On Finite Groups and Their Associated Graphs

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[2024]

Dedicated to My Loving Parents

&

All My Teachers

BIRLA INSTITUTE OF TECHNOLOGY AND SCIENCE, PILANI
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CERTIFICATE

This is to certify that the thesis entitled, **"On Finite Groups and Their Associated Graphs"** submitted by **Mr. Parveen** ID No. **2019PHXF0406P** for the award of Ph.D. degree of the institute embodies original work done by him under my supervision.

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Abstract

The study of graphs associated to groups, viz. Cayley graphs and commuting graphs, have been extensively studied in the literature because they have valuable applications and are related to automata theory. Motivated by the study of such graphs, the present thesis aims to investigate some more interconnections between finite groups and their associated graphs. In this context, the enhanced power graphs and power graphs of finite groups have been explored further. The *enhanced power graph* of a group G is the simple graph with the vertex set G and two distinct vertices x, y are adjacent if they belong to the same cyclic subgroup of G. The *power graph* of a group G is the simple graph with the vertex set G and two vertices a, b are adjacent if one is a power of the other.

This thesis begins with a characterization of finite groups such that the minimum degree and the vertex connectivity of the enhanced power graph are equal. Also, we give a description of finite groups with regular and strongly regular (proper) enhanced power graphs. Then we obtain an upper and a lower bound for the Wiener index of the enhanced power graph of nilpotent groups. Also, the finite nilpotent groups achieving these bounds have been characterized. Recently, the lambda number of the power graphs has been studied. In this thesis, we study the lambda number of the enhanced power graph of a finite group. We also show that the lambda number of the enhanced power graph of a finite simple group G is equal to $|G|$ if and only if G is not a cyclic group of order $n \geq 3$. Further, we discuss some graph-theoretic parameters including metric dimension, resolving polynomials and detour distant properties of enhanced power graphs of certain non-abelian groups. This thesis also gives an affirmative answer to the question posed by Cameron about the connected components of the complement of the enhanced power graph of a non-cyclic group and we proved that for a non-cyclic finite group G, the complement of the enhanced power graph of G has just one connected component apart from isolated vertices. Then we study a connected subgraph $\overline{\mathcal{P}_E(G^*)}$ induced by all the non-isolated vertices of the complement of the enhanced power graph of G. We proved that there does not exist any finite group for which the graph $\overline{\mathcal{P}_E(G^*)}$ is bicyclic, tricyclic or tetracyclic, respectively. We also characterized all the finite groups G such that $\overline{\mathcal{P}_E(G^*)}$ is unicyclic, pentacyclic, outerplanar, planar, projective planar and toroidal, respectively.

This thesis extends the results on the line graph characterization of power graphs of finite nilpotent groups to arbitrary finite groups. We establish a necessary and sufficient condition for finite groups whose enhanced power graphs are line graphs and classify all the finite nilpotent groups (except non-abelian 2-groups) whose proper enhanced power graphs are line graphs. Moreover, we determine all the finite groups whose (proper) power graphs and (proper) enhanced power graphs are the complement of line graphs, respectively. In this thesis, we also continue to study the difference graph of a finite group which is the difference of the enhanced power graph and the power graph of a finite group with all isolated vertices removed. We characterize arbitrary finite groups such that the difference graph is a chordal graph, star graph, dominatable, threshold graph and split graph, respectively. Then we precisely characterize all the nilpotent groups for which the difference graph belongs to the aforementioned five graph classes. We also determine all the finite nilpotent groups whose difference graph is cograph, bipartite, Eulerian, planar and outerplanar, respectively. Further, we characterize all the finite nilpotent groups such that the genus (or cross-cap) of the difference graph is at most 2. Finally, we study the difference graphs of certain non-nilpotent groups.

List of Symbols

$\mathbb{N}$	the set of natural numbers
$\mathbb{Z}$	the set of integers
$[n]$	$\{1, 2, ..., n\}$
$\gcd(a,\ b)$	greatest common divisor of a and b
$\operatorname{lcm}(a,\ b)$	least common mutliple of a and b
G	group
$\lvert X \rvert$	cardinality of a set X
$\langle a \rangle$	cyclic group generated by a
$\mathbb{Z}_n$	cyclic group of order n
$o(a)$	order of an element a
π_G	$\{o(a) :\ a \neq e \in G\}$
$\mathcal{M}(G)$	the set of all maximal cyclic subgroups of G
$\exp(G)$	the exponent of G
S_n	symmetric group
A_n	alternating group

$\overline{\Gamma}$	complement of a graph Γ
$\mathcal{L}(\Gamma)$	line graph of a graph Γ
$a \sim b$	a and b are adjacent
$a \nsim b$	a and b are not adjacent
$V(\Gamma)$	vertex set of a graph Γ
$E(\Gamma)$	edge set of a graph Γ
$\mathrm{N}(x)$	$\{y \in V(\Gamma) : y \sim x\}$
$\mathrm{N}[x]$	$\mathrm{N}(x) \cup \{x\}$
$\deg(x)$	the degree of x
$\mathrm{Dom}(\Gamma)$	the set of all dominating vertices of Γ
$\alpha(\Gamma)$	independence number of Γ
$\beta'(\Gamma)$	edge covering number of Γ
$\omega(\Gamma)$	clique number of a graph Γ
$\delta(\Gamma)$	minimum degree of a graph Γ
$\chi(\Gamma)$	chromatic number of Γ
$\kappa(\Gamma)$	vertex connectivity of Γ
$\kappa'(\Gamma)$	edge connectivity of Γ
$\mathcal{P}(G)$	power graph of the group G
$\mathcal{P}_E(G)$	enhanced power graph of the group G
$\mathcal{C}(G)$	commuting graph of the group G

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Introduction

Algebraic graph theory is a branch of mathematics which provides connections between algebra and graph theory. Another sub-branch of algebraic graph theory is spectral graph theory which emphasizes on the study of spectra of matrices (adjacency matrix or Laplacian matrix) associated with graphs. One of the broad research problems in algebraic graph theory is to study graphs associated with algebraic structures. Such problems have been extensively studied by various researchers into three categories: (i) study of graph-theoretic invariants (ii) study of underlying algebraic structures using its associated graph-theoretic invariants (iii) interconnections between algebraic graph and underlying algebraic structure.

The study of graphs associated with algebraic structures, namely: semigroups, groups, rings and vector spaces etc., is a large research area and has attracted considerable attention of various researchers. The study of graphs associated with groups become important due to its valuable applications and connections to automata theory (see Kelarev [2003, 2004]; Kelarev et al. [2009]). In this connection, Cayley graphs and commuting graphs have been studied extensively. Cayley graph was introduced by Arthur Cayley in 1878. The *Cayley digraph* of a finite group G with a generating set S, is a directed graph with vertex set G and for $x, y \in G$ there is an arc from x to y if and only if $xs = y$ for some $s \in S$. Cayley graph is used to design and analyse the topologies of interconnection networks that are helpful to connect processors in a supercomputer (see Cooperman et al. [1991]). In physics, Cayley graphs are useful into the study of quantum walks. For more details about the Cayley graphs, one can refer to Budden [1985]; Droms et al. [1998]; Li [2002]; Lanel et al. [2019] and references therein. Moreover, undirected Cayley graphs have also been studied in the literature (see Abdollahi and Jazaeri [2014]; Zhu et al. [2023]). The commuting graph was introduced by Brauer and Fowler [1955]. For a group G and $S \subseteq G$, the *commuting graph* $\mathcal{C}(G, S)$ is the simple graph whose vertex set is S and two distinct vertices x, y are adjacent if $xy = yx$. For $S = G$, the graph $\mathcal{C}(G, S)$ is denoted by $\mathcal{C}(G)$. This graph has also been studied by considering $G \backslash Z(G)$ as the vertex set, where $Z(G)$ is the center of G. Segev [1999, 2001]; Segev and Seitz [2002] used combinatorial parameters of certain commuting graphs to establish long-standing conjectures in the theory of division algebras. Commuting graphs on groups have played a crucial role in the classification of finite simple groups (see Aschbacher [2000]). Also, Hayat et al. [2019] utilized commuting graphs associated with groups to establish some Non-Singular with a Singular Deck (NSSD) molecular graphs. The commuting graph of a group has been studied extensively, see Iranmanesh and Jafarzadeh [2008]; Britnell and Gill [2017]; Kumar et al. [2021b] and references therein. Motivated by the several applications of these graphs associated to groups, numerous authors introduced and then studied various other graphs associated to groups. To name a few: intersection graph (Zelinka [1973]), prime graph (Tong-Viet [2014]), power graph (Chakrabarty et al. [2009]), enhanced power graph (Aalipour et al. [2017]). Other graphs associated to groups and their literature review can be found in Cameron [2022]. This thesis aims to study enhanced power graph and power graph of a finite group. In what follows, we present the literature review and the main results of the thesis on power graphs and enhanced power graphs associated to finite groups.

The *power graph* $\mathcal{P}(G)$ of a group G is the simple graph with vertex set G, and two vertices a, b are adjacent if one is a power of the other or equivalently: either $a \in \langle b \rangle$ or $b \in \langle a \rangle$. Kelarev and Quinn [2004] introduced the concept of the directed power graph. Later on, the undirected power graphs of groups have been studied by researchers from different perspectives, see Cameron and Ghosh [2011]; Feng et al. [2015]; Chattopadhyay et al. [2021]; Santiago Arguello et al. [2023] and references therein. The graph $\mathcal{P}^*(G)$ is the subgraph of $\mathcal{P}_E(G)$ induced by the set $G \setminus \{e\}$. Regularity of the proper power graph $\mathcal{P}^*(G)$ is studied by Moghaddamfar et al. [2014]. They proved that for a finite non-trivial group G, the proper power graph of G is strongly regular graph if and only if G is a p-group of order p^m for which the exponent of G is either p or p^m. Panda et al. [2024] characterized all the finite nilpotent groups G such that the minimum degree and vertex connectivity of $\mathcal{P}(G)$ are equal. Ali et al. [2022] obtained the metric dimension, resolving polynomials, distant properties and detour distant properties of the power graph $\mathcal{P}(G)$, when G is Q_{4n}, where $n = 2^k$ or D_{2m} for $m = p^\alpha$. Ma, Feng and Wang [2021] studied the $L(2,1)$-labeling of the power graph of a finite group. Recently, Sarkar [2022] investigated the lambda number of power graphs of finite simple groups. Also, Sarkar and Mishra [2023], studied the lambda number of power graphs of finite p-groups. Graph classes such as cographs, chordal graphs, split graphs, and threshold graphs, can be defined in terms of forbidden induced subgraphs. Graphs with forbidden subgraphs appear in extremal graph theory and in some other contexts. Certain forbidden subgraphs of undirected power graphs of groups have been studied (see Doostabadi et al. [2014] and Manna et al. [2021]). Mirzargar et al. [2012] classified all the finite groups with planar power graphs. Further, Doostabadi and Farrokhi D. G. [2017], characterized the finite groups whose power graphs are of (non)orientable genus one. Then all the finite groups with (non)orientable genus two power graphs have been characterized by Ma et al. [2019]. Recently, Bera [2022] characterized finite nilpotent groups G such that $\mathcal{P}(G)$ and $\mathcal{P}^{**}(G)$ are line graphs. Here, the graph $\mathcal{P}^{**}(G)$ is the subgraph of $\mathcal{P}(G)$ induced by the set of all non-dominating vertices. For more results on power graphs of groups, one can refer to the survey paper Kumar et al. [2021a] and references therein.

To measure how much the power graph is close to the commuting graph of a group G, Aalipour et al. [2017] introduced a novel graph called the enhanced power graph. The *enhanced power graph* $\mathcal{P}_E(G)$ of a group G is the simple graph with the vertex set G, and two distinct vertices x, y are adjacent if $x, y \in \langle z \rangle$ for some $z \in G$. Note that the power graph is a spanning subgraph of the enhanced power graph and the enhanced power graph is a spanning subgraph of the commuting graph. Aalipour et al. [2017] provided a characterization for finite groups G where equality holds for any two of the three graphs: power graph, enhanced power graph, and commuting graph of G.

Subsequent research has witnessed significant attention towards enhanced power graphs. Bera and Bhuniya [2018] characterized abelian groups and non-abelian p-groups possessing dominatable enhanced power graphs. Dupont et al. [2023] determined the rainbow connection number of the enhanced power graph of a finite group G. Moreover, they investigated graph-theoretic properties of the enhanced quotient graph of a finite group G (see Dupont et al. [2017]). The study of finite groups whose enhanced power graphs admit a perfect code has been explored by Ma et al. [2017]. Hamzeh and Ashrafi [2017a] derived the automorphism groups of enhanced power graphs of finite groups, while Ma and She [2020] explored the metric dimension of the enhanced power graph of a finite group. In particular, they determined the metric dimension of the enhanced power graph of an elementary abelian p-group, a dihedral group and a generalized quaternion group. Zahirović et al. [2020] proved that two finite abelian groups are isomorphic if their enhanced power graphs are isomorphic and also provided a characterization of finite nilpotent groups with perfect enhanced power graphs. P. Panda et al. [2021] investigated the minimum degree, independence number and matching number of $\mathcal{P}_E(G)$. Also, they determined them when G is finite abelian p-group, the dihedral group and the semidihedral group. Additionally, they proved that if G is one of these groups, then $\mathcal{P}_E(G)$ is perfect and then obtained their strong metric dimension. Further, the enhanced power graphs associated to certain non-abelian groups including generalized quaternion group, U_{6n} and V_{8n} etc., have been explored by Dalal and Kumar [2021]. Forbidden subgraphs of enhanced power graphs of finite groups have been studied by Ma et al. [2021a]. Then Bera et al. [2021] provided an upper bound for the vertex connectivity of the enhanced power graph of any finite abelian group and classified finite abelian groups with connected proper enhanced power graphs. Moreover, they studied the connectivity and the vertex connectivity of proper enhanced power graphs of certain non-abelian groups. The notion of the complement of the enhanced power graph has also been explored by Ma, Doostabadi and Wang [2021]. Mahmoudifar and Babai [2022] proved the enhanced power graph $\mathcal{P}_E(G)$ of a finite group G is dominatable if and only if there exists a prime p such that p divides the center of G and the Sylow p-subgroup of G is either cyclic or a generalized quaternion group. The graph $\mathcal{P}_E^*(G)$ is the subgraph of $\mathcal{P}_E(G)$ induced by the set $G \setminus \{e\}$. The graph $\mathcal{P}_E^*(G)$ is also known as the *deleted enhanced power graph* in the literature. Costanzo et al. [2021] studied the connectedness and diameter of the deleted enhanced power graph of certain finite groups. Bera and Dey [2022] classified all the nilpotent groups whose proper enhanced power graph $\mathcal{P}_E^{**}(G)$ is connected and calculated their diameter. Moreover, they determined the domination number of proper enhanced power graphs of finite nilpotent groups. They also calculated the multiplicity of the Laplacian spectral radius of $\mathcal{P}_E(G)$ when G is a finite nilpotent group. Dalal et al. [2023] calculated the number of connected components of $\mathcal{P}_E^*(G)$ when G is symmetric or alternating group. Cameron and Phan [2023] proved that the enhanced power graph of a finite group is weakly perfect. For a comprehensive overview of results and open questions on enhanced power graphs of groups, one can refer to the survey paper Ma et al. [2022] and references therein.

In this thesis, we study some more interconnections between enhanced power graphs and underlying groups. Also, some results of power graph of finite nilpotent groups are extended to arbitrary finite groups. Since the power graph is a spanning subgraph of the enhanced power graph, therefore we also study the difference graph of the enhanced power graph and the power graph of a finite group. The thesis is arranged into the following six chapters.

Chapter 1: Background

Chapter 2: Enhanced Power Graphs

Chapter 3: The Complement of Enhanced Power Graph

Chapter 4: Line Graph Characterization of Power Graphs

Chapter 5: Difference Graph of Power Graphs

Chapter 6: Conclusion and Future Research Work

In Chapter 1, we recall the necessary definitions and required results. Also, we fix our notations and derive some essential results which are useful in the sequel.

In Chapter 2, we develop some further results on the enhanced power graph of finite groups. In this connection, we characterize finite groups such that the minimum degree and the vertex connectivity of their enhanced power graphs are equal (see Theorem 2.1.2). We also study the regularity of (proper) enhanced power graphs of groups. All the finite groups are characterized for which (proper) enhanced power graphs are (strongly) regular (see Theorems 2.1.4, 2.1.6). In order to study the Wiener index of enhanced power graphs of nilpotent groups, we derive an upper and lower bounds of the Wiener index of enhanced power graphs of nilpotent groups. We also characterize finite nilpotent groups attaining these bounds (see Theorem 2.1.20).

The lambda number of the power graph has been studied in the literature. In this chapter, we study the λ-number of the enhanced power graph of a finite group. We obtain an upper and a lower bound for the lambda number of the enhanced power graphs of finite groups (see Theorem 2.2.5). Notably, for non-trivial simple groups, we establish conditions under which the λ-number is equal to the order of the group. We prove that $\lambda(\mathcal{P}_E(G)) = n$ if and only if G is not a cyclic group of order $n \geq 3$ (see Theorem 2.2.9). Also, we obtain the lambda number of the enhanced power graphs of nilpotent groups (see Theorems 2.2.16 and 2.2.19).

In the final part of this chapter, we obtain the Laplacian spectrum of the enhanced power graph of certain non-abelian groups namely: semidihedral groups, dihedral groups and generalized quaternion groups (see Theorems 2.3.2, 2.3.5 and 2.3.8). We determine the metric dimension and resolving polynomial of the enhanced power graph of semidihedral group (see Proposition 2.3.13 and Theorem 2.3.14). Moreover, we study the distant properties and detour distant properties, namely: closure, interior, distance degree sequence, and eccentric subgraph of the enhanced power graphs of these groups.

The results of Chapter 2 are published in the journals *"Acta Mathematica Hungarica"* (SCIE), *"Journal of Algebra and Its Applications"* (SCIE) and *"Discrete Mathematics, Algorithms and Applications"* (ESCI).

The motivation of Chapter 3 is to answer the question posed by Cameron [2022]: *Is it true that the complement of the enhanced power graph of a finite non-cyclic group has just one connected component apart from isolated vertices*? In Chapter 3, we gave an affirmative answer to this question (see Theorem 3.1.2). We classify all finite groups G such that the graph $\overline{\mathcal{P}_E(G)}$ is bipartite. We determine the girth of $\overline{\mathcal{P}_E(G)}$ and we also prove that the graph $\overline{\mathcal{P}_E(G)}$ is weakly perfect (see Theorem 3.1.3). It is natural to study the subgraph $\overline{\mathcal{P}_E(G^*)}$ of $\overline{\mathcal{P}_E(G)}$ induced by all the non-isolated vertices in $\overline{\mathcal{P}_E(G)}$. Thus, we characterize the groups G such that the graph $\overline{\mathcal{P}_E(G^*)}$ is dominatable and Eulerian, respectively. We classify all the finite groups G such that the graph $\overline{\mathcal{P}_E(G^*)}$ is unicyclic and pentacyclic, respectively. Additionally, we prove that the graph $\overline{\mathcal{P}_E(G^*)}$ cannot be bicyclic, tricyclic and tetracyclic, respectively (see Theorem 3.1.11).

All the graphs are not planar but they can be embedded on topological surfaces like g-hole torus, projective plane and Klein bottle etc. In view of this, we also study various embeddings of the graph $\overline{\mathcal{P}_E(G^*)}$ on certain surfaces. We classify all the finite groups G such that the graph $\overline{\mathcal{P}_E(G^*)}$ is outerplanar, planar, projective-planar and toroidal, respectively. We prove that there does not exist a group G such that the cross-cap of the graph $\overline{\mathcal{P}_E(G^*)}$ is two (see Theorem 3.2.1).

The results of this chapter are accepted for publication in SCIE journal *"Algebra Colloquium"*.

Bera [2022] characterized certain finite nilpotent groups where both power graphs and proper power graphs are line graphs. In Chapter 4, we extend his results to arbitrary finite groups (see Theorems 4.1.2 and 4.1.3). We also correct one of his results related to proper power graphs of dihedral groups (see Corollary 4.1.5). Furthermore, we characterize finite groups whose enhanced power graphs are line graphs (see Theorem 4.2.2). For finite nilpotent groups (excluding non-abelian 2-groups), we classify groups whose proper enhanced power graphs are line graphs (see Theorem 4.2.9). Finally, we determine all the finite groups whose (proper) power graphs (see Theorems 4.3.3 and 4.3.5) and (proper) enhanced power graphs (see Theorem 4.3.6) are the complement of line graphs, respectively.

The results of this chapter are accepted for publication in SCIE journal *"Journal of Algebra and Its Applications"*.

Biswas et al. [2022] studied the difference graph $\mathcal{D}(G) := \mathcal{P}_E(G) - \mathcal{P}(G)$ of the enhanced power graph and the power graph of G with all isolated vertices removed. They have investigated the connectedness and perfectness of $\mathcal{D}(G)$ for certain group classes. In Chapter 5, we continue the study of the difference graph $\mathcal{D}(G)$. Among other results, we characterize the finite groups G such that $\mathcal{D}(G)$ is a graph with forbidden induced subgraphs. After ascertaining the dominating vertices of $\mathcal{D}(G)$, we provide equivalent conditions on the finite group G such that $\mathcal{D}(G)$ is a chordal graph, star graph, dominatable, threshold graph, and split graph, respectively (see Theorems 5.1.8, 5.1.10 and 5.1.11). We classify all the finite nilpotent groups G such that $\mathcal{D}(G)$ belongs to the above five graph classes (see Theorem 5.2.2). Furthermore, we characterize the nilpotent groups whose difference graphs are cograph (see Theorem 5.2.3), bipartite (see Theorem 5.2.4), Eulerian (see Theorem 5.2.6), planar (see Theorem 5.2.8), and outerplanar (see Theorem 5.2.10), respectively. Further, we classify all the finite nilpotent groups such that the difference graph $\mathcal{D}(G)$ is of genus (or cross-cap) at most two (see Theorem 5.3.1). Finally, in Chapter 5, we study the difference graph of non-nilpotent groups. We classify all the values of n for which the difference graph of the symmetric group S_n (or alternating group A_n) is cograph, chordal, spilt, and threshold, respectively.

The results of Chapter 5 are published in SCIE journals *"Quaestiones Mathematicae"* and *"Ricerche di Matematica"*.

The thesis is summarized in Chapter 6 and concluded with some future research work.

Chapter 1

Background

In this chapter, we recall the necessary definitions and derive some required results which will be used in the thesis. Necessary definitions and essential results of group theory and graph theory have been presented in Section 1.1 and Section 1.2, respectively. In this chapter, we also fix our notations used throughout the thesis.

1.1 Groups

Let G be a non-empty set and $* : G \times G \to G$ be a binary operation such that the following properties hold for all elements a, b, c in G:

1. **Closure:** $a * b \in G$.

2. **Associativity:** $(a * b) * c = a * (b * c)$.

3. **Identity element:** There exists an element $e \in G$ such that $a * e = e * a = a$ for all a in G.

4. **Inverse element:** For each a in G, there exists an element a^{-1} in G such that $a * a^{-1} = a^{-1} * a = e$.

Then $(G, *)$ is called a group with respect to $*$. The group $(G, *)$ is denoted by G when the operation $*$ is clear in the context and $a * b$ will be denoted by ab. A group G is said to be *abelian* (or *commutative*) if, for all elements a, b in G, we have $ab = ba$. Otherwise, G is called a *non-abelian* group. A non-empty subset H of a group G is called a *subgroup* if H itself forms a group under the same binary operation in G. If H is a subgroup of G, then we write it as $H \leq G$. A subgroup H of G is called a *trivial* subgroup if $H = \{e\}$ and H is a *proper* subgroup of G if $H \neq G$. A group G is called *cyclic* if for every element b in G there exists $a \in G$ such that $b = a^n$ for some $n \in \mathbb{Z}$. Here a^n represents the result of applying the group operation n times on a. Moreover, the element a is called a *generator* of G. If a subgroup H of a group G is cyclic, then H is called a *cyclic subgroup* of G. For a subgroup H of a group G and $g \in G$, the set $gH = \{gh \mid h \in H\}$ is called a *left coset* of H in G. Moreover, gH is an equivalence class under the equivalence relation $a \equiv b (\text{mod } H)$ if and only if $ab^{-1} \in H$. Analogously, the set $Hg = \{hg \mid h \in H\}$ is defined as the *right coset* of H in G. A subgroup N of a group G is called a *normal subgroup*, denoted by $N \trianglelefteq G$, if for each element g in G, the left coset gN is equal to the right coset Ng. Alternatively, N is a normal subgroup of G if and only if $gNg^{-1} = N$ for all g in G, where $gNg^{-1} = \{gng^{-1} \mid n \in N\}$. A group G is called a *simple group* if it has no non-trivial proper normal subgroup. Let N be a normal subgroup of a group G. The set G/N of all the left (or right) cosets of N in G forms a group with respect to the operation $\cdot$, where $\cdot$ is defined by $(gN) \cdot (hN) = (gh)N$ for all g, h in G. The group G/N is also known as *quotient group.* The set of elements in the group G that commute with every element of G is called the *center* of the group G and it is denoted by $Z(G)$. Indeed, $Z(G)$ is a normal subgroup of the group G. If A is any subset of the group G, then the subgroup $\langle A \rangle$, generated by A, is the intersection of all the subgroups of G containing A. In fact, $\langle A \rangle$ is the unique smallest subgroup of G containing A.

Examples

- The group $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$ of integers modulo n is a cyclic group with respect to the addition modulo n. Moreover, $\mathbb{Z}_n = \langle 1 \rangle$.

- The set of all bijections (permutations) from a set of n elements to itself forms a group with respect to the composition of mappings and it is denoted by S_n. Moreover, $|S_n| = n!$ and it is a non-abelian group for $n \geq 3$. A *cycle* $(a_1\ a_2\ \ldots\ a_m)$ in the group S_n is the permutation that sends a_i to $a_{i+1}, 1 \leq i \leq m-1$ and a_m to a_1. The *length* of a cycle is the number of symbols which appear in it. For example, (2 1 3) is a cycle of length 3 that maps 2 to 1, 1 to 3 and 3 to 2. In particular, $S_3 = \{I, (1\ 2), (2\ 3), (1\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$, where I is the identity map on the set $\{1, 2, 3\}$. For each $\sigma \in S_n$, σ can be expressed as a product of transpositions (cycles of length two).

- An *even permutation* is a permutation that can be expressed as the product of an even number of transpositions. The *alternating group* A_n is a subgroup of the symmetric group S_n consisting of even permutations.

Remark 1.1.1. (i) Every finite cyclic group of order n is isomorphic to $\mathbb{Z}_n$.
(ii) The number of generators of a finite cyclic group of order n is $\phi(n)$, where $\phi(n)$ denotes the Euler's totient function of n.
(iii) For $n \geq 2$, A_n is a normal subgroup of S_n. Indeed, $|(A_n)| = n!/2$.
(iv) The quotient group S_n/A_n is isomorphic to the cyclic group of order 2.
(v) A_n is a non-abelian simple group for all $n \geq 5$.

The *order* $o(a)$ *of an element* a in G is the cardinality of the subgroup generated by a. The *exponent* of a finite group is defined as the least common multiple of the orders of all elements of the group and it is denoted by exp(G). A group G is called a *torsion group* if every element of G is of finite order. The set of orders of all the non-identity elements in G is denoted by π_G. For example, $\pi_{S_3} = \{2, 3\}$. If $|G| = p^n$

for some prime p, then G is called a *p-group*. If the order of every element of a group G is a power of a prime, then G is called an EPPO-*group*. Note that every p-group is an EPPO-group. Also, S_3 is an EPPO-group but not a p-group. A subgroup P of a group G is called a *Sylow p-subgroup* if $|P| = p^\alpha$ and $p^{\alpha+1}$ does not divide $|G|$.

Theorem 1.1.2 ([Dummit and Foote, 1991, Theorem 8 (Lagrange's Theorem)]). *If H is a subgroup of a finite group G, then the order of H divides the order of G and the number of left cosets of H in G equals $\frac{|G|}{|H|}$.*

Corollary 1.1.3 ([Dummit and Foote, 1991, Corollary 9]). *If G is a finite group and $x \in G$, then the order of x divides the order of G, i.e. $o(x)||G|$.*

Let $(G, \cdot)$ and $(H, *)$ be two groups. A function $\psi : G \to H$ is called a *group homomorphism* if, for all elements a, b in G, we have $\psi(a \cdot b) = \psi(a) * \psi(b)$. A group homomorphism $\psi : G \to H$ is called a *group isomorphism* if it is a bijection. In this case, we say that the groups G and H are *isomorphic* and we write it as $G \cong H$. The *direct product* $G \times H$ of the groups $(G, \cdot)$ and $(H, *)$ is the set of all ordered pairs (g, h), where $g \in G$ and $h \in H$. The set $G \times H$ form a group with respect to the operation $\circ$, where $\circ$ is defined by

$$(g_1, h_1) \circ (g_2, h_2) = (g_1 \cdot g_2, h_1 * h_2)$$

for all $(g_1, h_1), (g_2, h_2) \in G \times H$. Analogously, the direct product $G_1 \times G_2 \times \ldots \times G_n$ of the groups $G_1, G_2, \ldots, G_n$ can be defined.

1.1.1 Maximal Cyclic Subgroups and Nilpotent Groups

A cyclic subgroup of a group G is called a *maximal cyclic subgroup* if it is not properly contained in any cyclic subgroup of G. If G is a cyclic group, then G is the only maximal cyclic subgroup of G. If $G = \mathbb{Z}_2 \times \mathbb{Z}_4$, then $M_1 = \{(0, 1), (0, 2), (0, 3), (0, 0)\}$, $M_2 = \{(1, 1), (0, 2), (1, 3), (0, 0)\}$, $M_3 = \{(1, 2), (0, 0)\}$ and $M_4 = \{(1, 0), (0, 0)\}$ are the maximal cyclic subgroups of G. We denote $\mathcal{M}(G)$ by the set of all maximal cyclic subgroups of G. Also, $M \in \mathcal{M}(G)$, we write $\mathcal{G}_M = \{x \in G : \langle x \rangle = M\}$ and $\mathcal{G}_{\mathcal{M}(G)} = \{x \in G : \langle x \rangle \in \mathcal{M}(G)\}$.

For $n \geq 3$, the *dihedral group* D_{2n} is a group of order $2n$ is defined as

$$D_{2n} = \langle x, y : x^n = y^2 = e, xy = yx^{-1} \rangle.$$

The group D_{2n} is dihedral 2-group if $n = 2^\alpha$. Note that the group D_{2n} has one maximal cyclic subgroup $M = \langle x \rangle$ of order n and n maximal cyclic subgroups $M_i = \langle x^i y \rangle$, where $1 \leq i \leq n$, of order 2.

For $n \geq 2$, the *generalized quaternion group* Q_{4n} is a group of order $4n$ is defined as

$$Q_{4n} = \langle a, b : a^{2n} = e, a^n = b^2, ab = ba^{-1} \rangle.$$

The group Q_{4n} is generalized quaternion 2-group if $n = 2^\alpha$. Observe that the group Q_{4n} has one maximal cyclic subgroup $M = \langle a \rangle$ of order $2n$ and n maximal cyclic subgroups $M_i = \langle a^i b \rangle$, where $1 \leq i \leq n$, of order 4.

For $n \geq 2$, the *semidihedral group* SD_{8n} is a group of order $8n$ is defined as

$$SD_{8n} = \langle a, b : a^{4n} = b^2 = e, ba = a^{2n-1} b \rangle.$$

The group SD_{8n} is semidihedral 2-group if $n = 2^\alpha$. Note that the group SD_{8n} has one maximal cyclic subgroup $M = \langle x \rangle$ of order $4n$, $2n$ maximal cyclic subgroups $M_i = \langle a^{2i} b \rangle = \{e, a^{2i} b\}$, where $1 \leq i \leq 2n$, of order 2 and n maximal cyclic subgroups $M_j = \langle a^{2j+1} b \rangle = \{e, a^{2n}, a^{2j+1} b, a^{2n+2j+1} b\}$, where $1 \leq j \leq n$, of order 4.

For non-isomorphic groups up to order 15, the information about the number of maximal cyclic subgroups and their orders is given in Table 1.1.

Remark 1.1.4. Let G be a finite group. Then $G = \bigcup_{M \in \mathcal{M}(G)} M$ and the generators of a maximal cyclic subgroup do not belong to any other maximal cyclic subgroup of G. Consequently, if $|M_i|$ is a prime then for any $M_j \in \mathcal{M}(G) \setminus \{M_i\}$, we have $M_i \cap M_j = \{e\}$.

The following lemma is essential for further use.

Lemma 1.1.5. *If G is a finite group, then $|\mathcal{M}(G)| \neq 2$.*

Proof. On the contrary, assume that the group G has exactly two maximal cyclic subgroups M_1 and M_2. Then every element of G belongs to at least one of the maximal cyclic subgroup of G and $e \in M_1 \cap M_2$. It follows that $|M_1|+|M_2| \geq |G|+1$. Since M_1 and M_2 are proper subgroups of a finite group G, by Lagrange's theorem, we have

$$|M_1| \leq \frac{|G|}{2} \text{ and } |M_2| \leq \frac{|G|}{2}.$$

Consequently, we get $|G| + 1 \leq |M_1| + |M_2| \leq |G|$, which is not possible. Hence, $|\mathcal{M}(G)| \neq 2$. □

Let G be a group and H, K be subgroups of G. The subgroup $[H, K]$ of G is defined as the subgroup generated by all elements of the form $[h, k] := h^{-1}k^{-1}hk$, where $h \in H, k \in K$. The lower central series of subgroups of G is the descending sequence

$$G \geq G^{(2)} \geq G^{(3)} \geq \cdots \geq G^{(i)} \geq G^{(i+1)} \geq \cdots$$

of normal subgroups of G given by $G^{(2)} := [G, G]$ and $G^{(i+1)} := \left[G^{(i)}, G\right]$ for every $i \geq 2$. If $G^n = \{e\}$ for some $n \geq 0$, then G is said to be a *nilpotent group*. Every finite p-group is a nilpotent group. A finite p-group of order p^n is said to be of maximal class if $G^{(n-1)} \neq \{e\}$ and $G^{(n)} = \{e\}$. In this case, $G/G^{(2)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$ and $G^{(i)}/G^{(i+1)} \cong \mathbb{Z}_p$ for all $2 \leq i \leq n-1$.

For $x, y \in G$, define a relation ρ such that $x \,\rho\, y$ if and only if $\langle x \rangle = \langle y \rangle$. Note that ρ is an equivalence relation. The equivalence class of x is denoted by ρ_x. Let $d \in \pi_G$. Then the number of equivalence classes that consist of elements of order d is denoted by C_d. Note that C_d represents the number of cyclic subgroups of order d in G. Moreover, we write $\tau_d = \{g \in G : o(g) = d\}$. The following result discusses a lot more about the class numbers of a finite p-group G.

Theorem 1.1.6 ([Sarkar and Mishra, 2023, Theorem 1]). *Let G be a finite p-group of exponent p^k. Assume that G is not cyclic for an odd prime p, and for $p = 2$, it is neither cyclic nor of maximal class. Then*

(i) $\mathsf{C}_p \equiv 1 + p \,(\mathrm{mod} p^2)$.

(ii) $p \mid \mathsf{C}_{p^i}$ *for every* $2 \leq i \leq k$.

Corollary 1.1.7 ([Sarkar and Mishra, 2023, Corollary 1]). *Let G be a finite p-group of exponent p^k. Then $\mathsf{C}_{p^i} = 1$ for some $1 \leq i \leq k$ if and only if one of the following occurs:*

(1) $G \cong \mathbb{Z}_{p^k}$ *and* $\mathsf{C}_{p^j} = 1$ *for all* $1 \leq j \leq k$, *or*

(2) $p = 2$ *and G is isomorphic to one of the following 2-groups:*

(i) *dihedral 2-group* $D_{2^{k+1}}$. *Moreover, we have* $\mathsf{C}_2 = 1+2^k$, *and* $\mathsf{C}_{2^j} = 1$ *for all* ($2 \leq j \leq k$).

(ii) *generalized quaternion 2-group* $Q_{2^{k+1}}$. *Moreover, we have* $\mathsf{C}_4 = 1 + 2^{k-1}$ *and* $\mathsf{C}_{2^j} = 1$ *for all* $1 \leq j \leq k$ *and* $j \neq 2$.

(iii) *semi-dihedral 2-group* $SD_{2^{k+1}}$. *Moreover, we have* $\mathsf{C}_2 = 1 + 2^{k-1}, \mathsf{C}_4 = 1 + 2^{k-2}$ *and* $\mathsf{C}_{2^j} = 1$ *for all* $3 \leq j \leq k$.

The following lemma is a consequence of Corollary 1.1.7.

Lemma 1.1.8. *Let G be a finite p-group with exponent p^2 and G contains exactly one cyclic subgroup of order p^2. Then the following holds:*

(i) *If $p = 2$, then G is isomorphic to $\mathbb{Z}_4$ or D_8.*

(ii) *If $p > 2$, then G is isomorphic to $\mathbb{Z}_{p^2}$.*

The following lemma will be useful in the sequel.

Lemma 1.1.9. *Let G be a finite p-group with exponent p^2. Then either G has exactly one cyclic subgroup of order p^2 or G contains at least two cyclic subgroups M and N of order p^2 such that $|M \cap N| = p$.*

Proof. First note that $|Z(G)| \geq p$. If G has exactly one cyclic subgroup of order p^2, then there is nothing to prove. Now, suppose that G has two cyclic subgroups M and N of order p^2. Further, let $x \in Z(G)$ such that $o(x) = p$. If $x \in M \cap N$, then $|M \cap N| = p$. Thus, the result holds. We may now suppose that $x \notin M \cap N$. Without loss of generality, assume that $x \notin M$. Consider $M = \langle y \rangle$. Since $(xy)^p = y^p$, it implies that $o(xy) = p^2$. If $\langle xy \rangle = \langle y \rangle$, then $xy = y^k$ for some positive integer k and so $x = y^{k-1}$, which is not possible. Thus, $M' = \langle xy \rangle$ is a cyclic subgroup of order p^2. Also, $M \cap M' = \{e, y^p, y^{2p}, \ldots, y^{(p-1)p}\}$. This completes our proof. □

Some equivalent characterizations of nilpotent groups are given in the following theorem.

Theorem 1.1.10 ([Dummit and Foote, 1991, p. 193])**.** *Let G be a finite group. Then the following statements are equivalent:*

(i) *G is a nilpotent group.*

(ii) *Every Sylow subgroup of G is normal.*

(iii) *G is the direct product of its Sylow subgroups.*

(iv) *For $x, y \in G$, x and y commute whenever $o(x)$ and $o(y)$ are relatively primes.*

Let G be a finite nilpotent group and $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, where $p_1 < p_2 < \cdots < p_r$ are primes and $\alpha_1, \alpha_2, \ldots, \alpha_r$ are positive integers. For $i \in [r] = \{1, 2, ..., r\}$, we denote the Sylow p_i-subgroup of G by P_i. Then by Theorem 1.1.10, we have $G = P_1 P_2 \cdots P_r$ and that $|P_i| = p_i^{\alpha_i}$. For $x \in G$, there exists a unique element $x_i \in P_i$ for each $i \in [r]$ such that $x = x_1 x_2 \cdots x_r$. Since the mapping $(x_1, x_2, \ldots, x_r) \longmapsto x_1 x_2 \cdots x_r$ is a group isomorphism from $P_1 \times P_2 \times \cdots \times P_r$ to $P_1 P_2 \cdots P_r$, we sometimes write $P_1 \times P_2 \times \cdots \times P_r$ instead of $P_1P_2\cdots P_r$. Throughout this thesis, we use these notations frequently for nilpotent groups without mentioning them explicitly.

Lemma 1.1.11. *Let $G = P_1 \times P_2 \times \cdots \times P_r$ be a finite nilpotent group of order $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$. Suppose $x, y \in G$ such that $o(x) = s$ and $o(y) = t$. Then there exists an element $z \in G$ such that $o(z) = \text{lcm}(s,t)$.*

Proof. Let $x = (x_1, x_2, \ldots, x_r)$, $y = (y_1, y_2, \ldots, y_r) \in G$. It follows that $s = p_1^{\beta_1}p_2^{\beta_2}\cdots p_r^{\beta_r}$ and $t = p_1^{\gamma_1}p_2^{\gamma_2}\cdots p_r^{\gamma_r}$, where $p_i^{\beta_i} = o(x_i)$, $p_i^{\gamma_i} = o(y_i)$ and $0 \leq \beta_i, \gamma_i \leq \alpha_i$. Consequently, $\text{lcm}(s,t) = p_1^{\delta_1}p_2^{\delta_2}\cdots p_r^{\delta_r}$, where $\delta_i = \max\{\beta_i, \gamma_i\}$. Consider $z = (z_1, z_2, \ldots, z_r)$ such that

$$z_i = \begin{cases} x_i & \text{if } \beta_i \geq \gamma_i, \\ y_i & \text{if } \beta_i < \gamma_i. \end{cases}$$

Clearly, $z \in G$ and $o(z) = \prod_{i=1}^{r} o(z_i) = \prod_{i=1}^{r} p_i^{\delta_i}$. Thus, the result holds. □

Lemma 1.1.12 ([Chattopadhyay et al., 2021, Lemma 2.11])**.** *Any maximal cyclic subgroup of a finite nilpotent group $G = P_1P_2\cdots P_r$ is of the form $M_1M_2\cdots M_r$, where M_i is a maximal cyclic subgroup of P_i $(1 \leq i \leq r)$.*

Corollary 1.1.13. *Let $G = P_1P_2\cdots P_r$ be a finite nilpotent group and let P_i be a cyclic group for some i. Then P_i is contained in every maximal cyclic subgroup of G.*

1.2 Graphs

In this section, we recall all the necessary definitions, notations and results of graph theory from West [1996]. A *graph* Γ is defined by a set of vertices $V(\Gamma)$ and a set of edges $E(\Gamma)$, where $E(\Gamma)$ is an unordered pair of elements (not necessarily distinct) of $V(\Gamma)$. If $\{u,v\} \in E(\Gamma)$, then u and v are adjacent in Γ and it is denoted by $u \sim v$. Otherwise, we express it as $u \nsim v$. If $\{u,v\} \in E(\Gamma)$, then the vertices u and v are called *endpoints* of the edge $\{u, v\}$. Two edges e_1 and e_2 are said to be *incident* if they have a common endpoint. The *order* of a graph Γ is the cardinality of $V(\Gamma)$. An edge $\{u, v\}$ is called a *loop* if $u = v$. *Multiple edges* are the edges having the same endpoints. In this thesis, we are considering only simple graphs (graphs with no loops or multiple edges). A graph Γ is called a *null graph* if $E(\Gamma)$ is an empty set. If $V(\Gamma)$ is an empty set, then Γ is called an *empty* graph. The set $\mathrm{N}(x)$ of all the vertices which are adjacent to the vertex x in Γ is called the *neighbourhood* of x. Furthermore, we denote $\mathrm{N}[x] = \mathrm{N}(x) \cup \{x\}$. A *subgraph* Γ' of a graph Γ is a graph such that $V(\Gamma') \subseteq V(\Gamma)$ and $E(\Gamma') \subseteq E(\Gamma)$. A subgraph Γ' is called a *spanning subgraph* of the graph Γ if $V(\Gamma') = V(\Gamma)$. Let $X \subseteq V(\Gamma)$. Then the subgraph Γ' *induced by the set* X is a graph such that $V(\Gamma') = X$ and $u, v \in X$ are adjacent in Γ' if and only if they are adjacent in Γ. By $\Gamma \setminus X$, we mean that the subgraph of Γ induced by the set $V(\Gamma) \setminus X$. Two graphs Γ_1 and Γ_2 are said to be *isomorphic*, denoted as $\Gamma_1 \cong \Gamma_2$, if there exists a bijective function $f : V(\Gamma_1) \to V(\Gamma_2)$ such that $\{u, v\} \in E(\Gamma_1)$ if and only if $\{f(u), f(v)\} \in E(\Gamma_2)$. An *automorphism* of a graph Γ is an isomorphism from Γ to itself. A graph Γ is called Γ'-*free* is Γ does not contain Γ' as its induced subgraph. Let Γ' be a spanning subgraph of a graph Γ. Then the *difference* $\Gamma - \Gamma'$ is a graph such that $V(\Gamma - \Gamma') = V(\Gamma)$ and $E(\Gamma - \Gamma') = E(\Gamma) \setminus E(\Gamma')$. The union $\Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_n$ of the graphs $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ is a graph such that $V(\Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_n) = V(\Gamma_1) \cup V(\Gamma_2) \cup \cdots \cup V(\Gamma_n)$ and $E(\Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_n) = E(\Gamma_1) \cup E(\Gamma_2) \cup \cdots \cup E(\Gamma_n)$. We denote $n\Gamma$ by $\Gamma \cup \Gamma \cup \cdots \cup \Gamma$ (n-times). The *join* $\Gamma_1 \vee \Gamma_2$ of graphs Γ_1 and Γ_2 is a graph with vertex set $V(\Gamma \vee \Gamma') = V(\Gamma) \cup V(\Gamma')$ and edge set $E(\Gamma \vee \Gamma') = E(\Gamma) \cup E(\Gamma') \cup \{\{u, v\} \mid u \in V(\Gamma), v \in V(\Gamma')\}$. The *strong product* $\Gamma_1 \boxtimes \Gamma_2 \boxtimes \cdots \boxtimes \Gamma_n$ of n graphs $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ is a graph such that the vertex set $V(\Gamma_1 \boxtimes \Gamma_2 \boxtimes \cdots \boxtimes \Gamma_n) = V(\Gamma_1) \times V(\Gamma_2) \times \cdots \times V(\Gamma_n)$ and distinct vertices $(u_1, u_2, ..., u_n)$ and $(v_1, v_2, ..., v_n)$ are adjacent in $\Gamma_1 \boxtimes \Gamma_2 \boxtimes \cdots \boxtimes \Gamma_n$ if and only if either $u_i = v_i$ or $u_i \sim v_i$ in Γ_i for each $i \in [n]$.

A *walk* in a graph Γ from the vertex v_1 to the vertex v_m is defined as a sequence of vertices: $v_1, v_2, \ldots, v_k$ $(k > 1)$, where $v_i \sim v_{i+1}$ for every $i \in \{1, 2, \ldots, m-1\}$. The length of the walk is given by the number of edges in the sequence. A *trail* is a walk in which no edge is repeated. A *closed trail* is a trail whose initial and end vertex are identical. If no vertex in a walk is repeated, then it is called a *path*. If there exists a path between any two vertices of a graph Γ, then Γ is said to be a *connected graph*. Otherwise, we say Γ is *disconnected*. A connected subgraph Γ_1 of a graph Γ is called a *component (maximal)* if there exists a connected subgraph Γ_2 such that Γ_1 is a subgraph of Γ_2, then $\Gamma_1 = \Gamma_2$. The path graph P_n is defined by the set of vertices $\{v_1, v_2, \ldots, v_n\}$ and the set of edges $\{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}\}$. A vertex u is said to be a *dominating vertex* of a graph Γ if u is adjacent to all the other vertices of Γ. The set of all dominating vertices of Γ is denoted by $\mathrm{Dom}(\Gamma)$. A graph Γ is called *complete* if every vertex of Γ is a dominating vertex. The complete graph on n vertices is denoted by K_n. The *complement* of a graph Γ is the graph $\overline{\Gamma}$ such that $V(\overline{\Gamma}) = V(\Gamma)$ and two vertices are adjacent in $\overline{\Gamma}$ if and only if they are not adjacent in Γ. A path whose initial and end vertex are the same is called a *cycle*. A graph Γ is called *acyclic* if it contains no cycle. A simple connected acyclic graph is called a *tree*. A spanning subgraph Γ' of a graph Γ is called a *spanning tree* if Γ' is a tree. A cycle in a graph Γ is called *Hamiltonian cycle* if it includes all the vertices of Γ. If Γ contains a Hamiltonian cycle, then Γ is called a *Hamiltonian graph*. The *girth* of a graph Γ is the length of the shortest cycle in Γ. If Γ is acyclic, then we say the girth is ∞. The *cycle graph* C_n is a graph with vertex set $\{v_1, v_2, \ldots, v_n\}$ and the set of edges $\{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}\}$. The *distance* $d(u, v)$ between two vertices u and v in a graph Γ, is defined as follows:

$$d(u,v) = \begin{cases} 0 & \text{if } u = v, \\ \infty & \text{if there is no path from } u \text{ to } v, \\ \ell(P) & \text{where } P \text{ is a shortest path from } u \text{ to } v, \end{cases}$$

where $\ell(P)$ is the length of the path P. The maximum distance between two vertices of a connected graph Γ is called the *diameter* of Γ and it is denoted by $\mathrm{diam}(\Gamma)$. The Wiener index is one of the most frequently used topological indices in chemistry as a molecular shape descriptor. This was first used by H. Wiener in 1947 and then the formal definition of the Wiener index was introduced by Hosoya [1971]. For a connected graph Γ, the *Wiener index* $W(\Gamma)$ is defined by

$$W(\Gamma) = \sum_{x \in V(\Gamma)} \sum_{y \in V(\Gamma)} \frac{d(x,y)}{2}.$$

The Wiener index was also employed in crystallography, communication theory, facility location, cryptography, etc. (see Bonchev [2002]; Gutman et al. [1993]; Nikolić and Trinajstić [1995]). The number of vertices adjacent to a vertex u is called the *degree* of u and we write is as $\deg(u)$. The *minimum degree* $\delta(\Gamma)$ of a graph Γ is the smallest degree among all the vertices of Γ. A *regular* graph is a graph in which each vertex has the same degree. A graph Γ is said to be *strongly regular graph* with parameters (n, k, λ, μ) if it is k-regular graph on n vertices such that each pair of adjacent vertices has exactly λ common neighbours, and each pair of non-adjacent vertices has exactly μ common neighbours. If a connected graph Γ has a closed trail which contains all the edges of Γ, then Γ is said to be *Eulerian*.

Theorem 1.2.1 ([West, 1996, Theorem 1.2.26])**.** *A connected graph is Eulerian if and only if its every vertex is of even degree.*

The *chromatic number* $\chi(\Gamma)$ of a graph Γ is the minimum number of colors needed to color the vertices of the graph in such a way that no two adjacent vertices have the same color. A *clique* of a graph Γ is a complete subgraph of Γ. The *clique* *number* $\omega(\Gamma)$ of a graph Γ is the order of the largest complete subgraph. Let Γ be a graph and let X be a non-empty subset of $V(\Gamma)$. Then X is called an *independent set* of Γ if no two vertices of X are adjacent in Γ. The cardinality of maximum size independent set of a graph Γ is called the *independence number* of Γ and it is denoted by $\alpha(\Gamma)$. A graph Γ is said to be a *k-partite graph* if $V(\Gamma)$ can be partitioned into k independent subset. If $k = 2$, then Γ is called *bipartite graph.* A *complete k-partite* graph, denoted by $K_{n_1,n_2,\ldots,n_k}$, is a k-partite graph having its parts sizes $n_1, n_2, \ldots, n_k$ such that every vertex in each part is adjacent to all the vertices of all other parts of $K_{n_1,n_2,\ldots,n_k}$.

Theorem 1.2.2 ([West, 1996, Theorem 1.2.18]). *A graph Γ is bipartite if and only if Γ does not contain an odd cycle.*

Let Γ be a graph. A *dominating set* D of Γ is a subset of $V(\Gamma)$ such that for every vertex $v \in V(\Gamma)$, either $v \in D$ or v is adjacent to at least one vertex in D. The *domination number* is the cardinality of a smallest dominating set. A graph Γ is said to be a *threshold graph* if it can be constructed from a one vertex graph by repeated applications of the following two operations:

- Addition of a single isolated vertex to the graph.
- Addition of a single dominating vertex to the graph.

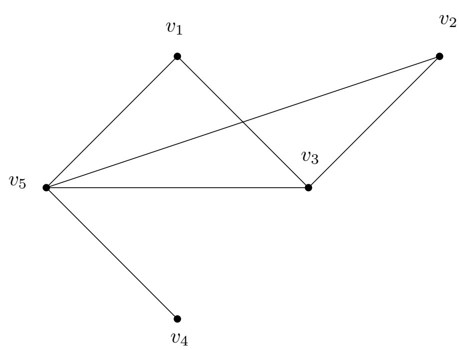

FIGURE 1.1: A threshold graph of five vertices.

An example of a threshold graph is given in Figure 1.1. Initially, the isolated vertex v_2 is included with the vertex v_1. Then the dominating vertex v_3 is added. Further, the isolated vertex v_4 is added to the graph whose vertices are v_1, v_2, v_3. Finally, the dominating vertex v_5 is added.

There are various equivalent characterizations for threshold graphs in the literature. One of them, which is used in the present thesis, is that a graph is a threshold graph if it has no induced subgraph isomorphic to C_4, P_4 and $2K_2$. The other important graph class is a cograph. A *cograph* can be defined recursively as follows:

- A single vertex graph is a cograph.
- If Γ_1 and Γ_2 are disjoint cographs then so is their union $\Gamma_1 \cup \Gamma_2$.
- If Γ is a cograph, then so is its complement $\overline{\Gamma}$.

The graph C_4 is a cograph because $C_4 = \overline{K_2 \cup K_2}$. In this thesis, we shall use an equivalent characterization of cographs which states that a graph Γ is a *cograph* if Γ is P_4-free. A graph Γ is a *chordal graph* if every cycle of length greater than 3 has a chord, which is an edge that is not part of the cycle but connects two vertices of the cycle. Notice that if a graph Γ is P_4-free and C_4-free, then Γ is a chordal graph. However, the converse need not be true. A *split graph* is a graph where the vertex set can partitioned into a maximal clique and an independent set.

Lemma 1.2.3 (Foldes and Hammer [1977]). *A graph Γ is a split graph if and only if Γ is C_4-free, C_5-free and $2K_2$-free.*

Notice that every threshold graph is also a cograph, split graph, and chordal graph. A graph Γ is *perfect* if $\omega(\Gamma') = \chi(\Gamma')$ for every induced subgraph Γ' of Γ. It is well known that cographs, split, threshold, and chordal graphs are all perfect graphs. A graph Γ is *weakly perfect* if $\chi(\Gamma) = \omega(\Gamma)$. A subgraph Γ' of Γ is called a *hole* if Γ' is a cycle as an induced subgraph, and Γ' is called an *antihole* of Γ if $\overline{\Gamma'}$ is a hole in $\overline{\Gamma}$.

