Hybridizable Discontinuous Galerkin Method for Integro-Differential Equations

THESIS

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CERTIFICATE

This is to certify that the thesis entitled, **"Hybridizable Discontinuous Galerkin Method for Integro-Differential Equations"** and submitted by **Ms. Riya Jain** ID No. **2018PHXF0422P** for the award of Ph.D. Degree of the institute embodies original work done by her under my supervision.

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Dedicated to

My Father

Late Mr. Rajesh Kumar Jain

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ABSTRACT

This thesis primarily focuses on exploring the hybridizable discontinuous Galerkin (HDG) method for parabolic and hyperbolic integro-differential equations of linear and nonlinear type. Emphasis has been given to the semi-discrete and fully- discrete a priori error analysis. In the existing literature, only sub-optimality was achieved for the flux and trace variables, whereas, we have proved the optimal order of convergence for the scalar, flux and trace variables. We have introduced a mixed type Ritz-Volterra projection for the model problems, which is one of the most crucial components in achieving optimality. Further, we have used the HDG projection, elliptic dual problem and Gronwall's lemma to derive the optimal convergence rates for the Ritz-Volterra projection. These estimates then give us the a priori error estimates.

We have also proved the super-convergence results for the scalar variable by defining a new approximation to the semi-discrete HDG approximation, known as the post-processed solution. To demonstrate the super-convergence, we have made use of the duality argument and related regularity results.

We also have discretized the HDG scheme in the time direction. For parabolic problems, the backward Euler method and the left rectangle rule have been used to approximate the derivative and integral terms, respectively, which help us to achieve the first order of convergence. While, for hyperbolic problems, the central difference scheme and mid-point rule have been used to approximate the derivative and integral terms, respectively, which, in turn, help us in achieving the second order of convergence in the temporal direction.

Subsequently, the theory has been validated through various examples on the two-dimensional domain. It has been verified that the optimal order of convergence is achieved for scalar and vector variables. In contrast, super-convergence is achieved for the post-processed solution for different degrees of polynomials. Finally, possible extensions of the work and scope for future investigations are discussed in the concluding Chapter.

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Chapter 1

Introduction

Partial Differential Equations (PDEs) hold significant importance in various scientific, engineering, and mathematical fields due to their ability to describe complex phenomena and systems. Understanding the importance of PDEs and the significance of solving them can be explained in the following ways:

- **Fundamental Laws of Physics:** PDEs are integral in formulating and expressing the fundamental laws of physics. Equations like heat equation, wave equation, and the Schrödinger equation in quantum mechanics rely on PDEs to accurately model physical systems' behaviour.

- **Engineering Applications:** PDEs find extensive use in engineering disciplines for modelling various processes like heat transfer, fluid dynamics, electromagnetism, and structural mechanics. Solving PDEs allows engineers to analyze and design systems effectively.

- **Boundary Value Problems:** PDEs are essential for handling boundary value problems, where the behaviour of a system is constrained by specific conditions at its boundaries. These problems arise in many practical applications, such as electronics, acoustics and geophysics.

- **Image and Signal Processing:** PDE-based techniques are used in image and signal processing for tasks like denoising, inpainting, and segmentation. Solving PDEs in these applications helps improve the quality of images and extract meaningful information from signals.

- **Finance and Economics:** PDEs are employed in finance and economics to model the behaviour of financial derivatives, option pricing, and risk management. Accurate solutions to these equations are vital for making informed decisions in financial markets.

- **Medical Imaging:** PDEs play a crucial role in medical imaging, such as MRI and CT scans, which are used to reconstruct images from collected data, aiding in accurate diagnoses and treatment plans.

- **Quantum Mechanics:** PDEs are fundamental to quantum mechanics, describing the evolution of quantum systems over time. They are used to understand particle behaviour and predict probabilities and states.

Hence, the importance of PDEs lies in their wide range of applications across scientific and engineering domains. They provide essential tools for modelling complex systems, gaining insights, making predictions, and optimizing designs, leading to advancements in technology and scientific knowledge.

An ”integro-differential equation” refers to an equation that encompasses both integrals and derivatives of an unknown function. Few of its applications are as follows [84]:

- They are often used in scientific and technical domains, serving as effective models for several phenomena, including circuit analysis.

- The behavior of interacting inhibitory and excitatory neurons can be elucidated through a set of integro-differential equations.

- The are widely used in the field of epidemiology, particularly in the mathematical modelling of epidemics. These equations are particularly useful when the models include age structure or geographical characteristics of the epidemics.
- Integro-differential equations are encountered as models in many fields, such as population growth, one-dimensional visco-elasticity and reactor dynamics.

Kirchhoff-type elliptic and parabolic equations are mathematical models with applications in various scientific and engineering disciplines. These equations typically involve differential operators that account for both Laplacian terms and non-linearities. Kirchhoff-type equations frequently arise in numerous applications, such as in elasticity and structural mechanics, image denoising and restoration and financial mathematics, see, [9, 117].

1.1 Motivation

There are several significant methods for finding the solution of PDE, including separation of variables, method of characteristic, integral transform, superposition principle, change of variables, Lie group method, semi-analytical methods, etc. In general, it is not easy to find the analytical solution for most of the PDEs. Hence, it becomes important to find their numerical solution. While the analytical solution is often challenging or even impossible to obtain for complex PDEs, numerical methods offer efficient and accurate approaches to approximate the solution. There are various methods that are used to find numerical solution of PDEs. Some of the commonly employed techniques include:

- **Finite Difference Method (FDM):** In FDM, the spatial and temporal derivatives in the PDE are approximated using finite differences. The problem domain is discretized, and the PDE get transformed into a system of algebraic equations, which are then solved using iterative or direct methods.
- **Finite Element Method (FEM):** FEM involves dividing the problem domain into smaller finite elements, and the solution of PDE is approximated over each element

using shape functions. The equations are then assembled, and boundary conditions are applied to solve for the unknowns.

- **Finite Volume Method (FVM):** FVM focuses on dividing the domain into control volumes, where the PDE is integrated over each control volume. This method is prevalent for problems with strong conservation properties, such as fluid dynamics.

- **Spectral Methods:** Spectral methods approximate the solution using a series of basis functions, such as Fourier series or Chebyshev polynomials. These methods offer high accuracy and convergence rates, especially for smooth solutions.

- **Boundary Element Method (BEM):** BEM involves converting the PDEs into integral equations defined along the boundary of the domain. These integral equations are used to approximate the boundary values, ultimately leading to a solvable system of equations.

- **Meshless Methods:** Meshless methods, like the Radial Basis Function (RBF) method, use scattered data points to construct approximations of the solution. These methods avoid the need for explicit meshes, simplifying the solution process for complex domains.

The FEM is a potent numerical approach employed for approximating solutions to PDEs and addressing various engineering problems effectively. It is widely employed in various fields, including mechanical, civil, aerospace, and biomedical engineering. The method involves dividing a complex problem domain into smaller, simpler subdomains called finite elements. These elements are interconnected at specific points called nodes. The main steps in the FEM are as follows:

- **Discretization:** The continuous problem domain is discretized into a finite number of elements, where each element is characterized by a set of mathematical functions known as shape functions.

- **Formulation of Element Equations:** For each element, the governing PDE is approximated using the shape functions. As a result, this process yields a set of algebraic equations specific to each element.

- **Assembly:** In this step, the individual element equations are merged to create a comprehensive global system of equations.

- **Application of Boundary Conditions:** Boundary conditions are applied to the global system to account for constraints and external influences on the problem.

- **Solution:** During the solution phase, the system of equations is effectively solved to determine the unknown values, such as displacements or temperatures, at each node.

FEM analysis is widely adopted and appreciated. A few of the advantages of this method are as follows:

- The process of representing complex geometries and irregular shapes in a mathematical model is facilitated by the availability of various finite elements that can discretize the domain.

- Boundary conditions can be readily integrated into the model.

- Different types of material properties can be easily incorporated into the model, either on an element-by-element basis or even within a single element.

- It is also possible to implement higher order elements in the model.

- It is characterised by its simplicity, compactness, and focus on achieving desired outcomes, which has contributed to its widespread adoption within the engineering community.

- The versatility and power of the FEM is enhanced by the availability of wide range of computer software packages.

In the traditional FEM, the solution is approximated by continuous basis functions over each element, and the element boundaries are shared smoothly with neighbouring elements. This leads to a continuous representation of the solution across the entire domain. However, in problems with discontinuities or sharp gradients, FEM may suffer from numerical diffusion and lack of accuracy. Hence, the discontinuous Galerkin method (DGM) was introduced to solve this issue. The DGM is a numerical technique for solving PDEs. It is a variant of the traditional Galerkin FEM that allows for discontinuities in the solution across element boundaries. This method is notably advantageous for addressing problems characterized by shocks, material interfaces, and various forms of discontinuities. In DGM, the key idea is to allow the solution to be discontinuous across element interfaces. The domain is still discretized into smaller finite elements, but the basis functions used within each element are constructed to be different on either side of the element boundary. This allows the method to represent discontinuities accurately. The DGM offers several advantages, making it a powerful and attractive numerical technique for solving PDEs, especially in problems with discontinuities and complex features. Some of the key advantages of the DGM include:

- **Handling Discontinuities:** DGM is particularly well-suited for problems with sharp gradients, shocks, and material interfaces. The method allows for discontinuous solutions across element boundaries, which enables accurate representation and capturing of these features without introducing numerical diffusion.

- **High Accuracy:** DGM achieves high-order accuracy using polynomial basis functions within each element. This allows for a better approximation of the solution and reduces the requirement for a fine mesh, leading to more efficient simulations.

- **Local Adaptivity:** DGM allows for local refinement and mesh adaptation. This means that elements can have varying sizes and degrees of basis functions in different regions of the domain, providing more flexibility in resolving localized phenomena.

- **Conservation Properties:** DGM maintains local conservation properties, making it suitable for problems requiring accurate mass conservation, momentum, energy, or other quantities.

Despite these advantages, it is worth noting that DGM also comes with challenges and potential drawbacks. Some of the main drawbacks of DGM are:

- **Global Coupling:** In the standard DGM, the numerical fluxes are typically computed element by element, leading to a global coupling of unknowns at element interfaces. This can result in a dense and large global linear system, making the method computationally expensive, especially for high-order approximations.

- **High Memory Requirements:** The global coupling in DGM can lead to increased memory requirements, as the method stores additional degrees of freedom associated with the numerical fluxes at element interfaces.

- **Stability and Time Stepping:** DGMs can be more sensitive to time-step restrictions for time-dependent problems, especially in the presence of stiff terms. This can impact the efficiency and stability of the simulations.

- **Lack of Hybridization:** Standard DGMs employ a single set of numerical fluxes for enforcing continuity across element interfaces. While this approach permits the handling of solution discontinuities, it may lead to reduced accuracy, particularly in cases featuring strong gradients or shock phenomena.

To address these limitations of DGM and improve its performance, the Hybridizable Discontinuous Galerkin (HDG) method was proposed. HDG method combines aspects of both DGM and continuous Galerkin (CG) methods, resulting in a more efficient and stable approach. Some of the key features of the HDG method that address the limitations of DGM include:

- **Hybridization:** HDG method introduces additional hybrid variables at element interfaces to enforce continuity more efficiently. This reduces global coupling and results in a smaller, sparser global linear system.

- **Reduced Number of Globally Coupled Degrees of Freedom:** In contrast to other alternative DGMs, which yield a final system encompassing all degrees of freedom associated with the approximate field variables, the HDG technique generates a final system expressed in terms of degrees of freedom related to the approximate traces of the field variables. This characteristic of the HDG technique leads to a reduction in globally coupled unknowns compared to other DGMs, as the approximation traces are solely defined along the element boundary.

- **Super-convergence:** The HDG method is renowned for demonstrating optimal convergence when approximating gradients in convection-diffusion problems. This exceptional convergence property is a distinctive feature of HDG methods, particularly evident in diffusion problems. In contrast, both DGMs and the standard continuous Galerkin approach exhibit sub-optimal convergence when approximating gradients. Within the realm of incompressible flows, research has showcased that the HDG technique achieves higher-order convergence for approximating velocity, pressure, velocity gradient, and vorticity compared to DGMs. To be precise, the HDG method attains an optimal order of convergence, whereas DGMs only attain a sub-optimal order of convergence. It is also noteworthy that the HDG approach exhibits remarkable convergence characteristics concerning numerical traces and the averaging of approximation variables.

- **Local Post-processing:** The HDG approach exhibits optimum convergence and super-convergence qualities, which may be leveraged to construct a local post-processing technique aimed at enhancing the spatial order of convergence for the numerical solution. In the context of incompressible flows, it is possible to use local post-processing techniques to derive an alternative approximation of the velocity field. This new approximation has the desirable property of being exactly divergence-free and exhibits a greater rate of convergence. In the context of time-dependent issues, it is sufficient to do post-processing just at the time levels where a higher degree of accuracy in the results is sought. Additionally, due to the fact that the post-processing is conducted at the individual element level, it incurs lower costs compared to the solution approach.

- **Geometric Flexibility and Mesh Adaptation:** The HDG method is capable of being implemented on unstructured meshes of a general nature. It is particularly well-suited for accommodating hp-adaptivity due to its ability to refine or coarsen the grid without being constrained by the continuity requirements commonly associated with conforming methods. Additionally, the HDG method allows for the utilisation of different orders of approximations on various elements or subdomains.

- **Local Solvers:** The HDG method utilizes local solvers to eliminate hybrid variables and directly represent the solution in terms of the primary unknowns. This further reduces memory requirements and computational costs.

- **Stabilization Techniques:** The HDG method incorporates stabilization techniques that enhance the stability and robustness of the method, particularly for problems with strong gradients and shocks.

- **Reduced Time-step Sensitivity:** The HDG method can be less sensitive to time-step restrictions, making it more efficient for time-dependent problems, even with large time steps.

Overall, the HDG method addresses the limitations of the standard DGM, providing a more efficient, accurate, and stable numerical approach. It has gained popularity in various scientific and engineering applications, particularly for problems that require high-order accuracy and deal with complex phenomena such as shocks, interfaces, and discontinuities. In the literature there are various higher-order methods which are used to find the numerical solution of PDEs, a few of them are as follows:

- **Weak Galerkin Method [136]:** It aims to simplify the implementation of finite element methods by weakening the continuity requirements on the solution across element boundaries. The method introduces weakly enforced continuity conditions, which can be beneficial for problems with complex geometries and irregular meshes. WG has been applied to a variety of PDEs, and its flexibility makes it suitable for parabolic and hyperbolic integro-differential equations.

- **Virtual Element Method [12]:** This method generalizes the concept of finite elements to arbitrary polygonal or polyhedral meshes. It allows for the approximation of the solution using functions defined locally on the mesh elements. It provides flexibility in handling complex geometries and is applicable to a wide range of PDEs, including parabolic and hyperbolic integro-differential equations.

- **Hybrid High-Order Methods [49]:** They combine the advantages of both finite element and finite volume methods. These methods aim to achieve high-order accuracy while maintaining stability and efficiency. They have been successfully applied to various PDEs, including parabolic and hyperbolic problems.

While each method has its unique features, there are common principles and goals that connect them. These methods share the objective of achieving high-order accuracy, handling complex geometries, and providing efficient numerical solutions. The relation between these methods can be explored through the study of their underlying mathematical principles, such as weak formulations, variational principles, and stability conditions. While each method has its strengths and is suitable for specific types of problems, the HDG method offers advantages in terms of reduced global unknowns, improved stability, simplicity in implementation, and compatibility with existing tools. The choice between methods depends on the specific requirements of the problem at hand, the computational resources available, and the preferences of the researcher.

1.2 Literature Survey

FEMs were first introduced to solve complex elasticity and structural analysis problems in civil and aeronautical engineering [69, 93]. Later on, it was developed and analyzed for the elliptic equations [5, 10, 14, 57, 67, 70, 83, 110], parabolic equations [1, 6, 28, 48, 54, 62, 75, 78, 131] and hyperbolic equations [7, 40, 45, 61, 72, 74, 77, 92, 97, 118, 132].

In [70], Hou *et al.* have investigated a multi scale FEM approach for solving a range of elliptic problems stemming from composite materials and porous media flow. They have developed adaptive multi scale finite element basis functions tailored to the local properties of the differential operator. In [57], Farago *et al.* have developed a coupling of the Sobolev space gradient method and the FEM. The Sobolev space gradient method reduces the solution of a quasi-linear elliptic problem to a sequence of linear Poisson equations which are further solved numerically by an appropriate FEM. In [14], Cai *et al.* have proposed a new FEM to compute singular solutions of Poisson equations on a polygonal domain subject to mixed boundary conditions. In [67], Guzman *et al.* have discussed a higher order piece-wise continuous FEMs for solving a class of interface problems which is based on correction terms added to the right hand side in the standard variational formulation. Further, they have derived optimal error estimates for the method in maximum norms. In [83, 110], the FEM is developed for elliptic equations on surfaces and fully nonlinear elliptic equations, respectively.

In [6], Babuvska *et al.* have derived a posteriori error estimates of finite element solutions for one-dimensional parabolic problems in an asymptotic form with the approach similar to the residual method. In [78], Johnson *et al.* have developed a class of mixed FEMs for parabolic problems which yield optimal order of convergence. They have also obtained the results for stationary and evolutionary Stokes equation. In [75], Jin *et al.* have studied the standard Galerkin FEM for a fractional-order parabolic equation with a space fractional derivative of Riemann-Liouville type. They have derived the error estimates for both semi and fully discrete schemes. In [62], Gao *et al.* have proposed a weak Galerkin FEM with stabilization term for parabolic equations by weakly defined gradient operators over discontinuous functions. They have further derived optimal order error estimates in L^2 norm.

In [45], Cowsat *et al.* have developed a mixed finite element scheme for second order hyperbolic equation. The question of convergence of the method for the hyperbolic equation is reduced to the associated elliptic equation. They have also discussed the stability conditions for the scheme, along with numerical examples. In [72], Hulbert *et al.* have developed a FEM to solve elasto-dynamics problems. They have used the FEM to approximate the solution in both spatial and temporal domains. Further, they have extended the analysis to structural dynamics problems. In [132], Šuli *et al.* have discussed the recent developments in a posteriori error estimates of hyperbolic equations. They have further done the global a posteriori error analysis in H^{-1} norm for the FEM, taking hyperbolic equation as the model problem. In [40], Cockburn *et al.* have developed a simple post-processing scheme that enhances the accuracy of the finite element approximation to transient hyperbolic equations. They have shown a significant improvement in the order of convergence of the post-processed solution.

To address the limitations of the FEM, DGMs were devised. They were initially developed for the hyperbolic equations [4, 8, 13, 30, 31, 51, 60, 71, 76, 82, 95, 122], and then further extended to the elliptic [2, 3, 18, 20, 21, 39, 50, 59, 68, 137, 140, 141] and the parabolic equations [15, 19, 29, 55, 56, 63, 108, 121, 126, 133, 138].

In [8], Bey *et al.* have developed an hp-version DGM for hyperbolic conservation laws. They have derived a priori error estimates using a new mesh-dependent norm. The results extend the previously known results for the mesh-dependent norm to the hp-version DGM. They have also derived the a posteriori error estimates and given several numerical examples. In [4], Atkins *et al.* have developed a DGM that does not require discrete quadrature formula for hyperbolic equations. This approach requires fewer operations and less storage but preserves the compactness and robustness of the classical DGM. In [30], Cockburn has developed two DGMs for nonlinear hyperbolic conservation laws. The first method developed is called shock-capturing DGM, an implicit method; the second is the Runge–Kutta DGM, an explicit method. In [31], Cockburn has given an overview of the DGM. He has shown that the DGMs can capture highly complex solutions presenting discontinuities with high resolution for nonlinear hyperbolic problems.

In [21], Castillo *et al.* have developed the local DGM for an elliptic problem. They have derived the error estimates for meshes with hanging nodes. Their analysis illustrates that for the stabilization parameter of order one, the potential and flux variables achieve convergence of order $k + 1/2$ and k, respectively, whereas, for the stabilization parameter of order h^{-1}, the potential variable achieves convergence of order $k + 1$. In [39], Cockburn *et al.* have developed a DGM for second-order elliptic problems which is super-convergent. They have shown that the flux variable achieves optimal order of convergence, whereas the potential variables achieve super-convergence for the model problem. Further, they have performed element-by-element post-processing to obtain new approximations. Based on this analysis, Yadav *et al.* [140], have developed the super-convergent DGM for non-selfadjoint linear elliptic problems and quasi-linear elliptic problems. They have proved that for a polynomial of degree $k \geq 1$, the flux variables converge with order $k+1$. They have also performed element-by-element post-processing of the potential variables and proved that the post-processed solution converges with order $k + 2$.

In [121], Riviere *et al.* have developed a time locally conservative DGM to approximate nonlinear parabolic equations. Optimal error estimates are derived. In [63], Georgoulis *et al.* have developed a posteriori error estimates for linear parabolic problems. They have used interior penalty DGM to spatially semi-discretize the problem and an implicit Euler time stepping scheme to completely discretize the problem in the temporal direction. In [19], Cao *et al.* have studied super-convergence properties of the local DGM for linear parabolic equations when alternating fluxes are used. They have proved that for any polynomial of degree k, the numerical fluxes converge at a rate of $2k + 1$ for all mesh nodes and the domain average under some suitable initial discretization.

The HDG method was initially developed by Cockburn for elliptic equations [34, 35, 37]. Due to various theoretical and computational advantages of the HDG method it was further developed for various types of problem, such as, the heat equation [23], the wave equation [42, 58, 105, 130], steady-state and time dependent convection-diffusion problems [32, 33, 41, 101, 102], elasticity problems [129], the Navier Stoke's equation [22, 96, 107, 116, 119, 120], equations in fluid mechanics [104], Stoke's flow [36, 103], the Maxwell's equation [52, 106], equations in continuum mechanics [100], the Helmholtz's equation [27, 66], among others [38, 43, 44, 53, 73, 109, 127, 128].

In [34], Cockburn *et al.* have developed LDG-hybridizable Galerkin method for second order elliptic problems. In contrast to all the known DGMs, this method is proven to have remarkable convergence properties. For a polynomial of degree $k \geq 0$, the potential and flux variables achieve convergence of order $k + 1$. Further element-by-element post-processing is also possible in this method, which leads to super-convergence results for the potential variable. In [35], Cockburn *et al.* have introduced a novel characterization of the approximate solution provided by hybridized mixed methods when addressing second-order self-adjoint elliptic problems. They have applied this characterization to obtain an explicit formula for the entries of the matrix equation for the Lagrange multiplier unknowns resulting from hybridization. In [37], Cockburn *et al.* have formulated a comprehensive framework for hybridizing FEMs when dealing with second-order elliptic problems. The methods considered in this framework are hybridized mixed, continuous Galerkin, nonconforming, and HDG methods. The framework facilitates the use of various methods within different elements or subdomains of the computational domain in a single implementation, with automatic coupling between them. In [38], Cockburn *et al.* have developed a new technique for the error analysis of the HDG methods to a model second-order elliptic problem. It employs a new projection whose design is inspired by the numerical traces of the methods. This makes the analysis of the discretization error projections straightforward and concise. They have demonstrated that these error projections are bounded by the distance between the solution and its projection. In [44], Cockburn *et al.* have presented a unified a posteriori error analysis for HDG methods for second order elliptic PDEs, which helped to derive new estimates for the methods.

Further, in [23], Chabaud *et al.* have developed the HDG method for the heat equation. They have demonstrated that if the solution is sufficiently smooth, the convergence properties of the elliptic equation also hold for the heat equation. As a result, they have demonstrated that for a polynomial of degree $k \geq 1$, the post-processed approximation of the scalar variable converges with an order of $\sqrt{\log(T/h^2)}h^{k+2}$. In [105], Nguyen *et al.* have introduced a category of HDG methods designed for the numerical simulation of wave phenomena in acoustics and elasto-dynamics. They have proved that all the unknown variables achieve the convergence of order $k + 1$ when the polynomial of degree $k \geq 0$ is used to approximate the solution. Further, they have proved super-convergence using local post-processing for displacement and velocity. In [42], Cockburn *et al.* have developed the HDG method for the wave equation in continuous time. They have analyzed the a priori error estimates for the method and proved that for a polynomial of degree $k \geq 0$, both velocity and gradient achieve the optimal rate of convergence. Additionally, they have proposed the local post-processing for the problem and proved that for a polynomial of degree $k \geq 1$, the post-processed solution achieves super-convergence. In [130], Stanglmeier *et al.* have developed the HDG method for the acoustic wave equation. They have proved that the method achieves the optimal order of convergence for all the unknown variables. They have also proved some super-convergence properties to improve the order of convergence of the approximate solution. They have extended the method to deal with the wave equation with perfectly matched layers.

In [101, 102], Nguyen *et al.* have developed the HDG method for steady and unsteady linear and nonlinear convection diffusion equations, respectively. They have developed the method by expressing the scalar and flux variables in terms of an approximate trace of the scalar variable. They have used the backward difference scheme to approximate the time derivative and Newton Raphson's method to solve the nonlinear system of equations. They have proved the optimal rate of convergence for scalar and flux variables and the super-convergence for the local post-processed solution of the scalar variable. In [33], Cockburn *et al.* have developed the HDG method for the convection-diffusion-reaction problem. They have focused on the computational aspect of the method and hence, have performed various numerical experiments for the method and compared their results with the other methods relevant to the diffusion-dominated regime. In [41], Cockburn *et al.* have developed the HDG method for fractional diffusion equations of order $-\alpha$ where $-1 < \alpha < 0$. They have proven the optimal order of convergence for all the unknowns, along with the super-convergence of the post-processed approximation, as expected by the method. In [22], Casmelioglu *et al.* have developed the HDG method for the Navier-

Stokes equation. They have proved that the velocity gradient, velocity and pressure converge with order $k+1$ for $k \geq 0$. They have also proved the super-convergence results for the velocity variable. Further, they have proved that these results depend only on the inverse of the stabilization parameter of the jump of the normal component of the velocity. In [106], Nguyen *et al.* have developed two HDG methods specifically designed for addressing time-harmonic Maxwell's equations. The first method actively enforces the divergence-free condition, while the second variant does not explicitly impose this condition. They have then proved that the vector variable achieves optimal order of convergence in both cases. In [100], Nguyen *et al.* have developed the HDG method for PDEs in continuum mechanics. They have taken into account both steady and time-dependent problems. They used the local HDG projections to derive the error estimates and achieved the expected outcomes. They have also illustrated the results computationally and compared them with the results of the continuous Galerkin method.

1.3 Objectives

The literature survey suggests that the study of the HDG method is a very active area for research. By keeping in mind the applications of integro-differential equations and the advantages of the HDG method, we planned to propose and analyze the HDG method for integro-differential equations. Although the DGM has been developed for linear parabolic and hyperbolic integro-differential equations [79, 115], going through the literature survey, we expect improvement in the estimates. Hence, based on the literature survey, we have set the following objectives for our thesis:

1. To develop HDG method for linear parabolic integro-differential equation with smooth data.

2. To propose HDG method for nonlinear parabolic integro-differential equation with smooth data.

3. To develop HDG method for linear hyperbolic integro-differential equation with smooth data.

4. To propose HDG method for nonlinear hyperbolic integro-differential equation with smooth data.

1.4 Preliminaries

In this section, we have stated and discussed various preliminaries that will be used throughout this thesis. For the sake of clarity, we have divided this section into various subsections. The first subsection, that is, subsection 1.4.1, introduces the finite element subdivision of the domain that is taken into consideration, along with its properties. It also discusses various types of edges that will be used in the thesis along with their properties. Subsection 1.4.2, consists of the types of finite element subspaces along with norms that are used in the definitions of the subspaces. In subsection 1.4.3, we discuss one of the most important projection used in this thesis, that is, the HDG projection. This projection will be used throughout this thesis for the a priori error analysis of the HDG method. We will also go through the properties of the projection followed by their estimates. Finally, in subsection 1.4.4, we have stated a few well known definitions, results and theorem that will be used further.

1.4.1 Finite Element Subdivision

Let $\Omega \subset \mathbb{R}^2$ be the domain in which we will be working throughout this thesis. Ω is a Lipschitz, convex, bounded domain with polygonal boundary $\partial\Omega$ [38]. Let $\mathcal{T}_h$ be finite element subdivision of Ω, that is, $\mathcal{T}_h$ consists of finite number of simplex K, such that $\Omega = \bigcup\limits_{K \in \mathcal{T}_h} K$. Simplex K can either be a triangle or a rectangle. Now, to measure the size of the simplex K, we denote the diameter of the K by h_K. Next, we define ρ_K as the diameter of the inscribed circle in K. Next we will define the following terms:

Chunkiness Parameter: For a simplex K, the chunkiness parameter, denoted by α_K is defined as:

$$\alpha_K = \frac{h_K}{\rho_K}.$$

Shape Regular: A finite element subdivision $\mathcal{T}_h$ is considered shape-regular if it possesses a positive constant α_0, satisfying the condition that:

$$\alpha_K \geq \alpha_0, \ \ \forall K \in \mathcal{T}_h.$$

Throughout this thesis, the subdivision $\mathcal{T}_h$ is considered to be shape regular. This condition means that the shape of the simplex cannot be too bad in the sense that the angles can neither be very wide nor very narrow, see, [85, pp. 46].

Lastly, we will end this subsection after defining the following notations:

- $\boldsymbol{\partial \mathcal{T}_h}$ **:** $= \{\partial K : K \in \mathcal{T}_h\}$
- $\boldsymbol{\Gamma_I}$ **:** set of interior edges of $\mathcal{T}_h$
- $\boldsymbol{\Gamma_\partial}$ **:** set of boundary edges of $\mathcal{T}_h$
- $\boldsymbol{\Gamma}$ **:** $= \Gamma_I \cup \Gamma_\partial$
- $\boldsymbol{h}$ **:** $= \max_{K \in \mathcal{T}_h} h_K$
- $\boldsymbol{\rho}$ **:** $= \min_{K \in \mathcal{T}_h} \rho_K$.

1.4.2 Finite Element Subspace

This part presents an introduction to the broken Sobolev spaces that are necessary for our analysis. We also describe the finite element spaces that are used in the HDG approximations. The broken Sobolev space of composite order s and exponent r, with $1 \leq r \leq \infty$, is defined on the subdivision $\mathcal{T}_h$, as follows:

$$W_r^s(\mathcal{T}_h) = \{v \in L^r(\Omega) : v|_K \in W_r^s(K), \forall K \in \mathcal{T}_h\},$$

where, $W_r^s(K)$ is the standard Sobolev space of order s with exponent r for each K. The associated norm and semi-norm for $1 \leq r < \infty$ are defined respectively, as follows:

$$\|v\|_{W_r^s(\mathcal{T}_h)} = \left(\sum_{K \in \mathcal{T}_h} \|v\|^r_{W_r^s(K)} \right)^{1/r} \quad \text{and} \quad |v|_{W_r^s(\mathcal{T}_h)} = \left(\sum_{K \in \mathcal{T}_h} |v|^r_{W_r^s(K)} \right)^{1/r},$$

whereas, for $r = \infty$, it is defined as:

$$\|v\|_{W^s_\infty(\mathcal{T}_h)} = \max_{K\in\mathcal{T}_h} \|v\|_{W^s_\infty(K)} \text{ and } |v|_{W^s_\infty(\mathcal{T}_h)} = \max_{K\in\mathcal{T}_h} |v|_{W^s_\infty(K)},$$

where, $\|v\|_{W^s_r(K)}$ and $|v|_{W^s_r(K)}$ are standard norm and semi-norm on K. When $r = 2$, we write $W^s_2(\mathcal{T}_h)$ as $H^s(\mathcal{T}_h)$ and similar changes are made for other notations.
Next, we introduce the following broken Sobolev spaces:

$$V = \{v \in L^2(\Omega) : v|_K \in H^1(K)\ \forall K \in \mathcal{T}_h\}$$

$$\boldsymbol{W} = \{\boldsymbol{w} \in \boldsymbol{L}^2(\Omega) : \boldsymbol{w}|_K \in \boldsymbol{H}^1(K)\ \forall K \in \mathcal{T}_h\},$$

where, $\boldsymbol{L}^2(\Omega) = (L^2(\Omega))^2$ and $\boldsymbol{H}^1(K) = (H^1(K))^2$.
We now introduce the following finite element spaces :

$$V_h = \{v \in L^2(\Omega) : v|_K \in P_k(K), \forall K \in \mathcal{T}_h\},$$
$$\boldsymbol{W}_h = \{\boldsymbol{w} \in \boldsymbol{L}^2(\Omega) : \boldsymbol{w}|_K \in \boldsymbol{P}_k(K), \forall K \in \mathcal{T}_h\},$$
$$M_h = \{\mu \in L^2(\Gamma) : \mu|_F \in P_k(F), \forall F \in \Gamma\}.$$

In this case, $\boldsymbol{P}_k(K) = [P_k(K)]^2$, whereas the space of polynomials defined on K with a maximum degree k is denoted by $P_k(K)$.
Next, let $u, v \in L^2(D)$, define $(u, v)_D = \int_D uv$, when the domain D is a subset of $\mathbb{R}^n$. For the boundary ∂D of D, define $\langle u, v\rangle_{\partial D} = \int_{\partial D} uvds$. Then, we introduce the following notations:

$$(u, v) = \sum_{K\in\mathcal{T}_h} (u, v)_K \text{ with norm } \|v\|^2 = \sum_{K\in\mathcal{T}_h} \|v\|^2_{L^2(K)},$$
$$\langle u, v\rangle_{\partial\mathcal{T}_h} = \sum_{K\in\mathcal{T}_h} \langle u, v\rangle_{\partial K} \text{ with norm } \|\mu\|^2_\tau = \sum_{K\in\mathcal{T}_h} \tau\|\mu\|^2_{L^2(\partial K)}.$$

Finally, we define the following space:

$$\boldsymbol{H}_{div}(\mathcal{T}_h) = \{\boldsymbol{w} \in \boldsymbol{L}^2(\Omega) : \nabla\cdot\boldsymbol{w}|_K \in L^2(K),\ \forall K \in \mathcal{T}_h\},$$

with norm:

$$\|\boldsymbol{w}\|_{\boldsymbol{H}_{div}(\mathcal{T}_h)} = \left(\sum_{K\in\mathcal{T}_h} \left(\|\boldsymbol{w}\|^2_{\boldsymbol{L}^2(K)} + \|\nabla\cdot\boldsymbol{w}\|^2_{L^2(K)}\right)\right)^{\frac{1}{2}}.$$

1.4.3 The HDG Projection

In this section, we will state the definition of the HDG projection along with its estimates. This projection will be used further for the error estimates of the HDG method.

The HDG projection [38], $\Pi_h : H^1(\mathcal{T}_h)\times\boldsymbol{H}_{div}(\mathcal{T}_h) \to V_h\times\boldsymbol{W}_h$, is denoted by $\Pi_h(u,\boldsymbol{\sigma}) = (\Pi_V u, \boldsymbol{\Pi_W \sigma})$ for any $(u,\boldsymbol{\sigma}) \in H^1(\mathcal{T}_h) \times \boldsymbol{H}_{div}(\mathcal{T}_h)$. For any simplex $K \in \mathcal{T}_h$, the projection is defined as follows:

$$(\Pi_V u, v)_K = (u, v)_K, \qquad \forall\, v \in P_{k-1}(K) \tag{1.1a}$$

$$(\boldsymbol{\Pi_W \sigma}, \boldsymbol{w})_K = (\boldsymbol{\sigma}, \boldsymbol{w})_K, \qquad \forall\, \boldsymbol{w} \in \boldsymbol{P}_{k-1}(K) \tag{1.1b}$$

$$\langle \boldsymbol{\Pi_W \sigma} \cdot \boldsymbol{\nu} + \tau \Pi_V u, \mu\rangle_F = \langle \boldsymbol{\sigma}\cdot\boldsymbol{\nu} + \tau u, \mu\rangle_F, \qquad \forall\, \mu \in P_k(F), \tag{1.1c}$$

for all faces F of the simplex K. When k is set to 0, equations (1.1a) and (1.1b) lose their significance, making the projection defined solely by equation (1.1c). It's important to note that despite being denoted as $\Pi_V u$, the first component of the projection depends on both variables, namely, u and $\boldsymbol{\sigma}$. The same holds true for the second component of the projection.

