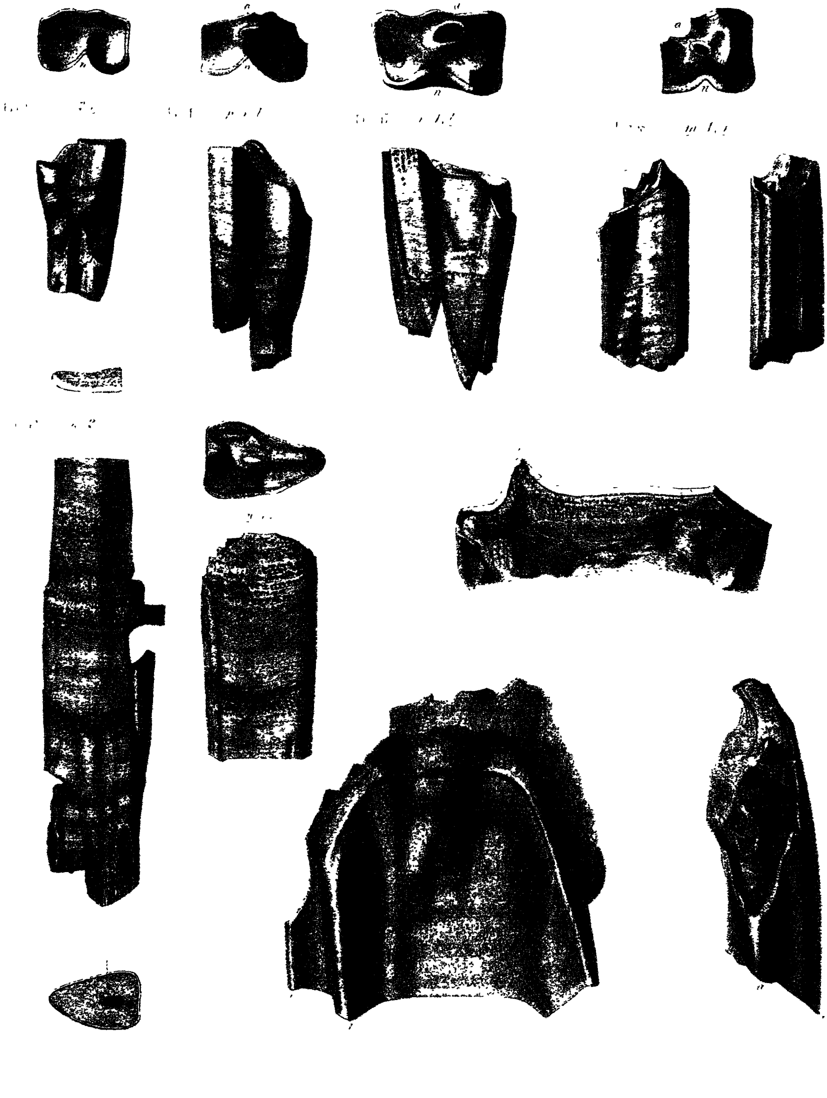
Fig. 10

Fig. 13









PHILOSOPHICAL

TRANSACTIONS

OF THE

ROYAL SOCIETY

OF

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FOR THE YEAR MDCCCLIII.

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CONTENTS.

APPENDIX.

PHILOSOPHICAL TRANSACTIONS.

XII. *An Account of Meteorological Observations in Four Balloon Ascents, made under the direction of the Kew Observatory Committee of the British Association for the Advancement of Science. By* JOHN WELSH, *Esq. Communicated by Colonel* SABINE, *R.A., Treas. and V.P.R.S., by the request of the Council of the British Association for the Advancement of Science.*

Received April 27,—Read May 26, 1853.

IN July 1852, the Committee of the Kew Observatory resolved to institute a series of balloon ascents, with the view of investigating such meteorological and physical phenomena as require the presence of an observer at a great height in the atmosphere. The arrangements made for carrying out this resolution have been stated by the Committee in their report to the Council of the British Association, a short account being at the same time given of some of the results derived from the ascents already made. Having been to a great extent entrusted by the Committee with the conduct of the observations and with the instrumental arrangements, I now, at their request, proceed to give a more detailed statement of the mode in which the experiments have been made, and of such results as may most readily be deduced from the observations recorded in the ascents.

The object to which especial attention was devoted, was the determination of the temperature and hygrometric condition of the air at different elevations above the earth's surface. Besides this, the observers were furnished with the means of procuring specimens of the air at different heights for the purpose of analysis, and of examining, if opportunity offered, whether the light reflected from the upper surface of the clouds was polarized.

§ 1. *Instruments and Apparatus.*

The instruments required for the investigations contemplated were—a barometer; dry and wet thermometers; an aspirator; REGNAULT's condensing hygrometer; DANIELL's dew-point hygrometer; a polariscope; and glass tubes, furnished with stopcocks, from which the air had been exhausted. All the instruments which were at all liable to accident were supplied in duplicate. The construction of the meteorological instruments was confided by the Committee to Mr. P. ADIE of London, under my own general superintendence. They were executed by him in a very satisfactory manner, having been made with much accuracy and with an anxious wish to promote the success of the experiments; many of the mechanical arrangements for the convenience of observation having also been devised by him.

Barometers.—The barometer employed was of the siphon form, on the construction generally known as GAY-LUSSAC'S. The tube was affixed to a brass scale in much the same way as a thermometer is attached to its scale. The brass scale was fixed within a stout rosewood frame furnished with a door which could be closed during carriage. The diameter of the tube was 0·25 inch. The graduation was made from the middle point upwards and downwards; each division being $\frac{1}{20}$th of an inch long, but representing twice that value; so that an observation of either branch of the siphon would give the length of the column of mercury, subject to a correction for inequality of the tube and error in the position of the zero-point of the scale. A complete observation of the instrument required however readings of both branches of the siphon, the true height of the mercury being the mean of the two. In order to facilitate rapidity of observation, verniers were dispensed with, the height of the mercury being merely estimated with reference to the scale placed behind it, just as if it had been a thermometer of large calibre. As it would have been nearly impossible to obtain in the car of the balloon a complete reading of both branches of the siphon for each observation, the corrections to the readings of the upper branch alone were previously obtained, throughout the anticipated range of the mercury, by the help of a large vacuum apparatus at the Kew Observatory, which has been employed in the pendulum experiments of Colonel SABINE and Professor STOKES. The barometers having been suspended within the receiver, the air was exhausted by about half an inch of pressure at a time, and readings taken from which tables of corrections were computed for different heights of the mercury. These corrections have been applied to all the observations. The difference between the indications of the siphon barometers and those of the Kew standard was also observed: both barometers were found to read 0·025 inch *higher* than the standard. It was found, by intercomparisons made last year, that the standard barometer at the Royal Observatory, Greenwich, reads *lower* than the Kew standard by 0·003 inch. The balloon barometers thus read 0·028 inch *higher* than the Greenwich standard; and, as that barometer has been generally referred to in the computations of height, the equation +0·028 has been applied to the terrestrial observations to render them comparable with those of the balloon barometers. Each barometer was provided with a thermometer to indicate the temperature of the mercury. In order to obtain this temperature more accurately, the bulb of the thermometer (which was cylindrical, about $1\frac{1}{2}$ inch long and $\frac{1}{8}$th of an inch diameter) was immersed in mercury contained in a tube of the same diameter as that of the barometer. The necessity for this precaution was found to be great, as very large differences sometimes existed between the temperature of the air thermometer and that of the mercury.

Dry and Wet Thermometers.—Two pairs of dry and wet thermometers were employed. One pair was mounted with the bulbs protected from radiation by a double conical shade, having highly polished silver surfaces, open at top and bottom to allow the circulation of the air. The inner shade was 2 inches high, $1\frac{3}{4}$ inch wide at the lower, and half an inch at the upper end: the outer shade was also 2 inches high, $2\frac{3}{4}$ inches wide at the lower, and $1\frac{3}{8}$ inch at the upper end*. Both thermometers were furnished with shades exactly similar; the bulbs being thus in the same circumstances, and completely protected from direct radiation. The thermometers were supported, $3\frac{1}{2}$ inches apart, by the arms of a light brass frame, also with a polished silver surface. A small brass-cistern was fixed near the wet thermometer, from which water was conveyed to the bulb by a conducting string of floss silk; when however the temperature fell below the freezing-point, the string was cut away and the bulb occasionally dipped in water.

As it was of essential importance that the thermometers should acquire with the utmost possible rapidity the temperature of the surrounding air, an arrangement was made, in connection with the second pair of dry and wet thermometers, for producing artificially a more rapid current over the bulbs than they would be exposed to by the mere vertical motion of the balloon. It was also thought desirable to avoid any tendency to a stagnation of the vapour of water in the neighbourhood of the wet bulb owing to the want of a sufficient circulation of air to carry it off, as might be the case when the balloon was nearly stationary or moving very slowly. An increased velocity in the circulation of the air would also tend to remove the effects of radiation, if the thermometers were not already sufficiently protected by the shades. With these objects the following contrivance was adopted. The thermometers were fixed vertically with their bulbs enclosed in two tubes placed side by side, and connected with each other by a cross tube joining their upper ends; these tubes having silver surfaces, and being further protected by a silver shade of the same dimensions as the outer shade of the other pair of thermometers. The first tube, in which was the bulb of the dry thermometer, had at its lower end a communication with the air: by means of an aspirator a current was produced from this opening, upwards over the dry bulb, then passing, by the communication at the top, into the second tube down which it moved over the wet bulb, leaving it by an opening connected by a flexible pipe with the aspirator. By this means, the temperature of the air was determined in its passage over the dry bulb, and afterwards its hygrometric condition on coming in contact with the wet; the vapour of water formed at the latter being carried off immediately into the aspirator. The whole distance which the air had to

* It might have been preferable to make the inner shade cylindrical instead of conical, as the air would have circulated more freely.

travel, between its entrance into the tubes and its leaving the wet thermometer, was about 4 inches: the diameter of the tubes enclosing the bulbs was 0·4 inch, and that of the connecting tube 0·25 inch. The aspirator was a cylindrical bellows; the valves being so arranged that, when the aspirator was forced open, the air could only enter it by passing over the thermometers: it was worked by attaching a weight to the lower end which pulled it open, the upper end being fixed; when it had opened to nearly its full extent, it was closed by means of a cord passing over pulleys and drawn up by the hand; a large valve allowing the air to escape rapidly from the aspirator as it was closed, and a second valve preventing the air from being driven backwards over the thermometers. Care was taken, in the construction of the different parts, that the aperture of the tubes should not be smaller between the thermometers and the external air, than between them and the aspirator; otherwise the air might, by undergoing a certain degree of expansion, have come in contact with the bulbs in different conditions with respect to temperature and capacity for moisture from those of the external air. This was guarded against by applying a stopcock near the aspirator, whose aperture was sufficiently small. A second flexible tube, with a stopcock, connected the aspirator with REGNAULT's hygrometer; so that the same aspirator might be used simultaneously for both instruments. Two different sizes of aspirator were used in the different ascents; the one being 12 inches diameter, and extending to about 18 inches, occupying about $1\frac{1}{2}$ minute in its descent; the other was 9 inches diameter, extending to 12 inches in 30 to 40 seconds. This was sufficient to produce a current of air over the bulbs at the rate of 12 to 14 feet in a second; the vertical velocity of the balloon seldom exceeding 4 or 5 feet. The thermometers employed were of great sensibility; the bulbs being cylindrical, the diameter not exceeding $\frac{1}{12}$th of an inch, and the length varying from a half to three-quarters of an inch. The length of one degree of the scale was from $\frac{1}{25}$th to $\frac{1}{20}$th of an inch, so that they could readily be read by estimation to $0^{\circ}\cdot1$. The graduation extended to 30° or 40° below zero of FAHRENHEIT. The scales of those used in the first ascent were of brass, but afterwards of ivory, in order to render the column of mercury more visible. The errors of all the thermometers were determined throughout the scale, from about 0° to 70°, by comparison with standards at the Kew Observatory; the comparisons below the freezing-point being made in mixtures of ice and salt. The corrections have been applied to all the readings. These thermometers were found to acquire the temperature of the air very rapidly: when heated 20° above the temperature of the air, and allowed to cool *at rest* in a confined room, they returned to within $0^{\circ}\cdot5$ of the previous reading in about 100 seconds; when gently fanned, by being carried through the room at the rate of 5 or 6 feet in a second, they returned to within $0^{\circ}\cdot5$ in 40 seconds; when under the action of the aspirator they returned to within $0^{\circ}\cdot5$ in 30 seconds, and exactly to the original reading in 45 seconds. Any correction on account of sluggishness in the thermometers must thus be very small: this is shown by the observations of October 21, when the descent took place with about the same velocity as the ascent, and observations were continued to within 3000 feet of the earth. The differences of temperature at the same height are scarcely appreciable, and even frequently in the opposite direction from what would result from insensibility in the thermometer. A few observations were taken during the *descent* on August 17, which, when compared with those made at the same height in the *ascent*, show a difference of about four degrees. The rapidity of the descent was on that occasion about twice as great as that of the ascent, which was also considerable, and the thermometers were not under aspiration. The protection from radiation has been examined by observing the thermometers within a room when alternately exposed to strong sunshine and shade—the effect upon the aspirated thermometers did not exceed 1°·5: in the open air, and with a gentle breeze, the effect was considerably less. The effect upon the free thermometer appeared to be greater; and the difference between its indications and those of the aspirated thermometer during some of the ascents is probably to be ascribed to this cause. It would appear from some portions of the series of August 26, that the long-continued exposure to the sun, in a nearly calm air, has produced an appreciable effect upon the readings of the thermometer, whether aspirated or free. Fortunately, with the exception of the ascent of August 26, the sun's radiation was never powerful; whilst on August 17, when the free thermometers were alone observed, the sun was scarcely ever visible. When the radiation was feeble, and the vertical motion of the balloon considerable, the two thermometers agreed very closely*. As hygrometers there is probably less difference in their value. In the examination of the results of the temperature observations, I have been led to prefer the indications of the aspirated thermometer.

REGNAULT's *Condensing Hygrometer*†.—The only difference in the construction of this instrument, from that usually adopted by M. REGNAULT, was that the small tube, by which the air enters the reservoir to agitate the ether, had a funnel-shaped opening at top to facilitate the supply of ether. The bulb of the thermometer was cylindrical, $1\frac{1}{2}$ inch long, and $\frac{1}{10}$th of an inch diameter. The scale was of ivory, and the thermometer was fixed into the reservoir by a cork.

No use was made of DANIELL's hygrometer as that of M. REGNAULT was found much more convenient for such observations, being to a great extent self-acting.

Polariscope.—This instrument was supplied by Mr. DARKER of Lambeth. Its principle is the same as that employed in Mr. WHEATSTONE's "Polar Clock;" the parts of the polariscope used having, I believe, actually formed portions of one of those instruments. It exhibits the existence of polarization in a conspicuous manner.

Exhausted Tubes for Collecting Air.—These tubes, which were constructed by Messrs. NEGRETTI and ZAMBRA, were about 9 inches long and $\frac{3}{4}$ inch diameter, fitted

* It would be advisable in any future experiments to apply additional shades to all the thermometers, and if possible to use a larger general screen at some distance from them.

† This instrument is described in the "Annuaire Météorologique de la France" for 1849, p. 221.

with stopcocks. They were prepared by Dr. MILLER previously to each ascent, and hermetically sealed immediately after their return to King's College.

§ 2. *Observing Arrangements, Personal and Instrumental.*

It was deemed advisable that, in the first ascents at least, two observers should take part in the work. Mr. R. B. NICKLIN, who for upwards of two years had been employed at the Kew Observatory and been practised in the observation of instruments, acted as my assistant in the first two ascents. Mr. NICKLIN's aid was of essential service, and I wish to express my acknowledgement for the careful manner in which he took the observations with which he was entrusted, and for the readiness with which he assisted me on several occasions, sometimes at considerable personal inconvenience, when unforeseen difficulties arose. Having in these two ascents acquired experience in the observations, and having got the instruments into better working order, in the last two ascents I undertook the observations alone, thus obtaining the power of reaching a greater elevation.

The car attached to the balloon was an oblong basket of wicker-work, about 6 feet long, 3 feet wide, and $2\frac{1}{2}$ feet deep. One end of this was occupied by the observers with the instruments, and the other by Mr. GREEN, who managed the balloon. A light board, a foot wide, was fixed across the car in front of the observers: at the extremity of this board, and projecting nearly a foot over the side of the car, was erected a light horizontal bar of wood, raised about 9 inches above the board, and inclined at an angle of about 45° to its length, the board being cut away beyond the bar so as not to present any resistance to the circulation of the air. Upon the bar were fixed the thermometers and hygrometers. The aspirator was fixed to the lower side of the board, in which a hole was cut to admit the connecting tubes. On the first ascent the barometers were suspended from the hoop by which the car is attached to the netting of the balloon; this was however found to be inconvenient; and in the subsequent ascents they were suspended by gimbals from the cross board, their verticality being secured by weights attached to the lower ends of the cases. When seated in the car for observation, Mr. NICKLIN occupying the right-hand corner and I the left, the stand supporting the thermometers was to my left, at a distance of about 18 inches; the aspirator being underneath the board, which served as a table before us: one barometer was immediately in front of Mr. NICKLIN, and the other before myself; the observations could thus be readily taken without rising from our places.

In order to obtain as continuous a record as possible of the variations of temperature and humidity, the observations were taken at very short intervals, generally at every minute, but frequently twice in a minute. In the first two ascents Mr. NICKLIN observed one barometer, whilst I observed the thermometers and hygrometers, taking an occasional observation of the second barometer as a check upon the indications of the other. A watch which had been set to Greenwich time was placed in sight of both observers. In the last two ascents, when I was the only observer, the barometer was always read first, and immediately afterwards the thermometers and hygrometers; the whole time occupied being only a few seconds, the error, arising from the observations not being strictly simultaneous, must be very small. Note-books were provided with columns ruled and headed for the different instruments.

§ 3. *Circumstances of the Ascents and General Observations.*

The ascents were made with Mr. C. GREEN's large balloon, well known by the name of the "Royal Nassau." It has been fortunate, for the success of these experiments, that the Kew Committee obtained the cooperation of Mr. GREEN, whose pre-eminence as a skilful aëronaut has been established by upwards of 500 ascents; and whose control over his balloon is so complete, that no one who accompanies him can be otherwise than relieved from apprehension, and free to devote his attention calmly to the work before him. Mr. GREEN on all occasions showed the most anxious desire to contribute to the success of experiments, in which he took great interest.

The ascents took place from the Royal Vauxhall Gardens, which were liberally placed at the disposal of the Committee by Mr. WARDELL, the Lessee. The balloon was inflated with carburetted hydrogen gas, obtained from the Vauxhall and Phœnix Gas-works.

First Ascent, August 17.—The weather, previously to the 17th, had been somewhat variable; on the 16th the wind changed from S.E. to S.W., and on the day of ascent it was from south. Clouds covered about three-fourths of the sky, the lowest stratum being a few detached masses of loose cumulus; a dense mass of cirrostratus (or stratus) being above, with perhaps occasional patches of cumulus intermediate. The ascent commenced at 3^h 49^m P.M., after considerable difficulty had been experienced in the preliminary arrangements, owing to the force of the wind. A short time was lost at first in the attempt to put the instruments into more convenient order, and also from the novelty of the situation. The lowest clouds, which extended only over a small area, and were not near the balloon, were passed before they were noticed; their height was estimated at about 2500 feet. Between this height and about 13,000 feet, the air seemed free of clouds; after this, although the balloon was never in actual cloud, there seemed to be occasional masses of loose cumulus at no great distance. When at the greatest elevation, there was, at apparently a short distance above us, a thick mass of cloud, which was probably the cirrostratus which had been seen from the earth. About this time, and while still rising, a few small star-shaped crystals of snow about $\frac{1}{25}$th of an inch diameter fell upon us. The sun was almost constantly obscured throughout the ascent. The descent commenced at 4^h 46^m P.M., and the earth was reached about 5^h 20^m P.M., near Swavesey in Cambridgeshire, about 57 miles north of London. There seems to have been little, if any variation in the direction of the balloon's flight: it would thus appear that, within the height reached by us, the air was moving from south, at an average rate of about 38 miles an hour. A violent thunder-storm, with heavy rain, occurred about two hours after the descent took place, some symptoms of which were at one time noticed from the balloon at a great distance.

In this ascent it was found impossible to use the aspirator, which was too large when two persons were seated. The free dry and wet thermometers were regularly observed. Some specimens of air were collected during the descent, and supplied to Dr. MILLER.

Second Ascent, August 26.—The wind on the 25th blew strongly from the west, but lulled in the evening. On the 26th it blew from east with moderate force; the sky was to a small extent obscured by detached masses of cumulus, and the sun shone brightly. The ascent took place at $4^h\ 43^m$ P.M., and observations were commenced at $4^h\ 46^m$. The clouds were again passed without being perceived, their height, however, was estimated at 3000 feet: above this height no clouds were met with, the sky being exceedingly clear and of a very deep blue colour. The currents of air passed through seem to have been from various directions, but generally moving with little rapidity. On leaving Vauxhall the balloon was at first carried towards the west for about 2 miles: when it reached the height of 5000 or 6000 feet it began to move slowly towards N.N.E. for about 4 miles, until about $5^h\ 25^m$, at an elevation of 12,000 feet, the direction of its motion, which was still for some time very slow, became W.N.W.; this direction it seems to have maintained during the remainder of the ascent, and probably with increased rapidity. The descent commenced at $7^h\ 0^m$, and the balloon reached the earth at $7^h\ 35^m$ P.M. near Chesham in the county of Bucks, about 25 miles W.N.W. of London. On this occasion all the instruments were regularly observed: some difficulty was experienced in the observation of REGNAULT's hygrometer, as the force of the aspirator was not sufficient to produce the great degree of cold required for the deposition of dew. This was remedied by Mr. NICKLIN, who, at the cost of some exertion, maintained an increased strain upon the aspirator during the observations. The sun shone brightly throughout the ascent. Specimens of air were again collected during the descent.

Third Ascent, October 21.—The weather had for a fortnight previously been fine, with an easterly wind; on the 19th the barometer began to fall and the east wind ceased; on the 20th the weather was fine, the air at the surface being calm, and the high clouds moving from S.W.: a fog existed on the night of the 20th, which slowly disappeared on the morning of the 21st, leaving the air in a very calm state and with some haze. A dense mass of cloud covered the sky, one or two slight showers falling about 10 A.M. I was the only observer on this occasion. The ascent commenced at $2^h\ 45^m$ P.M., and the balloon rose at first nearly vertically, but soon began to move towards E.N.E. Between the heights of 1000 and 2800 feet various detached and irregular masses of loose scud were encountered, but the balloon had not completely entered the dense mass of cloud till the height of nearly 3000 feet. At a height of 3700 feet the upper surface of the cloud was reached, and the sun was seen shining through thin cirrous clouds, at a great height. The height of the upper surface of the cloud was again observed during the descent at 4^h 6^m to be 3450 feet. When the balloon was close to the clouds, it was remarked that the general level of the surface was very uniform, presenting, however, a hillocky appearance; the irregularities being small, apparently not exceeding a very few feet. Shortly after clearing the clouds, a shadow of the balloon was seen on the surface fringed with a glory; with this shadow as a centre, there was also observed a circle of whitish light, the outer edge of it slightly tinged with yellow; its diameter being estimated at 80°. About this time there was noticed, stretching for a considerable length in a serpentine course over the surface of the cloud, a well-defined belt having the appearance of a broad road, both sides being strikingly distinct. When the balloon had attained a height of above 12,000 feet, Mr. GREEN, who had been watching its motion with reference to the clouds below, decided that, as it seemed to be moving rapidly from N.W., it would be prudent to descend below the clouds, to ascertain our position with reference to the sea, and if there should be space enough to ascend a second time to a greater height. It was found, however, on descending, that we were already very near the sea, indeed, moving along the river Thames within a short distance of its mouth. A second ascent being thus unadvisable, the descent was made at 4^h 20^m P.M. on the North bank of the Thames, between the villages of South Benfleet and Rayleigh in Essex, about 30 miles east of London. The average rate of motion was thus about 18 miles an hour, but in the higher part of our course it must have been considerably more.

When about 3000 to 4000 feet above the clouds they were examined with the polariscope. The reflected light from the clouds *next* the sun showed no trace whatever of polarization: the slightly bluish-grey clouds on the side *from* the sun showed *very* slight symptoms of polarization, the light of the sky being strongly polarized.

Fourth Ascent, November 10.—This ascent had been delayed for some days, owing to the unfavourable state of the weather, the wind having been generally from a westerly quarter. On the 10th the surface wind and the lower current of scud were moving very slowly from about N.E.: the upper clouds were only occasionally visible, and seemed to proceed from about N.N.W. The ascent commenced at 2^h 21^m 40^s P.M. At a height of 500 feet the first cloud, thin scud, was entered, the *upper* surface being 1970 feet high. A space of 2000 feet was clear of clouds, and at 4000 feet the second stratum of clouds was reached, its *upper* surface being found to be 4900 feet high. After this no clouds were met with, the sun shining through thin cirrous clouds, which must have been at a very great height. From notes taken at Vauxhall by Mr. GASSIOT, it appears that, at starting, the balloon moved towards south-west until 2^h 26^m, when, just as it had reached the *upper* surface of the *first* stratum of clouds, or at a height of about 2000 feet, the direction became easterly. Bearings and altitudes taken by Mr. GLAISHER show that at $2^h\ 44^m$, when the height was 11,000 feet, the balloon was 5 miles S. by E. of Greenwich Observatory. The greatest elevation (22,930 feet) was reached at $3^h\ 16^m$ P.M.; about which time the clouds, which had hitherto obscured the earth, had disappeared, and we perceived that the balloon was rapidly approaching the sea. Mr. GREEN discharged gas copiously, and the descent became very rapid; a landing being effected within 4 miles of the sea, accompanied by a considerable shock which broke several of the instruments. The descent took place, between $3^h\ 40^m$ and $3^h\ 45^m$, at Acryse near Folkstone, about 57 miles E.S.E. from London. The time occupied in moving from a little S.W. of Vauxhall to 5 miles S. by E. of Greenwich, or about 9 miles, was 18 minutes; the remainder of the distance to Acryse, about 50 miles, being accomplished in from 55 to 60 minutes, or at the rate of fully 50 miles an hour.

As the height reached on this occasion was considerably greater than in the previous ascents, the effect of the diminished pressure was more severely felt; both Mr. GREEN and myself having experienced considerable difficulty in respiration, with much breathlessness and fatigue after any muscular exertion.

§ 4. *Description of the Table of Observations.*

All the meteorological observations taken during the ascents are contained in Table I.

Column 1 contains the times at which the observations were made. Column 2 contains the readings of the thermometer attached to the barometer. Column 3 contains the observations of the barometer corrected for temperature, by SCHUMACHER's tables, and for scale error. The numbers, to which the mark † is affixed, in the observations of August 17 and 26, are the occasional readings by myself of the second barometer. The readings of the barometer were made by estimation to 0·01 inch; but the probable error of an observation, from various causes,—such as rapid change in the height, and the occasional oscillation of the mercury from agitation of the car,—is perhaps 0·03 inch, or even sometimes more. This degree of accuracy appears, however, to be quite sufficient with reference to the changes of the temperature and humidity; an error of 30 or 40 feet in the resulting height being equivalent in general to a change of only one-tenth of a degree of temperature.

Column 4 contains the height above the level of the sea, as deduced from the barometric readings by the formula of LAPLACE. The formula actually employed was

$$z=\log\left(\frac{h}{h'}\right)\times 18336\left(1+\frac{2(t+t')}{1000}\right)\left(1+0{\cdot}002837\cos 2L\right)\left(1+\frac{z+15926}{6366200}\right)*;$$

or expressed in English feet and FAHRENHEIT's degrees,

$$z=\log\left(\frac{h}{h'}\right)\times 60159\left(1+\frac{t+t'-64}{900}\right)\left(1+0{\cdot}002837\cos 2L\right)\left(1+\frac{z+52251}{20886900}\right),$$

where z is the height required; h and h', t and t' the height of the barometer cor-

* Annuaire Météorologique de la France, 1849, page 54.

rected for temperature, and the temperature of the air, at the lower and upper stations respectively; L, the latitude. The temperature of the air for the position of the balloon has been derived from the readings of the aspirated dry thermometer (column 5), except on August 17, when the free thermometer only was observed. The temperature and barometric height at the earth's surface have been taken by interpolation from the comparative observations at different stations; the mean height above the sea, of the stations referred to, having been included. The numbers, it will be seen, have been given only to the nearest 10 feet.

Many observers in different parts of the country made corresponding meteorological observations, generally at hourly intervals, on the days of the several ascents. These have been arranged in compact tabular order by Colonel Sykes, Chairman of the Kew Committee, and are appended to this report. The stations selected for comparison with the different days' observations have been those which lay nearest to the course of the balloon. The temperature of the air at the surface of the earth, has been derived from the mean of the observations at all the selected stations, both as regards its *absolute value* and *hourly change*. The *hourly change* of the barometer has been taken from the observations at all the selected stations; but its *absolute height* has always been derived from the mean of the observations at the Royal Observatory, Greenwich, and at the residence of James Glaisher, Esq., Lewisham. The error likely to result from adopting the height at these two stations as the standard of reference will be in any case very small, and can only affect the *absolute* and not the *relative* heights of the balloon by a few feet; while any uncertainty with regard to the index errors of other barometers is obviated. The quantity +0·028 has been added to the readings of the terrestrial barometers, on account of the index errors of the balloon barometers.

The following are the stations whose observations have been employed, and the resulting mean values for each day of ascent.

August 17, 5 stations, viz.—Greenwich; Lewisham; Enfield; St. John's Wood; Cambridge.

Mean temperature of the air at 4^h P.M. $=71^\circ{\cdot}2$; hourly change $=-1^\circ{\cdot}1$.

Mean height of the barometer at 4^h P.M. $=29{\cdot}740$ in.; hourly change $=-0{\cdot}036$ in.

August 26, 5 stations, viz.:—Greenwich; Lewisham; St. John's Wood; Kew Observatory; Stone Rectory, Bucks.

Time.	Temperature of the Air. Mean.	Temperature of the Air. Hourly Change.	Barometer. Mean Height.	Barometer. Hourly Change.
h	°	°	in.	in.
4 P.M.	69·7		29·949	
		−2·2		+0·010
5	67·5		·959	
		−2·3		+0·005
6	65·2		·964	
		−2·5		+0·021
7	62·7		·985	

October 21, 2 stations, viz.—Greenwich and Lewisham.

Time.	Temperature of the Air.		Barometer.	
	Mean.	Half-hourly Change.	Mean Height.	Half-hourly Change.
h m	°	°	in.	in.
2 30 P.M.	58·7		29·900	
		0·0		−0·001
3 0	58·7		·899	
		−1·0		− ·004
3 30	57·7		·895	
		−0·8		− ·007
4 0	56·9		·888	
		−0·8		·000
4 30	56·1		·888	

November 10, 2 stations,—Greenwich and Lewisham.

Time.	Temperature of the Air.			Barometer.	
	Greenwich.	Lewisham.	Mean.	Mean Height.	Half-hourly Change.
h m	°	°	°	in.	in.
2 30 P.M.	48·6	50·7	49·7	29·978	
					−0·003
3 0	48·0	49·3	48·7	·975	
					− ·004
3 30	49·0	49·6	49·3	·971	
	Mean		49·2		

As the progress of the temperature at these two stations has been very irregular and indefinite, a mean result has been adopted, and no allowance made for hourly change.

The height, above the mean sea level, of Greenwich =159 feet.
The height, above the mean sea level, of Lewisham = 80 feet.
Mean of both stations . . . . . . =120 feet.

Columns 5–10 contain the results of observations with the aspirated dry and wet thermometers; the tension of vapour, relative humidity (100 being complete saturation), and the calculated temperature of the dew-point having been deduced by Dr. APJOHN'S formula and DALTON'S Tables of the elasticity of vapour. Column 11 contains the readings of the dry thermometer, corrected for hourly change by means of the numbers deduced above from observations at different stations. The numbers in this column have been employed in the subsequent discussions and in the projected results.

Columns 12–17 contain the observations of the free dry and wet thermometers similarly reduced. Columns 18 and 19 contain the results of the direct dew-point observations with REGNAULT'S Hygrometer, and the corresponding tension of vapour derived from DALTON'S Table. When numbers are entered in column 18 with the sign — after them, it is meant that the temperature in the hygrometer had been lowered to the degree stated, but that no dew was deposited.

All the readings of both pairs of dry and wet thermometers have been corrected for index error; the corrections to the dew-point thermometer were very small, and have been neglected.

TABLE I.—Meteorological Observations made in the Four Balloon Ascents of August 17, August 26, October 21, and November 10, 1852.

	Greenwich Mean Time.	Barometer.		Height above sea-level.	Dry and Wet Thermometers, *aspirated*.						Temperature corrected for change.	Dry and Wet Thermometers, *free*.						REGNAULT'S Hygrometer.	
		Therm.	Reading corrected.		Dry.	Wet.	Diff.	Tension of vapour.	Relative humidity.	Dew-point.		Dry.	Wet.	Diff.	Tension of vapour.	Relative humidity.	Dew-point.	Dew-point.	Tension of vapour.
	h m s	°	in.	feet.	°	°	°	in.		°	°	°	°	°	in.		°	°	in.
August 17, 1852.	3 52 0		27·40	2,440							62·8	63·0	60·4	2·6	·503	87	58·8		
	54 0		26·40	3,470							59·2	59·4	57·1	2·3	·451	88	55·6		
	54 0	70	26·42†	3,450															
	55 0		25·80	4,110							58·1	58·2	55·3	2·9	·418	85	53·3		
	59 0		24·22	5,880							57·8	57·8	43·4	14·4	·164	34	26·6		
	59 18	66·5	24·10†	6,020															
	59 30		23·62	6,580															
	4 0 30										54·1	54·1	40·5	13·6	·147	34	23·5		
	1 0		23·42	6,800							54·1	54·1	40·8	13·3	·153	36	24·7		
	1 30		23·33	6,910							53·7	53·7	40·1	13·6	·144	34	23·0		
	2 0		23·13	7,140							53·1	53·1	39·8	13·3	·145	35	23·2		
	2 0	63·0	22·96†	7,350															
	2 30		22·93	7,380							52·3	52·3	39·3	13·0	·144	36	23·0		
	3 0		22·78	7,550							51·4	51·3	38·4	12·9	·138	35	21·8		
	3 30		22·83	7,480							50·5	50·4	37·7	12·7	·133	35	20·8		
	4 0		22·43	7,970							49·8	49·7	37·4	12·3	·136	37	21·4		
	4 30		22·33	8,100							49·8	49·7	38·3	11·4	·152	41	24·5		
	5 0		22·14	8,340							50·5	50·4	39·2	11·2	·162	43	26·2		
	5 0	61·0	22·13†	8,350															
	5 30		22·04	8,460							50·2	50·1	39·6	10·5	·172	46	27·9		
	6 0		21·94	8,590							49·8	49·7	39·1	10·6	·167	45	27·1		
	6 30		21·84	8,710							49·2	49·2	38·3	10·9	·158	44	25·5		
	7 0		21·84	8,700							48·0	47·9	37·3	10·6	·152	44	24·5		
	7 30		21·84	8,690							47·3	47·2	39·4	7·8	·194	57	31·3		
	8 0		21·74	8,810							47·5	47·3	40·5	6·8	·212	63	33·8		
	9 30		21·49	9,130							47·0	46·8	37·1	9·7	·159	48	25·7		
	10 0		21·49	9,120							45·9	45·7	36·7	9·0	·161	50	26·1		
	10 30		21·34	9,310							44·9	44·7	34·5	10·2	·135	43	21·2		
	11 0		21·24	9,430							44·0	43·8	33·2	10·6	·122	40	18·4		
	11 30		21·13	9,560							43·3	43·1	32·5	10·6	·117	40	17·3		
	12 0		21·00	9,730							42·8	42·6	32·3	10·3	·119	41	17·8		
	12 35	57·0	20·78†	10,020							43·1	42·8	32·5	10·3	·121	42	18·2		
	13 0		20·65	10,180							41·8	41·5	31·7	9·8	·127	46	19·5		
	13 0		20·62†	10,220															
	13 30		20·45	10,440															
	14 0		20·25	10,700							40·4	40·1	31·2	8·9	·132	50	20·6		
	14 0	54·0	20·27†	10,680															
	14 30		20·10	10,900							40·0	39·7	30·3	9·4	·122	47	18·4		
	15 0		20·06	10,950							39·4	39·1	31·9	7·2	·149	58	23·9		
	15 15		19·76†	11,360															
	15 30		19·80	11,290							38·1	37·8	32·0	5·8	·160	65	25·9		
	16 0		19·52	11,680							37·3	37·0	31·7	5·3	·162	68	26·2		
	16 15		19·57†	11,600															
	16 30		19·51	11,690							37·0	36·7	31·5	5·2	·161	68	26·1		
	17 0		19·40	11,840							36·5	36·2	31·2	5·0	·160	69	25·9		
	17 30		19·35	11,900															
	18 0		19·25	12,040							36·1	35·7	30·8	4·9	·158	69	25·5		
	18 30		19·15	12,180							35·7	35·3	30·8	4·5	·161	70	26·1		
	19 0		19·14	12,200															
	19 30		19·15	12,190							36·5	36·1	32·9	3·2	·182	79	29·5		
	20 0		19·16	12,150							34·8	34·4	31·4	3·0	·175	81	28·4		
	20 20	48·0	19·15†	12,160															
	20 30		19·16	12,140							34·1	33·7	29·9	3·8	·160	76	25·9		
	21 0		19·15	12,160							34·5	34·1	30·1	4·0	·160	75	25·9		
	21 30		19·06	12,280							33·7	33·3	31·5	1·8	·184	88	29·8		
	22 0		18·91	12,500															
	22 20	47·0	18·87†	12,550															
	22 30		18·66	12,860							33·9	33·4	31·4	2·0	·182	87	29·5		

Table I. (Continued.)

Date	Greenwich Mean Time.	Barometer. Therm.	Barometer. Reading corrected.	Height above sea-level.	Dry and Wet Thermometers, *aspirated*. Dry.	Wet.	Diff.	Tension of vapour.	Relative humidity.	Dew-point.	Temperature corrected for change.	Dry and Wet Thermometers, *free*. Dry.	Wet.	Diff.	Tension of vapour.	Relative humidity.	Dew-point.	Regnault's Hygrometer. Dew-point.	Tension of vapour.
	h m s	°	in.	feet.	°	°	°	in.		°	°	°	°	°	in.		°	°	in.
August 17, 1852.	4 23 0	......	18·56	13,000	......	......	......	......	......	......	33·7	33·2	31·3	1·9	·182	88	29·5		
	24 0	......	18·21	13,470	......	......	......	......	......	......	30·3	29·8	28·4	1·4	·166	90	26·9		
	24 15	44·0	18·13†	13,590															
	24 30	......	18·11	13,610															
	25 0	......	17·87	13,960	......	......	......	......	......	......	28·4	27·9	26·6	1·3	·156	89	25·2		
	25 30	......	17·77	14,100	......	......	......	......	......	......	27·5	27·0	25·9	1·1	·153	92	24·7		
	25 30	......	17·82†	14,020															
	26 0	......	17·47	14.550	......	......	......	......	......	......	27·0	26·5	25·0	1·5	·146	89	23·4		
	26 20	......	17·39†	14,680															
	26 30	......	17·37	14,710	......	......	......	......	......	......	27·1	26·6	24·6	2·0	·141	86	22·4		
	27 0	......	17·27	14,850	......	......	......	......	......	......	26·3	25·8	24·2	1·6	·141	88	22·4		
	27 15	39·0	17·08†	15,150															
	27 30	......	17·06	15,170	......	......	......	......	......	......	25·5	24·9	23·4	1·5	·137	89	21·6		
	27 45	38·0	16·97†	15,310															
	28 0	......	16·97	15,310	......	......	......	......	......	......	24·9	24·3	23·0	1·3	·136	90	21·4		
	28 20	......	16·98†	15,290															
	28 30	......	16·96	15,330	......	......	......	......	......	......	25·3	24·7	23·0	1·7	·134	87	21·0		
	29 0	......	16·86	15,480	......	......	......	......	......	......	24·9	24·3	22·6	1·7	·132	87	20·6		
	29 30	......	16·67	15,780	......	......	......	......	......	......	24·6	24·0	(27·)						
	30 0	......	16·67	15,780															
	31 0	37·0	16·52†	15,970	......	......	......	......	......	......	21·5	20·9							
	31 30	......	16·37†	16,200	......	......	......	......	......	......	20·4	19·8	18·4	1·4	·114	89	16·6		
	32 0	35·0	16·32†	16,280	......	......	......	......	......	......	20·0	19·4	18·0	1·4	·112	88	16·1		
	33 0	......	16·26	16,380	......	......	......	......	......	......	20·5	19·8	18·3	1·5	·113	88	16·4		
	33 30	......	16·16	16,550	......	......	......	......	......	......	20·5	19·8	17·7	2·1	·107	84	14·9		
	34 0	......	16·16	16.560	......	......	......	......	......	......	21·6	20·9	18·8	2·1	·112	84	16·1		
	34 15	......	16·06†	16.730															
	34 30	......	16·11	16,650	......	......	......	......	......	......	21·7	21·0	16·1	4·9	·085	63	8·7		
	35 0	......	16·06	16,730															
	35 30	......	16·06	16,720	......	......	......	......	......	......	21·0	20·3							
	36 0	32·0	15·96	16,880	......	......	......	......	......	......	20·4	19·7							
	36 30	......	15·96	16,870	......	......	......	......	......	......	20·0	19·3							
	37 0	......	15·96	16,870	......	......	......	......	......	......	19·6	18·9							
	37 0	......	15·92†	16,930															
	37 30	......	15·87	17,020	......	......	......	......	......	......	20·0	19·3							
	38 0	......	15·86	17,050	......	......	......	......	......	......	20·4	19·7	17·7	2·0	·108	84	15·1		
	38 0	......	15·80†	17,150															
	38 30	......	15·67	17,370	......	......	......	......	......	......	20·5	19·8	17·0	2·8	·101	79	13·3		
	39 0	......	15·57	17,530	......	......	......	......	......	......	19·5	18·8	16·0	2·8	·096	78	12·0		
	39 30	......	15·47	17,680	......	......	......	......	......	......	18·8	18·1	14·1	4·0	·083	69	8·1		
	40 0	30·0	15·37	17,860	......	......	......	......	......	......	18·6	17·9	14·4	3·5	·087	73	9·3		
	40 0	... ..	15·32†	17,950															
	40 30	......	15·17	18,200	......	......	......	......	......	......	17·9	17·2							
	41 0	......	15·17	18,170	......	......	......	......	......	......	16·4	15·7	9·4	6·3	·054	49	−3·2		
	41 30	... ..	15·07	18,310	......	......	......	......	......	......	14·5	13·8	7·7	6·1	·050	49	-5·2		
	42 0	......	14·86	18,670	......	......	......	......	......	......	14·0	13·3	7·2	6·1	·049	49	−5·8		
	42 15	......	14·67†	19,010															
	42 30	......	14·81	18,750	......	......	......	......	......	......	13·6	12·8	7·3	5·5	·053	53	−3·7		
	43 0	......	14·76	18,840	......	......	......	......	......	......	13·5	12·7	8·1	4·6	·060	61	−0·5		
	43 15	28·0	14·62†	19,090															
	43 30	......	14·66	19,020	......	......	......	......	......	......	13·2	12·4	7·5	4·9	·057	58	−1·8		
	43 45	......	14·54†	19,220															
	44 0	......	14·56	19,180	......	......	......	......	......	......	12·1	11·3	7·2	4·1	·060	64	−0·5		
	44 30	......	14·46	19,340	......	......	......	......	......	......	11·2	10·4	6·7	3·7	·061	67	−0·1		
	45 0	......	14·46	19,310	......	......	......	......	......	......	9·8	9·0	6·4	2·6	·065	76	+1·6		
	45 0	......	14·36†	19,500															
	45 30	......	14·41	19,420	......	......	......	......	......	......	10·5	9·7	7·7	2·0	·072	82	4·3		
	46 0	......	14·35	19,510	......	......	......	......	......	......	9·5	8·7	6·6	2·1	·068	80	2·8		
	46 30	......	14·41†	19,380	......	......	......	......	......	......	8·7	7·9	6·3	1·6	·070	85	3·6		

Table I. (Continued.)

Date	Greenwich Mean Time.	Barometer. Therm.	Barometer. Reading corrected.	Height above sea-level.	Dry and Wet Thermometers, *aspirated*. Dry.	Wet.	Diff.	Tension of vapour.	Relative humidity.	Dew-point.	Temperature corrected for change.	Dry and Wet Thermometers, *free*. Dry.	Wet.	Diff.	Tension of vapour.	Relative humidity.	Dew-point.	Regnault's Hygrometer. Dew-point.	Regnault's Hygrometer. Tension of vapour.
	h m s	°	in.	feet.	°	°	°	in.		°	°	°	°	°	in.		°	°	in.
August 17, 1852.	4 47 0										8·9	8·1							
	47 30		14·46	19,300															
	49 0		14·86	18,630							12·3	11·4							
	50 0		15·17	18.090							13·3	12·4							
	50 30		15·27	17,920							13·3	12·4							
	51 0		15·48	17,580							14·6	13·7							
	51 30	24·0	15·58	17,420							15·4	14·5							
	51 30		15·63†	17,330															
	52 0		15·88	16,920							15·9	15·0							
	52 15	24·0	15·83†	17,010															
	52 30		16·08	16,590															
	53 0		16·39	16,090							17·2	16·2	15·2	1·0	·103	92	13·9		
	54 0		16·59	15.790							19·0	18·0	16·2	1·8	·102	85	13·6		
	54 30		16·89	15 320							19·5	18·5							
	55 0		17·10	15,000							20·5	19·5	18·8	0·7	·120	94	18·0		
	56 45		17·77†	14,020															
	57 0										25·0	24·0							
	57 40		18·11†	13,530							26·0	25·0							
	58 0	24·0	18·36†	13,170							27·3	26·2							
	58 30		18·71†	12.680							28·4	27·3							
	5 0 30		19·57†	11,500							32·0	30·9	29·8	1·1	·177	92	28·7		
	1 30		20·28†	10.570							36·5	35·4	29·8	5·6	·145	64	23·2		
	3 0		20·88†	9,7?0							38·0	36·9	29·8	7·1	·133	56	20·8		
	5 0		22·31	?.020							43·0	41·8	30·8	11·0	·106	38	14·6		
	5 30		22·51	7,780							45·4	44·2	31·5	12·7	·096	31	12·0		
	6 0		22·91	7,310							48·0	46·8							
August 26, 1852.	4 46 0				65·9	61·5	4·4	·500	79	58·7	65·9	66·0	61·7	4·3	·505	79	59·0		
	47 0		29·34	710	64·5	60·9	3·6	·498	82	58·5	64·5	65·3	61·3	4·0	·501	80	58·7		
	47 30	71·0	29·18†	860															
	48 0		29·19	850	63·8	59·9	3·9	·478	80	57·3	63·9	64·8	60·7	4·1	·490	80	58·1		
	49 0		28·89	1,140	63·2	59·5	3·7	·473	81	57·0	63·3	63·8	59·7	4·1	·472	79	56·9		
	50 0		28·40	1,620	61·1	57·8	3·3	·450	83	55·5	61·2	62·5	58·1	4·4	·443	78	55·1		
	51 0		28·10	1,920															
	51 40																	50·0	·373
	52 0		27·85	2,160								60·9	57·3	3·6	·440	82	54·9		
	52 30				59·9	56·2	3·7	·421	80	53·5	60·1								
	53 0		27·70	2,310	58·9	55·8	3·1	·421	83	53·5	59·1	60·9	56·9	4·0	·429	79	54·1		
	53 30				58·9	55·5	3·4	·414	82	53·0	59·2								
	54 0		27·40	2,620	58·3	55·3	3·0	·415	84	53·1	58·6	60·3	55·7	4·6	·405	77	52·4		
	54 30				58·0	54·8	3·2	·406	83	52·5	58·3	59·6	55·6	4·0	·410	80	52·8		
	55 0		27·11	2,910															
	55 30				56·3	54·6	1·7	·419	91	53·4	56·7								
	56 0		26·90	3,120	55·8	53·2	2·6	·389	86	51·2	56·2	57·7	53·7	4·0	·382	79	50·7		
	57 0		26·41	3,630	54·8	52·0	2·8	·371	84	49·8	55·2	56·8	52·9	3·9	·372	79	49·9		
	57 30				54·4	51·6	2·8	·366	84	49·4	54·8								
	58 0		26·16	3,900															
	58 0	65·3	26·17†	3,890															
	58 30				52·5	50·0	2·5	·348	86	48·0	52·9	54·9	50·7	4·2	·341	77	47·4		
	59 0		25·77	4,300	51·5	48·7	2·8	·329	84	46·3	52·0	54·1	49·6	4·5	·324	76	45·9	44·3	·307
	59 30				51·0	47·9	3·1	·317	82	45·2	51·5								
	5 0 0		25·32	4,780	49·8	46·9	2·9	·307	83	44·3	50·3	52·4	48·6	3·8	·319	79	45·4	40·5	·269
	1 0		24·92	5,230	49·8	46·6	3·2	·302	82	43·8	50·3	50·5	46·8	3·7	·299	79	43·6		
	2 0		24·47	5,710	46·7	45·4	1·3	·307	92	44·3	47·3								
	2 30				45·7	44·3	1·4	·294	91	43·1	46·3							41·?	·274
	3 0		24·12	6,100	44·7	43·0	1·7	·278	89	41·5	45·3	47·1	44·2	2·9	·279	83	41·6		
	3 30				43·8	42·4	1·4	·274	91	41·0	44·4	46·1	43·3	2·8	·271	83	40·7	39·0	·255
	4 0		23·68	6,590	42·7	41·1	1·6	·260	89	39·5	43·3	44·9	42·7	2·2	·270	86	40·6		
	4 40				41·8	40·0	1·8	·248	88	38·2	42·5							35·5	·226

Table I. (Continued.)

	Greenwich Mean Time.	Barometer. Therm.	Barometer. Reading corrected.	Height above sea-level.	Dry and Wet Thermometers, *aspirated*. Dry.	Wet.	Diff.	Tension of vapour.	Relative humidity.	Dew-point.	Temperature corrected for change.	Dry and Wet Thermometers, *free*. Dry.	Wet.	Diff.	Tension of vapour.	Relative humidity.	Dew-point.	Regnault's Hygrometer. Dew-point.	Tension of vapour.
	h m s	°	in.	feet.	°	°	°	in.		°	°	°	°	°	in.		°	°	in.
August 26, 1852.	5 5 0		23·13	7,230	41·0	39·8	1·2	·252	92	38·7	41·7	43·6	40·9	2·7	·249	83	38·3		
	6 0		22·78	7,650															
	6 30				40·8	35·7	5·1	·183	67	29·6	41·5								
	7 0		22·33	8,200															
	8 0		22·08	8,510	42·3	31·0	11·3	·107	37	14·9	43·1								
	8 30				42·4	30·0	12·4	·092	32	10·8	43·2							14·—	
	9 0		21·74	8,940	41·7	29·0	12·7	·073	26	4·7	42·5	42·7	32·0	10·7	·110	38	15·7		
	9 30				41·1	28·5	12·6	·082	30	7·8	42·0							10·—	
	10 0		21·54	9,190	41·3	29·0	12·3	·088	32	9·6	42·2	41·7	31·7	10·0	·123	44	18·7	7·—	
	10 30				41·7	30·0	11·7	·099	35	12·8	42·6							6·0	·077
	11 0		21·44	9,320								43·6	34·9	8·7	·150	50	24·1		
	11 30				42·5	32·0	10·5	·113	39	16·4	43·4	43·6	33·0	10·6	·120	40	18·0		
	12 0		21·32	9,480															
	12 30				41·8	29·5	12·3	·092	33	10·8	42·8	44·0	32·0	12·0	·102	34	13·6		
	13 0		21·24	9,580	42·3	29·5	12·8	·088	31	9·6	43·3								
	13 30	58·4			40·9	27·2	13·7	·068	25	2·8	41·9								
	14 0		21·14	9,690	39·8	26·6	13·2	·069	26	3·2	40·8	42·7	30·1	12·6	·095	33	11·7		
	15 0		21·04	9,820	39·9	27·1	12·8	·075	28	5·4	41·0	43·6	31·2	12·4	·104	35	14·1		
	16 0		20·94	9,970	42·0	30·6	11·4	·107	38	14·9	43·1	44·5	33·4	11·1	·120	39	18·0		
	16 30				42·4	31·2	11·2	·113	39	16·4	43·5								
	17 0		20·75	10,200	39·8	28·5	11·3	·095	36	11·7	40·9	42·3	31·6	10·7	·120	42	18·0		
	17 0		20·74†	10,210															
	18 0		20·70	10,260	40·0	29·7	10·3	·108	41	15·1	41·2	42·3	32·0	10·3	·118	41	17·5		
	18 30				41·8	31·0	10·8	·113	40	16·4	43·0								
	19 0		20·70	10,290	42·7	32·0	10·7	·114	39	16·6	43·9	43·4	34·6	8·8	·149	50	23·9		
	20 0		20·85	10,100	43·8	32·6	11·2	·114	38	16·6	45·1								
	21 0		20·75	10,200	40·5	28·8	11·7	·094	35	11·4	41·8								
	22 0	58·0	20·65	10,340	40·8	29·5	11·3	·102	38	13·6	42·1							12·—	
	22 0		20·59†	10,420														10·8	·092
	22 30				40·8	29·4	11·4	·101	37	13·4	42·1								
	23 0		20·43	10,620	40·2	28·4	11·8	·092	35	10·8	41·6	42·8	32·4	10·4	·121	41	18·2		
	23 30				39·5	28·1	11·4	·094	36	11·4	40·9								
	24 0		20·25	10,840	37·6	27·3	10·3	·099	41	12·8	39·0							6·0	·077
	24 30				37·2	27·1	10·1	·097	40	12·2	38·6							6·0	·077
	24 40											38·9	29·4	9·5	·116	46	17·1		
	25 0		20·10	11,050	38·5	28·1	10·4	·102	41	13·6	40·0	39·4	30·5	8·9	·127	49	19·5		
	26 0		20·15	10,980	38·4	28·1	10·3	·102	41	13·6	39·9	39·0	30·2	8·8	·126	49	19·3		
	26 30				38·8	29·1	9·7	·113	45	16·4	40·3								
	27 0		20·20	10,920	39·4	29·1	10·3	·109	42	15·4	40·9								
	27 30				39·1	28·5	10·6	·102	40	13·6	40·6	39·4	30·9	8·5	·133	52	20·3		
	28 0		20·28	10,810	38·0	26·6	11·4	·085	34	8·7	39·6								
	28 30				36·8	26·6	10·2	·093	39	11·1	38·4								
	29 0		20·14	10,970	35·4	25·9	9·5	·094	42	11·4	37·0							10·—	
	29 30				34·0	25·1	8·9	·094	44	11·4	35·6	36·2	27·1	9·1	·104	45	14·1		
	30 0		19·95	11,210	33·9	25·0	8·9	·094	44	11·4	35·5							8·0	·083
	30 30				36·0	26·9	9·1	·104	45	14·1	37·7								
	31 0		19·80	11,440	35·9	27·0	8·9	·106	46	14·6	37·6	35·6	28·1	7·5	·122	54	18·4		
	31 30				35·4	26·4	9·0	·102	45	13·6	37·1	36·7	29·4	7·3	·132	56	20·6		
	32 0		19·73	11,540	36·8	27·2	9·6	·103	44	13·9	38·5								
	32 30				37·5	27·6	9·9	·103	43	13·9	39·2	39·5	32·0	7·5	·142	55	22·6		
	33 0		19·65	11,650	37·2	26·8	10·4	·095	40	11·7	39·0								
	34 0		19·59	11,740	37·3	27·2	10·1	·100	42	13·1	39·1	40·0	31·6	8·4	·140	53	22·2		
	35 0		19·59	11,740	38·0	27·7	10·3	·102	41	13·6	39·8								
	35 30				38·8	28·1	10·7	·101	40	13·3	40·6	38·0	30·0	8·0	·131	53	20·4	12·0	·096
	36 0		19·60	11,730															
	36 50				36·8	27·2	9·6	·103	44	13·9	38·7								
	37 0		19·64	11,650								40·7	33·3	7·4	·153	56	24·7		
	37 30				35·7	25·8	9·9	·093	41	11·1	37·6								
	38 0		19·60	11,690	35·0	25·1	9·9	·089	40	9·9	36·9								

Table I. (Continued.)

August 26, 1852.

Greenwich Mean Time.	Barometer. Therm.	Barometer. Reading corrected.	Height above sea-level.	Dry and Wet Thermometers, *aspirated*. Dry.	Wet.	Diff.	Tension of vapour.	Relative humidity.	Dew-point.	Temperature corrected for change.	Dry and Wet Thermometers, *free*. Dry.	Wet.	Diff.	Tension of vapour.	Relative humidity.	Dew-point.	Regnault's Hygrometer. Dew-point.	Tension of vapour.
h m s	°	in.	feet	°	°	°	in.		°	°	°	°	°	in.		°	°	in.
5 38 30	......	....	....	33·6	23·8	9·8	·082	39	7·7	35·6								
39 0	......	19·55	11,730	32·5	22·8	9·7	·077	38	6·1	34·5								
39 30	......	....	....	32·9	22·8	10·1	·075	36	5·4	34·9								
40 0	54·7	19·52	11,790	34·0	23·3	10·7	·074	35	5·0	36·0	......	......	......	......	......	......	6·0	·077
40 30	......	19·50†	11,840	35·8	25·6	10·2	·090	40	10·2	37·8								
41 0	......	19·50	11,830	34·7	23·6	11·1	·073	33	4·7	36·8								
41 30	......	....	....	33·9	23·2	10·7	·074	35	5·0	36·0								
42 0	......	19·32	12,050	32·5	22·3	10·2	·073	36	4·7	34·6	37·5	28·9	8·6	·121	50	18·2		
42 30	51·5	19·27†	12,150	34·9	24·7	10·2	·085	38	8·7	37·0								
43 0	......	19·20	12,220	32·7	22·3	10·4	·072	35	4·3	34·8	36·8	28·1	8·7	·116	49	17·1		
43 30	......	....	....	31·9	21·9	10·0	·073	37	4·7	34·1	......	......	......	......	......	......	0·—	
44 0	52·5	19·15	12,280	31·9	22·1	9·8	·075	38	5·4	34·1	......	......	......	......	......	......	−3·—	
44 30	......	....	....	31·2	21·2	10·0	·069	36	3·2	33·4	33·3	......	......	......	......	......	−5·0	·050
45 0	52·5	19·10	12,330															
46 0	52·5	19·07	12,360	29·0	21·3	7·7	·085	47	8·7	31·3								
46 30	......	....	....	29·2	21·1	8·1	·081	45	7·4	31·5								
47 0	......	19·05	12,390	29·0	21·1	7·9	·083	46	8·1	31·3								
47 30	......	....	....	30·9	23·2	7·7	·091	47	10·5	33·2	......	......	......	......	......	......	+4·0	·071
48 0	53·6	19·05	12,420	31·5	23·4	8·1	·093	47	11·1	33·8								
48 30	......	....	....	32·5	24·2	8·3	·096	47	12·0	34·8								
49 0	......	19·04	12,470	34·7	26·2	8·5	·106	48	14·6	37·1								
49 30	......	....	....	34·1	26·1	8·0	·109	51	15·4	36·5								
50 0	......	19·00	12,480	30·6	23·9	6·7	·105	55	14·4	33·0								
50 30	......	....	....	30·9	24·7	6·2	·113	59	16·4	33·3	......	......	......	......	......	......	0·0	·061
51 0	54·5	18·90	12,610	29·9	......	......	......	......	......	32·3	32·7	25·2	7·5	·107	53	14·9		
51 30	......	....	....	29·6	21·9	7·7	·088	48	9·6	32·1								
52 0	......	18·80	12,740	28·9	21·3	7·6	·086	48	9·0	31·4								
52 30	......	....	....	27·6	20·5	7·1	·086	50	9·0	30·1								
53 0	54·6	18·71	12,840	27·0	20·3	6·7	·087	52	9·3	29·5								
53 30	......	....	....	27·2	20·4	6·8	·087	52	9·3	29·7	......	......	......	......	......	......	−4·0	·052
54 0	54·8	18·70	12,890	29·5	22·5	7·0	·096	53	12·0	32·1								
55 0	......	....	....	28·7	21·8	6·9	·093	52	11·1	31·3								
56 0	......	18·68	12,930	30·0	23·2	6·8	·101	54	13·3	32·6								
56 30	......	18·87†	12,690															
57 0	55·4	18·78	12,860	35·4	27·1	8·3	·114	51	16·6	38·1								
57 30	......	....	....	35·1	27·1	8·0	·115	52	16·8	37·8								
58 0	56·5	18·94	12,610	34·2	25·9	8·3	·106	49	14·6	36·9								
58 30	......	....	....	34·9	26·4	8·5	·107	49	14·9	37·6								
59 0	......	19·20	12,260	35·8	27·3	8·5	·113	50	16·4	38·6								
59 30	......	....	....	36·4	26·0	10·4	·091	39	10·5	39·2								
6 0 0	......	19·48	11,860	......	......	......	......	......	......	......	35·7	25·2	10·5	·086	38	9·0		
1 0	57·0	19·59	11,700	35·0	24·7	10·3	·084	38	8·4	37·8								
2 0	58·2	19·59	11,690	34·6	24·9	9·7	·089	41	9·9	37·5								
2 30	......	....	....	32·5	24·2	8·3	·095	47	11·7	35·4								
3 0	......	19·45	11,870	33·1	25·2	7·9	·103	50	13·9	36·0								
4 0	51·0	19·25	12,140	32·7	25·2	7·5	·106	50	14·6	35·7								
4 0	......	19·23†	12,170															
5 0	......	19·17†	12,220	29·5	23·2	6·3	·103	57	13·9	32·5								
6 0	......	19·17†	12,210	29·0	22·8	6·2	·102	57	13·6	32·0								
7 30	......	19·28†	12,110	33·5	26·8	6·7	·122	58	18·5	36·6								
9 0	......	....	....	33·9	27·0	6·9	·121	57	18·2	37·0	35·8	24·7	11·1	·079	35	6·8		
11 0	......	....	....	......	......	......	......	......	......	......	36·3	25·7	10·6	·088	38	9·6		
12 0	......	19·60	11,650															
13 0	49·1	19·40	11,920	32·1	23·7	8·4	·091	45	10·5	35·4								
13 30	......	....	....	33·6	25·0	8·6	·098	46	12·5	37·0	34·7	24·5	10·2	·084	38	8·4		
14 0	......	19·26	12,130	32·4	24·4	8·0	·099	49	12·8	35·8	......	......	......	......	......	......	+5—	
15 0	47·8	19·16	12,270	32·9	25·2	7·7	·105	51	14·4	36·3	......	......	......	......	......	......	+2·5	·067
15 50	......	....	....	31·2	24·2	7·0	·105	54	14·4	34·7	......	......	......	......	......	......	+1·0	·064
16 0	......	19·01	12,460															

Table I. (Continued.)

Greenwich Mean Time.	Barometer.		Height above sea-level.	Dry and Wet Thermometers, *aspirated*.						Temperature corrected for change.	Dry and Wet Thermometers, *free*.						Regnault's Hygrometer.	
	Therm.	Reading corrected.		Dry.	Wet.	Diff.	Tension of vapour.	Relative humidity.	Dew-point.		Dry.	Wet.	Diff.	Tension of vapour.	Relative humidity.	Dew-point.	Dew-point.	Tension of vapour.
h m s	°	in.	feet	°	°	°	in.		°	°	°	°	°	in.		°	°	in.
August 26, 1852. 6 16 10	......	......	......	......	......	......	......	......	......	......	35·3	23·7	11·6	·072	32	4·3		
16 30	......	......	......	29·0	23·0	6·0	·105	59	14·4	32·5								
17 0	47·3	18·95	12,530	29·9	23·2	6·7	·102	55	13·6	33·4	32·2	22·5	9·7	·078	39	6·4		
18 0	......	18·85	12,670	29·7	23·2	6·5	·103	56	13·9	33·2	30·9	22·5	8·4	·087	45	9·3		
18 30	......	......	......	29·0	22·7	6·3	·102	57	13·6	32·6	31·3	22·7	8·6	·087	45	9·3		
19 0	46·6	18·71	12,880	30·0	23·3	6·7	·103	56	13·9	33·6	32·3	23·2	9·1	·087	43	9·3	− 1·0	·059
19 40	......	......	......	30·0	23·2	6·8	·101	55	13·3	33·6	32·1	22·7	9·4	·082	41	7·7	− 4·0	·052
20 0	45·9	18·66	12,950	30·1	23·2	6·9	·101	54	13·3	33·7								
20 30	......	......	......	30·5	23·4	7·1	·101	53	13·3	34·1								
21 0	45·0	18·55	13,110	29·9	23·2	6·7	·103	56	13·9	33·6	31·6	22·2	9·4	·080	41	7·1		
21 40	44·6	18·49†	13,170															
22 0	......	18·46	13,200	26·7	21·3	5·4	·101	61	13·3	30·4	29·0	20·5	8·5	·077	43	6·1		
23 0	44·5	18·26	13,500	27·4	21·9	5·5	·104	61	14·1	31·1	29·0	21·2	7·8	·086	48	9·0	+ 2·0	·066
24 0	......	18·11	13,720	......	......	......	......	......	......	......	29·5	21·9	7·6	·091	50	10·5		
25 0	43·0	17·97	13,920	26·7	20·1	6·6	·089	54	9·9	30·5	28·0	21·2	6·8	·093	54	11·1		
25 40	......	......	......	26·0	19·4	6·6	·086	53	9·0	29·9	27·0	20·4	6·6	·090	54	10·2	0·0	·061
26 0	42·6	17·85	14,080	25·6	19·3	6·3	·087	55	9·3	29·5	26·4	20·5	5·9	·095	58	11·7		
26 30	......	17·78†	14,220	27·7	20·5	7·2	·088	51	9·6	31·6	26·3	20·3	6·0	·094	58	11·4		
27 0	......	17·77	14.180	24·2	18·5	5·7	·087	58	9·3	28·1	25·8	20·2	5·6	·096	60	12·0		
27 0	......	17·75†	14,210															
28 0	......	17·62	14,440	26·1	20·3	5·8	·095	59	11·7	30·1	26·1	20·4	5·7	·096	60	12·0		
28 30	......	......	......	22·9	18·3	4·6	·094	65	11·4	26·9	25·1	19·6	5·5	·091	58	10·5		
29 0	......	17·45	14,640	22·5	18·3	4·2	·087	62	9·3	26·5	24·8	19·4	5·4	·094	61	11·3		
29 30	......	17·39†	14,750	23·1	18·9	4·2	·090	62	10·2	27·1	24·5	19·3	5·2	·095	62	11·7		
30 0	......	17·35	14.790	21·9	18·9	3·0	·106	76	14·6	25·9	24·5	19·5	5·0	·097	64	12·2		
30 0	......	17·33†	14,820															
31 0	39·8	17·31	14,870	23·1	19·3	3·8	·103	72	13·9	27·2								
32 0	39·0	17·26	14,930	21·9	18·5	3·4	·102	74	13·6	26·0								
32 30	......	......	......	22·4	19·1	3·3	·106	75	14·6	26·5	24·0							
33 0	......	17·16	15,090	21·6	18·9	2·7	·108	79	15·1	25·8	......	......	......	......	......	......	+ 12·0	·096
34 0	......	17·12	15,130	20·6	18·2	2·4	·107	81	14·9	24·8	22·5							
35 0	......	17·12	15,130															
35 20	......	......	......	20·8	18·3	2·5	·107	80	14·9	25·1	22·2	......	......	......	......	......	+ 6·0	·077
36 0	......	17·17	15,060															
36 30	......	......	......	22·2	18·9	3·3	·105	75	14·4	26·5								
37 0	......	17·13	15,110	20·8	17·9	2·9	·103	78	13·9	25·1								
38 0																		
39 0	......	16·92	15,410	......	......	......	......	......	......	......	19·4	16·4	3·0	·090	71	10·2		
39 30	......	......	......	......	......	......	......	......	......	......	18·8							
40 0	......	16·61	15,900	19·0	......	......	......	......	......	23·5	19·3	11·9	7·4	·054	43	− 3·2		
41 0	......	16·41	16,180	16·3	10·5	5·8	·058	51	− 1·4	20·8	17·5	9·7	7·8	·044	38	− 8·4		
42 0	......	16·26	16,400	14·6	8·7	5·9	·051	48	− 4·7	19·1	......	......	......	......	......	......	−10−	
43 0	......	16·14	16,600	14·8	9·6	5·2	·059	55	− 0·9	19·4	19·5	10·8	8·7	·044	35	− 8·4	−15−	
43 40	......	......	......	17·0	11·3	5·7	·062	54	+ 0·4	21·6	19·5	12·1	7·4	·056	44	− 2·3	−17·0	·032
44 0	......	16·06	16,760															
45 0	......	15·96	16,900	15·3	10·3	5·0	·063	58	+ 0·8	20·0								
46 0	......	15·95	16,940	16·3	10·7	5·6	·061	54	− 0·1	21·0	18·3	11·7	6·6	·059	49	− 0·9		
46 10	......	......	......	16·9	11·2	5·7	·062	54	+ 0·4	21·6								
46 20	......	......	......	......	......	......	......	......	......	......	15·6	9·4	6·2	·053	48	− 3·7		
47 0	......	15·96	16,930															
48 0	......	15·96	16,920	16·5	11·0	5·5	·062	55	+ 0·4	21·3	16·6	10·5	6·1	·057	50	− 1·8		
49 0	......	......	......	13·4	8·6	4·8	·059	58	− 0·9	18·2	15·6	9·0	6·6	·050	46	− 5·2		
50 0	33·4	15·86	17,010	12·4	7·6	4·8	·056	57	− 2·3	17·3	15·1	8·5	6·6	·049	45	− 5·8		
51 0	......	15·64	17,400	13·2	8·2	5·0	·056	56	− 2·3	18·1								
52 0	......	15·46	17,660	10·6	6·7	3·9	·058	63	− 1·4	15·6	14·0	7·8	6·2	·049	47	− 5·8		
52 30	......	......	......	10·2	6·8	3·4	·061	68	− 0·1	15·1								
53 0	......	15·31	17,890															
54 0	......	15·21	18,060	9·1	6·7	2·4	·066	77	+ 2·0	14·1								
55 0	......	15·01	18,370	7·3	5·7	1·6	·068	85	2·8	12·4								

TABLE I. (Continued.)

Date	Greenwich Mean Time.	Barometer. Therm.	Barometer. Reading corrected.	Height above sea-level.	Dry and Wet Thermometers, *aspirated*. Dry.	Wet.	Diff.	Tension of vapour.	Relative humidity.	Dew-point.	Temperature corrected for change.	Dry and Wet Thermometers, *free*. Dry.	Wet.	Diff.	Tension of vapour.	Relative humidity.	Dew-point.	REGNAULT'S Hygrometer. Dew-point.	Tension of vapour.
	h m s	°	in.	feet	°	°	°	in.		°	°	°	°	°	in.		°	°	in.
August 26, 1852.	6 56 0	......	14·91	18,590	9·5	6·5	3·0	·063	72	0·8	14·6								
	56 15	......	14·88†	18,650															
	57 0	......	14·76	18,870	9·9	6·7	3·2	·063	71	0·8	15·1	12·4	7·7	4·7	·058	59	− 1·4		
	57 30	......	......	......	9·7	6·6	3·1	·063	72	0·8	14·9	12·4	7·4	5·0	·055	56	− 2·8		
	58 0	......	14·65	19,070	10·1	7·1	3·0	·065	73	1·6	15·3	12·1	8·0	4·1	·062	64	+ 0·4		
	58 30	......	14·68†	19,000															
	59 0	......	14·61	19,100	8·3	6·3	2·0	·067	80	2·4	13·6								
	59 0	......	14·66†	19,010															
	7 0 0	......	14·69†	19,000	10·5	7·5	3·0	·066	73	2·0	15·8								
October 21, 1852.	2 47 0	......	29·73	280	57·3	50·6	6·7	·305	64	44·1	57·3								
	48 0	......	29·34	640	55·7	50·0	5·7	·309	68	44·5	55·7								
	48 30	62·2	29·12	850	54·8	49·6	5·2	·310	70	44·6	54·8								
	49 0	......	28·97	990	54·3	49·1	5·2	·304	70	44·0	54·3	54·6	49·4	5·2	·308	71	44·4		
	49 30	......	28·82	1,130	53·6	48·6	5·0	·301	71	43·7	53·6	53·7	48·7	5·0	·302	71	43·8		
	50 0	......	28·50	1.440	52·4	48·0	4·4	·301	74	43·7	52·4	52·6	48·2	4·4	·303	74	43·9		
	51 0	......	28·05	1,880	50·3	46·4	3·9	·288	76	42·5	50·3	50·3	46·5	3·8	·290	77	42·7		
	51 30	......	......	......	49·8	46·0	3·8	·285	77	42·2	49·8								
	52 0	......	27·61	2,310	49·3	46·1	3·2	·293	81	43·0	49·3	49·6	46·3	3·3	·294	80	43·1		
	53 0	......	27·25	2,670	49·5	48·6	0·9	·346	94	47·8	49·5	50·0	48·7	1·3	·343	92	47·5		
	53 35	......	......	......	......	......	......	......	......	......	......	50·4	49·4	1·0	·355	94	48·5		
	54 0	......	27·10	2,820	51·0	50·0	1·0	·363	94	49·2	51·0	51·4	50·7	0·7	·375	96	50·1		
	55 0	......	26·95	2,980	51·1	50·6	0·5	·376	97	50·2	51·1	51·7	51·4	0·3	·388	98	51·1		
	56 0	59·0	26·83	3,100	50·7	50·5	0·2	·378	99	50·4	50·7								
	56 30	......	......	......	......	......	......	......	......	......	......	51·7	51·0	0·7	·379	96	50·5		
	57 0	......	26·67	3.260	......	......	......	......	......	......	......	51·6	50·9	0·7	·378	96	50·4		
	58 0	......	26·52	3,420	50·1	50·1	0·0	·374	100	50·1	50·1	......	......	......	......	......	......	48·0	·349
	3 0 0	......	26·19	3,760	50·7	50·1	0·6	·368	96	49·6	50·7								
	0 35	......	......	......	51·3	50·2	1·1	·365	93	49·3	51·3								
	1 0	......	25·96	4,010	51·7	50·1	1·6	·359	91	48·9	51·7	54·1	51·0	3·1	·355	83	48·5		
	1 30	60·6	25·84	4,140	52·3	49·9	2·4	·348	86	48·0	52·3								
	2 0	......	25·73	4,260	......	......	......	......	......	......	......	53·8	50·3	3·5	·343	81	47·5		
	2 30	......	25·55	4.450	52·1	49·3	2·8	·337	84	47·0	52·2								
	3 0	......	25·48	4,520	52·0	49·1	2·9	·334	83	46·8	52·1	52·6	49·1	3·5	·328	80	46·2		
	4 0	......	25·07	4,960	50·3	45·9	4·4	·282	75	41·9	50·4	52·0	46·4	5·6	·276	69	41·3		
	4 35	......	......	......	49·9	45·0	4·9	·268	72	40·4	50·0	51·4	45·6	5·8	·266	68	40·2		
	5 0	58·5	24·70	5,370	49·6	44·5	5·1	·261	71	39·7	49·8	50·6	45·0	5·6	·261	68	39·7		
	5 30	......	......	......	48·9	44·0	4·9	·258	72	39·3	49·1								
	6 0	......	24·38	5,710	48·1	43·5	4·6	·255	73	39·0	48·3	49·2	43·7	5·5	·249	69	38·3		
	6 30	......	......	......	47·3	42·8	4·5	·250	73	38·4	47·5								
	6 40	......	24·08	6,050															
	7 0	......	23·95	6,190	46·6	42·3	4·3	·247	74	38·1	46·8								
	7 30	......	......	......	46·3	42·0	4·3	·245	75	37·9	46·5								
	8 0	......	23·46	6,750	44·9	40·6	4·3	·231	74	36·2	45·2	45·9	41·1	4·8	·232	72	36·3		
	9 0	......	23·29	6,940	44·5	39·0	5·5	·206	67	32·9	44·8								
	10 0	55·0	22·92	7,380	44·3	37·9	6·4	·189	61	30·5	44·6	......	......	......	......	......	......	27·0	·167
	11 0	......	22·73	7,590	42·9	36·2	6·7	·173	59	28·1	43·3								
	11 30	53·6	22·49	7,880	42·4	34·7	7·7	·153	53	24·7	42·8								
	12 0	......	22·31	8,090	41·6	34·2	7·4	·152	55	24·5	42·0								
	12 40	......	......	......	40·8	33·8	7·0	·153	57	24·7	41·2	42·7	34·8	7·9	·153	53	24·7		
	13 0	......	22·07	8,370	40·0	32·9	7·1	·146	55	23·4	40·4	39·6	32·7	6·9	·146	56	23·4		
	13 30	......	......	......	39·6	32·4	7·2	·142	55	22·6	40·1								
	14 30	......	21·46	9,110	37·8	30·8	7·0	·139	57	22·0	38·3								
	15 0	......	21·31	9,290	36·7	30·6	6·1	·145	62	23·2	37·2	36·7	30·4	6·3	·142	60	22·6		
	16 0	......	21·08	9,550	35·7	30·9	4·8	·157	69	25·4	36·2	37·1	31·2	5·9	·151	63	24·3		
	17 0	......	20·78	9,950	34·5	29·8	4·7	·150	69	24·1	35·1	35·6	30·1	5·5	·147	65	23·6		
	17 30	......	......	......	33·8	29·4	4·4	·150	71	24·1	34·4								
	18 0	......	20·54	10,240	32·7	28·9	3·8	·151	74	24·3	33·3	33·9	28·7	5·2	·140	66	22·2		
	18 30	......	......	......	32·1	28·4	3·7	·149	74	23·9	32·7	......	......	......	......	......	......	20·0	·129

Table I. (Continued.)

Date	Greenwich Mean Time.	Barometer. Therm.	Barometer. Reading corrected.	Height above sea-level.	Dry and Wet Thermometers, *aspirated.* Dry.	Wet.	Diff.	Tension of vapour.	Relative humidity.	Dew-point.	Temperature corrected for change.	Dry and Wet Thermometers, *free.* Dry.	Wet.	Diff.	Tension of vapour.	Relative humidity.	Dew-point.	Regnault's Hygrometer. Dew-point.	Tension of vapour.
	h m s	°	in.	feet	°	°	°	in.		°	°	°	°	°	in.		°	°	in.
October 21, 1852.	3 19 0	……	20·22	10,650	31·7	27·6	4·1	·142	72	22·6	32·3								
	19 30	……	……	……	31·2	27·0	4·2	·138	71	21·8	31·8								
	20 0	……	20·03	10,900	31·1	26·2	4·9	·129	67	20·0	31·8	31·0	26·3	4·7	·130	68	20·2		
	21 0	……	19·85	11,130	29·8	25·7	4·1	·131	71	20·4	30·5	31·6	26·8	4·8	·132	67	20·6		
	22 0	……	19·63	11,420	29·4	25·2	4·2	·128	71	19·8	30·1	30·4	25·9	4·5	·130	69	20·2		
	23 0	……	19·41	11,700	27·8	24·1	3·7	·126	73	19·3	28·6	29·2	24·7	4·5	·123	68	18·7		
	24 0	……	19·23	11,940	26·8	23·2	3·6	·121	73	18·2	27·6								
	25 0	……	19·10	12,130	26·8	22·6	4·2	·115	69	16·8	27·6	28·2	23·8	4·4	·119	70	17·8		
	26 0	40·4	18·95	12,320	25·9	……	……	……	……	……	26·8	……	……	……	……	……	……	17·0	·116
	27 0	……	18·85	12,450	24·9	……	……	……	……	……	25·8	27·5	24·2	3·3	·129	76	20·0		
	28 0	……	18·73	12,630	25·9						26·8								
	28 30	……	……	……	25·8	22·0	3·8	·114	71	16·6	26·7								
	29 0	……	18·72	12,640	25·7	22·0	3·7	·115	72	16·8	26·7	27·6	24·3	3·3	·130	76	20·2		
	29 30	……	……	……	26·0	22·0	4·0	·113	70	16·4	27·0								
	30 0	……	……	……	26·9	23·0	3·9	·119	72	17·8	27·9	27·7	24·1	3·6	·127	74	19·5	19·0	·124
	30 25	……	18·85	12,470															
	31 25	……	19·15	12,050	26·5	23·4	3·1	·126	77	19·3	27·5	27·8	24·2	3·6	·127	74	19·5	18·0	·120
	33 50	……	19·97	10,960	29·7	25·7	4·0	·131	71	20·4	30·8	……	……	……	……	……	……	19·0	·124
	36 0	……	20·67	10,070	33·4	29·2	4·2	·150	72	24·1	34·6								
	37 30	……	21·24	9,360	35·5	30·2	5·3	·149	66	23·9	36·7	36·2	31·3	4·9	·159	69	25·7		
	38 0	……	21·50	9,040	36·5	30·2	6·3	·140	60	22·2	37·7								
	39 0	43·6	21·81	8,660	……	……	……	……	……	……	……	38·6	32·3	6·3	·149	59	23·9		
	39 30	……	21·92	8,530	38·4	31·0	7·4	·136	55	21·4	39·7								
	40 0	……	22·05	8,380	39·3	31·2	8·1	·132	51	20·6	40·6								
	41 0	……	22·25	8,140	40·8	31·5	9·3	·125	46	19·1	42·1	……	……	……	……	……	……	25·0	·155
	42 30	44·5	22·46	7,900	42·5	……	……	……	……	……	43·8							24·0	·149
	43 30	……	22·55	7,800	43·2	35·1	8·1	·153	52	24·7	44·6	43·4	37·7	5·7	·195	66	31·4		
	45 0	……	22·49	7,870	42·5	35·2	7·3	·160	55	25·9	43·9								
	46 0	……	22·40	7,970	42·4	34·7	7·7	·153	53	24·7	43·8								
	48 0	……	22·31	8,090	42·9	34·7	8·2	·149	51	23·9	44·4	……	……	……	……	……	……	20·0	·129
	49 0	……	22·34	8,060	43·7	36·3	7·4	·169	56	27·4	45·2	43·1	35·7	7·4	·164	56	26·6		
	50 0	……	22·55	7,810	44·3	35·9	8·4	·156	51	25·2	45·7	44·6	37·3	7·3	·177	57	28·7		
	53 0	……	23·42	6,780	46·1	39·8	6·3	·206	63	32·9	47·7	46·6	40·6	6·0	·216	65	34·3		
	54 0	……	23·62	6,550	46·4	41·2	5·2	·229	69	35·9	48·0								
	54 20	……	……	……	……	……	……	……	……	……	……	46·5	41·5	5·0	·233	70	36·4		
	57 30	47·5	23·70	6,450															
	58 30	……	23·74	6,400	44·7	40·8	3·9	·236	76	36·8	46·5	……	……	……	……	……	……	35·0	·222
	4 1 30	……	24·62	5,420	46·7	42·4	4·3	·247	74	38·1	48·5								
	3 30	……	25·51	4,460	49·4	45·6	3·8	·284	78	42·1	51·3								
	4 0	……	25·75	4,210	……	……	……	……	……	……	……	49·5	46·2	3·3	·295	80	43·2		
	5 0	……	26·12	3,820	50·6	47·6	3·0	·314	82	45·0	52·5	50·6	48·1	2·5	·325	85	46·0		
	5 45	……	26·47	3,450	50·6	48·6	2·0	·335	88	46·8	52·5	50·6	48·7	1·9	·338	89	47·1		
	6 30	51·0	26·73	3,180	50·2	48·7	1·5	·342	91	47·5	52·1								
	8 0	……	27·03	2,880	50·1	49·1	1·0	·352	94	48·3	52·1								
November 10, 1852.	2 22 30	52·0	29·85	240	49·6	44·3	5·3	·247	67	38·1	49·6								
	23 0	……	29·55	510															
	23 30	……	29·47	580	48·2	43·5	4·7	·245	70	37·9	48·2	48·1	43·2	4·9	·240	69	37·3		
	24 30	……	28·96	1,060	46·4	42·2	4·2	·239	72	37·1	46·4	46·3	41·9	4·4	·233	71	36·4		
	25 0	52·0	28·66	1,330	45·0	41·6	3·4	·242	77	37·5	45·0								
	25 30	……	28·35	1,630	43·6	40·8	2·8	·241	80	37·4	43·6								
	26 10	……	27·99	1,970	42·3	38·7	3·6	·214	75	34·0	42·3	42·0	38·0	4·0	·203	72	32·5		
	27 10	……	27·18	2,760	39·6	37·4	2·2	·218	83	34·5	39·6	39·5	37·1	2·4	·214	82	34·0		
	28 0	……	26·78	3,150	38·0	36·5	1·5	·218	88	34·5	38·0	37·8	36·3	1·5	·217	89	34·4		
	29 0	……	26·23	3,700	37·2	36·2	1·0	·221	92	34·9	37·2	37·2	36·2	1·0	·221	92	34·9		
	29 30	……	25·93	4,010	36·8	36·0	0·8	·222	94	35·1	36·8								
	30 40	……	25·40	4,560	35·5	34·7	0·8	·211	94	33·6	35·5	……	……	……	……	……	……	33·0	·206
	31 0	……	25·08	4,890	34·3	34·0	0·3	·211	98	33·6	34·3								
	32 0	……	24·65	5,350	33·7	33·4	0·3	·206	97	32·9	33·7	33·8	33·4	0·4	·206	97	32·9		

Table I. (Continued.)

Date	Greenwich Mean Time.	Barometer. Therm.	Barometer. Reading corrected.	Height above sea-level.	Dry and Wet Thermometers, *aspirated.* Dry.	Wet.	Diff.	Tension of vapour.	Relative humidity.	Dew-point.	Temperature corrected for change.	Dry and Wet Thermometers, *free.* Dry.	Wet.	Diff.	Tension of vapour.	Relative humidity.	Dew-point.	Regnault's Hygrometer. Dew-point.	Tension of vapour.
	h m s	°	in.	feet	°	°	°	in.		°	°	°	°	°	in.		°	°	in.
November 10, 1852.	2 33 10	......	24·17	5,880	34·7	33·4	1·3	·197	90	31·7	34·7								
	34 0	44·3	23·94	6,140	35·6	31·4	4·2	·160	71	25·9	35·6								
	34 30	......	23·76	6,350	35·8	30·8	5·0	·150	66	24·1	35·8	36·3	31·8	4·5	·161	69	26·1		
	35 0	......	23·63	6,500	35·8	30·8	5·0	·150	66	24·1	35·8								
	35 30	......	23·45	6,700	35·6	31·4	4·2	·161	71	26·1	35·6								
	36 30	43·0	23·12	7,070	34·7	31·2	3·5	·167	76	27·1	34·7	35·1	31·4	3·7	·166	75	26·9		
	37 30	......	22·72	7,530	33·9	31·6	2·3	·178	84	28·9	33·9	33·7	31·3	2·4	·176	83	28·5		
	38 0	......	....	....	32·7	31·0	1·7	·179	88	29·0	32·7								
	38 30	......	22·29	8,030	32·7	31·4	1·3	·186	91	30·1	32·7								
	39 0	......	21·97	8,420	32·5	31·4	1·1	·187	92	30·2	32·5	32·2	31·2	1·0	·186	93	30·1		
	40 0	......	21·58	8,890	31·4	29·7	1·7	·171	88	27·7	31·4	31·3	29·7	1·6	·172	89	27·9		
	41 0	40·5	21·07	9,510	29·8	27·2	2·6	·149	81	23·9	29·8								
	42 0	......	20·71	9,970	29·5	26·2	3·3	·138	76	21·8	29·5	28·8	25·9	2·9	·139	78	22·0		
	43 0	......	20·17	10,630	26·9	24·2	2·7	·132	79	20·6	26·9	27·2	24·1	3·1	·129	77	20·0		
	44 0	......	19·65	11,310	24·9	22·4	2·5	·124	80	18·9	24·9	......	......	......	......	......	......	16·0	·111
	45 30	......	18·74	12,500	20·3	(32)	......	......	......	......	20·3	20·6	(32)	......	......	......	......	10·0	·089
	46 30	36·0	18·30	13,080	17·4	......	......	......	......	......	17·4								
	47 0	......	17·90	13,650	16·3	......	......	......	......	......	16·3	16·8							
	47 30	......	17·75	13,860	15·9	12·4	3·5	·076	68	5·8	15·9								
	49 0	......	16·66	15,480	13·4	8·9	4·5	·060	59	− 0·5	13·4								
	49 40	29·5	16·44	15,820	12·8	8·1	4·7	·057	58	− 1·8	12·8								
	50 30	......	16·23	16,150	......	......	......	......	......	......	......	12·7	8·4	4·3	·060	61	− 0·5		
	51 0	......	15·96	16,580	11·9	10·3	1·6	·081	84	+ 7·4	11·9								
	52 0	......	15·28	17,620	7·0	5·6	1·4	·068	85	+ 2·8	7·0	9·1	5·9	3·2	·060	69	− 0·5		
	52 30																		
	54 0	......	14·72	18,520	3·8	1·3	2·5	·052	74	− 4·2	3·8								
	55 0	......	14·44	18,990	2·7	0·6	2·1	·052	77	− 4·2	2·7	4·2	0·4	3·8	·043	60	− 9·1	− 30(?)	·019
	56 0	......	14·24	19,330	1·5	− 0·1	1·6	·053	82	− 3·7	1·5								
	57 0	......	14·01	19,730	0·4	− 0·5	0·9	·056	90	− 2·3	0·4	1·1	− 0·8	1·9	·050	78	− 5·2		
	58 0	......	14·04	19,690	1·0	+ 0·2	0·8	·058	91	− 1·4	1·0								
	59 0	......	14·20	19,400	1·5	0·0	1·5	·054	83	− 3·2	1·5	2·6	+ 1·1	1·5	·056	83	− 2·2		
	3 1 0	......	14·78	18,430	4·4	+ 0·9	3·5	·046	64	− 7·4	4·4	5·2	2·5	2·7	·053	71	− 3·7		
	2 0	18·2	14·95	18,110	3·4	− 0·2	3·6	·042	60	− 9·7	3·4								
	3 30	......	14·94	18,130	3·4	− 1·8	5·2	·031	44	−18·0	3·4	4·8	− 0·2	5·0	·035	48	−14·0		
	5 0	......	14·60	18,700	2·2	− 2·1	4·3	·035	53	−14·0	2·2	3·9	− 0·9	4·8	·035	49	−14·0		
	6 0	......	14·40	19,020	0·5	......	......	......	......	......	0·5	3·0	− 1·0	4·0	·039	57	−12·0		
	7 0	14·0	14·08	19,560	− 1·3	......	......	......	......	......	− 1·3								
	8 30	11·5	13·60	20,430	− 2·3	......	......	......	......	......	− 2·3	......	......	......	......	......	......	−22·0	·027
	9 0	......	13·45	20,650	− 5·1	......	......	......	......	......	− 5·1								
	10 30	......	13·02	21,380	− 9·3	......	......	......	......	......	− 9·3								
	12 0	......	12·65	22,110	− 9·6	......	......	......	......	......	− 9·6								
	13 0	......	12·51	22,370	−10·5	......	......	......	......	......	−10·5								
	14 30	......	12·40	22,640	− 8·9	......	......	......	......	......	− 8·9								
	15 0	......	12·24	22,930															
	16 0	......	12·24	22,930															
	17 0	......	....	....	− 6·5	......	......	......	......	......	− 6·5								
	19 0	......	12·94	21,640	− 5·3	......	......	......	......	......	− 5·3								

§ 5. *Variation of Temperature with Height.*

The observations of temperature given in the preceding table, with the corresponding heights, have been divided into groups, each group being composed of the observations within 1000 feet*. The numbers employed are those in column 11 of the table, which have been corrected for the change occurring in the temperature during the continuance of the experiments, as given by corresponding observations at the earth's surface. This correction is very probably inaccurate to some extent; but our information is as yet so imperfect with regard to the diurnal variations of temperature in the upper parts of the atmosphere, that no other course has appeared open to me. Any error arising from this cause is probably small in any of the series now under consideration, with the exception perhaps of August 26, when the hourly changes, as well as the time occupied in the ascent, were considerable. These groups are contained in the following table:—

TABLE II.—Means of Groups of the Observations of Temperature at different heights in the four Balloon Ascents of 1852, with the differences between the observed temperatures and those calculated by equations (1.) and (2.) from each whole series, and from the adopted divisions of each series.

Date.	Groups.			Temperature, observed − calculated.					
	No. of obs.	Height.	Temperature.	Whole Series. By eq. (1.).	Whole Series. By eq. (2.).	Lower Division. By eq. (1.).	Lower Division. By eq. (2.).	Upper Division. By eq. (1.).	Upper Division. By eq. (2.).
		feet	°	°	°	°	°	°	°
August 17	(1)	(120)	(71·2)	−1·4	+1·3	0·0	......	(−5·1)	(−7·5)
	1	2,440	62·8	−2·4	−1·4	0·0	......	(−5·6)	(−7·3)
	1	3,460	59·2	−2·8	−2·4	0·0	......	(−5·8)	(−7·1)
	1	4,110	58·1	−1·9	−1·7	(+1·3)	......	(−4·7)	(−5·8)
	1	5,880	57·8	+3·2	+2·8	(+7·3)	......	(+0·9)	(+0·3)
	3	6,800	54·0	+3·2	+1·6	(+6·8)	......	(+0·2)	(−0·2)
	5	7,530	51·4	+2·0	+1·1	(+6·9)	......	+0·1	−0·2
	8	8,550	49·0	+2·7	+1·7	(+8·1)	......	+1·1	+1·0
	7	9,470	44·4	+1·0	−0·1	(+6·9)	......	−0·4	−0·4
	4	10,680	40·4	+0·7	−0·4	(+7·2)	......	−0·3	−0·2
	4	11,620	37·2	+0·4	−0·6	......	......	−0·3	−0·1
	8	12,250	34·9	+0·1	−0·9	......	......	−0·5	−0·3
	3	13,480	30·8	−0·2	−0·9	......	......	−0·4	−0·2
	4	14,550	27·0	−0·7	−1·1	......	......	−0·6	−0·4
	6	15,510	24·4	−0·3	−0·4	......	......	0·0	+0·1
	10	16,600	20·6	−0·8	−0·4	......	......	−0·1	−0·1
	6	17,440	19·6	+0·9	+1·6	......	......	+1·7	+1·7
	6	18,490	15·0	−0·5	+0·9	......	......	+0·6	+0·5
	8	19,320	10·5	−2·4	−0·5	......	......	−1·0	−1·3

* The third group of October 21 extends only from 2000 to 2670 feet; the two observations between the latter height and 3000 feet, showing a marked change which refers them more intimately to the succeeding group. The lowest group in each series depends solely upon observations taken in the car, with the exception of that of August 17, when no observations having been recorded below 2000 feet, the general temperature at the earth has been adopted as the first result.

Table II. (Continued.)

Date.	Groups.			Temperature, observed − calculated.					
	No. of obs.	Height.	Temperature.	Whole Series.		Lower Division.		Upper Division.	
				By eq. (1.).	By eq. (2.).	By eq. (1.).	By eq. (2.).	By eq. (1.).	By eq. (2.).
		feet.	°	°	°	°	°	°	°
August 26	3	700	64·8	+1·6	+2·3	− 0·2	+ 0·2		
	2	1,380	62·2	+0·8	+1·4	− 0·4	− 0·3		
	5	2,480	59·1	+0·6	+0·9	+ 0·4	+ 0·2		
	4	3,390	55·7	−0·4	−0·3	+ 0·3	0·0		
	4	4,430	51·7	−1·7	−1·7	0·0	− 0·3	(− 9·8)	(−12·3)
	3	5,620	48·0	−2·3	−2·5	+ 0·5	+ 0·5	(− 9·5)	(−11·4)
	3	6,350	44·3	−4·1	−4·3	− 0·6	− 0·3	(−10·8)	(−12·3)
	3	7,390	41·9	−3·7	−4·1	(+ 0·7)	(+ 1·6)	(− 9·7)	(−10·8)
	3	8,730	42·9	+0·8	+0·4	(+ 6·4)	(+ 8·3)	(− 4·2)	(− 4·9)
	10	9,510	42·3	+2·2	+1·8	(+ 8·6)	(+11·2)	(− 2·2)	(− 2·1)
	20	10,590	41·0	+3·7	+3·3	(+11·1)	(+14·9)	+ 0·2	− 0·1
	25	11,630	37·4	+2·9	+2·5	(+11·2)	(+16·3)	0·0	0·0
	46	12,490	34·0	+1·7	+1·4	(+10·9)	(+17·1)	− 0·5	− 0·4
	5	13,350	31·9	+1·9	+1·6	......	......	+ 0·3	+ 0·4
	11	14,500	28·1	+1·1	+1·0	......	......	+ 0·4	+ 0·6
	7	15,200	25·3	+0·1	+0·1	......	......	− 0·1	+ 0·1
	8	16,700	20·2	−1·1	−0·8	......	......	− 0·1	− 0·1
	4	17,460	16·5	−2·8	−2·3	......	......	− 1·3	− 1·3
	8	18,750	14·5	−1·4	−0·6	......	......	+ 1·0	+ 0·8
October 21......	4	690	55·5	−1·4	+2·1	+ 0·2	......	(−10·9)	(−10·1)
	3	1,480	52·1	−3·1	−1·0	− 0·4	......	(−11·7)	(−11·0)
	3	2,360	49·5	−3·6	−3·1	+ 0·2	......	(−11·3)	(−10·8)
	10	3,330	51·4	+0·5	−0·3	(+ 5·5)	......	(− 6·1)	(− 5·8)
	6	4,420	51·7	+3·3	+1·5	(+ 9·8)	......	(− 2·1)	(− 1·9)
	6	5,530	48·9	+3·1	+0·6	(+10·9)	......	− 1·2	− 1·1
	7	6,580	46·5	+3·1	+0·5	(+12·3)	......	− 0·1	0·0
	8	7,770	44·1	+3·4	+1·1	(+14·1)	......	+ 1·6	+ 1·5
	9	8,280	41·7	+2·2	+ 0·2	(+13·6)	......	+ 0·9	+ 0·8
	6	9,380	36·9	−0·1	−1·1	(+12·7)	......	− 0·2	− 0·3
	8	10,520	32·7	−1·7	−1·2	......	......	− 0·6	− 0·6
	4	11,550	29·2	−2·8	−0·5	......	......	− 0·6	− 0·6
	9	12,430	27·0	−3·0	+1·1	......	......	+ 0·2	+ 0·2
November 10...	2	410	48·9	−0·1	+4·7	+ 0·2	......	(− 9·3)	(+ 1·5)
	4	1,500	44·3	−2·0	+1·4	− 0·3	......	(−10·6)	(− 1·4)
	1	2,760	39·6	−3·5	−1·6	− 0·2	......	(−11·4)	(− 3·9)
	2	3,430	37·6	−3·9	−2·6	+ 0·3	......	(−11·4)	(− 4·7)
	3	4,490	35·5	−3·3	−3·0	(+ 1·2)	......	(−10·3)	(− 4·8)
	2	5,620	34·2	−1·8	−2·4	(+ 5·1)	......	(− 8·1)	(− 3·8)
	4	6,420	35·7	+1·7	+0·5	(+ 9·6)	......	(− 4·2)	(− 0·6)
	3	7,460	33·8	+2·4	+0·7	(+11·6)	......	(− 2·9)	(− 0·3)
	3	8,440	32·2	+3·2	+1·1	(+13·7)	......	(− 2·5)	(+ 0·4)
	2	9,740	29·6	+3·9	+1·3	......	......	− 0·2	+ 0·9
	1	10,630	26·9	+3·4	+0·7	......	......	− 0·2	+ 0·4
	1	11,300	24·9	+3·1	+0·3	......	......	− 0·1	+ 0·1
	1	12,500	20·3	+1·5	−1·2	......	......	− 1·1	− 1·3
	3	13,530	16·5	+0·3	−2·3	......	......	− 1·7	− 2·3
	2	15,650	13·1	+2·1	+0·4	......	......	+ 1·3	+ 0·5
	1	16,580	11·9	+3·3	+2·0	......	......	+ 3·0	+ 2·2
	1	17,620	7·0	+1·0	+0·4	......	......	+ 1·2	+ 0·5
	6	18,480	3·3	−0·6	−0·4	......	......	+ 0·1	− 0·4
	6	19,460	0·6	−0·8	+0·2	......	......	+ 0·4	+ 0·2
	2	20,540	− 3·7	−2·4	−0·3	......	......	− 0·6	− 0·4
	2	21,510	− 7·3	−3·6	−0·4	......	......	− 1·2	− 0·6
	3	22,370	− 9·7	−3·9	+0·5	......	......	− 1·0	+ 0·1

In order to deduce from these numbers an approximation to the normal progression of temperature, freed from accidental irregularities, each series was in the first instance arranged in equations of the form—

$$T=X+YH, \quad \ldots \quad (1.)$$

T being the observed temperature at the height H; Y the change in degrees of temperature due to 1000 feet of height; and X the temperature at the level of the sea, which, with the addition of the quantity YH, would best represent the observed temperatures throughout the series, on the supposition of the change being uniform with the height. X and Y were eliminated by the method of least squares, and the following values obtained for the different series:—

	Aug. 17.	Aug. 26.	Oct. 21.	Nov. 10.
X=	72°·76	64°·98	58°·49	50°·02
Y=	− 3·097	− 2·617	− 2·291	− 2·496
Mean error .	1·72	2·16	2·65	2·56

On the supposition that the rate of change is not constant, but that it varies with the height, the following interpolating equation was employed,—

$$T=x+yH+zH^2+ \&c., \quad \ldots \quad (2.)$$

omitting higher powers of H than the second. When the same method of elimination was adopted, the following values were found:—

	Aug. 17.	Aug. 26.	Oct. 21.	Nov. 10.
x=	70°·17	64°·11	53°·36	44°·69
y=	− 2·363	− 2·346	+ 0·1132	− 1·095
z=	− 0·03613	− 0·01424	− 1·868	− 0·06070
Mean error	1·30	2·13	1·35	1·71

In Table II. will be found the differences between the observed temperatures, and those resulting from the two forms of equation employed. The progress of those differences in each series seems to follow a distinct law; there being in all cases a maximum of negative differences at a short distance from the earth, varying from about 2500 feet on October 21, to 6000 feet on August 26, followed also in each by a maximum of positive differences at an additional height of 3000 to 5000 feet. This peculiar departure from a regular progression will be distinctly traced in the projected results (Plates XIX.—XXII.). It is there seen, in all the four series, that after a steady decrease of the temperature in the lower portion of the curve, this decrease becomes arrested, and, for a space of about 2000 feet, the temperature remains almost constant, or even increases by a small amount; the decrease being afterwards resumed and continued, without much variation, throughout the upper portion of the curve. In the series of August 17 and 26 this fact is strikingly coincident with a large and abrupt diminution in the amount of aqueous vapour; the same coincidence being exhibited, in a less marked manner, on November 10. On October 21, the departure from a uniform decrease is very decidedly shown in connection with the stratum of dense cloud passed through. The temperature had been uniformly decreasing until the thick cloud was reached, when a decided rise commenced, which continued through the cloud, and for a space of about 600 feet above it; after which height the decrease was resumed, at first slowly, and afterwards with more rapidity.