Theorem 1.2.4 (Chudnovsky et al. [2006]). *A finite graph* Γ *is perfect if and only if it does not contain hole or antihole of odd length at least* 5.

If a graph Γ can be drawn on a plane such that its edges intersect only at their end points, then the graph Γ is called *planar*. The *face* of a planar graph is the area which is surrounded by edges. An *outerplanar graph* is a planar graph that can be embedded in the plane in such a way that all vertices are on the outer face. All the graphs are not planar but they can be embedded on topological surfaces like g-hole torus, projective plane and Klein bottle etc. Its applications lie in electronic printing circuits where the purpose is to embed a circuit, that is, the graph on a circuit board (the surface), without two connections crossing each other, resulting in a short circuit. A graph is said to be *embeddable* on a topological surface if it can be drawn on a surface without edge crossing. The *genus* $\gamma(\Gamma)$ of a graph Γ is the minimum integer g such that Γ can be embedded in an orientable surface with g handles. The graphs having genus 0 and genus 1 are called planar graphs and toroidal graphs, respectively. Let $\mathbb{N}_k$ be the non-orientable surface formed by the connected sum of k projective planes. The *cross-cap* $\overline{\gamma}(\Gamma)$ of a graph Γ is the minimum non-negative integer k such that Γ can be embedded in $\mathbb{N}_k$. A graph is called *projective planar* if its cross-cap is 1. An embedding in which every face is triangular is called a *triangular embedding*. In topological graph theory determining the genus and crosscap of a graph is a basic but a very complicated problem, it is indeed NP-complete. In a graph Γ, the *subdivision* of an edge $\{u, v\}$ is the operation of replacing $\{u, v\}$ with a path $u \sim w \sim v$ through a new vertex w. A *subdivision* of Γ is a graph obtained from Γ by successive edge subdivisions.

The following characterizations of planar and outerplanar graphs are useful.

Theorem 1.2.5 ([West, 1996, Theorem 6.2.2, Exercise 6.2.7]). *Let* Γ *be a graph. Then*

(i) Γ *is planar if and only if it does not contain a subdivision of* K_5 *or* $K_{3,3}$.

(ii) Γ *is outerplanar if and only if it does not contain a subdivision of* K_4 *or* $K_{2,3}$.

Theorem 1.2.6 ([White, 1973, p. 68, p. 185])**.** *The genus and cross-cap of the complete graph* K_n *and the complete bipartite graph* $K_{m,n}$ *are given below:*

(i) $\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$, $n \geq 3$.

(ii) $\gamma(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil$, $m, n \geq 2$.

(iii) $\overline{\gamma}(K_n) = \left\lceil \frac{(n-3)(n-4)}{6} \right\rceil$, $n \geq 3$, $n \neq 7$; $\overline{\gamma}(K_n) = 3$ *if* $n = 7$.

(iv) $\overline{\gamma}(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{2} \right\rceil$, $m, n \geq 2$.

Theorem 1.2.7 ([White, 1973, Corollary 6.14])**.** *Let* Γ *be a simple connected graph with* n *vertices and* m *edges, where* $n \geq 3$. *Then* $\gamma(\Gamma) \geq \frac{m}{6} - \frac{n}{2} + 1$. *Furthermore, equality holds if and only if* Γ *has a triangular embedding.*

Lemma 1.2.8 ([Mohar and Thomassen, 2001, Lemma 3.1.4])**.** *Let* $\psi : \Gamma \to \mathbb{N}_k$ *be a* 2*-cell embedding of a connected graph* Γ *to the non-orientable surface* $\mathbb{N}_k$. *Then* $v - e + f = 2 - k$, *where* v, e *and* f *are number of vertices, edges and faces of* $\psi(\Gamma)$ *respectively, and* k *is a cross-cap of* $\mathbb{N}_k$.

A *vertex (edge) cut-set* in a connected graph Γ is a set S of vertices (edges) such that the remaining subgraph $\Gamma \setminus S$, by removing the set S, is disconnected or has only one vertex. The *vertex connectivity* (*edge connectivity*) of a connected graph Γ is the minimum size of a vertex (edge) cut-set and it is denoted by $\kappa(\Gamma)$ ($\kappa'(\Gamma)$). For $k \geq 1$, graph Γ is *k-connected* if $\kappa(\Gamma) \geq k$.

Theorem 1.2.9 ([Plesník, 1975, Theorem 6])**.** *If the diameter of any graph is at most* 2*, then its edge connectivity and minimum degree are equal.*

Theorem 1.2.10 ([West, 1996, Theorem 4.1.9])**.** *If* Γ *is a simple graph, then*

$$\kappa(\Gamma) \leq \kappa'(\Gamma) \leq \delta(\Gamma).$$

1.2.1 Power Graphs and Enhanced Power Graphs of Finite Groups

The *commuting graph* $\mathcal{C}(G)$ of a finite group G is the simple undirected graph with vertex set G whose two vertices x and y are adjacent if $xy = yx$. Certain subgraphs, namely: power graph and enhanced power graph, of $\mathcal{C}(G)$ have been studied in the literature. Recall that the *power graph* $\mathcal{P}(G)$ of a group G is the simple graph with vertex set G and two vertices a, b are adjacent if one is a power of the other or equivalently: either $a \in \langle b \rangle$ or $b \in \langle a \rangle$. To measure how much the power graph

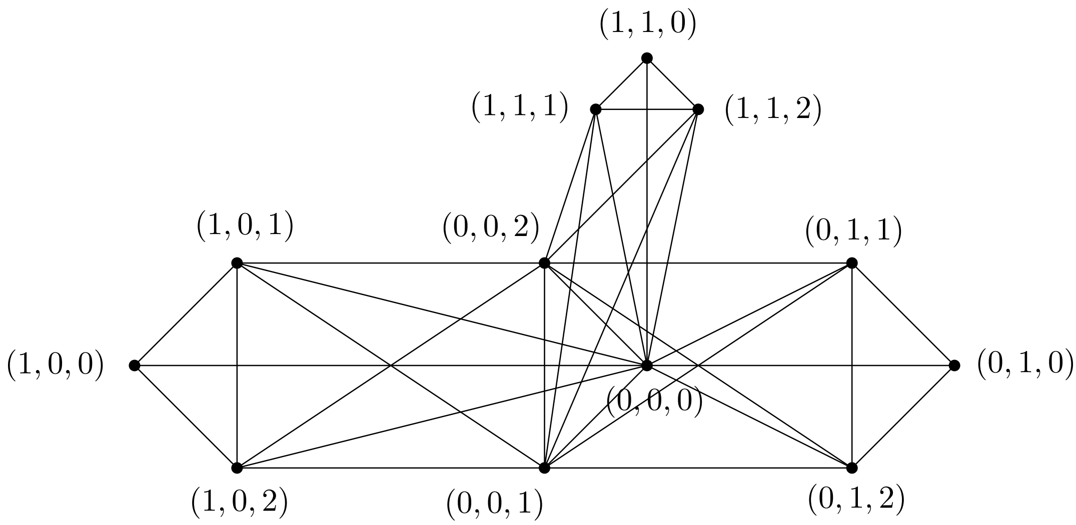

Figure 1.2: $\mathcal{P}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)$.

is close to the commuting graph of a group G, Aalipour et al. [2017] introduced a new graph called the enhanced power graph. The *enhanced power graph* $\mathcal{P}_E(G)$ of a group G is the simple graph with the vertex set G and two distinct vertices x, y are adjacent if $x, y \in \langle z \rangle$ for some $z \in G$. Note that the power graph is a spanning subgraph of the enhanced power graph and the enhanced power graph is a spanning subgraph of the commuting graph.

For a non-empty subset X of a group G, $\mathcal{P}_E(X)$ denotes the subgraph of $\mathcal{P}_E(G)$ induced by the set X. By proper enhanced power graphs $\mathcal{P}_E^*(G)$ and $\mathcal{P}_E^{**}(G)$, we

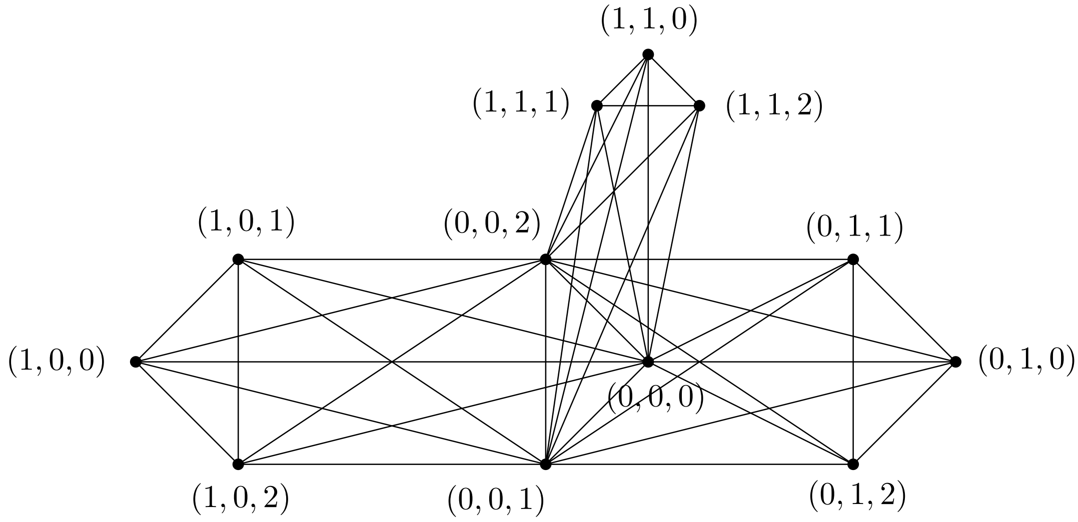

FIGURE 1.3: $\mathcal{P}_E(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)$.

mean that the subgraph of $\mathcal{P}_E(G)$ induced by the sets $G \setminus \{e\}$ and $G \setminus \mathrm{Dom}(\mathcal{P}_E(G))$, respectively. Analogously, proper power graphs $\mathcal{P}^*(G)$ and $\mathcal{P}^{**}(G)$ can be defined. Further, note that the identity element $e \in G$ is a dominating vertex of the graph $\Delta(G)$, where $\Delta(G) \in \{\mathcal{P}(G), \mathcal{P}_E(G)\}$. The graph $\Delta(G) \in \{\mathcal{P}(G), \mathcal{P}_E(G)\}$ is called *dominatable* if it contains a dominating vertex other than the identity element $e \in G$. The following results on power graphs and enhanced power graphs are useful for later use.

Theorem 1.2.11 ([Chakrabarty et al., 2009, Theorem 2.12])**.** *For a finite group G, the power graph $\mathcal{P}(G)$ is complete if and only if G is a cyclic group of order 1 or p^m, for some prime p and $m \in \mathbb{N}$.*

Theorem 1.2.12 ([Aalipour et al., 2017, Theorem 28])**.** *For a finite group G, the following conditions are equivalent:*

(i) *the power graph of G is equal to the enhanced power graph;*

(ii) *every cyclic subgroup of G has prime power order.*

Theorem 1.2.13 ([Cameron and Jafari, 2020, Theorem 4]). *Let G be a finite group. Suppose that $x \in G$ has the property that for all $y \in G$, either x is a power of y or vice versa. Then one of the following holds:*

(i) $x = e$;

(ii) *G is cyclic and x is a generator;*

(iii) *G is a cyclic p-group for some prime p and x is arbitrary;*

(iv) *G is a generalized quaternion group and x is of order 2.*

Theorem 1.2.14 ([Moghaddamfar et al., 2014, Theorem 4.2]). *Let G be a non-trivial finite group. Then the proper power graph $\mathcal{P}^*(G)$ is strongly regular if and only if G is a p-group of order p^m for which $\exp(G) = p$ or p^m.*

Theorem 1.2.15 ([Bera and Bhuniya, 2018, Theorem 2.4]). *The enhanced power graph $\mathcal{P}_E(G)$ is complete if and only if G is cyclic.*

Theorem 1.2.16 ([P. Panda et al., 2021, Theorem 3.2]). *For a finite group G, the minimum degree $\delta(\mathcal{P}_E(G)) = m - 1$, where m is the order of a maximal cyclic subgroup of minimum possible order.*

Let G' be a nilpotent group having no Sylow subgroups that are either cyclic or generalized quaternion and e' be the identity element of G'. Further, let e'' be the identity element and y be the element of order 2 of the generalized quaternion 2-group Q_{2^k}.

Theorem 1.2.17 ([Bera and Dey, 2022, Corollary 4.2]). *Let G be a finite nilpotent group. Then*

$$\mathrm{Dom}\,(\mathcal{P}_E(G)) = \begin{cases} \{e\} & \text{if } G = G', \\ \{(e', x) : x \in \mathbb{Z}_n\} & \text{if } G = G' \times \mathbb{Z}_n \text{ and } \gcd(|G'|, n) = 1, \\ \{(e', e''), (e', y)\} & \text{if } G = G' \times Q_{2^k} \text{ and } \gcd(|G'|, 2) = 1, \\ T_1 \cup T_2 & \text{if } G = G' \times \mathbb{Z}_n \times Q_{2^k} \text{ and } \gcd(|G'|, n) = \gcd(|G'|, 2) = \gcd(n, 2) = 1, \end{cases}$$

where $T_1 = \{(e', x, e'') : x \in \mathbb{Z}_n\}$ and $T_2 = \{(e', x, y) : x \in \mathbb{Z}_n\}$.

$\|G\|$	No. of Groups	Type of Groups	$\|\mathcal{M}(G)\|$	$\|M_i\|$
1	1	$\mathbb{Z}_1$	1	$\|M_1\| = 1$
2	1	$\mathbb{Z}_2$	1	$\|M_1\| = 2$
3	1	$\mathbb{Z}_3$	1	$\|M_1\| = 3$
4	2	$\mathbb{Z}_4$	1	$\|M_1\| = 4$
		$\mathbb{Z}_2 \times \mathbb{Z}_2$	3	$\|M_1\| = \|M_2\| = \|M_3\| = 2$
5	1	$\mathbb{Z}_5$	1	$\|M_1\| = 5$
6	2	$\mathbb{Z}_6$	1	$\|M_1\| = 6$
		S_3	4	$\|M_1\| = 3, \|M_2\| = \|M_3\| = \|M_4\| = 2$
7	1	$\mathbb{Z}_7$	1	$\|M_1\| = 7$
8	5	$\mathbb{Z}_8$	1	$\|M_1\| = 8$
		$\mathbb{Z}_2 \times \mathbb{Z}_4$	4	$\|M_1\| = \|M_2\| = 4,\ \|M_3\| = \|M_4\| = 2$
		$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	7	$\|M_i\| = 2$ for $i \in \{1,2,3,4,5,6,7\}$
		D_8	5	$\|M_1\| = 4,\ \|M_i\| = 2$ for $i \in \{2,3,4,5\}$
		Q_8	3	$\|M_1\| = \|M_2\| = \|M_3\| = 4$
9	2	$\mathbb{Z}_9$	1	$\|M_1\| = 9$
		$\mathbb{Z}_3 \times \mathbb{Z}_3$	4	$\|M_1\| = \|M_2\| = \|M_3\| = \|M_4\| = 3$
10	2	$\mathbb{Z}_{10}$	1	$\|M_1\| = 10$
		D_{10}	6	$\|M_1\| = 5,\ \|M_i\| = 2$ for $i \in \{2,3,4,5,6\}$
11	1	$\mathbb{Z}_{11}$	1	$\|M_1\| = 11$
12	5	$\mathbb{Z}_{12}$	1	$\|M_1\| = 12$
		$\mathbb{Z}_2 \times \mathbb{Z}_6$	3	$\|M_1\| = \|M_2\| = \|M_3\| = 6$
		A_4	7	$\|M_i\| = 3,\ \|M_j\| = 2$ for $i \in \{1,2,3,4\},\ j \in \{5,6,7\}$
		D_{12}	7	$\|M_1\| = 6,\ \|M_i\| = 2$ for $i \in \{2,3,4,5,6,7\}$
		Q_6	4	$\|M_1\| = 6,\ \|M_2\| = \|M_3\| = \|M_4\| = 4$
13	1	$\mathbb{Z}_{13}$	1	$\|M_1\| = 13$
14	2	$\mathbb{Z}_{14}$	1	$\|M_1\| = 14$
		D_{14}	8	$\|M_1\| = 7,\ \|M_i\| = 2$ for $i \in \{2,3,4,5,6,7,8\}$
15	1	$\mathbb{Z}_{15}$	1	$\|M_1\| = 15$

Table 1.1: The maximal cyclic subgroups of non-isomorphic groups of order upto 15.

Chapter 2

Enhanced Power Graphs

In order to see how close the power graph is to the commuting graph, Aalipour et al. [2017] introduced the notion of the enhanced power graph of a group. The enhanced power graph $\mathcal{P}_E(G)$ of a group G is a simple graph whose vertex set is the group G and two distinct vertices x, y are adjacent if $x, y \in \langle z \rangle$ for some $z \in G$. Aalipour et al. [2017] showed that the clique number of the enhanced power graph of a group G is finite if and only if G is of finite exponent. Bera and Bhuniya [2018] established that $\mathcal{P}_E(G)$ is planar if and only if the order of each element of G is at most 4. Zahirović et al. [2020] proved that two finite abelian groups are isomorphic if and only if their enhanced power graphs are isomorphic. Additionally, they provided a characterization of finite nilpotent groups whose enhanced power graphs exhibit perfection. Recently, P. Panda et al. [2021] explored the graph-theoretic properties, including minimum degree, independence number, matching number, strong metric dimension, and perfectness of enhanced power graphs over finite abelian groups. Furthermore, investigations of enhanced power graphs associated with certain non-abelian groups including semidihedral groups, dihedral groups, generalized quaternion groups, have been carried out by Dalal and Kumar [2021]. For existing results and open problems on enhanced power graphs of groups, we refer the reader to the survey paper Ma et al. [2022] and references therein.

In this chapter, we aim to contribute novel insights and results of enhanced power graphs and underlying groups. This chapter is arranged as follows. In Section 2.1, we characterize finite groups for which the minimum degree is equal to the vertex connectivity of the enhanced power graph $\mathcal{P}_E(G)$. Further, we classify groups with (strongly) regular enhanced power graphs. Moreover, in this section, we study the Wiener index of the enhanced power graph $\mathcal{P}_E(G)$ for nilpotent groups. Building on previous results on the λ-number of power graphs of finite groups, we extend these findings to enhanced power graphs in Section 2.2. Section 2.3 determines the Laplacian spectrum of the enhanced power graph for certain non-abelian groups including semidihedral groups, dihedral groups, and generalized quaternion groups. Moreover, the metric dimension and resolving polynomials of enhanced power graphs of these groups are explored in Section 2.3.

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2.1 Certain Properties of the Enhanced Power Graph $\mathcal{P}_E(G)$

This section begins with a characterization of finite groups such that the minimum degree and the vertex connectivity of the enhanced power graph are equal. Also, we give a description of finite groups with regular and strongly regular (proper) enhanced power graphs. Further, we discuss the vertex connectivity of the enhanced power graph of certain nilpotent groups. This section is concluded with an investigation of an upper and a lower bound for the Wiener index of the enhanced power graph of nilpotent groups. Moreover, the finite nilpotent groups achieving these bounds have been characterized.

2.1.1 Groups for which Minimum Degree and the Vertex Connectivity of $\mathcal{P}_E(G)$ are Equal

Panda et al. [2024] characterized finite nilpotent groups based on the equality of vertex connectivity and minimum degree in their power graphs. In this subsection, we characterize finite groups such that the minimum degree and the vertex connectivity of $\mathcal{P}_E(G)$ are equal. We begin with the following lemma.

Lemma 2.1.1. *Let G be a non-cyclic group and $M \in \mathcal{M}(G)$. Then $\overline{M}$ is a cut-set of $\mathcal{P}_E(G)$, where $\overline{M}$ is the union of all the sets of the form $M \cap \langle x \rangle$, for $x \in G \setminus M$.*

Proof. Let $M = \langle a \rangle$ and $M' = \langle b \rangle$ be two maximal cyclic subgroups of G. Then we claim that there is no path between a and b in $\mathcal{P}_E(G \setminus \overline{M})$. If possible, suppose that there is a path $a \sim x_1 \sim x_2 \sim \cdots \sim x_k \sim b$ from a to b in $\mathcal{P}_E(G \setminus \overline{M})$. Then $x_1 \in M$. Otherwise, $\langle a, x_1 \rangle$ is a cyclic subgroup which is not contained in M, which is impossible. We may now suppose that $x_1, x_2, \ldots, x_{r-1} \in M$ and $x_r \notin M$ for some $r \in [k] \setminus \{1\}$. Note that such r exists because $x_k \sim b$ and if $x_r \in M$ for each $r \in [k]$, then $x_k \in \overline{M}$ which is impossible. Now, if $x_{r-1} \in M$, then by using a similar argument, we obtain $x_{r-1} \in \overline{M}$. It follows that no such path exists between a and b. Thus, $\overline{M}$ is a cut-set. □

Theorem 2.1.2. *In the enhanced power graph $\mathcal{P}_E(G)$, we have $\delta(\mathcal{P}_E(G)) = \kappa(\mathcal{P}_E(G))$ if and only if one of the following holds:*

(i) *G is a cyclic group of order n.*

(ii) *G is non-cyclic and contains a maximal cyclic subgroup of order* 2.

Proof. First suppose that $\delta(\mathcal{P}_E(G)) = \kappa(\mathcal{P}_E(G))$. If G is cyclic, then we have nothing to prove. If possible, let G be non-cyclic group and it does not have a maximal cyclic subgroup of order 2. By Theorem 1.2.16, $\delta(\mathcal{P}_E(G)) = |M| - 1$, where $M \in \mathcal{M}(G)$ such that $|M|$ is minimum. By Lemma 2.1.1, $\overline{M}$ is a cut-set. Note that every generator of M does not belong to $\overline{M}$. Consequently, we get

$$\kappa(\mathcal{P}_E(G)) \leq |\overline{M}| < |M| - 1 = \delta(\mathcal{P}_E(G)),$$

which is impossible. Thus, G must have a maximal cyclic subgroup of order 2.

To prove the converse part, suppose that G is a cyclic group of order n. Then by Theorem 1.2.15, we have

$$\delta(\mathcal{P}_E(G)) = \kappa(\mathcal{P}_E(G)) = n - 1.$$

If G is non-cyclic and has a maximal cyclic subgroup M of order 2, then by Lemma 2.1.1, we obtain $\overline{M} = \{e\}$ is a cut-set. It follows that $\kappa(\mathcal{P}_E(G)) = 1$. By Theorem 1.2.16, we get $\delta(\mathcal{P}_E(G)) = |M| - 1 = 1$ and so $\delta(\mathcal{P}_E(G)) = \kappa(\mathcal{P}_E(G))$. □

The following example illustrates Theorem 2.1.2.

Example 2.1.3. For the non-cyclic group $G = \mathbb{Z}_2 \times \mathbb{Z}_4$, note that G has two maximal cyclic subgroups $\langle(1,2)\rangle$ and $\langle(1,0)\rangle$ of order 2. By Figure 2.1, observe that the minimum degree is 1 and $\{(0,0)\}$ is the smallest cut-set of $\mathcal{P}_E(G)$. Thus, $\kappa(\mathcal{P}_E(G)) = \delta(\mathcal{P}_E(G)) = 1$.

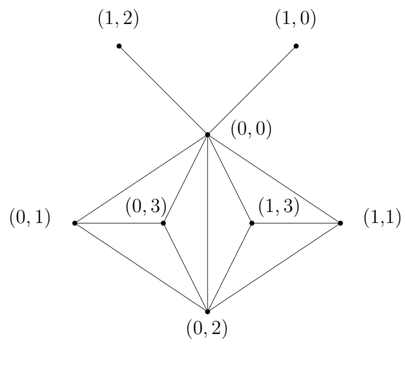


FIGURE 2.1: $\mathcal{P}_E(\mathbb{Z}_2 \times \mathbb{Z}_4)$.

2.1.2 Groups with Regular Enhanced Power Graphs

Moghaddamfar et al. [2014] provided characterizations for finite groups whose power graphs are regular. The identity element of the group G is adjacent to all the other elements of G in $\mathcal{P}_E(G)$. Thus, $\mathcal{P}_E(G)$ is regular if and only if G is a finite cyclic group (cf. Theorem 1.2.15). Recall that, the proper enhanced power graph $\mathcal{P}_E^*(G)$ is the subgraph of $\mathcal{P}_E(G)$ induced by $G \setminus \{e\}$. In this subsection, we classify the group G such that the graph $\mathcal{P}_E^*(G)$ is (strongly) regular.

Theorem 2.1.4. *Let G be a finite group. Then $\mathcal{P}_E^*(G)$ is regular if and only if one of the following holds:*

(i) *G is a cyclic group.*

(ii) *$|M_i| = |M_j|$ and $M_i \cap M_j = \{e\}$, where $M_i, M_j \in \mathcal{M}(G)$.*

Proof. Suppose that $\mathcal{P}_E^*(G)$ is regular. If G is cyclic, then there is nothing to prove. We may now suppose that G is a non-cyclic group. Assume that $M_i = \langle x \rangle$ and $M_j = \langle y \rangle$ are two maximal cyclic subgroups of G. Since

$$|M_i| - 2 = \deg(x) = \deg(y) = |M_j| - 2,$$

we deduce that $|M_i| = |M_j|$. Moreover, if $M_i \cap M_j \neq \{e\}$ for some distinct i, j then for a non-identity element $a \in M_i \cap M_j$, we obtain

$$\deg(a) \geq |M_i \cup M_j| - 2 > |M_i| - 2 = \deg(x).$$

Consequently, $\mathcal{P}_E^*(G)$ is not regular; a contradiction.
Conversely, suppose that G is a cyclic group. Then by Theorem 1.2.15, $\mathcal{P}_E^*(G)$ is complete and so is regular. We may now suppose that G is non-cyclic. If G satisfies condition (ii) then note that every element of $G \setminus \{e\}$ lies in exactly one maximal cyclic subgroup of G. Consequently for each $x \in G \setminus \{e\}$, we have $\deg(x) = |M_i| - 2$, where M_i is the maximal cyclic subgroup of G containing x. Hence, $\mathcal{P}_E^*(G)$ is regular. □

Remark 2.1.5. There are several groups which satisfy the condition (ii) of the Theorem 2.1.4. For instance, an elementary abelian p-group, a non-abelian group $G = \langle x, y, z; x^p = y^p = z^p = e, yz = zyx, xy = yx, xz = zx \rangle$, etc., where p is an odd prime.

Clearly, a strongly regular graph is always regular. However, the converse need not be true. We show that the converse is also true for $\mathcal{P}_E(G)$ in the following theorem.

Theorem 2.1.6. *Let G be a finite group. Then $\mathcal{P}_E^*(G)$ is regular if and only if $\mathcal{P}_E^*(G)$ is strongly regular.*

Proof. To prove the result, it is sufficient to show that if $\mathcal{P}_E^*(G)$ is regular then $\mathcal{P}_E^*(G)$ is strongly regular. Suppose that $\mathcal{P}_E^*(G)$ is regular. Then G must satisfy one of the conditions given in Theorem 2.1.4. If G is cyclic then being a complete graph, $\mathcal{P}_E^*(G)$ is strongly regular. If G satisfies condition (ii), then by the proof of Theorem 2.1.4, for each $x \in V(\mathcal{P}_E^*(G))$, we obtain $\deg(x) = m - 2$, where m is the order of a maximal cyclic subgroup containing x. For $m = 2$, $\mathcal{P}_E^*(G)$ is a null graph and so is strongly regular. If $m \geq 3$, then observe that in $\mathcal{P}_E^*(G)$, each pair of adjacent vertices has exactly $m - 3$ common neighbours and each pair of non-adjacent vertices has no common neighbour. Hence, $\mathcal{P}_E^*(G)$ is strongly regular with parameters $(n, m - 2, m - 3, 0)$. □

In view of [Aalipour et al., 2017, Theorem 28] and Theorem 1.2.14, we have the following corollary.

Corollary 2.1.7. *If G is a non-cyclic p-group then $\mathcal{P}_E^*(G)$ is regular if and only if the exponent of G is p.*

Theorem 2.1.8. *Let G be a non-cyclic nilpotent group. Then $\mathcal{P}_E^*(G)$ is regular if and only if G is a p-group with exponent p.*

Proof. Let $G = P_1P_2\cdots P_r$ be a non-cyclic nilpotent group of order $n = p_1^{\lambda_1}p_2^{\lambda_2}\cdot\cdot\cdot p_r^{\lambda_r}$. To prove our result it is sufficient to prove that if $\mathcal{P}_E^*(G)$ is regular then G is a p-group. If possible, let $r \geq 2$. Since G is a non-cyclic group there exists a non-cyclic Sylow subgroup P_i. Consequently, P_i has at least two maximal cyclic subgroups, namely: M_i and M_i'. Consider the maximal cyclic subgroups $M = M_1M_2\cdots M_i\cdots M_r$ and $M' = M_1M_2\cdots M_i'\cdots M_r$ of G, where M_j is a maximal cyclic subgroup of P_j for $j \in [r]\setminus\{i\}$. By Lemma 1.1.12, we obtain that M and M' are maximal cyclic subgroups of G such that $M \cap M' \neq \{e\}$; a contradiction of Theorem 2.1.4. Thus, $r = 1$ and so G is p-group. □

Corollary 2.1.9. *Let G be a finite non-cyclic abelian group. Then $\mathcal{P}_E^*(G)$ is regular if and only if G is an elementary abelian p-group.*

Based on the results obtained in this subsection, we posed the following conjecture which we are not able to prove.

Conjecture: Let G be a finite non-cyclic group. If $\mathcal{P}_E^*(G)$ is regular then G is a p-group with exponent p.

2.1.3 The Vertex Connectivity of $\mathcal{P}_E(G)$

In this subsection, we investigate the vertex connectivity of the enhanced power graph of some nilpotent groups. Recall that if G and H are two torsion groups then $\mathcal{P}_E(G \times H) \cong \mathcal{P}_E(G) \boxtimes \mathcal{P}_E(H)$ if and only if $\gcd(o(g), o(h)) = 1$ for all $g \in G$ and $h \in H$ (see [Zahirović et al., 2020, Lemma 2.1]). Let $G = P_1P_2\cdots P_r$ be a nilpotent group. For our purpose, first we show that the enhanced power graph of a finite nilpotent group is isomorphic to the strong product of the enhanced power graph of its Sylow subgroups (see Theorem 2.1.13). Using this and ascertaining a minimum cut-set, we obtain the vertex connectivity of $\mathcal{P}_E(G)$, where G is a nilpotent group such that each of its Sylow subgroups is cyclic except P_k for some $k \in [r]$.

Let $G = P_1P_2\cdots P_r$ be a nilpotent group. For $x = x_1x_2\cdots x_r \in G$, where $x_i \in P_i$, define $\eta_x = \{j \in [r] :\ x_j \neq e\}$. Note that if $\langle x\rangle \in \mathcal{M}(G)$ then $\eta_x = [r]$.

Lemma 2.1.10. *Let $H = \langle x\rangle$ and $x = \prod\limits_{i\in\eta_x} x_i$. Then $\langle x_i\rangle \subseteq \langle x\rangle$ for all $i \in \eta_x$.*

Proof. Consider $i_0 \in \eta_x$ and $l = \prod\limits_{i\in[r]\setminus\{i_0\}} o(x_i)$. Then by Theorem 1.1.10, we obtain $x^l = x_{i_0}^l$. Since $\gcd(l, o(x_{i_0})) = 1$, we have $\langle x^l\rangle = \langle x_{i_0}^l\rangle = \langle x_{i_0}\rangle$ and so $x_{i_0} \in \langle x^l\rangle$. Hence, $\langle x_{i_0}\rangle \subseteq \langle x\rangle$. □

Lemma 2.1.11. *Let G be a nilpotent group. Then $\langle x\rangle = \langle \prod\limits_{i\in\eta_x} x_i\rangle = \prod\limits_{i\in\eta_x}\langle x_i\rangle$, where $\langle x_i\rangle\langle x_j\rangle = \{ab : a \in \langle x_i\rangle \text{ and } b \in \langle x_j\rangle\}$.*

Proof. Clearly, $\langle \prod\limits_{i\in\eta_x} x_i\rangle \subseteq \prod\limits_{i\in\eta_x}\langle x_i\rangle$. If $a \in \prod\limits_{i\in\eta_x}\langle x_i\rangle$, then $a = \prod\limits_{i\in\eta_x} a_i$ such that $a_i \in \langle x_i\rangle$. Thus, $a_i = x_i^{k_i}$ for some $k_i \in \mathbb{N}$. By Lemma 2.1.10, $a_i = x^{\lambda_i k_i}$ for some $\lambda_i \in \mathbb{N}$ and so $a = x^{\sum_{i\in\eta_x}\lambda_i k_i}$. Consequently, we get $a \in \langle x\rangle$. Thus, the result holds. □

Lemma 2.1.12. *Let G be a nilpotent group such that $x = \prod\limits_{i=1}^{r} x_i$ and $y = \prod\limits_{i=1}^{r} y_i$. Then $x \sim y$ in $\mathcal{P}_E(G)$ if and only if $x_i \sim y_i$ in $\mathcal{P}_E(P_i)$ whenever $x_i \neq y_i$.*

Proof. First suppose that $x \sim y$ in $\mathcal{P}_E(G)$. Then there exists $z \in G$ such that $x, y \in \langle z\rangle$. We may now suppose that $x_i \neq y_i$ for some i. By Lemma 2.1.10, $x_i \in \langle x\rangle \subseteq \langle z\rangle$. Similarly, $y_i \in \langle z\rangle$. Thus, $\langle x_i, y_i\rangle \subseteq \langle z\rangle$ follows that $\langle x_i, y_i\rangle$ is a cyclic subgroup of P_i. Thus, $x_i \sim y_i$ in $\mathcal{P}_E(P_i)$. Conversely, suppose that $x_i \sim y_i$ in $\mathcal{P}_E(P_i)$ for $x_i \neq y_i$. Consider $K = \{j \in [r] : x_j \sim y_j \text{ in } \mathcal{P}_E(P_j)\}$. Consequently, for $i \in K$, we have $\langle x_i, y_i\rangle \subseteq \langle z_i\rangle$ for some $z_i \in P_i$. Choose $z = \prod\limits_{i\in K} z_i \cdot \prod\limits_{i\in[r]\setminus K} x_i$. Thus, by Lemma 2.1.11, $\langle z\rangle = \prod\limits_{i\in K}\langle z_i\rangle \cdot \prod\limits_{i\in[r]\setminus K}\langle x_i\rangle$. Consequently, $x = \prod\limits_{i\in[r]} x_i \in \langle z\rangle$ and $y = \prod\limits_{i\in[r]} y_i \in \langle z\rangle$. Hence, $x \sim y$ in $\mathcal{P}_E(G)$. □

Theorem 2.1.13. *Let $G = P_1P_2\cdots P_r$ be a nilpotent group. Then*

$$\mathcal{P}_E(G) \cong \mathcal{P}_E(P_1) \boxtimes \mathcal{P}_E(P_2) \boxtimes \cdots \boxtimes \mathcal{P}_E(P_r)$$

where P_i is the Sylow p_i-subgroup of G.

Proof. Let $x = x_1x_2\cdots x_r \in G$. Then define $\psi : V(\mathcal{P}_E(G)) \to V(\mathcal{P}_E(P_1)\boxtimes\mathcal{P}_E(P_2)\boxtimes \cdots \boxtimes \mathcal{P}_E(P_r))$ such that $x \longmapsto (x_1, x_2, \ldots, x_r)$, where $x_i \in P_i$. In view of Lemma 2.1.12, note that ψ is a graph isomorphism. □

Lemma 2.1.14. *Let G be a non-cyclic group and $\mathcal{T}(G) = \bigcap\limits_{M\in\mathcal{M}(G)} M$. Then $\mathcal{T}(G)$ is contained in every cut-set of $\mathcal{P}_E(G)$.*

Proof. Let $x \in \mathcal{T}(G)$ and $y(\neq x) \in G$. Since $y \in M$ for some $M \in \mathcal{M}(G)$ and $x \in \mathcal{T}(G)$, we have $x \in M$. Consequently, $x \sim y$. It follows that x is adjacent to every vertex of $\mathcal{P}_E(G)$. Thus, x must belong to every cut-set of $\mathcal{P}_E(G)$ and so is $\mathcal{T}(G)$. □

Theorem 2.1.15. *Let $G = P_1P_2\cdots P_r$ be a non-cyclic nilpotent group of order $n = p_1^{\lambda_1}p_2^{\lambda_2}...p_r^{\lambda_r}$ with $r \geq 2$. Suppose that each Sylow subgroup P_i of G is cyclic except P_k for some $k \in [r]$.*

(i) *If P_k is not a generalized quaternion 2-group, then $Q = P_1P_2\cdots P_{k-1}P_{k+1}\cdots P_r$ is the only minimum cut-set of $\mathcal{P}_E(G)$ and hence $\kappa(\mathcal{P}_E(G)) = \frac{n}{p_k^{\lambda_k}}$.*

(ii) *If P_k is a generalized quaternion 2-group, then the set $Q' = Z(Q_{2^\alpha})P_2\cdots P_r$ is the only minimum cut-set of $\mathcal{P}_E(G)$ and hence $\kappa(\mathcal{P}_E(G)) = \frac{n}{2^{\lambda_1-1}}$.*

Proof. (i) First, suppose that P_k is not a generalized quaternion 2-group. By Corollary 1.1.13, Q is contained in every maximal cyclic subgroup of G. By Lemma 2.1.14, Q is contained in every cut-set of $\mathcal{P}_E(G)$. Now, to prove our result we first prove the following claim.

Claim: Let T_i be a cut-set of $\mathcal{P}_E(P_i)$. Then $T = P_1\cdots P_{i-1}T_iP_{i+1}\cdots P_r$ is a cut-set of $\mathcal{P}_E(G)$.

Proof of the claim: Let T_i be a cut-set of $\mathcal{P}_E(P_i)$ and let $a, b \in P_i$ such that there exists no path between a and b in $\mathcal{P}_E(P_i \setminus T_i)$. It follows that, for the isomorphism ψ defined in the proof of Theorem 2.1.13, there is no path between $\psi(a)$ and $\psi(b)$ in the subgraph induced by $V(\boxtimes_{i=1}^{r}\mathcal{P}_E(P_i)) \setminus \psi(T)$. Consequently, there is no path between a and b in $\mathcal{P}_E(G \setminus T)$. Hence, T is a cut-set of $\mathcal{P}_E(G)$.

Now, by Theorem 1 of Bera et al. [2021], we obtain $\kappa(\mathcal{P}_E(P_k)) = 1$ and $\{e\}$ is the only cut-set of $\mathcal{P}_E(P_k)$. Thus, above claim follows that the set Q is the only minimum cut-set of $\mathcal{P}_E(G)$. Hence, $\kappa(\mathcal{P}_E(G)) = \frac{n}{p_k^{\lambda_k}}$.

(ii) Now, suppose that $P_k = Q_{2^\alpha}$ is a generalized quaternion 2-group. Note that the center $Z(Q_{2^\alpha})$ of Q_{2^α} is contained in every maximal cyclic subgroup of Q_{2^α}. Consequently, by Lemma 1.1.12, Q' is contained in every maximal cyclic subgroup of G. Thus, by Lemma 2.1.14, Q' is contained in every cut-set of $\mathcal{P}_E(G)$. By claim, Q' is a cut-set of $\mathcal{P}_E(G)$. Hence, Q' is the only minimum cut-set of $\mathcal{P}_E(G)$ and so $\kappa(\mathcal{P}_E(G)) = \frac{n}{2^{\lambda_1 - 1}}$. □

2.1.4 The Wiener Index of $\mathcal{P}_E(G)$

In this subsection, we study the Wiener index of $\mathcal{P}_E(G)$, where G is a finite nilpotent group. We obtain a lower bound and an upper bound of $W(\mathcal{P}_E(G))$. We also characterize the finite nilpotent groups attaining these bounds. Let $G = P_1 P_2 \cdots P_r$ be a finite nilpotent group. Then define

- $S_{0,i} = \{(x, x) : x \in \ P_i\}$.
- $S_{1,i} = \{(x, y) : x \sim y \text{ in } \mathcal{P}_E(P_i)\}$.
- $S_{2,i} = \{(x, y) : x \nsim y \text{ in } \mathcal{P}_E(P_i)\}$ with $|S_{2,i}| = m_i$.
- $S_0 = \{(x, x) : x \in \ G\}$.
- $S_1 = \{(x, y) : x \sim y \text{ in } \mathcal{P}_E(G)\}$.
- $S_2 = \{(x, y) : x \nsim y \text{ in } \mathcal{P}_E(G)\}$.

By the definition of Wiener index, we obtain

$$W(\mathcal{P}_E(G)) = \frac{|S_1| + 2|S_2|}{2}.$$

Now, we obtain the Wiener index of $\mathcal{P}_E(G)$, where G is a nilpotent group.

Theorem 2.1.16. *Let G be a nilpotent group of order $n = p_1^{\lambda_1} p_2^{\lambda_2} \cdots p_r^{\lambda_r}$ and let $|S_{2,i}| = m_i$. Then*

$$W(\mathcal{P}_E(G)) = \frac{2n^2 - n - \prod_{i=1}^{r}(p_i^{2\lambda_i} - m_i)}{2}.$$

Proof. Let $G = P_1 P_2 \cdots P_r$ be a nilpotent group such that $|P_i| = p_i^{\lambda_i}$ and let $x = x_1 x_2 \cdots x_r$, $y = y_1 y_2 \cdots y_r \in G$. By Theorem 2.1.13, note that $S_1 = \{(x, y) :$ either $x_i = y_i$ or $x_i \sim y_i$ in $\mathcal{P}_E(P_i)\} \setminus S_0$. Then

$$\begin{aligned}
|S_1| &= \prod_{i=1}^{r}(|S_{1,i}| + |S_{0,i}|) - n \\
&= \prod_{i=1}^{r}(p_i^{2\lambda_i} - m_i - p_i^{\lambda_i} + p_i^{\lambda_i}) - n \\
&= \prod_{i=1}^{r}(p_i^{2\lambda_i} - m_i) - n
\end{aligned}$$

and $|S_2| = n^2 - |S_0| - |S_1| = n^2 - \prod_{i=1}^{r}(p_i^{2\lambda_i} - m_i)$. Hence, $W(\mathcal{P}_E(G)) = \frac{2n^2 - n - \prod_{i=1}^{r}(p_i^{2\lambda_i} - m_i)}{2}$. □

Corollary 2.1.17. *Let G and G' be two nilpotent groups such that $|G| = |G'| = p_1^{\lambda_1} p_2^{\lambda_2} \cdots p_r^{\lambda_r}$. If $|S_{2,i}| \leq |S'_{2,i}|$ for all $i \in [r]$, then $W(\mathcal{P}_E(G)) \leq W(\mathcal{P}_E(G'))$.*

Lemma 2.1.18. *Let G be a p-group. Then $|S_2| \leq (|G| - p)(|G| - 1)$.*

Proof. Let $x \neq e \in G$. Since G is a p-group, we have $o(x) \geq p$. Thus, x is adjacent to at least $p-1$ vertices in $\mathcal{P}_E(G)$. It follows that x is not adjacent to at most $|G| - p + 1$ elements in $\mathcal{P}_E(G)$. Since x is at distance 0 from itself, it implies that the number of elements at distance two from x is at most $|G| - p$. Note that the identity element is adjacent to all other vertices in $\mathcal{P}_E(G)$. Thus, for $S_2 = \{(x, y) : x \nsim y \text{ in } \mathcal{P}_E(G)\}$, we have $|S_2| \leq (|G| - p)(|G| - 1)$. □

In view of Theorem 2.1.16 and Lemma 2.1.18, we have the following corollary.

Corollary 2.1.19. *Let G be a p-group. Then* $W(\mathcal{P}_E(G)) \leq \frac{(|G|-1)(2|G|-p)}{2}$.

For the nilpotent group G, now we give a sharp lower bound and an upper bound of $W(\mathcal{P}_E(G))$ (independent from m_i) in the following theorem.

Theorem 2.1.20. *Let G be a nilpotent group of order $n = p_1^{\lambda_1} p_2^{\lambda_2} \cdots p_r^{\lambda_r}$. Then*

(i)

$$\frac{n(n-1)}{2} \leq W(\mathcal{P}_E(G)) \leq \frac{2n^2 - n - \prod_{i=1}^{r} (p_i^{\lambda_i+1} + p_i^{\lambda_i} - p_i)}{2}.$$

(ii) *$W(\mathcal{P}_E(G))$ attains its lower bound if and only if G is cyclic.*

(iii) *$W(\mathcal{P}_E(G))$ attains its upper bound if and only if $|M| = p_1 p_2 \cdots p_r$ for every $M \in \mathcal{M}(G)$.*

Proof. (i)-(ii) From Lemma 2.1.18, it follows that $m_i \leq (p_i^{\lambda_i} - p_i)(p_i^{\lambda_i} - 1)$ for all $i \in [r]$. Consequently, by Theorem 2.1.16 and Corollary 2.1.17, we get $W(\mathcal{P}_E(G)) \leq \frac{2n^2 - n - \prod_{i=1}^{r} (p_i^{\lambda_i+1} + p_i^{\lambda_i} - p_i)}{2}$. Notice that $W(\mathcal{P}_E(G))$ is smallest if and only if $\mathcal{P}_E(G)$ is complete if and only if G is cyclic (cf. Theorem 1.2.15). Since the Wiener index of the complete graph on n vertices is $\frac{n(n-1)}{2}$, we obtain $\frac{n(n-1)}{2} \leq W(\mathcal{P}_E(G))$.

(iii) By Theorem 2.1.16, observe that $W(\mathcal{P}_E(G))$ is maximum if and only if m_i is maximum for all $i \in [r]$. First, we prove that m_i is maximum if and only if $|M'| = p_i$ for every $M' \in \mathcal{M}(P_i)$.

For simplicity, we write here $p_i = p$ and $\lambda_i = \lambda$ so that $m_i \leq (p^\lambda - p)(p^\lambda - 1)$. Now, let $|M'| = p$ for every $M' \in \mathcal{M}(P_i)$. Then for any non-identity element $x \in P_i$, $o(x) = p$. Since x is a generator of a maximal cyclic subgroup H of P_i, it implies that x is adjacent to all the other vertices of H and $x \nsim y$ for any $y \in P_i \setminus H$. It follows that x is adjacent to $p-1$ vertices of $\mathcal{P}_E(P_i)$. Consequently, x is at distance 2 from $p^\lambda - p$ vertices of $\mathcal{P}_E(P_i)$. Since x is an arbitrary non-identity element of P_i, we have $m_i = (p^\lambda - 1)(p^\lambda - p)$. Thus, m_i is maximum. Conversely, suppose that the m_i is maximum. On contrary, suppose $|M'| = p^\alpha$ for some $\alpha \geq 2$

and $M' \in \mathcal{M}(P_i)$. Further, assume that $x \in P_i$. Clearly, $o(x) \geq p$. If $x \in M'$ then x is adjacent to at least $p^{\alpha} - 1$ vertices of $\mathcal{P}_E(P_i)$ and so at most $p^{\lambda} - p^{\alpha}$ vertices are at distance 2 from x in $\mathcal{P}_E(P_i)$. Similarly, if $x \in P_i \setminus M'$ then there are at most $p^{\lambda} - p$ elements at distance 2 from x in $\mathcal{P}_E(P_i)$. Consequently, we get $m_i \leq (p^{\alpha}-1)(p^{\lambda}-p^{\alpha})+(p^{\lambda}-p)(p^{\lambda}-p^{\alpha}) < (p^{\lambda}-1)(p^{\lambda}-p)$; a contradiction. Hence, m_i is maximum if and only if $|M'| = p_i$ for all $M' \in \mathcal{M}(P_i)$.
Thus, by Lemma 1.1.12, we get $W(\mathcal{P}_E(G))$ is maximum if and only if $|M| = p_1p_2\cdots p_r$ for every $M \in \mathcal{M}(G)$. □

Note that the given upper bound is tight and it is attained by the group $G = \mathbb{Z}_{p_1}^{\lambda_1} \times \mathbb{Z}_{p_2}^{\lambda_2} \times \cdots \times \mathbb{Z}_{p_r}^{\lambda_r}$. Moreover, in this case, the graph $\mathcal{P}_E(G)$ has minimum number of edges.

2.2 Lambda Number of the Enhanced Power Graph

For non-negative integers j and k, an $L(j,k)$-*labeling* for the graph Γ is an integer valued function f on the vertex set $V(\Gamma)$ such that $|f(u) - f(v)| \geq k$ whenever u and v are vertices of distance two and $|f(u) - f(v)| \geq j$ whenever u and v are adjacent. The span of f is the difference between the maximum and minimum of f. It is convenient to assume that the minimum of f is 0 , we regard the span of f as the maximum of f. The $L(j,k)$-*labeling number* $\lambda_{j,k}(\Gamma)$ of the graph Γ is the minimum span over all $L(j,k)$-labeling for Γ. The radio channel assignment problem (Hale [1980]) and the study of the scalability of optical networks (Roberts [1991]), motivated the researchers to investigate the problem related to $L(j,k)$-labelings of a graph. The classical work of the $L(j,k)$-labeling problem is when $j = 2$ and $k = 1$. The $L(2,1)$-labeling number of a graph Γ is also called the λ-*number* of Γ. The λ-number of the power graph has been investigated by Ma, Feng and Wang [2021]. In this section, we study the λ-number of the enhanced power graphs of finite groups.

Recall that any non-cyclic nilpotent group G is of one of the following forms

(i) $G \cong G' \times \mathbb{Z}_n$, where G' is a non-trivial nilpotent group of odd order having no cyclic Sylow subgroup and $\gcd(n, |G'|) = 1$.

(ii) $G \cong G' \times P \times \mathbb{Z}_n$, where G' is a nilpotent group of odd order having no cyclic Sylow subgroup, P is a 2-group which is neither cyclic nor of maximal class and $\gcd(n, |G'|) = \gcd(2, n) = 1$.

(iii) $G \cong G' \times Q_{2^{k+1}} \times \mathbb{Z}_n$, where G' is described as in (ii), $Q_{2^{k+1}}$ is a generalized quaternion 2-group of order 2^{k+1} and $\gcd(n, |G'|) = \gcd(2, n) = 1$.

(iv) $G \cong G' \times D_{2^{k+1}} \times \mathbb{Z}_n$, where G' is described as in (ii), $D_{2^{k+1}}$ is a dihedral 2-group of order 2^{k+1} and $\gcd(n, |G'|) = \gcd(2, n) = 1$.

(v) $G \cong G' \times SD_{2^{k+1}} \times \mathbb{Z}_n$, where G' is described as in (ii), $SD_{2^{k+1}}$ is a semi-dihedral 2-group of order 2^{k+1} and $\gcd(n, |G'|) = \gcd(2, n) = 1$.

The following result characterizes the dominating vertices of the enhanced power graph of a finite nilpotent group and we use this result explicitly in this section without referring to it.

Theorem 2.2.1 ([Bera and Dey, 2022, Corollary 4.2]). *Let G be a finite non-cyclic nilpotent group and let $T_1 = \{(e', e_2, x) : x \in \mathbb{Z}_n\}$, $T_2 = \{(e', y, x) : y \in Q_{2^{k+1}}, x \in \mathbb{Z}_n$ and $\mathrm{o}(y) = 2\}$. Then*

$$\mathrm{Dom}(\mathcal{P}_E(G)) = \begin{cases} \{(e', x) : x \in \mathbb{Z}_n\}, & \textit{if } G = G' \times \mathbb{Z}_n \textit{ and } \gcd(|G'|, n) = 1 \\ \{(e', e_1, x) : x \in \mathbb{Z}_n\}, & \textit{if } G = G' \times P \times \mathbb{Z}_n \textit{ and } \gcd(|G'|, n) = \gcd(n, 2) = 1 \\ T_1 \cup T_2, & \textit{if } G = G' \times Q_{2^{k+1}} \times \mathbb{Z}_n \textit{ and } \gcd(|G'|, n) = \gcd(n, 2) = 1 \\ \{(e', e_3, x) : x \in \mathbb{Z}_n\}, & \textit{if } G = G' \times D_{2^{k+1}} \times \mathbb{Z}_n \textit{ and } \gcd(|G'|, n) = \gcd(n, 2) = 1 \\ \{(e', e_4, x) : x \in \mathbb{Z}_n\}, & \textit{if } G = G' \times SD_{2^{k+1}} \times \mathbb{Z}_n \textit{ and } \gcd(|G'|, n) = \gcd(n, 2) = 1, \end{cases}$$

where e', e_i's, $1 \le i \le 4$, are the identity elements of the respective groups in G.

Recall that $\mathtt{C}_d$ is the number of cyclic subgroups of order d in the group.

The following results will be useful for later use.

Lemma 2.2.2 ([Sarkar, 2022, Lemma 2.1]). *Let G be a finite non-cyclic simple group. Then for any $d \in \pi_G$, we have $\mathtt{C}_d \geq 2$.*

A *path covering* $\mathbb{S}(\Gamma)$ of a graph Γ is a collection of vertex-disjoint paths in Γ such that each vertex in $V(\Gamma)$ is contained in a path of $\mathbb{S}(\Gamma)$. The *path covering number* $\mu(\Gamma)$ of Γ is the minimum cardinality of a path covering of Γ

Theorem 2.2.3 ([Georges et al., 1994, Theorem 14]). *Let Γ be a graph of order n.*

(i) *Then $\lambda(\Gamma) \leq n-1$ if and only if $\mu(\overline{\Gamma}) = 1$.*

(ii) *Let $r \geq 2$ be an integer. Then $\lambda(\Gamma) = n + r - 2$ if and only if $\mu(\overline{\Gamma}) = r$.*

Now, first we obtain the bounds for $\lambda(\mathcal{P}_E(G))$, where G is a finite group. Then we classify finite simple groups G such that $\lambda(\mathcal{P}_E(G)) = |G|$. In fact, for the symmetric group S_n of degree n, we prove that $\lambda(\mathcal{P}_E(S_n)) = n!$ (see Theorem 2.2.13). For the nilpotent group G, the set of all dominating vertices of $\mathcal{P}_E(G)$ is obtained in Theorem 2.2.1. We obtain the lambda number of the enhanced power graphs of nilpotent groups (see Theorems 2.2.16 and 2.2.19).