From [38], we have the following estimates: For $k \geq 0$ and $\tau|_{\partial K}$ non-negative, $\tau = \max \tau|_{\partial K}$ a positive constant and $\tau^* = \max \tau|_{\partial K \setminus F^*}$, where F^* is a face of K at which $\tau|_{\partial K}$ is maximum, the systems (1.1) is uniquely solvable for $\Pi_V u$ and $\boldsymbol{\Pi_W \sigma}$. Additionally, there exists, C independent of K and τ such that, for all $1 \leq \alpha, \beta \leq k+1$,

$$\|\boldsymbol{\Pi_W \sigma} - \boldsymbol{\sigma}\|_K \leq C h_K^{\alpha} |\boldsymbol{\sigma}|_{\boldsymbol{H}^{\alpha}(K)} + C\tau^* h_K^{\beta} |u|_{H^{\beta}(K)}, \tag{1.2a}$$

$$\|\Pi_V u - u\|_K \leq C h_K^{\beta} |u|_{H^{\beta}(K)} + C h_K^{\alpha} \tau^{-1} |\nabla\cdot\boldsymbol{\sigma}|_{\boldsymbol{H}^{\alpha-1}(K)}. \tag{1.2b}$$

1.4.4 Some Established Results

Lemma 1.4.1. ***(Estimates of L^2-projection, [115])*** *Let $\boldsymbol{I}_h^k$ denote the L^2-projection. If $\boldsymbol{w} \in \boldsymbol{H}^{\boldsymbol{r+1}}(K)$ and $\boldsymbol{I}_h^k \boldsymbol{w} \in \boldsymbol{P}_{\boldsymbol{k}}(K)$, the subsequent approximation property holds:*

$$\|\boldsymbol{w} - \boldsymbol{I}_h^k \boldsymbol{w}\|_{\boldsymbol{L^2}(K)} + h^{\frac{1}{2}} \|\boldsymbol{w} - \boldsymbol{I}_h^k \boldsymbol{w}\|_{\boldsymbol{L^2}(\partial K)} \leq C h^{min(r,k)+1} \|\boldsymbol{w}\|_{\boldsymbol{H^{r+1}}(K)}.$$

Lemma 1.4.2. ***(Cauchy–Schwarz inequality, [81])*** *For all vectors $\mathbf{u}$ and $\mathbf{v}$ in an inner-product space, the following inequality holds true:*

$$|(\boldsymbol{u}, \boldsymbol{v})| \leq \|\boldsymbol{u}\| \|\boldsymbol{v}\|,$$

where, the inner-product and norm are associated with each other and to the inner-product space.

Lemma 1.4.3. ***(Young's inequality, [80])*** *If $u \geq 0$ and $v \geq 0$ are non-negative real numbers, then for all $\epsilon > 0$, we have the following inequality*

$$uv \leq \frac{u^2}{2\epsilon} + \frac{\epsilon v^2}{2}.$$

Lemma 1.4.4. ***(Poincare's inequality, [80])*** *Consider a real number p such that $1 \leq p < \infty$. Let C denote a constant, which relies solely on the domain Ω and the exponent p. For any function u that belongs to the Sobolev space $W_0^{1,p}(\Omega)$ and possesses a zero trace, the subsequent condition is valid:*

$$\|u\|_{L^p(\Omega)} \leq Ch\|\nabla u\|_{L^p(\Omega)}.$$

Lemma 1.4.5. ***(Inverse estimates, [11])*** *Let $\boldsymbol{w} \in \boldsymbol{W}_h$, then there exists a constant $C > 0$, such that*

$$\|\boldsymbol{w}\|_\infty \leq Ch^{-1}\|\boldsymbol{w}\|,$$

where $\| \cdot \|_\infty$ is the usual sup norm.

Lemma 1.4.6. ***(Weak commutative property, [38])*** *For any v in V_h and any $(\psi, \boldsymbol{p})$ in the domain of Π_h, we have the following equality, $\forall\ K \in \mathcal{T}_h$*

$$(v, \nabla \cdot \boldsymbol{p})_K = (v, \nabla \cdot \boldsymbol{\Pi_W p})_K + \langle v, \tau(\Pi_V \psi - \psi)\rangle_{\partial K}.$$

Lemma 1.4.7. ***(Gronwall's inequality, [28])*** *Let us consider the assumption that the function $G(t)$ is more than or equal to zero and the function $F(t)$ is absolutely integrable.*

Additionally, we have an integrable function $y(t)$ that is greater than or equal to zero. We will now assume the following inequality

$$y(t) \leq \int_0^t G(s)y(s)ds + F(t),$$

then,

$$y(t) \leq F(t) + \int_0^t G(\gamma)F(\gamma)e^{\int_0^t G(s)ds}d\gamma.$$

Lemma 1.4.8. ***(Discrete Gronwall's inequality, [28])*** *If y_n, f_n, and g_n are assumed to be non-negative sequences along with the following inequality*

$$y_n \leq f_n + \sum_{0\leq k<n} g_k y_k, \ \ n \geq 0$$

then,

$$y_n \leq f_n + \sum_{0\leq k<n} f_k g_k \, e^{\left(\sum_{k<j<n} g_j\right)}, \ \ n \geq 0.$$

Definition 1.4.9. ***(Elliptic projection with memory, [17])*** *An elliptic projection is defined by $\boldsymbol{Rq} \in \boldsymbol{W}_h$, that satisfies the following equation*

$$(a(u)(\boldsymbol{q} - \boldsymbol{Rq}), \boldsymbol{w}) = 0, \quad \ \forall\, \boldsymbol{w} \in \boldsymbol{W}_h,$$

where, a is a positive and bounded function.

Lemma 1.4.10. ***(Estimates for elliptic projection with memory, [17])*** *For the elliptic projection of above type, the following estimates hold true:*

$$\|\boldsymbol{q} - \boldsymbol{Rq}\|_p \leq C(u)h^{k+1-p}\|\boldsymbol{q}\|_{H^{k+1}(\mathcal{T}_h)}, \ \ p \leq k+1, \ \ p = 0, 1,$$

where $C = C(u)$ is a positive constant dependent on u.

Definition 1.4.11. ***(Raviart-Thomas projection, [125])*** *Given a function $\boldsymbol{\sigma} \in \boldsymbol{H}^1(\mathcal{T}_h)$ and an arbitrary simplex $K \in \mathcal{T}_h$, the restriction of $\boldsymbol{\Pi}_k^{RT}$, $k \geq 1$ to K is defined as the element of $\boldsymbol{P}_k(K) \bigoplus \boldsymbol{x}P_k(K)$ that satisfies*

$$(\boldsymbol{\Pi}_k^{RT}\boldsymbol{\sigma} - \boldsymbol{\sigma}, \boldsymbol{w})_K = 0, \ \ \forall\, \boldsymbol{w} \in \boldsymbol{P}_{k-1}(K), \textit{ for } \ k \geq 1,$$

$$\langle(\mathbf{\Pi}_k^{RT}\boldsymbol{\sigma}-\boldsymbol{\sigma}\rangle\cdot\boldsymbol{\nu},\mu)_F=0,\ \ \forall\,\mu\in P_k(F),$$

for all faces F of K.

Lemma 1.4.12. ***(Estimates of the Raviart-Thomas projection, [125])*** *For the Raviart-Thomas projection of above type, the following estimates hold true:*

$$\|\mathbf{\Pi}_k^{RT}\boldsymbol{\sigma}-\boldsymbol{\sigma}\|_K\leq Ch^{k+1}|\boldsymbol{\sigma}|_{H^{k+1}(K)}.$$

Definition 1.4.13. ***(Lipschitz continuity, [81])*** *In the context of metric spaces (X,d_X) and (Y,d_Y), where d_X and d_Y represent the metrics on the sets X and Y a function $f:X\to Y$ is considered Lipschitz continuous if there exists a non-negative real constant C such that for every x_1 and x_2 in X, the following inequality holds:*

$$d_Y(f(x_1),f(x_2))\leq Cd_X(x_1,x_2).$$

1.5 Organization of the Thesis

In Chapter 2, we develop the HDG method for linear parabolic integro-differential equation and derive uniform in time a priori error bounds. To handle the integral term, we introduce an extended Ritz-Volterra projection, which helps in achieving optimal order convergence of $O(h^{k+1})$ for the semi discrete problem when polynomials of degree $k\geq 0$ are used to approximate both the solution and the flux variables. Further, we propose an element-by-element post-processing and establish that it achieves convergence of the order $O(h^{k+2})$ for $k\geq 1$. We derive a fully discrete scheme using the backward Euler method and the left rectangular rule to discretize the derivative and integral term, respectively. Finally, we conclude the chapter by demonstrating the numerical results in two-dimensional domains to validate the theory.

Chapter 3 discusses the HDG method for a nonlinear parabolic integro-differential equation. We consider the nonlinear functions as Lipschitz continuous to analyze uniform in time a priori bounds. We introduce an extended type Ritz-Volterra projection and use it, along with the HDG projection, as an intermediate projection to achieve optimal order convergence of $O(h^{k+1})$ when polynomials of degree $k \geq 0$ are used to approximate both the solution and the flux variables. By relaxing the assumptions in the nonlinear variable, we achieve super-convergence by element-by-element post-processing. With the help of the backward Euler method in temporal direction and quadrature rule to discretize the integral term, we derive a fully discrete scheme along with its error estimates. Finally, with the help of numerical examples in two-dimensional domains, we obtain computational results, which verify our theoretical findings.

Chapter 4 introduces the HDG approach for a linear hyperbolic integro-differential equation. This chapter includes the development and thorough analysis of a priori error estimates for both semi-discrete and fully discrete schemes. In our analysis, we employ the Ritz-Volterra projection method and its associated estimates for error assessment in the semi-discrete case. Notably, we demonstrate super-convergence for the scalar variable by employing element-by-element post-processing techniques. For the fully discrete error analysis, we employ the central difference scheme to approximate the derivative and the mid-point rule to handle the integral term. As a result, we achieve a second-order convergence rate in the temporal direction. To validate our theoretical findings, we conduct a series of numerical experiments.

Chapter 5 applies the HDG method to a nonlinear hyperbolic integro-differential equation. We consider the nonlinear functions as Lipschitz continuous to analyze uniform in time a priori bounds. By relaxing the assumptions in the nonlinear variable, we achieve super-convergence by element-by-element post-processing. We use the central difference scheme in temporal direction and the mid-point rule to discretize the integral term to derive a fully discrete scheme and its error estimates. Finally, with the help of numerical examples in two-dimensional domains, we obtain the computational results, which verify the theory.

In Chapter 6, a comprehensive critical examination of the obtained results is presented, accompanied by a discussion on potential avenues for further exploration and the potential scope for future research topics.

Chapter 2

HDG Method for Linear Parabolic Integro-Differential Equations

2.1 Introduction

This chapter discusses the HDG method for the following problem: Find $u(x,t)$ such that

$$u_t(x,t) - \nabla \cdot \left(a(x)\nabla u(x,t) + \int_0^t b(x,t,s)\nabla u(x,s)ds \right) = f(x,t) \quad \text{in } \Omega \times (0,T], \tag{2.1a}$$

$$u(x,t) = 0 \qquad \text{on } \partial\Omega \times (0,T], \tag{2.1b}$$

$$u(x,0) = u_0(x) \quad \text{for } x \in \Omega, \tag{2.1c}$$

where $u : \Omega \times (0,T] \to \mathbb{R}$. The coefficients $a : \Omega \to \mathbb{R}$, $b : \Omega \times (0,T] \times (0,T] \to \mathbb{R}$ and $f : \Omega \times (0,T] \to \mathbb{R}$ are smooth functions with bounded derivatives. Additionally, we have $\alpha_0, M \geq 0$ such that $0 < \alpha_0 \leq a \leq M$ and $|b| \leq M$.

The utilization of parabolic integro-differential equations (PIDEs) is prevalent across numerous practical contexts. These equations find application in scenarios such as modeling heat conduction within materials possessing memory characteristics, characterizing non-local reactive flows within porous media, and describing non-Fickian fluid flow in porous media. For an in-depth exploration of this subject and additional references, see [90, 112, 135] and the associated sources.

In the academic literature, various researchers have made significant contributions to the analysis and estimation of errors associated with PIDEs. Cannon *et al.* [16] have undertaken a comprehensive examination of the Galerkin method for nonlinear integro-differential equations of parabolic nature. They have achieved optimal L^2 error estimates by employing a non-classical H^1-projection technique. Lin *et al.* [91] have focused on investigating the stability of Ritz-Volterra projection and derived maximum norm estimates, subsequently using them to establish L^∞ error estimates for FEMs applied to PIDEs. Furthermore, Lin *et al.* [90] have explored the convergence of finite element approximations in the context of both parabolic and hyperbolic integro-differential equations, leveraging the concept of Ritz-Volterra projection. Larson *et al.* [86] have described the numerical solution of PIDEs with memory using the DGM in the temporal domain. Mustapha *et al.* [99] have introduced an hp-version of DGM tailored for integro-differential equations of parabolic nature, providing optimal hp-version error estimates. Additionally, Pani *et al.* [115] have derived a priori error bounds for an hp-local Discontinuous Galerkin (LDG) approximation applied to a PIDE. Their analysis revealed that error estimates in the L^2 norm of the gradient and the potential exhibit optimality concerning the discretization parameter h while remaining sub-optimal in the degree of the polynomial p. Mustapha [98] has developed a super-convergent DGM designed for Volterra integro-differential equations, considering both smooth and non-smooth kernels. Goswami *et al.* [64] have obtained optimal error estimates for mixed FEMs employed in solving PIDEs, particularly when dealing with non-smooth initial data. Their approach combined energy arguments with repeated use of an integral operator. They have also proposed and analyzed an alternative approach for a priori error estimates in the context of semi-discrete Galerkin approximation to time-dependent PIDEs with non-smooth initial data [65], utilizing a similar methodology. Recently, Chen *et al.* [26] have introduced a two-grid FEM tailored for nonlinear PIDEs. In their work, they achieved optimal error estimates in the H^1 norm for spatially semi-discrete two-grid FEM, contributing to the field's understanding of error analysis for these complex equations."

The major contributions of this chapter are as follows:

- The use of the extended mixed type Ritz-Volterra projection is employed in order to get optimum estimates, owing to the inclusion of the integral component.
- Dual problem of the PIDE is used to perform element-by-element post-processing, which plays a crucial role in achieving super-convergence result.
- Based on backward Euler's method, a complete discrete scheme and corresponding error estimates are derived.
- Numerical experiments have been conducted to evaluate the performance of the HDG approach using various degrees of polynomial approximation. These experiments establish that optimum order of convergence for both the unknown variable and its associated flux is achieved. Additionally, it has been shown that the post-processed solution exhibits super-convergence properties.

We note that for simple presentation, we have used the backward Euler's method but higher order methods can be easily applied to derive higher order convergence in temporal direction. In this current chapter, the symbol C is employed to denote a positive constant, the specific value of which remains unspecified. Importantly, this constant is independent of both the discretization parameter h and the degree of the polynomial k. Also, argument x of functions will not be written explicitly, whereas t and s will be written as and when required.

The rest of the chapter is organized as follows: In section 2.2 the HDG method is discussed. Section 2.3 is devoted to the Ritz-Volterra projection and its estimates. Section 2.4 addresses the topic of a priori error estimations. In section 2.5, the element-by-element post-processing of the scalar variable is discussed. Section 2.6 pertains to the completely discrete scheme. Finally, numerical experiments are conducted in section 2.7

to provide visual demonstrations of the theoretical findings. Section 82 is concluded by a set of observations.

2.2 HDG Method

To define the method for PIDEs (2.1), we first introduce the following auxiliary variables:

$$\boldsymbol{q} = -\nabla u, \quad \boldsymbol{\sigma} = a\boldsymbol{q} + \int_0^t b(t,s)\boldsymbol{q}(s)ds,$$

and then, rewrite it as the following system of equations:

$$\boldsymbol{q} = -\nabla u \qquad \text{in } \Omega, \tag{2.2a}$$

$$\boldsymbol{\sigma} = a\boldsymbol{q} + \int_0^t b(t,s)\boldsymbol{q}(s)ds \qquad \text{in } \Omega, \tag{2.2b}$$

$$u_t + \nabla \cdot \boldsymbol{\sigma} = f \qquad \text{in } \Omega. \tag{2.2c}$$

At each time t within the interval $(0, T]$, the method provides an approximation $u_h(t)$ of the scalar function $u(t)$, an approximation $\boldsymbol{q}_h(t)$ and $\boldsymbol{\sigma}_h(t)$ of the vector function $\boldsymbol{q}(t)$ and $\boldsymbol{\sigma}(t)$, respectively, and an approximation $\hat{u}_h(t)$ of the trace of $u(t)$ on the boundaries of the elements. These approximations are computed in the function spaces V_h, $\boldsymbol{W_h}$, $\boldsymbol{W_h}$, and M_h, respectively. With these spaces, the HDG formulation seeks approximation $(u_h, \boldsymbol{q}_h, \boldsymbol{\sigma}_h, \hat{u}_h)(t) \in (V_h \times \boldsymbol{W}_h \times \boldsymbol{W}_h \times M_h)$, for $t \in (0, T]$, for any $(v_h, \boldsymbol{w}_h, \boldsymbol{\tau}_h, \mu_h, m_h) \in (V_h \times \boldsymbol{W}_h \times \boldsymbol{W}_h \times M_h \times M_h)$, satisfying

$$(\boldsymbol{q}_h, \boldsymbol{w}_h) - (u_h, \nabla \cdot \boldsymbol{w}_h) + \langle \hat{u}_h, \boldsymbol{w}_h \cdot \boldsymbol{\nu} \rangle_{\partial \mathcal{T}_h} = 0, \tag{2.3a}$$

$$(a\boldsymbol{q}_h, \boldsymbol{\tau}_h) - (\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + \int_0^t (b(t,s)\boldsymbol{q}_h(s), \boldsymbol{\tau}_h)ds = 0, \tag{2.3b}$$

$$(u_{ht}, v_h) - (\boldsymbol{\sigma}_h, \nabla v_h) + \langle \hat{\boldsymbol{\sigma}}_h \cdot \boldsymbol{\nu}, v_h \rangle_{\partial \mathcal{T}_h} = (f, v_h), \tag{2.3c}$$

$$\langle \hat{u}_h, \mu_h \rangle_{\partial \Omega} = 0, \tag{2.3d}$$

$$\langle \hat{\boldsymbol{\sigma}}_h \cdot \boldsymbol{\nu}, m_h \rangle_{\partial \mathcal{T}_h \setminus \partial \Omega} = 0, \tag{2.3e}$$

$$u_h(0) = \Pi_V u_0, \tag{2.3f}$$

where the numerical trace for flux is defined by

$$\hat{\boldsymbol{\sigma}}_h \cdot \boldsymbol{\nu} = \boldsymbol{\sigma}_h \cdot \boldsymbol{\nu} + \tau(u_h - \hat{u}_h) \ \text{ on } \partial \mathcal{T}_h,$$

for some non-negative stabilization parameter τ defined on Γ, which is assumed to be piecewise constant on the faces. We note that the exact solutions u, $\boldsymbol{q}$ and $\boldsymbol{\sigma}$ satisfy (2.3). Hence we obtain the following error equations:

$$(\boldsymbol{q}-\boldsymbol{q}_h, \boldsymbol{w}_h) - (u-u_h, \nabla\cdot\boldsymbol{w}_h) + \langle u-\hat{u}_h, \boldsymbol{w}_h\cdot\boldsymbol{\nu}\rangle_{\partial\mathcal{T}_h} = 0, \tag{2.4a}$$

$$(a(\boldsymbol{q}-\boldsymbol{q}_h), \boldsymbol{\tau}_h) - (\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + \int_0^t (b(t,s)(\boldsymbol{q}-\boldsymbol{q}_h)(s), \boldsymbol{\tau}_h)ds = 0, \tag{2.4b}$$

$$(u_t-u_{h_t}, v_h) - (\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, \nabla v_h) + \langle(\boldsymbol{\sigma}-\hat{\boldsymbol{\sigma}}_h)\cdot\boldsymbol{\nu}, v_h\rangle_{\partial\mathcal{T}_h} = 0, \tag{2.4c}$$

$$\langle u-\hat{u}_h, \mu_h\rangle_{\partial\Omega} = 0, \tag{2.4d}$$

$$\langle(\boldsymbol{\sigma}-\hat{\boldsymbol{\sigma}}_h)\cdot\boldsymbol{\nu}, m_h\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega} = 0, \tag{2.4e}$$

for all $(v_h, \boldsymbol{w}_h, \boldsymbol{\tau}_h, \mu_h, m_h) \in (V_h\times\boldsymbol{W}_h\times\boldsymbol{W}_h\times M_h\times M_h)$.

In our a priori error analysis, we introduce and thoroughly examine an extended mixed Ritz-Volterra projection.

2.3 Extended Mixed Ritz-Volterra Projection and Related Estimates

We define the following Ritz-Volterra projection: For each t, find $(\tilde{u}_h, \tilde{\boldsymbol{q}}_h, \tilde{\boldsymbol{\sigma}}_h, \hat{\tilde{u}}_h) \in (V_h\times \boldsymbol{W}_h\times\boldsymbol{W}_h\times M_h)$ satisfying

$$(\boldsymbol{q}-\tilde{\boldsymbol{q}}_h, \boldsymbol{w}_h) - (u-\tilde{u}_h, \nabla\cdot\boldsymbol{w}_h) + \langle u-\hat{\tilde{u}}_h, \boldsymbol{w}_h\cdot\boldsymbol{\nu}\rangle_{\partial\mathcal{T}_h} = 0, \tag{2.5a}$$

$$(a(\boldsymbol{q}-\tilde{\boldsymbol{q}}_h), \boldsymbol{\tau}_h) - (\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}}_h, \boldsymbol{\tau}_h) + \int_0^t (b(t,s)(\boldsymbol{q}-\tilde{\boldsymbol{q}}_h)(s), \boldsymbol{\tau}_h)ds = 0, \tag{2.5b}$$

$$-(\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}}_h, \nabla v_h) + \langle(\boldsymbol{\sigma}-\hat{\tilde{\boldsymbol{\sigma}}}_h)\cdot\boldsymbol{\nu}, v_h\rangle_{\partial\mathcal{T}_h} = 0, \tag{2.5c}$$

$$\langle u-\hat{\tilde{u}}_h, \mu_h\rangle_{\partial\Omega} = 0, \tag{2.5d}$$

$$\langle(\boldsymbol{\sigma}-\hat{\tilde{\boldsymbol{\sigma}}}_h)\cdot\boldsymbol{\nu}, m_h\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega} = 0, \tag{2.5e}$$

for all $(v_h, \boldsymbol{w}_h, \boldsymbol{\tau}_h, \mu_h, m_h) \in (V_h\times\boldsymbol{W}_h\times\boldsymbol{W}_h\times M_h\times M_h)$, where

$$\hat{\tilde{\boldsymbol{\sigma}}}_h\cdot\boldsymbol{\nu} = \tilde{\boldsymbol{\sigma}}_h\cdot\boldsymbol{\nu} + \tau(\tilde{u}_h - \hat{\tilde{u}}_h) \text{ on } \partial\mathcal{T}_h.$$

We write the errors in terms of the projection $\boldsymbol{I}_h^k$ (L^2-projection onto $\boldsymbol{W}_h$) and P_M (L^2-projection onto M_h) as

$$
\begin{aligned}
u - \tilde{u}_h &= (u - \Pi_V u) - (\tilde{u}_h - \Pi_V u) = \theta_u - \rho_u, \\
\boldsymbol{q} - \tilde{\boldsymbol{q}}_h &= (\boldsymbol{q} - \boldsymbol{I}_h^k \boldsymbol{q}) - (\tilde{\boldsymbol{q}}_h - \boldsymbol{I}_h^k \boldsymbol{q}) = \boldsymbol{\theta_q} - \boldsymbol{\rho_q}, \\
\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h &= (\boldsymbol{\sigma} - \boldsymbol{\Pi_W \sigma}) - (\tilde{\boldsymbol{\sigma}}_h - \boldsymbol{\Pi_W \sigma}) = \boldsymbol{\theta_\sigma} - \boldsymbol{\rho_\sigma}, \\
u - \hat{\tilde{u}}_h &= (u - P_M u) - (\hat{\tilde{u}}_h - P_M u) = \hat{\theta}_u - \hat{\rho}_u, \\
\boldsymbol{\sigma} - \hat{\tilde{\boldsymbol{\sigma}}}_h &= (\boldsymbol{\sigma} - \boldsymbol{P_M \sigma}) - (\hat{\tilde{\boldsymbol{\sigma}}}_h - \boldsymbol{P_M \sigma}) = \hat{\boldsymbol{\theta}}_{\boldsymbol{\sigma}} - \hat{\boldsymbol{\rho}}_{\boldsymbol{\sigma}}.
\end{aligned}
$$

Therefore, the system of equations become

$$
(\boldsymbol{\rho_q}, \boldsymbol{w_h}) - (\rho_u, \nabla \cdot \boldsymbol{w_h}) + \langle \hat{\rho}_u, \boldsymbol{w_h} \cdot \boldsymbol{\nu} \rangle_{\partial \mathcal{T}_h} = 0, \tag{2.6a}
$$

$$
\begin{aligned}
(a\boldsymbol{\rho_q}, \boldsymbol{\tau_h}) - (\boldsymbol{\rho_\sigma}, \boldsymbol{\tau}_h) + \int_0^t (b(t,s)\boldsymbol{\rho_q}(s), \boldsymbol{\tau}_h) ds &= (a\boldsymbol{\theta_q}, \boldsymbol{\tau}_h) - (\boldsymbol{\theta_\sigma}, \boldsymbol{\tau}_h) \\
&\quad + \int_0^t (b(t,s)\boldsymbol{\theta_q}(s), \boldsymbol{\tau}_h) ds,
\end{aligned} \tag{2.6b}
$$

$$
-(\boldsymbol{\rho_\sigma}, \nabla v_h) + \langle \hat{\boldsymbol{\rho}}_{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu}, v_h \rangle_{\partial \mathcal{T}_h} = 0, \tag{2.6c}
$$

$$
\langle \hat{\rho}_u, \mu_h \rangle_{\partial \Omega} = 0, \tag{2.6d}
$$

$$
\langle \hat{\boldsymbol{\rho}}_{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu}, m_h \rangle_{\partial \mathcal{T}_h \setminus \partial \Omega} = 0, \tag{2.6e}
$$

for all $(v_h, \boldsymbol{w}_h, \boldsymbol{\tau}_h, \mu_h, m_h) \in (V_h \times \boldsymbol{W}_h \times \boldsymbol{W}_h \times M_h \times M_h)$.

Lemma 2.3.1. *For all $\mu_h \in P_k(F)$, we have the following equality:*

$$
\langle \hat{\boldsymbol{\rho}}_{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu}, \mu_h \rangle_{\partial \mathcal{T}_h} = \langle \boldsymbol{\rho_\sigma} \cdot \boldsymbol{\nu} + \tau(\rho_u - \hat{\rho}_u), \mu_h \rangle_{\partial \mathcal{T}_h}.
$$

Proof. Consider,

$$
\langle \hat{\boldsymbol{\rho}}_{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu}, \mu_h \rangle_{\partial \mathcal{T}_h} = \langle (\hat{\tilde{\boldsymbol{\sigma}}}_h - P_M \boldsymbol{\sigma} + \boldsymbol{\sigma} - \boldsymbol{\sigma}) \cdot \boldsymbol{\nu}, \mu_h \rangle_{\partial \mathcal{T}_h}.
$$

Using the definition of $\hat{\tilde{\boldsymbol{\sigma}}}_h$, adding and subtracting the terms $\boldsymbol{\Pi}_W \boldsymbol{\sigma} \cdot \boldsymbol{\nu}$ and $\tau(\Pi_V u - u)$ and using the definition of HDG projection, we obtain

$$
\langle \hat{\boldsymbol{\rho}}_{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu}, \mu_h \rangle_{\partial \mathcal{T}_h} = \langle (\tilde{\boldsymbol{\sigma}} - \boldsymbol{\Pi}_W \boldsymbol{\sigma}) \cdot \boldsymbol{\nu} + \tau(\tilde{u}_h - \hat{\tilde{u}}_h - \Pi_V u + u), \mu_h \rangle_{\partial \mathcal{T}_h}.
$$

Finally, adding and subtracting $P_M u$, we arrive at the desired result. □

Below, we present the estimates for $\|\boldsymbol{\rho_\sigma}\|$ and $\|\boldsymbol{\rho_q}\|$.

Lemma 2.3.2. *For $t \in (0, T]$, a positive constant C that is unaffected by the values of h and k exists, such that it ensures the validity of the following inequality:*

$$\|\boldsymbol{\rho_\sigma}(t)\| + \|\boldsymbol{\rho_q}(t)\| + \|\hat{\rho}_u - \rho_u\|_\tau \leq C\left[\|\boldsymbol{\theta_q}(t)\| + \|\boldsymbol{\theta_\sigma}(t)\| + \int_0^t \|\boldsymbol{\theta_q}(s)\| ds\right].$$

Proof. For the estimates of $\|\boldsymbol{\rho_\sigma}\|$, we will choose $\boldsymbol{\tau}_h = \boldsymbol{\rho_\sigma}$ in (2.6b) to get

$$(a\boldsymbol{\rho_q}, \boldsymbol{\rho_\sigma}) - (\boldsymbol{\rho_\sigma}, \boldsymbol{\rho_\sigma}) + \int_0^t (b(t,s)\boldsymbol{\rho_q}(s), \boldsymbol{\rho_\sigma}) ds = (a\boldsymbol{\theta_q}, \boldsymbol{\rho_\sigma}) - (\boldsymbol{\theta_\sigma}, \boldsymbol{\rho_\sigma}) + \int_0^t (b(t,s)\boldsymbol{\theta_q}(s), \boldsymbol{\rho_\sigma}) ds.$$

Then, use of Lemma 1.4.2 and the fact that a and b are bounded, show

$$\begin{aligned}\|\boldsymbol{\rho_\sigma}\|^2 &= (a\boldsymbol{\rho_q}, \boldsymbol{\rho_\sigma}) + \int_0^t (b(t,s)\boldsymbol{\rho_q}(s), \boldsymbol{\rho_\sigma}) ds - (a\boldsymbol{\theta_q}, \boldsymbol{\rho_\sigma}) + (\boldsymbol{\theta_\sigma}, \boldsymbol{\rho_\sigma}) - \int_0^t (b(t,s)\boldsymbol{\theta_q}(s), \boldsymbol{\rho_\sigma}) ds, \\ &\leq C\left[\|\boldsymbol{\rho_q}\| + \|\boldsymbol{\theta_q}\| + \|\boldsymbol{\theta_\sigma}\| + \int_0^t (\|\boldsymbol{\rho_q}(s)\| + \|\boldsymbol{\theta_q}(s)\|) ds\right]\|\boldsymbol{\rho_\sigma}\|,\end{aligned}$$

and hence,

$$\|\boldsymbol{\rho_\sigma}\| \leq C\left[\|\boldsymbol{\rho_q}\| + \|\boldsymbol{\theta_q}\| + \|\boldsymbol{\theta_\sigma}\| + \int_0^t (\|\boldsymbol{\rho_q}(s)\| + \|\boldsymbol{\theta_q}(s)\|)\, ds\right]. \tag{2.7}$$

Next, for the estimates of $\boldsymbol{\rho_q}$, we will take $\boldsymbol{w}_h = \boldsymbol{\rho_\sigma}$, $\boldsymbol{\tau}_h = \boldsymbol{\rho_q}$, $v_h = \rho_u$, $\mu_h = -\hat{\boldsymbol{\rho}}_{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu}$ and $m_h = -\hat{\rho}_u$ in (2.6a), (2.6b), (2.6c), (2.6d) and (2.6e), respectively. Then, add the resulting equations, to get

$$\begin{aligned}&(\boldsymbol{\rho_q}, \boldsymbol{\rho_\sigma}) - (\rho_u, \nabla\cdot\boldsymbol{\rho_\sigma}) + \langle\hat{\rho}_u, \boldsymbol{\rho_\sigma}\cdot\boldsymbol{\nu}\rangle_{\partial\mathcal{T}_h} + (a\boldsymbol{\rho_q}, \boldsymbol{\rho_q}) - (\boldsymbol{\rho_\sigma}, \boldsymbol{\rho_q}) + \int_0^t (b(t,s)\boldsymbol{\rho_q}, \boldsymbol{\rho_q}) ds \\ &- (\boldsymbol{\rho_\sigma}, \nabla\rho_u) + \langle\hat{\boldsymbol{\rho}}_{\boldsymbol{\sigma}}\cdot\boldsymbol{\nu}, \rho_u\rangle_{\partial\mathcal{T}_h} - \langle\hat{\rho}_u, \hat{\boldsymbol{\rho}}_{\hat{\sigma}}\cdot\boldsymbol{\nu}\rangle_{\partial\mathcal{T}_h} = (a\boldsymbol{\theta_q}, \boldsymbol{\rho_q}) - (\boldsymbol{\theta_\sigma}, \boldsymbol{\rho_q}) + \int_0^t (b(t,s)\boldsymbol{\theta_q}, \boldsymbol{\rho_q}) ds.\end{aligned}$$

Now, combining the terms and using Lemma 2.3.1, we get

$$\|a^{1/2}\boldsymbol{\rho_q}\|^2 + \|\hat{\rho}_u - \rho_u\|_\tau^2 = (a\boldsymbol{\theta_q}, \boldsymbol{\rho_q}) - (\boldsymbol{\theta_\sigma}, \boldsymbol{\rho_q}) + \int_0^t \left[(b(t,s)\boldsymbol{\theta_q}(s), \boldsymbol{\rho_q}) - (b(t,s)\boldsymbol{\rho_q}(s), \boldsymbol{\rho_q})\right] ds.$$

Further, using boundedness of a and Lemma 1.4.2, we arrive at

$$\|\boldsymbol{\rho_q}\| \leq C\left[\|\boldsymbol{\theta_q}\| + \|\boldsymbol{\theta_\sigma}\| + \int_0^t (\|\boldsymbol{\theta_q}(s)\| + \|\boldsymbol{\rho_q}(s)\|) ds\right].$$

Finally, application of Gronwall's lemma along with (2.7) conclude the rest of the proof.

□

Below, we prove a lemma which provides the estimate for $\|\rho_u\|$.