The disturbance in the variation of the temperature now noticed, is in each series exhibited in such a systematic manner, that the hypothesis of a regular progression at all heights can scarcely be maintained. In order therefore to arrive at some approximate value of the normal variation of temperature in the atmosphere, it appears necessary to make abstraction of the disturbing cause. This I have endeavoured to do by dividing each series into two divisions; 1st, between the earth and the height where the diminution of temperature appears to be arrested; 2nd, above the point where the regular diminution of temperature seems to be resumed, omitting the space which is under the influence of the disturbance. The divisions adopted for the four series are as follows:—

	Aug. 17. Feet.	Aug. 26. Feet.	Oct. 21. Feet.	Nov. 10. Feet.
Lower division	0 to 4000	0 to 7000	0 to 2700	0 to 4000
Upper division	7000 to 20,000	10,000 to 19,000	5000 to 13,000	9000 to 23,000

These partial series have been examined by the same methods as the entire series; the number of groups in the lower division being, however, with the exception of August 26, too small to admit of the application with any advantage of equation (2.). The results for the different series are as follows:—

		By Equation (1.).			By Equation (2.).			
		X	Y	Mean error.				
Aug. 17	Lower division	71·62	−3·598	0·00				
	Upper division	76·68	−3·371	0·73	79·17	−3·771	+0·01484	0·71
Aug. 26	Lower division	67·46	−3·549	0·39	66·75	−2·969	−0·08220	0·30
	Upper division	76·36	−3·355	0·60	81·68	−4·104	+0·02552	0·58
Oct. 21	Lower division	57·77	−3·581	0·26				
	Upper division	68·77	−3·376	0·82	67·81	−3·162	−0·01119	0·80
Nov. 10	Lower division	50·21	−3·760	0·26				
	Upper division	59·45	−3·046	1·22	48·05	−1·516	−0·04791	1·02

The values of the constants in equation (2.), deduced from the higher divisions, show that, in the two series of August 17 and 26, the temperature decreases *less* rapidly as we ascend; whilst the values for October 21 and November 10 indicate a contrary result. The value of the second term (z) is, with the exception of the series of November 10, very small, and the amounts of the mean errors show that the observations are little better represented than by the single constant of equation (1.).

On the whole, we are scarcely at liberty to conclude from these results that the progress of the temperature, when free from disturbing influences, is other than *uniform* with the height.

Confining our attention now to the results deduced by equation (1.), we infer from them that in each series the rate of decrease of temperature, *below* the stratum where the disturbing influence exists, is *greater* than above that stratum; the ratio of the rate of decrease in the lower division to that in the higher, being—

On Aug. 17, 1·067; on Aug. 26, 1·058; on Oct. 21, 1·061; and on Nov. 10, 1·234.

If, in order to obtain the mean rate of decrease of temperature in the atmosphere, freed from disturbing causes, we allow to the lower and upper series values proportional to the spaces within which the observations for each division occur, we have the following numbers representing the decrease of temperature for 1000 feet of height:—

Aug. 17	3°·434	Oct. 21 . . .	3°·431
Aug. 26	3·440	Nov. 10 . . .	3·205

the values for the first three series being almost identical; that for the fourth differing from them by $\frac{1}{15}$th of the whole.

It may be convenient to give here the results for the rate of diminution of temperature obtained by different methods, expressed in the form usually adopted by meteorologists, viz. the height in feet equivalent to a decrease of one degree FAHRENHEIT.

	Aug. 17.	Aug. 26.	Oct. 21.	Nov. 10.
From the whole series	322·9	382·0	436·5	400·6
From first and last groups only . .	316·3	358·8	411·9	374·7
From lower division	277·9	281·8	279·3	266·0
From upper division	296·5	298·1	296·2	328·3
Mean of two divisions	292·0	290·7	291·5	312·0

The amount of distortion in the curve representing the diminution of temperature, produced by the disturbing influence which has been noticed, may be approximately stated at 7° on Aug. 17; 10°½ on Aug. 26; 11°½ on Oct. 21, and 12° on Nov. 10.

§ 6. *Variation of the Hygrometric Condition of the Air.*

As the amount of aqueous vapour in the air must necessarily decrease with the temperature, even although the proportion to the whole capacity of the air for moisture should remain constant, the changes at different heights may probably be most conveniently studied by examining the results for the "Relative Humidity," or the proportion which the amount of vapour present in the air bears to that which it would contain were it completely saturated. Since these changes do not, as in the case of temperature, appear to follow any regular course from which normal results might be derived, I shall here only state briefly the most prominent peculiarities presented by each series, referring for further information to the table of observations and to the projected results.

August 17.—We see by the curve of relative humidity for this day, that, from the earth's surface to the height of about 4000 feet, the humidity slightly increased; the presence of a considerable quantity of moisture being also shown by the existence of a partial stratum of cloud at the height of about 2500 feet. Between the heights of 4000 and 5880 the humidity decreased *with great rapidity* from about 85 to less than 35. For a considerable space little alteration took place, with the exception of a sudden increase at the height of about 9000 feet, which was confined to a stratum of not more than 400 feet; but as the evidence of its existence depends upon only one or two observations it may perhaps be doubtful. From 10,000 feet to 12,300, the humidity gradually increased to about 90, which value it retained very constantly through fully 4000 feet. After 16,500 feet there were considerable irregularities, there being however a comparatively dry stratum between 18,000 and 19,000 feet, which was followed by a decided increase in the humidity. These indications agree well with what is stated in § 3. with regard to the occasional existence of cloud above the height of 13,000 feet, and with the fact that at the highest point reached a mass of cloud was seen at a short distance above. In this series we can trace the existence of two distinct strata of moist air, besides a third, which undoubtedly existed at a greater height, but which was not quite reached.

August 26.—As on the first ascent, the humidity steadily increased from the earth's surface. Between the heights of 7200 and 8950 feet it also *rapidly* diminished from 92 to 26. For some distance the variations were no greater than might be supposed to arise from uncertainty of observation in such extreme circumstances. It will be remarked, on examining the curve of the tension of vapour, that whilst the indications of REGNAULT's hygrometer did not differ much from those of the wet thermometer at the height of about 11,000 feet, the difference became considerable at about 12,000 or 13,000 feet; thus rendering it probable that at the latter heights the relative humidity, as deduced from the dry and wet thermometers, was too great. The general accordance between the two hygrometers was however nearly restored at about 15,000 feet, confirming the rise which there took place in the amount of vapour. We may therefore consider that there was little change in the humidity from 9000 to 14,000 feet, a decided increase having however occurred at 15,000 feet, followed by a diminution till 16,400 feet; an increase having been again indicated in the remainder of the curve. The principal stratum of vapour on this day extended from the earth to 7200 feet, a second and perhaps a third of smaller thickness existing at 15,000 and 18,000 feet.

October 21.—The amount of moisture in the air on this occasion was considerable. The relative humidity increased as we left the earth, at first slowly till the height of 2000 feet, when irregular masses of cloud became frequent, and afterwards with more rapidity, till within the principal cloudy mass, at a height of 3450 feet, it attained the point of complete saturation. After leaving the cloud the humidity diminished steadily but not very rapidly till 5300 feet, where a slight rise commenced, continuing till 6700 feet; it then decreased till 8300 feet, when it rose again and remained nearly constant at 70 for the last 3000 feet of the ascent. The changes occurring in this series were neither to the same extent nor so abrupt in their character as those shown in the first two.

November 10.—The humidity, again, as in all the previous series, increased from the earth to the first cloud, which was at a low elevation and of but little density; upon leaving it, at about 1900 feet, a slight depression took place. Immediately above this low cloud a different current of air existed, shortly after entering which the humidity again increased until, in the second cloud, it became nearly complete; the decrease, after leaving the cloud at 5000 feet, becoming rapid and attaining a minimum at 6500 feet. A second well-defined maximum was reached at 8300 feet, followed at 10,000 feet by a secondary minimum. The humidity diminished on the whole till about 15,800 feet, when a sudden increase commenced, which continued from 16,500 to 17,600 feet, followed by an equally sudden decrease at 18,000 feet, the humidity subsequently increasing. The fluctuations in this series were numerous, there having been no fewer than four or perhaps five different strata of vapour.

§ 7. *General Remarks.*

The principal results deduced from the experiments described may be thus generally stated.

The temperature of the air decreases uniformly with the height above the earth's surface, until at a certain elevation, varying on different days, the decrease is arrested, and for a space of from 2000 to 3000 feet the temperature remains nearly constant, or even increases by a small amount; the regular diminution being afterwards resumed and generally maintained, at a rate slightly less rapid than in the lower part of the atmosphere, and commencing from a higher temperature than would have existed but for the interruption noticed. This interruption in the decrease of temperature is accompanied by a large and abrupt fall in the temperature of the dew-point, or by actual condensation of vapour, from which it may be inferred that the disturbance in the progression of temperature arises from a development of heat in the neighbourhood of the plane of condensation. The subsequent falls in the temperature of the dew-point are generally of an abrupt character, and corresponding interruptions in the decreasing progression of temperature are sometimes distinguishable, but in a less degree; as might indeed be expected from the fact, that at lower temperatures the variations in the absolute amount of aqueous vapour are necessarily smaller, and their thermic effects consequently diminished.

Dr. MILLER's *Analysis of Air collected in the Ascents.*

"King's College, London, 5 May, 1853.

"MY DEAR SIR,—The following particulars of my examinations of some of the specimens of air collected by Mr. WELSH in the course of the balloon ascents made under the superintendence of the Kew Committee of the British Association, may not be unacceptable to the Fellows of the Royal Society as supplementary to a part of Mr. WELSH's report and observations.

"The samples of air collected upon the 26th of August appear to have been taken in the most unexceptionable manner, and it was upon these only that my experiments were made. The recipients for the air were wide glass tubes, about 5 cubic inches in capacity, to each of which a portion of barometric tubing, 3 or 4 inches in length, was attached, as a neck that might receive a cap and stopcock, and which would admit also of being hermetically sealed afterwards by the blowpipe. Two of these tubes were furnished with excellent stopcocks, and were found able to support without leakage for twenty-four hours the exhaustion obtained by an air-pump, the gauge of which indicated a pressure of 0·5 inch.

"Having been thus tested they were exhausted to this extent immediately before the ascent took place, and were filled with the specimens to be examined by simply opening and then closing the stopcock, the altitude being determined by an observation of the barometer at the moment. In the third tube, a Torricellian vacuum was obtained, the tube being then sealed and drawn off, so as to admit of being broken at a filemark when the air was to be collected; after the specimen had been thus obtained, the aperture was closed by thrusting the neck of the tube into a cap filled with softened wax.

"The tubes were within twenty hours after the air had been collected hermetically sealed by myself, and the proportions of oxygen and nitrogen determined with great care by detonation with hydrogen in 'REGNAULT's Eudiometer.'

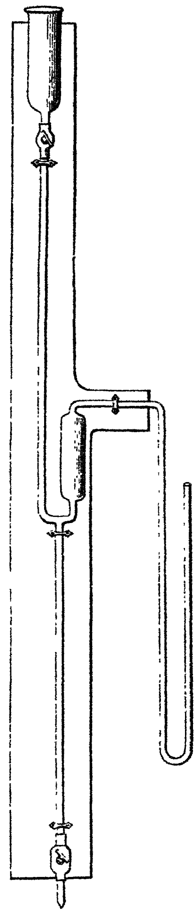
"The volumes of oxygen found in the air collected at different altitudes are given in the following table:—

	Altitude.	Volume of oxygen.
Air collected at King's College		20·920
Tube 2	13,460 feet . . .	20·888
Tube 3	18,000 feet . . .	20·747
Tube (G 1), Torricellian vacuum . . .	18,630 feet . . .	20·888

"From these observations it would appear that the composition of the atmosphere, as regards the proportion of oxygen and nitrogen, scarcely varies more as we ascend through the first half of that atmosphere (for at an altitude of about 3½ miles one-half of the atmosphere lies beneath us), than it is found to vary at different spots upon the surface: that there is, in fact (as GAY-LUSSAC had long since announced as the result of his experiments, made at a time when the methods of gaseous analysis were less perfect than at present), no sensible difference in the composition of the atmosphere upon the surface, and at the greatest heights accessible to man.

"In quantities of air so limited as those at my disposal, it was not possible to determine accurately the proportion of carbonic acid which they contained. Its presence however was distinctly shown by the formation of a film of carbonate of lead upon a solution of the subacetate which was introduced to a portion of the air confined over mercury.

"I have found a form of pipette, a sketch of which I subjoin, very useful for transferring small quantities of gases over mercury. It saves a great deal of fatigue, and I think contributes to precision in the results obtained. Its working is so simple as hardly to require description. It is first completely filled with mercury by closing the lower steel stopcock and opening the upper one, then pouring in mercury by the funnel until the metal escapes by the open end of the long bent tube; the upper stopcock is now closed, the bent tube introduced into the jar containing the gas to be transferred, and the end of the tube is lifted above the level of the metal in the jar of gas; the lower stopcock is then opened, mercury runs out, and gas takes its place; when a sufficient quantity has entered, the end of the tube is depressed beneath the mercury, a little of the metal enters and seals the opening, the lower stopcock is closed, and the pipette with its contents is withdrawn: the bent tube is now introduced beneath the jar which is to receive the gas. The funnel at top is filled with mercury, the upper stopcock opened, and the descending column of mercury expels the gas into the vessel destined to receive it.



"I am, my dear Sir,

"Yours very truly,

"William Allen Miller."

"*To Colonel Sykes,*

Chairman of the Kew Committee."

Meteorological Observations during the Ascents at different places.

TABLE III.—Places at which Meteorological Observations were taken in connection with the Balloon Ascents of 1852, with the Geographical coordinates, and name of the Observer or Authority.

Name of Place.	Latitude.	Longitude.	Height above sea level.	Authority.
	° ′ ″	° ′ ″		
Armagh	54 21 13	6 38 52 W.	209	Rev. Dr. Robinson, F.R.S.
Aylesbury	51 49	0 49 15 W.	284	Thomas Dell, Esq.
Bedford	52 8	0 28 W.	100	Dr. S. Herbert Barker.
Cambridge	52 12 52	0 5 53 E.	80	Professor Challis, F.R.S.
Cardington	52 7	0 24 W.	......	S. C. Whitbread, Esq.
Derby*	52 55	1 28 16 W.	100?	Mr. Davis, Optician.
Diss*	52 22	1 6 E.	130	Thomas E. Amyot, Esq.
Dublin	53 21	6 15 W.	24½	Rev. R. V. Dixon, A.M.
Edinburgh	55 27 23	3 10 45 W.	354	Professor Smyth, F.R.S.E.
Enfield	51 39	0 4 57 W.	76½	Rev. J. M. Heath.
Grantham	52 54 52	0 39 0 W.	190	J. W. Jeans, Esq.
Greenwich	51 28 38	0 0 0	159	The Astronomer Royal.
Hartwell House	51 49	0 51 W.	250	Dr. Lee, F.R.S.
Hartwell Rectory	51 48 36	0 51 W.	290	Rev. C. Lowndes, M.A., F.R.A.S.
Haverhill	52 5	0 26 36 E.	......	Wm. W. Boreham, Esq., F.R.A.S.
Hawerden	53 11 0	3 2 0 W.	260	Dr. T. Moffat, F.R.A.S.
Highfield House	52 57 30	1 10 W.	204	E. J. Lowe, Esq., F.R.A.S.
Holkham	52 57 10	0 48 E.	39	Samuel Shellabear, Esq.
Kew Observatory	51 28 37	0 15 45 W.	40?	Captain Younghusband, R.A., F.R.S.
Lewisham	51 28	0 1 W.	80	James Glaisher, Esq., F.R.S.
Linslade	51 55	0 40 W.	313	John Osborn, Jun., Esq.
Marboué	48 6 57	1 20 3 E.	361	M. Le Commandant Delcros.
Norwich	52 37	1 16 E.	33	Wm. Brooke, Esq.
Oxford	51 45 40	1 15 30 W.	210	M. J. Johnson, Esq., F.R.S.
Rosehill, Oxford	51 43 50	1 14 W.	270?	John Slatter, Esq.
Les Rousseaux	47 30 58	2 20 10 E.	......	M. Le Commandant Delcros.
Royston	52 2 40	0 0 30 W.	271	Hale Wortham, Esq.
Ryde	50 45	1 11 30 W.	110	Benjamin Barrow, Esq.
Southampton	50 54 34	1 24 25 W.	60	Dr. Drew, F.R.A.S.
Stone	51 47 57	0 52 16 W.	320	Rev. J. B. Reade, M.A., F.R.S.
St. Ives*	52 20	0 5 W.	......	John King Watts, Esq.
St. John's Wood	51 31	0 15 W.	150	George Leach, Esq.
Ventnor*	50 36	1 13 W.	150	Dr. Martin.
York	53 57 48	1 4 W.	50	John Ford, Esq.

* The barometrical observations at Derby, Diss, St. Ives, and Ventnor have not been corrected for temperature.

TABLE IV.—Meteorological Observations, made at Various Places on the days of the Four Balloon Ascents in 1852.

Observations on August 17, 1852.

Hour.	Barom.	Therm. Dry.	Therm. Wet.	Tension of Vapour.	Barom.	Therm. Dry.	Therm. Wet.	Tension of Vapour.	Barom.	Therm. Dry.	Therm. Wet.	Tension of Vapour.	Barom.	Therm. Dry.	Therm. Wet.	Tension of Vapour.
h	in.	°	°	in.	in.	°	°	in.	in.	°	°	in.	in.	°	°	in.
	Aylesbury.				Bedford.				Cambridge.				Cardington.			
9 A.M.	......	...	...	......	29·79	67·4	62·0	0·498	29·912	65·5	63·0	0·550	29·793	65·0	60·5	0·482
2 P.M.	29·568	72·3	66·7	0·590												
3	·523	73·0	68·5	·642	......	...	...	......	·813	72·0	68·6	·656	·725	70·8	65·0	·552
4	·515	69·8	65·7	·585	·70	73·1	67·0	·590	·814	71·3	68·5	·660	·717	70·0	66·0	·593
5	·502	69·3	66·0	·601	·66	70·6	66·8	·611	·789	70·3	68·1	·658	·669	69·2	66·0	·602
6	·485	67·0	65·7	·616	·65	69·6	67·0	·630	·774	69·3	67·7	·656	·678	67·4	66·0	·622
7	·477	66·0	65·0	·606	·64	68·4	66·5	·627	·756	67·8	66·5	·622	·668	67·0	65·0	·595
8	·487	62·0	62·0	·559	·62	66·1	65·0	·605	·730	67·5	66·3	·631	·623			
	Derby.				Dublin.				Diss.				Enfield.			
2 P.M.	29·74	71·0	64·0	0·518	......	...	...	......	29·85	70·0	66·0	0·593				
3	·74	72·0	65·0	·538	29·679	69·2	62·7	0·499	·80	70·0	66·0	·593	29·754	72·8	66·0	0·561
4	·71	70·0	64·5	·545	·682	68·5	62·0	·486	·78	69·5	66·0	·599	·735	71·7	66·6	·593
5	·71	68·0	65·0	·583	·682	67·6	61·8	·490	·78	69·0	66·0	·604	·707	69·9	66·0	·594
6	·69	67·0	64·0	·563	·686	66·9	60·9	·472	·76	68·5	66·5	·626	·696	69·5	66·0	·599
7	·68	65·5	63·5	·565	·689	63·5	59·0	·456	·74	67·0	65·5	·610	·657	68·9	66·0	·605
8	·68	64·5	63·0	·561	·709	62·5	58·9	·464	·73	65·0	64·0	·586	·650	66·8	64·8	·591
	Edinburgh.				Haverhill.				Hawerden, Chester.				Highfield House.			
9 A.M.	......	...	...	......	......	...	...	......	......	...	...	......	29·722	69·2	63·3	0·517
2 P.M.	......	...	...	......	29·750	72·0	65·5	0·554	29·450	67·5	63·0	0·528				
3	29·279	67·7	...	......	·734	72·0	66·3	·580	·454	67·8	64·0	·554	·653	75·0	67·0	·569
4	·264	67·5	...	......	·720	73·0	67·3	·601	·430	68·0	64·0	·552	·652	71·5	65·3	·553
5	·282	66·2	...	......	·705	71·0	67·5	·631	·428	66·9	63·9	·561	·639	68·8	65·2	·581
6	......	...	...	......	·674	69·8	66·8	·588	·420	66·5	64·0	·569	·623	67·0	64·5	·579
7	......	...	...	......	·663	68·0	66·5	·631	·420	65·8	63·5	·561	·607	65·0	63·9	·583
8	......	...	...	......	·629	66·2	64·8	·597	·414	...	...	......	·594	64·2	63·0	·565
	Holkham.				Linslade.				Norwich.				Oxford.			
9 A.M.	29·888	66·5	62·4	0·520	29·595	64·0	59·3	0·558	29·906	68·0	64·0	·552				
2 P.M.	......	...	...	......	·525	69·3	63·3	·516	......	...	...	......	29·630	72·1	66·4	0·583
3	·796	74·4	67·2	·582	·505	71·3	64·2	·521	·786	73·0	68·0	·625	·615	70·4	66·0	·589
4	·785	72·3	66·4	·581	......	70·2	64·8	·553	·766	72·0	68·0	·636	·577	67·6	65·1	·591
5	·784	72·2	67·0	·601	·457	68·5	64·9	·575	·756	72·0	68·0	·636	·587	67·7	65·0	·587
6	·763	68·8	65·2	·581	·454	67·4	65·1	·593	·748	70·0	67·0	·625	·507	66·0	64·8	·600
7	·756	67·2	64·7	·583	......	...	...	......	·716	68·0	66·0	·616	·528	65·8	64·8	·602
8	·735	66·3	64·0	·571	·420	63·9	62·3	·546	·709	65·0	64·5	·601	·512	65·1	64·7	·607
	Rosehill, Oxford.				Ryde.				Southampton.				St. Ives.			
9 A.M.	29·639	65·0	61·0	0·497	29·850	63·8	61·9	0·536	29·812	66·1	63·0	0·544				
2 P.M.	......	...	...	......	·748	65·9	65·7	·629	·729	71·0	67·0	·615	29·70	73·0		
3	·554	68·6	64·7	·568	·731	66·8	66·2	·635	·711	67·7	66·0	·619	·69	72·0		
4	·507	66·5	64·5	·585	·667	65·8	64·2	·583	·674	67·9	65·6	·603	·68	72·0		
5	·529	65·0	64·8	·611	·667	65·8	63·7	·568	·656	67·0	66·0	·627	·68	72·0		
6	·443	65·0	64·6	·605	·639	64·9	64·2	·593	·626	66·0	65·0	·606	·65	70·0		
7	......	...	...	......	·622	64·8	63·7	·579	·624	64·0	62·9	·574	·65	68·0		
8	......	...	...	......	·623	64·0	63·5	·581	(?) ·867	63·5	62·0	·542	·58	67·0		

Table IV. (Continued.)

Date	Hour		Barom.	Therm. Dry.	Therm. Wet.	Tension of Vapour.	Barom.	Therm. Dry.	Therm. Wet.	Tension of Vapour.	Barom.	Therm. Dry.	Therm. Wet.	Tension of Vapour.	Barom.	Therm. Dry.	Therm. Wet.	Tension of Vapour.
	h		in.	°	°	in.	in.	°	°	in.	in.	°	°	in.	in.	°	°	in.
Observations on August 17, 1852.			St. John's Wood.				York.											
	9	A.M.	29·785	62·8	60·3	0·500	29·721	66·0	63·0	0·545								
	2	P.M.																
	3		·663	69·8	66·3	·605	·734	71·0	64·5	·535								
	4		·647	69·5	66·5	·615	·710	71·0	64·0	·519								
	5		·608	68·8	65·8	·600	·701	69·0	64·0	·542								
	6		·596	68·6	66·2	·615	·701	69·0	64·0	·542								
	7		·552	68·8	66·4	·620	·713	68·0	63·5	·537								
	8		·580	65·0	64·0	·586	·707	66·0	64·0	·575								
			Armagh.				Grantham.				Greenwich.				Lewisham.			
	9	A.M.	29·297	65·5	62·6	0·538	29·688	66·4	61·4	0.493	29·804	64·0	58·1	0·426				
	3	P.M.	......	...	...	......	·623	70·3	64·0	·527	·692	75·4	65·6	·520	29·762	74·9	69·2	0·645
	3½		......	...	...	......	·615	69·6	63·6	·522	·689	74·0	66·1	·552	·755	72·3	67·0	·600
	4		......	...	...	......	·617	69·0	63·6	·529	·683	71·7	65·0	·543	·751	71·7	67·7	·630
	4½		......	...	...	......	·618	68·5	63·7	·538	·655	71·1	64·8	·543	·725	71·6	67·4	·621
	5		·437	64·2	58·6	·437	·600	68·0	63·5	·537	·636	70·6	65·0	·555	·714	70·7	67·5	·635
	5½		·440	64·6	58·8	·438	·571	67·5	63·5	·544	·623	69·8	65·0	·564	·686	70·0	67·1	·629
	6		·449	65·0	59·1	·442	·579	67·4	63·5	·544	·621	69·1	65·1	·575	·688	69·1	67·0	·636
	6½		·446	64·5	59·1	·442	·584	66·9	63·1	·538	·619	69·1	65·1	·575	·678	68·8	66·9	·636
	7		·446	64·1	58·6	·438	·561	66·2	62·8	·536	·602	68·6	65·4	·590	·648	68·4	67·6	·652
	7½		·464	61·4	57·2	·429	·551	65·9	62·6	·533	·607	61·4	60·5	·521	·656	64·0	63·7	·588
	8		·451	59·6	57·0	·444	·546	65·6	62·6	·537	·593	62·5	61·0	·524	·672	63·2	62·8	·570
	8½		......	...	...	......	......	...	...	......	·593	63·5	61·6	·530				
	9		......	...	...	......	·548	64·8	63·0	·558	·593	64·2	62·9	·562				
Observations on August 26, 1852.			Aylesbury.				Bedford.				Cambridge.				Derby.			
	9	A.M.	......	...	...	......	30·030	62·6	58·8	0·459	30·023	64·2	60·0	0·476				
	2	P.M.	29·721	72·3	66·0	0·568												
	3		·722	73·5	66·3	·564	·043	72·1	65·8	·562	·020	68·4	64·9	·576	29·97	68·0	62·0	0·492
	4		·727	70·6	65·0	·555	·062	70·0	65·0	·561	·017	67·8	64·8	·579	·98	68·0	63·0	·522
	5		·734	69·9	65·4	·575	·070	69·4	64·5	·552	·018	67·7	64·1	·559	·98	67·0	62·0	·503
	6		·744	66·0	63·3	·553	·070	66·5	63·0	·539	·031	66·4	63·0	·540	29·98	65·0	61·0	·496
	7		·759	60·7	59·7	·507	·079	64·0	61·5	·522	·056	63·2	60·9	·513	30·00	62·0	59·0	·472
	8		·775	58·7	58·2	·486	·079	61·2	60·0	·510	·065	61·0	58·8	·477	30·00			
			Diss.				Edinburgh.				Enfield.				Holkham.			
	9	A.M.	......	...	...	......	......	...	...	......	29·916	63·0	59·5	0·475	30·004	58·6	57·4	0·465
	2	P.M.	29·97	64·0	60·0	0·478	......	...	...	......								
	3		·97	65·5	61·0	·491	......	...	...	......	·941	70·9	64·6	·538	·037	67·0	60·8	·468
	4		·97	65·0	61·0	·496	......	...	...	......	·942	69·8	63·9	·529	·038	66·0	59·2	·433
	5		·98	63·5	61·5	·528	29·634	66·7	...	......	·945	69·2	63·4	·520	·037	63·6	58·1	·430
	6		·98	62·5	59·5	·480	......	...	...	......	·955	67·8	63·0	·524	·055	61·7	58·0	·448
	7		·99	59·5	57·0	·445	......	...	...	......	......	...	...	......	·064	60·4	57·8	·457
	8		......	...	...	......	......	...	...	......	......	...	...	......	·080	59·8	57·2	·447
			Norwich.				Oxford.				Royston.				Southampton.			
	9	A.M.	......	...	...	......	......	...	...	......	......	...	...	......	29·911	64·1	62·4	0·547
	2	P.M.	30·039	66·0	60·0	0·456	29·820	72·1	65·1	0·541								
	3		·040	66·0	60·0	·456	·822	71·0	64·6	·538	29·784	70·1	63·3	0·507				
	4		·044	65·5	60·0	·461	·820	70·1	63·3	·507	·789	69·4	63·2	·512				
	5		·046	65·0	60·0	·467	·832	68·4	64·0	·548	·799	66·8	62·2	·512				
	6		·052	62·0	59·0	·472	·830	67·0	63·8	·557	·798	63·7	60·0	·482	·925	69·0	66·5	·620
	7		·058	59·0	57·5	·464	·848	65·3	62·2	·528	·822	60·3	58·5	·477	·929	65·0	63·7	·576
	8		·066	55·0	54·5	·429	·857	63·0	61·0	·519	·827	58·0	56·8	·457				

Table IV. (Continued.)

Hour.		Barom.	Therm. Dry.	Therm. Wet.	Tension of Vapour.	Barom.	Therm. Dry.	Therm. Wet.	Tension of Vapour.	Barom.	Therm. Dry.	Therm. Wet.	Tension of Vapour.	Barom.	Therm. Dry.	Therm. Wet.	Tension of Vapour.
h		in.	°	°	in.	in.	°	°	in.	in.	°	°	in.	in.	°	°	in.
Observations on August 26, 1852.																	
		Stone.				St. Ives.				St. John's Wood.							
9	A.M.	29·637	67·4	64·0	0·559	......	...	...	......	29·862	62·8	61·3	0·529				
2	P.M.	·646	72·6	65·6	·551	29·85	72·0	...	......	......	...	...	......				
3		·648	68·9	64·0	·543	·85	71·0	...	......	·866	68·5	64·2	·553				
4		·646	68·9	64·5	·558	·85	69·0	...	......	·874	67·8	63·8	·548				
5		·656	68·7	63·5	·529	·86	69·0	...	......	·879	66·8	63·8	·560				
6		·664	66·6	62·9	·535	·87	68·0	...	......	·888	64·8	62·8	·552				
7		·689	63·0	60·3	·498	·89	64·0	...	......	·911	62·8	60·4	·503				
8		·694	60·9	58·8	·479	·90	61·0	...	......	·920	61·8	59·8	·497				
		Grantham.				Greenwich.				Kew.				Lewisham.			
9	A.M.	29·850	61·8	57·4	0·430	29·873	66·8	62·2	0·512								
Noon		......	...	...	......	·879	69·6	63·6	·522								
3	P.M.	·869	64·5	58·7	·436	·880	71·1	64·1	·521	......	...	...	......	29·958	72·7	66·5	0·579
3½		·872	65·0	59·3	·447	·880	71·1	64·1	·521	29·978	69·6	64·4	0·547	·952	72·5	66·2	·571
4		·873	65·4	59·7	·454	·876	70·9	63·8	·514	·992	69·1	64·0	·540	·956	71·8	66·0	·573
4½		·872	65·9	60·2	·462	·888	69·6	63·1	·507	·995	68·2	63·9	·547	·968	68·3	64·0	·549
5		·875	65·4	60·0	·463	·893	67·1	61·6	·489	·993	67·2	63·1	·534	·970	67·8	63·7	·545
5½		·879	64·5	58·5	·430	·887	66·6	62·0	·507	·992	66·7	63·0	·537	·967	67·5	63·5	·542
6		·883	64·0	57·9	·419	·890	64·5	60·2	·478	29·996	65·1	61·4	·507	·982	65·5	62·6	·538
6½		·884	63·0	57·7	·425	·897	63·8	59·6	·469	30·006	63·5	60·9	·510	·983	64·0	61·6	·524
6¾		......	...	...	......	......	...	...	......	·019	63·1	60·5	·503				
7		·885	61·0	57·3	·437	·911	62·7	59·1	·466	......	...	...	......	·989	62·7	61·0	·522
7½		·893	60·6	56·7	·424	·913	62·0	58·8	·466	......	...	...	......	29·994	62·2	60·5	·513
8		·897	59·6	56·3	·425	......	...	...	......	......	...	...	......	30·001	61·5	60·0	·506
8½		......	...	...	......	......	...	...	......	......	...	...	......	·009	60·6	59·2	·493
9		·903	57·9	55·6	·425	·929	59·3	57·3	·456	......	...	...	......	·013	59·6	58·5	·485
Observations on October 21, 1852.																	
		Bedford.				Enfield.				Hartwell House.				Hartwell Rectory.			
9	A.M.	29·94	46·6	46·0	0·318	29·984	46·5	45·9	0·317	29·810	49·3	47·8	0·329	29·695	47·8	47·8	0·346
1	P.M.	·88	56·6	53·5	·385	·917	52·0	50·2	·356								
2		·86	57·0	54·0	·394	·906	52·0	50·9	·373	·766	56·8	51·0	·321				
3		·85	56·4	53·0	·375	·891	52·0	51·5	·387	·766	55·8	51·0	·332				
4		·85	55·0	51·8	·361	·871	52·1	51·8	·394	·750	55·3	51·2	·343				
5		·85	54·0	51·7	·369	·871	52·0	52·0	·400	·742	54·0	51·0	·352				
6		·87	54·0	51·7	·369	·886	51·5	51·0	·380	·754	54·9	51·9	·364	29·626	54·0	52·5	·389
7		·87	54·1	52·0	·376	......	...	...	......	·734	54·6	53·0	·395				
		Linslade.				Norwich.				Oxford.				Rosehill, Oxford.			
9	A.M.	29·728	48·1	47·5	0·336	......	...	...	......	......	...	...	......	29·759	48·4	48·3	0·351
1	P.M.	·693	54·5	51·3	·354	......	...	...	......	29·789	55·0	52·8	0·386	·720	53·5	51·8	·378
2		·665	55·2	50·5	·327	29·932	53·0	51·0	0·364	·763	54·9	52·8	·387	·705	53·4	51·4	·369
3		·659	55·0	50·9	·339	·932	53·0	51·0	·364	·758	54·7	52·3	·377	·694	53·8	52·0	·380
4		·645	53·9	50·5	·342	·916	52·5	51·4	·379	·736	54·1	52·5	·388	·672	53·8	52·2	·384
5		·639	53·1	50·1	·340	·906	51·0	50·0	·362	·736	54·0	52·6	·392	·677	53·1	52·1	·390
6		·639	52·6	50·1	·346	·906	48·0	47·0	·326	·736	54·0	53·0	·402				
7		·623	51·9	51·2	·381	·906	47·8	47·7	·344	·731	54·0	53·2	·407				
		Ryde.				Stone.				St. John's Wood.				Ventnor.			
9	A.M.	29·933	53·1	52·0	0·388	29·649	48·3	47·8	0·340	29·892	46·3	45·6	0·313				
1	P.M.	·897	59·8	55·5	·402	·617	54·6	52·4	·380	·827	56·0	52·5	·367	29·912	60·0	56·0	0·412
2		......	...	...	......	·597	56·3	53·5	·389	·815	55·8	53·3	·390	·892	60·0	57·0	·440
3		·889	59·0	55·0	·397	·586	54·4	51·6	·362	·810	57·3	53·8	·386	·874	60·0	57·0	·440
4		......	...	...	......	·574	53·7	51·6	·370	·795	54·5	51·9	·369	·872	60·0	57·0	·440
5		......	...	...	......	·571	53·0	51·3	·371	·795	54·9	52·8	·387	·872	60·0	57·0	·440
6		·887	56·0	54·8	·426	·578	53·4	51·5	·372	·804	54·8	52·8	·389	·874	60·0	57·0	·440
7		......	...	...	......	·562	52·7	51·5	·379	·792	54·0	52·5	·389	·874	60·0	57·0	·440
8		......	...	...	......	·554	52·7	51·9	·389	......	...	...	......				
9		......	...	...	......	·550	52·7	51·9	·389	·783	53·0	52·0	·389				

Table IV. (Continued.)

Observations on October 21, 1852.

Hour.	Barom.	Therm. Dry.	Therm. Wet.	Tension of Vapour.
h	in.	°	°	in.
Grantham.				
9 A.M.	29·851	47·2	46·4	0·321
Noon	......	...	...	
1 P.M.	·794	51·6	49·5	·343
1½	......	...	...	......
2	·786	53·0	50·4	·349
2½	......	...	...	......
3	·760	52·5	49·4	·331
3½	......	...	...	......
4	·768	51·7	49·8	·350
4½	......	...	...	......
5	·747	51·0	49·6	·352
5½	......	...	...	......
6	·748	51·0	49·5	·350
6½	......	...	...	......
7	·743	51·0	49·9	·360
9	·717	51·6	50·3	·362

Hour.	Barom.	Therm. Dry.	Therm. Wet.	Tension of Vapour.
h	in.	°	°	in.
Greenwich.				
9 A.M.	29·887	48·3	46·0	0·299
Noon	·860	58·2	53·4	·365
1 P.M.	·845	59·7	53·7	·356
1½	......	...	...	......
2	·838	57·9	51·6	·323
2½	·836	58·7	51·9	·322
3	·835	58·7	52·4	·334
3½	·832	57·4	52·4	·349
4	·828	56·9	52·1	·347
4½	·826	56·4	51·9	·348
5	·831	55·9	51·9	·353
5½	·826	55·4	51·9	·359
6	·813	55·9	51·9	·353
6½	·806	54·8	51·4	·353
7	......	54·2	51·0	·350
9	·796	52·8	50·0	·341

Hour.	Barom.	Therm. Dry.	Therm. Wet.	Tension of Vapour.
h	in.	°	°	in.
Lewisham.				
9 A.M.	29·958	45·5	45·0	0·308
Noon				
1 P.M.	·907	59·1	53·8	·366
1½	·907	58·6	52·8	·346
2	·908	58·0	52·0	·333
2½	·908	58·6	52·1	·328
3	·907	58·7	53·7	·367
3½	·901	58·0	53·6	·373
4	·893	56·9	54·1	·397
4½	·893	55·7	54·0	·409
5	·891	54·7	53·3	·402
5½	·892	54·2	53·2	·405
6	·891	55·0	52·2	·370
6½	·895	55·3	53·0	·387
7	·892	53·0	52·0	·389
9				

Observations on November 10, 1852.

Hour.	Barom.	Therm. Dry.	Therm. Wet.	Tension of Vapour.
Bedford.				
9 A.M.	29·97	46·6	43·6	0·265
3 P.M.	·96	47·6	44·0	·263
Cambridge.				
1 P.M.	30·061	47·8	44·2	0·265
2	·058	48·2	44·0	·256
3	·056	47·1	43·8	·264
4	·047	46·5	44·0	·275
5	·033	46·1	43·5	·268
6	·029	44·6	43·5	·285
7	·025	44·2	43·5	·290
Diss.				
10 A.M.	29·99	46·0	44·5	0·292
1¼ P.M.	·98	48·5	46·0	·296
2¼	·98	48·0	45·0	·280
3½	·98	47·5	44·5	·275
5½	·96	44·5	43·5	·287
7	·96	43·5	43·0	·287
Enfield.				
9 A.M.	29·991	49·8	46·8	0·300
6 P.M.	·940	...	...	......
Grantham.				
9 A.M.	29·922	41·6	39·5	0·236
1 P.M.	·904	44·7	42·5	·263
2	·893	44·4	42·3	·262
3	·884	44·3	42·3	·263
4	·880	43·7	42·0	·264
5	·869	43·2	41·8	·265
6	·871	42·4	41·2	·262
7	·860	42·2	40·8	·256
9	·834	41·5	40·4	·256
Greenwich.				
9 A.M.	29·937	48·6	45·9	0·293
Noon	·928	51·0	44·6	·237
1 P.M.	·918	50·5	45·7	·267
1½	·920	50·5	45·7	·267

Hour.	Barom.	Therm. Dry.	Therm. Wet.	Tension of Vapour.
Greenwich.				
2 P.M.	29·918	50·1	45·4	0·265
2½	·917	48·6	45·0	·273
3	·914	48·0	44·5	·258
3½	·911	49·0	44·7	·262
4	·910	48·9	44·6	·261
4½	·910	47·7	44·0	·262
5	·905	47·3	43·6	·257
5½	·893	46·3	42·5	·244
6	·893	46·1	42·6	·249
6½	·900	46·1	42·5	·247
7	·894	46·1	42·6	·249
Hartwell House.				
9 A.M.	29·756	47·0	45·8	0·309
3 P.M.	·712	48·5	44·5	·263
Lewisham.				
9 A.M.	30·015	49·2	47·6	0·326
1 P.M.	·005	51·0	47·8	·309
1½	30·001	51·0	47·6	·305
2	29·987	50·8	47·5	·305
2½	·984	50·7	47·6	·308
3	·980	49·3	46·3	·295
3½	·975	49·6	46·5	·295
4	·975	49·0	45·7	·284
4½	·974	47·9	45·4	·291
5	·974	47·7	45·5	·295
5½	·966	47·0	44·8	·287
6	·964	47·0	44·7	·285
6½	·962	46·8	44·5	·283
7	·951	46·6	44·2	·279
Norwich.				
9 A.M.	30·013	46·7	43·7	0·266
Noon	29·997	48·4	43·4	·240
3 P.M.	·993	48·3	42·3	·218
6	......	44·7	42·4	·261
9	......	43·7	42·7	·279

Hour.	Barom.	Therm. Dry.	Therm. Wet.	Tension of Vapour.
Oxford.				
10 A.M.	29·891	48·3	45·4	0·286
1 P.M.	·863	49·5	46·2	·291
2	·859	49·0	46·0	·291
3	·852	48·9	45·3	·277
4	·836	48·1	45·3	·286
5½	·832	47·0	44·1	·272
6	·836	46·9	44·0	·271
7	·821	46·1	43·5	·269
Rosehill, Oxford.				
9 A.M.	29·829	47·2	44·8	0·285
Royston.				
10 A.M.	29·836	47·5	44·7	0·279
6 P.M.	·789	43·5	42·4	·275
Ryde.				
9 A.M.	30·023	51·6	50·5	0·368
Southampton.				
1⅓ P.M.	29·966	52·6	50·0	0·343
2	·959	52·9	50·0	·340
3	·942	52·9	50·3	·347
4	·945	51·5	50·0	·356
6	·938	50·6	49·6	·357
Stone.				
9 A.M.	29·707	46·3	44·2	0·282
9 P.M.	·627	44·7	42·6	·265
St. Ives.				
1 P.M.	29·44	44·0		
2	·42	44·0		
3	·37	45·0		
4	·35	48·0		
5	·28	48·0		
6	·27	48·0		
7	·27	48·0		
St. John's Wood.				
9 A.M.	29·932	48·3	46·3	0·306
5 P.M.	·891	46·6	44·1	·277
6	·884	46·3	43·6	·268

TABLE IV. (Continued.)

Hour.	Barom.	Therm.		Tension of Vapour.	Barom.	Therm.		Tension of Vapour.	Barom.	Therm.		Tension of Vapour.	Barom.	Therm.		Tension of Vapour.
		Dry.	Wet.			Dry.	Wet.			Dry.	Wet.			Dry.	Wet.	
h	in.	°	°	in.	in.	°	°	in.	in.	°	°	in.	in.	°	°	in.
	Les Rousseaux, Aug. 17.				Les Rousseaux, Aug. 26.				Marboué, Oct. 21.				Marboué, Nov. 10.			
7 A.M.	29·475	67·1	62·0	0·502	29·443	63·1	59·0	0·460	29·752	34·9	32·7	0·179	29·719	43·0	41·2	0·255
9	·456	69·1	63·1	·513	·445	72·0	61·3	·427	·752	52·0	47·1	·283	·703	49·3	46·4	·297
Noon	·395	76·7	66·4	·534	·449	78·8	66·9	·525	·691	60·1	53·2	·338	·668	57·6	52·7	·354
3 P.M.	·296	80·8	68·2	·546	·454	78·3	67·6	·553	·667	59·2	54·1	·371	·650	56·3	50·7	·319
9	·297	70·5	67·3	·630	·494	67·8	63·0	·524	·635	49·5	46·8	·304	·628	44·6	41·9	·252

DESCRIPTION OF THE PLATES.

PLATE XIX. XX. XXI. and XXII.

The results for each ascent of the observations of temperature, tension of vapour, and relative humidity are projected in these Plates. For the most part each individual observation is given, except when they were very numerous and occurring at too close intervals of height to be easily represented. In such cases groups have been taken, but no group ever contains more observations than were recorded within 200 feet.

The ordinates represent the height above the level of the sea, one division being equivalent to 200 feet; the abscissæ representing the temperature of the air, the tension of vapour, or the relative humidity.

The scale employed for the temperature is one division to 2° FAHR.; for the tension of vapour, ten divisions to 0·2 inch of pressure of mercury; and for the relative humidity, twenty divisions to the whole range 0—100.

The straight lines drawn through the curves of temperature are deduced from the results of equation (1.) for the upper and lower divisions in each series (see p. 25).

The points ⊙ in the curves of tension of vapour are from the indications of REGNAULT's hygrometer.

In the divisions occupied by the relative humidity the strong vertical lines correspond to the heights at which clouds existed in the air, dotted lines being drawn when the cloud was only partial.

XIII. *On certain Functions of the Spinal Chord, with further Investigations into its Structure. By* J. Lockhart Clarke, *Esq. Communicated by* Samuel Solly, *Esq., F.R.S.*

Received April 6,—Read April 14, 1853.

When I had the honour of laying before the Royal Society my former researches on the structure of the spinal chord, I intimated an intention of preparing another communication on the structure of the medulla oblongata and cerebellum; but as many important points in the minute anatomy of the chord still remained in obscurity, I thought it advisable to make them first the subject of special inquiry, as far as the new method I employed would enable me to proceed. Moreover, as all investigations into the structure of any organ have, or ought to have, for their object a clearer and better knowledge of its functions, I have undertaken also to communicate in this paper whatever physiological deductions may appear to follow from my observations. Having no particular theory to support, and being influenced in these inquiries by no other feeling than the simple desire to elicit truth, the greatest care has been taken to verify my facts, and caution has been exercised in drawing conclusions from them.

It is a question of great interest and physiological importance, whether the roots of the spinal nerves belong exclusively to the spinal chord, or whether part of them ascend within either the white or the grey columns, and form the channels by which impressions are transmitted to and from the brain. On account of its interest and importance, I have employed much time and labour in endeavouring to arrive at some well-grounded and settled conclusion on this very difficult subject, having devoted to it alone many hours daily for nearly five months. So extremely intricate, however, is the internal structure of the chord; so numerous are the planes in which the nerve-roots enter the grey substance; and so various are the directions which they pursue within it, that notwithstanding the perfect transparency of my preparations, and the sharp outline which their fibres retain, my efforts to determine the exact relation between these roots and the white and grey columns appeared for some time almost hopeless; but by varying my dissections according to the exigencies of each case of difficulty, I succeeded in arriving at several results which I believe will be considered important.

That part of the fibres composing the anterior and posterior roots of the spinal nerves ascend longitudinally with the white columns, without entering the grey substance of the chord, and transmit to and from the brain impressions which give rise to sensation and voluntary motion, is the opinion maintained by many eminent physiologists and excellent observers.

In support of this doctrine, Mr. GRAINGER*, Mr. SOLLY†, and Dr. J. BUDGE‡ have adduced the fact of having *traced* these fibres in the spinal chord of the vertebrata. A similar fact is said to have been established by Mr. NEWPORT§ and Dr. CARPENTER in the invertebrata. On the other hand, the opinion that all the fibres of the spinal nerves enter the grey substance and belong exclusively to the spinal chord, is held by TODD and BOWMAN‖, STILLING¶, VOLKMANN** and others.

In making the preparations for the investigation of this subject, I employed the second method described in my former communication on the Spinal Chord††. The animals selected for this purpose were the Ox, Calf, Cat, Rat, Mouse, and Frog. After many fruitless attempts I succeeded in rendering perfectly transparent the entire chords of the Mouse and Frog, but found that their cylindrical form interfered considerably with a distinct view of their several parts, and rendered the examination unsatisfactory. Having succeeded, however, in a similar way, with longitudinal sections of the spinal chord of the Cat, of at least $\frac{1}{12}$th of an inch in thickness and 2 inches in length, with the anterior and posterior roots attached, I was delighted, on examining these sections, with a view of its internal structure which far surpassed any I had hitherto been able to obtain.

On a former occasion, I showed unequivocally, that to the posterior white columns the posterior roots, and to the anterior white columns the anterior roots, of the spinal nerves are exclusively attached; while the lateral columns to which both these roots were formerly supposed to be connected, are in immediate connection only with the spinal-accessory nerve‡‡.

Of the Posterior Roots.—Plate XXIII. exactly represents a longitudinal section through the cervical enlargement of the spinal chord of the Cat, from the eighth to the twelfth pair of nerves. In this section the bundles which form the posterior roots P, P, P, P, are observed to consist of three kinds, which differ from each other partly in direction, and partly in the size of their component filaments.

The first kind, *a, a, a, a,* enter the chord transversely, and pursue a very remarkable course. I have not seen them distinctly below the cervical enlargement. Each bundle, after traversing the longitudinal fibres of the posterior columns P, C, in a compact form, and at right angles, continues in the same direction to a considerable but variable depth within the grey substance G, dilating and again contracting, so as to assume a fusiform appearance. It then bends round upon itself, at a right or more obtuse angle, and running for a considerable distance in a longitudinal direction

* Spinal Chord. † Human Brain. ‡ MÜLLER's Archiv, 1844. § Philosophical Transactions, 1843.

‖ Physiological Anatomy, vol. i. 1845. ¶ Untersuchungen über die Textur des Rückenmarks, 1842.

** Nervenphysiologie, in WAGNER. †† Philosophical Transactions, 1851, p. 608, Part II.

‡‡ Many of my preparations are in the possession of the Microscopical Society of London, and the Royal College of Surgeons.

down the chord, sends forwards, at short intervals, into the anterior grey substance, a series of fibres like those issuing from the roots of plants. In this longitudinal course it is joined by corresponding fibres from bundles above and below it, which thus contribute to form a continuous band.

The fibres projecting from this band into the anterior grey substance have the following distribution. Part of them form loops with each other within the grey substance, particularly near its border; others extend directly into the anterior white columns A, C, and bending round both *upwards* and *downwards*, are seen sometimes to re-enter the grey substance and form with each other a series of loops, and sometimes to continue a longitudinal course within the anterior white columns, amongst the fibres of which they become lost. Whether the latter, also, ultimately form broader loops with corresponding fibres from the grey substance, it is impossible to ascertain. But even if those which *ascend* in the anterior columns are continued upwards to the brain, one can scarcely avoid inferring that those which *descend* re-enter the grey substance, either to form loops, or to become continuous with the fibres of the anterior roots, since the whole of the latter, as we shall presently see, proceed directly to the grey substance. Indeed, I have sometimes felt almost persuaded that a great number of the fibres of these posterior roots are directly continuous, in the grey substance, with those of the anterior roots; but I cannot make this statement with absolute certainty; and as the question is one of extreme difficulty, I shall hereafter endeavour to make it a subject of special attention.

The second kind of bundles which form the posterior roots *b, b, b*, Plates XXIII. and XXIV. traverse the posterior columns transversely, and with different degrees of obliquity from without inwards, extending nearly as far as the posterior median fissure: see also Philosophical Transactions, 1851, Plate XXIII. fig. 14. Their component filaments are finer than those of the other kind of bundles, measuring, in a recent state, about the $\frac{1}{7000}$th of an inch in diameter. They enter and pass through the posterior grey substance at various angles, and in compact bundles which decussate and interlace each other in the most complicated manner. Some of their fibres cross over to the opposite side, through the posterior commissure, behind the spinal canal; others extend into the posterior and lateral white columns; and the rest may be traced deeply into the anterior grey substance, where they separate in various directions, and are ultimately lost to view.

The bundles which compose the third kind of posterior roots enter the chord obliquely, *c, c, c*, Plates I. and II. A few of their fibres proceed near the surface both upwards and downwards, and pass out again with the roots above and below them. The rest cross the posterior white columns obliquely, and chiefly *upwards*, a small number only passing downwards. Interlacing at the same time with each other and the roots already described, they diverge, and for the most part reach the grey substance at points successively more distant from their entrance in proportion to the obliquity of their course; the remainder or most divergent taking a longitudinal course with the fibres of the posterior white columns, amongst which they are lost. It is impossible to say whether any of these longitudinal fibres are continued as far as the brain, or whether they ultimately reach the grey substance of the chord. It is also extremely difficult to trace the other fibres of these roots after they have reached the posterior grey substance. In some of my finest preparations, however, they may be seen to interlace each other in a kind of network: see Plate XXIV. A large proportion diverge abruptly in various directions, so that in any section they are always divided. Many of them, both singly and in small bundles, may be observed to form loops by returning to the white columns.

Of the Anterior Roots.—The anterior roots of the spinal nerves, as I formerly described them, traverse the anterior part of the antero-lateral columns in distinct and nearly straight bundles, Plates XXIII. and XXIV. A, A, A, A. They form no interlacement with each other, like the posterior roots, until they reach the grey substance. Here their fibres diverge in every direction, like the expanded hairs of a brush. Some, near the margin, are easily seen to form loops with those of contiguous bundles: others run outwards to the lateral columns, and inwards to the anterior columns after decussating in the anterior commissure with corresponding fibres from the opposite side. A large number diverge equally *downwards* and *upwards* for some distance in the grey substance, while the remainder pass more deeply backwards and are lost. In no single instance have I seen any portion of these roots take a longitudinal course on directly entering the anterior white columns.

But besides the transverse bundles which form the anterior roots, a continuous system of exceedingly fine transverse fibres may be seen to issue from the anterior grey substance. They pass through, nearly all at right angles to, the anterior white columns, and disappear as they proceed towards the surface of the chord; but as many of them may be observed to turn round and take a longitudinal direction, it is probable that at the points where they disappear they all follow the same course. Within the grey substance they wind about and are gradually lost, mingling with the fibres of the anterior roots, and with those proceeding from the fine bundles of the posterior roots, which, perhaps, are continuous with them.

It may then, I think, be fairly laid down as a well-established fact, that nearly all, if not the whole of, the fibres composing the roots of the spinal nerves, after passing through the anterior and posterior white columns of the chord, proceed at once to its grey substance; and that if any of them ascend *directly* to the brain, it must be *those only* of the *posterior* roots which run longitudinally in the posterior columns.

That many excellent observers, with inferior means of observation, have arrived at a different conclusion on this extremely difficult subject, is not at all surprising. The opinions of Mr. Grainger are expressed in the following quotations from his excellent work on the spinal chord:—"After the *two* roots have perforated the theca vertebralis, and so reached the surface of the chord, it is well known that their fibres begin to separate from each other; of these fibres some are lost in the white substance, whilst others, entering more deeply into the lateral furrows, are found to continue their course, nearly at right angles with the spinal chord itself, as far as the grey substance, in which they are lost. . . . . In examining the roots of the nerves, I have always relied on the assistance of the naked eye only, avoiding, for fear of deception, the use of a lens. . . . . From careful dissection, I am convinced that it is only a part of the fibres belonging to the *two* roots which are attached to the grey substance, and that a considerable number of threads are lost in the fibrous part of the chord. . . . . The exact mode of their connection, however, with this latter substance is not known." Only a part, therefore, of Mr. GRAINGER's statements is really correct; for though some of the fibres of the posterior roots may be confined in their course within the white columns, this is certainly not the case with regard to the anterior roots; and many of the fibres of the *posterior* roots, which appeared to him to be longitudinal, were probably those which take a very oblique course towards the grey substance. The fibres, also, of the third pair of cerebral nerves, of which some have been said by Mr. GRAINGER to be continuous with the white columns, may be all traced to the vesicular substance situated below the iter a tertio ad quartum ventriculum. Great merit, however, is due to Mr. GRAINGER for having made out so much with so little assistance.

The question, then, arises,—Do the fibres of the posterior roots which ascend longitudinally in the posterior fasciculi transmit impressions to the sensorium? In answer to this question it may be stated,—1st, that, even if they reach the brain, their number appears insufficient to convey such impressions from all parts of the body; and 2ndly, that the anatomical connection of the posterior columns exclusively with the cerebellum would, *à priori*, lead us to a negative conclusion, unless we regard this latter organ as the common centre of sensation, which we have no grounds for believing it to be. But the same anatomical connection of the posterior columns with the cerebellum clearly indicates that they establish *some* physiological relation between this organ and the spinal chord; and if the cerebellum were proved to be, according to the generally received opinion, a centre of *motor* power *only*, destined for the control and coordination of complex muscular movements, it would follow that these columns were motor, and not sensitive, whether or not we admit them to be constituted wholly, or in part, of the longitudinal fibres of the posterior roots*.

Upon anatomical grounds, then, it would appear, that the posterior white columns are not the channels of communication between the sensorium and the posterior spinal nerves. Experimental evidence tends to the same conclusion. Both VAN DEEN† and STILLING found that irritation of these columns excited no sensation

* That the cerebellum is in *some way* concerned in the regulation and coordination of muscular movements appears to follow from experiment and pathological investigation; but I think there are reasons for believing that the *mode of action* usually assigned to it is unsatisfactory, and at variance with many well-established facts. The discussion of this question, however, would be foreign to the present subject.

† Traités et Découvertes sur la Physiologie de la Moëlle Epinière. Leide, 1841.

when their connection with the grey substance was interrupted. Nothing, however, is more certain than that the posterior roots are in *immediate* connection exclusively with these columns, which are separated from the antero-lateral columns by the posterior lateral fissure, through which a few only of the roots proceed to the grey substance. It is difficult, therefore, to conceive how sensory impressions can be conveyed after destruction of the posterior columns, as *appears* to have taken place in cases put on record by several observers. The only explanation of which these cases seem to admit, is that either these columns were only partially destroyed, or that sensory fibres are contained also in the anterior roots.

The question, then, remains,—By what means *are* impressions which are received by the chord transmitted upwards to the brain?

By VOLKMANN it is believed that all, or nearly all the fibres of the nerve-roots terminate in the grey substance of the chord. In support of this opinion, he endeavours to show, both by measurement and weight, that there is no progressive increase, from below upwards, in the size of the white columns; that, on the contrary, he found these columns absolutely smaller in the cervical region than in the lower parts, and everywhere in direct proportion to the quantity of the grey substance. He further states, that the total area of all the spinal nerves cut transversely is many times greater than the area of a transverse section of the chord above the first cervical nerve. Opposed to that of VOLKMANN is the opinion of KÖLLIKER*, who holds that the white columns of the chord are composed entirely of fibres derived from the spinal nerves. This opinion is founded on the fact pointed out by him,—and about the same time by myself,—that at least a large proportion of both roots, after passing through various parts of the grey substance, pursue a longitudinal course within the white columns. He further maintains, in opposition to VOLKMANN, that these columns do really increase progressively towards the brain, and that the nerve-fibres of which they mainly consist, do not recover their original diameter which they lost in passing through the grey substance. From these facts he calculates that the white columns are sufficiently large in the cervical region to give passage to all the spinal nerves, and that they form the channels of communication to and from the brain.

The results of my own observations are opposed to those of VOLKMANN, and partly in favour of those of KÖLLIKER. In all the Mammalia examined by myself, I have found the quantity of the white substance of the chord absolutely greater in the cervical region than elsewhere. It cannot therefore be, as stated by VOLKMANN, in direct proportion to the grey substance, since the latter is far more abundant in the lumbar enlargement. Again, his opinion that all, or nearly all the spinal nerves terminate in the grey substance of the chord, is refuted by the fact, shown by both KÖLLIKER and myself, that at least a great number of them do really pass out of the grey substance as longitudinal fibres of the white columns. But on the other hand,

* Mikroskopische Anatomie, Band ii.

I cannot agree with the statement of Kölliker, that all these fibres continue the same course within these columns in the direction of the brain; for it is certain, as I have already observed, that many of them not only pursue a *descending* course, but that others which actually *ascend* re-enter the grey substance to form loops. It is certain, also, that of the anterior roots in particular, the fibres, on reaching the grey substance, diverge in a direction as much *downwards* as upwards.

It seems therefore that many of the fibres which belong respectively to the anterior and posterior roots, in different regions of the chord, terminate there by forming with each other a series of loops, partly within the grey substance, and partly after extending through the latter into the white columns; and that these loops are of various sizes and lengths. Nor is it improbable that some of them may reach even as far as the brain, since it is well known that the formation of loops is at least one mode in which nerve-fibres terminate there. I am far from denying, however, that a portion of the roots may be connected with the vesicles of the chord.

Does the *grey substance* of the chord transmit impressions to and from the brain?—Stilling divided, in a living animal, the anterior white columns through their whole thickness, and found that voluntary power was still conveyed to parts below the section; but when the incision extended deeply into the grey substance, all voluntary movements were interrupted. From this experiment he infers that the grey substance alone is the conductor of voluntary power from the brain to the anterior roots; and the means of communication he explains by a system of fine longitudinal fibres which he found in the anterior grey substance. I thought formerly that I also had seen such fibres in comparatively small numbers; but further and more careful observation has led me to the conclusion, that excepting those in the *substantia gelatinosa* and those near the border of the *substantia spongiosa*, there is no regular system of fine longitudinal fibres in the grey substance. In looking for such fibres I have always selected a perfectly fresh chord. An exceedingly thin section, which by practice may be made with a very sharp instrument, was carefully laid on a glass slide and treated with acetic acid, which increases its transparency without producing any material alteration in the appearance of its fibres. It was then covered with thin glass, and examined, without any pressure, under sufficiently high powers. In such a section, fibres of various diameters—many exceedingly fine—may be seen to cross each other in every direction, but not in one direction more than another. It is probable that even the longitudinal fibres of the *substantia gelatinosa* are not continued, *individually*, as far as the brain, but pass out at intervals into the white columns, since their number, as well as the breadth of the *substantia gelatinosa*, does not increase regularly from below upwards. I believe that they are all derived from the fine bundles of the posterior roots*.

* Kölliker supposes that the nucleated fibres of the *substantia gelatinosa*, described by Remak, are processes of the nerve-vesicles; but these processes contain no nuclei. It is more probable that the fibres of Remak are really small blood-vessels, which have precisely the same appearance, and of which a large number enter with the posterior roots and pursue the same longitudinal course in the *substantia gelatinosa*.

The results obtained from the experiments of STILLING above described are involved in much obscurity; for in addition to other reasons, it would seem extremely difficult, if not impossible, to avoid leaving a portion of the anterior columns undivided,—that portion, namely, which is situated internal to the anterior grey substance. But even if they tended to show that the power of volition is not transmitted by these columns, they by no means prove it to be communicated by the grey substance, since a deep incision into the latter would divide not only many fibres of the anterior roots extending upwards, but a portion also of the lateral white columns into which these roots are prolonged.

That the grey substance of the chord does transmit impressions *from one side to the other, has been* fully proved by the experiments of VAN DEEN and STILLING; and the means by which the communication is effected is satisfactorily explained by the structure of the transverse commissure described by both STILLING and myself*. I believe, however, that the fibres of this commissure do not form a distinct set connecting the grey substance of one side with that of the other, but that they are all continuous with the roots of the nerves and the white columns†.

From certain facts, then, described in this communication, it would appear that the white columns are mainly constituted of fibres derived from both roots of the nerves. It is true that the processes of some of the vesicles situated near the margin of the grey substance may be seen to extend into these columns, with fibres of which they perhaps are continuous; and it is not improbable that others more deeply seated may have the same connection. But if there be, as some physiologists believe, a

* The structure immediately surrounding the spinal canal is described by STILLING as a *circular commissure* composed of fine grey fibres. By REMAK it is spoken of as the *commissura gelatinosa*, and as a continuation of the *substantia gelatinosa* of the posterior grey substance. KÖLLIKER, who declares that this structure was mistaken by STILLING for a spinal canal—the existence of which he denies—also considers it as a peculiar kind of vesicular substance, enclosing a central nucleus, (*grauer centraler Kern*) and which he calls the *substantia grisea centralis*. It appears, however, as described in my former communication, to consist of a circular layer of fine fibrous tissue surrounding and supporting the columnar epithelium which forms the wall of the spinal canal. The existence of this canal is unquestionable; but in the human chord it is often closed, and reduced, as if by lateral pressure, to a mere line; and around this the columnar epithelium is arranged in the form of an ellipse, which KÖLLIKER appears to have mistaken for a double nucleus.

† There seems to be a great correspondence in the fibrous arrangement between the grey substance of the spinal chord and the chiasma of the optic nerves. In a few transparent preparations that I have made of the latter:, the structure first pointed out by MAYO is very clearly discerned; but the course of the fibres is found to be still more intricate. There is a remarkable circumstance that I have observed, and which I do not recollect to have seen recorded, respecting the cross action of the optic nerves. After looking through the microscope, if the eye employed be directed to other objects, it must be well known that the *definition* is found to be unimpaired, although the *colour* is rendered darker than natural, in direct proportion to the previous stimulus of light; but if the same objects be viewed by the other eye which has not been employed at the microscope, I have invariably noticed that an opposite effect is produced, and in the same proportion,—the *colour* remains *natural*, but the *definition* is more or less *impaired*. It may be stated then as a law, that, *By close observation with magnifying glasses, under the stimulus of a strong light,—in the eye employed, the perception of colour, but not the defining power, is affected; while in the opposite eye, the defining power, but not the perception of colour, is impaired, and in a direct and equal ratio.*

direct continuity between the cerebral fibres and the caudate vesicles of the chord, I am disposed to think it is established by means of the system of fine nerve-tubes which are seen to proceed from the grey substance of the latter as longitudinal fibres of the white columns. In opposition to Wagner, I agree with Kölliker in the opinion that the process of a nerve-vesicle never becomes the axis-cylinder of a double-contoured nerve-tube, and that what Wagner considered as the process of a nerve-vesicle was probably the axis-cylinder itself. The evidence of any direct connection between these processes and the roots of the nerves is very unsatisfactory; for amongst thousands of preparations examined by myself with the greatest care, I met with a few cases only in which it was at all probable.

If any of those fibres of the posterior roots, which, after traversing the grey substance, enter the anterior white columns, could be proved to extend as far as the brain, the indications of sensation which are said to have followed irritation of these columns might be thus explained. It is quite certain, however, as already stated, that an equal number of those fibres pursue a *downward* course. On the other hand, if the anterior columns really be, as I believe they are, one of the channels which transmit downwards the influence of the will, they can perform this office only by first entering the grey substance, since into this all the fibres of the anterior roots may be traced. The same inference may be drawn with regard to the lateral white columns, which from their connection, through the grey substance, with the roots of the nerves, would appear to be both sensitive and motor.

The fact that many fibres of each root, on entering the grey substance, not only extend both upwards and downwards to a considerable distance beyond their point of entrance, but intermingle also in the most intricate manner with those of other roots, may serve to explain how impressions made at one particular spot are communicated in different directions to distant parts of the chord, so as to excite a simultaneous and sympathetic action in classes of muscles which otherwise would appear unconnected. It is probable that the fibres which quit the grey substance and return to it by the formation of loops within the white columns, may take that course for the purpose of stimulating particular parts of that substance, without affecting those which are intermediate.

In concluding this series of investigations, it has appeared to me, that, considering the beauty and transparency of my preparations, the distinctness with which their several parts are preserved, and the persevering labour which has been bestowed upon them, I might almost venture to think we have obtained nearly all that it is possible to know—with our present means—concerning the minute anatomy of the spinal chord. Our further progress is arrested chiefly by the continual and intricate deviations observed in the course of its fibres, and the consequent disorder in which its elements appear to be involved, when viewed in section. The confusion, however, is doubtless only apparent; for in this wonderful organ, with all its complexity of structure, we have no reason to doubt the constant and uniform existence of that beautiful order and design so conspicuously manifested by organic life in the formation of every part which science has revealed to the inquiring eye. Every fibre, probably, has its own fixed and appointed course, in accordance with the particular function it is destined to perform.

Explanation of the Plates.

PLATE XXIII.

Represents a longitudinal section through the cervical enlargement of the spinal chord of the Cat, from the 8th to the 12th pair of nerves. Magnified 30 diameters*.

P, C. Posterior white columns. A, C. Anterior white columns. G. Grey substance between the white columns. P. Posterior roots of the nerves, consisting of three kinds,—*a*, *b* and *c*. The second kind, *b*, traverse the posterior columns at first obliquely towards the mesial line, that is, in a plane oblique to that of the section, and were therefore divided near the surface of the chord. A. Anterior roots of the nerves. A, C′. A portion of the anterior column, showing the arrangement of their longitudinal fibres.

PLATE XXIV.

Represents a longitudinal section through the lumbar enlargement of the spinal chord of the Ox. Magnified 30 diameters.

P, C. Posterior columns, showing the intricate interlacement formed by the posterior roots of the nerves. P. Posterior roots. A, C. Anterior white columns. A. Anterior roots of the nerves. G. The anterior and posterior grey substance traversed by the fibres of the anterior and posterior roots of the nerves. *g*. The *substantia gelatinosa*.

In both Plates the vesicles have been omitted in order to avoid obscuring the fibres, which were observed by means of Ross's half-inch object-glass, with No. 2 eye-piece, giving a power of 150 diameters.

* The outlines of the drawings were made by means of a power of 40 diameters, but were reduced nearly one-third by the engraver.

XIV. *On the Thermal Effects of Fluids in Motion.* By WILLIAM THOMSON, *M.A., F.R.S., F.R.S.E., &c., Professor of Natural Philosophy in the University of Glasgow, For. Mem. of the Royal Swedish Academy of Sciences; and* J. P. JOULE, *F.R.S., F.C.S., Corr. Mem. R.A. Turin, Vice-President of the Literary and Philosophical Society of Manchester, &c.*

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IN a paper communicated to the Royal Society, June 20, 1844, "On the Changes of Temperature produced by the Rarefaction and Condensation of Air*," Mr. JOULE pointed out the dynamical cause of the principal phenomena, and described the experiments upon which his conclusions were founded. Subsequently Professor THOMSON pointed out that the accordance discovered in that investigation between the work spent and the mechanical equivalent of the heat evolved in the compression of air may be only approximate, and in a paper communicated to the Royal Society of Edinburgh in April 1851, "On a Method of discovering experimentally the relation between the Mechanical Work spent, and the Heat produced by the compression of a Gaseous Fluid†," proposed the method of experimenting adopted in the present investigation, by means of which we have already arrived at partial results‡. This method consists in forcing the compressed elastic fluid through a mass of porous non-conducting material, and observing the consequent change of temperature in the elastic fluid. The porous plug was adopted instead of a single orifice, in order that the work done by the expanding fluid may be immediately spent in friction, without any appreciable portion of it being even temporarily employed to generate ordinary *vis viva*, or being devoted to produce sound. The non-conducting material was chosen to diminish as much as possible all loss of thermal effect by conduction, either from the air on one side to the air on the other side of the plug, or between the plug and the surrounding matter.

A principal object of the researches is to determine the value of μ, CARNOT's function. If the gas fulfilled perfectly the laws of compression and expansion ordinarily assumed, we should have§

$$\frac{1}{\mu}=\frac{\frac{1}{\mathrm{E}}+t}{\mathrm{J}}+\frac{\mathrm{K}\delta}{\mathrm{E}p_0u_0\log\mathrm{P}},$$

where J is the mechanical equivalent of the thermal unit; p_0u_0 the product of the

* Philosophical Magazine, S. 3, vol. xxvi. p. 369.

† Transactions of the Royal Society, Edinburgh, vol. xx. Part II.

‡ Philosophical Magazine, S. 4, vol. iv. p. 481.

§ Dynamical Theory of Heat, equation (7), § 80, Transactions of the Royal Society of Edinburgh, vol. xx p. 297.

pressure in pounds on the square foot into the volume in cubic feet of a pound of the gas at 0° Cent.; P is the ratio of the pressure on the high pressure side to that on the other side of the plug; δ is the observed cooling effect; t the temperature Cent. of the bath, and K the thermal capacity of a pound of the gas under constant pressure equal to that on the low pressure side of the gas. To establish this equation it is only necessary to remark that $K\delta$ is the heat that would have to be added to each pound of the exit stream of air, to bring it to the temperature of the bath, and is the same (according to the general principle of mechanical energy) as would have to be added to it in passing through the plug, to make it leave the plug with its temperature unaltered. We have therefore $K\delta=-H$, in terms of the notation used in the passage referred to.

On the above hypothesis (that the gas fulfils the laws of compression and expansion ordinarily assumed) $\frac{\delta}{\log P}$ would be the same for all values of P; but REGNAULT has shown that the hypothesis is not rigorously true for atmospheric air, and our experiments show that $\frac{\delta}{\log P}$ increases with P. Hence, in reducing the experiments, a correction must be first applied to take into account the deviations, as far as they are known, of the fluid used, from the gaseous laws, and then the value of μ may be determined. The formula by which this is to be done is the following (Dynamical Theory of Heat, equation (f), § 74, or equation (17), § 95, and (8), § 89)—

$$\frac{1}{\mu}=\frac{\frac{1}{J}\{w-(p'u'-pu)\}+K\delta}{\frac{dw}{dt}},$$

where

$$w=\int_u^{u'}pdv,$$

u and u' denoting the volumes of a pound of the gas at the high pressure and low pressure respectively, and at the same temperature (that of the bath), and v the volume of a pound of it at that temperature, when at any intermediate pressure p. An expression for w for any temperature may be derived from an empirical formula for the compressibility of air at that temperature, and between the limits of pressure in the experiment.

The apparatus, which we have been enabled to provide by the assistance of a grant from the Royal Society, consists mainly of a pump, by which air may be forced into a series of tubes acting at once as a receiver of the elastic fluid, and as a means of communicating to it any required temperature; nozles, and plugs of porous material being employed to discharge the air against the bulb of a thermometer.

The pump a, fig. 1, consists of a cast-iron cylinder of 6 inches internal diameter, in which a piston, fig. 2, fitted with spiral metallic packing (of antifriction metal), works by the direct action of the beam of a steam-engine through a stroke of 22 inches. The pump is single-acting, the air entering at the base of the cylinder during the up-stroke, and being expelled thence into the receiving tubes by the down-stroke. The governor of the steam-engine limits the number of complete strokes of the pump to 27 per minute. The valves, fig. 3, consist of loose spheres of brass 0·6 of an inch in diameter, which fall by their own gravity over orifices 0·45 of an inch diameter. The cylinder and valves in connection with it are immersed in water to prevent the wear and tear which might arise from a variable or too elevated temperature.

Fig. 1.

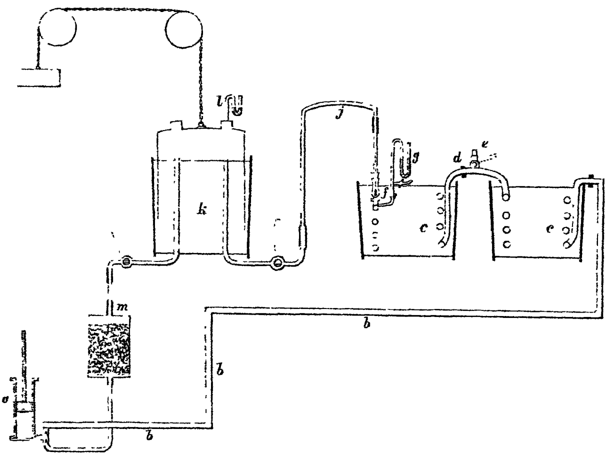


Fig. 2.

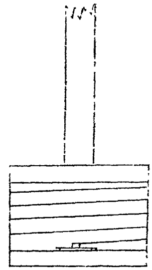


Fig. 3.

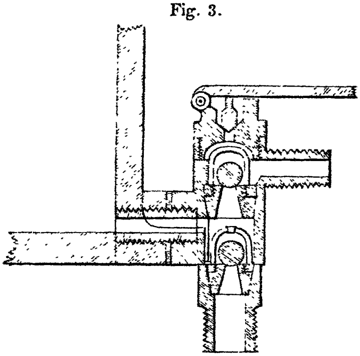
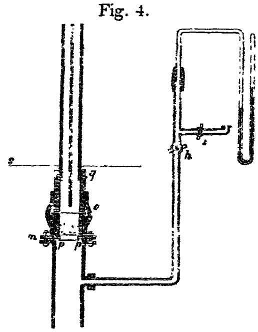


Fig. 4.



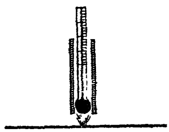
Wrought-iron tubing, *b b*, fig. 1, of 2 inches internal diameter, conducts the compressed air horizontally a distance of 6 feet, thence vertically to an elevation of 18 feet, where another length of 23 feet conveys it to the copper tubing, *c c*; the junction being effected by means of a coupling-joint. The copper tubing, which is of 2 inches internal diameter and 74 feet in length, is arranged in two coils, each being immersed in a wooden vessel of 4 feet diameter, from the bottom and sides of which it is kept at a distance of 6 inches. The coils are connected by means of a coupling joint *d*, near which a stopcock, *e*, is placed, in order to let a portion of air escape when it is wanted to reduce the pressure. The terminal coil has a flange, *f*, to which any required nozle may be attached by means of screw-bolts. Near the flange, a small pipe, *g*, is screwed, at the termination of which a calibrated glass tube bent (as shown in fig. 4), and partly filled with mercury, is tightly secured. A stopcock at *h*, and another in a small branch pipe at *i*, permit the air at any time to be let off, so as to examine the state of the gauge when uninfluenced by any except atmospheric pressure. The branch pipe is also employed in collecting a small portion of air for chemical analysis during each experiment. A pipe, *j*, is so suspended, that by means of india-rubber junctions, a communication can readily be made to convey the air issuing from the nozle into the gas-meter, *k*, which has a capacity of 40 cubic feet, and is carefully graduated by calibration. A bent glass tube, *l*, inserted in the top of the meter, and containing a little water, indicates the slight difference which sometimes exists between the pressure of air in the meter and that of the external atmosphere. When required, a wrought-iron pipe, *m*, 1 inch in diameter, is used to convey the elastic fluid from the meter to the desiccating apparatus, and thence to the pump so as to circulate through the entire apparatus.

We have already pointed out the different thermal effects to be anticipated from the rushing of air from a single narrow orifice. They are *cold*, on the one hand, from the expenditure of heat in labouring force to communicate rapid motion to the air by means of expansion; and *heat*, on the other, in consequence of the *vis viva* of the rushing air being reconverted into heat. The two opposite effects nearly neutralize each other at 2 or 3 inches distance from the orifice, leaving however a slight preponderance of cooling effect; but close to the orifice the variations of temperature are excessive, as will be made manifest by the following experiments.

A thin plate of copper, having a hole of $\frac{1}{20}$th of an inch diameter, drilled in the centre, was bolted to the flange, an india-rubber washer making the joint air-tight. At the ordinary velocity of the pump the orifice was sufficient to discharge the whole quantity of air when its pressure arrived at 124 lbs. on the square inch. When however lower pressures were tried, the stopcock *e* was kept partially open. The thermometer used was one with a spherical bulb 0·15 of an inch in diameter. Holding it as close to the orifice as possible without touching the metal, the following observations were made at various pressures, the temperature of the water in which the coils were immersed being 22° Cent. The air was dried and deprived of carbonic acid by passing it, previous to entering the pump, through a vessel $4\frac{1}{2}$ feet long and 20 inches diameter, filled with quicklime.

Total pressure of the air in lbs. on the square inch.	Temperature Centigrade.	Depression below temperature of bath.
124	8°·58	13°·42
72	11·65	10·35
31	16·25	5·75

Fig. 5.

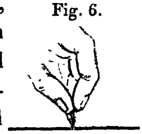


The heating effect was exhibited as follows:—The bulb of the thermometer was inserted into a piece of conical gutta percha pipe in such a manner that an extremely narrow passage was allowed between the interior surface of the pipe and the bulb. Thus armed, the thermometer was held, as represented by fig. 5, at half an inch distance from the orifice, when the following results were obtained:—

Total pressure of the air in lbs. on the square inch.	Temperature Centigrade.	Elevation above temperature of bath.
124	45°·75	23°·75
71	39·23	17·23
31	26·2	4·20

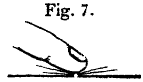
It must be remarked, that the above recorded thermal effects are not to be taken as representing the maximum results to be derived from the rushing air at the pressures named. The determination of these, in the form of experiment above given, is prevented by several circumstances. In particular it must be observed, that the cooling effects must have been reduced in consequence of the heat evolved by the friction of the rushing air against the bulb of the thermometer. The heating effects, resulting as they do from the absorption and conversion into heat of the *vis viva* of the rushing air, depend very much upon the narrowness of the space between the thermometer and gutta percha pipe. We intend further on to return to this subject, but in the mean time will mention three forms of experiment whereby the heating effect is very strikingly and instructively exhibited.

Fig. 6.



Experiment 1.—The finger and thumb are brought over the orifice, as represented in fig. 6, so that by gradually closing them the stream of air is pinched. It is found that the effort to close the finger and thumb is opposed by considerable force, which increases with the pressure applied. At the same time a strong tremulous motion is felt and a shrill noise is heard, whilst the heat produced in five or six seconds necessitates the termination of the experiment.

Fig. 7.



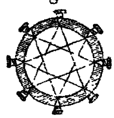
Experiment 2.—Fig. 7. The finger is placed over the orifice and pressed until a thin stratum of air escapes between the copper-plate and the finger. In this case the burning heat of the rushing air is equally remarkable in spite of the proximity of the finger to the cold metal.

Fig. 8.

Experiment 3.—Fig. 8. A piece of thick india-rubber is pressed by the finger over the narrow orifice so as to allow a thin stream of air to rush between the india-rubber and the plate of copper. In this case the india-rubber is speedily raised to a temperature which prevents its being handled comfortably.

We have now adduced enough to illustrate the immense and sudden changes of temperature which exist in the "rapids" of a current of air, changes which point out the necessity of employing a porous plug, in order that when the air arrives at the thermometer its state may be reduced to a uniform condition. Figs. 4 and 9 represent our first arrangement for the porous plug, where *n* is a brass casting with flange to bolt to the copper tube. It has eight studs, *o*, and eight holes, *pp*, drilled into the inner part of the flange. The studs and holes furnish the means of securing the porous material (in the present instance of cotton wool) in its place, by binding it down tightly with twine. Immediate contact between the cotton and metal is prevented by the insertion of a piece of india-rubber tubing; *qqq* are three pieces of india-rubber tube inserted within each other, the inner one communicating with a glass tube *r*, through which the divisions of the thermometer may be seen, and which serves to convey the air to the meter. In the experiments about to be given, the thermometer was in immediate contact with the cotton plug as represented in the figure, and the nozle was immersed in the bath up to the line *s*. The weight of the cotton wool in a dry state was 251 grs., its specific gravity 1·404, and being compressed into a space 1½ inch in diameter and 1·9 inch long, the opening left for the passage of air must have been equal in volume to a pipe to 1·33 of an inch diameter.

Fig. 9.



First series of experiments. Atmospheric air dried and deprived of carbonic acid by quicklime. Gauge 73·6; barom. 30·04=14·695 lbs. pressure per square inch.

Gauge.		Total pressure in lbs. per square inch.	Cubic inches of air passed per minute reduced to atmospheric pressure.	Temperature of bath* ascertained by Thermometer No. 1, in Centigrade degrees.		Temperature of the issuing air, ascertained by Thermometer No. 2.		Cooling effect.
37·5 37·5 38 37·8	37·7	35·854	12703	445 445·5 445·9 446	445·6 =18·2676	414 414 414·6 414·8	414·35=17·8298	0·4378
38 38 37·8 38	37·9	35·647	12703	446·1 446·6 446·8 447·1	446·65=18·3128	415·4 416 416·8 417·6	416·45=17·9295	0·3833
38 37·75 37·5 37·5	37·69	35·866	12703	447·2 447·5 447·8 448	447·62=18·3545	418 418·2 418·4 418	418·15=18·0110	0·3435

A Liebig tube containing sulphuric acid, specific gravity 1·8, gained 0·03 of a grain by passing through it, during the experiment, 100 cubic inches of air.

* By varying the temperature of the water in which the coils were immersed, it was found that the temperature of the water surrounding the first coil exercised no perceptible influence, the temperature of the rushing air being entirely regulated by that of the terminal coil. However, the precaution was taken of keeping both coils at nearly the same temperature.

The observations above tabulated were made at intervals of two or three minutes. It will be observed that the cooling effect appeared to be greater at the commencement than at the termination of the series. This may be attributed in a great measure to the drying of the cotton, which was found to contain at least 5 per cent. of moisture after exposure to the atmosphere. There was also another source of interference with the accuracy of the results owing to a considerable oscillation of pressure arising from the action of the pump. We had remarked that when the number of strokes of the engine was suddenly reduced from twenty-seven to twenty-five per minute, a depression of the thermometer equal to some hundredths of a degree Cent. took place, a circumstance evidently owing to the entire mass of air in the coils and cotton plugs suffering dilatation without allowing time for the escape of the consequent thermal effect. Hence it was found absolutely essential to keep the pump working at a perfectly uniform rate. For a similar reason it was also most important to prevent the oscillations of pressure due to the action of the pump, particularly as it appeared obvious that the heat evolved by the sudden increase of pressure, on the admission of a fresh supply of air from the pump, would arrive at the thermometer in a larger proportion than the cold produced by the subsequent gradual dilatation. In fact, on making an experiment in which the air was kept at a low pressure, by opening a stopcock provided for the purpose, the oscillations of pressure amounting to $\frac{1}{20}$th of the whole, it was found that an apparent heating effect, equal to 0°·2 Cent., was produced instead of a small cooling effect.

It became therefore necessary to obviate the above source of error, and the method first employed with that view, was to place a diaphragm of copper with a hole in its centre $\frac{1}{7}$th of an inch in diameter at the junction between the iron and copper pipes. The oscillation being thus reduced, so as to be hardly perceptible, we made the following observations.

Second series of experiments. Atmospheric air dried and deprived of carbonic acid by quicklime. Gauge 73·75; barometer 30·162=14·755 lbs. pressure per square inch; thermometer 19°·3 Cent.

Gauge.		Total pressure, in lbs. per square inch.	Cubic inches of air issuing per minute at atmospheric pressure.	Temperature of bath by Thermometer No. 1, degrees Centigrade.		Temperature of issuing air by Thermometer No. 2, degrees Centigrade.		Cooling effect.
39 38·6 38·5 38·5	38·65	36·069	11796	467 467 467 467·1	467·02=19°·186	434·6 435 435 435	434·9 =18°·810	0°·377
38·5 38·8 38·8 38·75	38·79	35·912	11796	467·1 467·2 467·2 467·3	467·2 =19·194	435·1 435·4 435·6 435·4	435·37=18·832	0·362
38·8 38·8 38·8 38·8	38·8	35·900	11796	467·3 467·4 467·4 467·4	467·37=19·202	435·6 435·8 435·9 436	435·82=18·854	0·348

Suspecting that particles of the sperm oil employed for lubricating the pump were carried mechanically to the cotton plug and interfered with the results, we now substituted a box with perforated caps, filled with cotton wool, for the diaphragm used in the last series. With this arrangement the pressure was kept as uniform as with the other, and all solid and liquid particles were kept back by filtration.

Third series of experiments. Atmospheric air dried and deprived of carbonic acid by quicklime*, and filtered through cotton. Gauge 73·7; thermometer 21°·7 Cent.; barometer 30·10=14·71 lbs. on the square inch.

Time of observation.	Gauge.		Total pressure in lbs. per square inch.	Cubic inches of air issuing per minute at atmospheric pressure.	Temperature of bath by Thermometer No. 1, in degrees Centigrade.		Temperature of the issuing air by Thermometer No. 2, in degrees Centigrade.		Cooling effect.
m									
3	39	39·2	34·410	11784	357·7	357·92=14°·506	337·35	337·89=14°·183	0°·323
6	39·1				357·8		337·8		
9	39·5				358		338		
12	39·2				358·2		338·4		
15	39·1	39·19	34·418	11784	358·7	358·97=14·552	338·8	338·85=14·230	0·322
16	39·35				358·9		338·7		
18	39·1				359·1		339		
21	39·2				359·2		338·9		
23	39·2	39·18	34·426	11784	359·4	359·72=14·584	339·25	339·69=14·270	0·314
25	39·1				359·7		339·8		
28	39·2				359·8		339·7		
30	39·2				360		340		
32	39·5	39·34	34·279	11784	360·1	360·27=14·607	340	340·25=14·296	0·311
34	39·3				360·2		340·2		
36	39·25				360·4		340·4		
38	39·3				360·4		340·4		

* The use of quicklime as a desiccating agent was suggested to us by Mr. THOMAS RANSOME. It answered its purpose admirably after it had fallen a little by use, so as to be finely subdivided. The perfection of its action was shown by the desiccating cylinder remaining, after having been used two hours, cold at the lower part, while the upper part for about 9 inches was made very hot. The analysis of the air passed during the third series of experiments showed that one of the Liebig tubes had gained no weight whatever; and in one instance we have observed that the sulphuric acid of 1·8 specific gravity, actually lost weight, apparently indicating that the air dried by quicklime was able to remove water from acid of that density.

The stopcock for reducing pressure being now partially opened, the observations were continued as follows:—

Time of observation.	Gauge.		Total pressure in lbs. per square inch.	Temperature of bath by Thermometer No. 1, in degrees Centigrade.		Temperature of the issuing air by Thermometer No. 2, in degrees Centigrade.		Cooling effect.
h m								
50	55·1			361·7		344		
52	55·1			361·9		344·8		
54	55·1			361·9		345·3		
55	55·1			361·9		345·8		
57	55·1			362·1		346·0		
59	55·1	55·12	22·876	362·3	362·26 = 14°·693	346·4	346·19 = 14°·579	0°·114
1 1	55·1			362·4		346·9		
3	55·1			362·7		347·2		
5	55·1			362·7		347·6		
7	55·3			363		347·9		
11	54·3			363·3		348·9		
13	54·4			363·3		348·9		
15	54·4			363·5		349·2		
17	54·7			363·7		349·4		
19	54·5	54·51	23·217	363·9	363·82 = 14·760	350	349·74 = 14·749	0·011
20	54·5			364·1		350		
22	54·6			364·2		350·3		
24	54·6			364·2		350·4		
26	54·6			364·2		350·6		
30	54·6			375		356·4		
32	54·6			375·4		358·2		
33	54·2			375·4		359·4		
35	54·3			375·5		359·8		
37	54·4	54·38	23·277	375·8	375·7 = 15·270	360	360 = 15·238	0·032
39	54·6			375·7		360·1		
40	54·3			375·8		360·3		
42	54·5			376		360·4		

During the above experiment 100 cubic inches of the air was slowly passed through two Liebig tubes containing sulphuric acid, specific gravity 1·8. The first tube gained 0·006 of a grain, the second remained at exactly the same weight.

P.S. Oct. 14, 1853.—The apparently anomalous results contained in the last Table have been fully explained, and shown to depend on the alteration of pressure which took place towards the beginning of the interval of time from 42^{m} to 50^{m}, by subsequent researches which we hope soon to lay before the Royal Society.

XV. *On the Anatomy and Physiology of* Cordylophora*, *a contribution to our knowledge of the Tubularian Zoophytes. By* George James Allman, *M.D., M.R.I.A., Professor of Botany in the University of Dublin, and Examiner in Zoology and Botany in the Queen's University in Ireland. Communicated by Professor* Edward Forbes, *F.R.S.*

Received May 31,—Read June 16, 1853.

THOUGH the attention of several physiologists has been directed to the Tubularian Zoophytes, and though the elucidation of many points of interest in their structure and in the physiological phenomena presented by them has been the result, yet in numerous most important particulars our knowledge of the *Tubulariadæ* is still very imperfect, notwithstanding the great value which a more thorough acquaintance with these simple organisms must possess in its bearing upon some of the leading questions in physiology.

It was with the view of filling up some of the deficiencies which still exist in our knowledge of the lower zoophytes, that the following researches were undertaken; and *Cordylophora lacustris,* affording as it does a fine typical example of tubularian structure, and being easily kept alive for months, was deemed peculiarly favourable for prolonged and careful observation; the results, moreover, which have been obtained from the investigation of this zoophyte, have been in many instances confirmed and extended by corresponding inquiries, instituted at the same time into the structure of allied genera.

It is certain that considerable confusion as well as much cumbrous circumlocution might be avoided, by giving greater precision and completeness to the terminology employed in the description of structure among zoophytes. With this view, I propose in the first place to employ the word *polype* in its most restricted sense.

* The genus *Cordylophora* may be characterized as follows:—

Fam. Tubulariadæ.

Genus Cordylophora, Allman.

Char. Gen.—*Polypi* tentaculis numerosis sparsis teretibus. *Capsulæ genitales* subsessiles in ramulis ultimis pone polypos affixæ. *Polyparium* pergamentaceum, ramosum, stolone fistuloso repente fixum. *Embryo* liber, subcylindricus, universè ciliatus.

Nomen.—Κορδύλη clava et φορέω fero.

Species unica, *C. lacustris.*

Habitat.—In aquis dulcibus quietis, corpora varia submersa obducens, et locos obscuros amans.

This word is used ambiguously by writers, being often employed to designate the entire composite fabric, the aggregate result of gemmation; while at other times it is intended to indicate each of those peculiar organisms which, almost always furnished with a mouth and tentacula, are developed upon various points of a common living basis, and are eminently characteristic of zoophytic form. In the following paper I shall use the term *polype* strictly in the latter signification; and for the entire mass, whether consisting of a single polype, as in *Hydra*, or of many united into a more or less definite assemblage, it will be sufficiently convenient to employ merely the term *zoophyte**.

Besides the more definite limits within which it is thus necessary to confine the term polype, the requirements of precise description demand the use of a few additional terms. To the common living basis by which the several polypes in a composite zoophyte are connected with one another, I propose to give the name of *cœnosarc*†, and every composite zoophyte will thus consist of a variable number of polypes, developing themselves from certain more or less definite points of a common cœnosarc. The term *polypary* has been used with just as little precision as *polype*, being sometimes employed to express this common connecting basis, and at other times being applied exclusively to the solid protective structures, whether forming for the zoophyte an external covering or constituting an internal axis; the ambiguity which thus results will be got rid of by using the word cœnosarc as here defined, and restricting the term polypary to the solid protective structures of the zoophyte. All the hydroid zoophytes can be proved to consist essentially of two distinct membranes; to the external of these membranes I shall give the name of *ectoderm*‡, and to the internal that of *endoderm*§.

ORGANS FOR THE PRESERVATION OF THE INDIVIDUAL.

1. *Ectoderm and Polypary.*

(*a.*) *Ectoderm.*—The ectoderm (Plates XXV. and XXVI. figs. 3, 4, 9*a*, *a*, *a*) is a well-defined membrane; it is composed of cells, and forms the external layer of the polypes and cœnosarc. Multitudes of *thread-cells* are developed in the substance of the ectoderm. The thread-cells, when in a quiescent state (fig. 5), present the appearance of minute ovate capsules, slightly curved at one end, and with a transparent cylinder occupying about two-thirds of the axis. Under the influence of excitement, one extremity of

* In the following paper the term *zoophyte* is thus used in its restricted application, by which it is confined to the true polype-bearing *Radiata*, and no more convenient or expressive word can be employed for the purpose. If we except the changes which modern research has rendered necessary in removing from the zoophytes of the older authors, the Sponges, Corallines and *Polyzoa*, this term was used long ago in almost exactly the same sense by PALLAS, and by ELLIS and SOLANDER, and some of our best living zoologists are now employing it with similar limitations.

† κοινὸς, σάρξ. ‡ ἐκτὸς, δέρμα. § ἔνδον, δέρμα.

the capsule becomes suddenly prolonged into a conical projection, surrounded near its apex by a circle of very minute curved spicula, the capsule at the same instant appearing empty (fig. 6). This phenomenon seems to consist in the sudden eversion through one end of the capsule of a delicate sac which had previously lain invaginated within it, and is generally immediately followed by the projection of a long and fine filament from the free end of the everted sac. The structure of these thread-cells, and the phenomena consequent on excitement, closely resemble what we find in the "hastigerous organs" of *Hydra*. The greater minuteness however of the thread-cells in *Cordylophora* renders it much more difficult to obtain a satisfactory examination of their structure. The thread-cells are developed in the interior of mother-cells (fig. 7), which are themselves secondary cells formed within the ectodermal cells; but I was not able to determine how far their development devolves exclusively upon certain cells of the ectoderm specially devoted to this office. The ectoderm of the tentacula differs slightly from that of the rest of the zoophyte, in the fact of the thread-cells being for the most part collected into groups, which present the appearance of little wart-like excrescences resembling the disposition of the corresponding organs in the tentacula of *Hydra*. At the extremity of the tentacula numerous thread-cells are crowded together, but there is no approach to the capitate terminations which are so striking in *Syncoryne*, and which, among other characters, zoologically distinguish that genus from *Cordylophora*.

(*b.*) *Polypary*.—The polypary (fig. 3 *b, b*) is a cylindrical tube investing the stems and branches of the zoophyte, and terminating just behind a short fleshy neck which immediately supports the polypes. On the younger branches it forms a thin, and almost colourless pergamentaceous investment; but in older parts of the zoophyte it is of a yellowish-brown colour, and composed of numerous layers. At the origin of the branches from the main stem, the polypary presents a number of annular corrugations. The cavity of the polypary appears by no means accurately filled by the cœnosarc, and we almost always find a considerable interval between the ectoderm of the cœnosarc and the walls of the polypary; this interval is crossed transversely by numerous processes of the ectoderm (fig. 3 *c, c*). It seems, however, almost certain that a very delicate living membrane is in actual contact throughout with the inner surface of the polypary, at least in all its younger portions; and the space which appears to exist between the polypary and the cœnosarc is then nothing but a large *lacuna* of the ectoderm, crossed by fleshy processes, which keep up a communication between its opposite walls. I have distinctly demonstrated the presence of such a membrane in *Syncoryne*, where I have traced the formation of the space in question from a simple lacuna of the ectoderm, though I have not succeeded in so decidedly making out these points in *Cordylophora*. No trace of organization can be detected in the polypary, which in all respects resembles a mere secretion deposited in layers from the ectoderm. Where the polypary terminates anteriorly, an exceedingly delicate transparent pellicle may be traced in continuity with it over the neck and clavate body of the polype, at least as far as the roots of the posterior tentacula.

It is certain that the cavity of the polypary increases in diameter, up to a certain point, with the growth of the animal, a fact which we can reconcile with the account just given of its formation, only by attributing to it a capacity of distension subsequently to its original deposition.

2. *The Endoderm and its Cavity.*

The endoderm (fig. 3, *d*, *d*) constitutes a very distinctly cellular layer; it is in contact throughout with the ectoderm, but is nevertheless distinguishable from it by a very decided boundary, and can only be considered as directly continuous with it at the mouths of the polypes. It forms the walls of an uninterrupted cavity, which, extending through the axis of the entire zoophyte, opens externally at the polype-mouths. This cavity may be divided into four distinct regions, namely, the *post-buccal cavity*, the *stomach*, the *tentacular canals*, and the *canal of the cœnosarc*, a division which, though in some degree arbitrary, will be found very useful in description.

(*a.*) *Post-buccal cavity.*—The post-buccal cavity (fig. 3 *f*) is situated immediately behind the mouth (fig. 3 *e*), which is a simple unarmed orifice placed at the extremity of a conical projection, into which the body of the polype is continued anteriorly. The post-buccal cavity lies in the axis of this projection; its walls are formed of elongated cells, with their long diameter perpendicular to the surface, and its interior is quite destitute of rugæ. It is usually in the form of a narrow tube, and is capable of complete obliteration by the temporary approximation of its walls; it is sometimes, however, dilated into a globular space, and sometimes spread out by a kind of semi-eversion into a nearly flat disc, which I have seen employed by the animal as an organ of adhesion.

(*b.*) *Stomach.*—The stomach (fig. 3 *g*) occupies the whole of the interior of the clavate body of the polype; as the endoderm passes backward from the post-buccal cavity to constitute the walls of the stomach, we find it forming a peculiar tissue composed of elongated cells, whose rounded ends project in prominent masses into the cavity of the stomach, where they constitute large irregular rugæ. The tissue of these rugæ is very remarkable; the cells composing it (fig. 4 *b*) are mother-cells, giving origin in their interior to several free secondary cells with distinct nuclei. In some of these secondary cells the contents are colourless and transparent, and the nuclei are then very evident (fig. 4 *c*); others contain a brown granular matter, by which the nucleus is more or less obscured (fig. 4 *d*); while others may occasionally be found filled with a brood of young cells (fig. 8). Lying apparently free in the mother-cells may also be generally seen, besides the free secondary cells, some rather irregular masses of a deep brown granular substance (fig. 4 *e*). I have found in *Hydra* a structure in all its essential characters entirely similar to this, and we cannot refuse to recognise in it an example of true glandular structure. The secondary cells are obviously true secreting cells, destined by their own rupture and that of the mother-cells to discharge their contents into the cavity of the stomach; while the free granular masses in the interior of the mother-cells were most probably originally contained in secondary cells, of whose secreting action they are the product, and which at a subsequent period had disappeared.

(*c.*) *Tentacular canals.*—The tentacula are hollow processes from the sides of the polypes, and their cavities are simple continuations of that of the stomach. This view is certainly opposed to the appearances presented on a superficial examination, and is at variance with the accounts given by other observers of the structure of the tentacula in the marine *Hydroida*; very careful and repeated observations, however, have convinced me of its truth. It must be admitted, that at first sight, the tentacula, even under well-managed microscopical examination, have exactly the appearance of tubes whose cavity is interrupted at regular intervals by completely formed transverse septa: I have, however, satisfied myself that in *Cordylophora*, and probably also in all the other *Hydroida*, the tube of the tentacula is perfectly continuous. The tentacula consist in reality, like all other parts of the animal, of an external or ectodermic layer (fig. 9 *a*, *a*), which is a simple continuation of the general ectoderm of the body, and of an internal or endodermic layer (fig. 9 *b*, *b*), which is in the same way a continuation of the endoderm of the stomach. The tentacular ectoderm has been already described. The endoderm consists of rather large cells with very delicate walls; it constitutes a thick lining of the tube, and nearly fills up the entire cavity; and indeed this cavity, by a temporary approximation of its walls, appears capable of occasional obliteration. The cells of the tentacular endoderm are glandular, and may be generally seen to contain the peculiar brown granules characteristic of the general endodermic secretion. The appearance of transverse septa is probably due to the occurrence at regular intervals, of interruptions in the continuity of the endodermic layer.

Though the tentacula of *Hydra* present no appearance of the transverse septa, their structure is nevertheless essentially the same as that of the tentacula of *Cordylophora*, the only difference being in the fact that the endoderm forms a thinner stratum in *Hydra*, and thus encroaches less on the cavity of the tube, and does not present the successive interruptions which in *Cordylophora* give rise to the fallacious appearance of transverse septa.

(*d.*) *Canal of the Cœnosarc.*—The cavity of the stomach is continuous posteriorly with the canal of the cœnosarc, fig. 3 *h*. The rounded masses of cells constituting the large rugæ into which the endoderm of the stomach is thrown, cannot be traced further back than the posterior extremity of the clavate body of the polype, and from this point, which may be conveniently though somewhat arbitrarily assumed as the posterior termination of the stomach, the canal of the cœnosarc commences, and thence extends as a continuous tube through all the ramifications of the cœnosarc. In that portion of the canal which immediately succeeds to the stomach, and which is here much contracted, the endoderm resembles in all respects that of the stomach, except in the absence of distinct rugæ; but as the cœnosarc passes backwards under cover of the polypary, the cells lose their elongated form, the endoderm becomes thinner and the canal proportionately wider. Throughout all the rest of the cœnosarc the endoderm is formed of a compact tissue destitute of rugæ, but whose component cells continue to retain their glandular character (fig. 10).