Theorem 2.2.4. *Let G be a finite group of order n. Then $\lambda(\mathcal{P}_E(G)) \geq n$ with equality holds if and only if $\overline{\mathcal{P}_E(G) \setminus \{e\}}$ contains a Hamiltonian path.*

Proof. It is well known that for a finite group G, the graph $\mathcal{P}_E(G)$ is a spanning subgraph of the power graph $\mathcal{P}(G)$ and note that the lambda number is a monotone parameter. By [Ma, Feng and Wang, 2021, Theorem 3.1], the result holds. □

Theorem 2.2.5. *Let G be a finite non-cyclic group of order n. Suppose that $M_1, M_2, \ldots, M_r$ are the maximal cyclic subgroups of G such that $m_1 \geq m_2 \geq \cdots \geq m_r$, where $m_i = \phi(|M_i|)$ for $1 \leq i \leq r$. Then*

$$\lambda(\mathcal{P}_E(G)) \leq \begin{cases} 2n - |\mathcal{G}_{\mathcal{M}(G)}| - 1; & \text{if } m_1 \leq \sum\limits_{i=2}^{r} m_i, \\ 2(n - m_1 - 1); & \text{Otherwise.} \end{cases}$$

Proof. We prove this result by finding an upper bound of the path covering number of $\overline{\mathcal{P}_E(G)}$. We discuss the following two possible cases.

Case-1: $m_1 \leq \sum_{i=2}^{r} m_i$. We discuss this case into two subcases.

Subcase-1.1: $m_1 = m_2$. Now, we provide a Hamiltonian path in the subgraph of $\overline{\mathcal{P}_E(G)}$ induced by the set $\mathcal{G}_{\mathcal{M}(G)}$. Note that, in $\overline{\mathcal{P}_E(G)}$, each generator of one maximal cyclic subgroup is adjacent to all the generators of other maximal cyclic subgroups. Consequently, we obtain a path $P : x_{1,1} \sim x_{1,2} \sim \cdots \sim x_{1,r} \sim x_{2,1} \sim x_{2,2} \sim \cdots \sim x_{m_s,s}$, where $\langle x_{i,j} \rangle = M_j$, $1 \leq i \leq m_j$ and $s = \max\{t : 2 \leq t \leq r,\ m_t = m_1\}$, covers all the vertices of $\mathcal{G}_{\mathcal{M}(G)}$ in $\overline{\mathcal{P}_E(G)}$. It follows that $\mu(\overline{\mathcal{P}_E(G)}) \leq n - |\mathcal{G}_{\mathcal{M}(G)}| + 1$. By Theorem 2.2.3, we have $\lambda(\mathcal{P}_E(G)) \leq 2n - |\mathcal{G}_{\mathcal{M}(G)}| - 1$.

Subcase-1.2: $m_1 > m_2$. Since $\mathcal{G}_{\mathcal{M}(G)} = \bigcup_{i=1}^{r} \mathcal{G}_{M_i}$, we consider $A_1 = \{a_1, a_2, \ldots, a_{m_1-m_2}\} \subseteq \mathcal{G}_{M_1}$. In A_2, we collect the $m_1 - m_2$ elements starting from $\mathcal{G}_{M_r}$. If $m_r \geq m_1 - m_2$, then we take $A_2 \subseteq \mathcal{G}_{M_r}$ such that $|A_2| = m_1 - m_2$. Otherwise, we collect remaining $(m_1 - m_2) - m_r$ elements from $\mathcal{G}_{M_{r-1}}$ and then choose remaining elements, if required, such that $|A_2| = m_1 - m_2$, from $\mathcal{G}_{M_{r-2}}, \mathcal{G}_{M_{r-3}}$ and so on. We write $A_2 = \{b_1, b_2, \ldots, b_{m_1-m_2}\}$. Further, consider the set $A_3 = \mathcal{G}_{\mathcal{M}(G)} \setminus (A_1 \cup A_2)$. In view of the above given partition of $\mathcal{G}_{\mathcal{M}(G)}$, Now, we provide a Hamiltonian path in the subgraph of $\overline{\mathcal{P}_E(G)}$ induced by the set $\mathcal{G}_{\mathcal{M}(G)}$. Note that $a_i \sim b_j$ in $\overline{\mathcal{P}_E(G)}$ for all $i, j \in \{1, 2, \ldots, m_1 - m_2\}$. Thus, we have a Hamiltonian path $P : a_1 \sim b_1 \sim a_2 \sim \cdots \sim a_{m_1-m_2} \sim b_{m_1-m_2}$ in the subgraph induced by the set $A_1 \cup A_2$. Notice that the subgraph Γ induced by the set A_3 in $\overline{\mathcal{P}_E(G)}$ is a complete t-partite graph, where $t = \max\{i : \mathcal{G}_{M_i} \cap A_3 \neq \varnothing\}$ and the partition set of Γ is $\mathcal{G}_{M_1} \setminus A_1, \mathcal{G}_{M_2}, \ldots, \mathcal{G}_{M_t} \setminus A_2$. Since $|\mathcal{G}_{M_1} \setminus A_1| = |\mathcal{G}_{M_2}| \geq |\mathcal{G}_{M_i}|$ for $3 \leq i \leq t$, we have a Hamiltonian path H' of Γ with initial vertex x belongs to $\mathcal{G}_{M_1}$. Since $b_{m_1-m_2} \in \mathcal{G}_{M_k}$ for some k, where $t \leq k \leq r$, we have $b_{m_1-m_2} \sim x$. Consequently, we get a Hamiltonian path in the subgraph induced by the set $\mathcal{G}_{\mathcal{M}(G)}$ in $\overline{\mathcal{P}_E(G)}$. Thus $\mu(\overline{\mathcal{P}_E(G)}) \leq n - |\mathcal{G}_{\mathcal{M}(G)}| + 1$. By Theorem 2.2.3, $\lambda(\mathcal{P}_E(G)) \leq 2n - |\mathcal{G}_{\mathcal{M}(G)}| - 1$.

Case-2: $m_1 > \sum_{i=2}^{r} m_i$. Since G is a non-cyclic group, it implies that M_1 is a proper subgroup of G. By consequence of Lagrange's theorem, $|M_1| \leq \frac{n}{2}$ and so $|\mathcal{G}_{M_1}| < \frac{n}{2}$. Notice that each element of $\mathcal{G}_{M_1}$ is adjacent to every element of $G \setminus M_1$ in $\overline{\mathcal{P}_E(G)}$. Thus, for $\langle x_i \rangle = M_1$ and $y_i \in G \setminus M_1$, we have a path $P : y_1 \sim x_1 \sim y_2 \sim \cdots \sim x_{m_1} \sim y_{m_1+1}$ of length $2m_1 + 1$ in $\overline{\mathcal{P}_E(G)}$. Consequently, $\mu(\overline{\mathcal{P}_E(G)}) \leq n - 2m_1$. Hence, by Theorem 2.2.3, $\lambda(\mathcal{P}_E(G)) \leq 2n - 2m_1 - 2$. □

In view of Lemma 1.1.5, we have the following corollary of Theorem 2.2.5.

Corollary 2.2.6. *Let G be a finite non-cyclic group of order n. Then $\lambda(\mathcal{P}_E(G)) \leq 2n - 4$, with equality holds if and only if G is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof. Since G is a non-cyclic group, by Lemma 1.1.5, we get $|\mathcal{G}_{\mathcal{M}(G)}| \geq 3$. Consequently, by Theorem 2.2.5, $\lambda(\mathcal{P}_E(G)) \leq 2n - 4$. If G is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, then $\lambda(\mathcal{P}_E(G)) \geq 4 = 2|G| - 4$ and so $\lambda(\mathcal{P}_E(G)) = 2n - 4$. We now suppose that $\lambda(\mathcal{P}_E(G)) = 2n - 4$. This is possible only when $|\mathcal{G}_{\mathcal{M}(G)}| = 3$. Note that for a non-cyclic group G, $|\mathcal{G}_{\mathcal{M}(G)}| = 3$ if and only if G has exactly three maximal cyclic subgroups each with having only one generator. By Remark 1.1.4, $|G| = 4$. Thus, we must have $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. □

Now, we classify finite simple groups G such that $\lambda(\mathcal{P}_E(G)) = |G|$. For this purpose, first we derive the following two lemmas. Recall that for $x, y \in G$, the equivalence relation ρ is defined as $x\rho y$ if and only if $\langle x \rangle = \langle y \rangle$.

Lemma 2.2.7. *Let G be a finite non-cyclic simple group. Then for any $d \in \pi_G$, there exists a Hamiltonian path in subgraph of $\overline{\mathcal{P}_E(G)}$ induced by the set τ_d.*

Proof. In view of Lemma 2.2.2, suppose $\complement_d = s$, where $s \geq 2$. Let $\mathcal{H}_d = \{\rho_1, \rho_2, \ldots, \rho_s\}$ be the set of all cyclic classes of elements of order d. Let $x, y \in \rho_i$, where $i \in \{1, 2, \ldots, s\}$. Then $x, y \in \langle x \rangle$ and so $x \sim y$ in $\mathcal{P}_E(G)$. Consequently, $x \nsim y$ in $\overline{\mathcal{P}_E(G)}$. Also, for $i \neq j$, let $x \in \rho_i$ and $y \in \rho_j$. Let $x \sim y$ in $\mathcal{P}_E(G)$. Then there exists $z \in G$ such that $x, y \in \langle z \rangle$. Since $o(x) = o(y) = d$, we obtain $\langle x \rangle = \langle y \rangle$, which is not possible. Thus, $x \nsim y$ in $\mathcal{P}_E(G)$ and so $x \sim y$ in $\overline{\mathcal{P}_E(G)}$. It follows that the subgraph of $\overline{\mathcal{P}_E(G)}$ induced by the set τ_d is a complete s-partite graph such that the size of each partition set is $\phi(d)$. Hence, the result holds. □

Lemma 2.2.8. *Let G be a finite non-cyclic simple group and $d_1, d_2 \in \pi_G$. For each $x \in \tau_{d_1}$, there exists $y \in \tau_{d_2}$ such that $x \sim y$ in $\overline{\mathcal{P}_E(G)}$.*

Proof. To prove this result, it is sufficient to show that there exists $y \in \tau_{d_2}$ such that $xy \neq yx$ so that $x \sim y$ in $\overline{\mathcal{P}_E(G)}$. Let $xy' = y'x$ for all $y' \in \tau_{d_2}$. Note that $\langle \tau_{d_2} \rangle$ is a subgroup of G. For $g \in G$ and $x' = x_1x_2 \cdots x_k \in \langle \tau_{d_2} \rangle$, where $x_i \in \tau_{d_2}$, we have $g^{-1}x'g = g^{-1}x_1x_2 \cdots x_kg = g^{-1}x_1gg^{-1}x_2g \cdots gg^{-1}x_kg$. Note that $g^{-1}x_ig \in \tau_{d_2}$ and so $g^{-1}x'g \in \langle \tau_{d_2} \rangle$. Therefore, $\langle \tau_{d_2} \rangle$ is normal subgroup of G. But G is simple and $\langle \tau_{d_2} \rangle \neq \{e\}$ gives $G = \langle \tau_{d_2} \rangle$. Since x commutes with every element of τ_{d_2}, it follows that x belongs to the center $Z(G)$ of G. Consequently, we get $G = Z(G)$ and so G is an abelian simple group. Therefore, G must be a cyclic group of prime order, a contradiction. Thus, the result holds. □

Theorem 2.2.9. *Let G be a non-trivial finite simple group of order n. Then $\lambda(\mathcal{P}_E(G)) = n$ if and only if G is not a cyclic group of order $n \geq 3$.*

Proof. If G is cyclic, then by Theorem 1.2.15, $\mathcal{P}_E(G)$ is a complete graph. Consequently, $\lambda(\mathcal{P}_E(G)) = 2n - 2$. Thus $\lambda(\mathcal{P}_E(G)) = n$ if and only if $n = 2$. We may now suppose that G is a non-cyclic group. To prove our result, it is sufficient to show that the graph $\overline{\mathcal{P}_E(G) \setminus \{e\}}$ has a Hamiltonian cycle (see Theorem 2.2.4). Let $\pi_G = \{d_1, d_2, \ldots, d_k\}$. Then $G \setminus \{e\} = \bigcup_{i=1}^{k} \tau_{d_i}$. By Lemma 2.2.7, for each $i \in [k] = \{1, 2, \ldots, k\}$, we have a Hamiltonian path in the subgraph induced by the set τ_{d_i} in $\overline{\mathcal{P}_E(G)}$ and by Lemma 2.2.8, we get a Hamiltonian path in $\overline{\mathcal{P}_E(G) \setminus \{e\}}$. Thus, the result holds. □

Now, we obtain the lambda number of enhanced power graphs of the symmetric groups S_n. We begin with the following lemma.

Lemma 2.2.10. *For any $d \in \pi_{S_n}$, where $n \geq 4$, we have $\mathtt{C}_d \geq 2$.*

Proof. On the contrary, for some $d \in \pi_{S_n}$, assume that $\mathsf{C}_d = 1$. Let $x \in \tau_d$. Then for any $g \in S_n$, we have $g^{-1}xg \in \rho_x$. Consequently, $g^{-1}\langle x\rangle g = \langle x\rangle$. Thus, $\langle x\rangle$ is a normal subgroup of S_n. But S_n has no non-trivial cyclic normal subgroup. Since $d \geq 2$, we obtain $\langle x\rangle \neq \{e\}$. Thus, the result holds. □

Lemma 2.2.11. *Let $d \in \pi_{S_n}$, where $n \geq 4$. Then there exists a Hamiltonian path in the subgraph of $\overline{\mathcal{P}_E(S_n)}$ induced by the set τ_d.*

Proof. By Lemma 2.2.10, and by the similar argument used in the proof of Lemma 2.2.7, the result holds. □

Lemma 2.2.12. *Suppose $d_1, d_2 \in \pi_{S_n}$, where $n \geq 4$. Then for each $x \in \tau_{d_1}$, there exists $y \in \tau_{d_2}$ such that $x \sim y$ in $\overline{\mathcal{P}_E(S_n)}$.*

Proof. To prove this result, it is sufficient to prove that there exists $y \in \tau_{d_2}$ such that $xy \neq yx$ and so $x \sim y$ in $\overline{\mathcal{P}_E(S_n)}$. If possible, let $xy' = y'x$ for all $y' \in \tau_{d_2}$. By the proof of Lemma 2.2.8, notice that $\langle \tau_{d_2}\rangle = G'$ is a normal subgroup of S_n. Since x commutes with every element of τ_{d_2}, it follows that x commutes with every element of G'. Now, if $n = 4$, then $G' = H$ or A_4 or S_4, where $H = \{e, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$. Note that $\pi_H = \{2\}$ and H does not contain all the elements of order 2 of S_4. It follows that $G' \neq H$. If $G' = A_4$, then x commutes with every element of A_4. It implies that $x = e$; a contradiction. Now, if $G' = S_4$ then $x \in Z(S_4)$. It follows that $x = e$, which is not possible. Thus there exists $y \in \tau_{d_2}$ such that $xy \neq yx$. We may now suppose that $n \geq 5$. Since G' is a non-trivial normal subgroup of S_n, then either $G' = A_n$ or $G' = S_n$. In both these cases, we obtain $x = e$, which is not possible. Thus, the result holds. □

Theorem 2.2.13. *For $n \geq 2$, we have $\lambda(\mathcal{P}_E(S_n)) = n!$.*

Proof. By Theorem 2.2.4, we obtain $\lambda(\mathcal{P}_E(S_n)) \geq n!$. Define a function f from S_2 to the set of all non-negative integers such that $f(e) = 0, \quad f((1\ 2)) = 2$. Clearly, f is an $L(2,1)$-labeling of $\mathcal{P}_E(S_2)$. It follows that $\lambda(\mathcal{P}_E(S_2)) \leq 2$ and so the result holds for $n = 2$. Further note that the function g from S_3 to the set of all non-negative integers defined by $g(e) = 0$, $g((1\ 2)) = 2$, $g((1\ 2\ 3)) = 3$, $g((1\ 3)) = 4$, $g((1\ 3\ 2)) = 5$ and $g((2\ 3)) = 6$, is an $L(2,1)$-labeling of $\mathcal{P}_E(S_3)$. It implies that $\lambda(\mathcal{P}_E(S_3)) \leq 6$. Therefore, $\lambda(\mathcal{P}_E(S_3)) = 3!$. For $n \geq 4$, let $\pi_{S_n} = \{d_1, d_2, \ldots, d_k\}$. Then $S_n \setminus \{e\} = \bigcup_{i=1}^{k} \tau_{d_i}$. Consequently, by Lemmas 2.2.11 and 2.2.12, we get a Hamiltonian path in $\overline{\mathcal{P}_E(S_n) \setminus \{e\}}$. Thus, the result holds. □

In the remaining part of this section, we obtain the lambda number of enhanced power graphs of finite nilpotent groups. We start with the following lemma

Lemma 2.2.14. *Let G' be a non-trivial nilpotent group of odd order having no cyclic Sylow subgroups. If $G \cong G' \times \mathbb{Z}_n$, where $\gcd(n, |G'|) = 1$, then $\complement_{o(x)} \geq 3$ for each $x \in G \setminus \mathrm{Dom}(\mathcal{P}_E(G))$.*

Proof. Let $x = (x', y')$ be an arbitrary element of $G \setminus \mathrm{Dom}(\mathcal{P}_E(G))$. Since G' is a nilpotent group, and so $G' = P_1 \times P_2 \times \cdots \times P_r$, where $P_i's$ are Sylow subgroups of G'. Consequently, $x = (x_1, x_2, \ldots, x_r, y')$, where $x_i \in P_i$ for each $i \in [r]$. It follows that $x_j \neq e$ for some $j \in [r]$ because $x \notin \mathrm{Dom}(\mathcal{P}_E(G))$. Consider $y = (x_1, x_2, \ldots, x_{j-1}, y_j, x_{j+1}, \ldots, x_r, y')$, $z = (x_1, x_2, \ldots, x_{j-1}, z_j, x_{j+1}, \ldots, x_r, y')$, where $y_j, z_j \in P_j$ such that $o(x_i) = o(y_j) = o(z_j)$ and the cyclic subgroups $\langle x_j \rangle, \langle y_j \rangle$ and $\langle z_j \rangle$ of P_j are distinct [cf. Theorem 1.1.6]. Clearly, $o(x) = o(y) = o(z)$. Note that the cyclic subgroups $\langle x \rangle, \langle y \rangle$ and $\langle z \rangle$ of G are distinct. Without loss of generality, let if possible, $\langle x \rangle = \langle y \rangle$. Then there exists $m \in \mathbb{N}$ such that $x^m = y$. Now, consider $l = o(x_1)o(x_2)\cdots o(x_{j-1})o(x_{j+1})\cdots o(x_r)o(y')$. Then $x^{ml} = y^l$ and it follows that $x_j^{ml} = y_j^l$. Since $\gcd(o(y_j), l) = 1$, we obtain $o(y_j) = o(y_j^l)$. Consequently, $\langle y_j \rangle = \langle y_j^l \rangle = \langle x_j^{ml} \rangle \subseteq \langle x_j \rangle$. Thus, $\langle y_j \rangle = \langle x_j \rangle$; a contradiction. Thus, the result holds. □

Lemma 2.2.15. *Let G' be a non-trivial nilpotent group of odd order having no cyclic Sylow subgroups. If $G \cong G' \times \mathbb{Z}_n$, where $\gcd(n, |G'|) = 1$, then for each $d \in D$, there* *exists a Hamiltonian path in the subgraph of* $\overline{\mathcal{P}_E(G)}$ *induced by the set* τ_d*, where* $D = \{o(x) : x \in G \setminus \mathrm{Dom}(\mathcal{P}_E(G))\}$.

Proof. Let $d \in D$. Then by Lemma 2.2.14, $\mathtt{C}_d = s$, where $s \geq 3$. Notice that the subgraph induced by the set τ_d in $\overline{\mathcal{P}_E(G)}$ is a complete s-partite graph with exactly $\phi(d)$ vertices in each partition set. Thus, we get a Hamiltonian path between any two elements of τ_d. □

Theorem 2.2.16. *Let* G' *be a non-trivial nilpotent group of odd order having no cyclic Sylow subgroups. If* $G \cong G' \times \mathbb{Z}_n$*, where* $\gcd(n, |G'|) = 1$*, then*

$$\lambda(\mathcal{P}_E(G)) = |G| + |\mathrm{Dom}(\mathcal{P}_E(G))| - 1.$$

Proof. In view of Theorem 2.2.3, to prove our result it is sufficient to show that $\mu(\overline{\mathcal{P}_E(G)}) = |\mathrm{Dom}(\mathcal{P}_E(G))| + 1$. For $x \in \mathrm{Dom}(\mathcal{P}_E(G))$, clearly x is an isolated vertex in $\overline{\mathcal{P}_E(G)}$. Thus, it is sufficient to show that the subgraph of $\overline{\mathcal{P}_E(G)}$ induced by the non-dominating vertices of $\mathcal{P}_E(G)$ has a Hamiltonian path. Let $G \setminus \mathrm{Dom}(\mathcal{P}_E(G))$ has elements of order $d_1, d_2, \ldots, d_t$. Consider $S = \{d_1, d_2, \ldots, d_t\}$, where $d_1 < d_2 < \cdots < d_t$.

Claim: There exists an ordered set $S' = \{\beta_1, \beta_2, \ldots, \beta_t\}$, where $\beta_i \in S$, such that either $\beta_i | \beta_{i+1}$ or $\beta_{i+1} | \beta_i$ for each $i \in [t-1]$.

Proof of claim: If for each $i \in [t-1]$, either $d_i | d_{i+1}$ or $d_{i+1} | d_i$ then $S = S'$. Otherwise, choose the smallest l such that neither $d_l | d_{l+1}$ nor $d_{l+1} | d_l$. By Lemma 1.1.11, $d_{l+j} = \mathrm{lcm}(d_l, d_{l+1}) \in \pi_G$ for some $j \geq 2$. Now, let $x = (x_1, x_2) \in G \setminus \mathrm{Dom}(\mathcal{P}_E(G))$ such that $o(x) = d_l$. Clearly, $o(x_1) > 1$ and $o(x) = o(x_1)o(x_2)$. Suppose $z = (z_1, z_2) \in G$ such that $o(z) = d_{l+j}$. Since $d_l | d_{l+j}$, it follows that $o(x_1) | o(z_1)$. Consequently, $o(z_1) > 1$ and so $d_{l+j} \in S$ (cf. Theorem 2.2.1). Thus, by taking i-th element of the ordered set $\{d_1, d_2, \ldots, d_l, d_{l+j}, d_{l+1}, \ldots, d_{l+j-1}, d_{l+j+1}, \ldots, d_t\}$ as γ_i, we get an ordered set $S_1 = \{\gamma_1, \gamma_2, \ldots, \gamma_l, \gamma_{l+1}, \gamma_{l+2}, \ldots, \gamma_t\}$ such that either $\gamma_i | \gamma_{i+1}$ or $\gamma_{i+1} | \gamma_i$ for each $i \in \{1, 2, \ldots, l+1\}$. If for each $i \in \{l+2, l+3, \ldots, t-1\}$, either $\gamma_i | \gamma_{i+1}$ or $\gamma_{i+1} | \gamma_i$, then $S_1 = S'$. Otherwise, choose the smallest $l' \in \{l+2, l+3, \ldots, t-1\}$ and repeat the above process. On continuing this process, we get desired ordered set S'. Now, by Lemma 2.2.15, for each $i \in [t]$, the subgraph of $\overline{\mathcal{P}_E(G)}$ induced by the set τ_{β_i} is a complete s-partite graph with $\phi(\beta_i)$ vertices in each partition set. Observe that $G \setminus \mathrm{Dom}(\mathcal{P}_E(G)) = \bigcup_{i=1}^{t} \tau_{\beta_i}$. Then there exist paths $H_1, H_2, \ldots, H_t$ which covers all the vertices of $\tau_{\beta_1}, \tau_{\beta_2}, \ldots, \tau_{\beta_t}$, respectively. Now, we shall show that for each $i \in [t-1]$, the end vertex of H_i is adjacent to the initial vertex of H_{i+1} in $\overline{\mathcal{P}_E(G)}$ through the following two cases:

Case-1: $\beta_i | \beta_{i+1}$. Let $x \in \tau_{\beta_i}$ and y be the initial vertex of H_{i+1}. If $x \nsim y$ in $\mathcal{P}_E(G)$, then we choose x to be the end vertex of H_i so that $x \sim y \in \overline{\mathcal{P}_E(G)}$. Now, we may assume that $x \sim y$ in $\mathcal{P}_E(G)$. Then there exists $z \in G$ such that $x, y \in \langle z \rangle$. Since $\beta_i | \beta_{i+1}$, we get $\langle x \rangle \subset \langle y \rangle$. Let $x' \in \tau_{\beta_i}$ such that $\langle x \rangle \neq \langle x' \rangle$. Now, if $x' \sim y$ in $\mathcal{P}_E(G)$ then $\langle x' \rangle \subset \langle y \rangle$, which is not possible as $\langle x \rangle \neq \langle x' \rangle$. Thus $x' \sim y$ in $\overline{\mathcal{P}_E(G)}$. Therefore, we can choose x' as the end vertex of H_i.

Case-2: $\beta_{i+1} | \beta_i$. Let x be the end vertex of H_i and $y \in \tau_{\beta_{i+1}}$. If $x \nsim y$ in $\mathcal{P}_E(G)$, then we choose y to be the initial vertex of H_{i+1} so that $x \sim y \in \overline{\mathcal{P}_E(G)}$. Otherwise, there exists $z \in G$ such that $x, y \in \langle z \rangle$. Since $\beta_{i+1} | \beta_i$, it follows that $\langle y \rangle \subset \langle x \rangle$. Let $y' \in \tau_{\beta_{i+1}}$ such that $\langle y \rangle \neq \langle y' \rangle$. Now, if $x \sim y'$ in $\mathcal{P}_E(G)$ then $\langle y' \rangle \subset \langle x \rangle$, which is not possible as $\langle y \rangle \neq \langle y' \rangle$. Thus $x \sim y'$ in $\overline{\mathcal{P}_E(G)}$. Therefore, consider y' as the initial vertex of H_{i+1}.

Hence, we get a Hamiltonian path in subgraph of $\overline{\mathcal{P}_E(G)}$ induced by the set $G \setminus \mathrm{Dom}(\mathcal{P}_E(G))$. □

Lemma 2.2.17. *Let G' be a nilpotent group of odd order having no cyclic Sylow subgroups. If $G \cong G' \times P \times \mathbb{Z}_n$, where P is a non-cyclic 2-group and $\gcd(n, |G'|) = \gcd(2, n) = 1$, then for each $x \in \mathcal{S}' = \{(g_1, g_2, g_3) \in G \mid g_1 \neq e_{G'}\}$, we have $\mathsf{C}_{o(x)} \geq 3$.*

Proof. After taking $\mathcal{S}'$ in place of $G \setminus \mathrm{Dom}(\mathcal{P}_E(G))$, the proof is similar to the proof of Lemma 2.2.14. Hence, we omit the details. □

Theorem 2.2.18. *Let G' be a nilpotent group of odd order having no cyclic Sylow subgroups. If $G \cong G' \times P \times \mathbb{Z}_n$, where P is a non-cyclic 2-group and $\gcd(n, |G'|) =$*

$\gcd(2, n) = 1$, *then there exists a Hamiltonian path in the subgraph of* $\overline{\mathcal{P}_E(G)}$ *induced by the set* $\mathcal{S}'$.

Proof. By Lemma 2.2.17, and by the similar argument used in the proof of Theorem 2.2.16, the result holds. □

Theorem 2.2.19. *Let* G' *be a nilpotent group of odd order having no cyclic Sylow subgroups. If* $G \cong G' \times P \times \mathbb{Z}_n$, *where* P *is a non-cyclic* 2*-group and* $\gcd(n, |G'|) = \gcd(2, n) = 1$, *then*

$$\lambda(\mathcal{P}_E(G)) = |G| + |\mathrm{Dom}(\mathcal{P}_E(G))| - 1.$$

Proof. In view of Theorem 2.2.3, we show that the subgraph of $\overline{\mathcal{P}_E(G)}$ induced by the set of all non-dominating vertices of $\mathcal{P}_E(G)$ has a Hamiltonian path. Let e_1, e_2 and e_3 be the identity elements of the groups G', P and $\mathbb{Z}_n$, respectively. By Theorem 2.2.18, let H' be a Hamiltonian path in the subgraph of $\overline{\mathcal{P}_E(G)}$ induced by the set $\mathcal{S}'$ and let $g' = (x', y', z')$ be the end vertex of H'. By the proof of Theorem 2.2.18, notice that the order of $g' = (x', y', z')$ is maximum. Suppose that the exponent of the group P is 2^k. Consequently, $o(y') = 2^k$. Further, observe that if $y \nsim y''$ in the graph $\mathcal{P}_E(P)$ then $(x, y, z) \nsim (x'', y'', z'')$ in $\mathcal{P}_E(G)$. In view of Theorem 1.1.6 and Corollary 1.1.7, we have the following cases:

Case-1: *P is not of maximal class.* Consider the set $\mathcal{S}'' = \{(e_1, y, z) \in G : y \neq e_2\}$. Notice that the sets $\mathcal{S}'$, defined in Lemma 2.2.17, $\mathcal{S}''$ and $\mathrm{Dom}(\mathcal{P}_E(G))$ forms a partition of G. Now, we provide a Hamiltonian path of the subgraph of $\overline{\mathcal{P}_E(G)}$ induced by the set $\mathcal{S}' \cup \mathcal{S}''$. To do this, first we drive a Hamiltonian path of the subgraph of $\overline{\mathcal{P}_E(G)}$ induced by the set $\mathcal{S}''$. By Theorem 1.1.6, for $2 \leq j \leq k$, we have $\mathsf{C}_{2^j} \geq 2$ and $\mathsf{C}_2 \geq 3$ in P. For $1 \leq j \leq k$, let $t_j = \mathsf{C}_{2^j}$ and let $\mathcal{T}_j = \{\mathcal{C}_1^{(j)}, \mathcal{C}_2^{(j)}, \ldots, \mathcal{C}_{t_j}^{(j)}\}$ denotes the cyclic classes of P containing the elements of order 2^j. Observe that each class in $\mathcal{T}_j$ is of cardinality 2^{j-1}. Notice that each element of P belongs to exactly one cyclic class of $\mathcal{T}_j$ for $j \in [k]$. Further note that, for $i \neq s$, if $x_i \in \mathcal{C}_i^{(j)}$ and $y_s \in \mathcal{C}_s^{(j)}$, then $x_i \nsim y_s$ in $\mathcal{P}_E(P)$. We label a class in $\mathcal{T}_k$ by $\mathcal{C}_1^{(k)}$ such that $y' \notin \mathcal{C}_1^{(k)}$.

For $2 \leq j \leq k$, let $u_j \in \mathcal{C}_{t_j}^{(j)}$ be an arbitrary element. Now, u_j can be adjacent to at most one of the cyclic classes in $\mathcal{T}_{j-1}$ in $\mathcal{P}_E(P)$. If possible, let $u_j \sim v_1$ and $u_j \sim v_2$, where $v_1 \in \mathcal{C}_{i_1}^{(j-1)}$, $v_2 \in \mathcal{C}_{i_2}^{(j-1)}$. Then there exist elements $w_1, w_2 \in P$ such that $u_j, v_1 \in \langle w_1 \rangle$ and $u_j, v_2 \in \langle w_2 \rangle$. Note that $o(v_1)|o(u_j)$ and $o(v_2)|o(u_j)$, we get $\langle v_1 \rangle = \langle v_2 \rangle$. It follows that $i_1 = i_2$. By $t_{j-1} \geq 2$, we obtain that the elements of $\mathcal{C}_{t_j}^{(j)}$ is not adjacent to at least one of the cyclic class in $\mathcal{T}_{j-1}$ in $\mathcal{P}_E(P)$. By relabelling, if necessary, we may assume that each element of $\mathcal{C}_{t_j}^{(j)}$ is not adjacent to every element of $\mathcal{C}_1^{(j-1)}$ in $\mathcal{P}_E(P)$. Since $t_1 \geq 3$, we can label a class in $\mathcal{T}_1 \setminus \mathcal{C}_1^{(1)}$ by $\mathcal{C}_{t_1}^{(1)}$ in which for $x \in \mathcal{C}_{t_1}^{(1)}$ and $y \in \mathcal{C}_1^{(k)}$, we have $\langle x \rangle \nsubseteq \langle y \rangle$. It implies that $x \nsim y$ in $\mathcal{P}_E(P)$. Let $z_1, z_2, \ldots, z_n$ be the elements of $\mathbb{Z}_n$. Then for $y_{p,q}^{(r)}$, the q-th element of $\mathcal{C}_p^{(r)}$, the path H'' given below

$(e_1, y_{1,1}^{(k)}, z_1) \sim (e_1, y_{2,1}^{(k)}, z_1) \sim \cdots \sim (e_1, y_{t_k,1}^{(k)}, z_1) \sim (e_1, y_{1,2}^{(k)}, z_1) \sim (e_1, y_{2,2}^{(k)}, z_1) \sim \cdots \sim (e_1, y_{t_k,2^{k-1}}^{(k)}, z_1) \sim (e_1, y_{1,1}^{(k-1)}, z_1) \sim (e_1, y_{2,1}^{(k-1)}, z_1) \sim \cdots \sim (e_1, y_{t_1,1}^{(1)}, z_1) \sim (e_1, y_{1,1}^{(k)}, z_2) \sim (e_1, y_{2,1}^{(k)}, z_2) \sim \cdots \sim (e_1, y_{t_1,1}^{(1)}, z_n)$,

where $1 \leq r \leq k$, $1 \leq p \leq t_r$ and $1 \leq q \leq 2^{r-1}$, is a Hamiltonian path in the subgraph of $\overline{\mathcal{P}_E(G)}$ induced by the set $\mathcal{S}''$. Since $y' \notin \mathcal{C}_1^{(k)}$, we get $y' \nsim y_{1,1}^{(k)}$ in $\mathcal{P}_E(P)$. Consequently, $(x', y', z') \nsim (e_1, y_{1,1}^{(k)}, z_1)$ in $\mathcal{P}_E(G)$ and so $(x', y', z') \sim (e_1, y_{1,1}^{(k)}, z_1)$ in $\overline{\mathcal{P}_E(G)}$. Thus, we get a Hamiltonian path in the subgraph of $\overline{\mathcal{P}_E(G)}$ induced by the set $\mathcal{S}' \cup \mathcal{S}''$.

Case-2: *P is of maximal class.* In view of Corollary 1.1.7, we discuss this case into three subcases.

Subcase-2.1: $P = Q_{2^{k+1}} = \left\langle x, y : x^{2^k} = e_2, x^{2^{k-1}} = y^2, y^{-1}xy = x^{-1} \right\rangle$, where $k \geq 2$. Consider the set $\mathcal{S}'' = \{(e_1, b, c) \in G : b \neq e_2, x^{2^{k-1}}\}$. Note that the sets $\mathcal{S}'$, $\mathcal{S}''$ and $\mathrm{Dom}(\mathcal{P}_E(G))$ forms a partition of the group G. Observe that $Q_{2^{k+1}}$ has one maximal cyclic subgroup of order 2^k and 2^{k-1} maximal cyclic subgroups of order 4 (see Dalal and Kumar [2021]). Let $M' = \langle x \rangle$ be the maximal cyclic subgroup of order 2^k and let for $1 \leq i \leq 2^{k-1}$, $M_i = \{e_2, x^{2^{k-1}}, x^i y, x^{2^{k-1}+i} y\}$ be the maximal cyclic subgroups of order 4. For $1 \leq j \leq 2^{k-1}$, note that $x^j y$ is a generator of a maximal cyclic subgroup of $Q_{2^{k+1}}$. Consequently, $x^j y \nsim a$ in $\mathcal{P}_E(Q_{2^{k+1}})$, where $a \in Q_{2^{k+1}} \setminus \langle x^j y \rangle$. Since $o(y') = 2^k$, for $k \geq 3$, we have $y' \in M'$. Thus, the Hamiltonian path H'' in the subgraph of $\overline{\mathcal{P}_E(G)}$ induced by the set $\mathcal{S}''$ can be given as $(e_1, xy, z_1) \sim (e_1, x, z_1) \sim (e_1, x^2y, z_1) \sim (e_1, x^2, z_1) \sim \cdots \sim (e_1, x^{2^{k-1}-1}y, z_1) \sim (e_1, x^{2^{k-1}-1}, z_1) \sim (e_1, x^{2^{k-1}}y, z_1) \sim (e_1, x^{2^{k-1}+1}, z_1) \sim \cdots \sim (e_1, x^{2^k-1}, z_1) \sim (e_1, x^{2^k-1}y, z_1) \sim (e_1, y, z_1) \sim (e_1, xy, z_2) \sim (e_1, x, z_2) \sim \cdots \sim (e_1, y, z_n)$, where $z_1, z_2, \ldots, z_n \in \mathbb{Z}_n$. We have a Hamiltonian path H' in the subgraph induced by the set $\mathcal{S}'$ with end vertex (x', y', z') and also, H'' is a Hamiltonian path induced by $\mathcal{S}''$ with initial vertex (e_1, xy, z_1). Moreover, $(x', y', z') \sim (e_1, xy, z_1)$. Thus, we get a Hamiltonian path H in the subgraph induced by $\mathcal{S}' \cup \mathcal{S}''$. If $k = 2$ and $y' \in M_1$ then again we have a Hamiltonian path by interchanging the vertices (e_1, xy, z_1) and (e_1, x^2y, z_1) of H.

Subcase-2.2: $P = D_{2^{k+1}} = \left\langle x, y : x^{2^k} = e_2 = y^2, y^{-1}xy = x^{-1} \right\rangle$, where $k \geq 1$. Consider the set $\mathcal{S}'' = \{(e_1, b, c) \in G : b \neq e_2\}$. Observe that the sets $\mathcal{S}'$, $\mathcal{S}''$ and $\mathrm{Dom}(\mathcal{P}_E(G))$ forms a partition of the group G. Also, notice that $M' = \langle x \rangle$ is the only maximal cyclic subgroup of order 2^k in $D_{2^{k+1}}$ and for $1 \leq i \leq 2^k$, $M_i = \{e_2, x^i y\}$ are the maximal cyclic subgroups of order 2 in $D_{2^{k+1}}$. By Figure 1 of P. Panda et al. [2021], $x^j y$, where $1 \leq j \leq 2^k$, is not adjacent to any non-identity element of $D_{2^{k+1}}$ in $\mathcal{P}_E(D_{2^{k+1}})$. Since $o(y') = 2^k$, for $k \geq 2$, we have $y' \in M'$. Thus, the Hamiltonian path in the subgraph of $\overline{\mathcal{P}_E(G)}$ induced by the set $\mathcal{S}''$ is $H'' : (e_1, xy, z_1) \sim (e_1, x, z_1) \sim (e_1, x^2y, z_1) \sim (e_1, x^2, z_1) \sim \cdots \sim (e_1, y, z_1) \sim (e_1, xy, z_2) \sim \cdots \sim (e_1, y, z_n)$, where $z_1, z_2, \ldots, z_n \in \mathbb{Z}_n$. We have a Hamiltonian path H' in the subgraph induced by the set $\mathcal{S}'$ with end vertex (x', y', z') and also, H'' is a Hamiltonian path induced by $\mathcal{S}''$ with initial vertex (e_1, xy, z_1). Furthermore, $(x', y', z') \sim (e_1, xy, z_1)$. Consequently, we get a Hamiltonian path H in the subgraph induced by $\mathcal{S}' \cup \mathcal{S}''$. If $k = 1$ and $y' \in M_1$, then again we have a Hamiltonian path by interchanging the vertices (e_1, xy, z_1) in (e_1, x^2y, z_1) of H.

Subcase-2.3: $P = SD_{2^{k+1}} = \left\langle x, y : x^{2^k} = e_2, y^2 = e_2, y^{-1}xy = x^{-1+2^{k-1}} \right\rangle$, where

$k \geq 3$. Consider the set $\mathcal{S}'' = \{(e_1, b, c) \in G : b \neq e_2\}$. Notice that the sets $\mathcal{S}'$, $\mathcal{S}''$ and $\mathrm{Dom}(\mathcal{P}_E(G))$ forms a partition of the group G. Also, note that $SD_{2^{k+1}}$ has one maximal cyclic subgroup of order 2^k, 2^{k-1} maximal cyclic subgroups of order 2 and 2^{k-2} maximal cyclic subgroups of order 4. Let $M' = \langle x \rangle$ be the maximal cyclic subgroup of order 2^k and for $1 \leq i \leq 2^{k-2}$, $M_i = \{e_2, x^{2^{k-1}}, x^{2i+1}y, x^{2^{k-1}+2i+1}y\}$ be the maximal cyclic subgroups of $SD_{2^{k+1}}$ of order 4 and for $1 \leq j \leq 2^{k-1}$, $M_j'' = \{e_2, x^{2j}y\}$ be the maximal cyclic subgroups of $SD_{2^{k+1}}$ of order 2. Now, for $1 \leq t \leq 2^k$, $x^t y$ is a generator of a maximal cyclic subgroup of $SD_{2^{k+1}}$. It follows that $x^j y \nsim b$ in $\mathcal{P}_E(SD_{2^{k+1}})$, where $b \in SD_{2^{k+1}} \setminus \langle x^j y \rangle$ (see Figure 2 of P. Panda et al. [2021]). Let $z_1, z_2, \ldots, z_n$ be the elements of $\mathbb{Z}_n$. Thus, the Hamiltonian path H'' in the subgraph of $\overline{\mathcal{P}_E(G)}$ induced by the set $\mathcal{S}''$ is given by
$(e_1, xy, z_1) \sim (e_1, x, z_1) \sim (e_1, x^3y, z_1) \sim (e_1, x^3, z_1) \sim \cdots \sim (e_1, x^{2^k-1}y, z_1) \sim (e_1, x^{2^k-1}, z_1) \sim (e_1, y, z_1) \sim (e_1, x^2y, z_1) \sim (e_1, x^2, z_1) \sim (e_1, x^4y, z_1) \sim \cdots \sim (e_1, x^{2^k-2}, z_1) \sim (e_1, x^{2^k-2}y, z_1) \sim (e_1, xy, z_2) \sim (e_1, x, z_2) \sim (e_1, x^3, z_2) \sim \cdots \sim (e_1, x^{2^k-2}y, z_n)$.

Moreover, we have a Hamiltonian path H' in the subgraph induced by the set $\mathcal{S}'$ with end vertex (x', y', z'). Since $o(y') = 2^k$, we have $y' \in M'$. It follows that $y' \nsim xy$ and so $(x', y', z') \sim (e_1, xy, z_1)$ in $\overline{\mathcal{P}_E(G)}$. Thus, the subgraph of $\overline{\mathcal{P}_E(G)}$ induced by the set $G \setminus \mathrm{Dom}(\mathcal{P}_E(G))$ has a Hamiltonian path. □

Corollary 2.2.20. *For the group $G = Q_{2^{k+1}}$, we have $\lambda(\mathcal{P}_E(G)) = 2^{k+1} + 1$.*

Corollary 2.2.21. *For the group $G \in \{D_{2^{k+1}}, SD_{2^{k+1}}\}$, we have $\lambda(\mathcal{P}_E(G)) = 2^{k+1}$.*

2.3 Enhanced Power Graphs of Certain Non-abelian Groups

In this section, we obtain the Laplacian spectrum of the enhanced power graph of certain non-abelian groups, namely: semidihedral groups, dihedral groups and generalized quaternion groups. Also, we obtained the metric dimension and the resolving polynomial of the enhanced power graphs of these groups. At the final part of this section, we study the distant properties and the detour distant properties, namely: closure, interior, distance degree sequence, eccentric subgraph of the enhanced power graph of these groups. We begin with the investigation of the enhanced power graph of semidihedral group. Recall that the *semidihedral group* SD_{8n} $(n \geq 2)$ is a group of order $8n$ is defined as

$$SD_{8n} = \langle a, b : a^{4n} = b^2 = e, ba = a^{2n-1}b \rangle.$$

We have

$$ba^i = \begin{cases} a^{4n-i}b & \text{if } i \text{ is even,} \\ a^{2n-i}b & \text{if } i \text{ is odd,} \end{cases}$$

Note that every element of $SD_{8n} \backslash \langle a \rangle$ is of the form $a^i b$ for some $0 \leq i \leq 4n-1$. We denote the subgroups $P_i = \langle a^{2i}b \rangle = \{e, a^{2i}b\}$ and $Q_j = \langle a^{2j+1}b \rangle = \{e, a^{2n}, a^{2j+1}b, a^{2n+2j+1}b\}$. Then

$$SD_{8n} = \langle a \rangle \cup \left(\bigcup_{i=0}^{2n-1} P_i \right) \cup \left(\bigcup_{j=0}^{n-1} Q_j \right). \tag{2.1}$$

In the following lemma, we obtain the neighbourhood of all the vertices of $\mathcal{P}_E(SD_{8n})$.

Lemma 2.3.1. *In $\mathcal{P}_E(SD_{8n})$, we have*

(i) $\mathrm{N}[e] = SD_{8n}$.

(ii) $\mathrm{N}[a^{2n}] = \langle a \rangle \cup \{a^{2i+1}b : 0 \leq i \leq 2n-1\}$.

(iii) $\mathrm{N}[a^i] = \langle a \rangle$, *where* $1 \leq i \leq 4n-1$ *and* $i \neq 2n$.

(iv) $\mathrm{N}[a^{2i+1}b] = \langle a^{2i+1}b \rangle = \{e, a^{2n}, a^{2i+1}b, a^{2n+2i+1}b\}$, *where* $0 \leq i \leq 2n-1$.

(v) $\mathrm{N}[a^{2i}b] = \{e, a^{2i}b\}$, *where* $1 \leq i \leq 2n$.

Proof. The proof of (i) is straightforward.

(ii) Note that $a^{2n} \sim x$ for all $x \in \langle a \rangle$ and $a^{2n} \sim a^{2i+1}b$ for all i, where $0 \leq i \leq 2n-1$

as $a^{2n} \in \langle a^{2i+1}b \rangle$. This implies that $\langle a \rangle \cup \{a^{2i+1}b : 0 \leq i \leq 2n-1\} \subseteq \mathrm{N}[a^{2n}]$. If possible, let $x \in \mathrm{N}[a^{2n}]$ such that $x = a^{2i}b$ for some i, where $0 \leq i \leq 2n-1$. In view of Equation (2.1), observe that x belongs to exactly one cyclic subgroup P_i and $x \sim a^{2n}$ gives $a^{2n} \in P_i$; a contradiction.

(iii) Let $i \neq 2n$ and $1 \leq i \leq 4n-1$. Then clearly $\langle a \rangle \subseteq \mathrm{N}[a^i]$. If $a^i \sim x$ for some $x \in SD_{8n} \setminus \langle a \rangle$, then either $x, a^i \in P_j$ or $x, a^i \in Q_k$ for some j, k, where $0 \leq j \leq 2n-1$ and $0 \leq k \leq n-1$ which is not possible. Further, note that if $\langle x \rangle$ is a maximal cyclic subgroup, then $\mathrm{N}[x] = \langle x \rangle$.

(iv)-(v) Since P_i and Q_j are maximal cyclic subgroups generated by $a^{2i}b$ and $a^{2j+1}b$, respectively, where $0 \leq i \leq 2n-1$ and $0 \leq j \leq n-1$, it follows that $\mathrm{N}[a^{2i}b] = \{e, a^{2i}b\}$ and $\mathrm{N}[a^{2j+1}b] = \langle a^{2j+1}b \rangle = \{e, a^{2n}, a^{2j+1}b, a^{2n+2j+1}b\}$. Thus, the result holds. □

2.3.1 Laplacian Spectrum

Hamzeh and Ashrafi [2017b] obtained the Laplacian spectrum of the power graph of SD_{8n}, when $n = 2^{\alpha}$. In this subsection, for arbitrary $n \in \mathbb{N}$, we investigate the Laplacian spectrum of the enhanced power graphs of the semidihedral group SD_{8n}, generalized quaternion group Q_{4n} and dihedral group D_{2n}, respectively. First, we recall the notation of the Laplacian matrix of a graph. Let Γ be a simple graph such that $V(\Gamma) = \{v_1, v_2, \ldots, v_n\}$. The *adjacency matrix* $A(\Gamma)$ of Γ is an $n \times n$ matrix whose $(i,j)th$ entry is 1 if there is an edge between the vertices v_i and v_j, and 0 otherwise. The *degree matrix* $D(\Gamma)$ of Γ is a diagonal matrix whose $(i,i)th$ entry is the degree of vertex v_i in Γ. The *Laplacian matrix* $L(\Gamma)$ of Γ is the matrix $D(\Gamma) - A(\Gamma)$. We denote $\Phi(L(\Gamma), x)$ by the *characteristic polynomial* of $L(\Gamma)$. Let $\lambda_{n_1}(\Gamma), \lambda_{n_2}(\Gamma), \ldots, \lambda_{n_r}(\Gamma)$ be the distinct eigenvalues of $L(\Gamma)$ such that $0 = \lambda_{n_1}(\Gamma) \leq \lambda_{n_2}(\Gamma) \leq \cdots \leq \lambda_{n_r}(\Gamma)$ and let $m_1, m_2, \ldots, m_r$ be the corresponding multiplicities. Then the *Laplacian spectrum* of Γ is given by $\begin{pmatrix} \lambda_{n_1}(\Gamma) & \lambda_{n_2}(\Gamma) & \cdots & \lambda_{n_r}(\Gamma) \\ m_1 & m_2 & \cdots & m_r \end{pmatrix}$. Let

I_n be the identity matrix of order n and let J_n be the square matrix of order n such that its each entry is 1. Further, for a matrix A, its transpose is denoted by A'.

We obtain the Laplacian eigenvalues of the enhanced power graph of the semidihedral group SD_{8n} in the following theorem.

Theorem 2.3.2. *The characteristic polynomial of the Laplacian matrix of* $\mathcal{P}_E(SD_{8n})$ *is*

$$\Phi(L(\mathcal{P}_E(SD_{8n})), x) = x(x-8n)(x-6n)(x-4n)^{4n-3}(x-2)^n(x-4)^n(x-1)^{2n}.$$

Proof. The Laplacian matrix $L(\mathcal{P}_E(SD_{8n}))$ is the $8n \times 8n$ matrix given below, where the rows and columns are indexed in order by the vertices $e = a^{4n}, a^{2n}, a, a^2, \ldots, a^{2n-1}$, $a^{2n+1}, a^{2n+2}, \ldots, a^{4n-1}$ and then $ab, a^{2n+1}b, a^3b, a^{2n+3}b, \ldots, a^{2n-1}b, a^{4n-1}b, a^2b, a^4b, \ldots$, $a^4nb = b$.

$$L(\mathcal{P}_E(SD_{8n})) = \begin{pmatrix} 8n-1 & -1 & -1 & -1 & \cdots\cdots & -1 & -1 & \cdots\cdots & -1 \\ -1 & & & & & & & & \\ -1 & & & & & & & & \\ -1 & & & A & & & & \mathcal{O} & \\ \vdots & & & & & & & & \\ \vdots & & & & & & & & \\ -1 & & & & & & & & \\ -1 & & & & & & & & \\ \vdots & & & \mathcal{O}' & & & & B & \\ \vdots & & & & & & & & \\ -1 & & & & & & & & \end{pmatrix}_{8n \times 8n},$$

where

$$A = \begin{pmatrix} 6n-1 & -1 & -1 & -1 & \cdots\cdots & -1 & -1 & \cdots\cdots & -1 \\ -1 & & & & & & & & \\ -1 & & & & & & & & \\ -1 & & & A_1 & & & & \mathcal{O}_1 & \\ \vdots & & & & & & & & \\ \vdots & & & & & & & & \\ -1 & & & & & & & & \\ -1 & & & & & & & & \\ \vdots & & & \mathcal{O}_1' & & & & A_2 & \\ \vdots & & & & & & & & \\ -1 & & & & & & & & \end{pmatrix}_{6n-1\times 6n-1}$$

such that $A_1 = 4nI_{4n-2} - J_{4n-2}$ and

$$A_2 = \begin{pmatrix} C & O_2 & \cdots\cdots & O_2 \\ O_2 & C & \cdots\cdots & O_2 \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ O_2 & O_2 & \cdots\cdots & C \end{pmatrix}_{2n\times 2n}$$

with

$$C = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix},$$

O_2 is the zero matrix of order 2×2, $B = I_{2n}$, $\mathcal{O}$ and $\mathcal{O}_1$ are the zero matrices of size $(6n-1) \times (2n)$ and $(4n-2) \times (2n)$, respectively. Then

$$\Phi(L(\mathcal{P}_E(SD_{8n})),x)=\begin{vmatrix} x-(8n-1) & 1 & 1 & 1 & \cdots\cdots & 1 & 1 & \cdots\cdots & 1\\ 1 & & & & & & & & \\ 1 & & & & & & & & \\ 1 & & & (xI_{6n-1}-A) & & & & \mathcal{O} & \\ \vdots & & & & & & & & \\ \vdots & & & & & & & & \\ 1 & & & & & & & & \\ 1 & & & & & & & & \\ \vdots & & & \mathcal{O}' & & & & (xI_{2n}-B) & \\ \vdots & & & & & & & & \\ 1 & & & & & & & & \end{vmatrix}.$$

Apply row operation $R_1 \to (x-1)R_1 - R_2 - \cdots - R_{8n}$ and then along the first row, we get

$$\Phi(L(\mathcal{P}_E(SD_{8n})),x)=\tfrac{x(x-8n)}{(x-1)}\begin{vmatrix}(xI_{6n-1}-A) & \mathcal{O}\\ \mathcal{O}' & (xI_{2n}-B)\end{vmatrix}$$

Using the fact that if M, Q are square matrices and the matrices O and O' are zero matrices of appropriate sizes, then $\begin{vmatrix} M & O\\ O' & Q\end{vmatrix} = |M|\cdot|Q|$ (see [Zhang, 1996, p.32]), we get

$$\Phi(L(\mathcal{P}_E(SD_{8n})),x)=\frac{x(x-8n)}{(x-1)}|xI_{6n-1}-A|\cdot|xI_{2n}-B|.$$

Observe that $|xI_{2n} - B| = (x-1)^{2n}$. Now, we obtain

$$|xI_{6n-1} - A| = \begin{vmatrix} x-(6n-1) & 1 \quad 1 \quad 1 \quad \cdots\cdots \quad 1 & 1 \quad \cdots\cdots \quad 1 \\ 1 & & \\ 1 & & \\ 1 & (xI_{4n-2} - A_1) & \mathcal{O}_1 \\ \vdots & & \\ \vdots & & \\ 1 & & \\ 1 & & \\ \vdots & \mathcal{O}_1' & (xI_{2n} - A_2) \\ \vdots & & \\ 1 & & \end{vmatrix}$$

Apply row operation $R_1 \to (x-2)R_1 - R_2 - R_3 - \cdots - R_{6n-1}$ and then along the first row, we obtain

$$|xI_{6n-1} - A| = \frac{(x-1)(x-6n)}{(x-2)}|xI_{4n-2} - A_1| \cdot |xI_{2n} - A_2|.$$

Clearly, $|xI_{2n} - A_2| = (x-2)^n(x-4)^n$. Now,

$$|xI_{4n-2} - A_1| = |xI_{4n-2} - (4nI_{4n-2} - J_{4n-2})| = |(x-4n)I_{4n-2} + J_{4n-2})|.$$

It is well known that if $\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ are eigenvalues of the matrix M then the eigenvalues of the matrix $\alpha I_m + M$ are $\{\alpha + \lambda_1, \alpha + \lambda_2, \ldots, \alpha + \lambda_m\}$. Since J_{4n-2} is the square matrix with all entries equal to 1, the eigenvalues of J_{4n-2} are $4n-2$ with multiplicity 1 and 0 with multiplicity $4n-3$. It follows that $x-2$ and $x-4n$ are the eigenvalues of the matrix $(x-4n)I_{4n-2} + J_{4n-2}$ with multiplicities 1 and $4n-3$, respectively. Consequently, $|xI_{4n-2} - A_1| = (x-4n)^{4n-3}(x-2)$ and this completes our proof. □

Corollary 2.3.3. *The Laplacian spectrum of $\mathcal{P}_E(SD_{8n})$ is given by*

$$\begin{pmatrix} 0 & 1 & 2 & 4 & 4n & 6n & 8n \\ 1 & 2n & n & n & 4n-3 & 1 & 1 \end{pmatrix}.$$

Consequently, due to Corollary 4.2 of Mohar [1991], we count the spanning trees of $\mathcal{P}_E(SD_{8n})$ in the following corollary.