Lemma 2.3.3. *For $t \in (0, T]$, a positive constant C that is unaffected by the values of h and k exists, such that it ensures the validity of the following inequality:*

$$\|\rho_u(t)\| \leq Ch^{k+1}\Big[\|u(t)\|_{H^{k+2}(\Omega)} + \int_0^t \|u(s)\|_{H^{k+2}(\Omega)} ds\Big].$$

Proof. For this estimate, we make use of the following auxiliary problem:

$$-\nabla \cdot (a\nabla\psi) = \rho_u \qquad \text{in } \Omega,$$
$$\psi = 0 \qquad \text{on } \partial\Omega,$$

with the following elliptic regularity result

$$\|\psi\|_{H^2(\Omega)} \leq \|\rho_u\|.$$

We will write the above problem in the following mixed form:

$$\boldsymbol{\phi} = -\nabla\psi \qquad \text{in } \Omega, \tag{2.8a}$$
$$\boldsymbol{p} = a\boldsymbol{\phi} \qquad \text{in } \Omega, \tag{2.8b}$$
$$\nabla \cdot \boldsymbol{p} = \rho_u \qquad \text{in } \Omega. \tag{2.8c}$$

Then, using L^2 inner product between (2.8c) and ρ_u, yields

$$\|\rho_u\|^2 = (\rho_u, \rho_u) = (\rho_u, \nabla \cdot \boldsymbol{p}),$$

using Lemma 1.4.6, we obtain

$$\begin{aligned}\|\rho_u\|^2 &= (\rho_u, \nabla \cdot \boldsymbol{\Pi_W p}) + \langle \rho_u, \tau(\Pi_V\psi - \psi)\rangle_{\partial\mathcal{T}_h} \\ &= (\boldsymbol{\rho_q}, \boldsymbol{\Pi_W p}) + \langle \hat{\rho}_u, \boldsymbol{\Pi_W p} \cdot \boldsymbol{\nu}\rangle_{\partial\mathcal{T}_h} + \langle \rho_u, \tau(\Pi_V\psi - \psi)\rangle_{\partial\mathcal{T}_h}. \qquad \text{by (2.6a)}\end{aligned}$$

By continuity of $\boldsymbol{p} \cdot \boldsymbol{\nu}$ and (2.6d), we arrive at

$$\begin{aligned}\|\rho_u\|^2 &= (\boldsymbol{\rho_q}, \boldsymbol{\Pi_W p}) + \langle \hat{\rho}_u, (\boldsymbol{\Pi_W p} - \boldsymbol{p}) \cdot \boldsymbol{\nu}\rangle_{\partial\mathcal{T}_h} + \langle \rho_u, \tau(\Pi_V\psi - \psi)\rangle_{\partial\mathcal{T}_h} \\ &= (\boldsymbol{\rho_q}, \boldsymbol{\Pi_W p}) + \langle \tau(\rho_u - \hat{\rho}_u), \Pi_V\psi\rangle_{\partial\mathcal{T}_h} + \langle \boldsymbol{\rho_\sigma} \cdot \boldsymbol{\nu}, P_M\psi\rangle_{\partial\mathcal{T}_h} \qquad \text{by (1.1c), (2.6e)}\end{aligned}$$

$$
\begin{aligned}
&= (\boldsymbol{\rho_q}, \boldsymbol{\Pi_W p}) + \langle \tau(\rho_u - \hat{\rho}_u) - \boldsymbol{\rho_\sigma} \cdot \boldsymbol{\nu}, \Pi_V \psi \rangle_{\partial \mathcal{T}_h} + (\boldsymbol{\rho_\sigma}, \nabla \Pi_V \psi) \\
&\quad + \langle \boldsymbol{\rho_\sigma} \cdot \boldsymbol{\nu}, \psi \rangle_{\partial \mathcal{T}_h} && \text{by (2.6c)} \\
&= (\boldsymbol{\rho_q}, \boldsymbol{\Pi_W p}) + (\boldsymbol{\rho_\sigma}, \nabla \psi) && \text{by (1.1a)} \\
&= (\boldsymbol{\rho_q}, \boldsymbol{\Pi_W p} - \boldsymbol{p}) + (\boldsymbol{\rho_q}, \boldsymbol{p}) - (\boldsymbol{\rho_\sigma}, \boldsymbol{\phi}) && \text{by (2.8a)} \\
&= (\boldsymbol{\rho_q}, \boldsymbol{\Pi_W p} - \boldsymbol{p}) + (\boldsymbol{\rho_q}, \boldsymbol{p}) - (\boldsymbol{\rho_\sigma}, \boldsymbol{\phi} - \boldsymbol{I}_h^k \boldsymbol{\phi}) - (\boldsymbol{\rho_\sigma}, \boldsymbol{I}_h^k \boldsymbol{\phi}).
\end{aligned}
$$

Finally, we get

$$
\begin{aligned}
\|\rho_u\|^2 &= (\boldsymbol{\rho_q}, \boldsymbol{\Pi_W p} - \boldsymbol{p}) + (\boldsymbol{\rho_q}, \boldsymbol{p}) - (a\boldsymbol{\rho_q}, \boldsymbol{I}_h^k \boldsymbol{\phi}) + \int_0^t (b(t,s)(\boldsymbol{\theta_q}(s) - \boldsymbol{\rho_q}(s)), \boldsymbol{I}_h^k \boldsymbol{\phi}) ds \\
&\quad + (a\boldsymbol{\theta_q}, \boldsymbol{I}_h^k \boldsymbol{\phi}) - (\boldsymbol{\theta_\sigma}, \boldsymbol{I}_h^k \boldsymbol{\phi}) && \text{by (2.6b)} \\
&= (\boldsymbol{\rho_q}, \boldsymbol{\Pi_W p} - \boldsymbol{p}) + (a\boldsymbol{\rho_q}, \boldsymbol{\phi}) - (a\boldsymbol{\rho_q}, \boldsymbol{I}_h^k \boldsymbol{\phi}) + \int_0^t (b(t,s)(\boldsymbol{\theta_q}(s) - \boldsymbol{\rho_q}(s)), \boldsymbol{I}_h^k \boldsymbol{\phi}) ds \\
&\quad + (a\boldsymbol{\theta_q}, \boldsymbol{I}_h^k \boldsymbol{\phi}) - (\boldsymbol{\theta_\sigma}, \boldsymbol{I}_h^k \boldsymbol{\phi}) && \text{by (2.8b)} \\
&= (\boldsymbol{\rho_q}, \boldsymbol{\Pi_W p} - \boldsymbol{p}) + (a\boldsymbol{\rho_q}, \boldsymbol{\phi} - \boldsymbol{I}_h^k \boldsymbol{\phi}) + (a\boldsymbol{\theta_q}, \boldsymbol{I}_h^k \boldsymbol{\phi}) - (\boldsymbol{\theta_\sigma}, \boldsymbol{I}_h^k \boldsymbol{\phi}) \\
&\quad + \int_0^t (b(t,s)(\boldsymbol{\theta_q}(s) - \boldsymbol{\rho_q}(s)), \boldsymbol{I}_h^k \boldsymbol{\phi}) ds. && (2.9)
\end{aligned}
$$

Next, using the Cauchy Schwarz inequality, we get the following inequality:

$$
\begin{aligned}
\|\rho_u\|^2 &\leq \|\boldsymbol{\rho_q}\| \|\boldsymbol{\Pi_W p} - \boldsymbol{p}\| + C\|\boldsymbol{\rho_q}\| \|\boldsymbol{\phi} - \boldsymbol{I}_h^k \boldsymbol{\phi}\| + C\|\boldsymbol{\theta_q}\| \|\boldsymbol{I}_h^k \boldsymbol{\phi}\|_{\boldsymbol{H}^1(\Omega)} + \|\boldsymbol{\theta_\sigma}\| \|\boldsymbol{I}_h^k \boldsymbol{\phi}\|_{\boldsymbol{H}^1(\Omega)} \\
&\quad + C \int_0^t \Big(\|\boldsymbol{\theta_q}(s)\| + \|\boldsymbol{\rho_q}(s)\| \Big) \|\boldsymbol{I}_h^k \boldsymbol{\phi}\|_{\boldsymbol{H}^1(\Omega)}.
\end{aligned}
$$

Now, a use of the estimates of HDG projection, estimate of $\|\boldsymbol{\rho_q}\|$, Lemma 1.4.1, elliptic regularity and the fact that $\|\boldsymbol{\phi}\|_{\boldsymbol{H}^1(\Omega)} \leq M\|\boldsymbol{p}\|_{\boldsymbol{H}^1(\Omega)}$ and $\|\boldsymbol{p}\|_{\boldsymbol{H}^1(\Omega)} \leq \|\psi\|_{H^2(\Omega)}$ will yield the desired result. □

Remark: The order of convergence of $\|\rho_u\|$ can be further increased to $k + 3/2$, using dual norm estimates. This additional result is stated in the form of the following lemma:

Lemma 2.3.4. *For $t \in (0, T]$, a positive constant C that is unaffected by the values of h and k exists, such that it ensures the validity of the following inequality:*

$$
\|\rho_u(t)\| \leq Ch^{k+3/2} \bigg[\|u(t)\|_{H^{k+2}(\Omega)} + \int_0^t \|u(s)\|_{H^{k+2}(\Omega)} ds \bigg].
$$

Proof. We begin by defining the following dual norm:

$$\|\boldsymbol{v}\|_{(\boldsymbol{H}^1(\Omega))^*} = \sup_{\boldsymbol{w}\in\boldsymbol{H}^1(\Omega), w\neq 0} \frac{(\boldsymbol{v},\boldsymbol{w})}{\|\boldsymbol{w}\|_{\boldsymbol{H}^1(\Omega)}}.$$

Now, from (2.9), we have the following inequality:

$$\begin{aligned}\|\rho_u\|^2 &\leq \|\boldsymbol{\rho_q}\|\|\boldsymbol{\Pi_W p}-\boldsymbol{p}\| + C\|\boldsymbol{\rho_q}\|\|\boldsymbol{\phi}-\boldsymbol{I}_h^k\boldsymbol{\phi}\| + C\|\boldsymbol{\theta_q}\|_{\boldsymbol{H}^1(\Omega)^*}\|\boldsymbol{I}_h^k\boldsymbol{\phi}\|_{\boldsymbol{H}^1(\Omega)} \\ &+ \|\boldsymbol{\theta_\sigma}\|_{\boldsymbol{H}^1(\Omega)^*}\|\boldsymbol{I}_h^k\boldsymbol{\phi}\|_{\boldsymbol{H}^1(\Omega)} + C\int_0^t \Big(\|\boldsymbol{\theta_q}(s)\|_{\boldsymbol{H}^1(\Omega)^*} + \|\boldsymbol{\rho_q}(s)\|_{\boldsymbol{H}^1(\Omega)^*}\Big)\|\boldsymbol{I}_h^k\boldsymbol{\phi}\|_{\boldsymbol{H}^1(\Omega)}.\end{aligned} \tag{2.10}$$

Hence, we require the estimates of $\|\boldsymbol{\theta_q}\|_{\boldsymbol{H}^1(\Omega)^*}$, $\|\boldsymbol{\theta_\sigma}\|_{\boldsymbol{H}^1(\Omega)^*}$ and $\|\boldsymbol{\rho_q}\|_{\boldsymbol{H}^1(\Omega)^*}$. For the estimates of $\|\boldsymbol{\theta_q}\|_{\boldsymbol{H}^1(\Omega)^*}$, we will proceed as follows:

$$\begin{aligned}(\boldsymbol{\theta_q},\boldsymbol{w}) &= (\boldsymbol{\theta_q},\boldsymbol{w}-\boldsymbol{I}_h^k\boldsymbol{w}) + (\boldsymbol{\theta_q},\boldsymbol{I}_h^k\boldsymbol{w}) \\ &\leq \|\boldsymbol{\theta_q}\|\|\boldsymbol{w}-\boldsymbol{I}_h^k\boldsymbol{w}\| \\ &\leq Ch\|\boldsymbol{\theta_q}\|\|\boldsymbol{w}\|_{\boldsymbol{H}^1(\Omega)}.\end{aligned}$$

Therefore, we have

$$\|\boldsymbol{\theta_q}\|_{\boldsymbol{H}^1(\Omega)^*} \leq Ch\|\boldsymbol{\theta_q}\|. \tag{2.11}$$

Now, for $\|\boldsymbol{\theta_\sigma}\|_{\boldsymbol{H}^1(\Omega)^*}$, we have for $k \geq 1$

$$\begin{aligned}(\boldsymbol{\theta_\sigma},\boldsymbol{w}) &= (\boldsymbol{\theta_\sigma},\boldsymbol{w}-\boldsymbol{I}_h^{k-1}\boldsymbol{w}) + (\boldsymbol{\theta_\sigma},\boldsymbol{I}_h^{k-1}\boldsymbol{w}) \\ &\leq \|\boldsymbol{\theta_\sigma}\|\|\boldsymbol{w}-\boldsymbol{I}_h^{k-1}\boldsymbol{w}\| \\ &\leq Ch\|\boldsymbol{\theta_\sigma}\|\|\boldsymbol{w}\|_{\boldsymbol{H}^1(\Omega)}.\end{aligned}$$

Therefore, we have

$$\|\boldsymbol{\theta_\sigma}\|_{\boldsymbol{H}^1(\Omega)^*} \leq Ch\|\boldsymbol{\theta_\sigma}\|. \tag{2.12}$$

Finally, for the estimates of $\|\boldsymbol{\rho_q}\|_{\boldsymbol{H}^1(\Omega)^*}$, we have

$$(\boldsymbol{\rho_q},\boldsymbol{w}) = (\boldsymbol{\rho_q},\boldsymbol{w}-\boldsymbol{I}_h^k\boldsymbol{w}) + (\boldsymbol{\rho_q},\boldsymbol{I}_h^k\boldsymbol{w})$$

$$
\begin{aligned}
&= (\rho_u, \nabla \cdot \boldsymbol{w_h}) - \langle \hat{\rho}_u, \boldsymbol{w_h} \cdot \boldsymbol{\nu} \rangle \quad \text{by (2.6a)} \\
&= (\rho_u, \boldsymbol{\nabla} \cdot \boldsymbol{w}) + \langle \rho_u - \hat{\rho}_u, (\boldsymbol{I}_h^k \boldsymbol{w} - \boldsymbol{w}) \cdot \boldsymbol{\nu} \rangle \\
&\leq C(\|\rho_u\| + h^{1/2} \|\rho_u - \hat{\rho}_u\|) \|\boldsymbol{w}\|_{\boldsymbol{H}^1(\Omega)}.
\end{aligned}
$$

Therefore, we have

$$
\|\boldsymbol{\rho_q}\|_{\boldsymbol{H}^1(\Omega)^*} \leq C(\|\rho_u\| + h^{1/2} \|\rho_u - \hat{\rho}_u\|). \tag{2.13}
$$

Use of (2.11), (2.12) and (2.13) in (2.10), will give the desired improved estimates of $\|\rho_u\|$, and hence, conclude the lemma. □

Lemma 2.3.5. *For $t \in (0, T]$, a positive constant C that is unaffected by the values of h and k exists, such that it ensures the validity of the following inequality:*

$$
\begin{aligned}
\|\rho_{u_t}(t)\| \leq C h^{k+1} \Bigg[&\|u(t)\|_{H^{k+2}(\mathcal{T}_h)} + \|u_t(t)\|_{H^{k+2}(\mathcal{T}_h)} + \int_0^t \big(\|u(s)\|_{H^{k+2}(\mathcal{T}_h)} \\
&+ \|u_s(s)\|_{H^{k+2}(\mathcal{T}_h)} \big) ds \Bigg].
\end{aligned}
$$

Proof. We begin by differentiating (2.6a)-(2.6e) with respect to t, to obtain

$$
\begin{aligned}
(\boldsymbol{\rho_{q_t}}, \boldsymbol{w_h}) - (\rho_{u_t}, \nabla \cdot \boldsymbol{w_h}) + \langle \hat{\rho}_{u_t}, \boldsymbol{w_h} \cdot \boldsymbol{\nu} \rangle_{\partial \mathcal{T}_h} &= 0, \\
(a \boldsymbol{\rho_{q_t}}, \boldsymbol{\tau_h}) - (\boldsymbol{\rho_{\sigma_t}}, \boldsymbol{\tau}_h) + (b(t,t) \boldsymbol{\rho_q}(t), \boldsymbol{\tau_h}) + \int_0^t \left(\frac{\partial}{\partial s} (b(t,s) \boldsymbol{\rho_q}(s)), \boldsymbol{\tau}_h \right) ds & \\
= (a \boldsymbol{\theta_{q_t}}, \boldsymbol{\tau}_h) - (\boldsymbol{\theta_{\sigma_t}}, \boldsymbol{\tau}_h) + (b(t,t) \boldsymbol{\theta_q}(t), \boldsymbol{\tau_h}) + \int_0^t \left(\frac{\partial}{\partial s} (b(t,s) \boldsymbol{\theta_q}(s)), \boldsymbol{\tau}_h \right) ds&, \\
-(\boldsymbol{\rho_{\sigma_t}}, \nabla v_h) + \langle \boldsymbol{\rho_{\sigma_t}} \cdot \boldsymbol{\nu}, v_h \rangle_{\partial \mathcal{T}_h} &= 0, \\
\langle \hat{\rho}_{u_t}, \mu_h \rangle_{\partial \Omega} &= 0, \\
\langle \hat{\boldsymbol{\rho}}_{\boldsymbol{\sigma_t}} \cdot \boldsymbol{\nu}, m_h \rangle_{\partial \mathcal{T}_h \setminus \partial \mathcal{T}_h} &= 0,
\end{aligned}
$$

for all $(v_h, \boldsymbol{w}_h, \boldsymbol{\tau}_h, \mu_h, m_h) \in (V_h \times \boldsymbol{W}_h \times \boldsymbol{W}_h \times M_h)$. Then, following the similar steps as we did in the proofs of Lemma 2.3.2 and 2.3.3, we can derive the estimates of ρ_{u_t}, $\boldsymbol{\rho_{q_t}}$ and $\boldsymbol{\rho_{\sigma_t}}$. □

Theorem 2.3.6. *When $u, u_t \in L^\infty(H^{k+2}(\mathcal{T}_h))$, a positive constant C that is unaffected by the values of h and k exists, such that it ensures the validity of the following inequality:*

$$\begin{aligned}\|(u-\tilde{u}_h)(t)\|+\|(\boldsymbol{q}-\tilde{\boldsymbol{q}}_h)(t)\|+\|(\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}}_h)(t)\| &\le Ch^{k+1}\Big[\|u(t)\|_{H^{k+2}(\mathcal{T}_h)} \\ &\quad + \int_0^t \|u(s)\|_{H^{k+2}(\mathcal{T}_h)}ds\Big], \\ \|(u_t-\tilde{u}_{h_t})(t)\| \le Ch^{k+1}\Big[\|u(t)\|_{H^{k+2}(\mathcal{T}_h)} &+ \|u_t(t)\|_{H^{k+2}(\mathcal{T}_h)} + \int_0^t \big(\|u(s)\|_{H^{k+2}(\mathcal{T}_h)} \\ &+ \|u_s(s)\|_{H^{k+2}(\mathcal{T}_h)}\big)ds\Big].\end{aligned}$$

Proof. With the help of (1.2b), Lemmas 2.3.3 and 2.3.5 and application of the triangle inequality, we get the desired result. □

2.4 A Priori Error Estimates for Semidiscrete Scheme

This section deals with the following theorem:

Theorem 2.4.1. *Let $(u, \boldsymbol{q}, \boldsymbol{\sigma})$ be the solution of* (2.2) *and $(u_h, \boldsymbol{q}_h, \boldsymbol{\sigma}_h) \in V_h \times \boldsymbol{W}_h \times \boldsymbol{W}_h$ be the solution of* (2.3). *If $u, u_t \in L^\infty(H^{k+2}(\mathcal{T}_h))$, $u_h(0) = \Pi_V u_0$ and $\boldsymbol{q}_h(0) = -\boldsymbol{I}_h^k \nabla u_0$, then, $\forall\, t \in (0, T]$, we have the following estimates:*

$$\begin{aligned}&\|(u-u_h)(t)\|+\|(\boldsymbol{q}-\boldsymbol{q}_h)(t)\|+\|(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)(t)\| \\ &\qquad \le Ch^{k+1}\Big[\|u(t)\|_{H^{k+2}(\mathcal{T}_h)} + \int_0^t \big(\|u(s)\|_{H^{k+2}(\mathcal{T}_h)} + \|u_s(s)\|_{H^{k+2}(\mathcal{T}_h)}\big)ds\Big], \\ &\|(u_t-u_{h_t})(t)\| \le Ch^{k+1}\Big[\|u(t)\|_{H^{k+2}(\mathcal{T}_h)} + \|u_t(t)\|_{H^{k+2}(\mathcal{T}_h)} + \|u_{tt}(t)\|_{H^{k+2}(\mathcal{T}_h)} \\ &\qquad + \int_0^t \big(\|u(s)\|_{H^{k+2}(\mathcal{T}_h)} + \|u_s(s)\|_{H^{k+2}(\mathcal{T}_h)} + \|u_{ss}(s)\|_{H^{k+2}(\mathcal{T}_h)}\big)\, ds\Big].\end{aligned}$$

We now rewrite the errors in terms of η and ξ in the following manner

$$e_u = u - u_h = (u - \tilde{u}_h) - (u_h - \tilde{u}_h) = \eta_u - \xi_u,$$

similarly, we will decompose $\boldsymbol{e_q}$, $\boldsymbol{e_\sigma}$, $\hat{e}_u$ and $\hat{\boldsymbol{e}}_{\boldsymbol{\sigma}}$ in terms of η's and ξ's. Since, from Section 2.3, we can compute the estimates of η_u, $\boldsymbol{\eta_q}$, $\boldsymbol{\eta_\sigma}$, $\hat{\eta}_u$ and $\hat{\boldsymbol{\eta}}_\sigma$, therefore, it remains to derive the estimates of ξ_u, $\boldsymbol{\xi_q}$, $\boldsymbol{\xi_\sigma}$, $\hat{\xi}_u$ and $\hat{\boldsymbol{\xi}}_\sigma$. For that, we proceed as follow:

$$(\boldsymbol{\xi_q}, \boldsymbol{w_h}) - (\xi_u, \nabla \cdot \boldsymbol{w_h}) + \langle \hat{\xi}_u, \boldsymbol{w_h} \cdot \boldsymbol{\nu} \rangle_{\partial \mathcal{T}_h} = 0 \qquad \forall \boldsymbol{w}_h \in \boldsymbol{W}_h, \tag{2.14a}$$

$$(a\boldsymbol{\xi_q}, \boldsymbol{\tau_h}) - (\boldsymbol{\xi_\sigma}, \boldsymbol{\tau}_h) + \int_0^t (b(t,s)\boldsymbol{\xi_q}(s), \boldsymbol{\tau}_h) ds = 0 \qquad \forall \boldsymbol{\tau_h} \in \boldsymbol{W}_h, \tag{2.14b}$$

$$(\xi_{ut}, v_h) - (\boldsymbol{\xi_\sigma}, \nabla v_h) + \langle \hat{\boldsymbol{\xi}}_{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu}, v_h \rangle_{\partial \mathcal{T}_h} = (\eta_{ut}, v_h) \qquad \forall v_h \in V_h, \tag{2.14c}$$

$$\langle \hat{\xi}_u, \mu_h \rangle_{\partial \Omega} = 0 \qquad \forall \mu_h \in M_h, \tag{2.14d}$$

$$\langle \hat{\boldsymbol{\xi}}_{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu}, m_h \rangle_{\partial \mathcal{T}_h \setminus \partial \Omega} = 0 \qquad \forall m_h \in M_h. \tag{2.14e}$$

Above system of equations is obtained using (2.4) and (2.5)

Lemma 2.4.2. *A positive constant C that is unaffected by the values of h and k exists, such that it ensures the validity of the following inequality:*

$$\|\xi_u(t)\|^2 + \int_0^t \Big(\|\boldsymbol{\xi_q}(s)\|^2 + \|\hat{\xi}_u - \xi_u\|_\tau^2 \Big) ds \le C \bigg(\|\xi_u(0)\|^2 + \int_0^T \|\eta_{u_t}(t)\|^2 dt \bigg).$$

Proof. Choose $\boldsymbol{w}_h = \boldsymbol{\xi_\sigma}$, $\boldsymbol{\tau}_h = \boldsymbol{\xi_q}$, $v_h = \xi_u$, $\mu_h = -\hat{\boldsymbol{\xi}}_{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu}$ and $m_h = -\hat{\xi}_u$ in (2.14a), (2.14b), (2.14c), (2.14d) and (2.14e), respectively, and then, adding the resulting equations, we obtain

$$(a\boldsymbol{\xi_q}, \boldsymbol{\xi_q}) + \int_0^t (b(t,s)\boldsymbol{\xi_q}(s), \boldsymbol{\xi_q}) ds + \frac{1}{2}\frac{d}{dt}\|\xi_u\|^2 + \|\hat{\xi}_u - \xi_u\|_\tau^2 = (\eta_{u_t}, \xi_u),$$

and hence,

$$\|\boldsymbol{\xi_q}\|^2 + \frac{d}{dt}\|\xi_u\|^2 \le \|\eta_{u_t}\|^2 + \|\xi_u\|^2 + C\int_0^t \|\boldsymbol{\xi_q}(s)\|^2 ds.$$

On integrating the above inequality from 0 to t, it follows that

$$\int_0^t \|\boldsymbol{\xi_q}(s)\|^2 ds + \|\xi_u\|^2 \le \|\xi_u(0)\|^2 + \int_0^t \bigg[\|\xi_u(s)\|^2 + C\int_0^s \|\boldsymbol{\xi_q}(\gamma)\|^2 d\gamma \bigg] ds + \int_0^t \|\eta_{u_s}(s)\|^2 ds.$$

Finally, application of the Gronwall's lemma gives the desired inequality. □

Lemma 2.4.3. *A positive constant C that is unaffected by the values of h and k exists, such that it ensures the validity of the following inequality:*

$$\|\boldsymbol{\xi_\sigma}(t)\|^2 + \|\boldsymbol{\xi_q}(t)\|^2 + \|\hat{\xi}_u - \xi_u\|_\tau^2 \le C\bigg(\|\boldsymbol{\xi_q}(0)\|^2 + \int_0^T \|\eta_{u_t}(t)\|^2 dt \bigg).$$

Proof. To begin with, we differentiate (2.14a) with respect to t and then choose $\boldsymbol{w_h} = \boldsymbol{\xi_\sigma}$, $\boldsymbol{\tau_h} = \boldsymbol{\xi_{q_t}}$, $v_h = \xi_{u_t}$ in (2.14a), (2.14b), (2.14c) respectively. Now, differentiate (2.14d) with respect to t and choose $\mu_h = -\hat{\boldsymbol{\xi}}_{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu}$ and $m_h = -\hat{\xi}_{u_t}$ in (2.14d) and (2.14e) respectively. Then adding the resulting equations, we obtain

$$(a\boldsymbol{\xi_q}, \boldsymbol{\xi_{q_t}}) + \frac{1}{2}\frac{d}{dt}\|\hat{\xi}_u - \xi_u\|_\tau^2 + \|\xi_{u_t}\|^2 + \int_0^t (b(t,s)\boldsymbol{\xi_q}(s), \boldsymbol{\xi_{q_t}})ds = (\eta_{u_t}, \xi_{u_t}).$$

Application of the Cauchy Schwarz inequality and Leibniz's theorem show,

$$\begin{aligned}\frac{d}{dt}\|a^{1/2}\boldsymbol{\xi}_q\|^2 + \frac{1}{2}\|\xi_{u_t}\|^2 \le \frac{1}{2}\|\eta_{u_t}\|^2 - \frac{d}{dt}\int_0^t (b(t,s)\boldsymbol{\xi_q}(s), \boldsymbol{\xi_q})ds + (b(t,t)\boldsymbol{\xi_q}, \boldsymbol{\xi_q}) \\ + \int_0^t (b_t(t,s)\boldsymbol{\xi_q}(s), \boldsymbol{\xi_q})ds.\end{aligned}$$

On integrating the above inequality from 0 to t, we obtain

$$\begin{aligned}\|a^{1/2}\boldsymbol{\xi}_q\|^2 + \int_0^t \|\xi_{u_s}(s)\|^2 \le \|a^{1/2}\boldsymbol{\xi_q}(0)\| + \int_0^t \|\eta_{u_s}(s)\|^2 - \int_0^t (b(t,s)\boldsymbol{\xi_q}(s), \boldsymbol{\xi_q})ds \\ + \int_0^t (b(s,s)\boldsymbol{\xi_q}(s), \boldsymbol{\xi_q}(s))ds + \int_0^t\int_0^s (b_s(s,\gamma)\boldsymbol{\xi_q}(\gamma), \boldsymbol{\xi_q}(s))d\gamma ds.\end{aligned}$$

Finally, a use of the Young's inequality and Gronwall's lemma along with the boundedness of a and b, will give the following estimate

$$\|\boldsymbol{\xi_q}\|^2 \le C\left(\|\boldsymbol{\xi_q}(0)\|^2 + \int_0^T \|\eta_{u_t}(t)\|^2 dt\right).$$

Now, choosing $\boldsymbol{\tau_h} = \boldsymbol{\xi_\sigma}$ in (2.14b) and then using Cauchy Schwarz inequality and boundedness of a will give

$$\|\boldsymbol{\xi_\sigma}\| \le C\left(\|\boldsymbol{\xi_q}\| + \int_0^t \|\boldsymbol{\xi_q}(s)\|ds\right).$$

Combining last two inequalities will give the desired result. □

Proof of Theorem 2.4.1: It simply follows by triangle's inequality, Theorem 2.3.6, Lemma 2.4.2 and Lemma 2.4.3.

□

2.5 Post-processing

We begin by defining the new approximation $u_h^* \in P_{k+1}(K)$, on the element K, as

$$u_h^* = u_h^p + \frac{1}{|K|}\int_K u_h, \;\; u_h^p \in P_{k+1}^0,$$

where u_h^p satisfies

$$(a\nabla u_h^p, \nabla v) = -(a\boldsymbol{q}_h, \nabla v), \;\; \forall v \in P_{k+1}^0,$$

where $P_k^0(K)$ has all the polynomials of $P_k(K)$ whose average value is zero.

Then, we have the following inequality:

$$\begin{aligned}\|u - u_h^*\|_{L^2(K)} &\leq \left\| u - u_h^p - \frac{1}{|K|}\int_K u_h dx \right\|_{L^2(K)} \\ &\leq \left\| u^p - u_h^p + \frac{1}{|K|}\int_K (u - u_h) dx \right\|_{L^2(K)} \\ &\leq \|I_h^{k-1} e_u\|_{L^2(K)} + \|u^p - u_h^p\|_{L^2(K)}, \end{aligned} \tag{2.15}$$

where, $u^p = u - \frac{1}{|K|}\displaystyle\int_K u dx$.
Below, we present lemmas that give the estimates of the terms in (2.15).

Lemma 2.5.1. *For the method of the form (2.3), a positive constant C that is unaffected by the values of h and k exists, such that it ensures the validity of the following inequality:*

$$\|I_h^{k-1} e_u\|_{L^2(K)} \leq C\sqrt{log\left(\frac{T}{\rho^2}\right)}\, h^{k+2}. \tag{2.16}$$

Proof. We begin the proof with the help of the following dual problem [113]. For fixed $t \in (0, T)$, let $\psi(s) \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfy the following system of equations

$$\begin{aligned} \boldsymbol{\phi}(s) &= \nabla\psi(s) \;\; \text{in } \Omega, \; s \leq t, \\ \boldsymbol{p}(s) &= a\boldsymbol{\phi}(s) + \int_s^t b(\gamma, s)\boldsymbol{\phi}(\gamma) d\gamma \;\; \text{in } \Omega, \; s \leq t, \\ \psi_s(s) + \nabla\cdot\boldsymbol{p}(s) &= 0 \;\; \text{in } \Omega, \; s \leq t, \\ \psi(s) &= 0 \;\; \text{on } \partial\Omega, \; s \leq t, \end{aligned}$$

$$\psi(t) = \lambda \quad \text{in } \Omega,$$

with the regularity results [112, 113] :

$$\int_0^t \|\psi(s)\|_2^2 \, ds \leq C\|\nabla\lambda\|^2, \tag{2.17}$$

$$\int_0^t (t-s)\|\psi(s)\|_2^2 \, ds \leq C\|\lambda\|^2. \tag{2.18}$$

We use the similar procedure as done in [139] to get the following equality

$$\begin{aligned}
\frac{d}{ds}(\psi(s), I_h^{k-1}e_u(s)) = (e_{u_s}(s), I_h^{k-1}\psi(s) - \psi(s)) - (e_{u_s}(s), I_h^k\psi(s) - \psi(s)) - (\boldsymbol{e_q}(s), \\
\boldsymbol{\Pi}_{k-1}^{RT}\boldsymbol{p}(s) - \boldsymbol{p}(s)) - (a\boldsymbol{e_q}(s), \boldsymbol{\phi}(s) - \boldsymbol{I}_h^k\boldsymbol{\phi}(s)) - (\boldsymbol{e_\sigma}(s), \boldsymbol{I}_h^k\boldsymbol{\phi}(s) \\
- \boldsymbol{\phi}(s)) - (\boldsymbol{e_\sigma}(s), \nabla(\psi - I_h^k\psi)(s)) - \langle \hat{\boldsymbol{e}}_{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu}, I_h^k\psi\rangle \\
+ \int_0^s (b(s,\gamma)\boldsymbol{e_q}(\gamma), \boldsymbol{I}_h^k\boldsymbol{\phi}(s))d\gamma - \int_s^t (b(\gamma,s)\boldsymbol{\phi}(\gamma), \boldsymbol{e_q}(s))d\gamma.
\end{aligned}$$

Integrating this equation from 0 to t and taking $e_u(0) = 0$, we obtain

$$\begin{aligned}
(\lambda, I_h^{k-1}e_u(t)) = \int_0^t \Big[(e_{u_s}(s), I_h^{k-1}\psi(s) - \psi(s)) - (e_{u_s}(s), I_h^k\psi(s) - \psi(s)) - (\boldsymbol{e_q}(s), \\
\boldsymbol{\Pi}_{k-1}^{RT}\boldsymbol{p}(s) - \boldsymbol{p}(s)) - (a\boldsymbol{e_q}(s), \boldsymbol{\phi}(s) - \boldsymbol{I}_h^k\boldsymbol{\phi}(s)) - (\boldsymbol{e_\sigma}(s), \boldsymbol{I}_h^k\boldsymbol{\phi}(s) - \boldsymbol{\phi}(s)) \\
- (\boldsymbol{e_\sigma}(s), \nabla(\psi - I_h^k\psi)(s)) - \langle \hat{\boldsymbol{e}}_{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu}, I_h^k\psi\rangle\Big] ds + \int_0^t \int_0^s (b(s,\gamma)\boldsymbol{e_q}(\gamma), \\
\boldsymbol{I}_h^k\boldsymbol{\phi}(s))d\gamma ds - \int_0^t \int_s^t (b(\gamma,s)\boldsymbol{\phi}(\gamma), \boldsymbol{e_q}(s))d\gamma ds.
\end{aligned}$$

Now, changing the order of integration in the last term, we obtain

$$\begin{aligned}
(\lambda, I_h^{k-1}e_u(t)) = \int_0^t \Big[(e_{u_s}(s), I_h^{k-1}\psi(s) - \psi(s)) - (e_{u_s}(s), I_h^k\psi(s) - \psi(s)) - (\boldsymbol{e_q}(s), \\
\boldsymbol{\Pi}_{k-1}^{RT}\boldsymbol{p}(s) - \boldsymbol{p}(s)) - (a\boldsymbol{e_q}(s), \boldsymbol{\phi}(s) - \boldsymbol{I}_h^k\boldsymbol{\phi}(s)) - (\boldsymbol{e_\sigma}(s), \boldsymbol{I}_h^k\boldsymbol{\phi}(s) - \boldsymbol{\phi}(s)) \\
- (\boldsymbol{e_\sigma}(s), \nabla(\psi - I_h^k\psi)(s)) - \langle \hat{\boldsymbol{e}}_{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu}, I_h^k\psi\rangle\Big] ds + \int_0^t \int_0^s (b(s,\gamma)\boldsymbol{e_q}(\gamma), \\
\boldsymbol{I}_h^k\boldsymbol{\phi}(s) - \boldsymbol{\phi}(s) d\gamma ds. \\
= \int_0^t [E_1 + E_2 + E_3 + E_4 + E_5 + E_6 + E_7] \, ds + E_8. \qquad (2.19)
\end{aligned}$$

Use of Cauchy Schwarz's inequality along with the properties of I_h^{k-1} and $\boldsymbol{I}_h^k$ yield

$$|E_1 + E_2 + E_3 + E_4 + E_5 + E_6| \leq Ch\left(\|e_{u_s}(s)\| + \|\boldsymbol{e_q}(s)\| + \|\boldsymbol{e_\sigma}(s)\|\right)\|\psi\|_2.$$

Use of (2.4e) and properties of the projection I_h^k gives

$$|E_7| \leq \|\hat{\boldsymbol{e}}_{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu}\|_{\partial K}\|I_h^k\psi - \psi\|_{\partial K} \leq Ch^{3/2}\|\hat{\boldsymbol{e}}_{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu}\|_{\partial K}\|\psi\|_2.$$

Finally, use of Cauchy Schwarz's inequality with boundedness of b and approximation property of $\boldsymbol{I}_h^k$ shows

$$|E_8| \leq Ch\int_0^t\int_0^s \|\boldsymbol{e_q}(\gamma)\|\|\psi\|_2 d\gamma ds.$$

Now, on substitution in (2.19), we arrive at

$$(\lambda, I_h^{k-1}e_u(t)) \leq Ch^{k+2}\int_0^t \|\psi(s)\|_2\, ds.$$

Next, for any $\delta \in (0, t)$, we have

$$\begin{aligned}
\int_0^t \|\psi(s)\|_2\, ds &= \int_0^{t-\delta} \|\psi(s)\|_2\, ds + \int_{t-\delta}^t \|\psi(s)\|_2\, ds \\
&= \int_0^{t-\delta} \sqrt{t-s}\, \|\psi(s)\|_2 \frac{1}{\sqrt{t-s}}\, ds + \int_{t-\delta}^t \|\psi(s)\|_2\, ds \\
&\leq \sqrt{\log\frac{t}{\delta}}\Big(\int_0^t (t-s)\|\psi(s)\|_2^2\, ds\Big)^{\frac{1}{2}} + \sqrt{\delta}\Big(\int_0^t \|\psi(s)\|_2^2\, ds\Big)^{\frac{1}{2}}
\end{aligned}$$

Now, using the regularity result (2.17) and (2.18), we obtain the desired estimates. This completes the rest of the proof. □

Lemma 2.5.2. *For the method of the form (2.3), a positive constant C that is unaffected by the values of h and k exists, such that it ensures the validity of the following inequality:*

$$\|u^p - u_h^p\| \leq Ch^{k+2}. \tag{2.20}$$

Proof. A use of the Poincare's inequality 1.4.4 shows

$$\|u^p - u_h^p\| \leq Ch\|\nabla(u^p - u_h^p)\|. \tag{2.21}$$

Now, for any $v \in P^0_{k+1}(K)$, there hold:

$$(a\nabla(u_h^p - u^p), \nabla v) + (a\nabla u^p, \nabla v) = -(a\boldsymbol{q}_h, \nabla v),$$

and hence,

$$\begin{aligned}(a\nabla(u_h^p - u^p), \nabla v) &= (a(\boldsymbol{q} - \boldsymbol{q}_h), \nabla v) \\ &\leq C\|\boldsymbol{q} - \boldsymbol{q}_h\|\|\nabla v\|. \qquad (2.22)\end{aligned}$$

Now, we note that

$$\begin{aligned}\alpha_0\|\nabla(I_h^{k+1}u^p - u_h^p)\|^2 \leq (a\nabla(I_h^{k+1}u^p - u^p), \nabla(I_h^{k+1}u^p - u_h^p)) + (a\nabla(u^p - u_h^p),\\ \nabla(I_h^{k+1}u^p - u_h^p)).\end{aligned}$$

Using Cauchy Schwarz's inequality and (2.22) with $v = I_h^{k+1}u^p - u_h^p$, we arrive at

$$\|\nabla(I_h^{k+1}u^p - u_h^p)\| \leq C(\|\nabla(I_h^{k+1}u^p - u^p)\| + \|\boldsymbol{q} - \boldsymbol{q}_h\|).$$

Lastly, we apply traingle's inequality to arrive at the following inequality

$$\begin{aligned}\|\nabla(u^p - u_h^p)\| &\leq \|\nabla(I_h^{k+1}u^p - u^p)\| + \|\nabla(I_h^{k+1}u^p - u_h^p)\| \\ &\leq C(\|\nabla(I_h^{k+1}u^p - u^p)\| + \|\boldsymbol{q} - \boldsymbol{q}_h\|). \qquad (2.23)\end{aligned}$$

A substitution of (2.23) in (2.21) concludes the proof. □

Theorem 2.5.3. *For the method of the form (2.3), when $u, u_t \in L^\infty(H^{k+2}(\mathcal{T}_h))$ a positive constant C that is unaffected by the values of h and $k \geq 1$ exists, such that it ensures the validity of the following inequality:*

$$\|u - u_h^*\| \leq C\sqrt{\log\left(\frac{T}{\rho^2}\right)}\, h^{k+2}.$$

Proof. Substitute (2.16) and (2.20) in (2.15) to prove the theorem. □

2.6 Fully Discrete Scheme

In this section, we will discretize equation (2.3) in time direction, based on backward difference scheme, along with left rectangle rule to approximate the integral term. We first divide the interval $[0, T]$ into M equally spaced sub-intervals by the following points

$$0 = t_0 < t_1 < ... < t_M = T,$$

with $t_n = n\Delta t$, $\Delta t = T/M$, as the time step.