The canal of the cœnosarc is filled with a fluid containing globular and irregular corpuscles of various sizes; these corpuscles may be observed to exhibit a kind of circulatory movement, which however is very irregular; they may be sometimes seen running from below upwards and again turning round in order to take a retrograde course; the motion of each corpuscle is generally confined to a small circuit, frequently not exceeding the diameter of the canal; sometimes many of the corpuscles stop altogether and allow others to pass them, while sometimes no motion whatever can be observed in any of the contents. The corpuscles while moving in their linear paths have besides a peculiar vibratile or tremulous motion in themselves, and this motion may generally be witnessed while the corpuscles are otherwise entirely at rest. The motion of the corpuscles may be traced all through the stems and branches, and even into the stomach. The cause of this phenomenon is very obscure; the existence of vibratile cilia on the walls of the canal might be adduced as an explanation, but these organs have never been made the subject of direct observation, notwithstanding the most careful attempts to demonstrate them; it appears to me that the true cause is to be found in currents produced by the active vital processes going on in the secreting cells of the endoderm, processes which we can hardly imagine to take place without causing local changes in the chemical constitution of the fluids in immediate contact with the cells, and a consequent disturbance of the stability of these fluids.

3. *Muscular System.*

Under this head may be described the only specialized motor apparatus which these animals present. It consists of numerous longitudinal fibres, which are in close contact with the inner surface of the ectoderm. These fibres (fig. 3 *i*, *i*, fig. 4 *f*, fig. 9 *c*) may be seen in all the naked portion of the cœnosarc, and in the body of the polype, and may be thence easily traced into the tentacula, in which they can be followed to the remote extremities of these organs, but they disappear where the cœnosarc passes under cover of the polypary. There can be little doubt that we are correct in considering them as the representatives of a muscular system, and in attributing to them the chief share in effecting the various changes of form which the polypes are perpetually presenting. Similar fibres may be witnessed in *Coryne*, *Syncoryne*, and other marine *Tubulariadæ*, in some of which they are even more distinct than in *Cordylophora*. In none could I detect the least trace of transverse striæ.

ORGANS FOR THE PRESERVATION OF THE SPECIES—EMBRYOLOGY.

The researches of modern naturalists, especially of Ehrenberg, Loven, Van Beneden, Sars, and Dujardin, have thrown much light on the embryology of the inferior zoophytes, and their observations have brought to our knowledge many remarkable and unexpected facts, though physiologists are by no means of one mind as to the true signification of several of the facts with which we have thus become acquainted.

During the later summer and autumn months *Cordylophora lacustris* may be seen with numerous oval capsules borne upon the ultimate or polypiferous ramuli, fig. 2. Each ramulus generally carries one, two, or three of these capsules, which are situated alternately on either side of the ramulus at some distance behind the body of the polype, and where the branch is still covered by the polypary, the more advanced capsules being always nearer to the main stems. They are nearly sessile on the supporting branches, and when fully developed are somewhat larger than the club-shaped bodies of the polypes. They are surrounded externally by a delicate continuation of the polypary, and as they approach towards maturity are found to contain within them either ova or spermatozoa.

The structure and development of these capsules and their contents are exceedingly interesting. The capsule on its first appearance presents itself as a small tubercle, consisting of a simple hernia-like protrusion, or diverticulum of the cœnosarc (fig. 11 *a*), enveloped by a delicate production of the polypary (fig. 11 *b*), and as the cavity of the diverticulum continues to communicate with the common canal of the branch, the contents of the latter pass freely into its interior. As the tubercle increases in size, we perceive that the external investment from the polypary immediately surrounds a sac of distinctly cellular structure, having the diverticulum from the cœnosarc projecting into it below, and having a remarkable system of branched tubes developed on its internal surface (figs. 12, 13, 14). These tubes spring from the sides of the diverticulum with whose cavity their own communicates; they occasionally inosculate with one another, and passing upwards in close connection with the walls of the cellular sac, extend to the summit of the sac where their branches terminate in cæca; they are lined with a layer of cells, which contain secondary cells and brown granules, and which thus resemble the general endoderm of the zoophyte. Thread-cells occur in the walls of the cellular sac, which is plainly referable to the ectodermal system. Between the cellular sac and the diverticulum is a considerable interval, which is filled with a granular fluid.

We next find that in those capsules which are to contain ova a number of spherical dark grey or bluish bodies (fig. 14 *f*) have become apparent in the midst of the granular fluid: these are *ova*; in most of them we may perceive the germinal vesicle in the form of a minute clear point, but the germinal spot is not apparent. The germinal vesicle soon disappears, and when the ova are now examined, the phenomenon of yelk-cleavage may be distinctly perceived in them, fig. 16. In the mean time a delicate structureless pellicle (fig. 13 *e*) has been formed on the external surface of the cellular sac; it is evidently a mere secretion from the surface of the sac, resembling the secretion of the polypary from the cœnosarc. Soon after the commencement of segmentation of the vitellus, the cellular sac with its system of ramified tubes disappears, and the cluster of ova is now distinctly exposed (fig. 15) lying upon the diverticulum from the cœnosarc, and immediately enclosed by the delicate structureless sac formed by the pellicle which had been secreted on the outside of the cellular sac, while the whole is included within the external pergamentaceous investment from the polypary. The ova vary in number; in many capsules I have counted ten, twelve, or even more, while some few contained only two or three.

The mulberry-like condition of the ova at length disappears, and these bodies now begin to elongate themselves, and when viewed by transmitted light, present a transparent margin (fig. 17). As development advances the ova become more and more elongated, and they soon exhibit a slow but evident motion in the interior of the containing sac (fig. 18). They are now in the condition of free embryos, only waiting for the rupture of the enclosing structures to escape into the surrounding medium and enjoy an independent existence. We accordingly soon find that these structures no longer confine them, and the embryos may now be seen escaping from the torn summit of the capsule in the form of infusorial animalcules clothed with short cilia, and swimming freely through the surrounding water (fig. 19).

The embryos on their escape from the capsule are usually of an elongated oval figure (figs. 19, 20), but very contractile, and capable of undergoing considerable change of form; they frequently assume a pyriform shape (fig. 21), but they never appear compressed, and the comparison with a *Planaria*, to which Dalyell and others have likened the embryos of *Campanularia*, will not here strictly apply. They are included in a distinct ectoderm, and contain in their interior a cavity which is separated from the ectoderm by a cellular interval representing the endoderm of the adult zoophyte, but in which the brown granules have not yet shown themselves. No mouth is yet evident, and the ectoderm seems in this early stage destitute of thread-cells.

After the embryo has continued for some time in the condition now described, the period of its final fixation approaches; the cilia disappear; one extremity becomes expanded into a kind of disc, by which it soon attaches itself to some fixed object; the mouth has now become apparent, and thread-cells are developed in the ectoderm (fig. 22); the embryo increases in length and thickness, and from the free extremity, which has begun to assume a clavate form, a single series of tentacula soon shoot forth (fig. 23); these are about four in number, and are situated a little behind the anterior end; the portion of the body which lies before them becomes the post-buccal cavity of the adult; the gastric cavity has become prolonged through the young stem; a delicate polypary has begun to invest it; and its walls begin to be coloured by the characteristic secretion of the endoderm; other tentacula soon shoot forth behind those first formed, and the little *Cordylophora* resembles in all respects, except in size, a full-grown stem with a solitary polype. Its solitary condition however is not long retained; prolongations have already been sent out from its base; these attach themselves to the body on which it grows, and constitute the system of prostrate adherent tubes, while *gemmæ* are soon formed from these tubes and from the free stem, and convert the young single-polyped zoophyte into the adult many-polyped *Cordylophora**.

But besides the ovigerous capsules, a second kind (fig. 24) is developed on certain branches. These differ neither in situation nor in visible organization from the capsules destined to contain ova; their component sacs, internal diverticulum, and ramified canals entirely correspond with those of the ovigerous capsules; but instead of containing ova, they are filled with a turbid fluid, in the midst of which innumerable minute corpuscles may be seen to exhibit a peculiar vibratory movement occasionally visible under slight pressure through the transparent walls, and the same motion is continued in these corpuscles after they are liberated from the containing capsule and spread over the field of the microscope.

That these capsules represent a male system with spermatozoa, there cannot be the least doubt. When liberated from the capsule, the moving bodies show themselves under two different forms; they are either oval corpuscles with an excessively delicate caudal filament (fig. 24 *c*), or they are minute spherical cells (fig. 24 *b*) with a similar caudal filament (by whose undulations the cells are moved about through the surrounding fluid), and enclosing a nucleus-like corpuscle, with which the filament appears to be connected. The former would seem to be the spermatozoa in their free and fully developed condition, while the latter appear to be "vesicles of evolution," in which the body of the spermatozoon is still confined, the tail alone being disengaged. It still remains difficult to explain the mode in which the spermatozoa gain access to the ova. In some cases I have succeeded, under slight pressure, in forcing out the contents through a minute orifice which made its appearance in the summit of the capsule, but this orifice, notwithstanding its definite position, was probably the result of rupture, and I could not detect anything corresponding to it in the ovigerous capsules. It would perhaps be more in accordance with analogy to suppose that the spermatozoa make their way into the cavity of the diverticulum, that they are thus conveyed into that of the cœnosarc, and ultimately reach the ova through the diverticulum of the ovigerous capsules. The male and female capsules appear to be always borne on different stems.

* See an interesting paper by the Rev. THOMAS HINCKS, entitled 'Further Notes on British Zoophytes, with Descriptions of New Species,' published in the Annals and Magazine of Natural History, March 1853.

GENERAL CONSIDERATIONS.

From the description now given, it is manifest that *Cordylophora* (and there is little doubt that we are justified in extending the generalization to all the hydroid zoophytes) consists essentially of two distinct membranes, enclosing a cavity which opens externally by one or more orifices (polype-mouths), and which is prolonged into the interior of a variable number of filiform processes (tentacula)*. The two membranes appear to be essentially secreting structures, but the products of their respective secretory action strikingly differ from one another; for while the cells of the endoderm give origin to secondary cells which secrete a coloured granular substance, which may be fairly assumed as representing the biliary secretion of higher animals, the ectodermal cells give origin to cells of a very remarkable kind, whose characteristic secretion is a peculiar filament, whose import and office are as yet but very imperfectly understood.

In none of these zoophytes can any approach to a specialized circulatory system be discovered; the function of circulation appearing to be represented merely by a gradual transmission of the fluids from cell to cell, depending wholly on the permeability of tissue. A respiratory system is equally without any special representative. Whatever aërating influence the surrounding water exerts must be received through the entire surface of the zoophyte, though doubtless most powerfully through the portion not covered by the polypary. The muscular system has begun to be specialized, but no trace of a nervous system can yet be detected.

The existence of an extensive system of *lacunæ*, as described by DUJARDIN†, in some marine *Tubulariadæ*, and by ECKER‡ in *Hydra*, where this anatomist describes them as constituting a network of ramified canals, has not been confirmed by my own researches.

The true signification of the reproductive capsules of the hydroid zoophytes is a matter of considerable interest. On this subject much additional light will be thrown by comparing the embryological phenomena of *Cordylophora* with those which are known to occur in the allied genera of the sea. In the marine *Tubulariadæ*, two distinct kinds of bodies are developed besides the ordinary terminal polypes. Of these one consists in medusa-like bodies, which are generally developed at the bases of the tentacula§, and which after a time have been seen to detach themselves from the supporting polype and swim freely away. The other kind of bodies met with in these polypes are peculiar sacs, occupying a similar position to that of the medusoid

* The composition of the hydroid zoophytes out of two distinct membranes has been already pointed out by HUXLEY (Philosophical Transactions, 1849, Part II.), who has shown that the organs of the *Medusæ* are in the same way essentially composed of two distinct membranes. A very important affinity of structure is thus established between the hydroid zoophytes and the *Medusæ*.

† Ann. Sci. Nat. 3me sér. tome iv. 1845.

‡ Ann. des Sc. Nat. 3me sér. tome 10, 1848.

§ In *Eudendrium* the medusoid bodies are situated at a short distance behind the terminal polypes, and in *Perigonymus*, SARS, they are borne upon every part of the stems and branches.

bodies, but never detaching themselves from the parent, and always after a time filled with ova or spermatozoa*.

In *Campanularia* bodies of highly developed medusan conformation are produced, generally in considerable numbers, in external capsules situated on parts of the polypary always at a distance from the terminal polypes. When arrived at a certain degree of maturity, they escape from the capsule and swim freely through the surrounding water. In the capsules of *Campanularia* another kind of body has been also observed, in which the medusan structure is less complete, and these, after disengaging themselves from the cavity of the capsule, continue attached to its mouth, never becoming entirely free; while a third kind of bodies, corresponding with the fixed sacs of the *Tubulariadæ*, is also borne in the capsules of *Campanularia*†.

In some instances the free medusoid bodies of the *Tubulariadæ* have been found to contain ova, as has been observed by WAGNER‡ in a tubularian polype from the Adriatic, apparently referable to *Hydractinia*, VAN BENEDEN, by LOVEN§ in a *Syncoryne*, and by DUJARDIN‖ in another species of *Syncoryne*, whose medusoid was seen to produce ova in the thickness of the stomach-walls. The less perfectly developed medusoids which remain attached to the mouth of the capsule in *Campanularia* have also been observed to contain ova as noted by LOVEN¶; while the previous observation of LISTER**, who saw multitudes of moving bodies escaping from these medusoids, is possibly also an instance of a similar phenomenon, though the form of the moving bodies as described by LISTER leaves a doubt of their being truly embryos; they are probably spermatozoa, and then LISTER's medusoids would be males: a view supported by SHULTZ's†† recent observation, that these attached medusoids of *Campanularia* sometimes contain spermatozoa instead of ova.

Now in *Cordylophora* I have never witnessed the production of medusoids, but I think nevertheless that it will be easily seen that the reproductive capsules of this genus have essentially the same organization as the medusoids of the marine genera. These capsules in an advanced stage exhibit but little of a medusoid conformation, but at an earlier period of their development, as we have just seen, they present us with an organized sac having a hollow central and fleshy column projecting into it below, and furnished with a system of branched canals which are developed on its walls, and communicate at their origin with the cavity of the central column. In this structure we can have no hesitation in recognising a true medusoid type; the organized cellular sac is homologous with the disc of the Medusa, the central fleshy column will represent the peduncle or proboscidiform stomach, while the system of branched tubes will correspond to the gastro-vascular canals‡‡. It is true that neither

* See KROHN in MÜLLER's Archiv. † Vide *infra*, p. 378. ‡ WAGNER in the Isis, 1833. Heft iii

§ LOVEN in WIEGMANN's Archiv, 1837.

‖ DUJARDIN in Ann. des Sci. Nat. 3ième sér. tom. iv. 1843. ¶ LOVEN, *l. c.*

** LISTER in Philosophical Transactions, 1844. †† SHULTZ in MÜLLER's Archiv, 1851.

‡‡ The ramified tubes of the reproductive capsules must, I think, be also assumed as homologous with the tentacles of the terminal polypes. The tentacles of a hydroid polype will thus find their homologues in the gastrovascular canals of a medusa.

disc nor stomach presents any obvious opening in the capsules of the polype, but this is merely a particular and non-essential modification of structure adapted to the function which more especially devolves on them, and it must be recollected that being in direct communication with the common cavity of the parent stem, they are in no need of the admission of nutritive matter from without. Neither does the ramified condition of the tubes, nor the absence of the marginal canal, invalidate the view here taken, for we know that the gastro-vascular canals in certain true Medusæ are ramified, while the closed condition of the disc brings with it the absence of the marginal canal*.

There is a difficulty in the determination of the exact part of the reproductive capsules, which is immediately concerned in the production of the ova or spermatozoa. As soon as ever these bodies become apparent, they are seen to occupy the entire space between the diverticulum and the walls of the inner vesicle. The observation already mentioned, as made by Dujardin on a *Syncoryne*, in whose free medusoid the ova were produced in the substance of the stomach-walls, would render it probable that in *Cordylophora* also they had their origin in the part homologous, namely, the central diverticulum, and that from this they became liberated at a very early age to undergo subsequent development in the cavity of the capsule; and this view is further supported by what I have myself observed in the fixed egg-bearing organs of *Sertularia argentea*, to be presently referred to.

With the reproductive capsules of *Cordylophora* we may now compare the fixed ovigerous sacs of the marine genera. It will be seen that these also possess a true medusoid structure, though this is perhaps not quite so easily made out in every case as in *Cordylophora*. In the marine *Tubulariadæ*, the sacs in question consist of a closed vesicle composed of cells, and surrounded externally by a delicate structureless capsule; into its bottom projects a large, hollow, cylindrical and contractile organ, whose cavity is in connection with that of the supporting polype, and round which the ova are developed. Now this central organ corresponds exactly to the central diverticulum in the reproductive capsules of *Cordylophora*, and plainly represents the proboscidiform stomach of a Medusa, while the cellular sac is homologous with the disc. It is true there is nothing here which evidently represents the gastro-vascular canals of the Medusæ and the ramified tubes in the capsules of *Cordylophora*, but there is yet quite sufficient essential resemblance to justify us in referring the reproductive capsules of the marine *Tubulariadæ* to the type of medusoid structure, to which we have just seen that those of *Cordylophora* admit of a close comparison.

In the *Campanulariæ*, besides the bodies of obvious medusoid structure which have been noticed by so many observers to escape from the capsule and then swim freely away, and those of less completely developed, though still obvious medusoid structure which remain attached at the mouth of the capsule, others are commonly seen in the

* The medusoids of the marine *Tubulariadæ* are, during the early stages of their development, contained in a delicate external sac which afterwards disappears. This is the homologue of the external investment of the reproductive capsules of *Cordylophora* and the marine *Tubulariadæ*.

interior of the capsule clustering round the central axis, and described by various naturalists as eggs. In these so-called eggs, however, there may be detected a medusoid structure quite as manifest as in the ovigerous sacs of the marine *Tubulariadæ*; each of them is in fact a fixed Medusa developing within it true ova which possess the germinal vesicle and germinal spot, and present the phenomenon of yelk-cleavage, and after a time escape as locomotive ciliated embryos. It is probable however that in some cases the production of spermatozoa instead of ova is the true office of these bodies.

In *Sertularia argentea* I have found the axis developed in the interior of the capsule into a body of very obvious medusoid conformation. In this body the medusoid structure has experienced an advance over that of the reproductive sacs of the *Tubulariadæ*; both mouth and disc are open, and four unbranched gastro-vascular canals are present, but it is permanently fixed, no marginal tentacula are developed, nor could I observe the least motion of the disc corresponding to the systole and diastole of the medusoids of the *Tubulariadæ* and the *Campanulariæ*. It is supported on a short stem, which springs from the bottom of the capsule, and is directly continuous with its stomach. In this stem the ova originate and appear to escape from it into the stomach of the medusoid, to be discharged from the mouth into an external delicate vesicle, where they are retained during subsequent stages of their development.

If the views now taken be correct, the reproductive capsules of *Cordylophora*, and those of the marine *Tubulariadæ*, as well as the parts immediately concerned in the sexual reproduction of the *Sertulariadæ*, must be regarded as distinct zooids*, presenting a more or less degraded type of medusoid structure†.

On investigations into the embryology of the lower polypes, an important influence was exercised by EHRENBERG's determination of the signification of the ovigerous capsules of *Campanularia* and *Coryne*. The existence of these capsules was, as is well known, supposed by the celebrated Berlin micrographist‡ to indicate a distinction of sex in the polypes of a zoophyte, the ordinary terminal polypes being considered by him as non-sexual, while the ovigerous capsules were maintained to be female polypes. This view was ardently adopted by LOVEN§, who, observing

* The introduction of the term *zooid* into the language of zoology is of very recent date. This term is intended to indicate each of the distinct organisms which, with various degrees of independence, express when taken together the total result of the development of a single *ovum*. It is a valuable addition to our terminology, enabling us to avoid the ambiguous sense which attaches itself to the word *individual* when this word is used in its biological signification as the logical element of a *species*. See HUXLEY, observations on Salpa, &c., in Philosophical Transactions, 1851, and Lecture on Animal Individuality, Ann. Nat. Hist. June 1852. See also CARPENTER, Princ. Gen. and Comp. Physiol. p. 906.

† Our knowledge of the egg-bearing and spermatozoa-bearing bodies of *Hydra* is not yet sufficiently accurate to enable us to decide with certainty how far the reproductive organization of this animal should be included in the same type with that of *Cordylophora* and the marine hydroid zoophytes. I believe that indications of a medusoid type may be here also witnessed, but further observations are necessary for the complete elucidation of this point.

‡ Corallenthiere des Rothen Meeres.

§ WIEGMANN's Archiv, 1837.

some of the free medusoids of a *Syncoryne* to be filled with ova, supposed these medusoids to possess the same signification as that which EHRENBERG had attributed to the fixed capsules. STEENSTRUP, adopting with some slight modification the same view, has found in it one of the most striking of the facts which he has so skilfully collated in his essay on "The Alternations of Generations;" while VAN BENEDEN*, in two elaborate memoirs full of beautiful and accurate observations, has strenuously opposed it, maintaining that the ovigerous fixed capsules are merely organs, "ovisacs," while the free medusoids are larvæ destined to undergo a series of transformations during a process of development into the form of the parent zoophyte. M. VAN BENEDEN, however, though he has had the medusoids long under his eyes, has never witnessed an actual development into the form of the parent polype, and the changes which he has figured as occurring in them do not appear to show any tendency to such a transformation. I have had the medusoids of a *Syncoryne* for more than a fortnight under constant inspection without perceiving the slightest change towards the form of the adult polype; at the end of this period they perished.

DUJARDIN† opposes the views both of EHRENBERG and VAN BENEDEN, and taking the medusoids for true *Medusæ*, and viewing them even as distinct genera of *Acalepha*, he generalizes the observed instances of their production, and maintains that the claviform polypes are universally only inferior phases of the development of the *Acalepha*; while he considers the bodies produced in the fixed sacs only as "*bulbilles*," by which he understands gemmæ, which become detached at an early stage, and are then capable of an independent development. The strong resemblance however between true *Medusæ* and the medusoids produced by a process of gemmation from polypes does not afford sufficient grounds for maintaining that all *Medusæ* are only advanced stages of tubularian zoophytes and of *Campanulariæ*, or that all these zoophytes are only earlier phases in the development of *Medusæ*; and M. DUJARDIN's generalization, from which would result the complete abolition of the *Campanulariæ* and tubularian zoophytes as distinct groups of the animal kingdom, is certainly destitute of sufficient foundation, while the bodies contained in the fixed sacs are undoubtedly not *gemmæ* but *ova*, as their structure and the whole history of their development must render evident. I cannot think, then, that either VAN BENEDEN or DUJARDIN has succeeded in overthrowing the theory of EHRENBERG.

The conclusion to which the facts sought to be demonstrated in the present paper would seem to lead, is that in the tubularian zoophytes there exist three kinds of zooids, all produced by a process of gemmation from an original stolon, which is itself the immediate product of a true ovum. These are,—1. the ordinary terminal

* Mém. sur les Campanulaires, and Recherches sur l'Embryogenie des Tubulaires, Nouv. Mém. de l'Acad. Roy. de Bruxelles, t. vii.

† DUJARDIN, Sur le Développement des Méduses et des Polypes Hydraires, Ann des Sci. Nat. 3^{me} sér. t. iv. 1845.

zooids, on which devolves more especially the function of the reception, digestion, and elaboration of the nutrient material; 2. the fixed reproductive zooids, in which every function is rendered subordinate to the production and development of the ova or spermatozoa, and in which a medusoid structure is more or less disguised; and 3. the free medusiform zooids, in which a complete medusoid structure is obvious. The office of these last is less exactly determined than that of either of the others; sexual reproduction is probably, as in the fixed reproductive zooids, their characteristic function; for as we have already seen, they have in several instances been found to contain ova; yet in the greater number of cases no ova have been detected in them, and observations are still wanting to enable us to decide whether at one period or another of their existence a true generative office is not, under favourable circumstances, always performed by them. It is certain that the other functions are not here so completely subordinated to reproduction as in the fixed sacs, for since the medusoid has become entirely detached from the parent, a more elaborate organization is necessary to enable it to maintain an independent existence. Uniting the functions of reproduction with extensive locomotive powers, it is probably destined to carry the ova to a distance from the parent stock, and thus provide for the dispersion of the species, for the motion of the ciliated embryo is too slow to be of any use in this respect. In the *Campanulariæ*, besides these three forms of zooids there exists a fourth, in which the medusoid structure is also obvious, though not so completely developed as in the free medusoids: the bodies belonging to this group remain permanently fixed at the mouth of the capsule; their function is that of true sexual generation, and after giving birth to ova or spermatozoa they wither away. The ovigerous sacs of degraded medusoid structure concealed in the interior of the capsules of *Campanularia* are quite different from these, and belong to the second kind of zooid enumerated above. In *Sertularia* and its immediate allies no locomotive zooids have as yet been found, and the generative zooids of *Sertularia argentea* already described admit of comparison rather with the fixed medusoids attached to the mouth of the capsule in *Campanularia*, than with the ovigerous sacs of the *Tubulariadæ*.

It would seem then that Ehrenberg struck upon the true determination of the ovigerous sacs in *Coryne* when he called them "female polypes," though he erred when, supposing the "ovarian vesicles" of *Campanularia* to be the homologues of the ovigerous sacs of *Coryne*, he called these vesicles also by the same name; the true homologues of the reproductive sacs of the *Tubulariadæ* being, not the external capsules of the *Campanulariæ*, but fixed bodies with a disguised medusoid structure contained in the interior of these capsules*.

The present state of our knowledge of the Tubularian and Sertularian polypes would seem to justify the generalization, that for the production of ova in these

* Krohn has already pointed out that the capsules in *Campanularia* have a signification different from that of the ovigerous sacs of *Coryne*, *l. c.*

polypes a more or less perfectly developed medusoid organization is required, this organization being demonstrable not only in the free medusoids themselves, which are occasionally observed to give origin to ova, but in those reproductive capsules and imperfectly developed medusoids which remain permanently adherent to the parent.

Description of the Plates.

PLATE XXV.

Fig. 1. *Cordylophora lacustris*, attached to a dead valve of *Anodon cygneus*. Natural size.

Fig. 2. A branch magnified with the polypes in various states of expansion, and with the reproductive capsules more or less developed.

Fig. 3. Longitudinal section of polype to show the details of its structure.

- *a.* Ectoderm.
- *b.* Polypary.
- *c.* Processes from the ectoderm attached to inner surface of the polypary.
- *d.* Endoderm.
- *e.* Mouth.
- *f.* Post-buccal cavity.
- *g.* Stomach.
- *h.* Common canal of the cœnosarc.
- *i.* Muscles.

PLATE XXVI.

Fig. 4. Portion of the walls of the stomach more highly magnified.

- *a.* Ectoderm, its cells containing thread-cells.
- *b.* Endoderm composed of elongated cells, with true secreting cells in their interior.
- *c.* Secreting cells with evident nucleus.
- *d.* Secreting cells with the nucleus obscured by the opake contents.
- *e.* Granular mass.
- *f.* Muscles.

Fig. 5. Thread-cells previous to the exsertion of the filament.

Fig. 6. Thread-cells after the exsertion of the filament.

Fig. 7. A group of cells liberated under pressure from the ectoderm. Some contain a single thread-cell, others a nucleus-like body, probably an undeveloped thread-cell. They appear to have been originally contained as secondary cells within the proper cells of the ectoderm.

Fig. 8. Cells liberated by pressure from the endoderm of the stomach. They are filled with smaller cells, which contain coloured granules, and seem to have been originally contained within the proper cells of the endoderm.

a. Nuclei of contained cells distinct.

b. Nuclei concealed by the coloured granules.

Fig. 9. Portion of a tentacle near its root, to show its structure.

a. Ectoderm with thread-cells.

b. Endoderm, its cells containing coloured granules.

c. Muscular fibres.

Fig. 10. Cells containing secondary cells with granular contents from the endoderm, when covered by the polypary.

*** In figs. 11–14 the same letters are used to indicate corresponding portions of structure, viz. *a*. Diverticulum from the cœnosarc. *b*. External investment of the reproductive capsule continued from the polypary. *c*. Cellular sac. *d*. Ramified canals. *e*. Structureless sac secreted on the outside of the cellular sac.

Fig. 11. Reproductive capsule—very early stage.

Fig. 12. The same—more advanced stage; the cellular sac and ramified canals are now visible.

Fig. 13. Ideal longitudinal section of a reproductive capsule of about the same period as fig. 12; showing the relation of the ramified canals to the diverticulum and cellular sac, and the presence of a delicate structureless sac which has been secreted on the outside of the cellular sac.

Fig. 14. Reproductive capsule in a more advanced stage; the ova are now visible in its interior.

Fig. 15. Reproductive capsule still further advanced; the cellular sac and ramified canals have disappeared, and the ova are now seen lying upon the extremity of the diverticulum, and immediately included within the structureless sac which had been formed upon the outside of the cellular sac.

Fig. 16. A more magnified view of an ovum from fig. 15; the germinal vesicle has disappeared, and the segmentation of the vitellus has converted the ovum into a mulberry-like mass.

Fig. 17. A capsule still further advanced; the mulberry-like condition has disappeared, and the ova have begun to elongate themselves.

Fig. 18. A capsule in a still more advanced stage; the ova have become still more elongated, and they now present a swarming motion in the interior of the capsule.

Fig. 19. Termination of the intra-capsular life of the embryo; the enclosing sacs have become ruptured, and the embryos are escaping in the form of free ciliated infusoria.

Fig. 20. Embryo just after its escape from the capsule, more highly magnified; it presents an internal closed cavity with ectoderm and endoderm.

Fig. 21. The same, showing the pyriform figure which it frequently assumes.

Fig. 22. Embryo subsequently to its locomotive stage; the cilia have disappeared, and it has become permanently fixed by one extremity.

Fig. 23. Further progress in the development of the young polype; the tentacula have begun to bud forth, and the stem is already surrounded by a delicate polypary.

Fig. 24. Male capsule with its contents.

a. Capsule with the contents escaping under slight pressure.

b. Caudate cells liberated from the capsule.

c. Spermatozoa.

XVI. *On the Change of Refrangibility of Light.*—No. II. *By* G. G. Stokes, *M.A., F.R.S., Fellow of Pembroke College, and Lucasian Professor of Mathematics in the University of Cambridge.*

Received June 16,—Read June 26, 1853.

THE chief object of the present communication is to describe a mode of observation, which occurred to me after the publication of my former paper, which is so convenient, and at the same time so delicate, as to supersede for many purposes methods requiring the use of sun-light. On account of the easiness of the new method, the cheapness of the small quantity of apparatus required, and above all, on account of its rendering the observer independent of the state of the weather, it might be immediately employed by chemists in discriminating between different substances.

I have taken the present opportunity of mentioning some other matters connected with the subject of these researches. The articles are numbered in continuation of those of the former paper.

Method of observing by the use of Absorbing Media.

241. Conceive that we had the power of producing at will media which should be perfectly opake with regard to rays belonging to any desired regions of the spectrum, from the extreme red to the most refrangible invisible rays, and perfectly transparent with regard to the remainder. Imagine two such media prepared, of which the second was opake with regard to those rays of the *visible* spectrum with regard to which the first was transparent, and *vice versâ*. It is clear that if both media were held in front of the eye no light would be perceived. The same would still be the case if the first medium were removed from the eye, and placed so as to intercept all the rays which fell on certain objects, which were then viewed through the second, provided the objects did nothing more than reflect, refract, scatter, or absorb the incident rays. But if any of the objects had the property of emitting rays of one refrangibility under the influence of rays of another, it might happen that some of the rays so emitted were capable of passing through the second medium, in which case the object would appear luminous in a dark field.

242. Let us consider now how the media must be arranged so as to bring out to the utmost the sensibility of a given substance. To take a particular instance, suppose the substance to be glass coloured by uranium. In this case the sensibility of the medium begins, with almost absolute abruptness, near the fixed line *b* of Fraunhofer, and continues from thence onwards. The dispersed light has the same,

or at least almost rigorously the same, composition throughout, and consists exclusively of rays less refrangible than *b*. Consequently, we should have to prepare a first medium which was opake with regard to the visible rays less refrangible than *b*, and transparent with respect to the rays, whether visible or invisible, more refrangible, and a second medium complementary to the former in the manner described in the preceding article. If the pair of media were still strictly complementary in this manner, but the point of the spectrum at which the transparency of the first medium began and that of the second ended were situated at some distance from *b*, the sensibility of the glass would be exhibited as before, only the maximum effect would not be produced, on account of the absorption of a portion either of the active or of the dispersed rays, according as the point in question was situated above or below *b*.

Now, although the commencement of the sensibility of canary glass is unusually abrupt, it generally happens that the sensibility of a medium, or at least the main part of it, comes on with great rapidity, and lasts throughout the rest of the spectrum, though frequently it is most considerable in a region extending not very greatly beyond the point where it commenced. In those cases in which the dispersion of different tints commenced at two or three different places in the spectrum, I have almost always had evidence of the independent presence of different sensitive principles, to which the observed effects were respectively due.

Hence, if we could prepare absorbing media at pleasure, we should get ready for general use in these observations a few pairs of media complementary in the particular manner already described, but having the points of the spectrum at which the transparency of the first medium commenced and that of the second ended different in different pairs, situated say in the yellow for one pair, in the blue for another, in the extreme violet for a third.

243. It is not of course possible to prepare media in this manner at pleasure, and all we can do is to select from among those which occur in nature. Nevertheless it is useful, as a guide in the selection, to consider what constitutes the ideal perfection of absorbing media for this particular purpose. But before proceeding to mention the media which I have found convenient, I will describe the arrangement which I have adopted for admitting the light.

A hole was cut in the window-shutter of a darkened room, and through this the light of the clouds and external objects entered in all directions. The diameter of the hole was four inches, and it might perhaps have been still larger with advantage. A small shelf, blackened on the top, which could be screwed on to the shutter immediately underneath the hole, served to support the objects to be examined, as well as the first absorbing medium. This, with a few coloured glasses, forms all the apparatus which it is absolutely necessary to employ, though for the sake of some experiments it is well to be provided also with a small tablet of white porcelain, and an ordinary prism, and likewise with one or two vessels for holding fluids.

244. In the observation, the first medium is placed resting on the shelf so as to cover the hole; the object is placed on the shelf immediately in front of the hole; the second medium is held anywhere between the eyes and the object. As it is not possible to obtain media which are strictly complementary, it will happen that a certain quantity of light is capable of passing through both media. This might no doubt be greatly reduced by increasing the absorbing power of the media, but it is by no means advisable to do so to any great extent, because it is important that the second medium should transmit as many as possible of the rays which are of such refrangibilities as to be stopped by the first. Accordingly, it might sometimes be doubtful whether the illumination perceived on the object were due merely to scattered light which had passed through both media, or to really "degraded" light*. To remove all doubt, it is generally sufficient to transfer the second medium from before the eyes to the front of the hole. The light merely scattered by the object will necessarily be the same as before, if the room be free from stray light; and even if there be a little stray light, the illumination, so far as it is concerned, will be increased instead of diminished; whereas if the illumination previously observed were due to fluorescence, and the media were properly chosen, the object which before was luminous will now be comparatively dark.

Sometimes, in the case of substances which have only a low degree of sensibility, it is better to leave the second medium in front of the eyes, and use a third medium, which is held alternately in the path of the rays incident on the object and between the object and the eyes. Such a medium, though not at all necessary, may be used

* This term, which was suggested to me by my friend Prof. THOMSON, appears to me highly significant. The expression *degradation of light* might be substituted with advantage for *true internal dispersion* to designate the general phenomenon; but it is perhaps a little too wide in its signification, and might be taken to include phosphorescence (if indeed in this case the refrangibility be really always lowered), as well as the emission of non-luminous radiant heat by a body which had been exposed to the red rays of the spectrum. As to the term *internal dispersion*, though I employed it, following Sir DAVID BREWSTER, I confess I never liked it. It seems especially awkward when applied to a washed paper or dyed cloth; it was adopted at a time when the phenomenon was confounded with opalescence; and, so far as it implies theoretical notions at all, it seems rather to point to a theory now no longer tenable: I allude to the theory of suspended particles. Indeed, this theory, as it seems to me, ceased to be tenable as soon as Sir JOHN HERSCHEL had discovered the peculiar analysis of light connected with epipolic dispersion, and Sir DAVID BREWSTER had connected the phenomenon with internal dispersion, so far at least as the common appearance of a continuous and coloured dispersed beam formed a connexion. The expression *dependent emission* is awkward, but would be significant, because the light is emitted in the manner of self-luminous bodies, but only in dependence upon the active rays, and so long as the body is under their influence. In this respect the phenomenon differs notably from phosphorescence. It is quite conceivable that a continuous transition may hereafter be traced by experiment from the one phenomenon to the other, but no such transition has yet been traced, nor is it by any means certain that the phenomena are not radically distinct. On this account it would, I conceive, be highly objectionable to call true internal dispersion *phosphorescence*. In my former paper I suggested the term *fluorescence*, to denote the general appearance of a solution of sulphate of quinine and similar media. I have been encouraged to give this expression a wider signification, and henceforth, instead of true internal dispersion, I intend to use the term fluorescence, which is a single word not implying the adoption of any theory.

also in the case of highly sensitive substances, for the sake of varying the experiment and rendering the result more striking.

As it will be convenient to have names for the media fulfilling these different offices, I will call the first medium, or that with which the hole is covered, the *principal absorbent*, the second medium the *complementary absorbent*, and the third medium, when such is employed, the *transfer medium*. For the transfer medium we may choose a medium of the same nature as the complementary absorbent, but paler. This is perhaps the best kind to employ in the methodical examination of various substances; but if the object of the observer be merely to illustrate the phenomena of the change of refrangibility of light, he may vary the experiments by using other media.

245. I have hitherto spoken only of the increase of illumination due to the sensibility of the substance under examination. But independently of illumination, the colour of the emitted light affords in most cases a ready means of detecting fluorescence. Thus, suppose the principal absorbent to transmit no visible rays but deep blue and violet, and the substance examined to appear, when viewed through the complementary absorbent, of a bright orange colour. Since no combination of the rays transmitted by the principal absorbent can make an orange, we may instantly conclude that the substance is sensitive. However, I do not consider it safe, at least for a beginner, to trust very much to *absolute* colour, for few who have not been used to optical experiments can be aware to what an extent the eye under certain circumstances is liable to be deceived. The *relative* colour of two objects seen at the same time under similar circumstances may usually be judged of safely enough; that is, of two such colours it may be possible to say with certainty that one inclines more to blue or to red than the other. Of course in many cases the change of colour is so great that there can be no mistake; still I think it a safe rule for a person employing these modes of observation without having been previously used to optical experiments, to require some other proof of a change of refrangibility than merely a change of colour. Experience will soon show what appearances may safely be relied on.

246. If it be desired to view the object isolated as much as possible, it may be placed directly on the shelf, or better still, on black velvet. But it is generally preferable to have for comparison a standard object which reflects freely the visible rays, of whatever kind, incident upon it, and does not possess any sensible degree of fluorescence. It is in this way that the white porcelain tablet is useful; and in observing, I generally place the tablet on the shelf and the object on it. A white plate would answer, but a tablet is better, on account both of its shape and of the comparative dullness of its surface. It is true that the tablet used exhibited a very sensible amount of fluorescence when examined in a linear spectrum formed by a quartz train; still the effect was so small, and so much of it was due to those highly refrangible rays which are stopped by glass, that for practical purposes the tablet may be regarded as insensible. However, an observer is not obliged either to assume that all tablets are alike, or to apply to the particular tablet which he proposes to use, methods of

observation requiring the use of apparatus which he is not supposed to possess. The methods of observation described in the present paper are complete in themselves; the observer has it in his power to test for himself the tablet he proposes to employ; and he is bound to do so before taking it for a standard of comparison. It may easily be tested by means of a prism, as will be explained presently.

247. The following are the combinations of media which I have chiefly employed:—

First combination.—In this combination the principal absorbent is a glass coloured deep violet by manganese with a little cobalt, or else a glass coloured deep purple by manganese alone, combined with a rather pale blue glass coloured by cobalt, or with a deeper blue glass in case the day be bright. It is very easy to tell by means of a prism, with candle-light, whether a purple glass contains any sensible quantity of cobalt, on account of the very peculiar mode of absorption which is characteristic of this metal. In the examination of substances by this combination no complementary absorbent is usually required; but if it be wished to employ one, a pale yellow glass, of the kind mentioned in connexion with the next combination, may be used.

Second combination.—In this case the principal absorbent is a solution of the ammoniaco-sulphate of copper, employed in such thickness as to give a deep blue. In my experiments the fluid was contained in a cell with parallel sides of glass, which was closed at the top for greater convenience; but a very broad flat bottle would answer as well, because in the case of the principal absorbent the regularity of refraction of rays across it is of no consequence. Such a bottle, however, would have to be ordered expressly. The complementary absorbent in this combination is a yellow glass coloured by silver, and slightly overburnt. These glasses, as commonly prepared, are opake with regard to most of the violet, but become transparent again with regard to the invisible rays beyond; and, in the case of a pale glass, the commencing transparency in the extreme violet may even be perceived by means of light received directly into the eye. I have got a glass of a pretty deep orange-yellow colour, which is more transparent than common window-glass with regard to rays of such high refrangibility as to be situated near the end of the region of the solar spectrum which it requires a quartz train to show. But when too much heat is used in the preparation, the glass acquires, on the interior of the coloured face, a delicate blue appearance, having a good deal of the aspect of a solution of sulphate of quinine, though it has in reality nothing to do with fluorescence; and in this state the glass is nearly opake with regard to the invisible rays of the solar spectrum beyond the violet, though it still transmits a few among those which are nearly the most refrangible. Of course, if the complementary absorbent were always left in its position between the eyes and the object, its transparency or opacity with regard to invisible rays would be a matter of indifference; but as it is desirable that its transference from that position to the front of the hole should produce as much difference as possible, it is important that it should be opake, or nearly so, with regard to the ultra-violet rays transmitted by the principal absorbent. Hence one of these slightly over-burnt glasses should be selected for the present purpose, and such are very commonly met with.

THIRD COMBINATION.—In this case everything is the same as in the preceding combination, except that the fluid is replaced by a glass of a pretty deep blue, coloured by cobalt.

FOURTH COMBINATION.—In this the principal absorbent is a solution of nitrate of copper, and the complementary absorbent a light red or deep orange glass.

248. In the first combination the darkness is tolerably complete without the use of any complementary absorbent, since no visible rays are transmitted except violet and some extreme red. The latter are no inconvenience, but rather help to set off the tint of the light due to fluorescence. This is, I think, the best combination to employ when the fluorescent light is blue, or at least deep blue; because in that case much of the light is lost by absorption in the yellow glass employed in the second combination. It has the advantage, too, of allowing the fluorescent light to enter the eye without being modified by absorption. Nevertheless no correct estimate can be formed of the absolute colour of the fluorescent light without making very great allowance for the effects of contrast, especially when the body, instead of being isolated as far as possible, is placed on the porcelain tablet.

The second combination is on the whole the most powerful. The media in this case make a very fair approach towards the ideal perfection explained in Art. 242. The darkness is so far complete, or else may easily be made so by increasing a little the strength of either absorbent, that if the tablet be written on with ink, and placed on the shelf between the media, the writing cannot be read. It forms a striking experiment, after having treated the tablet in this manner, to introduce between the media a piece of canary glass or a similar medium. The glass is not only luminous itself, but it emits so much light as to illuminate the whole tablet, so that the writing is instantly visible. In those cases in which the fluorescent light is yellow, orange, or red, it is shown a good deal more strongly by this combination than by the preceding.

The third combination is applicable to the same cases as the second. The blue glass answers extremely well, but is not quite so good as the blue fluid. The darkness is less complete, on account of the red and yellow transmitted by the glass. Nevertheless this combination is sometimes useful in observing with a prism, and at any rate it may very well be employed by a person who does not happen to have a vessel of the proper shape for holding the fluid.

In the second and third combinations the point of the spectrum at which the transparency of the principal absorbent begins, and that of the complementary absorbent ends, or rather the point which most nearly possesses this character, is situated in the blue. Thinking that the fluorescence of those substances which emit light of low refrangibility might be better brought out if this point were situated lower down in the spectrum, I tried the fourth combination. In this case the media have very fairly the required complementary character; the darkness is pretty complete, and the fluorescence of scarlet cloth and similar substances is very well exhibited. However, the effect in these cases is shown so well by the second combination, that, except it be for the sake of varying the experiment, I do not think it worth while to employ the fourth combination, more especially as it has the disadvantage of leaving the observer in doubt whether the red or orange light perceived constitutes the whole of the fluorescent light, or only that part of it which alone has been able to get through the complementary absorbent.

249. The mode of observation may be altered in various ways which afford pleasing illustrations of the theory, though in the regular examination of a set of substances it is best to proceed in a more methodical manner. Thus, if nothing but a violet or blue glass or a blue fluid be used as a principal absorbent, and the substances under examination be highly sensitive, their appearance will be remarkably changed if the coloured medium be transferred from before the hole to before the eyes. Again, if the complementary absorbent be made to exchange places with the principal absorbent, the result will be similar, although the very same media are merely interposed in different parts of the compound path of the light from the clouds to the eye. If a transfer medium be employed, and it be, as has hitherto been supposed, of the same general nature as the complementary absorbent, it will not produce much effect when it is interposed between the object and the eyes, but when it is placed in the path of the rays incident on the object, the fluorescent light will be nearly if not entirely cut off. If, however, we take for a transfer medium a glass or fluid having the same general character as the principal absorbent, the effect will be just the reverse. This is strikingly shown in the case of a substance, which, like scarlet cloth, emits a red fluorescent light, by taking for a transfer medium a solution of nitrate of copper, and in the case of turmeric paper or yellow uranite, by taking the same solution, or else a blue glass. In the case of the two substances last mentioned, if we take for a transfer medium a red solution of mineral chameleon, diluted so as to be merely pink, the intensity of the light emitted will, under certain conditions, be not much different in the two positions of the medium, because a portion of the active rays in one position and a portion of the degraded rays in the other will be absorbed; but the colour of the portion of the emitted light which reaches the eye will be altogether different in the two positions of the transfer medium.

Mode of observing by means of a Prism.

250. In this method no absorbing medium is required except the principal absorbent. The white tablet being laid on the shelf, a slit is first held in such a position as to be seen projected against the sky, and the light thus coming directly into the eye, after having passed through the principal absorbent, is analysed by a prism held in the other hand. The slit is now held so that the tablet is seen through it, and the light coming from the tablet is analysed. It will be found that the spectrum seen in the first instance is faithfully reproduced, being merely less lumi-

nous, as must necessarily happen. At least, this was the case in those tablets which I have examined; and in this way each observer ought to test for himself the tablet he proposes to employ. After having been thus tested, the tablet may be used as a standard of comparison.

Suppose now that it is wished to examine a slip of turmeric paper, or a riband, or other similar object. The object is laid on the tablet, and the slit held immediately in front of it, in such a manner that one part, suppose the central portion, of the slit is seen projected on the object, and the remainder on the tablet. The light coming through the slit is then analysed by the prism, and the fluorescence, if any, of the object is indicated by light appearing in those regions of the spectrum in which, in the case of the light scattered by the tablet, there is nothing but darkness.

Occasionally in these observations a blue glass is preferable to a solution of the ammoniaco-sulphate of copper, because the extreme red and the greenish yellow bands transmitted by the glass, while too faint to interfere with the fluorescent light, are useful as points of reference.

Fig. 1.

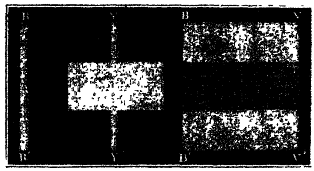
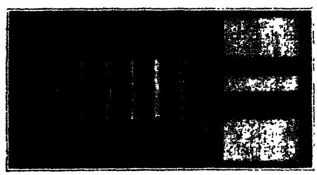


Fig. 2.



251. The general appearance of the spectrum in this mode of observation may be gathered from the accompanying figures, of which the first represents turmeric paper seen under the blue glass, and the second represents a mass of crystals of nitrate of uranium seen under the copper solution. In fig. 1, RR′, YY′ are the red and yellow bands transmitted by the glass, which are seen equally in the light scattered by the tablet and that scattered by the paper. BVB′V′ is the blue and violet light transmitted by the glass. Of this a considerable portion, especially in the more refrangible part, is absorbed by the turmeric paper, which on the other hand emits a quantity of red, yellow, and green light, not found among the incident rays. Fig. 2 sufficiently explains itself. In this case the fluorescent light is decomposed by the prism into bright bands, of which six may be readily made out. No blue or violet light enters the eye from the part of the slit which is seen projected on the mass of crystals, except where a crystalline face happens to be situated in such a position as to reflect the light of the sky into the eye, as represented in the figure. In the case of a substance so highly sensitive as nitrate of uranium, and which does not, like a slip of paper, lie flat on the tablet, the spectrum of the fluorescent light in reality extends, at least on the side next the window, though with less intensity, to some distance beyond the part of the slit which corresponds to the object, because the tablet is lighted up by the rays emitted by the object; but this is not represented in the figure.

252. The mode of using the prism just explained is that by which the phenomenon

of the change of refrangibility is most strikingly illustrated; but in the actual examination of substances the chief use of the prism is to determine, in the case of substances which are sufficiently sensitive to admit with advantage of such a mode of observation, the composition of the fluorescent light. For this purpose it is often better to isolate the object by placing it on black velvet. This is especially the case with very minute crystals, or other objects, which are best placed on black velvet, and viewed through the prism as a whole, no slit being required.

Examples of the application of the preceding methods of observation.

253. The peculiar properties of paper washed with tincture of turmeric or stramonium seeds, of yellow uranite, and other highly sensitive substances, come out in a remarkable manner under the modes of examination described in this paper. I need not say that such is the case with solutions of sulphate of quinine, or horse-chestnut bark, or other clear and highly sensitive media, seeing that in this case the appearance due to fluorescence is obvious to common observation. If a piece of horse-chestnut bark be put to float in a glass of water close to the hole covered by the principal absorbent, the appearance of the descending streams of solution of esculine is very singular and beautiful. My present object is however rather to illustrate the power of these methods by their application to substances which stand much lower in the scale.

By the use of absorbing media alone, as well by a principal absorbent and a prism, I have been able to detect without difficulty the sensibility of white paper on a day of continuous clouds and rain. Even cotton wool, which stands very much lower in the scale, is shown by the use of absorbing media with ordinary daylight to be sensitive. In the case of such substances as bone, ivory, white leather, the white part of a quill, which stand much higher in the scale, the most inexperienced observer could hardly fail instantly to detect the fluorescence. All plates of colourless glass which I have examined, and other pieces which were of such a shape as to admit of being looked into edgeways to a considerable depth, were found by the second combination to be sensitive. Crystals of sulphate of quinine, which may be readily prepared from the disulphate of commerce, show their fluorescence extremely well by the first combination. These crystals are much less sensitive than their solution, and the light which they emit is of a much deeper blue. It must in reality be of a very deep blue colour: for it nearly matches the fluorescent light of fluor-spar, although when the crystals are viewed under the violet glass the tint in both cases appears comparatively pale, from contrast with the violet. A solution of nitrate of uranium on the other hand has only a low degree of sensibility compared with the crystals of that salt. If a drop of the solution be placed on the porcelain tablet when the hole is covered with the deep blue copper solution, it appears comparatively dark, because much more illumination is lost by the absorption of the indigo and violet than is gained by the fluorescence of the solution. But when the tablet is viewed through the complementary absorbent, the solution is seen to be more luminous than the tablet, and to emit yellow rays, which are not found in the incident light.

The reactions of quinine mentioned in my former paper (Arts. 205–208), may very conveniently be observed by means of drops of the solution placed on the tablet; and in this way it is possible to work in a perfectly satisfactory manner with excessively minute quantities of quinine. The statement there made, that the blue colour was *destroyed* by hydrochloric acid, &c., must be understood only with reference to the mode of observation there supposed to be adopted, which was sufficient for the object in view. When the solutions are examined in a pure spectrum formed by sunlight, or even by the method described in the present paper, it is seen that the blue colour is not *absolutely* destroyed by hydrochloric acid, and is even developed to a *slight* extent on the addition of hydrochloric acid to a previously alkaline solution. Still there is a broad distinction between the two classes of solutions, which is all that is required. I have since extended a good deal these results, and mean to pursue the subject further. Meanwhile I may be permitted to correct an error in Art. 205, relative to the effect of hydrocyanic acid, which was there stated to develope the blue colour. The experiment was made with the acid of commerce, containing a foreign acid, to which the effect was probably due.

Comparison of the relative advantages of different modes of observation.

254. At first sight it might have been supposed that daylight could never be more than a poor substitute for sunlight in any observations relating to fluorescence. Such, however, I consider to be by no means the case. In the first place, when sunlight is used it is made to enter a room in a definite direction; whereas in using absorbing media all the rays are employed whose directions lie within a solid angle having the object examined for vertex, and the hole for base. If we leave out the part of this solid angle which corresponds to trees or houses, the part which corresponds to sky will still be so large as to make up in a good measure for the superior brilliancy of the light of the sun. In the second place, stray light is much more perfectly excluded than when a beam of sunlight, containing rays of all kinds, is admitted into a room. When indeed the use of sunlight is combined with that of absorbing media, it is possible to detect very minute degrees of sensibility. Still for general purposes I consider the methods depending upon the use of absorbing media with ordinary daylight quite comparable with, if not equal to, those methods involving the use of sunlight which are applicable to opake bodies; I allude especially to the method of a linear spectrum. The peculiarities in the composition of the fluorescent light, when such exist, can be made out about equally well by both methods.

But when the substance to be investigated is a solution, or a clear solid of sufficient size to be examined as such, methods of observation can be put in practice with sunlight which surpass anything that can be done merely by the use of absorbing media. In consequence of the absence of stray light, which would otherwise dazzle the eye, an amount of concentration of the rays can be brought to bear on the object, which enables the observer to detect excessively minute degrees of sensibility. Thus, when the sun's light is condensed by a rather large lens, and made to pass through a strong solution of the ammoniaco-sulphate of copper, the condensed beam of violet and invisible rays serves to detect fluorescence in almost all fluids. This, however, is no great advantage; the method is in fact too powerful; and the observer is left in doubt whether the effect perceived be due to the fluid deemed to be examined, or to some impurity which it contains in an amount otherwise perhaps inappreciable. The great advantage which sunlight observations possess in the examination of substances, which, however, is only applicable to clear media, is that they enable the observer to make out the distribution of activity in the incident spectrum. In some cases this constitutes the chief peculiarity in the mode of fluorescence of a particular substance; in other cases it enables the observer to see, as it were, independently of each other, different sensitive substances existing together in solution. Another advantage of sunlight, which applies equally to clear and to opake media, is that it enables the observer, with the assistance of a quartz train, to make out fluorescence which does not commence till that region of the spectrum which it requires a quartz train to show. But such cases are too rare to render this a point of much consequence. Of course there are observations such as those which relate to the fixed lines of the invisible rays, or to the determination of the absorbing action of a medium with regard to invisible rays of each degree of refrangibility in particular, which imperatively require sunlight: I am speaking at present only with reference to those observations of which the object is to investigate the mode of fluorescence of a particular substance.

As to the method of observation in which a prism is combined with a principal absorbent, its chief use is to determine, in the case of the more sensitive substances, the composition of the fluorescent light. It is not generally so convenient as the method which involves the use of absorbing media alone for determining which among a group of objects are sensitive, and which not, especially when the objects are minute.

255. Although the description of the mode of observing by means of absorbing media has run to some little length, the reader must not suppose that the observations are at all difficult. Of course observations of all kinds become more or less difficult when they are pushed to the extreme limit of refinement of which they are susceptible. But in the case of substances which are at all highly sensitive, and this comprises almost all the more interesting instances, the observations are extremely easy. I have spoken of a darkened room, which is certainly the most convenient when it can be had. But I have no doubt that an observer who could not procure such might easily arrange for himself a darkened box, which would answer the purpose. Indeed the fluorescence of highly sensitive substances, though they be opake, may be exhibited by means of absorbing media in broad daylight.

Platinocyanides.

256. In the Report of the Twentieth Meeting of the British Association (Edinburgh, 1850, Transactions of the Sections, p. 5), is a notice by Sir David Brewster

of "The Optical Properties of the Cyanurets of Platinum and Magnesia, and of Barytes and Platinum," salts which he had received from M. HAIDINGER of Vienna. The notice is chiefly devoted to the properties of the reflected light; but with respect to the latter of the salts, Sir DAVID remarks that "it possesses the property of internal dispersion, the dispersed light being a *brilliant green*, while the transmitted light is *yellow*." Although the distinction between true internal dispersion and opalescence was not at the time understood, there could be little doubt from the nature of the case that the internal dispersion mentioned by Sir DAVID BREWSTER was, in fact, an instance of the former of these phenomena; but I could not try for want of a specimen of the salt. Some months ago I received from M. HAIDINGER a specimen of the first of the salts mentioned at the beginning of this paragraph, namely M. QUADRAT's cyanide of platinum and magnesium, a salt of great optical interest on account of the remarkable metallic reflexion which it exhibits. On examining the salt, I was greatly interested by finding that it was highly sensitive, the fluorescent light being red. This induced me to form some of GMELIN's platinocyanide of potassium, and I found that the blue light which this salt exhibits in certain aspects is, in fact, due to fluorescence, a property which the salt possesses in an eminent degree. Having afterwards received some of the same salt pure from Professor GREGORY, I applied it to the formation, on a small scale, of the platinocyanides of calcium, barium, strontium, and two or three others. The three salts last named, of which the second is that mentioned by Sir DAVID BREWSTER, are all eminently sensitive, the fluorescent light being of different shades of green. It is only in the solid state that the platinocyanides are sensitive; their solutions look like mere water. The precipitates which a solution of platinocyanide of potassium gives with salts of the heavy metals are, in most cases that I have yet observed, insensible. With a solution of pernitrate of mercury, however, a bright yellow precipitate is produced which is exceedingly sensitive, so as to look brighter than even yellow uranite. The light emitted is yellow. It forms a very striking experiment to place side by side on the tablet with the second combination a drop of a solution of platinocyanide of potassium, and another of pernitrate of mercury. The drops look like water on the dark field of view; but when they are mixed, a precipitate is produced which glows like a self-luminous body with a yellow light. The precipitate with nitrate of silver is also sensitive, though not in so high a degree. The platinocyanides are extremely interesting; first, as forming a third case, or rather class of cases, in which the property of fluorescence is attached to substances chemically isolated in a satisfactory manner (though I believe chemists are acquainted with a few other organic compounds to which the property belongs), the other two cases being salts of quinine and of peroxide of uranium; and secondly, as exhibiting a new and remarkable feature, which consists in the polarization of the fluorescent light. I content myself at present with this notice; the salts require a more extended study.

XVII. *On the Secular Variation of the Moon's Mean Motion.*

By J. C. Adams, *Esq., M.A., F.R.S. &c.*

Received and Read June 16, 1853.

1. In treating a great problem of approximation, such as that presented to us by the investigation of the moon's motion, experience shows that nothing is more easy than to neglect, as insignificant, considerations which ultimately prove to be of the greatest importance. One instance of this occurs with reference to the secular acceleration of the moon's mean motion. Although this acceleration, and the diminution of the eccentricity of the earth's orbit, on which it depends, had been made known by observation as separate facts, yet many of the first geometers altogether failed to trace any connexion between them, and it was only after making repeated attempts to explain the phenomenon by other means, that Laplace himself succeeded in referring it to its true cause.

2. The accurate determination of the amount of the acceleration is a matter of very great importance. The effect of an error in any of the periodic inequalities upon the moon's place, is always confined within certain limits, and takes place alternately in opposite directions within very moderate intervals of time, whereas the effect of an error in the acceleration goes on increasing for an almost indefinite period, so that the calculation of the moon's place for a very distant epoch, such as that of the eclipse of Thales, may be seriously vitiated by it.

In the Mécanique Céleste, the approximation to the value of the acceleration is confined to the principal term, but in the theories of Damoiseau and Plana the developments are carried to an immense extent, particularly in the latter, where the multiplier of the change in the square of the eccentricity of the earth's orbit, which occurs in the expression of the secular acceleration, is developed to terms of the seventh order.

As these theories agree in principle, and only differ slightly in the numerical value which they assign to the acceleration, and as they passed under the examination of Laplace, with especial reference to this subject, it might be supposed that at most only some small numerical corrections would be required in order to obtain a very exact determination of the amount of this acceleration.

It has therefore not been without some surprise, that I have lately found that Laplace's explanation of the phenomenon in question is essentially incomplete, and that the numerical results of Damoiseau's and Plana's theories, with reference to it, consequently require to be very sensibly altered.

3. Laplace's explanation may be briefly stated as follows. He shows that the mean central disturbing force of the sun, by which the moon's gravity towards the earth is diminished, depends not only on the sun's mean distance, but also on the eccentricity of the earth's orbit. Now this eccentricity is at present, and for many ages has been, diminishing, while the mean distance remains unaltered. In consequence of this the mean disturbing force is also diminishing, and therefore the moon's gravity towards the earth at a given distance is, on the whole, increasing. Also, the area described in a given time by the moon about the earth is not affected by this alteration of the central force; whence it readily follows that the moon's mean distance from the earth will be diminished in the same ratio as the force at a given distance is increased, and that the mean angular motion will be increased in double the same ratio.

4. This is the main principle of Laplace's analytical method, in which he is followed by Damoiseau and Plana; but it will be observed, that this reasoning supposes that the area described by the moon in a given time is not permanently altered, or in other words, that the tangential disturbing force produces no permanent effect. On examination, however, it will be found that this is not strictly true, and I will endeavour briefly to point out the manner in which the inequalities of the moon's motion are modified by a gradual change of the central disturbing force, so as to give rise to such an alteration of the areal velocity.

As an example, I will take the *Variation*, the most direct effect of the disturbing force.

In the ordinary theory, the orbit of the moon as affected by this inequality only, would be symmetrical with respect to the line of conjunction with the sun, and the areal velocity generated while the moon was moving from quadrature to syzygy, would be exactly destroyed while it was moving from syzygy to quadrature, so that no permanent alteration of areal velocity would be produced.

In reality, however, the magnitude of the disturbing force by which this inequality is caused, depends in some degree on the eccentricity of the earth's orbit, and as this is continually diminishing, the central disturbing forces at equal angular distances on opposite sides of conjunction will not be exactly equal. Hence the orbit will no longer be symmetrically situated with respect to the line of conjunction. Now the change of areal velocity produced by the tangential force at any point, depends partly on the value of the radius vector at that point, and consequently the effects of the tangential force before and after conjunction will no longer exactly balance each other.

The other inequalities of the moon's motion will be similarly modified, especially those which depend, more directly, on the eccentricity of the earth's orbit, so that each of them gives rise to an uncompensated change of the areal velocity.

Since the distortion in the form of the orbit just pointed out is due to the alteration of the disturbing force consequent upon a change in the eccentricity of the earth's orbit, and it is by virtue of this distortion that the tangential force produces a permanent change in the rate of description of areas, it follows that this alteration of the areal velocity will be of the order of the square of the disturbing force multiplied by the rate of change of the earth's eccentricity.

It is evident that the amount of the acceleration of the moon's mean motion will be directly affected by this alteration of areal velocity.

5. Having thus briefly indicated the way in which the effect now treated of originates, I will proceed with the analytical investigation of its amount.

In the present communication, however, I shall confine my attention to the principal term of the change thus produced in the acceleration of the moon's motion, deferring to another, though I hope not a distant, opportunity, the fuller development of this subject, as well as the consideration of the secular variations of the other elements of the moon's orbit arising from the same cause.

In what follows, the notation, except when otherwise explained, is the same as that of DAMOISEAU'S "Théorie de la Lune."

6. If we suppose the moon to move in the plane of the ecliptic, and also neglect the terms depending on the sun's parallax, the differential equations of the moon's motion become

$$0=\frac{d^2u}{dv^2}+u-\frac{1}{h^2}+\frac{m'u'^3}{2h^2u^3}+\frac{3}{2}\frac{m'u'^3}{h^2u^3}\cos(2v-2v')$$

$$-\frac{3}{2}\frac{m'u'^3}{h^2u^4}\frac{du}{dv}\sin(2v-2v')-\frac{3m'}{h^2}\left(u+\frac{d^2u}{dv^2}\right)\int\frac{u'^3dv}{u^4}\sin(2v-2v')$$

$$\frac{dt}{dv}=\frac{1}{hu^2}+\frac{3}{2}\frac{m'}{h^3u^2}\int\frac{u'^3dv}{u^4}\sin(2v-2v')+\frac{27}{8}\frac{m'^2}{h^5u^2}\left[\int\frac{u'^3dv}{u^4}\sin(2v-2v')\right]^2.$$

In the solution usually given of these equations, u is expressed by means of a constant part and a series involving *cosines* of angles composed of multiples of $2v-2mv$, $cv-\varpi$, and $c'mv-\varpi'$; also t is expressed by means of a part proportional to v and a series involving *sines* of the same angles; the coefficients of the periodic terms being functions of m, e and e'. Now if e' be a constant quantity, this is the true form of the solution, but if e' be variable, it is impossible to satisfy the differential equations without adding to the expression for u a series of small supplementary terms depending on the *sines* of the angles whose cosines are already involved in it, and to that for t, similar terms depending on the *cosines* of the same angles, the coefficients of these new terms involving $\frac{de'}{dt}$ as a factor.

The quantity $\int\frac{u'^3dv}{u^4}\sin(2v-2v')$, which occurs in the above equations, is proportional to the variable part of the square of the areal velocity, and consists, in the ordinary theory, of a series of periodic terms involving *cosines* of the angles above mentioned. In consequence, however, of the existence of the new terms just described, there will be added to it a series of small terms involving *sines* of the same angles, together with a non-periodic part of the form $\int He'de'$ or $\frac{1}{2}He'^2$. The introduction of this term will evidently change the relation between the non-periodic part of $\frac{dt}{dv}$ and e'^2, upon which the secular acceleration depends.

7. We must commence by finding the new terms to be added to the ordinary expression for u.

For the sake of simplification we will neglect the eccentricity of the moon's orbit.

Let $\frac{1}{a}$ denote the non-periodic part of u, and $\frac{1}{a}+\delta u$ the complete value.

Then by substitution in the equation for u, making use of DAMOISEAU's developments of the undisturbed values of the several functions of u, u', and $v-v'$ which occur in it, putting $h^2=a_{,}$, and writing, for convenience, mv instead of $\int mdv+\lambda$, and $c'mv$ instead of $c'\int mdv+\lambda-\varpi'$ (as in PLANA, vol. i. p. 322), we obtain

$$
\begin{aligned}
0=&\frac{d^2\left(\frac{1}{a}\right)}{dv^2}+\frac{1}{a}-\frac{1}{a_{,}}+\frac{d^2\delta u}{dv^2}+\delta u \\
&+\frac{1}{2}\frac{\overline{m}^2}{a_{,}}\left(1+\frac{3}{2}e'^2\right)+\frac{3}{2}\frac{\overline{m}^2}{a_{,}}a'\delta u'+\frac{3}{2}\frac{\overline{m}^2}{a_{,}}e'\cos c'mv-\frac{3}{2}\frac{\overline{m}^2}{a_{,}}\{1+3e'\cos c'mv\}a\delta u \\
&-\frac{3}{2}\overline{m}^2\frac{a}{a_{,}}\frac{d\left(\frac{1}{a}\right)}{dv}\sin(2v-2mv)+\frac{3}{2}\frac{\overline{m}^2}{a_{,}}\left(1-\frac{5}{2}e'^2\right)\cos(2v-2mv) \\
&+\frac{21}{4}\frac{\overline{m}^2}{a_{,}}e'\cos(2v-2mv-c'mv)-\frac{3}{4}\frac{\overline{m}^2}{a_{,}}e'\cos(2v-2mv+c'mv) \\
&-\frac{3\overline{m}^2}{a_{,}}\int dv\left\{\left(1-\frac{5}{2}e'^2\right)\sin(2v-2mv)+\frac{7}{2}e'\sin(2v-2mv-c'mv)\right. \\
&\qquad\qquad\left.-\frac{1}{2}e'\sin(2v-2mv+c'mv)\right\} \\
&-\frac{9}{2}\frac{\overline{m}^2}{a_{,}}\left\{\left(1-\frac{5}{2}e'^2\right)\cos(2v-2mv)+\frac{7}{2}e'\cos(2v-2mv-c'mv)\right. \\
&\qquad\qquad\left.-\frac{1}{2}e'\cos(2v-2mv+c'mv)\right\}a\delta u \\
&-\frac{3}{2}\frac{\overline{m}^2}{a_{,}}\left\{\left(1-\frac{5}{2}e'^2\right)\sin(2v-2mv)+\frac{7}{2}e'\sin(2v-2mv-c'mv)\right. \\
&\qquad\qquad\left.-\frac{1}{2}e'\sin(2v-2mv+c'mv)\right\}\frac{d(a\delta u)}{dv} \\
&+12\frac{\overline{m}^2}{a_{,}}\int dv\left\{\left(1-\frac{5}{2}e'^2\right)\sin(2v-2mv)+\frac{7}{2}e'\sin(2v-2mv-c'mv)\right. \\
&\qquad\qquad\left.-\frac{1}{2}e'\sin(2v-2mv+c'mv)\right\}a\delta u \\
&-\frac{3\overline{m}^2}{a_{,}}\left\{\frac{d^2(a\delta u)}{dv^2}+a\delta u\right\}\int dv\left\{\left(1-\frac{5}{2}e'^2\right)\sin(2v-2mv)+\frac{7}{2}e'\sin(2v-2mv-c'mv)\right. \\
&\qquad\qquad\left.-\frac{1}{2}e'\sin(2v-2mv+c'mv)\right\}.
\end{aligned}
$$

8. **Also, assume**

$$a\delta u = m^2\left(1-\frac{5}{2}e'^2\right)\cos(2\nu-2m\nu)+a_{30}\frac{e'de'}{ndt}\sin(2\nu-2m\nu)$$

$$-\frac{3}{2}m^2e'\cos c'm\nu+a_{16}\frac{de'}{ndt}\sin c'm\nu$$

$$+\frac{7}{2}m^2e'\cos(2\nu-2m\nu-c'm\nu)+a_{33}\frac{de'}{ndt}\sin(2\nu-2m\nu-c'm\nu)$$

$$-\frac{1}{2}m^2e'\cos(2\nu-2m\nu+c'm\nu)+a_{34}\frac{de'}{ndt}\sin(2\nu-2m\nu+c'm\nu),$$

where the coefficients of the terms involving cosines are those given by the ordinary theory, and a_{30}, a_{16}, a_{33}, and a_{34} are numerical quantities to be determined.

9. In developing the terms of the above equation, by the substitution of this value of $a\delta u$, the quantity $\frac{de'}{dt}$ may be considered constant, and $\frac{de'}{d\nu}$ must be expressed in terms of it.

Thus $\frac{de'}{d\nu}=\frac{ndt}{d\nu}\,\frac{de'}{ndt}$

$$=\frac{de'}{ndt}\left\{1-\frac{11}{4}m^2\cos(2\nu-2m\nu)-\frac{77}{8}m^2e'\cos(2\nu-2m\nu-c'm\nu)\right.$$
$$\left.+\frac{11}{8}m^2e'\cos(2\nu-2m\nu+c'm\nu)\right\}.$$

Also, integrating by parts, and putting 2 instead of $2-2m$, $2-3m$, and $2-m$ in the divisors introduced by integration, since we only want to find the terms of the lowest order which are multiplied by $\frac{de'}{dt}$, we obtain

$$-\frac{3\overline{m}^2}{a_{\prime}}\int d\nu\left\{\left(1-\frac{5}{2}e'^2\right)\sin(2\nu-2m\nu)+\frac{7}{2}e'\sin(2\nu-2m\nu-c'm\nu)\right.$$
$$\left.-\frac{1}{2}e'\sin(2\nu-2m\nu+c'm\nu)\right\}$$

$$=\frac{3}{2}\frac{\overline{m}^2}{a_{\prime}}\left(1-\frac{5}{2}e'^2\right)\cos(2\nu-2m\nu)+\frac{21}{4}\frac{\overline{m}^2}{a_{\prime}}e'\cos(2\nu-2m\nu-c'm\nu)$$
$$-\frac{3}{4}\frac{\overline{m}^2}{a_{\prime}}e'\cos(2\nu-2m\nu+c'm\nu)$$
$$+\frac{15}{2}\frac{\overline{m}^2}{a_{\prime}}\int d\nu\frac{e'de'}{ndt}\frac{ndt}{d\nu}\cos(2\nu-2m\nu)-\frac{21}{4}\frac{\overline{m}^2}{a_{\prime}}\int d\nu\frac{de'}{ndt}\frac{ndt}{d\nu}\cos(2\nu-2m\nu-c'm\nu)$$
$$+\frac{3}{4}\frac{\overline{m}^2}{a_{\prime}}\int d\nu\frac{de'}{ndt}\frac{ndt}{d\nu}\cos(2\nu-2m\nu+c'm\nu).$$

And $\quad a'\delta u'=3m^2e'\sin c'm\nu[-e'\sin c'm\nu]$

$$=-\frac{3}{2}m^2e'^2,$$

retaining only the term which will be required.

10. When the proper substitutions are made, the terms involving cosines destroy each other, as in the usual theory, and by equating to zero the terms involving the sines, we obtain

$$20m^2-3a_{30}+\frac{15}{4}m^2=0,$$

or

$$3a_{30}=\frac{95}{4}m^2 \quad \therefore a_{30}=\frac{95}{12}m^2$$

$$3m^3+a_{16}=0 \quad \therefore a_{16}=-3m^3$$

$$-14m^2-3a_{33}-\frac{21}{8}m^2=0,$$

or

$$3a_{33}=-\frac{133}{8}m^2 \quad \therefore a_{33}=-\frac{133}{24}m^2$$

$$2m^2-3a_{34}+\frac{3}{8}m^2=0,$$

or

$$3a_{34}=\frac{19}{8}m^2 \quad \therefore a_{34}=\frac{19}{24}m^2.$$

11. In order to obtain the relation between a and $a_{\prime}$, we must substitute the value just found for $a\delta u$, in the same equation, and equate to zero the non-periodic part, observing that the terms

$$12\frac{\overline{m}^2}{a_{\prime}}\int dv\left\{(1-\frac{5}{2}e'^2)\sin(2v-2mv)+\frac{7}{2}e'\sin(2v-2mv-c'mv)\right.$$
$$\left.-\frac{1}{2}e'\sin(2v-2mv+c'mv)\right\}a\delta u$$

give

$$\frac{12\overline{m}^2}{a_{\prime}}\int dv\left\{\frac{95}{24}m^2\frac{e'de'}{ndt}-\frac{931}{96}m^2\frac{e'de'}{ndt}-\frac{19}{96}m^2\frac{e'de'}{ndt}\right\}$$
$$=-\frac{285}{4}\frac{m^4}{a_{\prime}}\int ndt\frac{e'de'}{ndt} \text{ nearly,}$$
$$=-\frac{285}{8}\frac{m^4}{a_{\prime}}e'^2 \text{ as their non-periodic part.}$$

Also the terms

$$\frac{15}{2}\frac{\overline{m}^2}{a_{\prime}}\int dv\frac{e'de'}{ndt}\frac{ndt}{dv}\cos(2v-2mv)-\frac{21}{4}\frac{\overline{m}^2}{a_{\prime}}\int dv\frac{de'}{ndt}\frac{ndt}{dv}\cos(2v-2mv-c'mv)$$
$$+\frac{3}{4}\frac{\overline{m}^2}{a_{\prime}}\int dv\frac{de'}{ndt}\frac{ndt}{dv}\cos(2v-2mv+c'mv)$$

of art. 9. similarly give

$$\frac{15}{2}\frac{\overline{m}^2}{a_{\prime}}\int dv\left(-\frac{11}{8}m^2\frac{e'de'}{ndt}\right)-\frac{21}{4}\frac{\overline{m}^2}{a_{\prime}}\int dv\left(-\frac{77}{16}m^2\frac{e'de'}{ndt}\right)+\frac{3}{4}\frac{\overline{m}^2}{a_{\prime}}\int dv\left(\frac{11}{16}m^2\frac{e'de'}{ndt}\right)$$
$$=-\frac{165}{32}\frac{m^4}{a_{\prime}}e'^2+\frac{1617}{128}\frac{m^4}{a_{\prime}}e'^2+\frac{33}{128}\frac{m^4}{a_{\prime}}e'^2 \text{ nearly}$$
$$=\frac{495}{64}\frac{m^4}{a_{\prime}}e'^2 \text{ as their non-periodic part.}$$

12. Hence we obtain

$$0=\frac{1}{a}-\frac{1}{a_{\prime}}+\frac{1}{2}\frac{\overline{m}^2}{a_{\prime}}\left(1+\frac{3}{2}e'^2\right)-\frac{9}{4}\frac{m^4}{a_{\prime}}e'^2+\frac{495}{64}\frac{m^4}{a_{\prime}}e'^2+\frac{27}{8}\frac{m^4}{a_{\prime}}e'^2$$
$$-\frac{9}{4}\frac{m^4}{a_{\prime}}(1-5e'^2)-\frac{441}{16}\frac{m^4}{a_{\prime}}e'^2-\frac{9}{16}\frac{m^4}{a_{\prime}}e'^2$$
$$+\frac{3}{2}\frac{m^4}{a_{\prime}}(1-5e'^2)+\frac{147}{8}\frac{m^4}{a_{\prime}}e'^2+\frac{3}{8}\frac{m^4}{a_{\prime}}e'^2-\frac{285}{8}\frac{m^4}{a_{\prime}}e'^2$$
$$-\frac{9}{4}\frac{m^4}{a_{\prime}}(1-5e'^2)-\frac{441}{16}\frac{m^4}{a_{\prime}}e'^2-\frac{9}{16}\frac{m^4}{a_{\prime}}e'^2,$$

or

$$0=\frac{1}{a}-\frac{1}{a_{\prime}}\left\{1-\frac{1}{2}\overline{m}^2-\frac{3}{4}\overline{m}^2e'^2+3m^4+\frac{3153}{64}m^4e'^2\right\}.$$

Now $\overline{m}^2=\frac{m^2}{(1+p)^3}$ in PLANA's notation, or (substituting the value of p given in PLANA, vol. ii. p. 855).

$$\overline{m}^2=m^2\left(1-\frac{1}{2}m^2-\frac{3}{4}m^2e'^2\right)\text{ nearly,}$$

$$\therefore\ \frac{1}{a}=\frac{1}{a_{\prime}}\left\{1-\frac{1}{2}m^2+\frac{13}{4}m^4-\frac{3}{4}m^2e'^2+\frac{3201}{64}m^4e'^2\right\}$$

and

$$a^2=a_{\prime}^2\left\{1+m^2-\frac{23}{4}m^4+\frac{3}{2}m^2e'^2-\frac{3129}{32}m^4e'^2\right\}.$$

13. Again, by substitution in the equation for $\frac{dt}{dv}$, we obtain

$$\frac{dt}{dv}=\frac{a^2}{\sqrt{a_{\prime}}}\Big\{1-2a\delta u+\frac{3}{2}m^4(1-5e'^2)+\frac{27}{8}m^4e'^2+\frac{147}{8}m^4e'^2+\frac{3}{8}m^4e'^2$$
$$+\frac{3}{2}\,\overline{m}^2\frac{a}{a_{\prime}}\int dv\Big[\left(1-\frac{5}{2}e'^2\right)\sin(2v-2mv)+\frac{7}{2}e'\sin(2v-2mv-c'mv)$$
$$-\frac{1}{2}e'\sin(2v-2mv+c'mv)\Big]$$
$$-3\overline{m}^2\frac{a}{a_{\prime}}\,a\delta u\int dv\Big[\left(1-\frac{5}{2}e'^2\right)\sin(2v-2mv)+\frac{7}{2}e'\sin(2v-2mv-c'mv)$$
$$-\frac{1}{2}e'\sin(2v-2mv+c'mv)\Big]$$
$$-6\,\overline{m}^2\frac{a}{a_{\prime}}\int dv\Big[\left(1-\frac{5}{2}e'^2\right)\sin(2v-2mv)+\frac{7}{2}e'\sin(2v-2mv-c'mv)$$
$$-\frac{1}{2}e'\sin(2v-2mv+c'mv)\Big]a\delta u$$
$$+\frac{27}{8}\,\overline{m}^4\left(\frac{a}{a_{\prime}}\right)^2\Big\{\int dv\Big[\left(1-\frac{5}{2}e'^2\right)\sin(2v-2mv)+\frac{7}{2}e'\sin(2v-2mv-c'mv)$$
$$-\frac{1}{2}e'\sin(2v-2mv+c'mv)\Big]\Big\}^2\Big\}.$$

14. Develope this equation as before, retaining m^4 only when it occurs in the non-periodic part, and we have

$$\frac{dt}{dv}=\frac{a^2}{\sqrt{a_{\prime}}}\Big\{1-2a\delta u+\frac{3}{2}m^4+\frac{3}{4}m^4(1-5e'^2)+\frac{27}{64}m^4(1-5e'^2)-\frac{495}{128}m^4e'^2$$
$$+\frac{117}{8}m^4e'^2+\frac{147}{16}m^4e'^2+\frac{3}{16}m^4e'^2+\frac{285}{16}m^4e'^2+\frac{1323}{256}m^4e'^2+\frac{27}{256}m^4e'^2$$
$$-\frac{3}{4}m^2\Big(1-\frac{5}{2}e'^2\Big)\cos(2v-2mv)-\frac{21}{8}m^2e'\cos(2v-2mv-c'mv)$$
$$+\frac{3}{8}m^2e'\cos(2v-2mv+c'mv)$$
$$-\frac{15}{8}m^2\frac{e'de'}{ndt}\sin(2v-2mv)+\frac{21}{16}m^2\frac{de'}{ndt}\sin(2v-2mv-c'mv)$$
$$-\frac{3}{16}m^2\frac{de'}{ndt}\sin(2v-2mv+c'mv)\Big\},$$

or

$$\frac{dt}{dv}=\frac{a^2}{\sqrt{a_{\prime}}}\Big\{1+\frac{171}{64}m^4+\frac{2391}{64}m^4e'^2$$
$$-\frac{11}{4}m^2\Big(1-\frac{5}{2}e'^2\Big)\cos(2v-2mv)-\frac{425}{24}m^2\frac{e'de'}{ndt}\sin(2v-2mv)$$
$$+3m^2e'\cos c'mv+6m^3\frac{de'}{ndt}\sin c'mv$$
$$-\frac{77}{8}m^2e'\cos(2v-2mv-c'mv)+\frac{595}{48}m^2\frac{de'}{ndt}\sin(2v-2mv-c'mv)$$
$$+\frac{11}{8}m^2e'\cos(2v-2mv+c'mv)-\frac{85}{48}m^2\frac{de'}{ndt}\sin(2v-2mv+c'mv)\Big\}.$$

15. Substitute the value before found for a^2 in terms of $a_{\prime}^2$;

$$\therefore\quad \frac{dt}{dv}=a_{\prime}^{\frac{3}{2}}\Big\{1+m^2-\frac{197}{64}m^4+\frac{3}{2}m^2e'^2-\frac{3867}{64}m^4e'^2$$
$$-\frac{11}{4}m^2\Big(1-\frac{5}{2}e'^2\Big)\cos(2v-2mv)-\frac{425}{24}m^2\frac{e'de'}{ndt}\sin(2v-2mv)$$
$$+3m^2e'\cos c'mv+6m^3\frac{de'}{ndt}\sin c'mv$$
$$-\frac{77}{8}m^2e'\cos(2v-2mv-c'mv)+\frac{595}{48}m^2\frac{de'}{ndt}\sin(2v-2mv-c'mv)$$
$$+\frac{11}{8}m^2e'\cos(2v-2mv+c'mv)-\frac{85}{48}m^2\frac{de'}{ndt}\sin(2v-2mv+c'mv)\Big\}.$$

16. Now, put $\frac{1}{n}=a_{\prime}^{\frac{3}{2}}\Big\{1+m^2-\frac{197}{64}m^4+\frac{3}{2}m^2e'^2-\frac{3867}{64}m^4e'^2\Big\}$,

multiply by n, and integrate;

$$\therefore\quad \int ndt=v-\frac{11}{8}m^2\Big(1-\frac{5}{2}e'^2\Big)\sin(2v-2mv)+\frac{295}{24}m^2\frac{e'de'}{ndt}\cos(2v-2mv)$$
$$+3me'\sin c'mv+3\frac{de'}{ndt}\cos c'mv$$
$$-\frac{77}{16}m^2e'\sin(2v-2mv-c'mv)-\frac{413}{48}m^2\frac{de'}{ndt}\cos(2v-2mv-c'mv)$$
$$+\frac{11}{16}m^2e'\sin(2v-2mv+c'mv)+\frac{59}{48}m^2\frac{de'}{ndt}\cos(2v-2mv+c'mv).$$

17. In the expression for $\frac{1}{n}$ just found, $a_{,}$ is absolutely constant, but e' is variable, consequently n will vary, and therefore m likewise, which is connected with it by the equation $m=\frac{n'}{n}$.

Taking the variation of the equation for n, and observing that $\frac{\delta m}{m}=-\frac{\delta n}{n}$, we have

$$0=\frac{\delta n}{n}(1-m^2)+\left(\frac{3}{2}m^2-\frac{3867}{64}m^4\right)\delta(e'^2),$$

$$\therefore \quad \frac{\delta n}{n}=-\left(\frac{3}{2}m^2-\frac{3771}{64}m^4\right)\delta(e'^2).$$

Therefore, if N be the initial value of n, and E′ the corresponding value of e',

$$n=\mathrm{N}-\left(\frac{3}{2}m^2-\frac{3771}{64}m^4\right)n(e'^2-\mathrm{E}'^2),$$

and

$$\int ndt=\mathrm{N}t+\varepsilon-\left(\frac{3}{2}m^2-\frac{3771}{64}m^4\right)\int(e'^2-\mathrm{E}'^2)ndt.$$

Hence the expression for the true longitude in terms of the mean, contains the secular equation

$$-\left(\frac{3}{2}m^2-\frac{3771}{64}m^4\right)\int(e'^2-\mathrm{E}'^2)ndt.$$

18. According to Plana, the corresponding terms in the expression for the secular equation are

$$-\left(\frac{3}{2}m^2-\frac{2187}{128}m^4\right)\int(e'^2-\mathrm{E}'^2)ndt.$$

Hence we see that the terms now taken into consideration have the effect of making the second term of the secular equation more than three times as great as it would otherwise be. Of course, the succeeding terms will also be materially changed.

The principal term of the correction to be applied to Plana's value of the secular acceleration is therefore

$$\frac{5355}{128}m^4\int(e'^2-\mathrm{E}'^2)ndt.$$

Now

$$\int(e'^2-\mathrm{E}'^2)ndt=-1270''\left(\frac{t}{100}\right)^2 \text{ nearly,}$$

where t is expressed in years; therefore the numerical value of this term is

$$-1''\cdot66\left(\frac{t}{100}\right)^2.$$

This result will serve to give an idea of the numerical importance of the new terms to be added to the received value of the secular acceleration, and probably will not differ widely from the complete correction; though in order to obtain a value sufficiently accurate to be definitively used in the calculation of ancient eclipses, the approximation must be carried considerably further.

The new periodic terms added to the moon's longitude are perfectly insignificant, the coefficient of that involving cos $c'm\nu$, which is by far the largest of them, only amounting to $0''\cdot003$.

19. Transforming the expressions found above, so as to obtain the moon's longitude and radius vector in terms of the time, and writing for convenience nt instead of $\int ndt+\varepsilon$, mnt instead of $mnt+\varepsilon'$, and $c'mnt$ instead of $c'mnt+\varepsilon'-\varpi'$, we have

$$\nu=nt+\frac{11}{8}m^2\left(1-\frac{5}{2}e'^2\right)\sin(2-2m)nt-\frac{74}{3}m^2\frac{e'de'}{ndt}\cos(2-2m)nt$$

$$-3me'\sin c'mnt-3\frac{de'}{ndt}\cos c'mnt$$

$$+\frac{77}{16}m^2e'\sin(2-2m-c'm)nt+\frac{215}{48}m^2\frac{de'}{ndt}\cos(2-2m-c'm)nt$$

$$-\frac{11}{16}m^2e'\sin(2-2m+c'm)nt-\frac{257}{48}m^2\frac{de'}{ndt}\cos(2-2m+c'm)nt$$

$$\frac{a}{r}=au=1-\frac{11}{8}m^4-\frac{201}{16}m^4e'^2$$

$$+m^2\left(1-\frac{5}{2}e'^2\right)\cos(2-2m)nt+\frac{203}{12}m^2\frac{e'de'}{ndt}\sin(2-2m)nt$$

$$-\frac{3}{2}m^2e'\cos c'mnt-3m^3\frac{de'}{ndt}\sin c'mnt$$

$$+\frac{7}{2}m^2e'\cos(2-2m-c'm)nt-\frac{61}{24}m^2\frac{de'}{ndt}\sin(2-2m-c'm)nt$$

$$-\frac{1}{2}m^2e'\cos(2-2m+c'm)nt+\frac{91}{24}m^2\frac{de'}{ndt}\sin(2-2m+c'm)nt.$$

20. The existence of the new terms in the expressions for the moon's coordinates occurred to me some time since, when I was engaged in thinking over a new method of treating the lunar theory, though I did not then perceive their important bearing on the value of the secular equation.

My attention was first directed to this latter subject while endeavouring to supply an omission in the theory of the moon given by PONTÉCOULANT in his "Théorie Analytique." In this valuable work, the author, following the example originally set by Sir J. LUBBOCK in his Tracts on the Lunar Theory, obtains directly the expressions for the moon's coordinates in terms of the time, which are found in PLANA's theory by means of the reversion of series. With respect to the secular acceleration of the mean motion, however, PONTÉCOULANT unfortunately adopts PLANA's result without examination. On performing the calculation requisite to complete this part of the theory, I was surprised to find that the second term of the expression for the secular acceleration thus obtained, not only differed totally in magnitude from the corresponding term given by PLANA, but was even of a contrary sign. My previous researches, however, immediately led me to suspect what was the origin of this discordance, and when both processes were corrected by taking into account the new terms whose existence I had already recognized, I had the satisfaction of finding a perfect agreement between the results.

June 16th, 1853.

XVIII. *On a Theory of the Syzygetic* relations of two rational integral functions, comprising an application to the Theory of* Sturm's *Functions, and that of the greatest Algebraical Common Measure.*

By J. J. Sylvester, *M.A. Dub., F.R.S., Barrister at Law, and formerly Professor of Natural Philosophy in University College, London.*

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Introduction.

"How charming is divine philosophy!
Not harsh and crabbed as dull fools suppose,
But musical as is Apollo's lute,
And a perpetual feast of nectar'd sweets,
Where no crude surfeit reigns!"—Comus.

In the first section of the ensuing memoir, which is divided into five sections, I consider the nature and properties of the residues which result from the ordinary process of successive division (such as is employed for the purpose of finding the greatest common measure) applied to $f(x)$ and $\phi(x)$, two perfectly independent rational integral functions of x. Every such residue, as will be evident from considering the mode in which it arises, is a syzygetic function of the two given functions; that is to say, each of the given functions being multiplied by an appropriate other function of a given degree in x, the sum of the two products will express a corresponding residue. These multipliers, in fact, are the numerators and denominators to the successive convergents to $\frac{\phi x}{fx}$ expressed under the form of a continued fraction. If now we proceed *à priori* by means of the given conditions as to the degree in (x) of the multipliers and of any residue, to determine such residue, we find, as shown in art. (2.), that there are as many homogeneous equations to be solved as there are constants to be determined; accordingly, with the exception of one arbitrary factor which enters into the solution, the problem is definite; and if it be further agreed that the quantities entering into the solution shall be of the lowest possible dimensions in respect of the coefficients of f and ϕ, and also of the lowest numerical denomination, then the problem (save as to the algebraical sign of *plus* or *minus*) becomes absolutely determinate, and we can assign the numbers of the dimensions for the respective residues and syzygetic multipliers. The residues given by the method of successive division are easily seen not

* *Conjugate* would imply something very different from *Syzygetic*, viz. a theory of the Invariantive properties of a system of two algebraical functions.

to be of these lowest dimensions; accordingly there must enter into each of them a certain unnecessary factor, which, however, as it cannot be properly called irrelevant, I distinguish by the name of the Allotrious Factor. The successive residues, when divested of these allotrious factors, I term the Simplified Residues, and in article (3.) and (4.) I express the allotrious factors of each residue in terms of the leading coefficients of the preceding simplified residues of f and φ. In article (5.) I proceed to determine by a direct method these simplified residues in terms of the coefficients of f and φ. Beginning with the case where f and φ are of the same dimensions (m) in x, I observe that we may deduce from f and φ m linearly independent functions of x each of the degree $(m-1)$ in x, all of them syzygetic functions of f and φ (vanishing when these two simultaneously vanish), and with coefficients which are made up of terms, each of which is the product of one coefficient of f and one coefficient of φ. These, in fact, are the very same (m) functions as are employed in the method which goes by the name of BEZOUT's abridged method to obtain the resultant to (*i. e.* the result of the elimination of x performed upon) f and φ. As these derived functions are of frequent occurrence, I find it necessary to give them a name, and I term them the (m) Bezoutics or Bezoutian Primaries; from these (m) primaries m Bezoutian secondaries may be deduced by eliminating linearly between them in the order in which they are generated,—first, the highest power of x between two, then the two highest powers of x between three, and finally, all the powers of x between them all: along with the system thus formed it is necessary to include the first Bezoutian primary, and to consider it accordingly as being also the first Bezoutian secondary; the last Bezoutian secondary is a constant identical with the Resultant of f and φ. When the m times m coefficients of the Bezoutian primaries are conceived as separated from the powers of x and arranged in a square, I term such square the Bezoutic square. This square, as shown in art. (7.), is symmetrical above one of its diagonals, and corresponds therefore (as every symmetrical matrix must do) to a homogeneous quadratic function of (m) variables of which it expresses the determinant. This quadratic function, which plays a great part in the last section and in the theory of real roots, I term the Bezoutiant; it may be regarded as a species of generating function. Returning to the Bezoutic system, I prove that the Bezoutian secondaries are identical in form with the successive simplified residues. In art. (6.) I extend these results to the case of f and φ being of different dimensions in x. In art. (7.) I give a mechanical rule for the construction of the Bezoutic square. In art. (8.) I show how the theory of $f(x)$ and $\varphi(x)$, where the latter is of an inferior degree to f, may be brought under the operation of the rule applicable to two functions of the same degree at the expense of the introduction of a known and very simple factor, which in fact will be a constant power of the leading coefficient in $f(x)$. In art. (9.) I give another method of obtaining directly the simplified residues in all cases. In art. (10.) I present the process of successive division under its most general aspect. In arts. (11.) and (12.) I demonstrate the identity of the *algebraical sign* of the Bezoutian secondaries with that of the simplified residues, generated by a process corresponding to the development of $\frac{\phi x}{fx}$ under the form of an *improper* continued fraction (where the negative sign takes the place of the positive sign which connects the several terms of an ordinary continual function). As the simplified residue is obtained by driving out an allotrious factor, the signs of the former will of course be governed by the signs accorded by previous convention to the latter; the convention made is, that the allotrious factors shall be taken with a sign which renders them always *essentially positive* when the coefficients of the given functions are real. I close the section with remarking the relation of the syzygetic factors and the residues to the convergents of the continued fraction which expresses $\frac{\phi x}{fx}$, and of the continued fraction which is formed by reversing the order of the quotients in the first named fraction.

In the second section I proceed to express the residues and syzygetic multipliers in terms of the roots and factors of the given functions; the method becoming as it may be said *endoscopic* instead of being *exoscopic**, as in the first section. I begin in arts. (14.) and (15.) with obtaining in this way, under the form of a sum or double sum of terms involving factors and roots of f and ϕ, and certain arbitrary functions of the roots in each term, a general representative, or to speak more precisely, a group of general representatives for a conjunctive of any given degree in x to f and ϕ, *i. e.* a rational integral function of x, which is the sum of the products of f and ϕ multiplied respectively by rational integral functions of x, so as to vanish of necessity when f and ϕ simultaneously vanish. This variety of representatives refers not merely to the appearance of arbitrary functions, but to an essential and precedent difference of representation quite irrespective of such arbitrariness.

In articles (16.), (17.), (18.), (19.), (20.), (21.), I show how the arbitrary form of function entering into the several terms of any one (at pleasure) of the formulæ that represent a conjunctive of any given degree may be assigned, so as to make such conjunctive identical in form with a simplified residue of the same degree. The form of arbitrary function so assigned, it may be noticed, is a fractional function of the roots, so that the expression becomes a sum or double sum of fractions. I first prove in arts. (16.), (17.) that such sum is essentially integral, and I determine the *weight* of its leading coefficient in respect of the roots of f and ϕ (this weight being measured

* These words admit of an extensive and important application in analysis. Thus the methods for resolving an equation (or to speak more accurately, for making one equation depend upon another of a simpler form) furnished by TSCHIRNHAUSEN and Mr. JERRARD (although not so presented by the latter) are essentially exoscopic; on the other hand, the methods of LAGRANGE and ABEL for effecting similar objects are endoscopic. So again, the memoir of JACOBI, "De Eliminatione," hereinafter referred to, takes the exoscopic, and the valuable "Nota ad Eliminationem pertinens" of Professor RICHELOT in CRELLE's Journal, the endoscopic view of the subject. In the present memoir (in which the two trains of thought arising out of these distinct views are brought into mutual relation) the subject is treated (chiefly but not exclusively) under its endoscopic aspect in the second, third and fourth sections, and exoscopically in the first and last sections.

by the number of roots of f and φ conjointly, which appear in any term of such coefficient). Now in the succeeding articles I revert to the Bezoutic system of the first section, and beginning with the supposition of m and n being equal, I demonstrate that the most general form of a conjunctive of any degree in x will be a linear function of the Bezoutics, from which it is easy to deduce that the simplified residues of any given degree in x are the conjunctives whose weight in respect of the roots is a *minimum*; so that all conjunctives having that weight must be identical (to a numerical factor *près*), and any integral form of less weight apparently representing a conjunctive must be nugatory, every term vanishing identically. These results are then extended to the case of two functions of unlike degrees. The conclusion is, that the weight of the forms assumed in (16.) and (17.) being equal to the minimum weight, they must (unless they were to vanish, which is easily disproved) represent the simplified residues, or which is the same thing, the Bezoutian secondaries.

We thus obtain for each simplified residue a number of essentially distinct forms of representation, but all of which must be identical to a numerical factor *près*, a result which leads to remarkable algebraical theorems.

The number of these different formulæ depends upon the degree of the residue; there being only one for the last or constant residue, two for the last but one, three for the last but two, and so on. The formulæ continue to have a meaning when their degree in x exceeds that of f or φ; but then, as although always representing conjunctives, they no longer represent residues, this identity no longer continues to subsist. In articles (22.), (23.), (24.), (25.), I enter into some developments connected with the general formulæ in question: these, it may be observed, are all expressed by means of fractions containing in the numerator and denominator products of differences; the differences in the numerator products being taken between groups of roots of f and groups of roots of φ; and in the denominator between roots of f *inter se* and roots of φ *inter se*. A great enlargement is thus opened out to the ordinary theory of partial fractions.

In art. (26.) I find the numerical ratios between the different formulæ which represent (to a numerical factor *près*) the same simplified residue, and in arts. (27.) and (28.) I determine the relations of algebraical sign of these formulæ to the simplified residues or Bezoutian secondaries. In art. (29.) I determine the syzygetic multipliers corresponding to any given residue in terms of the factors and roots of the given functions; but the expressions for these, which are closely analogous to those for the residues, cease to be polymorphic. They are obtained separately from the syzygetic equation, and it is worthy of notice, that to obtain the one we use the first of the polymorphic expressions for the residue, and to obtain the other the opposite extremity of the polymorphic scale. In the subsequent articles of this section, by aid of certain general properties of continued fractions, I establish a theorem of reciprocity between the series of residues and either series of syzygetic multipliers.

Section III. is devoted to a determination of the values of the preceding formulæ for the residues and multipliers in the case applicable to M. STURM's theorem, where $\varphi(x)$ becomes the differential derivative of fx. It becomes of importance to express the formulæ for this case in terms of their roots and factors of fx alone, without the use of the roots and factors of $f'x$, which will of course be functions of the former.

By selecting a proper form out of the polymorphic scale, the fractional terms of the series for each residue in this case become separately integral, and we obtain my well-known formulæ for the simplified residues (STURM's reduced auxiliary functions) in terms of the factors and the squared differences of partial groups of roots. This is shown in art. (35.). In art. (36.) the multiplier of $f'x$ in the syzygetic equation is expressed by formulæ of equal simplicity, and in a certain sense complementary to the former. This method, however, does not apply to obtaining expressions for the multiplier of fx in the same equation in terms of the roots and factors of fx; for the separate fractions whose sum represents any one of these factors it will be found do not admit of being expressed as integral functions of the roots and factors. To obviate this difficulty I look to the syzygetic equation itself, which contains five quantities, viz. the given function, its first differential derivative, the residue of a given degree, and the two multipliers, all of which, except the multiplier of fx, are known, or have been previously determined as rational integral functions of the roots and factors of fx. I use this equation itself for determining the fifth quantity, the multiplier in question. To perform the general operations by a direct method required for this would be impossible; the difficulty is got over by finding, by means of the syzygetic equation, the particular form that the result must assume when certain relations of equality spring up between the roots of fx; and then, by aid of these particular determinations, the general form is demonstratively inferred.

This investigation extends over arts. (38.), (39.), (40.), (41.), (42.), (43.). It turns out that the expressions for the multipliers of fx are of much greater complexity than for the multipliers of $f'x$ or for the residues. Any such multiplier consists of a sum of parts, each of which, as in the case of the residues and of the factors of $f'x$, is affected with a factor consisting of the squared differences of a group of roots; but the other factor, instead of being simply (as for the residues and factors before mentioned) a product of certain factors of fx, consists of the sum of a series of products of sums of powers by products of combinations of factors of fx, each of which series is affected with the curious anomaly of its last term, becoming augmented in a certain numerical ratio beyond what it should be, in order to be conformable to the regular flow of the preceding terms in the series*.