Corollary 2.3.4. *The number of spanning trees of* $\mathcal{P}_E(SD_{8n})$ *is* $2^{11n-5}3n^{4n-2}$.

Hamzeh and Ashrafi [2017b] obtained the Laplacian spectrum of the power graph of Q_{4n} when $n = 2^{\alpha}$. Now, we determine the Laplacian spectrum of the enhanced power graph of generalized quaternion group Q_{4n} and dihedral group D_{2n} for arbitrary $n \in \mathbb{N}$. Recall that for $n \geq 2$, the *generalized quaternion group* Q_{4n} of order $4n$ is defined by

$$Q_{4n} = \langle a, b : a^{2n} = e, a^n = b^2, ab = ba^{-1} \rangle.$$

Theorem 2.3.5. *The characteristic polynomial of the Laplacian matrix of* $\mathcal{P}_E(Q_{4n})$ *is*

$$\Phi(L(\mathcal{P}_E(Q_{4n})), x) = x(x-4n)^2(x-4)^n(x-2)^n(x-2n)^{2n-3}.$$

Proof. The Laplacian matrix $L(\mathcal{P}_E(Q_{4n}))$ is the $4n \times 4n$ matrix given below, where the rows and columns are indexed by the vertices $e = a^{2n}, a^n, a, a^2, \ldots, a^{n-1}, a^{n+1}, a^{n+2}, \ldots, a^{2n-1}$, and then $ab, a^{n+1}b, a^2b, a^{n+2}b \ldots, a^nb, a^{2n}b = b$.

$$L(\mathcal{P}_E(Q_{4n})) = \begin{pmatrix} 4n-1 & -1 & \cdots\cdots & -1 & -1 & \cdots\cdots & -1 \\ -1 & 4n-1 & \cdots\cdots & -1 & -1 & \cdots\cdots & -1 \\ -1 & -1 & & & & & \\ -1 & -1 & A & & & \mathcal{O} & \\ \vdots & \vdots & & & & & \\ -1 & -1 & & & & & \\ \vdots & \vdots & \mathcal{O}' & & & B & \\ -1 & -1 & & & & & \end{pmatrix}_{4n \times 4n},$$

where $A = 2nI_{2n-2} - J_{2n-2}$ and

$$B = \begin{pmatrix} C & O_2 & \cdots\cdots & O_2 \\ O_2 & C & \cdots\cdots & O_2 \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ O_2 & O_2 & \cdots\cdots & C \end{pmatrix}_{2n\times 2n}$$

with

$$C = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix},$$

O_2 is the zero matrix of order 2×2 and $\mathcal{O}$ is the zero matrix of order $(2n-2) \times (2n)$. Then

$$\Phi(L(\mathcal{P}_E(Q_{4n})), x) = \begin{vmatrix} x-(4n-1) & 1 & \cdots & 1 \;\; 1 & \cdots 1 \\ 1 & x-(4n-1) & \cdots & 1 \;\; 1 & \cdots 1 \\ 1 & 1 & & & \\ 1 & 1 & xI_{2n-2} - A & & \mathcal{O} \\ \vdots & \vdots & & & \\ 1 & 1 & & & \\ 1 & 1 & & & \\ \vdots & \vdots & \mathcal{O}' & & xI_{2n} - B \\ 1 & 1 & & & \end{vmatrix}$$

Apply the following row operations consecutively

- $R_1 \to (x-1)R_1 - R_2 - \cdots - R_{4n}$
- $R_2 \to (x-2)R_2 - R_3 - \cdots - R_{4n}$

and then expand, we get

$$\Phi(L(\mathcal{P}_E(Q_{4n})), x) = \frac{x(x-4n)^2}{(x-2)} \begin{vmatrix} xI_{2n-2} - A & \mathcal{O} \\ \mathcal{O}' & xI_{2n} - B \end{vmatrix} = \frac{x(x-4n)^2}{(x-2)} |xI_{2n-2} - A| \cdot |xI_{2n} - B|.$$

In the similar lines of the proof of Theorem 2.3.2, we get

$$|xI_{2n-2} - A| = (x-2n)^{2n-3}(x-2) \text{ and } |xI_{2n} - B| = (x-2)^n(x-4)^n$$

This completes our proof. □

Corollary 2.3.6. *The Laplacian spectrum of $\mathcal{P}_E(Q_{4n})$ is given by*

$$\begin{pmatrix} 0 & 2 & 4 & 2n & 4n \\ 1 & n & n & 2n-3 & 2 \end{pmatrix}.$$

Consequently, due to Corollary 4.2 of Mohar [1991], we count the spanning trees of $\mathcal{P}_E(Q_{4n})$ in the following corollary.

Corollary 2.3.7. *The number of spanning trees of $\mathcal{P}_E(Q_{4n})$ is $2^{5n-1}n^{2n-2}$.*

Recall that for $n \geq 3$, the *dihedral group* D_{2n} of order $2n$ is defined by

$$D_{2n} = \langle a, b : a^n = b^2 = e, ab = ba^{-1} \rangle.$$

In the similar lines of the Laplacian spectrum of $\mathcal{P}_E(SD_{8n})$ obtained in Theorem 2.3.5, one can obtain the following theorem.

Theorem 2.3.8. *The Laplacian spectrum of $\mathcal{P}_E(D_{2n})$ is given by*

$$\begin{pmatrix} 0 & 1 & n & 2n \\ 1 & n & n-2 & 1 \end{pmatrix}$$

Consequently, due to Corollary 4.2 of Mohar [1991], we count the spanning trees of $\mathcal{P}_E(D_{2n})$ in the following corollary.

Corollary 2.3.9. *The number of spanning trees of $\mathcal{P}_E(D_{2n})$ is n^{n-2}.*

Remark 2.3.10. Let $k, \alpha \in \mathbb{N}$ and p be an odd prime. Ali et al. [2022] obtained the distant properties and detour distant properties of the power graph $\mathcal{P}(G)$, when G is Q_{4n}, where $n = 2^k$ or D_{2m} for $m = p^\alpha$. Note that, for arbitrary $n, m \in \mathbb{N}$, we have

$$\mathcal{P}_E(Q_{4n}) = K_2 \vee (K_{2n-2} \cup nK_2), \text{ and}$$

$$\mathcal{P}_E(D_{2m}) = K_1 \vee (K_{m-1} \cup \overline{K}_m).$$

Thus, in view of Proposition 3.1 and Proposition 3.3 of Ali et al. [2022], the results of metric dimension, resolving polynomial, distant properties and detour distant properties of $\mathcal{P}(Q_{4n})$ and $\mathcal{P}(D_{2m})$ will also hold for $\mathcal{P}_E(Q_{4n})$ and $\mathcal{P}_E(D_{2m})$, respectively, for arbitrary $n, m \in \mathbb{N}$.

2.3.2 Metric Dimension and Resolving Polynomial

The metric dimension of a graph has several applications, see Cameron and Van Lint [1991]; Chartranda et al. [2000] and Khuller et al. [1994]. Now, we obtain the metric dimension and resolving polynomial of the enhanced power graph of the semidihedral group. First, we recall some basic definitions and necessary results from Kumar et al. [2021b]. Let Γ be a graph. Then a vertex z of Γ *resolves* the vertices u and v if $d(z, u) \neq d(z, v)$. A subset $U \subseteq V(\Gamma)$ is said to be a *resolving set* in Γ if every pair of vertices in Γ is resolved by some vertex of U. The *metric dimension* $\dim(\Gamma)$ of a graph Γ is the size of a smallest resolving set. A subset $S \subseteq V(\Gamma)$ is said to be an *i-subset* if $|S| = i$. Let $r_i = |\mathcal{R}(\Gamma, i)|$ such that $\mathcal{R}(\Gamma, i)$ is the family of resolving sets each of cardinality i. Recall that the *resolving polynomial* $\beta(\Gamma, x)$ of a graph Γ is defined by $\beta(\Gamma, x) = \sum_{i=\dim(\Gamma)}^{n} r_i x^i$. The *resolving sequence* $(r_{\dim(\Gamma)}, r_{\dim(\Gamma)+1}, \ldots, r_n)$ is the sequence of coefficients of $\beta(\Gamma, x)$. If $\mathrm{N}[u] = \mathrm{N}[v]$ for distinct $u, v \in V(\Gamma)$, then u are v are called *true twins*. However, $\mathrm{N}(u) = \mathrm{N}(v)$ then u are v are called *false twins*. If u and v are either true twins or false twins then they are called *twins*. If any two vertices of $U \subseteq V(\Gamma)$ are twins, then U is called a *twin-set*. The following result is useful to obtain the resolving polynomial of a graph.

Remark 2.3.11 ([Ali et al., 2016, Remark 3.3]). Let U be a twin-set in a connected graph Γ such that $|U| = l \geq 2$. Then each resolving set for Γ contains at least $l - 1$ vertices of U.

Proposition 2.3.12 ([Ali et al., 2016, Proposition 3.5]). *Let Γ be a connected graph of order n. Then the only resolving set of cardinality n is the set $V(\Gamma)$ and a resolving set of cardinality $n-1$ can be chosen in n possible different ways.*

Proposition 2.3.13. *The metric dimension of $\mathcal{P}_E(SD_{8n})$ is $7n - 4$.*

Proof. In view of Lemma 2.3.1, we get twin-sets $\langle a\rangle \setminus \{e, a^{2n}\}$, $\{a^{2i}b : 1 \leq i \leq 2n\}$ and $\{a^{2j+1}b, a^{2n+2j+1}b\}$ where $0 \leq j \leq n-1$. By Remark 2.3.11, any resolving set of $\mathcal{P}_E(SD_{8n})$ contains at least $7n - 4$ vertices. Now, we produce a resolving set of cardinality $7n - 4$. By Lemma 2.3.1, observe that the set

$$R = \{a^{2i}b : 1 \leq i \leq 2n-1\} \cup \{a^i : i \neq 1, 2n, 4n\} \cup \{a^{2i+1}b : 0 \leq i \leq n-1\}$$

is a resolving set of cardinality $7n-4$. Consequently, $\dim(\mathcal{P}_E(SD_{8n})) = 7n - 4$. □

Theorem 2.3.14. *The resolving polynomial of $\mathcal{P}_E(SD_{8n})$ is given below:*

$$\beta(\mathcal{P}_E(SD_{8n}), x) = x^{8n} + 8nx^{8n-1} + \sum_{i=7n-4}^{8n-2} r_i x^i,$$

where

$$r_i = \begin{cases} \sum_{j=0}^{i-(7n-4)} \binom{n}{j} 2^{n-j} k_{7n+j-i} & 7n-4 \leq i \leq 7n-1 \\ \sum_{j=0}^{4} \binom{n}{i-7n+j} 2^{n-(i-7n+j)} k_j & 7n \leq i \leq 8n-4 \\ \sum_{j=0}^{8n-i} \binom{n}{n+j-(8n-i)} 2^{n-(n+j-(8n-i))} k_j & 8n-3 \leq i \leq 8n-2 \end{cases}$$

and $k_0 = 1, k_1 = 6n, k_2 = 8n^2 + 8n - 3, k_3 = 16n^2 - 2n - 2, k_4 = 8n^2 - 4n.$

Proof. In view of Proposition 2.3.13, we have $\dim(\mathcal{P}_E(SD_{8n})) = 7n - 4$. It is sufficient to find the resolving sequence $(r_{7n-4}, r_{7n-3}, \ldots, r_{8n-2}, r_{8n-1}, r_{8n})$. By the proof of Proposition 2.3.13, any resolving set R satisfies the following:

- $|R \cap (\langle a \rangle \setminus \{e, a^{2n}\})| \geq 4n - 3$;
- $|R \cap H_l| \geq 1$, where $H_l = \{a^{2l+1}b, a^{2n+2l+1}\}$ for $0 \leq l \leq n - 1$;
- $|R \cap \{a^{2i}b : 1 \leq i \leq 2n\}| \geq 2n - 1$.

Let $T = SD_{8n} \setminus \bigcup_{l=0}^{n-1} H_l$. In view of Remark 2.3.11, to get $|R| = 7n - 4$, we must choose exactly one element from each H_l (where $|H_l| = 2$) and $4n-3$ elements from $(\langle a \rangle \setminus \{e, a^{2n}\})$ and $2n - 1$ elements from $\{a^{2i}b : 1 \leq i \leq 2n\}$. Therefore, we have

$$r_{7n-4} = \binom{2}{1}^n \binom{4n-2}{4n-3} \binom{2n}{2n-1} = 2^{n+1}n(4n-2)$$

For $7n - 3 \leq |R| = i \leq 7n - 1$, with above restriction we have one of the following cases:

- Choose $6n - (7n - i)$ elements from T and n elements from $\bigcup_{l=0}^{n-1} H_l$.
- Choose $6n - (7n - i) - 1$ elements from T and $n + 1$ elements from $\bigcup_{l=0}^{n-1} H_l$.

$$\vdots$$

$$\vdots$$

- Choose $6n - 4$ elements from T and $n + (i - 7n + 4)$ elements from $\bigcup_{l=0}^{n-1} H_l$.

Thus,

$$r_i = \sum_{j=0}^{i-(7n-4)} \binom{n}{j} 2^{n-j} k_{7n+j-i},$$

where $k_0 = 1, k_1 = 6n, k_2 = 8n^2 + 8n - 3, k_3 = 16n^2 - 2n - 2, k_4 = 8n^2 - 4n$ and k_t is the number of ways of selecting $6n-t$ elements from T in R for where, $0 \leq t \leq 4$.

For $7n \leq |R| = i \leq 8n-4$, with above restriction we have one of the following cases:

- Choose $6n$ elements from T and $n + (i - 7n)$ elements from $\bigcup_{l=0}^{n-1} H_l$.
- Choose $6n-1$ elements from T and $n + 1 + (i - 7n)$ elements from $\bigcup_{l=0}^{n-1} H_l$.
- Choose $6n-2$ elements from T and $n + 2 + (i - 7n)$ elements from $\bigcup_{l=0}^{n-1} H_l$.
- Choose $6n-3$ elements from T and $n + 3 + (i - 7n)$ elements from $\bigcup_{l=0}^{n-1} H_l$.
- Choose $6n-4$ elements from T and $n + 4 + (i - 7n)$ elements from $\bigcup_{l=0}^{n-1} H_l$.

Therefore, we have

$$r_i = \sum_{j=0}^{4} \binom{n}{i-7n+j} 2^{n-(i-7n+j)} k_j.$$

For $8n-3 \leq |R| = i \leq 8n-2$, with above restriction we have one of the following cases:

- Choose $6n$ elements from T and $n + n - (8n - i)$ elements from $\bigcup_{l=0}^{n-1} H_l$.
- Choose $6n-1$ elements from T and $n + n - (8n - i) + 1$ elements from $\bigcup_{l=0}^{n-1} H_l$.

$$\vdots$$
$$\vdots$$

- Choose $6n - (8n - i)$ elements from T and $2n$ elements from $\bigcup_{l=0}^{n-1} H_l$.

Thus, we get

$$r_i = \sum_{j=0}^{8n-i} \binom{n}{n+j-(8n-i)} 2^{n-(n+j-(8n-i))} k_j.$$

By Proposition 2.3.12, $r_{8n-1} = 8n$ and $r_{8n} = 1$. Hence, the result holds. □

2.3.3 Distant Properties and Detour Distant Properties

The detour distance of a graph has many applications in channel assignment in FM and it is also widely used in chemical theory (see Zhou and Cai [2010]). In this subsection, we obtain the detour radius, detour eccentricity, detour degree, detour degree sequence and detour distance degree sequence of each vertex of $\mathcal{P}_E(SD_{8n})$. The following definitions are useful in the sequel. Let Γ be a connected graph. The *detour distance*, $d_D(u, v)$, between any two vertices u and v of Γ is the length of longest $u - v$ path in Γ. The *eccentricity* $ecc(u)$ (*detour eccentricity* $ecc_D(u)$) of the vertex u in Γ is the maximum distance (detour distance) between u to any other vertex of Γ. The *radius* $rad(\Gamma)$ (*detour radius* $rad_D(\Gamma)$) of a graph Γ is the minimum eccentricity (detour eccentricity) among all the vertices of Γ. The maximum detour eccentricity of Γ among all the vertices of Γ is called the *detour diameter* $diam_D(G)$ of Γ. For a vertex u of Γ, if $d(u, v) = ecc(u)$ then v is called an *eccentric vertex* for u. If v is an eccentric vertex for some vertex of Γ, then v is called an eccentric vertex of the graph Γ. The subgraph induced by the set of all eccentric vertices of Γ is said to be an *eccentric subgraph* of Γ and it is denoted by $Ecc(\Gamma)$. If each vertex of Γ is eccentric, then Γ is said to be *eccentric graph*. Then *centre* $Cen(\Gamma)$ of Γ is the subgraph of Γ induced by the set of all the vertices having the minimum eccentricity. The *closure* $Cl(\Gamma)$ of Γ is the graph obtained by repeatedly adding edges between non-adjacent vertices whose sum of degree is at least $|V(\Gamma)|$ until no such pair remains. If $\Gamma = Cl(\Gamma)$, then Γ is called a *closed graph* (see Chartrand and Zhang [2004]). A vertex v of Γ is called a *boundary vertex* of a vertex u if

$d(u,w) \leq d(u,v)$ for $w \in \mathrm{N}(v)$. If v is a boundary vertex of some vertex of Γ, then v is called a boundary vertex of Γ. Then *boundary* $\partial(\Gamma)$ of the graph Γ is the subgraph of Γ induced by the set of all the boundary vertices of Γ. For a vertex v in Γ, if the subgraph of Γ induced by the set $\mathrm{N}(v)$ is complete, then v is called a *complete vertex*. For each $u \neq v$, if there exists a vertex w and a path $u - w$ such that v lies in the path at the same distance from both u and w, then the vertex v is known as an *interior vertex* of Γ. The subgraph $Int(\Gamma)$ induced by the set of all the interior vertices of Γ is called the *interior* of Γ.

Theorem 2.3.15. *In $\mathcal{P}_E(SD_{8n})$, we have*

$$ecc_D(v) = \begin{cases} 4n+1, & \textit{when } v \in \{e, a^{2n}\}; \\ 4n+2, & \textit{when } v \in \{a^{2i}b : 1 \leq i \leq 2n\}; \\ 4n+3, & \textit{when } v \in \{a^i : 1 \leq i \leq 4n-1, i \neq 2n\} \cup \{a^{2i+1}b : 1 \leq i \leq 2n-1\}. \end{cases}$$

Proof. In view of Lemma 2.3.1, we have the following:

- $d_D(e, a^i) = \begin{cases} 4n+1, & \text{if } i \neq 2n \\ 4n-1, & \text{if } i = 2n. \end{cases}$
- $d_D(e, a^{2i+1}b) = 4n+1$ for all i, where $0 \leq i \leq 2n-1$;
- $d_D(e, a^{2i}b) = 1$ for all i, where $1 \leq i \leq 2n$;
- $d_D(a^{2n}, a^j) = 4n+1$ for all j, where $1 \leq j \leq 4n-1$ and $j \neq 2n$;
- $d_D(a^{2n}, a^{2i+1}b) = 4n+1$ for all i, where $0 \leq i \leq 2n-1$;
- $d_D(a^{2n}, a^{2i}b) = 4n$ for all i, where $1 \leq i \leq 2n$;
- $d_D(a^i, a^j) = 4n+1$ for all $i \neq j$, where $1 \leq i, j \leq 4n-1$ and $i, j \neq 2n$;
- $d_D(a^i, a^{2j+1}b) = 4n+3$, where $1 \leq i \leq 4n-1$, $i \neq 2n$ and $0 \leq j \leq 2n-1$;
- $d_D(a^i, a^{2j}b) = 4n+2$, where $1 \leq i \leq 4n-1$, $i \neq 2n$ and $0 \leq j \leq 2n$;

- $d_D(a^{2i+1}b, a^{2n+2i+1}b) = 4n+1$, where $0 \leq i \leq n-1$;
- $d_D(a^{2i+1}b, a^{2j+1}b) = 4n+3$ for all $i \neq j$, where $0 \leq i \leq 2n-1$ and $j \neq n+i$;
- $d_D(a^{2i+1}b, a^{2j}b) = 4n+2$, where $0 \leq i \leq 2n-1$ and $0 \leq j \leq 2n-1$;
- $d_D(a^{2i}b, a^{2j}b) = 2$ for all $i \neq j$, where $1 \leq i, j \leq 2n$.

This completes our proof. □

Corollary 2.3.16. *For $\mathcal{P}_E(SD_{8n})$, we have*

(i) $\text{rad}_D(\mathcal{P}_E(SD_{8n}) = 4n+1$

(ii) $\text{diam}_D(\mathcal{P}_E(SD_{8n}) = 4n+3$.

The number of elements in the set $D(v) = \{u \in V(\Gamma) : d_D(u,v) = ecc_D(v)\}$ is called the *detour degree* $d_D(v)$ of the vertex v of Γ. The *average detour degree* $D_{av}(\Gamma)$ of the graph Γ is defined as the quotient $\frac{\sum\limits_{v \in V(\Gamma)} d_D(v)}{|V(\Gamma)|}$. The *detour degree sequence* $D(\Gamma)$ of a graph Γ is the detour degrees of vertices of Γ written in non-increasing order. For a vertex $u \in V(\Gamma)$, the number of vertices at detour distance i from the vertex u is denoted by $D_i(u)$. Then the sequence $D_0(u), D_1(u), D_2(u), \ldots, D_{ecc_D(u)}(u)$ is called the *detour distance degree sequence of a vertex u*. We write it by $dds_D(u)$. By (a^r, b^s, c^t) we mean a occurs r times, b occurs s times and c occurs t times in its degree sequence. Further, $(a^r, b^s, c^t)^k = \underbrace{(a^r, b^s, c^t), (a^r, b^s, c^t), \ldots, (a^r, b^s, c^t)}_{k-\text{times}}$. The following remark is useful in the sequel.

Remark 2.3.17 ([Ali et al., 2016, Remark 2.6]). Let v be a vertex of a graph Γ. Then

(i) $D_0(v) = 1$ and $D_{ecc_D}(v) = d_D(v)$.

(ii) The length of sequence $dds_D(v)$ is one more than the detour eccentricity of v.

(iii) $\sum_{i=0}^{ecc_D(v)} D_i(v) = |\Gamma|$.

Proposition 2.3.18. *In $\mathcal{P}_E(SD_{8n})$, we have*

$$d_D(x) = \begin{cases} 6n-4, & \text{when } x = a^{2i+1}b, \text{ where } 0 \le i \le 2n-1; \\ 2n, & \text{when } x = a^i \text{ for all } i \ne 2n, \text{ where } 1 \le i \le 4n-1; \\ 6n-2, & \text{when } x \in \{a^{2i}b : 0 \le i \le 2n-1\} \cup \{e, a^{2n}\}. \end{cases}$$

Proof. Let $x = a^{2i+1}b$ for some i, where $0 \le i \le 2n-1$. In view of Theorem 2.3.15, we get $ecc_D(x) = 4n+3$. By the proof of Theorem 2.3.15, observe that

$$D(x) = \{a^{2j+1}b : 0 \le j \le 2n-1, j \ne i, j \ne n+i\} \cup \left(\langle a \rangle \setminus \{e, a^{2n}\}\right).$$

For $x = a^i$, where $1 \le i \le 4n-1$ and $i \ne 2n$, we obtain $ecc_D(x) = 4n+3$ (see Theorem 2.3.15). Again by the proof of Theorem 2.3.15, we get

$$D(x) = \{a^{2j+1}b : 0 \le j \le 2n-1\}.$$

Now, let $x \in \{a^{2i}b : 0 \le i \le 2n-1\} \cup \{e, a^{2n}\}$. Then $ecc_D(x) = 4n+1$ when $x \in \{e, a^{2n}\}$, and $ecc_D(x) = 4n+2$ when $x = a^{2i}b$ for some i, where $0 \le i \le 2n-1$ (cf. Theorem 2.3.15). By the proof of Theorem 2.3.15, we get

$$D(x) = \{a^{2i+1}b : 0 \le i \le 2n-1\} \cup \left(\langle a \rangle \setminus \{e, a^{2n}\}\right).$$

Thus, the result holds. □

Corollary 2.3.19. *In $\mathcal{P}_E(SD_{8n})$, we have*

(i)
$$D(\mathcal{P}_E(SD_{8n})) = ((6n-4)^{2n}, (2n)^{4n-2}, (6n-2)^{2n+2})$$

(ii)
$$D_{av}(\mathcal{P}_E(SD_{8n})) = \frac{8n^2-n-1}{2n}$$

Theorem 2.3.20. *In $\mathcal{P}_E(SD_{8n})$, we have*

$$dds_D(\mathcal{P}_E(SD_{8n})) = \begin{cases} (1, 2n, 0^{4n-3}, 1, 0, 6n-2), (1, 0^{4n-2}, 1, 2n, 6n-2), \\ (1, 0^{4n}, 4n-1, (2n)^2)^{4n-2}, (1^2, 2n-1, 0^{4n-3}, 1, 0, 6n-2)^{2n}, \\ (1, 0^{4n}, 3, 2n, 6n-4)^{2n}. \end{cases}$$

Proof. In view of Theorem 2.3.15, we get

$$ecc_D(v) = \begin{cases} 4n+1, & \text{when } v \in \{e, a^{2n}\}; \\ 4n+2, & \text{when } v \in \{a^{2i}b : 1 \leq i \leq 2n\}; \\ 4n+3, & \text{when } v \in \{a^i : 1 \leq i \leq 4n-1, i \neq 2n\} \cup \{a^{2i+1}b : 1 \leq i \leq 2n-1\}. \end{cases}$$

By the proof of Theorem 2.3.15, we have the following:

- $dds_D(e) = (1, 2n, \underbrace{0, 0, \ldots, 0}_{4n-3}, 1, 0, 6n-2)$,
- $dds_D(a^{2n}) = (1, \underbrace{0, 0, \ldots, 0}_{4n-2}, 1, 2n, 6n-2)$,
- $dds_D(a^i) = (1, \underbrace{0, 0, \ldots, 0}_{4n}, 4n-1, 2n, 2n)$,
- $dds_D(a^{2i}b) = (1, 1, 2n-1, \underbrace{0, 0, \ldots, 0}_{4n-3}, 1, 0, 6n-2)$,
- $dds_D(a^{2i+1}b) = (1, \underbrace{0, 0, \ldots, 0}_{4n}, 3, 2n, 6n-4)$.

Thus, the result holds. □

Theorem 2.3.21 (Chartrand and Zhang [2004], p.337)**.** *Let Γ be a connected graph and $v \in V(\Gamma)$. Then v is a complete vertex of Γ if and only if v is a boundary vertex of x for all $x \in V(\Gamma) \setminus \{v\}$.*

Theorem 2.3.22 (Chartrand and Zhang [2004], p.339)**.** *Let Γ be a connected graph and $v \in V(\Gamma)$. Then v is a boundary vertex of Γ if and only if v is not an interior vertex of Γ.*

Proposition 2.3.23. *In $\mathcal{P}_E(SD_{8n})$, we have*

(i) $Int(\mathcal{P}_E(SD_{8n}) = K_2$,

(ii) $Cl(\mathcal{P}_E(SD_{8n}) = \mathcal{P}_E(SD_{8n})$.

Proof. (i) In view of Lemma 2.3.1, x is not a complete vertex if and only if $x \in \{e, a^{2n}\}$. By Theorems 2.3.21 and 2.3.22, x is an interior vertex if and only if $x \in \{e, a^{2n}\}$. Thus, we have $Int(\mathcal{P}_E(SD_{8n}) = K_2$.
(ii) In view of Lemma 2.3.1, we have

$$\deg(v) = \begin{cases} 1, & \text{when } v = a^{2i}b \text{ for some } i, \text{ where } 1 \leq i \leq 2n; \\ 3, & \text{when } v = a^{2i+1}b \text{ for some } i, \text{ where } 0 \leq i \leq 2n-1; \\ 4n-1, & \text{when } v \in \langle a \rangle \setminus \{e, a^{2n}\}; \end{cases}$$

Now, we observe that for any pair of non-adjacent vertices x and y, we have

$$\deg(x) + \deg(y) \leq 6n + 1 < 8n.$$

Thus, we have $Cl(\mathcal{P}_E(SD_{8n})) = \mathcal{P}_E(SD_{8n})$. □

Theorem 2.3.24. *In $\mathcal{P}_E(SD_{8n})$, we have*

$$Ecc(\mathcal{P}_E(SD_{8n})) = \mathcal{P}_E(SD_{8n}) \setminus \{e\}.$$

Proof. Since e is adjacent to all the vertices of $\mathcal{P}_E(SD_{8n})$ we get $ecc(e) = 1$. For $v \neq e$, we have $d(v, e) = ecc(e)$. Therefore, v is an eccentric vertex in $\mathcal{P}_E(G)$. If e is an eccentric vertex in $\mathcal{P}_E(SD_{8n})$, then $1 = d(v, e) = ecc(v)$ for some non-identity vertex says v. This implies that v is adjacent to all the vertices of $\mathcal{P}_E(SD_{8n})$. It follows that $v = e$ (see Lemma 2.3.1); a contradiction. □

Chapter 3

The Complement of Enhanced Power Graph

The motivation of this chapter is the following question posed by Cameron [2022].
Question: Is it true that the complement of the enhanced power graph of a finite non-cyclic group has just one connected component apart from isolated vertices?

In this chapter, we give an affirmative answer to the above question. Further, a natural question arises is to study graph theoretic properties of the connected graph obtained by removing isolated vertices. This chapter is arranged as follows. In Section 3.1, we have proved that the complement of the enhanced power graph of a finite group has just one connected component apart from isolated vertices. Moreover, we obtain the girth and the chromatic number of $\overline{\mathcal{P}_E(G)}$. Further, we classify all finite groups such that the subgraph $\overline{\mathcal{P}_E(G^*)}$, obtained by deleting isolated vertices of $\overline{\mathcal{P}_E(G)}$, is dominatable, Eulerian, unicyclic and pentacyclic, respectively. In Section 3.2, we classify all finite groups G such that the graph $\overline{\mathcal{P}_E(G^*)}$ is outerplanar, planar, projective-planar and toroidal, respectively. The content of this chapter is accepted for publication in SCIE journal *"Algebra Colloquium"*.

3.1 The Structure and Graph Invariants of $\overline{\mathcal{P}_E(G)}$

In this section, first we give an affirmative answer to the question posed by Cameron [2022] (see Question 20). Recall that $\overline{\mathcal{P}_E(G)}$ denotes the complement of the enhanced power graph $\mathcal{P}_E(G)$. We classify all finite groups G such that the graph $\overline{\mathcal{P}_E(G)}$ is bipartite. We obtain the girth of $\overline{\mathcal{P}_E(G)}$ and also we prove that the graph $\overline{\mathcal{P}_E(G)}$ is weakly perfect. Further, we study the subgraph $\overline{\mathcal{P}_E(G^*)}$ of $\overline{\mathcal{P}_E(G)}$ induced by all the non-isolated vertices in $\overline{\mathcal{P}_E(G)}$. Then we classify the groups G such that the graph $\overline{\mathcal{P}_E(G^*)}$ is dominatable and Eulerian, respectively. Finally, we classify all finite groups G such that the graph is $\overline{\mathcal{P}_E(G^*)}$ is unicyclic and pentacyclic. We also prove that the graph $\overline{\mathcal{P}_E(G^*)}$ cannot be bicyclic, tricyclic and tetracyclic.

Lemma 3.1.1. *Let G be a finite group. Then x is an isolated vertex of $\overline{\mathcal{P}_E(G)}$ if and only if x lies in every maximal cyclic subgroup of G.*

Proof. First suppose that x lies in every maximal cyclic subgroup of G. This implies that x is adjacent to every element of G in $\mathcal{P}_E(G)$ and so x is an isolated vertex in $\overline{\mathcal{P}_E(G)}$. Conversely, let x be an isolated vertex in $\overline{\mathcal{P}_E(G)}$. Then x is a dominatable vertex in $\mathcal{P}_E(G)$. Consequently, x belongs to every maximal cyclic subgroup of G. □

Theorem 3.1.2. *Let G be a finite non-cyclic group. Then $\overline{\mathcal{P}_E(G)}$ has just one connected component, apart from isolated vertices.*

Proof. Let M be a maximal cyclic subgroup of G and let $x \in \widetilde{M}$, where $\widetilde{M}$ is the set of generators of M. First note that x is adjacent to every element of $G \setminus M$ in $\overline{\mathcal{P}_E(G)}$. If possible, assume that $x \nsim y$ in $\overline{\mathcal{P}_E(G)}$, for some $y \in G \setminus M$. Then x is adjacent to y in $\mathcal{P}_E(G)$. Therefore, $x, y \in \langle z \rangle$ for some $z \in G$. This contradicts the maximality of M. Thus, $(G \setminus M) \cup \widetilde{M}$ is a connected component of $\overline{\mathcal{P}_E(G)}$. Now, for $z \in M \setminus \widetilde{M}$, if $z \in \cap M_i$ for every i, then by Lemma 3.1.1, z is an isolated vertex. Now, if $z \notin M_j$ for some $M_j \in \mathcal{M}(G)$, then z is adjacent to every element of $\widetilde{M_j} \subseteq (G \setminus M)$ in $\overline{\mathcal{P}_E(G)}$. Thus, the result holds. □

Theorem 3.1.3. *Let G be a finite group. Then the following hold:*

(i) *The graph $\overline{\mathcal{P}_E(G)}$ is bipartite if and only if G is cyclic.*

(ii) *The girth of $\overline{\mathcal{P}_E(G)}$ is either 3 or ∞.*

(iii) *The graph $\overline{\mathcal{P}_E(G)}$ is weakly perfect.*

Proof. (i) Let $\overline{\mathcal{P}_E(G)}$ be a bipartite graph such that $|\mathcal{M}(G)| \geq 3$. Suppose $M_1 = \langle x \rangle$, $M_2 = \langle y \rangle$ and $M_3 = \langle z \rangle$. Then we get a cycle $x \backsim y \backsim z \backsim x$ of odd length in $\overline{\mathcal{P}_E(G)}$; a contradiction (cf. Theorem 1.2.2). Consequently, $|\mathcal{M}(G)| \leq 2$. By Lemma 1.1.5, we get $|\mathcal{M}(G)| = 1$. Thus, G is a cyclic group. Conversely, assume that G is a cyclic group of order n. By Theorem 1.2.15, $\overline{\mathcal{P}_E(G)}$ is a null graph. Hence, $\overline{\mathcal{P}_E(G)}$ is a bipartite graph.
(ii) If G is a cyclic group, then $\overline{\mathcal{P}_E(G)}$ is a null graph and hence the girth of $\overline{\mathcal{P}_E(G)}$ is ∞. We may now suppose that G is a non-cyclic group. By the proof of part (i), notice that $\overline{\mathcal{P}_E(G)}$ contains a cycle of length 3. Thus, the girth of $\overline{\mathcal{P}_E(G)}$ is 3.
(iii) By Theorem 3.3 of P. Panda et al. [2021], $\alpha(\mathcal{P}_E(G)) = |\mathcal{M}(G)|$. For a graph Γ, we have $\alpha(\Gamma) = \omega(\overline{\Gamma})$. Thus, $\omega(\overline{\mathcal{P}_E(G)}) = |\mathcal{M}(G)|$. Also, $\chi(\overline{\mathcal{P}_E(G)}) \geq \omega(\overline{\mathcal{P}_E(G)}) = |\mathcal{M}(G)|$. Notice that if G is a non-cyclic group then $\overline{\mathcal{P}_E(G)}$ is a $k-$partite graph, where $k = |\mathcal{M}(G)|$. Consequently, $\chi(\overline{\mathcal{P}_E(G)}) \leq |\mathcal{M}(G)|$. Thus, the result holds. □

Note that the subgraph $\overline{\mathcal{P}_E(G^*)}$ of $\overline{\mathcal{P}_E(G)}$ induced by all the non-isolated vertices in $\overline{\mathcal{P}_E(G)}$ is connected. In what follows, we investigate the subgraph $\overline{\mathcal{P}_E(G^*)}$. We begin with the classification of the groups G such that $\overline{\mathcal{P}_E(G^*)}$ is dominatable and Eulerian, respectively.

Theorem 3.1.4. *The graph $\overline{\mathcal{P}_E(G^*)}$ is dominatable if and only if G has a maximal cyclic subgroup of order 2.*

Proof. Let $\overline{\mathcal{P}_E(G^*)}$ be a dominatable graph. Then there exists a vertex x of $\overline{\mathcal{P}_E(G^*)}$ such that x is adjacent to every vertex of $\overline{\mathcal{P}_E(G^*)}$. Note that if $o(x) \geq 3$, then

$x \nsim x^{-1}$ in $\overline{\mathcal{P}_E(G^*)}$; a contradiction. It follows that $o(x) = 2$. Moreover, note that $\langle x \rangle \in \mathcal{M}(G)$. Otherwise, the generator of a maximal cyclic subgroup containing x is not adjacent to x in $\overline{\mathcal{P}_E(G^*)}$, which is not possible. Conversely, let $M = \langle x \rangle$ be a maximal cyclic subgroup of order 2. Then $x \sim y$ for every $y \in G \setminus M$. Consequently, x is a dominating vertex of $\overline{\mathcal{P}_E(G^*)}$. Thus, the result holds. □

Corollary 3.1.5. *The graph $\overline{\mathcal{P}_E(G^*)}$ is complete if and only if $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$.*

Theorem 3.1.6. *For $x \in G$, let M_x be the union of all the maximal cyclic subgroups of G containing x. Then the graph $\overline{\mathcal{P}_E(G^*)}$ is Eulerian if and only if either $|G|$ is odd or $|\mathrm{M}_x|$ is even for every $x \in V(\overline{\mathcal{P}_E(G^*)})$.*

Proof. If $|G|$ is odd, then the enhanced power graph $\mathcal{P}_E(G)$ is Eulerian (see Theorem 2.5 of Bera and Bhuniya [2018]). Thus, the degree of every vertex of $\mathcal{P}_E(G)$ is even (see Theorem 1.2.1). Let $x \in V(\overline{\mathcal{P}_E(G^*)})$ such that $\deg(x) = m$ in $\mathcal{P}_E(G)$ and let $|G| = n$. Thus, in $\overline{\mathcal{P}_E(G^*)}$, we have $\deg(x) = n - m - 1$ which is an even number. Consequently, $\overline{\mathcal{P}_E(G^*)}$ is Eulerian. We may now suppose that $|\mathrm{M}_x|$ is even for every vertex of $\overline{\mathcal{P}_E(G^*)}$. Now, let y be an arbitrary vertex of $\overline{\mathcal{P}_E(G^*)}$. Note that in $\overline{\mathcal{P}_E(G^*)}$, $\deg(y) = |G| - |\mathrm{M}_y|$ is even. Thus, $\overline{\mathcal{P}_E(G^*)}$ is Eulerian. Conversely, suppose that $\overline{\mathcal{P}_E(G^*)}$ is Eulerian. If $|G|$ is odd, then there is nothing to prove. If $|G|$ is not odd and $|\mathrm{M}_x|$ is odd for some $x \in V(\overline{\mathcal{P}_E(G^*)})$, then $\deg(x) = |G| - |\mathrm{M}_x|$ is odd; a contradiction. Thus, $|\mathrm{M}_x|$ must be even for every $x \in V(\overline{\mathcal{P}_E(G^*)})$. □

If G is a 2-group, then every non-trivial subgroup of G is of even order. Consequently, $|\mathrm{M}_x|$ is even for every $x \in V(\overline{\mathcal{P}_E(G^*)})$. Thus, we have the following corollary of Theorem 3.1.6.

Corollary 3.1.7. *If G is a 2-group, then the graph $\overline{\mathcal{P}_E(G^*)}$ is Eulerian.*

Now, we give examples of even order groups G such that the graph $\overline{\mathcal{P}_E(G^*)}$ is Eulerian.

Example 3.1.8. Consider G to be the dihedral group D_{2n}. Notice that G has a maximal cyclic subgroup $M = \langle x \rangle$ of order n and every element of G belongs to exactly one maximal cyclic subgroup of G. If n is odd, then $\deg(x) = |G| - |M|$ which is an odd number. Hence, $\overline{\mathcal{P}_E(D_{2n}^*)}$ is not Eulerian. If n is even, then every maximal cyclic subgroup of D_{2n} is of even order. Consequently, $|\mathrm{M}_x|$ is even for every $x \in V(\overline{\mathcal{P}_E(D_{2n}^*)})$. It follows that $\overline{\mathcal{P}_E(D_{2n}^*)}$ is Eulerian. Thus, $\overline{\mathcal{P}_E(D_{2n}^*)}$ is Eulerian if and only if n is even.

Example 3.1.9. Consider G to be the generalized quaternion group Q_{4n}. Observe that the center of Q_{4n} is contained in every maximal cyclic subgroup of G and $x \in Q_{4n} \setminus Z(Q_{4n})$ belongs to exactly one maximal cyclic subgroup of Q_{4n}. Also, notice that Q_{4n} has 1 maximal cyclic subgroup of order $2n$ and n maximal cyclic subgroup of order 4. Consequently, $|\mathrm{M}_x|$ is even for every $x \in V(\overline{\mathcal{P}_E(Q_{4n}^*)})$. Thus, $\overline{\mathcal{P}_E(Q_{4n}^*)}$ is Eulerian for all $n \geq 2$.

Suppose Γ is a connected graph with n vertices and m edges. If $c(\Gamma) = m-n+1$ then Γ is called c-cyclic. We call Γ to be unicyclic, bicyclic, tricyclic, tetracyclic and pentacyclic if $c(\Gamma) = 1, 2, 3, 4$ and 5, respectively. Clearly, Γ is a tree if and only if $c(\Gamma) = 0$. The following lemma is easy to prove.

Lemma 3.1.10. *Let Γ' be a connected subgraph of a connected graph Γ. Then $c(\Gamma') \leq c(\Gamma)$.*

Now, we classify all the finite groups G such that $c(\overline{\mathcal{P}_E(G^*)}) \in \{1, 2, 3, 4, 5\}$.

Theorem 3.1.11. *Let G be a finite non-cyclic group. Then the following hold:*

(i) *The graph $\overline{\mathcal{P}_E(G^*)}$ is unicyclic if and only if G is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.*

(ii) *The graph $\overline{\mathcal{P}_E(G^*)}$ is pentacyclic if and only if G is isomorphic to S_3.*

(iii) *The graph $\overline{\mathcal{P}_E(G^*)}$ cannot be bicyclic, tricyclic, and tetracyclic.*

Proof. In view of Lemma 1.1.5, we prove the result through the following cases on the cardinality of the set $\mathcal{M}(G)$.

Case-1: $|\mathcal{M}(G)| = 3$. Let M_1, M_2 and M_3 be the maximal cyclic subgroups of G such that $|M_i| = m_i$ for $i \in \{1, 2, 3\}$. Without loss of generality, assume that $\phi(m_1) \geq \phi(m_2) \geq \phi(m_3)$. Now, we have the following subcases:

Subcase-1.1: $\phi(m_1) = \phi(m_2) = \phi(m_3) = 1$. It follows that $m_1 = m_2 = m_3 = 2$. The identity element belongs to every maximal cyclic subgroup of G. Hence, we get $|G| = 4$ and by Table 1.1, $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Consequently, $\overline{\mathcal{P}_E(G^*)} \cong K_3$ and $c(K_3) = 1$. Thus, $\overline{\mathcal{P}_E(G^*)}$ is unicyclic.

Subcase-1.2: $\phi(m_1) = 2$ and $\phi(m_2) = 1 = \phi(m_3)$. Then $m_2 = m_3 = 2$. Consequently, by Remark 1.1.4, $m_1 + 2 = |G|$ and by Lagrange's theorem, $m_1 \leq \frac{|G|}{2}$. It follows that $|G| \leq 4$. But $\phi(m_1) = 2$ implies that $m_1 \geq 3$, which is not possible.

Subcase-1.3: $\phi(m_1) = 2 = \phi(m_2)$ and $\phi(m_3) = 1$. It follows that $m_1, m_2 \in \{3, 4, 6\}$ and $m_3 = 2$. Now, let us assume that $m_j = 3$ and $m_k = t$, for distinct $j, k \in \{1, 2\}$ and $t \in \{3, 4, 6\}$. Then by Remark 1.1.4, $|G| \in \{6, 7, 9\}$. By Table 1.1, no such group exists. Now, if $m_1 = m_2 = 4$ and $m_3 = 2$, then $|M_1 \cap M_2| \in \{1, 2\}$. Consequently, by Remark 1.1.4, $|G| \in \{7, 8\}$. The existence of a subgroup of order 4 implies that $|G| = 8$. By Table 1.1, no such group exists.
For distinct $i, j \in \{1, 2\}$, if $m_i = 4, m_j = 6$ and $m_3 = 2$, then $|M_1 \cap M_2| \in \{1, 2\}$. Consequently, by Remark 1.1.4, $|G| \in \{9, 10\}$. But the existence of a subgroup of order 6 follows that no such group exists. If $m_1 = m_2 = 6$ and $m_3 = 2$ then $|M_1 \cap M_2| \in \{1, 2, 3\}$. Consequently, by Remark 1.1.4, $|G| \in \{10, 11, 12\}$. Since there exists a subgroup of order 6, we get $|G| = 12$. By Table 1.1, no such group exists.

Subcase-1.4: $\phi(m_1) = \phi(m_2) = \phi(m_3) = 2$. Let Γ be the subgraph of $\overline{\mathcal{P}_E(G^*)}$ induced by the set $\mathcal{G}_{\mathcal{M}(G)}$. Then note that $c(\Gamma) = 7$. By Lemma 3.1.10, $c(\overline{\mathcal{P}_E(G^*)}) \geq 7$.

Subcase-1.5: $\phi(m_1) \geq 4$ and $\phi(m_2) = \phi(m_3) = 1$. Then $m_2 = m_3 = 2$. By

Remark 1.1.4, $m_1 + 2 = |G|$ and by Lagrange's theorem, $m_1 \leq \frac{|G|}{2}$. It follows that $|G| \leq 4$, and so $m_1 \leq 2$, which is not possible.

Subcase-1.6: $\phi(m_1) \geq 4$, $\phi(m_2) \geq 2$ and $\phi(m_3) \geq 1$. Let a_1, a_2, a_3, a_4 be generators of M_1 and b_1, b_2 be generators of M_2. Further, suppose that $M_3 = \langle c_1 \rangle$. Let Γ be the subgraph of $\overline{\mathcal{P}_E(G^*)}$ induced by the set $S = \{a_1, a_2, a_3, a_4, b_1, b_2, c_1\}$. Then $c(\Gamma) = 8$ and by Lemma 3.1.10, we get $c(\overline{\mathcal{P}_E(G^*)}) \geq 8$.

Case-2: $|\mathcal{M}(G)| = 4$. Let M_1, M_2, M_3 and M_4 be the maximal cyclic subgroups of G such that $|M_i| = m_i$ for $i \in \{1, 2, 3, 4\}$. Without loss of generality, assume that $\phi(m_1) \geq \phi(m_2) \geq \phi(m_3) \geq \phi(m_4)$. Now, we have the following subcases:

Subcase-2.1: $\phi(m_1) = \phi(m_2) = \phi(m_3) = \phi(m_4) = 1$. Then $m_1 = m_2 = m_3 = m_4 = 2$. The identity element belongs to every maximal cyclic subgroup of G. Hence, we obtain $|G| = 5$. By Table 1.1, no such group exists.

Subcase-2.2: $\phi(m_1) = 2$ and $\phi(m_2) = \phi(m_3) = \phi(m_4) = 1$. Then $m_1 \in \{3, 4, 6\}$ and $m_2 = m_3 = m_4 = 2$. Consequently, by Remark 1.1.4, $|G| \in \{6, 7, 9\}$. The existence of a subgroup of order 2 gives $|G| = 6$. By Table 1.1, $G \cong S_3$. By Figure 3.1, we get $c(\overline{\mathcal{P}_E(S_3^*})= 5$. Thus, $\overline{\mathcal{P}_E(S_3^*)}$ is pentacyclic.

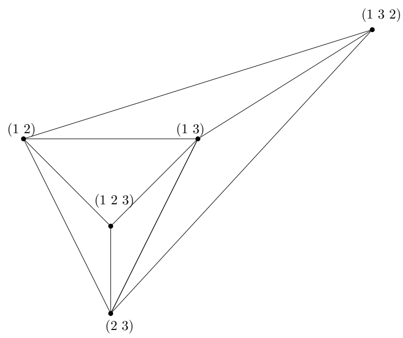


Figure 3.1: A planar drawing of $\overline{\mathcal{P}_E(S_3^*)}$.

Subcase-2.3: $\phi(m_1) \geq 2$, $\phi(m_2) \geq 2$, $\phi(m_3) \geq 1$ and $\phi(m_4) \geq 1$. Let a_1, a_2 be generators of M_1 and b_1, b_2 be generators of M_2. Further, let $M_3 = \langle c_1 \rangle$ and $M_4 = \langle d_1 \rangle$. Let Γ be the subgraph of $\overline{\mathcal{P}_E(G^*)}$ induced by the set $S = \{a_1, a_2, b_1, b_2, c_1, d_1\}$. Then $c(\Gamma) = 8$ and by Lemma 3.1.10, we get $c(\overline{\mathcal{P}_E(G^*)}) \geq 8$.

Subcase-2.4: $\phi(m_1) \geq 4$ and $\phi(m_i) \geq 1$ for $i \in \{2,3,4\}$. Let a_1, a_2, a_3, a_4 be generators of M_1, and let $M_2 = \langle b_1 \rangle, M_3 = \langle c_1 \rangle$ and $M_4 = \langle d_1 \rangle$. Let Γ be the subgraph of $\overline{\mathcal{P}_E(G^*)}$ induced by the set $S = \{a_1, a_2, a_3, a_4, b_1, c_1, d_1\}$. Then $c(\Gamma) = 9$. By Lemma 3.1.10, we get $c(\overline{\mathcal{P}_E(G^*)}) \geq 9$.

Case-3: $|\mathcal{M}(G)| \geq 5$. Consider the set $S = \{g_1, g_2, \ldots, g_5\}$, where $M_i = \langle g_i \rangle \in \mathcal{M}(G)$ for $i \in \{1, 2, \ldots, 5\}$. Note that the subgraph of $\overline{\mathcal{P}_E(G^*)}$ induced by the set S is isomorphic to the complete graph K_5. Since $c(K_5) = 6$, by Lemma 3.1.10, we obtain $c(\overline{\mathcal{P}_E(G^*)}) \geq 6$.

Thus, the result holds. □

3.2 Embedding of $\overline{\mathcal{P}_E(G^*)}$ on Surfaces

In this section, we study certain embeddings of the graph $\overline{\mathcal{P}_E(G^*)}$ on various surfaces. We classify all finite groups G such that the graph $\overline{\mathcal{P}_E(G^*)}$ is outerplanar, planar, projective-planar and toroidal, respectively. Moreover, we show that there does not exist a group G such that the cross-cap of the graph $\overline{\mathcal{P}_E(G^*)}$ is two.

Theorem 3.2.1. *Let G be a finite non-cyclic group. Then*

(i) *The graph $\overline{\mathcal{P}_E(G^*)}$ is outerplanar if and only if G is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.*

(ii) *The graph $\overline{\mathcal{P}_E(G^*)}$ is planar if and only if G is isomorphic to one of the three groups: $\mathbb{Z}_2 \times \mathbb{Z}_2$, S_3, Q_8.*

(iii) *The graph $\overline{\mathcal{P}_E(G^*)}$ is projective-planar if and only if G is isomorphic to either D_8 or $\mathbb{Z}_2 \times \mathbb{Z}_4$.*

(iv) *The graph $\overline{\mathcal{P}_E(G^*)}$ cannot have cross-cap 2.*

(v) *The graph $\overline{\mathcal{P}_E(G^*)}$ is toroidal if and only if G is isomorphic to one of the following 5 groups:*

$$D_8, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_6, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Proof. In view of Lemma 1.1.5, we prove the result through the following cases on the cardinality of the set $\mathcal{M}(G)$.