The fully discrete approximation to the problem (2.3) is defined as follows: For $1 \leq n \leq M$, find $(U^n, \boldsymbol{Q}^n, \boldsymbol{S}^n, \hat{U}^n) \in (V_h \times \boldsymbol{W}_h \times \boldsymbol{W}_h \times M_h)$, such that, for any $(v_h, \boldsymbol{w}_h, \boldsymbol{\tau}_h, \mu_h, m_h) \in (V_h \times \boldsymbol{W}_h \times \boldsymbol{W}_h \times M_h \times M_h)$, we require

$$(\boldsymbol{Q}^n, \boldsymbol{w}_h) - (U^n, \nabla \cdot \boldsymbol{w}_h) + \langle \hat{U}^n, \boldsymbol{w}_h \cdot \boldsymbol{\nu} \rangle_{\partial \mathcal{T}_h} = 0 \qquad \forall \boldsymbol{w}_h \in \boldsymbol{W}_h, \quad (2.24a)$$

$$(a\boldsymbol{Q}^n, \boldsymbol{\tau}_h) - (\boldsymbol{S}^n, \boldsymbol{\tau}_h) + \big(\Delta t \sum_{i=0}^{n-1} b(t_n, t_i)\boldsymbol{Q}^i, \boldsymbol{\tau}_h\big) = 0 \qquad \forall \boldsymbol{\tau}_h \in \boldsymbol{W}_h, \quad (2.24b)$$

$$\left(\frac{U^n - U^{n-1}}{\Delta t}, v_h\right) - (\boldsymbol{S}^n, \nabla v_h) + \langle \hat{\boldsymbol{S}}^n \cdot \boldsymbol{\nu}, v_h \rangle_{\partial \mathcal{T}_h} = (f, v_h) \quad \forall v_h \in V_h, \quad (2.24c)$$

$$\langle \hat{U}^n, \mu_h \rangle_{\partial \Omega} = 0 \qquad \forall \mu_h \in M_h, \quad (2.24d)$$

$$\langle \hat{\boldsymbol{S}}^n \cdot \boldsymbol{\nu}, m_h \rangle_{\partial \mathcal{T}_h \setminus \partial \Omega} = 0 \qquad \forall m_h \in M_h, \quad (2.24e)$$

where numerical trace for flux is defined by:

$$\hat{\boldsymbol{S}}^n \cdot \boldsymbol{\nu} = \boldsymbol{S}^n \cdot \boldsymbol{\nu} + \tau(U^n - \hat{U}^n) \text{ on } \partial \mathcal{T}_h.$$

Theorem 2.6.1. *Let u be the solution of (2.1), such that $u, u_t \in L^\infty(H^{k+2}(\mathcal{T}_h))$ and $u_h(0) = U^0 = \Pi_V u_0$, then for all $1 \leq n \leq M$,*

$$\|u(t_n) - U^n\| \leq O(h^{k+1} + \Delta t).$$

Proof. We begin by writing $\|u(t_n) - U^n\| \leq \|u(t_n) - u_h(t_n)\| + \|u_h(t_n) - U^n\|$. We only need to derive the estimate of the second term in the right hand side. We will use ζ_u^n to denote $u_h(t_n) - U^n$. Similarly, $\boldsymbol{\zeta}_{\boldsymbol{q}}^n$, $\boldsymbol{\zeta}_{\boldsymbol{\sigma}}^n$ and $\hat{\zeta}_u^n$.

Now, using (2.3) and (2.24), we have the following

$$(\boldsymbol{\zeta}_{\boldsymbol{q}}^n, \boldsymbol{w}_h) - (\zeta_u^n, \nabla \cdot \boldsymbol{w}_h) + \langle \hat{\zeta}_u^n, \boldsymbol{w}_h \cdot \boldsymbol{\nu} \rangle_{\partial \mathcal{T}_h} = 0, \tag{2.25a}$$

$$(a\boldsymbol{\zeta}_{\boldsymbol{q}}^n, \boldsymbol{\tau}_h) - (\boldsymbol{\zeta}_{\boldsymbol{\sigma}}^n, \boldsymbol{\tau}_h) + \int_0^{t_n} (b(t_n, s)\boldsymbol{q}_h(s), \boldsymbol{\tau}_h) ds = \left(\Delta t \sum_{i=0}^{n-1} b(t_n, t_i) \boldsymbol{Q}^i, \boldsymbol{\tau}_h \right), \tag{2.25b}$$

$$\left(u_{h_t}(t_n) - \frac{U^n - U^{n-1}}{\Delta t}, v_h \right) - (\boldsymbol{\zeta}_{\boldsymbol{\sigma}}^n, \nabla v_h) + \langle \hat{\boldsymbol{\zeta}}_{\boldsymbol{\sigma}}^n \cdot \boldsymbol{\nu}, v_h \rangle_{\partial \mathcal{T}_h} = 0, \tag{2.25c}$$

$$\langle \hat{\zeta}_u^n, \mu_h \rangle_{\partial \Omega} = 0, \tag{2.25d}$$

$$\langle \hat{\boldsymbol{\zeta}}_{\boldsymbol{\sigma}}^n \cdot \boldsymbol{\nu}, m_h \rangle_{\partial \mathcal{T}_h \setminus \partial \Omega} = 0, \tag{2.25e}$$

for all $(v_h, \boldsymbol{w}_h, \boldsymbol{\tau}_h, \mu_h, m_h) \in (V_h \times \boldsymbol{W}_h \times \boldsymbol{W}_h \times M_h \times M_h)$. Where numerical trace for flux is defined by:

$$\hat{\boldsymbol{\zeta}}_{\boldsymbol{\sigma}}^n \cdot \boldsymbol{\nu} = \boldsymbol{\zeta}_{\boldsymbol{\sigma}}^n \cdot \boldsymbol{\nu} + \tau(\zeta_u^n - \hat{\zeta}_u^n) \ \text{ on } \partial \mathcal{T}_h.$$

Taking $\boldsymbol{w}_h = \boldsymbol{\zeta}_{\boldsymbol{\sigma}}^n$, $\boldsymbol{\tau}_h = \boldsymbol{\zeta}_{\boldsymbol{q}}^n$, $v_h = \zeta_u^n$, $\mu_h = -\hat{\boldsymbol{\zeta}}_{\boldsymbol{\sigma}}^n \cdot \boldsymbol{\nu}$ and $m_h = -\hat{\zeta}_u^n$ in (2.25a), (2.25b), (2.25c), (2.25d) and (2.25e), respectively, and then, adding the resulting equations to obtain

$$\begin{aligned} \|a^{1/2}\boldsymbol{\zeta}_{\boldsymbol{q}}^n\|^2 + \left(\frac{\zeta_u^n - \zeta_u^{n-1}}{\Delta t}, \zeta_u^n \right) + \|\hat{\zeta}_u^n - \zeta_u^n\|_\tau^2 + (J^n, \zeta_u^n) &= \left(E_h^n(\boldsymbol{q}_h), \boldsymbol{\zeta}_{\boldsymbol{q}}^n \right) \\ &\quad - \left(\Delta t \sum_{i=0}^{n-1} b(t_n, t_i) \boldsymbol{\zeta}_{\boldsymbol{q}}^i, \boldsymbol{\zeta}_{\boldsymbol{q}}^n \right), \end{aligned}$$

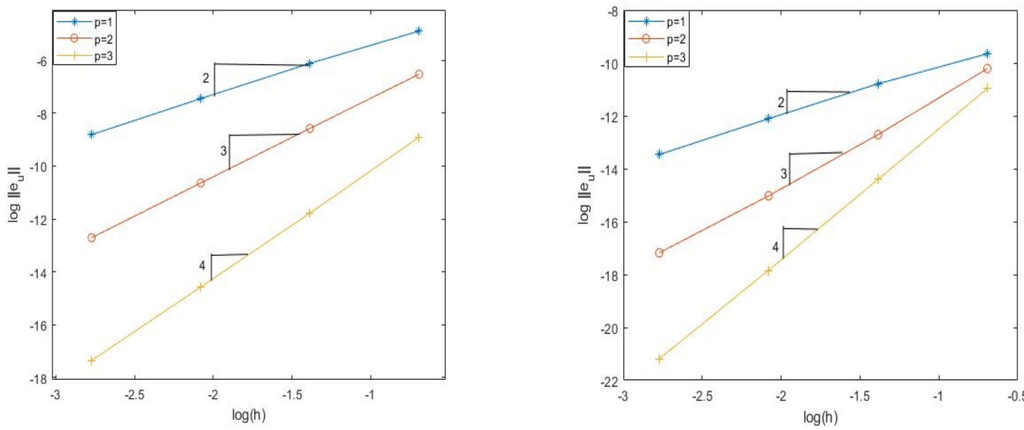
$$\begin{aligned} \implies \quad & \|a^{1/2}\boldsymbol{\zeta}_q^n\|^2 + \frac{1}{2\Delta t} \left(\|\zeta_u^n\|^2 - \|\zeta_u^{n-1}\|^2 \right) + \frac{1}{2} \left\| \frac{\zeta_u^n - \zeta_u^{n-1}}{\Delta t} \right\|^2 + \|\hat{\zeta}_u^n - \zeta_u^n\|_\tau^2 \\ & + (J^n, \zeta_u^n) = \left(E_h^n(\boldsymbol{q}_h), \boldsymbol{\zeta}_q^n \right) - \left(\Delta t \sum_{i=0}^{n-1} b(t_n, t_i) \boldsymbol{\zeta}_q^i, \boldsymbol{\zeta}_q^n \right), \end{aligned}$$

where

$$J^n = u_{h_t}(t_n) - \frac{u_h(t_n) - u_h(t_{n-1})}{\Delta t},$$

and

$$E_h^n(\boldsymbol{q}_h) = \Delta t \sum_{i=0}^{n-1} b(t_n, t_i) \boldsymbol{q}_h(t_i) - \int_0^{t_n} b(t_n, s) \boldsymbol{q}_h(s) ds.$$



(a) Example 1

(b) Example 2

Figure 2.1: Convergence behaviour of $\|e_u\|$ at $t = 1$

Now, summing over n from $n = 1$ to $n = m$, where $1 \leq m \leq M$ to obtain

$$\Delta t \sum_{n=1}^{m} \|\boldsymbol{\zeta}_{\boldsymbol{q}}^n\|^2 + \frac{1}{2}\|\zeta_u^m\|^2 + \frac{\Delta t}{2}\sum_{n=1}^{m}\left\|\frac{\zeta_u^n - \zeta_u^{n-1}}{\Delta t}\right\|^2 \leq \frac{1}{2}\|\zeta_u^0\|^2 + \Delta t \sum_{n=1}^{m}(I_1^n + I_2^n + I_3^n), \tag{2.26}$$

where,

$$I_1^n = (J^n, \zeta_u^n)\,,\; I_2^n = \left(E_h^n(\boldsymbol{q}_h), \boldsymbol{\zeta}_{\boldsymbol{q}}^n\right)\,,\; I_3^n = \left(\Delta t \sum_{i=0}^{n-1} b(t_n, t_i)\boldsymbol{\zeta}_{\boldsymbol{q}}^i, \boldsymbol{\zeta}_{\boldsymbol{q}}^n\right).$$

Using Taylor's series approximation, we note that

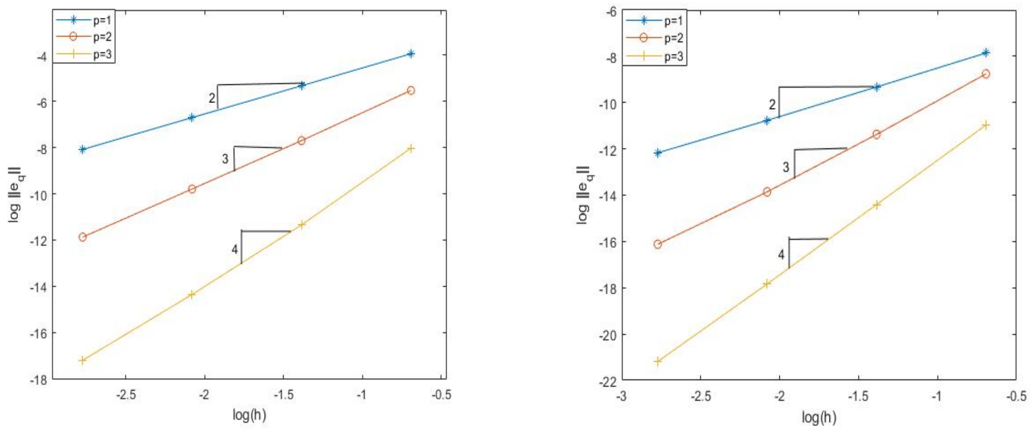
$$J^n = \frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}(s - t_{n-1})u_{h_{ss}}(s)ds,$$
$$\text{and,}\quad \|J^n\|^2 \leq \frac{\Delta t}{3}\int_{t_{n-1}}^{t_n}\|u_{h_{ss}}(s)\|^2 ds.$$

Therefore,

$$|I_1^n| \leq \frac{\Delta t}{6}\int_{t_{n-1}}^{t_n}\|u_{h_{ss}}(s)\|^2 ds + \frac{1}{2}\|\zeta_u^n\|^2. \tag{2.27}$$

Next, a use of quadrature error yields

$$\|E_h(\boldsymbol{q}_h)\| \leq \Delta t \int_0^{t_n}\|\boldsymbol{q}_{h_s}(s)\|ds.$$



(a) Example 1

(b) Example 2

Figure 2.2: Convergence behaviour of $\|\boldsymbol{e_q}\|$ at $t = 1$

Then, a use of Young's inequality yields

$$|I_2^n| \leq C \left(\Delta t^2 \int_0^{t_n} \|\boldsymbol{q}_{h_s}\|^2 ds + \|\zeta_{\boldsymbol{q}}^n\|^2 \right). \tag{2.28}$$

Using (2.27) and (2.28) in (2.26) and setting

$$\tilde{C} = \frac{1}{2}\|\zeta_u^0\|^2 + C\Delta t \sum_{n=1}^{m} \left(\Delta t \int_{t_{n-1}}^{t_n} \|u_{h_{ss}}(s)\|^2 ds + \Delta t^2 \int_0^{t_n} \|\boldsymbol{q}_{h_s}\|^2 ds \right),$$

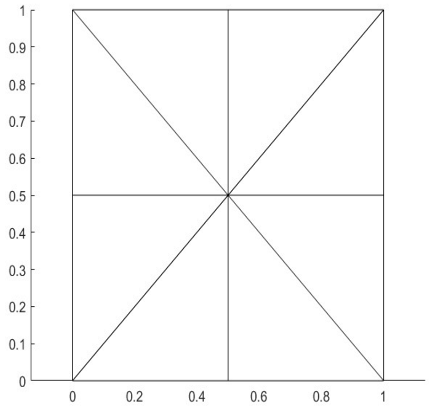
we obtain

$$C\Delta t \sum_{n=1}^{m} \|\boldsymbol{\zeta}_{\boldsymbol{q}}^n\|^2 + C\|\zeta_u^m\|^2 + \frac{\Delta t}{2} \sum_{n=1}^{m} \left\| \frac{\zeta_u^n - \zeta_u^{n-1}}{\Delta t} \right\|^2 \leq \tilde{C} + C\Delta t \sum_{n=1}^{m} \sum_{i=1}^{n-1} \|\boldsymbol{\zeta}_{\boldsymbol{q}}^i\|^2.$$

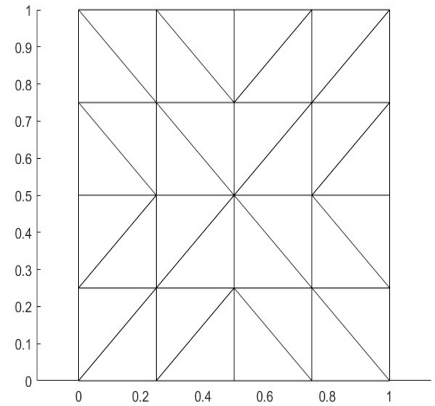
Finally, use of discrete Gronwall's lemma along with theorem 2.4.1 will give the desired estimate. For more details, see [47]. □

2.7 Numerical Experiments

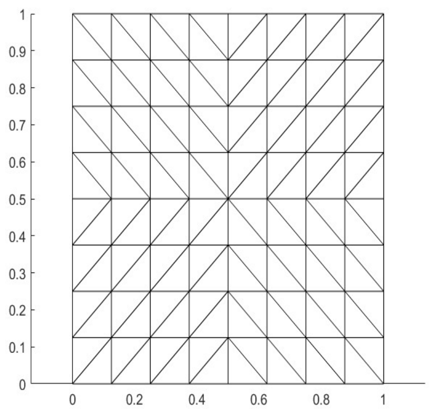
This section consists of two numerical examples which are used to verify the theoretical results that are proved in the chapter. The examples consist of (2.1a)-(2.1c) with $a = 1$, for $\Omega = (0,1) \times (0,1)$ and $T = 1$. Figure 2.3 shows the domain discretization used for different mesh sizes. We have used the backward Euler's method to completely discretize



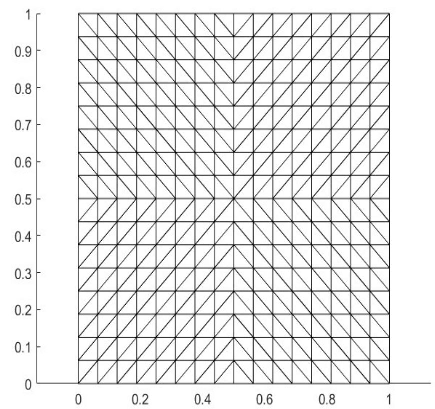
(a) $h = \dfrac{1}{2}$



(b) $h = \dfrac{1}{4}$



(c) $h = \dfrac{1}{8}$

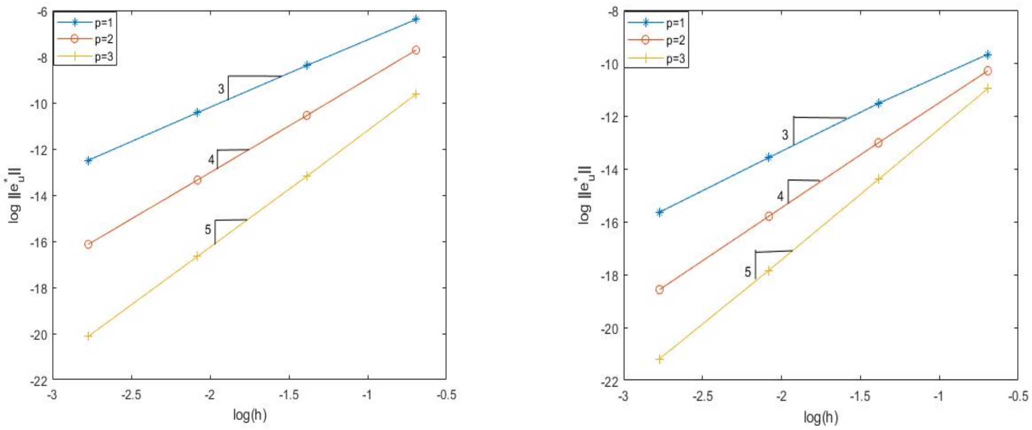


(d) $h = \dfrac{1}{16}$

Figure 2.3: domain discretization for different values of h

the problem, along with left rectangle rule to approximate the integral term. We observe that the optimal order of convergence in case of u and $\boldsymbol{q}$ and the super-convergence in the case of u_h^* as predicted by our theory, is achieved.

Example 1: Choose $u(x, y, t) = e^{-t}x(1-x)y(1-y)$ and $b(x, t, s) = e^{t-s}$. We have used MATLAB codes to compute L^2 error estimates for the three unknowns, that is, u, $\boldsymbol{q}$ and $\boldsymbol{\sigma}$, for different mesh sizes, that is, for $h = \dfrac{1}{2}$, $h = \dfrac{1}{4}$, $h = \dfrac{1}{8}$ and $h = \dfrac{1}{16}$. Next, we have computed their orders of convergence and found that they match with the theoretical findings. In Figures 2.1a, 2.2a and 2.4a, we plot the computed error with the mesh sizes



(a) Example 1

(b) Example 2

Figure 2.4: Convergence behaviour of $\|e_u^*\|$ at $t = 1$

for different degrees of polynomials. Table 2.1 gives the time convergence for u for the example for different time steps.

$\boldsymbol{\Delta t}$ $(h = 1/4)$	**Order** (Ex. 1)	**Order** (Ex. 2)
0.25	0.9845	0.8956
0.125	1.0076	0.9823
0.0625	1.3045	1.1263
0.03125	1.2134	1.1943

Table 2.1: Order of convergence for time

Example 2: Choose $u(x, y, t) = (0.01)\cos(\pi t)x(1-x)y(1-y)$ and $b(x, t, s) = \sin(\pi t)\cos(\pi s)$. We have used MATLAB codes to compute L^2 error estimates for the three unknowns, that is, u, $\boldsymbol{q}$ and $\boldsymbol{\sigma}$, for different mesh sizes, that is, for $h = \dfrac{1}{2}$, $h = \dfrac{1}{4}$, $h = \dfrac{1}{8}$ and $h = \dfrac{1}{16}$. Next, we have computed their orders of convergence and found that they match with the theoretical findings. In Figures 2.1b, 2.2b and 2.4b, we plot the computed error with the mesh sizes for different degrees of polynomials. Table 2.1 gives the time convergence for u for the example for different time steps.

2.8 Conclusions

Due to various theoretical and computational benefits of the HDG method, in this chapter, it has been proposed and analyzed for equation (2.1). Throughout this chapter, HDG and Ritz-Volterra projections has been used to derive the error estimates. Further, element-by-element post-processing has been proposed. It has been shown that the solution and its gradient achieve optimal rate of convergence, that is, of order $k + 1$, $k \geq 0$ in the discretizing parameter h, whereas, super-convergence has been achieved, that is, of order $k + 2$, $k \geq 1$, for the post-processed solution. Finally, numerical results have been discussed. This analysis can be extended to 3-dimensional domain, by incorporating the changes accordingly. We can easily achieve higher order scheme for fully discrete case by applying a higher order scheme to approximate the time derivative and the integral term.

Chapter 3

HDG Method for Nonlinear Parabolic Integro-Differential Equations

3.1 Introduction

This chapter discusses the HDG method for the following problem: Find $u(x,t)$ such that

$$u_t(x,t) - \nabla \cdot \left(a(u)\nabla u(x,t) + \int_0^t b(u(s))\nabla u(x,s)ds \right) = f(u) \quad \text{in } \Omega \times (0,T], \tag{3.1a}$$

$$u(x,t) = 0 \qquad \text{on } \partial\Omega \times (0,T], \tag{3.1b}$$

$$u(x,0) = u_0(x) \quad \text{for } x \in \Omega. \tag{3.1c}$$

Here, $u : \Omega \times (0,T] \to \mathbb{R}$. For the functions $a : \mathbb{R} \to \mathbb{R}$ and $b : \mathbb{R} \to \mathbb{R}$, we consider that $0 < a_* \leq a(u) \leq M$ and $|b(u)| \leq M$, where a_* and M are positive constants. Furthermore, the functions $a(u)$, $b(u)$, their derivatives and $f(u)$ satisfy the Lipschitz continuity condition near 'u'. For the existence and uniqueness of the solution of (3.1), we refer to [24].

PIDEs of the above type occur in numerous physical models and engineering problems, for example, gas diffusion problems, heat transfer problems, and the non-local reactive flows in porous media. For further details, we refer to [26, 64, 115] and references therein. In literature, the FEM has been applied to the nonlinear parabolic integro-differential equations, see, [16, 17, 87]. In [16, 17], Ritz-Volterra projection has been introduced to deal with the integral term, whereas, in [89], Lin has derived the maximum norm estimates for the same. In [88], he has analyzed the Galerkin method for (3.1a) along with nonlinear boundary conditions. In [26], Chen *et al.* presented a two-grid FEM for PIDEs of type (3.1) and derived the error estimates by solving the nonlinear problem on a coarser grid and then a linearized problem on a finer grid.

In this chapter, the HDG method is implemented on (3.1). The major contributions of this chapter are as follows:

- For the error analysis, only the first order derivative of the nonlinear variables a and b, along with the Lipschitz continuity condition, has been considered, without taking into consideration, their second order derivative.

- To deal with the integral term, Ritz-Volterra projection of extended type is introduced and analyzed. This helps to achieve optimal estimates of order $O(h^{k+1})$ when polynomials of degree $k \geq 0$ are used to approximate both 'u' and '∇u'.

- Dual problem is used for element-by-element post-processing to achieve super-convergence results for the post-processed solution. Super-convergence is achieved by considering the derivative of order only up to one of the nonlinear variables f, a, and b.

- Using backward Euler's method for time derivative, a complete discrete scheme is proposed, and corresponding error estimates are derived.

- With the help of different numerical examples, it has been verified that the unknown variable and the flux achieve optimal order of convergence, whereas the post-processed solution attains the super-convergence.

We have used backward Euler's method for the time derivative, but higher-order methods can also be applied to derive higher-order convergence in the temporal direction. For the sake of simplicity, C is used to denote an inclusive, positive constant independent of discretizing parameter h as well as the degree of polynomial k. Also, argument $\boldsymbol{x}$ of functions will not be written explicitly, whereas t and s will be written as and when required.

The chapter's structure is as follows: Section 3.2 defines the HDG method for nonlinear PIDE (3.1). It also introduces an intermediate projection, along with its estimates. Section 3.3 analyses the error for the semi-discrete problem. In Section 3.4, the post-processed solution is introduced, along with its estimates. Section 3.5 deals with the fully discrete scheme. Section 3.6 validates the theoretical results with the help of a few numerical examples. Finally, Section 3.7 gives some concluding remarks.

3.2 HDG Method

To implement the HDG method on (3.1), we begin by introducing the following variables:

$$\boldsymbol{q} = -\nabla u, \quad \boldsymbol{\sigma} = a(u)\boldsymbol{q} + \int_0^t b(u(s))\boldsymbol{q}(s)ds,$$

and then, using these variables, we write (3.1) as follows:

$$\boldsymbol{q} = -\nabla u, \qquad \text{in } \Omega \times (0,T], \tag{3.2a}$$

$$\boldsymbol{\sigma} = a(u)\boldsymbol{q} + \int_0^t b(u(s))\boldsymbol{q}(s)ds, \qquad \text{in } \Omega \times (0,T], \tag{3.2b}$$

$$u_t + \nabla\cdot\boldsymbol{\sigma} = f(u), \qquad \text{in } \Omega \times (0,T]. \tag{3.2c}$$

The HDG formulation seeks approximation $(u_h, \boldsymbol{q}_h, \boldsymbol{\sigma}_h, \hat{u}_h)(t) \in (V_h, \boldsymbol{W}_h, \boldsymbol{W}_h, M_h)$, for $t \in (0,T]$, for any $(v_h, \boldsymbol{w}_h, \boldsymbol{\tau}_h, \mu_h, m_h) \in (V_h, \boldsymbol{W}_h, \boldsymbol{W}_h, M_h, M_h)$, such that the following equations are satisfied

$$(\boldsymbol{q}_h, \boldsymbol{w}_h) - (u_h, \nabla\cdot\boldsymbol{w}_h) + \langle \hat{u}_h, \boldsymbol{w}_h\cdot\boldsymbol{\nu}\rangle_{\partial\mathcal{T}_h} = 0, \tag{3.3a}$$

$$(a(u_h)\boldsymbol{q}_h, \boldsymbol{\tau}_h) - (\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + \int_0^t (b(u_h(s))\boldsymbol{q}_h(s), \boldsymbol{\tau}_h)ds = 0, \tag{3.3b}$$

$$(u_{ht}, v_h) - (\boldsymbol{\sigma}_h, \nabla v_h) + \langle \hat{\boldsymbol{\sigma}}_h \cdot \boldsymbol{\nu}, v_h \rangle_{\partial \mathcal{T}_h} = (f(u_h), v_h), \tag{3.3c}$$

$$\langle \hat{u}_h, \mu_h \rangle_{\partial \Omega} = 0, \tag{3.3d}$$

$$\langle \hat{\boldsymbol{\sigma}}_h \cdot \boldsymbol{\nu}, m_h \rangle_{\partial \mathcal{T}_h \setminus \partial \Omega} = 0, \tag{3.3e}$$

$$u_h(0) = \Pi_V u_0, \tag{3.3f}$$

where the numerical trace for flux is defined by

$$\hat{\boldsymbol{\sigma}}_h \cdot \boldsymbol{\nu} = \boldsymbol{\sigma}_h \cdot \boldsymbol{\nu} + \tau(u_h - \hat{u}_h) \ \text{ on } \partial \mathcal{T}_h.$$

Here, $\tau \geq 0$ is defined on Γ and is known as the stabilization parameter, which is assumed to be a piece-wise constant on the faces. We note that the exact solutions u, $\boldsymbol{q}$ and $\boldsymbol{\sigma}$ satisfy (3.3). Hence, we obtain the following error equations:

$$(\boldsymbol{q} - \boldsymbol{q}_h, \boldsymbol{w}_h) - (u - u_h, \nabla \cdot \boldsymbol{w}_h) + \langle u - \hat{u}_h, \boldsymbol{w}_h \cdot \boldsymbol{\nu} \rangle_{\partial \mathcal{T}_h} = 0, \tag{3.4a}$$

$$(a(u)\boldsymbol{q} - a(u_h)\boldsymbol{q}_h, \boldsymbol{\tau}_h) - (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + \int_0^t ((b(u(s))\boldsymbol{q} - b(u_h(s))\boldsymbol{q}_h)(s), \boldsymbol{\tau}_h) ds = 0, \tag{3.4b}$$

$$(u_t - u_{h_t}, v_h) - (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \nabla v_h) + \langle (\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}_h) \cdot \boldsymbol{\nu}, v_h \rangle_{\partial \mathcal{T}_h} = (f(u) - f(u_h), v_h), \tag{3.4c}$$

$$\langle u - \hat{u}_h, \mu_h \rangle_{\partial \Omega} = 0, \tag{3.4d}$$

$$\langle (\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}_h) \cdot \boldsymbol{\nu}, m_h \rangle_{\partial \mathcal{T}_h \setminus \partial \Omega} = 0, \tag{3.4e}$$

for all $(v_h, \boldsymbol{w}_h, \boldsymbol{\tau}_h, \mu_h, m_h) \in (V_h, \boldsymbol{W}_h, \boldsymbol{W}_h, M_h, M_h)$.
For the further analysis, we add and subtract $a(u)\boldsymbol{q}_h + \int_0^t b(u(s))\boldsymbol{q}_h(s) ds$ in (3.4b), to get the error equations in the following form:

$$(\boldsymbol{q} - \boldsymbol{q}_h, \boldsymbol{w}_h) - (u - u_h, \nabla \cdot \boldsymbol{w}_h) + \langle u - \hat{u}_h, \boldsymbol{w}_h \cdot \boldsymbol{\nu} \rangle_{\partial \mathcal{T}_h} = 0, \tag{3.5a}$$

$$\begin{aligned} &(a(u)(\boldsymbol{q} - \boldsymbol{q}_h), \boldsymbol{\tau}_h) - (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + \int_0^t (b(u(s))(\boldsymbol{q} - \boldsymbol{q}_h)(s), \boldsymbol{\tau}_h) \, ds \\ &\quad = ((a(u_h) - a(u))\boldsymbol{q}_h, \boldsymbol{\tau}_h) + \int_0^t ((b(u_h(s)) - b(u(s)))\boldsymbol{q}_h(s), \boldsymbol{\tau}_h) ds, \end{aligned} \tag{3.5b}$$

$$(u_t - u_{h_t}, v_h) - (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \nabla v_h) + \langle (\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}_h) \cdot \boldsymbol{\nu}, v_h \rangle_{\partial \mathcal{T}_h} = (f(u) - f(u_h), v_h), \tag{3.5c}$$

$$\langle u - \hat{u}_h, \mu_h \rangle_{\partial \Omega} = 0, \tag{3.5d}$$

$$\langle(\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}_h) \cdot \boldsymbol{\nu}, m_h\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega} = 0, \tag{3.5e}$$

for all $(v_h, \boldsymbol{w}_h, \boldsymbol{\tau}_h, \mu_h, m_h) \in (V_h, \boldsymbol{W}_h, \boldsymbol{W}_h, M_h, M_h)$.

3.2.1 An Intermediate Projection and Related Estimates

We define the following Ritz-Volterra projection: For each t and given $(u, \boldsymbol{q}, \boldsymbol{\sigma})$, find $(\tilde{u}_h, \tilde{\boldsymbol{q}}_h, \tilde{\boldsymbol{\sigma}}_h, \hat{\tilde{u}}_h) \in (V_h \times \boldsymbol{W}_h \times \boldsymbol{W}_h \times M_h)$ satisfying

$$(\boldsymbol{q} - \tilde{\boldsymbol{q}}_h, \boldsymbol{w}_h) - (u - \tilde{u}_h, \nabla \cdot \boldsymbol{w}_h) + \langle u - \hat{\tilde{u}}_h, \boldsymbol{w}_h \cdot \boldsymbol{\nu}\rangle_{\partial\mathcal{T}_h} = 0, \tag{3.6a}$$

$$(a(u)(\boldsymbol{q} - \tilde{\boldsymbol{q}}_h), \boldsymbol{\tau}_h) - (\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h, \boldsymbol{\tau}_h) + \int_0^t (b(u(s))(\boldsymbol{q} - \tilde{\boldsymbol{q}}_h)(s), \boldsymbol{\tau}_h)ds = 0, \tag{3.6b}$$

$$-(\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h, \nabla v_h) + \langle(\boldsymbol{\sigma} - \hat{\tilde{\boldsymbol{\sigma}}}_h) \cdot \boldsymbol{\nu}, v_h\rangle_{\partial\mathcal{T}_h} = 0, \tag{3.6c}$$

$$\langle u - \hat{\tilde{u}}_h, \mu_h\rangle_{\partial\Omega} = 0, \tag{3.6d}$$

$$\langle(\boldsymbol{\sigma} - \hat{\tilde{\boldsymbol{\sigma}}}_h) \cdot \boldsymbol{\nu}, m_h\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega} = 0, \tag{3.6e}$$

for all $(v_h, \boldsymbol{w}_h, \boldsymbol{\tau}_h, \mu_h, m_h) \in (V_h \times \boldsymbol{W}_h \times \boldsymbol{W}_h \times M_h \times M_h)$, where

$$\hat{\tilde{\boldsymbol{\sigma}}}_h \cdot \boldsymbol{\nu} = \tilde{\boldsymbol{\sigma}}_h \cdot \boldsymbol{\nu} + \tau(\tilde{u}_h - \hat{\tilde{u}}_h) \text{ on } \partial\mathcal{T}_h.$$

We decompose the errors using $\boldsymbol{I}_h^k$ (L^2-projection onto $\boldsymbol{W}_h$) and P_M (L^2-projection onto M_h) into θ's and ρ's as done in Chapter 2. Now, the system of equations (3.6) become

$$(\boldsymbol{\rho_q}, \boldsymbol{w}_h) - (\rho_u, \nabla \cdot \boldsymbol{w}_h) + \langle\hat{\rho}_u, \boldsymbol{w}_h \cdot \boldsymbol{\nu}\rangle_{\partial\mathcal{T}_h} = 0, \tag{3.7a}$$

$$\begin{aligned}(a(u)\boldsymbol{\rho_q}, \boldsymbol{\tau_h}) - (\boldsymbol{\rho_\sigma}, \boldsymbol{\tau}_h) + \int_0^t (b(u(s))\boldsymbol{\rho_q}(s), \boldsymbol{\tau}_h)ds = (a(u)\boldsymbol{\theta_q}, \boldsymbol{\tau}_h) - (\boldsymbol{\theta_\sigma}, \boldsymbol{\tau}_h)\\ + \int_0^t (b(u(s))\boldsymbol{\theta_q}(s), \boldsymbol{\tau}_h)ds,\end{aligned} \tag{3.7b}$$

$$-(\boldsymbol{\rho_\sigma}, \nabla v_h) + \langle\hat{\boldsymbol{\rho}}_{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu}, v_h\rangle_{\partial\mathcal{T}_h} = 0, \tag{3.7c}$$

$$\langle\hat{\rho}_u, \mu_h\rangle_{\partial\Omega} = 0, \tag{3.7d}$$

$$\langle\hat{\boldsymbol{\rho}}_{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu}, m_h\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega} = 0, \tag{3.7e}$$

for all $(v_h, \boldsymbol{w}_h, \boldsymbol{\tau}_h, \mu_h, m_h) \in (V_h \times \boldsymbol{W}_h \times \boldsymbol{W}_h \times M_h \times M_h)$.