The fourth section opens with the establishment of two propositions concerning

* The syzygetic multipliers are identical with the numerators and denominators (expressed in their simplest form) of the successive convergents to the continued fraction which expresses $\frac{f'x}{fx}$.

quadratic functions which are made use of in the sequel. Art. (28.) contains the proof of a law which, although of extreme simplicity, I do not remember to have seen, and with which I have not found that analysts are familiar: I mean the law of the constancy of signs (as regards the number of positive and negative signs) in any sum of positive and negative squares into which a given quadratic function admits of being transformed by substituting for the variables linear functions of the variables with real coefficients. This constant *number* of positive signs which attaches to a quadratic function under all its transformations, and which is a transcendental function of the coefficients invariable for *real* substitutions, may be termed conveniently its *inertia*, until a better word be found. This inertia it is shown in art. (26.), by aid of a theorem identical with one formerly given by M. CAUCHY, is measured by the number of combinations of sign in the series of determinants of which the first is the complete determinant of the function; the second, the determinant when one variable is made zero; the next, the determinant when another variable as well as the first is made zero, and so on, until all the variables are exhausted, and the determinant becomes positive unity. In art. (46.) I give some curious and interesting expressions for the residues and syzygetic multipliers, under the form of determinants communicated to me by M. HERMITE; and in art. (47.) I show how. by aid of the generating function which M. HERMITE employs, and of the law of inertia stated at the opening of the section, an *instantaneous* demonstration may be given of the applicability of my formulæ for M. STURM's functions for discovering the number of real roots of fx, without any reference to the rule of common measure; and moreover, that these formulæ may be indefinitely varied, and give the generating function, out of which they may be evolved in its most general form. Had the law of inertia been familiar to mathematicians, this constructive and instantaneous method of finding formulæ for determining the number of real roots within prescribed limits would, in all probability, have been discovered long ago, as an obvious consequence of such law. I then proceed in arts. (48.) and (49.), to inquire as to the nature of the indications afforded by the successive simplified residues to two general functions f and ϕ; and I find that the succession of signs of these residues serves to determine the number of roots of f or ϕ, comprised between given limits after all pairs of roots of either function, contained within the given limits, not separated by roots of the other function, have been removed, and the operation, if necessary, repeated *toties quoties* until no two roots of either function are left unseparated by roots of the other; or in other words, until every root finally retained in one function is followed by a root of the other, or else by one of the assigned limits. The system of roots comprised between given limits thus reduced I call the effective scale of intercalations; such a scale may begin with a root of the numerator or of the denominator of $\frac{\phi x}{fx}$; and upon this and the relative magnitudes of the greatest root of ϕx and fx it will depend whether in the series of residues (among which fx and ϕx are for this purpose to be counted) changes will be lost or gained as x passes from positive infinity to negative infinity. In art. (50.) I observe that the theory of real roots of a single function given by M. STURM's theorem is a corollary to this theory of the intercalations of real roots of two functions, depending upon the well-known law, that odd groups of the limiting function $f'x$ lie between every two consecutive real roots of fx. In art. (51.) I verify the law of reciprocity, already stated to exist between the residues of fx and ϕx, by an *à posteriori* method founded on the theory of intercalations. In arts. (52.), (53.), (54.), I obtain a remarkable rule, founded upon the process of common measure, for finding a superior and inferior limit in an infinite variety of ways to the roots of any given function. This method stands in a singular relation of contrast to those previously known. All previous methods (including those derived through NEWTON's Rule) proceed upon the idea of treating the function whose roots are to be limited as made up of the *sum* of parts, each of which retains a constant sign for all values of the variable external to the quantities which are to be shown to limit the roots. My method, on the other hand, proceeds upon the idea of treating the function as the product of *factors* retaining a constant sign for such values of the variable. In art. (55.), the concluding article of the fourth section, I point out a conceivable mode in which the theory of intercalations may be extended to systems of three or more functions.

In Section V. arts. (56.), (57.), I show how the *total* number of effective intercalations between the roots of two functions of the same degree is given by the *inertia* of that quadratic form which we agreed to term the Bezoutiant to f and ϕ; and in the following article (58.) the result is extended to embrace the case contemplated in M. STURM's theorem; that is to say, I show, that on replacing the function of x by a homogeneous function of x and y, the Bezoutiant to the two functions, which are respectively the differential derivatives of f with respect to x and with respect to y, will serve to determine by its form or *inertia* the total number of *real* roots and of *equal* roots in $f(x)$. The subject is pursued in the following arts. (59.), (60.). The concluding portion of this section is devoted to a consideration of the properties of the Bezoutiant under a purely morphological point of view; for this purpose f and ϕ are treated as homogeneous functions of two variables x, y, instead of being regarded as functions of x alone. In arts. (61.), (62.), (63.), it is proved that the Bezoutiant is an invariantive function of the functions from which it is derived: and in art. (64.) the important remark is added, that it is an invariant of that particular class to which I have given the name of Combinants, which have the property of remaining unaltered, not only for linear transformations of the variables, but also for linear combinations of the functions containing the variables, possessing thus a character of double invariability. In arts. (65.), (66.), I consider the relation of the Bezoutiant to the differential determinant, so called by JACOBI, but which for greater brevity I call the Jacobian. On proper substitutions being made in the Bezoutiant of the (m) variables which it contains (m being the degree in x, y of f and ϕ), the Bezoutiant becomes identical with the Jacobian to f and ϕ; but as it is afterwards shown, this is not a property peculiar to the Bezoutiant; in fact there exists a whole family of quadratic forms of m variables, lineo-linear (like the Bezoutiant) in respect of the coefficients in f and φ, all of which enjoy the same property. The number of individuals of such family must evidently be infinite, because any linear combination of any two of them must possess a similar property; I have discovered, however, that the number of independent forms of this kind is limited, being equal to the number of odd integers not greater than the degree of the two functions f and φ. In arts. (67.) and (68.), I give the means of constructing the scale of forms, which I term the constituent or fundamental scale, of which all others of the kind are merely numerico-linear combinations. This scale does not directly include the Bezoutiant within it, and it becomes an object of interest to determine the numbers which connect the Bezoutiant with the fundamental forms; this calculation I have carried on (in arts. (69.), (70.), (71.)) from $m=1$ to $m=6$ inclusive, and added an easy method of continuing indefinitely. In this method the numbers in the linear equation corresponding to any value of m are determined successively, and each made subject to a verification before the next is determined, there being always pairs of equations which ought to bring out the same result for each coefficient.

In the next and concluding art. (72.), I remark upon the different directions in which a generalization may be sought of the subject-matter of the ideas involved in M. STURM's theorem, and of which the most promising is, in my opinion, that which leads through the theory of intercalations. Some of the theorems given by me in this paper have been enunciated by me many years ago, but the demonstrations have not been published, nor have they ever before been put together and embodied in that compact and organic order in which they are arranged in this memoir,—the fruit of much thought and patient toil, which I have now the honour of presenting to the Royal Society.

June 16, 1853.

In a supplemental part to the third section I have given expressions in terms of the roots of φx *and* fx *for the* quotients *which arise in developing* $\frac{\varphi x}{fx}$ *under the form of a continued fraction, and some remarkable properties concerning these* quotients. *In a supplemental part to the fourth section I have given an extended theory of my new method of finding limits to the real roots of any algebraical equation. This method, so extended, possesses a marked feature of distinction from all preceding methods used for the same purpose, inasmuch as it admits in every case of the limits being brought up into actual coincidence with the extreme roots, whereas in other methods a wide and arbitrary interval is in general necessarily left between the roots and the limits.*

SECTION I.

On the complete and simplified residues generated in the process of developing under the form of a continued fraction, an ordinary rational algebraical fraction.

Art. (1.). Let P and Q be two rational integral functions of x, and suppose that the process of continued successive division leads to the equations

$$\left.\begin{array}{l} P\ -M_0Q\ +R_1=0 \\ Q\ -M_1R_1+R_2=0 \\ R_1-M_2R_2+R_3=0 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{array}\right\} \quad \ldots\ldots\ldots\ldots \quad (1.)$$

so that

$$\frac{Q}{P}=\frac{1}{M_0-}\,\frac{1}{M_1-}\,\frac{1}{M_2-}\ \&c. \quad \ldots\ldots\ldots\ldots \quad (2.)$$

which is what I propose to call an improper continued fraction, differing from a proper only in the circumstance of the successive terms being connected by negative instead of positive signs.

M_0, M_1, M_2, &c., R_1, R_2, R_3, &c. are, of course, functions of x: the latter we may agree to call the 1st, 2nd, 3rd, &c. residues (in order to avoid the use of the longer term "residues with the signs changed"); and by way of distinction from what they become when certain factors are rejected, we may call R_1, R_2, R_3, &c. the complete residues. Each such complete residue will in general be of the form $\frac{N_\iota \cdot \varrho_\iota}{D_\iota}$, N_ι and D_ι being integral functions of the coefficients only of P and Q, but ϱ_ι an integral function of these coefficients, and of x: ρ_ι may then be termed the ιth simplified residue, and $\frac{N_\iota}{D_\iota}$ the ιth *allotrious* factor. Suppose P to be of m and Q of n dimensions in x, and $m-n=e$, the process of continued division may be so conducted, that all the residues may contain only integer powers of x; and we may upon this supposition make M_0 of e dimensions, and M_1, M_2, M_3, &c. each of one dimension only in x; so that R_1, R_2, R_3, ... will be respectively of $(n-1)$, $(n-2)$, $(n-3)$, &c. dimensions in x.

P and Q are supposed to be perfectly unrelated, and each the most general function that can be formed of the same degree. From (1.) we obtain

$$\left.\begin{array}{l} R_1=M_0.Q-P \\ R_2=M_1R_1-Q \\ \quad=(M_0M_1-1)Q-M_1.P \\ R_3=(M_0M_1M_2+M_0+M_2)Q-(M_1M_2-1)P \\ \&c.=\&c. \end{array}\right\} \quad \ldots\ldots\ldots \quad (3.)$$

and in general we shall have

$$R_{\iota}=Q_{\iota}.Q+P_{\iota}.P, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

where it is evident that Q_{ι} will be of $e+(\iota-1)$, and P_{ι} of $(\iota-1)$ dimensions in x.

Art. (2.) Hence it follows that the ratios $P_{\iota}:Q_{\iota}:R_{\iota}$ may be ascertained by the direct application of the method of indeterminate coefficients, for Q_{ι} will contain $e+\iota$, and P_{ι} will contain ι disposable constants, making $e+2\iota$ disposable constants in all. Again, $Q_{\iota}.Q$ and $P_{\iota}P$ will each rise to the degree $n+e+2-1$ in x; but their sum R_{ι} is to be only of $n-\iota$ dimensions in x. Hence we have to make $(n+e+\iota-1)-(n-\iota)$, *i. e.* $e+2\iota-1$ quantities (which are linear in respect to the given coefficients in P and Q, as well as in respect to the new disposable constants in P_{ι} and Q_{ι}) all vanish, that is to say, there will be $e+2\iota-1$ linear homogeneous equations to be satisfied by means of $e+2\iota$ disposable quantities; the ratios of these latter are, therefore, *determinate*, so that we may write

$$\left.\begin{aligned} P_{\iota}&=\lambda_{\iota}.(P_{\iota}) \\ Q_{\iota}&=\lambda_{\iota}(Q_{\iota}) \\ R_{\iota}&=\lambda_{\iota}(R_{\iota}) \end{aligned}\right\}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.)$$

and when (P_{ι}), (Q_{ι}), (R_{ι}) are taken prime to one another, it is obvious that (R_{ι}) will be in all of $e+2\iota$ dimensions in the given coefficients, *i. e.* of ι in respect of the coefficients of P, and of $e+\iota$ in respect of those of Q: λ_{ι} will correspond to what I have previously called the allotrious factor; being in fact foreign to the value of R_{ι} as determined by means of the equation (4.), and arising only from the particular method employed to obtain it through the medium of the system (1.): it becomes a matter of some interest and importance to determine the values of this allotrious factor for different values of ι*.

* These are identical with what I termed quotients of succession in the London and Edinburgh Philosophical Magazine (December, 1839); but by an easily explicable error of inadvertence, the quantities "Q_1," "Q_2," &c. therein set out are not as they are therein stated to be, the quotients of succession or allotrious factors themselves, but the ratios of each such to the one preceding, if in the series; so that—

"Q_1" is λ_1

"Q_2" is $\frac{\lambda_2}{\lambda_1}$

"Q_3" is $\frac{\lambda_3}{\lambda_2}$

&c. . . .

This error is corrected by my distinguished friend M. STURM (LIOUVILLE's Journal, tom. viii. 1842. Sur un théorème d'Algèbre de M. SYLVESTER), who appears, however, to have overlooked that I was obviously well acquainted with the existence and nature of these factors, and their essential character, of being perfect squares in the case contemplated in his memoir and my own. MM. BORCHARDT, TERQUEM, and other writers, in quoting my formulæ for M. STURM's auxiliary functions, have thus been led into the error of alluding to them as completed by M. STURM.

Art. (3.). This may be done by the following method, which is extremely simple, and would admit of a considerable extension in its applications, were it not beside my immediate purpose to digress from the objects set out in the title to the memoir, by entering upon an investigation of the special or singular cases which may arise in the process of forming the continued fraction, when one or more of the leading coefficients in any of the residues vanish; such an inquiry would require a more general character to be imparted to the values of the quotients and residues than I shall for my present purposes care to suppose.

Let us begin with supposing $e=1$, and write

$$\left.\begin{aligned} f&=ax^n+bx^{n-1}+cx^{n-2}+\text{ \&c.} \\ \varphi&=\alpha x^{n-1}+\beta x^{n-2}+\gamma x^{n-3}+\text{ \&c.}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.)$$

Let ψ be the first residue of $\frac{f}{\varphi}$, and ω of $\frac{\varphi}{\psi}$, and therefore of $\frac{\varphi}{\alpha^2\psi}$, so that ω is the second residue of $\frac{f}{\varphi}$.

Let $\omega=\lambda(\omega)$, ω being entirely integer, and λ a function of the coefficients in f and φ. If we make $\lambda=\frac{\text{N}}{\text{D}}$, N and D being integer functions, D will evidently be L^2; where L denotes the first coefficient in the simplified residue $\alpha^2\varphi$, and is evidently of two dimensions in α, β, &c., and of one in a, b, &c.; $\text{D}\omega$ is therefore of $2\times 2+1$, *i. e.* five dimensions in α, β, &c., and of two dimensions in a, b, &c.; but ω (by virtue of what has been observed of the equations in system (5.)) is of three dimensions in α, β, &c., and of two in a, b, &c. Hence N is of two dimensions in α, β, &c., and of none in a, b, &c. This enables us at once to perceive that $\text{N}=a^2$,

$$\left.\begin{aligned} &\text{for } \psi \text{ is of the form } f-(px+q)\varphi, \\ &\text{and } \omega \text{ is of the form } \varphi-(p'x+q')\psi\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.)$$

But $\text{N}=0$ makes ω vanish, and therefore, upon this supposition, f and φ would appear to have a common algebraical factor ψ, that is to say, N vanishing, would appear to imply that the resultant of f and φ must vanish, so that N would appear to be contained as a factor in this general resultant, which latter is, however, clearly indecomposable into factors—a seeming paradox—the solution of which must be sought for in the fact, that the equation $\text{N}=0$ is *incompatible* with the existence of the usual equations (7.) connecting f, φ, ψ and ω: but this failure of the existence of the equations (7.) (bearing in mind that N has been shown to be a function only of the set of coefficients α, β, &c.), can only happen by reason of α vanishing whenever N vanishes; α must therefore be a root of N, or which is the same thing, N a power of (α) and hence $\text{N}=\alpha^2$.

The same result may be obtained *à posteriori* by actually performing the successive divisions; if the coefficients of any dividend be a, b, c, d, &c., and of the divisor

α, β, γ, δ, &c., the first remainder forming the second divisor will be easily seen to have for its coefficients—

$$\frac{1}{\alpha^2}\cdot\begin{vmatrix} a & b & c \\ 0 & \alpha & \beta \\ \alpha & \beta & \gamma \end{vmatrix}, \quad \frac{1}{\alpha^2}\cdot\begin{vmatrix} a & b & d \\ 0 & \alpha & \beta \\ \alpha & \beta & \delta \end{vmatrix}, \quad \frac{1}{\alpha^2}\cdot\begin{vmatrix} a & b & e \\ 0 & \alpha & \beta \\ \alpha & \beta & \varepsilon \end{vmatrix} \quad \text{\&c.}$$

Hence the coefficients in the next remainder (making $\begin{vmatrix} a & b & c \\ 0 & \alpha & \beta \\ \alpha & \beta & \gamma \end{vmatrix} = m$) will be each of the form of the compound determinant,—

$$\frac{1}{m^2}\cdot\left\{\begin{array}{ccc} \alpha & \beta & \gamma \\ 0 & \begin{matrix} a & b & c \\ 0 & \alpha & \beta \\ \alpha & \beta & \gamma \end{matrix} & \begin{matrix} a & b & d \\ 0 & \alpha & \gamma \\ \alpha & \beta & \delta \end{matrix} \\ \begin{matrix} a & b & c \\ 0 & \alpha & \beta \\ \alpha & \beta & \gamma \end{matrix} & \begin{matrix} a & b & d \\ 0 & \alpha & \gamma \\ \alpha & \beta & \delta \end{matrix} & \begin{matrix} a & b & e \\ 0 & \alpha & \delta \\ \alpha & \beta & \varepsilon \end{matrix} \end{array}\right\}.$$

The compound determinant above written will be the first coefficient in the remainder under consideration; the subsequent coefficients will be represented by writing f, ϕ; g, γ, &c., respectively in lieu of e, ε. Omitting the common multiplier $\frac{1}{m^2}$, the determinant above written is equal to

$$\alpha\left\{\begin{matrix} a & b & e \\ 0 & \alpha & \beta \\ \alpha & \beta & \gamma \end{matrix} \times \begin{matrix} a & b & e \\ 0 & \alpha & d \\ \alpha & \beta & \varepsilon \end{matrix} - \begin{matrix} a & b & d \\ 0 & \alpha & \gamma \\ \alpha & \beta & \delta \end{matrix} \times \begin{matrix} a & b & d \\ 0 & \alpha & \gamma \\ \alpha & \beta & \delta \end{matrix}\right\}$$

$$+\begin{matrix} a & b & c \\ 0 & \alpha & \beta \\ \alpha & \beta & \gamma \end{matrix} \times \left\{\beta\,.\begin{matrix} a & b & d \\ 0 & \alpha & \gamma \\ \alpha & \beta & \delta \end{matrix} - \gamma\,.\begin{matrix} a & b & c \\ 0 & \alpha & \beta \\ \alpha & \beta & \gamma \end{matrix}\right\}.$$

The last written pair of terms are together equal to

$$\begin{matrix} a & b & c \\ 0 & \alpha & \beta \\ a & \beta & \gamma \end{matrix} \times \left\{ -d\beta\alpha^2 + c\gamma\alpha^2 + a\alpha(\beta\delta - \gamma^2) \right\},$$

which is of the form $\alpha^2\text{A} - a^2\beta^2(\beta\delta - \gamma^2)\alpha$, and the sum of the first written pair is of the form $\alpha^2\text{B} + (a\beta^2 . a\beta\delta - a\gamma\beta . a\gamma\beta)\alpha$. Hence the entire determinant is of the form $\alpha^2(\text{A}+\text{B})$, showing that α^2 will enter as a factor into this and every subsequent coefficient in the second remainder, as previously demonstrated above.

It may, moreover, be noticed, that this remainder, when α^2 has been expelled, will for general values of the coefficients be numerically as well as literally in its lowest terms, as evinced by the fact that there exist terms (*ex. gr.* $\alpha a^2\gamma\epsilon$) having ± 1 for their numerical part. The same explicit method might be applied to show, that if the first divisor were e degrees instead of being only one degree in x lower than the first dividend, α^{e+1} would be contained in every term of the second residue; the difficulty, however, of the proof by this method augments with the value of e; but the same result springs as an immediate consequence from the method first given, which remains good *mutatis mutandis* for the general case, as may easily be verified by the reader. Applying now this result to the functions P and Q, supposed to be of the respective degrees n and $n-e$ in x, and calling the coefficients of the leading terms in the successive simplified residues α_1, α_2, α_3, &c., and the leading coefficient in Q α, and before denoting the successive allotrious factors by λ_1, λ_2, &c., it will readily be seen that

$$\lambda_1=\frac{1}{\alpha^{e+1}}\quad \lambda_2.\lambda_1=\frac{1}{\alpha_1^2}\quad \lambda_3.\lambda_2=\frac{1}{\alpha_2^2}\quad \lambda_4.\lambda_3=\frac{1}{\alpha_3^2},\ \&c.,$$

$$i.\,e.\ \lambda_1=\frac{1}{\alpha^{e+1}}\quad \lambda_2=\frac{\alpha^{e+1}}{\alpha_1^2}\quad \lambda_3=\frac{\alpha_1^2}{\alpha^{e+1}\alpha_2^2}\quad \lambda_4=\frac{\alpha^{e+1}.\alpha_2^2}{\alpha_1^2.\alpha_3^2},$$

and in general

$$\left.\begin{array}{l}\lambda_{2m+1}=\dfrac{1}{\alpha^{e+1}}.\dfrac{\alpha_1^2.\alpha_3^2\ldots.\alpha_{2m-1}^2}{\alpha_2^2.\alpha_4^2\ldots.\alpha_{2m}^2}\\[2ex] \lambda_{2m}=\alpha^{e+1}.\dfrac{1.\alpha_2^2.\alpha_4^2\ldots\alpha_{2m-2}^2}{\alpha_1^2.\alpha_3^2.\alpha_5^2\ldots\alpha_{2m-1}^2}\end{array}\right\}\ \ldots\ldots\ldots\ldots \quad (8.)$$

Art. (4.). Strictly speaking, we have not yet fully demonstrated that the complete allotrious factors are represented by the values above given for λ, but only that these latter are contained as factors in the allotrious factors; we must further prove that there exist no other such factors. This may be shown as follows: it is obvious from the nature of the process that the complete residues will always remain of one dimension in respect of the given coefficients, *i. e.* first of one dimension in the set a, b, c, &c., and of zero dimensions in α, β, γ, &c.; then conversely, of one dimension in α, β, γ, &c., and of zero dimensions in a, b, c, &c., and so on, the residues being evidently required to conform in their dimensions to those of the first dividend and the first divisor alternately. These coefficients then are always of unit dimensions in respect to the given coefficients; whereas it has been shown (art. 2.) that the simplified residues in respect to these coefficients are successively of the dimensions $2+e$, $4+e$, $6+e$, &c.

Let the complete residue corresponding to λ_{2m} be $M.\lambda_{2m}.\alpha_{2m}$,

$$i.\,e.\ M.\frac{\alpha^{e+1}}{\alpha_1^2}.\frac{\alpha_2^2}{\alpha_3^2}.\frac{\alpha_4^2}{\alpha_5^2}\cdots\cdots\frac{\alpha_{2m-2}^2}{\alpha_{2m-1}}.\alpha_{2m},$$

or say M.L; in passing from α_{2q} to α_{2q+1} the dimensions rise 2 units for all values of q except zero, and when $q=0$ the dimensions increase *per saltum* from 1 to $2+e$; hence the total dimensions of L in the joint coefficients will be

$$\big((e+1)-(2e+2)\big)-(m-1)4+4m+e=1,$$

and therefore M is of zero dimensions, and λ_{2m} is the complete allotrious factor. In like manner if the complete residue corresponding to λ_{2m+1} be $M.\lambda_{2m+1}.\alpha_{2m+1}$,

$$i.\,e.\ M\frac{1}{\alpha^{e+1}}.\frac{\alpha_1^2}{\alpha_2^2}.\frac{\alpha_3^2}{\alpha_4^2}\ldots\ldots\frac{\alpha_{2m-1}^2}{\alpha_{2m}^2}.\alpha_{2m+1},$$

or say M.L, the dimensions of L will be

$$-(e+1)-m.4+\big(e+2.(2m+1)\big),\ i.\,e.\ 1,$$

and hence, as in the preceding case, M is of zero dimensions, and λ_{2m+1} is the complete allotrious factor.

Art. (5.). I proceed to show how the simplified residues may be most conveniently obtained by a direct process, identical with that which comes into operation in applying to the two given functions of x the method familiarly known under the name of BEZOUT's abridged method of elimination. Let us call the two given functions U and V, and commence with the case where U and V are of equal dimensions (n) in x. The simplified ιth residue will then be a function of $n-\iota$ dimensions in x, and of ι dimensions in respect of each given set of coefficients, and may be taken equal to $V_\iota.U+U_\iota.V$, where V_ι and U_ι are each of $(\iota-1)$ dimensions in x.

Let

$$U=a_0.x^n+a_1.x^{n-1}+a_2.x^{n-2}+\ldots\ldots+a_n,$$

$$V=b_0.x^n+b_1.x^{n-1}+b_2.x^{n-2}+\ldots\ldots+b_n,$$

we may write in general {m being taken any positive integer not exceeding n},

$$U=(a_0x^m+a_1x^{m-1}+\ldots.+a_m)x^{n-m}+(a_{m+1}x^{n-m-1}+a_{m+2}x^{n-m-2}+\ldots.+a_n)$$

$$V=(b_0x^m+b_1x^{m-1}+\ldots.+b_m)x^{n-m}+(b_{m+1}x^{n-m-1}+b_{m+2}x^{n-m-2}+\ldots.+b_n).$$

Hence

$$\left.\begin{array}{l}(b_0x^m+b_1.x^{m-1}+\ldots.+b_m)U-(a_0x^m+a_1x^{m-1}+\ldots.+a_m)V\\ \quad={}_m K_1x^{n-1}+{}_mK_2.x^{n-2}+{}_mK_3x^{n-3}+\ldots\ldots+{}_mK_n,\end{array}\right\} \quad \ldots\ldots \quad (9.)$$

where if we use (r, s) to denote $a_r.b_s-a_s.b_r$ for all values of r and s, we have

${}_mK_1=(0,m+1)$ ${}_mK_2=(0,m+2)+(1,m+1)$ ${}_mK_3=(0,m+3)+(1,m+2)+(2,m+1)$, and in general ${}_mK_i=\Sigma(r, s)$, the values of r and s admissible within the sign of summation being subject to the two conditions, one the equality $r+s=m+i$, the other the inequality r *less than* i. By giving to m all the different values from 0 to $m-1$ in succession, and calling $b_0x^m+b_1x^{m-1}+\ldots.+b_m$, $a_0x^m+a_1.x^{m-1}+..+a_m$ respectively Q_m and P_m, we have

$$\left.\begin{array}{l}Q_0\ .U-P_0\ .V=K_1\ x^{n-1}+\ K_2x^{n-2}+\ldots\ldots+\ K_n\\ Q_1\ .U-P_1\ .V={}_1K_1.x^{n-1}+\ {}_1K_2x^{n-2}+\ldots\ldots+\ {}_1K_n\\ Q_2\ .U-P_2\ .V={}_2K_1\ x^{n-1}+\ {}_2K_2x^{n-2}+\ldots\ldots+\ {}_2K_n\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ Q_{n-1}.U-P_{n-1}.V={}_{n-1}K_1\ x^{n-1}+{}_{n-1}K_2x^{n-2}+\ldots\ldots+{}_{n-1}K_n\end{array}\right\} \quad \ldots\ldots \quad (10.)$$

The right-hand members of these (n) equations I shall henceforth term the Bezoutians to U and V.

[The determinant formed by arranging in a square the n sets of coefficients of the n Bezoutians, and which I shall term the Bezoutian matrix, gives, as is well known, the Resultant (meaning thereby the Result in its simplest form of eliminating the variables out) of U and V.]

Eliminating dialytically, first x^{n-1} between the first and second, then x^{n-1} and x^{n-2} between the first, second and third, and so on, and finally, all the powers of x between the 1st, 2nd, 3rd, nth of these Bezoutians, and repeating the first of them, we obtain a derived set of (n) equations, the right-hand members of which I shall term the secondary Bezoutians to U and V, this secondary system of equations being

$$\left.\begin{array}{l} Q_0.U-P_0.V=K_1.x^{n-1}+K_2.x^{n-2}+K_3x^{n-3}+\ldots+K_n \\ ({}_1K_1Q_0-K_1Q_1)U-({}_1K_1.P_0-K_1.P_1)V=L_1.x^{n-2}+L_2.x^{n-3}+\ldots+L_{n-1} \\ \big(({}_1K_1.{}_2K_2-{}_2K_1.{}_1K_2)Q_0+({}_2K_1.K_2-K_1.{}_2K_2)Q_1+(K_1.{}_1K_2-{}_1K_1.{}_1K_2).Q_2\big)U \\ -\big(({}_1K_1.{}_2K_2-{}_2K_1.{}_1K_2)P_0+({}_2K_1.K_2-K_1.{}_2K_2)P_1+(K_1.{}_1K_2-{}_1K_1.{}_1K_2)P_2\big)V \\ =M_1.x^{n-3}+M_2.x^{n-4}+\ldots.+M_{n-2} \\ \&c.=\&c. \end{array}\right\} \quad (11.)$$

And we can now already without difficulty establish the important proposition, that the successive simplified residues to $\frac{U}{V}$, expanded under the form of an improper continued fraction, abstracting from the algebraical sign (the correctness of which also will be established subsequently), will be represented by the n successive *Secondary Bezoutians* to the system U, V.

For if we write the system of equations (11.) under the general form

$$\vartheta_\iota.U-\Pi_\iota.V=A_\iota.x^{n-\iota}+B_\iota x^{n-\iota-1}+\ \&c.,$$

the degree of ϑ_ι and H_ι in x will be that of $Q_{\iota-1}$ and $P_{\iota-1}$, *i. e.* $\iota-1$; and the dimensions of A_ι, B_ι, &c., in respect of each set of coefficients is evidently (ι); consequently, by virtue of art. (2.), $A_\iota x^{n-1}+B_\iota x^{\iota-2}+$ &c., which is the ith Bezoutian, will (saving at least a numerical factor of a magnitude and algebraical sign to be determined, but which (when proper conventions are made) will be subsequently proved to be $+1$) represent the ιth simplified residue to $\frac{U}{V}$*, as was to be shown.

Art. (6.). More generally, suppose U and V to be respectively of $n+e$ and n dimensions in x.

* V is supposed to be taken as the first divisor, and the term residue is used, as hitherto in this paper, throughout in the sense appertaining to the expansion conducted, so as to lead to an *improper* continued fraction, in that sense, in fact, in which it would, more strictly speaking, be entitled to the appellation of *excess* rather than that of *residue*.

Let $$U=a_0.x^{n+e}+a_0.x^{n+e-1}+a_0.x^{n+e-2}+ \&c.$$
$$V=b_0.x^n+b_1x^{n-1}+ \&c.$$

Making
$$U=(a_0x^{e+m}+a_1.x^{e+m-1}+\&c. +a_{e+m})x^{n-m}+(a_{e+m+1}x^{n-m-1}+\&c. +a_{n+e})$$
$$V=(b_0x^m+b_1x^{m-1}+ \ldots +b_m)x^{n-m}+(b_{m+1}x^{n-m-1}+\&c. +b_n),$$

we obtain the equation
$$Q_m.U-P_{e+m}.V={}_mK_1.x^{n+e-1}+{}_mK_2.x^{n+e-2}+\&c. +{}_mK_m, \quad \ldots \ldots \quad (12.)$$

where
$$Q_m=(b_0x^m+ \ldots +b_m)P_{e+m}=(a_0.x^{e+m}+ \ldots +a_{e+m})$$
$${}_mK_1=a_0.b_{m+1}\,;\ {}_mK_2=a_0.b_{m+2}+a_1.b_{m+1};\ \ldots {}_mK_e=a_0.b_{m+e}+a_1.b_{m+e-1}+\&c. +a_e.b_m$$
$${}_mK_{\iota+1}=a_0.b_{m+\iota+1}+\&c. +a_{\iota+1}.b_m-a_{\iota+m+1}.b_0\ \&c.=\&c.$$

By giving to m every integer value from 0 to $(n-1)$ inclusive, we thus obtain n equations of the form of (12.), each of the degree $n+e-1$ in x, and of one dimension in regard to each set of coefficients.

In addition to these equations we have the (e) equations of the form
$$x^\mu.V=b_0.x^{n+\mu}+b_1.x^{n+\mu-1}+ \&c. +b_nx^\mu, \quad \ldots \ldots \quad (13.)$$
in which μ may be made to assume every value from 0 to $(e-1)$ inclusive, and the left right-hand side of the equation for all such values of μ will remain of a degree in x not exceeding $n+e-1$, the degree of the equations of the system above described. There will thus be (e) equations in which only the (b) set of coefficients appear, and (n) equations containing in every term one coefficient out of each of the two sets.

The total number of equations is of course $n+e$. Between the (e) equations of the second system (13.) and the (r) occurring first in order of the first system (12.), we may eliminate dialytically the $e+r-1$ highest powers of x, and there will thus arise an equation of the form
$$\theta_{r-1}U-\omega_{e+r-1}.V=Lx^{n-r}+L'x^{n-r-1}+ \&c. +L \ (14.),$$
where θ_{r-1} and ω_{e+r-1} are respectively of the degrees $r-1$ and $e+r-1$ in x, and $L, L', \ldots (L)$ are of (r) dimensions in the (a) set, and of $(e+r)$ dimensions in the (b) set of coefficients, and consequently $Lx^{n-1}+L'x^{n-r-1}+\ldots+(L)$ must satisfy the conditions necessary and sufficient to prove its being (to a numerical factor *près*) a simplified residue to (U, V).

Thus suppose $$U=a_0.x^4+a_1.x^3+a_2.x^2+a_3.x+a_4$$
$$V= \qquad b_0x^2 +b_1x +b_2.$$

Then, corresponding to the system of which equation (13.) is the type, we have
$$V=b_0.x^2+b_1.x+b_2$$
$$xV=b_0.x^3+b_1.x^2+b_2.x.$$

Again, to form the system of which equation (12.) is the type, we write

$$b_0.\mathrm{U}-(a_0x^2+a_1x+a_2)\mathrm{V}=b_0(a_3x+a_4)-(a_0x^2+a_1x+a_2)(b_1x+b_2)$$
$$=-a_0b_1.x^3-(a_0b_2+a_1b_1)x^2+(b_0a_3-a_1.b_2-a_2.b_1)x+(b_0a_4-a_2.b_2)$$
$$(b_0x+b_1).\mathrm{U}-(a_0x^3+a_1x^2+a_2x+a_3)\mathrm{V}=(b_0x+b_1)a_4-(a_0x^3+a_1x^2+a_2x+a_3)b_2$$
$$=-a_0.b_2x^3-a_1b_2x^2+(b_0.a_4-a_2.b_2)x+(b_1a_4-b_2.a_3).$$

Combining the two equations of the first system with the first of the second system, we obtain the first simplified residue $\mathrm{L}x+\mathrm{L}'$, where

$$-\mathrm{L}=\begin{vmatrix} 0 & b_0 & b_1 \\ b_0 & b_1 & b_2 \\ a_0.b_1 & a_0.b_2+a_1.b_1 & a_1.b_2+a_2.b_1-b_0.a_3 \end{vmatrix}$$

and

$$\mathrm{L}'=\begin{matrix} 0 & b_0 & b_2 \\ b_0 & b_1 & 0 \\ a_0.b_1 & a_0b_2+a_1b_1 & a_2.b_2-b_0.a_4. \end{matrix}$$

By again combining the two equations of the first system with both of the second system, we have the determinant

$$\mathrm{R}=\begin{vmatrix} 0 & b_0 & b_1 & b_2 \\ b_0 & b_1 & b_2 & 0 \\ a_0.b_1 & a_0.b_2+a_1.b_1 & a_1.b_2+a_2.b_1-b_1a_3 & a_2.b_2-b_0.a_4 \\ a_0.b_2 & a_1.\ b_2 & a_2.b_2-b_0.a_4 & a_0.b_2-a_4.b_1 \end{vmatrix}$$

which is the last simplified residue, or in other terms, the resultant to the system U, V.

Art. (7.). It is most important to observe that the Bezoutian matrix to two functions of the same degree (n) is a symmetrical matrix, the terms similarly disposed in respect to one of the diagonals being equal.

Thus retaining the notation of art. (5.), so that

$$\begin{matrix} (0,1)=a\beta-b\alpha & (1,2)=b\gamma-c\alpha & (2,3)=c\delta-d\gamma \\ (0,2)=a\gamma-c\alpha & (1,3)=b\delta-d\beta & \&c. \\ (0.3)=a\delta-d\alpha & \&c. & \\ \&c. & & \end{matrix}$$

&c. &c., when $n=1$ the Bezoutian matrix consists of a single term (0, 1); when $n=2$, it becomes

$$\begin{matrix} (0,1) & (0,2) \\ (0,2) & (1,2); \end{matrix}$$

when $n=3$, it becomes

$$\begin{matrix} (0,1) & (0,2) & (0,3) \\ (0,2) & \begin{pmatrix}(0,3)\\+\\(1,2)\end{pmatrix} & (1,3) \\ (0,3) & (1,3) & (2,3); \end{matrix}$$

when $n=4$, it becomes

$$\begin{array}{cccc}
(0,1) & (0,2) & (0,3) & (0,4)\\
(0,2) & \begin{pmatrix}(0,3)\\+\\(1,2)\end{pmatrix} & \begin{pmatrix}(0,4)\\+\\1,3\end{pmatrix} & (1,4)\\
(0,3) & \begin{pmatrix}(0,4)\\+\\(1,3)\end{pmatrix} & \begin{pmatrix}(1,4)\\+\\(2,3)\end{pmatrix} & (2,4)\\
(0,4) & (1,4) & (2,4) & (3,4);
\end{array}$$

when $n=5$, it becomes

$$\begin{array}{ccccc}
(0,1) & (0,2) & (0,3) & (0,4) & (0,5)\\
(0,2) & \begin{pmatrix}(0,3)\\+\\(1,2)\end{pmatrix} & \begin{pmatrix}(0,4)\\+\\(1,3)\end{pmatrix} & \begin{pmatrix}(0,5)\\+\\(1,4)\end{pmatrix} & (1,5)\\
(0,3) & \begin{pmatrix}(0,4)\\+\\(1,3)\end{pmatrix} & \begin{pmatrix}(0,5)\\+\\(2,4)\\+\\(2,3)\end{pmatrix} & \begin{pmatrix}(1,5)\\+\\(2,4)\end{pmatrix} & (2,5)\\
(0,4) & \begin{pmatrix}(0,5)\\+\\(1,4)\end{pmatrix} & \begin{pmatrix}(1,5)\\+\\(2,4)\end{pmatrix} & \begin{pmatrix}(2,5)\\+\\(3,4)\end{pmatrix} & (3,5)\\
(0,5) & (1,5) & (2,5) & (3,5) & (4,5),
\end{array}$$

and so forth. Every such square it is apparent may be conceived as a sort of sloped pyramid, formed by the successive superposition of square layers, which layers possess not merely a simple symmetry about a diagonal (such as is proper to a *multiplication* table), but the higher symmetry (such as exists in an *addition* table), evinced in all the terms in any line of terms parallel to the diagonal transverse to the axis of symmetry being alike*. Thus for $n=5$, the three layers or stages in question will be seen to be, the first—

$$\begin{array}{ccccc}
(0,1) & (0,2) & (0,3) & (0,4) & (0,5)\\
(0,2) & (0,3) & (0,4) & (0,5) & (1,5)\\
(0,3) & (0,4) & (0,5) & (1,5) & (2,5)\\
(0,4) & (0,5) & (1,5) & (2,5) & (3,5)\\
(0,5) & (1,5) & (2,5) & (3,5) & (4,5);
\end{array}$$

the second—

$$\begin{array}{ccc}
(1,2) & (1,3) & (1,4)\\
(1,3) & (1,4) & (2,4)\\
(1,4) & (2,4) & (3,4);
\end{array}$$

and the third—

$$(2,3).$$

In general, when (n) is odd, say $2p+1$, the pyramid will end with a single term

* A square arrangement having this kind of symmetry, viz. such as obtains in the so-called Pythagorean addition table as distinguished from that which obtains in the multiplication table, may be universally called Persymmetric.

$(p, (p+1))$, and when even, as $2p$, with a square of 4 terms,

$$((p-2), (p-1)), \quad ((p-2), p)$$
$$((p-2), p), \qquad ((p-1), p).$$

Each stage may be considered as consisting of three parts, a diagonal set of equal terms transverse to the axis of symmetry, and two triangular wings, one to the left, and the other to the right of this diagonal; the terms in each such diagonal for the respective stages will be

$$(0, n); \quad (1, n-1); \quad (2, (n-2)); \ \ldots; \quad (p, (p+1)),$$

p being $\frac{n}{2}-1$ when n is even, and $\frac{n-1}{2}$ when n is odd.

If we change the order of the coefficients in each of the two given functions, it will be seen that the only effect will be to make the left and right triangular wings to change places, the diagonals in each stage remaining unaltered. The mode of forming these triangles is an operation of the most simple and mechanical nature, too obvious to need to be further insisted on here.

Art. (8.). When we are dealing with two functions of unequal degrees, n and $n+e$, we can still form a square matrix with the coefficients of the two systems of (e) and (n) equations respectively, but this will no longer be symmetrical about a diagonal: it is obvious, however, that if we treat the function of the lower degree, as if it were of the same degree as the other function, which we may do by filling up the vacant places with terms affected with zero coefficients, the symmetry will be recovered; and it is somewhat important (as will appear hereafter) to compare the values of the Bezoutian secondaries as obtained, first in their simplest form by treating each of the two functions as complete in itself, and secondly, as they come out, when that of the functions, which is of the lower degree, is looked upon as a defective form of a function of the same degree as the other. A single example will suffice to make the nature of the relation between the two sets of results apparent.

Take
$$fx = a\,x^4 + b\,x^3 + cx^2 + dx + e$$
$$\phi x = 0.x^4 + 0.x^3 + \gamma x^2 + \delta x + \varepsilon.$$

The general method of art. (7.) then gives for the Bezoutian matrix

$$\begin{matrix}
0 & a\gamma & a\delta & a\varepsilon \\[6pt]
a\gamma & \begin{pmatrix} a\delta \\ + \\ b\gamma \end{pmatrix} & \begin{pmatrix} a\varepsilon \\ + \\ b\delta \end{pmatrix} & b\varepsilon \\[6pt]
a\delta & \begin{pmatrix} a\varepsilon \\ + \\ b\delta \end{pmatrix} & \begin{pmatrix} b\varepsilon \\ + \\ c\delta - d\gamma \end{pmatrix} & c\varepsilon - e\gamma \\[6pt]
a\varepsilon & b\varepsilon & c\varepsilon - e\gamma & d\varepsilon - e\delta.
\end{matrix}$$

We shall not affect the value either of the complete determinant, or of any of the minor determinants appertaining to the above matrix, by subtracting the second line of terms, each increased in the ratio of $b:a$ from the first line of terms respectively;

the matrix so modified becomes

$$\begin{matrix} 0; & a\gamma; & a\delta; & a\varepsilon \\ a\gamma; & a\delta; & a\varepsilon; & 0 \\ a\delta; & a\varepsilon+b\delta; & \begin{pmatrix} b\varepsilon \\ + \\ c\delta-d\gamma \end{pmatrix}; & c\varepsilon-e\gamma \\ a\varepsilon; & b\varepsilon; & c\varepsilon-e\gamma; & d\varepsilon-e\delta. \end{matrix}$$

Again, adopting the method of art. (6.), we should obtain the matrix

$$\begin{matrix} 0; & \gamma; & \delta; & \varepsilon \\ \gamma; & \delta; & \varepsilon; & 0 \\ \delta; & a\varepsilon-b\delta; & \begin{pmatrix} b\varepsilon \\ + \\ c\delta-d\gamma \end{pmatrix}; & c\varepsilon-e\gamma \\ a\varepsilon; & b\varepsilon; & c\varepsilon-e\gamma; & d\varepsilon-e\delta. \end{matrix}$$

Hence it is apparent that the secondary Bezoutians obtained by the symmetrizing method will differ from those obtained by the unsymmetrical method by a constant factor a^2; and so in general it may readily be shown that the secondary Bezoutians, by the use of the symmetrizing method, will each become affected with a constant irrelevant factor a^ω, where (ω) is the difference of the degrees of the two functions, and (a) the leading coefficient of the higher one of the two. When (a) is taken unity, the Bezoutian secondaries, as obtained by either method, will of course be identical.

Art. (9.). There is another method* of obtaining the simplified residues to any two functions U and V of the degrees n and $n+e$ respectively, which, although less elegant, ought not to be passed over in silence. This method consists in forming the identical equations (of which for greater brevity the right-hand members are suppressed).

$$\begin{aligned} V &= \&c. \\ xV &= \&c. \\ &\vdots \\ x^{e-1}.V &= \&c. \\ U &= \&c. \\ x^{e}.V &= \&c. \\ x.U &= \&c. \\ x^{e+1}.V &= \&c. \\ x^{2}.U &= \&c. \\ x^{e+2}.V &= \&c. \\ \&c. &= \&c. \\ x^{n-1}.U &= \&c. \\ x^{e+n-1}.V &= \&c. \end{aligned}$$

* Originally given by myself in the London and Edinburgh Philosophical Magazine, as long ago as 1839 or 1840; and some years subsequently in unconsciousness of that fact, reproduced by my friend Mr. CAYLEY, to whom the method is sometimes erroneously ascribed, and who arrived at the same equations by an entirely different circle of reasoning.

If we equate the right-hand members of $(e+2\iota)$ of the above equations to zero, and then eliminate dialytically the several powers of x from $x^{n+e+\iota-1}$ to $x^{n-\iota+1}$ (both inclusive), the result of this process will evidently be of $(e+\iota)$ dimensions in respect of the coefficients in V, and of ι dimensions in respect of the coefficients in U; and of the degree $x^{n-\iota}$ in x it will also be of the form

$$(A+Bx+\dots Lx^{e-1})U+(F+Gx+\dots+Qx^{e+\iota-1}),$$

and by virtue of art. (2.) must consequently be the ιth simplified residue to the system U, V.

Art. (10.). The most general view of the subject of expansion by the method of continued division, consists in treating the process as having reference solely to the two systems of coefficients in U and V, which themselves are to be regarded in the light of generating functions. To carry out this conception, we ought to write

$$U=a_0+a_1.y+a_2.y^2+a_3y^3+\&c.\ ad\ inf.$$
$$V=b_0+b_1.y+b_2.y^2+b_3y^3+\&c.\ ad\ inf.,$$

and might then suppose the process of successive division applied to U and V, so as to obtain the successive equations

$$\begin{aligned}U-M_1V+R_1&=0\\ V-M_2R_1+R_2&=0\\ R_1-M_3R_2+R_3&=0\\ &\&c.\ \&c.,\end{aligned}$$

M_1, M_2, M_3, &c. being each severally of any degree whatever in y, and in general the degree of y in M_ι being any given arbitrary function $\phi(\iota)$ of ι. The values of the coefficients of the residues R_1, R_2, R_3 ..., or of these forms simplified by the rejection of detachable factors, becomes then the distinct object of the inquiry, and will, of course, depend only upon the coefficients in P and Q and the nature of the arbitrary continuous or discontinuous function $\phi(\iota)$, which regulates the number of *steps* through which each successive process of division is to be pursued. Following out this idea in a particular case, if we again reduce to our two initial functions the forms previously employed, and write

$$U=a_0.x^n+a_1.x^{n-1}+\&c.$$
$$V=b_0.x^n+b_1.x^{n-1}+\&c.;$$

and if, instead of making, according to the more usual course of proceeding, the divisions proceed first through one step and ever after through two steps at a time which is tantamount to making $\phi 1=1$ $\phi(1+\omega)=2$, we push each division through one step only at a time, and no more (so that in fact $\phi(i)$ is always 1), we shall have

$$\begin{aligned}U-m_1.\ V+R_1&=0\\ V-m_2x.\ R_1+R_2&=0\\ R_1-m_3.\ R_2+R_3&=0\\ R_2-m_4.x.R_3+R_4&=0\\ &\&c.\ \&c.,\end{aligned}$$

m_1, m_2, m_3, &c. being functions of the coefficients only of U and V; and it is not without interest to observe (which is capable of an easy demonstration) that the simplified residues contained in R_1, R_2, &c., found according to this mode of development, will be the successive dialytic resultants obtained by eliminating the $(\iota-1)$th highest powers of x between the ι first of the system of the annexed equations (supposed to be expressed in terms of x)

$$U=0$$
$$V=0$$
$$x.U=0$$
$$x.V=0$$
$$x^2 U=0$$
$$x^2 V=0$$
&c. &c.
$$x^{n-1}.U=0$$
$$x^{n-1}.V=0.$$

If we combine together $2i+1$ of the above equations, the highest power of x entering on the left-hand side will be x^{n+i}, and we shall be able to eliminate $2i$ of these factors, leaving x^{n-i} the highest power remaining uneliminated. If we take $2i$, *i.e.* i pairs of the equations, the highest power of x appearing in any of them will be x^{n+i-1}, and we shall be able to eliminate between them so as still to leave $x^{n+i-1-2i-1}$, *i.e.* x^{n-i} as before, the highest power of x remaining uneliminated; and it will be readily seen that such of the simplified residues corresponding to this mode of development as occupy the odd places in the series of such residues, will be identical with the successive simplified residues resulting from the ordinary mode of developing $\frac{U}{V}$ under the form of a continued fraction.

Art. (11.). It has been shown that the simplified residues of fx and ϕx resulting from the process of continued division are identical in point of form with the secondary Bezoutians of these functions, but it remains to assign the numerical relations between any such residue and the corresponding secondary.

To determine this numerical relation, it will of course be sufficient to compare the magnitude of the coefficient of any one power of x in the one, with that of the same power in the other; and for this purpose I shall make choice of the leading coefficients in each. In what follows, and throughout this paper, it will always be understood that in calculating the determinant corresponding to any square the product of the terms situated in the diagonal descending from left to right will always be taken with the positive sign, which convention will serve to determine the sign of all the other products entering into such determinant. Now adopting the umbral notation for determinants*, we have, by virtue of a much more general theorem for compound

* See London and Edinburgh Philosophical Magazine, April 1851.

determinants, the following identical equation:—

$$\begin{matrix}a_1a_2a_3\ldots a_{m-1}\\ \alpha_1\alpha_2\alpha_3\ldots\alpha_{m-1}\end{matrix}\times\begin{matrix}a_1a_2a_3\ldots a_{m+1}\\ \alpha_1\alpha_2\alpha_3\ldots\alpha_{m+1}\end{matrix}$$

$$=\left(\begin{matrix}a_1a_2a_3\ldots a_{m-1}a_m\\ \alpha_1\alpha_2\alpha_3\ldots\alpha_{m-1}\alpha_m\end{matrix}\right)\times\left(\begin{matrix}a_1a_2\ldots a_{m-1}a_{m+1}\\ \alpha_1\alpha_2\ldots\alpha_{m-1}\alpha_{m+1}\end{matrix}\right)$$

$$-\left(\begin{matrix}a_1a_2a_3\ldots a_{m-1}a_m\\ \alpha_1\alpha_2\alpha_3\ldots\alpha_{m-1}\alpha_{m+1}\end{matrix}\right)\times\left(\begin{matrix}a_1a_2\ldots a_{m-1}a_{m+1}\\ \alpha_1\alpha_2\ldots\alpha_{m-1}\alpha_m\end{matrix}\right)$$

and consequently

$$\begin{matrix}a_1a_2a_3\ldots a_{m-1}\\ \alpha_1\alpha_2\alpha_3\ldots\alpha_{m-1}\end{matrix}\times\begin{matrix}a_1a_2a_3\ldots a_{m-1}\ a_m.a_{m+1}\\ \alpha_1\alpha_2\alpha_3\ldots\alpha_{m-1}.\alpha_m.\alpha_{m+1}\end{matrix}$$

$$=\left(\begin{matrix}a_1a_2a_3\ldots a_{m-1}a_m\\ \alpha_1\alpha_2\alpha_3\ldots\alpha_{m-1}\alpha_m\end{matrix}\right)\times\left(\begin{matrix}a_1a_2\ldots a_{m-1}a_{m+1}\\ \alpha_1\alpha_2\ldots\alpha_{m-1}\alpha_{m+1}\end{matrix}\right)$$

$$-\left(\begin{matrix}a_1a_2\ldots a_{m-1}\ a_m\\ \alpha_1\alpha_2\ldots\alpha_{m-1}.\alpha_{m+1}\end{matrix}\right)^2$$

and consequently when

$$\left\{\begin{matrix}a_1a_2\ldots a_{m-1}\ a_m\\ \alpha_1\alpha_2\ldots\alpha_{m-1}.\alpha_{m+1}\end{matrix}\right\}=0$$

$$\begin{matrix}a_1a_2\ldots a_{m-1}\\ \alpha_1\alpha_2\ldots\alpha_{m-1}\end{matrix}\text{ and }\begin{matrix}a_1a_2\ldots a_{m-1}.a_ma_{m+1}\\ \alpha_1\alpha_2\ldots\alpha_{m-1}.\alpha_m\alpha_{m+1}\end{matrix}$$

will have different algebraical signs, it being of course understood that all the quantities entering into the determinants thus *umbrally* represented above are supposed to be real quantities. This theorem, translated into the ordinary language of determinants, may be stated as follows:—Begin with any square of terms whether symmetrical or otherwise, say of r lines and r columns; let this square be bordered laterally and longitudinally by the same r new quantities symmetrically disposed in respect to one of the diagonals, the term common to the superadded line and column being filled up with any quantity whatever: we thus obtain a square of $(r+1)$ lines and columns: let this be again bordered laterally and longitudinally by $(r+1)$ quantities symmetrically disposed above the same diagonal as that last selected, the place in which this new line and column meet being also filled up with any arbitrary quantity; and proceeding in this manner, let the determinants corresponding to the square matrices thus formed be called D_{r-1}, D_r, D_{r+1}, D_{r+2}, ... this series of quantities will possess the property, that no term in it can vanish without the terms on either side of that so vanishing having contrary signs. Thus if we begin with a square consisting of one single term, we may suppose that by accretions formed after the above rule it has been developed into the square (M) below written, and which of course may be indefinitely extended:—

$$\begin{matrix} a & l & m & p & s \\ l & b & n & q & t \\ m & n & c & r & u; \\ p & q & r & d & v \\ s & t & u & v & e \end{matrix} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad \text{(M.)}$$

r here begins with the value (1), and D_0, D_1, D_2, D_3, D_4, D_5 will represent the progression,

$$1: \quad a: \quad \begin{matrix} a & l \\ l & b \end{matrix}: \quad \begin{matrix} a & l & m \\ l & b & n \\ m & n & c \end{matrix}; \quad \begin{matrix} a & l & m & p \\ l & b & n & q \\ m & n & c & r \\ p & q & r & d \end{matrix}; \quad \begin{matrix} a & l & m & p & s \\ l & b & n & q & t \\ m & n & c & r & u \\ p & q & r & d & v \\ s & t & u & v & e \end{matrix} \ . \ . \quad \text{(II.)}$$

so if we use the matrix

$$\begin{matrix} a & l & m & p & s \\ l & b & n & q & t \\ m & n & c & r & u \\ p & q & r & d & v \\ s & t & u & v & e \end{matrix}$$

the determinants D_1, D_2. D_3, D_4, D_5 representing

$$a; \quad \begin{matrix} a & l \\ l' & b \end{matrix}; \quad \begin{matrix} a & l & m \\ l' & b & n \\ m & n & c \end{matrix}; \quad \begin{matrix} a & l & m & p \\ l' & b & n & q \\ m & n & c & r \\ p & q & r & d \end{matrix}; \ \&c.$$

will possess the property in question; the line and column l, b; l', b not being identical, the first determinant D_0 representing (1) must not be included in the progression.

We shall have occasion to use this theorem as applicable to the case of a matrix symmetrical throughout, and we may term the progression (II) above written a progression of the successive principal determinants about the axis of symmetry of the square matrix (M), and so in general. Now it is obvious that the leading coefficients of the successive Bezoutian secondaries are the successive principal determinants about the axis of symmetry of the Bezoutian squares; they will therefore have the property which has been demonstrated of such progressions; to wit, if the first of them vanishes, the second will have a sign contrary to that of $+1$; if the second vanishes, the third will have a sign contrary to that of the first, and so on.

Art. (12.). Now let fx and ϕx be any two algebraical functions of x with the leading coefficients in each, for greater simplicity supposed positive: and in the course of developing $\frac{\phi x}{fx}$ under the form of an improper continued fraction by the common process of successive division, let any two consecutive residues (the word residue being used in the same conventional sense as employed throughout) be

$$Ax+Bx^{\iota-1}+Cx^{\iota-2}+\&c.$$

$$B'x^{\iota-1}+C'x^{\iota-2}+D'x^{\iota-3}+\&c.$$

The residue next following, obtained by actually performing the division and duly changing the sign of the remainder will be

$$\left\{\left(\frac{AD}{B'}-C\right)-\left(\frac{AC'}{B'}-B\right)\frac{C'}{B'}\right\}x^{\iota-2}+\&c.,$$

which is of the form

$$\frac{1}{B'^2}\left\{B'M-AC'^2\right\}x^{\iota-2}+\&c.$$

Thus the leading coefficients in the complete unreduced residues will be

$$A\,;\ B'\,;\ \frac{1}{B'^2}\left\{B'M-AC'^2\right\},$$

and when reduced by the expulsion of the allotrious factor will become A; B'; $B'.M-A\,C'^2$, and consequently, when B' the leading coefficient of one of the simplified residues vanishes, the leading coefficients of the residues immediately preceding and following that one will have contrary signs.

First, let

$$fx=ax^n+bx^{n-1}+\&c.,\quad \phi x=\alpha x^n+\beta x^{n-1}+\&c.$$

As regards the numerical ratio of each Bezoutian secondary to the corresponding simplified residue, it has been already observed that there are always unit coefficients in the latter of these, and the same is obviously true of the former; hence if we call the progression of the leading coefficients of the simplified residues

$$R_1;\quad R_2;\quad R_3;\quad R_4,\ \&c.,$$

and that of the leading coefficients of the Bezoutian secondaries

$$B_1;\quad B_2;\quad B_3;\quad B_4,\ \&c.,$$

we have

$$B_1=\pm R_1\quad B_2=\pm R_2\quad B_3=\pm R_3\quad B_4=\pm R_4,\ \&c.$$

It may be proved by actual trial that $B_1=R_1$ and $B_2=R_2$. Moreover, since the signs are invariable, and do not depend upon the values of the coefficients, we may suppose $B_2=0$ (which may always be satisfied by real values of the quantities, of which B_2 is a function); we shall also, therefore, have $R_2=0$, and consequently B_3 has the opposite sign to that of B_1, and R_3 the opposite sign to that of R_1, which is equal to B_1: hence when $B_2=0$, B_3 and R_3 are equal, and consequently are always equal; in like manner we can prove that R_4 and B_4 have the same sign when R_3 and B_3 vanish, and consequently are always equal, and so on *ad libitum*, which proves that the series $B_1, B_2, \ldots B_n$ is identical with the series $R_1, R_2, \ldots R_n$, and consequently that the Bezoutian secondaries are identical in form, magnitude and algebraical sign with the simplified residues. Secondly, when fx and ϕx are not of the same degree, it has been shown that the secondaries formed from the nonsymmetrical matrix corresponding to this case will be the same as those formed from the *symmetrical* matrix corresponding to fx and $\Phi(x)$ (where Φx is $\phi(x)$ treated by aid of evanescent terms, as of the same degree as fx), with the exception merely of a constant multiplier (a power of the leading coefficient of fx) being introduced into each secondary. By aid of this observation, the proposition established for the case of two functions of the same degree may be readily seen to be capable of being extended, from the case of f and ϕ being of the equal dimensions in x, to the general case of their dimensions being any whatever.

Art. (13.). Before closing this section, it may be well to call attention to the nature of the relation which connects the successive residues of fx and ϕx with these functions themselves, and with the improper continued fractional form into which $\frac{\phi x}{fx}$ is supposed to be developed in the process of obtaining these residues.

If ϕx be of n degrees, and fx of $n+e$ degrees in (n), we shall have

$$\frac{\phi x}{fx}=\frac{1}{Q_1-}\frac{1}{q_2-}\frac{1}{q_3-}\cdots\frac{1}{q_n},$$

where Q_1 may be supposed to be a function of x of the degree (e), and $q_2, q_3, \ldots q_n$, are all linear functions of x; the total number of the quotients $Q_1, q_2, \ldots q_n$ being of course (n) when the process of continued division is supposed to be carried out until the last residue is zero. Upon this supposition the last but one residue is a constant, the preceding one a function of x of the first degree, the one preceding that a function of x of the second degree, and so on.

Let us call the residue of the degree ε in x, ϑ_ε; it will readily be seen that the successive complete residues arranged in an ascending order will be

$$\vartheta_0,\ \vartheta_0.q_n,\ \vartheta_0(q_{n-1}.q_n-1),\quad \vartheta_0(q_{n-2}.q_{n-1}.q_n-q_{n-2}-q_n)\ ;\ \&c.,$$

being in the ratios of the quantities

$$1\ ;\quad q_n\ ;\quad q_{n-1}-\frac{1}{q_n}\ ;\quad q_{n-2}-\frac{1}{q_{n-1}}\frac{1}{q_n}\ ;\ \&c.$$

Again, we shall have in general

$$\Lambda_\iota f-L_\iota\phi=\vartheta_\iota, \quad \ldots\ldots\ldots\ldots \quad (15.)$$

Λ_ι being an integral function of x of the degree $n-\iota-1$, and L_ι an integral function of x of the degree $(n+e)-\iota-1$; and it is easy to see that the successive convergents to the continued fraction—

$$\frac{1}{Q_1-}\frac{1}{q_2-}\frac{1}{q_3-}\ \&c.$$

have their respective numerators and denominators identical with those of the fractions

$$\frac{\Lambda_{n-1}}{L_{n-1}},\ \frac{\Lambda_{n-2}}{L_{n-2}},\ \frac{\Lambda_{n-3}}{L_{n-3}}\ \&c.$$

Adopting the language which I have frequently employed elsewhere, I call ϑ_ι a syzygetic function, or more briefly, a conjunctive of f and ϕ, and Λ_ι and L_ι may be termed the syzygetic factors to ϑ_ι so considered. If we divide each term of the equation (15.) by the allotrious factor (M), we have

$$\frac{\Lambda_\iota}{M}f-\frac{L_\iota}{M}\phi=R_\iota,$$

where R_ι is the ιth simplified residue to (f, ϕ); and if we call $\frac{\Lambda_\iota}{M}=\tau_\iota$, and $\frac{L_\iota}{M}=t_\iota$, so as to obtain the equation

$$\tau_\iota.f-t_\iota.\phi=R_\iota, \qquad (16.)$$

we see that $\frac{\tau_\iota}{t_\iota}$, the fraction formed by the component factors to any simplified residue of (f, ϕ), will be identical in value (although no longer in its separate terms) with one of the corresponding convergents to $\frac{\phi}{f}$, exhibited under the form of an improper continued fraction. I shall in the next section show how, not only the successive simplified residues, but also the component syzygetic factors of each of them, and consequently the successive convergents, may be expressed in terms of the roots of the two given functions.

Since the preceding section was composed the valuable memoir of the lamented JACOBI, entitled "De Eliminatione Variabilis è duabus Equationibus Algebraicis," CRELLE, vol. xvi., has fallen under my notice. That memoir is restricted to the consideration of two equations of the same degree, and the principal results in this section as regards the Bezoutic square and the allotrious factors applicable to that case will be found contained therein. The mode of treatment however is sufficiently dissimilar to justify this section being preserved unaltered under its original form.

SECTION II.

On the general solution in terms of the roots of any two given algebraical functions of x *of the syzygetic equation, which connects them with a third function, whose degree in* (x) *is given, but whose form is to be determined.*

Art. (14.). Let f and ϕ be two given functions in x of the degrees m and n respectively in x, and for the sake of greater simplicity let the coefficients of the highest power of x in f and ϕ be each taken unity, and let it be proposed to solve the syzygetic equation

$$\tau_\iota f-t_\iota.\phi+\vartheta_\iota=0, \qquad (17.)$$

where ϑ_ι is given only in the number of its dimensions in x, which I suppose to be (ι); but the forms of τ_ι, t_ι, ϑ_ι are all to be determined in terms of $h_1, h_2 \ldots h_m$ the roots of f and $\eta_1, \eta_2, \ldots \eta_n$ the roots of ϕ.

I shall begin with finding ϑ_ι; and before giving a more general representation of ϑ_ι, I propose now to demonstrate that we may make

$$\vartheta_\iota=\Sigma\{P_{q_1 q_2 \ldots q_\iota}\times(x-h_{q_1})(x-h_{q_2})\ldots(x-h_{q_\iota})\}, \qquad (18.)$$

where $P_{q_1, q_2 \ldots q_\iota}$ is used to denote

$$\left\{\begin{array}{l}(h_{q_{\iota+1}}-\eta_1)(h_{q_{\iota+1}}-\eta_2) \;.\;.\;.\; (h_{q_{\iota+1}}-\eta_n) \\ \times(h_{q_{\iota+2}}-\eta_1)(h_{q_{\iota+2}}-\eta_2) \;.\;.\;.\; (h_{q_{\iota+2}}-\eta_n) \\ \times(h_{q_{\iota+3}}-\eta_1)(h_{q_{\iota+3}}-\eta_2) \;.\;.\;.\; (h_{q_{\iota+3}}-\eta_n) \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \times(h_{q_m}-\eta_1)(h_{q_m}-\eta_2) \;.\;.\;.\;.\; (h_{q_m}-\eta_n)\end{array}\right\} R.(h_{q_1}\, h_{q_2} \ldots h_{q}),$$

$R(h_{q_1}\, h_{q_2} \ldots h_{q_\iota})$ denoting any rational symmetrical form of function whatever of the quantities preceded by the symbol R, and $q_1\, q_2 \ldots q_\iota\, q_{\iota+1} \ldots q_m$ being any permutation of the m indices $1, 2, \ldots m$.

Suppose $f=0$ and $\varphi=0$, then x is equal to one of the series of roots

$$h_1\, h_2 \ldots h_m,$$

and also to one of the series of roots

$$\eta_1\, \eta_2 \ldots \eta_n.$$

Suppose then that $\qquad x=h_a=\eta_\omega,$

and consider any term of ϑ_ι.

If in any such term (a) is found in the series $q_1\, q_2 \ldots q_\iota$, then

$$(x-h_{q_1})(x-h_{q_2}) \ldots (x-h_{q_\iota})=0.$$

But if not, then (a) must be found in the complementary series $h_{q_{\iota+1}}, h_{q_{\iota+2}}, \ldots, h_{q_m}$, and consequently $P_{q_1 q_2 \ldots q_\iota}$ will contain a factor $h_a-\eta_\omega$ and $P_{q_1 q_2 \ldots q_\iota}=0$; in every case therefore

$$P_{q_1 q_2 \ldots q_\iota} \times (x-h_{q_1})(x-h_{q_2}) \ldots (x-h_{q_\iota})=0,$$

and therefore ϑ_ι as expressed in equation (18.) is a syzygetic function of f and φ; accordingly we have found a function of the ιth degree in x, and of course expressible by calculating the symmetric functions as a function only of x and of the coefficients of f and φ, which will satisfy the equation

$$\tau_\iota . f - t_\iota . \varphi + \vartheta_\iota = 0.$$

[It will be remembered that by virtue of art. (2) we know *à priori* that all the values of ϑ_ι satisfying this equation are identical, save as to an allotrious factor, which is a function only of the coefficients in f and φ.] It is clear that we may interchange the h and η, m and n, and thus another representation of a value of ϑ_ι satisfying the equation (17.) will be

$$\vartheta_\iota = \Sigma R(\eta_{q_1}\eta_{q_2} \ldots \eta_{q_\iota}) \times \left\{\begin{array}{l}(\eta_{q+1}-h_1)(\eta_{q+2}-h_2) \ldots (\eta_{q+1}-h_m) \\ (\eta_{q+2}-h_1)(\eta_{q+2}-h_2) \ldots (\eta_{q+2}-h_m) \\ (\eta_{q+3}-h_1)(\eta_{q+3}-h_2) \ldots (\eta_{q+3}-h_m) \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ (\eta_{q_m}-h_1)\;(\eta_{q_m}-h_2) \ldots (\eta_{q_m}-h_m)\end{array}\right\} (x-\eta_{q_1})(x-\eta_{q_2}) \ldots (x\;\;\; \eta_{q_\iota}).$$

Art. (15.). If we employ in general the condensed notation

$$\begin{bmatrix} l, m, n, \ldots p \\ \lambda, \mu, \ldots \nu \end{bmatrix}$$

to denote the product of the differences resulting from the subtraction of each of the quantities $\lambda, \mu, \ldots \nu$ in the lower line from all of those in the upper line $l, m, n, \ldots p$, the two values above given for ϑ_ι may be written under the respective forms

$$\Sigma R(h_{q_1}h_{q_2}\ldots h_{q_\iota}) \cdot \begin{bmatrix} h_{q_\iota+1}h_{q_\iota+2}\ldots h_{q_m} \\ \eta_1 \quad \eta_2 \quad \ldots \eta_n \end{bmatrix} (x-h_{q_1})(x-h_{q_2})\ldots(x-h_{q_\iota})$$

and

$$\Sigma R(\eta_{q_1}\eta_{q_2}\ldots\eta_{q_\iota}) \cdot \begin{bmatrix} \eta_{\xi_\iota+1}\eta_{\xi_\iota+2}\ldots\eta_{\xi_n} \\ h_1 \quad h_2 \quad \ldots h_m \end{bmatrix} \times (x-\eta_{\xi_1})(x-\eta_{\xi_2})\ldots(x-\eta_{\xi_\iota})$$

in each of which equations disjunctively and in some order of relation each with each

$$q_1, q_2, q_3 \ldots, q_m = 1, 2, 3, \ldots, m,$$

and

$$\xi_1, \xi_2, \xi_3 \ldots, \xi_n = 1, 2, 3, \ldots, n.$$

These two forms are only the two extremities of a scale of forms all equally well adapted to express ϑ_ι; for let v and ν be any two integers so taken as to satisfy the equation

$$v + \nu = \iota,$$

and let $R(\overline{\ldots\ldots}; \overline{\ldots\ldots})$, where the dots denote any quantities whatever, be used to denote a rational form of function which remains unaltered in value when any two of the quantities under each and either (the same one) of the two bars are mutually interchanged, then we may write

$$\vartheta_\iota = \Sigma \left\{ \begin{array}{c} R(\overline{h_{q_1}h_{q_2}\ldots h_{q_v}}; \overline{\eta_{\xi_1}\eta_{\xi_2}\ldots\eta_{\xi_\nu}}) \times \begin{bmatrix} h_{q_{v+1}}h_{q_{v+2}}\ldots h_{q_m} \\ \eta_{\xi_{\nu+1}}\eta_{\xi_{\nu+2}}\ldots\eta_{\xi_n} \end{bmatrix} \\ \times (x-h_{q_1})(x-h_{q_2})\ldots(x-h_{q_v}) \times (x-\eta_{\xi_1})(x-\eta_{\xi_2})\ldots(x-\eta_{\xi_\nu}) \end{array} \right\} \quad . \quad . \quad (19.)$$

For if, as above, we suppose $x=h_a=\eta_\omega$, any term of ϑ_ι in which $q_1, q_2, \ldots q_v$ comprise among them h_a, or in which $\xi_1\xi_2\ldots\xi_\nu$ comprise among them η_ω, will vanish by virtue of the factors $(x-h_{q_1})(x-h_{q_2})\ldots(x-h_{q_v}) \times (x-\eta_{\xi_1})(x-\eta_{\xi_2})\ldots(x-\eta_{\xi_\nu})$; but if neither h_a nor η_ω is so comprised, then h_a must be one of the terms in the complementary series $q_{v+1}, q_{v+2}, \ldots q_m$, and η_ω one of the terms in the complementary series $\xi_{\nu+1}, \xi_{\nu+2}, \ldots \xi_n$, and therefore one of the quantities $h_{q_{v+1}}, h_{q_{v+2}}, \ldots h_{q_m}$ will equal one of the quantities $\eta_{\xi_{\nu+1}}, \eta_{\xi_{\nu+2}}, \ldots \eta_{\xi_n}$, and consequently the term of ϑ_ι in question will vanish by virtue of the factor $\begin{bmatrix} h_{q_{v+1}}h_{q_{v+2}}\ldots h_{q_m} \\ \eta_{\xi_{\nu+1}}\eta_{\xi_{\nu+2}}\ldots\eta_{\xi_n} \end{bmatrix}$ vanishing. In either case therefore every term included within the sign of summation vanishes when $x=h_a=\eta_\omega$, *i. e.* whenever $f(x)=0$ and $\phi=(x)=0$. Hence ϑ_ι, as given by equation (19.), will satisfy the syzygetic equation

$\tau_\iota f - t_\iota.\phi + \vartheta_\iota = 0$ for all values of v and υ which make $v+\upsilon=\iota$, and for all symmetrical forms of the function denoted by the symbol $R(\overline{\ldots\ldots};\ \overline{\ldots\ldots})$.

Art. (16.). I shall now proceed to show how to assign the arbitrary function whose form is denoted by this symbol in such a manner as to make ϑ_ι become identical with a simplified residue to f and ϕ. To this end I take for $R(h_{q_1}h_{q_2}\ldots h_{q_\iota};\ k_{\xi_1}k_{\xi_2}\ldots k_{\xi_\upsilon})$ the value

$$R=\frac{\begin{bmatrix} h_{q_1}h_{q_2}\ldots h_{q_v} \\ \eta_{\xi_1}\eta_{\xi_2}\ldots\eta_{\xi} \end{bmatrix}}{\begin{bmatrix} h_{q_1} & h_{q_2} & \ldots h_{q_v} \\ h_{q_{v+1}} & h_{q_{v+2}} & \ldots h_{q_m} \end{bmatrix}\times\begin{bmatrix} k_{\xi_1} & k_{\xi_2} & \ldots k_{\xi_v} \\ k_{\xi_{v+1}} & k_{\xi_{v+2}} & \ldots k_{\xi_n} \end{bmatrix}}, \quad . \quad . \quad . \quad . \quad . \quad (20.)$$

we shall then have

$$\vartheta_\iota=\Sigma\frac{\begin{bmatrix} h_{q_1}h_{q_2}\ldots h_{q_v} \\ \eta_{\xi_1}\eta_{\xi_2}\ldots\eta_{\xi_v} \end{bmatrix}\times\begin{bmatrix} h_{q_{v+1}}h_{q_{v+2}}\ldots h_{q_m} \\ \eta_{\xi_{v+1}}\eta_{\xi_{v+2}}\ldots\eta_{\xi_n} \end{bmatrix}}{\begin{bmatrix} h_{q_1} & h_{q_2} & \ldots h_{q_v} \\ h_{q_{v+1}} & h_{q_{v+2}} & \ldots h_{q_m} \end{bmatrix}\times\begin{bmatrix} \eta_{\xi_1} & \eta_{\xi_2} & \ldots \eta_{\xi_v} \\ \eta_{\xi_{v+1}} & \eta_{\xi_{v+2}} & \ldots\eta_{\xi_n} \end{bmatrix}}\{(x-h_{q_1})(x-h_{q_2})\ldots(x-h_{q_v})\}\{(x-\eta_{\xi_1})(x-\eta_{\xi_2})\ldots(x-\eta_{\xi_n})\}\ (21.)$$

I shall first show this sum of fractions is in substance an integral function of the quantities $h_1h_2\ldots h_m$; $k_1k_2\ldots k_m$. For greater conciseness write in general $x-h=E$, $x-\eta=H$, we have then, since $h-\eta=H-E$, $h_{q_2}-h_{q_s}=E_{q_s}-E_{q_2}$ $\eta_{\xi_r}-\eta_{\xi_s}=H_{\xi_s}-H_{\xi_r}$,

$$\vartheta_\iota=\Sigma\left\{\frac{\begin{bmatrix} H_{\xi_1}H_{\xi_2}\ldots H_{\xi_v} \\ E_{q_1}E_{q_2}\ldots E_{q_v} \end{bmatrix}\times\begin{bmatrix} H_{\xi_{v+1}}H_{\xi_{v+2}}\ldots H_{\xi_n} \\ E_{q_{v+1}}E_{q_{v+2}}\ldots E_{q_m} \end{bmatrix}}{\begin{bmatrix} E_{q_{v+1}} & E_{q_{v+2}} & \ldots E_{q_m} \\ E_{q_1} & E_{q_2} & \ldots E_{q_v} \end{bmatrix}\times\begin{bmatrix} H_{\xi_{v+1}} & H_{\xi_{v+2}} & \ldots H_m \\ H_{\xi_1} & H_{\xi_2} & \ldots H_{\xi_v} \end{bmatrix}}\right..E_{q_1}E_{q_2}\ldots E_{q_m}. \quad . \quad . \quad . \quad (22.)$$

On reducing the fractions contained within the sign of summation to a common denominator, ϑ_ι will take the form $\frac{N}{D.\Delta}$, where D will be the product of the $m.\frac{m-1}{2}$ differences of E_1, E_2, $\ldots E_m$ subtracted each from each, and Δ the corresponding product of the differences *inter se* of H_1, H_2, $\ldots H_n$.

Hence, unless the sum in question is an integral function of the E's and H's, it will become infinite when any two of the E series, or any two of the H series of quantities are made equal. Suppose now $E_1=E_2$; the terms in (22.) which contain E_1-E_2 in the denominator will evidently group themselves into pairs of the respective forms,

$$\frac{(E_1.E_{q_3}\ldots E_{q_v})\times(H_{\xi_1}H_{\xi_2}\ldots H_{\xi_v})\times\begin{bmatrix} E_1\ E_{q_3}\ldots E_{q_v} \\ H_{\xi_1}H_{\xi_2}\ldots H_{\xi_v} \end{bmatrix}\times\begin{bmatrix} E_2 & E_{q_{v+2}}\ldots E_{q_m} \\ H_{\xi_{v+1}} & H_{\xi_{v+2}}\ldots H_{\xi_n} \end{bmatrix}}{\begin{bmatrix} E_1E_{q_3} & \ldots E_{q_v} \\ E_2E_{q_{v+1}} & \ldots E_{q_m} \end{bmatrix}\times\begin{bmatrix} H_{\xi_1} & H_{\xi_2} & \ldots H_{\xi_v} \\ H_{\xi_{v+1}} & H_{\xi_{v+2}} & \ldots H_{\xi_n} \end{bmatrix}}$$

and

$$\frac{(E_2.E_{q_3}\dots E_{q_v})\times(H_{\xi_1}H_{\xi_2}\dots H_{\xi_v})\times\begin{bmatrix}E_2\ E_{q_3}\dots E_{q_v}\\ H_{\xi_1}H_{\xi_2}\dots H_{\xi_v}\end{bmatrix}\times\begin{bmatrix}E_1\quad E_{q_{v+1}}\dots E_{q_m}\\ H_{\xi_{v+1}}H_{\xi_{v+2}}\dots H_{\xi_n}\end{bmatrix}}{\begin{bmatrix}E_2E_{q_3}\ \dots E_{q_v}\\ E_1E_{q_{v+1}}\dots E_{q_m}\end{bmatrix}\times\begin{bmatrix}H_{\xi_1}\ H_{\xi_2}\ \dots H_{\xi_v}\\ H_{\xi_{v+1}}H_{\xi_{v+2}}\dots H_{\xi_n}\end{bmatrix}};$$

the sum of this pair of terms will be of the form

$$\frac{P}{Q}\cdot\left\{\frac{E_1}{E_1-E_2}\cdot\frac{\begin{bmatrix}E_1\\ H_{\xi_1}H_{\xi_2}\dots H_{\xi_v}\end{bmatrix}\times\begin{bmatrix}E_2\\ H_{\xi_{v+1}}H_{\xi_{v+2}}\dots H_{\xi_n}\end{bmatrix}}{\begin{bmatrix}E_1\\ E_{q_{v+1}}E_{q_{v+2}}\dots E_{q_m}\end{bmatrix}}\right\}$$

$$+\frac{P}{Q}\cdot\left\{\frac{E_2}{E_2-E_1}\cdot\frac{\begin{bmatrix}E_2\\ H_{\xi_1}H_{\xi_2}\dots H_{\xi_v}\end{bmatrix}\times\begin{bmatrix}E_1\\ H_{\xi_{v+1}}H_{\xi_{v+2}}\dots H_{\xi_n}\end{bmatrix}}{\begin{bmatrix}E_2\\ E_{q_{v+1}}E_{q_{v+2}}\dots E_{q_m}\end{bmatrix}}\right\},$$

where Q, it may be observed, does not contain H_1-H_2, so that $\frac{P}{Q}$ remains finite when $H_1=H_2$.

The above pair of terms together make up a sum of the form

$$\frac{P}{Q}\cdot\frac{1}{E_1-E_2}\cdot\frac{\phi(E_1, E_2)\psi E_2-\phi(E_2, E_1)\psi E_1}{\psi E_1\times\psi E_2},$$

which (as the numerator of the third factor vanishes when $E_1=E_2$) remains finite on that supposition. Hence the whole sum of terms in (22.) which is made up of such pairs of terms, and of other terms in which E_1-E_2 does not enter, remains finite when $E_1-E_2=0$, and therefore generally when $D=0$, and similarly when $H_1-H_2=0$, and therefore also when $\Delta=0$; hence the expression for ϑ in (22.) is an integral function of the E and H series of quantities, as was to be proved.

Art. (17.). Let us now proceed to determine the dimensions of the coefficient of x^{ι}, the highest power of x in this value of ϑ_{ι}, when supposed to be expressed under the form of an integral function (as it has been proved to be capable of being expressed) of $h_1\ h_2\dots h_m;\quad \eta_1\ \eta_2\dots\eta_n;\quad x$.

This coefficient is the sum of fractions the numerators of each of which consist of two factors, which are respectively of $v\times\nu$ and of $(m-v)\times(n-\nu)$ dimensions in respect of the two sets of roots taken conjointly, and the denominators of two factors respectively of $v.(m-v)$ and $\nu\times(n-\nu)$ dimensions in respect of the same.

Consequently, the exponent of the total dimensions of the coefficient in question

$$=v\times\nu+(m-v)(n-\nu)-v(m-v)-\big(\nu.(n-\nu)\big)$$

$$=(m-v-\nu)\times(n-v-\nu)$$

$$=(m-\iota).(n-\iota),$$

and thus is seen to depend only on the degree ι in x of $\vartheta_{,,}$ and not upon the mode of partitioning ι into two parts v and ν, for the purpose of representing $\vartheta_{,,}$ by means of formula (19.)

Art. (18.). I shall now demonstrate that every form in this scale (to a numerical factor *près*) is identical with a simplified residue to f, ϕ of the same degree ι in x. Any such simplified residue is like $\vartheta_{,,}$ a syzygetic function, or to use a briefer form of speech, a conjunctive of f, ϕ; and if we agree to understand by the "weight" of any function of the coefficients of f and ϕ its joint dimensions in respect of the roots of f and ϕ combined, I shall prove,—1st, that any simplified residue of f and ϕ of a given degree in x is that conjunctive, whose weight in respect of the roots of f and ϕ is less than the weight of any other such conjunctive; and 2nd, that $\vartheta_{,,}$ as determined above (in equation 24.), is of the same weight as the simplified residue, and can therefore only differ from it by some numerical factor. For the purpose of comparison of weights, it will of course be sufficient to confine our attention to the coefficients of the highest (or any other, the same power, for each) in x of the forms whose weights are to be compared.

Suppose f to be of m dimensions, and ϕ to be of n dimensions in x; and let $m=n+e$.

Suppose

$$\Lambda.f+\mathrm{L}.\phi=\mathrm{A}x^i+\mathrm{B}x^{i-1}+\&c.+\mathrm{K} \quad \ldots\ldots\ldots\ldots \quad (23.)$$

$$\Lambda=\lambda_0.x^q+\lambda_1.x^{q-1}+\&c.+\lambda_q$$

$$\mathrm{L}=l_0x^{q+e}+l_1.x^{q+\delta-1}+\&c.+l_{q+e},$$

the number of homogeneous equations to be satisfied by the $q+1$ quantities $\lambda_0\lambda_1\ldots\lambda_q$, and the $q+\delta+1$ quantities μ_0, $\mu_1,\ldots\mu_{q+e}$ will be $m+q-i$, and therefore $q+1$ and $q+e+1$ taken together must be not less than $m+q-i+1$, *i. e.* $2q+e+2$ must be not less than $q+m-i+1$, *i. e.* q not less than $m-i-e-1$; and if this inequality be satisfied $2q+e+2-(q+m-i-1)+1$, *i. e.* $q+i+e-m+2$ will be the number of arbitrary constants entering into the solution of equation (23.).

If q be greater than $(n-1)$, let $q=(n-1)+t$;

and let $\quad (\Lambda)=(\lambda_0).x^{n-1}+(\lambda_1)x^{n-2}+\ldots+(\lambda_{n-1})$

$$(\mathrm{L})=l_0.x^{n+e-1}+l_1.x^{n+e-2}+\ldots+l(x_{e+n-1});$$

and let (Λ), (L) be so taken as to satisfy the equation

$$(\Lambda)f+(\mathrm{L}).\phi=\mathrm{A}x^{\iota}+\mathrm{B}x^{\iota-1}+\ldots+\mathrm{K};$$

and make $\quad \Xi=(\Lambda)+(f+gx+\ldots+hx^{t-1}).\phi$

$$\mathrm{X}=(\mathrm{L})-(f+gx+\ldots+hx^{t-1})f,$$

f, g,...h being arbitrary constants;

then $\quad \Xi f+\mathrm{X}.\phi=(\Lambda)f+(\mathrm{L})\phi=\mathrm{A}x^{\iota}+\mathrm{B}x^{\iota-1}+\ldots+\mathrm{K}.$

Now the total number of arbitrary constants in the system (Λ) and (L) will be $n-1+i+e-m+2$, *i. e.* $i+1$; hence the total number of arbitrary constants in Ξ and

X will be $i+1+t$, *i. e.* $q-n+i+2$, which is equal to $q+i+e-m+2$, the number of arbitrary constants in the most general values of Λ and L. Hence $\{\Lambda=\Xi\,;\ L=X\}$ is the general solution of $\Lambda.f+L.\phi=Ax^i+Bx^{i-1}+\ldots+K$; and consequently the most general form of $Ax^i+Bx^{i-1}+\ldots+K$, which is evidently independent of the (t) arbitrary quantities $f, g\ldots h$, will contain the same number of arbitrary constants as enter into the system (Λ) and (L), *i. e.* $i+1$.

Art. (19.). Let us now begin with the case of greater simplicity when $m=n$, *i. e.* $e=0$; and let us revert to the system of equations marked (10.) in Section I., in which U and V are to be replaced by f and ϕ.

1st. Let $i=n-1$, and therefore $i+1$, the number of arbitrary quantities in the conjunctive is n.

From the system of equations (10.), we have for all values of $\varrho_1, \varrho_2, \varrho_3\ldots\varrho_n$,

$$(\varrho_1 Q_0+\varrho_2 Q_1+\ldots+\varrho_n . Q_{n-1})f$$
$$-(\varrho_1 P_0+\varrho_2 . P_1+\ldots+\varrho_n . P_{n-1})\phi$$
$$=(\varrho_1 . K_1+\varrho_2 . {}_1K_1+\ldots+\varrho_n . {}_{n-1}K_1)x^{n-1}+\&c.,$$

and consequently the most *general* value of ϑ_{n-1} in the equation

$$\tau_{n-1} . f-t_{n-1} . \phi+\vartheta_{n-1}=0,$$

where $$\vartheta_{n-1}=Ax^{n-1}+Bx^{n-2}+\ldots+L$$

will be obtained by making

$$\tau_{n-1}=\varrho_1 . Q_0+\varrho_2 . Q_1+\ldots+\varrho_n . Q_n$$
$$t_{n-1}=-\varrho_1 . P_0-\varrho_2 . P_1\ldots-\varrho_n . P_n,$$

which solution contains n, *i. e.* the proper number of arbitrary contants.

Again, if $i=n-2$ $i+1=n-1$, which will therefore be the number of arbitrary constants in the most general value of ϑ_{n-2} of the equation

$$\tau_{n-2} . f-t_{n-2} . \phi+\vartheta_{n-2}=0.$$

This most general value of ϑ_{n-2} is therefore found by making

$$\tau_{n-2}=\varrho'_1 Q_0+\varrho'_2 . Q_1+\ldots+\varrho'_n . Q_n$$
$$t_{n-1}=-\varrho'_1 P_0-\varrho'_2 . P_1\ldots-\varrho'_n . P_n,$$

where $\varrho'_1, \varrho'_2,\ldots\varrho'_n$ are no longer entirely independent, but subject to the equation

$$\varrho'_1 . K_1+\varrho'_2 . {}_1K_1+\ldots+\varrho'_n . {}_{n-1}K_1=0,$$

so as to leave $(n-1)$ constants arbitrary.

We thus obtain $\vartheta_{n-2}=(\varrho'_1 K_2+\varrho'_2 . {}_1K_2+\ldots+\varrho'_n . {}_{n-1}K_2)x^{n-2}+\&c.$ In like manner, and for the same reasons, the most general values of ϑ_{n-3} in the equation

$$\tau_{n-3} . f-t_{n-3} . \phi+\vartheta_{n-3}=0$$

will be found by making

$$\tau_{n-3}=\varrho''_1 . Q_0+\varrho''_2 . Q_1+\ldots+\varrho''_n . Q_{n-1}$$
$$t_{n-3}=-\varrho''_1 . P_0-\varrho''_2 . Q_1\ldots-\varrho''_n . P_{n-1},$$

where ϱ_1'', ϱ_2'', ...ϱ_n'' are subject to satisfying the two equations

$$\varrho_1''.\mathrm{K}_1+\varrho_2''.{}_1\mathrm{K}_1+\ldots+\varrho_n''.{}_{n-1}\mathrm{K}_1=0$$
$$\varrho_1''.\mathrm{K}_2+\varrho_2''\ {}_1\mathrm{K}_2+\ldots+\varrho_n''.{}_{n-1}\mathrm{K}_2=0,$$

so as to leave $(n-2)$ constants arbitrary; and we thus obtain

$$\vartheta_{n-3}=(\varrho_1''.\mathrm{K}_3+\varrho_2''.{}_1\mathrm{K}_3+\ldots+\varrho_n''.{}_{n-1}\mathrm{K}_3)x^{n-3}+\&c.,$$

and so on, the number of independent arbitrary constants in ϑ decreasing (as it ought) each time by one unit as the degree of ϑ descends, until finally, if $\tau_0.f-t_0.\varphi+\vartheta_0=0$, ϑ_0 being a constant, the general value for ϑ_0 is found by making

$$\tau_0=(\varrho_1).\mathrm{Q}_0+(\varrho_2)\mathrm{Q}_1+\ldots+(\varrho_n).\mathrm{Q}_{n-1}$$
$$t_0=-(\varrho_1)\mathrm{P}_0-(\varrho_2)\mathrm{P}_1-\ldots-(\varrho_n).\mathrm{P}_{n-1},$$

where ϱ_1, ϱ_2, ...ϱ_n are subject to satisfy the $(n-1)$ equations

$$(\varrho_1).\mathrm{K}_1+\&c.=0$$
$$(\varrho_1).\mathrm{K}_2+\&c.=0$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$
$$(\varrho_1).\mathrm{K}_{n-1}+\&c.=0,$$

which gives $$\vartheta_0=\mathrm{K}_n(\varrho)_1+{}_2\mathrm{K}_n(\varrho)_n+\ldots+{}_{n-1}\mathrm{K}_n(\varrho)_n.$$

Now evidently the lowest weight in respect to the roots of U and V that can be given to $(\varrho_1\mathrm{K}_1+\varrho_2\ {}_1\mathrm{K}_1+\ldots+\varrho_n\ {}_{n-1}\mathrm{K}_1)x^{n-1}+\&c.$, when the multipliers ϱ_1, ϱ_2, ...ϱ_n are absolutely independent, is found by taking $\varrho_1=1$ $\varrho_2=0$ $\varrho_3=0\ldots\varrho_n=0$, which makes the weight of the leading coefficient in ϑ_{n-1}, the same as that of K_1, *i. e.* 1.

Again, when one equation,

$$\varrho_1\ \mathrm{K}_1+\varrho_2\ {}_1\mathrm{K}_1+\ldots+\varrho_n'\ {}_{n-1}\mathrm{K}_1=0,$$

exists between the (ϱ)'s, the lowest weight will be found by making

$$\varrho_1'={}_1\mathrm{K}_1\quad \varrho_2'=-\mathrm{K}_1\quad \varrho_3=0\quad \varrho_4'=0\ldots\varrho_n'=0,$$

which makes the weight of the leading coefficient in ϑ_{n-2} depend on

$${}_1\mathrm{K}_1\ \mathrm{K}_2-\mathrm{K}_1\ {}_1\mathrm{K}_2,$$

which is of the weight $1+3$, *i. e.* 4 in respect of the roots of f and φ.

Similarly, ϑ_{n-3} will have its lowest weight when its leading coefficient is the determinant

$$\begin{matrix}\mathrm{K}_1 & \mathrm{K}_2 & \mathrm{K}_3\\ {}_1\mathrm{K}_1 & {}_1\mathrm{K}_2 & {}_1\mathrm{K}_3\\ {}_2\mathrm{K}_1 & {}_2\mathrm{K}_2 & {}_2\mathrm{K}_3,\end{matrix}$$

the weight of which is $1+3+5=9$; and finally, the lowest weighted value of ϑ_0 is the determinant represented by the complete Bezoutian square; the weight in general of ϑ_{n-i} being $1+3+\ldots+(2\iota-1)$, *i. e.* i^2, or which is the same thing otherwise expressed, the weight of the leading coefficient of the lowest-weighted conjunctive of f and φ of the degree ι in x is $(n-\iota)(m-\iota)$*. It will of course have been seen in the fore-

* n and m are supposed equal and $\iota=n-i$.

going demonstration, that the weight of ${}_rK'_s$ [which means $\Sigma(a_r.b_s-a_s.b_r)$ a_r, a_s being the coefficients of x^{n-r}, x^{n-s} in f, and b_r, b_s of the same in ϕ] has been correctly taken to be $r+s$ in respect of the roots of f and ϕ conjoined.

Art. (20.). If now we proceed in like manner with the general case of $m=n+e$, it may be shown, in precisely the same way as in the preceding article, that the most general value of any conjunctive of f and ϕ will be a linear function of (e) functions,

$$\begin{array}{l} x^n \ +a_1.x^{n-1}+a_2\,x^{n-2}+\ldots+a_n \\ x^{n+1}+a_1.x^n \ +a_2.x^{n-1}+\ldots+a_n.x \\ x^{n+2}+a_1.x^{n+1}+a_2.x^n \ +\ldots+a_n\,x^2 \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ x^{m-1}+a_1.x^{m-2}+\&c. \qquad +a_nx^{e-1}, \end{array}$$

and of the (n) functions,

$$\begin{array}{l} {}_0K_1.x^{n-1}+\ {}_0K_2.x^{n-2}+\ldots+\ {}_0K_n \\ {}_1K_1.x^{n-1}+\ {}_1K_2.x^{n-2}+\ldots+\ {}_1K_n \\ \qquad \&c. \qquad\qquad \&c. \\ {}_{n-1}K_1.x^{n-1}+{}_{n-1}K_2\,x^{n-2}+\ldots+{}_{n-1}K_n, \end{array}$$

and that consequently, if the degree of such conjunctive in x be $(n-i)$, it will be of the lowest weight when it is a linear function of the entire (e) upper set of functions, and (ι) of the lower set; and consequently, the coefficient of the highest power of x in such conjunctive will be the determinant

$$\begin{array}{llllll} {}_0K_1 & {}_0K_2 & {}_0K_3\ldots\ldots{}_0K_i & \ldots\ldots\ldots\ldots\ldots\ldots{}_0K_{i+e} \\ {}_1K_1 & {}_1K_2 & {}_1K_3\ldots\ldots{}_1K_i & \ldots\ldots\ldots\ldots\ldots\ldots{}_1K_{i+e} \\ {}_2K_1 & {}_2K_2 & {}_2K_3\ldots\ldots{}_2K_i & \ldots\ldots\ldots\ldots\ldots\ldots{}_2K_{i+e} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ {}_{i-1}K_1 & {}_{i-1}K_2 & {}_{i-1}K_3\ldots{}_{i-1}K_2 & \ldots\ldots\ldots\ldots\ldots{}_{i-1}K_{i+e} \\ 1 & a_1 & a_2\ldots\ldots.a_{i-1} & a_i\ldots\ldots\ldots\ldots a_{i+e} \\ & 1 & a_1\ldots\ldots.a_{i-1} & a_i\ldots\ldots\ldots\ldots a_{i+e-1} \\ & & 1\ \ldots\ldots.a_{i-2} & a_i\ldots\ldots\ldots\ldots a_{i+e-2} \\ & & \cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot \\ & & \cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot \\ & & & 1\quad .\quad .\quad .\quad .\quad a_i, \end{array}$$

the weight of which is evidently that of

$${}_0K_1\times{}_1K_2\times{}_2K_3\ldots\times{}_{i-1}K_i\times(a_i)^e,$$

$$i.\ e.\ 1+3+5+\ldots+(2i-1)+e.i$$

$$i.\ e.\ i^2+ei,\ \text{or}\ i(e+i),\ \text{which is}\ (n-\iota)(m-\iota)\ \text{if}\ \iota=n-i.$$

Hence the weight of the leading coefficient in the lowest-weighted conjunctive of f and φ of the degree ι in x is $(m-\iota)(n-\iota)$, m being the degree of f and n of φ.

From this we infer that any conjunctive of f and φ of the degree ι, of which the leading coefficient is of the weight $(m-\iota)(n-\iota)$ (all the coefficients being of course understood to be integral functions of the roots of f and φ), must, to a numerical factor *près*, be equivalent to any other of the same weight; and furthermore, any supposed function of x of the ιth degree which possesses the property characteristic of a conjunctive of vanishing, when f and φ vanish simultaneously, but of which the weight of the leading coefficient would be *less* than $(m-\iota)(n-\iota)$, must be a mere nugatory form and have all its terms *identically zero**.

Art. (21.). We have previously shown, art. (16.), that ϑ, as defined by equation (21.), is an integral function of the roots f and φ, and vanishes when f and φ vanish. Moreover, its weight in the roots has been proved to be $(m-\iota)(n-\iota)$, and consequently, if by way of distinguishing the several forms of ϑ, we name that one where ι in the equation above cited is supposed to be divided into two parts, v and ν, $\vartheta_{v,\nu}$, we have for all values of v and ν, such that $v+\nu$ is not greater than n, $\vartheta_{v,\nu}$ to a constant numerical factor *près* identical with the $(v+\nu)$th simplified residue to (f, φ), so that the form of $\vartheta_{v,\nu}$ depends only upon the value of $v+\nu$.

Art. (22.). It must be well borne in mind that this permanency of the value of $\vartheta_{v,\iota-v}$ for different values of v has only been established for the case where i can be the degree of a residue to f and φ, that is to say, when i is less than the lesser of the two indices m and n. When i does not satisfy this condition of inequality, the theorem ceases to be true. It is clear that when $m=n$ and $v+\nu=m=n$, $\vartheta_{v,\nu}$ which always remains a conjunctive of f and φ, can only be a numerical linear function of f and φ; and I have ascertained when $m=n$ on giving to v and ν the respective values successively $(0, n)$, $(1, n-1)$, $(2, (n-2))$, ... $(n, 0)$

that $\quad \vartheta_{0,n}=f;\ \vartheta_{1,n-1}=(n-1)f+\varphi;\ \vartheta_{2,n-2}=\frac{(n-1)(n-2)}{1.2}f+(n-1)\varphi\ldots$

$$\vartheta_{n-1,1}=f+(n-1)\varphi,\ \vartheta_{n,0}=\varphi.$$

Thus, by way of a simple example, let

$$f=x^2+ax+b=(x-h_1)(x-h_2)$$

$$\varphi=x^2+\alpha x+\beta=(x-\eta_1)(x-\eta_2)$$

$$\vartheta_{0,2}=(x-h_1)(x-h_2)\left\{\frac{\begin{bmatrix}h_1h_2\\ \cdot\cdot\end{bmatrix}\times\begin{bmatrix}\cdot\cdot\\ k_1k_2\end{bmatrix}}{\begin{bmatrix}h_1h_2\\ \cdot\cdot\end{bmatrix}\quad\begin{bmatrix}\cdot\cdot\\ k_1k_2\end{bmatrix}}\right.=(x-h_1)(x-h_2)=f$$

* And more generally it admits of being demonstrated by precisely the same course of reasoning, that the number of arbitrary parameters in a conjunctive of the degree i, and of the weight $(m-i)(n-i)+\epsilon$ in the roots cannot (abstraction being supposed to be made of an arbitrary numerical multiplier) exceed the number ϵ.

$$\vartheta_{1,1}=\Sigma(x-h_1)(x-k_1)\frac{\begin{bmatrix}h_1\\k_1\end{bmatrix}\times\begin{bmatrix}h_2\\k_2\end{bmatrix}}{\begin{bmatrix}h_1\\h_2\end{bmatrix}\times\begin{bmatrix}k_1\\k_2\end{bmatrix}}$$

$$=\Sigma\frac{x-h_1}{h_1-h_2}\Sigma\frac{x-k_1}{k_1-k_2}.\{(h_1-k_1)(h_2-k_2)\},$$

$$i.e. =\Sigma\frac{x-h_1}{h_1-h_2}.\left\{\frac{1}{k_1-k_2}\begin{pmatrix}(x-k_1)(h_1-k_1)(h_2-k_2)\\-(x-k_2)(h_1-k_2)(h_2-k_1)\end{pmatrix}\right\}$$

$$=\Sigma\frac{x-h_1}{h_1-h_2}.\left\{(h_1-h_2)x+\{(k_1+k_2)h_2-(h_1h_2+k_1k_2)\}\right\}$$

$$=(x-h_1)x+(x-h_2)x-(k_1+k_2)x+(h_1h_2+k_1k_2)$$

$$=(x^2-(h_1+h_2)x+h_1h_2)+(x^2-(k_1+k_2)x+k_1k_2)$$

$$=(x^2+ax+b)+(x^2+\alpha x+\beta)$$

$$=f+\phi;$$

so we find also $\vartheta_{2,0}=\phi$.

Art. (23.). The expression $\vartheta_{v,\nu}$, which is universally a conjunctive of f and ϕ, continues algebraically interpretable so long as $v+\nu$ has any value intermediate between (0) and $m+n$; when $v+\nu=0$, we must of course have $v=0$ and $\nu=0$, and $\vartheta_{0,0}$ becomes the resultant of f and ϕ when $v+\nu=m+n$; we must also have the unique solution $v=m$ and $\nu=n$, and $\vartheta_{m,n}$ becomes necessarily $f\times\phi$, which we thus see stands in a sort of antithetical relation to the resultant of f and ϕ, say (f,ϕ). Nor is it without interest to remark that $f\times\phi=0$ implies that a root of f or else of ϕ is zero; and $(f,\phi)=0$ implies that if a root of the one of the functions is zero, so also is a root of the other, *i. e.* that a root of each or of neither is zero. As i increases from 0 to n or decreases from $m+n$ to $m-1$, the number of solutions of the equation $v+\nu=i$ in the one case, and the number of admissible solutions of the equation $v+\nu=i$ in the other case, which is subject to the condition that ν must not exceed n, continues to increase by a unit at each step; there being thus $n+1$ different forms $\vartheta_{v,\nu}$ when $v+\nu=n$, and the same number when $v+\nu=m-1$. For all values of i intermediate between n and $(m-1)$ (both taken exclusively) it is very remarkable that $\vartheta_{v,\nu}$ will vanish, as I proceed to demonstrate.

Art. (24.). The weight of the coefficient of the highest power of $\vartheta_{v,\nu}$ ($v+\nu$ being equal to i) is $(m-i)(n-i)$, and consequently, when i is greater than n, and less than m, $\vartheta_{v,\nu}$ would contain fractional functions of the roots of f and ϕ, if there were in it a power x^i, but $\vartheta_{v,\nu}$ has been proved to be always an integer function of the roots. Hence the coefficient of x^i will be zero, and so more generally the first power of x in $\vartheta_{v,\nu}$, of which the coefficient is not zero, will be x^ω, subject to the condition (since evidently the weight of the several coefficients goes on increasing by units as the degree of the terms in x decreases by the same) that ω be not less than $(m-i)(i-n)$; let then $\omega=(m-i)(i-n)$, $\vartheta_{v,\nu}$ becomes of the form $Ax^{i-\omega}+Bx^{i-\omega-1}+$ &c., where A is of zero dimensions; but this is impossible if $i-\omega<n$, for then $Ax^{i-\omega}+$ &c. is a conjunctive of

weight lower than the lowest-weighted simplified residue of the degree $i-\omega$. Hence ω is not greater than $i-n$ or $(m-i)(i-n)$ is not greater than $i-n$, *i. e.* $m-i$ cannot be greater than 1, *i. e.* i when intermediate between m and n cannot be less than $m-1$, otherwise $\vartheta_{v,\nu}$ will vanish identically. Moreover, when $i=m-1$, $\omega=i-n$, and $i-\omega=n$, and accordingly $\vartheta_{v,m+1-v}$ is not merely, as we might know, *à priori* an algebraical, but more simply a numerical multiple of φ for all values of v. The same is of course true also, m being greater than n, for every form $\vartheta_{v,n-v}$, since this is always a conjunctive of f and φ, of which the former is of a degree higher than the ϑ in question, so that the multiplier of f in this conjunctive must be zero*.