Case-1: $|\mathcal{M}(G)| = 3$. Let M_1, M_2 and M_3 be the maximal cyclic subgroups of G such that $|M_i| = m_i$ for $i \in \{1,2,3\}$. Without loss of generality, assume that $\phi(m_1) \geq \phi(m_2) \geq \phi(m_3)$. Now, we have the following subcases:

Subcase-1.1: $\phi(m_1) = \phi(m_2) = \phi(m_3) = 1$. By **Subcase-1.1** of the Theorem 3.1.11, we get $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Subcase-1.2: $\phi(m_1) = 2$ and $\phi(m_2) = \phi(m_3) = 1$. By **Subcase-1.2** of the Theorem 3.1.11, no such group exists.

Subcase-1.3: $\phi(m_1) = 2 = \phi(m_2)$ and $\phi(m_3) = 1$. By **Subcase-1.3** of the Theorem 3.1.11, no such group exists.

Subcase-1.4: $\phi(m_1) = \phi(m_2) = \phi(m_3) = 2$. Then $m_i \in \{3,4,6\}$ for each $i \in \{1,2,3\}$. For distinct $i,j,k \in \{1,2,3\}$, if $m_i = m_j = 3$ and $m_k = t$, where $t \in \{3,4,6\}$, then by Remark 1.1.4, $|G| \in \{7,8,10\}$. But the existence of a subgroup of order 3 follows that no such group exists. For distinct $i,j,k \in \{1,2,3\}$, if $m_i = 3$ and $m_j = m_k = 4$, then $|M_j \cap M_k| \in \{1,2\}$. Consequently, by Remark 1.1.4, we get $|G| \in \{8,9\}$, which is not possible because G has subgroups of order 3 and 4. For distinct $i,j,k \in \{1,2,3\}$, if $m_i = m_j = 4$ and $m_k = 6$, then notice that the cardinality of the intersection of any two of these maximal cyclic subgroups is at most 2. Consequently, by Remark 1.1.4, $|G| \in \{10,11,12\}$. Since there exists a maximal cyclic subgroup of order 6, we obtain $|G| = 12$. By Table 1.1, no such group exists. If $m_1 = m_2 = m_3 = 4$, then by Remark 1.1.4, $|G| \in \{8,9,10\}$. The existence of a subgroup of order 4 gives $|G| = 8$. By Table 1.1, we have $G \cong Q_8$. For distinct $i,j,k \in \{1,2,3\}$, if $m_i = m_j = 6$ and $m_k = 3$, then $|M_i \cap M_j| \in \{1,2,3\}$. Consequently, we get $|G| \in \{11,12,13\}$ (see Remark 1.1.4), and therefore $|G| = 12$. By Table 1.1, no such group exists. For distinct $i,j,k \in \{1,2,3\}$, if $m_i = m_j = 6$ and $m_k = 4$, then by Remark 1.1.4, $|G| \in \{11,12,13,14\}$. The existence of a subgroup of order 4 gives $|G| = 12$. By Table 1.1, no such group exists. If $m_1 = m_2 = m_3 = 6$, then by Remark 1.1.4, and the existence of order 6 gives $|G| = 12$. By Table 1.1, $G \cong \mathbb{Z}_2 \times \mathbb{Z}_6$. For distinct $i, j, k \in \{1, 2, 3\}$, if $m_i = 3, m_j = 4$ and $m_k = 6$, then $\text{lcm}(3, 4, 6) = 12$ divides $|G|$. The identity element belongs to every maximal cyclic subgroup of a group G. Consequently, by Remark 1.1.4, $|G| \leq 11$; a contradiction.

Subcase-1.5: $\phi(m_1) \geq 4$ and $\phi(m_2) = \phi(m_3) = 1$. By **Subcase-1.5** of the Theorem 3.1.11, no such group exists.

Subcase-1.6: $\phi(m_1) = 4$, $\phi(m_2) = 2$ and $\phi(m_3) = 1$. Then $m_1 \in \{5, 8, 10, 12\}$, $m_2 \in \{3, 4, 6\}$ and $m_3 = 2$. If $m_1 = 5$ and $m_2 = t$, where $t \in \{3, 4, 6\}$, then by Remark 1.1.4, we obtain $|G| \in \{8, 9, 11\}$. The existence of a subgroup of order 5 implies that no such group exists. If $m_1 = s$, $m_2 = t$, where $s \in \{8, 10, 12\}$ and $t \in \{3, 4, 6\}$ then by Remark 1.1.4, we obtain $|G| \leq s + t$ as the identity element belongs to every maximal cyclic subgroup of G. By Lagrange's theorem, $2s \leq |G|$. It implies that $s \leq t$; a contradiction.

Subcase-1.7: $\phi(m_1) = 4$ and $\phi(m_2) = \phi(m_3) = 2$. Then $m_1 \in \{5, 8, 10, 12\}$ and $m_2, m_3 \in \{3, 4, 6\}$. If $m_1 = 5$ and $m_2 = m_3 = 3$, then by Remark 1.1.4, $|G| = 9$. The existence of a subgroup of order 5 implies that no such group exists. Now, assume that at least one of m_2 and m_3 is not equal to 3. Let x be an element of order 2 in $M_2 \cup M_3$. Then the subgraph induced by the set $\mathcal{G}_{\mathcal{M}(G)} \cup \{x\}$ contains a subgraph isomorphic to $K_{4,5}$. Then by Theorem 1.2.6, we get $\gamma(\overline{\mathcal{P}_E(G^*)}) \geq 2$ and $\overline{\gamma}(\overline{\mathcal{P}_E(G^*)}) \geq 3$.
If $m_1 \in \{8, 12\}$ and m_2, $m_3 \in \{3, 4, 6\}$, then the subgraph induced by the set $\mathcal{G}_{\mathcal{M}(G)} \cup \{x, y\}$ contains a subgraph isomorphic to $K_{6,4}$, where x and y are elements of order 4 in M_1. Then by Theorem 1.2.6, we obtain $\gamma(\overline{\mathcal{P}_E(G^*)}) \geq 2$ and $\overline{\gamma}(\overline{\mathcal{P}_E(G^*)}) \geq 4$. If $m_1 = 10$ and m_2, $m_3 \in \{3, 4, 6\}$, then the subgraph induced by the set $\mathcal{G}_{\mathcal{M}(G)} \cup \{x, y, z, t\}$, where x, y, z and t are elements of order 5 in M_1, contains $K_{8,4}$ as a subgraph. It follows that $\gamma(\overline{\mathcal{P}_E(G^*)}) \geq 3$ and $\overline{\gamma}(\overline{\mathcal{P}_E(G^*)}) \geq 6$.

Subcase-1.8: $\phi(m_1) \geq 4$, $\phi(m_2) \geq 4$ and $\phi(m_3) \geq 1$. Then the subgraph induced by the set $\mathcal{G}_{\mathcal{M}(G)}$ contains a subgraph isomorphic to $K_{4,5}$. By Theorem

1.2.6, we get $\gamma(\overline{\mathcal{P}_E(G^*)}) \geq 2$ and $\overline{\gamma}(\overline{\mathcal{P}_E(G^*)}) \geq 3$.

Subcase-1.9: $\phi(m_1) \geq 6$, $\phi(m_2) = 2$ and $\phi(m_3) = 1$. Then $m_2 \in \{3, 4, 6\}$ and $m_3 = 2$. By Remark 1.1.4, $|G| \leq m_1 + 6$. The existence of a subgroup of order m_1 follows that $m_1 \leq 6$; which is not possible.

Subcase-1.10: $\phi(m_1) \geq 6$, $\phi(m_2) = 2$ and $\phi(m_3) = 2$. Then the subgraph induced by the set $\mathcal{G}_{\mathcal{M}(G)}$ contains a subgraph isomorphic to $K_{4,5}$. Then by Theorem 1.2.6, we get $\gamma(\overline{\mathcal{P}_E(G^*)}) \geq 2$ and $\overline{\gamma}(\overline{\mathcal{P}_E(G^*)}) \geq 3$.

Case-2: $|\mathcal{M}(G)| = 4$. Let M_1, M_2, M_3 and M_4 be the maximal cyclic subgroups of G such that $|M_i| = m_i$ for $i \in \{1, 2, 3, 4\}$. Without loss of generality, assume that $\phi(m_1) \geq \phi(m_2) \geq \phi(m_3) \geq \phi(m_4)$. Now, we have the following subcases:

Subcase-2.1: $\phi(m_1) = \phi(m_2) = \phi(m_3) = \phi(m_4) = 1$. By **Subcase-2.1** of the Theorem 3.1.11, no such group exists.

Subcase-2.2: $\phi(m_1) = 2$ and $\phi(m_2) = \phi(m_3) = \phi(m_4) = 1$. Then $m_1 \in \{3, 4, 6\}$ and $m_i = 2$ for every $i \in \{2, 3, 4\}$. By Remark 1.1.4, we obtain $|G| \in \{6, 7, 9\}$. Since there exists a subgroup of order 2, we have $|G| = 6$. By Table 1.1, we obtain $G \cong S_3$.

Subcase-2.3: $\phi(m_1) = \phi(m_2) = 2$ and $\phi(m_3) = \phi(m_4) = 1$. Then $m_1, m_2 \in \{3, 4, 6\}$ and $m_3 = m_4 = 2$. If $m_1 = 3$ and $m_2 = t$, where $t \in \{3, 4, 6\}$, then by Remark 1.1.4, we obtain $|G| \in \{7, 8, 10\}$. The existence of a subgroup of order 3 implies that no such group exists. If $m_1 = 4$ and $m_2 = t$, where $t \in \{3, 4, 6\}$, then by Remark 1.1.4, we get $|G| \in \{8, 9, 10, 11\}$. Since there exists a subgroup of order 4, we have $|G| = 8$. By Table 1.1, we obtain $G \cong \mathbb{Z}_2 \times \mathbb{Z}_4$. If $m_1 = 6$ and $m_2 = t$, where $t \in \{3, 4, 6\}$, then $|G| \in \{10, 11, 12, 13\}$ (cf. Remark 1.1.4). The existence of a subgroup of order 6 gives $|G| = 12$. By Table 1.1, no such group exists.

Subcase-2.4: $\phi(m_1) = \phi(m_2) = \phi(m_3) = 2$ and $\phi(m_4) = 1$. Then $m_1, m_2, m_3 \in \{3, 4, 6\}$ and $m_4 = 2$. For distinct $i, j, k \in \{1, 2, 3\}$, if $m_i = m_j = 3$ and $m_k = t$, where $t \in \{3, 4, 6\}$, then by Remark 1.1.4, $|G| \in \{8, 9, 11\}$. The existence of subgroups of order 2 and 3, follows that that no such group exists.

For distinct $i, j, k \in \{1, 2, 3\}$, if $m_i = 3$ and $m_j = m_k = 4$, then $|M_j \cap M_k| \in \{1, 2\}$. Consequently, we get $|G| \in \{9, 10\}$ (see Remark 1.1.4), which is not possible because G has a subgroup of order 4. For distinct $i, j, k \in \{1, 2, 3\}$, if $m_i = m_j = 4$ and $m_k = 6$, then notice that the cardinality of intersection of any two of these maximal cyclic subgroups is at most 2. Consequently, by Remark 1.1.4, $|G| \in \{11, 12, 13\}$. Since G has a subgroup of order 2, we have $|G| = 12$. By Table 1.1, no such group exists. If $m_1 = m_2 = m_3 = 4$, then by Remark 1.1.4, $|G| \in \{9, 10, 11\}$. The existence of a subgroup of order 4 implies that no such group exists.
For distinct $i, j, k \in \{1, 2, 3\}$, if $m_i = m_j = 6$ and $m_k = 3$, then $|M_i \cap M_j| \in \{1, 2, 3\}$. Consequently, by Remark 1.1.4, $|G| \in \{12, 13, 14\}$. The existence of a subgroup of order 3 implies that $|G| = 12$. By Table 1.1, no such group exists. For distinct $i, j, k \in \{1, 2, 3\}$, if $m_i = m_j = 6$ and $m_k = 4$, then by Remark 1.1.4, $|G| \in \{12, 13, 14, 15\}$ and therefore $|G| = 12$. By Table 1.1, no such group exists. If $m_1 = m_2 = m_3 = 6$, then by Remark 1.1.4, $13 \leq |G| \leq 17$. Consequently, the existence of a subgroup of order 6 implies that $|G| = 12$. By Table 1.1, no such group exists. For distinct $i, j, k \in \{1, 2, 3\}$, if $m_i = 3, m_j = 4$ and $m_k = 6$, then $\mathrm{lcm}(3, 4, 6)$ divides $|G|$. By Remark 1.1.4, $|G| \leq 12$. Thus, we obtain $|G| = 12$. By Table 1.1, no such group exists.

Subcase-2.5: $\phi(m_1) = \phi(m_2) = \phi(m_3) = \phi(m_4) = 2$. If $m_i = 3$ for every $i \in \{1, 2, 3, 4\}$, then $|G| = 9$ and therefore by Table 1.1, $G \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.

Notice that if $x \notin \mathcal{G}_{\mathcal{M}(G)}$ is an element of M_i, for $i \in \{1, 2, 3, 4\}$, such that x belongs to at most two maximal cyclic subgroups, then the subgraph induced by $\mathcal{G}_{\mathcal{M}(G)} \cup \{x\}$ contains a subgraph isomorphic to $K_{4,5}$. For distinct $i, j, k, l \in \{1, 2, 3, 4\}$, let $m_i = 6$, $m_j, m_k, m_l \in \{3, 4, 6\}$ and let $a, b \in M_i$ such that $o(a) = 2$, $o(b) = 3$. Then either a or b belongs to at most two maximal cyclic subgroups. Consequently, G contains a subgraph isomorphic to $K_{4,5}$. Then by Theorem 1.2.6, we get $\gamma(\overline{\mathcal{P}_E(G^*)}) \geq 2$ and $\overline{\gamma}(\overline{\mathcal{P}_E(G^*)}) \geq 3$. For distinct $i, j, k, l \in \{1, 2, 3, 4\}$, if $m_i = 3$ and $m_j = m_k = m_l = 4$, then by Remark 1.1.4, we get $|G| \in \{10, 11, 12\}$.

The existence of a subgroup of order 3 follows that $|G| = 12$. By Table 1.1, no such group exists. If $m_i = 4$ for every $i \in \{1, 2, 3, 4\}$, then by Remark 1.1.4, we get $|G| \in \{10, 11, 12, 13\}$. The existence of a subgroup of order 4 implies that $|G| = 12$. By Table 1.1, no such group exists.

Subcase-2.6: $\phi(m_1) \geq 4$ and $\phi(m_2) = \phi(m_3) = \phi(m_4) = 1$. By the similar argument used in **Subcase-1.5** of the Theorem 3.1.11, no such group exists.

Subcase-2.7: $\phi(m_1) = 4$, $\phi(m_2) = 2$ and $\phi(m_3) = \phi(m_4) = 1$. Then $m_1 \in \{5, 8, 10, 12\}$, $m_2 \in \{3, 4, 6\}$ and $m_3 = m_4 = 2$. If $m_1 = 5$, $m_2 = t$, where $t \in \{3, 4, 6\}$, then $|G| \in \{9, 10, 12\}$ (see Remark 1.1.4). Since there exists a subgroup of order 5, we get $|G| = 10$. By Table 1.1, no such group exists. If $m_1 = s$, $m_2 = t$, where $s \in \{8, 10, 12\}$, and $t \in \{3, 4, 6\}$, then M_1 has an element x such that $\langle x \rangle \neq M_1$ and $x \notin M_2 \cup M_3 \cup M_4$. The graph induced by the set $\mathcal{G}_{\mathcal{M}(G)} \cup \{x\}$ contains a subgraph isomorphic to $K_{4,5}$. By Theorem 1.2.6, we obtain $\gamma(\overline{\mathcal{P}_E(G^*)}) \geq 2$ and $\overline{\gamma}(\overline{\mathcal{P}_E(G^*)}) \geq 3$.

Subcase-2.8: $\phi(m_1) \geq 4$, $\phi(m_2) \geq 2$, $\phi(m_3) \geq 2$ and $\phi(m_4) \geq 1$. Then the graph induced by the set $\mathcal{G}_{\mathcal{M}(G)}$ contains a subgraph isomorphic to $K_{4,5}$. By Theorem 1.2.6, we obtain $\gamma(\overline{\mathcal{P}_E(G^*)}) \geq 2$ and $\overline{\gamma}(\overline{\mathcal{P}_E(G^*)}) \geq 3$.

Subcase-2.9: $\phi(m_1) \geq 4$, $\phi(m_2) \geq 4$, $\phi(m_3) = 1$ and $\phi(m_4) = 1$. Then the graph induced by the set $\mathcal{G}_{\mathcal{M}(G)}$ contains a subgraph isomorphic to $K_{6,5}$. By Theorem 1.2.6, we obtain $\gamma(\overline{\mathcal{P}_E(G^*)}) \geq 2$ and $\overline{\gamma}(\overline{\mathcal{P}_E(G^*)}) \geq 4$.

Subcase-2.10: $\phi(m_1) \geq 6$, $\phi(m_2) = 2$ and $\phi(m_3) = \phi(m_4) = 1$. Then the subgraph induced by the set $\mathcal{G}_{\mathcal{M}(G)}$ contains a subgraph isomorphic to $K_{4,6}$. Consequently, by Theorem 1.2.6, we obtain $\gamma(\overline{\mathcal{P}_E(G^*)}) \geq 2$ and $\overline{\gamma}(\overline{\mathcal{P}_E(G^*)}) \geq 4$.

Case-3: $|\mathcal{M}(G)| = 5$. Let M_1, M_2, M_3, M_4 and M_5 be the maximal cyclic subgroups of G such that $|M_i| = m_i$ for $i \in \{1, 2, 3, 4, 5\}$. Without loss of generality, assume that $\phi(m_1) \geq \phi(m_2) \geq \phi(m_3) \geq \phi(m_4) \geq \phi(m_5)$. Now, we have the following subcases:

Subcase-3.1: $\phi(m_1) = \phi(m_2) = \phi(m_3) = \phi(m_4) = \phi(m_5) = 1$. Then $m_i = 2$ for every $i \in \{1, 2, 3, 4, 5\}$. Consequently by Remark 1.1.4, $|G| = 6$. By Table 1.1, no such groups exist.

Subcase-3.2: $\phi(m_1) = 2$ and $\phi(m_2) = \phi(m_3) = \phi(m_4) = \phi(m_5) = 1$. It follows that $m_1 \in \{3, 4, 6\}$ and $m_i = 2$ for every $i \in \{2, 3, 4, 5\}$. Consequently, by Remark 1.1.4, we have $|G| \in \{7, 8, 10\}$. Note that $|G| = 7$ if $m_1 = 3$, but 3 does not divide 7 and $|G| = 10$ if $m_1 = 6$ but 6 does not divide 10. It follows that $|G| = 8$. By Table 1.1, $G \cong D_8$.

Subcase-3.3: $\phi(m_1) = \phi(m_2) = 2$ and $\phi(m_3) = \phi(m_4) = \phi(m_5) = 1$. Then $m_1, m_2 \in \{3, 4, 6\}$ and $m_3 = m_4 = m_5 = 2$. For distinct $i, j \in \{1, 2\}$, if $m_i = 3$ and $m_j = t$, where $t \in \{3, 4, 6\}$, then $|G| \in \{8, 9, 11\}$. Since there exist subgroups of order 2 and 3, it follows that no such group exists. For distinct $i, j \in \{1, 2\}$, if $m_i = 4$ and $m_j = t$, where $t \in \{4, 6\}$, then $|G| \in \{9, 10, 11, 12\}$ (see Remark 1.1.4). The existence of a subgroup of order 4 implies that $|G| = 12$. By Table 1.1, no such groups exist. If $m_1 = m_2 = 6$, then by Remark 1.1.4, $|G| \in \{12, 13, 14\}$ and therefore $|G| = 12$. By Table 1.1, no such groups exists.

Subcase-3.4: $\phi(m_1) = \phi(m_2) = \phi(m_3) = 2$ and $\phi(m_4) = \phi(m_5) = 1$. It follows that $m_1, m_2, m_3 \in \{3, 4, 6\}$ and $m_4 = m_5 = 2$. If $m_1 = m_2 = m_3 = 3$, then by Remark 1.1.4, $|G| = 9$. But there exists a maximal cyclic subgroup of order 2. Thus, no such group exists. For distinct $i, j, k \in \{1, 2, 3\}$, let $m_i = 3$, $m_j = s$ and $m_k = t$, where $s \in \{3, 4, 6\}$ and $t \in \{4, 6\}$. Further, suppose that x is the element of order 2 in M_k. Then the graph induced by the set $\mathcal{G}_{\mathcal{M}(G)} \cup \{x\}$ contains $K_{4,5}$ as a subgraph. Consequently, by Theorem 1.2.6, we obtain $\gamma(\overline{\mathcal{P}_E(G^*)}) \geq 2$ and $\overline{\gamma}(\overline{\mathcal{P}_E(G^*)}) \geq 3$. For distinct $i, j, k \in \{1, 2, 3\}$, let $m_i = 4$, $m_j = 6$ and $m_k = t$, where $t \in \{4, 6\}$. Further, suppose that x is an element of order 3 in M_j. Then the subgraph of $\overline{\mathcal{P}_E(G^*)}$ induced by the set $\mathcal{G}_{\mathcal{M}(G)} \cup \{x\}$ contains a subgraph isomorphic to $K_{4,5}$. Consequently, by Theorem 1.2.6, we obtain $\gamma(\overline{\mathcal{P}_E(G^*)}) \geq 2$ and $\overline{\gamma}(\overline{\mathcal{P}_E(G^*)}) \geq 3$.
If $m_1 = m_2 = m_3 = 4$, then $|G| \in \{10, 11, 12\}$ (see Remark 1.1.4). The existence of a subgroup of order 4 implies that $|G| = 12$. By Table 1.1, no such group exists. If

$m_1 = m_2 = m_3 = 6$, then the subgraph induced by the set $\mathcal{G}_{\mathcal{M}(G)} \cup \{x, y\}$ contains $K_{4,5}$ as a subgraph, where x and y are elements of order 2 and 3, respectively, in M_1.

Subcase-3.5: $\phi(m_1) \geq 2$, $\phi(m_2) \geq 2$, $\phi(m_3) \geq 2$, $\phi(m_4) \geq 2$ and $\phi(m_5) \geq 1$. It follows that the subgraph of $\overline{\mathcal{P}_E(G^*)}$ induced by the set $\mathcal{G}_{\mathcal{M}(G)}$ contains a subgraph isomorphic to $K_{4,5}$. Consequently, by Theorem 1.2.6, we obtain $\gamma(\overline{\mathcal{P}_E(G^*)}) \geq 2$ and $\overline{\gamma}(\overline{\mathcal{P}_E(G^*)}) \geq 3$.

Subcase-3.6: $\phi(m_1) \geq 4$ and $\phi(m_i) = 1$ for every $i \in \{2, 3, 4, 5\}$. By using the similar argument given in **Subcase-1.5** of Theorem 3.1.11, no such group exists.

Subcase-3.7: $\phi(m_1) \geq 4$, $\phi(m_2) \geq 2$ and $\phi(m_i) \geq 1$ for every $i \in \{3, 4, 5\}$. By the similar argument used in **Subcase-3.5**, we get $\gamma(\overline{\mathcal{P}_E(G^*)}) \geq 2$ and $\overline{\gamma}(\overline{\mathcal{P}_E(G^*)}) \geq 3$.

Case-4: $|\mathcal{M}(G)| = 6$. Let M_1, M_2, M_3, M_4, M_5 and M_6 be the maximal cyclic subgroups of G such that $|M_i| = m_i$ for $i \in \{1, 2, 3, 4, 5, 6\}$. Without loss of generality, assume that $\phi(m_1) \geq \phi(m_2) \geq \phi(m_3) \geq \phi(m_4) \geq \phi(m_5) \geq \phi(m_6)$. Now, we have the following subcases:

Subcase-4.1: $\phi(m_i) = 1$ for each $i \in \{1, 2, 3, 4, 5, 6\}$. It follows that $|G| = 7$ which is not possible because $|M_i| = 2$.

Subcase-4.2: $\phi(m_1) = 2$, $\phi(m_i) = 1$ for each $i \in \{2, 3, 4, 5, 6\}$. Then $m_1 \in \{3, 4, 6\}$. Consequently, by Remark 1.1.4, $|G| \in \{8, 9, 11\}$. Note that $|G| = 8$ if $m_1 = 3$ but 3 does not divide 8, $|G| = 9$ if $m_1 = 4$ but 4 does not divide 9 and $|G| = 11$ if $m_1 = 6$ but 6 does not divide 11. Thus, no such group exists.

Subcase-4.3: $\phi(m_1) = \phi(m_2) = 2$ and $\phi(m_i) = 1$ for every $i \in \{3, 4, 5, 6\}$. It follows that $m_1, m_2 \in \{3, 4, 6\}$ and $m_i = 2$. If $m_1 = m_2 = 3$, then by Remark 1.1.4, we get $|G| = 9$. No such groups exist because we have a maximal cyclic subgroup of order 2. For distinct $i, j \in \{1, 2\}$, if $m_i \geq 3$ and $m_j = t$, where $t \in \{4, 6\}$, then the subgraph of $\overline{\mathcal{P}_E(G^*)}$ induced by the set $\mathcal{G}_{\mathcal{M}(G)} \cup \{x\}$, where x is the elements of order 2 in M_j, contains a subgraph isomorphic to $K_{4,5}$. Consequently, we get

$\gamma(\overline{\mathcal{P}_E(G^*)}) \geq 2$ and $\overline{\gamma}(\overline{\mathcal{P}_E(G^*)}) \geq 3$.

Subcase-4.4: $\phi(m_1) \geq 2,\ \phi(m_2) \geq 2,\ \phi(m_3) \geq 2$ and $\phi(m_i) \geq 1$ for every $i \in \{4, 5, 6\}$. By the similar argument given in **Subcase-3.5**, $\gamma(\overline{\mathcal{P}_E(G^*)}) \geq 2$ and $\overline{\gamma}(\overline{\mathcal{P}_E(G^*)}) \geq 3$.

Subcase-4.5: $\phi(m_1) \geq 4,\ \phi(m_i) \geq 1$ for every $i \in \{2, 3, 4, 5, 6\}$. By the similar argument given in **Subcase-3.5**, $\gamma(\overline{\mathcal{P}_E(G^*)}) \geq 2$ and $\overline{\gamma}(\overline{\mathcal{P}_E(G^*)}) \geq 3$.

Case-5: $|\mathcal{M}(G)| = 7$. Now, we have the following subcases:

Subcase-5.1: $\phi(m_i) = 1$ for every $i \in \{1, 2, 3, 4, 5, 6, 7\}$. Then by Remark 1.1.4, $|G| = 8$. By Table 1.1, $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Subcase-5.2: $\phi(m_1) = 2,\ \phi(m_i) = 1$ for every $i \in \{2, 3, 4, 5, 6, 7\}$. It follows that $m_1 \in \{3, 4, 6\}$ and $m_i = 2$. Consequently, by Remark 1.1.4, $|G| \in \{9, 10, 12\}$. Note that $|G| = 9$ when $m_1 = 3$. But the existence of a subgroup of order 2 implies $|G| \neq 9$. Also, $|G| = 10$ when $m_1 = 4$. But this is not possible because of the existence of a subgroup of order 4. If $|G| = 12$, then by Table 1.1, we get $G \cong D_{12}$.

Subcase-5.3: $\phi(m_1) \geq 2,\ \phi(m_2) \geq 2$, and $\phi(m_i) \geq 1$ for every $i \in \{3, 4, 5, 6, 7\}$. By the similar argument used in **Subcase-3.5**, $\gamma(\overline{\mathcal{P}_E(G^*)}) \geq 2$ and $\overline{\gamma}(\overline{\mathcal{P}_E(G^*)}) \geq 3$.

Subcase-5.4: $\phi(m_1) \geq 4,\ \phi(m_i) \geq 1$ for every $i \in \{2, 3, 4, 5, 6, 7\}$. Similar to the **Subcase-3.5**, we obtain $\gamma(\overline{\mathcal{P}_E(G^*)}) \geq 2$ and $\overline{\gamma}(\overline{\mathcal{P}_E(G^*)}) \geq 3$.

Case-6: $|\mathcal{M}(G)| \geq 8$. In this case $\overline{\mathcal{P}_E(G^*)}$ contains a subgraph isomorphic to K_8. Thus, $\gamma(\overline{\mathcal{P}_E(G^*)}) \geq 2$ and $\overline{\gamma}(\overline{\mathcal{P}_E(G^*)}) \geq 4$.

Conversely, if $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ then $\overline{\mathcal{P}_E(G^*)} \cong K_3$. Thus, $\overline{\mathcal{P}_E(G^*)}$ is an outerplanar graph.

If $G \cong S_3$, then $\overline{\mathcal{P}_E(G^*)}$ contains a subgraph isomorphic to K_4. Consequently, $\overline{\mathcal{P}_E(G^*)}$ is not an outerplanar graph. The graph $\overline{\mathcal{P}_E(S_3^*)}$ is planar (see Figure 3.1).

If $G \cong Q_8$, then note that the subgraph induced by the set $\{i, -i, j, -j, k\}$ contains $K_{2,3}$ as a subgraph. It follows that $\overline{\mathcal{P}_E(G^*)}$ is not an outerplanar graph. Moreover, the graph $\overline{\mathcal{P}_E(Q_8^*)}$ is planar (cf. Figure 3.2).

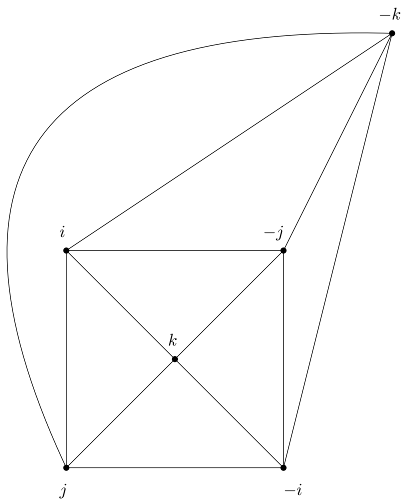


FIGURE 3.2: A planar drawing of $\overline{\mathcal{P}_E(Q_8^*)}$.

If $G \cong \mathbb{Z}_2 \times \mathbb{Z}_4$, then $\overline{\mathcal{P}_E(G^*)}$ contains a subgraph isomorphic to $K_{3,3}$. Consequently, $\overline{\mathcal{P}_E(G^*)}$ is not a planar graph. A toroidal and projective embedding of $\overline{\mathcal{P}_E(G^*)}$ are given in Figures 3.3 and 3.4, respectively.

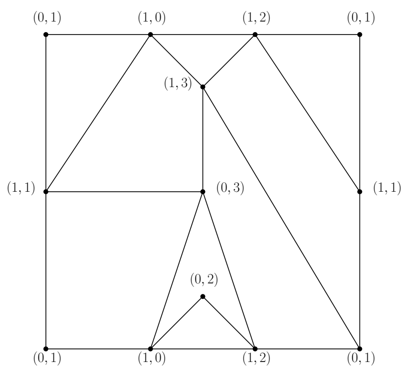


FIGURE 3.3: A toroidal embedding of $\overline{\mathcal{P}_E((\mathbb{Z}_2 \times \mathbb{Z}_4)^*)}$.

If $G \cong D_8$, then notice that the subgraph of $\overline{\mathcal{P}_E(G^*)}$ induced by the set $\{x, y, xy, x^2y, x^3y\}$ contains K_5 as a subgraph. Consequently, $\overline{\mathcal{P}_E(G^*)}$ is not a planar graph. A toroidal and projective embedding of $\overline{\mathcal{P}_E(G^*)}$ are in Figure 3.5 and Figure 3.6, respectively.

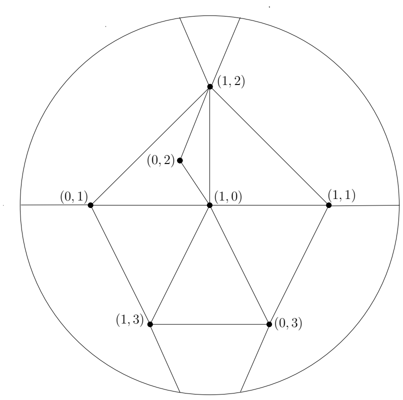


FIGURE 3.4: A projective embedding of $\overline{\mathcal{P}_E((\mathbb{Z}_2 \times \mathbb{Z}_4)^*)}$.

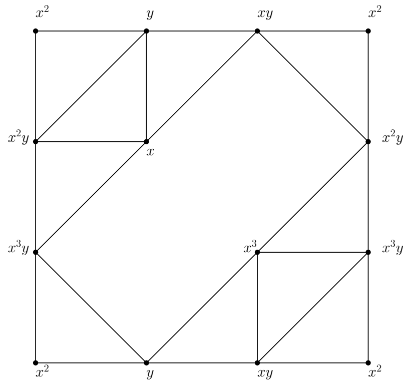


FIGURE 3.5: A toroidal embedding of $\overline{\mathcal{P}_E(D_8^*)}$.

If $G \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ then $\overline{\mathcal{P}_E(G^*)} \cong K_{2,2,2,2}$ and it contains a subgraph isomorphic to $K_{4,4}$. Consequently, $\gamma(\overline{\mathcal{P}_E(G^*)}) \geq 1$ and by Jungerman [1979], we obtain $\overline{\gamma}(\overline{\mathcal{P}_E(G^*)}) = 3$. A toroidal embedding of $\overline{\mathcal{P}_E(G^*)}$ is given in Figure 3.7.

If $G \cong \mathbb{Z}_2 \times \mathbb{Z}_6$ then $\overline{\mathcal{P}_E(G^*)} \cong K_{3,3,3}$. By Section 2 of Ringel and Youngs [1970], we get $\gamma(\overline{\mathcal{P}_E(G^*)}) = 1$ and by Theorem 10 of Ellingham et al. [2006], we get $\overline{\gamma}(\overline{\mathcal{P}_E(G^*)}) = 3$.

If $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, then $\overline{\mathcal{P}_E(G^*)} \cong K_7$. Consequently, by Theorem 1.2.6, we have $\overline{\gamma}(\overline{\mathcal{P}_E(G^*)}) = 3$ and $\gamma(\overline{\mathcal{P}_E(G^*)}) = 1$. If $G \cong D_{12}$, then $\overline{\mathcal{P}_E(G^*)}$ contains a subgraph isomorphic to $K_{5,6}$. Consequently, by Theorem 1.2.6, we get $\gamma(\overline{\mathcal{P}_E(G^*)}) \geq 3$ and $\overline{\gamma}(\overline{\mathcal{P}_E(G^*)}) \geq 6$. □

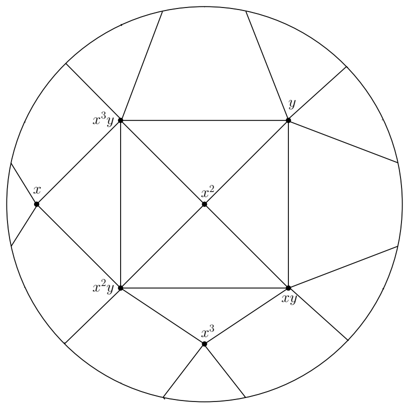


Figure 3.6: A projective embedding of $\overline{\mathcal{P}_E(D_8^*)}$.

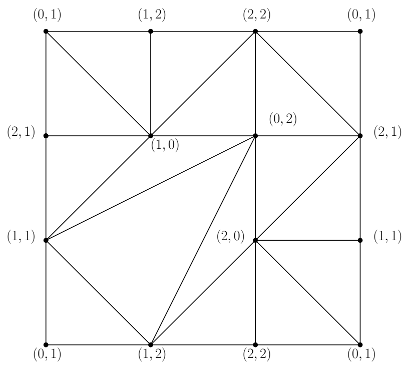


Figure 3.7: A toroidal embedding of $\overline{\mathcal{P}_E((\mathbb{Z}_3 \times \mathbb{Z}_3)^*)}$.

Chapter 4

Line Graph Characterization of Power Graphs

Bera [2022] characterized finite nilpotent groups whose power graphs and proper power graphs are line graphs. The proper enhanced power graph $\mathcal{P}_E^{**}(G)$ is the graph obtained from $\mathcal{P}_E(G)$ after deleting its dominating vertices. Analogously, we can define the proper power graph $\mathcal{P}^{**}(G)$. In this chapter, we extend the results of Bera [2022] to arbitrary finite groups. Also, we correct his corresponding result of the proper power graphs of dihedral groups. Moreover, we classify all the finite groups whose enhanced power graphs are line graphs. We precisely characterize all the finite nilpotent groups (except non-abelian 2-groups) whose proper enhanced power graphs are line graphs of some graphs. Finally, we determine all the finite groups whose (proper) power graphs and (proper) enhanced power graphs are the complement of line graphs, respectively. The content of this chapter is accepted for publication in SCIE journal *"Journal of Algebra and Its Applications"*.

Recall that the *line graph* $\mathcal{L}(\Gamma)$ of the graph Γ is the graph whose vertices are all the edges of Γ and two vertices of $\mathcal{L}(\Gamma)$ are adjacent if they are incident in Γ. An example of $\mathcal{L}(\Gamma)$ of the graph Γ is shown in Figure 4.1. Line graphs are described by

FIGURE 4.1: Line graph of a graph Γ.

nine forbidden subgraphs. A characterization of the line graph and its complement is given in the following two theorems. We shall use them frequently in this chapter.

Theorem 4.0.1 ([Beineke, 1970, p. 130, Theorem 1]). *Let Γ be a graph. Then Γ is the line graph of some graph if and only if none of the nine graphs in* Figure 4.2 *is an induced subgraph of Γ.*

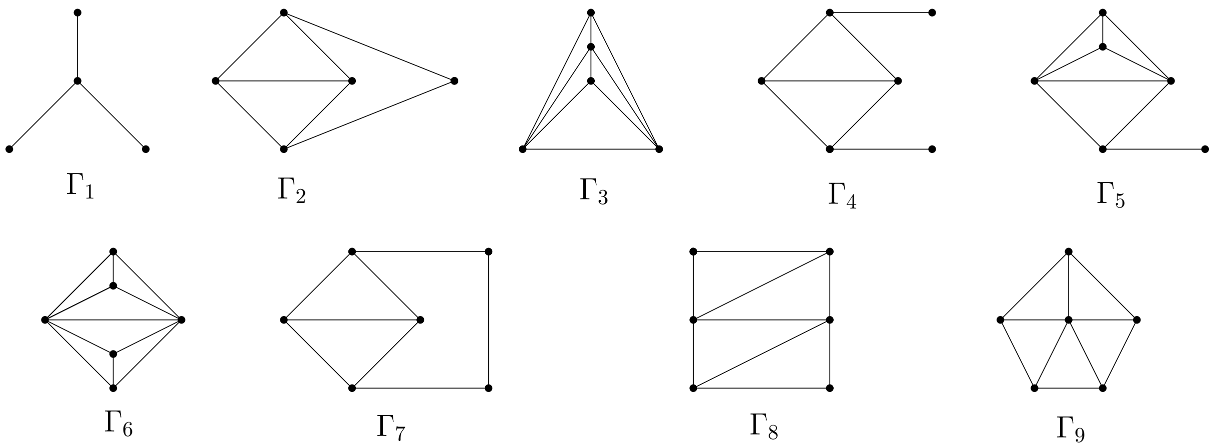


FIGURE 4.2: Forbidden induced subgraphs of line graphs.

Theorem 4.0.2 ([Barati, 2021, Theorem 3.1]). *A graph Γ is the complement of a line graph if and only if none of the nine graphs $\overline{\Gamma_i}$ in* Figure 4.3 *is an induced subgraph of Γ.*

We shall use Γ_i (or $\overline{\Gamma_i}$), where $1 \leq i \leq 9$, explicitly in this chapter without referring to it.

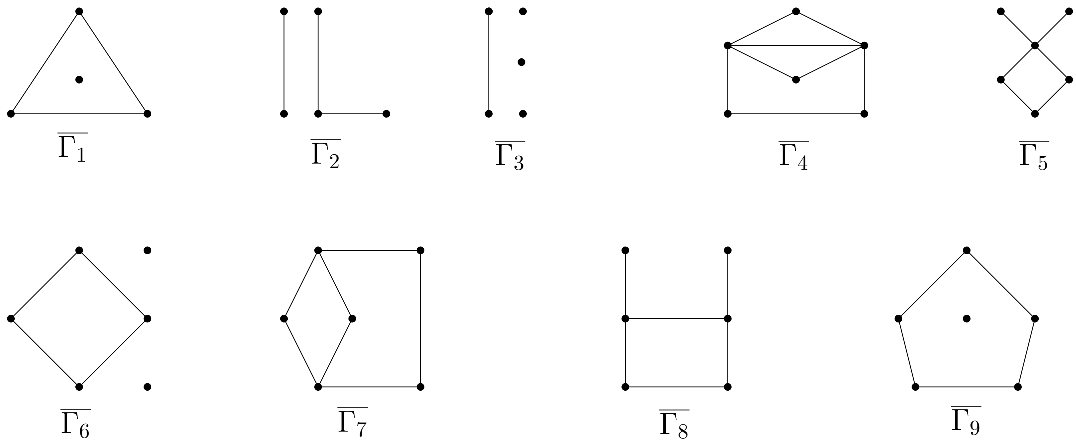


FIGURE 4.3: Forbidden induced subgraphs of the complement of line graphs.

Recently, Bera [2022] classified all the finite nilpotent groups whose power graphs and proper power graphs are line graphs in the following theorems.

Theorem 4.0.3 ([Bera, 2022, Theorem 1.7])**.** *Let G be a nilpotent group. Then the power graph $\mathcal{P}(G)$ is a line graph of some graph Γ if and only if G is cyclic p-group.*

A line graph characterization of the proper power graph $\mathcal{P}^{**}(G)$ has been obtained in Theorem 1.8 of Bera [2022], which can be rewritten in the following form to avoid any ambiguity.

Theorem 4.0.4. *Let G be a nilpotent group (except non-abelian 2-groups). Then the proper power graph $\mathcal{P}^{**}(G)$ is a line graph of some graph Γ if and only if one of the following holds:*

(i) $G \cong \mathbb{Z}_{p^t}, t \geq 1$

(ii) $G \cong \mathbb{Z}_{pq}$

(iii) $G \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^2}$

(iv) $G \cong \mathbb{Z}_{2^2} \times \mathbb{Z}_{2^2}$

(v) $G \cong \underbrace{\mathbb{Z}_p \times \cdots \times \mathbb{Z}_p}_{k \text{ times}, \, k \geq 2}$

(vi) *G is a non-abelian p-group and the intersection of any two maximal cyclic subgroups of G is trivial.*

Theorem 4.0.5 ([Bera, 2022, Theorem 2.1]). *Let G be a finite cyclic group. Then there exists a graph Γ such that $\mathcal{P}(G) = \mathcal{L}(\Gamma)$ if and only if G is a p-group.*

Motivated by the work of Bera [2022], in this chapter, we intend to study the line graphs of certain power graphs associated to finite groups, viz: power graph, proper power graph, enhanced power graph, proper enhanced power graph. In order to extend the results of Bera [2022], we study the following problems.

- Classification of finite groups G such that $\Delta(G) \in \{\mathcal{P}(G), \mathcal{P}^{**}(G), \mathcal{P}_E(G), \mathcal{P}_E^{**}(G)\}$ is a line graph.
- Classification of finite groups G such that $\Delta(G) \in \{\mathcal{P}(G), \mathcal{P}^{**}(G), \mathcal{P}_E(G), \mathcal{P}_E^{**}(G)\}$ is the complement of a line graph.

This chapter is arranged as follows. In Section 4.1, we extend the results of Bera [2022] from nilpotent groups to arbitrary finite groups. In Section 4.2, we classify all the finite groups whose enhanced power graphs are line graphs. Moreover, we precisely characterize all the finite nilpotent groups (except non-abelian 2-groups) whose proper enhanced power graphs are line graphs. Section 4.3 comprises the classification of all the finite groups whose (proper) power graphs and (proper) enhanced power graphs are the complement of line graphs, respectively.

In this chapter, the following notations and remark are useful.

Notations:

- $\mathcal{M}^{(p)}(G) = \{M \in \mathcal{M}(G) : M \text{ is a } p\text{-group}\}$; the set of all maximal cyclic subgroups which are p-groups.
- $\mathcal{T}(G) = \bigcap_{M \in \mathcal{M}(G)} M$; the intersection of all the maximal cyclic subgroups of G.

Remark 4.0.6. (i) Let $x, y \in G$. Then $x \sim y$ in $\mathcal{P}_E(G)$ if and only if $x, y \in M$ for some $M \in \mathcal{M}(G)$. Consequently, $\mathrm{Dom}(\mathcal{P}_E(G)) = \mathcal{T}(G)$.
(ii) For $x, y \in G$, we have $x \sim y$ in $\mathcal{P}(G)$ if and only if $x, y \in M$ for some $M \in \mathcal{M}(G)$, and either $o(x)|o(y)$ or $o(y)|o(x)$.
(iii) Let $M \in \mathcal{M}(G)$ such that $M = \langle m \rangle$. Then $\mathrm{N}[m] = M$ in both $\mathcal{P}_E(G)$ and $\mathcal{P}(G)$. Moreover, if G is non-cyclic then $m \notin \mathcal{T}(G)$ and so $m \in V(\mathcal{P}_E^{**}(G))$.
(iv) Let $M, M' \in \mathcal{M}(G)$ such that $M = \langle m \rangle$ and $M' = \langle m' \rangle$. Then $m \nsim m'$ in $\mathcal{P}_E(G)$ and so $m \nsim m'$ in $\Delta(G) \in \{\mathcal{P}(G), \mathcal{P}^{**}(G), \mathcal{P}_E^{**}(G)\}$.

4.1 Groups whose Power Graphs are Line Graphs

In this section, the results of Bera [2022] are extended from nilpotent groups to arbitrary finite groups. We begin with the following proposition.

Proposition 4.1.1. *Let G be a finite non-cyclic group and let $\Delta(G) \in \{\mathcal{P}(G), \mathcal{P}_E(G)\}$. Then there does not exist any graph Γ such that $\Delta(G) = \mathcal{L}(\Gamma)$.*

Proof. Since G is a finite non-cyclic group, by Lemma 1.1.5, we have $|\mathcal{M}(G)| \geq 3$. Let $M_1 = \langle x \rangle$, $M_2 = \langle y \rangle$ and $M_3 = \langle z \rangle$ be maximal cyclic subgroups of G. Then the subgraph induced by the set $\{x, y, z, e\}$ of $\Delta(G)$ is isomorphic to Γ_1 (see Remark 4.0.6(iv)). Consequently, $\Delta(G)$ is not a line graph of any graph (cf. Theorem 4.0.1). □

Theorem 4.1.2. *Let G be a finite group. Then the power graph $\mathcal{P}(G)$ is a line graph of some graph Γ if and only if G is a cyclic group of prime power order.*

Proof. The result holds by combining Theorem 4.0.5 and Proposition 4.1.1. □

Now, we give a description of the group G such that the graph $\mathcal{P}^{**}(G)$ is a line graph in the following theorem.

Theorem 4.1.3. *Let G be a finite non-cyclic group which is not a generalized quaternion group. Then the proper power graph $\mathcal{P}^{**}(G)$ is a line graph if and only if G satisfies the following conditions:*

(i) *For each $M \in \mathcal{M}(G)$, we have $|M| \in \{6, p^{\alpha}\}$ for some prime p.*

(ii) (a) *If $M_i, M_j, M_k \in \mathcal{M}^{(2)}(G)$, then $|M_i \cap M_j| \leq 2$ and $|M_i \cap M_j \cap M_k| = 1$.*
(b) *If $M_s \in \mathcal{M}(G) \setminus \mathcal{M}^{(2)}(G)$ and $M_t \in \mathcal{M}(G)$, then $|M_s \cap M_t| = 1$.*

Proof. Let G be a finite non-cyclic group which is not a generalized quaternion group. Then by Theorem 1.2.13, $V(\mathcal{P}^{**}(G)) = G \setminus \{e\}$. Suppose $\mathcal{P}^{**}(G)$ is a line graph of some graph Γ. On the contrary, assume that G does not satisfy condition (i). Then G has a maximal cyclic subgroup M such that neither $|M| = 6$ nor $|M|$ is a prime power. Let $|M| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} (k \geq 2)$ be the prime power factorization of $|M|$. If $p_i > 3$ for some $i \in [k]$, then M contains at least 4 elements of order p_i and 4 elements of order $p_i p_j$ for some $j \in [k] \setminus \{i\}$. Let $x_1, x_2, x_3, y_1, z_1 \in M$ such that $o(x_1) = o(x_2) = o(x_3) = p_i p_j$, $o(y_1) = p_j$ and $o(z) = p_i$. Then by Remark 4.0.6(ii), the subgraph of $\mathcal{P}^{**}(G)$ induced by the set $\{x_1, x_2, x_3, y_1, z_1\}$ is isomorphic to Γ_3; a contradiction (see Theorem 4.0.1). Thus, $p_i \leq 3$ for all $i \in [k]$. Therefore, $|M| = 2^{\alpha} 3^{\beta}$ for some $\alpha \geq 1, \beta \geq 1$. Since $|M| \neq 6$, we get either $\alpha \geq 2$ or $\beta \geq 2$. First suppose that $\alpha \geq 2$. Let $x_1, x_2, y_1, y_2, z_1, z_2 \in M$ such that $o(x_1) = o(x_2) = 4$, $o(y_1) = o(y_2) = 3$ and $o(z_1) = o(z_2) = 12$. Observe that the subgraph of $\mathcal{P}^{**}(G)$ induced by the set $\{x_1, x_2, y_1, y_2, z_1, z_2\}$ is isomorphic to Γ_6; again a contradiction. Similarly, we get a contradiction for $\beta \geq 2$. Thus, for each $M \in \mathcal{M}(G)$, we obtain $|M| \in \{6, p^{\alpha}\}$.

If possible, assume that G does not satisfy condition (ii). Further, we have the following cases:

Case-1: *G does not satisfy* (ii)(a). Then we have the following two subcases:

Subcase-1.1: *$|M_1 \cap M_2 \cap M_3| \geq 2$ for some $M_1, M_2, M_3 \in \mathcal{M}^{(2)}(G)$.* Let $x\ (\neq e) \in M_1 \cap M_2 \cap M_3$ and $M_i = \langle m_i \rangle$ for each $i \in \{1, 2, 3\}$. Then by Remark 4.0.6(ii), the subgraph of $\mathcal{P}^{**}(G)$ induced by the vertex set $\{x, m_1, m_2, m_3\}$ is isomorphic to Γ_1, which is not possible.

Subcase-1.2: $|M_1 \cap M_2| \geq 3$ *for some* $M_1, M_2 \in \mathcal{M}^{(2)}(G)$. Suppose $x, y \in (M_1 \cap M_2) \setminus \{e\}$. Let $M_1 = \langle m_1 \rangle$ and $M_2 = \langle m_2 \rangle$. By Remark 4.0.6(ii), the subgraph of $\mathcal{P}^{**}(G)$ induced by the set $\left\{x, y, m_1, m_1^{-1}, m_2, m_2^{-1}\right\}$ is isomorphic to Γ_6; a contradiction.

Thus, G must satisfy the condition (ii)(a).

Case-2: *G does not satisfy* (ii)(b). Then there exist two maximal cyclic subgroups $M_1 \in \mathcal{M}(G) \setminus \mathcal{M}^{(2)}(G)$ and $M_2 \in \mathcal{M}(G)$ such that $|M_1 \cap M_2| \geq 2$. In view of the condition (i), we have the following subcases.

Subcase-2.1: $|M_1| = 6$. Then $|M_1 \cap M_2| \in \{2, 3\}$. Let $|M_1 \cap M_2| = 2$, $M_1 = \langle x \rangle$ and $M_2 = \langle y \rangle$. Then $x^3 \in M_1 \cap M_2$ because x^3 is the only element of order 2 in M_1. The subgraph of $\mathcal{P}^{**}(G)$ induced by the set $\{x, x^2, x^3, x^4, x^5, y\}$ is isomorphic to Γ_5; a contradiction. If $|M_1 \cap M_2| = 3$, then $x^2, x^4 \in M_1 \cap M_2$. The subgraph induced by the set $\{x, x^2, x^4, x^5, y, y^{-1}\}$ is isomorphic to Γ_6, which is not possible.

Subcase-2.2: $|M_1| = p^\alpha (p > 2)$. Let $M_1 = \langle x \rangle$, $M_2 = \langle y \rangle$ and let m be a non-identity element of $M_1 \cap M_2$. Then the subgraph of $\mathcal{P}^{**}(G)$ induced by the set $\{x, x^{-1}, y, y^{-1}, m, m^{-1}\}$ is isomorphic to Γ_6; a contradiction.

Conversely, suppose that G satisfies both the given conditions. On the contrary, assume that $\mathcal{P}^{**}(G)$ is not a line graph and so it has an induced subgraph Γ isomorphic to one of the nine graphs given in Figure 4.2. In view of Remark 1.1.4, first we prove the following claim.

Claim 2.3: If $x \in V(\Gamma)$ such that $x \in M$ for some $M \in \mathcal{M}(G)$, then M must be a 2-group.

Proof of claim: If possible, let $x \in M$ and $M \in \mathcal{M}(G) \setminus \mathcal{M}^{(2)}(G)$. Then by condition (ii)(b), we have $M \cap M' = \{e\}$ for all $M' (\neq M) \in \mathcal{M}(G)$. Consequently, by Remark 4.0.6(ii), we get $\mathrm{N}(x) \subseteq M$ in $\mathcal{P}^{**}(G)$. Therefore, if $x \sim y$, then $y \in M$ and

so $\mathrm{N}(y) \subseteq M$. Connectedness of Γ implies that $V(\Gamma) \subseteq M$. For $|M| = p^{\alpha}$, where p is an odd prime, note that the subgraph of $\mathcal{P}^{**}(G)$ induced by any non-empty subset of M is a complete graph. It implies that Γ is a complete subgraph of $\mathcal{P}^{**}(G)$, which is not true because $\Gamma \cong \Gamma_i$ for some i, where $1 \leq i \leq 9$ (see Figure 4.2).
If $|M| = 6$, then $M \cong \mathbb{Z}_6$. Observe that the subgraph of $\mathcal{P}^{**}(G)$ induced by the set $M \setminus \{e\}$, shown in Figure 4.4, cannot contain Γ as an induced subgraph. Thus, the claim holds.