Note that, $\langle\hat{\boldsymbol{\rho}}_{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu}, \mu_h\rangle_{\partial\mathcal{T}_h} = \langle\boldsymbol{\rho_\sigma} \cdot \boldsymbol{\nu} + \tau(\rho_u - \hat{\rho}_u), \mu\rangle_{\partial\mathcal{T}_h}$, for all $\mu_h \in P_k(F)$.

Below, we present the estimates for $\|\boldsymbol{\rho_\sigma}\|$ and $\|\boldsymbol{\rho_q}\|$.

Lemma 3.2.1. *For $t \in (0,T]$, a positive constant C that is unaffected by the values of h and k exists, such that it ensures the validity of the following inequality:*

$$\|\hat{\rho}_u - \rho_u\|_\tau + \|\boldsymbol{\rho_\sigma}(t)\| + \|\boldsymbol{\rho_q}(t)\| \le C\bigg[\|\boldsymbol{\theta_q}(t)\| + \|\boldsymbol{\theta_\sigma}(t)\| + \int_0^t \|\boldsymbol{\theta_q}(s)\|ds\bigg].$$

Proof. Choose $\boldsymbol{\tau}_h = \boldsymbol{\rho_\sigma}$ in (3.7b). Use of Cauchy-Schwarz inequality along with $0 < a_* \le a \le M$, $|b(u)| \le M$ yield

$$\begin{aligned}\|\boldsymbol{\rho_\sigma}\|^2 &= (a(u)\boldsymbol{\rho_q}, \boldsymbol{\rho_\sigma}) + \int_0^t (b(u(s))\boldsymbol{\rho_q}(s), \boldsymbol{\rho_\sigma})ds - (a(u)\boldsymbol{\theta_q}, \boldsymbol{\rho_\sigma}) + (\boldsymbol{\theta_\sigma}, \boldsymbol{\rho_\sigma}) \\ &\quad - \int_0^t (b(u(s))\boldsymbol{\theta_q}(s), \boldsymbol{\rho_\sigma})ds, \\ &\le C\bigg[\|\boldsymbol{\rho_q}\| + \|\boldsymbol{\theta_q}\| + \|\theta_\sigma\| + \int_0^t (\|\boldsymbol{\rho_q}(s)\| + \|\boldsymbol{\theta_q}(s)\|)ds\bigg]\|\boldsymbol{\rho_\sigma}\|,\end{aligned}$$

and hence,

$$\|\boldsymbol{\rho_\sigma}\| \le C\bigg[\|\boldsymbol{\rho_q}\| + \|\boldsymbol{\theta_q}\| + \|\theta_\sigma\| + \int_0^t (\|\boldsymbol{\rho_q}(s)\| + \|\boldsymbol{\theta_q}(s)\|)ds\bigg]. \tag{3.8}$$

Next, take $\boldsymbol{w}_h = \boldsymbol{\rho_\sigma}$, $\boldsymbol{\tau}_h = \boldsymbol{\rho_q}$, $v_h = \rho_u$, $\mu_h = -\hat{\boldsymbol{\rho}}_{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu}$ and $m_h = -\hat{\rho}_u$ in (3.7a), (3.7b), (3.7c), (3.7d) and (3.7e), respectively. Then, add the resulting equations to arrive at

$$\begin{aligned}(a(u)\boldsymbol{\rho_q}, \boldsymbol{\rho_q}) + \|\hat{\rho}_u - \rho_u\|_\tau^2 = (a(u)\boldsymbol{\theta_q}, \boldsymbol{\rho_q}) - (\boldsymbol{\theta_\sigma}, \boldsymbol{\rho_q}) + \int_0^t \big[(b(u(s))\boldsymbol{\theta_q}(s), \boldsymbol{\rho_q}) \\ - (b(u(s))\boldsymbol{\rho_q}(s), \boldsymbol{\rho_q})\big]ds.\end{aligned}$$

Further, use of the boundedness of a and b yield

$$\|\hat{\rho}_u - \rho_u\|_\tau + \|\boldsymbol{\rho_q}\| \le C\bigg[\|\boldsymbol{\theta_q}\| + \|\boldsymbol{\theta_\sigma}\| + \int_0^t (\|\boldsymbol{\theta_q}(s)\| + \|\boldsymbol{\rho_q}(s)\|)ds\bigg].$$

Finally, use of Gronwall's lemma along with (3.8) will prove the Lemma. □

Next, we state the lemma, which provides the estimate for $\|\rho_u\|$.

Lemma 3.2.2. *For $t \in (0, T]$, a positive constant C that is unaffected by the values of h and k exists, such that it ensures the validity of the following inequality:*

$$\|\rho_u(t)\| \leq Ch^{k+1}\Big[\|u(t)\|_{H^{k+2}(\Omega)} + \int_0^t \|u(s)\|_{H^{k+2}(\Omega)} ds\Big].$$

Proof. For this estimate, we use the duality argument. We make use of the following problem:

$$\begin{aligned} -\nabla \cdot (a(u)\nabla\psi) &= \rho_u && \text{in } \Omega, \\ \psi &= 0 && \text{on } \partial\Omega, \end{aligned}$$

with the following regularity

$$\|\psi\|_{H^2(\Omega)} \leq \|\rho_u\|.$$

Mixed formulation for the dual problem is written as follows:

$$\boldsymbol{\phi} = -\nabla\psi \qquad \text{in } \Omega, \tag{3.9a}$$

$$\boldsymbol{p} = a(u)\boldsymbol{\phi} \qquad \text{in } \Omega, \tag{3.9b}$$

$$\nabla \cdot \boldsymbol{p} = \rho_u \qquad \text{in } \Omega. \tag{3.9c}$$

Then, using L^2 inner product between (3.9c) and ρ_u, yields

$$\|\rho_u\|^2 = (\rho_u, \rho_u) = (\rho_u, \nabla \cdot \boldsymbol{p}),$$

and using Lemma 1.4.6, we obtain

$$\begin{aligned} \|\rho_u\|^2 &= (\rho_u, \nabla \cdot \boldsymbol{\Pi_W p}) + \langle \rho_u, \tau(\Pi_V\psi - \psi)\rangle_{\partial\mathcal{T}_h} \\ &= (\boldsymbol{\rho_q}, \boldsymbol{\Pi_W p}) + \langle \hat{\rho}_u, \boldsymbol{\Pi_W p} \cdot \boldsymbol{\nu}\rangle_{\partial\mathcal{T}_h} + \langle \rho_u, \tau(\Pi_V\psi - \psi)\rangle_{\partial\mathcal{T}_h}. \qquad \text{by(3.7a)} \end{aligned}$$

By continuity of $\boldsymbol{p} \cdot \boldsymbol{\nu}$ and (3.7d), we arrive at

$$\begin{aligned} \|\rho_u\|^2 &= (\boldsymbol{\rho_q}, \boldsymbol{\Pi_W p}) + \langle \hat{\rho}_u, (\boldsymbol{\Pi_W p} - \boldsymbol{p}) \cdot \boldsymbol{\nu}\rangle_{\partial\mathcal{T}_h} + \langle \rho_u, \tau(\Pi_V\psi - \psi)\rangle_{\partial\mathcal{T}_h} \\ &= (\boldsymbol{\rho_q}, \boldsymbol{\Pi_W p}) + \langle \tau(\rho_u - \hat{\rho}_u), \Pi_V\psi\rangle_{\partial\mathcal{T}_h} + \langle \boldsymbol{\rho_\sigma} \cdot \boldsymbol{\nu}, P_M\psi\rangle_{\partial\mathcal{T}_h} \qquad \text{by (1.1c), (3.7e)} \\ &= (\boldsymbol{\rho_q}, \boldsymbol{\Pi_W p}) + \langle \tau(\rho_u - \hat{\rho}_u) - \boldsymbol{\rho_\sigma} \cdot \boldsymbol{\nu}, \Pi_V\psi\rangle_{\partial\mathcal{T}_h} + \langle \boldsymbol{\rho_\sigma} \cdot \boldsymbol{\nu}, \psi\rangle_{\partial\mathcal{T}_h} \end{aligned}$$

$$+ (\boldsymbol{\rho_\sigma}, \nabla \Pi_V \psi) \qquad \text{by (3.7c)}$$
$$= (\boldsymbol{\rho_q}, \boldsymbol{\Pi_W p}) + (\boldsymbol{\rho_\sigma}, \nabla \psi) \qquad \text{by(1.1a)}$$
$$= (\boldsymbol{\rho_q}, \boldsymbol{\Pi_W p} - \boldsymbol{p}) + (\boldsymbol{\rho_q}, \boldsymbol{p}) - (\boldsymbol{\rho_\sigma}, \boldsymbol{\phi}) \qquad \text{by(3.9a)}$$
$$= (\boldsymbol{\rho_q}, \boldsymbol{\Pi_W p} - \boldsymbol{p}) + (\boldsymbol{\rho_q}, \boldsymbol{p}) - (\boldsymbol{\rho_\sigma}, \boldsymbol{\phi} - \boldsymbol{I}_h^k \boldsymbol{\phi}) - (\boldsymbol{\rho_\sigma}, \boldsymbol{I}_h^k \boldsymbol{\phi}).$$

Now, use (3.7b) with $\boldsymbol{\tau}_h = \boldsymbol{I}_h^k \boldsymbol{\phi}$, to obtain

$$\begin{aligned}
\|\rho_u\|^2 &= (\boldsymbol{\rho_q}, \boldsymbol{\Pi_W p} - \boldsymbol{p}) + (\boldsymbol{\rho_q}, \boldsymbol{p}) - (a(u)\boldsymbol{\rho_q}, \boldsymbol{I}_h^k \boldsymbol{\phi}) + \int_0^t (b(u(s))(\boldsymbol{\theta_q}(s) - \boldsymbol{\rho_q}(s)), \boldsymbol{I}_h^k \boldsymbol{\phi}) ds \\
&\quad + (a(u)\boldsymbol{\theta_q}, \boldsymbol{I}_h^k \boldsymbol{\phi}) - (\boldsymbol{\theta_\sigma}, \boldsymbol{I}_h^k \boldsymbol{\phi}) \\
&= (\boldsymbol{\rho_q}, \boldsymbol{\Pi_W p} - \boldsymbol{p}) + (a(u)\boldsymbol{\rho_q}, \boldsymbol{\phi}) - (a(u)\boldsymbol{\rho_q}, \boldsymbol{I}_h^k \boldsymbol{\phi}) + \int_0^t (b(u(s))(\boldsymbol{\theta_q}(s) - \boldsymbol{\rho_q}(s)), \boldsymbol{I}_h^k \boldsymbol{\phi}) ds \\
&\quad + (a(u)\boldsymbol{\theta_q}, \boldsymbol{I}_h^k \boldsymbol{\phi}) - (\boldsymbol{\theta_\sigma}, \boldsymbol{I}_h^k \boldsymbol{\phi}) \qquad \text{by(3.9b)} \\
&= (\boldsymbol{\rho_q}, \boldsymbol{\Pi_W p} - \boldsymbol{p}) + (a(u)\boldsymbol{\rho_q}, \boldsymbol{\phi} - \boldsymbol{I}_h^k \boldsymbol{\phi}) + (a(u)\boldsymbol{\theta_q}, \boldsymbol{I}_h^k \boldsymbol{\phi}) - (\boldsymbol{\theta_\sigma}, \boldsymbol{I}_h^k \boldsymbol{\phi}) \\
&\quad + \int_0^t (b(u(s))(\boldsymbol{\theta_q}(s) - \boldsymbol{\rho_q}(s)), \boldsymbol{I}_h^k \boldsymbol{\phi}) ds.
\end{aligned}$$

Next, using the Cauchy Schwarz inequality, the following inequality is obtained.

$$\begin{aligned}
\|\rho_u\|^2 \le{}& \|\boldsymbol{\rho_q}\| \|\boldsymbol{\Pi_W p} - \boldsymbol{p}\| + C\|\boldsymbol{\rho_q}\| \|\boldsymbol{\phi} - \boldsymbol{I}_h^k \boldsymbol{\phi}\| + C\|\boldsymbol{\theta_q}\| \|\boldsymbol{I}_h^k \boldsymbol{\phi}\|_{\boldsymbol{H}^1(\Omega)} + \|\boldsymbol{\theta_\sigma}\| \|\boldsymbol{I}_h^k \boldsymbol{\phi}\|_{\boldsymbol{H}^1(\Omega)} \\
&+ C \int_0^t \Big(\|\boldsymbol{\theta_q}(s)\| + \|\boldsymbol{\rho_q}(s)\| \Big) \|\boldsymbol{I}_h^k \boldsymbol{\phi}\|_{\boldsymbol{H}^1(\Omega)}.
\end{aligned}$$

Now, use of (1.2), Lemma 3.2.1, Lemma 1.4.1, elliptic regularity, $\|\boldsymbol{\phi}\|_{\boldsymbol{H}^1(\Omega)} \le M \|\boldsymbol{p}\|_{\boldsymbol{H}^1(\Omega)}$ and $\|\boldsymbol{p}\|_{\boldsymbol{H}^1(\Omega)} \le \|\psi\|_{H^2(\Omega)}$, yield the desired result. □

Lemma 3.2.3. *For $t \in (0, T]$, a positive constant C that is unaffected by the values of h and k exists, such that it ensures the validity of the following inequality:*

$$\|\rho_{u_t}(t)\| \le C h^{k+1} \left[\|u(t)\|_{H^{k+2}(\Omega)} + \|u_t(t)\|_{H^{k+2}(\Omega)} + \int_0^t \left\{ \|u(s)\|_{H^{k+2}(\Omega)} + \|u_s(s)\|_{H^{k+2}(\Omega)} \right\} ds \right].$$

Proof. We will begin by differentiating (3.7a)-(3.7e) w.r.t. t, to obtain

$$(\boldsymbol{\rho_{q_t}}, \boldsymbol{w_h}) - (\rho_{u_t}, \nabla \cdot \boldsymbol{w_h}) + \langle \hat{\rho}_{u_t}, \boldsymbol{w_h} \cdot \boldsymbol{\nu} \rangle_{\partial \mathcal{T}_h} = 0,$$
$$(a_u(u) u_t \boldsymbol{\rho_q} + a(u) \boldsymbol{\rho_{q_t}}, \boldsymbol{\tau_h}) - (\boldsymbol{\rho_{\sigma_t}}, \boldsymbol{\tau}_h) + (b(u) \boldsymbol{\rho_q}(t), \boldsymbol{\tau_h}) = (a_u(u) u_t \boldsymbol{\theta_q} + a(u) \boldsymbol{\theta_{q_t}}, \boldsymbol{\tau}_h)$$

$$
\begin{aligned}
&- (\boldsymbol{\theta}_{\boldsymbol{\sigma_t}}, \boldsymbol{\tau}_h) + (b(u)\boldsymbol{\theta}_{\boldsymbol{q}}(t), \boldsymbol{\tau_h}),\\
-(\boldsymbol{\rho}_{\boldsymbol{\sigma_t}}, \nabla v_h) + \langle \boldsymbol{\rho}_{\boldsymbol{\sigma_t}} \cdot \boldsymbol{\nu}, v_h\rangle_{\partial\mathcal{T}_h} &= 0,\\
\langle \hat{\rho}_{u_t}, \mu_h\rangle_{\partial\Omega} &= 0,\\
\langle \hat{\boldsymbol{\rho}}_{\boldsymbol{\sigma_t}} \cdot \boldsymbol{\nu}, m_h\rangle_{\partial\mathcal{T}_h\setminus\partial\mathcal{T}_h} &= 0,
\end{aligned}
$$

for all $(v_h, \boldsymbol{w}_h, \boldsymbol{\tau}_h, \mu_h, m_h) \in (V_h \times \boldsymbol{W}_h \times \boldsymbol{W}_h \times M_h)$.

Now, using a similar approach as above, desired estimates are obtained. □

Theorem 3.2.4. *When $u, u_t \in L^\infty(H^{k+2}(\mathcal{T}_h))$, a positive constant C that is unaffected by the values of h and k exists, such that it ensures the validity of the following inequality:*

$$
\|(u-\tilde{u}_h)(t)\|+\|(\boldsymbol{q}-\tilde{\boldsymbol{q}}_h)(t)\|+\|(\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}}_h)(t)\| \leq Ch^{k+1}\left[\|u(t)\|_{H^{k+2}(\Omega)}+\int_0^t \|u(s)\|_{H^{k+2}(\Omega)}ds\right],
$$

$$
\begin{aligned}
\|(u_t - \tilde{u}_{h_t})(t)\| \leq Ch^{k+1}\Bigg[&\|u(t)\|_{H^{k+2}(\Omega)} + \|u_t(t)\|_{H^{k+2}(\Omega)} + \int_0^t \Big\{\|u(s)\|_{H^{k+2}(\Omega)}\\
&+ \|u_s(s)\|_{H^{k+2}(\Omega)}\Big\}ds\Bigg].
\end{aligned}
$$

Proof. With the help of (1.2b), Lemma 3.2.2 and Lemma 3.2.3 and application of triangle inequality, we obtain the desired result. □

3.3 A priori Error Estimates for Semi-discrete Scheme

In this section, we derive the error estimates for the approximate solution obtained by (3.3). Below, we present the main theorem.

Theorem 3.3.1. *If u, u_t and q are bounded, $u, u_t \in L^\infty(H^{k+2}(\mathcal{T}_h))$, $u_h(0) = \Pi_V u_0$ and $\boldsymbol{q}_h(0) = -I_h\nabla u_0$, then, the following estimates hold true:*

$$
\begin{aligned}
&\|(u - u_h)(t)\| + \|(\boldsymbol{q} - \boldsymbol{q}_h)(t)\| + \|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)(t)\|\\
&\qquad \leq Ch^{k+1}\left[\|u(t)\|_{H^{k+2}(\mathcal{T}_h)} + \int_0^t \left\{\|u(s)\|_{H^{k+2}(\mathcal{T}_h)} + \|u_s(s)\|_{H^{k+2}(\mathcal{T}_h)}\right\}ds\right],
\end{aligned}
$$

$$
\begin{aligned}
\|(u_t - u_{h_t})(t)\| \leq Ch^{k+1}\Bigg[&\|u(t)\|_{H^{k+2}(\mathcal{T}_h)} + \|u_t(t)\|_{H^{k+2}(\mathcal{T}_h)} + \int_0^t \big\{\|u(s)\|_{H^{k+2}(\mathcal{T}_h)}\\
&+ \|u_s(s)\|_{H^{k+2}(\mathcal{T}_h)}\big\}ds\Bigg].
\end{aligned}
$$

To prove Theorem 3.3.1, we decompose the error terms into η's and ξ's as done in Chapter 2. With the help of this decomposition, (3.5) can be rewritten as

$$(\boldsymbol{\xi_q}, \boldsymbol{w_h}) - (\xi_u, \nabla \cdot \boldsymbol{w_h}) + \langle \hat{\xi}_u, \boldsymbol{w_h} \cdot \boldsymbol{\nu} \rangle_{\partial \mathcal{T}_h} = 0, \tag{3.10a}$$

$$\begin{aligned}(a(u_h)\boldsymbol{\xi_q}, \boldsymbol{\tau_h}) - (\boldsymbol{\xi_\sigma}, \boldsymbol{\tau}_h) + \int_0^t (b(u_h(s))\boldsymbol{\xi_q}(s), \boldsymbol{\tau}_h) ds &= -((a(u) - a(u_h))\tilde{\boldsymbol{q}}_h, \boldsymbol{\tau}_h) \\ &\quad - \int_0^t ((b(u(s)) - b(u_h(s)))\tilde{\boldsymbol{q}}_h, \boldsymbol{\tau}_h),\end{aligned} \tag{3.10b}$$

$$(\xi_{ut}, v_h) - (\boldsymbol{\xi_\sigma}, \nabla v_h) + \langle \hat{\boldsymbol{\xi}}_{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu}, v_h \rangle_{\partial \mathcal{T}_h} = (f(u) - f(u_h), v_h) - (\eta_{ut}, v_h), \tag{3.10c}$$

$$\langle \hat{\xi}_u, \mu_h \rangle_{\partial \Omega} = 0, \tag{3.10d}$$

$$\langle \hat{\boldsymbol{\xi}}_{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu}, m_h \rangle_{\partial \mathcal{T}_h \setminus \partial \Omega} = 0, \tag{3.10e}$$

for all $(v_h, \boldsymbol{w}_h, \boldsymbol{\tau}_h, \mu_h, m_h) \in (V_h \times \boldsymbol{W}_h \times \boldsymbol{W}_h \times M_h \times M_h)$. Now, we present a series of lemmas that will help to prove Theorem 3.3.1.

Lemma 3.3.2. *([26]) If $u \in L^\infty(H^2)$ and $\boldsymbol{q} \in L^\infty(\boldsymbol{H}^1)$, then there exists $C = C(\boldsymbol{q})$, such that*

$$\|\tilde{\boldsymbol{q}}_h\|_\infty + \|\tilde{\boldsymbol{q}}_{h_t}\|_\infty \leq C(\boldsymbol{q}).$$

Proof. From the elliptic projection with memory, Lemma 1.4.5 and Lemma 1.4.10, we have the following

$$\begin{aligned}\|\tilde{\boldsymbol{q}}_h\|_\infty &\leq \|\tilde{\boldsymbol{q}}_h - \boldsymbol{Rq}\|_\infty + \|\boldsymbol{Rq}\|_\infty \\ &\leq Ch^{-1}\|\tilde{\boldsymbol{q}}_h - \boldsymbol{Rq}\| + \|\boldsymbol{q}\|_\infty \\ &\leq Ch^{-1}\left(\|\tilde{\boldsymbol{q}}_h - \boldsymbol{q}\| + \|\boldsymbol{q} - \boldsymbol{Rq}\|\right) + C \\ &\leq Ch^{-1}\left(h + h\right) + C \\ &\leq C.\end{aligned}$$

Similarly, the estimates of $\|\tilde{\boldsymbol{q}}_{h_t}\|_\infty$ can be derived. □

Lemma 3.3.3. *A positive constant C that is unaffected by the values of h and k exists, such that it ensures the validity of the following inequality:*

$$\|\xi_u(t)\|^2 + \int_0^t \|\boldsymbol{\xi_q}(s)\|^2 \, ds \leq C\bigg(\|\xi_u(0)\|^2 + \int_0^T \left(\|\eta_u(t)\|^2 + \|\eta_{u_t}(t)\|^2\right) dt\bigg).$$

Proof. Take $\boldsymbol{w}_h = \boldsymbol{\xi_\sigma}$, $\boldsymbol{\tau}_h = \boldsymbol{\xi_q}$, $v_h = \xi_u$, $\mu_h = -\hat{\boldsymbol{\xi}}_{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu}$ and $m_h = -\hat{\xi}_u$ in (3.10a), (3.10b), (3.10c), (3.10d) and (3.10e), respectively and then, add the resulting equations to obtain

$$(a(u_h)\boldsymbol{\xi_q}, \boldsymbol{\xi_q}) + \int_0^t (b(u_h(s))\boldsymbol{\xi_q}(s), \boldsymbol{\xi_q})ds + \frac{1}{2}\frac{d}{dt}\|\xi_u\|^2 + \|\hat{\xi}_u - \xi_u\|_\tau^2 = (f(u) - f(u_h), \xi_u)$$
$$- (\eta_{u_t}, \xi_u) - ((a(u) - a(u_h))\tilde{\boldsymbol{q}}_h, \boldsymbol{\xi_q}) - \int_0^t ((b(u(s)) - b(u_h(s)))\tilde{\boldsymbol{q}}_h, \boldsymbol{\xi_q}).$$

Next, use of boundedness of a, b as well as Lipschitz continuity of f, a and b along with Lemma 3.3.2, yield

$$\|\boldsymbol{\xi_q}\|^2 + \frac{d}{dt}\|\xi_u\|^2 + \|\hat{\xi}_u - \xi_u\|_\tau^2 \le C\Big[\int_0^t \|\boldsymbol{\xi_q}(s)\|ds + \|\xi_u + \eta_u\| + \int_0^t \|\xi_u(s) + \eta_u(s)\|ds\Big]$$
$$\|\boldsymbol{\xi_q}\| + C\left(\|\xi_u + \eta_u\| + \|\eta_{u_t}\|\right)\|\xi_u\|.$$

Use of Young's inequality yields

$$\|\boldsymbol{\xi_q}\|^2 + \frac{d}{dt}\|\xi_u\|^2 + \|\hat{\xi}_u - \xi_u\|_\tau^2 \le C\bigg(\|\eta_u\|^2 + \int_0^t (\|\xi_u(s)\|^2 + \|\eta_u(s)\|^2)ds + \|\eta_{u_t}\|^2$$
$$+ \|\xi_u\|^2 + \int_0^t \|\boldsymbol{\xi_q}(s)\|^2 ds\bigg).$$

On integrating the above inequality from 0 to t, it follows that

$$\int_0^t (\|\boldsymbol{\xi_q}(s)\|^2 + \|(\hat{\xi}_u - \xi_u)(s)\|_\tau^2)ds + \|\xi_u(t)\|^2 \le C\bigg(\|\xi_u(0)\|^2 + \int_0^t \bigg(\|\xi_u(s)\|^2 + \|\eta_u(s)\|^2$$
$$+ \|\eta_{u_s}(s)\|^2 + \int_0^s \|\boldsymbol{\xi_q}(\gamma)\|^2 d\gamma\bigg)ds\bigg).$$

Finally, use of the Gronwall's lemma gives the desired inequality. □

Lemma 3.3.4. *A positive constant C that is unaffected by the values of h and k exists, such that it ensures the validity of the following inequality:*

$$\|\boldsymbol{\xi_\sigma}(t)\|^2 + \|\boldsymbol{\xi_q}(t)\|^2 + \|\hat{\xi}_u - \xi_u\|_\tau^2 \le C\left(\|\boldsymbol{\xi_q}(0)\|^2 + \|\xi_u(0)\|^2 + \int_0^T \left(\|\eta_u(t)\|^2 + \|\eta_{u_t}(t)\|^2\right)dt\right).$$

Proof. To begin with, we differentiate (3.10a) with respect to t and then choose $\boldsymbol{w_h} = \boldsymbol{\xi_\sigma}$, $\boldsymbol{\tau_h} = \boldsymbol{\xi_{q_t}}$, $v_h = \xi_{u_t}$ in (3.10a), (3.10b), (3.10c) respectively. Now, differentiate (3.10d) with respect to t and choose $\mu_h = -\hat{\boldsymbol{\xi}}_{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu}$ and $m_h = -\hat{\xi}_{u_t}$ in (3.10d) and (3.10e) respectively. Then adding the resulting equations, we obtain

$$(a(u_h)\boldsymbol{\xi_q}, \boldsymbol{\xi_{q_t}}) + \frac{1}{2}\frac{d}{dt}\|\hat{\xi}_u - \xi_u\|_\tau^2 + \|\xi_{u_t}\|^2 + \int_0^t (b(u_h(s))\boldsymbol{\xi_q}(s), \boldsymbol{\xi_{q_t}})ds = -(\eta_{u_t}, \xi_{u_t})$$
$$- ((a(u) - a(u_h))\tilde{\boldsymbol{q}}_h, \boldsymbol{\xi_{q_t}}) - \int_0^t ((b(u(s)) - b(u_h(s)))\tilde{\boldsymbol{q}}_h(s), \boldsymbol{\xi_{q_t}})ds + (f(u) - f(u_h), \xi_{u_t}),$$

which further yields,

$$C\frac{d}{dt}(\|\boldsymbol{\xi_q}\|^2 + \|\hat{\xi}_u - \xi_u\|_\tau^2) + \|\xi_{u_t}\|^2 \leq -\frac{d}{dt}((a(u) - a(u_h))\tilde{\boldsymbol{q}}_h, \boldsymbol{\xi_q}) + ((a(u) - a(u_h))\tilde{\boldsymbol{q}}_{h_t}, \boldsymbol{\xi_q})$$
$$+ ((a_u(u)u_t - a_u(u_h)u_{h_t})\tilde{\boldsymbol{q}}_h, \boldsymbol{\xi_q}) - \int_0^t \frac{d}{dt}((b(u(s)) - b(u_h(s)))\tilde{\boldsymbol{q}}_h(s), \boldsymbol{\xi_q})ds$$
$$- \int_0^t \frac{d}{dt}(b(u_h(s))\boldsymbol{\xi_q}(s), \boldsymbol{\xi_q})ds + (f(u) - f(u_h), \xi_{u_t}) - (\eta_{u_t}, \xi_{u_t}).$$

Next, we will use the Cauchy Schwarz inequality, Leibniz's theorem and Lemma 3.3.2 along with the fact that a, a_u, b, b_u and f are Lipschitz continuous with respect to u. Then, integrating the resulting equations from 0 to t, yields the following inequality

$$\|\boldsymbol{\xi_q}\|^2 + \|\hat{\xi}_u - \xi_u\|_\tau^2 + \int_0^t \|\xi_{u_s}\|^2 ds \leq C(\|\boldsymbol{\xi_q}(0)\|^2 + \|\xi_u(0)\|^2 + \|\eta_u\|^2 + \|\xi_u\|^2)$$
$$+ C\int_0^t \left(\|\eta_u\|^2 + \|\xi_u\|^2 + \|\boldsymbol{\xi_q}\|^2 + \|\eta_{u_s}\|^2 + \|\xi_{u_s}\|^2\right) ds.$$

Finally, use of the Gronwall's lemma yields the following result

$$\|\boldsymbol{\xi_q}\|^2 + \|\hat{\xi}_u - \xi_u\|_\tau^2 \leq C\left(\|\boldsymbol{\xi_q}(0)\|^2 + \|\xi_u(0)\|^2 + \int_0^T \left(\|\eta_u(s)\|^2 + \|\eta_{u_s}(s)\|^2\right) ds\right).$$

Now, choosing $\boldsymbol{\tau_h} = \boldsymbol{\xi_\sigma}$ in (3.10b) and then proceeding as above will give

$$\|\boldsymbol{\xi_\sigma}\| \leq C\left(\int_0^T \left(\|\eta_u(s)\| + \|\eta_{u_s}(s)\|\right) ds\right).$$

Combining the last two inequalities completes the proof. □

Proof of Theorem 3.3.1: To prove the theorem, we use triangle inequality, Theorem 3.2.4, Lemma 3.3.3 and Lemma 3.3.4.

□

3.4 Post-processing

We begin by defining the new approximation $u_h^* \in P_{k+1}(K)$, on the element K, as

$$u_h^* = u_h^p + \frac{1}{|K|}\int_K u_h, \;\; u_h^p \in P_{k+1}^0,$$

where u_h^p satisfies

$$(a(u_h)\nabla u_h^p, \nabla v) = -(a(u_h)\boldsymbol{q}_h, \nabla v), \;\; \forall v \in P_{k+1}^0,$$

where $P_{k+1}^0(K) \subset P_{k+1}(K)$, which has all the polynomials whose mean is zero.
By the definition of u_h^* and adding and subtracting $\frac{1}{|K|}\int_K u dx$, on any $K \in \mathcal{T}_h$, we obtain

$$\|u - u_h^*\|_{L^2(K)} \le \|I_h^{k-1} e_u\|_{L^2(K)} + \|u^p - u_h^p\|_{L^2(K)}, \tag{3.11}$$

where, $u^p = u - \frac{1}{|K|}\int_K u dx$.
Remark: For the estimates of the post-processed solution, the function f is considered to be differentiable and its derivative to be Lipschitz continuous along 'u', while for a priori error estimates, the Lipschitz continuity of f was sufficient.
Below, we present two lemmas that help to obtain the estimates for the terms on the right-hand side of (3.11).

Lemma 3.4.1. *A positive constant C that is unaffected by the values of h and k exists, such that it ensures the validity of the following inequality:*

$$\|I_h^{k-1} e_u\| \le C\sqrt{\log\left(\frac{T}{\rho^2}\right)}\, h^{k+2}. \tag{3.12}$$

Proof. We start by recalling the following dual problem [113]. For $\psi(s) \in H^2(\Omega) \cap H_0^1(\Omega)$, we have the following equations, when $t \in (0, T)$ is fixed,

$$\boldsymbol{\phi}(s) = \nabla\psi(s) \quad \text{in } \Omega, \; s \le t,$$
$$\boldsymbol{p}(s) = a(u)\boldsymbol{\phi}(s) + \int_s^t b(u(\gamma))\boldsymbol{\phi}(\gamma)d\gamma \quad \text{in } \Omega, \; s \le t,$$

$$\psi_s(s) + \nabla \cdot \boldsymbol{p}(s) = a_u(u)\boldsymbol{q} \cdot \boldsymbol{\phi} + \int_s^t b_u(u(\gamma))\boldsymbol{q}(\gamma) \cdot \boldsymbol{\phi}(\gamma) d\gamma + f_u(u)\psi \quad \text{in } \Omega,\ s \leq t,$$

$$\psi(s) = 0 \quad \text{on } \partial\Omega,\ s \leq t,$$

$$\psi(t) = \lambda \quad \text{in } \Omega,$$

ψ satisfies the following regularity results, see [112, 113]:

$$\int_0^t \|\psi(s)\|_2^2 \, ds \leq C\|\nabla\lambda\|^2 \tag{3.13}$$

$$\int_0^t (t-s)\|\psi(s)\|_2^2 \, ds \leq C\|\lambda\|^2. \tag{3.14}$$

The following proposition can easily be proved with the help of the regularity results (3.13) and (3.14). It is used to prove the super-convergence of $\|I_h^{k-1}e_u\|$.