Art. (25.). To enter into a further or more detailed examination of the values assumed by $\vartheta_{v,\nu}$ for the most general values of m, n, i, would be to transcend the limits I have proposed to myself in drawing up the present memoir. What we have established is, that to every form of $\vartheta_{v,i-v}$ appertaining to a value of i between 0 and n, there is a sort of conjugate form for which i lies between $m+n$ and m; that for $i=m-1$ or $i=n$, $\vartheta_{v,i-v}$ becomes a numerical multiplier of φ; and that when i lies in the intermediate region between n and $m-1$, $\vartheta_{v,i-v}$ vanishes for all values of v. I pause only for a moment to put together for the purpose of comparison the forms corresponding to i and to $m+n-i$. By art. (16.), making $i=v+\nu$,

$$\vartheta_i=\Sigma(x-h_{q_1})(x-h_{q_2})\dots(x-h_{q_v})\times(x-\eta_{\xi_1})(x-\eta_{\xi_2})\dots(x-\eta_{\xi_\nu})$$

$$\times\frac{\begin{bmatrix}h_{q_1}\,h_{q_2}\dots h_{q_v}\\ \eta_{\xi_1}\,\eta_{\xi_2}\dots\eta_{\xi_\nu}\end{bmatrix}\times\begin{bmatrix}h_{q_{v+1}}\,h_{q_{v+2}}\dots h_{q_m}\\ \eta_{\xi_{\nu+1}}\,\eta_{\xi_{\nu+2}}\dots\eta_{\xi_n}\end{bmatrix}}{\begin{bmatrix}h_{q_1}\quad h_{q_2}\quad\dots h_{q_v}\\ h_{q_{v+1}}\,h_{q_{v+2}}\dots h_{q_m}\end{bmatrix}\times\begin{bmatrix}\eta_{\xi_1}\quad\eta_{\xi_2}\quad\dots\eta_{\xi_\nu}\\ \eta_{\xi_{\nu+1}}\,\eta_{\xi_{\nu+2}}\dots\eta_{\xi_n}\end{bmatrix}}.$$

The conjugate form for which $i'=m+n-i$ and $m-v$ $n-\nu$ take the places of v and ν $(m-v)(n-\nu)$ will be got by taking

$$\vartheta_{i'}=\Sigma(x-h_{q_{v+1}})(x-h_{q_{v+2}})\dots(x-h_{q_m})\times(x-\eta_{\xi_{\nu+1}})(x-\eta_{\xi_{\nu+2}})\dots(x-\eta_{\xi_n})$$

$$\times\frac{\begin{bmatrix}h_{q_1}\,h_{q_2}\dots h_{q_v}\\ \eta_{\xi_1}\,\eta_{\xi_2}\dots\eta_{\xi_\nu}\end{bmatrix}\times\begin{bmatrix}h_{q_{v+1}}\,h_{q_{v+2}}\dots h_{q_m}\\ \eta_{\xi_{\nu+1}}\,\eta_{\xi_{\nu+2}}\dots\eta_{\xi_n}\end{bmatrix}}{\begin{bmatrix}h_{q_1}\quad h_{q_2}\quad\dots h_{q_v}\\ h_{q_{v+1}}h_{q_{v+2}}\dots h_{q_m}\end{bmatrix}\times\begin{bmatrix}\eta_{\xi_1}\quad\eta_{\xi_2}\quad\dots\eta_{\xi_\nu}\\ \eta_{\xi_{\nu+1}}\,\eta_{\xi_{\nu+2}}\dots\eta_{\xi_n}\end{bmatrix}},$$

which it will be perceived are identical, term for term, in the fractional constant factor, and differ only in the linear functions of x, which in ϑ_i and in $\vartheta_{i'}$ are complementary to one another. Our proper business is only with those forms for which $i<n$.

Art. (26.). It will presently be seen to be necessary to ascertain the numerical relations between $\vartheta_{0,i}$ and $\vartheta_{i,0}$ when $i<n$, and this naturally brings under our notice the

* It thus appears that if the indices m and n do not differ by at least 3 units, ϑ_i will have an actual quantitative existence for all values of i between 0 and $m+n$; or in other words, the failure in the quantitative existence of the forms ϑ_i only begins to show itself when this difference is 3; thus if $m=n+3$, ϑ_n exists, and ϑ_{n+2} exists, but $\vartheta_{n+1}=0$.

inquiry into the numerical relations which exist between the entire series of forms $\vartheta_{v,\,i-v}$ for a given value of i, corresponding to all values of v between 0 and i inclusive.

In order to avoid a somewhat oppressive complication of symbols, I shall take a particular numerical example, *i. e.* $m=7$ $n=6$ $i=4$, and compare the values of $\vartheta_{0,4}$; $\vartheta_{1,3}$; $\vartheta_{2,2}$; $\vartheta_{3,1}$; $\vartheta_{4,0}$, all of which we know to be identical, [to a numerical factor *près*] with one another and with the second simplified residue to f and ϕ, that being of the fourth degree in x; our object in the subjoined investigation is to determine the numerical ratios of these several forms of ϑ to one another.

First. Let $v=0$ $\nu=4$. The leading coefficient $\vartheta_{0,4}$ is

$$\Sigma\frac{\genfrac{}{}{0pt}{}{\eta_5\,\eta_6}{h_1\,h_2\,h_3\,h_4\,h_5\,h_6\,h_7}}{\genfrac{}{}{0pt}{}{\eta_5\,\eta_6}{\eta_1\,\eta_2\,\eta_3\,\eta_4}}$$

which we know *à priori* (it should be observed) to be essentially an *integral* function of the h and the η system. In this, the term containing η_6^3 will be evidently

$$\text{(A.)}\qquad \Sigma\frac{\genfrac{}{}{0pt}{}{\eta_5}{h_1\,h_2\,n_3\,h_4\,h_5\,h_6\,h_7}}{\genfrac{}{}{0pt}{}{\eta_5}{\eta_1\,\eta_2\,\eta_3\,\eta_4}},$$

the η system to which the latter summation relates being now reduced to consist of $\eta_1\,\eta_2\,\eta_3\,\eta_4\,\eta_5$. In this expression, again, the coefficient of η_5^3 is evidently 1. Hence, therefore, the leading coefficient in $\vartheta_{0,4}$ contains the term $\eta_5^3.\eta_6^3$.

Secondly. Let $v=1$ $\nu=3$. The leading coefficient in $\vartheta_{1,3}$ becomes

$$\Sigma\frac{\genfrac{[}{]}{0pt}{}{\eta_1\,\eta_2\,\eta_3}{h_1}\times\genfrac{[}{]}{0pt}{}{\eta_4\,\eta_5\,\eta_6}{h_2\,h_3\,h_4\,h_5\,h_6\,h_7}}{\genfrac{[}{]}{0pt}{}{h_2\,h_3\,h_4\,h_5\,h_6\,h_7}{h_1}\times\genfrac{[}{]}{0pt}{}{\eta_4\,\eta_5\,\eta_6}{\eta_1\,\eta_2\,\eta_3}}.$$

In this, the factor affecting η_6^3 will be

$$\Sigma\frac{\genfrac{[}{]}{0pt}{}{\eta_1\,\eta_2\,\eta_3}{h_1}\times\genfrac{[}{]}{0pt}{}{\eta_4\,\eta_5}{h_2\,h_3\,h_4\,h_5\,h_6\,h_2}}{\genfrac{[}{]}{0pt}{}{h_2\,h_3\,h_4\,h_5\,h_6\,h_7}{h_1}\times\genfrac{[}{]}{0pt}{}{\eta_4\,\eta_5}{\eta_1\,\eta_2\,\eta_3}},$$

η_6 being now understood to be eliminated out of the η system included within the above summation. Again, in this latter sum the factor affecting η_5^3 will be

$$\text{(B.)}\qquad \Sigma\frac{\genfrac{[}{]}{0pt}{}{\eta_1\,\eta_2\,\eta_3}{h_1}\times\genfrac{[}{]}{0pt}{}{\eta_4}{h_2\,h_3\,h_4\,h_5\,h_6\,h_7}}{\genfrac{[}{]}{0pt}{}{h_2\,h_3\,h_4\,h_5\,h_6\,h_7}{h_1}\times\genfrac{[}{]}{0pt}{}{\eta_4}{\eta_1\,\eta_2\,\eta_3}},$$

η_5 and η_6 being now both eliminated out of the η system. This last sum can of course only represent a numerical quantity.

So in like manner, again, if $v=2$ $\nu=2$, the coefficient of $\eta_6^3.\eta_5^3$ will be similarly reducible to the form

(C.) $$\Sigma\frac{\left[\begin{matrix}\eta_1\ \eta_2\\ h_1\ h_2\end{matrix}\right]\times\left[\begin{matrix}\eta_3\ \eta_4\\ h_3\ h_4\ h_5\ h_6\ h_7\end{matrix}\right]}{\left[\begin{matrix}h_3\ h_4\ h_5\ h_6\ h_7\\ h_1\ h_2\end{matrix}\right]\times\left[\begin{matrix}\eta_3\ \eta_4\\ h_1\ h_2\end{matrix}\right]}\text{ in }\vartheta_{2,\,2}.$$

So, again, when $v=3$ $\nu=1$, the coefficient of $\eta_6^3.\eta_5^3$ will be

(D.) $$\Sigma\frac{\left[\begin{matrix}\eta_1\\ h_1\ h_2\ h_3\end{matrix}\right]\times\left[\begin{matrix}\eta_2\ \eta_3\ \eta_4\\ h_4\ h_5\ h_6\ h_7\end{matrix}\right]}{\left[\begin{matrix}h_4\ h_5\ h_6\ h_7\\ h_1\ h_2\ h_3\end{matrix}\right]\times\left[\begin{matrix}\eta_2\ \eta_3\ \eta_4\\ \eta_1\end{matrix}\right]}\text{ in }\vartheta_{3,\,1};$$

and finally, the coefficient of $\eta_6^3.\eta_5^3$ will be

(E.) $$\Sigma\frac{\left[\begin{matrix}\eta_1\ \eta_2\ \eta_3\ \eta_4\\ h_5\ h_6\ h_7\end{matrix}\right]}{\left[\begin{matrix}h_5\ h_6\ h_7\\ h_1\ h_2\ h_3\ h_4\end{matrix}\right]}\text{ in }\vartheta_{0,\,4};$$

out of all which sums it is to be remembered that η_5 and$_6$ are supposed excluded from appearing. All these several coefficients being numbers in disguise, we may determine them by giving any values at pleasure to the terms in the h and η system.

Let now $\eta_1=h_1$ $\eta_2=h_2$ $\eta_3=h_3$ $\eta_4=h_4$, then in (B.) it will readily be seen that all the terms included within the sign of summation vanish identically, except the following, viz.—

$$\frac{\left[\begin{matrix}\eta_1\ \eta_2\ \eta_3\\ h_4\end{matrix}\right]\times\left[\begin{matrix}\eta_4\\ h_1\ h_2\ h_3\ h_5\ h_6\ h_7\end{matrix}\right]}{\left[\begin{matrix}h_1\ h_2\ h_3\ h_5\ h_6\ h_7\\ h_4\end{matrix}\right]\times\left[\begin{matrix}\eta_4\\ \eta_1\ \eta_2\ \eta_3\end{matrix}\right]}$$

$$\frac{\left[\begin{matrix}\eta_1\ \eta_2\ \eta_4\\ h_3\end{matrix}\right]\times\left[\begin{matrix}\eta_3\\ h_1\ h_2\ h_4\ h_5\ h_6\ h_7\end{matrix}\right]}{\left[\begin{matrix}h_1\ h_2\ h_4\ h_5\ h_6\ h_7\\ h_3\end{matrix}\right]\times\left[\begin{matrix}\eta_3\\ \eta_1\ \eta_2\ \eta_4\end{matrix}\right]}$$

$$\frac{\left[\begin{matrix}\eta_1\ \eta_3\ \eta_4\\ h_2\end{matrix}\right]\times\left[\begin{matrix}\eta_2\\ h_1\ h_3\ h_4\ h_5\ h_6\ h_7\end{matrix}\right]}{\left[\begin{matrix}h_1\ h_3\ h_4\ h_5\ h_6\ h_7\\ h_2\end{matrix}\right]\times\left[\begin{matrix}\eta_2\\ \eta_1\ \eta_3\ \eta_4\end{matrix}\right]}$$

$$\frac{\left[\begin{matrix}\eta_2\ \eta_3\ \eta_4\\ h_1\end{matrix}\right]\times\left[\begin{matrix}\eta_1\\ h_2\ h_3\ h_4\ h_5\ h_6\ h_7\end{matrix}\right]}{\left[\begin{matrix}h_2\ h_3\ h_4\ h_5\ h_6\ h_7\\ h_1\end{matrix}\right]\times\left[\begin{matrix}\eta_1\\ \eta_2\ \eta_3\ \eta_4\end{matrix}\right]}.$$

In each of these expressions the first factor of the numerator is identical in value (by reason of the equations $h_1=\eta_1$ $h_2=\eta_2$ $h_3=\eta_3$ $h_4=\eta_4$) with $(-)^3\times$ the second factor of the denominator, and the second factor of the numerator with $(-)^6\times$ the first factor of the denominator; hence the coefficient of $\eta_5^3.\eta_6^3$ in $\vartheta_{1,3}$ is -4.

In like manner the only effective terms of $\vartheta_{2,2}$ will be

$$\frac{\begin{bmatrix}\eta_1\ \eta_2\\ h_3\,h_4\end{bmatrix}\times\begin{bmatrix}\eta_3\ \eta_4\\ h_1\,h_2\,h_5\,h_6\,h_7\end{bmatrix}}{\begin{bmatrix}h_1\,h_2\,h_5\,h_6\,h_7\\ h_3\,h_4\end{bmatrix}\times\begin{bmatrix}\eta_3\,\eta_4\\ \eta_1\,\eta_2\end{bmatrix}},\quad \frac{\begin{bmatrix}\eta_3\ \eta_4\\ h_1\,h_2\end{bmatrix}\times\begin{bmatrix}\eta_1\ \eta_2\\ h_3\,h_4\,h_5\,h_6\,h_7\end{bmatrix}}{\begin{bmatrix}h_3\,h_4\,h_5\,h_6\,h_7\\ h_1\,h_2\end{bmatrix}\times\begin{bmatrix}\eta_1\,\eta_2\\ \eta_3\,\eta_4\end{bmatrix}}$$

$$\frac{\begin{bmatrix}\eta_1\ \eta_3\\ h_2\,h_4\end{bmatrix}\times\begin{bmatrix}\eta_2\ \eta_4\\ h_1\,h_3\,h_5\,h_6\,h_7\end{bmatrix}}{\begin{bmatrix}h_1\,h_3\,h_5\,h_6\,h_7\\ h_2\,h_4\end{bmatrix}\times\begin{bmatrix}\eta_2\,\eta_4\\ \eta_1\,\eta_3\end{bmatrix}},\quad \frac{\begin{bmatrix}\eta_2\ \eta_4\\ h_1\,h_3\end{bmatrix}\times\begin{bmatrix}\eta_1\ \eta_3\\ h_2\,h_4\,h_5\,h_6\,h_7\end{bmatrix}}{\begin{bmatrix}h_2\,h_4\,h_5\,h_6\,h_7\\ h_1\,h_3\end{bmatrix}\times\begin{bmatrix}\eta_1\,\eta_3\\ \eta_2\,\eta_4\end{bmatrix}}$$

$$\frac{\begin{bmatrix}\eta_1\ \eta_4\\ h_2\,h_3\end{bmatrix}\times\begin{bmatrix}\eta_2\ \eta_3\\ h_1\,h_4\,h_5\,h_6\,h_7\end{bmatrix}}{\begin{bmatrix}h_1\,h_4\,h_5\,h_6\,h_7\\ h_2\,h_3\end{bmatrix}\times\begin{bmatrix}\eta_1\,\eta_4\\ \eta_2\,\eta_3\end{bmatrix}},\quad \frac{\begin{bmatrix}\eta_2\ \eta_3\\ h_1\,h_4\end{bmatrix}\times\begin{bmatrix}\eta_1\ \eta_4\\ h_2\,h_3\,h_5\,h_6\,h_7\end{bmatrix}}{\begin{bmatrix}h_2\,h_4\,h_5\,h_6\,h_7\\ h_2\,h_3\end{bmatrix}\times\begin{bmatrix}\eta_2\,\eta_3\\ \eta_1\,\eta_4\end{bmatrix}}.$$

Any other term will necessarily contain in the numerator a factor, whose symbolical representation will contain one of the quantities η_1 η_2 η_3 η_4 in the upper line, and one of the quantities h_1 h_2 h_3 h_4, having the same subscript index in the lower line, and which will therefore vanish; the number of effective terms being evidently the number of ways in which four things can be combined 2 and 2 together, and the value of each term is evidently $(-)^{2.2}.(-1)^{2.5}.1$, so that the entire value of the coefficient of $\eta_5^3.\eta_6^3$ in $\vartheta_{2,2}$ is $+6$.

Precisely in the same manner, we shall find that the leading coefficient in $\vartheta_{3,1}$ will contain the term $-4\eta_5^3.\eta_6^3$, the (-1) resulting from the operation $(-1)^{1.3}.(-1)^{3.4}$, and in $\vartheta_{4,0}$ the term $+\eta_5^3.\eta_6^3$, the $+1$ resulting from the operation $(-1)^{4.3}$. Hence it appears that $\vartheta_{0,4}$; $\vartheta_{1,3}$; $\vartheta_{2,2}$; $\vartheta_{3,1}$; $\vartheta_{4,0}$ are to one another in the ratios of 1; -4; 6; -4; 1; and so in general for any values of m, n, i (i being less than m and less than n) it will be found that

$$\vartheta_{0,i},\quad \vartheta_{1,i-1},\quad \vartheta_{2,i-2},\ldots\vartheta_{i,0}$$

will be in the ratios of the numbers

$$1;\quad (-1)^{m-1}.i;\quad (-1)^{2(m-2)}.i.\frac{i-1}{2};\quad (-1)^{3(m-3)}.i.\frac{i-1}{2}.\frac{i-2}{3},\ldots;\quad (-1)^{i(m-i)}.$$

Art. (27.). The method employed in the preceding investigation will enable us to affix the proper sign and numerical factor to $\vartheta_{0,i}$ or $\vartheta_{i,0}$, or in general to $\vartheta_{v,i-v}$, in order that it may represent the Bezoutian secondary of the degree i in x. [This latter has been already identified with the simplified residue obtained by expanding $\frac{\phi x}{fx}$ under the form of an improper continued fraction.] For this purpose, it will be sufficient to compare a single term of any such ϑ with the corresponding one in the Symmorphic Bezoutian secondary. Let us first suppose that $m=n$, f and Φ being of the same degree. A glance at the form of the Bezoutian square will show that if we form the Bezoutian secondary of the degree $(n-i)$ in x, the coefficient of its leading term will contain the term $(-)^{(i-1)\frac{i}{2}}(0,i)^i$; $(0,i)$ as usual denoting the product of the coefficient of x^n in f by the coefficient of x^{n-i} in φ, less the product of the coefficient of x^n in φ by that of x^{n-i} in f; and as we suppose the first coefficients in f and φ to be each 1, if we term the other coefficients last spoken of a_i and α_i respectively, this said coefficient of the leading term of the ith Bezoutian secondary will contain the term $(-)^{(i-1)\frac{i}{2}}(a_i-\alpha_i)^i$, and consequently $(-1)^{(i-1)\frac{i}{2}}.\alpha_i^i$ and $(-)^{i\frac{i+1}{2}}a_i^i$.

Now by the like reasoning as that employed in the preceding article, the coefficient of the leading term in $\vartheta_{m-i,0}$ *i. e.*

$$\Sigma(x-h_{q_{i+1}})(x-h_{q_{i+2}})\dots(x-h_{q_m})\frac{\begin{bmatrix} h_{q_1} h_{q_2}\dots h_{q_i} \\ \eta_1\ \eta_2\ \dots\eta_m \end{bmatrix}}{\begin{bmatrix} h_{q_1} & h_{q_2} & \dots h_{q_i} \\ h_{q_{i+1}} & h_{q_{i+2}} & \dots h_{q_m} \end{bmatrix}}$$

will contain the quantity $\Sigma(h_1.h_2.h_3\dots h_i)^i$, and therefore will contain a term $\left(\Sigma(h_1.h_2.h_3\dots h_i)\right)^i$, *i. e.* $(-)^{ii}a_i^i$, which is equal to $(-)^i a_i^i$, since $(i-1)i$ is always even. Hence $\vartheta_{m-i,0}=(-)^{i.\frac{i-1}{2}}\times$ the corresponding Bezoutian secondary.

Art. (28.). The above applies to the case where we have supposed $m=n$. When this equality does not exist we may proceed as follows. Prefix to $\varphi(x)$, the first coefficient of which is still supposed to be 1, a term $\varepsilon-x^m$, where ε is positive and indefinitely small, and let φx so augmented be called $\Phi(x)$. Then if $k_1k_2\dots k_n$ are the roots of φx, $k_1k_2\dots k_n$, together with the $(m-n)$ values of $\left(\frac{1}{\varepsilon}\right)^{\frac{1}{m-n}}$, will be the roots of $\Phi(x)$.

But it has already been proved that when (as here supposed) the first coefficient of fx is 1, the Bezoutian secondaries to f and φ will be identical with those to f and Φ respectively; at least it has been proved that these latter, when $\varepsilon=0$, but the form of Φ is preserved, become identical with the former, and consequently the same is true when ε is taken indefinitely small. Now if we call the $(m-n)$ roots of Φ which do not belong to φ, $\eta_{n+1}, \eta_{n+2}\dots\eta_m$, and make

$$\Psi_{m-i,0}=\Sigma(x-h_{q_{i+1}})(x-h_{q_{i+2}})\dots(x-h_{q_m}).\frac{\begin{bmatrix} h_{q_1} h_{q_2}\dots h_{q_i} \\ \eta_1\ \eta_2\ \dots\eta_m \end{bmatrix}}{\begin{bmatrix} h_{q_1} & h_{q_2} & \dots h_{q_i} \\ h_{q_{i+1}} & h_{q_{i+2}} & \dots h_{q_m} \end{bmatrix}},$$

we have $\Psi_{m-i,0}=\Sigma P(h_{q_1}h_{q_2}\dots h_{q_i})\begin{bmatrix} h_{q_1} & h_{q_2} & \dots h_{q_i} \\ \eta_{n+1} & \eta_{n+2} & \dots\eta_m \end{bmatrix}$,

where $$P(h_{q_1}h_{q_2}\dots h_{q_i})=(x-h_{q_{i+1}})(x-h_{q_{i+2}})\dots(x-h_{q_m})\frac{\begin{bmatrix} h_{q_1} h_{q_2}\dots h_{q_i} \\ \eta_1\ \eta_2\ \dots\eta_n \end{bmatrix}}{\begin{bmatrix} h_{q_1} & h_{q_2} & \dots h_{q_i} \\ h_{q_{i+1}} & h_{q_{i+2}} & \dots h_{q_m} \end{bmatrix}}.$$

But since $k_{n+1}, k_{n+2}\ldots k_m$ are infinite in value,

$$\begin{bmatrix} h_{q_1} & h_{q_2} & \ldots h_{q_i} \\ k_{n+1} & k_{n+2} & \ldots k_m \end{bmatrix} = \left((-k_{n+1}).(-k_{n+2})\ldots(-k_m)\right)^i\left(\frac{1}{\epsilon}\right)^i.$$

Hence
$$\Psi_{m-i,0} = \left(\frac{1}{\epsilon}\right)^i \Sigma P(h_{q_1} h_{q_2} \ldots h_{q_i})$$
$$= \left(\frac{1}{\epsilon}\right)^i \vartheta_{m-i,0}$$

and
$$\vartheta_{m-i,0} = \epsilon^i \Psi_{m-i,0}.$$

But by what has been shown antecedently [taking account of the fact of the leading coefficient of Φ being ϵ in place of 1, which introduces the factor ϵ^i], we have

$$\epsilon^i \Psi_{m-i,0} = (-)^{(i-1)\frac{i}{2}} B'_i,$$

where B'_i is the Bezoutian secondary of the $(m-i-1)$th degree in x to f and ϕ; but B'_i it has been proved $=B_i$. the Bezoutian secondary of the same degree to f and ϕ; hence $\vartheta_{m-i,0} = (-)^{i.\frac{i-1}{2}}.B_i$.

Art. (29.). If now we return to the syzygetic equation, $\tau.f - t\phi + \vartheta = 0$, ϑ may be treated as known, having in fact been completely determined as a function of the roots, as well in its most general form, as also so as to represent the simplified residues to f and ϕ in the preceding articles; it remains to determine the values of τ and t as functions of the roots corresponding to any allowable form of ϑ, but I shall confine the investigation to the case where ϑ is the lowest-weighted conjunctive, or which is the same thing, a simplified residue to t and ϕ of any given degree in x; each value of $\frac{\tau}{t}$ will then represent one of the convergents to $\frac{\phi}{f}$ when expanded under the form of a continued fraction. If ϑ be of the ith degree in x, τ is of the degree $(n-i-1)$ and t of the degree $(m-i-1)$. This being supposed, and calling $n-i-1=\nu$, $m-i-1=\mu$, I say that t will be represented by G and τ by Γ, where

$$G = (-)^i \Sigma (x-h_{q_1})(x-h_{q_2})\ldots(x-h_{q_\mu}) \frac{\begin{bmatrix} h_{q_1} h_{q_2} \ldots h_{q_m} \\ \eta_1 \; \eta_2 \; \ldots \eta_n \end{bmatrix}}{\begin{bmatrix} h_{q_1} & h_{q_2} & \ldots h_{q_\mu} \\ h_{q_{\mu+1}} & h_{q_{\mu+2}} & \ldots h_{q_m} \end{bmatrix}},$$

and τ is an analogous form Γ; $h_1\, h_2 \ldots h_m$, as heretofore, being the roots of f, and $\eta_1\, \eta_2 \ldots \eta_n$ of ϕ. To fix the ideas and make the demonstration more immediately seizable, give m and n specific values; thus let $m=5$, $n=4$, $i=2$, so that $\mu=5-2-1=2$. Put ϑ under the form $\vartheta_{i,0}$, so that ϑ in the case before us

$$= \Sigma (x-h_{q_1})(x-h_{q_2}) \frac{\begin{bmatrix} h_{q_3} h_{q_4} h_{q_5} \\ \eta_1 \; \eta_2 \; \eta_3 \; \eta_4 \end{bmatrix}}{\begin{bmatrix} h_{q_3} h_{q_4} h_{q_5} \\ h_{q_1} h_{q_2} \end{bmatrix}}.$$

Now make $x=h_1$, then $f=0$, and ϑ becomes

$$\Sigma(h_1-h_{q_1})(h_1-h_{q_2})\frac{\begin{bmatrix} h_{q_3}\,h_{q_4}\,h_{q_5} & \\ \eta_1\ \eta_2\ \eta_3\ \eta_4 \end{bmatrix}}{\begin{bmatrix} h_{q_3}\,h_{q_4}\,h_{q_5} \\ h_{q_1}\,h_{q_2} \end{bmatrix}},$$

$$i.\,e.\ \Sigma\frac{\begin{bmatrix} h_1 \\ h_4\,h_5 \end{bmatrix}\begin{bmatrix} h_1\,h_2\,h_3 \\ \eta_1\ \eta_2\ \eta_3\ \eta_4 \end{bmatrix}}{\begin{bmatrix} h_1\,h_2\,h_3 \\ h_4\,h_5 \end{bmatrix}},$$

h_1 being kept constant in the above sum, but h_2, h_3, h_4, h_5 being partitionable in all the six possible ways into two groups, as into h_4, h_5, h_2, h_3 in the term above expressed. This sum is evidently identical with

$$\Sigma\frac{\begin{bmatrix} h_1\,h_2\,h_3 \\ \eta_1\ \eta_2\ \eta_3\ \eta_4 \end{bmatrix}}{\begin{bmatrix} h_2\,h_3 \\ h_4\,h_5 \end{bmatrix}},\ i.\,e. \begin{bmatrix} h_1 \\ \eta_1\,\eta_2\,\eta_3\,\eta_4 \end{bmatrix}\times\Sigma\frac{\begin{bmatrix} h_2\,h_3 \\ \eta_1\,\eta_2\,\eta_3\,\eta_4 \end{bmatrix}}{\begin{bmatrix} h_2\,h_3 \\ h_4\,h_5 \end{bmatrix}}.$$

Again, ϕ becomes

$$\begin{bmatrix} h_1 \\ \eta_1\,\eta_2\,\eta_3\,\eta_4 \end{bmatrix}.$$

Hence $t=\frac{\vartheta}{\phi}$ becomes

$$\Sigma\frac{\begin{bmatrix} h_2\,h_3 \\ \eta_1\,\eta_2\,\eta_3\,\eta_4 \end{bmatrix}}{\begin{bmatrix} h_2\,h_3 \\ h_4\,h_5 \end{bmatrix}}.$$

But when $x=h_1$, $\frac{\mathrm{G}}{(-)^i}$ becomes

$$\begin{bmatrix} h_1 \\ h_2\,h_3 \end{bmatrix}\frac{\begin{bmatrix} h_2\,h_3 \\ \eta_1\,\eta_2\,\eta_3\,\eta_4 \end{bmatrix}}{\begin{bmatrix} h_2\,h_3 \\ h_1\,h_4\,h_5 \end{bmatrix}},$$

$$i.\,e. =\frac{\begin{bmatrix} h_1 \\ h_2\,h_3 \end{bmatrix}}{\begin{bmatrix} h_2\,h_3 \\ h_1 \end{bmatrix}}.\frac{\begin{bmatrix} h_2\,h_3 \\ \eta_1\,\eta_2\,\eta_3\,\eta_4 \end{bmatrix}}{\begin{bmatrix} h_2\,h_3 \\ h_4\,h_5 \end{bmatrix}},$$

$$=(-1)^i.t.$$

Thus when $x=h_1$, $t=\mathrm{G}$. In like manner, when $x=h_2$, or h_3, or h_4, or h_5, t always $=\mathrm{G}$; but t and G are both functions of x of the same degree, and of only two dimensions in x. Hence t is identical with G. So in general it may be proved, that whenever $x=h_1$, or h_2 or h_3... or h_n, t and G, which are each of only $(n-1-i)$ dimensions in x, are equal. Hence universally $t=G$, as was to be shown. To find τ we must avail ourselves of the symmorphic, or as we may better say (it being at the opposite extremity of the scale of forms, the antimorphic), value of ϑ represented by $\vartheta_{0,i}$, taking care to preserve ϑ strictly identical under both forms of representation, in point of sign as well as quantity. That is to say, we must make

$$\vartheta_{0,i}=(-)^{i(m-i)}\Sigma(x-\eta_{q_1})(x-\eta_{q_2})\ldots(x-\eta_{q_i})\frac{\begin{bmatrix} h_1 & h_2 & \ldots h_m \\ \eta_{q_{i+1}} & \eta_{q_{i+2}} & \ldots h_{q_n}\end{bmatrix}}{\begin{bmatrix} \eta_{q_{i+1}} & \eta_{q_{i+2}} & \ldots \eta_{q_n} \\ \eta_{q_1} & \eta_{q_2} & \ldots \eta_{q_i}\end{bmatrix}}$$

$$=(-)^{\omega}\Sigma(x-\eta_{q_1})(x-\eta_{q_2})\ldots(x-\eta_{q_i})\frac{\begin{bmatrix} \eta_{q_{i+1}} & \eta_{q_{i+2}} & \ldots \eta_{q_n} \\ h_1 & h_2 & \ldots h_m\end{bmatrix}}{\begin{bmatrix} \eta_{q_{i+1}} & \eta_{q_{i+2}} & \ldots \eta_{q_n} \\ \eta_{q_1} & \eta_{q_2} & \ldots \eta_{q_i}\end{bmatrix}},$$

where
$$\omega=i(m-i)+m(n-i),$$

so that
$$(-)^{\omega}=(-)^{mi-i+mn-mi}=(-)^{mn-i};$$

and consequently the same reasoning as was applied to t to prove $t=G$, will serve to show that $-\tau=\Gamma$, where

$$\Gamma=(-)^{mn}.\Sigma(x-\eta_{\xi_1})(x-\eta_{\xi_2})\ldots(x-\eta_{\xi_\nu})\frac{\begin{bmatrix} \eta_{\xi_1} & \eta_{\xi_2} & \ldots \eta_{\xi_\nu} \\ h_1 & h_2 & \ldots h_m\end{bmatrix}}{\begin{bmatrix} \eta_{\xi_1} & \eta_{\xi_2} & \ldots \eta_{\xi_\nu} \\ \eta_{\xi_{\nu+1}} & \eta_{\xi_{\nu+2}} & \ldots \eta_{\xi_n}\end{bmatrix}},$$

or

$$\tau=(-)^{\omega}\Sigma(x-\eta_{\xi_1})(x-\eta_{\xi_2})\ldots(x-\eta_{\xi_\nu})\frac{\begin{bmatrix} h_1 & h_2 & \ldots h_m \\ \eta_{\xi_1} & \eta_{\xi_2} & \ldots \eta_{\xi_\nu}\end{bmatrix}}{\begin{bmatrix} \eta_{\xi_1} & \eta_{\xi_2} & \ldots \eta_{\xi_\nu} \\ \eta_{\xi_{\nu+1}} & \eta_{\xi_{\nu+2}} & \ldots \eta_{\xi_n}\end{bmatrix}},$$

where

$$\omega=mn-1-m\nu=mn-1-m(n-i-1)$$
$$=mi-m-1.$$

Art. (30.). I have not succeeded in throwing t and τ under any other than the single forms for each above given, and it is remarkable that whilst apparently t and τ admit only of this single representation, ϑ admits of the variety of forms included under the general symbol $\vartheta_{v,i-v}$ for a given value of i; and it ought to be remarked that these forms (although the most perfectly symmetrical and exactly balanced representations) [and for that reason possibly the most commodious for the ascertainment of the allotrious factor belonging to them respectively] by no means exhaust the almost infinite variety of modes by which the simplified residues, *i. e.* the hekistobarytic, or if we like so to call them, the prime conjunctives, admit of being represented as functions of the roots of the given functions; but if in art. (16.), instead of writing

$$R=\frac{\begin{bmatrix} h_{q_1} h_{q_2} \dots h_{q_\nu} \\ \eta_{\xi_1} \eta_{\xi_2} \dots \eta_{\xi_\nu} \end{bmatrix}}{\begin{bmatrix} h_{q_1} & h_{q_2} & \dots h_{q_v} \\ h_{q_{v+1}} & h_{q_{v+2}} & \dots h_{q_m} \end{bmatrix} \times \begin{bmatrix} \eta_{\xi_1} & \eta_{\xi_2} & \dots \eta_{\xi_\nu} \\ \eta_{\xi_{\nu+1}} & \eta_{\xi_{\nu+2}} & \dots \eta_{\xi_n} \end{bmatrix}},$$

we had made

$$R=\frac{P(h_{q_1} h_{q_2} \dots h_{q_v};\ \eta_{\xi_1} \eta_{\xi_2} \dots \eta_{\xi_\nu})}{\begin{bmatrix} h_{q_1} & h_{q_2} & \dots h_{q_v} \\ h_{q_{v+1}} & h_{q_{v+2}} & \dots h_{q_m} \end{bmatrix} \times \begin{bmatrix} \eta_{\xi_1} & \eta_{\xi_2} & \dots \eta_{\xi_\nu} \\ \eta_{\xi_{\nu+1}} & \eta_{\xi_{\nu+2}} & \dots \eta_{\xi_m} \end{bmatrix}},$$

where P represents any function symmetrical in respect of h_{q_1}, $h_{q_2} \dots h_{q_v}$, and also in respect of η_{ξ_1}, $\eta_{\xi_2} \dots \eta_{\nu}$, (the interchanges, that is to say, between one h and another h, or between one η and another η, leaving P unaltered), it might be shown that the value of $\vartheta_{v,\nu}$ resulting from the introduction of this more general value of R would (as for the particular value assumed) always be expressible as an integral function of the roots, and consequently, if P be taken of the same dimensions in the roots as the numerator of R previously assumed, *i. e.* $v\nu$, $\vartheta_{v,\nu}$ would continue to be (unless indeed it vanish) identical (to some numerical factor *près*) with the corresponding simplified residue. If, on the other hand, P be taken of less than $v\nu$ dimensions in h and k, we know *à priori* that $\vartheta_{v,\nu}$ must vanish, as otherwise we should have a conjunctive of a weight less than the minimum weight. When P is of the proper *amount* of weight $v\nu$, it is I think probable that another condition as to the *distribution* of the weight will be found to be necessary in order that $\vartheta_{v,\nu}$ may not vanish, viz. that the highest power of any single (h) in P shall not exceed v, nor the highest power of any single η exceed ν. But as I have not had leisure to enter upon the inquiry, the verification or disproval of this supposed law, and more generally the evolution of the allotrious numerical factor introduced into $\vartheta_{v,\nu}$ by assigning any particular form to (P) satisfying the necessary conditions of amount and distribution of weight, must be reserved, amongst other points connected with the theory of the remarkable forms (19.) art. (15.), as a subject for future investigation.

Art. (31.). A property of continued fractions, which, if known, I have not met with in any treatise on the subject (but which has been already cursorily alluded to in these pages), gives rise to a remarkable property of reciprocity connecting τ and t severally with ϑ in the syzygetic equation $\tau f - t\phi + \vartheta = 0$.

Let the successive convergents to the ordinary continued fraction

$$\frac{1}{q_1+}\ \frac{1}{q_2+}\ \frac{1}{q_3+} \dots \frac{1}{q_{i-1}+}\ \frac{1}{q_i}$$

be called

$$\frac{l_1}{m_1},\ \frac{l_2}{m_2} \dots \frac{l_{i-1}}{m_{i-1}}\ \frac{l_i}{m_i}$$

respectively, it is well known that

$$m_{i-1}.l_i - m_i\, l_{i-1} = (-)^{i-1}.1\,;$$

but I believe that it has not been observed that this is only the extreme cases of a much more general equation, viz.

$$m_{i-\rho}\, l_i - m_i\, l_{i-\rho} = (-)^{i-\rho}.\mu_{\rho-1}{}^{*},$$

where $\mu_1, \mu_2, \ldots \mu_i$ denote respectively the denominators to the convergents to the continued fractions formed with the quotients taken in a reverse order, *i. e.* the continued fraction

$$\frac{1}{q_i+}\ \frac{1}{q_{i-1}+}\ \frac{1}{q_{i-2}+}\cdots+\frac{1}{q_2+}\ \frac{1}{q_1}.$$

This is easily proved when $\rho=1$; μ_0 is of course (as usual) to be considered 1. So more simply for the improper continued fraction,

$$\frac{l_i}{m_i}=\frac{1}{q_1-}\ \frac{1}{q_2-}\cdots\frac{1}{q_{i-1}-}\ \frac{1}{q_i},$$

of which the convergents are supposed to be

$$\frac{l_1}{m_1}\ \frac{l_2}{m_2}\cdots\frac{l_{i-1}}{m_{i-1}}\ \frac{l_i}{m_i},$$

and the reverse fraction

$$\frac{1}{q_i-}\ \frac{1}{q_{i-1}}\cdots\frac{1}{q_2-}\ \frac{1}{q_1},$$

of which the convergents are supposed to be

$$\frac{\lambda_1}{\lambda_2},\ \frac{\lambda_2}{\mu_2},\cdots\frac{\lambda_i}{\mu_i},$$

we have the more simple equation

$$l_i.m_{i-\rho}-l_{i-\rho}.m_i+\mu_{\rho-1}=0.$$

And it is well known, or at all events easily demonstrable, that

$$\frac{l_{i-1}}{l_i}=\frac{1}{q_i-}\ \frac{1}{q_{i-1}-}\ \frac{1}{q_{i-2}}\cdots\frac{1}{q_2}$$

$$\frac{m_{i-1}}{m_i}=\frac{1}{q_i-}\ \frac{1}{q_{i-1}-}\ \frac{1}{q_{i-2}}\cdots\frac{1}{q_2}\ \frac{1}{q_1}.$$

Art. (32.). If now we use subscript indices to denote the degree in x of the quantities to which they are affixed, we have the general syzygetic equation

$$K\tau_{m-i-1}.f_m - Kt_{m-i-1}.\varphi_n + K\vartheta_i = 0,$$

where K, a constant (which I have given the means of determining in the first section), being rightly assumed $K.\tau_{n-i-1}$, $K\tau_{m-i-1}$, become the numerator and denominator respectively of one of the convergents to $\frac{\varphi}{f}$, expressed as an improper continued fraction, and $K\vartheta_i$ becomes the denominator to one of the convergents to $\frac{t^{m-1}}{f}$, or,

* See London and Edinburgh Philosophical Magazine, "On a Fundamental Theorem in the Theory of Continued Fractions," October, 1853.

which is the same thing, to $\frac{\tau^{n-1}}{\phi}$*. Conversely, it is obvious that if we adopt as our primitive functions $cf(m)$ and t_{m-1}, (c) being the value of K when $i=0$, we shall obtain as the general form of our syzygetic equation, bearing in mind that $(m-1)$ now replaces (n),

$$c.\mathrm{K}'(\tau)_{m-i}fm-\mathrm{K}'\vartheta_{m-i-1}t_{m-1}+\mathrm{K}'\tau_i=0\,;$$

and similarly, if we adopt as our primitive functions τ_{n-1} and $c\phi_n$, we obtain for our general syzygetic equation, observing that $(n-1)$ now replaces (m),

$$\mathrm{K}'\vartheta_{n-i-1}.\tau_{m-1}-c\mathrm{K}'.\vartheta_{n-i-1}.\phi_n+\mathrm{K}'\tau_i=0\,;$$

so that (making abstraction of the constant factors and looking merely to the forms of the several functions which enter into the equations) we see that on the first hypothesis, viz. of t_{m-1} being substituted for ϕ_n, the conjunctives of each degree in x change places with the second conjunctive factors, *i. e.* the original multipliers of ϕ of the same degree in x, and *vice versâ*; and in the second hypothesis, where τ_{n-1} takes the place of fm, the conjunctives of each degree in x change places with the first conjunctive factors, *i. e.* the original multipliers of f of the same degree in x, and *vice versâ*; t_{m-1} and τ_{n-1} being respectively multipliers of ϕ and f, such that the difference of the respective products is independent of x. These results ought to be capable of being verified by aid of our general formulæ for t, τ, ϑ, and as this verification will serve to exhibit in a clearer light the nature of the reciprocity between the conjunctives and the conjunctive factors, it may be not uninteresting to set it out.

Art. (33.). As usual, let $h_1\ h_2 \ldots h_m$ be the roots of $f(x)$, and $\eta_1\ \eta_2\ldots\eta_{m-1}$ the roots of $\phi(x)$, the last conjunctive factor to ϕ, which is of the degree $(m-1)$ in x, will be represented, neglecting powers of $(-)$, by t_{m-1}, where

$$t_{m-1}=\Sigma(x-h_{q_1})(x-h_{q_2})\ldots(x-h_{q_{m-1}})\frac{\begin{bmatrix}h_{q_1}\ h_{q_2}\ldots h_{q_{m-1}}\\ \eta_1\ \eta_2\ \ldots\eta_{q_n}\end{bmatrix}}{\begin{bmatrix}h_{q_n}\\ h_{q_1}\ h_{q_2}\ldots h_{q_{m-1}}\end{bmatrix}}$$

If now we for greater simplicity make $t_{n-1}=t(x)$, and call the roots of t, $\eta'_1\ \eta'_2\ldots\eta'_{n-1}$ any such quantity as

$$\begin{bmatrix}h_{q_m}\\ \eta'_1\ \eta'_2\ldots\eta'_{m-1}\end{bmatrix}=t(h_{q_m})=(h_{q_m}-h_{q_1})(h_{q_m}-h_{q_2})\ldots(h_{q_m}-h_{q_{m-1}})\times\frac{\phi(h_{q_1}).\phi(h_{q_2})\ldots\phi(h_{q_{m-1}})}{(h_{q_m}-h_{q_1})(h_{q_m}-h_{q_2})\ldots(h_{q_m}-h_{q_{m-1}})}$$
$$=\phi(h_{q_1})\phi(h_{q_2})\ldots\phi(h_{q_{m-1}})$$
$$=\mathrm{R}\frac{1}{\phi(h_{q_m})},$$

* Since i is always supposed less than n (n being the degree of the lower degreed of the two functions f and ϕ), the fact of the last quotient to $\frac{t_{m-1}}{f}$ being wanting to $\frac{\tau_{n-1}}{\phi}$ will not affect the accuracy of the statement in the text above, since this latter will contain as many quotients as can in any case be required for expressing ϑ_i.

R denoting a constant independent of the root h_{q_m} selected (and which constant is in fact the resultant of the two functions $f(x)$ and $\varphi(x)$), that is to say,

$$\varphi(h_1)\varphi(h_2)\varphi(h_3)\ldots\varphi(h_m).$$

But by our general formulæ (8.) the simplified residue to $f(x)$ and $t(x)$ of the ith degree in x will be represented by

$$\vartheta'_{i,0}=\Sigma(x-h_{q_1})(x-h_{q_2})\ldots(x-h_{q_i})\left\{\frac{\begin{bmatrix}h_{q_{i+1}} & h_{q_{i+2}}\ldots h_{q_m}\\ \eta'_1 & \eta'_2 \ldots\eta'_{m-1}\end{bmatrix}}{\begin{bmatrix}h_{q_1} & h_{q_2} & \ldots h_{q_i}\\ h_{q_{i+1}} & h_{q_{i+2}}\ldots h_{q_m}\end{bmatrix}}\right\};$$

$$\therefore \vartheta'_{i,0}=\Sigma(x-h_{q_1})(x-h_{q_2})\ldots(x-h_{q_i})\times\left\{\mathrm{R}^{m-i}.\frac{(\varphi h_{q_{i+1}})^{-1}.\varphi(h_{q_{i+2}})^{-1}\ldots\varphi(h_{q_m})^{-1}}{\begin{bmatrix}h_{q_1} & h_{q_2} & \ldots h_{q_i}\\ h_{q_{i+1}} & h_{q_{i+2}}\ldots h_{q_m}\end{bmatrix}}\right\}$$

$$=\mathrm{R}^{m-i-1}\Sigma(x-h_{q_1})(x-h_{q_2})\ldots(x-h_{q_i})\frac{\varphi(h_{q_1})\varphi(h_{q_2})\ldots\varphi(h_{q_i})}{\begin{bmatrix}h_{q_1} & h_{q_2} & \ldots h_{q_i}\\ h_{q_{i+1}} & h_{q_{i+2}}\ldots h_{q_m}\end{bmatrix}},$$

or $\vartheta'_i=\mathrm{R}^{m-i-1}t_i,$

the relation which was to be obtained. So conversely, in precisely the same manner, calling t'_i the conjunctive factor of the degree i in x to $t(x)$ in the syzygetic equation, which connects $f(x)$ and $t(x)$ with a corresponding simplified residue, we have

$$t'_i=\Sigma(x-h_{q_1})(x-h_{q_2})\ldots(x-h_{q_i})\frac{\begin{bmatrix}h_{q_1} & h_{q_2}\ldots h_{q_i}\\ \eta'_1 & \eta'_2\ldots\eta'_{m-1}\end{bmatrix}}{\begin{bmatrix}h_{q_1} & h_{q_2} & \ldots h_{q_i}\\ h_{q_{i+1}} & h_{q_{i+2}}\ldots h_{q_m}\end{bmatrix}}$$

$$=\mathrm{R}^{i-1}\Sigma(x-h_{q_1})(x-h_{q_2})\ldots(x-h_{q_i})\frac{\varphi h_{q_{i+1}}\varphi h_{q_{i+2}}\ldots\varphi h_{q_m}}{\begin{bmatrix}h_{q_1} & h_{q_2} & \ldots h_{q_i}\\ h_{q_{i+1}} & h_{q_{i+2}}\ldots h_{q_m}\end{bmatrix}}$$

$$=\mathrm{R}^{i-1}.\vartheta_i,$$

the conjugate equation to the one previously obtained*.

And evidently the same reasoning serves to establish the reciprocity, or rather reciprocal convertibility, between the ϑ series and the τ series, when in lieu of the original primitives $f(x)$ and $\varphi(x)$ we take as our primitives $\tau(x)$ and $\varphi(x)$, $\tau(x)$ being the function which satisfies the equation

$$\tau(x)fx-t(x)\varphi x+\mathrm{R}=0.$$

Art. (34.). It may be remarked that if $n=m-1$ (the last syzygetic equation being

* M. HERMITE, by a peculiar method, first discovered one of these two conjugate relations of reciprocity, applicable to the case of STURM's theorem, where $\varphi x=f'x$, and I am indebted to him for bringing the subject under my notice.

thus $t_{m-1}.\phi_{m-1}-\tau_{m-2}.f_m+\vartheta_0=0$), when t_{m-1} and f_m are taken as the primitives, the corresponding equation will be of the form

$$t'_{m-1}.t_{m-1}-\tau'_{m-2}f_m+\vartheta'_0=0;$$

these two equations must therefore be identical, and consequently $t'_{m-1}=\phi_{m-1}$ (to a numerical factor *près*), so that t_{m-1} and ϕ_{m-1} are reciprocal forms; this is also obvious from the consideration that t'_{m-1} must, by the general law of reciprocity (established above), be a residue to (f_m, ϕ_{m-1}), which the latter function itself may be considered to be. Or the same thing is obvious directly, by writing

$$t_{m-1}=t(x)=\Sigma(x-h_{q_1})(x-h_{q_2})\ldots(x-h_{q_{m-1}}).\frac{\phi(h_{q_1})\phi(h_{q_2})\ldots\phi(h_{q_{m-1}})}{(h_{q_m}-h_{q_1})(h_{q_m}-h_{q_2})\ldots(h_{q_m}-h_{q_{m-1}})},$$

and then making

$$t'_{m-1}=\Sigma(x-h_{q_1})(x-h_{q_2})\ldots(x-h_{q_{m-1}}).\frac{t(h_{q_1})t(h_{q_2})\ldots t(h_{q_{m-1}})}{(h_{q_m}-h_{q_1})(h_{q_m}-h_{q_2})\ldots(h_{q_m}-h_{q_{m-1}}}$$

$$=\Sigma(x-h_{q_1})(x-h_{q_2})\ldots(x-h_{q_{m-1}}).\frac{(\phi(h_{q_1}).\phi(h_{q_2})\ldots\phi h_{q_{m-1}}.h_{q_m})}{\Delta},$$

where

$$\Delta=(-)^{m.\frac{m-1}{2}}(h_1-h_2)^2.(h_1-h_3)^2\ldots(h_1-h_m)^2$$
$$\times(h_2-h_3)^2\ldots(h_2-h_m)^2$$
$$\cdots\cdots$$
$$\times(h_{m-1}-h_m)^2$$

$=(-1)^{m.\frac{m-1}{2}}$ D (D being the Discriminant, more commonly called the Determinant to f); or finally,

$t'_{m-1}=\frac{R^{m-1}}{D}\phi$, as was to be shown.

SECTION III.

On the application of the Theorems in the preceding Section to the expression in terms of the roots of any primitive function of STURM'S *auxiliary functions, and the other functions which connect these with the primitive function and its first differential derivative.*

Art. (35.). The formulæ in the preceding Section had reference to the case of two absolutely independent functions and their respective systems of roots: when the functions become so related that the roots of the one system become explicitly or implicitly functions of the roots of the other system, the formulæ will become expressible in terms of these latter alone, and in some cases the terms (of which the sum is always essentially integral) will become separately and individually representable under an integral form. Such, as I shall proceed to show, is the case for two functions, of which one is the differential derivative of the other. When f and ϕ are thus related, so that $\varphi=\frac{df}{dx}$, calling as before $h_1\,h_2\ldots h_m$ the roots of f, and $\eta_1\,\eta_2\ldots\eta_{m-1}$ the roots of φ, we shall have in general

$$\begin{bmatrix} h_{q_{i+1}} \\ \eta_1\,\eta_2\ldots\eta_{m-1} \end{bmatrix} =(h_{q_{i+1}}-\eta_1)(h_{q_i}-\eta_2)\ldots(h_{q_{i+1}}-\eta_{m-1})$$

$$=f'h_{q_{i+1}}=\begin{bmatrix} h_{q_{i+1}} \\ h_{q_1}\,h_{q_2}\ldots h_{q_i}\,h_{q_{i+2}}\ldots h_{q_m} \end{bmatrix}=\begin{bmatrix} h_{q_{i+1}} \\ h_{q_1}h_{q_2}\ldots h_{q_i} \end{bmatrix}\times\begin{bmatrix} h_{q_{i+1}} \\ h_{q_{i+2}}\,h_{q_{i+3}}\ldots h_{q_m} \end{bmatrix}.$$

Consequently

$$\begin{bmatrix} h_{q_{i+1}}\,h_{q_{i+2}}\ldots h_{q_m} \\ \eta_1\quad \eta_2\quad \ldots\eta_{m-1} \end{bmatrix}=\begin{bmatrix} h_{q_{i+1}} \\ \eta_1\eta_2\ldots\eta_{m-1} \end{bmatrix}\times\begin{bmatrix} h_{q_{i+2}} \\ \eta_1\quad \eta_2\ldots\eta_{m-1} \end{bmatrix}\times\&\text{c.}\times\begin{bmatrix} h_{q_m} \\ \eta_1\quad \eta_2\ldots\eta_{m-1} \end{bmatrix}$$

$$=\begin{bmatrix} h_{q_{i+1}} \\ h_{q_1}h_{q_2}\ldots h_{q_i} \end{bmatrix}\times\begin{bmatrix} h_{q_{i+1}} \\ h_{q_{i+2}}\,h_{q_{i+3}}\ldots h_{q_{m-1}} \end{bmatrix}$$

$$\times\begin{bmatrix} h_{q_{i+2}} \\ h_{q_1}h_{q_2}\ldots h_{q_i} \end{bmatrix}\times\begin{bmatrix} h_{q_{i+2}} \\ h_{q_{i+1}}\,h_{q_{i+3}}\ldots h_{q_{m-1}} \end{bmatrix}$$

$$\times\&\text{c.}\quad .\quad .\quad .\quad .$$

$$\times\begin{bmatrix} h_{q_m} \\ h_{q_1}h_{q_2}\ldots h_{q_i} \end{bmatrix}\times\begin{bmatrix} h_{q_m} \\ h_{q_{i+1}}\,h_{q_{i+2}}\ldots h_{q_{m-1}} \end{bmatrix}.$$

Hence

$$\frac{\begin{bmatrix} h_{q_{i+1}}\,h_{q_{i+2}}\ldots h_{q_m} \\ \eta_1\eta_2\quad\ldots\eta_{m-1} \end{bmatrix}}{\begin{bmatrix} h_{q_{i+1}}\,h_{q_{i+2}}\ldots h_{q_m} \\ h_{q_1}\quad h_{q_2}\quad\ldots h_{q_i} \end{bmatrix}}=$$

$$\begin{bmatrix} h_{q_{i+1}} \\ h_{q_{i+2}}\,h_{q_{i+3}}\ldots h_{q_m} \end{bmatrix}\times\begin{bmatrix} h_{q_{i+2}} \\ h_{q_i}\quad h_{q_{i+1}}\ldots h_{q_m} \end{bmatrix}\times\ldots\ldots\ldots\times\begin{bmatrix} h_{q_m} \\ h_{q_1}\;h_{q_2}\ldots h_{q_{m-1}} \end{bmatrix}$$

$$=(-)^{\frac{1}{2}(m-i)(m-i-1)}.\zeta(h_{q_{i+1}}\,h_{q_{i+2}}\ldots h_{q_m}),$$

the ζ denoting the operation of taking the product of the squares of the differences of the quantities which this symbol governs. Hence the Bezoutian secondary to f and f' of the $(m-i-1)$th degree in x, viz.—

$$(-)^{i.\frac{i-1}{2}}\Sigma(x-h_{q_{i+1}})(x-h_{q_{i+2}})\ldots(x-h_{q_m})\frac{\begin{bmatrix} h_{q_1}\,h_{q_2}\ldots h_{q_i} \\ \eta_1\quad\eta_2\;\ldots\eta_{m-1} \end{bmatrix}}{\begin{bmatrix} h_{q_1}\quad h_{q_2}\quad\ldots h_{q_i} \\ h_{q_{i+1}}\,h_{q_{i+2}}\ldots h_{q_m} \end{bmatrix}},$$

becomes

$$(-)^{i(i-1)}.\zeta(h_{q_1}\,h_{q_2}\ldots h_{q_i})\Sigma(x-h_{q_{i+1}})(x-h_{q_{i+2}})\ldots(x-h_{q_m})$$

$$=\zeta(h_{q_1}\,h_{q_2}\ldots h_{q_i})\Sigma(x-h_{q_{i+1}})(x-h_{q_{i+2}})\ldots(x-h_{q_m}),$$

since $(-)^{i(i-1)}=1$, which gives the well-known formulæ (enunciated by me in the London and Edinburgh Philosophical Magazine for 1839) for expressing M. STURM's auxiliary functions in terms of the roots of the primitive, and which I therein stated were immediately deducible from the general formulæ (also enunciated in the same paper) applicable to any two functions. These more general formulæ appear to have completely escaped the notice of M. STURM and others, who have used the special formulæ applicable to the case of one function becoming the first differential derivative of the other.

Art. (36.). In precisely the same manner, if we form as usual the ordinary syzygetic equation

$$t.f'x-\tau fx+\vartheta=0,$$

we may find the different values of t given by the complementary formulæ; and using t_i to denote the multiplier of the degree i in x, *i. e.* appertaining to the *residue* of the degree $(m-i-1)$ in x, we have

$$t_i=\Sigma\frac{\begin{bmatrix} h_{q_1}\,h_{q_2}\ldots h_{q_i} \\ \eta_1\ \ \eta_2\ \ldots\eta_{m-1}\end{bmatrix}}{\begin{bmatrix} h_{q_1}\ \ h_{q_2}\ \ \ldots h_{q_i} \\ h_{q_{i+1}}\,h_{q_{i+2}}\ldots h_{q_m}\end{bmatrix}}(x-h_{q_1})(x-h_{q_2})\ldots(x-h_{q_i})$$

$$=\zeta(h_{q_1}\,h_{q_2}\ldots h_{q_i})(x-h_{q_1})(x-h_{q_2})\ldots(x-h_{q_i}).$$

Art. (37.). Thus, if we make $i=m-1$,

$$f_1'(x)=t_{m-1}=\zeta(h_{q_1}\,h_{q_2}\ldots h_{q_{m-1}})(x-h_{q_1})(x-h_{q_2})\ldots(x-h_{q_{m-1}}).$$

It is evident from the form of $f_1''.x$ that it possesses relative to fx, the same property as fx, I mean the property that when x is indefinitely near to a real root of fx, and is passing from the inferior to the superior side of such root, $\frac{f_1'x}{fx}$ like $\frac{f'(x)}{fx}$ will pass from being negative to being positive, or in other words, $f_1'x$ and $f'x$ have always the same sign in the immediate vicinity to a real root of fx. Hence it follows that $f_1'(x)$ might be used instead of $f'x$, to produce, by the Sturmian process of common measure, a series of auxiliary functions, which with fx and $f_1'.x$ would form a rhizoristic series, *i. e.* a series for determining (as in the manner of M. STURM's ordinary auxiliaries) the number of real roots of fx comprised within given limits. The rhizoristic series generated by this process will, it is easily seen, be (to a constant factor *près*) the denominators (reckoning $+1$ as the denominator in the zero place) of the successive convergents to $\frac{f'x}{fx}$ thrown under the form of a continued fraction $\frac{1}{q_1-}\ \frac{1}{q_2-}\ldots\frac{1}{q_{n-1}-}\ \frac{1}{q_n}$; M. STURM's own rhizoristic series, on the contrary (will be to a constant factor *près*), the denominators of the convergents to the inverse fraction $\frac{f_1x}{fx}$, which will be of the form $K\cdot\frac{1}{q_n-}\ \frac{1}{q_{n-1}-}\ldots\frac{1}{q_2-}\ \frac{1}{q_1}$; accordingly these two rhizoristic series will be equivalent as regards the number of changes and of combinations of sign (afforded byeach) corresponding to any given value of x, of which of course the q's are linear functions. This result agrees with what has been demonstrated by me by a more general method (in the London and Edinburgh Philosophical Magazine, June and July 1853), where it has been proved, by means of a very simple theorem of determinants, that the two series

$$\frac{1}{q_1};\quad \frac{1}{q_1-}\ \frac{1}{q_2-};\quad \frac{1}{q_1-}\ \frac{1}{q_2-}\ \frac{1}{q_3};\ \ldots\frac{1}{q_1-}\ \frac{1}{q_2-}\quad \frac{1}{q_3}\ \ldots\frac{1}{q_n}$$

and

$$\frac{1}{q_n};\quad \frac{1}{q_n-}\ \frac{1}{q_{n-1}};\quad \frac{1}{q_n-}\ \frac{1}{q_{n-1}}\ \frac{1}{q_{n-2}};\ \ldots\frac{1}{q_n-}\ \frac{1}{q_{n-1}-}\ \frac{1}{q_{n-2}}\ldots\frac{1}{q_1}$$

always contain (for real values of $q_1, q_2, q_3 \ldots q_n$) the same number of positive and negative signs.

Art. (38.). Having now determined the general values of ϑ and t in the equation $tf'(x)-\tau fx+\vartheta=0$ as explicit integral functions of the roots of fx, the more difficult task remains to assign to τ its value similarly expressed. This cannot readily be effected by means of substitutions in the general formulæ, the method we adopted for finding t and ϑ; but all the other quantities except τ in the syzyzetic equation being integral functions of the roots, it is evident that τ also must be an integral function of the same, and to obtain it we may use the expression $\tau=\frac{tf'x-\vartheta}{fx}$.

To obtain the general form of τ by direct calculation from this formula would however be found to be impracticable; the mode I adopt therefore to discover the general expression for τ corresponding to different values of ϑ, is to ascertain its value on the hypothesis of particular relations existing between the roots of fx, and then from the particular values of τ thus obtained to infer demonstratively its general form, as will be seen below. The demonstration of τ is unavoidably somewhat long, τ being in fact represented by a double sum of partial symmetrical functions.

Using the subscript indices of each function as the syzygetic equation to denote its degree in x, we have in general

$$t_{m-i-1}f'x-\tau_{m-i-2}fx+\vartheta_i=0,$$

where if we make

$$h_1-x=k_1 \quad h_2-x=k_2 \ldots\ldots\ldots h_m-x=k_m,$$

so that

$$h_i-h_\omega=k_i-k_\omega,$$

and therefore

$$\zeta(h_{\theta_1}\ h_{\theta_2}\ldots h_{\theta_\rho})=\zeta(k_{\theta_1}\ k_{\theta_2}\ldots k_{\theta_\rho}),$$

we have in effect found

$$\vartheta_i=\Sigma(k_{q_1}.k_{q_2}\ldots k_{q_i})\zeta(k_{q_{i+1}}\ k_{q_{i+2}}\ldots k_{q_m})$$

and

$$t_{m-i-1}=\pm\Sigma(k_{q_1}\ k_{q_2}\ldots k_{q_{m-i-1}})\zeta(k_{q_1}\ k_{q_2}\ldots k_{q_{m-i-1}});$$

we have also $f'(x)=(-)^{m-1}.\Sigma k_1\ k_2\ldots k_{m-1}$.

Let us commence with the case where $i=0$, we have then

$$\vartheta_0=\zeta(k_1\, k_2 \ldots k_m)$$

$$t_{m-1}=\Sigma(k_{q_1}\, k_{q_2} \ldots k_{q_{m-1}})\zeta(k_{q_1}\, k_{q_2} \ldots k_{q_{m-1}}),$$

we have thus

$$(-)^m.\tau_{m-2}(k_1.k_2\ldots k_m)=-\Sigma(k_{q_1}\, k_{q_2}\ldots k_{q_{m-1}})\times\Sigma(k_{q_1}\, k_{q_2}\ldots k_{q_{m-1}}\zeta(k_{q_1}\, k_{q_2}\ldots k_{q_{m-1}}))+\zeta(k_1\, k_2\ldots k_m)$$

[It may easily be verified that the negative sign interposed between the two parts of the right-hand member of the equation has been correctly taken, for

$$\zeta(k_1\, k_2 \ldots k_m \quad) \text{ contains a term } k_1^{2(m-1)}.k_2^{2(m-2)}\ldots k_{m-2}^2.k_{m-1}^2,$$

$$\Sigma(k_{q_1}\, k_{q_2}\ldots k_{q_{m-1}}) \text{ contains a term } k_1 \quad .k_2 \quad \ldots k_{m-2}.k_{m-1},$$

and

$$\Sigma(k_{q_1}\, k_{q_2}\ldots k_{q_{m-1}})\zeta(k_{q_1}\, k_{q_2}\ldots k_{q_{m-1}}) \text{ contains a term } k_1^{2m-3}\, .k_2^{2m-5}\ldots k_{m-2}^3.k_{m-1}^1,$$

and thus the term $k_1^{2(m-1)}.k_2^{2(m-2)}\ldots k_{m-2}^4.k_{m-1}^2$, which does not contain $k_1\, k_2\ldots k_m$, will (as it ought to do) disappear from the right-hand side of the equation.]

Now suppose

$$k_1=k_2,$$

then

$$\zeta(k_1\, k_2\ldots k_m)=0,$$

and also

$$\zeta(k_{q_1}\, k_{q_2}\ldots k_{q_{m-1}})=0,$$

except when one or the other of the two disjunctive equations

$$q_1,\, q_2,\, q_3\ldots q_{m-1}=1,\, 3,\, 4\ldots m$$

$$q_1,\, q_2,\, q_3\ldots q_{m-1}=2,\, 3,\, 4\ldots m$$

is satisfied (by a disjunctive equation, meaning an equation which affirms the equality of one set of quantities with another set the same in number, each with each, but in some unassigned order).

Hence

$$\Sigma k_{q_1}\, k_{q_2}\ldots k_{q_{m-1}}\zeta(k_{q_1}\, k_{q_2}\ldots k_{q_{m-1}})$$

$$=2k_1\, k_3\ldots k_m\zeta(k_1\, k_3\ldots k_m).$$

Hence when

$$k_1=(-)^m k_2 \quad \tau_{m-2} \text{ becomes } \frac{2}{k_1}\Sigma(k_{q_1}\, k_{q_2}\ldots k_{q_{m-1}})\zeta(k_1\, k_3\ldots k_m),$$

$$\textit{i. e. } 2\zeta(k_1\, k_3\ldots k_m)\{k_1\Sigma k_{r_3}\, k_{r_4}\ldots k_{r_{m-1}}+2k_3\, k_4\ldots k_m\},$$

the Σ referring to $r_3, r_4, \ldots, r_m$ supposed to be disjunctively equal to $3, 4, \ldots m$.

Now τ_{m-2} is of $(m-2)$ dimensions in x, and whenever more than one equality exists between the k's, ϑ_0 and t_{m-1} both vanish (in fact every term in each vanishes separately), and therefore τ_{m-2}, which $=\frac{\vartheta_0+t_{m-1}f'x}{k_1.k_2\ldots k_m}$, will vanish.

Hence $(-)^m\tau_{m-2}$ must be always of the form

$$\Sigma\zeta(h_{q_1}\, h_{q_2}\ldots h_{q_{m-1}})\times\Psi(k_{q_1}\, k_{q_2}\ldots k_{q_{m-1}}\, ;\ k_{q_m}),$$

Ψ denoting some integral function of $(m-2)$ dimensions in respect of the system of quantities $k_{q_1} k_{q_2} \ldots k_{q_m}$. The result above obtained enables us to assign the value of

$$\Psi(k_1, k_3 \ldots k_m, k_2)$$

when $k_1 = k_2$,

$$\text{viz. } k_1 \Sigma(k_{r_3} k_{r_4} \ldots k_{r_{m-1}}) + 2k_3 . k_4 \ldots k_m.$$

Now for a moment suppose, selecting $(m-1)$ terms k_1, k_3, $k_4 \ldots k_m$ out of the m terms of the k series, that

$$\Omega(k_1\, k_3\, k_4 \ldots k_m,\, k_2) = k_2^{m-2} - k_2^{m-3} . S_1(\eta_1\, \eta_3 \ldots \eta_m) + k_2^{m-4} S_2(\eta_1\, \eta_3 \ldots \eta_m)$$
$$\pm \text{ \&c. } \mp k_2\, S_{m-3}(k_1\, k_3 \ldots k_m) \pm 2S_{m-2}(k_1\, k_3 \ldots k_m),$$

where S_1 means that the quantities which it governs are to be simply added together, S_2 denotes that their binary, S_3 that their ternary, and in general S_r that their r-ary products are to be added together.

When $k_1 = k_2$, Ω becomes

$$k_1^{m-2} - k_1^{m-3}\big(k_1 + S_1(k_3\, k_4 \ldots k_m)\big) + k_1^{m-4} . \big(k_1\, S_1(k_3\, k_4 \ldots k_m) + S_2(k_3\, k_4 \ldots k_m)\big)$$
$$- k_1^{m-5} . \big(k_1\, S_2(k_3\, k_4 \ldots k_m) + S_3(k_3\, k_4 \ldots k_m)\big) \pm \text{\&c.} \mp k_1\big(k_1 S_{m-4}(k_3 k_4 \ldots k_m) + S_{m-3}(k_3\, k_4 \ldots k_m)\big)$$
$$\pm 2S_{m-2}(k_3\, k_4 \ldots k_m),$$

which evidently equals

$$\pm \{2S_{m-2}(k_3\, k_4 \ldots k_m) + k_1\, S_{m-3}(k_3\, k_4 \ldots k_m)\},$$
$$\textit{i. e. } \pm \{k_1\, \Sigma(k_{r_3}\, k_{r_4} \ldots k_{r_{m-1}}) + 2k_3\, k_4 \ldots k_m\}.$$

Hence when $k_1 = k_2$, $\Psi = \Omega$, and

$$(-)^m \tau_{m-2} = \Sigma \zeta(h_{q_1}\, h_{q_2} \ldots h_{q_{m-1}}) \times \Omega(k_{q_1}\, k_{q_2} \ldots k_{q_{m-1}} k_{q_m});$$

and so in like manner, when k_1 is equal to any one of the $(m-1)$ quantities k_2, $k_3 \ldots k_m$, the form of τ_{m-2} above written will have been correctly assumed. But τ_{m-2} may be treated as a function of $(m-2)$ dimensions in k_1, and consequently any form of $(m-2)$ dimensions in k_1, which fits it for $(m-1)$ different values of k_1, must be its general form, and accordingly we have universally,

$$(-)^m \tau_{m-2} = \Sigma \zeta(h_{q_1}\, h_{q_2} \ldots h_{q_{m-1}}) \times \{(x - h_{q_m})^{m-2} - (x - h_{q_m})^{m-3} S_1(x - h_{q_1},\, x - h_{q_2} \ldots x - h_{q_{m-1}})$$
$$+ (x - h_{q_m})^{m-4} S_2(x - h_{q_1},\, x - h_{q_2} \ldots x - h_{q_{m-1}}) \pm \text{ \&c.}$$
$$\mp (x - h_{q_m}) S_{m-3}(x - h_{q_1},\, x - h_{q_2} \ldots x - h_{q_{m-1}}) \pm 2S_{m-2}(x - h_{q_1},\, x - h_{q_2} \ldots x - h_{q_{m-1}})\}.$$

Art. (39.). With a view to better paving our way to the general form of τ for all values of i, let us pass over the case of $i=1$ and go at once to the equation

$$t_{m-3} . f'x - \tau_{m-4} fx + \vartheta_2 = 0;$$

and to better fix our ideas let $m=7$, so that the equation becomes

$$t_4 . f'x - \tau_3 . fx + \vartheta_2 = 0;$$

we have then, preserving the same relation as before [*i. e.* using h to denote any root of fx, and k to denote $h-x$], the equation

$$\pm k_1 k_2 k_3 k_4 k_5 k_6 k_7 . \tau_3 = \Sigma k_{q_1} . k_{q_2} \zeta(k_{q_3} k_{q_4} k_{q_5} k_{q_6} k_{q_7})$$
$$-\Sigma(k_{q_1} k_{q_2} k_{q_3} k_{q_4} k_{q_5} k_{q_6}) \times \Sigma\{(k_{q_1} k_{q_2} k_{q_3} k_{q_4}) \zeta(k_{q_1} k_{q_2} k_{q_3} k_{q_4})\};$$

and τ_3 will vanish whenever more than three relations of equality exist between the k's, for then each term in *both* of the two sums in the right-hand member of the equation above written will separately vanish; and of course three relations of equality between the same are sufficient to make all the terms in the first of these sums vanish. This relationship between the different k's corresponding to a multiplicity 3 may arise in different ways; the multiplicity 3 may be divided into 3 units corresponding to 3 pairs of equal roots, or into 2 and 1 corresponding one set of 3 equal roots, and a second set of 2 equal roots, or may be taken "en bloc," which corresponds to the case of one set of 4 equal roots. I shall make the first of these suppositions, which will sufficiently well answer our purpose in the case before us.

Thus I shall suppose $\quad k_1 = k_4 \quad k_2 = k_5 \quad k_3 = k_6,$

then, as above remarked, $\zeta(k_{q_3} k_{q_4} k_{q_5} k_{q_6} k_{q_7}) = 0$ for all values of $q_3 q_4 q_5 q_6 q_7$, and therefore

$$\Sigma k_{q_1} k_{q_2} \zeta(k_{q_3} k_{q_4} k_{q_5} k_{q_6} k_{q_7}) = 0;$$

also $\Sigma k_{q_1} k_{q_2} k_{q_3} k_{q_4} k_{q_5} k_{q_6}$ becomes

$$k_1 k_2 k_3 (k_1 . k_2 . k_3 + 2k_7 . \overline{k_1 k_2 + k_1 k_3 + k_2 k_3}),$$

and $\zeta(\eta_{q_1} \eta_{q_2} \eta_{q_3} \eta_{q_4})$ vanishes, except for the cases where $q_1 q_2 q_3 q_4$ represent respectively, q_1 the index 1 or 4, q_2 the index 2 or 5, q_3 the index 3 or 6, and q_4 the index 7.

Hence $\quad \Sigma k_{q_1} k_{q_2} k_{q_3} k_{q_4} \zeta(k_{q_1} k_{q_2} k_{q_3} k_{q_4}) = 2^3 k_1 k_2 k_3 k_7 \zeta(k_1 k_2 k_3 k_7),$

and consequently τ_3 becomes

$$\pm 8\zeta(k_1 k_2 k_3 k_7) \times \{k_1 k_2 k_3 + 2k_7(k_1 k_2 + k_1 k_3 + k_2 k_3)\}.$$

Hence we are able to predict that the general expression for our τ in the case before us will be

$$\tau_3 = \mp \Sigma\{\zeta(k_{q_1} k_{q_2} k_{q_3} k_{q_7}) \times \left\{ \begin{array}{l} (k^3_{q_4} + k^3_{q_5} + k^3_{q_6}) - (k^2_{q_4} + k^2_{q_5} + k^4_{q_6})(k_{q_1} + k_{q_2} + k_{q_3} + k_{q_7}) \\ \quad + (k_{q_4} + k_{q_5} + k_{q_6})(k_{q_1} . k_{q_2} + k_{q_1} . k_{q_3} + k_{q_1} k_{q_7} + k_{q_2} . k_{q_3} + k_{q_2} . k_{q_7} + k_{q_3} . k_{q_7}) \\ \quad - 4(k_{q_1} k_{q_2} k_{q_3} + k_{q_1} k_{q_2} k_{q_7} + k_{q_1} k_{q_3} k_{q_7} + k_{q_2} k_{q_3} . k_{q_7}) \end{array} \right\}.$$

For in the first place, the fact that the τ vanishes when more than three relations of equality exist between the k's, proves that we may assume τ_3 of the form

$$\Sigma\{\zeta(k_{q_1} k_{q_2} k_{q_3} k_{q_7}) \times \phi\{k_{q_1} k_{q_2} k_{q_3} k_{q_7};\ k_{q_4} k_{q_5} k_{q_6}\},$$

the semicolon (;) separating the k's into two groups, in respect of each of which severally ϕ is a symmetrical form. But if in the expression last above written for τ_3 we make $\quad k_1 = k_4 \quad k_2 = k_5 \quad k_3 = k_6,$

it becomes

$$\mp 8\zeta(k_1 k_2 k_3 k_7) \left\{ \begin{array}{l} (k^3_1 + k^3_2 + k^3_3) - (k^2_1 + k^2_2 + k^2_3)(k_1 + k_2 + k_3 + k_7) \\ \quad + (k_1 + k_2 + k_3)(k_1 k_2 + k_1 k_3 + k_2 k_3 + k_1 k_7 + k_2 k_7 + k_3 k_7) \\ \quad - 4(k_1 k_2 k_3 + k_1 k_2 k_7 + k_1 k_3 k_7 + k_2 k_3 k_7) \end{array} \right\}.$$

Now in general if

$$\sigma_r=a_1^r+a_2^r+a_3^r+\ldots+a_t^r,$$

and

$$S_r=\Sigma_t^0(a_1.a_2.a_3\ldots a_r),$$

$$\sigma_r-\sigma_{r-1}S_1+\sigma_{r-2}S_2\pm\ldots\pm rS_r=0.$$

Consequently the sum of the terms constituting the second factor in the above expression

$$=(3-4)k_1.k_2k_3+(2-4)k_7(k_1k_2+k_1k_3+k_2k_3).$$

Hence the above expression becomes

$$\pm 8\zeta(k_1k_2k_3k_7)\{k_1.k_2.k_3+2(k_1k_2+k_1k_3+k_2k_3)k_7\}.$$

Thus, then, whenever $k_1k_2k_3$ are respectively equal to any three of the quantities $k_4k_5k_6k_7$, which may take place in twenty-four different ways {twenty-four being the number of permutations of four things}, our τ_3 will have been correctly assumed; but $\zeta(k_{q_1}k_{q_2}k_{q_3}k_{q_7})$ being replaceable by $\zeta(h_{q_1}h_{q_2}h_{q_3}h_{q_7})$, the τ_3 may be treated as a cubic function in k_1, k_2, k_3, and arranged according to the powers of $k_1k_2k_3$ will contain only twenty terms; hence, since the assumed form is verified for more than twenty, *i. e.* for twenty-four values of h_1, h_2, h_3, it follows that the assumed form is universally identical with the form of τ, which was to be determined.

Art. (40.). Now, again, in order to facilitate the conception of the general proof, let us suppose fx to be of only five dimensions in x, i still remaining 3: it will no longer be possible when we suppose a multiplicity three to prevail among the roots. to conceive this multiplicity to be distributed into three parts, for that would require the existence of three pairs of roots, there being only five. But we may, if we please, make $h_1=h_2=h_3$, and $h_4=h_5$, or else $h_1=h_2=h_3=h_4$, or in any other mode conceive the multiplicity to be divided into two parts, 2 and 1 respectively, or to be taken collectively "*en bloc.*" As a mode of proceeding the more remote from that last employed, I shall choose the latter supposition. Then we obtain (τ now becoming τ_{5-2-2}, *i. e.* τ_1)

$$k_1k_2k_3k_4k_5.\tau_1=\pm\Sigma k_{q_1}k_{q_2}k_{q_3}k_{q_4}\times\{\Sigma k_{q_1}k_{q_2}\zeta(k_{q_1}k_{q_2})\},$$

and $\zeta(k_q\,k_{q_2})$ will vanish, except in the case where q_1 represent the indices 1 or 2 or 3 or 4, and q_2 the index 5; also

$$\Sigma k_{q_1}k_{q_2}k_{q_3}k_{q_4}=k_{q_1}^4+4k_1^3.k_5.$$

Hence our equation becomes

$$k_1^4.k_5.\tau=\pm(k_1^4+4k_1^3k_5)4k_1.k_5\zeta(k_1k_5),$$

and τ becomes

$$-4\zeta(k_1k_5)(k_1+4k_5).$$

If, now, we assume for the general value of τ in the case before us

$$\tau=\Sigma\zeta(k_{q_1}k_{q_5})\{(k_{q_2}+k_{q_3}+k_{q_4})-4(k_{q_1}+k_{q_5})\},$$

when $k_1=k_2=k_3=k_4$, τ becomes

$$\pm 4\zeta(k_1k_5)\big(3k_1-(4k_1+k_5)\big),$$

$$i.\,e.\ \pm 4\zeta(k_1k_5)(k_1+4k_5).$$

Hence then for the two systems of values of $h_1\,h_2\,h_3$, viz.

$$\begin{matrix} h_1=h_4 & & h_1=h_5 \\ h_2=h_4 & \text{or} & h_2=h_5 \\ h_3=h_4 & & h_3=h_5, \end{matrix}$$

the form of τ will have been correctly assumed. But since the derived form is a linear function of h_1, h_2, h_3, this is not enough to identify the assumed with the general form, since for such verification four systems of values must be taken, four being the number of terms in a function of three variables of the first degree. If, however, we had adopted a separation of the multiplicity three into two parts, and had started with supposing $k_1=k_2=k_3$ $k_4=k_5$, we should have found that τ would have become

$$=6\zeta(k_1,\,k_5)(2k_1+3k_5).$$

Moreover, when these equalities subsist,

$$k_1\,k_2\,k_3\,k_4+k_1\,k_2\,k_3\,k_5+k_1\,k_2\,k_4\,k_5+k_1\,k_3\,k_4\,k_5+k_2\,k_3\,k_4\,k_5$$

becomes $2k_1^3\,k_5+3k_1^2.k_5^2$, and the common factor $k_1^2.k_4$ disappears in the course of the operations for finding τ, and eventually we have to show (in order to support the universality of the previously assumed form for τ) that

$$\eta_{q_2}+\eta_{q_3}+\eta_{q_4}-4(\eta_{q_1}+\eta_{q})$$

becomes $-2\eta_{q_4}-3\eta_5$ when

$$\eta_{q_2}=\eta_{q_3}=\eta_{q.}=\eta_1$$

and

$$\eta_{q_4}=\eta_{q_5}=\eta_5,$$

which is evidently true. Hence then τ will have been correctly assumed for the following cases,

$$k_1=k_2=k_5=k_3$$
$$k_1=k_2=k_5=k_4:$$

and also for the cases

$$\left.\begin{matrix} k_1=k_2=k_3 \text{ and } k_5=k_4 \\ k_1=k_5=k_3 \text{ and } k_2=k_4 \\ k_2=k_5=k_3 \text{ and } k_1=k_4 \end{matrix}\right\}$$
$$\left.\begin{matrix} k_1=k_2=k_4 \text{ and } k_5=k_3 \\ k_1=k_5=k_4 \text{ and } k_2=k_3 \\ k_2=k_5=k_4 \text{ and } k_1=k_3 \end{matrix}\right\},$$

i.e. for eight cases in all, whereas four only would have sufficed. Hence, "*ex abundantiâ demonstrationis*," the form assumed for τ_1 is in the case before us the general form.

Art. (41.). We may now easily write down the general form which τ assumes for all values of i and prove its correctness. If the roots be $h_1\,h_2\,h_3\ldots h_m$, and

$$t_{m-i-1}f'x-\tau_{m-i-2}fx+\vartheta_i=0,$$

we shall have

$$\pm t_{m-i-1}=\Sigma\begin{Bmatrix}\zeta(h_{q_1}h_{q_2}h_{q_3}\ldots h_{q_{m-i-1}})\times[\sigma_{m-i-2}-\sigma_{m-i-3}.S_1+\sigma_{m-i-4}.S_2\mp \&c. \\ +(-)^{m-i-3}\sigma_1.S_{m-i-3}+(-)^{m-i-2}(\sigma_0+1)S_{m-i-2}]\end{Bmatrix},$$

where σ_r denotes in general the sum of the rth *powers* of the $(i+1)$ quantities

$$(x-h_{q_{m-i}}),\ (x-h_{q_{m-i+1}}),\ \ldots(x-h_{q_m}),$$

and S_r denotes in general the sum of the *products* of the *complementary* $(m-i-1)$ quantities

$$(x-h_{q_1}),\ (x-h_{q_2})\ldots(x-h_{q_{m-i-1}})$$

combined, r and r together. It will of course also be understood that $\sigma_0=i+1$, so that $\sigma_0+1=i+2$.

Art. (42.). To prove the correctness of this general determination of the form of τ_{m-i-1}, let us suppose in general that $i+1$ relations of equality spring up between the (m) quantities $k_1, k_2, \ldots k_m$, we shall then easily obtain (N representing a certain numerical multiplier)

$$\pm Q=N.\zeta(k_1\,k_2\ldots k_{m-i-1})\frac{\Sigma k_{q_1}k_{q_2}\ldots k_{q_{m-1}}}{k_1^{\mu_1-1}.k_2^{\mu_2-1}\ldots k_{m-i-1}^{\mu_{m-i-1}}},$$

$k_1, k_2\ldots k_{m-i-1}$ being what the (k) system becomes when repetitions are excluded, and being respectively supposed to occur $\mu_1, \mu_2, \ldots \mu_{m-i-1}$ times respectively, so that

$$\mu_1+\mu_2+\ldots+\mu_{m-i-1}=m;$$

the fractional part of the right-hand member of the equation immediately above written will be readily seen to be equivalent to

$$\Sigma\mu_{\theta_{m-i-1}}(k_{\theta_1}.k_{\theta_2}\ldots k_{\theta_{m-i-2}}).$$

To establish the correctness of the assumed form, we must be able, as in the particular cases previously selected, to prove two things; the one, and the more difficult thing to be proved is, that when the series of distinct quantities $k_1, k_2, k_3\ldots k_m$ become converted into μ_1 groups of k_1; μ_2 groups of $k_2\ldots\mu_{m-i-1}$ groups of k_{m-i-1}, then that

$$\Sigma\mu_{\theta_1}.(k_{\theta_2}k_{\theta_3}k_{\theta_4}\ldots k_{\theta_{m-i-1}}),$$

or in other terms,

$$\Sigma\pm k_{\theta_1}k_{\theta_2}k_{\theta_3}\ldots k_{\theta_{m-i-1}}\Sigma^1_{m-i-1}(\mu_\theta),$$

becomes identical with

$$\sigma_{m-i-2}-\sigma_{m-i-3}S_1\pm \&c.+(\sigma_0+1)S_{m-i-2}.$$

The other step to be made, and with which I shall commence, consists in showing that the number of terms in the expression last above written, considered as a function of $(m-i-2)$th degree of $(i+1)$ variables, is never greater than the entire number of ways in which $(2+1)$ quantities out of m quantities may be equated to the remaining $(m-i-1)$ quantities, viz. each of the first set respectively to all the same, or all different, or some the same and some different; in short, in any manner each of the $i+1$ quantities with some one or another (without restriction against repetitions) of the $m-i-1$ remaining quantities. This latter number being in fact the number of ways in which $(m-i-1)$ quantities may be combined $(i+1)$ together with repetitions admissible by a well-known arithmetical theorem is $(m-i-1)^{i+1}$, and the first number is $\frac{(i+1)(i+2)\ldots(m-2)}{1.2\ldots(m-i-2)}$, which is always less than the other. It remains then only to prove the remaining step of the demonstration*.

Art. (43.). To fix the ideas let $m=10$, $i=5$, and consider the expression

$$\begin{aligned}&(k_5^3+k_6^3+k_7^3+k_8^3+k_9^3+k_{10}^3)-(k_5^2+k_6^2+k_7^2+k_8^2+k_9^2+k_{10}^2)(k_1+k_2+k_3+k_4)\\&+(k_5+k_6+k_7+k_8+k_9+k_{10})(k_1.k_2+k_1\,k_3+k_1\,k_4+k_2\,k_3+k_2.k_4+k_3.k_4)\\&-7(k_1\,k_2\,k_3+k_1\,k_2+k_4+k_1.k_3.k_4+k_2.k_3.k_4).\end{aligned}$$

Now suppose the six quantities k_5, k_6, k_7, k_8, k_9, k_{10} to become respectively equal each to some one or another of the four quantities k_1, k_2, k_3, k_4, as for instance, I shall suppose

$$\begin{aligned}k_5&=k_6=k_7=k_1\\k_8&=k_9=k_2\\k_{10}&=k_3.\end{aligned}$$

Then $$\mu_1=4,\ \mu_2=3,\ \mu_3=2,\ \mu_4=1,$$

and the formula of art. (41) becomes

$$\begin{aligned}&(3k_1^3+2k_2^3+k_3^3)-(3k_1^2+2k_2^2+k_3^2)(k_1+k_2+k_3+k_4)\\&+(3k_1+2k_2+k_3)(k_1.k_2+k_1.k_3+k_1.k_4+k_2.k_3+k_4.k_4+k_3.k_4)\\&-7(k_1.k_2.k_3+k_1.k_2.k_4+k_1\,k_3\,k_4+k_2.k_3.k_4)\\&=3\{(k_1^3-k_1^2.(\overline{k_2+k_3+k_4}+k_1)+k_1(\overline{k_2\,k_3+k_2\,k_4+k_3.k_4}+k_4.\overline{k_2+k_3+k_4})\\&+2\{k_2^3-k_2^2.(\overline{k_1+k_3+k_4}+k_2)+k_2(\overline{k_1\,k_3+k_1\,k_4+k_3\,k_4}+k_2\,\overline{k_1+k_3+k_4})\\&+(k_3^3-k_3^2(\overline{k_1+k_2+k_4}+k_3)+k_3(\overline{k_1.k_2+k_1.k_4+k_2.k_4}+k_3\,\overline{k_1+k_2+k_4})\\&-\{k_2\,k_3\,k_4+k_1\,k_3\,k_4+k_1\,k_2\,k_4+k_1\,k_2\,k_3\}\\&=-k_1\,k_2\,k_3-2k_1\,k_2\,k_4-3k_1\,k_3\,k_4-4k_2\,k_3\,k_4\\&=-k_1\,k_2\,k_3\,k_4\left\{\frac{\mu_1}{k_1}+\frac{\mu_2}{k_2}+\frac{\mu_3}{k_3}+\frac{\mu_4}{k_4}\right\}.\end{aligned}$$

In the above investigation the quantities which with their repetitions make up the k's system, are k_4, k_1, k_2, k_3, *appearing* respectively 1, 2, 3, 4 times, that is to say *repeated* 0, 1, 2, 3 times; 7 is 1 more than the sum of the repetitions $0+1+2+3$, and the numbers 1, 2, 3, 4 arise from subtracting from 7 the sums $1+2+3$; $0+2+3$; $0+1+3$; $0+1+2$; respectively, so that the remainders 1, 2, 3, 4 denote respectively one more than the number of *repetitions* of k_4, k_1, k_2, k_3, *i. e.* are the number of *appear-*

* If this first step of the demonstration appear unsatisfactory or subject to doubt, it may be dispensed with, and the result obtained in the succeeding article (the demonstration of which is wholly unexceptionable) being assumed, it may be proved that the formula there obtained on a particular hypothesis must be universally true, in precisely the same way and by aid of the same Lemma in and by aid of which the formula obtained in the Supplement to this section for the simplified quotients to $\frac{f'x}{fx}$ upon a like particular hypothesis is shown to be of universal application, *i. e.* by showing that otherwise a function of $2i-1$ variables would contain a function of $2i$ variables as a factor.

ances of k_4, k_1, k_2, k_3; and thus with a slight degree of attention to the preceding process the reader may easily satisfy himself that the preceding demonstration (although not so expressed) is in essence universal, and the form of r as an explicit function of x and of the roots of $f(x)$ is thus completely established for all values of m and of i.

Supplement to SECTION III.

On the Quotients resulting from the process of continuous division ordinarily applied to two Algebraical Functions in order to determine their greatest Common Measure.

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Art. (*a.*)* We have now succeeded in exhibiting the forms of the numerators and denominators of $\frac{f'x}{fx}$ developed into a continued fraction in terms of the differences of the roots and factors of fx. It remains to exhibit the quotients themselves of this continued fraction under a similar form.

Lemma.—*An equation being supposed of an arbitrary degree* n, *there exists no function of* n *and of less than* 2i *of the coefficients*†, *which vanishes for all values of* n *whenever the* n *roots reduce in any manner to* i *distinct groups of equal roots; or in other words, any function of* n *and the first* 2i−1 *coefficients of an equation of the* n*th degree, which vanishes for all values of* n *in every case where the roots retain only* i *distinct names, must be identically zero.*

To render the statement of the proof more simple, let i be taken equal to 3. And let the roots be supposed to reduce to p roots a, q roots b, and r roots c. And let s_r in general denote the sum of the rth powers of the roots. Then we have evidently

$$\begin{aligned}
p+q+r&=s_0\\
pa+qb+rc&=s_1\\
pa^2+qb^2+rc^2&=s_2\\
pa^3+qb^3+rc^3&=s_3\\
pa_4+qb^4+rc^4&=s_4,
\end{aligned}$$

&c. &c., *ad infinitum.*

Eliminating p, q, r between the first, second, third and fourth equations, we obtain

$$\begin{vmatrix} 1 & 1 & 1 & s_0 \\ a & b & c & s_1 \\ a^2 & b^2 & c^2 & s_2 \\ a^3 & b^3 & c^3 & s_3 \end{vmatrix}=0.$$

* The articles in this and subsequent sections to which Latin or Greek letters are prefixed, although in strict connexion with the context, are supplementary in the sense of having been supplied since the date when the paper was presented for reading to the Royal Society. All the articles marked with numbers (from 1 to 72), and the Introduction, appeared in the memoir as originally presented to the Society, June 16, 1853.

† In the proposition thus enunciated the coefficient of the highest power of x is supposed to be a numerical quantity.

In like manner eliminating ap, bq, cr between the second, third, fourth and fifth equations, we have

$$\begin{vmatrix} 1 & 1 & 1 & s_1 \\ a & b & c & s_2 \\ a^2 & b^2 & c^2 & s_3 \\ a^3 & b^3 & c^3 & s_4 \end{vmatrix} = 0;$$

and so in general we have for all values of e,

$$\begin{vmatrix} 1 & 1 & 1 & s_e \\ a & b & c & s_{e+1} \\ a^2 & b^2 & c^2 & s_{e+2} \\ a^3 & b^3 & c^3 & s_{e+3} \end{vmatrix} = 0;$$

whence it may immediately be deduced, that, upon the given supposition of there being only three groups of distinct roots, we must have the following infinite system of coexisting equations satisfied, viz.—

$$\begin{array}{ll} s_0t+s_1u+s_2v+s_3w=0 & \text{say } L_0=0 \\ s_1t+s_2u+s_3v+s_4w=0 & L_1=0 \\ s_2t+s_3u+s_4v+s_5w=0 & L_2=0 \\ s_3t+s_4u+s_5v+s_6w=0 & L_3=0 \\ s_4t+s_5u+s_6v+s_7w=0, & L_4=0, \\ \&c.\ \&c.\ \&c. & \&c., \end{array}$$

and conversely, when this infinite system of equations is satisfied the roots must reduce themselves to three groups of equal roots.

Let now ϕ be any function of $s_0\ s_1\ s_2 \ldots s_\omega$, which vanishes when this is the case. Then ϕ must necessarily contain as a factor some derivee of the infinite system of equations above written, *i. e.* some function of $s_0\ s_1\ s_2$, &c., which vanishes when these equations are satisfied; *i. e.* some conjunctive of the quantities $L_0\ L_1\ L_2\ L_3$: but it is obviously impossible in any such conjunctive to exclude s_6 from appearing, unless by introducing some other s with an index higher than 6, and consequently ϕ cannot be merely a function of $s_0\ s_1\ s_2\ s_3\ s_4\ s_5$, nor consequently of n, and the first five coefficients; or if such, it is identically zero, and so in general any function of n, and only $2i-1$ of the coefficients, which vanishes when the roots reduce to i groups of equal roots, must be identically zero, as was to be proved.

Art. (*b*.) It ought to be observed that the preceding reasoning depends essentially upon the circumstance of n being left arbitrary. If n were given the proposition would no longer be true. In fact, on that supposition, the n roots reducing to i distinct roots would imply the existence of $n-i$ conditions between the n roots; and consequently $n-i$ independent equations would subsist between the n coefficients, and functions could be formed of i only of the coefficients, which would satisfy the prescribed condition of vanishing when the roots resolved themselves into i groups of distinct identities.

Art. (*c.*) Let $D_{r_1 r_2 \ldots r_i}$ be used in general to denote the determinant

$$\begin{matrix} s_{r_1} & s_{r_1+1} & s_{r_1+2} & \ldots & s_{r_1+i-1} \\ s_{r_2} & s_{r_2+1} & s_{r_2+2} & \ldots & s_{r_2+i-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ s_{r_i} & s_{r_i+1} & s_{r_i+2} & \ldots & s_{r_i+i-1}, \end{matrix}$$

then the simplified ith Sturmian residue R_i may be expressed under the form

$$D_{1,2,3\ldots i}\, x^{n-i-1} - D_{2,3,\ldots i+1}\, x^{n-i-2} + D_{3,4\ldots(i+2)}\, x^{n-i-3} \ldots \pm D_{n-i,n-i-1,\ldots n},$$

which is easily identifiable with the known expression for such residue.

Now obviously the necessary and sufficient conditions in order that the n roots may consist of only repetitions of i distinct roots is, that R_i shall be identically zero, that is to say, we must have

$$D_{1,2,\ldots i}=0 \quad D_{2,3\ldots i+1}=0 \;\ldots\ldots\; D_{n-i,n-i-1,\ldots n}=0.$$

But the reasoning of the preceding article shows that although these equations are necessary and sufficient, they are but a selected system of equations of an infinite number of similar equations which subsist*, and that, in fact, whatever be the value of (n), we may take $r_1\ r_2 \ldots r_i$ perfectly arbitrary and as great as we please, and the equation

$$D_{r_1 r_2 \ldots r_i}=0$$

must exist by virtue of the existence of the $n-i$ equations last above written.

Art. (*d.*) I now return to the question of expressing the successive quotients of $\frac{f'x}{fx}$ as functions of the differences of the roots and factors; that they must be capable of being so expressed is an obvious consequence of the fact, that the numerators and denominators of the convergents have been put under that form, since if

$$\frac{N_{i-2}}{D_{i-2}},\ \frac{N_{i-1}}{D_{i-1}},\ \frac{N_i}{D_i}$$

are any three consecutive convergents of the continued fraction

$$\frac{1}{Q_1-}\ \frac{1}{Q_2-}\cdots\cdots\frac{1}{Q_i},$$

we must have

$$D_{i-2}.N_i - N_{i-2}.D_i = Q_i.$$

It would not, however, be easy to perform the multiplications indicated in the above equation, so as to obtain Q_i under its reduced form as a linear function of x. I proceed therefore to find Q_i constructively in the following manner.

Let R_{i-2}, R_{i-1}, R_i be three consecutive residues, $f'x$ counting as the residue in the zero place, then $Q_i=\frac{R_{i-2}-R_i}{R_{i-1}}$ and is of the form $\frac{p}{q}x+\frac{p'}{q'}$.

* But *quære* whether any other *sufficient* system can be found of equations so few in number as this system.

Now in general if we call the n roots of fx, where the coefficient of x^n is supposed to be unity, $h_1\ h_2 \dots h_n$, and if we use Z_ι to denote $\Sigma\zeta(h_{\theta_1}\ h_{\theta_2}\dots h_{\theta_\iota})$*, with the convention that $Z_1=n$, $Z_0=1$, we have, employing (i) to denote $\frac{1}{2}\{(-1)^i+1\}$,

$$R_i=\frac{Z_{i-1}^2.Z_{i-3}^2\dots Z_{(i)}^2}{Z_{i-1}^2.Z_{i-2}^2\dots Z_{(i)+1}^2}.\Sigma\{\zeta(h_{\theta_1}\ h_{\theta_2}\dots h_{\theta_{i+1}})(x-h_{\theta_{i+2}})(x-h_{\theta_{i+3}})\dots(x-h_{\theta_n})\}$$

$$R_{i-1}=\frac{Z_{i-2}^2.Z_{i-4}^2\dots Z_{(i)+1}^2}{Z_{i-1}^2.Z_{i-3}^2\dots Z_{(i)}^2}\Sigma\{\zeta(h_{\theta_1}\ h_{\theta_2}\dots h_{\theta_i})(x-h_{\theta_{i+1}})(x-h_{\theta_{i+2}})\dots(x-h_{\theta_n})\}$$

$$R_{i-2}=\frac{Z_{i-3}^2.Z_{i-5}^2\dots Z_{(i)}^2}{Z_{i-2}^2.Z_{i-4}^2\dots Z_{(i)+1}^2}\Sigma\{\zeta(h_{\theta_1}\ h_{\theta_2}\dots h_{\theta_{i-1}})(x-h_{\theta_i})(x-h_{\theta_{i+1}})\dots(x-h_{\theta_n})\}.$$

The part of R_{i-1} within the sign of summation is

$$Z_i x^{n-i}-\Sigma(h_{\theta_{i+1}}+h_{\theta_{i+2}}+\dots+h_{\theta_n})\zeta(h_{\theta_1}\ h_{\theta_2}\dots h_{\theta_i})x^{n-i-1}+\&c.,$$

say

$$Z_i x^{n-i}-Z'_i x^{n-i-1}+\&c.,$$

and the part of R_{i-2} within the sign of summation is

$$Z_{i-1}x^{n-i+1}-Z'_{i-1}x^{n-i}+\&c.,$$

and $Z_i^2.\dfrac{Z_{i-1}x^{n-i+1}-Z'_{i-1}x^{n-i}}{Z_i x^{n-i}-Z'_i x^{n-i-1}}=Z_{i-1}\,Z_i\,x+(Z_{i-1}.Z'_i-Z_i\,Z'_{i-1})+$ an algebraic fraction.

Hence

$$Q_i=\frac{1}{Z_i^2}.\frac{Z_{i-3}^2.Z_{i-5}^2\dots Z_{(i)}^2}{Z_{i-2}^2\ Z_{i-3}^2\dots Z_{(i)+1}^2}.\left\{\frac{Z_{i-2}^2.Z_{i-4}^2\dots Z_{(i)+1}^2}{Z_{i-1}^2.Z_{i-3}^2\dots Z_{(i)}^2}\right\}^{-1}$$

$$\times\{Z_{i-1}.Z_i\,x+(Z_{i-1}\,Z'_i-Z_i\,Z'_{i-1})\}$$

$$=\frac{Z_{i-1}^2}{Z_i^2}.\frac{Z_{i-3}^4.Z_{i-5}^4\dots Z_{(i)}^4}{Z_{i-2}^4.Z_{i-4}^4\dots Z_{(i)+1}^4}.T_i$$

T_i denoting $Z_{i-1}.Z_i\,x+(Z_{i-1}\,Z'_i-Z_i.Z'_{i-1})$.