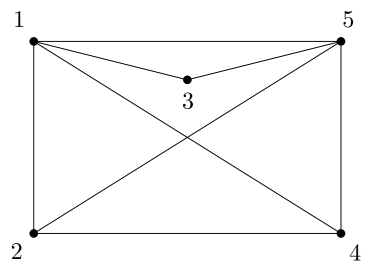


FIGURE 4.4: Subgraph of $\mathcal{P}^{**}(G)$ induced by $M \setminus \{e\}$.

Now, if Γ is isomorphic to $K_{1,3}$, (see Figure 4.5(a)), then by Remark 4.0.6(ii) and Claim 2.3, there exist maximal cyclic subgroups $M_1, M_2, M_3 \in \mathcal{M}^{(2)}(G)$ such that $a, d \in M_1,\ b, d \in M_2$ and $c, d \in M_3$. Note that $M_1 \neq M_2$. Otherwise, $a \sim b$ in $\mathcal{P}^{**}(G)$ (see Remark 4.0.6(ii)), which is not possible. Similarly, $M_2 \neq M_3$ and $M_1 \neq M_3$. Also, $d \in (M_1 \cap M_2 \cap M_3) \setminus \{e\}$; a contradiction to the condition (ii)(a). Thus, Γ cannot be isomorphic to $K_{1,3}$.

Now, suppose $\Gamma \cong \Gamma_i$ for some i, where $2 \leq i \leq 9$. Further, note that Γ has an induced subgraph isomorphic to Γ' as shown in Figure 4.5(b).

FIGURE 4.5: (a) $K_{1,3}$ (b) Γ'.

Since $x \sim y$, $y \sim z$ and $z \sim x$, then by the definition of $\mathcal{P}^{**}(G)$, it is easy to observe that one of the following three holds: (i) $x, y \in \langle z \rangle$, (ii) $y, z \in \langle x \rangle$, (iii) $x, z \in \langle y \rangle$. Consequently, there exists $M \in \mathcal{M}^{(2)}(G)$ such that $x, y, z \in M$. Similarly, there exists $M' \in \mathcal{M}^{(2)}(G)$ such that $y, z, w \in M'$. Notice that $M \neq M'$. Otherwise, $x \sim w$ in $\mathcal{P}^{**}(G)$, which is not possible. But $y, z \in M \cap M'$; a contradiction of condition (ii)(a). Thus, $\mathcal{P}^{**}(G)$ is a line graph. This completes our proof. □

Corollary 4.1.4. *Let G be a finite non-cyclic group of odd order and let $M_i \in \mathcal{M}(G)$. Then the proper power graph $\mathcal{P}^{**}(G)$ is a line graph if and only if G satisfies the following conditions:*

(i) *For each i, we have $|M_i| = p^\alpha$ for some prime p.*

(ii) *The intersection of any two maximal cyclic subgroups is trivial.*

For the dihedral group $D_{2n} = \langle a, b : a^n = b^2 = e, ab = ba^{-1} \rangle$, note that $\mathcal{P}^{**}(D_6)$ is a line graph because $\mathcal{P}^{**}(D_6) = \mathcal{L}(K_{1,2} \cup 3K_2)$. It follows that Theorem 1.10 of Bera [2022] is not correct. Moreover, we correct the same in the following corollary.

Corollary 4.1.5. *Let G be the dihedral group D_{2n} of order $2n$. Then $\mathcal{P}^{**}(G)$ is a line graph of some graph Γ if and only if $n \in \{6, p^\alpha\}$ for some prime p.*

Proof. First assume that $\mathcal{P}^{**}(D_{2n})$ is a line graph. Note that D_{2n} has one maximal cyclic subgroup $M = \langle a \rangle$ of order n, and n maximal cyclic subgroups $M_i = \langle a^i b \rangle$, where $1 \leq i \leq n$, of order 2. Then by Theorem 4.1.3, either $n = 6$ or $n = p^\alpha$ for some prime p. Conversely, if $n \in \{6, p^\alpha\}$, then G satisfies the condition (i). Note that the intersection of any two maximal cyclic subgroups of D_{2n} is trivial. Thus, condition (ii) holds. By Theorem 4.1.3, $\mathcal{P}^{**}(D_{2n})$ is a line graph. □

Corollary 4.1.6. *Let G be the semidihedral group*

$$SD_{8n} = \langle a, b : a^{4n} = b^2 = e, ba = a^{2n-1} b \rangle.$$

Then $\mathcal{P}^{**}(SD_{8n})$ is not a line graph of any graph.

Proof. Consider the maximal cyclic subgroups $M_1 = \langle a \rangle$, $M_2 = \langle ab \rangle$ and $M_3 = \langle a^3 b \rangle$ of SD_{8n}. Note that $M_1 \cap M_2 \cap M_3 = \{e, a^{2n}\}$. By Theorem 4.1.3, $\mathcal{P}^{**}(SD_{8n})$ is not a line graph of any graph. □

Theorem 4.1.7. *Let G be a generalized quaternion group Q_{4n}. Then $\mathcal{P}^{**}(G)$ is a line graph if and only if $n \in \{p, 2^k\}$ for some odd prime p and $k \geq 1$.*

Proof. For the generalized quaternion group $G = Q_{4n}$, we get $V(\mathcal{P}^{**}(G)) = G \setminus Z(G)$. Moreover, G has one maximal cyclic subgroup $M = \langle a \rangle$ of order $2n$ and n maximal cyclic subgroups, $M_i = \langle a^i b \rangle$ for each $i \in [n]$, of order 4. Suppose that $\mathcal{P}^{**}(G)$ is a line graph. If possible, assume that n is divisible by primes p and q such that $p < q$. Let $x_1, x_2, y_1, y_2, z_1, z_2 \in M$ such that $o(x_1) = o(x_2) = 2p$, $o(y_1) = o(y_2) = 2pq$ and $o(z_1) = o(z_2) = q$. By Remark 4.0.6(ii), the subgraph induced by the set $\{x_1, x_2, y_1, y_2, z_1, z_2\}$ is isomorphic to Γ_6; a contradiction. Thus, $n = p^\alpha$. If $p > 2$ and $\alpha \geq 2$, then note that M has at least two elements x, x' of order $2p$, two elements y, y' of order p^2 and two elements z, z' of order $2p^2$. The subgraph induced by the set $\{x, x', y, y', z, z'\}$ is isomorphic to Γ_6, again a contradiction. Thus, either $n = 2^k$ or n is an odd prime. Conversely, if either n is an odd prime or $n = 2^k$, then

$$\mathcal{P}^{**}(Q_{4n}) = K_{2n-2} \cup nK_2 = L(K_{1,2n-2} \cup nK_{1,2}).$$

□

4.2 Groups whose Enhanced Power Graphs are Line Graphs

In this section, we intend to classify all the groups G such that the enhanced power graph $\mathcal{P}_E(G)$ and the proper enhanced power graph $\mathcal{P}_E^{**}(G)$ are line graphs, respectively. First we prove the following lemma.

Lemma 4.2.1. *Let G be a finite cyclic group. Then $\mathcal{P}_E(G)$ is a line graph of some graph.*

Proof. If G is a cyclic group of order n, then $\mathcal{P}_E(G) \cong K_n$ (cf. Theorem 1.2.15). Further, note that $K_n = \mathcal{L}(K_{1,n})$. Thus, for a finite cyclic group G, the enhanced power graph $\mathcal{P}_E(G)$ is the line graph of the star graph $K_{1,n}$. □

Theorem 4.2.2. *Let G be a finite group. Then $\mathcal{P}_E(G)$ is a line graph of some graph Γ if and only if G is a cyclic group.*

Proof. The result holds by Proposition 4.1.1 and Lemma 4.2.1. □

If the group G is cyclic, then $\mathcal{P}_E^{**}(G)$ is an empty graph (cf. Theorem 1.2.15). Consequently, we give a description of finite non-cyclic groups G whose proper enhanced power graphs $\mathcal{P}_E^{**}(G)$ are line graphs (see Theorem 4.2.3).

Theorem 4.2.3. *Let G be a finite non-cyclic group and $\mathcal{T}(G) = \bigcap_{M_i \in \mathcal{M}(G)} M_i$. Then the proper enhanced power graph $\mathcal{P}_E^{**}(G)$ is a line graph of some graph Γ if and only if G satisfies the following conditions:*

(i) $|(M_i \cap M_j) \setminus \mathcal{T}(G)| \leq 1$.

(ii) $|(M_i \cap M_j \cap M_k) \setminus \mathcal{T}(G)| = 0$.

Proof. First, suppose that $\mathcal{P}_E^{**}(G)$ is a line graph of some graph Γ. On the contrary, assume that G does not satisfy condition (i). Then G has two maximal cyclic subgroups M_1 and M_2 such that $|(M_1 \cap M_2) \setminus \mathcal{T}(G)| \geq 2$. Since $e \in \mathcal{T}(G)$, we have $|M_1| \geq 3$ and $|M_2| \geq 3$. Suppose $M_1 = \langle x_1 \rangle = \langle y_1 \rangle$ and $M_2 = \langle x_2 \rangle = \langle y_2 \rangle$. Further, let $x, y \in (M_1 \cap M_2) \setminus \mathcal{T}(G)$. Then the subgraph induced by the set $\{x, y, x_1, y_1, x_2, y_2\}$ is isomorphic to Γ_6 (see Remark 4.0.6(i)); a contradiction. Thus, G must satisfy the condition (i). Now, suppose that G does not satisfy the condition (ii). Then G has three maximal cyclic subgroups M', M'' and M''' such that $|(M' \cap M'' \cap M''') \setminus \mathcal{T}(G)| \geq 1$. Assume that $m \in (M' \cap M'' \cap M''') \setminus \mathcal{T}(G)$. Consider $M' = \langle m' \rangle$, $M'' = \langle m'' \rangle$ and $M''' = \langle m''' \rangle$. Then the subgraph induced by the set $\{m, m', m'', m'''\}$ is isomorphic to Γ_1, which is not possible.

Conversely, suppose that G satisfies (i) and (ii). On the contrary, assume that $\mathcal{P}_E^{**}(G)$ is not a line graph. Then by Theorem 4.0.1, $\mathcal{P}_E^{**}(G)$ has an induced subgraph isomorphic to one of the nine graphs given in Figure 4.2. Let $\mathcal{P}_E^{**}(G)$ has an induced subgraph isomorphic to $K_{1,3}$ given in Figure 4.5. Consequently, $\langle a, d\rangle, \langle b, d\rangle$ and $\langle c, d\rangle$ are cyclic subgroups of G. Let M_1, M_2 and M_3 be maximal cyclic subgroups containing $\langle a, d\rangle, \langle b, d\rangle$ and $\langle c, d\rangle$, respectively. Note that $M_1 \neq M_2$. Otherwise, $a \sim b$ in $\mathcal{P}_E^{**}(G)$. Similarly, $M_2 \neq M_3$ and $M_3 \neq M_1$. Since $d \in V(\mathcal{P}_E^{**}(G))$, we obtain $d \notin \mathcal{T}(G)$. It follows that $d \in (M_1 \cap M_2 \cap M_3) \setminus \mathcal{T}(G)$; a contradiction of condition (ii). Thus, $\mathcal{P}_E^{**}(G)$ cannot contain an induced subgraph isomorphic to $K_{1,3}$. Now, suppose that $\mathcal{P}_E^{**}(G)$ has an induced subgraph isomorphic to one of the remaining eight graphs given in Figure 4.2. Then observe that $\mathcal{P}_E^{**}(G)$ has an induced subgraph isomorphic to Γ' as shown in Figure 4.5. Note that x, y and z belong to a maximal cyclic subgroup of G. On contrary, assume that $x, y, z \notin M$ for any $M \in \mathcal{M}(G)$. Since $x \sim y, y \sim z$ and $z \sim x$ in $\mathcal{P}_E^{**}(G)$, by Remark 4.0.6(i), we have three maximal cyclic subgroups M_4, M_5 and M_6 such that $x, y \in M_4,\ y, z \in M_5$ and $z, x \in M_6$. Thus, $x \in (M_4 \cap M_6) \setminus \mathcal{T}(G)$. If $o(x) \geq 3$, then $x^{-1}\ (\neq x) \in M_4 \cap M_6$. Further, note that $\mathrm{N}(x) = \mathrm{N}(x^{-1})$ and so $x^{-1} \notin \mathcal{T}(G)$. It follows that $x^{-1} \in (M_4 \cap M_6) \setminus \mathcal{T}(G)$; a contradiction of condition (i). Consequently, $o(x) = 2$. Similarly, $o(y) = o(z) = 2$. But M_4 cannot contain two elements of order 2. Thus, $x, y, z \in M'$ for some $M' \in \mathcal{M}(G)$. By the similar argument, we get $y, z, w \in M''$ for some $M'' \in \mathcal{M}(G)$. Note that $M' \neq M''$. Otherwise, $x \sim w$ in $\mathcal{P}_E^{**}(G)$. Also, $y, z \in (M' \cap M'') \setminus \mathcal{T}(G)$; again a contradiction. Thus, $\mathcal{P}_E^{**}(G)$ cannot contain an induced subgraph isomorphic to the graph Γ' (see Figure 4.5). This completes our proof. □

Example 4.2.4. For $n \geq 2$, consider the *semidihedral group*

$$SD_{8n} = \langle a, b : a^{4n} = b^2 = e, ba = a^{2n-1}b\rangle.$$

Since SD_{8n} has a maximal cyclic subgroup $M = \langle a^2 b\rangle$ of order 2, we obtain

$\mathcal{T}(SD_{8n}) = \{e\}$. Consider the maximal cyclic subgroups $M_1 = \langle a \rangle$, $M_2 = \langle ab \rangle$ and $M_3 = \langle a^3 b \rangle$ of SD_{8n}. Then note that $M_1 \cap M_2 \cap M_3 = \{e, a^{2n}\}$. Thus, SD_{8n} does not satisfy the condition (ii) of Theorem 4.2.3, and so $\mathcal{P}_E^{**}(SD_{8n})$ is not a line graph of any graph.

Corollary 4.2.5. *Let G be a finite non-cyclic group such that the intersection of any two maximal cyclic subgroups is equal to $\mathcal{T}(G)$. Then $\mathcal{P}_E^{**}(G) = \mathcal{L}(\Gamma)$ for some graph Γ.*

Example 4.2.6. For $n \geq 2$, the *generalized quaternion group*

$$Q_{4n} = \langle a, b : a^{2n} = e, a^n = b^2, ab = ba^{-1} \rangle.$$

Observe that Q_{4n} has $n+1$ maximal cyclic subgroup, viz. $M = \langle a \rangle$ of order $2n$ and for each $i \in [n]$, $M_i = \langle a^i b \rangle$ of order 4. One can see that the intersection of any two maximal cyclic subgroups of Q_{4n} is $\{e, a^n\}$ and so $\mathcal{T}(Q_{4n}) = \{e, a^n\}$. Consequently, $P_E^{**}(Q_{4n})$ is a line graph of some graph Γ. Indeed,

$$P_E^{**}(Q_{4n}) = K_{2n-2} \cup nK_2 = \mathcal{L}(K_{1,2n-2} \cup nK_{1,2}).$$

Example 4.2.7. For $n \geq 3$, consider the *dihedral group*

$$D_{2n} = \langle a, b : a^n = b^2 = e, ab = ba^{-1} \rangle.$$

One can observe that D_{2n} has $n+1$ maximal cyclic subgroups, viz. $M = \langle a \rangle$ of order n and for each $i \in [n]$, $M_i = \langle a^i b \rangle$ of order 2. Consequently, the intersection of any two maximal cyclic subgroups of D_{2n} is trivial. It follows that $\mathcal{T}(D_{2n}) = \{e\}$. Thus, by Corollary 4.2.5, $P_E^{**}(D_{2n})$ is a line graph of some graph Γ. In fact, $P_E^{**}(D_{2n}) = K_{n-1} \cup nK_1 = \mathcal{L}(K_{1,n-1} \cup nK_2)$.

The converse of Corollary 4.2.5 need not be true in general. For instance, if $G = \mathbb{Z}_2 \times \mathbb{Z}_{2^2}$, then $\mathcal{P}_E^{**}(G)$ is a line graph of some graph. Also, note that $M_1 = \langle (\overline{0}, \overline{1}) \rangle$ and $M_2 = \langle (\overline{1}, \overline{1}) \rangle$ are maximal cyclic subgroups of G such that $M_1 \cap M_2 \neq \{(\overline{0}, \overline{0})\}$. Whereas, $\mathcal{T}(\mathbb{Z}_2 \times \mathbb{Z}_{2^2}) = \{(\overline{0}, \overline{0})\}$.

However, the converse of Corollary 4.2.5 is true when G is of odd order.

Corollary 4.2.8. *Let G be a finite group of odd order. Then the proper enhanced power graph $\mathcal{P}_E^{**}(G)$ is a line graph of some graph Γ if and only if the intersection of any two maximal cyclic subgroups is equal to $\mathcal{T}(G)$.*

Proof. Suppose that $\mathcal{P}_E^{**}(G)$ is a line graph of some graph Γ. On the contrary, assume that there exist two maximal cyclic subgroups M_1 and M_2 such that $x \in (M_1 \cap M_2) \setminus \mathcal{T}(G)$. Since $M_1 \cap M_2$ is a subgroup of G, we obtain $x^{-1} \in M_1 \cap M_2$. Also, $\mathrm{N}(x) = \mathrm{N}(x^{-1})$. It follows that $x^{-1} \in (M_1 \cap M_2) \setminus \mathcal{T}(G)$; a contradiction (see Theorem 4.2.3). □

Theorem 4.2.9. *Let G be a finite non-cyclic nilpotent group (except non-abelian 2-groups). Then the proper enhanced power graph $\mathcal{P}_E^{**}(G)$ is a line graph of some graph Γ if and only if G is isomorphic to one of the following groups.*

(i) $\mathbb{Z}_2 \times \mathbb{Z}_{2^2}$

(ii) $\mathbb{Z}_{2^2} \times \mathbb{Z}_{2^2}$

(iii) $\mathbb{Z}_n \times \mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$, *where p is a prime and* $\gcd(n, p) = 1$

(iv) $\mathbb{Z}_n \times Q_{2^k}$ *such that* $\gcd(2, n) = 1$

(v) $\mathbb{Z}_n \times P$ *such that P is a non-abelian p-group with* $\gcd(n, p) = 1$ *and the intersection of any two maximal cyclic subgroups of P is trivial.*

In order to prove Theorem 4.2.9, first we prove some necessary results.

Lemma 4.2.10. *Let G be a finite non-cyclic nilpotent group. If the proper enhanced power graph $\mathcal{P}_E^{**}(G)$ is a line graph, then there exists a unique non-cyclic Sylow subgroup of G.*

Proof. Let $G = P_1P_2 \cdots P_r$ be a finite nilpotent group such that P_i's are Sylow p_i-subgroups of G. On the contrary, assume that G has two Sylow subgroups which are non-cyclic. Without loss of generality, suppose that P_1 and P_2 are non-cyclic. By Lemma 1.1.5, it implies that $|\mathcal{M}(P_i)| \geq 3$ for every $i \in \{1, 2\}$. Consider $M_1, M_1', M_1'' \in \mathcal{M}(P_1)$ such that $M_1 = \langle x_1 \rangle$, $M_1' = \langle y_1 \rangle$ and $M_1'' = \langle z_1 \rangle$. Since G is non-cyclic, we get $M, M', M'' \in \mathcal{M}(G)$, where $M = M_1 M_2 \cdots M_r$, $M' = M_1' M_2 \cdots M_r$ and $M'' = M_1'' M_2 \cdots M_r$ (cf. Lemma 1.1.12). Assume that $M_i = \langle x_i \rangle$ for $i \in [r] \setminus \{1\}$. Note that $M = \langle x \rangle$, $M' = \langle y \rangle$ and $M'' = \langle z \rangle$, where $x = x_1 x_2 \cdots x_r$, $y = y_1 x_2 \cdots x_r$ and $z = z_1 x_2 \cdots x_r$. By Remark 4.0.6, $x \nsim y$, $y \nsim z$ and $z \nsim x$. Therefore, $x, y, z \in V(\mathcal{P}_E^{**}(G))$. Now, consider $t = e x_2 \cdots x_r$ and $t' = e x_2' x_3 \cdots x_r$ such that $\langle x_2 \rangle \in \mathcal{M}(P_2)$ and $\langle x_2' \rangle \neq \langle x_2 \rangle$. By Remark 4.0.6(iv), we have $x_2 \nsim x_2'$ in $\mathcal{P}_E(P_2)$ and so $t' \nsim t$ in $\mathcal{P}_E(G)$ (cf. Theorem 2.1.13). Also, $x \sim t$, $y \sim t$ and $z \sim t$ in $\mathcal{P}_E(G)$ and so in $\mathcal{P}_E^{**}(G)$. Thus, the subgraph induced by the set $\{x, y, z, t\}$ is isomorphic to Γ_1 (see Figure 4.2); a contradiction. Thus, the result holds. □

Proposition 4.2.11. *Let G be a finite non-cyclic abelian group. Then the proper enhanced power graph $\mathcal{P}_E^{**}(G)$ is a line graph of some graph Γ if and only if G is isomorphic to one of the following groups:*

(i) $\mathbb{Z}_2 \times \mathbb{Z}_{2^2}$

(ii) $\mathbb{Z}_{2^2} \times \mathbb{Z}_{2^2}$

(iii) $\mathbb{Z}_n \times \mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$, *where p is a prime and* $\gcd(n, p) = 1$.

Proof. Let G be a finite non-cyclic abelian group. Then $G \cong P_1 \times P_2 \times \cdots \times P_r$, where P_i's are Sylow p_i-subgroups of G. Suppose that $\mathcal{P}_E^{**}(G)$ is a line graph. Then by Lemma 4.2.10, G has a unique non-cyclic Sylow subgroup. Consequently, the group $G \cong \mathbb{Z}_n \times P$, where P is a non-cyclic abelian Sylow p-subgroup of G and $\gcd(n, p) = 1$. Then by Theorem 1.2.17, $V(\mathcal{P}_E^{**}(G)) = G \setminus \{(a, e) : a \in \mathbb{Z}_n\}$, where e is the identity element of the group P. Observe that $\mathcal{P}_E(P)$ is an induced subgraph of $\mathcal{P}_E(G)$. Also, $\mathcal{P}_E(P) = \mathcal{P}(P)$ (see Theorem 1.2.12). By Proposition 3.5

of Bera [2022], if $P \ncong \mathbb{Z}_2 \times \mathbb{Z}_{2^2},\ \mathbb{Z}_{2^2} \times \mathbb{Z}_{2^2},\ \mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$, then $\mathcal{P}^{**}(P)$ has an induced subgraph isomorphic to Γ_1 and so $\mathcal{P}_E^{**}(G)$ contains an induced subgraph isomorphic to Γ_1; a contradiction. Thus, P is isomorphic to one of the three groups: $\mathbb{Z}_2 \times \mathbb{Z}_{2^2},\ \mathbb{Z}_{2^2} \times \mathbb{Z}_{2^2},\ \mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$. Suppose $P = \mathbb{Z}_2 \times \mathbb{Z}_{2^2}$. Then the group $G \cong \mathbb{Z}_n \times \mathbb{Z}_2 \times \mathbb{Z}_{2^2}$. If $n > 1$, then notice that $\mathcal{T}(G) = \{(a, \overline{0}, \overline{0}) : a \in \mathbb{Z}_n\}$ (see Theorem 1.2.17 and Remark 4.0.6(i)). Also, G has maximal cyclic subgroups $M_1 = \langle(\overline{1}, \overline{0}, \overline{1})\rangle$ and $M_2 = \langle(\overline{1}, \overline{1}, \overline{1})\rangle$ of order $4n$. Observe that the elements $(\overline{1}, \overline{0}, \overline{2}), (\overline{2}, \overline{0}, \overline{2}) \in (M_1 \cap M_2) \setminus \mathcal{T}(G)$, a contradiction to Theorem 4.2.3. Thus, $n = 1$.

Now, suppose $P \cong \mathbb{Z}_{2^2} \times \mathbb{Z}_{2^2}$. Then $G \cong \mathbb{Z}_n \times \mathbb{Z}_{2^2} \times \mathbb{Z}_{2^2}$. If $n > 1$, then observe that $\mathcal{T}(G) = \{(a, \overline{0}, \overline{0}) : a \in \mathbb{Z}_n\}$ (see Theorem 1.2.17 and Remark 4.0.6(i)). Also, G has maximal cyclic subgroups $M_3 = \langle(\overline{1}, \overline{1}, \overline{0})\rangle$ and $M_4 = \langle(\overline{1}, \overline{1}, \overline{2})\rangle$ of order $4n$. Notice that $(\overline{1}, \overline{2}, \overline{0}), (\overline{2}, \overline{2}, \overline{0}) \in (M_3 \cap M_4) \setminus \mathcal{T}(G)$, again a contradiction to Theorem 4.2.3.

Conversely, if either $G \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^2}$ or $G \cong \mathbb{Z}_{2^2} \times \mathbb{Z}_{2^2}$, then $\mathcal{P}_E^{**}(G) = \mathcal{P}^{**}(G)$. By Theorem 3.4 of Bera [2022], we obtain $\mathcal{P}_E^{**}(G) = \mathcal{L}(\Gamma)$ for some graph Γ. Now, suppose $G \cong \mathbb{Z}_n \times \mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$ (k-times), where $k \geq 2$. Then

$$\mathcal{P}_E^{**}(G) = \frac{p^k - 1}{p - 1} K_{(p-1)n} = L\left(\frac{p^k - 1}{p - 1} K_{1,(p-1)n}\right).$$

This completes our proof. □

Proposition 4.2.12. *Let G be a finite non-abelian nilpotent group (except non-abelian 2-group). Then $\mathcal{P}_E^{**}(G)$ is a line graph of some graph Γ if and only if G is isomorphic to one of the following groups:*

(i) $\mathbb{Z}_n \times Q_{2^k}$ *such that* $\gcd(2, n) = 1$.

(ii) $\mathbb{Z}_n \times P$ *such that P is a non-abelian p-group with* $\gcd(n, p) = 1$ *and the intersection of any two maximal cyclic subgroups of P is trivial.*

Proof. Let $G = P_1 \times P_2 \times \cdots \times P_r$ be a finite non-abelian nilpotent group which is not a 2-group. Suppose that $\mathcal{P}_E^{**}(G)$ is a line graph. By Lemma 4.2.10, exactly one P_i is non-cyclic. Consequently, $G \cong \mathbb{Z}_n \times P$ such that P is a non-abelian p-group and $\gcd(n, p) = 1$. If $P = Q_{2^k}$, then there is nothing to prove. We may now suppose that P is not a generalized quaternion 2-group and $n > 1$. On the contrary, assume that P has two maximal cyclic subgroups M' and M'' such that $x\ (\neq e) \in M' \cap M''$. Consequently, G has two maximal cyclic subgroup $M_1 = \mathbb{Z}_n \times M'$ and $M_2 = \mathbb{Z}_n \times M''$ (see Lemma 1.1.12) such that $(\overline{1}, x), (\overline{2}, x) \in M' \cap M''$. Since $\mathcal{T}(G) = \{(a, e) : a \in \mathbb{Z}_n\}$ (see Theorem 1.2.17 and Remark 4.0.6(i)), we get a contradiction of Theorem 4.2.3. Thus, G is isomorphic to the group described in (ii). Now, suppose $n = 1$. Then $G = \mathbb{Z}_1 \times P$ is a p-group, where p is an odd prime. Then by Corollary 4.2.8, the intersection of any two maximal cyclic subgroups of G is equal to $\mathcal{T}(G)$. By Theorem 1.2.17 and Remark 4.0.6(i), we get $\mathcal{T}(G) = \{e\}$. Thus, G is isomorphic to the groups described in (ii).

Conversely, suppose that $G \cong \mathbb{Z}_n \times Q_{2^k}$ such that $\gcd(2, n) = 1$. It is not difficult to observe that the intersection of any two maximal cyclic subgroups of Q_{2^k} is the center $Z(Q_{2^k})$. Consequently, the intersection of any two maximal cyclic subgroups of G is the set $\{(a, b) : a \in \mathbb{Z}_n, b \in Z(Q_{2^k})\}$ (see Lemma 1.1.12). Indeed, $\mathcal{T}(G) = \{(a, b) : a \in \mathbb{Z}_n, b \in Z(Q_{2^k})\}$. By Corollary 4.2.5, $\mathcal{P}_E^{**}(G)$ is a line graph of some graph Γ. If $G \cong \mathbb{Z}_n \times P$, where P is a non-abelian p-group such that $\gcd(n, p) = 1$ and the intersection of any two maximal cyclic subgroups of P is trivial, then by Lemma 1.1.12, the intersection of any two maximal cyclic subgroups of G is the set $\{(a, e) : a \in \mathbb{Z}_n\}$. Moreover, $\mathcal{T}(G) = \{(a, e) : a \in \mathbb{Z}_n\}$. By Corollary 4.2.5, $\mathcal{P}_E^{**}(G)$ is a line graph of some graph Γ. □

Proof of Theorem 4.2.9: The result holds by Propositions 4.2.11 and 4.2.12.

4.3 Groups whose Power Graphs and Enhanced Power Graphs are the Complement of Line Graphs

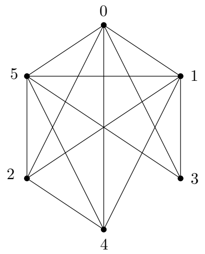
In this section, we classify all the groups G such that the graph $\Delta(G) \in \{\mathcal{P}_E(G), \mathcal{P}(G), \mathcal{P}_E^{**}(G), \mathcal{P}^{**}(G)\}$ is the complement of the line graph of some graph. In this connection, Theorem 4.0.2 will be useful. We begin with the following proposition.

Proposition 4.3.1. *Let G be a finite cyclic group. Then the power graph $\mathcal{P}(G)$ is the complement of a line graph of some graph Γ if and only if either $G \cong \mathbb{Z}_6$ or $G \cong \mathbb{Z}_{p^\alpha}$ for some prime p.*

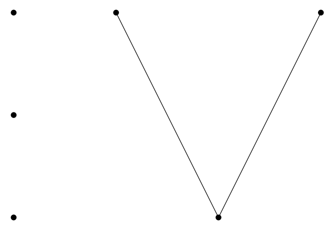
Proof. Let G be a cyclic group of order n. First suppose that $\mathcal{P}(G)$ is the complement of the line graph of some graph Γ. On the contrary, assume that neither $n = 6$ nor n is of prime power. Consider the prime factorization of $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ ($r \geq 2$) such that $p_1 < p_2 < \cdots < p_r$. If $p_i \geq 5$ for any $i \in [r]$, then G has at least 4 elements of order p_i. Let $x, y, z, w \in G$ such that $o(x) = o(y) = o(z) = p_i$ and $o(w) = p_j$ for some $j \in [r] \setminus \{i\}$. Then by Remark 4.0.6(ii), the subgraph induced by the set $\{x, y, z, w\}$ is isomorphic to $\overline{\Gamma_1}$ (see Theorem 4.0.1); a contradiction. Thus, $p_i \leq 3$ for all $i \in [r]$. Consequently, $r = 2$ and $p_1 = 2, p_2 = 3$. Since $n \neq 6$, we have either $\alpha_1 \geq 2$ or $\alpha_2 \geq 2$. If $\alpha_1 \geq 2$, then consider $x_1, x_2, x_3, x_4 \in G$ such that $o(x_1) = 2,\ o(x_2) = o(x_3) = 4$ and $o(x_4) = 3$. The subgraph of $\mathcal{P}(G)$ induced by the set $\{x_1, x_2, x_3, x_4\}$ is isomorphic to $\overline{\Gamma_1}$; a contradiction. Similarly, if $\alpha_2 \geq 2$, then again we get a contradiction. Thus, either $G \cong \mathbb{Z}_6$ or $G \cong \mathbb{Z}_{p^\alpha}$ for some prime p.

Conversely, if $G \cong \mathbb{Z}_{p^\alpha}$, then $\mathcal{P}(G) = K_{p^\alpha}$ (cf. Theorem 1.2.11). Observe that $K_n = \overline{\mathcal{L}(nK_2)}$ and so $\mathcal{P}(G) = \overline{\mathcal{L}(p^\alpha K_2)}$. If $G \cong \mathbb{Z}_6$, then by Figure 4.6, we obtain $\mathcal{P}(G) = \overline{\mathcal{L}(3K_2 \cup P_4)}$. This completes our proof. □

Proposition 4.3.2. *Let G be a finite non-cyclic group and $\Delta(G) \in \{\mathcal{P}_E(G), \mathcal{P}(G), \mathcal{P}_E^{**}(G), \mathcal{P}^{**}(G)\}$. Then $\Delta(G)$ is the complement of a line graph of some graph Γ if*



(a)



(b)

Figure 4.6: (a) $\mathcal{P}(\mathbb{Z}_6)$ (b) $\mathcal{L}(3K_2 \cup P_4)$.

and only if G is isomorphic to either Q_8 or $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$.

Proof. Let G be a finite non-cyclic group such that $\Delta(G)$ is the complement of a line graph of some graph Γ. Since G is non-cyclic, by Lemma 1.1.5, we have $|\mathcal{M}(G)| \geq 3$. We now discuss the following cases.

Case-1: $|\mathcal{M}(G)| \geq 4$. In this case, we show that $G \cong \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ (k-copies), where $k \geq 3$. On the contrary, if G is not isomorphic to $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, then G has a maximal cyclic subgroup M such that $|M| \geq 3$. Consequently, M has at least 2 generators. Let $x, y \in M$ such that $M = \langle x \rangle = \langle y \rangle$ and let z, t, w be generators of other three maximal cyclic subgroups of G. Then by Remark 4.0.6, the subgraph induced by the set $\{x, y, z, t, w\}$ is isomorphic to $\overline{\Gamma_3}$ (see Figure 4.3); which is not possible.

Case-2: $|\mathcal{M}(G)| = 3$. Consider $M_1, M_2, M_3 \in \mathcal{M}(G)$ such that $\phi(|M_1|) \geq \phi(|M_2|) \geq \phi(|M_3|)$. First note that $\phi(|M_1|) \geq 3$ is not possible. If $\phi(|M_1|) \geq 3$, then consider $M_1 = \langle x \rangle = \langle y \rangle = \langle z \rangle$ and $M_2 = \langle t \rangle$. The subgraph of $\Delta(G)$ induced by the set $\{x, y, z, t\}$ is isomorphic to $\overline{\Gamma_1}$; a contradiction. Thus, we may now suppose that $\phi(|M_1|) \leq 2$. Then $|M_1| \in \{2, 3, 4, 6\}$. Let $M_1 = \langle x \rangle$ such that $|M_1| = 6$. Then x^2 and x^3 are elements of order 3 and 2, respectively. Let $M_2 = \langle y \rangle$. Then M_2 cannot contain both the elements x^2 and x^3. Otherwise, $M_1 \subseteq M_2$ which is not possible. Without loss of generality, assume that $x^2 \notin M_2$. Then $x^2 \nsim y$ in $\mathcal{P}_E(G)$ and so $x^2 \nsim y$ in $\Delta(G)$. Consequently, the subgraph of $\Delta(G)$ induced by the set

$\{x, x^2, x^5, y\}$ is isomorphic to $\overline{\Gamma_1}$; again a contradiction. Thus, $|M_1| \leq 4$. Similarly, we get $|M_2| \leq 4$ and $|M_3| \leq 4$. It follows that $|G| \leq |M_1 \cup M_2 \cup M_3| \leq 10$. By Table 1.1, there exist only two groups Q_8 and $\mathbb{Z}_2 \times \mathbb{Z}_2$ (whose order is at most 10) with exactly three maximal cyclic subgroups. Thus, $G \cong Q_8$ or $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Conversely, let $G \cong Q_8$. For $\Delta(G) \in \{\mathcal{P}_E(G), \mathcal{P}(G)\}$, by Figure 4.7, we obtain $\Delta(G) = K_2 \vee 3K_2 = \overline{\mathcal{L}(2K_2 \cup K_4)}$. If $\Delta(G) \in \{\mathcal{P}_E^{**}(G), \mathcal{P}^{**}(G)\}$, then we have $\Delta(G) = 3K_2 = \overline{\mathcal{L}(K_4)}$. Now, assume that $G \cong \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ (k-times), where $k \geq 2$. For $\Delta(G) \in \{\mathcal{P}_E(G), \mathcal{P}(G)\}$, we obtain $\Delta(G) = K_{1,2^k-1} = \overline{\mathcal{L}(K_2 \cup K_{1,2^k-1})}$. If $\Delta(G) \in \{\mathcal{P}_E^{**}(G), \mathcal{P}^{**}(G)\}$, then $\Delta(G) = (2^k - 1)K_1 = \overline{\mathcal{L}(K_{1,2^k-1})}$. □

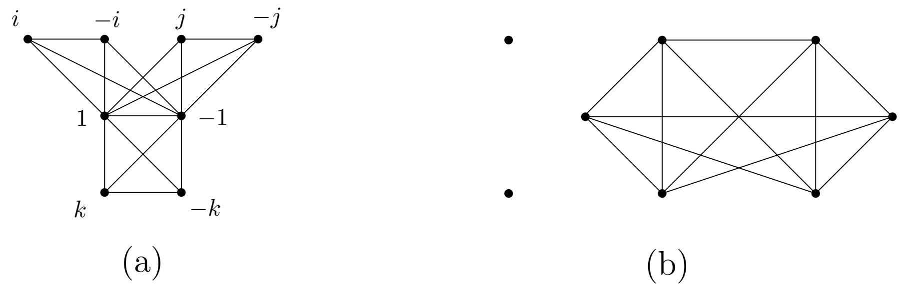


FIGURE 4.7: (a) $\mathcal{P}_E(Q_8)$ (or $\mathcal{P}(Q_8)$) (b) $\mathcal{L}(2K_2 \cup K_4)$.

Theorem 4.3.3. *The power graph $\mathcal{P}(G)$ of a finite group is the complement of a line graph of some graph Γ if and only if G is isomorphic to one of the groups:*

$$\mathbb{Z}_6,\ \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2,\ Q_8,\ \mathbb{Z}_{p^\alpha},\ \textit{where } p \textit{ is a prime.}$$

Proof. The result holds by Propositions 4.3.1 and 4.3.2. □

Proposition 4.3.4. *Let G be a finite cyclic group which is not a p-group. Then $\mathcal{P}^{**}(G)$ is the complement of a line graph of some graph Γ if and only if $G \cong \mathbb{Z}_6$.*

Proof. First suppose that $\mathcal{P}^{**}(G)$ is the complement of a line graph of some graph. Then in the similar lines of the proof of Proposition 4.3.1, we obtain $G \cong \mathbb{Z}_6$. Conversely, note that $\mathcal{P}^{**}(\mathbb{Z}_6) = \overline{\mathcal{L}(P_4)}$. This completes our proof. □

Theorem 4.3.5. *Let G be a finite group which is not a cyclic p-group. Then $\mathcal{P}^{**}(G)$ is the complement of a line graph of some graph Γ if and only if G is isomorphic to one of the groups: $\mathbb{Z}_6, \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2, Q_8$.*

Proof. The result holds by Proposition 4.3.2 and Proposition 4.3.4. □

Theorem 4.3.6. *Let G be a finite group of order n. Then the enhanced power graph $\mathcal{P}_E(G)$ is the complement of a line graph of some graph Γ if and only if G is isomorphic to one of the groups: $\mathbb{Z}_n, \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2,\ Q_8$.*

Proof. If G is a cyclic group of order n, then $\mathcal{P}_E(G) \cong K_n$ (cf. Theorem 1.2.15). Observe that $K_n = \overline{\mathcal{L}(nK_2)}$. Consequently, by Proposition 4.3.2, the result holds. □

Chapter 5

Difference Graph of Power Graphs

There is a hierarchy containing the power graph, the enhanced power graph, and the commuting graph: one is a spanning subgraph of the next. Hence, it is natural to study the difference graph of these graphs. Let G be a finite group. Aalipour et al. [2017], [Question 42] motivated the researchers to study the connectedness of the difference graph $\mathcal{C}(G)-\mathcal{P}(G)$ of the commuting graph and the power graph of G, i.e., the graph with vertex set G in which x and y are adjacent if they commute, but neither is a power of the other. Further, Cameron [2022] discussed some developments on the difference graph $\mathcal{C}(G) - \mathcal{P}(G)$. Moreover, some results were also developed on the difference graph $\mathcal{C}(G) - \mathcal{P}_E(G)$ of the commuting graph and the enhanced power graph in Cameron [2022]. Motivated by the above results, Biswas et al. [2022] studied the difference graph $\mathcal{D}(G) := \mathcal{P}_E(G) - \mathcal{P}(G)$ of the enhanced power graph and the power graph of G with all isolated vertices removed. For certain group classes, they have investigated the connectedness and perfectness of $\mathcal{D}(G)$. Further study on the difference graph $\mathcal{D}(G)$ has been explored in this chapter. We study some interplay between the algebraic properties of the group G and graph-theoretic properties of $\mathcal{D}(G)$. Among other results, we characterize the finite groups G such that $\mathcal{D}(G)$ is a graph with forbidden induced subgraphs.

This chapter is organized as follows. In Section 5.1, we first investigate all possible dominating vertices of $\mathcal{D}(G)$. We provide equivalent conditions on G such that $\mathcal{D}(G)$ is a chordal graph, star graph, dominatable, threshold graph, and split graph, respectively. Subsequently, we conclude that the latter four graph classes are equal for $\mathcal{D}(G)$. In Section 5.2, we classify the finite nilpotent groups G such that $\mathcal{D}(G)$ belong to the above five graph classes. In particular, we prove that all these graph classes are equal for $\mathcal{D}(G)$ when G is a nilpotent group. Furthermore, we characterize the nilpotent groups whose difference graphs are cograph, bipartite, Eulerian, planar, and outerplanar. Section 5.3 comprises the characterization of all the finite nilpotent groups G such that the genus (or cross-cap) of the difference graph $\mathcal{D}(G)$ is at most 2. We study the difference graph of non-nilpotent groups in Section 5.4. We classify the values of n for which the difference graphs of the symmetric group S_n and alternating group A_n are the aforementioned graphs with forbidden induced subgraphs. The content of Sections 5.1, 5.2 and 5.4 is published in SCIE journal *"Quaestiones Mathematicae"*. The content of Section 5.3 is published in SCIE journal *"Ricerche di Matematica"*.

We now introduce notations for adjacency on the graphs defined above on a group G. Hereafter in this chapter, for $x, y \in G$, we write $x \overset{p}{\sim} y$, $x \overset{e}{\sim} y$, and $x \overset{d}{\sim} y$ to denote the adjacency of x and y in $\mathcal{P}(G)$, $\mathcal{P}_E(G)$, and $\mathcal{D}(G)$, respectively. Likewise, their non-adjacencies are denoted by $x \overset{p}{\nsim} y$, $x \overset{e}{\nsim} y$, and $x \overset{d}{\nsim} y$, respectively. The following results are useful in the sequel.

Remark 5.0.1. Let x and y be two elements of a finite group G such that neither $o(x)|o(y)$ nor $o(y)|o(x)$. Then $x \overset{p}{\nsim} y$. The converse is also true if x and y belong to the same cyclic subgroup of G.

Proposition 5.0.2 ([Biswas et al., 2022, Proposition 2.1])**.** *Let G be a group with order greater than 1. Then a non-identity element $g \notin V(\mathcal{D}(G))$ if and only if either $\langle g \rangle$ is a maximal cyclic subgroup of G or that every cyclic subgroup of G containing g has prime-power order.*

Proposition 5.0.3 ([Biswas et al., 2022, Proposition 4.1]). *In a finite nilpotent group G, if $x, y \in G$ such that* $\gcd(o(x), o(y)) = 1$, *then* $x \overset{d}{\sim} y$.

Lemma 5.0.4 ([Biswas et al., 2022, Lemma 4.4]). *Let G be a group and H a subgroup of G. If H is not an EPPO-group itself, then $\mathcal{D}(H)$ is an induced subgraph of $\mathcal{D}(G)$.*

5.1 Difference Graph of a Finite Group

In this section, we study the difference graph $\mathcal{D}(G)$ of an arbitrary finite group G. Note that if G is an EPPO-group, then the enhanced power $\mathcal{P}_E(G)$ coincides with the power graph $\mathcal{P}(G)$ and so $\mathcal{D}(G)$ is a null graph. We first investigate the dominating vertices of $\mathcal{D}(G)$. Then, we give a necessary and sufficient condition for the difference graph of a finite group to be a chordal graph, star graph, dominatable, threshold graph, and split graph, respectively. We begin with the following lemma to study dominating vertices of $\mathcal{D}(G)$.

Lemma 5.1.1. *Let G be a finite group which is not an EPPO-group. If $x \in V(\mathcal{D}(G))$, then $x^{-1} \in V(\mathcal{D}(G))$. Moreover, $x \overset{d}{\nsim} x^{-1}$.*

Proof. Let $x \in V(\mathcal{D}(G))$. Then there exists a vertex $y \in G$ such that $x \overset{d}{\sim} y$. It follows that $x, y \in \langle z \rangle$ for some $z \in G$ and $x \notin \langle y \rangle$, $y \notin \langle x \rangle$. Now, $x \in \langle z \rangle$ implies that $x^{-1} \in \langle z \rangle$, and $\langle x \rangle = \langle x^{-1} \rangle$ implies that $x^{-1} \notin \langle y \rangle, y \notin \langle x^{-1} \rangle$. Consequently, $x^{-1} \overset{d}{\sim} y$ and so $x^{-1} \in V(\mathcal{D}(G))$. Furthermore, $x \overset{e}{\sim} x^{-1}$ and $x \overset{p}{\sim} x^{-1}$. It follows that $x \overset{d}{\nsim} x^{-1}$. □

Lemma 5.1.2. *Let G be a finite group such that G is not an EPPO-group. If $x \in V(\mathcal{D}(G))$ is a dominatable vertex, then $o(x) = 2$.*

Proof. Let $x \in G$ be a dominating vertex in $\mathcal{D}(G)$. By Lemma 5.1.1, $x \overset{d}{\nsim} x^{-1}$. Consequently, either $x = e$ or $o(x) = 2$. But $e \notin V(\mathcal{D}(G))$ and so $o(x) = 2$. □

Lemma 5.1.3. *Let G be a finite group which is not an EPPO-group. Then $\mathcal{D}(G)$ can have at most one dominating vertex. In particular, $\mathcal{D}(G)$ is not a complete graph.*

Proof. On the contrary, assume that x and y are two dominating vertices of $\mathcal{D}(G)$. By Lemma 5.1.2, $o(x) = o(y) = 2$. Since $x \overset{d}{\sim} y$, there exists $z \in G$ such that $x, y \in \langle z \rangle$, which contradicts the fact that a cyclic group can contain at most one element of order 2. Thus, $\mathcal{D}(G)$ has at most one dominating vertex. Since $\mathcal{D}(G)$ has more than one vertices, this implies that it is not complete. □

Proposition 5.1.4. *Let G be a finite group and x, y be two non-identity elements of G such that $o(x)|o(y)$. Then x is not adjacent to y in $\mathcal{D}(G)$.*

Proof. If $x \overset{e}{\nsim} y$, the result holds trivially. We may now assume that $x \overset{e}{\sim} y$. It implies that $x, y \in \langle z \rangle$ for some $z \in G$. Now, to prove our result, it is sufficient to show that $\langle x \rangle \subseteq \langle y \rangle$, so that $x \overset{p}{\sim} y$. Let $o(x) = m$. Since $o(x)|o(y)$, we have a cyclic subgroup H of order m contained in $\langle y \rangle$. Consequently, H is a subgroup of $\langle z \rangle$. Also, $\langle x \rangle$ is a cyclic subgroup of order m contained in $\langle z \rangle$. Since a cyclic group can contain at most one cyclic subgroup of a particular order, we obtain $\langle x \rangle = H$ and so $\langle x \rangle \subseteq \langle y \rangle$. □

Corollary 5.1.5. *Let G be a finite nilpotent group. If x are y are two non-identity elements of the same Sylow subgroup of G, then $x \overset{d}{\nsim} y$.*

Note that a maximum-order element of a finite group generates a maximal cyclic subgroup. Taking Proposition 5.0.2 into consideration, we have the following remark.

Remark 5.1.6. If x is an element of maximum possible order in a finite group G, then $x \notin V(\mathcal{D}(G))$.

For a finite group G, we say that

- *G satisfies the condition $\mathcal{A}$* if $\pi_G \subseteq \{2p_1, 2p_2, \ldots, 2p_k\} \bigcup \left(\bigcup_{i=0}^{k} \{p_i^{\alpha_i} : \alpha_i \in \mathbb{N}\} \right)$ for some $k \in \mathbb{N}$, where $p_0 = 2$ and $p_1, p_2, \ldots, p_k$ are distinct odd primes, and the cardinality of the intersection of any two cyclic subgroups of order $2p_i$, $1 \leq i \leq k$, is at most 2.
- *G satisfies the condition $\mathcal{B}$* if $\pi_G \subseteq \{2p_1, 2p_2, \ldots, 2p_k\} \bigcup \left(\bigcup_{i=0}^{k} \{p_i^{\alpha_i} : \alpha_i \in \mathbb{N}\} \right)$ for some $k \in \mathbb{N}$, where $p_0 = 2$ and $p_1, p_2, \ldots, p_k$ are distinct odd primes, and the cardinality of the intersection of any two cyclic subgroups of orders $2p_i$ and $2p_j$, $1 \leq i, j \leq k$, is 2.

For a finite group G, note that if G satisfies the condition $\mathcal{B}$, then it also satisfies the condition $\mathcal{A}$. However, the converse need not be true. For the symmetric group $G = S_5$, we have $\pi_G = \{2, 3, 4, 5, 6\}$. Also, one can observe that the intersection of any two cyclic subgroups of S_5 of order 6 is trivial. It follows that G does not satisfy the condition $\mathcal{B}$.

Proposition 5.1.7. *Let G be a finite group which is not an EPPO-group. Then $\mathcal{D}(G)$ is C_4-free if and only if G satisfies the condition $\mathcal{A}$.*

Proof. First, suppose that the graph $\mathcal{D}(G)$ is C_4-free. Consider an element $x \in G$ of order pq, where p, q are distinct odd primes. Then there exist $x_1, x_2, y_1, y_2 \in \langle x \rangle$ such that $o(x_1) = o(x_2) = p$ and $o(y_1) = o(y_2) = q$. Note that the subgraph of $\mathcal{P}_E(G)$ induced by these four elements is isomorphic to a complete graph K_4. By Remark 5.0.1 and Proposition 5.1.4, we get an induced cycle $x_1 \overset{d}{\sim} y_1 \overset{d}{\sim} x_2 \overset{d}{\sim} y_2 \overset{d}{\sim} x_1$ isomorphic to C_4 in $\mathcal{D}(G)$; a contradiction.

Now, suppose G has an element x of the order $2p^\alpha$ for some odd prime p and $\alpha \geq 2$. Then there exist $x_1, x_2, y_1, y_2 \in \langle x \rangle$ such that $o(x_1) = o(x_2) = p^2$ and $o(y_1) = o(y_2) = 2p$. Clearly, the subgraph of $\mathcal{P}_E(G)$ induced by the set $\{x_1, x_2, y_1, y_2\}$ is a complete graph K_4. By Proposition 5.1.4 and Remark 5.0.1, the subgraph of $\mathcal{D}(G)$ induced by the set $\{x_1, x_2, y_1, y_2\}$ is isomorphic to C_4; a contradiction.

Let G has an element x such that $o(x) = 2^\alpha p$, where p is an odd prime and $\alpha \geq 2$. Then there exist $x_1, x_2, y_1, y_2 \in \langle x \rangle$ such that $o(x_1) = o(x_2) = 4$ and $o(y_1) = o(y_2) = p$. Similar to the argument used previously, the subgraph of $\mathcal{D}(G)$ induced by the set $\{x_1, x_2, y_1, y_2\}$ is isomorphic to C_4; again a contradiction.

Now, assume that G has two cyclic subgroups M and N of order $2p$ for some odd prime p such that $|M \cap N| = p$. Then there exists $z_1, z_2 \in M \cap N$ such that $o(z_1) = o(z_2) = p$. Let x and y be the elements of order 2 in M and N, respectively. Notice that $x \overset{e}{\sim} z_1$, $x \overset{e}{\sim} z_2$, $y \overset{e}{\sim} z_1$ and $y \overset{e}{\sim} z_2$. By Remark 5.0.1 and Proposition 5.1.4, we obtain an induced cycle $x \overset{d}{\sim} z_1 \overset{d}{\sim} y \overset{d}{\sim} z_2 \overset{d}{\sim} x$ isomorphic to C_4 in $\mathcal{D}(G)$; again a contradiction. Consequently, G satisfies the condition $\mathcal{A}$.

Conversely, assume that G satisfies the condition $\mathcal{A}$. To prove $\mathcal{D}(G)$ is a C_4-free, on the contrary, we assume that G has an induced cycle $x \overset{d}{\sim} y \overset{d}{\sim} z \overset{d}{\sim} t \overset{d}{\sim} x$. If $o(x) = 2p_i$, then note that $\langle x \rangle$ is a maximal cyclic subgroup G. Also, if $o(x) = p_i^\alpha$, for some $\alpha \geq 2$, then every cyclic subgroup containing x is of prime power order. Consequently, by Proposition 5.0.2, we have either $o(x) = 2$ or $o(x) = p_i$ for some $i \in [k]$.