Proposition 3.4.2. *[23] Let $\lambda = P_{h'}\theta$, where $\theta \in V_h$ and $P_{h'}$ is defined as in [23], then*

$$\int_0^t \|\psi(s)\|_2^2 \, ds \leq C\|\nabla\theta\|^2,$$
$$\int_0^t (t-s)\|\psi(s)\|_2^2 \, ds \leq \frac{C}{\rho^2}\|\theta\|^2.$$

Note, using properties of the projection $\boldsymbol{I}_h^k$, I_h^{k-1} and $\boldsymbol{\Pi}_{k-1}^{RT}$, that

$$\begin{aligned}
&\frac{d}{ds}(\psi(s), I_h^{k-1}e_u(s)) = (e_{u_s}(s), I_h^{k-1}\psi(s) - \psi(s)) - (e_{u_s}(s), I_h^k\psi(s) - \psi(s)) \\
&- (\boldsymbol{e_q}(s), \boldsymbol{\Pi}_{k-1}^{RT}\boldsymbol{p}(s) - \boldsymbol{p}(s)) - (a(u)\boldsymbol{e_q}(s), \boldsymbol{\phi}(s) - \boldsymbol{I}_h^k\boldsymbol{\phi}(s)) - (\boldsymbol{e_\sigma}(s), \boldsymbol{I}_h^k\boldsymbol{\phi}(s) - \boldsymbol{\phi}(s)) \\
&- (\boldsymbol{e_\sigma}(s), \nabla(\psi - I_h^k\psi)(s)) - \langle \hat{\boldsymbol{e}}_{\boldsymbol{\sigma}}(s) \cdot \boldsymbol{\nu}, I_h^k\psi \rangle - ((a(u_h) - a(u))\boldsymbol{q}_h, \boldsymbol{I}_h^k\boldsymbol{\phi}(s)) \\
&+ (a_u(u)\boldsymbol{q} \cdot \boldsymbol{\phi}, I_h^{k-1}e_u(s)) - (f(u_h) - f(u), I_h^k\psi) + (f_u(u)\psi, I_h^{k-1}e_u) \\
&- \int_0^s ((b(u_h(\gamma)) - b(u(\gamma)))\boldsymbol{q}_h(\gamma), \boldsymbol{I}_h^k\boldsymbol{\phi}(s))d\gamma + \int_s^t (b_u(u(\gamma))\boldsymbol{q}(\gamma) \cdot \boldsymbol{\phi}(\gamma), I_h^{k-1}e_u(s))d\gamma \\
&+ \int_0^s (b(u(\gamma))\boldsymbol{e_q}(\gamma), \boldsymbol{I}_h^k\boldsymbol{\phi}(s))d\gamma - \int_s^t (b(u(\gamma))\boldsymbol{\phi}(\gamma), \boldsymbol{e_q}(s))d\gamma.
\end{aligned}$$

Integrating this equation from 0 to t, we obtain

$$\begin{aligned}
&(\lambda, I_h^{k-1}e_u(t)) = \int_0^t \Big[[(e_{u_s}(s), I_h^{k-1}\psi(s) - \psi(s)) - (e_{u_s}(s), I_h^k\psi(s) - \psi(s)) \\
&- (\boldsymbol{e_q}(s), \boldsymbol{\Pi}_{k-1}^{RT}\boldsymbol{p}(s) - \boldsymbol{p}(s)) - (a(u)\boldsymbol{e_q}(s), \boldsymbol{\phi}(s) - \boldsymbol{I}_h^k\boldsymbol{\phi}(s)) - (\boldsymbol{e_\sigma}(s), \boldsymbol{I}_h^k\boldsymbol{\phi}(s) - \boldsymbol{\phi}(s))
\end{aligned}$$

$$
\begin{aligned}
&-(\boldsymbol{e_\sigma}(s), \nabla(\psi - I_h^k\psi)(s))] - \langle \hat{\boldsymbol{e}}_{\boldsymbol{\sigma}}(s)\cdot\boldsymbol{\nu}, I_h^k\psi\rangle + \big[-((a(u_h)-a(u))\boldsymbol{q}_h, \boldsymbol{I}_h^k\boldsymbol{\phi}(s)) \\
&+ (a_u(u)\boldsymbol{q}\cdot\boldsymbol{\phi}, I_h^{k-1}e_u(s))\big] + \big[(f_u(u)\psi, I_h^{k-1}e_u) - (f(u_h)-f(u), I_h^k\psi)\big]\Big] ds \\
&+ \Big[\int_0^t\int_s^t (b_u(u(\gamma))\boldsymbol{q}(\gamma)\cdot\boldsymbol{\phi}(\gamma), I_h^{k-1}e_u(s))d\gamma ds - \int_0^t\int_0^s ((b(u_h(\gamma)) - b(u(\gamma)))\boldsymbol{q}_h(\gamma), \\
&\boldsymbol{I}_h^k\boldsymbol{\phi}(s))d\gamma ds\Big] + \Big[\int_0^t\int_0^s (b(u(\gamma))\boldsymbol{e_q}(\gamma), \boldsymbol{I}_h^k\boldsymbol{\phi}(s))d\gamma ds - \int_0^t\int_s^t (b(u(\gamma))\boldsymbol{\phi}(\gamma), \boldsymbol{e_q}(s))d\gamma ds\Big]. \\
&= \int_0^t [E_1 + E_2 + E_3 + E_4]\, ds + E_5 + E_6. \qquad (3.15)
\end{aligned}
$$

Use of Cauchy Schwarz's inequality and the properties of I_h^{k-1}, I_h^k, $\boldsymbol{\Pi}_{k-1}^{RT}$ and $\boldsymbol{I}_h^k$ show

$$|E_1| \le Ch\left(\|e_{u_s}(s)\| + \|\boldsymbol{e_q}(s)\| + \|\boldsymbol{e_\sigma}(s)\|\right)\|\psi(s)\|_2.$$

Use of (3.5e) with properties of I_h^k yield

$$|E_2| \le \|\hat{\boldsymbol{e}}_{\boldsymbol{\sigma}}(s)\cdot\boldsymbol{\nu}\|_{\partial K}\|I_h^k\psi - \psi\|_{\partial K} \le Ch^{3/2}\|\hat{\boldsymbol{e}}_{\boldsymbol{\sigma}}(s)\cdot\boldsymbol{\nu}\|_{\partial K}\|\psi(s)\|_2.$$

Next, we have

$$
\begin{aligned}
E_3 &= -((a(u_h)-a(u))\boldsymbol{q}_h, \boldsymbol{I}_h^k\boldsymbol{\phi}(s)) + (a_u(u)\boldsymbol{q}\cdot\boldsymbol{\phi}, I_h^{k-1}e_u(s)) \\
&= -((a(u_h)-a(u))(\boldsymbol{q}_h - \boldsymbol{q}), \boldsymbol{I}_h^k\boldsymbol{\phi}(s)) - ((a(u_h)-a(u))\boldsymbol{q}, \boldsymbol{I}_h^k\boldsymbol{\phi}(s) - \boldsymbol{\phi}(s)) \\
&\quad + (a_u(u)\boldsymbol{q}\cdot\boldsymbol{\phi}, I_h^{k-1}e_u(s) - e_u(s)) - ((a(u_h)-a(u)), \boldsymbol{q}\cdot\boldsymbol{\phi}(s)) + (a_u(u)\boldsymbol{q}\cdot\boldsymbol{\phi}, e_u(s)).
\end{aligned}
$$

Use of Taylor's series expansion yields

$$
\begin{aligned}
E_3 &= -((a(u_h)-a(u))(\boldsymbol{q}_h - \boldsymbol{q}), \boldsymbol{I}_h^k\boldsymbol{\phi}(s)) - ((a(u_h)-a(u))\boldsymbol{q}, \boldsymbol{I}_h^k\boldsymbol{\phi}(s) - \boldsymbol{\phi}(s)) \\
&\quad + (a_u(u)\boldsymbol{q}\cdot\boldsymbol{\phi}, I_h^{k-1}e_u(s) - e_u(s)) + (a_u(u) - a_u(u_h + \lambda(u_h - u))e_u(s), \boldsymbol{q}\cdot\boldsymbol{\phi}).
\end{aligned}
$$

Use of generalized Holder's inequality yields

$$|E_3| \le Ch^{k+2}\|\psi(s)\|_2.$$

For E_4, a similar procedure can be followed, whereas for E_5 and E_6, a change of order of integration followed by a similar procedure as for E_3 will give the estimates. Now, on substitution in (3.15), we arrive at

$$(\lambda, I_h^{k-1}e_u(t)) \le Ch^{k+2}\int_0^t \|\psi(s)\|_2\, ds.$$

Next, for any $\delta \in (0, t)$, there holds

$$\begin{aligned}\int_0^t \|\psi(s)\|_2 \, ds &= \int_0^{t-\delta} \|\psi(s)\|_2 \, ds + \int_{t-\delta}^t \|\psi(s)\|_2 \, ds \\ &= \int_0^{t-\delta} \sqrt{t-s}\, \|\psi(s)\|_2 \frac{1}{\sqrt{t-s}} \, ds + \int_{t-\delta}^t \|\psi(s)\|_2 \, ds \\ &\leq \sqrt{\log\frac{t}{\delta}} \Big(\int_0^t (t-s) \|\psi(s)\|_2^2 \, ds \Big)^{\frac{1}{2}} + \sqrt{\delta} \Big(\int_0^t \|\psi(s)\|_2^2 \, ds \Big)^{\frac{1}{2}}.\end{aligned}$$

Now, using Proposition 3.4.2 yields the required estimate. □

Lemma 3.4.3. *A positive constant C that is unaffected by the values of h and k exists, such that it ensures the validity of the following inequality:*

$$\|u^p - u_h^p\| \leq C h^{k+2}. \tag{3.16}$$

Proof. Use of the Poincare's inequality shows

$$\|u^p - u_h^p\| \leq C h \|\nabla(u^p - u_h^p)\|. \tag{3.17}$$

Now, for any $v \in P_0^{k+1}(K)$, there holds

$$(a(u_h)\nabla(u_h^p - u^p), \nabla v) + (a(u_h)\nabla u^p, \nabla v) = -(a(u_h)\boldsymbol{q}_h, \nabla v),$$

and hence,

$$\begin{aligned}(a(u_h)\nabla(u_h^p - u^p), \nabla v) &= (a(u_h)(\boldsymbol{q} - \boldsymbol{q}_h), \nabla v) \\ &\leq C\|\boldsymbol{q} - \boldsymbol{q}_h\| \|\nabla v\|.\end{aligned} \tag{3.18}$$

Now, we note that

$$\begin{aligned}a_* \|\nabla(I_h^{k+1}u^p - u_h^p)\|^2 \leq\ & (a(u_h)\nabla(I_h^{k+1}u^p - u^p), \nabla(I_h^{k+1}u^p - u_h^p)) \\ &+ (a(u_h)\nabla(u^p - u_h^p), \nabla(I_h^{k+1}u^p - u_h^p)).\end{aligned}$$

Use of Cauchy Schwarz's inequality and (3.18) and a replacement of v by $I_h^{k+1}u^p - u_h^p$ yield

$$\|\nabla(I_h^{k+1}u^p - u_h^p)\| \leq C(\|\nabla(I_h^{k+1}u^p - u^p)\| + \|\boldsymbol{q} - \boldsymbol{q}_h\|).$$

Now, by triangle inequality, we arrive at

$$\begin{aligned}\|\nabla(u^p - u_h^p)\| &\leq \|\nabla(I_h^{k+1}u^p - u^p)\| + \|\nabla(I_h^{k+1}u^p - u_h^p)\| \\ &\leq C(\|\nabla(I_h^{k+1}u^p - u^p)\| + \|\boldsymbol{q} - \boldsymbol{q}_h\|).\end{aligned}$$

Use of Theorem 3.3.1 and projection property of I_h^{k+1}, yield

$$\|\nabla(u^p - u_h^p)\| \leq Ch^{k+1}. \tag{3.19}$$

A substitution of (3.19) in (3.17) will conclude the proof. □

Theorem 3.4.4. *If $u, u_t \in L^\infty(H^{k+2}(\mathcal{T}_h))$, then, a positive constant C that is unaffected by the values of h and k exists, such that it ensures the validity of the following inequality:*

$$\|u - u_h^*\| \leq C\sqrt{\log\left(\frac{T}{\rho^2}\right)}\, h^{k+2}.$$

Proof. Substitute (3.12) and (3.16) in (3.11) to prove the theorem. □

3.5 Fully Discrete Scheme

In this section, a completely discrete scheme is derived for the problem (3.3), based on the backward difference method, along with the left rectangle rule to approximate the integral term. We first divide the interval $[0, T]$ into M equally spaced sub-intervals by the following points

$$0 = t_0 < t_1 < ... < t_M = T,$$

with $t_n = n\Delta t$, $\Delta t = T/M$, be the time step.

The fully discrete approximation for the problem (3.3) is defined as follows: For $1 \leq n \leq M$, find $(U^n, \boldsymbol{Q}^n, \boldsymbol{S}^n, \hat{U}^n) \in (V_h \times \boldsymbol{W}_h \times \boldsymbol{W}_h \times M_h)$, such that

$$(\boldsymbol{Q}^n, \boldsymbol{w}_h) - (U^n, \nabla \cdot \boldsymbol{w}_h) + \langle \hat{U}^n, \boldsymbol{w}_h \cdot \boldsymbol{\nu}\rangle_{\partial\mathcal{T}_h} = 0, \tag{3.20a}$$

$$(a(U^n)\boldsymbol{Q}^n, \boldsymbol{\tau}_h) - (\boldsymbol{S}^n, \boldsymbol{\tau}_h) + \big(\Delta t \sum_{i=0}^{n-1} b(U^i)\boldsymbol{Q}^i, \boldsymbol{\tau}_h\big) = 0, \tag{3.20b}$$

$$\left(\frac{U^n - U^{n-1}}{\Delta t}, v_h\right) - (\boldsymbol{S}^n, \nabla v_h) + \langle \hat{\boldsymbol{S}}^n \cdot \boldsymbol{\nu}, v_h\rangle_{\partial\mathcal{T}_h} = (f(U^n), v_h), \tag{3.20c}$$

$$\langle \hat{U}^n, \mu_h \rangle_{\partial\Omega} = 0, \tag{3.20d}$$

$$\langle \hat{\boldsymbol{S}}^n \cdot \boldsymbol{\nu}, m_h \rangle_{\partial\mathcal{T}_h \setminus \partial\Omega} = 0, \tag{3.20e}$$

for all $(v_h, \boldsymbol{w}_h, \boldsymbol{\tau}_h, \mu_h, m_h) \in (V_h \times \boldsymbol{W}_h \times \boldsymbol{W}_h \times M_h \times M_h)$. Here, numerical trace for flux is defined by:

$$\hat{\boldsymbol{S}}^n \cdot \boldsymbol{\nu} = \boldsymbol{S}^n \cdot \boldsymbol{\nu} + \tau(U^n - \hat{U}^n) \text{ on } \partial\mathcal{T}_h.$$

The nonlinear system of equation (3.20) has four unknowns; namely, U^n, $\boldsymbol{Q}^n$, $\boldsymbol{S}^n$ and $\hat{U}^n$ for $1 \le n \le M$. We will begin by using Newton's iterative method to derive the first iterative value of U, Q and S in terms of $\hat{U}$ using (3.20a)-(3.20c). Next, $\hat{U}$ is estimated using (3.20d)-(3.20e). Finally, we can get the next iterative values of U, Q and S by substituting the value of $\hat{U}$.

Theorem 3.5.1. *Let u be the solution of* (3.1)*, $u, u_t \in L^\infty(H^{k+2}(\mathcal{T}_h))$, $u_h(0) = U^0 = \Pi_V u_0$ and $\boldsymbol{q}_h(0) = \boldsymbol{Q}^0 = -I_h \nabla u_0$, then for all $1 \le n \le M$,*

$$\|u(t_n) - U^n\| + \|\boldsymbol{q}(t_n) - \boldsymbol{Q}^n\| + \|\boldsymbol{\sigma}(t_n) - \boldsymbol{S}^n\| \le O(h^{k+1} + \Delta t).$$

Proof. We begin by writing $\|u(t_n) - U^n\| \le \|u(t_n) - u_h(t_n)\| + \|u_h(t_n) - U^n\|$. We only need to derive the estimates of $\|u_h(t_n) - U^n\|$, which will be denoted by $\|\zeta_u^n\|$. Similarly, $\boldsymbol{\zeta}_{\boldsymbol{q}}^n$, $\boldsymbol{\zeta}_{\boldsymbol{\sigma}}^n$ and $\hat{\zeta}_u^n$.

Now, using (3.3) and (3.20), we have the following system of equations

$$(\boldsymbol{\zeta}_{\boldsymbol{q}}^n, \boldsymbol{w}_h) - (\zeta_u^n, \nabla \cdot \boldsymbol{w}_h) + \langle \hat{\zeta}_u^n, \boldsymbol{w}_h \cdot \boldsymbol{\nu} \rangle_{\partial\mathcal{T}_h} = 0, \tag{3.21a}$$

$$(a(U^n)\boldsymbol{\zeta}_{\boldsymbol{q}}^n, \boldsymbol{\tau}_h) - (\boldsymbol{\zeta}_{\boldsymbol{\sigma}}^n, \boldsymbol{\tau}_h) + \int_0^{t_n} (b(u_h(s))\boldsymbol{q}_h(s), \boldsymbol{\tau}_h) ds + ((a(u_h(t_n)) - a(U^n))\boldsymbol{q}_h, \boldsymbol{\tau}_h) = \left(\Delta t \sum_{i=0}^{n-1} b(U^i)\boldsymbol{Q}^i, \boldsymbol{\tau}_h \right), \tag{3.21b}$$

$$\left(u_{h_t}(t_n) - \frac{U^n - U^{n-1}}{\Delta t}, v_h \right) - (\boldsymbol{\zeta}_{\boldsymbol{\sigma}}^n, \nabla v_h) + \langle \hat{\boldsymbol{\zeta}}_{\boldsymbol{\sigma}}^n \cdot \boldsymbol{\nu}, v_h \rangle_{\partial\mathcal{T}_h} = (f(u_h(t_n)) - f(U^n), v_h), \tag{3.21c}$$

$$\langle \hat{\zeta}_u^n, \mu_h \rangle_{\partial\Omega} = 0, \tag{3.21d}$$

$$\langle \hat{\boldsymbol{\zeta}}_{\boldsymbol{\sigma}}^n \cdot \boldsymbol{\nu}, m_h \rangle_{\partial\mathcal{T}_h \setminus \partial\Omega} = 0, \tag{3.21e}$$

for all $(v_h, \boldsymbol{w}_h, \boldsymbol{\tau}_h, \mu_h, m_h) \in (V_h \times \boldsymbol{W}_h \times \boldsymbol{W}_h \times M_h \times M_h)$. Here numerical trace for flux is defined by:

$$\hat{\boldsymbol{\zeta}}_{\boldsymbol{\sigma}}^n \cdot \boldsymbol{\nu} = \boldsymbol{\zeta}_{\boldsymbol{\sigma}}^n \cdot \boldsymbol{\nu} + \tau(\zeta_u^n - \hat{\zeta}_u^n) \text{ on } \partial\mathcal{T}_h.$$

Take $\boldsymbol{w}_h = \boldsymbol{\zeta}_{\boldsymbol{\sigma}}^n$, $\boldsymbol{\tau}_h = \boldsymbol{\zeta}_{\boldsymbol{q}}^n$, $v_h = \zeta_u^n$, $\mu_h = -\hat{\boldsymbol{\zeta}}_{\boldsymbol{\sigma}}^n \cdot \boldsymbol{\nu}$ and $m_h = -\hat{\zeta}_u^n$ in (3.21a), (3.21b), (3.21c), (3.21d) and (3.21e), respectively and then, add the resulting equations to obtain

$$\begin{aligned}&(a(U^n)\boldsymbol{\zeta}_{\boldsymbol{q}}^n, \boldsymbol{\zeta}_{\boldsymbol{q}}^n) + ((a(u_h(t_n)) - a(U^n))\boldsymbol{q}_h, \boldsymbol{\zeta}_{\boldsymbol{q}}^n) + \left(\frac{\zeta_u^n - \zeta_u^{n-1}}{\Delta t}, \zeta_u^n\right) + \|\hat{\zeta}_u^n - \zeta_u^n\|_\tau^2 \\ &+ (J^n, \zeta_u^n) = \left(E_h^n(\boldsymbol{q}_h), \boldsymbol{\zeta}_{\boldsymbol{q}}^n\right) + (f(u_h(t_n)) - f(U^n), \zeta_u^n) - \left(\Delta t \sum_{i=0}^{n-1} b(U^i)\boldsymbol{\zeta}_{\boldsymbol{q}}^i, \boldsymbol{\zeta}_{\boldsymbol{q}}^n\right),\end{aligned}$$

here,

$$J^n = u_{h_t}(t_n) - \frac{u_h(t_n) - u_h(t_{n-1})}{\Delta t},$$

$$E_h^n(\boldsymbol{q}_h) = \Delta t \sum_{i=0}^{n-1} b(U^i)\boldsymbol{q}_h(t_i) - \int_0^{t_n} b(u_h(s))\boldsymbol{q}_h(s)ds.$$

Hence,

$$\begin{aligned}&(a(U^n)\boldsymbol{\zeta}_{\boldsymbol{q}}^n, \boldsymbol{\zeta}_{\boldsymbol{q}}^n) + ((a(u_h(t_n)) - a(U^n))\boldsymbol{q}_h, \boldsymbol{\zeta}_{\boldsymbol{q}}^n) + \frac{1}{2\Delta t}\left(\|\zeta_u^n\|^2 - \|\zeta_u^{n-1}\|^2\right) + \frac{1}{2}\left\|\frac{\zeta_u^n - \zeta_u^{n-1}}{\Delta t}\right\|^2 \\ &+ \|\hat{\zeta}_u^n - \zeta_u^n\|_\tau^2 + (J^n, \zeta_u^n) = \left(E_h^n(\boldsymbol{q}_h), \boldsymbol{\zeta}_q^n\right) + (f(u_h(t_n)) - f(U^n), \zeta_u^n) - \left(\Delta t \sum_{i=0}^{n-1} b(U^i)\boldsymbol{\zeta}_q^i, \boldsymbol{\zeta}_q^n\right).\end{aligned}$$

Now, summing over n from $n = 1$ to $n = m$, where $1 \le m \le M$ to obtain

$$\Delta t \sum_{n=1}^m \|\boldsymbol{\zeta}_{\boldsymbol{q}}^n\|^2 + \frac{1}{2}\|\zeta_u^m\|^2 + \frac{\Delta t}{2}\sum_{n=1}^m \left\|\frac{\zeta_u^n - \zeta_u^{n-1}}{\Delta t}\right\|^2 \le \frac{1}{2}\|\zeta_u^0\|^2 + \Delta t \sum_{n=1}^m (I_1^n + I_2^n + I_3^n + I_4^n + I_5^n), \tag{3.22}$$

where,

$$\begin{aligned}&I_1^n = (J^n, \zeta_u^n), \; I_2^n = \left(E_h^n(\boldsymbol{q}_h), \boldsymbol{\zeta}_{\boldsymbol{q}}^n\right), \; I_3^n = (f(u_h(t_n)) - f(U^n), \zeta_u^n), \\ &I_4^n = ((a(u_h(t_n)) - a(U^n))\boldsymbol{q}_h, \boldsymbol{\zeta}_{\boldsymbol{q}}^n) \text{ and } I_5^n = \left(\Delta t \sum_{i=0}^{n-1} b(U^i)\boldsymbol{\zeta}_{\boldsymbol{q}}^i, \boldsymbol{\zeta}_{\boldsymbol{q}}^n\right).\end{aligned}$$

Now,

$$J^n = -e_{u_t}(t_n) + \frac{e_u(t_n) - e_u(t_{n-1})}{\Delta t} + u_t(t_n) - \frac{u(t_n) - u(t_{n-1})}{\Delta t}.$$

Using Taylor's series approximation, we note that

$$J^n = -e_{u_t}(t_n) + \frac{1}{\Delta t}\int_{t_{n-1}}^{t_n} e_{u_s}(s)ds + \frac{1}{\Delta t}\int_{t_{n-1}}^{t_n} (s - t_{n-1})u_{ss}(s)ds,$$

$$\text{and,} \quad \|J^n\|^2 \le C\left(\|e_{u_t}(t_n)\|^2 + \frac{1}{\Delta t}\int_{t_{n-1}}^{t_n} \|e_{u_s}(s)\|^2 ds + \Delta t\int_{t_{n-1}}^{t_n} \|u_{ss}(s)\|^2 ds\right).$$

Therefore, use of Theorem 3.3.1, yields

$$\Delta t\sum_{n=1}^{m} |I_1^n| \le C(\Delta t^2 + h^{2(k+1)}) + \frac{1}{2}\sum_{n=1}^{m} \|\zeta_u^m\|^2. \tag{3.23}$$

Now,

$$E_h^n(\boldsymbol{q}_h) = -\Delta t\sum_{i=0}^{n-1} b(U^i)\boldsymbol{e_q}(t_i) + \int_0^{t_n} b(u_h(s))\boldsymbol{e_q}(s)ds + \Delta t\sum_{i=0}^{n-1} b(U^i)\boldsymbol{q}(t_i)$$
$$- \int_0^{t_n} b(u_h(s))\boldsymbol{q}(s)ds.$$

Next, again use of Theorem 3.3.1 and rectangle rule yield

$$\|E_h^n(\boldsymbol{q}_h)\| \le C(h^{k+1} + \Delta t).$$

Then, use of Young's inequality along with the boundedness of b and Lipschitz continuity of a and f show

$$\Delta t\sum_{n=1}^{m} |I_2^n| \le C(\Delta t^2 + h^{2(k+1)}) + \frac{\Delta t}{2}\sum_{n=1}^{m} \|\zeta_{\boldsymbol{q}}^m\|^2, \tag{3.24}$$

$$\Delta t\sum_{n=1}^{m} |I_3^n| \le C\sum_{n=1}^{m} \|\zeta_u^m\|^2, \tag{3.25}$$

$$\Delta t\sum_{n=1}^{m} |I_4^n| \le C\sum_{n=1}^{m} \left(\|\zeta_u^m\|^2 + \|\zeta_{\boldsymbol{q}}^m\|^2\right). \tag{3.26}$$

Using (3.23), (3.24), (3.25) and (3.26) in (3.22), we obtain

$$C\Delta t\sum_{n=1}^{m} \|\boldsymbol{\zeta_q^n}\|^2 + C\|\zeta_u^m\|^2 + \frac{\Delta t}{2}\sum_{n=1}^{m} \left\|\frac{\zeta_u^n - \zeta_u^{n-1}}{\Delta t}\right\|^2 \le C(\Delta t^2 + h^{2(k+1)}) + C\Bigg(\sum_{n=1}^{m} \|\zeta_u^m\|^2$$

$$+ \Delta t \sum_{n=1}^{m} \sum_{i=1}^{n-1} \|\boldsymbol{\zeta}_{\boldsymbol{q}}^i\|^2\Big).$$

Finally, use of discrete Gronwall's lemma yields

$$\|\zeta_u^m\|^2 \leq C(\Delta t^2 + h^{2(k+1)}).$$

To derive the estimates of $\|\zeta_{\boldsymbol{q}}^m\|$, we begin by subtracting (3.21a) by itself at $(n-1)^{th}$ time level and dividing it by Δt, to arrive that the following equation

$$\left(\frac{\boldsymbol{\zeta}_{\boldsymbol{q}}^n - \boldsymbol{\zeta}_{\boldsymbol{q}}^{n-1}}{\Delta t}, \boldsymbol{w}_h\right) - \left(\frac{\zeta_u^n - \zeta_u^{n-1}}{\Delta t}, \nabla \cdot \boldsymbol{w}_h\right) + \left\langle \frac{\hat{\zeta}_u^n - \hat{\zeta}_u^{n-1}}{\Delta t}, \boldsymbol{w}_h \cdot \boldsymbol{\nu} \right\rangle_{\partial \mathcal{T}_h} = 0. \quad (3.27)$$

Similarly, from (3.21d), we get

$$\left\langle \frac{\hat{\zeta}_u^n - \hat{\zeta}_u^{n-1}}{\Delta t}, \mu_h \right\rangle_{\partial \Omega} = 0. \quad (3.28)$$

Take $\boldsymbol{w}_h = \boldsymbol{\zeta}_{\boldsymbol{\sigma}}^n$, $\boldsymbol{\tau}_h = \dfrac{\boldsymbol{\zeta}_{\boldsymbol{q}}^n - \boldsymbol{\zeta}_{\boldsymbol{q}}^{n-1}}{\Delta t}$, $v_h = \dfrac{\zeta_u^n - \zeta_u^{n-1}}{\Delta t}$, $\mu_h = -\hat{\boldsymbol{\zeta}}_{\boldsymbol{\sigma}}^n \cdot \boldsymbol{\nu}$ and $m_h = -\dfrac{\hat{\zeta}_u^n - \hat{\zeta}_u^{n-1}}{\Delta t}$ in (3.27), (3.21b), (3.21c), (3.28) and (3.21e), respectively and then, add the resulting equations. Finally, we take summation over n from $n = 1$ to $n = m$, where $1 \leq m \leq M$ to obtain

$$\frac{1}{2}\|\boldsymbol{\zeta}_{\boldsymbol{q}}^m\|^2 + \frac{\Delta t}{2} \sum_{n=1}^{m} \left\| \frac{\boldsymbol{\zeta}_{\boldsymbol{q}}^n - \boldsymbol{\zeta}_{\boldsymbol{q}}^{n-1}}{\Delta t} \right\|^2 + \Delta t \sum_{n=1}^{m} \left\| \frac{\zeta_u^n - \zeta_u^{n-1}}{\Delta t} \right\|^2 \leq \Delta t \sum_{n=1}^{m} \Bigg[-\Bigg(\Delta t \sum_{i=0}^{n-1} b(U^i)\boldsymbol{\zeta}_{\boldsymbol{q}}^i,$$
$$\frac{\boldsymbol{\zeta}_{\boldsymbol{q}}^n - \boldsymbol{\zeta}_{\boldsymbol{q}}^{n-1}}{\Delta t}\Bigg) - \left(u_{h_t}(t_n) - \frac{u_h(t_n) - u_h(t_{n-1})}{\Delta t}, \frac{\zeta_u^n - \zeta_u^{n-1}}{\Delta t}\right) + \Bigg(\Delta t \sum_{i=0}^{n-1} b(U^i)\boldsymbol{q}_h(t_i),$$
$$\frac{\boldsymbol{\zeta}_{\boldsymbol{q}}^n - \boldsymbol{\zeta}_{\boldsymbol{q}}^{n-1}}{\Delta t}\Bigg) - \left(\int_0^{t_n} b(u_h(s))\boldsymbol{q}_h(s)ds, \frac{\boldsymbol{\zeta}_{\boldsymbol{q}}^n - \boldsymbol{\zeta}_{\boldsymbol{q}}^{n-1}}{\Delta t}\right) + \left(f(u_h(t_n)) - f(U^n), \frac{\zeta_u^n - \zeta_u^{n-1}}{\Delta t}\right)$$
$$-\left((a(u_h(t_n)) - a(U^n))\boldsymbol{q}_n, \frac{\boldsymbol{\zeta}_{\boldsymbol{q}}^n - \boldsymbol{\zeta}_{\boldsymbol{q}}^{n-1}}{\Delta t}\right)\Bigg].$$

Proceeding as above, will give the estimates for $\|\boldsymbol{\zeta}_{\boldsymbol{q}}^n\|$. Finally, for the estimates of $\|\boldsymbol{\zeta}_{\boldsymbol{\sigma}}^n\|$, we take $\boldsymbol{\tau}_h = \boldsymbol{\zeta}_{\boldsymbol{\sigma}}^n$ in (3.21b) and simplify, to get the desired result. This concludes the proof. □

		u_h		$\boldsymbol{q_h}$		u_h^*	
$\boldsymbol{k}$	$\boldsymbol{h}$	**Error**	**Order**	**Error**	**Order**	**Error**	**Order**
1	$\frac{1}{2}$	3.0168e-03		1.0547e-02		6.3117e-04	
	$\frac{1}{4}$	8.4684e-04	1.8370	2.8531e-03	1.8735	8.7584e-05	2.8493
	$\frac{1}{8}$	2.2262e-04	1.9275	7.3254e-04	1.9615	1.1266e-05	2.9588
	$\frac{1}{16}$	5.6846e-05	1.9694	1.8488e-04	1.9863	1.4222e-06	2.9857
2	$\frac{1}{2}$	1.2063e-03		3.5126e-03		2.0439e-04	
	$\frac{1}{4}$	1.6104e-04	2.9050	4.4836e-04	2.9698	1.2135e-05	4.0741
	$\frac{1}{8}$	2.0576e-05	2.9684	5.6550e-05	2.9871	7.4159e-07	4.0324
	$\frac{1}{16}$	2.5946e-06	2.9874	7.0953e-06	2.9946	4.1680e-08	4.1532
3	$\frac{1}{2}$	1.2034e-04		1.6065e-04		1.9335e-05	
	$\frac{1}{4}$	7.4686e-06	4.0102	9.6716e-06	4.0540	6.6795e-07	4.8554
	$\frac{1}{8}$	4.6405e-07	4.0085	5.9810e-07	4.0153	2.2307e-08	4.9042
	$\frac{1}{16}$	2.8911e-08	4.0046	3.7151e-08	4.0089	6.9700e-10	5.0002

Table 3.1: Computed convergence rates and L^2 error estimates in the context of $Example\ 1$

3.6 Numerical Results

This section comprises two numerical examples, aimed at validating the theoretical findings presented in this chapter. Specifically, the examples involve equations (3.1a)-(3.1c), where $\Omega = (0,1) \times (0,1)$ and $T = 1$. To discretize the problem, we employed the backward Euler's method and approximated the integral term using the left rectangle rule. Our observations confirm the attainment of the optimal order of convergence for u and $\boldsymbol{q}$, as well as the predicted super-convergence for u_h^* as per our theoretical framework.

Example 1:. Choose $u(x,y,t) = e^{-t}x(1-x)y(1-y)$ and the coefficients be $a(u) = 1 + u^2$, $b(u) = u$ and $f(u) = u - u^3 + g(x,y,t)$, where $g(x,y,t)$ is decided by the exact solution u. We have used MATLAB codes to compute L^2 error estimates for the three unknowns, that is, u, $\boldsymbol{q}$ and $\boldsymbol{\sigma}$, for different mesh sizes, that is, for $h = \frac{1}{2}$, $h = \frac{1}{4}$, $h = \frac{1}{8}$ and $h = \frac{1}{16}$. Next, we have computed their orders of convergence and found that they match with the theoretical findings. Table 3.1 gives the computed order of convergence along with L^2 error estimates for u, $\boldsymbol{q}$ and u_h^* at $t = 1$.Table 3.2 gives the time convergence for u for the example for different time steps.

$\boldsymbol{\Delta t}$ $(h = 1/2)$	**Order** (Ex. 1)	**Order** (Ex. 2)
0.25	0.9167	0.8965
0.125	0.9813	0.9921
0.0625	1.0431	0.9976
0.03125	1.1452	1.0673

Table 3.2: Order of convergence for time

Example 2:. Let $u(x, y, t) = e^t \sin(x) \sin(y)(1 - x)(1 - y)$ and the coefficients be $a(u) = 1 + u^2$, $b(u) = u$ and $f(u) = u - u^3 + g(x, y, t)$, where $g(x, y, t)$ is decided by the exact solution u. We have used MATLAB codes to compute L^2 error estimates for the three unknowns, that is, u, $\boldsymbol{q}$ and $\boldsymbol{\sigma}$, for different mesh sizes, that is, for $h = \frac{1}{2}$, $h = \frac{1}{4}$, $h = \frac{1}{8}$ and $h = \frac{1}{16}$. Next, we have computed their orders of convergence and found that they match with the theoretical findings. Table 3.3 gives the computed order of convergence along with L^2 error estimates for u, $\boldsymbol{q}$ and u_h^* at $t = 1$. Table 3.2 gives the time convergence for u for the example for different time steps.

3.7 Conclusions

Due to various theoretical and computational benefits of the HDG method, it has been proposed and analyzed for nonlinear parabolic integro-differential equation (3.1). The nonlinear functions have been considered to be Lipschitz continuous to prove the a priori error estimates. Throughout this chapter, HDG and Ritz-Volterra projections have been used to derive the error estimates. Further, element-by-element post-processing has

		u_h		$\boldsymbol{q_h}$		u_h^*	
$\boldsymbol{k}$	$\boldsymbol{h}$	**Error**	**Order**	**Error**	**Order**	**Error**	**Order**
1	$\frac{1}{2}$	1.2014e-02		4.3255e-02		2.6559e-03	
	$\frac{1}{4}$	3.5180e-03	1.7718	1.1598e-02	1.8990	3.8902e-04	2.7713
	$\frac{1}{8}$	9.3814e-04	1.9069	2.9598e-03	1.9703	5.2593e-05	2.8869
	$\frac{1}{16}$	2.4120e-04	1.9595	7.4533e-04	1.9895	7.2453e-06	2.8597
2	$\frac{1}{2}$	1.8172e-03		4.8681e-03		2.9255e-04	
	$\frac{1}{4}$	2.4448e-04	2.8939	6.1897e-04	2.9754	1.7182e-05	4.0897
	$\frac{1}{8}$	3.2532e-05	2.9098	7.7882e-05	2.9905	1.0012e-06	4.1011
	$\frac{1}{16}$	4.1104e-06	2.9845	9.7467e-06	2.9983	5.7465e-08	4.1229
3	$\frac{1}{2}$	1.6168e-04		3.3585e-04		2.6980e-05	
	$\frac{1}{4}$	1.0318e-05	3.9699	2.1050e-05	3.9959	9.3014e-07	4.8583
	$\frac{1}{8}$	6.5356e-07	3.9807	1.3076e-06	4.0089	2.9204e-08	4.9932
	$\frac{1}{16}$	4.1112e-08	3.9907	7.6083e-08	4.1032	9.1180e-10	5.0013

Table 3.3: Computed convergence rates and L^2 error estimates in the context of $Example\ 2$

been proposed. It has been shown that the solution and its gradient achieved the optimal rate of convergence, that is, of order $k+1$, $k \geq 0$ in the discretizing parameter h, whereas super-convergence has been achieved, that is, of order $k+2$, $k \geq 1$, for the post-processed solution, when the function f was differentiable and its derivative was Lipschitz continuous. A fully discrete scheme has also been discussed, which is of order $O(h^{k+1} + \Delta t)$. Higher order fully discrete scheme can be easily achieved by using higher order difference scheme for the derivative term and higher order quadrature rule for the integral term. Finally, numerical results have been discussed. This analysis can be extended to a 3-dimensional domain by incorporating the changes accordingly.

Chapter 4

HDG Method for Linear Hyperbolic Integro-Differential Equations

4.1 Introduction

This chapter discusses the HDG method for the following linear hyperbolic integro-differential equation: Find $u(x,t)$ such that

$$u_{tt}(x,t) - \nabla \cdot \left(a(x)\nabla u(x,t) + \int_0^t b(x,t,s)\nabla u(x,s)ds \right) = f(x,t) \text{ in } \Omega \times (0,T], \tag{4.1a}$$

$$u(x,t) = 0 \text{ on } \partial\Omega \times (0,T], \tag{4.1b}$$

$$u(x,0) = u_0(x) \; \forall \, x \in \Omega, \tag{4.1c}$$

$$u_t(x,0) = u_1(x) \; \forall \, x \in \Omega, \tag{4.1d}$$

where $u : \Omega \times (0,T] \to \mathbb{R}$, The functions $f : \Omega \times (0,T] \to \mathbb{R}$, $u_0 : \Omega \to \mathbb{R}$ and $u_1 : \Omega \to \mathbb{R}$ are known. The known functions $a : \Omega \to \mathbb{R}$ and $b : \Omega \times (0,T] \times (0,T] \to \mathbb{R}$ satisfy the following properties: function a is positive and bounded. There exist $\alpha_0 > 0$, $M > 0$ such that $0 < \alpha_0 \leq a \leq M$, whereas, b is smooth and twice differentiable with bounded derivatives and $|b| \leq M$.

The type of hyperbolic integral-differential equation which is stated above occur in many physical problems, such as visco-elasticity, fluid dynamics, epidemiology and population dynamics; see [79] and references therein.

In the literature, Pani *et al.* [114] have analyzed fully discrete schemes for time-dependent partial integro-differential equations, using energy methods, paying attention to the storage required during time-stepping. Further, errors are estimated in L^2 and H^1 norms. In [123], Saedpanah has formulated continuous space-time FEM of degree one for an integro-differential equation of hyperbolic type with mixed boundary conditions. Further, a posteriori error estimates are also established. Then, in [124], a first-order continuous space–time FEM for a hyperbolic integro-differential equation has been formulated. Moreover, the theory is illustrated with the help of an example. In [79], Karaa *et al.* have applied DGM to (4.1). A priori error estimates are derived for both scalar as well as for vector variables, and the optimal rate of convergence is derived for the scalar variable and sub-optimal rate of convergence for vector variables. Later, in [94], Merad *et al.* proposed a Galerkin method based on least squares for a two-dimensional hyperbolic integro-differential equation with purely integral conditions. They have also discussed the existence and uniqueness of the solution of the model problem under specific conditions. In [26], Chen *et al.* have proposed a two-grid finite element scheme for a semi-linear hyperbolic integro-differential equation, which uses two grids to deal with the semi-linearity of the problem and achieves the same order of accuracy as that of the ordinary FEM. Recently, Tan *et al.* [134] have applied a fully discrete two-grid FEM on a hyperbolic integro-differential equation and achieved optimal order of convergence. The scheme has reduced the computational cost while maintaining numerical accuracy.