Art. (*e*.) If the process of obtaining the successive quotients and residues be considered, it will easily be seen that each step in the process imports two new coefficients into the quotients, the first quotient containing no literal quotient in the part multiplying x and containing the first literal coefficient in the other part, the second quotient containing two literal coefficients in the one part and three in the other, and in general the ith quotient containing $2i-2$ of the letters in the one part and $2i-1$ of them in the other. Hence T_i being made equal to $L_i.x+M_i$, L_i contains $2i-2$ and M_i contains $2i-1$ of the literal coefficients of fx.

Moreover, we have

$$Z_i \text{ of the form } T_i^2.\frac{P_{i-2}-mP_i}{P_{i-1}},$$

where

$$P_{i-1}=\Sigma\zeta(h_{\theta_1}\ h_{\theta_2}\dots h_{\theta_i})\eta_{\theta_{i+1}}\ \eta_{\theta_{i+2}}\dots\eta_{\theta_n}$$

$$P_{i-2}=\Sigma\zeta(h_{\theta_1}\ h_{\theta_2}\dots h_{\theta_{i-1}})\eta_{\theta_i}\ \eta_{\theta_{i+1}}\dots\eta_{\theta_n},$$

* ζ it will be remembered is the symbol of the operation of taking the product of the squares of the differences of the quantities which it governs.

and P_i, which is the ith simplified residue, vanishes when the n roots in any manner become reduced to only i distinct groups.

I proceed to show that if we make

$$A_i x+B_i=U_i=A^2_{i,1}(x-h_1)+A^2_{i,2}(x-h_2)+\ldots+A^2_{i,n}(x-h_n),$$

where in general

$$A_{i,e} \text{ represents } \Sigma\zeta(h_{\theta_1}\, h_{\theta_2}\ldots h_{\theta_{i-1}})(h_e-h_{\theta_1})(h_e-h_{\theta_2})\ldots(h_e-h_{\theta_{i-1}}),$$

then will

$$T_i=U_i.$$

It will be observed that $A_{i,e}$ is identical with what the simplified denominator of the $(i-1)$th convergent becomes when we write h_e in place of x, and consequently, when arranged according to the powers of h_e, will be of the form

$$c_1.h_e^{i-1}+c_2\, h_e^{i-2}+\ldots+c_i$$

where $c_1, c_2, \ldots c_i$ are functions of the coefficients, but containing no more of them than enters into Q_{i-1}, *i.e.* containing only $2i-2$ of them.

Now A_i is made up of terms, each consisting of some binary product of

$$c_1,\ c_2,\ \ldots,\ c_i$$

combined with some term of the series

$$\Sigma h^{2i-2},\ \Sigma h^{2i-3}\ \ldots\ \Sigma h^0\,;$$

and any one of this latter set of terms expressed as a function of the coefficients of fx contains at most $2i-2$ of them.

Hence only $2i-2$ of the coefficients enter into A_i, and in like manner only $2i-1$ of them into B_i.

The number of letters, therefore, in A_i and in B_i is the same as in L_i and in M_i, viz. $2i-2$ and $2i-1$ respectively.

Now let the roots consist of only i distinct groups of equal roots, so that T_i becomes $=Z_i^2.\frac{P_{i-2}}{P_{i-1}}$.

I shall show that in whatever way the equal roots are supposed to be grouped upon this supposition, there will result the equation

$$T_i=U_i,$$

where

$$T_i=\{\Sigma\zeta(\eta_{\theta_1}\eta_{\theta_2}\ldots\eta_{\theta_i})\}^2.\frac{P_{i-2}}{P_{i-1}}$$

$$P_{i-2}=\Sigma\{\eta_{\theta_i}\,\eta_{\theta_{i+1}}\ldots\eta_{\theta_n}\,\zeta(\eta_{\theta_1}\,\eta_{\theta_2}\ldots\eta_{\theta_{i-1}})\}$$

$$P_{i-1}=\Sigma\{\eta_{\theta_{i+1}}\,\eta_{\theta_{i+2}}\ldots\eta_{\theta_n}\zeta(\eta_{\theta_1}\,\eta_{\theta_2}\ldots\eta_{\theta_i})\},$$

and

$$H_i=A_1^2.\eta_1+A_2^2.\eta_2+\ldots+A_n^2.\eta_n,$$

A_e meaning

$$\Sigma\{(\eta_e-\eta_{\theta_1})(\eta_e-\eta_{\theta_2})\ldots(\eta_e-\eta_{\theta_{i-1}})\zeta(\eta_{\theta_1}\,\eta_{\theta_2}\ldots\eta_{\theta_{i-1}})\},$$

and η_ω meaning $x-h_\omega$.

Let the n factors be constituted of m_1 factors η_1 m_2 factors $\eta_2 \dots m_i$ factors η_i. Then

$$Z_i = \mu \zeta(\eta_1 \eta_2 \dots \eta_i),$$

where

$$\mu = m_1 . m_2 \dots m_i$$

$$P_{i-1} = \mu \zeta(\eta_1 \eta_2 \dots \eta_i) \eta_1^{m_1-1} . \eta_2^{m_2-1} \dots \eta_i^{m_i-1},$$

and

$$P_{i-2} = \mu_1 . \zeta(\eta_2 \eta_3 \dots \eta_i) \eta_1^{m_1} . \eta_2^{m_2-1} \dots \eta_i^{m_i-1}$$
$$+ \mu_2 . \zeta(\eta_1 \eta_3 \dots \eta_i) \eta_1^{m_1-1} . \eta_2^{m_2} \dots \eta_i^{m_i-1}$$
$$+ \text{\&c. \&c.}$$
$$+ \mu_i \zeta(\eta_1 \eta_2 \dots \eta_{i-1}) \eta_1^{m_1-1} . \eta_2^{m_2-1} \dots \eta_i^{m_i},$$

where

$$\mu_1 = \frac{\mu}{m_1} \quad \mu_2 = \frac{\mu}{m_2} \dots\dots \mu_i = \frac{\mu}{m_i}.$$

Hence $$T_i = \mu^2 \zeta(\eta_1 \eta_2 \dots \eta_i) \left\{ \frac{\eta_1 \zeta(\eta_2 \eta_3 \dots \eta_i)}{m_1} + \frac{\eta_2 \zeta(\eta_1 \eta_3 \dots \eta_i)}{m_2} + \dots + \frac{\eta_i \zeta(\eta_1 \eta_2 \dots \eta_{i-1})}{m_i} \right\}.$$

Again, in U_i the term containing η_1 will be

$$m_1 \eta_1 \Sigma \{ (\eta_1 - \eta_2)(\eta_1 - \eta_3) \dots (\eta_1 - \eta_i) \zeta(\eta_2 \eta_3 \dots \eta_i) \}^2$$
$$= m_1 \eta_1 \times (m_2 . m_3 \dots m_i)^2 \times (\eta_1 - \eta_2)^2 (\eta_1 - \eta_3)^2 \dots (\eta_1 - \eta_i)^2 \{ \zeta \eta_2 . \eta_3 \dots \eta_i \}^2$$
$$= \frac{\mu^2}{m_1} \eta_1 \times \zeta(\eta_1 \eta_2 \dots \eta_i) \zeta(\eta_2 \eta_3 \dots \eta_i).$$

Hence $$U_i = \mu^2 \zeta(\eta_1 \eta_2 \dots \eta_i) \left\{ \frac{\eta_1 \zeta \eta_2 \eta_3 \dots \eta_i}{m_1} + \frac{\eta_2 \zeta \eta_1 \eta_3 \dots \eta_i}{m_2} + \text{\&c.} \right\} = T_i.$$

Hence, therefore, $U_i - T_i$ vanishes whenever the roots of fx contain only i distinct groups of equal roots, and it has been shown that U_i and T_i each contain only $2i-1$ of the coefficients of fx, so that $U_i - T_i$ is a function only of n and these $2i-1$ letters, and consequently by virtue of the Lemma in Art. (a.) $U_i - T_i$ is universally zero, *i. e.* U_i is identical with T_i, as was to be proved. In the same manner as observed in a preceding marginal note, the expression given in the antecedent articles for the numerator of the ith convergents having been verified for the case of the roots consisting of only i distinct groups, could have been at once inferred to be generally true by aid of the Lemma above quoted.

Art. (f.) Since the coefficient of x in T_i is $Z_{i-1} \times Z_i$, we deduce the unexpected relation

$$\Sigma \zeta(h_1 h_2 \dots h_{i-1}) \times \Sigma \zeta(h_1 h_2 \dots h_i) = P_1^2 + P_2^2 + \dots + P_n^2,$$

where

$$P_e = \Sigma \{ (h_e - h_{\theta_1})(h_e - h_{\theta_2}) \dots (h_e - h_{\theta_{i-1}}) \zeta(h_{\theta_1} h_{\theta_2} \dots h_{\theta_{i-1}}) \}.$$

So that every simplified Sturmian quotient to $\frac{f'x}{fx}$, when the (n) roots of fx are real, will be the sum of n squares. But the equation is otherwise remarkable, in exhibiting the product of the sum of $\frac{n.(n-1)\dots(n-i+2)}{1.2\dots(i-1)}$ squares by another sum of $\frac{n(n-1)\dots(n-i-1)}{1.2\dots i}$ squares under the form of the sum of n squares.

If we call the ith simplified denominator to the Sturmian convergents to $\frac{f'x}{fx}$, $D_i(x)$, and if we call the ith simplified quotient $X_i(x)$, we have

$$X_i x=\Sigma_n^1(D_{i-1}h_e)^2.(x-h_e).$$

If we construct the numerators and denominators of the convergents to

$$\frac{1}{Q_1-}\,\frac{1}{Q_2-}\,\frac{1}{Q_3}\cdots\frac{1}{Q_i}$$

according to the general rule for continued fractions as functions of Q_1, Q_2, Q_3, &c., so that calling the denominators Δ_1, Δ_2, Δ_3, &c. Δ_i,

$$\Delta_1=Q,\ \Delta_2=Q_1Q_2-1\ldots\ldots\Delta_i=Q_i\Delta_{i-1}-\Delta_{i-2},$$

we have
$$\Delta_{i-1}x=\frac{Z_{i-2}^2.Z_{i-4}^2\ldots Z_{(i-1)}}{Z_{i-1}^2.Z_{i-3}^2\ldots Z_1^2}D_{i-1}(x),$$

$\Delta_{i-1}.x$ being in fact the multiplier of $f'x$ in the equation which connects fx and $f'x$ with the $i-1$th *complete* residue, and consequently retaining $Q(x)$ to designate the complete ith quotient, we have

$$Q_i(x)=\frac{Z_{i-1}^2}{Z_i^2}.\frac{Z_{i-3}^4.Z_{i-5}^4\ldots Z_i^4}{Z_{i-2}^4.Z_{i-4}^4\ldots Z_{i+1}^4}\Sigma\{D_{i-1}.h_e\}^2(x-h_e)$$

$$=\frac{Z_{i-1}^6}{Z_i^2}.\frac{Z_{i-3}^8.Z_{i-5}^8\ldots Z_i^8}{Z_{i-2}^8.Z_{i-4}^8\ldots Z_{i+1}^8}\Sigma\Big\{\{\Delta_{i+1}.h_e\}^2(x-h_e)\Big\},$$

which equation gives the connexion between the form of any quotient and that of the immediately preceding convergent denominator of the continued fraction which expresses $\frac{f'x}{fx}$.

Art. (g.) I have found that the coefficients of the n factors of fx in the expression above given for the quotients possess the property that the sum of their square roots taken with the proper signs is zero for each quotient except the first (the coefficients for the first being all units), *i. e.* $D_ih_1+D_ih_2+\ldots D_ih_n=0$ for all values of i except $i=1$. Moreover I find that the determinant formed by the n sets of the n coefficients of the factors of fx in the complete set of n quotients is identically zero, *i. e.* the Determinant represented by the square matrix

$$\left\{\begin{matrix} 1 & 1 & 1 & 1 \\ (D_1\,h_1)^2 & (D_1\,h_2)^2 & (D_1\,h_3)^2 & \ldots(D_1\,h_n)^2 \\ (D_2.h_1)^2 & (D_2\,h_2)^2 & (D_2\,h_3)^2 & \ldots(D_2\,h_n)^2 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ (D_{n-1}h_1)^2 & (D_{n-1}\,h_2)^2 & (D_{n-1}h_3)^2 & \ldots(D_{n-1}h_n)^2 \end{matrix}\right\}=0.$$

Art. (h.) It should be observed that U_i is the form of the simplified quotients for all the quotients except the nth (*i. e.* the last), for which the simplified form is not U_n, but $U_n\div\zeta(h_1\,h_2\ldots h_n)$, which arises from the circumstance of the last divisor, which is the final Sturmian residue, not containing x; it being evidently the case that the division of a rational function of x by another one degree lower, introduces into the integral part of the quotient the square of the leading coefficient of the divisor, *subject to the exception* that when the divisor is of the degree zero, the simple power enters in lieu of the square. The general formula gives for the reduced nth quotient the expression

$$\Sigma\{(\overline{h_1-h_2}\,\overline{h_1-h_3}\ldots\overline{h_1-h_n}\zeta(h_2 h_3\ldots h_n))^2(x-h_1)\},$$

which equals

$$\zeta(h_1 h_2\ldots h_n)\Sigma\zeta(h_2 h_3\ldots h_n)(x-h_1).$$

Rejecting the first factor, we have

$$\Sigma\zeta(h_2 h_3\ldots h_n)(x-h_1),$$

which is equal to the penultimate residue, which residue is (as it evidently ought to be) identical with the simplified last quotient.

Art. (*i.*) We have thus succeeded in giving a perfect representation of $\frac{f'x}{fx}$, *i. e.* of

$$\frac{1}{x-h_1}+\frac{1}{x-h_2}+\ldots\ldots+\frac{1}{x-h_n},$$

under the form of a continued fraction of the form

$$\frac{1}{m_1(x-e_1)-}\;\frac{1}{m_2(x-e_2)-}\;\ldots\ldots\;\frac{1}{m_n(x-e_n)},$$

where $m_1\, m_2\ldots m_n$; $e_1\, e_2\ldots e_n$ are all determinate and known functions of $h_1\, h_2\ldots h_n$.

We may by means of this identity, differentiating any number of times with respect to x both sides of the equation, obtain analogous expressions for the series

$$\frac{1}{(x-h_1)^t}+\frac{1}{(x-h_2)^t}+\cdots+\frac{1}{(x-h_n)^t}.$$

But to do this we must be in possession of a rule for the differentiation of continued fractions whose quotients are linear functions of the variable. I subjoin here the first step only toward such investigation.

Let the denominator of

$$\frac{1}{q_1-}\;\frac{1}{q_2-}\;\ldots\ldots\frac{1}{q_n}$$

where $q_1\, q_2\ldots q_n$ are any n arbitrary quantities, be denoted by $[q_1, q_2, q_3\ldots q_n]$, so that the entire fraction will be equal to

$$\frac{[q_2\, q_3\ldots q_n]}{[q_1\, q_2\, q_3\ldots q_n]},$$

any such quantity as $[q_i\, q_{i+1}\ldots q_n]$ may be termed a Cumulant, of which $q_i, q_{i+1}\ldots q_n$ may be severally termed the elements or Components, and the complete arrangement of the elements may be termed the Type. The cumulant corresponding to any Type remains unaffected by the order of the elements in the Type being reversed, as is evident from any cumulant being in fact representable under the form of a symmetrical determinant, thus *ex. gr.* the cumulant $[q_1 q_2 q_3 q_4]$ may be represented by the determinant

$$\begin{matrix} q_1 & 1 & 0 & 0 \\ 1 & q_2 & 1 & 0 \\ 0 & 1 & q_3 & 1 \\ 0 & 0 & 1 & q_4, \end{matrix}$$

and $q_4 q_3 q_2 q_1$ will in like manner be represented by the determinant

$$\begin{matrix} q_4 & 1 & 0 & 0 \\ 1 & q_3 & 1 & 0 \\ 0 & 1 & q_2 & 1 \\ 0 & 0 & 1 & q_1, \end{matrix}$$

which is equal to the former.

Art. (j.) Let it be proposed in general to find the first differential coefficient in respect to x of the fraction

$$\frac{[q_i\, q_{i-1} \dots q_n]}{[q_1\, q_2\, q_3 \dots q_n]} = \mathrm{F}_i,$$

where each q is a function of one or more variables.

I find that the variation of F_i may be expressed as follows:—

$$-\delta \mathrm{F}_i = \{\delta[q_1, q_2 \dots q_{i-2}, q_n] + \delta[q_1, q_2 \dots q_{i-2}, q_{n-1}] . q_n^2$$

$$+\delta[q_1, q_2, q_3 \dots q_{i-2}, q_{n-2}] . [q_n, q_{n-1}]^2 + \&c. + \delta[q_1, q_2, q_3 \dots q_{i-2}, q_{i-1}] . [q_n, q_{n-1}, q_{n-2} \dots q_i]^2\}$$

$$\div [q_1, q_2, q_3 \dots q_n]^2.$$

Art. (k.) Suppose $i=2$, and $q_1 = a_1 x + b_1$ $q_2 = a_2 x + b_2 \cdots\cdots q_n = a_n x + b_n$, we shall have by virtue of the above equation,

$$\frac{d}{dx} \mathrm{F}_2, \ i.\ e.\ \frac{d}{dx} . \left\{ \frac{1}{q_1 -}\ \frac{1}{q_2 -}\ \frac{1}{q_3} \dots \frac{1}{q_n} \right\}$$

$$= -\frac{1}{[q_1\, q_2 \dots q_n]^2} \{ a_n . 1^2 + a_{n-1} . q_n^2 + a_{n-2} . [q_n, q_{n-1}]^2 + \&c. + a_1 [q_n, q_{n-1}, q_{n-2} \dots q_2]^2 \}.$$

If we call $\mathrm{F}_2 = \frac{\phi x}{fx}$ every such quantity as $[q_n, q_{n-1} \dots q_i]$ represents to a constant factor *près* the $(i-1)$th simplified residue (ϕx counting as the first of them) to $\frac{\phi x}{fx}$, and making certain obvious but somewhat tedious reductions, and rejecting the common factor $-\frac{1}{(fx^2)}$, we obtain the expression

$$\frac{\mathrm{C}_0 . \mathrm{R}_1^2}{\mathrm{C}_1} + \frac{\mathrm{R}_2^2}{\mathrm{C}_1 . \mathrm{C}_2} + \frac{\mathrm{R}_3^2}{\mathrm{C}_2 . \mathrm{C}_3} + \cdots\cdots + \frac{\mathrm{R}_n^2}{\mathrm{C}_{n-1} . \mathrm{C}_n} = (\phi x . f'x - \phi' x f x),$$

where $\mathrm{R}_1\ \mathrm{R}_2 \dots \mathrm{R}_n$ represent ϕx and the successive simplified residues to fx, ϕx, and

C_i means the coefficient of the highest power of x in R_i, and C_0 the first coefficient in fx*.

Art. (*l*.) If we take $g(x)$ of the same degree as $f(x)$ and for greater simplicity make the first coefficients in $f(x)$ and $g(x)$, each of them unity, the successive simplified residues to $\frac{gx}{fx}$ will be identical with the simplified residues to $\frac{-fx+gx}{gx}$ (including amongst them the quantity $gx-fx$ itself), and since

$$(fx-g(x))g'x-(fx-g(x))'gx=(g'xfx-f'x.gx),$$

the right-hand side of the equation above written, when the residues are made to refer to f and g, instead of referring to f and φ, are taken of the same degree in x, becomes equal to $f'xgx-fxg'x$; and if we now agree to consider f and g as homogeneous functions each of the nth degree in x and 1, the equation becomes

$$\frac{R_1^2}{C_1}+\frac{R_2^2}{C_1.C_2}+\frac{R_3^2}{C_2.C_3}+\dots\dots+\frac{R_n^2}{C_{n-1}.C_n}$$

$$=\left\{f(x,1)\frac{d}{dx}g(x,1)-g(x,1)\frac{d}{dx}f(x,1)\right\}=\frac{1}{n}\left(x\frac{d}{dx}f+\frac{d}{d1}f\right)\left(\frac{d}{dx}g\right)-\frac{1}{n}\left(x\frac{d}{dx}g+\frac{d}{d1}\cdot g\right)\left(\frac{d}{dx}f\right)$$

$$=\frac{1}{n}\left\{\frac{df}{d1}\cdot\frac{dg}{dx}-\frac{df}{dx}\cdot\frac{dg}{d1}\right\}=\frac{1}{n}J(f,g),$$

where J indicates the Jacobian of the given functions f and g in respect to the variables x and 1, meaning thereby the so-called *Functional Determinant* of JACOBI to f and g in respect of x and 1, which equation also obviously must continue to hold good when we restore to the coefficients of x^n in f and g their general values.

It may happen that for particular relations between the coefficients of f and g

* This result may be obtained directly as follows:—

Let fx, φx and the $(m-1)$ complete Sturmian residues be called $\rho_0\ \rho_1\ \rho_2\dots\rho_n$; let the n complete quotients be called $q_1\ q_2\dots q_n$, and let the allotrious factors to the residues $\rho_2, \rho_3, \dots \rho_n$ be called $\mu_2\ \mu_3\dots\mu_n$; then

$$\rho_0=q_1.\rho_1-\rho_2;\ \rho_1=q_2\,\rho_2-\rho_3;\ \rho_2=q_3\,\rho_3-\rho_4;\ \text{\&c.}$$

hence

$$\rho_1\delta\rho_0-\rho_0\delta\rho_1=\rho_1^2\delta q_1+(\rho_2\delta\rho_1-\rho_0\delta\rho_2)$$
$$=\rho_1^2\delta q_1+\rho_2^2\delta q_2+(\rho_3\delta\rho_2-\rho_2\delta\rho_3)$$
$$=\text{\&c.}$$
$$=\rho_1^2\delta q_1+\rho_2^2\delta q_2+\rho_3^2\delta q_3+\dots+\rho_n^2\delta q_n,$$

but we have in general $\rho_i=\mu_i.R_i$;

hence

$$\delta q_i=\frac{C_{i-1}}{C_i}.\frac{\mu_{i-1}}{\mu_i}\delta x$$

and

$$\rho_i^2\delta q_i=\frac{C_{i-1}}{C_i}.\mu_{i-1}.\mu_i R_i^2\delta x;$$

but it may be easily seen that

$$\mu_{i-1}.\mu_i=\frac{1}{C_{i-1}^2};\ \text{except when } i=1, \text{ for which case } \mu_{i-1}.\mu_i=1,$$

hence

$$\rho_i^2\delta q_i=\frac{1}{C_{i-1}.C_i}R_i^2\delta x,\ \text{when } i>1, \text{ and } =\frac{C_0}{C_1}R_1^2\delta x \text{ when } i=1,$$

which proves the theorem in the text.

certain of the residues may be wanting, which will be the case when any of the secondary Bezoutics have their first or successive first terms affected with the coefficient zero; the equation connecting the residues with the Jacobian will then change its form (as some of the quantities $C_1, C_2, \ldots C_n$ will become zero); but I do not propose to enter for the present into the theory of these failing, or as they may more properly be termed, Singular cases in the theory of elimination.

Art. (*m.*) The series last obtained for J (f, g) leads to a result of much interest in the theory, and of which great use is made in the concluding section of this memoir, viz. the identification of the Jacobian (abstraction made of the numerical factor n) with what the Bezoutiant becomes when in place of the n variables in it, $u_1\ u_2 \ldots u_n$, we write $x^{n-1}, x^{n-2}, \ldots x, 1$. Thus suppose f and g to be each of the third degree, and let

$$\begin{array}{l} Ax^2+Hx+G \\ Hx^2+Bx+F \\ Gx^2+Fx+C \end{array}$$

be the three primary Bezoutics; if we make

$$x^2=u \quad x=v \quad 1=w,$$

these may be written under the form

$$\begin{array}{l} Au+Hv+Gw=L \\ Hu+Bv+Fw=M \\ Gu+Fv+Cw=N; \end{array}$$

and if the Bezoutiant be called $\mathfrak{B}$, we have

$$L=\frac{d\mathfrak{B}}{du} \quad M=\frac{d\mathfrak{B}}{dv} \quad N=\frac{d\mathfrak{B}}{dw}.$$

The simplified residues to f and g are L, (L, M), (L, M, N), where (L, M) means the result of eliminating u between L and M, and (L, M, N) the result of eliminating u and v between L, M, N; and by a theorem (virtually implied in the direct method* of reducing a quadratic function to the form of a sum of squares), if we call the leading coefficients of these quantities C_1, C_2, C_3, we have

$$\frac{L^2}{C}+\frac{(L, M)^2}{C_1.C_2}+\frac{(L, M, N)^2}{C_2.C_3}=\mathfrak{B}.$$

Hence when $n=3$ $\frac{1}{3}.J(f, g)=\mathfrak{B}$ when in $\mathfrak{B}$, u, v, w are turned into $x^2, x, 1$, and so in general for any values of n, the Bezoutiant correspondingly modified, becomes $\frac{1}{n}.J(f, g)$, as was to be shown†.

* Viz. that of M. Cauchy, adverted to in Section IV. art. 44–45.

† Compare Jacobi, "De Eliminatione," § 2. The general expression for the allotrious factor, I may here incidentally mention, is given under the head Theorem a, § 16, which comes quite at the end of the same paper.

[Received Nov. 3, 1853.]

Art. (n.) The expressions obtained for the quotients to $\frac{f'x}{fx}$ may be generalized and extended to the quotients to $\frac{\phi x}{fx}$, where ϕx and fx are two functions of x of any degrees m and n, whose roots are respectively, $k_1\ k_2 \ldots k_m$, and $h_1\ h_2 \ldots h_n$. If we suppose

$$\frac{\phi x}{fx} = \frac{1}{Q(x)-}\ \frac{1}{q_2(x)-}\ \frac{1}{q_3(x)-} \cdots\cdots \frac{1}{q_{m+1}(x)},$$

where $Q(x)$ is of $n-m$ dimensions, and $q_2(x)$, $q_3(x) \ldots q_{m+1}(x)$, each of one dimension in x, it may be proved that on writing

$$\frac{1}{Q(x)-}\ \frac{1}{q_2(x)-} \cdots\cdots \frac{1}{q_i(x)} = \frac{N_i(x)}{D_i(x)},$$

we shall have

$$\Sigma^1_m \theta \left\{ (N_i\, k_\theta)^2 . \frac{fk_\theta}{\phi' k_\theta}(x-k_\theta) \right\} = Cq_{i+1}(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(A.)}$$

$$\Sigma^1_n \theta \left\{ (D_i\, h_\theta)^2 . \frac{\phi h_\theta}{f' h_\theta}(x-h_\theta) \right\} = C' q_{i+1}(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(B.)}$$

where $$C \pm C' = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(E.)}$$

$Cq_{i+1}(x)$ being the $(i+1)$th simplified quotient. When $Q(x)$ is a linear function of x, in finding $q_1 x$ from the formula B, we must take $D_0 x=1$. The proof of this theorem being generally true, may easily be shown to depend upon its being true in the special case*, when $m=\mu+i$, and $n=\mu+i'$ (m being supposed less than n), and $h_1, h_2, \ldots h_n$ become $l_1\ l_2 \ldots l_\mu\ h_1\ h_n \ldots h_{i'}$, and $k_1\ k_2 \ldots k_m$ become $l_1\ l_2 \ldots l_\mu\ k_1\ k_2 \ldots k_i$; and the truth of the theorem for this special case (if for instance we wish to prove the formula (B)) depends upon the expression

$$\left| \begin{matrix} h_1\, h_2 \ldots h_{i'-1} \\ k_1\, k_2 \ldots k_m \end{matrix} \right| \div \left| \begin{matrix} h_1\, h_2 \quad \ldots h_{i'-1} \\ h_{i'}\, h_{i'+1} \ldots h_n \end{matrix} \right|$$
$$\times \left| \begin{matrix} h_1\, h_2 \ldots h_{i'} \\ k_1\, k_2 \ldots k_m \end{matrix} \right| \div \left| \begin{matrix} h_1\, h_2 \quad \ldots h_{i'} \\ h_{i'-1}\, h_{i'+2} \ldots h_n \end{matrix} \right|$$

being identical with the expression

$$\left\{ \left| \begin{matrix} h_1\, h_2 \ldots h_{i'-1} \\ k_1\, k_2 \ldots k_m \end{matrix} \right| \div \left| \begin{matrix} h_1\, h_2 \quad \ldots h_{i'-1} \\ h_{i'}\, h_{i'+1} \ldots h_n \end{matrix} \right| \times (h_{i'}-h_1)(h_{i'}-h_2)\ldots(h_{i'}-h_{i'-1}) \right\}$$

$$\times \frac{\left| \begin{matrix} h_{i'} \\ k_1\, k_2 \ldots k_m \end{matrix} \right|}{\left| \begin{matrix} h_{i'} \\ h_1\, h_2 \ldots h_{i'-1}\, h_{i'+1} \ldots h_n \end{matrix} \right|}$$

* By virtue of the Lemma, that when ϕx and fx are two algebraical functions ($x^n + ax^n$ &c.) : ($x^{n+e} + ax^{n+(e)}$ &c.) no function of the coefficients vanishing identically when i roots of fx coincide with i roots of ϕx respectively can be formed, in which there are fewer of the coefficients of f and ϕ respectively than appear in the leading coefficient of the $(n-i+1)$th residue of $\frac{\phi}{f}$.

as it may readily be shown to be. And the formula (A.) may be verified in precisely the same manner. There is no difficulty in finding the values of C and C', which are products of powers, some positive and some negative, of the leading coefficients in the simplified residues, and recognising that they satisfy the equation (E.); when ϕx is of one degree below fx this equation is of the form $C+C'=0$.

Art. (o.) When $\phi x=f'x$, this expression for the $(i+1)$th simplified quotient becomes $\Sigma(D_i h)^2(x-h)$, as previously found; the correlative expression will be

$$-\Sigma(N_i k)^2\frac{fk}{f''k}(x-k),$$

k being any root of $f'x=0$, which is equal to the former expression. The general expressions above given for the simplified quantities are of course integral functions of h and k, although given under the form of the sums of fractions, by virtue of the well-known theorem that $\Sigma\frac{\vartheta(h)}{f'h}$, where ϑ is an integral function of h, and the summation comprises all the roots (h) of $fh=0$ is always integral.

Art. (p.) It will be found that for all values of i greater than unity

$$\Sigma'_m\theta(N_i.k_\theta)\frac{fk_\theta}{\phi' k_\theta}=0,$$

and that

$$\Sigma'_n\theta.(D_i h_\theta)\frac{\phi h_\theta}{f'h_\theta}=0.$$

The theorem of art $(n.)$ is in effect a theorem of Cumulants of the form

$$[Q_1(x),\ q_2(x),\ \ldots q_i(x)\ldots q_n(x)],$$

where the elements are all independent of one another, and where fx represents

$$[Q_1(x)\ q_2(x)\ q_i(x)\ldots q_n(x)] \text{ and } \phi x \text{ represents } [q_2 x,\ q_3(x),\ \ldots q_n(x)],$$

n being any *number whatever greater* than i; this makes the theorem still more remarkable. The urgency of the press precludes my investigating for the present the more general theorem which must be presumed to exist, whereby q_{i+1} can be connected with $[q_1\, q_2\, q_3 \ldots q_i]$, or $[q_2\, q_3 \ldots q_i]$, and with $[q_1\, q_2\, q_3 \ldots q_{i+e}]$ and $[q_2\, q_3 \ldots q_{i+e}]$, when each (q) represents a function of an *arbitrary* degree in x. The theorem so generalized would comprehend the complete theory of the quotients arising from the process of continued division, without exclusion of the singular cases (at present supposed to be excluded) where one or several consecutive principal coefficients in one or more of the residues, vanish.

Art. $(q.)$ The complete statement of two twin theorems suggested by and intimately connected with the biform representation of the quotients $\frac{\phi x}{fx}$, given in the preceding article, is too remarkable to be omitted.

Suppose $\phi x=f'x$, and let the successive convergents to $\frac{f'x}{fx}$ be called

$$\frac{1}{T_1.x},\ \frac{t_1 x}{T_2 x},\ \ldots\ \frac{t_{n-2}.x}{T_{n-1}.x},\ \frac{t_{n-1}.x}{T_n x}$$

where the subscrolet index to t or T indicates the degree in x. Then if we call the roots of fx h_1 $h_2 \ldots h_n$ the theorem already cited in a preceding article, concerning the denominators of the convergents, may be expressed as follows:—

$$\left\{\begin{matrix} \left(\frac{f'h_1}{\varphi h_1}\right)^2; & \left(\frac{f'h_2}{\varphi h_2}\right)^2; & \ldots; & \left(\frac{f'h_n}{\varphi h_n}\right)^2 \\ (T_1\, h_1)^2; & (T_1\, h_2)^2; & \ldots; & (T_1\, h_n)^2 \\ (T_2\, h_1)^2; & (T_2\, h_2)^2; & \ldots; & (T_2\, h_n)^2 \\ \cdot & \cdot & \cdot & \cdot \\ (T_{n-1}.h_1)^2; & (T_{n-1}.h_2)^2; & \ldots; & (T_{n-1}.h_n)^2 \end{matrix}\right\}=0$$

where it will be observed that the first line of terms consists exclusively of units, since $f'x=\varphi x$ by hypothesis.

Correlatively I have ascertained that preserving the same assumption that $\varphi x=f'x$. so that consequently $\frac{\varphi' k}{fk}$ means $\frac{f'' k}{fk}$, the following theorem obtains, viz. that if k_1 $k_2 \ldots k_{n-1}$ are the $(x-1)$ roots of φx.

$$\left\{\begin{matrix} \left(\frac{\varphi' k_1}{fk_1}\right)^2; & \left(\frac{\varphi' k_2}{fk_2}\right)^2; & \ldots\left(\frac{\varphi' k_{n-1}}{fk_{n-1}}\right)^2 \\ (t_1(k_1))^2; & (t_1(k^2))^2; & \ldots(t_1(k_{n-1}))^2 \\ (t_2(k_1))^2; & (t_2(k_2))^2; & \ldots(t_2(k_{n-1}))^2 \\ \cdot & \cdot & \cdot \\ (t_{n-2}(k_1))^2; & (t_{n-2}(k_2)); & \ldots(t_{n-2}(k_{n-1}))^2 \end{matrix}\right\}=0.$$

It may consequently be conjectured, when φ and f are independent functions of x and respectively of the degree $n-1$ and n, and $\frac{\varphi x}{fx}$ is expanded under the form of a continued fraction, of which, as before, $\frac{1}{T_1}$; $\frac{t_1}{T_2}$; $\ldots$; $\frac{t_{n-1}}{T_n}$ are the successive convergents, that we shall have analogous determinants to the twin forms above given, each separately vanishing, these more general determinants differing only from their model forms in respect of the uppermost line of terms in the one of them, being each multiplied by certain functions of h_1, $h_2, \ldots h_n$ respectively (all of which become units when $\varphi x=f'x$), and in the other of them by certain functions of k_1, $k_2, \ldots k_n$.

The exact form, however, of such functions, and even the possibility of such form being found capable of making the determinants vanish, remains open for further inquiry.

Section IV.

On some further Formulæ connected with M. Sturm's *theorem, and on the Theory of Intercalations whereof that theorem may be treated as a corollary.*

As preparatory to some remarks about to be made on the formulæ connected with M. Sturm's theorem, it is necessary to premise two theorems concerning quadratic functions of great importance, one which, notwithstanding its extreme simplicity, is as far as I know very little (if at all) known, and the other was given in part many years ago by M. Cauchy, but is also not generally known. The former of these two theorems is as follows. If a quadratic homogeneous function of any number of variables be (as it may be in an infinite variety of ways) transformed into a function of a new set of variables, linearly connected by real coefficients with the original set, in such a way that only positive and negative squares of the new variables appear in the transformed expression, the number of such positive and negative squares respectively will be constant for a given function whatever be the linear transformations employed. This evidently amounts to the proposition, that if we have $2n$ positive and negative squares of homogeneous real linear functions of n variables identically equal to zero, the number of positive squares and of negative squares must be equal to one another, so that *ex. gr.* we cannot have

$$\pm\{u_1^2+u_2^2+ \&c. \ldots +u_n^2+u_{n+1}^2-u_{n+2}^2-u_{n+3}^2- \&c. -u_{2n}^2\}$$

identically zero when n of the variables are linear functions of the remaining n; and this is obviously the case, for if the equation could be identically satisfied we might make

$$u_{n+2}=u_1 \quad u_{n+3}=u_2 \ldots\ldots u_{2n}=u_{n-1},$$

and we should then be able to find u_{n+1} as a real numerical multiple of u_n, and consequently should have the equation $u_n^2.\{1+k^2\}=0$, which is obviously impossible; *à fortiori* we may prove that in the identical equation existing between the sum of an even number of positive and of negative squares of real linear functions of half the number of independent variables, there cannot be *more* than a difference of two (as we have proved that there cannot be that difference) between the number of positive and negative squares. Hence there must be as many of one as of the other; and as a consequence, the number of positive squares or of negative squares in the transform of a given quadratic function of any number of variables effected by any set of real linear substitutions is constant, being in fact some unknown transcendental function of the coefficients of the given function. I quote this law (which I have enunciated before, but of which I for the first time publish the proof) under the name of the law of inertia for quadratic forms.

Art. (45.). The other theorem is the following. If any quadratic function be represented in the umbral notation* under the form of $(a_1x_1+a_2x_2+ \ldots +a_nx_n)^2$, where a_1, $a_2 \ldots a_n$ are the umbræ of the coefficients, and $x_1, x_2 \ldots x_n$ the variables, then by writing

$$\begin{vmatrix}a_1\\a_1\end{vmatrix}x_1+\begin{vmatrix}a_1\\a_2\end{vmatrix}x_2+\begin{vmatrix}a_1\\a_3\end{vmatrix}x_3+\begin{vmatrix}a_1\\a_4\end{vmatrix}x_4+\ldots+\begin{vmatrix}a_1\\a_n\end{vmatrix}x_n=y_1$$

$$\begin{vmatrix}a_1\,a_2\\a_1\,a_2\end{vmatrix}x_2+\begin{vmatrix}a_1\,a_2\\a_1\,a_3\end{vmatrix}x+\begin{vmatrix}a_1\,a_2\\a_1\,a_4\end{vmatrix}x_4+\&c.+\begin{vmatrix}a_1\,a_2\\a_1\,a_n\end{vmatrix}x_n=y_2$$

$$\begin{vmatrix}a_1\,a_2\,a_3\\a_1\,a_2\,a_3\end{vmatrix}x_3-\begin{vmatrix}a_1\,a_2\,a_3\\a_1\,a_2\,a_4\end{vmatrix}x_4+\&c.+\begin{vmatrix}a_1\,a_2\,a_3\\a_1\,a_2\,a_n\end{vmatrix}x_n=y_3$$

&c. &c. &c.

$$\begin{vmatrix}a_1\,a_2\ldots a_n\\a_1\,a_2\ldots a_n\end{vmatrix}x_n=y_n,$$

* For an explanation of the umbral notation, see London and Edinburgh Philosophical Magazine, April 1851, or thereabouts.

$(a_1x_1+a_2x_2+\ldots+a_nx_n)^2$ will assume the form

$$\frac{|a_1|}{|a_1|}y_1^2+\frac{\frac{|a_1\,a_2|}{|a_1\,a_2|}}{\frac{|a_1|}{|a_1|}}y_2^2+\frac{\frac{|a_1\,a_2\,a_3|}{|a_1\,a_2\,a_3|}}{\frac{|a_1\,a_2|}{|a_1\,a_2|}}y_3^2+\ldots+\frac{\frac{|a_1\,a_2\ldots a_{n-1}\,a_n|}{|a_1\,a_2\ldots a_{n-1}\,a_n|}}{\frac{|a_1\,a_2\ldots a_{n-1}|}{|a_1\,a_2\ldots a_{n-1}|}}y_n^2$$

and consequently the number of positive squares in the reduced form of the given function will always be the number of continuations or permanencies of sign of the series

$$1\,;\ \frac{a_1}{a_1}\,;\ \left|\begin{matrix}a_1\,a_2\\a_1\,a_2\end{matrix}\right|;\ \left|\begin{matrix}a_1\,a_2\,a_3\\a_1\,a_2\,a_3\end{matrix}\right|;\ \ldots\left|\begin{matrix}a_1\,a_2\ldots a_n\\a_1\,a_2\ldots a_n\end{matrix}\right|$$

the several terms of this progression being in fact the determinants of what the given function becomes when we obliterate successively all the variables but one, then all but that another, then all but these two and a third, until finally, the last term is the determinant of the given function with all the variables retained. This comes to saying that if we call the function (suppose of four variables) f, and we write

$$\begin{matrix}\frac{d^2f}{dx_1^2} & \frac{d^2f}{dx_1\,dx_2} & \frac{d^2f}{dx_1\,dx_3} & \frac{d^2f}{dx_1\,dx_4}\\[2mm] \frac{d^2f}{dx_2\,dx_1} & \frac{d^2f}{dx_2^2} & \frac{d^2f}{dx_2\,dx_3} & \frac{d^2f}{dx_2\,dx_4}\\[2mm] \frac{d^2f}{dx_3\,dx_1} & \frac{d^2f}{dx_3\,dx_2} & \frac{d^2f}{dx_3^2} & \frac{d^2f}{dx_3\,dx_4}\\[2mm] \frac{d^2f}{d_4\,dx_1} & \frac{d^2f}{dx_4\,dx_2} & \frac{d^2f}{dx_4\,dx_3} & \frac{d^2f}{dx_4^2},\end{matrix}$$

(where all the terms are of course coefficients of the given function expressed as above for greater symmetry of notation), the inertia of f will be measured by the number of continuations of sign in the series formed of the *successive principal minor coaxal* determinants $\left(\text{in writing which I shall use in general } (r, s) \text{ to denote } \frac{d^2f}{dx_r\,dx_s}\right)$,

$$1,\ (1,1),\ \begin{bmatrix}(1,1) & (1,2)\\(2,1) & (2,2)\end{bmatrix},\ \begin{bmatrix}1,1 & 1,2 & 1,3\\2,1 & 2,2 & 2,3\\3,1 & 3,2 & 3,3\end{bmatrix},\ \begin{bmatrix}(1,1) & (1,2) & (1,3) & (1,4)\\(2,1) & (2,2) & (2,3) & (2,4)\\(3,1) & (3,2) & (3,3) & (3,4)\\(4,1) & (4,2) & (4,3) & (4,4)\end{bmatrix}$$

and in like manner in general*.

* I have given a direct *à posteriori* demonstration in the London and Edinburgh Philosophical Magazine, that the number of continuations of sign in any series formed like the above form a symmetrical matrix, is unaffected by any permutations of the lines and columns thereof, which leaves the symmetry subsisting, that is to say (using the umbral notation), if $\theta_1, \theta_2, \theta_3 \ldots \theta_i$ are disjunctively equal, each to each, in any arbitrary order to 1, 2, 3....i, the number of continuations of sign in the series

$$1,\left|\begin{matrix}a_{\theta_1}\\a_{\theta_1}\end{matrix}\right|,\ \left|\begin{matrix}a_{\theta_1}\,a_{\theta_2}\\a_{\theta_1}\,a_{\theta_2}\end{matrix}\right|,\ \left|\begin{matrix}a_{\theta_1}\,a_{\theta_2}\,a_{\theta_3}\\a_{\theta_1}\,a_{\theta_2}\,a_{\theta_3}\end{matrix}\right|\ldots,\ \left|\begin{matrix}a_{\theta_1}\,a_{\theta_2}\,a_{\theta_3}\ldots a_{\theta_i}\\a_{\theta_1}\,a_{\theta_2}\,a_{\theta_3}\ldots a_{\theta_i}\end{matrix}\right|$$

is irrespective of the order of the natural numbers 1, 2, 3....i in the arrangement $\theta_1, \theta_2, \theta_3 \ldots \theta_i$.

Art. (46.). Reverting now to the simplified Sturmian residues, since by the theory set out in the first Section these differ from the unsimplified complete residues required by the Sturmian method only in the circumstance of their being divested of factors, which are necessarily perfect squares and therefore essentially positive, these simplified Sturmians may of course be substituted for the complete Sturmians for the purposes of M. STURM's theorem. The leading coefficients in these simplified Sturmians, reckoning $f'(x)$ as one of them, will be

$$m\Sigma\zeta(h_1 h_2),\ \Sigma\zeta(h_1 h_2 h_3)\ldots\zeta(h_1 h_2\ldots h_m),$$

which it is easily seen, as remarked long ago by Mr. CAYLEY, are the successive principal minor coaxal determinants of the matrix

$$\begin{matrix} \sigma_0, \sigma_1, \sigma_2, \sigma_3\ldots\sigma_{m-1} \\ \sigma_1, \sigma_2, \sigma_3\ldots\ldots\sigma_m \\ \sigma_2, \sigma_3\ldots\ldots\ldots\sigma_{m-1} \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \sigma_{m-1}\, \sigma_m\ldots\ldots\sigma_{2m-2}, \end{matrix}$$

where in general $\sigma_r = h_1^r + h_2^r + \ldots + h_m^r$, and of course $\sigma_0 = m$. M. HERMITE has improved upon this remark by observing, which is immediately obvious, that if we use σ_r to denote, not the quantity above written, but $\frac{h_1^r}{x-h_1}+\frac{h_2^r}{x-h_2}+\ldots\frac{h_m^r}{x-h_m}$, the successive coaxal determinants of the above matrix will become respectively

$$\Sigma\frac{1}{x-h_1};\ \Sigma\left\{\frac{\zeta(h_1 h_2)}{(x-h_1)(x-h_2)}\right\};\ \Sigma\frac{\zeta(h_1 h_2 h_3)}{(x-h_1)(x-h_2)(x-h_3)};\ \ldots\frac{\zeta(h_1 h_2\ldots h_m)}{(x-h_1)(x-h_2)\ldots(x-h_m)};$$

that is to say, these successive coaxal determinants, when multiplied up by fx, will become respectively

$$\Sigma(x-h_2)(x-h_3)\ldots(x-h_m);\ \Sigma\zeta(h_1 h_2)\{(x-h_3)(x-h_4)\ldots(x-h_m)\};\ \ldots\ldots;\ \Sigma\zeta(h_1 h_2\ldots h_m),$$

that is to say, will represent the simplified Sturmian series given by my general formulæ. M. HERMITE further remarks, that the matrix formed after this rule will evidently be that which represents the determinant of the quadratic function (which may be treated as a generating function)

$$\Sigma\frac{1}{x-h_1}.\{u_1+h_1.u_2+h_1^2.u_3+\ldots+h_1^{m-1}.u_n\}^2,$$

in which, since only the squared differences of the terms in the (h) series finally remain in the successive coaxal determinants, we may write $(x-h_1)$, $(x-h_2)\ldots(x-h_m)$ simultaneously in place of $h_1 h_2\ldots h_m$ without affecting the result, consequently the generating function above may be replaced by the generating function

$$\Sigma\frac{1}{x-h_1}.\{u_1+(x-h_1)u_2+(x-h_1)^2.u_3+\ldots+(x-h_1)^{m-1}.u_m\}^2;$$

the corresponding matrix to which becomes

$$\begin{matrix} \Sigma\frac{1}{x-h_1}, & \theta_0, & \theta_1, \ldots\theta_{m-2} \\ \theta_0, & \theta_1, & \theta_2, \ldots\theta_{m-1} \\ \theta_1, & \theta_2, & \ldots\theta_m \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \theta_{m-2}, & \theta_{m-1}, & \ldots\theta_{2m-3}, \end{matrix}$$

where θ_i denotes $\Sigma(x-a)^i$, and $\Sigma\frac{1}{x-h_1}=\frac{f'x}{fx}$. Hence every simplified residue is of the form

$$f'x\times\left\{\begin{matrix} \theta_1 & \theta_2 & \ldots\theta_r \\ \theta_2 & \theta_3 & \ldots\theta_{r+1} \\ \cdot & \cdot & \cdot \\ \theta_r & \theta_{r+1} & \ldots\theta_{2r-1} \end{matrix}\right\}+fx\times\left\{\begin{matrix} 0 & \theta_0 & \theta_1\ldots\theta_r \\ \theta_0 & \theta_1 & \ldots\theta_{r+1} \\ \cdot & \cdot & \cdot \\ \theta_r & \theta_{r+1} & \ldots\theta_{2r-1} \end{matrix}\right\}.$$

The residue in question will be of the degree $m-r-2$ in x, and consequently we have, according to the notation antecedently used for the syzygetic equations

$$t_{r+1}=\left\{\begin{matrix} \theta_1 & \theta_2 & \ldots\theta_r \\ \theta_2 & \theta_3 & \ldots\theta_{r+1} \\ \cdot & \cdot & \cdot \\ \theta_r & \theta_{r+1} & \ldots\theta_{2r-1} \end{matrix}\right\}$$

$$-\tau_r=\left\{\begin{matrix} 0 & \theta_0 & \theta_1\ldots\theta_r \\ \theta_0 & \theta_1 & \ldots\theta_{r+1} \\ \theta_1 & & \ldots\theta_{r+2} \\ \cdot & \cdot & \cdot \\ \theta_r & \theta_{r+1} & \ldots\theta_{2r-1} \end{matrix}\right\}.$$

Elegant and valuable for certain purposes as are these formulæ for t_{r+1} and τ_r, they are affected with the disadvantage of being expressed by means of formulæ of a much higher degree in the variable x than really appertains to them, the paradox (if it may be termed such) being explained by the circumstance of the coefficients of all the powers of x above the right degree being made up of terms which mutually destroy one another. Upon the face of the formulæ, t_{r+1} and τ_r which are in fact only of the degrees $r+1$, and r respectively in x would appear to be of the degree $1+3+5+\ldots+(2r-1)$, *i. e.* of the degree r^2.

Art. (47.). I may add the important remark, which does not appear to have occurred immediately to my friend M. Hermite when he communicated to me the above most interesting results, that in fact, by virtue of the law of inertia for quadratic forms, we may dispense with any identification of the successive coaxal determinants of the matrix to the generating function

$$\Sigma\frac{1}{\varrho-h_1}\{u_1+h_1u_2+h_1^2.u_3+\ldots+h_1^{m-1}.u_m\}^2$$

with my formulæ for the Sturmian functions, and prove *ab initio* in the most simple manner, that the successive ascending coaxal determinants (always of course supposed to be taken about the axis of symmetry) of the matrix to the form above written, or to the more general form (which I shall quote as G, viz.)

$$\Sigma(\varrho-h_1)^q\{\varphi_1.(h_1)u_1+\varphi_2(h_1)u_2+\ldots+\varphi_m.(h_m).u_m\}^2 \quad . \quad . \quad . \quad . \quad . \quad (G.)$$

(where $\varphi_1, \varphi_2\ldots\varphi_m$ are absolutely arbitrary integral forms of function with real coefficients), will form a rhizoristic series in regard to fx (*i. e.* a series, the difference between the number of the continuations of sign between the successive terms of which corresponding to two different values of ϱ will determine the number of real roots of ϱ lying between such two assumed values), provided only that q be an odd positive or negative integer. Nothing can be easier than the demonstration, for whenever ϱ is greater than any one of the real roots as (h_1)

1st. Any pair of imaginary roots will give rise to two terms of the form

$$(l+m\sqrt{-1})^q.(v+w\sqrt{-1})^2 \text{ and } (l-m\sqrt{-1})^q.(v-w\sqrt{-1})^2;$$

or more simply,

$$(L+M\sqrt{-1}).(v^2+w^2+2vw\sqrt{-1})$$

and $$(L-M\sqrt{-1}).(v^2-w^2-2vw\sqrt{-1}),$$

where v and w are real linear functions of $u_1, u_2, \ldots u_n$. The sum of which couple will be

$$2\{L.(w^2-v^2)-2Mwv\}=\frac{2}{L}.\{(Lw-Mv)^2-(L^2+M^2)v^2\}=p^2-q^2;$$

so that each such couple combined will for every value of x give rise to one positive and one negative square.

2ndly. Any real root of the series $h_1, h_2, \ldots h_m$, when ϱ is taken greater than such root, will give rise to a positive square of a real linear function of $u_1, u_2, \ldots u_n$.

3rdly. Any real root of the same series, when ϱ is beneath it in value (q being odd), will give rise to the negative of the square of a real linear function of the same. Hence the number of real roots between ϱ taken equal to one value (a), and ϱ taken equal to any other value (b), will be denoted by the loss of an equal number of positive squares in the reduced form of the expression (G.) when ϱ is taken (a) and when ϱ is taken (b); *i. e.* by virtue of art. (45.) will be denoted by the difference of the number of permanencies of sign in the successive minor determinants of the matrix corresponding to the quadratic form (G.)* (which we have taken as our generating function) resulting

* The *inertia* of the quadratic form G is the measure of the number of real roots of fx comprised between ∞ and ρ, and may be estimated in any manner that may be found most convenient. If ρ be made infinity, and $\varphi_i h$ be taken equal to h^{i-1}, and the inertia of the corresponding value of G be estimated by means of the formulæ in ordinary use by geometers for determining the nature of a surface of the second degree, the criteria of the number of real roots in fx will be, or may be made to be, symmetrical in respect to the two ends of the expression fx. This system of criteria, however, is not so good as that given by the Bezoutiant to the two differential coefficients of $f(x, 1)$ taken with regard to x and 1 respectively, which will also possess the like character of symmetrical indifference, and be one less in number than the former.

from the substitution respectively of a and b in place of ϱ, which gives a theorem equivalent to that of M. STURM, transformed by my formulæ, when we choose to adopt the particular suppositions

$$q=-1 \quad \varphi_1 h=1 \quad \varphi_2 h=h \quad \varphi_3.h=h^2 \ldots \varphi_m h=h^{m-1}.$$

This method of *constructing* a rhizoristic series to fx by a direct process is deserving of particular attention, because it does not involve the use of the notion of continuous variation, upon which all preceding proofs of STURM'S theorem proceed. It completes the cycle of the Sturmian ideas. Happily this cycle was commenced from the other end, for it would have been difficult to have suspected that the root-expressions for the terms in the rhizoristic series could be identified with the residues, had the former been the first to be discovered, and much of the theory of algebraical common measure laid open by means of this identification would probably have remained unknown.

Art. (48.). I proceed now to consider a theorem concerning the relative positions of the real roots of two independent algebraical functions as indicated by the succession of signs presented by their Bezoutian secondaries; this more general theory of intercalations or relative interpositions will be seen to include within it as a corollary the justly celebrated theorem of M. STURM.

Let the real roots of fx taken in descending order of magnitudes be $h_1\, h_2 \ldots h_p$, and the real roots of φx taken in the like order $\eta_1\, \eta_2 \ldots \eta_q$, so that

$$fx=(x-h_1)(x-h_2)\ldots(x-h_p)\mathrm{H}$$
$$\varphi x=(x-\eta_1)(x-\eta_2)\ldots(x-\eta_q)\mathrm{K},$$

H and K being functions of x incapable of changing their signs. Now, as in M. STURM'S method, let us inquire what takes place in respect to the sign of $\frac{\varphi(x)}{f(x)}$, which I shall call the Indicatrix, as x descends the scale of real magnitude from $+\infty$ to $-\infty$. If between $+\infty$ and h_1, i real roots of φx are contained, it is obvious that as x travels from $+\infty$ to the superior brink of h_1, the Indicatrix will change its sign from + to − and from − to + altogether i times, so that at the moment when x is about to pass through h_1, it will be positive if i is zero or even, and negative if i is odd; but the moment after x has passed through the value h_1, the indicatrix will be negative on the first supposition, and positive on the other supposition. Hence immediately after the passage of x through h_1 the indicatrix will have been once oftener negative than positive on the one supposition, and as often negative as positive on the other. Again, in like manner as x traverses the interval between h_1 and the inferior brink of h_2, if no η or an even number of η's occupy this interval, the sign which the Indicatrix had at the beginning of this interval will have been reversed once oftener than restored; but if there be an odd number of k's so interposed, the number of reversals and restorations will have been identical; and so for each successive interval, reckoned from a value for x immediately subsequent to one real root of fx, down to a value immediately subsequent to the next less real root of the same; and it is evident that the effect upon the sign of the Indicatrix at the end of every such interval depends, not upon the number of η's grouped together in such interval, but upon the form of the group as regards its being made up of an odd or even number of terms [the first interval will of course be understood to extend from $+\infty$ to a value immediately inferior to h_1, and the last from a value immediately inferior to h_p to $-\infty$]. Hence as regards the relation of the signs of the Indicatrix at the beginning to the sign at the end of every such interval, nothing will be altered by taking away any even number of η's that may be found therein. If we suppose this to be done, we shall then have in some of the intervals one η occurring and in the other intervals no η; that is to say, some of the h's will be separated by single η's, but other h's will come together. Again, by removing any even number of h's not separated by η's (and thus removing an even number of intervals), it is clear that as many changes of sign of the Indicatrix will have been done away with from $+$ to $-$ as from $-$ to $+$, and no effect upon the excess of the one kind of changes of sign over the other kind of changes of sign will have been produced. By removing pairs of h's in this manner, it may happen that η's will again be brought together, any even number of which, not separated by h's, may again be removed and then pairs of h's not separated by η's in their turn, and so continually *toties quoties* until at length we must arrive at a reduced system of h's and η's, where no two h's and no two η's come together, or else all the h's and all the η's will have disappeared. Let the scale of h's and η's thus simplified and reduced be called the effective scale of intercalations. The number of h's and the number of η's in any such scale will be equal, or will at most differ from one another by a unit, since at each part of the scale, except at the end, every h is followed by an η and every η by an h. If the scale begins and ends with an h, there will of course be one more h than η; if it begin and end with an η, there will be one more η than h; if it begin with an h or an η and end with an η or h, there will be as many of the one as of the other.

1st. Suppose the effective intercalation scale to commence with an h; then in passing from $+\infty$ to just beyond the first h the sign of the indicatrix $\frac{\varphi x}{fx}$ changes from $+$ to $-$; it changes again from $-$ to $+$ as it passes the first η, then again from $+$ to $-$ as it passes the second h, and so on; that is to say, there will be a change always in the same direction from $+$ to $-$ as x passes, from being just greater than to being just less than any h appearing in the effective scale. 2nd. If the effective scale begin with η, the indicatrix will conversely be negative after passing the first and every subsequent η, and change from $-$ to $+$ in the act of passing through the first and every subsequent h. So that on either supposition the changes of sign for the effective scale always take place in the same direction, and the number of h's in the effective scale will be measured by the number of such changes, and consequently will be measured by the difference between the number of times that the indicatrix $\frac{\varphi x}{fx}$ changes its sign from $+$ to $-$ as x passes through each in turn of the real roots of fx, and the number of times that in passing through any such root it changes its sign from $-$ to $+$; if the former number be greater than the latter, the effective scale of interpositions will begin with a root of fx; if it be less, the scale will begin with a root of φx. If instead of beginning with $+\infty$ and ending with $-\infty$ we begin and end with any two limits, a and b respectively (making abstraction of all roots of fx or of φx lying outside these limits, and forming the effective intercalation scale with the roots comprised within these limits exclusively), we shall obviously obtain a similar result, but with the condition that the changes from $+$ to $-$ will be *in excess* if an even number of h's and η's combined be cut off by the superior limit, and the effective scale begin with an h, or if an odd number of h's and η's combined be so cut off and the scale begin with an η; and in *defect* if an odd number of h's and η's combined be so cut off and the scale begin with an h, or an even number be so cut off and the scale begin with an η. If, now, supposing fx to be of n, and φx of not more than n, say (m) dimensions, we form the *signaletic* series fx, φx, B_1, B_2, ...B_m (where the B_1, B_2, ...B_m are the Bezoutian secondaries or simplified successive residues corresponding to $\frac{\varphi x}{fx}$ expanded under the form of an improper continued fraction), it may be shown, in the same way as for STURM's theorem, that whenever $\frac{\varphi x}{fx}$ changes from $+$ to $-$ a change of sign will be gained in the series, and when from $-$ to $+$ a change will be lost; and that no change can be gained or lost except as x passes through the successive real roots of fx. Hence the difference between the number of changes of sign in the above signaletic series when x is taken (a), and the number of the same when x is taken (b), will indicate the number of roots of fx remaining in the effective scale of interpositions formed between such of the roots of fx and of φx as lie between (a) and (b); calling the one number $I(a)$ and the other $I(b)$, the sign of $I(b)-I(a)$ depends not on the relative magnitudes of (a) and (b), but upon the manner in which the effective scale commences; if $I(a)-I(b)$ is positive, the effective scale formed between the (a) and (b) will commence with a root of fx; if negative, it will commence with a root of $\varphi(x)$.

Art. (49.). In forming the scale of effective interpositions, it is evidently not necessary to go on reducing the (h) series and the η series separately and alternately; the same result will be effected more expeditiously by eliding simultaneously any even number of h's that come together without being separated by an η, and any even number of η's that come together without being separated by an (h), and, repeating this process of simultaneous elision, as often as may be required, until no two h's or η's come together. Thus, for instance, denoting the magnitudes of the series of real roots of f and of φ by the distances of h and η points taken along a right line from a fixed point therein, and supposing such series of roots between the limits a and b to be

$$h\,h\,h\,\eta\,\eta\,h\,\eta\,\eta\,h\,\eta\,\eta\,h\,h\,\eta\,h\,\eta\,h\,h\,h\,h\,h\,\eta\,\eta\,h,$$

our first reduction brings this scale to the form

$$h\,\eta\,h\,h\,\eta\,\eta\,h\,\eta\,h\,h;$$

the next reduction brings it to the form

$$h\,\eta\,\eta\,h\,\eta;$$

and a third and final reduction brings it to the form

$$h\,\eta\,h\,\eta;$$

and accordingly we shall find for such an arrangement of the h and η system

$$\mathrm{I}(b)-\mathrm{I}(a)=\pm 2.$$

Art. (50.). If we suppose $\phi x=\frac{dfx}{dx}$, by a well-known theorem of algebra, any two consecutive roots of fx will contain between them an odd number of roots of ϕx, and the number of real roots of $f'x$ greater than the greatest root of fx, and the number of real roots of $f'x$ less than the least root of fx will each be even. Hence the effective intercalation scale between any two limits (a) and (b) will be formed by merely reducing the η groups to single units, and the number of h's in the scale so formed will be the total number of h's between the limits (a) and (b). Moreover, since such scale commences always with a root of fx, or with an even number of roots of $f'x$ followed by a root of fx, if the number of h's and η's cut off be even, and with a root of $f'x$ or an even number of roots of fx followed by a root of fx, if the number so cut off be odd, it follows that for this case $\mathrm{I}(a)-\mathrm{I}(b)$, (a) being the superior limit, will be always positive, and will measure the total number of real roots of $f(x)$ lying between (a) and (b); this, then, is STURM's theorem, treated as a corollary to the Theory of Intercalations.

Art. (51.). If we write down the last syzygetic equation between fx of m and $\phi(x)$ of n dimensions, viz.

$$\tau_{n-1}.(x)f(x)-t_{m-1}(x)\phi x+\vartheta_0=0,$$

it has been shown that the succession of signs in the series formed with fx, ϕx and their successive Bezoutian secondaries will contain the same number of continuations and variations as the series formed with $f(x)$, $t_{m-1}(x)$, and their successive Bezoutian secondaries. This indicates that the effective scale of interpositions for fx and ϕx will contain an equal number of roots of fx with the effective scale for fx and $t_{m-1}(x)$; the two scales however will not necessarily be identical, because the roots of ϕx will not necessarily be in the same order relative to the h's in the one scale as those of $t_{m-1}.x$ relative to the h's in the other scale. This equality is perfectly well explained *à posteriori* by the form of $t_{m-i}.x$, which by the formula in Section II. will be represented by

$$\Sigma(x-h_{q_1})(x-h_{q_2})\ldots(x-h_{q_{m-1}})\cdot\frac{\phi h_{q_1}.\phi h_{q_2}\ldots\phi h_{q_{m-1}}}{(h_{q_m}-h_{q_1})(h_{q_m}-h_{q_2})\ldots h_{q_m}-h_{q_{m-1}}}.$$

Now, whenever x is indefinitely near to any one of the roots of fx, as h_{q_m}, this sum reduces to the simple expression

$$\phi h_{q_1}\phi h_{q_2}\ldots\phi h_{q_{m-1}}=\{\phi h_1.\phi h_2\ldots\phi h_m\}\frac{1}{\phi h_{q_m}},$$

and consequently in the immediate neighbourhood of every real root of fx, $\phi(x)$ and $t_{m-1}.x$ will have always the same or always a contrary sign, according as $\phi h_{q_1}.\phi h_{q_2}\ldots\phi h_{q_m}$ is positive or negative, which will depend upon the relative disposition of the real roots in f and ϕ; in either case the effective scale of interpositions for fx with ϕx and for fx with $t_{m-1}.x$ must contain the same number of h's; but the difference will be, that if $\phi h_1.\phi h_2...\phi h_m$ is positive an h will occupy the first place in each scale, or the second place in each scale; but if negative, then in one scale an (h) will occupy the first place, and in the other scale the second place.

Art. (52.). The same process of common measure or residues which serves to furnish a rhizoristic series for $f(x)$ or a synrhizoristic series for fx and ϕx, will serve also to furnish superior and inferior limits to the real roots of any proposed equation. Thus suppose fx to be any rational integral function of (x) of the degree (n) and $\phi(x)$ any other function of x, which I shall begin with supposing to be of the degree $(n-1)$, and let the successive quotients resulting from the process of finding the greatest common measure of fx, ϕx continued until the last remainder is not a constant but zero, be supposed to be (as they may generally be taken, but subject to cases of exception, which will hereafter be alluded to) n linear functions $q_1\,q_2...q_n$, then we shall have

$$\frac{\phi x}{fx}=\frac{1}{q_1+}\;\frac{1}{q_2+}\cdots\frac{1}{q_{n-1}+}\;\frac{1}{q_n},$$

and therefore
$$\phi x=\text{K.N}$$
$$fx=\text{K.D},$$

where N is the numerator and D the denominator of the fraction

$$\frac{1}{q_1+}\;\frac{1}{q_2+}\cdots\frac{1}{q_n},$$

and K is a constant (the value of which is immaterial to be considered, but in fact equals

$$\pm\frac{L_0}{L_1^2}.\frac{L_2^2}{L_3^2}.\frac{L_4^2}{L_5^2}\ \&\text{c.},$$

L_0, L_1, L_2, L_3, &c. being the leading coefficients of the last, the last but one, the last but two, &c. of the Bezoutian secondaries to fx and ϕx). Accordingly,

if $n=1$, let $D=q_1=\mu_1$;

if $n=2$, let $D=q_2\,q_1+1=\mu_1\left\{q_2+\frac{1}{\mu_1}\right\}=\mu_1.\mu_2$;

if $n=3$, let $D=q_3\{q_2\,q_1+1\}+q_1=\mu_1\,\mu_2\left\{q_3+\frac{1}{\mu_2}\right\}=\mu_3$;

. . . .

and in general let $D=\mu_1.\mu_2.\mu_3...\mu_n$,

where $\mu_1=q_1\quad \mu_2=q_2+\frac{1}{\mu_1}\quad \mu_3=q_3+\frac{1}{\mu_2}\cdots\cdots\cdots\mu_n=q_n+\frac{1}{\mu_{n-1}}$.

Now suppose x to be so taken that

$$\left.\begin{array}{ll} q_1 & \text{does not lie between } +1 \text{ and } -1 \\ q_2 & \ldots\ldots\ldots +2 \text{ and } -2 \\ q_3 & \ldots\ldots\ldots +2 \text{ and } -2 \\ q_4 & \ldots\ldots\ldots +2 \text{ and } -2 \\ \ldots & \ldots\ldots\ldots \\ q_{n-1} & \ldots\ldots\ldots\ 2 \text{ and } -2 \\ q_n & \ldots\ldots\ldots\ 1 \text{ and } -1 \end{array}\right\}, \quad \ldots\ldots \quad (\omega.)$$

where it will be observed that the excluded region lies between $+2$ and -2 for all the intermediate quotients, but between only $+1$ and -1 for the first and last quotient. Then μ_1 is positively or negatively greater than 1, therefore $\frac{1}{\mu_1}$ is a positive or negative fraction, but q_2 is positively or negatively greater than 2; therefore μ_2 will be of the same sign in q_2, and also μ_2 will be positively or negatively greater than 1; therefore $\frac{1}{\mu_2}$ will be a positive or negative fraction, but q_3 is positively or negatively greater than 2; therefore μ_3 will be of the same sign as q_3, and also μ_3 will be positively or negatively greater than 1; and proceeding in this way, we find that all values of μ_i, from $i=1$ to $i=n-1$, will be of the same sign as q_i, and positively or negatively greater than 1. Finally, $\frac{1}{\mu_{n-1}}$ will be a fraction, and therefore, since q_n is positively or negatively greater than 1, $\mu_n=q_n+\frac{1}{\mu_{n-1}}$ will have the same sign as (q_n) (but of course is not necessarily greater than 1, nor would that condition serve any purpose were it satisfied). We infer consequently, that when the conditions (ω) are satisfied, $\mu_1, \mu_2, \mu_3 \ldots \mu_n$ will respectively have the same signs as $q_1\ q_2 \ldots q_n$; and therefore $D=\mu_1.\mu_2.\mu_3\ldots\mu_n$ has the same sign as $q_1.q_2.q_3\ldots q_n$. Now suppose

$$q_1=a_1x+b_1 \quad q_2=a_2.x+b_2\ldots q_n=a_n.x+b_n,$$

and solve the $2n$ equations

$$\begin{array}{llll} a_1x+b_1=+c_1 & a_2x+b_2=+c_2\ldots a_{n-1}.x+b_{n-1}= & c_{n-1} & a_n.x+b_n= \quad c_n \\ a_1x+b_1=-c_1 & a_2x+b_2=-c_2\ldots a_{n-1}.x+b_{n-1}= & -c_{n-1} & a_n.x+b_n=-c_n, \end{array}$$

where $\quad c_1=1 \quad c_2=2 \quad c_3=2 \ldots\ldots\ldots c_{n-1}=2 \quad c_n=1.$

Whenever in any one of the n pairs of equations above written the coefficient of x is positive, the upper equation of the pair will bring out the greater value of x; but when the coefficient is negative the lower equation will give the greater value. Take the pair

$$a_ix+b_i=c_i$$
$$a_ix+b_i=-c_i.$$

If a_i is positive a_ix+b_i will always be positive, and greater than c_i between $x=\infty$ and $x=$ the greater of the two values of x; if a_i is negative a_ix+b_i will always be negative, and less (*i. e.* nearer to $-\infty$) than $-c_i$ for all values of x between the same limits as before. So again it will be seen in like manner, that whether a_i be positive or negative between $x=-\infty$ and $x=$ the lesser of the two values of x corresponding to the above pair of equations, a_ix+b_i will always retain the same sign, and will be greater than $+c_i$, or less than $-c_i$, according as a_i is negative or positive. If, then, we take the greatest of the *greaters* of the n pairs of values of x, *i. e.* the absolute greatest of the $2n$ values, and the least of the *lessers*, *i. e.* the absolute least of the same, say L and Λ between L and Λ, $q_1, q_2, \ldots q_n$ will each always retain an invariable sign, and will then fall without the limits $\pm c_1\ \pm c_2 \ldots\ldots\ldots \pm c_{n-1}\ \pm c_n$, so that between $+\infty$ and L and between Λ and $-\infty$, $\mu_1.\mu_2\ldots\mu_n$, *i. e.* a constant multiple of $f(x)$, will retain the same sign as $q_1 . q_2 ... q_n$, *i. e.* will never change its sign from the beginning to the end of one interval, nor from the beginning to the end of the other; and consequently L and Λ will be a superior and inferior limit respectively to the real roots of fx. It will of course be observed that it is indifferent for the purposes of the foregoing theorem, whether $\frac{\phi x}{fx}$ be expanded under the form of a proper or an improper fraction, *i. e.* whether we employ the ordinary or the Sturmian process of successive division, for changing the signs of the residues will only have the effect of changing q_i into $(\pm) q_i$, and the pair of equations $(\pm) q_i = \pm c_i$ remains the same whether the + or the − sign be prefixed to q_i. The result is, that if we form the $2n$ quantities

$$\frac{\pm 1 - b_1}{a_1}; \quad \frac{\pm 2 - b_2}{a_2}; \quad \frac{\pm 3 - b_3}{a_3}; \ldots \frac{\pm 2 - b_{n-1}}{a_{n-1}}; \ldots \frac{\pm 1 - b_n}{a_2},$$

the greatest of them will be a superior, and the least of them an inferior limit to the roots of fx*.

It may be remarked, that if the successive dividends in the course of the process be multiplied respectively by $k_1, k_2, \ldots k_n$, $\frac{\phi x}{fx}$ will take the form

$$\frac{k_1}{q_1 +} \; \frac{k_2}{q_2 +} \; \frac{k_3}{q_3 +} \ldots \frac{k_n}{q_n};$$

and if we write $\quad a_1 x + b_1 = \pm c_1 \quad a_2 x + b_2 = \pm c_2 \ldots a_n x + b_n = \pm c_n$

and make $\quad c_1 = 1 \quad c_2 = 1 + k_2 \quad c_3 = 1 + k_3 \ldots c_n = 1 + k_n$,

the same reasoning as above will show the greatest and least of the $2n$ quantities

$$\frac{\pm 1 - b_1}{a_1}, \quad \frac{\pm (1 + k_2) - b_2}{a_2}, \ldots \frac{\pm (1 + k_n) - b_{n-1}}{a_{n-1}}; \quad \frac{\pm 1 - b_n}{a_n}$$

will be a superior and inferior limit to the roots of fx.

For greater simplicity, again, consider $k_1, k_2 \ldots k_n$ to be all equal to unity; we may make this addition to the theorem as above stated, viz. calling $L_1 \Lambda_1$; $L_2 \Lambda_2$; ... $L_n \Lambda_n$ the greatest and least values of the terms contained respectively in the series marked below 1, 2, 3...n, viz.—

$$
\begin{array}{llllll}
(1.) & \frac{\pm 1 - b_1}{a_1}; & \frac{\pm 2 - b_2}{a_2}; & \frac{\pm 2 - b_3}{a_3}; \ldots & \frac{\pm 2 - b_{n-1}}{a_{n-1}}; & \frac{\pm 1 - b_n}{a_n} \\
(2.) & \ldots & \frac{\pm 1 - b_2}{a_2}; & \frac{\pm 2 - b_3}{a_3}; \ldots & \frac{\pm 2 - b_{n-1}}{a_{n-1}}; & \frac{\pm 1 - b_n}{a_n} \\
(3.) & \ldots & \ldots & \frac{\pm 1 - b_3}{a_3}; \ldots & \frac{\pm 2 - b_{n-1}}{a_{n-1}}; & \frac{\pm 1 - b_n}{a_n} \\
 & \ldots & \ldots & \ldots & \ldots & \ldots \\
 & \ldots & \ldots & \ldots & \ldots & \ldots \\
(n-1) & \ldots & \ldots & \ldots & \frac{\pm 1 - b_{n-1}}{a_{n-1}}; & \frac{\pm 1 - b_n}{a_n} \\
(n.) & \ldots & \ldots & \ldots & \ldots & \frac{\pm 1 - b_n}{a_n},
\end{array}
$$

* For a generalization and improved form of statement of this theorem see Supplement to the present Section.

$L_1\ \Lambda_2$; $L_2\ \Lambda_2$; ... $L_n\ \Lambda_n$ will be respectively superior and inferior limits to fx, φx and their successive residues. As a corollary, we see, of course, that L and Λ, the superior and inferior limit to the roots of the given function fx, must always lie between $+\infty$ and the greatest root, and between $-\infty$ and the least root, of the arbitrarily assumed function φx.

Art. (53.). Let us now assume somewhat more generally that φx is any number of degrees θ_1 in x lower than fx, which will cause the first quotient q_{θ_1} to be of the degree θ_1 in x; and let us further suppose that φx stands in such a relation to fx that the following quotients, $q_{\theta_2}, q_{\theta_3}, \ldots q_{\theta_\rho}$, are of the degrees $\theta_2, \theta_3 \ldots \theta_\rho$ in x ($\theta_2, \theta_3 \ldots \theta_\nu$ being supposed not necessarily units, as they would generally be, but any positive integers whatever, as may happen in consequence of one or more of the leading coefficients in any residue vanishing), then

$$\frac{\varphi x}{fx}=\frac{1}{q_{\theta_1}+}\ \frac{1}{q_{\theta_2}+}\ \frac{1}{q_{\theta_3}+}\cdots+\frac{1}{q_{\theta_\rho}},$$

where $\theta_1+\theta_2+\theta_3+\ldots\theta_\rho=n$, and consequently fx will be equal to the denominator of the last convergent above written, multiplied by a constant, so that we have now $c.fx=m_1.m_2\ldots m_\rho$, where

$$m_1=q_{\theta_1}\quad m_2=q_{\theta_2}+\frac{1}{m_1}\ldots m_\rho=q_{\theta_{\rho-1}}+\frac{1}{m_{\rho-1}}.$$

And as in the case previously considered, so long as

$$q_{\theta_1}\ \begin{matrix}>1\\ \text{or}\\ <-1\end{matrix}\quad q_{\theta_2}\ \begin{matrix}>2\\ \text{or}\\ <-2\end{matrix}\quad q_{\theta_3}\ \begin{matrix}>2\\ \text{or}\\ <-2\end{matrix}\ \ldots q_{\theta_\rho}\ \begin{matrix}>1\\ \text{or}\\ <-1\end{matrix}$$

fx will have the same sign as $q_{\theta_1}.q_{\theta_2}\ldots q_{\theta_\rho}$.

Let now $\quad q_{\theta_1}=\pm c_1\quad q_{\theta_2}=\pm c_2\ldots q_{\theta_\rho}=\pm c_\rho$,

where $\quad c_1=1\quad c_2=2\ldots c_{\rho-1}=2\quad c_\rho=1$.

Consider any pair of the above equations as $q_{\theta_i}^2-c_i^2=0$.

1st. Suppose all the roots of this equation are impossible, $q_{\theta_i}^2-c_i^2$ must be positive for all values of x, and q_{θ_i} can never lie between $+c_i$ and $-c_i$; moreover, since upon the hypothesis made, $q_{\theta_i}+c_i$ and $q_{\theta_i}-c_i$ always retain the same sign, viz. that of the coefficient of the highest power of q_{θ_i}, it follows that q_{θ_i} must also always retain the same sign; for if we construct the two curves $y=q_{\theta_i}+c_i$ and $y=q_{\theta_i}-c_i$, these will both lie on the same side of the axis of x, and never cut the axis, consequently the curve $y=q_{\theta_i}$, which lies between them, must also lie on the same side as either of them, and never cut the axis.

Hence, then, if the roots of the equation are all impossible, q_{θ_i} will always retain the same sign, and will never fall within the region bounded on its two sides by $+c_i$ and $-c_i$.

2nd. Suppose the equation to have one or more possible roots, and l_i to the greatest, and λ_i the least (which of course, if there is but one possible root, will be identical). If the leading coefficient of q_i is positive, the greatest root (l) of the equation $q_i-c=0$ will exceed the greatest root of (l') of the equation $q_{\theta_i}+c_i=0$; for between $x=\infty$ and $x=l'$, q_{θ_i} must go through all values intermediate between ∞ and $-c_i$; hence there must be a quality l intermediate between l' and $+\infty$, which will make $q_{\theta_i}=c_i$. In like manner, if the leading coefficient of q_{θ_i} is negative, it will be seen that the greatest root of $q_{\theta_i}+c_i=0$ will exceed that of $q_{\theta_i}-c_i=0$. Moreover, in the one case q_i will be always positive and greater than c_i, and in the other always negative, and less than c_i. In every case, therefore, between $+\infty$ and l_i, q_{θ_i} retains the same sign, and does not fall within the region bounded by $+c_i$ and $-c_i$; the same thing may be shown to be true for all values of x between $-\infty$ and λ_i. Hence, then, by the same reasoning as that employed in the preceding article, we are enabled to affirm, that if we form the equation

$$(q^2_{\theta_1}-1)(q^2_{\theta_2}-4)(q^2_{\theta_3}-4)\dots(q^2_{\theta_{\rho-1}}-4)(q^2_{\theta_\rho}-1)=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (\psi.)$$

its greatest root will be a superior limit, and its least root an inferior limit to the roots of the equation $fx=0$, whatever be the value of the assumed function ϕx; and if the above equation (ψ.) has no real root, all the roots of fx will be imaginary.

Art. (54.). In the preceding two articles it has been supposed that all the quotients are taken integral functions of x; but the process of successive division may be so conducted as to give rise to quotients of the form

$$ax^i+bx^{i-1}+\dots+c+\frac{d}{x}+\dots+\frac{l}{x^{i'}}.$$

Suppose then that we have in general

$$\frac{\phi x}{fx}=\frac{1}{q_1+}\;\frac{1}{q_2+}\dots+\frac{1}{q_\omega},$$

where $q_1, q_2, \dots q_\omega$ are each of the general form above written (but of course i and i' being not necessarily the same for any two of the quotients), and suppose that the sum of the degrees in x of $q_1, q_2, \dots q_\omega$ is $n+t$, where t is essentially (as it must be) positive. Then we shall find, as in the last article, that L and Λ being called the greatest and least roots of $(q_1^2-1)(q_2^2-4)\dots(q^2_{n-1}-4)(q_n^2-1)$, D the denominator of the last convergent to the continued fraction above written, will never change its sign between $+\infty$ and L, nor between Λ and $-\infty$; but here we shall have

$$fx=\mathrm{K}x^t\times\mathrm{D}.$$

Hence $x^t.\mathrm{D}$ will be invariable in sign within each of these two intervals.

1st. Let t be even; then $f(x)$ will be invariable in sign, whatever L and Λ may be for each such interval.

2nd. Let t be odd; then if L is >0 and $\Lambda<0$, $f(x)$ cannot change its sign in either interval; but if L is <0 or $\Lambda>0$, fx will change its sign as x passes through zero, but will be invariable for each of the three regions contained between $+\infty$ and L, L and 0, or 0 and Λ (as the case may be), and Λ and $-\infty$; so that universally L and Λ will be a superior and inferior limit to the roots of fx, making abstraction of the roots (if any such there be in fx) whose value is zero.

Art. (55.). I shall close this section with offering (for what it is worth) a bare suggestion as to the mode in which the theory of Intercalations may hereafter be found to admit of being extended from a system of two general functions of x, to a system of three general functions of x, y, four general functions of x, y, z, and in general to a system of ε general functions of $\varepsilon-1$ variables, or which is the same thing, of ε homogeneous functions of ε variables. In the case of two functions of x, $f(x)$ and φx, $fx=0$ and $\varphi x=0$ may be considered to represent two systems of points in a right line; and the theory relates in this case to the relative positions of these two "Kenothemes" or point systems; and of course using x and y to denote the distances of any point in a line from two fixed points therein respectively, instead of fx and φx, we may employ two homogeneous functions of x and y, as $f(x, y)$ and $\varphi\ (x, y)$, to denote these two systems of points. So, similarly, if we have three functions of two variables, $f(x, y)$, $g(x, y)$, $h(x, y)$, which I shall suppose to be of the same degree, we may consider the mutual relations of the Monothemes, that is to say, the three plane curves, denoted by the equations $f(x, y)=0$, $g(x, y)=0$, $h(x, y)=0$. Now every two of these will intersect one another in a system of points, which we may call (f, g) for the intersections of f and g, (g, h) for those of (g and h), and (h, f) for those of h and f. If we take any two of these systems of intersections, as (f, g) and (g, h), they will both lie upon one of the given curves (g). And by reading off the two systems of points (f, g) and (g, h), arranged according to the order upon which they are disposed upon the curve g, we may, by following the course of such curve, form a scale of effective intercalations for these two systems, and in like manner for the two systems (g, h) and (h, f); (h, f) and (f, g). Now I believe that it will be found that when f, g, h represent any algebraical curves consisting of a single continuous line, either extending to infinity in both directions, or returning to itself (and I have fully satisfied myself of the truth of this for the case of ellipses), each effective scale of intercalation will contain the same number of pairs of points; if, however, the curves consist of more than one branch, as if hyperbolæ be considered, such is no longer necessarily the case; from these facts, conjoined with the light thrown upon the subject by its relation to the theory of combinants explained in the succeeding section, I am induced to infer the probability of the truth of the following law (which, for avoidance of further uncertainty, I confine to the case of functions of the same degree), viz. that if f, g, h be three homogeneous functions of x, y, and z of the same degree, and if U, V, W be any three linear functions of f, g, h, and if $U=0$, $V=0$, $W=0$ be treated as the equations to three cones, and if we form an effective scale of the intercalations of the lines of intersection of U and W, and V and W, according to the order in which they are disposed upon W (which seems to require that the lines shall be continuous, in order to admit of a fixed order of reading off the intersections of any two of them upon the third); then whatever value may have been given to the coefficients in the linear functions the number of elements remaining in any such scale will (as I conjecture) be constant, and some theory (to be discovered) for three functions analogous to that of Bezoutian residues for two functions will serve to determine the number of the elements so remaining. And so, in like manner, but with a difficulty increasing at each step (as at the next step we should have to pass into *quasi-space* of four dimensions), a theory of intercalations may be conjectured to exist for any (n) general functions of any ($n-1$) variables.

Development of the method of assigning a superior and inferior limit to the roots of any algebraical equation.

Art. (α.). Since the articles in the preceding part of this section on the method of discovering limits to the roots of an algebraical equation were written, the method of which the germ is therein contained has presented itself in a much more fully developed form, which I proceed to exhibit: for greater simplicity I shall suppose ϕx to be of $n-1$, and fx to be of n dimensions in x, and that by means of the ordinary process for common measure (except that as in STURM's theorem the sign of all the remainders are changed) $\frac{\phi x}{fx}$ has been thrown under the form of the improper continued fraction

$$\frac{1}{q_1-}\,\frac{1}{q_2-}\,\frac{1}{q_3-}\cdots\frac{1}{q_n},$$

where $q_1\,q_2\ldots q_n$ are all restricted to signify simple linear functions of x.

Suppose the series q_1, q_2, q_3, $\ldots q_n$ to be resolved into the distinct sequences

$$q_1\,q_2\ldots q_i;\quad q_{i+1}\,q_{i+2}\ldots q_{i'};\quad q_{i'+1}\ldots q_{i''}\ldots q_{i,+1}\ldots\ldots q_n,$$

in such a manner that in each sequence as $q_{i+1}\,q_{i+2}\ldots q_{i'}$ the coefficients of x have all the same sign, but that in any two adjoining sequences the coefficients of x have opposite signs, so that for instance in q_i and q_{i+1} the coefficients of x are unlike, as also in $q_{i'}$ and $q_{i'+1}$; there will of course be nothing to preclude any of these sequences becoming reduced to a single term.

The first theorem is, that the greatest and least roots of the product of the cumulants

$$[q_1\,q_2\ldots q_i]\times[q_{i+1}\,q_{i+2}\ldots q_{i'}]\ldots\times[q_{i,+1}\,q_{i,+2}\ldots q_n]$$

are superior and inferior limits to the roots of fx. To prove this theorem I begin with premising the two following lemmas, one virtually and the other expressly contained in the Philosophical Magazine for the months of September and October of the present year*.

* Each of these two lemmata flows readily from the faculty previously adverted to engaged by every cumulant of being representable under the form of a determinant. As to the second lemma, it becomes apparent immediately when the cumulant is so represented, by separating the matrix into two rectangles and expressing the entire determinant according to a well-known rule for the decomposition of determinants as a function of the determinants belonging to these two rectangles taken separately. As to the first lemma, by reason of the cumulant $[\omega_1\,\omega_2\ldots\ldots\omega_{i-1}\,\omega_i\,\omega_{i+1}]$ being so representable, we know that when $[\omega_1\,\omega_2\ldots\omega_{i-1}\,\omega_i]=0$, $[\omega_1\,\omega_2\ldots\omega_{i-1}]$ and $[\omega_1\,\omega_2\ldots\omega_{i+1}]$ must have opposite signs. Suppose, now, that the theorem is true when the number of elements in the type does not exceed i; then the roots of $[\omega_1\,\omega_2\ldots\omega_{i-1}]$, say of ψ_{i-1}, being called $h_1\,h_2\ldots h_{i-1}$, and of $[\omega_1\,\omega_2\ldots\omega_{i-1}\,\omega_i]$, say of ψ_i, being called $k_1\,k_2\ldots k_i$, these may be arranged in the following order of

Lemma A. The roots of the cumulant $[q_1, q_2 \dots q_i]$, in which each element is a linear function of x, and wherein the coefficient of x for each element has the like sign, are all real, and between every two of such roots is contained a root of the cumulant $[q_1 q_2 \dots q_{i-1}]$, and *ex converso* a root of the cumulant $[q_2 q_3 \dots q_i]$, and (as an evident corollary) for all values of ϱ and ϱ' intermediate between 1 and i the greatest root of $[q_1 q_2 \dots q_{i-1} q_i]$ will be greater, and the least root of the same will be less than the greatest and least roots respectively of $[q_\varrho q_{\varrho+1} \dots q_{\varrho'-1} q_{\varrho'}]$.

Lemma B. For all values of the elements $q_1 q_2 \dots q_n$, the cumulant

$$[q_1 q_2 \dots q_{\omega-1} q_\omega q_{\omega+1} q_{\omega+2} \dots q_n]$$
$$=[q_1 q_2 \dots q_{\omega-1} q_\omega] \times [q_{\omega+1} q_{\omega+2} \dots q_n]$$
$$-[q_1 q_2 \dots q_{\omega-1}] \times [q_{\omega+2} \dots q_n].$$

Thus *ex. gr.* the cumulant $[abcd]$, *i. e.* $abcd-ab-cd-ad+1$,

$$=[ab] \times [cd]-[a] \times [d]=(ab-1)(cd-1)-ad,$$

and $[abcde]$, i. e. $abcde-abc-abe-ade-cde+a+c+e=[abc][ae]-[ab][e]$,

$$\text{i. e.}=(abc-a-c)(de-1)-(ab-1)e.$$

Art. (β.). Now suppose that $q_1 q_2 \dots q_\omega q_{\omega+1} \dots q_n$ are all linear functions of x, and that the coefficients of x have all one (say the positive) sign in $q_1 q_2 \dots q_\omega$, and all the contrary signs in $q_{\omega+1} \dots q_n$, and let L be not less than the greatest root of $[q_1 q_2 \dots q_\omega]$ or of $[q_{\omega+1} \dots q_n]$, and also let Λ be not greater than the least root of each of these same two cumulants; then by lemma A, L and Λ will also be respectively greater than the greatest, and less than the least roots of $[q_1 q_2 \dots q_{\omega-1}]$ and of $[q_{\omega-2} \dots q_n]$. Now the coefficient of the highest power of x in both $[q_1 q_2 \dots q_\omega]$ and in $[q_1 q_2 \dots q_{\omega-1}]$ is positive, but as to $[q_{\omega+1} \dots q_n]$ and $[q_{\omega+2} \dots q_n]$ is of contrary signs in the two, viz. negative in that one of those cumulants which contains an odd, and positive in that one of the two which contains an even number of elements. Hence by virtue of Lemma B, L and any quantity greater than L substituted for x will make $[q_1 q_2 \dots q_n]$ to have always the same sign, and in like manner it may be shown that Λ and any quantity less than Λ substituted for x will also cause $[q_1 q_2 \dots q_n]$ to retain always the same sign. Hence L and Λ are superior and inferior limits to $[q_1 q_2 \dots q_n]$; and the same reasoning would

magnitude $k_1 h_1\ k_2 h_2\ k_3 \dots k_{i-1}\ h_{i-1}\ k_i$; and if the roots of $[\omega_1 \omega_2 \dots \omega_{i-1} \omega_i \omega_{i+1}]$, say of ψ_{i+1}, be called $l_1\ l_2 \dots l_{i+1}$, from the fact of the leading coefficients in ψ_{i-1} and ψ_{i+1} expanded according to the powers of x having the same sign, it follows that when $x=\infty$, ψ_{i-1} and ψ_{i+1} have the same sign, but they have contrary signs when $x=k$; but ψ_{i-1} does not change its sign between $x=\infty$ and $x=k$, hence ψ_{i+1} does change its sign between $x=\infty$ and $x=k_1$, and therefore a root of ψ_{i+1} lies between ∞ and k_1; in like manner precisely it may be shown that a root of ψ_{i+1} lies between $-\infty$ and k_i; and since ψ_{i-1} changes its sign between k_1 and k_2, between k_2 and $k_3 \dots k_i$ and between k_{i-1} and k_i, ψ_{i+1} must likewise change its sign between one and the other extremity of each of these intervals, and hence the roots $l_1\ l_2 \dots l_{i+1}$ are intercalated between ∞, $k_1, k_2, \dots k_i$, $-\infty$, or which is the same thing, $k_1, k_2, \dots k_i$ are respectively intercalated between $l_1, l_2, \dots l_{i+1}$; consequently, if the theorem is true up to i, it is true for $i+1$, and therefore true universally; but is manifestly true when $i=2$, for then $x=\pm\infty$ makes $[\omega_1, \omega_2]$, *i. e.* $\omega_1\ \omega_2-1$ positive: but $\omega_1=0$ makes it negative, which proves the theorem contained in Lemma A.

evidently apply if we had supposed the signs of the coefficients of x in the first partial series of elements to have been negative, and in the other series of elements to have been positive.

The greatest and least roots of $[q_1 q_2 \dots q_\omega] \times [q_{\omega+1} \dots q_n]$ evidently satisfy the condition to which L and Λ are subject, and may be taken in place of L and Λ respectively. They will accordingly be superior and inferior limits to the cumulant

$$[q_1 q_2 \dots q_\omega \; q_{\omega+1} \dots q_n].$$

Again, by virtue of theorem (B.) it may readily be shown that

$$\begin{aligned}
&[q_1 q_2 \dots q_{\omega_1} q_{\omega_1+1} q_{\omega_1+2} \dots q_{\omega_2} q_{\omega_2+1} \dots q_n] \\
&= [q_1 q_2 \dots q_{\omega_1}] \times [q_{\omega_1+1} q_{\omega_1+2} \dots q_{\omega_2}] \times [q_{\omega_2+1} \dots q_n] \\
&- [q_1 q_2 \dots q_{\omega_1-1}] \times [q_{\omega_1+2} \dots q_{\omega_2}] \times [q_{\omega_2+1} \dots q_n] \\
&- [q_1 q_2 \dots q_{\omega_1}] \times [q_{\omega_1+1} \dots q_{\omega_2-1}] \times [q_{\omega_2+2} \dots q_n] \\
&+ [q_1 q_2 \dots q_{\omega_1-1}] \times [q_{\omega_1+2} \dots q_{\omega_2-1}] \times [q_{\omega_2+2} \dots q_n];
\end{aligned}$$

and hence if $q_1 q_2 \dots q_n$ are all linear functions of x in which the coefficients of x have all the same algebraical sign in any one (taken *per se*) of the three series

$$q_1 q_2 \dots q_{\omega_1}; \quad q_{\omega_1+1} \dots q_{\omega_2}; \quad q_{\omega_2+1} \dots q_n;$$

but so that this sign changes in passing from one series to another, it is easily seen, by the same reasoning as in the preceding case, that the two positive and two negative products on the right-hand side of the equation all give the same sign to the coefficient of the highest power of x, and consequently that if L and Λ be superior and inferior limits to

$$[q_1 \dots q_{\omega_1}], \quad [q_{\omega_1+1} \dots q_{\omega_2}], \quad [q_{\omega_2+1} \dots q_n],$$

and consequently by Lemma A, to

$$[q_1 q_2 \dots q_{\omega_1-1}], \quad [q_{\omega_1+2} \dots q_{\omega_2}], \quad [q_{\omega_1+1} \dots q_{\omega_2-1}], \quad [q_{\omega_1+2} \dots q_{\omega_2-1}], \text{ and to } [q_{\omega_2+2} \dots q_n],$$

L or Λ substituted for x will cause $[q_1 q_2 \dots q_n]$ to retain always the same sign, and will consequently be superior and inferior limits thereto; and so in general; whence it follows, returning to the theorem to be demonstrated, that the greatest and least roots of

$$[q_1 q_2 \dots q_i] \times [q_{i+1} q_{i+2} \dots q_{i'}] \times \dots \times [q_{+1} \dots q_n],$$

will be superior and inferior limits to the cumulant $[q_1 q_2 \dots q_n]$, *i. e.* to C.fx*, and therefore to fx, as was to be proved.

* If $\frac{\phi x}{fx}$ expanded as a continued fraction by means of the common measure process gives rise to the quotients $q_1, q_2, \dots q_n$, and if $L_1, L_2, \dots L_{n-1}, L_n$ be the leading coefficients of the successive simplified residues, (L_n being, in fact, the final simplified residue, *i. e.* the resultant to $\phi x, fx$), we must have $\phi x = C[q_2, q_3 \dots q_n]$ $fx = C[q_1, q_2 \dots q_n]$, where (supposing ϕx to be of $n-1$, and fx of n dimensions in x),

$$C = \frac{1}{L_n} \left\{ \frac{L_n^2 \,.\, L_{n-2}^2 \,.\, L_{n-4}^2 \text{ \&c.}}{L_{n-1}^2 \,.\, L_{n-3}^2 \,.\, L_{n-5}^2 \text{ \&c.}} \right\}.$$

Art. (γ.). The second theorem is the following: if $q_1\, q_2, \ldots q_n$, be linear functions of x, say a_1x+b_1, $a_2x+b_2 \ldots a_nx+b_n$, in which the coefficients of x have all the same sign, and if we take the quantities $\mu_1, \mu_2, \ldots \mu_{n-1}$, all having the same sign as $a_1\, a_2 \ldots a_n$, but otherwise arbitrary, and make

$$k_1=\mu_1 \quad k_2=\mu_2+\frac{1}{\mu_1} \quad k_3=\mu_3+\frac{1}{\mu_2}\cdots k_{n-1}=\mu_{n-2}+\frac{1}{\mu_{n-2}} \quad k_n=\frac{1}{\mu_n},$$

then the greatest of the quantities

$$\frac{k_1-b_1}{a_1},\ \frac{k_2+b_2}{a_2},\ \ldots\frac{k_n-b_n}{a_n},$$

say L, is a superior limit, and the least of the quantities

$$\frac{-k_1-b_1}{a_1},\ \frac{-k_2-b_2}{a_2},\ \ldots\ \frac{-k_n-b_n}{a_n},$$

say Λ, is an inferior limit to the roots of fx.

L and any value greater than L substituted for x will evidently make q_1-k_1; q_2-k_2; ...; q_n-k_n, all of them positive.

Hence when $x=$ or $>$ L q_1 is positive and $>\mu_1$ and

$$q_2-\frac{1}{q_1}>k_2-\frac{1}{\mu_1}>\mu_2+\frac{1}{\mu_1}-\frac{1}{\mu_1}, \text{ i. e. is positive, and } >\mu_2,$$

$$q_3-\frac{1}{q_2-}\ \frac{1}{q_1}>k_3-\frac{1}{\mu_2}>\mu_3+\frac{1}{\mu_2}-\frac{1}{\mu_2}, \text{ i. e. is positive, and } >\mu_3,$$

. . . .

. . . .

$$\text{and } q_n-\frac{1}{q_{n-1}-}\ \frac{1}{q_{n-2}}\cdots\frac{1}{q_1}>\frac{1}{\mu_{n-1}}-\frac{1}{\mu_{n-1}}, \text{ i. e. is positive.}$$

and consequently the cumulant $[q_1q_2q_3\ldots q_n]$, which

$$=q_1\times\left(q_2-\frac{1}{q_1}\right)\times\left(q_3-\frac{1}{q_2-}\ \frac{1}{q_1}\right)\times \&c.,$$

remains of a constant sign when L and any quantity greater than L is substituted for x. Hence L is a superior limit. In like manner Λ and any quantity less than Λ will evidently make q_1+k_1, q_2+k_2; ... q_n+k_n all of them negative, so that when $x=$ or $<\Lambda$ q_1 is negative, and $<-\mu_1$

$$q_2-\frac{1}{q_1}<k_2-\frac{1}{\mu} \text{ is negative, and } <-\mu_2,$$

$$q_3-\frac{1}{q_2}<k_3-\frac{1}{\mu_2} \text{ is negative, and } <-\mu_3,$$

. . . .

. . . .

. . . .

$$\text{and } q_n-\frac{1}{q_{n-1}-}\ \frac{1}{q_{n-2}}\cdots\frac{1}{q_1}<\frac{1}{\mu_{n-1}}-\frac{1}{\mu_{n-1}} \text{ is negative.}$$

So that $[q_1\, q_2 \ldots q_n]$ for all values of x less than A will preserve an invariable sign, and consequently A is an inferior limit to fx.

Art. (δ.). It may be remarked that the quantities

$$\mu_1;\ \mu_2+\frac{1}{\mu_1};\ \mu_3+\frac{1}{\mu_2};\ \ldots\ \mu_{n-2}+\frac{1}{u_{n-3}};\ \mu_{n-1}+\frac{1}{\mu_{n-2}};\ \frac{1}{\mu_{n-1}}$$

may be derived successively from one another, according to the same law, from whichever end of the series we begin.

If we take any two consecutive terms as

$$\mu_i+\frac{1}{\mu_{i-1}};\ \mu_{i+1}+\frac{1}{\mu_i},$$

the effect of diminishing μ_i is to decrease the first of these two terms, and *pro tanto,* to tend to deduce the limit; but on the other hand, $\frac{1}{\mu_i}$ being increased, there is brought into play an opposite tendency, which operates *pro tanto* to increase the value of the limit.

Art. (ε.). It is of importance to remark, that by a right selection of the system of quantities $\mu_1\, \mu_2 \ldots \mu_{n-1}$, which enter into the composition of $k_1\, k_2 \ldots k_n$, L may be made to coincide with the greatest root of $[q_1\, q_2 \ldots q_n]$; and so in like manner by a right selection of another system of these quantities, whereby to form $k_1\, k_2 \ldots k_n$, A may be made to coincide with the least root of the same. Thus let $\mu_1\ \mu_2 \ldots \mu_{n-1}$ be so chosen, that

$$q_1-k_1=0 \quad q_2-k_2=0 \ldots q_n-k_n=0$$

are all satisfied by the same value of x.

Then $$q_1=\mu_1 \quad q_2=\mu_2+\frac{1}{\mu_1} \quad q_3=\mu_3+\frac{1}{\mu_2} \cdots q_n=\frac{1}{\mu_{n-1}}$$

exist simultaneously.

Hence $$\mu_2=q_2-\frac{1}{q_1} \quad \mu_3=q_3-\frac{1}{\mu_2}=q_3-\frac{1}{q_2-}\,\frac{1}{q_1}$$

$$\mu_{n-1}=q_{n-1}-\frac{1}{q_{n-2}-}\,\frac{1}{q_{n-3}}-\cdots\frac{1}{q_1}$$

$$q_n=\frac{1}{q_{n-1}-}\,\frac{1}{q_{n-2}-}\cdots\frac{1}{q_1},$$

which is satisfied by making $[q_n,\, q_{n-1},\, q_{n-2} \ldots q_1]=0$.

It remains then only to show that the greatest root of x in this equation substituted for x in $q_1, q_2, \ldots q_n$ will make $\mu_1\, \mu_2 \ldots \mu_{n-1}$ all of one sign, and that the least root of x similarly substituted, will also make them all of one, but a contrary sign, which may be proved as follows.

We have

$\mu_1=q_1$ $\mu_2=[q_1, q_2]\div q_1$ $\mu_3=[q_1\, q_2\, q_3]\div[q_1, q_2]$ &c. $\mu_{n-1}=[q_1\, q_2 \ldots q_{n-1}]\div[q_1, q_2 \ldots q_{n-2}]$; and by Lemma B the superior limit to $[q_1\, q_2 \ldots q_n]$ will be a superior limit also to $q_1, q_2\, q_3, \ldots, q_{n-2}$, and to $[q_1\, q_2], [q_1\, q_2\, q_3] \ldots, [q_1\, q_2 \ldots, q_{n-1}]$.