First let $o(x) = 2$. Since $x \overset{d}{\sim} y$, we get $x, y \in \langle g \rangle$ for some $g \in G$ and neither $o(y)|o(x)$ nor $o(x)|o(y)$. It follows that $|\langle g \rangle| = 2p_i$ and $o(y) = p_i$ for some $i \in [k]$. Since $y \overset{d}{\sim} z$, we obtain $y, z \in \langle g' \rangle$ for some $g' \in G$ and neither $o(y)|o(z)$ nor $o(z)|o(y)$. Consequently, $|\langle g' \rangle| = 2p_i$ and $o(z) = 2$. Thus, $y \in \langle g \rangle \cap \langle g' \rangle$. It follows that $|\langle g \rangle \cap \langle g' \rangle| = p_i$, which is not possible. Similarly, none of $o(y)$, $o(z)$, and $o(t)$ is 2. We may now assume that $o(x) = p_i$ for some $i \in [k]$. Since $x \overset{d}{\sim} y$, by Proposition 5.1.4, $o(y) = 2$; again a contradiction. Hence, $\mathcal{D}(G)$ is C_4-free. □

Theorem 5.1.8. *Let G be a finite group which is not an EPPO-group. Then $\mathcal{D}(G)$ is a chordal graph if and only if G satisfies the condition $\mathcal{A}$.*

Proof. In view of Proposition 5.1.7, we prove the desired result by showing that the graph $\mathcal{D}(G)$ does not contain an induced cycle of length greater than 4. Now, suppose G satisfies the condition $\mathcal{A}$. If G has an element x such that $o(x) = p^\alpha$, where p is a prime and $\alpha \geq 2$, then note that every cyclic subgroup of G containing x is of prime power order. Also, for an odd prime p', notice that each element whose order is of the form $2p'$, generates a maximal cyclic subgroup of G. Thus, by Proposition 5.0.2, the vertex set of $\mathcal{D}(G)$ contains only elements of the orders $2, p_1, p_2, \ldots, p_k$ (cf. Proposition 5.0.2). We claim that $\mathcal{D}(G)$ is a bipartite graph. Consider a partition of $V(\mathcal{D}(G))$ into two subsets A and B such that if $o(x) = 2$ then $x \in A$. Otherwise $x \in B$. By Proposition 5.1.4, no two elements of A are adjacent in $\mathcal{D}(G)$. For $a, b \in B$ such that $o(a) = o(b)$, we have $a \overset{d}{\nsim} b$. Now, let $x, y \in B$ such that $o(x) = p_i$ and $o(y) = p_j$ for distinct $i, j \in [k]$. Then $x \overset{d}{\nsim} y$ because G does not contain an element of the order $p_i p_j$. Thus, $\mathcal{D}(G)$ is a bipartite graph. Since the cardinality of the intersection of any two distinct cyclic subgroups of the order $2p_i$ is at most 2, we obtain that every element of order p_i, $1 \leq i \leq k$, in B is adjacent to exactly one element of order 2 in $\mathcal{D}(G)$. Thus, the length of any path in $\mathcal{D}(G)$ is at most 2. This completes the proof. $\square$

Corollary 5.1.9. *Let G be a finite group which is not an EPPO-group. Then $\mathcal{D}(G)$ is a chordal graph if and only if it is C_4-free.*

In the following theorem, we characterize finite groups whose difference graphs are star graphs and dominatable, respectively.

Theorem 5.1.10. *Let G be a finite group which is not an EPPO-group. Then the following conditions are equivalent:*

(i) *$\mathcal{D}(G)$ is a star graph.*

(ii) *$\mathcal{D}(G)$ is dominatable.*

(iii) *G satisfies the condition $\mathcal{B}$.*

Proof. Clearly, (i) $\Longrightarrow$ (ii). Now, we show that (ii) $\Longrightarrow$ (iii) and (iii) $\Longrightarrow$ (i).

We begin with the proof of (ii) $\Longrightarrow$ (iii). Let $\mathcal{D}(G)$ be a dominatable graph. By Lemmas 5.1.2 and 5.1.3, there exists a unique element x of order 2 such that

$x \overset{d}{\sim} y$ for every $y \neq x \in V(\mathcal{D}(G))$. Let $g \in G$ be such that $o(g) = p_1p_2$ for some odd primes p_1 and p_2. There exists $y, z \in \langle g \rangle$ such that $o(y) = p_1$ and $o(z) = p_2$. Then $y \overset{e}{\sim} z$ and by Remark 5.0.1, $y \overset{p}{\nsim} z$. It follows that $y \overset{d}{\sim} z$. Consequently, $y, z \in V(\mathcal{D}(G))$. Since x is a dominating vertex in $\mathcal{D}(G)$, we get $x \overset{d}{\sim} y$ and $x \overset{d}{\sim} z$. It implies that $\langle x, y \rangle$ and $\langle x, z \rangle$ are cyclic subgroups of G generated by xy and xz, respectively. Since y commutes with x and z, we obtain a cyclic subgroup $\langle xyz \rangle$ of G containing y and xz. Note that neither $o(y)|o(xz)$ nor $o(xz)|o(y)$. Thus, $y \overset{d}{\sim} xz$ and so $xz \in V(\mathcal{D}(G))$. By Proposition 5.1.4, $x \overset{d}{\nsim} xz$, which is a contradiction. Thus, G has no element of the order p_1p_2. If G has an element x' of order $2^\alpha p$ for some odd prime p and $\alpha \geq 2$, then note that there exist elements $g, g' \in \langle x' \rangle$ such that $o(g) = 4$ and $o(g') = p$. Then $g \overset{d}{\sim} g'$ (see Remark 5.0.1) and so $g \in V(\mathcal{D}(G))$, but $g \overset{d}{\nsim} x$ (cf. Proposition 5.1.4). This contradicts the fact that x is a dominating vertex.

Suppose G has an element y of order $2p^2$ for some odd prime p. Let $t, z, w \in \langle y \rangle$ be such that $o(t) = 2, o(z) = p$ and $o(w) = p^2$. Clearly, $z \overset{d}{\sim} t$ and so $t \in V(\mathcal{D}(G))$. Since $x \in V(\mathcal{D}(G))$ is a unique element of order 2, we obtain $x = t$. Therefore, $x \in \langle y \rangle$. Notice that $xz \in \langle y \rangle$ is an element of the order $2p$ and $w \overset{e}{\sim} xz$. Note that neither $o(w)|o(xz)$ nor $o(xz)|o(w)$. Thus, $w \overset{d}{\sim} xz$ and so $xz \in V(\mathcal{D}(G))$. By Proposition 5.1.4, $x \overset{d}{\nsim} xz$; again a contradiction. Consequently, G has no element of the order $2p^2$ for an odd prime p.

Let M and N be two cyclic subgroups of the order $2p_i$ and $2p_j$, respectively, for some $i, j \in [k]$. To prove the result, it is sufficient to show that $x \in M \cap N$. Consider $y, z \in M$ such that $o(y) = 2$ and $o(z) = p_i$. Note that $y \overset{d}{\sim} z$ and so $y \in V(\mathcal{D}(G))$. Consequently, $x = y \in M$. Similarly, we obtain $x \in N$. It follows that $|M \cap N| = 2$. Thus, G satisfies the condition $\mathcal{B}$.

We next prove (iii) $\implies$ (i). Let G satisfies the condition $\mathcal{B}$. Observe that if $x \in V(\mathcal{D}(G))$, then x belongs to a cyclic subgroup of the order $2p_i$ for some $i \in [k]$ (see Proposition 5.0.2). By the proof of the Theorem 5.1.8, notice that $\mathcal{D}(G)$ is a bipartite graph and the partition set A contains elements of the order 2 and the partition set B contains the elements of the orders $p_1, p_2, \dots, p_k$. First note that $|A| = 1$. Assume that $x, y \in A$. Then there exist two cyclic subgroups M and N of the order $2p_i$ and $2p_j$, respectively, such that $x \in M, y \in N$. Since the group G satisfies the condition $\mathcal{B}$, we obtain $|M \cap N| = 2$. Consequently, $M \cap N$ contains an element of order 2. It follows that $x = y$. Thus, the partition set A of $\mathcal{D}(G)$ contains exactly one element, say x. Since $V(\mathcal{D}(G)) = A \cup B$, we obtain that if $y \in B$, then $y \overset{d}{\sim} x$. Thus, $\mathcal{D}(G)$ is a star graph with x as the dominating vertex of $\mathcal{D}(G)$. □

Next, we characterize finite groups whose difference graphs are threshold graphs and split graphs, respectively.

Theorem 5.1.11. *Let G be a finite group which is not an EPPO-group. Then the following conditions are equivalent:*

(i) *$\mathcal{D}(G)$ is a threshold graph.*

(ii) *$\mathcal{D}(G)$ is a split graph.*

(iii) *G satisfies the condition $\mathcal{B}$.*

Proof. Clearly, (i) $\Longrightarrow$ (ii). By Theorem 5.1.10, we have (iii) $\Longrightarrow$ (i).

We now prove (ii) $\Longrightarrow$ (iii). Let $\mathcal{D}(G)$ be a split graph. Then by Proposition 5.1.7, G satisfies the condition $\mathcal{A}$. Now, to prove the result, we show that for $1 \leq i, j \leq k$, the cardinality of the intersection of any two cyclic subgroups of the order $2p_i$ and $2p_j$, respectively, is 2. On the contrary, assume there exist two cyclic subgroups M and N of the order $2p_i$ and $2p_j$, respectively, such that $|M \cap N| = 1$. Suppose $x \in M$ and $y \in N$ are the elements of order 2. Let $z \in M$ and $t \in N$ such that $o(z) = p_i$ and $o(t) = p_j$. Then $x \overset{e}{\sim} z$ and $y \overset{e}{\sim} t$. By Remark 5.0.1, $x \overset{p}{\nsim} z$ and $y \overset{p}{\nsim} t$. It follows that $x \overset{d}{\sim} z$ and $y \overset{d}{\sim} t$. Note that G has no element of the order $p_i p_j$ as G satisfies the condition $\mathcal{A}$. Consequently, $z \overset{e}{\nsim} t$ and so $z \overset{d}{\nsim} t$. Since G satisfies the condition $\mathcal{A}$, we get $x \overset{e}{\nsim} t$, $y \overset{e}{\nsim} z$. By Proposition 5.1.4, $x \overset{d}{\nsim} y$.

Thus, the subgraph of $\mathcal{D}(G)$ induced by the set $\{x, y, z, t\}$ is isomorphic to $2K_2$; a contradiction. Thus, G satisfies the condition $\mathcal{B}$. □

In view of the above two theorems, we deduce that all four classes addressed in them are equal for difference graphs of finite groups.

5.2 Difference Graph of Nilpotent Groups

Let G be a finite nilpotent group and $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$, where $p_1 < p_2 < \cdots < p_r$ are primes and $\alpha_1, \alpha_2, \ldots, \alpha_r$ are positive integers.

In this section, we classify the finite nilpotent groups G such that the difference graph $\mathcal{D}(G)$ is a chordal graph, star graph, dominatable, threshold graph, and split graph, respectively. Then we characterize the nilpotent groups whose difference graphs are cograph, bipartite, Eulerian, planar, and outerplanar, respectively. Note that a finite nilpotent group G is an EPPO-group if and only if G is a p-group.

Theorem 5.2.1. *Let G be a finite nilpotent group which is not a p-group. Then the following conditions are equivalent:*

(i) *G satisfies the condition $\mathcal{A}$.*

(ii) *G satisfies the condition $\mathcal{B}$.*

(iii) *$G \cong \mathbb{Z}_2 \times P$, where P is a p-group of exponent $p > 2$.*

Proof. (i) $\Longrightarrow$ (ii): Suppose that G satisfies the condition $\mathcal{A}$. First we claim that $\pi_G = \{2, p_1, 2p_1\}$. If possible, assume that G has elements of the order p_1 and p_2 for some odd primes p_1 and p_2. Since G is a nilpotent group, it implies that G has elements of the order p_1p_2 (cf. Lemma 1.1.11); a contradiction. Thus, π_G can have at most one odd prime. Since G is not a p-group, we have $2, p_1 \in \pi_G$, where p_1 is an odd prime. Consequently, $\{2, p_1, 2p_1\} \subseteq \pi_G$. Now suppose G contains an element of the order 2^α, where $\alpha \geq 2$. Then by Lemma 1.1.11, G contains an element of order $2^{\alpha}p_1$; a contradiction. Therefore, G does not contain elements of the order 2^{α}, where $\alpha \geq 2$. Similarly, if G contains an element of the order p_1^{α}, where $\alpha \geq 2$, then one can observe that the group G contains an element of the order $2p_1^{\alpha}$, which is not possible. Thus, $\pi_G = \{2, p_1, 2p_1\}$.

Suppose that M and N are two cyclic subgroups of G of the order $2p_1$ such that $|M \cap N| = 1$. Let $x \in M$ and $y \in N$ be elements of the order 2. Suppose $z \in M$ such that $o(z) = p_1$. Observe that $o(yz) = 2p_1$ and $|M \cap \langle yz \rangle| = p_1$; again a contradiction. Thus, $|M \cap N| = 2$ and so G satisfies the condition $\mathcal{B}$.
(ii) $\implies$ (iii): Let $G = P_1P_2 \cdots P_r$ be a nilpotent group such that $|P_i| = p_i^{\alpha_i}$ and $p_j < p_{j+1}$ for $j \in [r-1]$. Let $r \geq 3$. Then there exist $x, y, z \in G$ such that $o(x) = p_1$, $o(y) = p_2$ and $o(z) = p_3$. It follows that $o(yz) = p_2p_3$, which is not possible, as G satisfies the condition $\mathcal{B}$. Thus, $r = 2$. Consequently, we get $p_1 = 2$. Now, to prove the result, it is sufficient to show that $|P_1| = 2$ and $\exp(P_2) = p_2$. First we show that $\exp(P_1) = 2$. Assume that $\exp(P_1) = 2^{\alpha}$ for $\alpha \geq 2$. Then there exists $x \in P_1$ such that $o(x) = 2^{\alpha}$. Let $y \in P_2$ be such that $o(y) = p_2$. Consequently, $o(xy) = 2^{\alpha}p_2$, which is not possible. Thus, $\exp(P_1) = 2$. Similarly, we obtain $\exp(P_2) = p_2$. Now, we show that $|P_1| = 2$. On the contrary, assume that P_1 has two non-identity elements x and y. Let $z \in P_2$ such that $o(z) = p_2$. Then $\langle xz \rangle$ and $\langle yz \rangle$ are two cyclic subgroups of order $2p_2$. Since $z \in \langle xz \rangle \cap \langle yz \rangle$, we obtain $|\langle xz \rangle \cap \langle yz \rangle| = p_2$, which is a contradiction. This completes the proof.
(iii) $\implies$ (i): Let $G \cong \mathbb{Z}_2 \times P$, where P is a p-group of exponent $p > 2$. Then $\pi_G = \{2, p, 2p\}$ and the cardinality of the intersection of any two cyclic subgroups of the order $2p$ is 2. Thus, G satisfies the condition $\mathcal{A}$. □

In view of Theorems 5.1.8, 5.1.10, and 5.2.1, we have the following theorem. This theorem characterizes finite nilpotent groups G for which the classes into consideration of $\mathcal{D}(G)$ are equal.

Theorem 5.2.2. *Let G be a finite nilpotent group which is not a p-group. Then the following conditions are equivalent:*

(i) $\mathcal{D}(G)$ *is a chordal graph.*

(ii) $\mathcal{D}(G)$ *is a star graph.*

(iii) $\mathcal{D}(G)$ *is dominatable.*

(iv) $\mathcal{D}(G)$ *is a threshold graph.*

(v) $\mathcal{D}(G)$ *is a spilt graph.*

(vi) $G \cong \mathbb{Z}_2 \times P$*, where* P *is a* p*-group of exponent* $p > 2$*.*

We next investigate the difference graphs of finite nilpotent groups for cograph. Followed by this, we classify the finite nilpotent groups whose difference graphs are bipartite, Eulerian, planar, and outerplanar, respectively.

Let G be a finite group but not a p-group. Biswas et al. [2022] proved that when G is a cyclic group of order n, the graph $\mathcal{D}(G)$ is cograph if and only if n is a product of two distinct primes. In the following theorem, we generalize this result from a cyclic group to an arbitrary nilpotent group.

Theorem 5.2.3. *Let* G *be a finite nilpotent group which is not a* p*-group. Then* $\mathcal{D}(G)$ *is a cograph if and only if* $G \cong P_1 \times P_2$*, where each* P_i *is a* p_i*-group of exponent* p_i *for every* $i \in \{1, 2\}$*.*

Proof. Let us assume that $\mathcal{D}(G)$ is a cograph. Let $G = P_1 P_2 \cdots P_r$ such that $|P_i| = p_i^{\alpha_i}$ and $p_j < p_{j+1}$ for $j \in [r-1]$. Suppose $r \geq 3$. Let x, y, z be elements of order p_1, p_2 and p_3, respectively. Notice that $o(xz) = p_1 p_3$ and $o(xy) = p_1 p_2$. Now, by Proposition 5.0.3, $x \overset{d}{\sim} y$ and $y \overset{d}{\sim} xz$. Note that $x, y, z \in \langle xyz \rangle$. It follows that $xz, xy \in \langle xyz \rangle$. Note that neither $o(xy) | o(xz)$ nor $o(xz) | o(xy)$. Consequently, $xz \overset{d}{\sim} xy$. Since $\langle x \rangle \subseteq \langle xy \rangle$, $\langle x \rangle \subseteq \langle xz \rangle$ and $\langle y \rangle \subseteq \langle xy \rangle$, we have $x \overset{d}{\nsim} xy$, $x \overset{d}{\nsim} xz$ and $y \overset{d}{\nsim} xy$. Thus, the subgraph induced by the set $\{x, y, xz, xy\}$ is isomorphic to the path graph P_4; a contradiction. Therefore, $r = 2$.

Now, we have to show that the exponents of P_1 and P_2 are p_1 and p_2, respectively. We prove it for P_1, and a similar result holds for P_2 as well. Let $\exp(P_1) \geq p_1^2$. Then there exists an element x such that $o(x) = p_1^2$. Let $y \in \langle x \rangle$ such that $o(y) = p_1$. Let $z \in G$ such that $o(z) = p_2$. Then by Proposition 5.0.3, $y \overset{d}{\sim} z$ and $z \overset{d}{\sim} x$. Now, $x, z \in \langle xz \rangle$. It follows that $x, yz \in \langle xz \rangle$ and neither $o(x)|o(yz)$ nor $o(yz)|o(x)$. Consequently, $x \overset{d}{\sim} yz$. Also, $o(y)$ and $o(z)$ divides $o(yz)$. Then by Proposition 5.1.4, $y \overset{d}{\nsim} yz$ and $z \overset{d}{\nsim} yz$. Since $y \in \langle x \rangle$, we obtain $x \overset{d}{\nsim} y$. Thus, the graph induced by the set $\{y, z, x, yz\}$ is isomorphic to the path graph P_4.

Conversely, let us assume that $G \cong P_1 \times P_2$ such that $\exp(P_1) = p_1$ and $\exp(P_2) = p_2$. It follows that G has elements of order 1, p_1, p_2, and p_1p_2. Observe that $e \notin V(\mathcal{D}(G))$. By Remark 5.1.6, the elements of order p_1p_2 do not belong to the vertex set of $\mathcal{D}(G)$. By Proposition 5.0.3 and Corollary 5.1.5, $\mathcal{D}(G)$ is a complete bipartite graph $K_{p_1^{\alpha_1}-1, p_2^{\alpha_2}-1}$ and so $\mathcal{D}(G)$ is a cograph. Thus, the result holds. $\square$

Theorem 5.2.4. *Let G be a finite nilpotent group which is not a p-group. Then $\mathcal{D}(G)$ is a bipartite graph if and only if $G \cong P_1 \times P_2$ such that $\exp(P_i) = p_i$ for at least one i, where $1 \leq i \leq 2$.*

Proof. Let $G = P_1P_2 \cdots P_r$ be a nilpotent group such that $|P_i| = p_i^{\alpha_i}$ and $p_j < p_{j+1}$ for $j \in [r-1]$. First, assume that $\mathcal{D}(G)$ is a bipartite graph. Let $r \geq 3$. Let x, y, z be elements of G such that $o(x) = p_1$, $o(y) = p_2$ and $o(z) = p_3$. By Proposition 5.0.3, $x \overset{d}{\sim} y$, $y \overset{d}{\sim} z$ and $z \overset{d}{\sim} x$. Thus, we get a cycle of length 3 in $\mathcal{D}(G)$; a contradiction. Therefore, $r = 2$. Now, suppose that $\exp(P_1) \geq p_1^2$ and $\exp(P_2) \geq p_2^2$. Then there exist $x, y \in G$ such that $o(x) = p_1^2$ and $o(y) = p_2^2$. Consider $x_1 \in \langle x \rangle, y_1 \in \langle y \rangle$ such that $o(x_1) = p_1$ and $o(y_1) = p_2$. Then $o(x_1y_1) = p_1p_2$. Notice that $x, y \in \langle xy \rangle$ and so $x_1y_1 \in \langle xy \rangle$. It follows that $x \overset{e}{\sim} x_1y_1 \overset{e}{\sim} y \overset{e}{\sim} x$. Note that neither $o(x_1y_1)|o(x)$ nor $o(x)|o(x_1y_1)$. By Remark 5.0.1, $x \overset{p}{\nsim} x_1y_1$. Similarly, $y \overset{p}{\nsim} x_1y_1$ and $x \overset{p}{\nsim} y$. It follows that $x \overset{d}{\sim} x_1y_1 \overset{d}{\sim} y \overset{d}{\sim} x$; again a contradiction.

Conversely, let $G \cong P_1 \times P_2$ such that $\exp(P_i) = p_i$ for some i. Without loss of generality, assume that $\exp(P_1) = p_1$ and $\exp(P_2) = p_2^k$ for some $k \geq 1$. Then

$\pi_G = \{p_1, p_2, p_2^2, \ldots, p_2^k, p_1p_2, p_1p_2^2, \ldots, p_1p_2^k\}$. Consider a partition of $V(\mathcal{D}(G))$ into two sets A and B such that for $x \in V(\mathcal{D}(G))$, if $p_1|o(x)$ then $x \in A$. Otherwise, $x \in B$. If $x, y \in A$, then observe that either $o(x)|o(y)$ or $o(y)|o(x)$ and so $x \overset{d}{\nsim} y$ (see Proposition 5.1.4). Now, if $a, b \in B$ then $o(a) = p^\alpha$ and $o(b) = p^\beta$. By Proposition 5.1.4, $a \overset{d}{\nsim} b$. Thus, $\mathcal{D}(G)$ is a bipartite graph. □

Corollary 5.2.5. *Let G be a finite cyclic group which is not a p-group. Then $\mathcal{D}(G)$ is a bipartite graph if and only if $G \cong \mathbb{Z}_{p_1^\alpha p_2}$ for some distinct primes p_1, p_2 and $\alpha \geq 1$.*

Theorem 5.2.6. *Let G be a finite nilpotent group which is not a p-group. Then $\mathcal{D}(G)$ is Eulerian if and only if $|G|$ is odd.*

Proof. Let $\mathcal{D}(G)$ be an Eulerian graph. Suppose $|G|$ is even. Therefore G has an odd number of elements of order 2. Let $x \in G$ be an element of order $p(\neq 2)$ for some prime p. Notice that if x is adjacent to y, then x is adjacent to y^{-1}. By Proposition 5.0.3, x is adjacent to every element of order 2. Consequently, $\deg(x)$ is odd, which is a contradiction.

Conversely, suppose that $|G|$ is odd. Let x be an arbitrary element of $V(\mathcal{D}(G))$. If $x \overset{d}{\sim} y$, then $x \overset{d}{\sim} y^{-1}$. It follows that the degree of x in $\mathcal{D}(G)$ is even. Thus, the result holds. □

The following example shows that in the above theorem, the condition that G is a nilpotent group is indeed necessary.

Example 5.2.7. Consider the dihedral group $D_{30} = \langle x, y : x^{15} = y^2 = e, xy = yx^{-1}\rangle$ of order 30. By Figure 5.1, observe that $\mathcal{D}(D_{30}) \cong K_{2,4}$. Hence, $\mathcal{D}(D_{30})$ is Eulerian, but $|D_{30}|$ is not odd.

Theorem 5.2.8. *Let G be a finite nilpotent group which is not a p-group. Then $\mathcal{D}(G)$ is planar if and only if G is isomorphic to one of the following groups:*

$$\mathbb{Z}_{12},\ D_8 \times \mathbb{Z}_3,\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2 \times \mathbb{Z}_3,\ \mathbb{Z}_2 \times \mathcal{Q}_1,\ \mathbb{Z}_3 \times \mathcal{Q}_2,\ \textit{where}$$

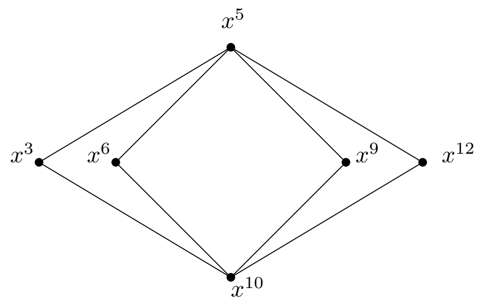


Figure 5.1: The difference graph of D_{30}.

$\mathcal{Q}_i$ is a q_i-group of prime exponent q_i such that $q_1 > 2$ and $q_2 > 3$.

Proof. Let $\mathcal{D}(G)$ be a planar graph. Let $G = P_1P_2\cdots P_r$ such that $|P_i| = p_i^{\alpha_i}$ and $p_j < p_{j+1}$ for $j \in [r-1]$. Let $r \geq 3$. Let $X = \{x_1, x_2, x_3\}$ such that $o(x_i) \in \{p_1, p_2\}$. Let $Y = \{y_1, y_2, y_3\}$ such that $o(y_i) = p_3$. By Proposition 5.0.3, each element of X is adjacent to every element of Y in $\mathcal{D}(G)$. Thus, the subgraph induced by the set $X \cup Y$ has a subgraph isomorphic to $K_{3,3}$; a contradiction. Thus, we obtain $r = 2$.

Now, let $p_1 \geq 5$. Then G has at least 4 elements of order p_1 and at least 6 elements of order p_2. Let $X = \{x_1, x_2, x_3\}$ such that $o(x_i) = p_1$ and $Y = \{y_1, y_2, y_3\}$ such that $o(y_i) = p_2$. Then the subgraph induced by the set $X \cup Y$ is isomorphic to $K_{3,3}$; a contradiction. Consequently, $p_1 \in \{2, 3\}$.

Now, we prove our result in the following two cases.

Case-1: $p_1 = 2$. We discuss this case in the following four subcases.

Subcase-1.1: $\exp(P_1) \geq 8$. It follows that G has at least one cyclic subgroup of order 8. Let x_1, x_2, x_3 and x_4 be the generators of a cyclic subgroup of order 8 in G. Let $y_1 \in \langle x_1 \rangle$ such that $o(y_1) = 4$. Let z_1, z_2 be elements of order p_2. Now, $x_1, z_1 \in \langle x_1z_1 \rangle$ implies that $y_1z_1 \in \langle x_1z_1 \rangle$. Note that neither $o(x_1)|o(y_1z_1)$ nor $o(y_1z_1)|o(x_1)$. Consequently, $y_1z_1 \overset{d}{\sim} x_1$. Similarly, $y_1z_1 \overset{d}{\sim} x_2$ and $y_1z_1 \overset{d}{\sim} x_3$. By Proposition 5.0.3, $x_i \overset{d}{\sim} z_j$ for each $i \in \{1, 2, 3\}$ and $j \in \{1, 2\}$. Consequently, the subgraph induced by the set $\{x_1, x_2, x_3, y_1z_1, z_1, z_2\}$ is isomorphic to $K_{3,3}$; a contradiction.

Subcase-1.2: $\exp(P_1) = 4$ *and* $|P_2| > 3$. Then the subgraph induced by the set $(P_1 \cup P_2) \setminus \{e\}$ has a subgraph isomorphic to $K_{3,3}$ (see Proposition 5.0.3 and Corollary 5.1.5); a contradiction.

Subcase-1.3: $\exp(P_1) = 4$ *and* $|P_2| = 3$. Clearly, $P_2 \cong \mathbb{Z}_3$. Since P_1 is a finite 2-group with exponent 4, by Lemma 1.1.9, either P_1 has exactly one cyclic subgroup of order 4 or P_1 contains at least two cyclic subgroups of order 4 such that the cardinality of their intersection is 2. Consequently, we discuss this subcase into the following two further subcases.

Subcase-1.3(a): P_1 *contains exactly one cyclic subgroup of order* 4. By Lemma 1.1.8, we obtain that P_1 is isomorphic to D_8 or $\mathbb{Z}_4$. Consequently, either $G \cong D_8 \times \mathbb{Z}_3$ or $G \cong \mathbb{Z}_{12}$.

Subcase-1.3(b): P_1 *contains at least two cyclic subgroups* M_1 *and* M_2 *of order* 4 *such that* $|M_1 \cap M_2| = 2$. Let y_1 and y_2 be elements of order 3 in P_2. Suppose $M_1 = \{e, x, x_1, x_2\}$ and $M_2 = \{e, x, x_3, x_4\}$. Now, $x_1, y_1 \in \langle x_1y_1 \rangle$ and so $xy_1 \in \langle x_1y_1 \rangle$. Also, neither $o(xy_1)|o(x_1)$ nor $o(x_1)|o(xy_1)$. Consequently, $x_1 \overset{d}{\sim} xy_1$. Similarly, $x_2 \overset{d}{\sim} xy_1$ and $x_3 \overset{d}{\sim} xy_1$. By Proposition 5.0.3, $y_i \overset{d}{\sim} x_j$ for each $i \in \{1, 2\}$ and $j \in \{1, 2, 3\}$. Thus, the subgraph of $\mathcal{D}(G)$ induced by the set $\{y_1, y_2, xy_1, x_1, x_2, x_3\}$ has a subgraph isomorphic to $K_{3,3}$; a contradiction.

Subcase-1.4: $P_1 \cong \mathbb{Z}_2$. Let $\exp(P_2) > p_2$. Then P_2 has at least one cyclic subgroup H of order p_2^2. Let x_1, x_2 and x_3 be generators of H and let y_1, y_2 be two elements of order p_2 in H. Suppose $x \in P_1$ is an element of order 2. Then $o(xy_1) = o(xy_2) = 2p_2$. Also, $x, y_1 \in \langle xx_1 \rangle$ implies that $xy_1 \in \langle xx_1 \rangle$ and so $xy_1 \overset{e}{\sim} x_1$. Indeed $xy_i \overset{e}{\sim} x_j$ for each $i \in \{1, 2\}$ and $j \in \{1, 2, 3\}$. But neither $o(xy_i)|o(x_j)$ nor $o(x_j)|o(xy_i)$ and so $xy_i \overset{d}{\sim} x_j$ for each $i \in \{1, 2\}$ and $j \in \{1, 2, 3\}$. By Proposition 5.0.3, $y \overset{d}{\sim} x_i$ for each $i \in \{1, 2, 3\}$. Thus, the subgraph induced by set $\{x_1, x_2, x_3, x, xy_1, xy_2\}$ is isomorphic to $K_{3,3}$; a contradiction. Thus, $\exp(P_2) = p_2$.

Subcase-1.5: $P_1 \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$. In this case, we show that $P_2 \cong \mathbb{Z}_3$. On the contrary, assume that $|P_2| > 3$. By Proposition 5.0.3, each non-identity element of P_1 is adjacent to every non-identity element of P_2, and both P_1, P_2 contain at least 3 non-identity elements. Thus, the subgraph induced by the set $(P_1 \cup P_2) \setminus \{e\}$ has a subgraph isomorphic to $K_{3,3}$; a contradiction. Thus, $P_2 \cong \mathbb{Z}_3$.

Case-2: $p_1 = 3$. We discuss this case in the following two subcases.

Subcase-2.1: $P_1 \cong \mathbb{Z}_3$. By replacing $p_1 = 2$ to $p_1 = 3$ in the **Subcase 1.4**, we obtain $G \cong \mathbb{Z}_3 \times P$, where P is a p-group of exponent $p > 3$.

Subcase-2.2: $|P_1| > 3$. In this case, $|P_2| > 3$. Both P_1 and P_2 have at least 3 non-identity elements. Thus, the subgraph induced by the set $(P_1 \cup P_2) \setminus \{e\}$ has a subgraph isomorphic to $K_{3,3}$.

Conversely, let $G \cong \mathbb{Z}_2 \times P$, where P is a p-group of exponent $p > 2$. Then by Theorem 5.2.2, $\mathcal{D}(G)$ is a star graph, and so $\mathcal{D}(G)$ is planar. If $G \cong \mathbb{Z}_3 \times P$,

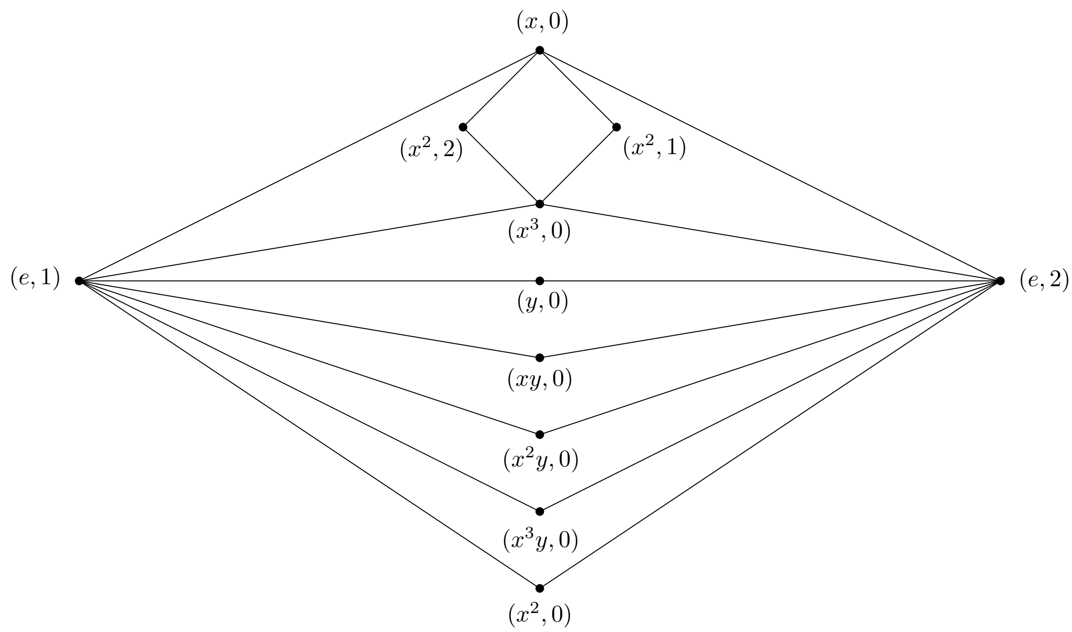


Figure 5.2: Planar drawing of $\mathcal{D}(D_8 \times \mathbb{Z}_3)$.

where P is a p-group of exponent $p > 3$, then G has elements of order 1, 3, p, and $3p$. Notice that $e \notin V(\mathcal{D}(G))$ and by Remark 5.1.6, the elements of the order $3p$ do not belong to the vertex set of $\mathcal{D}(G)$. By Proposition 5.0.3, in $\mathcal{D}(G)$, each element

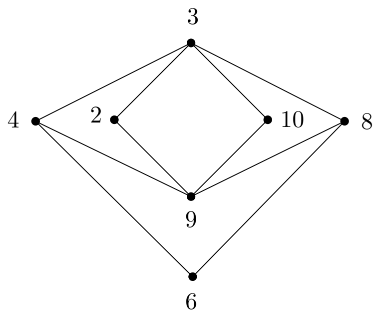


FIGURE 5.3: Planar drawing of $\mathcal{D}(\mathbb{Z}_{12})$.

of order 3 is adjacent to every element of order p. By Corollary 5.1.5, $\mathcal{D}(G)$ is a complete bipartite graph $K_{2,|P|-1}$. Thus, $\mathcal{D}(G)$ is planar.

Now, let $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2 \times \mathbb{Z}_3$. Then G has elements of order 1, 2, 3 and 6, respectively. Note that the elements of order 1 and 6 do not belong to the vertex set of $\mathcal{D}(G)$. By Proposition 5.0.3, each element of order 2 is adjacent to every element of order 3. By Corollary 5.1.5, the elements of the same order are not adjacent in $\mathcal{D}(G)$. Consequently, $\mathcal{D}(G)$ is a complete bipartite graph $K_{2,t-1}$, where $t = |\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2|$. Thus, $\mathcal{D}(G)$ is planar.

If $G \cong D_8 \times \mathbb{Z}_3$, where $D_8 = \langle x, y : x^4 = y^2 = e, xy = yx^{-1} \rangle$, then $\mathcal{D}(G)$ is a planar graph (cf. Figure 5.2). If $G \cong \mathbb{Z}_{12}$, then by Figure 5.3, the graph $\mathcal{D}(G)$ is planar. □

Corollary 5.2.9. *Let G be a finite cyclic group which is not a p-group. Then the difference graph $\mathcal{D}(G)$ is planar if and only if G is isomorphic to one of the groups: $\mathbb{Z}_{12}$, $\mathbb{Z}_{2p}$ or $\mathbb{Z}_{3q}$, where $p > 2$ and $q > 3$ are primes.*

Theorem 5.2.10. *Let G be a finite nilpotent group which is not a p-group. Then $\mathcal{D}(G)$ is an outerplanar graph if and only if $G \cong \mathbb{Z}_2 \times P$, where P is a p-group of exponent $p > 2$.*

Proof. We first assume that $\mathcal{D}(G)$ is an outerplanar graph. Recall that every outerplanar graph is a planar graph. In view of Theorem 5.2.8, if $G \cong \mathbb{Z}_3 \times P$, where

P is a p-group of exponent $p > 3$, then by Proposition 5.0.3, each non-identity element of $\mathbb{Z}_3$ is adjacent to every non-identity element of P. Since $|P| \geq 5$, we get a subgraph of $\mathcal{D}(G)$, which is isomorphic to $K_{2,3}$; a contradiction. Let $G \cong D_8 \times \mathbb{Z}_3$. Then by Figure 5.2, observe that the subgraph induced by the set $\{(e,1),(e,2),(x,0),(x^3,0),(y,0)\}$ is isomorphic to $K_{2,3}$, which is not possible. If $G \cong \mathbb{Z}_{12}$, then by Figure 5.3, notice that the subgraph induced by the set $\{4,8,3,6,9\}$ is isomorphic to $K_{2,3}$; a contradiction. If $G \cong P \times \mathbb{Z}_3$, where $P = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, then by Proposition 5.0.3, the subgraph of $\mathcal{D}(G)$ induced by the set $(P \cup \mathbb{Z}_3) \setminus \{e\}$ has a subgraph isomorphic to $K_{2,3}$. Consequently, $G \cong \mathbb{Z}_2 \times P$, where P is a p-group of exponent $p > 2$.

Converse follows from Theorem 5.2.2. □

5.3 Embeddings of Difference Graphs

This section aims to classify all the finite nilpotent groups such that the difference graph $\mathcal{D}(G)$ is of genus (or cross-cap) at most two. If G is a p-group, then it is well known that the power graph and enhanced power graph of G are equal. Thus, $\mathcal{D}(G)$ is an empty graph whenever G is a p-group.

Let $G \cong P \times \mathbb{Z}_3$ be a finite nilpotent group, where P is a 2-group with exponent 4. Then we say that

- *G satisfies the condition $\mathcal{C}_1$*, if P contains two maximal cyclic subgroups H and K of order 4 such that $|H \cap K| = 2$, and the intersection of any other pair of maximal cyclic subgroups of P is trivial.
- *G satisfies the condition $\mathcal{C}_2$*, if P contains four maximal cyclic subgroups H_1, H_2, H_3 and H_4 of order 4 such that $|H_1 \cap H_2| = |H_3 \cap H_4| = 2$, and the intersection of any other pair of maximal cyclic subgroups of P is trivial.
- *G satisfies the condition $\mathcal{C}_3$*, if P contains three maximal cyclic subgroups

H_1, H_2 and H_3 of order 4 such that $|H_1 \cap H_2 \cap H_3| = 2$, and the intersection of any other pair of maximal cyclic subgroups of P is trivial.

For $1 \le i \le 3$, if the group G satisfies the condition $\mathcal{C}_i$, then we denote it by the group $\mathcal{G}_i$. The main result of this section is as follows.

Theorem 5.3.1. *Let G be a nilpotent group which is not a p-group and let $\gamma(\mathcal{D}(G))$ and $\overline{\gamma}(\mathcal{D}(G))$ be the genus and cross-cap of $\mathcal{D}(G)$, respectively. Then*

(i) $\gamma(\mathcal{D}(G)) = 1$ *if and only if G is isomorphic to one of the following groups:*

$$\mathbb{Z}_{18},\ \mathbb{Z}_{20},\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5,\ \mathbb{Z}_{28},\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_7,\ \ \mathcal{G}_1.$$

(ii) $\gamma(\mathcal{D}(G)) = 2$ *if and only if G is isomorphic to one of the following groups:*

$$\mathbb{Z}_{35},\ \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3,\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3,\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{11},\ \mathbb{Z}_{44},\ \mathcal{G}_2,\ \mathcal{G}_3.$$

(iii) $\overline{\gamma}(\mathcal{D}(G)) = 1$ *if and only if G is isomorphic to $\mathbb{Z}_{20}$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5$.*

(iv) $\overline{\gamma}(\mathcal{D}(G)) = 2$ *if and only if G is isomorphic to one of the groups: $\mathbb{Z}_{18}$, $\mathbb{Z}_{28}$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_7$, $\mathcal{G}_1$.*

In order to prove Theorem 5.3.1, the following lemmas are useful.

Lemma 5.3.2. *Let $G = P_1P_2 \cdots P_r$ $(r \ge 2)$ be a finite nilpotent group. Then the subgraph of $\mathcal{D}(G)$ induced by the set $(P_1 \cup P_2 \cup \cdots \cup P_r) \setminus \{e\}$ is isomorphic to complete r-partite graph $K_{|P_1|-1, |P_2|-1, \ldots, |P_r|-1}$.*

Proof. The result holds by Propositions 5.0.3 and 5.1.5. □

Lemma 5.3.3. *Let $G \cong P_1 \times P_2$ be a finite nilpotent group, where $\exp(P_i) = p_i$ for each $i \in \{1, 2\}$. Then $\mathcal{D}(G)$ is a complete bipartite graph isomorphic to $K_{|P_1|-1, |P_2|-1}$.*

Proof. First note that $\pi_G = \{p_1, p_2, p_1p_2\}$ and the identity element of G does not belongs to $V(\mathcal{D}(G))$. By Proposition 5.0.2, the elements of the order p_1p_2 do not belong to $V(\mathcal{D}(G))$. Thus, $V(\mathcal{D}(G))$ contains the elements of orders p_1 and p_2 only. Consider the sets $A = \{x \in V(\mathcal{D}(G)) : o(x) = p_1\}$ and $B = \{y \in V(\mathcal{D}(G)) : o(y) = p_2\}$. Clearly, A and B form a partition of $V(\mathcal{D}(G))$. By Propositions 5.0.3 and 5.1.5, $\mathcal{D}(G)$ is a complete bipartite graph which is isomorphic to $K_{|P_1|-1,|P_2|-1}$. This completes our proof. □

Lemma 5.3.4. *Let $G = P_1P_2 \cdots P_r$ be a finite nilpotent group. If $r \geq 3$, then $\gamma(\mathcal{D}(G)) \geq 3$ and $\overline{\gamma}(\mathcal{D}(G)) \geq 6$.*

Proof. Let $G = P_1P_2 \cdots P_r$ be a nilpotent group such that $|P_i| = p_i^{\alpha_i}$ and $p_j < p_{j+1}$ for $j \in [r-1]$. We prove our result in the following two cases:

Case-1: $r \geq 4$. By Lemma 5.3.2, the subgraph of $\mathcal{D}(G)$ induced by the set $(P_1 \cup P_2 \cup \cdots \cup P_r) \setminus \{e\}$ contains a subgraph isomorphic to $K_{1,2,4,6}$. Note that $K_{1,2,4,6}$ has a subgraph which is isomorphic to $K_{7,6}$. By Theorem 1.2.6, we get $\gamma(\mathcal{D}(G)) \geq 5$ and $\overline{\gamma}(\mathcal{D}(G)) \geq 10$.

Case-2: $r = 3$. In this case, we have $p_3 \geq 5$. Now, we discuss the following subcases:

Subcase-2.1: $p_3 = 5$. Clearly, $p_1 = 2$ and $p_2 = 3$. By Lemma 1.1.11, G has an element x such that $\langle x \rangle \cong \mathbb{Z}_{30}$. Note that $\mathbb{Z}_{30}$ has 4 elements of order 10 and 8 elements of order 15. Suppose $x_i, y_j \in \langle x \rangle$ such that $o(x_i) = 10$ and $o(y_j) = 15$ for every $i \in \{1, 2, 3, 4\}$ and $j \in \{1, 2, \ldots, 8\}$. Note that $x_i \overset{e}{\sim} y_j$, but $x_i \overset{p}{\nsim} y_j$ (see Remark 5.0.1). It follows that for each $i \in \{1, 2, 3, 4\}$ and $j \in \{1, 2, \ldots, 8\}$, we have $x_i \overset{d}{\sim} y_j$. Consequently, the subgraph of $\mathcal{D}(G)$ induced by the set $\{x_1, x_2, x_3, x_4, y_1, \ldots, y_8\}$ is isomorphic to $K_{4,8}$. Thus, $\gamma(\mathcal{D}(G)) \geq 3$ and $\overline{\gamma}(\mathcal{D}(G)) \geq 6$.

Subcase-2.2: $p_3 \geq 7$. In this subcase G has at least 1 element of order p_1, 2 elements of order p_2, 2 elements of order p_1p_2 and 6 elements of order p_3. By Proposition 5.0.3, the elements of order p_1, p_2 and p_1p_2 will be adjacent to each element of order p_3. Consequently, $\mathcal{D}(G)$ contains a subgraph isomorphic to $K_{5,6}$

and so $\gamma(\mathcal{D}(G)) \geq 3$, $\overline{\gamma}(\mathcal{D}(G)) \geq 6$. □

Now, we prove our main result of this section.

Proof of Theorem 5.3.1. Let $G = P_1P_2 \cdots P_r$ be a nilpotent group such that $|P_i| = p_i^{\alpha_i}$ and $p_j < p_{j+1}$ for $j \in [r-1]$. First, suppose that $\gamma(\mathcal{D}(G)) \leq 2$ and $\overline{\gamma}(\mathcal{D}(G)) \leq 2$. By Lemma 5.3.4, we obtain $r = 2$. Thus, $G = P_1 \times P_2$. We prove our result through the following cases:

Case-1: $p_1 \geq 7$. It follows that $p_2 \geq 11$. By Lemma 5.3.2, the subgraph of $\mathcal{D}(G)$ induced by the set $(P_1 \cup P_2) \setminus \{e\}$ has a subgraph isomorphic to $K_{6,10}$. Consequently, $\gamma(\mathcal{D}(G)) \geq 8$ and $\overline{\gamma}(\mathcal{D}(G)) \geq 16$.

Case-2: $p_1 = 5$. Then we must have $p_2 \geq 7$.

Subcase-2.1: $|P_1| = 5$ *and* $|P_2| = p_2$. Then $G \cong \mathbb{Z}_5 \times \mathbb{Z}_{p_2}$. By Lemma 5.3.3, $\mathcal{D}(G) \cong K_{4,p_2-1}$. If $p_2 = 7$, then $\gamma(\mathcal{D}(\mathbb{Z}_{35})) = 2$ and $\overline{\gamma}(\mathcal{D}(\mathbb{Z}_{35})) = 4$. If $p_2 \geq 11$, then $\gamma(\mathcal{D}(G)) \geq 4$ and $\overline{\gamma}(\mathcal{D}(G)) \geq 8$.

Subcase-2.2: $|P_1| = 5^\alpha$ *and* $|P_2| = p_2^\beta$, *where both* α *and* β *are not equal to* 1. If $\alpha \geq 2$, then the graph induced by the set $(P_1 \cup P_2) \setminus \{e\}$ has a subgraph isomorphic to $K_{24,6}$. Consequently, $\gamma(\mathcal{D}(G)) \geq 22$ and $\overline{\gamma}(\mathcal{D}(G)) \geq 44$. Similarly, if $\alpha = 1$ and $\beta \geq 2$, then we get $\gamma(\mathcal{D}(G)) \geq 23$ and $\overline{\gamma}(\mathcal{D}(G)) \geq 46$.

Case-3: $p_1 = 3$. Clearly, $p_2 \geq 5$.

Subcase-3.1: $|P_1| = 3$. Note that $\exp(P_2) = p_2^\alpha$, where $\alpha \geq 1$. If $\alpha = 1$, then by Theorem 5.2.8, the graph $\mathcal{D}(G)$ is planar. Consequently, $\gamma(\mathcal{D}(G)) = 0 = \overline{\gamma}(\mathcal{D}(G))$. Now, we assume that $\alpha \geq 2$. Let $x \in P_1$ such that $o(x) = 3$ and $y \in P_2$ such that $o(y) = p_2^2$. Notice that $\langle xy \rangle$ is a cyclic subgroup of order $3p_2^2$ in G. Consider the sets $S = \{z \in \langle xy \rangle : o(z) = 3p_2\}$ and $T = \{z' \in \langle xy \rangle : o(z') = p_2^2\}$. Let $x' \in S$ and $y' \in T$. Clearly, $x' \overset{e}{\sim} y'$. Also, neither $o(x')|o(y')$ nor $o(y')|o(x')$. By Remark 5.0.1, $x' \overset{p}{\nsim} y'$. Thus, $x' \overset{d}{\sim} y'$. Consequently, $\mathcal{D}(G)$ contains a subgraph isomorphic to $K_{|S|,|T|}$. Since $p_2 \geq 5$, we obtain $|S| \geq 8$ and $|T| \geq 20$. Thus, $\gamma(\mathcal{D}(G)) \geq 27$ and $\overline{\gamma}(\mathcal{D}(G)) \geq 54$.

Subcase-3.2: $|P_1| = 3^\alpha$, *where* $\alpha \geq 2$. Notice that P_1 has at least 8 non-identity elements and P_2 has at least 4 non-identity elements. By Lemma 5.3.2, the graph induced by $(P_1 \cup P_2) \setminus \{e\}$ contains a subgraph isomorphic to $K_{8,4}$. Consequently, $\gamma(\mathcal{D}(G)) \geq 3$ and $\overline{\gamma}(\mathcal{D}(G)) \geq 6$.

Case-4: $p_1 = 2$. Now, we have the following possible subcases.

Subcase-4.1.1: $|P_1| = 2$ *and* $\exp(P_2) = p_2$. By Theorem 5.2.8, the graph $\mathcal{D}(G)$ is planar. Consequently, $\gamma(\mathcal{D}(G)) = 0 = \overline{\gamma}(\mathcal{D}(G))$.

Subcase-4.1.2: $|P_1| = 2$, $|P_2| = 3^\alpha$, *where* $\alpha \geq 2$ *and* $\exp(P_2) = 9$. In view of Lemma 1.1.9, we have the following two further subcases:

Subcase-4.1.2(a): P_2 *contains exactly one cyclic subgroup of order* 9. By Lemma 1.1.8, we get $G \cong \mathbb{Z}_{18}$. Observe that the subgraph of $\mathcal{D}(\mathbb{Z}_{18})$ induced by the set $\{3, 9, 15, 2, 4, 8, 10, 14, 16\}$ is isomorphic to $K_{3,6}$. Thus, $\gamma(\mathcal{D}(G)) \geq 1$ and $\overline{\gamma}(\mathcal{D}(G)) \geq 2$. The genus 1 and cross-cap 2 drawings of $\mathcal{D}(\mathbb{Z}_{18})$ are given in Figures 5.4 and 5.5, respectively.

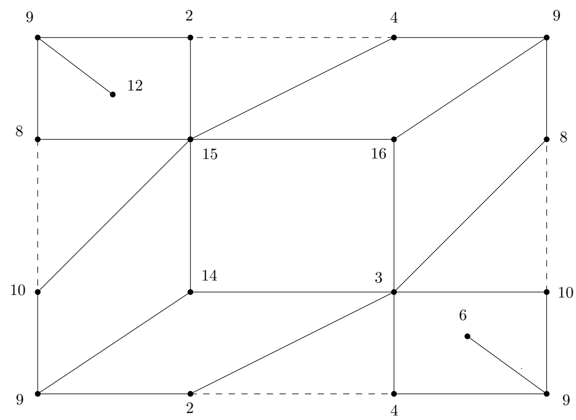


Figure 5.4: Embedding of $\mathcal{D}(\mathbb{Z}_{18})$ in $\mathbb{S}_1$.

Subcase-4.1.2(b): P_2 *contains two cyclic subgroups* H *and* K *of order* 9 *such that* $|H \cap K| = 3$. Let $P_1 = \langle x \rangle$, $H = \langle y \rangle$ and $K = \langle z \rangle$. Suppose $x_1, x_2 \in H \cap K$ such that $o(x_1) = o(x_2) = 3$. Then

$$P_1H = \{e, x, x_1, x_2, y, y^2, y^4, y^5, y^7, y^8, xx_1, xx_2, xy, xy^2, xy^4, xy^5, xy^7, xy^8\}, \text{ and}$$

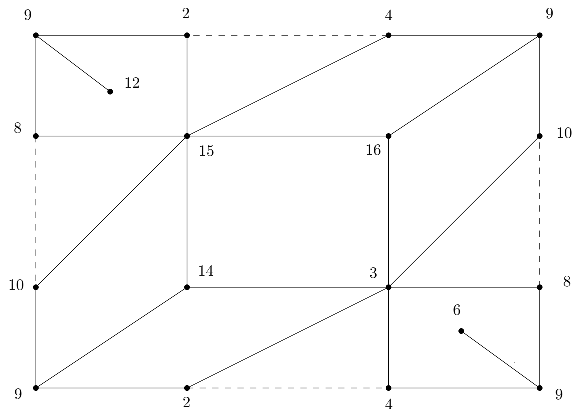


FIGURE 5.5: Embedding of $\mathcal{D}(\mathbb{Z}_{18})$ in $\mathbb{N}_2$.

$$P_1K = \{e, x, x_1, x_2, z, z^2, z^4, z^5, z^7, z^8, xx_1, xx_2, xz, xz^2, xz^4, xz^5, xz^7, xz^8\}$$

are maximal cyclic subgroups of order 18 (see Lemma 1.1.12). Consider the sets $S = \{x, xx_1, xx_2\}$ and $T = \{y, y^2, y^4, y^5, y^7, y^8, z, z^2, z^4, z^5, z^7, z^8\}$. Let $x' \in S$ and $y' \in T$. Notice that $x' \overset{e}{\sim} y'$. Also, neither $o(x')|o(y')$ nor $o(y')|o(x')$. It follows that $x' \overset{p}{\nsim} y'$ and so $x' \overset{d}{\sim} y'$. Thus, the subgraph induced by $S \cup T$ has a subgraph isomorphic to $K_{3,12}$. It implies that $\gamma(\mathcal{D}(G)) \geq 3$ and $\overline{\gamma}(\mathcal{D}(G)) \geq 5$.