This chapter analyzes HDG method for the model problem (4.1) and discusses a priori error estimates. The most significant points of this chapter are as follows:

- In contrast to DGMs, optimal convergence rates have been obtained for gradient and trace.

- For the scalar variable, a new post-processed approximation has been defined, which achieves the super-convergence.
- Mid-point rule and central difference scheme are used to approximate the integral and the derivative term, respectively, to achieve second order of convergence in the temporal direction.
- The theoretical results are verified by implementing HDG method for problems in the 2-dimensional domain.

The remainder of this chapter is structured as follows: Section 4.2 defines HDG formulation for the model problem (4.1). Section 4.3 provides the highlights of the chapter by stating all the essential results of the chapter. In Section 4.4, a priori error estimates are derived using various crucial steps. In Section 4.5, the superconvergence results for the scalar variable are analyzed. Section 4.6 is about discretizing the scheme in temporal direction. Ultimately, numerical data are presented in Section 4.7 to validate the theoretical conclusions. Several final observations are included in Section 4.8.

4.2 HDG Method

Throughout the chapter, we have used the following auxiliary variable in $\Omega \times (0, T]$:

$$\boldsymbol{q} = -\nabla u, \quad \boldsymbol{\sigma} = a\boldsymbol{q} + \int_0^t b(t,s)\boldsymbol{q}(s)ds.$$

Using these variables, (4.1) is rewritten as follows:

$$\boldsymbol{q} = -\nabla u \qquad \text{in } \Omega \times (0, T], \tag{4.2a}$$

$$\boldsymbol{\sigma} = a\boldsymbol{q} + \int_0^t b(t,s)\boldsymbol{q}(s)ds \qquad \text{in } \Omega \times (0, T], \tag{4.2b}$$

$$u_{tt} + \nabla \cdot \boldsymbol{\sigma} = f \qquad \text{in } \Omega \times (0, T]. \tag{4.2c}$$

At each time t within the interval $(0, T]$, the method provides an approximation $u_h(t)$ of the scalar function $u(t)$, an approximation $\boldsymbol{q}_h(t)$ and $\boldsymbol{\sigma}_h(t)$ of the vector function $\boldsymbol{q}(t)$ and $\boldsymbol{\sigma}(t)$, respectively, and an approximation $\hat{u}_h(t)$ of the trace of $u(t)$ on the boundaries of the elements. These approximations are computed in the function spaces V_h, $\boldsymbol{W_h}$, $\boldsymbol{W_h}$, and M_h, respectively. With these spaces, HDG formulation seeks approximation $(u_h, \boldsymbol{q}_h, \boldsymbol{\sigma}_h, \hat{u}_h)(t) \in (V_h \times \boldsymbol{W}_h \times \boldsymbol{W}_h \times M_h)$, for $t \in (0, T]$, that satisfy the following equations:

$$(\boldsymbol{q}_h, \boldsymbol{w}_h) - (u_h, \nabla \cdot \boldsymbol{w}_h) + \langle \hat{u}_h, \boldsymbol{w}_h \cdot \boldsymbol{\nu} \rangle_{\partial \mathcal{T}_h} = 0, \tag{4.3a}$$

$$(a\boldsymbol{q}_h, \boldsymbol{\tau}_h) - (\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + \int_0^t (b(t,s)\boldsymbol{q}_h(s), \boldsymbol{\tau}_h) ds = 0, \tag{4.3b}$$

$$(u_{h_{tt}}, v_h) - (\boldsymbol{\sigma}_h, \nabla v_h) + \langle \hat{\boldsymbol{\sigma}}_h \cdot \boldsymbol{\nu}, v_h \rangle_{\partial \mathcal{T}_h} = (f, v_h), \tag{4.3c}$$

$$\langle \hat{u}_h, \mu_h \rangle_{\partial \Omega} = 0, \tag{4.3d}$$

$$\langle \hat{\boldsymbol{\sigma}}_h \cdot \boldsymbol{\nu}, m_h \rangle_{\partial \mathcal{T}_h \backslash \partial \Omega} = 0, \tag{4.3e}$$

$$u_h(0) = \Pi_V u_0, \tag{4.3f}$$

$$u_{h_t}(0) = \Pi_V u_1, \tag{4.3g}$$

for any $(v_h, \boldsymbol{w}_h, \boldsymbol{\tau}_h, \mu_h, m_h) \in (V_h \times \boldsymbol{W}_h \times \boldsymbol{W}_h \times M_h \times M_h)$, along with the following relation:

$$\hat{\boldsymbol{\sigma}}_h \cdot \boldsymbol{\nu} = \boldsymbol{\sigma}_h \cdot \boldsymbol{\nu} + \tau(u_h - \hat{u}_h) \text{ on } \partial \mathcal{T}_h,$$

where, $\tau \geq 0$ on Γ and piece-wise constant on the faces. Now, with the help of (4.2) and (4.3), we have the following error equations:

$$(\boldsymbol{q} - \boldsymbol{q}_h, \boldsymbol{w}_h) - (u - u_h, \nabla \cdot \boldsymbol{w}_h) + \langle u - \hat{u}_h, \boldsymbol{w}_h \cdot \boldsymbol{\nu} \rangle_{\partial \mathcal{T}_h} = 0, \tag{4.4a}$$

$$(a(\boldsymbol{q} - \boldsymbol{q}_h), \boldsymbol{\tau}_h) - (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + \int_0^t (b(t,s)(\boldsymbol{q} - \boldsymbol{q}_h)(s), \boldsymbol{\tau}_h) ds = 0, \tag{4.4b}$$

$$(u_{tt} - u_{h_{tt}}, v_h) - (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \nabla v_h) + \langle (\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}_h) \cdot \boldsymbol{\nu}, v_h \rangle_{\partial \mathcal{T}_h} = 0, \tag{4.4c}$$

$$\langle u - \hat{u}_h, \mu_h \rangle_{\partial \Omega} = 0, \tag{4.4d}$$

$$\langle (\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}_h) \cdot \boldsymbol{\nu}, m_h \rangle_{\partial \mathcal{T}_h \backslash \partial \Omega} = 0, \tag{4.4e}$$

for all $(v_h, \boldsymbol{w}_h, \boldsymbol{\tau}_h, \mu_h, m_h) \in (V_h \times \boldsymbol{W}_h \times \boldsymbol{W}_h \times M_h \times M_h)$.

4.3 The Main Results

Within this part, we provide the primary outcomes of the chapter in the form of the following theorems:

Theorem 4.3.1. *Let* $(u, \boldsymbol{q}, \boldsymbol{\sigma})$ *be the solution of* (4.2) *with* $u \in L^\infty(H^{k+2}(\mathcal{T}_h))$, $u_t, u_{tt} \in L^2(H^{k+2}(\mathcal{T}_h))$ *and* $u_0, u_1 \in H^{k+2}(\mathcal{T}_h)$ *for* $k \geq 0$. *Additionally, let* $(u_h, \boldsymbol{q}_h, \boldsymbol{\sigma}_h, \hat{u}_h) \in (V_h \times \boldsymbol{W}_h \times \boldsymbol{W}_h \times M_h)$ *be the solution of* (4.3) *along with* $u_h(0) = \Pi_V u_0$, $u_{h_t}(0) = \Pi_V u_1$, $\boldsymbol{q}_h(0) = -\boldsymbol{I}_h^k \nabla u_0$ *and* $\boldsymbol{\sigma}_h(0) = \boldsymbol{\Pi}_{\boldsymbol{W}}(a\nabla u_0)$. *Consequently, the following estimations hold:*

$$\|(u - u_h)(t)\| + \|(\boldsymbol{q} - \boldsymbol{q}_h)(t)\| + \|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)(t)\| + \|(u - \hat{u}_h)(t)\|_\tau \leq Ch^{k+1},$$
$$\|(u_t - u_{h_t})(t)\| \leq Ch^{k+1}.$$

For the next result, we define the post- processed solution $u_h^* \in P_{k+1}(K)$ on the element K, as

$$u_h^* = u_h^p + \frac{1}{|K|}\int_K u_h, \;\; u_h^p \in P_{k+1}^0, \tag{4.5}$$

where u_h^p satisfies

$$(a\nabla u_h^p, \nabla v) = -(a\boldsymbol{q}_h, \nabla v), \;\; \forall v \in P_{k+1}^0, \tag{4.6}$$

where $P_{k+1}^0(K)$ represents the collection of polynomials in $P_{k+1}(K)$ whose average is zero. The next theorem gives the L^2 estimates of u_h^*.

Theorem 4.3.2. *Under the conditions of Theorem 4.3.1, there exists a positive constant* C *independent of* h *and* k *such that*

$$\|u - u_h^*\| \leq Ch^{k+2}.$$

Theorem 4.3.3. *Let* $\Delta t = \dfrac{T}{M}$, *for some positive integer* M, *and* $t_n = n\Delta t$, *for* $1 \leq n \leq M$. *Let* $(U^n, \boldsymbol{Q}^n, \boldsymbol{S}^n) \in (V_h \times \boldsymbol{W}_h \times \boldsymbol{W}_h)$ *be the fully discrete approximations of* u, $\boldsymbol{q}$ *and* $\boldsymbol{\sigma}$ *and* $\hat{U}^n \in M_h$ *be the approximation of* u *on* Γ. *Then, we have the following estimates:*

$$\|\partial_t \Upsilon \zeta_u^n\| + \|\Upsilon \boldsymbol{\zeta}_{\boldsymbol{q}}^n\| + \|\Upsilon \boldsymbol{\zeta}_{\boldsymbol{\sigma}}^n\| + \|\Upsilon \hat{\zeta}_u^n\|_\tau \leq O(h^{k+1} + \Delta t^2), \tag{4.7a}$$

$$\|\zeta_u^{n+1}\| \leq O(h^{k+1} + \Delta t^2), \tag{4.7b}$$

where, $\zeta' s$ are defined as: $\zeta_u^n = u_h(t_n) - U^n$. Similarly, $\boldsymbol{\zeta}_{\boldsymbol{q}}^n$, $\boldsymbol{\zeta}_{\boldsymbol{\sigma}}^n$ and $\hat{\zeta}_u^n$. Also, $\Upsilon U^n = \dfrac{U^{n+1} + U^n}{2}$ and $\partial_t \Upsilon U^n = \dfrac{U^{n+1} - U^n}{\Delta t}$, see Section 4.6.

4.4 Semi-Discrete Error Analysis

In this part, we provide comprehensive demonstrations of the assertions made in Theorem 4.3.1.

STEP I: Extended type Ritz-Voterra projection. For each $t \in (0, T]$, we define $(\tilde{u}_h, \tilde{\boldsymbol{q}}_h, \tilde{\boldsymbol{\sigma}}_h, \hat{\tilde{u}}_h) \in (V_h \times \boldsymbol{W}_h \times \boldsymbol{W}_h \times M_h)$ as the Ritz-Volterra projection, provided it satisfy the following equations:

$$(\boldsymbol{q} - \tilde{\boldsymbol{q}}_h, \boldsymbol{w}_h) - (u - \tilde{u}_h, \nabla \cdot \boldsymbol{w}_h) + \langle u - \hat{\tilde{u}}_h, \boldsymbol{w}_h \cdot \boldsymbol{\nu} \rangle_{\partial \mathcal{T}_h} = 0, \tag{4.8a}$$

$$(a(\boldsymbol{q} - \tilde{\boldsymbol{q}}_h), \boldsymbol{\tau}_h) - (\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h, \boldsymbol{\tau}_h) + \int_0^t (b(t,s)(\boldsymbol{q} - \tilde{\boldsymbol{q}}_h)(s), \boldsymbol{\tau}_h) ds = 0, \tag{4.8b}$$

$$-(\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h, \nabla v_h) + \langle (\boldsymbol{\sigma} - \hat{\tilde{\boldsymbol{\sigma}}}_h) \cdot \boldsymbol{\nu}, v_h \rangle_{\partial \mathcal{T}_h} = 0, \tag{4.8c}$$

$$\langle u - \hat{\tilde{u}}_h, \mu_h \rangle_{\partial \Omega} = 0, \tag{4.8d}$$

$$\langle (\boldsymbol{\sigma} - \hat{\tilde{\boldsymbol{\sigma}}}_h) \cdot \boldsymbol{\nu}, m_h \rangle_{\partial \mathcal{T}_h \setminus \partial \Omega} = 0, \tag{4.8e}$$

for all $(v_h, \boldsymbol{w}_h, \boldsymbol{\tau}_h, \mu_h, m_h) \in V_h \times \boldsymbol{W}_h \times \boldsymbol{W}_h \times M_h$, where

$$\hat{\tilde{\boldsymbol{\sigma}}}_h \cdot \boldsymbol{\nu} = \tilde{\boldsymbol{\sigma}}_h \cdot \boldsymbol{\nu} + \tau(\tilde{u}_h - \hat{\tilde{u}}_h) \text{ on } \partial \mathcal{T}_h.$$

In order to derive the estimates of the Ritz-Volterra projection, we disintegrate it in θ's and ρ's as done in Chapter 2.

Therefore, the system of equations (4.8) can be rewritten as

$$(\boldsymbol{\rho_q}, \boldsymbol{w_h}) - (\rho_u, \nabla \cdot \boldsymbol{w_h}) + \langle \hat{\rho}_u, \boldsymbol{w_h} \cdot \boldsymbol{\nu} \rangle_{\partial \mathcal{T}_h} = 0, \tag{4.9a}$$

$$(a\boldsymbol{\rho_q}, \boldsymbol{\tau_h}) - (\boldsymbol{\rho_\sigma}, \boldsymbol{\tau}_h) + \int_0^t (b(t,s)\boldsymbol{\rho_q}(s), \boldsymbol{\tau}_h) ds = (a\boldsymbol{\theta_q}, \boldsymbol{\tau}_h) - (\boldsymbol{\theta_\sigma}, \boldsymbol{\tau}_h) + \int_0^t (b(t,s)\boldsymbol{\theta_q}(s), \boldsymbol{\tau}_h) ds, \tag{4.9b}$$

$$-(\boldsymbol{\rho_\sigma}, \nabla v_h) + \langle \hat{\boldsymbol{\rho}}_{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu}, v_h \rangle_{\partial \mathcal{T}_h} = 0, \tag{4.9c}$$

$$\langle \hat{\rho}_u, \mu_h \rangle_{\partial \Omega} = 0, \tag{4.9d}$$

$$\langle \hat{\boldsymbol{\rho}}_{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu}, m_h \rangle_{\partial \mathcal{T}_h \setminus \partial \Omega} = 0, \tag{4.9e}$$

for all $(v_h, \boldsymbol{w}_h, \boldsymbol{\tau}_h, \mu_h, m_h) \in (V_h \times \boldsymbol{W}_h \times \boldsymbol{W}_h \times M_h)$.

STEP II: Estimates of $\left\|\frac{\partial^l \boldsymbol{\rho_\sigma}}{\partial t^l}\right\|$, $\left\|\frac{\partial^l \boldsymbol{\rho_q}}{\partial t^l}\right\|$ and $\left\|\frac{\partial^l \rho_u}{\partial t^l}\right\|$ for $l = 0, 1, 2$.

Lemma 4.4.1. *There exists a positive constant C which does not rely on h and k such that $\forall t \in (0, T]$, the inequality below is valid for $l = 0, 1, 2$*

$$\left\|\frac{\partial^l \rho_u}{\partial t^l}\right\| + \left\|\frac{\partial^l \boldsymbol{\rho_q}}{\partial t^l}\right\| + \left\|\frac{\partial^l \boldsymbol{\rho_\sigma}}{\partial t^l}\right\| \leq C h^{k+1}. \tag{4.10}$$

Proof. For $l = 0, 1$, see Chapter 2. For $l = 2$, we begin by differentiating (4.9a)-(4.9e) twice w.r.t. t, to obtain

$$(\boldsymbol{\rho_{q_{tt}}}, \boldsymbol{w_h}) - (\rho_{u_{tt}}, \nabla \cdot \boldsymbol{w_h}) + \langle \hat{\rho}_{u_{tt}}, \boldsymbol{w_h} \cdot \boldsymbol{\nu} \rangle_{\partial \mathcal{T}_h} = 0, \tag{4.11a}$$

$$\begin{aligned}(a\boldsymbol{\rho_{q_{tt}}}, \boldsymbol{\tau_h}) - (\boldsymbol{\rho_{\sigma_{tt}}}, \boldsymbol{\tau_h}) + \frac{\partial^2}{\partial t^2}\left(\int_0^t (b(t,s)\boldsymbol{\rho_q}(s), \boldsymbol{\tau}_h) ds\right) &= (a\boldsymbol{\theta_{q_{tt}}}, \boldsymbol{\tau}_h) - (\boldsymbol{\theta_{\sigma_{tt}}}, \boldsymbol{\tau}_h) \\ &\quad + \frac{\partial^2}{\partial t^2}\left(\int_0^t (b(t,s)\boldsymbol{\theta_q}(s), \boldsymbol{\tau}_h) ds\right),\end{aligned} \tag{4.11b}$$

$$-(\boldsymbol{\rho_{\sigma_{tt}}}, \nabla v_h) + \langle \boldsymbol{\rho_{\sigma_{tt}}} \cdot \boldsymbol{\nu}, v_h \rangle_{\partial \mathcal{T}_h} = 0, \tag{4.11c}$$

$$\langle \hat{\rho}_{u_{tt}}, \mu_h \rangle_{\partial \Omega} = 0, \tag{4.11d}$$

$$\langle \hat{\boldsymbol{\rho}}_{\boldsymbol{\sigma_{tt}}} \cdot \boldsymbol{\nu}, m_h \rangle_{\partial \mathcal{T}_h \setminus \partial \mathcal{T}_h} = 0, \tag{4.11e}$$

for all $(v_h, \boldsymbol{w}_h, \boldsymbol{\tau}_h, \mu_h, m_h) \in (V_h \times \boldsymbol{W}_h \times \boldsymbol{W}_h \times M_h \times M_h)$.

Now, adding (4.11) after taking $\boldsymbol{w}_h = \boldsymbol{\rho_{\sigma_{tt}}}$, $\boldsymbol{\tau}_h = \boldsymbol{\rho_{q_{tt}}}$, $v_h = \rho_{u_{tt}}$, $\mu_h = -\hat{\boldsymbol{\rho}}_{\boldsymbol{\sigma_{tt}}} \cdot \boldsymbol{\nu}$ and $m_h = -\hat{\rho}_{u_{tt}}$ and simplifying using Cauchy Schwarz inequality and the Gronwall's lemma, will give the estimate of $\|\boldsymbol{\rho_{q_{tt}}}\|$, whereas taking $\boldsymbol{\tau}_h = \boldsymbol{\rho_{\sigma_{tt}}}$ in (4.11b) gives the estimate of $\|\boldsymbol{\rho_{\sigma_{tt}}}\|$.

For the estimate of $\|\rho_{u_{tt}}\|$, we begin by taking into account the following dual problem:

$$\boldsymbol{\phi} = -\nabla \psi \qquad \text{in } \Omega, \tag{4.12a}$$

$$\boldsymbol{p} = a\boldsymbol{\phi} \qquad \text{in } \Omega, \tag{4.12b}$$

$$\nabla \cdot \boldsymbol{p} = \rho_{u_{tt}} \qquad \text{in } \Omega, \tag{4.12c}$$

$$\psi = 0 \qquad \text{on } \partial\Omega, \tag{4.12d}$$

along with:

$$\|\psi\|_{H^2(\Omega)} \leq \|\rho_{u_{tt}}\|. \tag{4.13}$$

Consider,

$$\|\rho_{u_{tt}}\|^2 = (\rho_{u_{tt}}, \nabla \cdot \boldsymbol{p}).$$

Lastly, we make use of (4.11), (4.12), (1.2) and (4.13) along with the estimates of $\|\boldsymbol{\rho_{q_{tt}}}\|$ and $\|\boldsymbol{\rho_{\sigma_{tt}}}\|$ to finish the proof. □

Remark: The order of convergence of $\|\rho_u\|$ can be further increased to $k + 3/2$, using dual norm estimates.

STEP III: Estimates of Ritz-Volterra projection.

Theorem 4.4.2. *For $t \in (0, T]$, if $u \in L^\infty(H^{k+2}(\mathcal{T}_h))$, $u_t, u_{tt} \in L^2(H^{k+2}(\mathcal{T}_h))$ and $l = 0, 1, 2$, then there is a positive constant C which does not rely on h and k such that*

$$\left\|\frac{\partial^l \eta_u}{\partial t^l}\right\| + \left\|\frac{\partial^l \boldsymbol{\eta_q}}{\partial t^l}\right\| + \left\|\frac{\partial^l \boldsymbol{\eta_\sigma}}{\partial t^l}\right\| \leq Ch^{k+1}, \tag{4.14}$$

$$\left\|I_h^{k-1}\left(\frac{\partial^l \eta_u}{\partial t^l}\right)\right\| \leq Ch^{k+2}. \tag{4.15}$$

Proof. The inequality (4.14) can be obtained with the help of (1.2), Lemma 4.4.1 and the triangle inequality.

For the estimates of $\|I_h^{k-1}\eta_u\|$, the following dual problem is considered in $\Omega \times (0, T]$

$$\boldsymbol{\phi} = -\nabla\psi,$$

$$\boldsymbol{p} = a\boldsymbol{\phi},$$

$$\nabla \cdot \boldsymbol{p} = \theta,$$

which satisfies the elliptic regularity

$$\|\psi\|_{H^2(\Omega)} \leq \|\theta\|.$$

Now, using (4.8a) and proceeding as in [39], concludes the proof for $k \geq 1$.

$$\begin{aligned}
(I_h^{k-1}\eta_u, \theta) &= (I_h^{k-1}\eta_u, \nabla \cdot \boldsymbol{p}) \\
&= (\eta_u, \nabla \cdot \boldsymbol{\Pi}_{k-1}^{RT}\boldsymbol{p}) \\
&= (\boldsymbol{\eta_q}, \nabla \cdot (\boldsymbol{\Pi}_{k-1}^{RT}\boldsymbol{p} - \boldsymbol{p})) - (a\boldsymbol{\eta_q}, \nabla\psi) \\
&\leq Ch^{k+2}\|\theta\|.
\end{aligned}$$

Similar procedure can be followed for $l = 1, 2$. □

STEP IV: Estimates of $\|\xi_u\|$, $\|\boldsymbol{\xi_q}\|$ and $\|\boldsymbol{\xi_\sigma}\|$

In order to derive the error estimates, we disintegrate them in the following form

$$e_u = u - u_h = (u - \tilde{u}_h) - (u_h - \tilde{u}_h) = \eta_u - \xi_u.$$

Similarly for $\boldsymbol{e_q}$, $\boldsymbol{e_\sigma}$, $\hat{e}_u$ and $\hat{\boldsymbol{e}}_{\boldsymbol{\sigma}}$. Hence, (4.4) can be written as

$$(\boldsymbol{\xi_q}, \boldsymbol{w_h}) - (\xi_u, \nabla \cdot \boldsymbol{w_h}) + \langle \hat{\xi}_u, \boldsymbol{w_h} \cdot \boldsymbol{\nu} \rangle_{\partial\mathcal{T}_h} = 0 \qquad \forall \boldsymbol{w}_h \in \boldsymbol{W}_h, \tag{4.16a}$$

$$(a\boldsymbol{\xi_q}, \boldsymbol{\tau_h}) - (\boldsymbol{\xi_\sigma}, \boldsymbol{\tau}_h) + \int_0^t (b(t,s)\boldsymbol{\xi_q}(s), \boldsymbol{\tau}_h)ds = 0 \qquad \forall \boldsymbol{\tau}_h \in \boldsymbol{W}_h, \tag{4.16b}$$

$$(\xi_{u_{tt}}, v_h) - (\boldsymbol{\xi_\sigma}, \nabla v_h) + \langle \hat{\boldsymbol{\xi}}_{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu}, v_h \rangle_{\partial\mathcal{T}_h} = (\eta_{u_{tt}}, v_h) \qquad \forall v_h \in V_h, \tag{4.16c}$$

$$\langle \hat{\xi}_u, \mu_h \rangle_{\partial\Omega} = 0 \qquad \forall \mu_h \in M_h, \tag{4.16d}$$

$$\langle \hat{\boldsymbol{\xi}}_{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu}, m_h \rangle_{\partial\mathcal{T}_h \setminus \partial\Omega} = 0 \qquad \forall m_h \in M_h. \tag{4.16e}$$

For any function w in $[0, t]$, let us define $\bar{w}$ as:

$$\bar{w}(t) = \int_0^t w(s)ds.$$

Clearly, $\bar{w}_t = w$ and $\bar{w}(0) = 0$.

Lemma 4.4.3. *There exists a positive constant C which does not rely on h and k such that $\forall t \in (0,T]$, the inequality below is valid*

$$\|\xi_u(t)\|^2 + \|\bar{\boldsymbol{\xi}}_q(t)\|^2 + \|(\bar{\hat{\xi}}_u - \bar{\xi}_u)(t)\|_\tau^2 \leq C\Big(\|\xi_u(0)\|^2 + \|a^{1/2}\bar{\boldsymbol{\xi}}_q(0)\|^2 + \|\bar{\hat{\xi}}_u(0) - \bar{\xi}_u(0)\|_\tau^2 + \int_0^T \|\eta_{u_t}(t)\|^2 dt\Big).$$

Proof. We integrate (4.16b), (4.16c) and (4.16e) from 0 to t and then, choose $\boldsymbol{w}_h = \bar{\boldsymbol{\xi}}_{\boldsymbol{\sigma}}$, $\boldsymbol{\tau}_h = \boldsymbol{\xi}_{\boldsymbol{q}}$, $v_h = \xi_u$, $\mu_h = -\bar{\hat{\boldsymbol{\xi}}}_{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu}$ and $m_h = -\hat{\xi}_u$ in (4.16) and add them, to obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|a^{1/2}\bar{\boldsymbol{\xi}}_q\|^2 + \|\xi_u\|^2 + \|\bar{\hat{\xi}}_u - \bar{\xi}_u\|_\tau^2\right) = (\eta_{u_t}, \xi_u) - \int_0^t \int_0^s (b(s,\gamma)\boldsymbol{\xi}_{\boldsymbol{q}}(\gamma), \boldsymbol{\xi}_{\boldsymbol{q}}(s)) d\gamma ds.$$

It follows from integrating aforementioned inequality

$$\|a^{1/2}\bar{\boldsymbol{\xi}}_q\|^2 + \|\xi_u\|^2 + \|\bar{\hat{\xi}}_u - \bar{\xi}_u\|_\tau^2 \leq \|\xi_u(0)\|^2 + \|a^{1/2}\bar{\boldsymbol{\xi}}_q(0)\|^2 + \|\bar{\hat{\xi}}_u(0) - \bar{\xi}_u(0)\|_\tau^2 + 2\int_0^t (\eta_{u_s}, \xi_u) ds - 2\int_0^t \int_0^s \int_0^\gamma (b(\gamma,\gamma^*)\boldsymbol{\xi}_{\boldsymbol{q}}(\gamma^*), \boldsymbol{\xi}_{\boldsymbol{q}}(\gamma)) d\gamma^* d\gamma ds.$$

Let the last term on the right-hand side of the above equation be denoted by I, then we have

$$I = 2\int_0^t \int_0^s (b(\gamma,\gamma)\bar{\boldsymbol{\xi}}_{\boldsymbol{q}}(\gamma), \boldsymbol{\xi}_{\boldsymbol{q}}(s)) d\gamma ds - 2\int_0^t \int_0^s \int_0^\gamma (b_{\gamma^*}(\gamma,\gamma^*)\bar{\boldsymbol{\xi}}_{\boldsymbol{q}}(\gamma^*), \boldsymbol{\xi}_{\boldsymbol{q}}(s)) d\gamma^* d\gamma ds.$$

Applying integration by parts again, on both the terms of the above equation and then, using Cauchy Schwarz inequality we arrive at

$$I \leq C\left(\|\bar{\boldsymbol{\xi}}_{\boldsymbol{q}}(t)\| \int_0^t \|\bar{\boldsymbol{\xi}}_{\boldsymbol{q}}(s)\| ds + \int_0^t \|\bar{\boldsymbol{\xi}}_{\boldsymbol{q}}(s)\|^2 ds\right).$$

Lastly, we use Young's inequality and Gronwall's lemma to finish the proof. □

Lemma 4.4.4. *There exists a positive constant C which does not rely on h and k such that $\forall t \in (0,T]$, the inequality below is valid*

$$\|\xi_{u_t}\|^2 + \|\boldsymbol{\xi}_{\boldsymbol{\sigma}}(t)\|^2 + \|\boldsymbol{\xi}_{\boldsymbol{q}}(t)\|^2 + \|\hat{\xi}_u - \xi_u\|_\tau^2 \leq C\Big(\|\boldsymbol{\xi}_{\boldsymbol{q}}(0)\|^2 + \|\xi_{u_t}(0)\|^2 + \|\hat{\xi}_u(0) - \xi_u(0)\|_\tau^2 + \int_0^T \|\eta_{u_{tt}}(t)\|^2 dt\Big).$$

Proof. Firstly, we differentiate (4.16a) with respect to t and then select $\boldsymbol{w_h} = \boldsymbol{\xi_\sigma}$, $\boldsymbol{\tau_h} = \boldsymbol{\xi_{q_t}}$, $v_h = \xi_{u_t}$ in (4.16a), (4.16b), (4.16c), respectively. Then, we differentiate (4.16d) and select $\mu_h = -\hat{\boldsymbol{\xi}}_{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu}$ and $m_h = -\hat{\xi}_{u_t}$ in (4.16d) and (4.16e), respectively. Finally, by combining the ensuing equations, we have

$$(a\boldsymbol{\xi_q}, \boldsymbol{\xi_{q_t}}) + (\xi_{u_{tt}}, \xi_{u_t}) + \frac{1}{2}\frac{d}{dt}\|\hat{\xi}_u - \xi_u\|_\tau^2 + \int_0^t (b(t,s)\boldsymbol{\xi_q}(s), \boldsymbol{\xi_{q_t}}(t))ds = (\eta_{u_{tt}}, \xi_{u_t}).$$

Application of Leibnitz's theorem shows,

$$\frac{1}{2}\frac{d}{dt}\left(\|a^{1/2}\boldsymbol{\xi_q}\|^2 + \|\xi_{u_t}\|^2 + \|\hat{\xi}_u - \xi_u\|_\tau^2\right) = (\eta_{u_{tt}}, \xi_{u_t}) + \frac{d}{dt}\int_0^t (b(t,s)\boldsymbol{\xi_q}(s), \boldsymbol{\xi_q}(t))ds$$
$$- (b(t,t)\boldsymbol{\xi_q}(t), \boldsymbol{\xi_q}(t)) - \int_0^t (b_t(t,s)\boldsymbol{\xi_q}(s), \boldsymbol{\xi_q}(t))ds.$$

Integrating the aforementioned inequality from 0 to t yields

$$\|a^{1/2}\boldsymbol{\xi_q}\|^2 + \|\xi_{u_t}\|^2 \le \|a^{1/2}\boldsymbol{\xi_q}(0)\| + \|\xi_{u_t}(0)\|^2 + \|\xi_{u_t}(0)\|^2 + \|\hat{\xi}_u(0) - \xi_u(0)\|_\tau^2 + \int_0^t (\eta_{u_{ss}}, \xi_{u_s})ds$$
$$+ \int_0^t (b(t,s)\boldsymbol{\xi_q}(s), \boldsymbol{\xi_q}(t))ds - \int_0^t (b(s,s)\boldsymbol{\xi_q}(s), \boldsymbol{\xi_q}(s))ds - \int_0^t \int_0^s (b_s(s,\gamma)\boldsymbol{\xi_q}(\gamma), \boldsymbol{\xi_q}(s))d\gamma ds.$$

The following estimate is obtained using Gronwall's theorem and the assumptions on a and b:

$$\|\boldsymbol{\xi_q}\|^2 + \|\xi_{u_t}\|^2 \le C\left(\|\boldsymbol{\xi_q}(0)\|^2 + \|\xi_{u_t}(0)\|^2 + \|\hat{\xi}_u(0) - \xi_u(0)\|_\tau^2 + \int_0^T \|\eta_{u_{tt}}(t)\|^2 dt\right). \tag{4.17}$$

Now, choosing $\boldsymbol{\tau_h} = \boldsymbol{\xi_\sigma}$ in (4.16b) yields

$$\|\boldsymbol{\xi_\sigma}\| \le C\left(\|\boldsymbol{\xi_q}\| + \int_0^t \|\boldsymbol{\xi_q}(s)\|ds\right). \tag{4.18}$$

Combining (4.17) and (4.18) will finish the proof. □

Proof of Theorem 4.3.1: If we chose $u_h(0) = \tilde{u}_h(0) = \Pi_v u_0$, $\boldsymbol{q}_h(0) = \tilde{\boldsymbol{q}}_h(0) = -\boldsymbol{I}_h^k \nabla u_0$ and $\boldsymbol{\sigma}_h(0) = \tilde{\boldsymbol{\sigma}}_h(0) = \boldsymbol{\Pi}_{\mathbf{W}}(a\nabla u_0)$, then triangle inequality, Lemma 4.4.2, Lemma 4.4.3 along with Lemma 4.4.4 yield the desired result. □

4.5 Post-processing

To begin with, we define the function $\psi(s) \in H^2(\Omega) \cap H_0^1(\Omega),\ s \leq t$ to be the solution of the following problem:

$$\psi_{ss} - \nabla \cdot \left(a(x)\nabla\psi + \int_s^t b(\gamma, s)\nabla\psi(\gamma)d\gamma \right) = 0, \tag{4.19}$$

with the following final and boundary conditions:

$$\begin{aligned} \psi(x,s) &= 0 && \text{on } \partial\Omega, s \leq t, \\ \psi(x,t) &= 0 && \text{in } x \in \Omega, \\ \psi_s(x,t) &= \lambda(x) && \text{in } x \in \Omega. \end{aligned}$$

Lemma 4.5.1. *(Regularity Results) There exists a constant C dependent on the data of the above problem, such that it satisfies the following inequality:*

$$\|\psi(s)\|_{L^\infty(H^1)} + \|\psi_s(s)\|_{L^\infty(L^2)} \leq C\|\lambda\|, \tag{4.20a}$$

$$\|\underline{\psi}(s)\|_2 \leq C\|\lambda\|, \tag{4.20b}$$

where, $\underline{\psi}(s) = \int_s^t \psi(\gamma)d\gamma$.

Proof. The first inequality can be proved using a simple kickback argument [42]. To prove the second inequality, we begin by integrating (4.19) from s to t, noting that $-\psi_s(s) = \underline{\psi}_{ss}(s)$ and using the boundary condition, to obtain

$$\underline{\psi}_{ss}(s) - \nabla \cdot \left(a(x)\nabla\underline{\psi} + \int_s^t \int_\gamma^t b(\gamma^*, \gamma)\nabla\psi(\gamma^*)d\gamma^* d\gamma \right) = -\lambda.$$

Next, we assume the following elliptic regularity on $\underline{\psi}$ [42], and use (4.20a) to get

$$\begin{aligned} \|\underline{\psi}\|_2 &\leq C \left\| \nabla \cdot \left(a(x)\nabla\underline{\psi} \right) \right\| \\ &\leq C \left(\|\underline{\psi}_{ss}(s)\| + \|\lambda\| + \left\| \int_s^t \int_\gamma^t b(\gamma^*, \gamma)\nabla\psi(\gamma^*)d\gamma^* d\gamma \right\| \right) \\ &\leq C \left(\|\psi_s(s)\| + \|\lambda\| + \left\| \int_s^t \int_\gamma^t b(\gamma^*, \gamma)\nabla\psi(\gamma^*)d\gamma^* d\gamma \right\| \right) \\ &\leq C \|\lambda\| . \end{aligned}$$

□

Lemma 4.5.2. *For the method represented by (4.3), there exists a positive constant C that remains independent of both h and k, such that for all $t \in (0,T]$, the following inequality holds:*

$$\|I_h^{k-1} e_u\|_{L^2(K)} \leq C h^{k+2}, \tag{4.21}$$

where, I_h^{k-1} is L^2-projection onto the space of polynomial for degree at most $k-1$.