Consequently this superior limit will make $\mu_1, \mu_2 \ldots \mu_{n-1}$ have all the same sign as that of the coefficients of x in $q_1, q_2, \ldots q_n$. And in like manner, the inferior limit to $[q_1, q_2, \ldots q_n]$ will cause $\mu_1, \mu_2, \ldots \mu_{n-1}$ to have all the contrary sign to that of these coefficients.

Thus then we see that when the coefficients of x in the partial quotients to $\frac{\phi x}{fx}$ expressed as an improper continued fraction form a single series of continuations of signs, by a right choice of the arbitrary constants $\mu_1, \mu_2, \ldots \mu_{n-1}$ the superior or inferior limit given by this new method may severally and separately be made to coincide with the greatest and least real root, or each in turn with the sole real root of fx, if there be but one.

Art. (ζ.). The general method of enclosing the roots of fx within limits is founded upon the combination of the two theorems above demonstrated. An arbitrary function ϕx one degree in x below, fx being assumed, and by aid of the auxiliary function $\phi x, fx$ being thrown under the form

$$C[q_1, q_2, \ldots q_i\ q'_1, q'_2, \ldots q'_{i'}\ q''_1 \ldots \ldots (q)_1, (q)_2 \ldots, (q)_{i^v}],$$

in which the coefficient of x is supposed to change sign in the passage from q_i to q'_1, from $q'_{i'}$ to q''_1, &c., a superior limit is found to each of the cumulants

$$[q_1\ q_2 \ldots q_i],\ [q'_1\ q'_2 \ldots q'_{i'}],\ \ldots\ [(q)_1\ (q)_2 \ldots (q)_{i^v}],$$

taken separately, by means of the second theorem, and then by virtue of the first theorem the greatest of these superior limits is a superior limit to the cumulant

$$[q_1\ q_2 \ldots q_i \ldots (q)_1 \ldots (q)_{i^v}],$$

and consequently to fx, and so *mutatis mutandis* the least of the inferior limits of the same partial cumulants is an inferior limit to the total cumulant

$$[q_1 q_2 \ldots q_i \ldots (q)_1 (q)_2 \ldots (q)_{i^v}].$$

Art. (η.). When all the roots of fx are real, if ϕx be so assumed that all its roots are intercalated between those of fx, the partial quotients to $\frac{\phi x}{fx}$ will form but one single series. In order that ϕx may fulfill this condition, it is necessary that the coefficients of ϕx shall be subject to certain conditions of inequality, not necessary here to be investigated; but no conditions of equality, *i. e.* no equations between the coefficients of ϕx, are introduced by this condition; or in other words, the coefficients* of ϕx, the auxiliary function, are independent and arbitrary within limits; and we have shown that in this case the auxiliary constants $\mu_1\ \mu_2 \ldots \mu_{n-1}$ may be so determined that the limits may be made to come separately and respectively into contact with the two extreme roots. When all the roots of fx are not real, the quotients (however ϕx is chosen) can no longer be made to form a single series. It still however remains true, that, by a due choice of the auxiliary function followed by a due choice of the

* It need scarcely be stated that $f'x$ is the simplest form of ϕx, which satisfies the condition in question.

auxiliary constants, this coincidence may be brought about, so long as there is a single real root in fx.

It is rather important to demonstrate this universal possibility of effecting a coincidence of the limits to the roots with the extreme roots themselves, because it is the most striking feature which distinguishes the method of limitation here developed from all others previously brought to light.

Art. (θ.). Before entering upon this demonstration I may make the passing remark, that every method of root-limitation is implicitly a method of root-approximation.

For instance, let e be any given quantity between which and $+\infty$ it is known that a root of fx lies. Then if we write $x=e+\frac{1}{y}$, and form the equation $y^n f\left(e+\frac{1}{y}\right)=0$, and find L a superior limit to y, it is clear that $e+\frac{1}{\mathrm{L}}$ will lie between e and the root of fx say E, next superior to e. Again, making $x=e+\frac{1}{\mathrm{L}}+\frac{1}{y'}$, and finding a superior limit L$'$ to y', we shall have $e+\frac{1}{\mathrm{L}}+\frac{1}{\mathrm{L}'}$ still nearer to E than $e+\frac{1}{\mathrm{L}}$ was; and so we may proceed advancing nearer and nearer, and always from the same side towards E at each step, and finally obtain E under the form $e+\frac{1}{\mathrm{L}}+\frac{1}{\mathrm{L}'}+\frac{1}{\mathrm{L}''}+$ &c. And in like manner calling $\mathrm{E}_{,}$ the root next below e, we may find $\mathrm{E}_{,}=e-\frac{1}{\Lambda}-\frac{1}{\Lambda'}-\frac{1}{\Lambda''}$, &c.

Art. (ι.). In establishing the theorem of coincidence above adverted to, the following notation will be found very advantageous. Let Ω denote a Type of any number of Elements, as $q_1\, q_2 \ldots q_{i-1}\, q_i$, and let Ω' denote this same type when the last element. and $'\Omega$ the same type when the first element is cut off, and Ω the same type when both extremes are cut off, so that the apocopated type Ω' will mean $[q_1\, q_2 \ldots q_{i-1}]$; the apocopated type $'\Omega$ will mean $[q_2\, q_3 \ldots q_i]$, and the doubly apocopated type $'\Omega'$ will mean $[q_2\, q_3 \ldots q_{i-1}]$.

If now a type Ω be made up of the types $\Omega_1\, \Omega_2 \ldots \Omega_i$ put in apposition, and if we use in general $[\Omega]$ to denote the cumulant corresponding to the type Ω, there will be a very simple law* connecting $[\Omega]$ with

$$[\Omega_1][\Omega_2][\Omega_3]\ldots[\Omega_{i-2}][\Omega_{i-1}][\Omega_i]$$
$$[\Omega'_1][\Omega'_2][\Omega'_3]\ldots[\Omega'_{i-2}][\Omega'_{i-1}]$$
$$['\Omega_2]['\Omega_3]\ldots['\Omega_{i-2}]['\Omega_{i-1}]['\Omega_i]$$
$$['\Omega'_2]['\Omega'_3]\ldots['\Omega'_{i-2}]['\Omega'_{i-1}].$$

This law will be seen to be obviously deducible by successive steps of expansion

* The cumulant corresponding to any portion or fragment of a type may be said to be a partial cumulant to the entire type, and a type whose elements are constituted out of the elements of two or more types placed in juxtaposition may be said to be the aggregate of these types; the law given in the text above may then be said to have for its object the expansion of the complete cumulant to any type in terms of complete and partial cumulants to the types of which the given type is the aggregate.

from the fundamental theorem given in Lemma (B.) art. ($i.$), for the case of $\Omega=\Omega_1\Omega_2$, and will be best understood by showing its operation in a few simple cases.

Thus let $\Omega=\Omega_1\,\Omega_2$*.

Then $$[\Omega]=[\Omega_1]\times[\Omega_2]-[\Omega_1']\times['\Omega_2].$$

Let $\Omega=\Omega_1\,\Omega_2\,\Omega_3$.

Then $$\begin{aligned}[\Omega]=&[\Omega_1]\times[\Omega_2]\times[\Omega_3]\\ &-[\Omega_1']\times['\Omega_2]\times[\Omega_3]-[\Omega_1]\times[\Omega_2']\times['\Omega_3]\\ &+[\Omega_1']\times['\Omega_2']\times['\Omega_3].\end{aligned}$$

Let $\Omega=\Omega_1\,\Omega_2\,\Omega_3\,\Omega_4$.

Then $$[\Omega]=[\Omega_1]\times[\Omega_2]\times[\Omega_3]\times[\Omega_4]$$
$$-[\Omega_1']\times['\Omega_2]\times[\Omega_3]\times[\Omega_4]-[\Omega_1]\times[\Omega_2']\times['\Omega_3]\times[\Omega_4]-[\Omega_1]\times[\Omega_2]\times[\Omega_3']\times['\Omega_4]\dagger$$
$$+[\Omega_1']\times['\Omega_2']\times['\Omega_3]\times[\Omega_4]-[\Omega_1']\times['\Omega_2]\times[\Omega_3']\times['\Omega_4]-[\Omega_1]\times[\Omega_2']\times['\Omega_3']\times['\Omega_4]$$
$$-[\Omega_1']\times['\Omega_2']\times['\Omega_3']\times[\Omega_4],$$

and so in general if $\Omega=\Omega_1\Omega_2\ldots\Omega_i$, $[\Omega]$ may be expanded under the form of the sum of 2^{i-1} products separable into i alternately positive and negative groups containing respectively 1, $(i-1)$, $(i-1)\frac{i-2}{2}$, $\ldots(i-1)$, 1 products.

Art. ($\varkappa.$). In every one of the above groups forming a product the accents enter in pairs and between contiguous factors, it being a condition that if any Ω have an accent on the right the next Ω must have one on the left, and if it have one on the left the preceding Ω must have an accent on the right, and the number of pairs of accents goes on increasing in each group from 0 to $i-1$. This rule serves completely to define the development in question‡.

* The sign of equality is employed here to denote the relation between a concrete whole and the aggregate of its parts.

† The number of distinct factors entering into these products, taken collectively, is evidently $i+2(i-1)+(i-2)$, i. e. $4(i-1)$.

‡ When each partial type Ω consists of a single element, every doubly accented Ω will vanish, and every singly accented Ω will become unity; hence we may derive the rule for the expansion of the cumulant $[a\ a_2 a_3\ldots a_i]$ in terms of $a_1\,a_2\ldots a_i$, which will accordingly consist of

$$a_1\cdot a_2\cdot a_3\ldots a_i-\Sigma\frac{1}{a_e\cdot a_{e+1}}(a_1\cdot a_2\ldots a_i)+\Sigma\frac{1}{a_e\cdot a_{e+1}\times a_f\cdot a_{f+1}}(a_1\cdot a_2\ldots\ldots a_i)\mp\&c.,$$

the indices e and f, $e+1$ and f, &c. being understood to be all distinct integers (which agrees with the known rule for the expression of the denominator of a continued fraction in terms of the quotients). The number of terms in this expansion, in consequence of the vanishing of the quantities affected with a double accent, reduces from 2^{i-1} down to the ith term in the series commencing with 1, 2, 3, &c. defined by the equation $u_{i+1}=u_i+u_{i-1}$,

$$i.\ e.\quad \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{i+1}-\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{i+1};$$

the number, therefore, of products in which double accents occur in the general expansion of $[\omega_1\,\omega_2\ldots\omega_i]$ is

$$2^{i-1}-\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{i+1}+\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{i+1}.$$

For greater brevity let $[\Omega_e][\Omega'_e]['\Omega_e][\Omega'_e]$ be denoted respectively by ω_e, ω'_e, $'\omega_e$, $'\omega'_e$, then when the type Ω_e consists of a single element,

$$\omega'_e=1 \quad '\omega_e=1 \quad '\omega'_e=0.$$

It should be observed that the two equations $\omega_e=0$ $\omega'_e=0$ cannot exist simultaneously, for if Ω_e represent $q_1 q_2 \dots q_i$,

$$\omega_e=q_i\omega'_e-\omega''_e \quad \omega'_e=q_{i-1}\omega''_e-\omega'''_e, \text{ \&c.},$$

so that if $\omega_e=0$ and $\omega'_e=0$, we have $\omega''_e=0$, $\omega'''_e=0$, &c., and thus, finally, $-1=0$, which is absurd.

Now, if we suppose $\Omega_1 \Omega_2 \dots \Omega_e$ to be types every element in each of which is a linear function of x, the coefficients of x in these elements being positive in Ω_1, negative in Ω_2, and so on alternately, and Ω is the aggregate of $\Omega_1 \Omega_2 \dots \Omega_e$, it may easily be made out that each term in the development of ω in terms of $\omega_1, \omega'_1, '\omega_1, '\omega'_1$; $\omega_2, \omega'_2, '\omega_2, '\omega'_2$: &c. will have the same sign when we give to x a value which is a superior limit, or an inferior limit to the roots of each of the cumulants $\omega_1, \omega_2, \dots \omega_e$, and consequently to those of the cumulants $\omega'_1, \omega'_2, \dots \omega'_e$; $'\omega_1, '\omega_2, \dots '\omega_e$; $'\omega'_1, '\omega'_2, \dots '\omega'_e$; the products affected with positive signs being all positive or negative in themselves, and those affected with negative signs being reversely all negative, or all positive.

Thus, *ex. gr.* if $\qquad \Omega=\Omega_1\Omega_2$

$$\omega=\omega_1 . \omega_2-\omega'_1 . '\omega_2,$$

and the sign of the leading coefficient in $'\omega_2$ will be the contrary of that in ω_2, but ω_1 and ω'_1 have both the same positive sign ; so again if $\Omega=\Omega_1\Omega_2\Omega_3$,

$$\omega=\omega_1 . \omega_2 . \omega_3-\omega'_1 . '\omega_2 . \omega_3-\omega_1 . \omega'_2 . '\omega_3+\omega'_1 . '\omega_2 . \omega_3,$$

where the leading coefficients in ω_2 and ω_2 have contrary signs, as have also those in ω_2 and ω'_2 between ω_2 and $'\omega'_2$ have the same sign ; and of course the leading coefficients in ω_1, ω_3, ω'_1, $'\omega_3$ have all the same sign, they being all positive, and so in general. But the superior limit to the roots of any integral algebraical function of x substituted in place of x causes the signs of the resulting values of the functions to coincide with the signs of the leading coefficients, so that in the example last above given, L a superior limit to all the factors in the several products in the equation substituted for x will make $\omega_1 . \omega_2 . \omega_3$, $-\omega'_1 . '\omega_2 . \omega_3$, $-\omega_1 . \omega'_2 . '\omega_3$, $\omega_1 '\omega'_2 . '\omega_3$ to have all the same sign. The like will be true of Λ the inferior limit ; for if Ω_1, Ω_2, Ω_3 contain respectively n_1, n_2, n_3 elements, the values of the four products last above written, when $x=-\infty$, will be to the values of the same when $x=+\infty$ in the respective ratios of

$$(-)^{m_1+m_2+m_3}:1 \; ; \quad (-)^{m_1+m_2+m_3-2}:1 \; ; \quad (-)^{m_1+m_2+m_3-2}:1 \; ; \quad (-)^{m_1+m_2+m_3-4}:1,$$

and so in general. Hence we deduce the theorem, that if the total type Ω represent the aggregate in apposition of the partial orders $\Omega_1\Omega_2 \dots \Omega_e$ (the elements being understood to be linear functions of x, which are subject to the law of alternation in the signs of the coefficients of x in passing from one partial type to another), no superior limit to $\omega_1, \omega_2, \ldots \omega_e$ can make ω vanish unless each separate product in the expansion of ω in terms of $\omega_1, \omega_2, \ldots \omega_e$ and the appurtenant apocopated cumulants vanish separately.

Art. (λ.). From the above theorem we may deduce the following law, viz. that if the roots of $\omega_1, \omega_2, \ldots \omega_e$ be supposed to be arranged in order of magnitude, and λ to be that one of them which is nearest to $+\infty$ or to $-\infty$, then if e is even it is impossible for λ to be a root of ω. Thus suppose $e=2$, and consequently $\omega=\omega_1.\omega_2-\omega'_1.{}'\omega_2$; if λ be a root of ω_1 and one of the two extremes of the roots of ω_1, ω_2 put in order of magnitude, λ cannot be a root of ${}'\omega_2$, for the roots of ${}'\omega_2$ are confined between the roots of ω_2; but if λ make ω and ω_1 each vanish, we must have $\omega'_1.{}'\omega_2=0$, hence $\omega'_1=0$ as well as $\omega_1=0$, which is impossible. In like manner if a root of ω_2 were the extreme root, the same impossibility could be in like manner established.

Again, suppose $e=4$, so that

$$\omega=\omega_1.\omega_2.\omega_3.\omega_4\left\{1-\frac{\omega'_1.{}'\omega_2}{\omega_1.\omega_2}-\frac{\omega'_2.{}'\omega_3}{\omega_2.\omega_3}-\frac{\omega'_3.{}'\omega_4}{\omega_3.\omega_4}+\frac{\omega'_1.{}'\omega'_2.{}'\omega_3}{\omega_1.\omega_2.\omega_3}+\frac{\omega'_1.{}'\omega_2.\omega'_3.{}'\omega_4}{\omega_1.\omega_2.\omega_3.\omega_4}+\frac{\omega_1.\omega'_2.{}'\omega'_3.{}'\omega_4}{\omega_2.\omega_3.\omega_4}-\frac{\omega'_1.{}'\omega'_2.{}'\omega'_3.{}'\omega_4}{\omega_1.\omega_2.\omega_3.\omega_4}\right\}.$$

Let λ continue to denote one or the other extreme of the roots of $\omega_1\ \omega_2\ \omega_3\ \omega_4$. We must in each case, if λ makes $\omega=0$, have

$$\omega_1.\omega_2.\omega_3.\omega_4=0;\ \ \omega'_1.\omega_2.\omega_3.\omega_4=0;\ \ \omega_1.\omega'_2.{}'\omega_3.\omega_4=0;\ \ \omega_1.\omega_2.\omega'_3.{}'\omega_4=0;$$

$$\omega'_1.{}'\omega'_2.{}'\omega_3=0;\ \ \omega'_1.{}'\omega_2.\omega'_3.{}'\omega_4=0;\ \ \omega_1.\omega'_2.{}'\omega'_3.{}'\omega_4=0;\ \ \omega'_1.\omega'_2.{}'\omega'_3.{}'\omega_4=0.$$

Now suppose that λ is a root of ω_1, then the equations remaining to be satisfied are

$$\omega'_1.{}'\omega_2.\omega_3.\omega_4=0;\ \ \omega'_1.{}'\omega'_2.{}'\omega_3=0;\ \ \omega'_1.{}'\omega_2.\omega'_3.{}'\omega_4=0;\ \ \omega'_1.{}'\omega_2.{}'\omega'_3.\omega_4=0.$$

Since ω_1 and ω'_1 cannot both be zero together, λ cannot make ω'_1 or ω_1 zero; and because λ is an extreme to the roots of $\omega_2, \omega_3, \omega_4$. λ cannot make ω_2 or ω'_2 or ω_3 or ${}'\omega_3$ or ω'_4 or ω_4 zero, so that in fact when $x=\lambda$ none of the singly accented quantities ω can be zero. As regards the doubly accented quantities ω, the same thing cannot be affirmed, because if any Ω contains only one element the corresponding value of ω with a double accent vanishes spontaneously. Again, any of the unaccented quantities ω may vanish, because we may suppose any of these to have an extreme root λ. Consequently the first, second and fourth of the equations remaining to be satisfied, might be satisfied on making the necessary suppositions as to the form of the quantities ω and the values of the extreme roots; but the third remaining equation $\omega'_1.\omega_2.\omega'_3.{}'\omega_4=0$, in which only singly accented quantities ω occur, remains incapable of being satisfied on any supposition whatever. And the same thing would be true if we suppose λ to be a root of any other ω instead of ω_1. Hence λ cannot make $\omega=0$ when $e=4$.

In like manner, if e be any even number 2ε, there will be an equation

$$\omega'_1.{}'\omega_2.\omega'_3.{}'\omega_4.\omega'_5.{}'\omega_6\ldots\omega_{2\varepsilon-1}.{}'\omega_{2\varepsilon}=0$$

to be satisfied by that value (if it exist) of x which, besides being an extreme (on either side) of the roots of $\omega_1, \omega_2, \ldots \omega_{2\varepsilon}$ arranged in order of magnitude, also makes $\omega=0$. But as such equation cannot be satisfied, neither extreme root of the roots of

$\omega_1.\omega_2...\omega_{2\iota}$ can be a root of ω, as was to be proved. Consequently, unless ϕx is so assumed that the number of changes of sign in the coefficients of x in the quotients resulting from $\frac{\phi x}{fx}$ expanded as an improper continued fraction is even (for if the *changes* from sequence to sequence are odd the number of *sequences* themselves is even), the method of limitation in the text cannot give the means of drawing either limit indefinitely near to one or the other extreme roots of fx.

Art. (μ.). It now remains to prove the converse, and to show, 1st, that when the number of changes is even, *i. e.* the number of sequences odd, this coincidence can always be effected; and 2ndly, that it is always possible when fx has one or more real roots, so to assume ϕx that the number of sequences shall be odd.

The first part of the proposition is easily proved. Thus suppose $e=3$, so that

$$\omega=\omega_1.\omega_2.\omega_3-\omega_1'.'\omega_2.\omega_3-\omega_1.\omega_2'.'\omega_3+\omega_1.'\omega_2.'\omega_3.$$

If we suppose λ either extreme of the scale formed by writing in order of magnitude, the roots of ω_1, ω_2, ω_3 to be a root common to ω_1 and to ω_3 and if $'\omega_2'=0$, which last equation may be satisfied by supposing the type Ω_2 to consist of a single element, the separate equations

$$\omega_1.\omega_2.\omega_3=0 \quad \omega_1'.'\omega_2.\omega_3=0 \quad \omega_1.\omega_2.'\omega_3=0 \quad \omega_1.'\omega_2'.'\omega_3=0$$

will all be satisfied; and so in general it may be shown without difficulty that if $e=2\iota+1$, and if λ be a root common to $\omega_1=0$ $\omega_3=0$ $\omega_5=0...\omega_{2\iota+1}=0$, and if ω_2, ω_4, ... $\omega_{2\iota}$ be all *simple linear functions* of x, so that consequently $'\omega_2'=0$ $'\omega_4'=0...'\omega_{2\iota}'=0$, each separate term in the development of ω will vanish singly and separately, and consequently λ will be a root of ω: for since λ makes $\omega_1=0$ $\omega_3=0...\omega_{2\iota+1}=0$, every product in the developed form ω, in which ω_1, ω_3, ...$\omega_{2\iota+1}$ do not each bear at least one accent, will vanish; and if we consider any product in which ω_1, ω_3...$\omega_{2\iota+1}$ are all accented, if in any two of these immediately following one after the other as ω_{2k-1}, ω_{2k+1}, an accent falls to the right of the first, and to the left of the second, the intervening term ω_{2k} will bear a double accent, and will therefore vanish, since Ω_{2k} is supposed to be a linear function of x; but it is impossible when every ω is accented to prevent two accents of contiguous odd terms in any such product, from falling to the right of the left, and to the left of the right, term of the two, since the contrary would imply that all the accents would fall to the right, or all to the left, which, as above remarked, is impossible, on accoun of the two extreme terms being only simply accentable, *i. e.* ω_1 only to the right, and $\omega_{2\iota+1}$ only to the left. Hence, when x substituted for λ makes ω_1 ω_3...$\omega_{2\iota+1}$ all vanish, and when ω_2, ω_4, ... ω_{2k} are all linear functions of x, $x=\lambda$ will be a root of ω.

Art. (ν.). I believe that the remaining part of the proposition may be rigorously demonstrated, viz. that when any of the roots of fx are real, and the number of odd integers not exceeding the index of the degree of fx is m, and the number of imaginary pairs of roots in fx is μ, ϕx may be so assumed that the quotients to $\frac{\phi x}{fx}$ expanded under the form of an improper continued fraction, may be made to take the form $\Omega_1 : \Omega_2 ; \Omega_3 ; \Omega_4 ; \ldots ; \Omega_{2i+1}$, where $\Omega_2 ; \Omega_4 ; \ldots \Omega_{2i}$ are linear functions of x, and i is any number assumed at will, not less than μ, and of course not greater than m; and where $\omega_1 ; \omega_3 ; \ldots \omega_{2i+1}$ will have in common a root λ, which may be made at will the greatest or the least root of $\omega_1 . \omega_2 . \omega_3 . \omega_{2i+1}$; the investigation, however, according to the present light which I possess on the subject, appears complicated and tedious, and therefore, in order that the press, which is waiting for the completion of these supplemental articles, may not be kept standing, must be adjourned to some future occasion. For the present I content myself with showing the truth of the law for the simple case where fx is a cubic function of x.

1st. If $\frac{\phi x}{fx}$ gives rise to a single sequence of quotients Ω, we know, from the theory of intercalations, that it is necessary that all the roots of fx shall be real, and in order that when this is the case the quotients may form a single sequence Ω, it is only necessary so to assume ϕx, that its roots may be intermediate between those of fx.

2nd. If the roots of fx are not all real, or if they are all real, but do not compose the roots of fx intercalated between them, and if for greater brevity of ratiocination we stipulate that ϕx shall have its leading coefficients of the same sign as that of the leading coefficient of fx, the leading coefficients of the three quotients will either bear the respective signs $++-$, or the respective signs $+-+$, or the respective signs $+--$; in the first and last of these cases there would be two sequences, and therefore, by what has been shown above, the method of limitation of the text could not give a limit coincident with a root. Let us then look to the remaining case, and inquire whether, and how, ϕx may be assumed so that fx shall become representable to a constant factor *près* by the cumulant $[p(x-a), -q(x-\beta), r(x-a)]$, where p, q, r are all positive, and a is a root of fx.

Let this cumulant be called hfx.

Nothing in point of generality will be lost if we suppose the leading coefficient of hfx to be -1. We then have

$$hfx = [p(x-a), \ -q(x-\beta) ; \ r(x-a)]$$
$$= -pqr(x-a)^2(x-b)-(p+r)(x-a)$$

and writing $\frac{hfx}{x-a} = x^2 + Bx + C$ and making $x=a$, we find from the above identity that

$$p+r = a^2 + Ba + C, \ i.e. \ p = a^2 + Ba + C - r,$$

and $$pqr(x-\beta) = x + a + B,$$

hence $$\beta + a + B = 0, \ i.e. \ \beta = -B - a,$$

and $$pqr = 1, \text{ and } \therefore \ qr = \frac{1}{p} = \frac{1}{a^2 + Ba + C - r}.$$

Hence if ϕx be so assumed that the quotients to $\frac{\phi x}{fx}$ are $p(x-a)$; $-q(x-\beta)$; $r(x-a)$.

we have

$$h\phi x=[-q(x-\beta),\ r(x-a)]=-qr(x+B+a)(x-a)-1$$
$$=-qr(x^2+Bx-a^2-aB)-1=-\frac{1}{p}\{x^2+Bx-a^2-aB+p\}.$$

Hence $\phi(x)$ is of the form

$$m(x^2+Bx-a^2-aB+(a^2+aB+C-r))=m(x^2+Bx+C-r).$$

If we call the three roots of fx, a, b, c respectively, we have

$$q=\frac{1}{r(a^2+Ba+C-r)}=\frac{1}{r((a-b)(a-c)+r)};$$

and since q and r are both to be positive, we see that (a) must be taken the greatest or least of the three roots if they are all real, so that a^2+Ba+C may be positive, which it will of course necessarily be if b and c are imaginary; we must also have $a^2+Ba+C-r$ positive, so that the form of ϕx is $m((x^2-a^2)+B(x-a)-t)$, t being necessarily positive, but otherwise arbitrary, a form containing two arbitrary constants, one of which is subject to satisfy a certain condition of inequality; whereas when fx is of such a form as to *admit*, and $\phi(x)$ is supposed to be so assumed as to *cause it to come to pass* that the quotients to $\frac{\phi x}{fx}$ form a single sequence, then the three coefficients in ϕx remain exempt from all conditions of equality but are subject to two conditions of inequality. And so in general when the degree of fx is x and the number of sequences $2i+1$, it is to be inferred that the n coefficients of ϕx will be subject to satisfy $n-i-1$ conditions of inequality and i conditions of equality.

Art. (ξ.). The theory of the determination of the minimum interval between either limit determinable by this method and the nearest root, or between the two limits so determinable when ϕx is so assumed that $\frac{\phi x}{fx}$ gives rise to a defined even number of sequences (which will include the theory of the case where all the roots of fx are imaginary), must be deferred to an opportunity more favourable for leisurely contemplation. As regards the application of the theory to the very interesting case of all the roots being imaginary, the principal point remaining to be cleared up is the determination of the least value that can be assigned to the greatest, and the greatest value that can be assigned to the least root of the algebraical product $X_1.X_2.X_3\ldots X_{2n}$, where $X_1, X_2,\ldots X_{2n}$ are all of them real linear functions of x, subject to the condition that the cumulant $[X_1, X_2, X_3\ldots X_{2n}]$ shall (to a numerical factor *près*) be equal to a given function of the degree $2n$ in x incapable of changing its sign, which condition implies, as a necessary consequence, that the coefficients of x in each of the terms $X_1, X_2,\ldots X_{2n}$ must be affected with the same algebraical sign.

Art. (o.). It should be observed that in the application of the above method, the division of the series of quotients into distinct sequences governed by the signs of the coefficients of x is introduced for the purpose of drawing the limits closer to the roots, but is not necessary for the mere object of assigning limits.

Thus, for instance, if there be two sequences so that

$$[q_1 q_2 \dots q_i; \quad q_{i+1} q_{i+2} \dots q_{i+i'}]$$

$$q_1^2=\mu_1^2 \quad q_1^2=\left(\mu_2+\frac{1}{\mu_1}\right)^2 \quad q_3^2=\left(\mu_3+\frac{1}{\mu_2}\right)^2 \dots q_i^2=\left(\frac{1}{\mu_{i-1}}\right)^2$$

and $$q_{i+1}^2=\nu_1^2 \quad q_{i+2}^2=\left(\nu_2+\frac{1}{\nu_1}\right)^2 \dots q_{i+i'}=\left(\frac{1}{\nu_{i'-1}}\right)^2$$

the greatest and least roots of x deduced from these equations will be superior and inferior limits respectively to the roots of fx; from which it is clear that if leaving all the other equations unaltered, except those which contain respectively q_i^2 and q_{i-1}^2, we write in place of these

$$q_i^2=\left(\varrho+\frac{1}{\mu_{i-1}}\right)^2$$

$$q_{i+1}^2=\left(\frac{1}{\varrho}+\nu_1\right)^2$$

the roots of the system of $i+i'$ equations thus modified will *à fortiori* be limits to the roots of fx, but then the quantities

$$\mu_1, \mu_2+\frac{1}{\mu_1} \dots, \mu_{i-1}+\frac{1}{\mu_{i-2}}, \varrho+\frac{1}{\mu_{i-1}}, \nu_1+\frac{1}{\varrho}, \nu_2+\frac{1}{\nu_1}, \dots, \frac{1}{\nu_{i'-1}}$$

form the same single series as would correspond to the two sequences

$$q_1\, q_2 \dots q_i\, q_{i+1} \dots q_{i+i'}$$

treated as a single sequence, and the same is obviously the case for any number of sequences*.

Art. (π.). If we consider a single sequence as $q_1\, q_2 \dots q_3$, and write

$$q_1=a_1(x-c_1) \quad q_2=a_2(x-c_2) \dots q_n=a_n(x-c_n)$$

where $a_1, a_2, \dots, a_n$ are supposed to have all the same sign, and write

$$a_1^2(x-c_1)^2=\mu_1^2 \quad a_2^2(x-c_2)^2=\left(\mu_2+\frac{1}{\mu_1}\right)^2 \dots a_n^2(x-c_n)^2=\left(\frac{1}{\mu_{n-1}}\right)^2$$

* It follows from this, that if $q_1, q_2, \dots q_n$ be all linear functions of x, and if

$$Q=(q_1^2-\mu_1^2)\left(q_2^2-\left(\mu_2+\frac{1}{\mu_1}\right)^2\right)\left(q_3^2-\left(\mu_3+\frac{1}{\mu_2}\right)^2\right)\dots\dots\left(q_n^2-\frac{1}{\mu_{n-1}^2}\right),$$

no root of Q can lie between the extreme roots of the function K, used to denote the cumulant

$$[\sqrt{q_1^2}, \quad -\sqrt{q_2^2}, \quad \sqrt{q_3^2} \dots\dots\dots, \pm\sqrt{q_n^2}],$$

the square roots being understood to be taken so as to make the sign of the coefficients of x all of them positive; and from a preceding article we know that either extreme root of Q can be made to coincide with a corresponding extreme root of K. Hence we have an *à priori* solution of the following question. viz. "To determine the $(n-1)$ positive quantities $\mu_1, \mu_2, \dots \mu_{n-1}$, so as to make the greatest root of Q a minimum and its least root a maximum;" for the greatest root of K will be the minimum greatest root of Q, and the least root of K the maximum least root of Q. Calling these respectively l and λ, the two systems of values of $\mu_1, \mu_2, \dots \mu_{n-1}$ required will be obtained by substituting respectively l and λ for x in the equations

$$\mu_1=\sqrt{q_1^2} \quad \mu_2=-\sqrt{q_2^2}-\frac{1}{\mu_1} \quad \mu_3=+\sqrt{q_3^2}-\frac{1}{\mu_2} \dots\dots\dots \mu_{n-1}=\pm\sqrt{q_{n-1}^2}-\frac{1}{\mu_{n-2}}.$$

it seems not unlikely that the interval between the greatest and least of the roots of the above equations will be a minimum when the intervals between any pair is the same for each pair, *i. e.* when

$$\frac{\mu_1}{a_1}=\frac{\mu_2+\frac{1}{\mu_1}}{a_1}=\frac{\mu_3+\frac{1}{\mu_2}}{a_3}=\ldots=\frac{\frac{1}{\mu_{n-1}}}{a_n}.$$

If we assume these equations, and write $\mu_1=a_1\xi$, the equation for determining ξ, will be

$$[a_1\xi,\ a_2\xi,\ a_3\xi,\ldots,\ a_n\xi]=0.$$

If $n=2$ this equation becomes $a_1a_2\xi_2-1=0$.
If $n=3$, rejecting the factor ξ, it becomes

$$a_1a_2a_3\xi^2-(a_1+a_3)=0.$$

If $n=4$ it becomes

$$a_1.a_2.a_3.a_4\ \xi^4-(a_1.a_2+a_3.a_4+a_1.a_4)\xi^2+1=0.$$

If $n=5$, rejecting the factor ξ, it becomes

$$a_1.a_2.a_3.a_4.a_5\ \xi^4-(a_1.a_2.a_3+a_1.a_2.a_5+a_1.a_4.a_5+a_3.a_4.a_5)\xi^2+(a_1+a_3+a_5)=0,$$

and so in general the equation in ξ^2 being always of a degree measured by the integer nearest to and not exceeding $\frac{n}{2}$; and it is easy to be seen that for all values of n, the second coefficient divided by the first will be an inferior limit to ξ^2 (of course actually coinciding with it for the cases of $n=2$ and $n=3$). Hence we have the following valuable practical rule for finding a superior and inferior limit to the cumulant

$$[a_1(x-c_1),\ a_2(x-c_2),\ldots,\ a_n(x-c_n)],$$

where $a_1, a_2,\ldots a_n$ have the same sign, viz. if C be the greatest, and K be the least of the quantities $c_1, c_2,\ldots c_n$, $C+\Delta$ will be a superior, and $K-\Delta$ an inferior limit, Δ being taken equal to the positive value of

$$\sqrt{\frac{1}{a_1.a_2}+\frac{1}{a_2.a_3}+\frac{1}{a_3.a_4}+\cdots\frac{1}{a_{n-1}.a_n}};$$

and it may be noticed that C and K are the quantities which would themselves be the superior and inferior limits to the given cumulant if the series of terms $a_1, a_2,\ldots a_n$, instead of presenting only a sequence of continuations or permanencies, presented only a sequence of changes or variations of sign.

SECTION V.

On the Theory of Intercalations as applicable to two functions of the same degree, and on the formal properties of the Bezoutiant with reference to the method of Invariants.

Art. (56.). If fx and ϕx be any two given functions of x of the same degree m, we may form a system of m Bezoutics to f and ϕ (as shown in the first section), the coefficients of the powers of x^{m-1}, x^{m-2}, ... x^1 x^0 in which will compose a square matrix of m lines of m terms each, which will be symmetrical in respect to the diagonal which passes through the first coefficient of the first Bezoutic and the last coefficient of the last Bezoutic; and we may construct a quadratic homogeneous function of m new variables, such that its determinantive matrix shall coincide with the Bezoutic square so formed. This quadratic form may be considered in the light of a generating function. All its coefficients will be formed of quantities obtained by taking any two coefficients in one of the given functions, and two corresponding coefficients in the other given function, multiplying them in cross order, and taking the difference: each coefficient of the generating function in question will consist of one or more such differences, and will thus be of two dimensions altogether, being linear in respect to the coefficients of f, and also linear in respect to the coefficients of φ. This generating function I term the *Bezoutiant*, and it may be denoted by the symbol $B(f, \varphi)$: the determinant of B is of course the resultant to f, φ, and the matrix to B is the Bezoutic square to f, φ. Now we have seen that the decrease in the number of continuations of sign in the series 1, $B_1(x)$, $B_2(x) \ldots B_m(x)$ (where $B_1(x)$, $B_2(x) \ldots B_m(x)$ are the (n) Bezoutics to f, φ), as x changes from a to b, measures the number of roots of fx retained in the effective scale of intercalations taken between the limits (a) and (b). If we take the entire scale between $+\infty$ and $-\infty$ the total number of effective intercalations will be the same, whether reckoned by the number of roots of f or of φ remaining; for these two numbers can never differ except by a unit, since no two of either can ever come together; but the number of each remaining in the effective scale will be $m-2i$ and $m-2i'$ respectively, i being the number of pairs of imaginary roots and pairs of unseparated real roots of f and i' being the similar number for φ; so that we must have $i=i'$.

Now obviously this number becomes measured by the number of continuations of sign in the *signaletic* series 1, (B_1), (B_2), ... (B_m), where in general (B_r) denotes the principal coefficient in $B_r(x)$.

But (B_1), (B_2), ... (B_m) are the successive ascending coaxal minor determinants about the axis of symmetry to the Bezoutic square; and accordingly the number of continuations just spoken of, measures the number of positive terms in the Bezoutiant when linearly transformed, so as to contain only positive and negative squares, or in other words, measures the *inertia* of the Bezoutiant, the constant integer which adheres to it under all its real linear transformations.

Art. (57.). This inertia is the same number as in the case of a homogeneous quadratic function of three variables, used to express a curve referred to trilinear coordinates, serves to determine whether such conic belongs to the impossible class or to the possible class of conics, being 3 or 0 in the former case, and 1 or 2 in the latter; or as in the case of a homogeneous quadratic function of four variables used to denote a surface referred to quadriplanar or tetrahedral coordinates, serves to determine whether such surface belongs to the impossible class or to the class consisting of the ellipsoid and the hyperboloid of two sheets (which are *descriptively* indistinguishable), or to the hyperboloid of one sheet, being 0 or 4 in the first case, 1 or 3 in the second, and 2 in the third. The most *symmetrical* (but least expeditious) method of finding the *inertia* of any quadratic form is that which corresponds to the method of orthogonal transformations, and is, in fact, the usual method employed in geometrical treatises on lines and surfaces of the second degree. If we apply this method to the Bezoutiant B considered as a homogeneous quadratic function of the (m) arbitrarily named variables $u_1, u_2, u_3, \ldots u_m$ in order to measure its inertia, that is to say, the number of effective interpositions between the two systems of roots, we must construct the determinant

$$D(\lambda)=\begin{vmatrix} \frac{d^2.B}{du_1^2}+\lambda; & \frac{d^2B}{du_1.du_2}; & \frac{d^2B}{du_1.du_3}; & \cdots & \frac{d^2B}{du_1.du_m} \\ \frac{d_2B}{du_2.du_1}; & \frac{d^2B}{du_2^2}+\lambda; & \frac{d^2B}{du_2.du_3}; & \cdots & \frac{d^2B}{du_2.du_m} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{d^2B}{du_m.du_1}; & \frac{d^2B_x}{du_m.du_2}; & \frac{d^2B}{du_m.du_3}; & \cdots & \frac{d^2B}{du_m^2}+\lambda \end{vmatrix}$$

All the roots of $D(\lambda)=0$, as is well known, are real; the inertia of B, being measured by the number of positive roots of $D(-\lambda)$, will be equal to the number of continuations of sign in $D(\lambda)$ expressed as a function of λ of the mth degree.

If in fx and ϕx we reverse the order of the coefficients, and fx and ϕx so transformed become $f_1(x)$ and $\phi_1(x)$, it is obvious that the roots of f_1 and ϕ_1 being the reciprocals of the roots of f and ϕ respectively, the number of effective intercalations to f_1 and ϕ_1 must be the same as for f and ϕ. Accordingly we find that the form of the Bezoutiant to f and ϕ is the same as that of the Bezoutiant to f_1 and ϕ_1, the sole difference (one only of *names*) being that $B(u_1, u_2, \ldots u_{m-1}, u_m)$ for the one becomes $B(u_m, u_{m-1}, \ldots u_2, u_1)$ for the other. The equation $D(\lambda)$, which determines the *inertia* of B, remains precisely the same as it ought to do for either of the two systems f and ϕ or f_1 and ϕ_1.

Art. (58.). The theory in the preceding articles of this section may be made to embrace the case involved in STURM's theorem; for if

$$fx=a_0.x^n+a_1.x^{n-1}+\ldots+a_{m-1}.x^{m-1}+a_m.x^m$$

$$f'x=ma_0.x^{n-1}+(n-1)a_1.x^{n-2}+\ldots+a_{m-1},$$

and

$$f_1x=mfx-f'x$$

$$=a_1.x^{n-1}+2a_2.x^{n-2}+\ldots+m.a_m$$

the Bezoutian secondaries, or which is the same thing, the simplified Sturmian residues to fx and $f'x$, will evidently be the same as those to f_1x and $f'x$. Accordingly, if we form the signaletic series

$$fx_1 f'x,\ B_1,\ B_2\ldots B_{m-1},$$

where B_1, $B_2 \ldots B_{m-1}$ are the Bezoutian secondaries to f_1x and $f'x$, the number of *variations* of sign between consecutive terms in this series, when x is made $+\infty$, will measure the number of pairs of imaginary roots in fx; and fx and $f'x$ forming always a continuation, and the coefficient of $f'(x)$ being supposed positive, we see that the terms of the rhizoristic series will be 1, (B_1), $(B_2) \ldots (B_{m-1})$ consisting of positive unity, and the successive ascending coaxal determinants of the Bezoutian matrix to f' and f_1x. Hence then the form of the Bezoutiant to $f'x$ and f_1x will serve to determine the number of pairs of imaginary, and consequently also the number of real roots to fx. It should be remarked that the form of the Bezoutiant to $f'x$ and f_1x, considered as a quadratic function of u_1, $u_2 \ldots u_{m-1}$ and of the coefficients in $f(x)$, will remain unaltered when for fx we write f_1x, for this will change the signs throughout of fx and f_1x, and consequently the coefficients in the Bezoutiant, which contain in every term one coefficient from $f'x$, and one from f_1x, will remain unaltered in sign.

Art. (59.). It appears then from the preceding article, that for every function of x of the degree m, there exists a homogeneous quadratic function of $(m-1)$ variables, the *inertia* of which augmented by unity will represent the number of real roots in the given function. Now this *inertia* itself may be measured by the number of positive roots of a certain equation in λ formed from the quadratic function (in fact the well-known equation for the secular inequalities of the planets), all whose roots will be real. Hence then we are led to the following remarkable statement. "*An algebraical equation of any degree being given, an equation whose degree is one unit lower may be formed, all the roots of which shall be real, and of which the number of positive roots shall be one less than the total number of real roots of the given equation.*"

Let us suppose fx written in its most general form, the first and last as well as all the intermediate coefficients being anything whatever: by reversing the order of the coefficients $f'x$ will become f_1x and f_1x will become $f'x$; the Bezoutiant to f_1x and $f'x$ (which we may term the Bezoutoid to fx) will remain unaltered except in sign, and the equation of the $(m-1)$th degree in λ formed from the Bezoutoid remain unchanged, consequently the equation in λ enables us to substitute, for the purpose of calculating the total number of real roots in $f(x)$ in lieu of STURM's auxiliary functions to $f(x)$, another set of functions which remain unaltered when the order of the coefficients is completely reversed, *i. e.* in effect, when we consider the number of real roots of $f\left(\frac{1}{x}\right)$ in lieu of those of $f(x)$. And of course more generally the equation of the mth degree in λ formed from the Bezoutiant to any two functions fx and ϕx of the mth degree each in x, supplies a set of functions for determining the total number of effective intercalations between the roots of $f(x)$ and $\phi(x)$, which do not alter when we consider in lieu of these, the roots of $f\left(\frac{1}{x}\right)$ and $\phi\left(\frac{1}{x}\right)$. This substitution of functions symmetrically formed in respect to the two ends of an equation for the purpose of assigning the total number of real roots in lieu of the unsymmetrical ones furnished by the ordinary method of M. STURM, had been long felt by me to be a desideratum, and as an object the accomplishment of which was indispensable to the ulterior development of the theory, and it is certain that I did not in anticipation exaggerate the importance of the result to be attained.

Art. (60.). It may happen that the Bezoutiant to f and φ (each of the mth degree) may become a quadratic function of less than m independent variables, or the Bezoutoid to f (a function in x of the mth degree) of less than $(m-1)$ independent variables. This will take place whenever f and φ have roots in common, or whenever F has equal roots. The number of independent relations of equality between the roots of f and φ, and the amount of multiplicity, however distributed, among the roots of F, will be indicated by the number of *orders* thus disappearing out of the general form of the Bezoutiant and Bezoutoid in the respective cases*. In what particular mode the form of each would be affected according to the manner of the distribution of the equalities and the multiplicity requires a specific discussion, which I must reserve for some future occasion.

Art. (61.). I shall devote the remainder of this memoir to a consideration of the properties and affinities of Bezoutiants or Bezoutoids, regarded from the point of view of the Calculus of Invariants. For this purpose it will be more convenient hereafter to convert all the functions which we are concerned with into homogeneous forms, and I shall accordingly for the future use f and φ to denote functions each of x and y, which I shall write under the form

$$f=a_0.x^m+ma_1.x^{m-1}.y+m.\frac{m-1}{2}a_2.x_{m-2}.y^2+\ldots+a_m.x^m$$

$$\varphi=b_0.x^m+b_1.x^{m-1}.y+m.\frac{m-1}{2}b_2.x_{m-2}.y^2+\ldots+b_m.x^m.$$

In what follows a knowledge of the general principles of the Method of Invariants is presupposed, but a perusal of my two papers on the Calculus of Forms in the Cambridge and Dublin Mathematical Journal, February and May 1852, will furnish nearly all the information that is strictly necessary for the present purpose. The first point to be established is, that B, the Bezoutian of fx and φx, is a Covariant to the system f, φ; the variables in B being in compound relation of cogredience with the combinations of powers of x and y,

$$x^{m-1};\ x^{m-2}.y;\ x^{m-3}\ldots y^{m-1}.$$

That is to say, I propose to show that if f, g, h, k be any four quantities, taken for greater simplicity subject to the relation $fk-gh=1$, and if on substituting $fx+gy$ for x and $hx+ky$ for y, $f(x, y)$ becomes

$$A_0.x^m+mA_1.x^{m-1}.y+m.\frac{m-1}{2}A_2^1.x^{m-2}.y^2+A_m.y^m,\text{ say } G(x, y),$$

* I have elsewhere defined how this word order, as here employed, is to be understood. If F, a homogeneous function of $x_1, x_2, \ldots x_n$, can be expressed as a function of $u_1, u_2, \ldots u_{n-i}$ (all linear functions of $x_1, x_2, \ldots x_n$), F is said to be a function of $n-i$ orders, or to have lost i of the orders belonging to the complete form.

and $\phi(x, y)$ becomes

$$B_0.x^m+B_1.x^{m-1}.y+m.\frac{m-1}{2}B_2x^{m-2}.y^2+B_m.y^m, \text{ say } T(x, y),$$

and if $B'(u_1^1, u_2^1\ldots u_m^1)$ be the Bezoutiant to G and T; $B(u_1, u_2\ldots u_m)$ being that to f and ϕ, then, on making $u_1, u_2\ldots u_m$, the same *linear* functions of $u_1', u_2'\ldots u_m$

as $(fx+gy)^m$; $(fx+gy)^{m-1}(hx+ky)$; ......$(fx+gy)(hx+ky)^{m-1}$; $(hx+ky)^{m-1}$

are respectively of

$$x^m,\ x^{m-1}.y\ldots x.y^{m-1};\ y^m,$$

B will become identical with B′. I was led to suspect the high probability of the truth of this proposition concerning the invariance of the Bezoutiant from the following considerations: 1st. That for the particular case where f and ϕ are the differential derivatives in respect to x and y respectively of the same function $F(x, y)$, the Bezoutiant of f and ϕ, which then becomes the Bezoutoid of F, determines the number of real factors in F, which obviously remains the same for all linear transformations of F. 2ndly. That taking f and ϕ in their most general form, the invariant to their Bezoutiant, *i. e.* the determinant of their Bezoutiant is an invariant of f and ϕ, being in fact the resultant of these two functions; now as every concomitant (an invariantive form of the most general kind) to a concomitant is itself a concomitant to the primitive, so it appeared to me, and is I believe true (although awaiting strict proof), that any form satisfying certain necessary and tolerably obvious conditions of homogeneity and isobarism, a concomitant to which is also a concomitant to a given form, will be itself a concomitant to such form; this principle, if admitted, would be of course at once conclusive as to the Bezoutiant being an invariantive concomitant to the functions from which it is derived.

Art. (61*.). Since the publication of the two papers above referred to on the Calculus of Forms, I have made the important observation that every species of concomitant, however complex, to a given system of functions, may be treated as a simple invariant of a system including the given system together with an appropriate superadded system of absolute functions: thus an ordinary covariant involving only one system of variables, as $u, v, w\ldots$ cogredient with $x, y, z\ldots$ the variables of a system S, is in fact an invariant of the system S combined with the system $ux-vy$, $vz-wy$, $wx-uz$, &c., $u, v, w\ldots$ being treated as constants; so again a simple contravariant of S is an invariant of S combined with the equation $ux+vy+wz+$&c.; so again, to meet the case before us, a covariant to the binary system f and ϕ expressed as a function of $u_1, u_2\ldots u_m$, where $u_1, u_2\ldots u_m$ are cogredient with $x^{m-1}, x^{m-2}.y\ldots y^{m-1}$, may be regarded as an invariant of the ternary system f, ϕ, Ω, where

$$\Omega=u_1.y^{m-1}-mu_2.y^{m-2}x+m.\frac{m-1}{2}u_3.y^{m-3}x^2\ldots+(-)^{m-1}.u_{m-1}.x^{m-1},$$

($u_1, u_2, \ldots u_{m-1}$ being here to be treated as constants), and accordingly the differential equations which serve to define in the most general and absolute manner such covariant of f, φ, or invariant to f, φ, Ω, say I, will take the form

$$\left\{\begin{aligned}&\left\{\left(a_0.\frac{d}{da_1}+b_0.\frac{d}{db_1}\right)+2\left(a_1.\frac{d}{da_2}+b_1.\frac{d}{db_2}\right)+3\left(a_2.\frac{d}{da_3}+b_2.\frac{d}{db_3}\right)+\ldots+m\left(a^{m-1}\frac{d}{da_m}+b_{m-1}.\frac{d}{db_m}\right)\right.\\ &\left.\quad-\left(u_1\frac{d}{du_2}+2u_2.\frac{d}{du_3}+3u_3.\frac{d}{du_4}+\ldots+(m-1)u_{m-1}.\frac{d}{du_m}\right)\right\}\end{aligned}\right\}\mathrm{I}=0$$

$$\left\{\begin{aligned}&\left\{a_m.\frac{d}{da_{m-1}}+b_m.\frac{d}{db_{m-1}}\right\}+2\left\{a_{m-1}.\frac{d}{da_{m-2}}+b_{m-1}.\frac{d}{db_{m-2}}\right\}\\ &\qquad+3\left\{a_{m-2}.\frac{d}{da_{m-3}}+b_{m-2}.\frac{d}{db_{m-3}}\right\}+\ldots+m\left(a_1.\frac{d}{da_0}+b_1.\frac{d}{db_0}\right)\\ &\quad-\left(u_{m-1}.\frac{d}{du_{m-2}}+2u_{m-2}.\frac{d}{du_{m-3}}+3u_{m-3}.\frac{d}{du_{m-4}}\ldots+(m-1)a_2.\frac{d}{du_1}\right.\end{aligned}\right\}\mathrm{I}=0.$$

These equations may be proved to be satisfied when I is taken $=$B, the Bezoutiant to f, φ, and thus B may be proved to be a covariant to f, φ, but the demonstration is long and tedious. An admirable suggestion, well worthy of its keen-witted author, for which I am indebted to Mr. CAYLEY, will enable us to prove the invariantive character of B by a much more expeditious method.

Art. (62.). For greater simplicity begin with considering functions of a single variable x; and in order to fix the ideas, suppose (m) to be taken 5, and write

$$fx=ax^5+bx^4+cx^3+dx^2+ex+l$$
$$\varphi x=\alpha x^5+\beta x^4+\gamma x^3+\delta x^2+\varepsilon x+\lambda,$$

and let $\vartheta=\frac{fx\varphi x'-fx'.\varphi x}{x-x'}$; this is of course an integral function of x and x', since the numerator vanishes when $x=x'$; and we have by performing the actual operations,

$$\begin{aligned}\vartheta=&\left\{\begin{aligned}&(a\beta-b\alpha)x^4.x'^4+(a\gamma+c\alpha)x^3.x'^3(x+x')+(a\delta-d\alpha)x^2x'^2(x^2+xx'+x'^2)+(a\varepsilon-e\alpha)\\ &xx'(x^3+x^2x'+xx'^2+x'^3)+(a\lambda-l\varepsilon)(x^4+x^3x'+x^2x'^2+xx'^3+x'^4)\end{aligned}\right\}\\ &+\left\{\begin{aligned}&(b\gamma-c\beta)x^3x'^3+(b\delta-d\beta)x^2x'^2(x+x')+(b\varepsilon-e\beta)xx'(x^2+xx'+x'^2)\\ &+(b\lambda-l\beta)(x^3+x^2x'+xx'^2+x'^3)\end{aligned}\right\}\\ &+\left((c\delta-d\gamma)x^2x'^2+(c\varepsilon-e\gamma)xx'(x+x')+(c\lambda-l\gamma).(x^2+xx'+x'^2)\right)\\ &+\left((d\varepsilon-e\delta)xx'+(d\lambda-l\delta)(x+x')\right)\\ &+(e\lambda-l\varepsilon);\end{aligned}$$

and if we arrange ϑ under the form

$$\begin{aligned}&\mathrm{A}_{4,4}\,x^4.x'^4+\mathrm{A}_{4,3}\,x^4.x'^3+\mathrm{A}_{4,2}\,x^4x'^2+\mathrm{A}_{4,1}\,x^4.x'+\mathrm{A}_{4,0}\,.x^4\\ &+\mathrm{A}_{3,4}\,x^3x'^4+\mathrm{A}_{3,3}\,x^3x'^3+\mathrm{A}_{3,2}\,x^3x'^2+\mathrm{A}_{3,1}\,x^3x'+\mathrm{A}_{3,0}\,x^3\\ &+\mathrm{A}_{2,4}\,x^2.x'^4+\mathrm{A}_{2,3}\,x^2.x'^3+\mathrm{A}_{2,2}\,x^2x'^2+\mathrm{A}_{2,1}\,x^2x'+\mathrm{A}_{2,0}\,x^2\\ &+\mathrm{A}_{1,4}\,xx'^4+\mathrm{A}_{1,3}\,xx'^3+\mathrm{A}_{1,2}\,xx'^2+\mathrm{A}_{1,0}\,xx'+\mathrm{A}_{1,0}\,x\\ &+\mathrm{A}_{0,4}\,x'^4+\mathrm{A}_{0,3}\,x'^3+\mathrm{A}_{0,2}\,x'^2+\mathrm{A}_{0,1}\,x'+\mathrm{A}_{0,0};\end{aligned}$$

it will readily be perceived that the matrix formed by the twenty-five coefficients, viz.—

$$
\begin{array}{ccccc}
A_{4,4} & A_{4,3} & A_{4,2} & A_{4,1} & A_{4,0} \\
A_{3,4} & A_{3,3} & A_{3,2} & A_{3,1} & A_{3,0} \\
A_{2,4} & A_{2,3} & A_{2,2} & A_{2,1} & A_{2,0} \\
A_{1,4} & A_{1,3} & A_{1,2} & A_{1,1} & A_{1,0} \\
A_{0,4} & A_{0,3} & A_{0,2} & A_{0,1} & A_{0,0},
\end{array}
$$

will be symmetrical about its dexter diagonal (that one, namely, which passes through $A_{4,4}$ and $A_{0,0}$), and will be identical with the Bezoutian square corresponding to the system f, ϕ; in fact, using the notation previously employed in the first section, it becomes

$$
(\alpha.) \quad \dots\dots \quad
\begin{array}{ccccc}
(0,1) & (0,2) & (0,3) & (0,4) & (0,5) \\
(0,2) & \left\{\begin{array}{c}(0,3)\\+\\(1,2)\end{array}\right\} & \left\{\begin{array}{c}(0,4)\\+\\(1,3)\end{array}\right\} & \left\{\begin{array}{c}(0,5)\\+\\(1,4)\end{array}\right\} & (1,5) \\
(0,3) & \left\{\begin{array}{c}(0,4)\\+\\(1,3)\end{array}\right\} & \left\{\begin{array}{c}(0,5)\\+\\(1,4)\\+\\(2,3)\end{array}\right\} & \left\{\begin{array}{c}(1,5)\\+\\(2,3)\end{array}\right\} & (2,5) \\
(0,4) & \left\{\begin{array}{c}(0,5)\\+\\(1,4)\end{array}\right\} & \left\{\begin{array}{c}(1,5)\\+\\(2,3)\end{array}\right\} & \left\{\begin{array}{c}2,5\\+\\3,4\end{array}\right\} & (3,5) \\
(0,5) & (1,5) & (2,5) & (3,5) & (4,5),
\end{array}
$$

(r, s) being used in general to denote the difference between the cross products of the coefficients of x^{5-r} and x^{5-s} in f and ϕ. Restoring now to m its general value, and taking f and ϕ homogeneous functions of x and y, and making

$$\vartheta=\frac{f(x, y)\phi(x', y')-f(x', y')\phi(x, y)}{xy'-x'y},$$

we see without difficulty that

$$\vartheta=\Sigma A_{r,s}\{x^r.y^{m-1-r}x'^s.y^{m-1-s}\},$$

where $A_{r,s}$ is the term in the rth line and sth column of the Bezoutiant matrix to f and ϕ. This is the identification, the idea of which, as before observed, is due to Mr. Cayley.

Art. (63.). If, now, we consider the system of functions

$$f(x, y)=a_0.x^m+ma_1.x^{m-1}.y+\dots\dots\dots+a_m.y^m$$
$$\phi(x, y)=b_0.x^m+mb_1.x^{m-1}.y+\dots\dots\dots+b_m.y^m$$
$$\Omega(x, y)=u_{m-1}.y^{m-1}-(m-1)u_{m-2}.y^{m-2}\pm\dots\dots+(-)^{m-1}.u_1.x^{m-1},$$

evidently $f(x, y)\phi(x', y')-f(x', y')\phi(x, y)$ is a covariant with f and ϕ, and therefore (which is a mere truism) with the entire system f, ϕ, Ω. So also is $xy'-x'y$, and therefore ϑ, the quotient of these two, is a covariant to the system. Hence, therefore, by virtue of a general theorem given in my Calculus of Forms,

$$\Omega\left(\frac{d}{dy}, -\frac{d}{dx}\right)\vartheta$$

is a covariant to the system; and again, therefore,

$$\Omega\left(\frac{d}{dy'}, -\frac{d}{dx'}\right).\Omega\left(\frac{d}{dy}, -\frac{d}{dx}\right)\vartheta$$

is a covariant thereto. Now ϑ is of $(m-1)$ dimensions in x, y and also of the same in x', y'. Consequently this latter form will contain only the quantities $u_1, u_2, \ldots u_{m-1}$, and the coefficients of f and ϕ, so that the powers of x, y; x', y' will not appear in it.

Now $$\vartheta=\Sigma_{m-1}^{0}\Sigma_{m-1}^{0}\mathrm{A}_{r,s}\{x^r.y^{m-1-r}.x'^s.y'^{m-1-s}\}$$

$$(-)^{m-1}\Omega\left(\frac{d}{dy}, -\frac{d}{dx}\right)=u_{m-1}\left(\frac{d}{dx}\right)^{m-1}+(m-1)u_{m-2}\left(\frac{d}{dx}\right)^{m-2}\frac{d}{dy}+\ldots+u_1\left(\frac{d}{dy}\right)^{m-1}$$

$$(-)^{m-1}\Omega\left(\frac{d}{dy'}, -\frac{d}{dx'}\right)=u_{m-1}\left(\frac{d}{dx'}\right)^{m-1}+(m-1)u_{m-2}\left(\frac{d}{dx'}\right)^{m-2}\frac{d}{dy'}+\ldots+u_1\left(\frac{d}{dy'}\right)^{m-1}.$$

$$\therefore \frac{1}{1.2.3\ldots(m-1)^2}.\Omega\left(\frac{d}{dy'}, -\frac{d}{dx'}\right).\Omega\left(\frac{d}{dy}, -\frac{d}{dx}\right)\vartheta$$

$$=\Sigma_{m-1}^{0}(\mathrm{A}_{r,r}.u_r^2)+2\Sigma_{m-1}^{0}\Sigma_{m-1}^{0}(\mathrm{A}_{r,s}.u_r.u_s),$$

r and s being excluded in the latter sum from being made equal; but this latter expression is the Bezoutiant to f, ϕ. Hence the Bezoutiant of f, ϕ is an invariant to f, ϕ, Ω, *i.e.* a covariant to the system f, ϕ, as was to be proved. The mode of obtaining the covariant ϑ, used in this and the preceding article, is very remarkable. I believe that the true *suggestive* view of the process for finding it, is to consider

$$f(x, y).\phi(x', y')-f(x', y').\phi(x, y)$$

as a concomitant capable of being expressed under the form of a function of ϑ and ω, ω standing for the universal covariant $xy'-x'y$; ϑ is then to be considered, not properly as a quotient, but rather as an invariant of the form $\vartheta.\omega$, a function of ω of the first degree, where ϑ is treated as constant.

Art. (64.). B is not an ordinary covariant of f and ϕ, it belongs to that special and most important family of invariants to a system to which I have given the name of Combinants*, viz. Invariants, which, besides the ordinary character of invariance, when linear substitutions are impressed upon the variables, possess the same character of invariance when linear substitutions are impressed upon the functions themselves containing the variables; combinants being, as it were, invariants to a system of

* For some remarks on the Classification of Combinants, see Cambridge and Dublin Mathematical Journal, November, 1853.

functions in their corporate combined capacity *quâ* system. That the Bezoutiant possesses this property is evident; for if instead of f and φ we write $kf+i\varphi$ and $k'f+i'\varphi$, any such quantity as $a_r.b_s-a_s.b_r$ (a_r, b_r being coefficients in f, and a_s, b_s the corresponding ones in φ) becomes

$$(ka_r+ib_r)(k'a_s+i'b_s)-(ka_s+ib_s)(k'a_r+i'b_r),\ i.\ e.\ (ki'-k'i)(a_r.b_s-a_s.b_r),$$

so that B, the Bezoutiant, becomes increased in the ratio of $(ki'-k'i)^m$, *i.e.* remains always unaltered in point of form and absolutely immutable, provided that $ki'-k'i$ be taken, as we may always suppose to be the case, equal to 1.

We derive immediately from this observation, the somewhat remarkable geometrical proposition, that the intersections with the axis of x made by any two curves of the family of curves $u=\lambda f(x)+\mu\varphi(x)$, ($f$ and φ being functions of x of the same degree) give rise to a constant number of effective intercalations, whatever values be given to λ or μ for the two curves so selected.

Art. (65.). $B(u_1, u_2, \dots u_m)$ being a covariant of the system f and φ, and $u_1, u_2, \dots u_m$ cogredient with x^{m-1}, $x^{m-2}.y_0 \dots y^{m-1}$, it follows from a general principle in the theory of invariants, that on making $u_1, u_2, \dots u_m$ respectively equal to the quantities with which they are cogredient, B will become an ordinary covariant to f and φ. By this transformation B becomes a function of x and y of the degree $2(m-1)$ in x and y conjointly, and linear in respect to the coefficients of f, and also in respect to those of φ. The only covariant capable of answering this description is what I am in the habit of calling the Jacobian (after the name of the late but ever-illustrious JACOBI), a term capable of application to any number of homogeneous functions of as many variables. In the case before us, where we have two functions of two variables, the Jacobian

$$J(f,\varphi)=\begin{Bmatrix}\frac{df}{dx}; & \frac{d\varphi}{dx}\\ \frac{df}{dy}; & \frac{d\varphi}{dy}\end{Bmatrix}=\frac{df}{dx}.\frac{d\varphi}{dy}-\frac{df}{dy}.\frac{d\varphi}{dx}.$$

We have then the interesting proposition*, that the Bezoutiant to two functions, when the variables in the former are replaced by the combinations of the variables in the latter, with which they are cogredient, becomes the Jacobian†. So in the case of a single function F of the degree m, the Bezoutiant, *i. e.* the Bezoutoid to $\frac{dF}{dx}$, $\frac{dF}{dy}$, on making the $(m-1)$ variables which it contains identical with x^{m-2}; $x^{m-3}.y$; $\dots y^{m-2}$ respectively, becomes identical with the Jacobian to $\frac{dF}{dx}$, $\frac{dF}{dy}$, *i. e.* the Hessian of F, viz.

$$\begin{Bmatrix}\frac{d^2F}{dx^2}; & \frac{d^2F}{dxdy}\\ \frac{d^2F}{dxdy}; & \frac{d^2F}{dy^2}\end{Bmatrix}.$$

* I have subsequently found that this proposition is contained under another mode of statement, at the end of Section 2 of the Memoir of JACOBI, "De Eliminatione," above referred to.

† For a strict proof of this proposition see Supplement to Third Section of this memoir.

As an example of this property of the Bezoutiant, suppose

$$f=ax^3+bx^2y+cxy^2+dy^3$$
$$\varphi=\alpha x^3+\beta x^2y+\gamma xy^2+\delta y^3.$$

The Bezoutiant matrix becomes

$$\begin{array}{ccc} a\beta-b\alpha\,; & a\gamma-c\alpha\,; & a\delta-d\alpha \\ a\gamma-c\alpha\,; & \begin{array}{c} a\delta-d\alpha \\ + \\ b\gamma-c\beta \end{array}; & b\gamma-c\beta \\ a\delta-d\alpha\,; & b\gamma-c\beta\,; & c\delta-d\gamma. \end{array}$$

The Bezoutiant accordingly will be the quadratic function

$$(a\beta-b\alpha)u_1^2+(\overline{a\delta-d\alpha}+\overline{b\gamma-c\beta})u_2^2+\overline{c\delta-d\gamma}u_3^2$$
$$+2(a\gamma-c\alpha)u_1.u_2+2(a\delta-d\alpha)u_3.u_1+2(b\gamma-c\beta)u_2.u_3,$$

which on making

$$u_1=x^2 \quad u_2=xy \quad u_3=y^2,$$

becomes

$$Lx^4+Mx^3y+Nx^2y^2+Pxy^3+Qy^4, \quad \ldots \ldots \ldots \quad (\beta.)$$

where L, M, N, P, Q respectively will be the sum of the terms lying in the successive bands drawn parallel to the sinister diagonal of the Bezoutiant matrix, *i. e.*

$$L=a\beta-b\alpha$$
$$M=2(a\gamma-c\alpha)$$
$$N=3(a\delta-d\alpha)+(b\gamma-c\beta)$$
$$P=2(b\gamma-c\beta)$$
$$Q=c\delta-d\gamma.$$

The biquadratic function in x and y (β.) above written will be found on computation to be identical in point of form with the Jacobian to f, φ, viz.

$$(3ax^2+2bxy+cy^2)(\beta x^2+2\gamma xy+3\delta y^2)-(3\alpha x^2+2\beta xy+\gamma y^2)(bx^2+2cxy+dy^2),$$

this latter being in fact

$$3Lx^4+3Mx^3y+3Nx^2y^2+3Pxy^3+3Qy^4.$$

The remark is not without some interest, that in fact the Bezoutiant, which is capable (as has been shown already) of being mechanically constructed, gives the best and readiest means of calculating the Jacobian; for in summing the sinister bands transverse to the axis of symmetry the only numerical operation to be performed is that of addition of positive integers, whereas the direct method involves the necessity of numerical subtractions as well as additions, inasmuch as the same terms will be repeated with different signs. Thus if

$$f=ax^5+bx^4y+cx^3y^2+dx^2y^3+exy^4+ly^5$$
$$\varphi=\alpha x^5+\beta x^4y+\gamma x^3y^2+\delta x^2y^3+\varepsilon xy^4+\lambda y^5,$$

using (r, s) in the ordinary sense that has been considered throughout, we obtain by taking the sum of the sinister bands in (α.)* for the value of B when we write x^4, x^3y, x^2y^2, xy^3, y^4 in place of u_1, u_2, u_3, u_4, u_5,

$$(0, 1)x^8+2(0, 2)x^7y+\big(3(0, 3)+(1, 2)\big)x^6y^2+\big(4(0, 4)+2(1, 3)\big)x^5y^3$$
$$+\big(5(0, 5)+3(1, 4)+(2, 3)\big)x^4y^4+\big(4(1, 5)+2(2, 4)\big)x^3y^5+\big(3(2, 5)+(3, 4)\big)x^2y^6$$
$$+2(3, 5)xy^7+(4, 5)y^8.$$

The direct process requires the calculation of

$$(5ax^4+4bx^3y+3cx^2y^2+2dxy^0+ey^4)(\beta x^4+2\gamma x^3y+3\delta x^2y^2+4\varepsilon xy^3+5\lambda y^4)$$
$$-(5\alpha x^4+4\beta x^3y+3\gamma x^2y^2+2\delta xy^3+\varepsilon y^4)(bx^4+2cx^3y+3dx^2y^2+4exy^3+5ly^4),$$

each coefficient of which will contain the numerical factor 5; so that to reduce the Jacobian to its simplest form each coefficient will necessitate the employment of additions, subtractions, and a division, instead of additions merely, as when the Bezoutic square is employed. For instance, to find the coefficient of $x^4.y$ from the above expression (α.), we have to calculate

$$\frac{1}{5}\big(25(0, 5)+16(1, 4)+9(2, 3)+4(3, 2)+(4, 1)\big),$$

$$i.e.\ \frac{1}{5}\big(25(0, 5)+(16-1)(1, 4)+(9-4)(2, 3)\big),$$

which is $5(0, 5)+3(1, 4)+(2, 3)$, agreeing with what has been found above for the value of such coefficient, by a simple process of counting. The same remark will, of course, also apply to the computation of the Hessian of F by means of its Bezoutoid.

(Art. 66.). This relation between the Bezoutiant and the Jacobian led me to inquire whether, as would at first sight appear probable, the Bezoutiant were the only lineo-linear quadratic function of (m) variables covariantive to f and φ (the word lineo-linear being used to denote the form of coefficients, such as those in the Bezoutiant, linear in respect of the coefficients in f and the coefficients of φ). If so, then there would have existed a method of performing the inverse process of recovering the Bezoutiant from the Jacobian, almost as simple as that of deriving the Jacobian from the Bezoutiant. On investigating the matter, however, I found that such is by no means the case†, but that there exists a whole family of independent lineo-

* *Vide* art. 62.

† This might have been concluded immediately from the following observation. Let J, the Jacobian of f and φ, be expressed under the form

$$A_0x^{2m-2}+(2m-2)A_1.x^{2m-1}.y+(2m-2)\frac{2m-3}{2}A_2x^{2m-2}.y^2+\ldots+A_{2m-2}.y^{2m-2},$$

then we know from the Calculus of Forms, that, D being taken to represent the persymmetrical Determinant

$$\begin{matrix} A_0; & A_1; & A_2; & \ldots\ldots; A_{m-1} \\ A_1; & A_2; & A_3; & \ldots\ldots; A_m \\ A_2; & A_3; & A_4: & \ldots\ldots; A_{m+1} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ A_{m-1}; & A_m; & A_{m+1}; & \ldots\ldots; A_{2m-2}, \end{matrix}$$

linear quadratic covariants of m variables to every two homogeneous functions of x and y of the mth degree. I have, moreover, I believe, succeeded in determining the number of such lineo-linear quadratic forms for any value of (m), of which all the rest, in whatever manner obtained, may be expressed as linear functions, the coefficients of the linear relations moreover being abstract numbers; in other words, I have succeeded in forming the fundamental or constituent scale of lineo-linear quadratic forms of m variables covariantive to f and ϕ; a result of too great interest, as exhibiting the affinities of the Bezoutiant to its cognate forms, to be altogether passed over in silence. Supposing the number of linearly independent forms of the kind to be ν, then speaking *à priori* any of the forms taken at random might seem to be equally eligible to form one of the ν included in the fundamental scale, combined with any $(\nu-1)$ others independent *inter se*, and of which the selected one is also independent. In fact, however, this is not so; for it will always be more satisfactory to contemplate the fundamental scale of forms as generated successively or simultaneously by a uniform process; and in the case before us, the process which I have hit upon, and which I believe is the simplest that can be employed for generating the fundamental scale, will be found not to include directly the Bezoutiant among the number. There will thus arise two subjects of inquiry; 1st, the mode of forming the fundamental scale, and proving its fundamental

D=0 is the condition to be satisfied in order that J may be representable under the form of the sum of the squares of $(m-1)$ linear functions of x and y, and D itself is an invariant to J, and consequently an invariant and (as is obvious from its form) a combinantive invariant to f and ϕ. Moreover, which is more immediately to the point, we know that the quadratic form Q

$$\left(A_0u_1^2+2A_1(u_1\,.\,(m-1)u_2)+A_2\left\{((m-1)u_2)^2+2u_1\,.\left(\frac{(m-1)(m-2)}{2}\right)u_3\right\}+\&c.+A_{2m-2}u_m^2\right)$$

will be an invariant to f, ϕ and Ω (this last quantity Ω being defined as in p. 524), and a combinantive covariant to f and ϕ in the same sense precisely as the Bezoutiant is a covariant to the same, and like the Bezoutiant is lineo-linear in respect of the coefficients of f and ϕ. If we operate with the symbol E, where E represents

$$\frac{d}{dA_0}v_1^2+2\frac{d}{dA_1}\,.\,v_1v_2+\frac{d}{dA_2}(v_2^2+2v_1\,.\,u_3)+\&c.+\frac{d}{dA_{3m-2}}\,.\,v_m^2,$$

upon K any invariant of f and ϕ, we shall obtain E.K, a quadratic function of $v_1v_2\ldots v_m$, which by the rules of the Calculus of Forms we know will be a contravariant to f and ϕ, and the matrix corresponding to which must evidently be persymmetrical. It is an interesting subject of inquiry, which I reserve for some future occasion, to determine the Co-bezoutiant, the Discriminant of which must be employed for K, so that when this discriminant is operated upon by E, the matrix corresponding to E.K may become identical (term for term) with the matrix which is the inverse to the Bezoutiant matrix, which inverse, as Jacobi has so simply and beautifully demonstrated, possesses this persymmetrical character. *Vide* the "De Eliminatione," section 5. The investigation of the arithmetical connexion between the Q of this note and the fundamental Co-bezoutiants must be also similarly reserved. I believe it to be generally true, and have verified the fact for the case of two cubic functions, that E.Q gives a quadratic form such that the corresponding matrix is the inverse to the matrix of Q. The calculations necessary for extending the verification of this remarkable proposition for functions of x, y exceeding the third degree (notwithstanding that they are much abbreviated by the application of the rules of the calculus) still remain excessively laborious. The abbreviation alluded to consists in confining the verification in question to the comparison of either one of the two unreiterated terms at opposite corners of the matrix to E.Q with the corresponding term in the inverse matrix of Q; if these coincide, it is easy to prove that every other pair of corresponding terms in the two matrices must also coincide respectively with one another.

character; 2ndly, determining the numerical relations which connect that very important form, perhaps of all of its kind, the most important with the forms comprised in the fundamental or constituent scale. These questions I propose to consider more fully at a future period. For the present I shall content myself with giving a method of forming the constituent scale (without, however, seeking the proof of all the forms *extra* to such assumed scale being linear functions of these comprised within it), and with determining the numerical relations between the forms in this scale and the Bezoutiant for a limited number of values of m. All the forms which we are seeking, besides being lineo-linear quadratics, must also be combinantive invariants to f and ϕ, remaining (as forms) unaltered for any linear substitutions impressed either upon the variables or upon the functions containing the variables.

Art. (67.). I must here premise that if there be any two forms of the same degree (and that degree odd) in x and y, a combinant may be formed from them, which will be linear in respect to each set of coefficients*. Thus calling the two functions

$$a_0.x^{2n+1}+(2n+1)a_1.x^{2n}.y+(2n+1).\frac{2n}{2}a_2.x^{2n-1}.y^2+\ldots+a_{2n+1}.y^{2n+1}$$

$$\alpha_0x^{2n+1}+(2n+1)\alpha_1.x^{2n}.y+(2n+1)\frac{2n}{2}\alpha_2.x^{2n-1}.y^2+\ldots+\alpha_{2n+1}.y^{2n+1},$$

the lineo-linear combinant in question will be

$$\mathrm{T}=\left\{a_0.\alpha_{2n+1}-(2n+1).a_1.\alpha_{2n}+(2n+1)2n\ a_2.\alpha_{2n-1}+\frac{(2n+1)(2n)(2n-1)}{1.2.3}a_3\alpha_{2n-2}\ \&\text{c.}-a_{2n+1}.\alpha_0\ \&\text{c.}\right\}$$

which, using our customary notation, will be of the form

$$(0,2n+1)-(2n+1)(1,2n)+\frac{(2n+1)2n}{1.2}(2,2n-1)\pm\&\text{c.}+(-)^n.\frac{(2n+1)(2n)(2n-1)\ldots(n+2)}{1.2.3\ldots n}(n,n+1).$$

As a corollary to this proposition (which, as well as the proposition itself, will be needed for the purposes of the ensuing determination), taking any function of an even degree in x, y, $\mathrm{F}(x,y)$, there will exist a combinant to $\frac{d\mathrm{F}}{dx}$ and $\frac{d\mathrm{F}}{dy}$, by virtue of what has been stated above, which will be Mr. CAYLEY's well-known quadrivariant to F; viz. if $\mathrm{F}=a_0.x^{2n}+a_1.x^{2n-1}+\ldots+a_{2n}.x^{2n}$, this will be

$$a_0.a_{2n}-2na_1.a_{2n}+\frac{2n(2n-1)}{2}a_2.a_{2n-2}+\ldots+\frac{1}{2}(-)^n.\frac{2n(2n-1)\ldots(n+1)}{1.2\ldots n}a_n^2.$$

The proposition itself is easily proved; first, the expression T being expressed entirely in terms of quantities of the form (r,s) remains unaltered for linear substitutions impressed upon the forms f and ϕ; it remains then only to show that T satisfies the differential equations to T treated as a mere invariant, viz.—

* I may add here incidentally (although not wanted for our present purposes) that as a combinant in which each set of coefficients enters linearly can always be formed to a system of functions 2 in number of as many variables and of any odd degree, so reciprocally can a combinant in which each set of coefficients enters linearly be always formed to a system of functions each of the degree 2, of which and of the variables contained in them, the number is any odd integer.

$$\left\{\begin{array}{l} a_0.\frac{d}{da_1}+2a_1.\frac{d}{da_2}+3a_2.\frac{d}{da_3}+\ldots+(2n+1)a_{2n}.\frac{d}{da_{2n+1}} \\ +\alpha_0.\frac{d}{d\alpha_1}+2\alpha_1.\frac{d}{d\alpha_2}+3\alpha_2.\frac{d}{d\alpha_3}+\ldots+(2n+1)\alpha_{2n}.\frac{d}{d\alpha_{2n+1}} \end{array}\right\}T=0,$$

and

$$\left\{\begin{array}{l} a_{2n+1}.\frac{d}{da_{2n+1}}+2a_{2n}.\frac{d}{da_{2n-1}}+\ldots+(2n+1)a_1.\frac{d}{da_0} \\ +\alpha_{2n+1}.\frac{d}{d\alpha_{2n+1}}+2\alpha_{2n}.\frac{d}{d\alpha_{2n-1}}+\ldots+(2n+1)\alpha_1.\frac{d}{d\alpha_0} \end{array}\right\}T=0.$$

From the hemihedral symmetry of T, which only changes its sign when the order of the coefficients in f and φ is simultaneously reversed, it is obvious that one of these equations cannot be satisfied without the other being so too. Looking then exclusively at the first of them, we see that this is satisfied by virtue of the equations

$$\left\{a_0.\frac{d}{da_1}+(2n+1)\alpha_{2n}\frac{d}{d\alpha_{2n+1}}\right\}T=0$$

$$\left\{2a_1.\frac{d}{da_2}+2n.\alpha_{2n-1}.\frac{d}{d\alpha_{2n}}\right\}T=0$$

. . . .

. . . .

$$\left\{(2n+1)a_{2n}\frac{d}{da_{2n+1}}+\alpha_0.\frac{d}{d\alpha_1}\right\}T=0.$$

Hence then the differential equations to T being satisfied proves that it is an invariant, and, as above observed, its form shows upon its face that it is a combinant.

Precisely in the same way it may be demonstrated, that to two functions each of the same even degree $(2m)$ as

$$a_0x^{2m}+2ma_1.x^{2m-1}y+\frac{2m.(2m-1)}{2}a_2.x^{2m-2}y^2+\ldots+a_{2m}.y^{2m}$$

and $\alpha_0.x^{2m}+2m\alpha_1.x^{2m-1}.y+2m.\frac{(2m-1)}{2}\alpha_2.x^{2m-2}.y^2+\ldots+\alpha_{2m}.y^{2m}$

there will be a quantity

$$G=a_0.\alpha_{2m}-2ma_1.\alpha_{2m-1}+\frac{2m.(2m-1)}{2}a_2.\alpha_{2m-2}\pm\&c.\ -2m\alpha_1a_{2m-1}+\alpha_0.a_{2m},$$

which, although not a combinant, will satisfy the differential equations necessary to prove it to be an ordinary invariant to the two given functions.

Art. (68.). Now let us consider the three forms f, φ and the subsidiary form

$$f=a_0x^m+ma_1.x^{m-1}.y+\ldots+a_m.y^m$$

$$\varphi=b.x^m+mb_1.x^{m-1}.y+\ldots+b_m.y^m$$

$$\Omega=u_1.y^{m-1}-(m-1)u_2.y^{m-2}.x\pm\&c.\ +(-)a_m^{m-1}x^{m-1},$$

where $u_1, u_2, \ldots u_m$ are to be treated as constants.

Make
$$E_{2i+1}.f=\frac{1.2\ldots(2i+1)}{m(m-1)\ldots(m-2i)}\left(\xi\frac{d}{dx}+\eta\frac{d}{dy}\right)^{2i+1}.f$$

$$E_{2i+1}.\varphi=\frac{1.2\ldots(2i+1)}{m(m-1)\ldots(m-2i)}\left(\xi\frac{d}{dx}+\eta\frac{d}{dy}\right)^{2i+1}.\varphi,$$

i being any integer such that $2i+1$ does not exceed m, and now consider $E_{2i+1}.f$, $E_{2i+1}.\phi$ as two functions of the degree $2i+1$ in ξ, η (x and y being regarded as constants); and by virtue of the formula in the last article, form T_i, the lineo-linear combinant of $E_{2i+1}.f$ and $E_{2i+1}.\phi$; T_i will then be lineo-linear in respect to the coefficients in f and ϕ, and of the degree $2\big(m-(2i+1)\big)$ in respect to x and y. Again, let

$$E_i.\Omega=\frac{1.2...2i}{m(m-1)...(m-2i+1)}.\left(\xi\frac{d}{dx}+\eta\frac{d}{dy}\right)^{2i}.\Omega.$$

$E_{2i}.\Omega$ treated as a function of ξ and η of the degree $2i$ will furnish a quadrinvariant Q_i of the degree $2(m-1-2i)$ in respect of x and y, and quadratic in respect of the system $u_1, u_2, \dots u_m$. We have thus two forms, T_i and Q_i, each of the same even degree $\big(2m-(2i+1)\big)$ in respect of x, y. Forming between these the lineo-linear invariant G_i, G_i will be a function lineo-linear in respect of the coefficients of f and ϕ, and quadratic in respect of the system $u_1, u_2, \dots u_m$. Moreover, G_i will (by the general principle of successive concomitance) be an invariant in respect to the system f, ϕ, Ω, and combinantive in respect to f and ϕ. Thus then G_i for all admissible values of i will belong to the family of forms to which the Bezoutiant is to be referred.

It requires to be noticed, that when i is taken (0), so that T_i and G_i are of the degree $2(m-1)$, E_i for this case must be taken equal to Ω^2, which evidently fulfills the required conditions of being of the degree $2(m-1)$ in (x, y), and quadratic in respect of the coefficients of Ω. If, now, m be even, we may take for $2i+1$ successively all the odd numbers from 1 to $(m-1)$ inclusively, and there will be $\frac{m}{2}$ forms G_i; when m is odd we may take for $2i+1$ successively all the odd numbers from 1 to m, and the number of forms of G_i will be $\frac{m+1}{2}$. It should be observed, that when m is odd and $2i+1=m$, T_i will become identical with the lineo-linear combinant to f and ϕ and Q_i with the quadrinvariant to Ω; and no power of x or y will enter into either, so that G_m will become simply $T_m\times Q_m$. I am now able to enunciate the proposition, that $G_0, G_1, \dots G_{\frac{m}{2}-1}$, when m is even, and $G_0, G_1, \dots G_{\frac{m-1}{2}}$, when m is odd, form the constituent scale of forms, of which the Bezoutiant and all other lineo-linear quadratic functions of m variables, which are combinants of the system f, ϕ, will be numerically-linear functions. I propose to term the members of this scale Co-bezoutiants.

As regards the present memoir, I shall content myself with exhibiting a partial verification of this law as regards the connection of the Bezoutiant with the G scale of Co-bezoutiants, and a complete determination of the numerical multipliers which express this connection for the cases comprised between $m=2$ and $m=6$ taken inclusively. It is impossible to predict for what ulterior purposes in the development of the Calculus of Invariants these numbers may or may not be required, and it seems to me desirable that a commencement of a table containing them should be made and placed on record. The remaining pages of this memoir will accordingly be devoted to the ascertainment of them.

The theory of the Bezoutoid being included within that of the Bezoutiant, need not hereafter call for any special attention; I may merely notice that the Bezoutoid to a function of the degree (m) will be a numerico-linear function of $\frac{m-3}{2}$ of the G's if m be odd, and $\frac{m-4}{2}$ of the G's if m be even.

It will be more convenient hereafter to denote the G's as G_1, G_3, G_5 respectively, in lieu of G_0, G_1, G_2, &c., and to continue at the same time to give to the T's and Q's the same subscripts as the corresponding G's.

Art. (69.). 1st. Suppose $m=2$,

$$f=ax^2+2bxy+cy^2$$
$$\phi=\alpha x^2+2\beta xy+\gamma y^2$$
$$\Omega=u_1.y-u_2.x.$$

Then

$$E_1.f=(ax+by)\xi+(bx+cy)\eta$$
$$E_1.\phi=(\alpha x+\beta y)\xi+(\beta x+\gamma y)\eta$$
$$T_1=(ax+by)(\beta x+\gamma y)-(bx+cy)(\alpha x+\beta y)$$
$$=(a\beta-b\alpha)x^2+(a\gamma-c\alpha)xy+(b\gamma-c\beta)y^2$$
$$Q_1=\Omega^2=u_1^2y^2-2u_1.u_2xy+u_2^2.x^2$$

and $\therefore\ G_1=(a\beta-b\alpha)u_1^2+(a\gamma-c\alpha)u_1u_2+(b\gamma-c\beta)u_2^2.$

Let us now form in the usual manner the Bezoutiant to f, ϕ; this is the quadratic function which corresponds to the matrix

$$\left.\begin{matrix}(2a\beta-2b\alpha); & (a\gamma-c\alpha)\\ (a\gamma-c\alpha); & (2b\gamma-c\beta)\end{matrix}\right\},$$

$$i.e.\ \tfrac{1}{2}B=(a\beta-b\alpha)u_1^2+(a\gamma-c\alpha)u_1.u_2+(b\gamma-c\beta)u_2^2=G_1 \text{ or } B=2G_1.$$

2nd. Suppose $m=3$.

$$f=ax^3+3bx^2y+3cxy^2+dy^3$$
$$\phi=\alpha x^3+3\beta x^2y+3\gamma xy^2+\delta y^3$$
$$\Omega=u_1y^2-2u_2.yx+u_3.x^2.$$

We have then

$$E_1.(f)=(ax^2+2bxy+cy^2)\xi+(bx^2+2cxy+dy^2)\eta$$
$$E_1.(\phi)=(\alpha x^2+2\beta xy+\gamma y^2)\xi+(\beta x^2+2\gamma xy+\delta y^2)\eta$$
$$T_1=(ax^2+2bxy+cy^2)(\beta x^2+2\gamma xy+\delta y^2)-(bx^2+2cxy+dy^2)(\alpha x^2+2\beta xy+\gamma y^2)$$
$$=(a\beta-b\alpha)x^4+2(a\gamma-c\alpha)x^3y+\big(3(\beta\gamma-c\beta)+(a\delta-d\alpha)\big)x^2y^2+2(b\delta-d\beta)x^2y^2+(c\delta-d\gamma)y^4$$
$$Q_1=\Omega^2=u_1^2y^4-4u_1.u_2.y^3u+(4u_2^2+2u_1.u_3)y^2x^2-4u_2.u_3yx^0+u_3^2.x^4.$$

Supplying for facility of computation the reciprocals of the binomial coefficients to the index 4, viz.—

$$1; \quad -\frac{1}{4}; \quad \frac{1}{6}; \quad -\frac{1}{4}; \quad 1,$$

we obtain

$$G_1=(a\beta-b\alpha)u_1^2+2(a\gamma-c\alpha)u_1.u_2+\left(2(b\gamma-c\beta)+\frac{2}{3}(a\delta-d\alpha)\right)u_2^2$$

$$+\left((b\gamma-c\beta)+\frac{1}{3}(a\delta-d\alpha)\right)u_1.u_3+2(b\delta-d\beta)u_2.u_3+(c\delta-d\gamma)u_3^2.$$

It will here and henceforth be more useful to employ $[r, s]$ to denote, not the difference of the cross products of the $(r+1)$th and $(s+1)$th *entire* coefficients in f and φ, but the difference of the cross products of these coefficients divided each by its appropriate binomial coefficient. We may then write

$$G_1=[0, 1]u_1^2+2[0, 2]u_1.u_2+([1, 2]+\frac{1}{3}[0, 3])u_1.u_3+(2[1, 2]+\frac{2}{3}[0, 3]).u_2^2$$
$$+2[1, 3]u_2.u_3+[2, 3]u_3^2.$$

Again,

$$G_3=\{(a\delta-d\alpha)-3(b\gamma-c\beta)\}+(u_1.u_3-u_2^2)=([0, 3]-3[1, 2])(u_1u_3)-([0, 3]-3[1, 2])u_2^2.$$

Hence

$$G_1-\frac{1}{3}G_3=[0, 1]u_1^2+2[0, 2]u\ .u_2+2[1, 2]u_1.u_3+([0, 3]+[1, 2])u_2^2+2[1, 3]u_2.u_3+[2, 3]u_3^2.$$

But, again, the Bezoutiant of f, φ corresponds to the matrix

$$\begin{matrix} 3[0, 1]; & 3[0, 2]; & [0, 3] \\ 3[0, 2]; & [0, 3]+9[1, 2]; & 3[1, 3] \\ [0, 3]; & 3[1, 3]; & [3, 4]. \end{matrix}$$

Hence summing the sinister bands to form the coefficients, we have

$$B=3[0, 1]u_1^2+6[0, 2]u_1^2.u_2+(3[0, 3]+9[1, 2])u_2^2+6[1, 3]u_2.u_3+[2, 3]u_3^2=3G_1-G_3.$$

3rd. Suppose $m=4$,

$$f=ax^4+4bx^3y+6cx^2y^2+4dxy^3+ey^4$$
$$\varphi=\alpha x^4+4\beta x^3y+6\gamma x^2y^2+4\delta xy^3+\varepsilon y^4$$
$$\Omega=u_1y^3-3u_2y^2x+3u_3yx^2-u_4.x^3.$$

Then

$$E_3f=(ax+by)\xi^3+3(bx+cy)\xi^2\eta+3(cx+dy)\xi\eta^2+(dx+ey)\eta^3,$$

$$\therefore \quad T_3f=\begin{Bmatrix}(ax+by)(\delta x+\varepsilon y) \\ -(\alpha x+\beta y)(dx+ey)\end{Bmatrix}-3\begin{Bmatrix}(bx+cy)(\gamma x+\delta y) \\ -(\beta x+\gamma y)(cx+dy)\end{Bmatrix}$$
$$=([0, 3]-3[1, 2])x^2+([0, 4]-2[1, 3])xy+([1, 4]-3[2, 3])y^2$$

and

$$Q_3=(u_1.y-u_2x)(u_3y-u_4x)-(u_2y-u_3x)^2$$
$$=(u_1.u_3-u_2^2)y^2(u_1.u_4-u_2u_3)xy+(u_2.u_4-u_3^2)x^2.$$

Hence supplying the binomial reciprocals

$$1\ ;\ -\frac{1}{2}\ ;\ 1,$$

we have

$$G_3=([0, 3]-3[1, 2])(u_1.u_3-u_2^2)+\frac{1}{2}([0, 4]-2[1, 3])(u_1.u_4-u_2.u_3)$$
$$+([1, 4]-3[2, 3])(u_2.u_4-u_3^2).$$

Again,

$$T_3=(ax^3+3bx^2y+3cxy^2+dy^3)(\beta x^3+3\gamma x^2y+3\delta xy^2+\varepsilon y^3)$$
$$-(\alpha x^3+3\beta x^2y+3\gamma xy^2+\delta y^3)(bx^3+3cx^2y+3dxy^2+ey^3)$$
$$=[0, 1]x^6+3[0, 2]x^5y+(3[0, 3]+6[1, 2])x^4y^2+([0, 4]+8[1, 3])x^3y^3$$
$$+(3[1, 4]+6[2, 3])x^2y^4+3[2, 4]xy^5+[3, 4]y^6,$$

and $Q_1=\Omega^2$

$$=u_1^2.y^6-6u_1.u_2.x^5y+(9u_2^2+6u_1.u_3)y^4x^2-(2u_1.u_4+18u_2.u_3)x^3y^3$$
$$+(9u_3^2+6u_2.u_4)y^2x^4-6u_3.u_4.yx^5+u_4^2.x^6.$$

Hence, supplying the reciprocal binomial coefficients,

$$1\ ;\ -\frac{1}{6}\ ;\ +\frac{1}{15}\ ;\ -\frac{1}{20}\ ;\ \frac{1}{15}\ ;\ -\frac{1}{6};\ 1,$$

we find

$$G_1=[0, 1]u_1^2+3[0, 2]u_1.u_2+\left(\frac{1}{5}[0, 3]+\frac{2}{5}[1, 2]\right)(9u_2^2+6u_1.u_3)$$
$$+\left(\frac{1}{10}[0, 4]+\frac{8}{10}[1, 3]\right)(u_1.u_4+9u_2.u_3)+\left(\frac{1}{5}[1, 4]+\frac{2}{5}[2, 3]\right)\times(9u_3^2+6u_2.u^4)$$
$$+3[2, 4]u_3.u_4+[3, 4]u_4^2.$$

Now the Bezoutic square, taking account of the binomial factors in f and φ, may be written under the form

$$\begin{array}{cccc}
4[0, 1]\ ; & 6[0, 2]\ ; & 4[0, 3]\ ; & [0, 4] \\
6[0, 2]\ ; & \begin{bmatrix} 4[0, 3] \\ +24[1, 2] \end{bmatrix}; & \begin{bmatrix} [0, 4] \\ +16[1, 3] \end{bmatrix}; & 4[1, 4] \\
4[0, 3]\ ; & \begin{bmatrix} [0, 4] \\ +16[1, 3] \end{bmatrix}; & \begin{bmatrix} [1, 4] \\ +24[2, 3] \end{bmatrix}; & 6[2, 4] \\
[0, 4]\ ; & 4[1, 4]\ ; & 6[2, 4]\ ; & [3, 4].
\end{array}$$

Hence the Bezoutiant B becomes

$$4[0, 1]u_1^2+12[0, 2]u_1.u_2+(4[0, 3]+24[1, 2])u_2^2+2[0, 4]u_1u_4$$
$$+(2[0, 4]+32[1, 3])u_2.u_3+8[1, 4]u_2.u_4+([1, 4]+24[2, 3])u_3^2$$
$$+12[2, 4]u_3.u_4+[3, 4]u_4^2.$$

And we ought to have $B=cG_1+eG_3$, to satisfy which equation we must manifestly have $c=4$; to find (e), compare the coefficients of u_2^2, this gives

$$4[0,3]+24[1,2]=\frac{36}{5}[0,3]+\frac{72}{5}[1,2]+e(3[1,2]-[0,3]);$$

accordingly we ought to be able to satisfy the two equations

$$\frac{36}{5}-e=4 \quad \frac{72}{5}+3e=24,$$

each of which accordingly we find is satisfied by the equality $e=\frac{16}{5}$.

Substituting in the equation for B above written, we thus obtain

$$B=4G_1+\frac{16}{5}G_3,$$

which will be found to be identically true.

Art (70.). We may now see our way to a more concise mode of obtaining the numerical coefficients [by which they may in fact be computed and verified with comparatively little labour], connecting the Bezoutiant with the co-bezoutiant forms of the constituent scale. It will not fail to have been remarked, that throughout the preceding determinations I have presumed the truth of the formulæ which admits of an immediate verification, that for all values of m and ω we have the identical equation

$$\left(\xi\frac{d}{dx}+\eta\frac{d}{dy}\right)^{\omega}\cdot\left\{c_0x^m+mc_1x^{m-1}y+m.\frac{m-1}{2}c_2x^{m-2}y^2+\ldots+mc_{m-1}yx^{m-1}+c_m.x^m\right\}$$
$$=\frac{(m.m-1)\ldots(m-\omega+1)}{1.2\ldots\omega}\left\{L_0.\xi^{\omega}+\omega L_1.\xi^{\omega-1}.\eta+\omega.\frac{\omega-1}{2}L_2.\xi^{\omega-2}.\eta^2+\ldots+L_{\omega}.\eta^{\omega}\right\},$$

where

$$\left.\begin{array}{l}
L_0=c_0.x^{m-\omega}+(m-\omega)c_1.x^{m-\omega-1}.y+(m-\omega)\frac{m-\omega-1}{2}c_2.x^{m-\omega-2}.y^2\ldots+c_{m-\omega}.y^m \\
L_1=c_1.x^{m-\omega}+(m-\omega)c_2.x^{m-\omega-1}.y+(m-\omega)\frac{m-\omega-1}{2}c_2.x^{m-\omega-2}.y^2+\ldots+c_{m-\omega+1}.y^m \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
L_{\omega}=c_{\omega}.x^{m-\omega}+(m-\omega)c_{\omega+1}x^{m-\omega-1}.y+(m-\omega)\frac{m-\omega-1}{2}c_2.x^{m-\omega-2}.y^2\ldots+c_m.y^m
\end{array}\right\}$$

Let us now proceed to determine by an abridged method the linear relations corresponding to the cases of $m=5$, $m=6$, and first for $m=5$.

Let

$$f=ax^5+5bx^4y+10cx^0y^2+16dx^2y^3+5exy^4+hy^5$$
$$\phi=\alpha x^5+5\beta x^4y+10\gamma x^3y^2+10\delta x^2y^3+5\varepsilon xy^4+\eta y^5$$
$$\Omega=u_1.y^4-4u_2.y^3x+6u_3.y^2x^2-4u_4.yx^3+u_5.y^4.$$

In forming G_5, G_3, G_1, let us confine our attention to the terms u_1^2; $u_1.u_3$; $u_1.u_4$.

A comparison of the coefficients of these with those in the Bezoutiant (B) will be sufficient for assigning the three numerical quantities which connect B with G_1, G_3, G_5. I omit $u_1.u_2$, because G_1 is the only one of the G's for any value of (m) which contains u_1^2 or $u_1.u_2$, and in G_1 the terms containing u_1^2 and $u_1.u_2$ are

$$[0, 1]u_1^2+(m-1)[0, 2].u_1.u_2,$$

and the corresponding part of the Bezoutiant is

$$m[0, 1]u_1^2+m.(m-1)[0, 2]u_1.u_2;$$

so that if we write

$$B=c_1.G_1+c_3.G_3+c_5.G_5+\&c.,$$

the two terms u_1^2 and $u_1.u_2$ will only enable us to form one equation with the c's, viz. $c_1=m$. Again, instead of considering the entire coefficients of $u_1.u_3$ and $u_1.u_4$, it will be sufficient to take a single argument of either of these coefficients (in the forms to be compared), as for instance [0, 3] and [1, 3]. Then c_1 being known, c_3, c_5 will be determined; but for the purposes of verification I shall furthermore compute the whole of the coefficient of $u_1.u_5$.

Accordingly [calculating the G system in reverse order] we have

$$G_5=\{[0, 5]-5[1, 4]+10[2, 3]\}\{u_1.u_5-4u_2.u_4+.3u_3^2\}$$
$$=\{[0, 5]-5[1, 4]+10[2, 3]\}u_1.u_5+\ldots$$

$$E_3.f=(ax^2+2bxy+cy^2)\xi^3+3(bx^2+2cxy+dy^2)\xi^2\eta+3(cx^2+2dxy+ey^2)\xi\eta^2+(dx^2+2exy+fy^2)\eta^3;$$

$E_3.\phi=$ &c. &c.;

$$\therefore\quad T_3=\{(ax^2+2bxy+cy^2)(\delta x^2+2\varepsilon xy+\eta y^2)-(\alpha x^2+2\beta xy+\gamma y^2)(dx^2+2exy+hy^2)\}$$
$$-\{3(bx^2+2cxy+dy^2)(\gamma x^2+2\delta xy+\varepsilon y^2)-(\beta x^2+2\gamma xy+\delta y^2)(cx^2+2dxy+ey^2)\}$$
$$=[0, 3]x^4+(2[0, 4]+\ldots)x^3y+\{[0, 5]+[1, 4]-8[2, 3]\}x^2y^2+\&c.$$

[The number -8 results from the calculation $1-3(4-1)=-8$.]

Again,

$$E_2\Omega=(u_1y^2-2u_2yx+u_3.x^2)\xi^2-2(u_2.y^2-2u_3.yx+u_4x^2)\xi\eta+(u_3y^2-2u_4.yx+u_5x^2)\eta^2,$$

$$\therefore\quad Q_3=(u_1.y^2-2u_2yx+u_3x^2)(u_3y^2-2u_4yx+u_5x^2)-(u_2.y^2-2u_3.yx+u_4.x^2)^2$$
$$=u_1.u_3.y^4-2u_1.u_4.y^3x+u_1.u_5y^2x^2+\&c.,$$

all the terms and parts of terms unexpressed being free of u_1, and therefore not necessary for our purpose. Hence supplying the reciprocal factors

$$1\ ;\ -\frac{1}{4}\ ;\ \frac{1}{6}\ ;\ldots,$$

we have

$$G_3=[0, 3]u_1.u_3+([0, 4]+)u_1.u_4+\frac{1}{6}\{[0, 5]+[1, 4]+[2, 3]\}u_1.u_5+\&c.$$

Again, expressing $E_1.f$ and $E_1.\phi$ in the usual way, we obtain

$$\mathrm{T}_1=(ax^4+4bx^3y+6cx^2y^2+4dxy^3+ey4)(\beta x^4+4\gamma x^3y+6\delta x^2y^2+4\varepsilon xy^3+\eta y^4)$$
$$-(\alpha x^4+4\beta x^3y+6\gamma x^2y^2+4\delta xy^3+\varepsilon y^4)(bx^4+4cx^3y+6dx^2y^2+4exy^3+hy^4)$$
$$=[0,\ 1]x^8+4[0,\ 2]x^7y+(6[0,\ 3]+)x^6y^2+(4[0,\ 4]+)x^5y^3+([0,\ 5]$$
$$+15[1,\ 4]+20[2,\ 3])x^4y^4+\ \&c.$$

(where it may be observed that the numbers 15 and 20 in the coefficient of $x^4.y^4$ arise from the quantities 4^2-1; 6^2-4^2).