Subcase-4.1.3: $|P_1| = 2$ *and* $\exp(P_2) = 3^{\alpha}$, *where* $\alpha \geq 3$. Then there exists an element $y \in P_2$ such that $o(y) = 27$. Let $P_1 = \langle x \rangle$. Observe that $\langle xy \rangle$ is a cyclic subgroup of order 54 in G. Consider the sets $S = \{s \in \langle xy \rangle : o(s) = 18\}$ and $T = \{t \in \langle xy \rangle : o(t) = 27\}$. Suppose $x' \in S$ and $y' \in T$. Observe that $x' \overset{e}{\sim} y'$ and $x' \overset{p}{\nsim} y'$ (cf. Remark 5.0.1). It follows that $x' \overset{d}{\sim} y'$. Thus, $\mathcal{D}(G)$ has a subgraph isomorphic to $K_{|S|,|T|}$. Since $|S| = 6$ and $|T| = 18$, we obtain $\gamma(\mathcal{D}(G)) \geq 16$ and $\overline{\gamma}(\mathcal{D}(G)) \geq 32$.

Subcase-4.1.4: $|P_1| = 2$ *and* $\exp(P_2) = p_2^{\alpha}$, *where* $p_2 \geq 5$, $\alpha \geq 2$. Then there exists an element $y \in P_2$ such that $o(y) = p_2^2$. Let $P_1 = \langle x \rangle$. Observe that $\langle xy \rangle$ is a cyclic subgroup of order $2p_2^2$ in G. Consider the sets $S = \{s \in \langle xy \rangle : o(s) = 2p_2\}$ and $T = \{t \in \langle xy \rangle : o(t) = p_2^2\}$. Similar to **Subcase 4.1.3**, we obtain a subgraph of $\mathcal{D}(G)$, which is isomorphic to $K_{|S|,|T|}$. Since $p_2 \geq 5$, we have $|S| \geq 4$ and $|T| \geq 20$.

It follows that $\gamma(\mathcal{D}(G)) \geq 9$ and $\overline{\gamma}(\mathcal{D}(G)) \geq 18$.

Subcase-4.2.1: $|P_1| = 4$ *and* $|P_2| = 3$. Then either $G \cong \mathbb{Z}_4 \times \mathbb{Z}_3$ or $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$. By Theorem 5.2.8, in both of these cases, $\mathcal{D}(G)$ is a planar graph. Consequently, $\gamma(\mathcal{D}(G)) = 0 = \overline{\gamma}(\mathcal{D}(G))$.

Subcase-4.2.2: $|P_1| = 4$ *and* $|P_2| = 5$. Then either $G \cong \mathbb{Z}_4 \times \mathbb{Z}_5 \cong \mathbb{Z}_{20}$ or $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5$. If $G \cong \mathbb{Z}_{20}$, then by Theorem 5.2.8, $\gamma(\mathcal{D}(G)) \geq 1$ and $\overline{\gamma}(\mathcal{D}(G)) \geq 1$. The genus 1 and cross-cap 1 drawings of $\mathcal{D}(\mathbb{Z}_{20})$ are given in Figures 5.6 and 5.7, respectively.

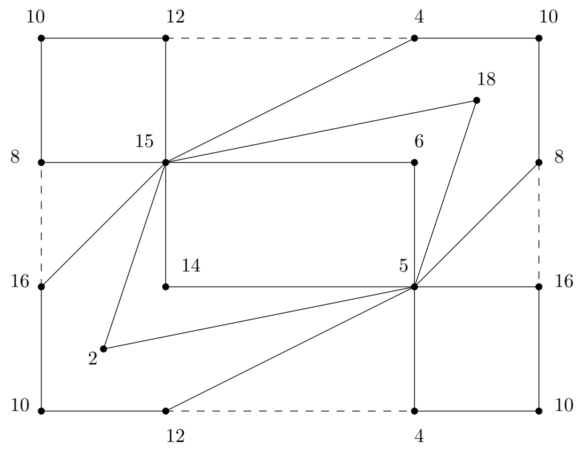


FIGURE 5.6: Embedding of $\mathcal{D}(\mathbb{Z}_{20})$ in $\mathbb{S}_1$.

If $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5$, then by Lemma 1.1.9, $\mathcal{D}(G) \cong K_{3,4}$. Consequently, $\gamma(\mathcal{D}(G)) = 1$ and $\overline{\gamma}(\mathcal{D}(G)) = 1$.

Subcase-4.2.3: $|P_1| = 4$ *and* $|P_2| = 7$. Then G is isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_7$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_7$. If $G \cong \mathbb{Z}_4 \times \mathbb{Z}_7$, then $\mathcal{D}(G)$ contains a subgraph which is isomorphic to $K_{3,6}$ (see Lemma 5.3.3). Consequently, $\gamma(\mathcal{D}(G)) \geq 1$ and $\overline{\gamma}(\mathcal{D}(G)) \geq 2$. The genus 1 and cross-cap 2 drawings of $\mathcal{D}(\mathbb{Z}_{28})$ are given in Figure 5.8 and 5.9, respectively.

If $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_7$, then by Lemma 5.3.3, $\mathcal{D}(G) \cong K_{3,6}$. It follows that $\gamma(\mathcal{D}(G)) = 1$ and $\overline{\gamma}(\mathcal{D}(G)) = 2$.

Subcase-4.2.4: $|P_1| = 4$ *and* $|P_2| = 9$. Then either $P_1 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $P_1 \cong \mathbb{Z}_4$. Also, P_2 is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$ or $\mathbb{Z}_9$. Consequently, G is isomorphic to one of

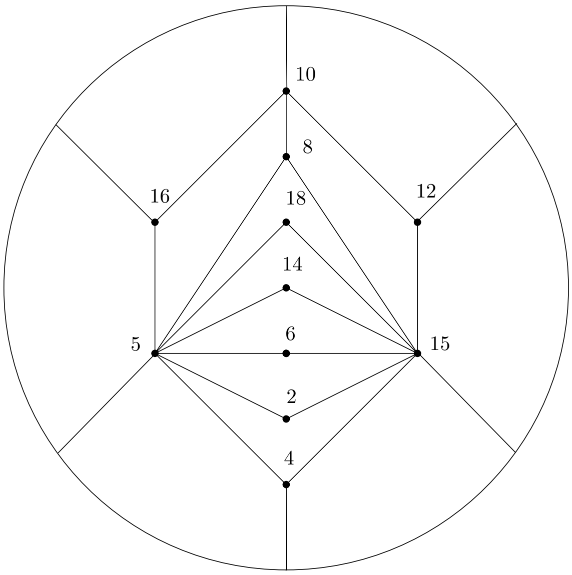


FIGURE 5.7: Embedding of $\mathcal{D}(\mathbb{Z}_{20})$ in $\mathbb{N}_1$.

the groups: $\mathbb{Z}_{36}$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9$, $\mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. If $G \cong \mathbb{Z}_{36}$, then $\mathcal{P}_E(G)$ is a complete graph. Consider the set $S = \{x \in G : o(x) = 9\}$ and $T = \{y \in G : o(y) \in \{2, 4, 6\}\}$. Let $x' \in S$ and $y' \in T$. Clearly, $x' \overset{e}{\sim} y'$ and by Remark 5.0.1, $x' \overset{p}{\nsim} y'$. Thus, $x' \overset{d}{\sim} y'$ and so $\mathcal{D}(G)$ contains a subgraph which is isomorphic to $K_{|S|,|T|}$. Since $|S| = 6$ and $|T| = 5$, we obtain $\gamma(\mathcal{D}(G)) \geq 3$ and $\overline{\gamma}(\mathcal{D}(G)) \geq 6$.

If $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9$, then G has exactly 3 maximal cyclic subgroups $H_1 = \langle(1, 0, 1)\rangle$, $H_2 = \langle(0, 1, 1)\rangle$ and $H_3 = \langle(1, 1, 1)\rangle$. Consider the set $S = \{x \in G : o(x) = 9\}$ and $T = \{y \in G : o(y) \in \{2, 6\}\}$. Then $|S| = 6$ and $|T| = 9$. Also, $S \subseteq (H_1 \cap H_2 \cap H_3)$. Suppose $x' \in S$ and $y' \in T$ are arbitrary elements. Then $x', y' \in H_i$ for some $i \in \{1, 2, 3\}$. It follows that $x' \overset{e}{\sim} y'$. Consequently, $x' \overset{d}{\sim} y'$ (cf. Remark 5.0.1). Thus, $\mathcal{D}(G)$ contains a subgraph isomorphic to $K_{6,9}$. Hence, $\gamma(\mathcal{D}(G)) \geq 7$ and $\overline{\gamma}(\mathcal{D}(G)) \geq 14$.

If $G \cong \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, then by Lemma 5.3.2, $\mathcal{D}(G)$ contains a subgraph which is isomorphic to $K_{3,8}$. Thus, $\gamma(\mathcal{D}(G)) \geq 2$ and $\overline{\gamma}(\mathcal{D}(G)) \geq 3$. The genus 2 drawing of $\mathcal{D}(G)$ is given in Figure 5.10.

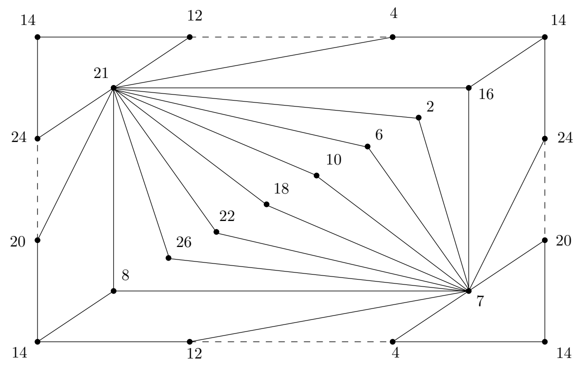


Figure 5.8: Embedding of $\mathcal{D}(\mathbb{Z}_{28})$ in $\mathbb{S}_1$.

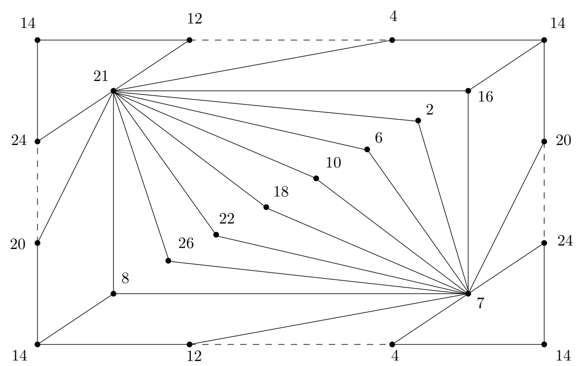


Figure 5.9: Embedding of $\mathcal{D}(\mathbb{Z}_{28})$ in $\mathbb{N}_2$.

If $G \cong \mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_3$, then by Lemma 5.3.3, we have $\mathcal{D}(G) \cong K_{3,8}$. Consequently, $\gamma(\mathcal{D}(G)) = 2$ and $\overline{\gamma}(\mathcal{D}(G)) = 3$.

Subcase-4.2.5: $|P_1| = 4$ *and* $|P_2| = 11$. Thus, either $G \cong \mathbb{Z}_4 \times \mathbb{Z}_{11} \cong \mathbb{Z}_{44}$ or $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{11}$. If $G \cong \mathbb{Z}_4 \times \mathbb{Z}_{11}$, then the subgraph of $\mathcal{D}(G)$ induced by the set $(P_1 \cup P_2) \setminus \{e\}$ contains a subgraph isomorphic to $K_{3,10}$. Consequently, $\gamma(\mathcal{D}(G)) \geq 2$ and $\overline{\gamma}(\mathcal{D}(G)) \geq 4$. The genus 2 drawing of $\mathcal{D}(\mathbb{Z}_{44})$ is given in Figure 5.11.
If $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{11}$, then $\mathcal{D}(G) \cong K_{3,10}$ (cf. Lemma 5.3.3) and so $\gamma(\mathcal{D}(G)) = 2$, $\overline{\gamma}(\mathcal{D}(G)) = 4$.

Subcase-4.2.6: $|P_1| = 4$ *and* $|P_2| = p_2$*, where* $p_2 \geq 13$. By Lemma 5.3.2, $\mathcal{D}(G)$ contains a subgraph isomorphic to $K_{3,12}$. Thus, $\gamma(\mathcal{D}(G)) \geq 3$ and $\overline{\gamma}(\mathcal{D}(G)) \geq 5$.

Subcase-4.2.7: $|P_1| = 4$ *and* $|P_2| = p_2^{\alpha}$*, where* $p_2 \geq 5$*,* $\alpha \geq 2$ *and* $\exp(P_2) = p_2$.

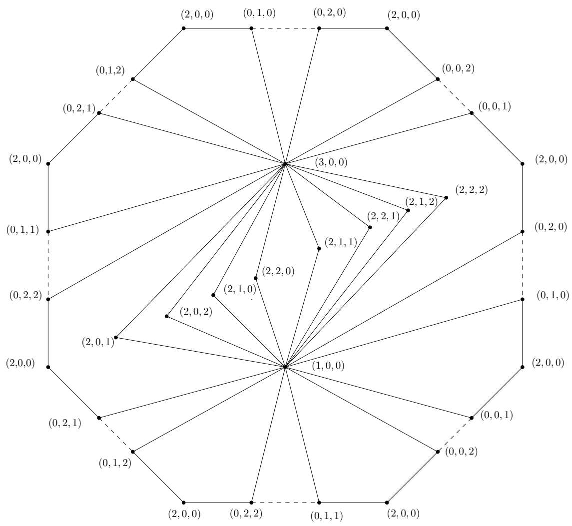


Figure 5.10: Embedding of $\mathcal{D}(\mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3)$ in $\mathbb{S}_2$.

By Lemma 5.3.2, $\mathcal{D}(G)$ contains a subgraph isomorphic to $K_{3,24}$. Thus, $\gamma(\mathcal{D}(G)) \geq 6$ and $\overline{\gamma}(\mathcal{D}(G)) \geq 11$.

Subcase-4.2.8: $|P_1| = 4$ *and* $|P_2| = 3^{\alpha}$*, where* $\alpha \geq 3$. It implies that the minimum number of non-identity elements in P_1 and P_2 are 3 and 26, respectively. Consequently, by Lemma 5.3.2, $\mathcal{D}(G)$ contains a subgraph isomorphic to $K_{3,26}$. It follows that $\gamma(\mathcal{D}(G)) \geq 6$ and $\overline{\gamma}(\mathcal{D}(G)) \geq 12$.

Subcase-4.3.1: $|P_1| = 2^{\alpha}$ *and* $|P_2| = 3^{\beta}$*, where* $\alpha \geq 3$*,* $\beta \geq 2$. By Lemma 5.3.2, the subgraph induced by the set $(P_1 \cup P_2) \setminus \{e\}$ contains a subgraph isomorphic to $K_{7,8}$. It follows that $\gamma(\mathcal{D}(G)) \geq 8$ and $\overline{\gamma}(\mathcal{D}(G)) \geq 15$.

Subcase-4.3.2: $|P_1| = 2^{\alpha}$ $(\alpha \geq 3)$ *with* $\exp(P_1) = 2$ *and* $|P_2| = 3$. By Theorem 5.2.8, the graph $\mathcal{D}(G)$ is planar and so $\gamma(\mathcal{D}(G)) = 0 = \overline{\gamma}(\mathcal{D}(G))$.

Subcase-4.3.3: $|P_1| = 2^{\alpha}$ $(\alpha \geq 3)$ *with* $\exp(P_1) = 4$ *and* $|P_2| = 3$. Consider $P_2 = \langle x \rangle$. Further, suppose that P_1 has t (≥ 1) maximal cyclic subgroups of order 4 and s (≥ 0) maximal cyclic subgroups of order 2. Consider the maximal cyclic subgroups $M_i = \langle y_i \rangle$, where $1 \leq i \leq t$, of order 4. If $s \geq 1$, then

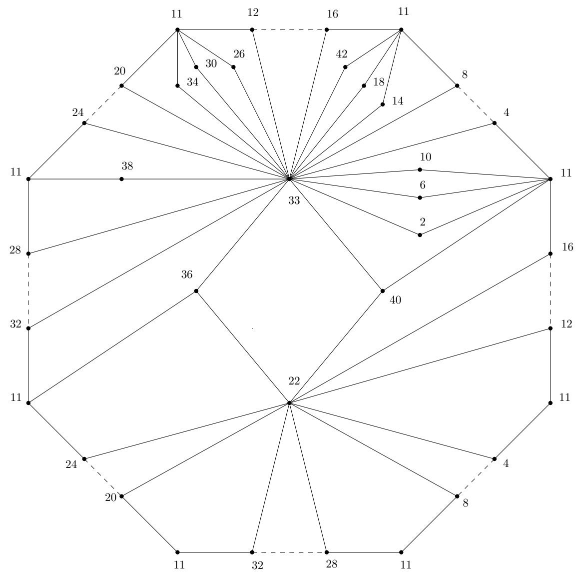


Figure 5.11: Embedding of $\mathcal{D}(\mathbb{Z}_{44})$ in $\mathbb{S}_2$.

consider $M'_j = \langle z_j \rangle$, where $1 \leq j \leq s$, as the maximal cyclic subgroup of order 2. Consequently, maximal cyclic subgroups of order 12 in G are of the form $M_iP_2 = \{e, x, x^2, y_i, y_i^2, y_i^3, y_ix, y_i^2x, y_i^3x, y_ix^2, y_i^2x^2, y_i^3x^2\}$. Also, maximal cyclic subgroups of order 6 are of the form $M'_jP_2 = \{e, x, x^2, z_j, z_jx, z_jx^2\}$ (see Lemma 1.1.12). Notice that P_2 is contained in every maximal cyclic subgroup of G. Also, the identity element e and the generators of M_iP_2 and M'_jP_2 do not belong to the vertex set of $\mathcal{D}(G)$ (cf. Proposition 5.0.2).

If $t = 1$, then by Lemma 1.1.8, we have $P_1 = D_8$. Thus, $G \cong D_8 \times \mathbb{Z}_3$. By Theorem 5.2.8, $\mathcal{D}(G)$ becomes planar and so $\gamma(\mathcal{D}(G)) = 0 = \overline{\gamma}(\mathcal{D}(G))$. We may now suppose that $t > 1$. In view of Lemma 1.1.9, now we discuss this subcase into the following six further subcases.

Subcase-4.3.3(a): *P_1 contains two maximal cyclic subgroups H and K of order 4 such that $|H \cap K| = 2$, and the intersection of any other pair of maximal cyclic subgroups of P_1 is trivial.* Without loss of generality, assume that $H = M_1$ and $K = M_2$. Since $|M_1 \cap M_2| = 2$, we get $M_1P_2 \cap M_2P_2 = \{e, x, x^2, y_1^2, y_1^2x, y_1^2x^2\}$. Consider the set $S = \{y \in M_1P_2 \cup M_2P_2 : o(y) = 4\}$ and $T = \{z \in M_1P_1 \cup M_2P_2 :$

$o(z) \in \{3, 6\}\}$. Let $x' \in S$ and $y' \in T$. Then $x', y' \in M_iP_2$ for some $i \in \{1, 2\}$. Thus, $x' \overset{e}{\sim} y'$. Consequently, $x' \overset{d}{\sim} y'$ (cf. Remark 5.0.1). Since $|S| = 4$ and $|T| = 4$, the subgraph induced by $S \cup T$ is isomorphic to $K_{4,4}$. Thus, $\gamma(\mathcal{D}(G)) \geq 1$ and $\overline{\gamma}(\mathcal{D}(G)) \geq 2$. The genus 1 and cross-cap 2 drawings of $\mathcal{D}(G)$ are given in Figures 5.12 and 5.13, respectively. The graph G_3, given in Figure 5.14, can be inserted in the face F of Figure 5.12 and 5.13, respectively, without edge crossings.

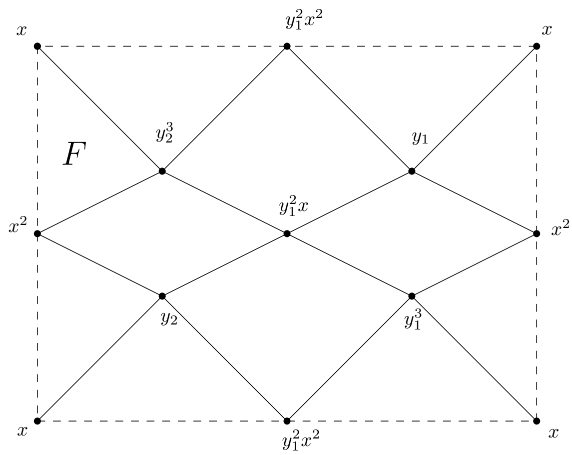


FIGURE 5.12: Embedding of $\mathcal{D}(P_1 \times P_2)$, where P_1 and P_2 are according to **Subcase-4.3.3(a)**, in $\mathbb{S}_1$.

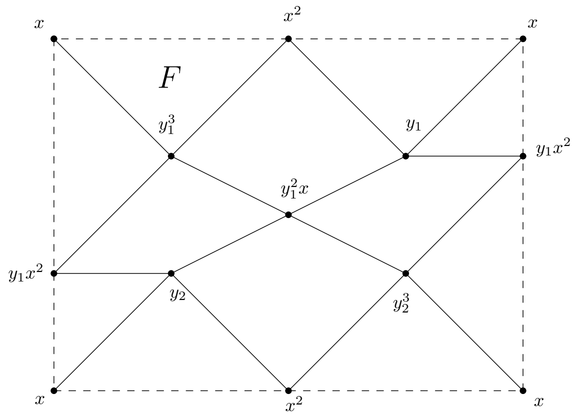


FIGURE 5.13: Embedding of $\mathcal{D}(P_1 \times P_2)$, where P_1 and P_2 are according to **Subcase-4.3.3(a)**, in $\mathbb{N}_2$.

Subcase-4.3.3(b): P_1 *contains four maximal cyclic subgroups* H_1, H_2, H_3 *and* H_4 *of order* 4 *such that* $|H_1 \cap H_2| = |H_3 \cap H_4| = 2$, *and the intersection of any* *other pair of maximal cyclic subgroups of P_1 is trivial.* Without loss of generality, assume that $H_i = M_i$ for $1 \leq i \leq 4$. Now, similar to the **Subcase-4.3.3(a)**, we get a subgraph Γ' of $\mathcal{D}(G)$, which is isomorphic to $K_{4,4}$. Moreover, $|M_3 \cap M_4| = 2$. It implies that $M_3P_2 \cap M_4P_2 = \{e, x, x^2, y_3^2, y_3^2x, y_3^2x^2\}$. Now, to embed $\mathcal{D}(G)$ through Γ' in $\mathbb{S}_1$, first we insert the vertices $y_3, y_3^3, y_4, y_4^3, y_3^2x, y_3^2x^2$ and their incident edges. Since $y_3 \overset{d}{\sim} y_3^2x \overset{d}{\sim} y_3^3 \overset{d}{\sim} y_3^2x^2 \overset{d}{\sim} y_4$ and $y_3^2x \overset{d}{\sim} y_4^3 \overset{d}{\sim} y_3^2x^2$; all these vertices must be inserted in the same face F'. Note that the vertices y_3, y_3^3, y_4 and y_4^3 are adjacent to both the vertices x and x^2 (see Proposition 5.0.2). Consequently, the face F' must contain the vertices x and x^2. After inserting the vertices y_3, y_3^3, y_4, y_4^3 and their incident edges, it is impossible to insert the vertices $y_3^2x, y_3^2x^2$ without edge crossing (see Figure 5.16). Thus, $\gamma(\mathcal{D}(G)) \geq 2$. The genus 2 drawing of $\mathcal{D}(G)$ is given in Figure 5.15, and the subgraph G_5, (given in Figure 5.14) can be inserted in the face F. By the similar argument used earlier in this subcase, any embedding of $\mathcal{D}(G)$ in $\mathbb{N}_2$ is also not possible without edge crossings. Hence, $\overline{\gamma}(\mathcal{D}(G)) \geq 3$.

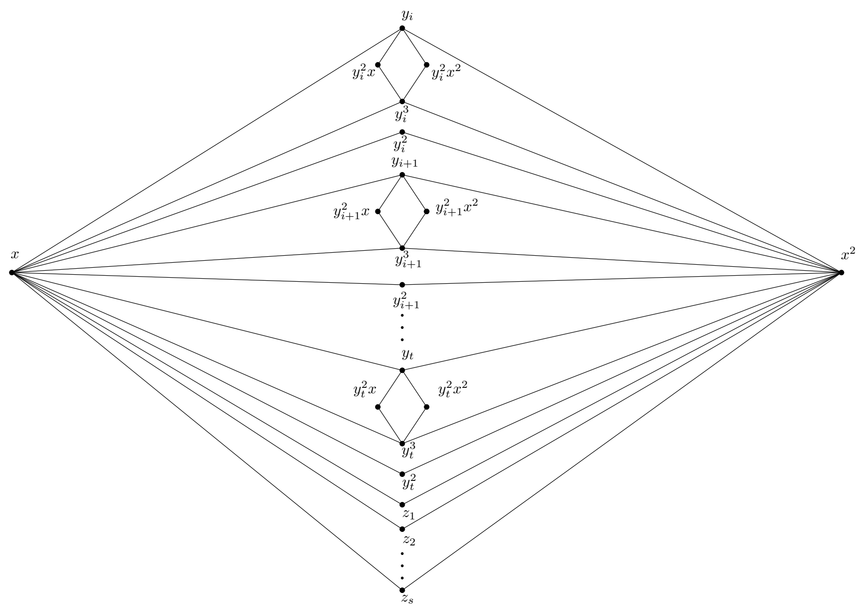


Figure 5.14: The subgraph G_i of $\mathcal{D}(G)$ induced by the set $(V(\mathcal{D}(G)) \setminus (\bigcup_{j=1}^{i-1} M_jP_2)) \cup \{x, x^2\}$.

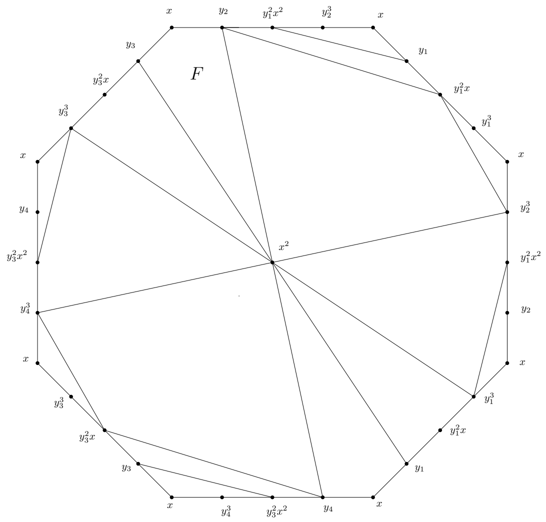


FIGURE 5.15: Embedding of $\mathcal{D}(P_1 \times P_2)$, where P_1 and P_2 are according to **Subcase-4.3.3(b)**, in $\mathbb{S}_2$.

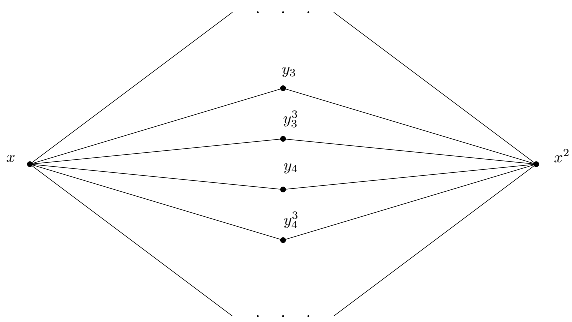


FIGURE 5.16: The face F'.

Subcase-4.3.3(c): *P_1 contains six maximal cyclic subgroups H_1, H_2, H_3, H_4, H_5 and H_6 of order 4 such that $|H_1 \cap H_2| = |H_3 \cap H_4| = |H_5 \cap H_6| = 2$, and the intersection of any three maximal cyclic subgroups of P_1 is trivial.* Without loss of generality, assume that $H_i = M_i$ for each $i \in \{1, 2, \ldots, 6\}$. Similar to the **Subcase-4.3.3(b)**, we obtain a subgraph Γ'' of $\mathcal{D}(G)$, induced by the set $V(\mathcal{D}(G)) \cap (M_1P_2 \cup M_2P_2 \cup M_3P_2 \cup M_4P_2)$, which cannot be embedded in $\mathbb{S}_1$ and $\mathbb{N}_2$. It follows that $\gamma(\mathcal{D}(G)) \geq 2$ and $\overline{\gamma}(\mathcal{D}(G)) \geq 3$. Also, we have $M_5P_2 \cap M_6P_2 = \{e, x, x^2, y_5^2, y_5^2x, y_5^2x^2\}$. Suppose $\gamma(\mathcal{D}(G)) = 2$. To embed $\mathcal{D}(G)$ in $\mathbb{S}_2$, first we insert the vertices $y_5, y_5^3, y_6, y_6^3, y_5^2x, y_5^2x^2$ and their incident edges in genus 2 drawing of Γ'.

By the similar argument used in **Subcase-4.3.3(b)** (by taking y_5 in place of y_3, and y_4 in place of y_6) it is impossible to insert these vertices in $\mathbb{S}_2$ without edge crossings. Thus, $\gamma(\mathcal{D}(G)) \geq 3$.

Subcase-4.3.3(d): *P_1 contains three maximal cyclic subgroups H_1, H_2 and H_3 of order 4 such that $|H_1 \cap H_2 \cap H_3| = 2$, and the intersection of any other pair of maximal cyclic subgroups of P_1 is trivial.* Without loss of generality, assume that $H_i = M_i$ for $1 \leq i \leq 3$. Since $|M_1 \cap M_2 \cap M_3| = 2$, we obtain $M_1P_2 \cap M_2P_2 \cap M_3P_2 = \{e, x, x^2, y_1^2, y_1^2x, y_1^2x^2\}$. Consider the set $S = \{y \in M_1P_2 \cup M_2P_2 \cup M_3P_2 : o(y) = 4\}$ and $T = \{z \in M_1P_2 \cup M_2P_2 \cup M_3P_2 : o(z) \in \{3, 6\}\}$. Note that the subgraph Γ', induced by the set $S \cup T$ is isomorphic to $K_{6,4}$. Thus, $\overline{\gamma}(\mathcal{D}(G)) \geq 4$ and $\gamma(\mathcal{D}(G)) \geq 2$. Now, we show that $\mathcal{D}(G)$ can be embedded in $\mathbb{S}_2$ without edge crossing. Since Γ' is a bipartite graph, it implies that each face of Γ' is of even length at least 4 in $\mathbb{S}_2$. Consequently, each face must contain at least two vertices of each partite set of Γ'. It follows that there exists an embedding of Γ' in $\mathbb{S}_2$ such that the face F_2 contains the vertices x and x^2. Now, one can embed $\mathcal{D}(G)$ in $\mathbb{S}_2$ through Γ' by inserting the subgraph G_4 into F_2. Therefore, $\gamma(\mathcal{D}(G)) = 2$.

Subcase-4.3.3(e): *P_1 contains five maximal cyclic subgroups H_1, H_2, H_3, H_4 and H_5 of order 4 such that $|H_1 \cap H_2 \cap H_3| = 2 = |H_4 \cap H_5|$, and the intersection of any four maximal cyclic subgroups of P_1 is trivial.* Without loss of generality, assume that $H_i = M_i$ for each $i \in \{1, 2, \ldots, 5\}$. Similar to the **Subcase-4.3.3(d)**, we obtain a subgraph Γ' of $\mathcal{D}(G)$, induced by the set $V(\mathcal{D}(G)) \cap (M_1P_2 \cup M_2P_2 \cup M_3P_2)$, which cannot be embedded in $\mathbb{S}_1$ and $\mathbb{N}_3$ without edge crossing. It follows that $\gamma(\mathcal{D}(G)) \geq 2$ and $\overline{\gamma}(\mathcal{D}(G)) \geq 3$. Moreover, $|M_4 \cap M_5| = 2$ implies that $M_4P_2 \cap M_5P_2 = \{e, x, x^2, y_4^2, y_4^2x, y_4^2x^2\}$. Suppose $\gamma(\mathcal{D}(G)) = 2$. Now, to embed $\mathcal{D}(G)$ in $\mathbb{S}_2$, first we insert the vertices $y_4, y_4^3, y_5, y_5^3, y_4^2x, y_4^2x^2$ and their incident edges in genus 2 drawing of Γ'. By the similar argument used in **Subcase-4.3.3(b)** (by taking y_4 in place of y_3, and y_5 in place of y_6) it is impossible to insert these vertices in $\mathbb{S}_2$ without edge crossings. It follows that $\gamma(\mathcal{D}(G)) \geq 3$.

Subcase-4.3.3(f): *P_1 contains four maximal cyclic subgroups H_1, H_2, H_3 and H_4 of order 4 such that* $|H_1 \cap H_2 \cap H_3 \cap H_4| = 2$. Without loss of generality, assume that $H_i = M_i$ for each $i \in \{1, 2, 3, 4\}$. Since $|M_1 \cap M_2 \cap M_3 \cap M_4| = 2$, we obtain $M_1P_2 \cap M_2P_2 \cap M_3P_2 \cap M_4P_2 = \{e, x, x^2, y_1^2, y_1^2x, y_1^2x^2\}$. Consider the set $S = \{y \in M_1P_2 \cup M_2P_2 \cup M_3P_2 \cup M_4P_2 : o(y) = 4\}$ and $T = \{z \in M_1P_2 \cup M_2P_2 \cup M_3P_2 \cup M_4P_2 : o(z) \in \{3, 6\}\}$. Note that the subgraph of $\mathcal{D}(G)$ induced by $S \cup T$ has a subgraph isomorphic to $K_{8,4}$. Thus, $\overline{\gamma}(\mathcal{D}(G)) \geq 3$ and $\gamma(\mathcal{D}(G)) \geq 6$.

Subcase-4.3.4: $|P_1| = 2^\alpha$ *and* $|P_2| = 3$*, where* $\alpha \geq 3$ *and* $\exp(P_1) \geq 8$. Then there exists an element $y \in P_2$ such that $o(y) = 8$. Suppose $x \in P_1$ such that $o(x) = 3$. Notice that $\langle xy \rangle$ is a cyclic subgroup of order 24 in G. Consider the sets $S = \{s \in \langle xy \rangle : o(s) = 8\}$ and $T = \{t \in \langle xy \rangle : o(t) \in \{3, 6, 12\}\}$. Let $x' \in S$ and $y' \in T$. Observe that, $x' \overset{e}{\sim} y'$. Consequently, by Remark 5.0.1, $x' \overset{d}{\sim} y'$. Thus, $\mathcal{D}(G)$ contains a subgraph isomorphic to $K_{|S|,|T|}$. Since $|S| = 4$ and $|T| = 8$, we obtain $\gamma(\mathcal{D}(G)) \geq 3$ and $\overline{\gamma}(\mathcal{D}(G)) \geq 6$.

Subcase-4.3.5: $|P_1| = 2^\alpha$ *and* $|P_2| = p_2$ *such that* $\alpha \geq 3$*,* $p_2 \geq 5$. Then P_1 has at least 7 non-identity elements and P_2 has at least 4 non-identity elements. Consequently, by Lemma 5.3.2, $\mathcal{D}(G)$ contains a subgraph which is isomorphic to $K_{4,7}$. It follows that $\gamma(\mathcal{D}(G)) \geq 3$ and $\overline{\gamma}(\mathcal{D}(G)) \geq 5$.

This completes our proof.

5.4 Difference Graph of Non-abelian Groups

In this section, we investigate the difference graphs of non-nilpotent groups S_n and A_n with some forbidden induced subgraphs. Note that for $1 \leq n \leq 4$ and $1 \leq n \leq 6$, the symmetric group S_n and the alternating group A_n, respectively, are EPPO-groups. Hence, their difference graphs are null graphs for the corresponding values of n. Therefore, we consider $\mathcal{D}(S_n)$ $(n \geq 5)$ and $\mathcal{D}(A_n)$ $(n \geq 7)$.

In the following theorems, we determine the values of n for which $\mathcal{D}(S_n)$ is cograph, chordal, split and threshold, respectively.

Theorem 5.4.1. *For $n \geq 5$, the difference graph $\mathcal{D}(S_n)$ is a chordal graph if and only if $n = 5$.*

Proof. First, we show that for $n \geq 6$, $\mathcal{D}(S_n)$ has an induced cycle isomorphic to C_4. Notice that $H_1 = \langle (12), (345) \rangle$, $H_2 = \langle (345), (16) \rangle$, $H_3 = \langle (16), (354) \rangle$ and $H_4 = \langle (354), (12) \rangle$ are cyclic subgroups of S_6. Thus, $(12) \overset{e}{\sim} (345) \overset{e}{\sim} (16) \overset{e}{\sim} (354) \overset{e}{\sim} (12)$. By Remark 5.0.1, $(12) \overset{p}{\nsim} (345) \overset{p}{\nsim} (16) \overset{p}{\nsim} (354) \overset{p}{\nsim} (12)$. By Proposition 5.1.4, $(12) \overset{d}{\nsim} (16)$ and $(345) \overset{d}{\nsim} (354)$. Consequently, the subgraph of $\mathcal{D}(S_6)$ induced by these four vertices is isomorphic to C_4. Hence, $\mathcal{D}(S_n)$ is not a chordal graph for $n \geq 6$.

Now, we shall prove that $\mathcal{D}(S_5)$ is a chordal graph. On the contrary, assume that $\mathcal{D}(S_5)$ has an induced subgraph isomorphic to a cycle graph of at least 4 vertices, say $x \overset{d}{\sim} y \overset{d}{\sim} z \overset{d}{\sim} \cdots \overset{d}{\sim} t \overset{d}{\sim} x$. Observe that $\pi_{S_5} = \{2, 3, 4, 5, 6\}$. Also, note that each element of order 4, 5, 6 generates a maximal cyclic subgroup of S_5. Consequently, $o(x), o(y), o(z), o(t) \in \{2, 3\}$. Notice that each element of order 6 in S_5 is of the form $(abc)(de)$, and each element of order 3 commutes with exactly one element of order 2. Now, let $o(x) = 2$. Then by Proposition 5.1.4, $o(y) = 3$ and $o(z) = 2$. Since $x \overset{d}{\sim} y$ and $y \overset{d}{\sim} z$, we get $\langle x, y \rangle$ and $\langle y, z \rangle$ are cyclic subgroups of S_5. It follows that $xy = yx$ and $yz = zy$. Consequently, $x = z$, which is not possible. Similarly, if $o(x) = 3$, then we obtain $y = t$, which is not possible. Hence, $\mathcal{D}(S_5)$ is a chordal graph. □

Theorem 5.4.2. *For $n \geq 5$, the difference graph $\mathcal{D}(S_n)$ is a cograph if and only if $n = 5$.*

Proof. For $n = 6$, notice that the induced path $(15)(24)(36) \overset{e}{\sim} (164)(253) \overset{e}{\sim} (12)(34)(56) \overset{e}{\sim} (153)(426)$ is isomorphic to P_4. By Remark 5.0.1, $(15)(24)(36) \overset{p}{\nsim} (164)(253)$, $(164)(253) \overset{p}{\nsim} (12)(34)(56)$ and $(12)(34)(56) \overset{p}{\nsim} (153)(426)$. We obtain an induced path isomorphic to P_4. Note that S_n, where $n \geq 6$, has a subgroup isomorphic to S_6. By Lemma 5.0.4, for $n \geq 6$, $\mathcal{D}(S_n)$ is not a cograph.

Conversely, if $n = 5$, then we show that $\mathcal{D}(S_5)$ is a cograph. On the contrary, assume that $\mathcal{D}(S_5)$ has an induced path $P : x \overset{d}{\sim} y \overset{d}{\sim} z \overset{d}{\sim} t$ isomorphic to P_4. Notice that $\pi_{S_5} = \{2, 3, 4, 5, 6\}$ and each element of the order 4, 5, 6 generates a maximal cyclic subgroup of S_5. Consequently, elements of these orders do not belong to the vertex set of $\mathcal{D}(S_5)$ (cf. Proposition 5.0.2). It follows that the order of each vertex in P is either 2 or 3. Let $o(x) = 2$. Then by Proposition 5.1.4, $o(y) = o(t) = 3$ and $o(z) = 2$. Note that $\langle x, y\rangle$ is a cyclic subgroup of order 6. Without loss of generality, assume that $x = (12)$ and $y = (345)$. Also, $\langle y, z\rangle$ is a cyclic subgroup of order 6. It follows that $z = (12)$, which is not possible.

Now, let $o(x) = 3$. Then by Proposition 5.1.4, $o(y) = o(t) = 2$ and $o(z) = 3$. Observe that $\langle x, y\rangle$ is a cyclic subgroup of order 6 in S_5. Without loss of generality, suppose $x = (123)$ and $y = (45)$. Note that $\langle y, z\rangle$ is a cyclic subgroup of order 6. It follows that either $z = (132)$ or $z = (123)$. But $x \neq z$ implies that $z = (132)$. Since $\langle z, t\rangle$ is a cyclic subgroup of order 6, we obtain $t = (45)$, which is not possible. Hence, $\mathcal{D}(S_5)$ is a cograph. □

Theorem 5.4.3. *For $n \geq 5$, the difference graph $\mathcal{D}(S_n)$ is neither a split graph nor a threshold graph.*

Proof. Notice that $H_1 = \langle(12), (345)\rangle$ and $H_2 = \langle(13), (245)\rangle$ are cyclic subgroups of order 6 of S_5. Thus, $(12) \overset{e}{\sim} (345)$ and $(13) \overset{e}{\sim} (245)$. By Remark 5.0.1, $(12) \overset{p}{\nsim} (345)$ and $(13) \overset{p}{\nsim} (245)$. Consequently, $(12) \overset{d}{\sim} (345)$ and $(13) \overset{d}{\sim} (245)$. Now, by Proposition 5.1.4, $(12) \overset{d}{\nsim} (13)$ and $(345) \overset{d}{\nsim} (245)$. Also, $\langle(12), (245)\rangle$ and $\langle(13), (345)\rangle$ are non-cyclic subgroups of S_5. Consequently, $(12) \overset{d}{\nsim} (245)$ and $(13) \overset{d}{\nsim} (345)$. Thus, the subgraph induced by the set $\{(12), (345), (13), (245)\}$ is isomorphic to $2K_2$. Hence, S_n, where $n \geq 5$, can neither be a split graph nor a threshold graph. □

We next study the difference graphs of alternating groups.

Theorem 5.4.4. *For $n \geq 7$, the difference graph $\mathcal{D}(A_n)$ is not a chordal graph.*

Proof. Notice that $(12)(34) \overset{d}{\sim} (567) \overset{d}{\sim} (13)(24) \overset{d}{\sim} (576) \overset{d}{\sim} (12)(34)$ is an induced cycle of $\mathcal{D}(A_7)$ which is isomorphic to C_4. It follows that $\mathcal{D}(A_7)$ is not a chordal graph. Hence, $\mathcal{D}(A_n)$, where $n \geq 7$, cannot be a chordal graph. □

By the proof of the above theorem, observe that the difference graph $\mathcal{D}(A_n)$, where $n \geq 7$, contains an induced cycle isomorphic to C_4. Thus, we have the following corollary.

Corollary 5.4.5. *For $n \geq 7$, the difference graph $\mathcal{D}(A_n)$ is neither a split graph nor a threshold graph.*

Theorem 5.4.6. *For $n \geq 7$, the difference graph $\mathcal{D}(A_n)$ is a cograph if and only if $n = 7$.*

Proof. Note that $(12)(34) \overset{d}{\sim} (567) \overset{d}{\sim} (12)(38) \overset{d}{\sim} (564)$ is an induced path in $\mathcal{D}(A_8)$ which is isomorphic to P_4. Thus, $\mathcal{D}(A_8)$ is not a cograph. It follows that for $n \geq 8$, $\mathcal{D}(A_n)$ is not a cograph. Now, we show that $\mathcal{D}(A_7)$ is a cograph. On contrary, assume that $\mathcal{D}(A_7)$ has an induced path $P : x \overset{d}{\sim} y \overset{d}{\sim} z \overset{d}{\sim} t$ isomorphic to P_4. Note that $\pi_{A_7} = \{2, 3, 4, 5, 6, 7\}$. Also, each element of the order 4,5,6,7 generates a maximal cyclic subgroup of A_7. Consequently, the elements of these order do not belong to the vertex set of $\mathcal{D}(A_7)$. It follows that $o(x), o(y), o(z), o(t) \in \{2, 3\}$. Let $o(x) = 2$. Then $o(y) = o(t) = 3, o(z) = 2$. Observe that $\langle x, y\rangle$ is a cyclic subgroup of order 6 and each element of order 6 in A_7 has the cycle decomposition $\{2, 2, 3\}$. Without loss of generality, assume that $x = (12)(34)$ and $y = (567)$. Since $\langle y, z\rangle$ and $\langle z, t\rangle$ are cyclic subgroups of order 6, we get either $z = (13)(24)$ or $(14)(23)$. It follows that $t = (576)$. In this case, $t \overset{d}{\sim} x$; a contradiction.

Now, let $o(x) = 3$. Then $o(y) = o(t) = 2$ and $o(z) = 3$. Note that $\langle x, y\rangle$ is a cyclic subgroup of order 6 and each element of order 6 in A_7 has the cycle decomposition

$\{2,2,3\}$. Without loss of generality, assume that $x = (123)$ and $y = (45)(67)$. Since $\langle y, z\rangle$ and $\langle z, t\rangle$ are cyclic subgroups of order 6, we get $z = (132)$. Consequently, either $t = (46)(57)$ or $(47)(65)$. In both of these cases, $t \overset{d}{\sim} x$; again a contradiction. Thus, the difference graph $\mathcal{D}(A_7)$ is a cograph. $\square$

Chapter 6

Conclusion and Future Research Work

6.1 Contribution of the Thesis

This thesis contributes on further developments on the enhanced power graphs of finite groups. The results obtained in this thesis reveal some interconnections between graph theoretic properties of the enhanced power graph and algebraic properties of underlying groups. The chapter wise contribution of the thesis are as follows.

In Chapter 2, various aspects of enhanced power graphs of finite groups have been explored. In this connection, we characterized finite groups G such that the minimum degree and the vertex connectivity of the enhanced power graph $\mathcal{P}_E(G)$ are equal. Additionally, finite groups for which the (proper) enhanced power graphs are (strongly) regular have been classified. The vertex connectivity of enhanced power graphs for certain nilpotent groups is also determined. Furthermore, we obtained upper and lower bounds for the Wiener index of the enhanced power graph of finite nilpotent groups. The nilpotent groups achieving these bounds are also characterized.

In this chapter, the existing results on the lambda number of power graphs have been extended to enhanced power graphs. Specifically, for a non-trivial simple group G of order n, we proved that $\lambda(\mathcal{P}_E(G)) = n$ if and only if G is not a cyclic group of order $n \geq 3$. We also determined the lambda number of enhanced power graphs for finite nilpotent groups.

In the final part of this chapter, we computed the Laplacian spectrum of enhanced power graphs for certain non-abelian groups such as semidihedral groups, dihedral groups, and generalized quaternion groups. We also obtained the metric dimension and resolving polynomials of enhanced power graphs for these groups. Moreover, we delve into distant properties and detour distant properties including closure, interior, distance degree sequence, and eccentric subgraph of the enhanced power graphs of semidihedral groups.

In Chapter 3, we supplied an affirmative answer to the question posed by Cameron [2022], which states that: Is it true that the complement of the enhanced power graph $\mathcal{P}_E(G)$ of a non-cyclic group G has only one connected component apart from isolated vertices? We classified all finite groups G such that the graph $\overline{\mathcal{P}_E(G)}$ is bipartite. We proved that the graph $\overline{\mathcal{P}_E(G)}$ is weakly perfect. Further, we studied the subgraph $\overline{\mathcal{P}_E(G^*)}$ of $\overline{\mathcal{P}_E(G)}$ induced by the set of all the non-isolated vertices of $\overline{\mathcal{P}_E(G)}$. We classified all finite groups G such that the graph $\overline{\mathcal{P}_E(G^*)}$ is unicyclic and pentacyclic, respectively. We proved the non-existence of finite groups G such that the graph $\overline{\mathcal{P}_E(G^*)}$ is bicyclic, tricyclic or tetracyclic. Finally, all finite groups G such that the graph $\overline{\mathcal{P}_E(G^*)}$ is outerplanar, planar, projective-planar and toroidal, respectively, have been characterized.

Bera [2022] characterized certain finite nilpotent groups where both power graphs and proper power graphs are line graphs. In Chapter 4, we extended the results of the above-mentioned article to arbitrary finite groups. Moreover, we classified all the finite groups whose enhanced power graphs are line graphs. We classified all the finite nilpotent groups (except non-abelian 2-groups) whose proper enhanced power graphs are line graphs of some graphs. Finally, we determined all the finite groups whose (proper) power graphs and (proper) enhanced power graphs are the complement of line graphs, respectively.

The study of the difference graph $\mathcal{D}(G)$ of a finite group G was initiated by Biswas et al. [2022]. In Chapter 5, we continued the study of the difference graph $\mathcal{D}(G)$ of a finite group G and classified all the nilpotent groups G such that $\mathcal{D}(G)$ is a chordal graph, cograph, threshold graph, bipartite, Eulerian, planar and outerplanar, respectively. Also, we characterize all the finite nilpotent groups G such that the genus (or cross-cap) of the difference graph $\mathcal{D}(G)$ is at most 2. We further studied the difference graph of some non-nilpotent groups and characterized all the values of n such that the difference graph of the symmetric group S_n (or alternating group A_n) is cograph and chordal, respectively.

6.2 Scope for Future Research

We conclude this thesis with some research questions which can be addressed in future.

- Classify all finite groups G such that the enhanced power graph $\mathcal{P}_E(G)$ has genus (cross-cap) at most two.

- Normal subgroup based power graph was introduced by Bhuniya and Bera [2017]. Analogously one can define normal subgroup based enhanced power graph. Let H be a normal subgroup of a finite group G. The normal subgroup based enhanced power graph $\Gamma_H(G)$ of G is the simple undirected graph with vertex set $V(\Gamma_H(G)) = (G \setminus H) \cup \{e\}$ and two distinct vertices a and b are adjacent if there exists an element $c \in G$ such that $aH = c^m H$ and $bH = c^n H$ for some $m, n \in \mathbb{N}$. Graph theoretical properties of the normal subgroup based enhanced power graph $\Gamma_H(G)$ and its interconnections with underlying groups (or vice versa) would be interesting to investigate.

- Classification of all finite groups G such that the enhanced power graph $\mathcal{P}_E(G)$ is Hamiltonian.
- Classification of all finite groups G such that $\mathcal{P}_E(G)$ is minimally (edge) connected.
- Complete characterization of finite groups G such that the proper enhanced power graph $\mathcal{P}_E^*(G)$ is connected. However, some partial answers are known in this case.
- The study of enhanced power graph of semigroup $\mathcal{P}_E(S)$ has been initiated in Dalal et al. [2024]. However, not much work has been done in this direction. One can reveal some more insights of $\mathcal{P}_E(S)$.

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List of Research Publications

The following research papers are published/accepted in international journals:

1. J. Kumar, X. Ma, **Parveen** and S. Singh, Certain properties of the enhanced power graph associated with a finite group, *Acta Mathematica Hungarica*, **169**(1):238–251, 2023.

2. **Parveen**, S. Dalal and J. Kumar, Enhanced power graphs of certain non-abelian groups, *Discrete Mathematics, Algorithms and Applications*, **16**(5):2350063, 2024.

3. J. Kumar, R.P. Panda and **Parveen**, On the difference graph of power graphs of finite groups, *Quaestiones Mathematicae*, **47**(5):997-1018, 2024.

4. **Parveen** and J. Kumar, The complement of enhanced power graph of a finite group, *Algebra Colloquium*, Accepted, 2023.
   `https://arxiv.org/abs/2207.04641`

5. **Parveen** and J. Kumar, Nilpotent groups whose Difference graphs have positive genus, *Ricerche di Matematica*, 2023.
   `https://doi.org/10.1007/s11587-023-00830-9`

6. **Parveen**, S. Dalal and J. Kumar, Lambda Number of the enhanced power graph of a finite group, *Journal of Algebra and Its Applications*, 2023.
   `https://doi.org/10.1142/S0219498825501828`

7. **Parveen** and J. Kumar, On finite groups whose power graphs are line graphs, *Journal of Algebra and Its Applications*, 2024.
   `https://doi.org/10.1142/S0219498825502858`

Paper Presentations in Conferences

1. *The Complement of Enhanced Power Graph of a Finite Group* at International Conference on Algebra & Discrete Mathematics, organized by Department of Mathematics, Savitribai Phule Pune University, Pune, India, during 26-28 May 2022.

2. *On the Genus of the Difference Graph of Finite Groups* at International Conference on Graphs, Networks and Combinatorics (ICGNC 2023), organized by the Department of Mathematics, Ramanujan College, University of Delhi, Delhi (India), during 10-12 January 2023.

3. *On the Difference Graph of Power Graphs of Finite Groups* at International Conference on Recent Advances in Graph Theory and Allied Areas (ICRAGAA-2023), organized by the Department of Mathematics, St. Aloysius College Elthuruth, Thrissur, Kerala, India, during 02-04 February, 2023.

4. *On the Enhanced Power Graph of Finite Groups* at the 4th International Conference on Computational Algebra, Computational Number Theory and Applications (CACNA-2023), organized by the University of Kashan, Iran, during 4-6 July 2023.

5. *Finite Groups whose Normal Subgroup Based Enhanced Power Graph have Positive Genus* at International Conference on Graph Theory and its Applications (ICGTA-2023), Organized by Amrita Vishwa Vidyapeetham, Coimbatore, India, during 18-20 December, 2023.

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