Again, $\mathrm{Q}_1=\Omega^2=u_1^2.x^8+8u_1.u_2x^7y+12u_1.u_3x^6y^2-8u_1.u_4x^5y^3+2u_1.u_5.x^4y^4+$ &c.

Hence supplying the multipliers

$$1;\quad \frac{-1}{8};\quad \frac{1}{28};\quad \frac{-1}{56};\quad +\frac{1}{70};\ \&c.$$

we have

$$\mathrm{G}_1=[0,\ 1]u_1^2+4[0,\ 2]u_1.u_2+\frac{18}{7}[0,\ 3]u_1.u_3+\frac{4}{7}[0,\ 4]u_1.u_4$$
$$+\frac{1}{35}([0,5]+15[1,\ 4]+20[2,3])u_1.u_5.$$

Again, the Bezoutiant

$\mathrm{B}=5[0,\ 1]u_1^2+2.10[0,\ 2]u_1.u_2+2.10[0,\ 3]u_1.u_3+2.5[0,\ 4]u_1.u_4+2.[0,5]u_1.u_5+$ &c.

Accordingly, if we write $\mathrm{B}=c_1.\mathrm{G}_1+c_3.\mathrm{G}_3+c_5.\mathrm{G}_5$, we have, as above remarked, $c_1=5$; and to determine c_3, c_5, we have, by comparing the coefficients of $u_1.u_3$, $u_1.u_4$ in B, G_1, G_3, G_5,

$$20=\frac{90}{7}+c_3$$
$$10=\frac{20}{7}+c_3.$$

These two equations, then, as it turns out, are not independent, but are satisfied simultaneously by

$$c_3=\frac{50}{7}.$$

Finally, equating the coefficients of the several arguments in $u_1.u_5$, we have

$$0=5\times\frac{1}{35}+\frac{50}{7}\times\frac{1}{6}+c_5 \text{ from the argument } [0,\ 5]$$
$$0=5\times\frac{15}{35}+\frac{50}{7}\times\frac{1}{6}+5c_5 \text{ from the argument } [1,\ 4]$$
$$0=5\times\frac{20}{35}+\frac{50}{7}\times\frac{8}{6}+10c_5 \text{ from the argument } [2,\ 3].$$

The 1st of which equations gives

$$c_5=2-\frac{1}{7}-\frac{25}{21}=\frac{14}{21}=\frac{2}{3};$$

the 2nd gives

$$c_5=\frac{3}{7}+\frac{5}{21}=\frac{2}{3},$$

and the 3rd gives

$$c_5=\frac{20}{21}+\frac{2}{7}=\frac{2}{3}.$$

We have thus abundantly verified the accuracy of the calculation, and there results the relation

$$B=5G_1+\frac{50}{7}G_3+\frac{2}{3}G_5.$$

Lastly, let $m=6$,

$$f=ax^6+6bx^5y+15cx^4y^2+20dx^3y^3+15ex^2y^4+6hxy^5+ly^6$$

$$\varphi=\alpha x^6+6\beta x^5y+15\gamma x^4y^2+20\delta x^3y^3+15\varepsilon x^2y^4+6\eta xy+\lambda y^6$$

$$\Omega=u_1.y^5-5u_2.y^4x+10u_3.y^3x^2-10u_4y^2x^3+50yx^4-u_6.x^6.$$

I shall here confine myself to the determination of a single argument in each of the terms u_1^2; $u_1.u_2$; $u_1.u_3$; $u_1.u_4$; $u_1.u_5$; $u_1.u_6$; this will be ample for the purpose of verification, as the equation to be assigned is of the form

$$B=c_5.G_1+c_3.G_3+c_5.G_5.$$

The arguments which I select as the most simple, will be those expressed by the symbols (0, 1); (0, 2); (0, 3); (0, 4); (0, 5); (0, 6) respectively, then we have

$$T_5=(ax+by)(\eta x+\lambda y)\mp\&c.-(hx+ly)(\alpha x+\beta y)$$

$$=([0, 5]+\ldots)x^2+([0, 6]+\ldots)xy+(\ldots)y^2$$

$$Q_5=(u_1.y-u_2x)(u_5y-u_6x)\mp\&c.$$

$$=(u_1.u_5+\ldots)y^2-(u_1.u_6+\ldots)yx+(\ldots)x^2.$$

Hence supplying the binomial reciprocals

$$1;\quad -\frac{1}{2};\quad 1,$$

$$G_5=([0, 5]+\ldots)u_1.u_5+\frac{1}{2}([0, 6]+\ldots)u_1.u_6+\&c.$$

Again,

$$T_3=(ax^3+\ldots)(\delta x^3+3\varepsilon x^2y+3\eta xy^2+\lambda y^3)\mp\&c.-(dx^3+3ex^2y+3hxy^2+ly^3)(\alpha x^3+\ldots)$$

$$=([0, 3]+\ldots)x^6+(3[0, 4]+\ldots)x^5y+(3[0, 5]+\ldots)x^4y^2+([0, 6]+\ldots)x^3y^3+\&c.$$

$$Q_3=(u_1.y^3\mp\&c.)(u_3y^3\mp3u_4y^3+3u_57x^2-u_6x^3)-\&c.$$

$$=(u_1.u_3+\ldots)y^6-(3u_1.u_4+\ldots)y^5x+(3u_1.u_5+\ldots)y^4x^2-(u_1.u_6+\ldots)y^3x^3+\&c.,$$

and the reciprocal binomial multipliers will be

$$1;\quad \frac{-1}{6};\quad \frac{+1}{15};\quad \frac{-1}{20};\ \&c.$$

Hence

$$G_3=[0, 3]u_1.u_2+\frac{3}{2}[0, 4]u_1.u_4+\frac{3}{5}[0, 5]u_1.u_5+\frac{1}{20}[0, 6]u_1.u_6\ \&c.\ \&c.$$

Finally,

$$T_1=(ax^5+\&c.)(\beta x^5+5\gamma x^4y+10\delta x^3y^2+10\varepsilon x^2y^3+5\eta xy^4+\lambda y^5)-\&c.$$
$$=([0,1]+\ldots)x^{10}+5([0,2]+\ldots)x^9y+(10[0,3]+\ldots)x^8y^2+(10[0,4]+\ldots)x^7y^3$$
$$+(5[0,5]+\ldots)x^6y^4+([0,6]+\ldots)x^5y^5+\&c.$$
$$Q_1=\Omega^2=u_1^2y^{10}+(10u_1.u_2+\ldots)y^9.x+(20u_1.u_3+\ldots)y^8x^2+(20u_1.u_4+)y^7x^3$$
$$+(10u_1.u_5+\ldots)y^6x^4+(2u_1.u_6+\ldots)y^5x^5+\&c.;$$

and supplying the numerical series

$$1;\quad -\frac{1}{10};\quad \frac{1}{45};\quad \frac{-1}{120};\quad \frac{1}{210};\quad \frac{-1}{252};\ \&c.,$$

we have

$$G_1=[0,1]u_1^2+5[0,2]u_1u_2+\frac{40}{9}[0,3]u_1u_3+\frac{5}{3}[0,4]\,u_1.u_4$$
$$+\frac{5}{21}[0,5]u_1.u_5+\frac{1}{126}[0,6]u_1.u_6+\&c.$$

Again, the Bezoutiant

$$=6[0,1]u_1^2+30[0,2]u_1.u_2+40[0,3]u_1.u_3+30[0,4]u_1.u_4$$
$$+12[0,5]u_1.u_5+2[0,6]u_1.u_6+\&c.\ \&c.=B.$$

Hence making

$$B=c_1.G_2+c_3.G_3+c_5.G_5,$$

from u_1^2 and $u_1.u_2$ we obtain respectively

$$c_1=6$$
$$5c_1=30;$$

hence from $u_1.u_3$ and $u_1.u_4$ we obtain respectively

$$\left.\begin{array}{r}\frac{240}{9}+\ c_3=40\\ \frac{30}{3}+\frac{3}{2}c_3=30\end{array}\right\}\text{ or } c_3=\frac{40}{3};$$

hence from $u_1.u_5$ and $u_1.u_6$ we obtain respectively

$$6\times\frac{5}{21}+\frac{40}{3}\cdot\frac{3}{5}+c_5=12,\ i.e.\ c_5=12-8-\frac{10}{7}=\frac{18}{7}$$
$$6\times\frac{1}{126}+\frac{40}{3}\cdot\frac{1}{20}+\frac{1}{2}c_5=2,\ i.e.\ \frac{1}{2}c_5=2-\frac{2}{3}-\frac{1}{21}=\frac{9}{7};$$

hence

$$c_5=\frac{18}{27},$$

and the equation sought for is

$$B=6G_1+\frac{40}{3}G_3+\frac{18}{7}G_5.$$

Art. (71.). The following table exhibits the relations between the Bezoutiant and the correspondent system of Co-bezoutiants for all values of m between 1 and 6 under a synoptical form.

$$m=1 \quad B=G_1$$
$$m=2 \quad B=2G_1$$
$$m=3 \quad B=3G_1-4G_3$$
$$m=4 \quad B=4G_1+\frac{16}{5}G_3$$
$$m=5 \quad B=5G_1+\frac{50}{7}G_3+\frac{2}{3}G_5$$
$$m=6 \quad B=6G_1+\frac{40}{3}G_3+\frac{18}{7}G_5.$$

These series could if wanted be easily extended, and the calculation of the coefficients reduced to a mere mechanical procedure.

If we suppose m to be $2i$ or $2i-1$, we have the equation

$$B=c_1.G_1+c_3.G_3+\ldots+c_{2i-1}\,G_{2i-1};$$

and it appears from the foregoing instances that the comparison of the coefficients, either of u_1^2, or of $u_1.u_2$ on the two sides of the equation, will serve to give c_1 and c_1 (which is always m being known), c_3 may be found by a comparison of the coefficients either of $u_1.u_3$, or of $u_1.u_4$, and so on for $c_5\ldots c_{2i-1}$; all the coefficients in the equation for B above given, thus admitting of being found separately and successively and in two modes, so that there is a check at each step upon the correctness of the computations: the only exception to this last remark is (when m is odd) for the last coefficient of which the above condensed method affords only a single determination. I need hardly add the remark, that in substituting x^{m-1}, $x^{m-2}.y$; $\ldots x.y^{m-2}\ldots y^{m-1}$ in place of u_1, u_2, $\ldots u_{m-1}$, u_m respectively, all the G's become (to a numerical factor *près*) identical with one another and with the Jacobian to the system $(f\varphi)$.

Art. (72.). The foregoing theory took its origin (as will have been readily imagined) in meditations growing out of the celebrated theorem of M. STURM. There appear to be several directions in which a development or extension of the subject matter of that theorem may be sought for. Thus a theory may be constructed relative to a single function of one or more variables, viewed in all cases as representing a geometrical locus. In the limiting case, when this locus becomes a system of points in a right line, we have the theorem of STURM; generally the theory will be that of contours. Or, again, a theory may be formed in which the number of functions is always kept equal to that of the variables. We have then a theory of discreet points corresponding to roots, the number of real ones of which comprised within given limits it is the object of such theory to determine. M. HERMITE, in a memoir recently presented to the French Institute, appears to have made a valuable addition to the Sturmian theory extended in this direction, to which the beautiful researches of M. CAUCHY and the joint labours of MM. LIOUVILLE and STURM, with reference to the disposition of the imaginary roots of equations appear to have led the way. Finally, the number of variables may be supposed to be arbitrarily increased, but made always inferior by a unit to the number of the functions in which they are contained, or which comes to the same thing, we may construct the theory of a system of homogeneous functions equal in number to the variables in them, which in its simplest case becomes the theory of Intercalations which has been here partially considered, and which (as has been shown) embraces (not as a particular case, but as an implied consequence and easily extricated result) the theorem of M. STURM.

London, June 25, 1853.

General and Concluding Supplement.

Art. (א.). The expressions given in art. (*n*.) for the partial quotients of the continued fraction represented by $\frac{\phi x}{fx}$, are restricted to the supposition of all these partial quotients (except the first) being linear in x; when the first partial quotient is linear the formula (B.) of that article continues applicable on replacing $(D_i\, h_\theta)$by 1. I was forcibly struck by the peculiarity of these formulæ not ceasing to be true in consequence of the first partial quotient being supposed non-linear; and reflecting upon this, I was soon led to perceive that all the partial quotients might be supposed to be arbitrary integral functions of x, and the formulæ would still continue to apply to any such of them as might happen to be linear, although, as it were, imbedded among a group of other non-linear partial quotients. From this it was but an easy step to perceive that the formulæ A and B must admit of extension to the representation of partial quotients of any form, and that the dimorphism of the representation of the linear partial quotients could only be a consequence of the equation in integers $u+\nu=1$ having two solutions $u=0$, $\nu=1$ and $u=1$, $\nu=0$. I now proceed to enunciate the very remarkable general theorem (or as it may perhaps not inappropriately be termed Algebraical Porism), by virtue of which any partial quotient of a given degree in x belonging to an infinite continued fraction, all of whose partial quotients are algebraical functions of x, may be expressed to a constant factor *près*, by means of the numerator and denominator (or if we please either one of these) of the convergent *immediately* antecedent to and of the numerator and denominator of *any* convergent *not* antecedent to the partial quotient which is to be determined.

Art. (ב.). *Theorem.* Let Q_1, Q_2, ...Q_i Q_{i+1}....Q_n, &c., each of an arbitrary degree in x, be the n first partial quotients of an algebraical continued fraction; let Q_{i+1} be the partial quotient to be determined and of the given degree ω_i; let

$$\frac{1}{Q_1-}\,\frac{1}{Q_2-}\,\frac{1}{Q_3-}\cdots\frac{1}{Q_i}=\frac{\phi_i(x)}{f_i(x)}$$

and

$$\frac{1}{Q_1-}\,\frac{1}{Q_2-}\,\frac{1}{Q_3-}\,\frac{1}{Q_i-}\,\frac{1}{Q_{i+1}}\cdots\frac{1}{Q_n}=\frac{\Phi(x)}{F(x)};$$

let u and ν be any couple of integers of the $\omega_{i+1}+1$ couples which satisfy the equation $u+\nu=\omega_{i+1}$; then, as usual, denoting the product of the differences of each of one set of terms from each of another set, by writing the former under the latter, and calling $\eta_1\,\eta_2\ldots\eta_\mu$ the μ roots of $\Phi(x)$, and $h_1\,h_2\ldots.h_m$ the m roots of $F(x)$, (Φ and F being supposed respectively of μ and m dimensions in x), and forming the disjunctive equations

$$\theta_1,\ \theta_2,\ \theta_3\ \ldots,\ \theta_\mu = 1,\ 2,\ 3,\ \ldots..\mu$$
$$t_1,\ t_2,\ t_3\ \ldots,\ t_m = 1,\ 2,\ 3,\ \ldots..m,$$

we have the following equation,

$$Q_{i+1} = K_{u,\nu} \times \Sigma\Big\{(\phi\eta_{\theta_1}.\phi\eta_{\theta_2}\ldots.\phi\eta_{\theta_\nu})^2 \times (fh_{t_1}.fh_{t_2}\ldots.fh_{t_u})^2$$
$$\times \frac{\begin{bmatrix}\eta_{\theta_1} & \eta_{\theta_2} & \ldots..\eta_{\theta_\nu} \\ h_{t_{u+1}} & h_{t_{u+2}} & \ldots..h_{t_u}\end{bmatrix} \times \begin{bmatrix} h_{t_1} & h_{t_2} & \ldots..h_{t_u} \\ \eta_{\theta_{\nu+1}} & \eta_{\theta_{\nu+2}} & \ldots..\eta_{\theta_\mu}\end{bmatrix}}{\begin{bmatrix}\eta_{\theta_1} & \eta_{\theta_2} & \ldots..\eta_{\theta_\nu} \\ \eta_{\theta_{\nu+1}} & \eta_{\theta_{\nu+2}} & \ldots..\eta_{\theta_\mu}\end{bmatrix} \times \begin{bmatrix} h_{t_1} & h_{t_2} & \ldots..h_{t_u} \\ h_{t_{u+1}} & h_{t_{u+2}} & \ldots..h_{t_m}\end{bmatrix}} \Big((x-\eta_{\theta_1})(x-\eta_{\theta_2})\ldots.(x-\eta_{\theta_\nu})\Big)\Big((x-h_{t_1})(x-h_{t_2})\ldots.(x-h_{t_u})\Big)\Big\}$$

and moreover the different values of $K_{u,\nu}$ depending upon the different modes of breaking up ω_i into two parts u and ν are all (to a numerical factor *près*) equal to one another. Thus then the theorem pointed at in art. (p.) is discovered, and the way laid open (by an unexpected channel) for a complete discussion of the theory of the singular cases which may occur in the expansion of any rational algebraical fraction under the form of a continued fraction.

Art. (ג.). In the above expression, if we suppose $\omega_i=1$, we have $u=1$ and $\nu=0$, or $u=0$ and $\nu=1$, and remembering that

$$\begin{bmatrix} h \\ \eta_1\,\eta_2\ldots.\eta_\mu \end{bmatrix} = \Phi h \text{ and } \begin{bmatrix} \eta \\ h_1\,h_2\ldots.h_m \end{bmatrix} = Fh$$
$$\begin{bmatrix} h_{t_1} \\ h_{t_2}\,h_{t_3}\ldots h_{tm} \end{bmatrix} = F'h, \text{ and } \begin{bmatrix} \eta_{\theta_1} \\ \eta_{\theta_2}\,\eta_{\theta_3}\ldots.\eta_{\theta_\mu} \end{bmatrix} = \Phi'h,$$

Q_{i+1} becomes by virtue of the general formula representable under either of the equivalent forms

$$K_{0,1}\ \Sigma_\mu^{1}{}_\theta\Big\{(\phi.\eta_\theta)^2\frac{F\eta_\theta}{\Phi'\eta_\theta}(x-\eta_\theta)\Big\} \text{ and } K_{1,0}\ \Sigma_m^{1}{}_t\Big\{(fh_t)^2.\frac{\Phi h_t}{F'h_t}(x-h_t)\Big\},$$

$K_{0,1}$ and $K_{1,0}$ being either equal, or differing only in the sign agreeably to the formulæ A and B.

Art. (ד.). It may be worth while to notice, that, although (of course) these formulæ and the general formulæ of (art. ב.), when supposed converted into functions of x and of the coefficients of F and of Φ by the reduction, integration and summation of the symmetrical functions of the roots which enter into them remain universally valid, and subject to no cases of exception, yet antecedently to these processes being performed the formulæ as they stand may become illusory when any relations of equality exist between the roots of Φ *inter se*, or between the roots of F *inter se*. Thus in the case before us, if Φ have equal roots the formulæ commencing with $K_{0,1}$ is illusory, and if F have equal roots the other of the two formulæ becomes illusory.

Let us take the second of these and suppose that $F(x)$ has

$$k_1 \text{ roots } c_1,\ k_2 \text{ roots } c_2,\ \ldots.k_p \text{ roots } c_p,$$

we may pass to the actual case from any case where the roots are infinitesimally near to the actual roots of $F(k)$, and all infinitesimally different from one another. Moreover the choice of the infinitesimal variations being arbitrary, let the k_1 roots c_1 be replaced by a group of roots

$$c_1+\delta;\ c_1+\delta\varrho_1;\ c_1+\delta\varrho_1^2;\ \ldots .\ c_1+\delta\varrho_1^{k_1-1},$$

where ϱ_1 is a prime root of the equation $\varrho_1^{k_1}=0$, and δ is an infinitesimal quantity, and suppose each of the other groups to be varied in an analogous manner. Then it may easily be shown from this that the one of the formulæ in question will become

$$K_0^1\Sigma_p^1 k_t \frac{\left(\frac{d}{dc}\right)^{k-1}\{(fc_t)^2(\Phi c_t)(x-c_t)\}}{\left(\frac{d}{dc_t}\right)^k Fc_t};$$

and similarly, the twin formula becomes

$$K_1^0\Sigma_\pi^1 \varkappa_\theta \frac{\left(\frac{d}{d\gamma}\right)^{\varkappa-1}\{(\varphi\gamma_\theta)^2(F\gamma_\theta)(x-\gamma_\theta)\}}{\left(\frac{d}{d\gamma_\theta}\right)^\varkappa \Phi\gamma_\theta}\,*.$$

Corresponding modifications will admit of being made by aid of a like method in the general formulæ of art. (ב.) upon a similar supposition as to equalities springing up between the roots of fx *per se* and of $\varphi(x)$ *per se*, or between the roots of fx and φx *inter se*.

Art. (ה.). If in (art. ב.) we take $i=0$, the formula for Q_{i+1} will become

$$Q_1=K_{u.\nu}\frac{\begin{bmatrix}\eta_{\theta_1} & \eta_{\theta_2} & \ldots\eta_{\theta_\nu}\\ h_{t_{u+1}} & h_{t_{u+2}} & \ldots h_{t_m}\end{bmatrix}\times\begin{bmatrix}h_{t_1} & h_{t_2} & \ldots h_{t_u}\\ \eta_{\theta_{\nu+1}} & \eta_{\theta_{\nu+2}} & \ldots\eta_{\theta_\mu}\end{bmatrix}}{\begin{bmatrix}\eta_{\theta_1} & \eta_{\theta_2} & \ldots\eta_{\theta_\nu}\\ \eta_{\theta_{\nu+1}} & \eta_{\theta_{\nu+2}} & \ldots\eta_{\theta_\mu}\end{bmatrix}\times\begin{bmatrix}h_{t_1} & h_{t_2} & \ldots h_{t_u}\\ h_{t_{u+1}} & h_{t_{u+2}} & \ldots h_{t_m}\end{bmatrix}}\Big((x-\eta_{\theta_1})\ldots(x-\eta_{\theta_\nu})\Big)\Big((x-h_{t_1})\ldots(x-h_{t_u})\Big),$$

u and ν being any two integers whose sum is ω_1, which is identical (as it ought to be) with the expression virtually contained in the formulæ of Section II. for the syzygetic multiplier of $\Phi(x)$ in the syzygetic equation connecting Fx and Φx with their first residue when Φx is supposed to be ω_1 dimensions in x lower than Fx identical, *videlicet*, in other words, with the integer part of the algebraical fraction $\frac{F(x)}{\Phi(x)}$.

* For in general if ρ is a prime root of the equation $\rho^\omega=1$, and if fx have ω roots all equal to c and ψx is any other function of x and if δ is an infinitesimal quantity, then rejecting all powers of δ higher than the $(\omega-1)$th degree,

$$\frac{\psi(c+\delta)}{f'(c+\delta)}+\frac{\psi(c+\rho\delta)}{f'(c+\rho\delta)}+\frac{\psi(c+\rho^2\delta)}{f'(c+\rho^2\delta)}+\ldots+\frac{\psi(c+\rho^{\omega-1}\delta)}{f'(c+\rho^{\omega-1}\delta)}$$

$$=\frac{1}{\left(\frac{d}{dc}\right)^\omega fc.\delta^{\omega-1}}\left\{\psi(c+\delta)+\rho\psi(c+\rho\delta)+\rho^2\psi(c+\rho^2\delta)+\ldots+\rho^{\omega-1}\psi(c+\rho^{\omega-1}\delta)\right\}$$

$$=\frac{\left(\frac{d}{dc}\right)^{\omega-1}\psi c\omega\delta^{\omega-1}}{\left(\frac{d}{dc}\right)^\omega fc\delta^{\omega-1}}=\omega\frac{\left(\frac{d}{dc}\right)^{\omega-1}\psi c}{\left(\frac{d}{dc}\right)^\omega fc}.$$

Art. (ו.). When $\Phi(x)=F'(x)$,

$$\frac{\Phi(h_1)\Phi(h_2)\ldots\Phi(h_{\omega_i+1})}{\begin{bmatrix} h_1 & h_2 & \ldots h_{\omega_{i+1}} \\ h_{1+\omega_{i+1}} & h_{2+\omega_{i+2}} & \ldots h_m\end{bmatrix}} \text{ becomes identical with } (-)^{\frac{1}{2}(\omega_{i+1}-1)\omega_{i+1}}\zeta(h_1\, h_2\ldots h_{\omega_{i+1}}),$$

and we may consequently (using an extreme term in the forms in the polymorphic scale of forms representing Q_{i+1}), write

$$Q_{i+1}=(-)^{\frac{1}{2}(\omega_{i+1}-1)\omega_{i+1}}K_{0,\,\omega_{i+1}}\Sigma\zeta(h_1\, h_2\ldots h_{\omega_{i+1}})(f_i h_1)^2(f_i h_2)^2\ldots(f_i h_{\omega_{i+1}})^2(x-h_1)(x-h_2)\ldots(x-h_{\omega_{i+1}}).$$

Art. (ז.). The following observations will serve to complete the theory of the singular cases in the expansion of an algebraical continued fraction.

Preserving the notation of art. (ב.), let

$$\sigma_i=m-(\omega_1+\omega_2+\ldots.+\omega_{i-1}+1),$$

Then (calling the roots of Fx, $h_1\, h_2\ldots h_m$) the (i)th simplified residue to $\frac{\Phi x}{F(x)}$, in accordance with the general formulæ for the residues in the second section (for greater simplicity selecting an extreme term of the polymorphic scale), will be represented by

$$\Sigma\frac{\Phi h_1\,\Phi h_2\,\Phi h_3\ldots\Phi(h_{\sigma_i})}{\begin{bmatrix} h_1 & h_2 & h_3 & \ldots h_{\sigma_i} \\ h_{1+\sigma_i} & h_{2+\sigma_i} & h_{3+\sigma_i} & \ldots h_m\end{bmatrix}}(x-h_1)(x-h_2)(x-h_3)\ldots.(x-h_{\sigma_i}),$$

which will be of the form $L_i x^{\sigma_i-\omega_i+1}+$ &c., all the terms containing powers of x superior to σ_i vanishing by the coefficients becoming zero. If in the above expression we should use σ'_i in lieu of σ_i, where σ'_i is σ_i diminished by any integer inferior to ω_i, we should get other forms of the same residue, but these will all be of higher dimensions in the roots or coefficients than the one just given, and in fact the forms thus obtained corresponding to the values σ_i, σ_i-1, σ_i-2, $\ldots\sigma'_i-\omega_i+1$ substituted for σ_i in succession, would by aid of the relations of condition between the coefficients of Φx and Fx implied in the value of ω_i admit of being exhibited as a scale in which each form would be an exact algebraical product of the form which precedes it, multiplied by a function of the coefficients, and did space permit thereof it would be perfectly easy to give the forms of these multiplicators. But I pass on to the representation of what is more material, viz. the form of the complete residue in the case supposed, merely observing (as an *obiter dictum*) that the existence of each singular partial quotient (meaning thereby a quotient non-linear in x) only affects the form of the single simplified residue in immediate connexion with itself, and not at all the form of the other residues antecedent or subsequent to that one.

Art. (ח.). Let the ith simplified residue be called R_i and the corresponding complete residue $[R_i]$, then applying a method similar to the method given in Section I., we shall find that

$$(-)^{\gamma}\,[R_i]=\frac{L_{i-2}^{\omega_{i-1}+1}.L_{i-2}^{\omega_{i-4}+1}.\&c.}{L_{i-3}^{\omega_{i-1}+1}.L_{i-3}^{\omega_{i-3}+1}.\&c.}R_i,$$

L_i representing the leading coefficient in the (ith) simplified residue, and the sign of interrogation (?) denoting some function of $\omega_1\,\omega_2\ldots\omega_\iota$ (possibly a constant) remaining to be determined. And reverting to art. (ב.), the quantity that would be called K_{0,ω_ι} according to the notation employed in the formulæ expressing Q_{i+1} in that article, will (abstraction being made of the algebraical sign and using for greater brevity (ι), ($\iota-1$), &c. to express $1+\omega_i$, $1+\omega_{i-1}$, &c.) come to be represented by

$$\frac{L_{i-1}^{3(\iota-1)}}{L_i^{\iota}}\cdot\frac{L_{i-3}^{4(\iota-3)}.L_{i-5}^{4(\iota-5)}\,\&c.}{L_{i-2}^{4(\iota-2)}.L_{i-4}^{4(\iota-4)}\,\&c.},$$

a similar convention being supposed to be made respecting the numerator and denominator of each convergent as was made respecting them in the particular case treated of in art. (f), page 473.

Art. (ט.). I will merely add a very few words in generalization of the method of limiting the roots of fx given in the Supplement to the fourth Section. As an inferior limit to fx is identical with a superior limit to $f(-x)$, we may confine our attention to superior limits alone. Suppose then that

$$\frac{\phi x}{fx}=\frac{1}{Q_1-}\,\frac{1}{Q_2-}\cdots\frac{1}{Q_\iota-}\,\frac{1}{Q'_1-}\,\frac{1}{Q'_2-}\cdots\frac{1}{Q'_{i'}}\cdots\cdots\frac{1}{(Q)_1-}\,\frac{1}{(Q)_2-}\cdots\frac{1}{(Q)_i},$$

where the partial quotients Q are each of any arbitrary degree in x, and have all one algebraical sign in the coefficients of the highest powers of x from Q_1 to Q_i, and all the same sign (contrary to the former), in the coefficients of the highest powers of x from Q'_ι to $Q'_{i'}$, and so on alternately, then 1°, a superior limit to the superior limits of the cumulants $[Q_1\,Q_2\ldots Q_i]$, $[Q'_1,\,Q'_2\ldots Q_{i'}]$, $\ldots[(Q)_1\,(Q)_2\ldots(Q)_{i}]$ will be a superior limit to fx, so that it remains only to give a rule for finding a superior limit to a cumulant $[Q_1, Q_2, Q_3\ldots Q_i]$, which, 2°, is to be found by making

$$Q_1-M_1=0,\ Q_2-M_2=0,\ Q_3-M_3=0\ldots Q_i-M_i=0,$$

where $$M_1=\mu_1\quad M_2=\mu_2+\frac{1}{\mu_1}\quad M_3=\mu_3+\frac{1}{\mu_2}\ldots M_i=\frac{1}{\mu_{i-1}},$$

$\mu_1, \mu_2, \ldots\mu_{i-1}$ being any quantities entirely independent and arbitrary except in regard to their being all of the same sign as the leading coefficient in the element $Q_1, Q_2\ldots Q_i$.

We may then find L_1, L_2, $\ldots L_i$ any superior limits to the roots of x in these i equations respectively; L, the greatest of these, will be a superior limit to the proposed cumulant $[Q_1\,Q_2\ldots Q_i]$; and it may be observed that $M_1\,M_2\ldots M_\iota$ are the general values which satisfy the equation

$$M_1-\frac{1}{M_2-}\,\frac{1}{M_3-}\cdots\frac{1}{M_i}=0,$$

subject to the condition that for all values of e

$$\frac{1}{M_e-}\,\frac{1}{M_{e-1}-}\,\frac{1}{M_{e-2}}\cdots\frac{1}{M_1}$$

shall have a given invariable sign. The first part of the process, as just shown, consists in separating the type of the total cumulant which represents fx into partial types, the point for each fracture of the total type being marked by a change of sign in the elements of the type for the value $x=+\infty$; it is easily seen therefore from this, that if $\frac{\Phi x}{Fx}$ is the generatrix of the cumulant in question, the number of such fractures (*i. e.* the number one less than the number of partial cumulants) will be the number of changes of algebraical sign in the signaletic series, consisting of the leading coefficients in Fx and in each of the odd-placed complete residues respectively, together with the number of changes of sign in the signaletic series, consisting of the leading coefficients in Φx and in each of the even-placed complete residues respectively.

The syzygetic theory of two algebraical functions, and the allied theory of algebraical continued fractions with their principal applications, may, I think, now be said to be completely made out, as well for the singular cases as for the general hypothesis.

Art. ('.). I will conclude with observing that the theory within developed gives the means of transforming (explicitly and without the aid of symmetrical functions) into an algebraical continued fraction, any given sum of algebraical fractions of the form

$$\frac{c_1}{x-h_1}+\frac{c_2}{x-h_2}+\frac{c_3}{x-h_3}\dots+\frac{c_n}{x-h_n},$$

where each c and h are supposed known. For let the above sum be called $\frac{\Phi x}{F(x)}$, then if h_θ, c_θ be used to denote any pair of corresponding terms of the h series and the c series, we have $\frac{\Phi h_\theta}{F'h_\theta}=c_\theta$, as is well known and easily proved. Again, if $D_i x$ represent the simplified denominator of the ith convergent to the continued fraction equal to $\frac{\Phi x}{(F'x)}$ which is to be found, say

$$\frac{1}{(A_1x+B_1)-}\ \frac{1}{(A_2x+B_2)-}\dots\frac{1}{A_nx+B_n},$$

we have $D_ix=\Sigma\frac{\Phi h_1\ \Phi h_2\dots\Phi h_i}{\left|\begin{matrix}h_1 & h_2 & \dots h_i\\ h_{i+1} & h_{i+2} & \dots h_n\end{matrix}\right|}(x-h_1)(x-h_2)\dots(x-h_n)$,

$$=\Sigma(-)^{i\frac{(i-1)}{2}}\frac{\zeta(h_1\,h_2\dots h_i)\Phi h_1\ \Phi h_2\dots\Phi h_i}{f'h_1 . f'h_2 . . f'h_i}(x-h_1)(x-h_2)\dots(x-h_i)$$

$$=(-)^{i\frac{i-1}{2}}\Sigma\{c_1c_2\dots c_i\zeta(h_1h_2\dots h_i)(x-h_1)(x-h_2)\dots(x-h_i)\}.$$

Therefore $(D_ih_1)^2=\{\Sigma(c_2c_3\dots c_{i+1})\zeta(h_2h_3\dots h_{i+1})(h_1-h_2)(h_1-h_3)\dots(h_1-h_{i+1})\}^2$

$$=\left\{\Sigma(c_2c_2\dots c_{i+1})\zeta^{\frac{1}{2}}(h_1h_2\dots h_{i+1})\zeta^{\frac{1}{2}}(h_2h_3\dots h_{i+1})\right\}^2;$$

and the simplified $(i+1)$th quotient, *i. e.* the value of $A_{i+1}x+B_{i+1}$, when divested of the allotrious factor, has been proved to be equal to

$$\Sigma(D_ih_1)^2\frac{\Phi h_1}{F'h_1}(x-h_1);$$

it is therefore now known as a rational and *integral* function of x; $h_1h_2\ldots h_n$; $c_1c_2\ldots c_n$. The allotrious factor itself is made up of the product of squares of quantities all of the same form as the leading coefficient in D_ix, which, from what has been shown above, is seen to be equal to

$$(-)^{\frac{i-1}{2}}\Sigma\{(c_1c_2\ldots c_i)\zeta(h_1h_2\ldots h_i)\}.$$

Hence each term in the continued fraction

$$\frac{1}{(A_1x+B_1)-}\frac{1}{(A_2x+B_2)-}\cdots\frac{1}{(A_nx+B_n)},$$

which is to be made equal to

$$\frac{c_1}{(x-h_1)}+\frac{c_2}{(x-h_2)}+\cdots\frac{c_n}{(x-h_n)},$$

is completely assigned in terms of x and the given quantities c and h.

Art. (ↄ.). The number of effective intercalations between the roots of Φx, Fx is easily seen to be equal to the excess of the number of positive real numerators over the number of negative real numerators in the partial fractions of which $\frac{\Phi x}{Fx}$ is the sum, and hence we see *à priori*, as an obvious consequence of a simple extension of the reasoning in art. (47.), that the inertia of the quadratic function

$$\Sigma\left\{c_\theta(u_1+h_\theta u_2+h_\theta^2u_3+\ldots.+h^{n-1}.u_n)^2\frac{1}{x-h_\theta}\right\},$$

where $c_\theta=\frac{\Phi h_\theta}{F'h_\theta}$ will represent the value of the index in question. So too we may see that the formulæ given for the residues to $fx, f'x$ in art. (46.) continue to apply to the residues Fx, Φx. That is to say, these residues when divided out by Fx will be respectively represented by the successive principal coaxal determinants to the matrix

$$\begin{matrix} S_0S_1 & S_2 & \ldots S_{m-1} \\ S_1S_2 & S_3 & \ldots S_m \\ S_2S_3S_3 & & \ldots S_{m+1} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ S_{m-1}S_m & S_{m+1} & \ldots S_{2m-2}, \end{matrix}$$

where in general $$S_r=\frac{c_1}{x-h_1}h_1^r+\frac{c_2}{x-h_2}h_2^r+\ldots+\frac{c_n}{x-h_n}h_n^r;$$

and using the same matrix as above written with S' substituted for S, where in general

$$S'_r=c_1(x-h)_1^r+c_2(x-h_2)h_2^r+\ldots+c_n(x-h_m)h_n^r;,$$

the successive principal coaxal determinants of the new matrix represent the successive denominators to the convergents of the continued fraction which expresses $\frac{\Phi x}{Fx}$.

The expression for the numerators to the convergents may also, there is no doubt, be obtained by some simple modification (dependent on introducing the quantities $c_1c_2\ldots c_n$) of the formula in art. (41.), p. 465.

I annex, more with the hope of suggesting than (in all instances) of conveying a full conception of the force of the definitions, a Glossary, or rather a Repertory of the principal terms of art employed in the preceding pages, which might otherwise be apt to occasion some difficulty to persons unfamiliar with the subject.

ERRATA AND ADDENDA.

Page 408, 410, 412, 414, in running head to page, *for* Conjugate *read* Syzygetic.
—— 408, line 16 from foot, *for* above *read* about.
—— 409, line 4 from top, *for* continual *read* continued.
—— 429, line 12 from foot, *for* the same r new, *read* the same number r of new.
—— 430, line 3 from foot, *after* simplicity *insert* a comma.
—— 432, line 2 above (15.), *for* $\frac{1}{q_{n-1}}$ *read* $\frac{1}{q_{\overline{n-1}}}$.
—— 432, line 3 under (15.), *after* fraction *dele* —
—— 434, at end of the equation nearest the foot, *for* $(x \quad \eta_{q_\iota})$ *read* $x-\eta_{q_\iota}$.
—— 436, in equation (21.), *for* $(x-\eta_{\xi_n})$ *read* $(x-\eta_{\xi_\nu})$.
—— 436, line 2 under (21.), *for* k_m *read* k_n.
—— 438, line 10 from foot, *for* (λ_0) (λ_1) (λ_{n-1}) *read* λ_0 λ_1 λ_{n-1}.
—— 439, line 3 from top, after the words "solution of" *insert* "the equation."
—— 439, line 10 from top, *for* and therefore *read* then.
—— 444, line 2 from top, *for* or *read* i. e.
—— 448, Art. 28, line 3, *for* $\varepsilon - x^m$ *read* εx^m.
—— 452, line 1, *for* but *read* for.
—— 454, lines 5 and 14, *for* fm *read* f_n.
—— 458, line 4 in Art. 37, *for* fx *read* f′x.
—— 459, line 7 in Art. 38, *for* $-\vartheta$ *read* $+\vartheta$.
—— 464, line 15 from foot, *for* $k_1=k_2-k_3$ *read* $k_1=k_2=k_3$.
—— 467, line 6 from foot, *for* Latin and Greek *read* Latin, Greek and Hebrew.
—— 479, Art. p, line 2, *for* Σ'_m and Σ'_n *read* Σ^1_m and Σ^1_n.
—— 479, last line, *for* subscrolet *read* subscript.
—— 481, in the value of y_3 near foot of page, *for* the sign $-$ *read* $+$
—— 482, middle of the page, *for* $\frac{d^2f}{dx_2^2}$ *read* $\frac{d^2f}{dx_2^2}$.
—— 485, line 10, *for* ρ *read* x.
—— 497, Art. β, *for* Now *read* Also.
—— 504, line 12, *dele* $'w_1$.
—— 514, Art. (61.), lines 7 and 8, *for* $a_m . x^m$ and $b_m x^m$ *read* $a_m . y^m$, $b_m y^m$.
—— 515, line 4, *for* $u'_1, u'_2 \ldots u_m$ *read* $u'_1, u'_2 \ldots u'_m$.
—— 518, near middle of page, *for* $\frac{1}{1.2.3\ldots(m-1)^3}$ *read* $\frac{1}{(1.2.3\ldots(m-1))^2}$.
—— 524, near middle of page, *for* $\alpha_0 . \frac{d}{da_1}$ *read* $\alpha_0 . \frac{d}{d\alpha_1}$.

Glossary of new or unusual Terms, or of Terms used in a new or unusual sense in the preceding Memoir.

Allotrious.—The allotrious factor to a residue or quotient in the process of common measure applied to two algebraical functions is the constant factor of which such residue or quotient must be divested in order to become an integral and irreducible function.

Apocopated.—Applied to a type in the Theory of Cumulants, denotes a type the final or initial element of which has been taken away. If both are taken away, the type is said to be doubly apocopated.

Bezoutic.—For definition of Primary and Secondary Bezoutics see first Section. *Bezoutiant* to two functions, each of degree n, is a homogeneous quadratic invariantive function of n variables, the form of which serves to assign the index of the scale of the effective intercalations of the real roots of the two given functions.

Bezoutoid.—The Bezoutiant to two homogeneous functions obtained by differentiation from one homogeneous function of two variables. The Bezoutoid to a given function of m dimensions in the variables is accordingly a quadratic function of $(m-1)$ variables, the form of which is sufficient for determining the number of real roots in the given function.

Characteristic.—The employment of this word has been avoided in the preceding memoir; but as it contains an idea of capital importance in analysis, and especially in all inquiries of the kind here treated of, I subjoin the definition of its meaning. The characteristic of a simple condition of any kind is the rational integral function (in its lowest terms) whose evanescence necessarily and universally implies and is implied by the satisfaction of such condition. A simple condition has always a single characteristic, abstraction being made of the algebraical sign, which remains indeterminate. In like manner, a multiple condition, or a system of conditions, will have for its characteristic a plexus of rational integral functions, whose evanescence necessarily and universally implies and is implied by the satisfaction of such multiple condition or system of conditions. The number of functions in the characteristic plexus will however in general greatly exceed the index of the multiplicity of the conditions, and need not always be a unique system. There are however exceptions to this: thus the duplex condition, that a biquadratic function of x shall contain a cubic factor, or that a curve of the third degree shall have a cusp, will each be definitely characterized by a plexus of two functions, and no more.

The spirit of the higher analysis resides, and is to be sought for, in the *logic* of characteristics.

Co-bezoutiant.—Any homogeneous quadratic function similar in form and in its property of invariance to the Bezoutiant.

Cogredient and Contragredient.—A system of variables is cogredient to another system when it is subject to undergo simultaneously therewith linear substitutions of a like kind, and contragredient when it is subject to undergo linear substitutions simultaneously therewith but of a contrary kind.

Combinant.—A function of the quantities appearing in a given set of functions which remains unaltered as well for linear substitutions impressed upon the variables as for linear combinations of the functions themselves.

Concomitant.—*Nomen generalissimum* for a form invariantively connected with a given form or system of forms.

Conjunctive.—A syzygetic function of a given set of functions. Any function which universally, and *subject to no cases of exception*, vanishes when a certain number of other functions all vanish together must be a conjunctive (*i. e.* a syzygetic function), or a root of a conjunctive of such functions. But if its vanishing is subject to cases of exception, then all that can be predicated of it is that it is *syzygetically related* to such functions, but it may, and usually does happen, that it will be syzygetically related to them in more than one way.

Contravariant.—A function which stands in the same relation to the primitive function from which it is derived as any of its linear transforms to an inversely derived transform of its primitive.

Covariant.—A function which stands in the same relation to the primitive function from which it is derived as any of its linear transforms to a similarly derived transform of its primitive.

Cumulant.—The denominator of the simple algebraical fraction which expresses the value of an improper continued fraction. See *Type*, infra.

Determinant.—This word is used throughout in the single sense, after which it denotes the alternate or hemihedral function the vanishing of which is the condition of the possibility of the coexistence of a system of a certain number of homogeneous linear equations of as many variables.

Dialytic.—If there be a system of functions containing in each term different combinations of the powers of the variables in number equal to the number of the functions, a resultant may be formed from these functions by, as it were, *dissolving* the relations which connect together the different combinations of the powers of the variables, and treating them as simple independent quantities linearly involved in the functions. The resultant so formed is called the Dialytic Resultant of the functions supposed; and any method by which the elimination between two or more equations can be made to depend on the formation of such a resultant is called a dialytic method of elimination. In such method accordingly the process of elimination between equations of a higher degree than the first is always reduced to a question of elimination between equations which are of the first degree only.

Discriminant.—The resultant of the n differential coefficients of a homogeneous function of n variables. See *Resultant*, infra.

Disjunctive.—A disjunctive equation is a relation between two sets of quantities such that each one of either set is equal according to some unspecified order of connexion with some one of the other set.

Effective scale of intercalations is the series of the real roots of two functions of x written in order of magnitude after repeated processes of removing pairs of roots belonging to either the same function (when not separated by roots of the other function): the roots of the two functions follow each other alternately.

Effluent.—From every homogeneous function of any number of variables i of the degree mm', where mm' are any two integers, may be formed (as shown in the Calculus of Forms, Section I.) a covariantive function of the degree m and of μ variables [where μ is the number of permutations that can be obtained by dividing m' into i parts (zeros admissible)], in which all the coefficients are numerical multiples of the given coefficients; covariants so formed may be termed effluents of their primitive. An example of this occurs in the foot note to Section V. p. 522, where the quantity there called Q is a quadratic effluent of the Jacobian.

Element.—A simple component of the type to a cumulant. See Cumulant, *supra*.

Emanant.—The result of operating any number of times (suppose i times) upon a given homogeneous function of any number of variables $x, y, z \ldots t$ with the operative symbol

$$\left(x'\frac{d}{dx}+y'\frac{d}{dy}+z'\frac{d}{dz}+\ldots+t'\frac{d}{dt}\right),$$

is called the ith emanant of the function operated upon. Every emanant is a covariant to its primitive, the new variables $x', y', z', \ldots t'$ being cogradient with the variables $x, y, z \ldots t$ with which they are respectively associated. $E_{2i+i}f$, $E_{2i+i}\phi$, page 522, are emanants of f and ϕ. The process of emanation is one of incessant occurrence in the theory of invariants. When the order of the emanation is the same as the degree of the function (supposed to be rational and integral) from which the emanation proceeds, the form of the original function is reproduced in the final emanant, the names only of the variables being changed.

Endoscopic, Exoscopic.—When the coefficients of the functions concerned in any investigation are regarded as integral indecomposable monads, the method is called exoscopic, and endoscopic when the coefficients are treated with reference to their internal constitution as composed of roots or other elements.

In addition to the examples in the foot note to Section 1, these words have a marked and most important application in the theory of Invariants, especially of two variables.

Form.—Any function may be regarded as an *opus operatum*; the matter operated upon being the variables, and the substance of the operations being the form, which resides in the function as the soul in the body. A form is always common to an infinity of functions, but for greater brevity may be and frequently is called by the name of some specified function in which it is contained.

Fundamental.—The fundamental scale of a system of Invariants or Concomitants is a set of the same, whereof every other is a Rational Integral Function.

Hessian or *Hessean*, named after Dr. Otto Hesse, of Konigsberg (the worthy pupil of his illustrious master, Jacobi, but who, to the scandal of the mathematical world, remains still without a Chair in the University which he adorns with his presence and his name), is the Jacobian to the differential coefficients of a homogeneous function of any number of variables. It is to a Jacobian what a Bezoutoid is to a Bezoutiant, or a Discriminant to a Resultant.

Hyperdeterminants.—See Memoir of Mr. Cayley, Cambridge and Dublin Mathematical Journal, May 1845, and Crelle's Journal of about the same date.

Improper, continued fraction is a continued fraction differing only from an ordinary one in the circumstance of negative signs being substituted for positive signs to connect the terms.

Inertia.—The unchangeable number of integers in the excess of positive over negative signs which adheres to a quadratic form expressed as the sum of positive and negative squares, notwithstanding any real linear transformations impressed upon such form.

Intercalations.—The theory of intercalations is the theory of the relative distribution of the real roots, or point-roots, of two or more equations, but in this theory the number of roots mutually interposed is to be taken only with reference to the number 2 as a modulus.

Invariance.—The property (under prescribed or implied conditions) of remaining invariable.

Invariant.—A function of the coefficients of one or more forms which remains unaltered when these undergo suitable linear transformations.

Inverse.—The inverse to a given square matrix is formed by selecting in its turn each component of the given matrix, substituting unity in its place, making all the other components in the same line and column therewith zero, and finally writing the value of the determinant corresponding to the matrix thus modified in lieu of the selected component. If the determinant to the matrix be equal to unity, its second inverse, *i. e.* the inverse to its inverse, will be identical, term for term, with the original matrix.

Jacobian.—The Jacobian to n homogeneous functions of n variables is the determinant represented by the symmetrical collocation in a square of the n differential coefficients of each of the n functions.

Kenotheme.—A finite system of discrete points defined by one or more homogeneous equations in number one less than the number of variables contained therein.

Limiting Series.—One set of quantities whose extreme values are exterior to the extreme values of a second set is set to limit the latter.

Matrix.—A square or rectangular arrangement of terms in lines and columns.

Minor Determinant.—Any determinant retained represented by a square group of terms arbitrarily chosen out of a matrix is a minor determinant thereto. The simple terms of the matrix are the last minors, and of course if the matrix is a square, it will itself in its totality represent a single complete determinant.

Monotheme.—A line, or finite system of lines, defined by one or more homogeneous equations two less in number than the numbers of the variables contained therein.

Order.—The orders of a homogeneous function are the linear functions of the variables the least in number by aid of which the function admits of being expressed.

Persymmetrical.—A symmetrical matrix, in which all the terms in the diagonal bands transverse to the axis of symmetry are identical, is said to be persymmetrical. *Ex.* An addition table.

Quadrinvariant.—An invariant of which the terms are quadratic functions of the coefficients of the primitive.

Relation (simple and compound). Vide *Substitution*, infra.

Resultant.—The resultant of n homogeneous general functions of n variables is that function of their coefficients which, equalled to zero, expresses in the simplest terms the condition of the possibility of their coexistence.

Rhizoristic.—A rhizoristic series is a series of disconnected functions which serve to fix the number of real roots of a given function lying between any assigned limits.

Signaletic.—A signaletic or *Semaphoretic* series is a sequence of disjunctive terms, considered solely with reference to the algebraical signs of *plus* and *minus* which they respectively carry.

Singular.—A proper algebraical function of a given degree, n, in one variable in its most general form, will, in respect to that variable, be of the nth degree in the denominator and the $(n-1)$th degree in the numerator, and will admit of being represented by a continued algebraical fraction of n terms, all of them linear.

But for particular values of, or relations among, the coefficients entering into the given fraction this mode of representation fails, and the continued fraction, instead of consisting of linear terms

n in number, will consist of terms, some of them at least, non-linear, and fewer than n in number. These then are the *singular* cases (or cases of singularity) in the theory of the development of an algebraical fraction under the continued fraction form; and it will be seen that according to this definition the case of the development of any proper algebraical fraction in which the degree of the numerator is more than one unit below that of the denominator, belongs (strictly speaking) to the class of singular cases; and this view of the case supposed is perfectly correct and conformable to the analogies of the subject.

Substitution (linear, *similar* or *contrary*).—A linear substitution is said to be impressed upon a system of variables when each variable is replaced by a linear conjunctive of all the variables. The matrix formed by the coefficients of substitution arranged in regular order is called the Matrix of Substitution, and is of course a square. When two substitutions (impressed in two systems of variables) have the same matrix, they are said to be *similar* and *contrary* when their matrices are contrary, *i. e.* mutually inverse to each other. When two systems of variables are supposed to be subject to the condition that their substitutions are always similar or always contrary, they are said to be related or in simple relation, the relation being of cogredience in the one case and of contragredience in the other.

When a linear substitution is impressed upon a system of independent variables, a corresponding linear substitution is necessarily impressed at the same time upon every complete system of homogeneous combinations (*i. e.* products and powers and products of powers) of these variables, the matrix to which latter substitution will consist of terms which will be functions (depending upon the degree of the homogeneous combinations) of the terms of the matrix to the primitive substitution. This matrix may be termed a compound matrix, having the primitive matrix for its base.

If, now, two systems of independent variables are subject to be synchronously impressed with substitutions, the matrices to which (not being both of them simple matrices) have for their bases matrices which are either similar or contrary, these two systems will be said to be in *compound relation* of cogredience in the one case, and of contragredience in the other.

Syrrhizoristic.—A syrrhizoristic series is a series of disconnected functions which serve to determine the effective intercalations of the real roots of two functions lying between any assigned limits.

Syzygetic.—A syzygetic function or conjunctive of a number of given rational integral functions is the sum of these affected respectively with arbitrary functional multipliers, which are termed the syzygetic multipliers. When a syzygetic function of a given set of functions can be made to vanish, they are said to be syzygetically related.

Transform.—Equivalent to the French noun substantive "*transformée.*"

Type.—The type of a cumulant is the series of the simple elements (or quotients), arranged in a fixed order, of which the cumulant is composed.

Umbral.—The umbral notation is a notation according to which simple quantities are denoted by syllables, instead of by single letters (the composition of these syllables being governed by the mode in which the quantities which they express are obtained); and the single letters of such syllables are termed umbral quantities or *umbræ*.

Weight.—In this memoir (throughout the earlier sections) the weight of any quantity composed of the product of the coefficients of any given function or functions of x is used to denote the number of roots of x appertaining to the given function or functions which must be employed to express such quantity. More generally, when dealing with a system of homogeneous functions, the *weight* of a quantity may be defined with respect to *any selected variable* therein as the sum of the weights in respect to such variable of the several coefficients of which the quantity is composed (the weight of each several coefficient meaning the index of the power of the selected variable in that term of the given function or functions which is affected with such coefficient). These two definitions of *weight* may be perfectly well reconciled with each other by understanding the weight of a quantity formed from the coefficients of a function or system of functions of x to mean the weight, in respect *to unity*, of such quantity when the given functions are treated as homogeneous functions of x and 1.

Zeta.—The symbol ζ (preceding a row of bracketed terms) is used to denote the product of the squared differences of the terms which it affects.

[ ]. A bracket of this form, when inclosing a superior and an inferior row of terms m and n in number respectively, indicates the mn products of the differences obtained by subtracting each term in the second row from each term in the first row; when enclosing an arrangement of terms in a single line, it is used to denote the cumulant of which such an arrangement is the type.

CONTENTS.

XIX. The Bakerian Lecture.—*On the Influence of the Moon on the Magnetic Declination at Toronto, St. Helena, and Hobarton.* *By Colonel* Edward Sabine, *R.A., Treas. and V.P.*

Received and Read November 17, 1853.

THE success which attended the endeavour to detect the influence of the moon on the pressure of the atmosphere, by a suitable arrangement of the hourly barometrical observations at St. Helena*, naturally suggested the idea that the influence of the moon on the direction of the magnetic needle, supposing such an influence to exist, might be manifested by an analogous arrangement of the hourly magnetical observations at that station; inasmuch as the magnetical disturbances due to other causes, and liable to mask so small an effect as that which might be anticipated from the moon, were, like those of the barometer, of inconsiderable amount at St. Helena when compared with those at many other stations.

An examination of observations of the Declination made at Milan, whilst M. Kreil was Director of that observatory, led him, in a memoir read in February 1841 to the Royal Bohemian Society of Sciences, to announce his belief that the moon does actually exercise an influence on the magnetic direction at the surface of our globe, cognisable by a variation in the Declination depending on the moon's hour-angle, and completing its period in a lunar day. M. Kreil has since confirmed the discovery thus announced by investigations based on a more extensive series of similar observations made under his direction at Prague, and discussed,—1st, in the 'Magnetische und Meteorologische Beobachtungen zu Prag' for 1841; and 2nd, in a memoir presented to the Imperial Academy of Sciences at Vienna in June 1850, and published in the Transactions of that Academy in 1852.

Meanwhile, in the course of the discussion of the results derivable from the Observations at the British Colonial Observatories, I had selected for a primary examination of this subject the series of hourly observations of the Declination at St. Helena, extending from September 1842 to August 1847 inclusive; and having determined on the process through which the observations should be passed, the work of reduction was commenced, and though occasionally interrupted by more pressing duties, was resumed from time to time, and was at length completed for the five years in the early part of the summer of 1852. The result was conclusive, in so far as a variation depending upon the moon's hour-angle was systematically and consistently manifested; but this variation differed so considerably in many important particulars, as

* Philosophical Transactions, 1847, Art. V.

well as in amount, from the conclusions which M. KREIL had derived from the Milan and Prague observations, as to make it plain to perceive that a knowledge of the phenomena at many different parts of the globe must be obtained, before a general theory of the nature and character of the moon's magnetic influence could be arrived at by inductive reasoning. Having, therefore, in my possession a similar series of hourly observations at Toronto and Hobarton, which stations might with St. Helena be deemed to a certain extent representatives of the phenomena in the middle latitudes of the two hemispheres and in the tropics, I thought it best to retain the conclusions at St. Helena until they should be accompanied by those at Toronto and Hobarton. I took with me, however, to the Belfast Meeting of the British Association in September 1852 an abstract of the St. Helena results, and showed them to several persons interested in magnetical researches who attended that meeting.

The investigation has since been completed for the three stations, so far at least as regards the magnetic *Declination*, and the results form the subject of this communication; to be followed, should circumstances permit, by the results of a similar investigation into the corresponding phenomena of the Inclination and Total Force. It has appeared the most convenient arrangement to treat the stations separately, and to take them in the order of their succession from North to South; commencing therefore with Toronto.

Toronto.—The results at this station are obtained from six years of hourly observation, extending from July 1842 to June 1848 inclusive. The observations were received at Woolwich in the form of Monthly Returns, in which the scale-reading of the Declinometer at every hour of mean Göttingen Time was inserted in tables of double entry, having the hours in vertical columns and the days in horizontal lines, with the daily means in the last vertical column, and the monthly means at the different hours of solar time in the bottom horizontal line. The process at Woolwich commenced by marking for omission all those disturbed observations, whensoever occurring, which exceeded a certain limit on either side of the monthly mean in the same month and at the same hour. When a disturbance was so considerable as materially to affect the amount of the monthly mean at the bottom of the page, a fresh monthly mean was taken, omitting the disturbed observation and giving fresh limiting values, by which the disturbances were marked afresh. The limits adopted, beyond which disturbances were omitted and within which observations were retained for the proposed investigation, were (at Toronto) four scale divisions (or 2′·9 in arc) above or below the monthly mean at the same hour. The observations retained were then marked in small figures on the face of the returns with the lunar hour to which each observation most nearly corresponded. For this purpose the time of the moon's passage of the meridian at Greenwich was taken from the Nautical Almanac, and corrected for the difference of longitude, so as to give the time of the moon's passage of the astronomical meridian at Toronto in the mean solar time of the station. The difference of time corresponding to the difference between the meridians of Toronto and Göttingen was then applied, so as to give the mean *Göttingen* time of the moon's passage of the astronomical meridian of Toronto. The observation at the Göttingen hour nearest to the time thus computed was then marked with 0^h, signifying that it was the nearest observation to the moon's upper culmination, and from which its distance could not exceed half an hour. The time of the moon's inferior passage was then computed in a similar manner, and the observation at the Göttingen hour nearest to it was marked 12^h. The intermediate hours received corresponding markings, except that on those occasions when thirteen solar hours, and consequently thirteen observations, were comprised within twelve lunar hours, that observation was omitted which fell most nearly equidistant between the epochs of two exact lunar hours. The observations were then considered to be prepared for arrangement in *lunar tables,* but instead of the observations themselves, the differences at each hour between the scale-reading observed and the mean scale-reading in the same month and at the same hour were entered in these tables, by which process the diurnal and other variations depending on the time of the solar year and the hour of the solar day were, in great part at least, eliminated. The differences were marked with a + or — sign according as the scale-reading at the time of observation was greater or less than the monthly mean at the same hour; the entries having a + sign implying a more westerly direction of the north end of the magnet than its mean direction in the same month and at the same hour; and those having a — sign implying the converse. The mean was then taken, in every month, of every lunar hour (taking the signs into account); the monthly means were collected into yearly means,—and finally, the means at each lunar hour in each of the six years of observation were collected in a single table.

The whole number of observations in the six years at Toronto which were thus treated amounted to 44,754, of which 6356 were put aside in consequence of the amount of disturbance (measured from the monthly mean at the same hour) equalling or exceeding $2'{\cdot}9$ in arc.

The following Table shows the amount of Easterly or Westerly deflection of the north end of the magnet at the different lunar hours derived from the observations of the six years by the process which has been described, and inferred to be due to the moon's action. The partial results, or those afforded by the separate years, will be printed in the introduction to the third volume of the Toronto Observations. They exhibit only such small partial differences from the mean of the six years as may reasonably be ascribed to the operation of other causes than the moon's influence; and present no notable trace of a progressive increase in the amount of the variation in the successive years from 1843 to 1847, such as that which has been found to characterize the variations depending upon the *solar* hours.

The deflections in the subjoined Table are in scale divisions, of which each $=0'{\cdot}721$. The readings to which the — signs are prefixed denote Easterly deflections, and those which have the + signs prefixed Westerly deflections.

TABLE I.

Lunar hours.	Deflections.	Lunar hours.	Deflections.	Lunar hours.	Deflections.	Lunar hours.	Deflections.
	Scale divisions.		Scale divisions.		Scale divisions.		Scale divisions.
21	−0·15	4	+0·14	9	−0·12	16	+0·03
22	−0·35	5	+0·175	10	−0·30	17	+0·17
23	−0·30	6	+0·17	11	−0·43	18	+0·31
0	−0·35	7	+0·11	12	−0·42	19	+0·11
1	−0·32	8	+0·01	13	−0·30	20	+0·06
2	−0·225			14	−0·19		
3	−0·14			15	−0·12		

We may represent these numbers by the general formula—

$$\begin{aligned}\Delta_x = -0{\cdot}10125 &+ {\cdot}0147 \sin(a+36^\circ\ 43') - {\cdot}2993 \sin(2a+276^\circ\ 51') \\ &- {\cdot}0045 \sin(3a+247^\circ\ 23') + {\cdot}0234 \sin(4a+\ 68^\circ\ 13') \\ &+ {\cdot}0376 \sin(5a+\ 49^\circ\ 19') - {\cdot}0195 \sin(6a+16^\circ\ 08') \\ &- {\cdot}0135 \sin(7a+345^\circ\ 25') - {\cdot}0109 \sin(8a+6^\circ\ 35') \\ &- {\cdot}0266 \sin(9a+307^\circ\ 22') + {\cdot}0244 \sin(10a+58^\circ\ 17') \\ &+ {\cdot}0043 \sin(11a+87^\circ\ 01') + {\cdot}00833 \cos 12a\,;\end{aligned}$$

a being the difference between x and 18 lunar hours expressed in hours and parts of an hour multiplied by 15°.

With this formula we obtain the times of greatest easterly deflection $0^h\ 22^m$ and $11^h\ 23^m$, and the amounts $15''{\cdot}7$ and $19''{\cdot}1$: and of greatest westerly deflection $5^h\ 0^m$ and $18^h\ 0^m$, and the amounts $7''{\cdot}6$ and $13''{\cdot}4$. If we suppose the differences between the times and amounts of the two extreme easterly deflections to be due, wholly or in part, to accidental irregularities in the amount of the deflections at the different hours, the exact values of which would probably require a longer period of observation to determine,—and if the same remark be also applicable to the times and amounts of the two extreme westerly deflections,—we find, as the mean of the times of extreme easterly deflection, $11^h\ 53^m{\cdot}5$, and $23^h\ 53^m{\cdot}5$, and the amount $17''{\cdot}4$; and of the extreme westerly deflections $5^h\ 30^m$, and $17^h\ 30^m$, and the amount $10''{\cdot}5$; making the total amount of the variation due to lunar influence $(17''{\cdot}4+10''{\cdot}5=)27''{\cdot}9$.

The progression being a double one in the 24 lunar hours the variation passes four times through zero when the lunar variation disappears or equals 0. These times as given by the formula are $3^h\ 29^m$; $8^h\ 08^m$; $15^h\ 45^m$; and $20^h\ 27^m$; or $3^h\ 33^m$ before, and $3^h\ 29^m$ after the moon's upper culmination; and $3^h\ 52^m$ before, and $3^h\ 45^m$ after the moon's lower culmination.

It is quite possible that when the facts shall have been more precisely determined, it may prove that neither the times nor the amounts of the two easterly deflections are strictly symmetrical; and so also in regard to the two extreme westerly deflections. With a full consideration of what may be due to accidental irregularities, and particularly to the influence on the precise time of the extreme elongation of even a very small error occurring near the turning hours, there still seems an indication of systematic difference, which is the more deserving of attention, because M. KREIL has inferred the existence of an inequality of the same kind, especially in the amounts of the similar deflections at opposite points of the hour-circle, in the Prague observations.

The disparity in the amounts of the deflections in *opposite* directions, viz. 17″·4 easterly, and 10″·5 westerly, is one which, from its magnitude, cannot be imagined to be occasioned by accidental irregularities, and must be regarded as a real feature of the phenomena under discussion.

M. Kreil has inferred from the Prague observations, that the variation due to the moon's influence is not equally cognisable in all parts of the year, being greatest in summer, and disappearing wholly in the three winter months. To examine this question, I have caused in the following table the results of the Toronto observations to be arranged in two divisions, one comprising the results of thirty-six months composed of the six summer months in each of the six years under examination, and the other also of thirty-six months composed of the six winter months in each of the same years.

Table II.

Lunar hours.	Summer.	Winter.	Table I.	Lunar hours.	Summer.	Winter.	Table I.
	Scale divisions.	Scale divisions.	Scale divisions.		Scale divisions.	Scale divisions.	Scale divisions.
0	−0·51	−0·18	−0·35	12	−0·43	−0·42	−0·42
1	−0·54	−0·10	−0·32	13	−0·40	−0·21	−0·30
2	−0·49	+0·05	−0·225	14	−0·32	−0·08	−0·19
3	−0·35	+0·07	−0·14	15	−0·37	+0·12	−0·12
4	−0·08	+0·36	+0·14	16	−0·09	+0·15	+0·03
5	+0·07	+0·29	+0·175	17	−0·01	+0·31	+0·17
6	+0·13	+0·21	+0·17	18	+0·27	+0·34	+0·31
7	+0·14	+0·09	+0·11	19	+0·15	+0·07	+0·11
8	+0 11	−0·09	+0·01	20	+0·07	+0·04	+0·06
9	+0·01	−0·25	−0·12	21	−0·15	−0·15	−0·15
10	−0·26	−0·33	−0·30	22	−0·45	−0·24	−0·35
11	−0·42	−0·42	−0·43	23	−0·43	−0·18	−0·30

The principal differences which show themselves in this comparison of opposite seasons are, that the easterly deflection at the upper culmination is less, and the westerly deflection on both sides of the meridian is greater, in the winter than in the summer half-period; at the inferior culmination the amount of extreme easterly deflection is nearly identical in the two seasons. For the purpose of examining how far similar partial differences would be shown when periods of similar duration are taken without reference to season, I arranged as in the following table, a comparison of the results in two half-periods, each comprising three complete years (including both summer and winter months), and each, therefore, containing the same number of months, viz. thirty-six, as in Table II.

Table III.

Lunar hours.	October 1842 to September 1845.	October 1845 to September 1848.	Table I.	Lunar hours.	October 1842 to September 1845.	October 1845 to September 1848.	Table I.
	Scale divisions.	Scale divisions.	Scale divisions.		Scale divisions.	Scale divisions.	Scale divisions.
0	−0·32	−0·39	−0·35	12	−0·49	−0·35	−0·42
1	−0·26	−0·38	−0·32	13	−0·34	−0·26	−0·30
2	−0·18	−0·27	−0·225	14	−0·27	−0·12	−0·19
3	−0·05	−0·23	−0·14	15	−0·11	−0·13	−0·12
4	+0·15	+0·14	+0·14	16	+0·07	−0·01	+0·03
5	+0·21	+0·14	+0·175	17	+0·20	+0·10	+0·17
6	+0·20	+0·13	+0·17	18	+0·31	+0·31	+0·31
7	+0·15	+0·08	+0·11	19	+0·13	+0·09	+0·11
8	−0·04	+0·05	+0·01	20	+0·07	+0·04	+0·06
9	−0·10	−0·14	−0·12	21	−0·17	−0·13	−0·15
10	−0·34	−0·26	−0·30	22	−0·31	−0·39	−0·35
11	−0·47	−0·37	−0·43	23	−0·39	−0·22	−0·30

The inference from these two comparisons, Tables II. and III., appears to be simply, that in consequence of the small amount of the moon's influence, which has to be brought out from amongst many disturbing causes of much greater influence, the mean result of six years is considerably more satisfactory than the mean results of three years of observation, whether the latter consist of three complete years, or of three years composed either of thirty-six summer or of thirty-six winter months; but that the variation due to the moon's influence is distinctly cognisable in all cases. The results may be more irregular in the winter months, in consequence possibly of the greater prevalence of the other disturbing causes at that season; and these, in a less complete system of observation, may in the extreme winter months, as appears to be the case at Prague, even wholly mask the effect of the moon's influence. In the present case, if we take a mean of opposite points of the hour-circle at Toronto in the summer and winter half-period, we find the amount of the whole deflection, between 0^h and 12^h on the one hand, and 6^h and 18^h on the other hand, to be 29″ in the three years consisting of thirty-six summer months, and 24″·9 in the three years consisting of thirty-six winter months. If we do the same by the half-periods, each of three complete years or of eighteen summer and eighteen winter months, we find the whole deflection between 0^h and 12^h on the one hand, and 6^h and 18^h on the other, 28″·5 in the years 1842–45, and 24″·4 in the years 1845–48. The differences from the results of the six years combined are nearly equal in both cases.

In order to examine whether any difference is to be traced in the lunar diurnal variation which it might be possible to attribute to the moon's change of declination, I caused two tables to be formed, consisting respectively of the lunar diurnal variation on the three days of highest northern, and on the three days of highest southern declination throughout the six years of observation; the three days in the first table consisted in every case of the day when the moon was at her extreme distance north of the equator, and of one day before and one day after; and in the second table of the day when she was at the extreme distance south of the equator, and of one day on either side of that day. The first table comprised 230 days and the second 226 days; the reason of the number of days falling short of the full complement being the intervention of Sundays, when no observations were recorded. Table IV. exhibits in one view the results of these two tables. It does not appear to present any satisfactory evidence of the existence of a difference either in the character or the amount of the lunar diurnal variation at the periods of the moon's extreme divergence on either side of the equator; at the same time the number of days (230 and 226) is too few to furnish *positive* evidence as to the existence of a *small* difference; the general character of the variation appears to be the same in both cases, when allowance is made for irregularities which must be expected to present themselves in such short periods of comparison.

TABLE IV.

Lunar hours.	Days of the moon's extreme north declination.	Days of the moon's extreme south declination.	Mean of six years, Table I.	Lunar hours.	Days of the moon's extreme north declination.	Days of the moon's extreme south declination.	Mean of six years, Table I.
	Scale divisions.	Scale divisions.	Scale divisions.		Scale divisions.	Scale divisions.	Scale divisions.
0	−·35	−·35	−·35	12	−·47	−·31	−·42
1	−·33	−·17	−·32	13	−·14	−·28	−·30
2	−·48	−·10	−·225	14	−·16	−·28	−·19
3	−·13	+·13	−·14	15	−·16	−·10	−·12
4	+·05	+·11	+·14	16	+·07	−·04	+·03
5	+·18	−·04	+·175	17	+·01	+·36	+·17
6	+·15	+·01	+·17	18	+·15	+·18	+·31
7	+·16	−·03	+·11	19	−·12	+·19	+·11
8	−·01	+·10	+·01	20	+·14	+·08	+·06
9	00	+·07	−·12	21	+·02	+·04	−·15
10	−·19	−·16	−·30	22	−·04	−·35	−·35
11	−·43	−·25	−·43	23	−·22	−·38	−·30

St. Helena.—The hourly observations at St. Helena, which have been employed in this investigation, extended from September 1842 to August 1847, comprehending five years. They have been treated in the manner already described in the discussion of the Toronto observations, with this difference only, that as the magnetic disturbances are less in amount at St. Helena, the limits of exclusion on account of disturbance were taken at 2·5 scale divisions, or 1′·78 in arc.

The whole number of observations at St. Helena, which have been thus treated, amounts to 36,449, of which 2678 have been put aside in consequence of the amount of disturbance (measured from the monthly mean at the same hour) equalling or exceeding 1′·78 in arc. The following Table exhibits the mean variation in the five years at the several lunar hours, due to the moon's influence. The deflections are those of the north end of the magnet, and are easterly when the − sign is used, and westerly when the + sign is used. The value of a scale division at St. Helena was 0′·711.

TABLE V.

Lunar hours.	Deflections.	Lunar hours.	Deflections.	Opposite points on the hour-circle combined.	
				Lunar hours.	Deflections.
	Scale divisions.		Scale divisions.		Scale divisions.
2	·00	14	·00	2 and 14	·00
3	−·10	15	−·09	3 and 15	−·095
4	−·08	16	−·11	4 and 16	−·095
5	−·10	17	−·08	5 and 17	−·09
6	−·07	18	−·06	6 and 18	−·065
7	+·01	19	+·01	7 and 19	+·01
8	+·09	20	+·06	8 and 20	+·075
9	+·11	21	+·11	9 and 21	+·11
10	+·14	22	+·16	10 and 22	+·15
11	+·16	23	+·10	11 and 23	+·13
12	+·14	0	+·08	12 and 0	+·11
13	+·09	1	+·06	13 and 1	+·075

The differences between the deflections at opposite points of the hour-circle at this station have so much the character of mere accidental irregularities that we may venture for greater simplicity to combine them, as is done in the two last columns of Table V., and to represent the deflections thus combined by a formula of fewer terms, in which *a*, the hour-angle (reckoned from eighteen hours), is multiplied by 30° instead of 15°. The formula is as follows:—

$$\Delta_r = +{\cdot}02625 + {\cdot}1259 \sin(a + 317^\circ\ 51') + {\cdot}01047 \sin(2a + 25^\circ\ 57')$$
$$- {\cdot}0083 \sin(3a + 5^\circ\ 43') + {\cdot}0118 \sin(4a + 287^\circ\ 48')$$
$$+ {\cdot}00235 \sin(5a + 291^\circ\ 26') + {\cdot}0029 \cos 6a.$$

The lunar diurnal variation at St. Helena is, as at Toronto, a double progression, having two westerly extremes at nearly opposite points of the hour-circle, and two easterly extremes also at nearly opposite points. The amounts of extreme deflection, both easterly and westerly, are considerably less than at Toronto, but exhibit the same peculiarity of the extreme elongations in the one direction being about one-third greater than the extreme elongations in the other direction; at Toronto, the easterly elongations are the greater, at St. Helena the westerly. But a still more remarkable distinction in the lunar variation at the two stations respects the lunar hours at which the elongations respectively occur. At Toronto, as we have seen, the two easterly extremes coincide nearly with the epochs of the moon's superior and inferior culminations, and the westerly extremes nearly with the quadrantal hours of 6 and 18. At St. Helena, on the other hand, neither the times of the easterly nor of the westerly elongation coincide either with the culminations or with the hours of 6 and 18; and are so far and so systematically distant from those hours as to leave no doubt of the reality of their differences from them. The above formula gives for the times of extreme easterly deflection at St. Helena $3^h\ 24^m$ and $15^h\ 24^m$; and for the times of extreme westerly deflection $10^h\ 6^m$ and $22^h\ 6^m$; and the four hours at which the lunar variation disappears, or becomes 0, are $2^h\ 0^m$, $6^h\ 53^m$, $14^h\ 0^m$ and

$18^h\ 53^m$. The mean amount of extreme easterly elongation is $4''\cdot48$, and of extreme westerly elongation $6''\cdot43$, making the whole range of variation in a lunar day due to the moon's influence $10''\cdot91$.

The question of any possible difference existing in the lunar diurnal variation at St. Helena when the moon is at her extreme distance on either side of the equator, has been examined in the same manner as has been already described in pages 554 and 555 in the case of the Toronto observations, and with the same general result. The lunar diurnal variation appears to be the same whether the moon is N. or S. of the equator (or in this case N. or S. of the station). The amount of the lunar diurnal variation being altogether less at St. Helena than at Toronto, small irregularities naturally appear larger in proportion. The number of days included in the comparison is also less, extending through five years only instead of six.

Hobarton.—The hourly observations at Hobarton which have been employed for this investigation are from January 1843 to December 1847 inclusive, comprehending five years; their number is 36,820, of which 3242 have been omitted as disturbed, being found to differ from the monthly mean at the same hour to an amount equalling or exceeding 3·4 scale divisions, or $2'\cdot41$ in arc.

The deflections are those of the north end of the magnet, and the signs imply, as at each of the other two stations, an easterly deflection when the — sign is used, and a westerly deflection when the + sign is used. The value of a scale division at Hobarton was $0'\cdot71$. The deflections are exhibited in the following Table, which may be advantageously compared with the similar table at Toronto, page 552.

TABLE VI.

Lunar hours.	Deflections.	Lunar hours.	Deflections.	Lunar hours.	Deflections.	Lunar hours.	Deflections.
	Scale divisions.		Scale divisions.		Scale divisions.		Scale divisions.
23	+·08	6	—·05	10	+·05	18	—·08
0	+·15	7	—·06	11	+·14	19	—·15
1	+·19	8	—·10	12	+·25	20	—·15
2	+·24	9	—·01	13	+·33	21	—·12
3	+·30			14	+·31	22	—·02
4	+·21			15	+·24		
5	+·07			16	+·15		
				17	+·01		

This table exhibits, as at Toronto and St. Helena, a double progression, having two westerly and two easterly extremes at nearly opposite points of the hour-circle; and still showing, as at the other two stations, the phænomenon of the elongations in one direction being considerably greater than in the other direction. At Hobarton, as at St. Helena, the westerly elongations are the greatest, whilst at Toronto the easterly are greatest. The hours of extreme elongation differ both from those at Toronto and those at St. Helena. The values in Table VI. may be represented by the following formula, in which a, the hour-angle reckoned as before from eighteen hours, is multiplied by 15°.

$$\begin{aligned}\Delta_s = +\cdot0825 &- \cdot0487 \sin(a+39°\ 15') - \cdot2056 \sin(2a+38°\ 00') \\ &+ \cdot0173 \sin(3a+339°\ 41') - \cdot0161 \sin(4a+68°\ 56') \\ &- \cdot0187 \sin(5a+274°\ 54') + \cdot0083 \sin(6a+323°\ 08') \\ &- \cdot0098 \sin(7a+52°\ 26') + \cdot0076 \sin(8a+19°\ 06') \\ &- \cdot0023 \sin(9a+334°\ 32') + \cdot0015 \sin(10a+323°\ 08') \\ &- \cdot0100 \sin(11a+274°\ 44') - \cdot0025 \cos 12a.\end{aligned}$$

This formula gives for the times of extreme westerly deflection $2^h\ 52^m$ and $13^h\ 21^m$, the amounts being $12''\cdot8$ and $14''\cdot9$; and for the times of extreme easterly deflection $8^h\ 0^m$ and $19^h\ 30^m$, the amounts being $4''\cdot3$ and $6''\cdot7$. The means of the two times of westerly elongation are $2^h\ 06^m\cdot5$ and $14^h\ 06^m\cdot5$, and of the amounts, $13''\cdot85$. The means of the two times of easterly elongation are $7^h\ 45^m$ and $19^h\ 45^m$, and of the amounts, $5''\cdot5$. The total range of the lunar variation is $(13''\cdot85+5''\cdot5=)19''\cdot35$.

The times at which the deflection due to the moon's action is 0 are $5^h\ 27^m$, $9^h\ 06^m$, $17^h\ 05^m$, and $22^h\ 09^m$. These times taken in pairs are equidistant from two points on the moon's hour-circle, of which one is $1^h\ 48^m$ after her upper culmination, and the other is $1^h\ 05^m\cdot5$ after her lower culmination.

General Remarks.—We learn from the results which have been thus stated, that the existence of a lunar diurnal variation in the magnetic declination is shown at each of the three stations, Toronto, St. Helena, and Hobarton, and that it has the same general character at each, viz. that of a double progression in the lunar day with two easterly maxima at nearly opposite points of the hour-circle, and two westerly maxima also at nearly opposite points of the hour-circle. The extreme elongations in the one direction are not at precisely opposite points of the hour-circle at any of the three stations, nor have the amounts of the two elongations, which take place in the same direction, always precisely the same value, but the slight inequalities in these respects are within the limits which might be occasioned by accidental irregularities. It is otherwise, however, with the difference in amount between the easterly and westerly deflections, which exhibit at each of the three stations a disparity too great to be ascribed to accident. At Hobarton and St. Helena the westerly elongations have the largest values, and at Toronto the easterly (speaking always of the north end of the magnet). At Toronto and St. Helena the lesser elongations are about two-thirds of the amount of the larger; at Hobarton the disparity is still greater; to obtain the exact proportions with a sufficient degree of assurance would require, no doubt, a longer period of observation.

The difference which presents itself at the different stations, in the times of the occurrence of the extreme easterly and westerly elongations of the variation, is a very important feature in its bearing on the questions which must arise in regard to the nature and character of the lunar magnetic influence. At Toronto the easterly elongations take place almost exactly at the hours of upper and lower culmination, and the westerly elongations at the intermediate hours of 6 and 18. At Hobarton the westerly elongations (which, as Hobarton and Toronto are in opposite hemispheres, may be regarded as corresponding to the easterly at Toronto) take place (speaking generally) about two hours after the upper and lower culminations, and the easterly elongations (corresponding to the westerly at Toronto) about two hours after the lunar hours of 6 and 18. At St. Helena, which is in the same hemisphere with Hobarton, the westerly elongations occur about two hours before the culminations, and the easterly extremes about two hours, or rather more, before the hours of 6 and 18. The differences in this respect at the different stations are a feature of such essential theoretical importance, that I should have delayed the presentation of this paper until it could have been accompanied by a similar investigation into the results obtainable from the five years of hourly observations printed in the first volume of the Magnetical Observations at the Cape of Good Hope, had I not the expectation that that portion of the task may be undertaken by Mr. PIERCE MORTON, who now conducts the magnetical observatory at that station.

The entire amount of the variation, reckoned from one extreme elongation to the other, is at Toronto about 27″; at Hobarton about 20″; and at St. Helena about 10″. The terrestrial force, retaining the magnet suspended horizontally in its mean position, may be taken approximately at Toronto at 3·54, at Hobarton at 4·51, and at St. Helena at 5·57. It may be desirable also to state that the amount of the magnetic Declination at the period under consideration was, at Toronto 1° 34′ West, at Hobarton 9° 55′ East, and at St. Helena 23° 25′ West.

The number of observations which have contributed to these results (exclusive of those omitted on account of disturbance) are, 38,398 at Toronto, 33,771 at St. Helena, and 33,578 at Hobarton; making, altogether, 105,747 observations.

With the exception of marking the observations for the lunar hours to which they most nearly correspond, and the calculation of the constants in the formulæ at Toronto and St. Helena, the processes to which the observations received from the stations have been submitted have been executed by Mr. JOHN MAGRATH, principal Clerk in the Woolwich Office, assisted by Sergeant CHARLES ORGAN and Corporal MATTHEW M'ILROY, of the Royal Artillery, employed in the office as military clerks.

Woolwich, November 16, 1853.

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PHILOSOPHICAL TRANSACTIONS

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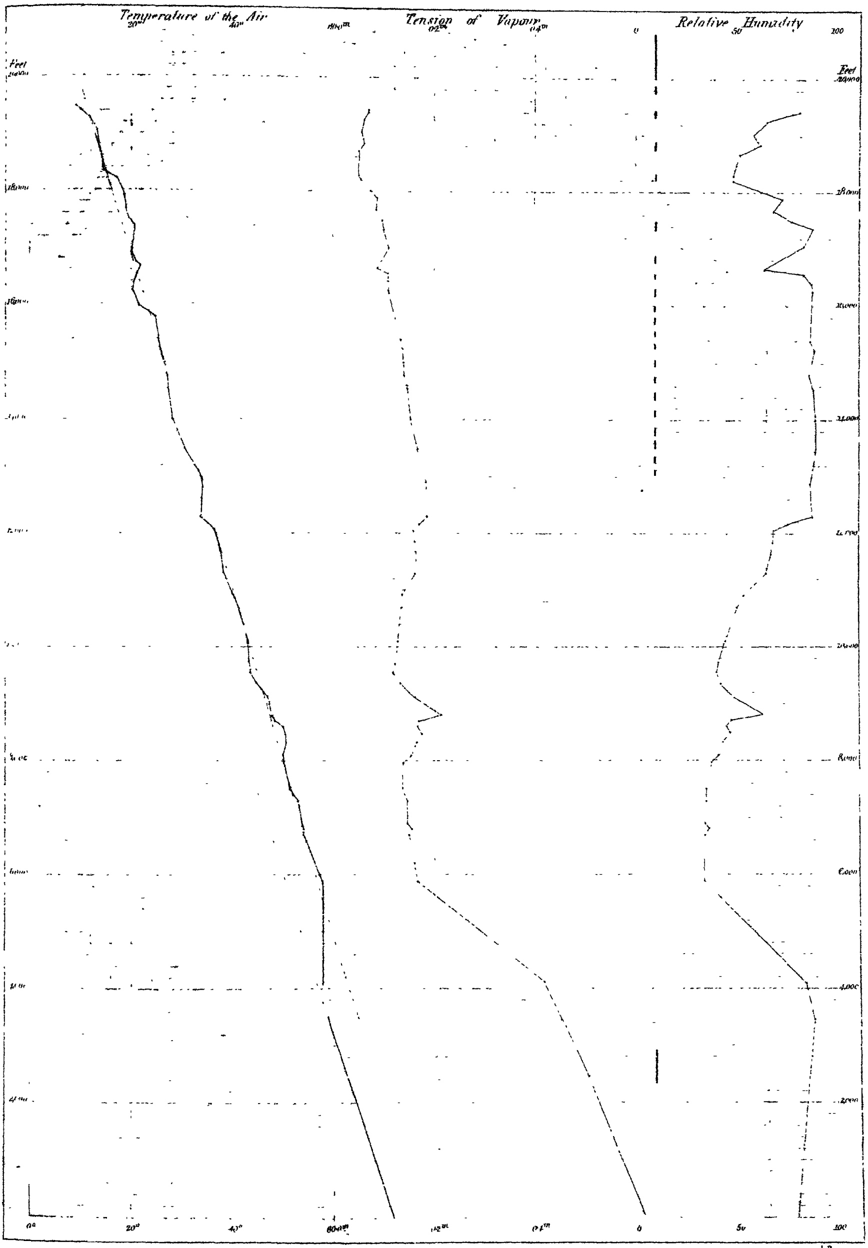

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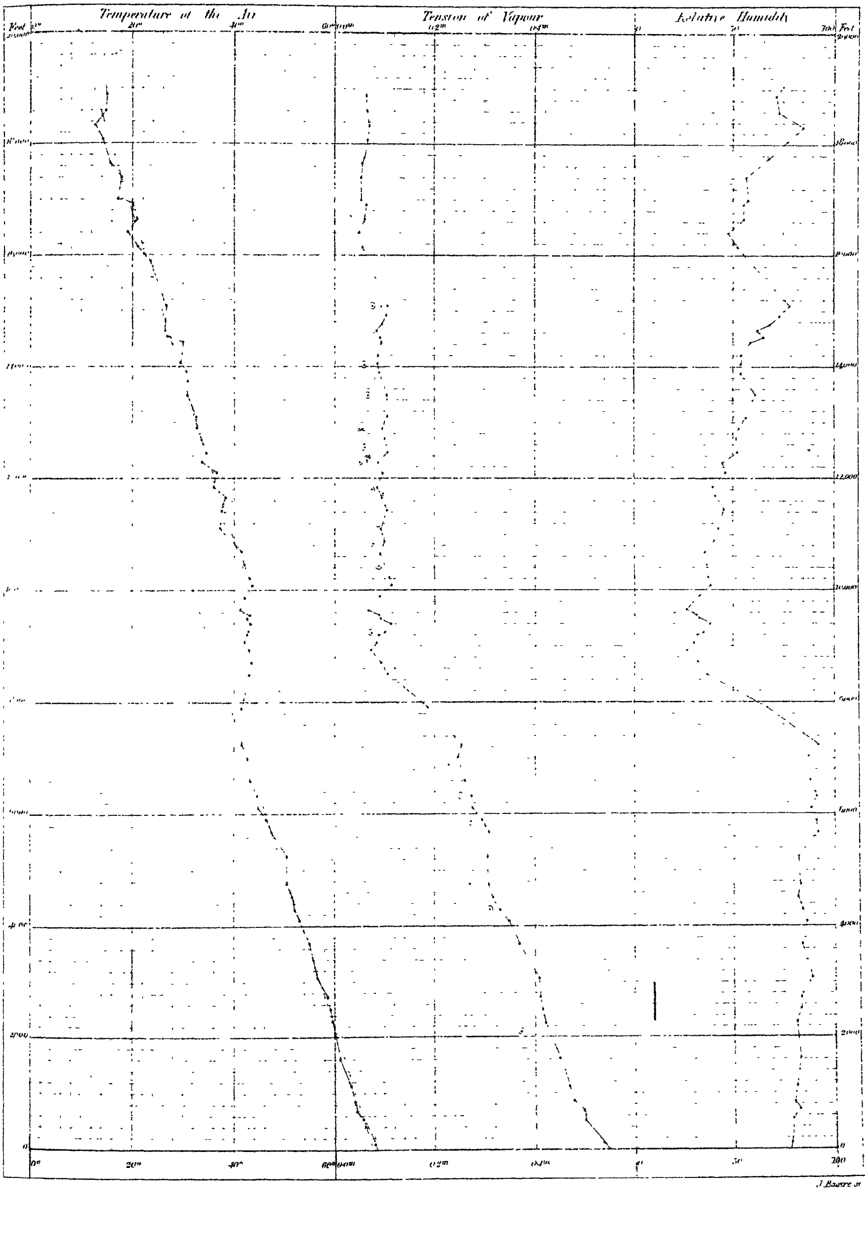

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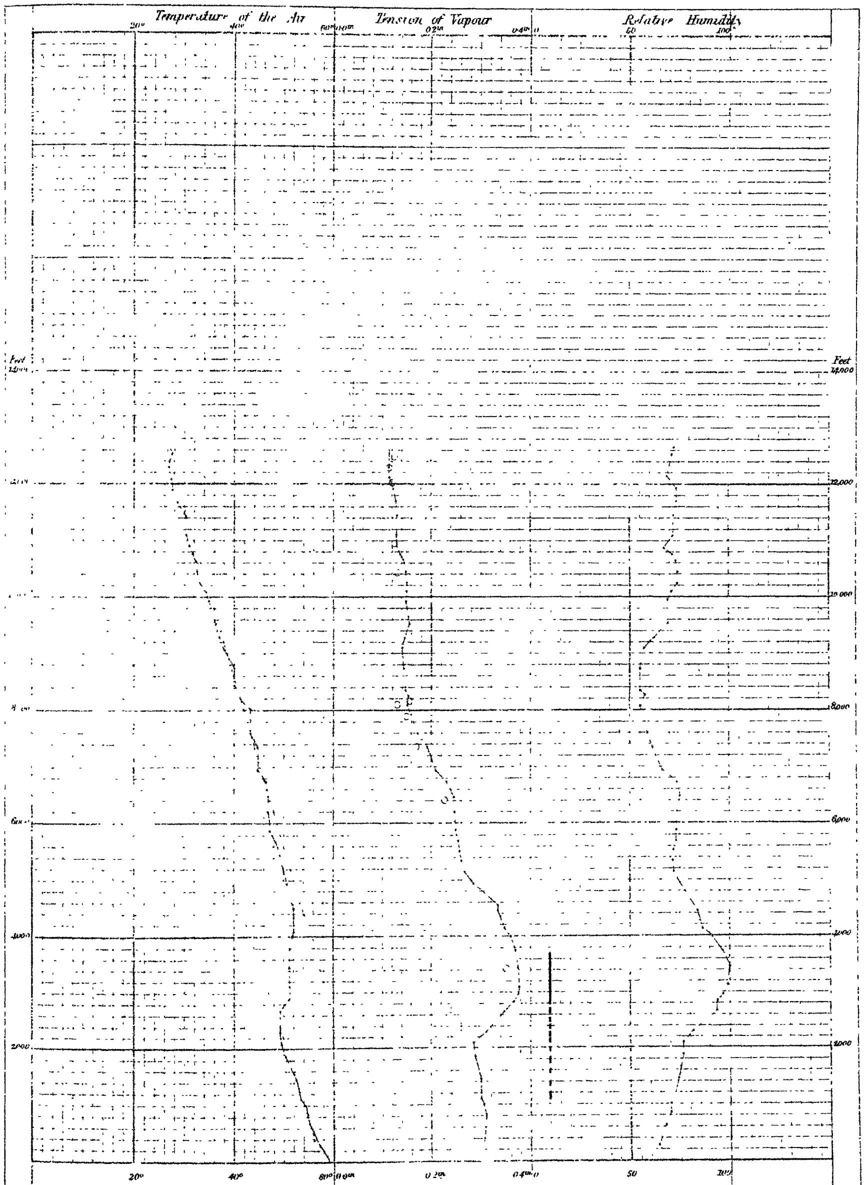

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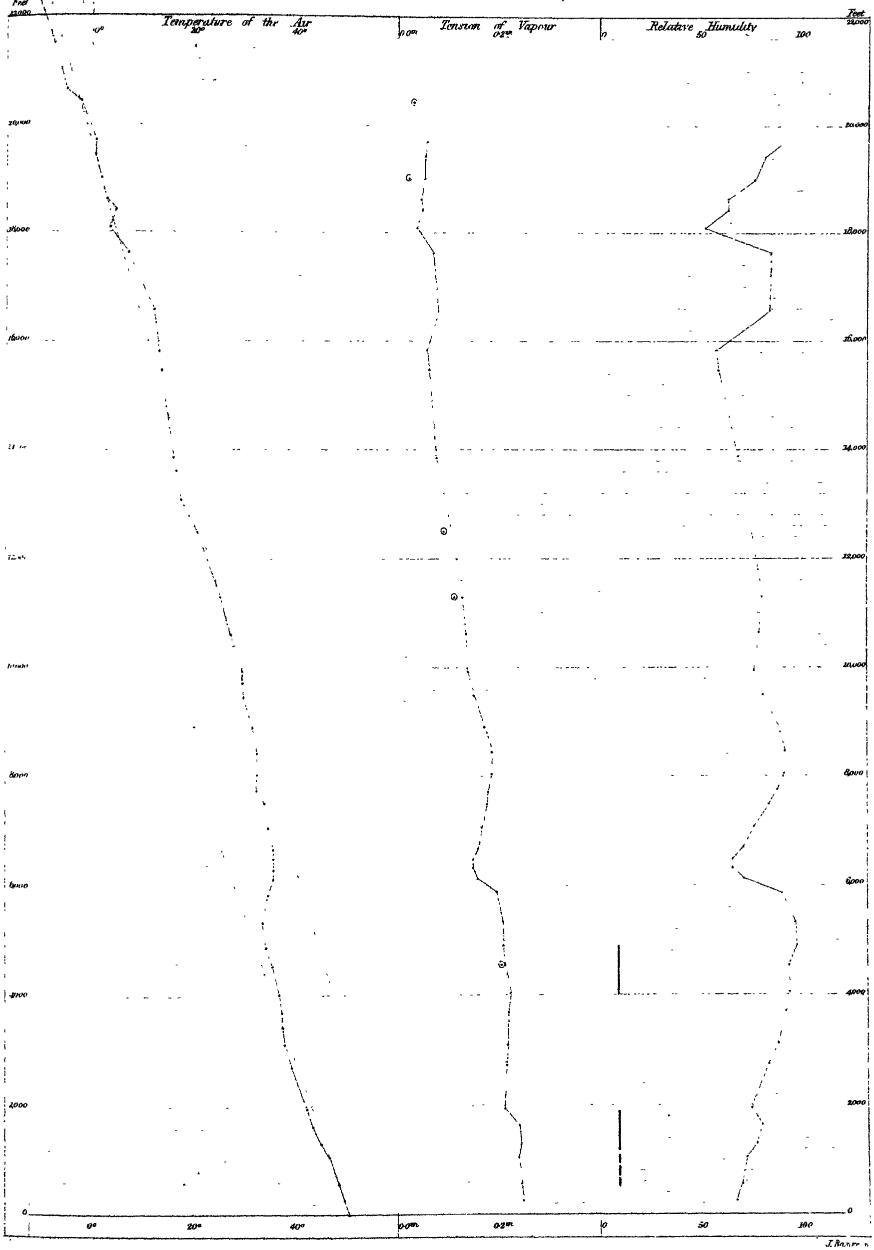

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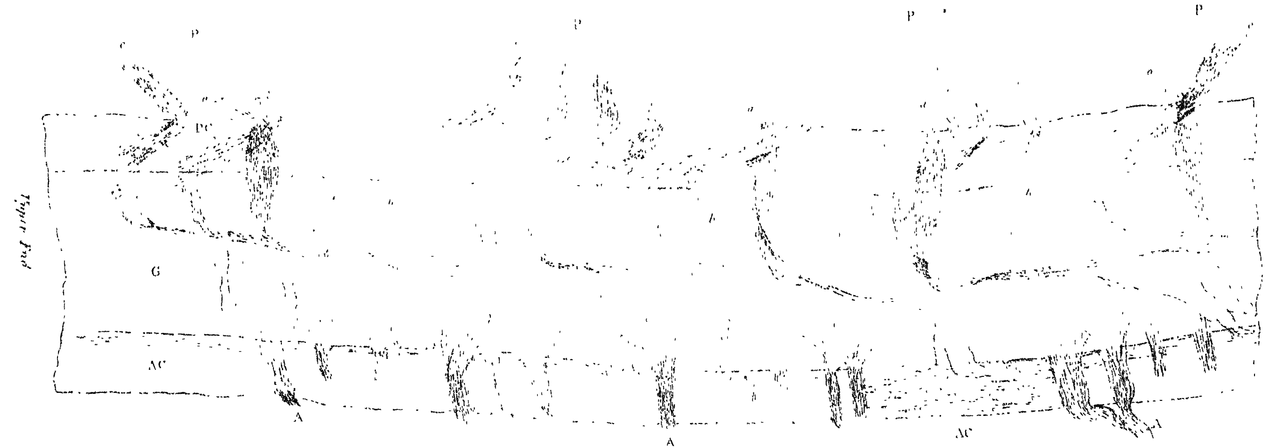


magnified 30 diameters


p
p
p
p
a
a
b'
d
a
d

[illegible]

[illegible]

[illegible]

Fig 2

Fig 3

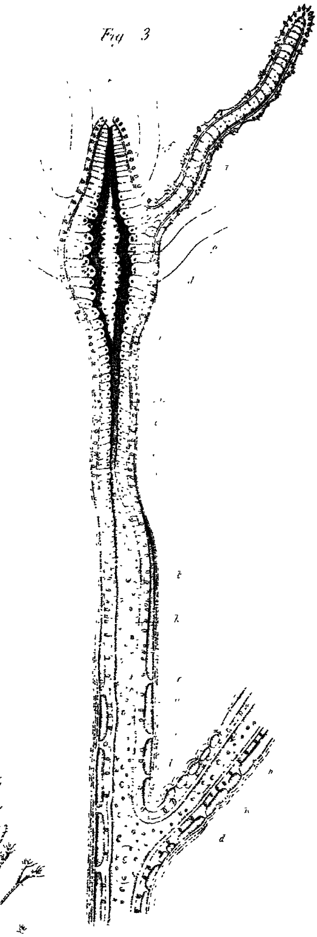
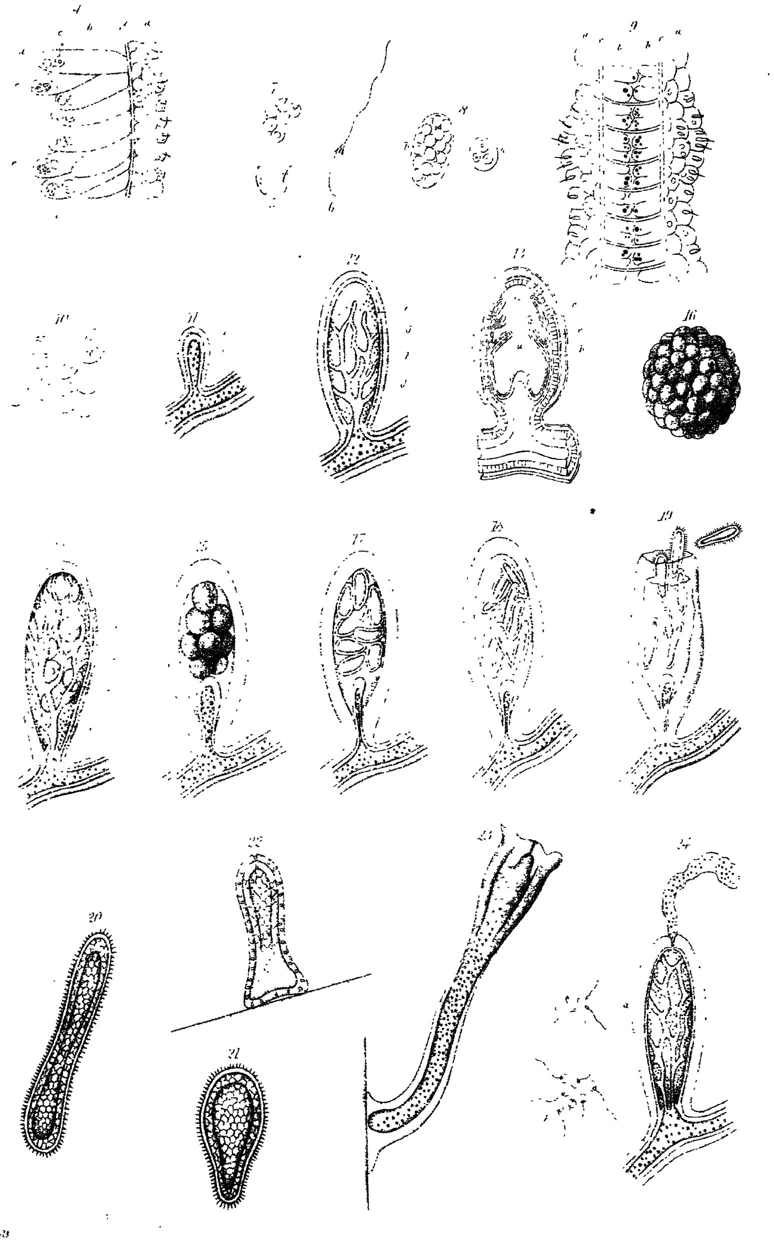


Fig 1

G. J. Allman, del.

J. Basire sc.



G.J Allman del.