Proof. Since, $e_u = \eta_u - \xi_u$, therefore, $\|I_h^{k-1} e_u\| \leq \|I_h^{k-1}\eta_u\| + \|I_h^{k-1}\xi_u\|$.
For the estimates of $\|I_h^{k-1}\xi_u\|$, we start by rewriting (4.19) in the following mixed form:

$$\boldsymbol{\phi}(s) = \nabla\psi(s) \quad \text{in } \Omega,\ s \leq t, \tag{4.22a}$$

$$\boldsymbol{p}(s) = a\boldsymbol{\phi}(s) + \int_s^t b(\gamma, s)\boldsymbol{\phi}(\gamma)d\gamma \quad \text{in } \Omega,\ s \leq t, \tag{4.22b}$$

$$\psi_{ss}(s) - \nabla \cdot \boldsymbol{p}(s) = 0 \quad \text{in } \Omega,\ s \leq t, \tag{4.22c}$$

$$\psi(s) = 0 \quad \text{on } \partial\Omega,\ s \leq t, \tag{4.22d}$$

$$\psi(t) = 0 \quad \text{in } \Omega, \tag{4.22e}$$

$$\psi_s(t) = I_h^{k-1}\xi_u(t) \quad \text{in } \Omega. \tag{4.22f}$$

We begin by taking the inner product of (4.22c) with $I_h^{k-1}\xi_u(s)$, to obtain

$$(\psi_{ss}(s), I_h^{k-1}\xi_u(s)) - (\nabla \cdot \boldsymbol{p}(s), I_h^{k-1}\xi_u(s)) = 0.$$

Now,

$$\begin{aligned}\frac{d}{ds}\left[(\psi_s(s), I_h^{k-1}\xi_u(s)) - (\psi(s), I_h^{k-1}\xi_{u_s}(s))\right] &= (\psi_{ss}(s), I_h^{k-1}\xi_u(s)) - (\psi(s), I_h^{k-1}\xi_{u_{ss}}(s)) \\ &= -(\psi(s), I_h^{k-1}\xi_{u_{ss}}(s)) + (\nabla \cdot \boldsymbol{p}(s), I_h^{k-1}\xi_u(s)).\end{aligned}$$

Use of (4.16) and intermediate projections, see [139], yield the following equality

$$\begin{aligned}&\frac{d}{ds}\left[(\psi_s(s), I_h^{k-1}\xi_u(s)) - (\psi(s), I_h^{k-1}\xi_{u_s}(s))\right] = (\xi_{u_{ss}}(s), I_h^{k}\psi(s) - \psi(s)) \\ &- (\xi_{u_{ss}}(s), I_h^{k-1}\psi(s) - \psi(s)) + (\boldsymbol{\xi_q}(s), \boldsymbol{\Pi}_{k-1}^{RT}\boldsymbol{p}(s) - \boldsymbol{p}(s)) + (a\boldsymbol{\xi_q}(s), \boldsymbol{\phi}(s) - \boldsymbol{I}_h^k\boldsymbol{\phi}(s)) \\ &+ (\boldsymbol{\xi_\sigma}(s), \boldsymbol{I}_h^k\boldsymbol{\phi}(s) - \boldsymbol{\phi}(s)) + (\boldsymbol{\xi_\sigma}(s), \nabla(\psi - I_h^k\psi)(s)) + \langle \hat{\boldsymbol{\xi}}_{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu}, I_h^k\psi\rangle - (\eta_{u_{ss}}, I_h^k\psi) \\ &- \int_0^s (b(s,\gamma)\boldsymbol{\xi_q}(\gamma), \boldsymbol{I}_h^k\boldsymbol{\phi}(s))d\gamma + \int_s^t (b(\gamma,s)\boldsymbol{\phi}(\gamma), \boldsymbol{\xi_q}(s))d\gamma.\end{aligned}$$

Taking $\xi_u(0) = \xi_{u_s}(0) = 0$ and integrating the equation from 0 to t followed by a change of order of integration of the last term, we obtain

$$\begin{aligned}
\|I_h^{k-1}\xi_u\|^2 = \int_0^t \Big[&(\xi_{u_{ss}}(s), I_h^k\psi(s) - \psi(s)) - (\xi_{u_{ss}}(s), I_h^{k-1}\psi(s) - \psi(s)) + (\boldsymbol{\xi_q}(s),\\
&\boldsymbol{\Pi}_{k-1}^{RT}\boldsymbol{p}(s) - \boldsymbol{p}(s)) + (a\boldsymbol{\xi_q}(s), \boldsymbol{\phi}(s) - \boldsymbol{I}_h^k\boldsymbol{\phi}(s)) + (\boldsymbol{\xi_\sigma}(s), \boldsymbol{I}_h^k\boldsymbol{\phi}(s) - \boldsymbol{\phi}(s))\\
&+ (\boldsymbol{\xi_\sigma}(s), \nabla(\psi - I_h^k\psi)(s)) + \langle \hat{\boldsymbol{\xi}}_{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu}, I_h^k\psi\rangle - (\eta_{u_{ss}}, I_h^k\psi)\Big] ds\\
&- \int_0^t \int_s^t (b(\gamma, s)\boldsymbol{\xi_q}(s), \boldsymbol{\phi}(\gamma) - \boldsymbol{I}_h^k\boldsymbol{\phi}(\gamma)) d\gamma ds\\
&= \int_0^t [E_1 + E_2 + E_3 + E_4 + E_5 + E_6 + E_7 + E_8]\, ds + E_9. \qquad (4.23)
\end{aligned}$$

Cauchy Schwarz's inequality and (4.20a) show

$$E_1 + E_2 \leq Ch^{k+2}\|I_h^{k-1}\xi_u(s)\|.$$

Next, a use of identity $\int_0^t f(z)g(z)dz = f(0)\bar{g}(0) + \int_0^t f_z(z)\bar{g}(z)dz$ along with (4.20b), yield

$$|E_3 + E_4 + E_5 + E_6| \leq Ch^{k+2}\|I_h^{k-1}\xi_u\|.$$

Use of (4.4e), properties of the projection I_h and (4.20a) give

$$|E_7| \leq \|\hat{\boldsymbol{\xi}}_{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu}\|_{\partial K}\|I_h^k\psi - \psi\|_{\partial K} \leq Ch^{k+2}\|I_h^{k-1}\xi_u\|.$$

We rewrite E_8 as follows

$$\begin{aligned}
(\eta_{u_{ss}}, I_h^k\psi) &= (\eta_{u_{ss}}, I_h^k\psi - I_h^{k-1}\psi) + (\eta_{u_{ss}}, I_h^{k-1}\psi)\\
&= (\eta_{u_{ss}}, I_h^k\psi - I_h^{k-1}\psi) + (I_h^{k-1}\eta_{u_{ss}}, I_h^{k-1}\psi)\\
&\leq \|\eta_{u_{ss}}\|\|I_h^k\psi - I_h^{k-1}\psi\| + \|I_h^{k-1}\eta_{u_{ss}}\|\|I_h^{k-1}\psi\|\\
&\leq Ch^{k+2}\|I_h^{k-1}\xi_u\|.
\end{aligned}$$

Finally, use of boundedness of b shows

$$|E_9| \leq M\left|\int_0^t \left(\int_s^t \boldsymbol{\phi}(\gamma) - \boldsymbol{I}_h^k\boldsymbol{\phi}(\gamma)\right) d\gamma, \boldsymbol{\xi_q}(s)ds\right|$$

$$= M \left| \int_0^t \bar{\boldsymbol{\phi}}(\gamma) - \boldsymbol{I}_h^k \bar{\boldsymbol{\phi}}(\gamma), \boldsymbol{\xi_q}(s) ds \right|$$
$$\leq Ch^{k+2} \|\bar{\psi}(s)\|_2$$
$$\leq Ch^{k+2} \|I_h^{k-1} \xi_u\|, \qquad \text{(by(4.20b))}.$$

Substituting in (4.23), we get

$$\|I_h^{k-1} \xi_u(t)\|^2 \leq Ch^{k+2} \int_0^t \|I_h^{k-1} \xi_u(s)\| ds.$$

Use of Young's inequality and Gronwall's Lemma yield the following estimate

$$\|I_h^{k-1} \xi_u(t)\| \leq Ch^{k+2}. \tag{4.24}$$

Finally, (4.24) and (4.15) conclude the proof of the theorem. □

Lemma 4.5.3. *There exists a positive constant C which does not rely on h and k such that $\forall t \in (0, T]$, the inequality below is valid for $k \geq 1$*

$$\|u^p - u_h^p\| \leq Ch^{k+2}, \tag{4.25}$$

where, $u^p = u - \frac{1}{|K|} \int_K u dx.$

Proof. See Chapter 2 (Lemma 5.2). □

Proof of Theorem 4.3.2: By the definition of u_h^* from (4.5), on any $K \in \mathcal{T}_h$, we obtain, as in Chapter 2 that

$$\|u - u_h^*\|_{L^2(K)} \leq \|I_h^{k-1} e_u\|_{L^2(K)} + \|u^p - u_h^p\|_{L^2(K)}, \tag{4.26}$$

where I_h^0 is L^2-projection onto the space of polynomials of total degree 0. A substitution of (4.21) and (4.25) in (4.26) finishes the proof. □

4.6 Fully Discrete Scheme

Here, we present a completely discrete method for approximating the solution to (4.1). To accomplish this, we discretize equation (4.3) in the time direction using a central difference scheme and the midpoint rule. First, we split the interval $[0, T]$ into M subintervals with equal spacing using the following points:

$$0 = t_0 < t_1 < ... < t_M = T,$$

with $t_n = n\Delta t$ where, $\Delta t = T/M$.

We begin by defining the following notations,

$$\Upsilon U^n = \frac{U^{n+1} + U^n}{2}, \quad \Phi U^n = \frac{U^{n+1} + 2U^n + U^{n-1}}{4} = \frac{\Upsilon U^n + \Upsilon U^{n-1}}{2},$$

$$\partial_t \Upsilon U^n = \frac{U^{n+1} - U^n}{\Delta t}, \; \partial_t^2 U^n = \frac{U^{n+1} - 2U^n + U^{n-1}}{\Delta t^2},$$

$$\delta_t U^n = \frac{\partial_t \Upsilon U^n + \partial_t \Upsilon U^{n-1}}{2} = \frac{U^{n+1} - U^{n-1}}{2\Delta t},$$

$$E_h^n(\boldsymbol{Q}) = \Delta t \sum_{j=0}^{n-1} b(t_n, t_{j+1/2}) \Upsilon \boldsymbol{Q}^j, \quad \Upsilon E_h^n(\boldsymbol{Q}) = \frac{E_h^{n+1}(\boldsymbol{Q}) + E_h^n(\boldsymbol{Q})}{2}.$$

For $1 \leq n \leq M$, find $(U^n, \boldsymbol{Q}^n, \boldsymbol{S}^n, \hat{U}^n) \in (V_h \times \boldsymbol{W}_h \times \boldsymbol{W}_h \times M_h)$, such that, for any $(v_h, \boldsymbol{w}_h, \boldsymbol{\tau}_h, \mu_h, m_h) \in (V_h \times \boldsymbol{W}_h \times \boldsymbol{W}_h \times M_h \times M_h)$, we require

$$\frac{2}{\Delta t}(\partial_t \Upsilon U^0, v_h) - (\Upsilon \boldsymbol{S}^0, \nabla v_h) + \langle \Upsilon \hat{\boldsymbol{S}}^0 \cdot \boldsymbol{\nu}, v_h \rangle_{\partial \mathcal{T}_h} = (\Upsilon f^0 + \frac{2}{\Delta t} u_1, v_h), \tag{4.27a}$$

$$\langle \Upsilon \hat{\boldsymbol{S}}^0 \cdot \boldsymbol{\nu}, \mu \rangle_{\partial \mathcal{T}_h \backslash \partial \Omega} = 0, \tag{4.27b}$$

$$(\Upsilon \boldsymbol{Q}^n, \boldsymbol{w}_h) - (\Upsilon U^n, \nabla \cdot \boldsymbol{w}_h) + \langle \Upsilon \hat{U}^n, \boldsymbol{w}_h \cdot \boldsymbol{\nu} \rangle_{\partial \mathcal{T}_h} = 0, \;\; n \geq 0, \tag{4.27c}$$

$$(a \Upsilon \boldsymbol{Q}^n, \boldsymbol{\tau}_h) - (\Upsilon \boldsymbol{S}^n, \boldsymbol{\tau}_h) + (\Upsilon E_h^n(\boldsymbol{Q}), \boldsymbol{\tau}_h) = 0, \;\; n \geq 0, \tag{4.27d}$$

$$(\partial_t^2 U^n, v_h) - (\Phi \boldsymbol{S}^n, \nabla v_h) + \langle \Phi \hat{\boldsymbol{S}}^n \cdot \boldsymbol{\nu}, v_h \rangle_{\partial \mathcal{T}_h} = (\Phi f^n, v_h), \;\; n \geq 1, \tag{4.27e}$$

$$\langle \Upsilon \hat{U}^n, \mu_h \rangle_{\partial \Omega} = 0, \;\; n \geq 0, \tag{4.27f}$$

$$\langle \Phi \hat{\boldsymbol{S}}^n \cdot \boldsymbol{\nu}, m_h \rangle_{\partial \mathcal{T}_h \backslash \partial \Omega} = 0, \;\; n \geq 1, \tag{4.27g}$$

Proof of Theorem 4.3.3: We begin by writing $\|u(t_n) - U^n\| \le \|u(t_n) - u_h(t_n)\| + \|u_h(t_n) - U^n\|$. We only need to derive the estimate $\|u_h(t_n) - U^n\|$. We will use ζ_u^n to denote $u_h(t_n) - U^n$. Similarly, $\boldsymbol{\zeta}_{\boldsymbol{q}}^n$, $\boldsymbol{\zeta}_{\boldsymbol{\sigma}}^n$ and $\hat{\zeta}_u^n$.

Now, using (4.3) and (4.27), we have the following system of equations

$$\frac{2}{\Delta t}(\partial_t \Upsilon \zeta_u^0, v_h) - (\Upsilon \boldsymbol{\zeta}_{\boldsymbol{\sigma}}^0, \nabla v_h) + \langle \Upsilon \hat{\boldsymbol{\zeta}}_{\boldsymbol{\sigma}}^0 \cdot \boldsymbol{\nu}, v_h \rangle_{\partial \mathcal{T}_h} = \left(\frac{2}{\Delta t}\left(\partial_t \Upsilon u_h^0 - u_1\right) - \Upsilon u_{h_{tt}}^0, v_h \right), \tag{4.28a}$$

$$\langle \Upsilon \hat{\boldsymbol{\zeta}}_{\boldsymbol{\sigma}}^0 \cdot \boldsymbol{\nu}, \mu \rangle_{\partial \mathcal{T}_h \setminus \partial\Omega} = 0, \tag{4.28b}$$

$$(\Upsilon \boldsymbol{\zeta}_{\boldsymbol{q}}^n, \boldsymbol{w}_h) - (\Upsilon \zeta_u^n, \nabla \cdot \boldsymbol{w}_h) + \langle \Upsilon \hat{\zeta}_u^n, \boldsymbol{w}_h \cdot \boldsymbol{\nu} \rangle_{\partial \mathcal{T}_h} = 0, \tag{4.28c}$$

$$(a \Upsilon \boldsymbol{\zeta}_{\boldsymbol{q}}^n, \boldsymbol{\tau}_h) - (\Upsilon \boldsymbol{\zeta}_{\boldsymbol{\sigma}}^n, \boldsymbol{\tau}_h) + (\Upsilon I^n(\boldsymbol{q_h}), \boldsymbol{\tau}_h) = (\Upsilon E_h^n(\boldsymbol{Q}), \boldsymbol{\tau}_h), \tag{4.28d}$$

$$(\partial_t^2 \zeta_u^n, v_h) - (\Phi \boldsymbol{\zeta}_{\boldsymbol{\sigma}}^n, \nabla v_h) + \langle \Phi \hat{\boldsymbol{\zeta}}_{\boldsymbol{\sigma}}^n \cdot \boldsymbol{\nu}, v_h \rangle_{\partial \mathcal{T}_h} = \left(\partial_t^2 u_h^n - \Phi u_{h_{tt}}^n, v_h \right), \tag{4.28e}$$

$$\langle \Upsilon \hat{\zeta}_u^n, \mu_h \rangle_{\partial\Omega} = 0, \tag{4.28f}$$

$$\langle \Phi \hat{\boldsymbol{\zeta}}_{\boldsymbol{\sigma}}^n \cdot \boldsymbol{\nu}, m_h \rangle_{\partial \mathcal{T}_h \setminus \partial\Omega} = 0, \tag{4.28g}$$

for all $(v_h, \boldsymbol{w}_h, \boldsymbol{\tau}_h, \mu_h, m_h) \in (V_h \times \boldsymbol{W}_h \times \boldsymbol{W}_h \times M_h \times M_h)$. Here,

$$I^n(\boldsymbol{q_h}) = \int_0^{t_n} b(t_n, s) \boldsymbol{q}_h(s) ds.$$

We begin with the proof of (4.7a). Let $n \ge 1$; then, we start by subtracting (4.28c) from itself after replacing n by $n-1$ and then, dividing the resulting equation by $2\Delta t$. Secondly, we will perform the same operations in (4.28f). Next, in (4.28d), we will replace n by $n-1$ and take the average of the resulting equation with itself. Now, take $\boldsymbol{w}_h = \Phi \boldsymbol{\zeta}_{\boldsymbol{\sigma}}^n$, $\boldsymbol{\tau}_h = \delta_t \boldsymbol{\zeta}_{\boldsymbol{q}}^n$, $v_h = \delta_t \zeta_u^n$, $\mu_h = -\Phi \hat{\boldsymbol{\zeta}}_{\boldsymbol{\sigma}}^n \cdot \boldsymbol{\nu}$ and $m_h = -\delta_t \hat{\zeta}_u^n$ in (4.28c), (4.28d), (4.28e), (4.28f) and (4.28g), respectively and then, add (4.28c)-(4.28e), (4.28f) and (4.28g) to obtain

$$\begin{aligned} \left(a \Phi \boldsymbol{\zeta}_{\boldsymbol{q}}^n, \delta_t \boldsymbol{\zeta}_{\boldsymbol{q}}^n\right) + \left(\partial_t^2 \zeta_u^n, \delta_t \zeta_u^n\right) + \left\langle \Phi \hat{\zeta}_u^n - \Phi \zeta_u^n, \tau(\delta_t \hat{\zeta}_u^n - \delta_t \zeta_u^n) \right\rangle &= \left(\Phi E_h^n(\boldsymbol{Q}), \delta_t \boldsymbol{\zeta}_{\boldsymbol{q}}^n\right) \\ &- (\Phi I^n(\boldsymbol{q_h}), \delta_t \boldsymbol{\zeta}_{\boldsymbol{q}}^n) + \left(\Phi u_{h_{tt}}^n - \partial_t^2 u_h^n, \delta_t \boldsymbol{\zeta}_{\boldsymbol{q}}^n\right). \end{aligned}$$

Now, we can write $\left(a \Phi \boldsymbol{\zeta}_{\boldsymbol{q}}^n, \delta_t \boldsymbol{\zeta}_{\boldsymbol{q}}^n\right)$ as

$$\left(a \Phi \boldsymbol{\zeta}_{\boldsymbol{q}}^n, \delta_t \boldsymbol{\zeta}_{\boldsymbol{q}}^n\right) = \left(a \left(\frac{\Upsilon \boldsymbol{\zeta}_{\boldsymbol{q}}^n + \Upsilon \boldsymbol{\zeta}_{\boldsymbol{q}}^{n-1}}{2} \right), \frac{\Upsilon \boldsymbol{\zeta}_{\boldsymbol{q}}^n - \Upsilon \boldsymbol{\zeta}_{\boldsymbol{q}}^{n-1}}{\Delta t} \right)$$

$$= \frac{1}{2\Delta t}\Big[(a\Upsilon\boldsymbol{\zeta}_{\boldsymbol{q}}^n, \Upsilon\boldsymbol{\zeta}_{\boldsymbol{q}}^n) - (a\Upsilon\boldsymbol{\zeta}_{\boldsymbol{q}}^{n-1}, \Upsilon\boldsymbol{\zeta}_{\boldsymbol{q}}^{n-1})\Big].$$

Using similar approach for other terms, the equation can be further written as

$$\begin{aligned}\frac{1}{2\Delta t}\Big[\Upsilon\|\partial_t\Upsilon\zeta_u^n\|^2 - \|\partial_t\Upsilon\zeta_u^{n-1}\|^2 + (a\Upsilon\boldsymbol{\zeta}_{\boldsymbol{q}}^n, \Upsilon\boldsymbol{\zeta}_{\boldsymbol{q}}^n) - (a\Upsilon\boldsymbol{\zeta}_{\boldsymbol{q}}^{n-1}, \Upsilon\boldsymbol{\zeta}_{\boldsymbol{q}}^{n-1}) + \|\Upsilon\hat{\zeta}_u^n - \Upsilon\zeta_u^n\|_\tau^2 \\ -\|\Upsilon\hat{\zeta}_u^{n-1} - \Upsilon\zeta_u^{n-1}\|_\tau^2\Big] = \big(\Phi E_h^n(\boldsymbol{q_h}), \delta_t\boldsymbol{\zeta}_{\boldsymbol{q}}^n\big) - (\Phi I^n(\boldsymbol{q_h}), \delta_t\boldsymbol{\zeta}_{\boldsymbol{q}}^n) + \big(\Phi E_h^n(\boldsymbol{\zeta_q}), \delta_t\boldsymbol{\zeta}_{\boldsymbol{q}}^n\big) \\ + \big(\Phi u_{h_{tt}}^n - \partial_t^2 u_h^n, \delta_t\boldsymbol{\zeta}_{\boldsymbol{q}}^n\big).\end{aligned}$$

Now, multiplying the equation by $2\Delta t$ and adding from $n = 1$ to $n = m$, we obtain the following inequality

$$\begin{aligned}\|\partial_t\Upsilon\zeta_u^m\|^2 + \|\Upsilon\boldsymbol{\zeta}_{\boldsymbol{q}}^m\|^2 + \|\Upsilon\hat{\zeta}_u^m - \Upsilon\zeta_u^m\|_\tau^2 \le \|\partial_t\Upsilon\zeta_u^0\|^2 + \|\Upsilon\boldsymbol{\zeta}_{\boldsymbol{q}}^0\|^2 + \|\Upsilon\hat{\zeta}_u^0 - \Upsilon\zeta_u^0\|_\tau^2 \\ + 2\Delta t\sum_{n=1}^{m}(J_1^n + J_2^n + J_3^n),\end{aligned} \tag{4.29}$$

where

$$\begin{aligned}&J_1^n = \big(\Phi E_h^n(\boldsymbol{q_h}), \delta_t\boldsymbol{\zeta}_{\boldsymbol{q}}^n\big) - (\Phi I^n(\boldsymbol{q_h}), \delta_t\boldsymbol{\zeta}_{\boldsymbol{q}}^n),\ J_2^n = \big(\Phi E_h^n(\boldsymbol{\zeta_q}), \delta_t\boldsymbol{\zeta}_{\boldsymbol{q}}^n\big), \\ &J_3^n = \big(\Phi u_{h_{tt}}^n - \partial_t^2 u_h^n, \delta_t\boldsymbol{\zeta}_{\boldsymbol{q}}^n\big).\end{aligned}$$

For the estimates of $\|\partial_t\Upsilon\zeta_u^0\|^2 + \|\Upsilon\boldsymbol{\zeta}_{\boldsymbol{q}}^0\|^2 + \|\Upsilon\hat{\zeta}_u^0 - \Upsilon\zeta_u^0\|_\tau^2$, we consider the following equations

$$\frac{2}{\Delta t}(\partial_t\Upsilon\zeta_u^0, v_h) - (\Upsilon\boldsymbol{\zeta}_{\boldsymbol{\sigma}}^0, \nabla v_h) + \langle\Upsilon\hat{\boldsymbol{\zeta}}_{\boldsymbol{\sigma}}^0\cdot\boldsymbol{\nu}, v_h\rangle_{\partial\mathcal{T}_h} = \left(\frac{2}{\Delta t}\big(\partial_t\Upsilon u_h^0 - u_1\big) - \Upsilon u_{h_{tt}}^0, v_h\right), \tag{4.30a}$$

$$(\Upsilon\boldsymbol{\zeta}_{\boldsymbol{q}}^0, \boldsymbol{w}_h) - (\Upsilon\zeta_u^0, \nabla\cdot\boldsymbol{w}_h) + \langle\Upsilon\hat{\zeta}_u^0, \boldsymbol{w}_h\cdot\boldsymbol{\nu}\rangle_{\partial\mathcal{T}_h} = 0, \tag{4.30b}$$

$$(a\Upsilon\boldsymbol{\zeta}_{\boldsymbol{q}}^0, \boldsymbol{\tau}_h) - (\Upsilon\boldsymbol{\zeta}_{\boldsymbol{\sigma}}^0, \boldsymbol{\tau}_h) + \big(I_1^0, \boldsymbol{\tau}_h\big)\,ds = \big(\Upsilon E_h^0(\boldsymbol{Q}), \boldsymbol{\tau}_h\big), \tag{4.30c}$$

$$\langle\Upsilon\hat{\zeta}_u^0, \mu_h\rangle_{\partial\Omega} = 0, \tag{4.30d}$$

$$\langle\Upsilon\hat{\boldsymbol{\zeta}}_{\boldsymbol{\sigma}}^0\cdot\boldsymbol{\nu}, m_h\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega} = 0, \tag{4.30e}$$

for all $(v_h, \boldsymbol{w}_h, \boldsymbol{\tau}_h, \mu_h, m_h) \in (V_h \times \boldsymbol{W}_h \times \boldsymbol{W}_h \times M_h \times M_h)$. We take $v_h = \Upsilon\zeta_u^0$, $\boldsymbol{w}_h = \Upsilon\boldsymbol{\sigma}^0$, $\boldsymbol{\tau}_h = \Upsilon\boldsymbol{\zeta}_{\boldsymbol{q}}^0$, $\mu_h = -\Upsilon\hat{\boldsymbol{\zeta}}_{\boldsymbol{\sigma}}^0\cdot\boldsymbol{\nu}$ and $m_h = -\Upsilon\delta_t\hat{\zeta}_u^0$ in (4.30a), (4.30b), (4.30c), (4.30d) and (4.30e), respectively and add the resulting equations, to get the following inequality

$$\|\partial_t \Upsilon \zeta_u^0\|^2 + \|\Upsilon \boldsymbol{\zeta}_{\boldsymbol{q}}^0\|^2 + \|\Upsilon \hat{\zeta}_u^0 - \Upsilon \zeta_u^0\|_\tau^2 \leq \frac{1}{2}\left(\Upsilon E_h^0(\boldsymbol{Q}), \Upsilon \boldsymbol{\zeta}_{\boldsymbol{q}}^0\right) - \frac{1}{2}\int_0^{t_1}\left(b(t_1,s)\boldsymbol{q}_h(s), \Upsilon \boldsymbol{\zeta}_{\boldsymbol{q}}^0\right) ds$$
$$+ \left(\frac{2}{\Delta t}\left(\partial_t \Upsilon u_h^0 - u_1\right) - \Upsilon u_{h_{tt}}^0, \Upsilon \zeta_u^0\right).$$

Now, proceeding in the similar way as to obtain (4.29) will prove that

$$\|\partial_t \Upsilon \zeta_u^0\|^2 + \|\Upsilon \boldsymbol{\zeta}_{\boldsymbol{q}}^0\|^2 + \|\Upsilon \hat{\zeta}_u^0 - \Upsilon \zeta_u^0\|_\tau^2 \leq C(h^{2(k+1)} + \Delta t^4).$$

Next, for J_1^n, a use of Theorem 4.3.1 along with quadrature error yield

$$\begin{aligned}\|\Phi E_h^n(\boldsymbol{q_h}) - \Phi I^n(\boldsymbol{q_h})\| &\leq \|\Phi E_h^n(\boldsymbol{q}) - \Phi I^n(\boldsymbol{q}) - \Phi E_h^n(\boldsymbol{e_q}) + \Phi I^n(\boldsymbol{e_q})\| \\ &\leq C(h^{k+1} + \Delta t^2).\end{aligned}$$

Further, use of Young's inequality yields

$$\Delta t \sum_{n=1}^m |J_1^n| \leq C\left(h^{2(k+1)} + \Delta t^4\right) + \frac{\Delta t}{2}\sum_{n=1}^m \left\|\frac{\Upsilon \boldsymbol{\zeta}_{\boldsymbol{q}}^n - \Upsilon \boldsymbol{\zeta}_{\boldsymbol{q}}^{n-1}}{\Delta t}\right\|^2. \tag{4.31}$$

Use of Taylor's series expansion, along with Young's inequality, yield

$$\Delta t \sum_{n=1}^m |J_3^n| \leq C\left(h^{2(k+1)} + \Delta t^4\right) + \frac{1}{2}\sum_{n=1}^m \left\|\frac{\partial_t \Upsilon \zeta_u^n + \partial_t \Upsilon \zeta_u^{n-1}}{2}\right\|^2. \tag{4.32}$$

Use of (4.31) and (4.32) in (4.29) along with discrete Gronwall's lemma yield

$$\|\partial_t \Upsilon \zeta_u^m\|^2 + \|\Upsilon \boldsymbol{\zeta}_{\boldsymbol{q}}^m\|^2 + \|\Upsilon \hat{\zeta}_u^m - \Upsilon \zeta_u^m\|_\tau^2 \leq C\left(h^{2(k+1)} + \Delta t^4\right).$$

Finally, use of triangle inequality and Theorem 4.3.1, finish the proof of (4.7a).

Now, for the proof of (4.7b), we introduce the following notations:

$$\underset{\sim}{\phi}^0 = 0, \quad \underset{\sim}{\phi}^n = \Delta t \sum_{j=0}^{n-1} \Upsilon \phi^j, \quad \partial_t \Upsilon \underset{\sim}{\phi}^n = \Upsilon \phi^n, \quad \Delta t \sum_{j=0}^{n} \Phi \phi^j = \Upsilon \underset{\sim}{\phi}^n - \frac{\Delta t}{2}\Upsilon \phi^0.$$

Next, we multiply (4.28d), (4.28e) and (4.28g) by k, take summation over n and use (4.28a) and (4.28b) to get the following system of equation

$$(\Upsilon \boldsymbol{\zeta}_{\boldsymbol{q}}^n, \boldsymbol{w}_h) - (\Upsilon \zeta_u^n, \nabla \cdot \boldsymbol{w}_h) + \langle \Upsilon \hat{\zeta}_u^n, \boldsymbol{w}_h \cdot \boldsymbol{\nu}\rangle_{\partial \mathcal{T}_h} = 0, \tag{4.33a}$$

$$(a\Upsilon \underset{\sim}{\boldsymbol{\zeta_q}}^n, \boldsymbol{\tau}_h) - (\Upsilon \underset{\sim}{\boldsymbol{\zeta_\sigma}}^n, \boldsymbol{\tau}_h) + \left(\Upsilon \underset{\sim}{E_h}^n(\boldsymbol{\zeta}_{\boldsymbol{q}}^n), \boldsymbol{\tau}_h\right) = \left(\Upsilon \underset{\sim}{F_h}^n(\boldsymbol{q_h}), \boldsymbol{\tau}_h\right), \tag{4.33b}$$

$$(\partial_t \Upsilon \zeta_u^n, v_h) - (\Upsilon \underset{\sim}{\boldsymbol{\zeta_\sigma}}^n, \nabla v_h) + \langle \Upsilon \underset{\sim}{\hat{\boldsymbol{\zeta}}_{\boldsymbol{\sigma}}}^n \cdot \boldsymbol{\nu}, v_h\rangle_{\partial\mathcal{T}_h} = \left(\Delta t \sum_{j=0}^{n} \left(\partial_t^2 u_h^n - \Phi u_{h_{tt}}^n\right), v_h\right), \tag{4.33c}$$

$$\langle \Upsilon \hat{\zeta}_u^n, \mu_h\rangle_{\partial\Omega} = 0, \tag{4.33d}$$

$$\langle \Upsilon \underset{\sim}{\hat{\boldsymbol{\zeta}}_{\boldsymbol{\sigma}}}^n \cdot \boldsymbol{\nu}, m_h\rangle_{\partial\mathcal{T}_h\backslash\partial\Omega} = 0. \tag{4.33e}$$

Choose $\boldsymbol{w}_h = \Upsilon \underset{\sim}{\hat{\boldsymbol{\zeta}}_{\boldsymbol{\sigma}}}^n$, $\boldsymbol{\tau}_h = \Upsilon \boldsymbol{\zeta}_{\boldsymbol{q}}^n$, $v_h = \Upsilon \zeta_u^n$, $\mu_h = -\Upsilon \underset{\sim}{\hat{\boldsymbol{\zeta}}_{\boldsymbol{\sigma}}}^n \cdot \boldsymbol{\nu}$ and $m_h = -\Upsilon \hat{\zeta}_u^n$ in (4.33a), (4.33b), (4.33c), (4.33d) and (4.33e), respectively, and add the resulting equations. After simplifying as above, we attain the desired estimate. For further details, see [79].

□

4.7 Numerical Experiments

The performance of the suggested HDG approach for equations (4.1a)-(4.1c) is discussed in this section. The problem has been discretized using the central difference technique, and the integral term has been approximated using the mid-point rule. For the sake of simplicity, the function a is chosen to be 1 throughout, with the problem domain being $\Omega = (0,1) \times (0,1)$. We demonstrate the order of convergence for the L^2 norm of the error in u, the gradient $\boldsymbol{q} = -\nabla u$, and post-processed solution u_h^*. We see that the super-convergence in the case of u_h^* and the optimal order of convergence for u and $\boldsymbol{q}$ are realized as anticipated by our derived results.

Example 1: Let $u(x,y,t) = t^2 e^t x(1-x)y(1-y)$ represents the precise solution with $b(x,t,s) = e^{t-s}$. Table 4.1 displays the computed order of convergence and L^2 error estimates for u and $\boldsymbol{q}$ and u_h^* at $t = 1$ for $k = 1$, $k = 2$, and $k = 3$ for a variety of h values. We observe that the convergence rates for $\|e_u\|$, $\|\boldsymbol{e_q}\|$ and $\|e_u^*\|$ are of the order $O(h^{k+1})$, $O(h^{k+1})$ and $O(h^{k+2})$, respectively. Table 4.2 gives the time convergence for u for the example for different time steps.

		u_h		$\boldsymbol{q_h}$		u_h^*	
$\boldsymbol{k}$	$\boldsymbol{h}$	**Error**	**Order**	**Error**	**Order**	**Error**	**Order**
1	$\frac{1}{2}$	3.1116e-02		1.1484e-01		9.0267e-03	
	$\frac{1}{4}$	6.9328e-03	2.1662	2.2498e-02	2.3518	8.3514e-04	3.4341
	$\frac{1}{8}$	1.8259e-03	1.9248	5.5337e-03	2.0235	9.9647e-05	3.0671
	$\frac{1}{16}$	4.7854e-04	1.9319	1.3897e-03	1.9934	1.2100e-05	3.0417
2	$\frac{1}{2}$	7.1740e-03		2.5155e-02		4.6657e-03	
	$\frac{1}{4}$	6.8389e-04	3.3909	1.9663e-03	3.6773	2.7192e-04	4.1009
	$\frac{1}{8}$	8.3343e-05	3.0366	2.0802e-04	3.2407	1.6997e-05	3.9998
	$\frac{1}{16}$	1.9478e-06	3.0635	3.6215e-06	3.3576	1.5014e-07	4.0361
3	$\frac{1}{2}$	2.2952e-03		1.0145e-02		2.2604e-03	
	$\frac{1}{4}$	7.1660e-05	5.0013	3.0543e-04	5.0538	6.7900e-05	5.0570
	$\frac{1}{8}$	2.5553e-06	4.8096	9.6065e-06	4.9907	2.1192e-06	5.0018
	$\frac{1}{16}$	1.3766e-08	4.5306	3.7858e-08	4.8432	7.1367e-09	5.0004

Table 4.1: Computed convergence rates and L^2 error estimates in the context of $Example$ 1

$\boldsymbol{\Delta t}$ $(h = 1/4)$	**Order** (Ex. 1)	**Order** (Ex. 2)
0.25	1.8134	1.7895
0.125	1.9145	1.8967
0.0625	2.0846	1.9932
0.03125	2.1783	2.0814

Table 4.2: Order of convergence for time

Example 2: Let $u(x, y, t) = t\sin(\pi t)\sin\left(\frac{\pi}{2}x\right)\sin\left(\frac{\pi}{2}y\right)$ represents the precise solution with $b(x, t, s) = \sin(\pi t)\cos(\pi s)$. Table 4.3 displays the computed order of convergence and L^2 error estimates for u and $\boldsymbol{q}$ and u_h^* at $t = 1$ for $k = 1$, $k = 2$, and $k = 3$ for a variety of h values. We observe that the convergence rates for $\|e_u\|$, $\|\boldsymbol{e_q}\|$ and $\|e_u^*\|$

		u_h		$\boldsymbol{q_h}$		u_h^*	
$\boldsymbol{k}$	$\boldsymbol{h}$	**Error**	**Order**	**Error**	**Order**	**Error**	**Order**
1	$\frac{1}{2}$	2.1155e-01		3.0941e-01		2.9528e-02	
	$\frac{1}{4}$	5.5961e-02	1.9185	7.0823e-02	2.1272	3.6281e-03	3.0248
	$\frac{1}{8}$	1.4394e-02	1.9589	1.7400e-02	2.0251	4.4006e-04	3.0434
	$\frac{1}{16}$	3.6385e-03	1.9841	4.3324e-03	2.0058	5.3611e-05	3.0371
2	$\frac{1}{2}$	2.4148e-02		4.6484e-02		7.1897e-03	
	$\frac{1}{4}$	4.2070e-03	2.5210	6.7201e-03	2.7902	3.7601e-04	4.2571
	$\frac{1}{8}$	5.5692e-04	2.9173	8.6377e-04	2.9598	2.1278e-05	4.1433
	$\frac{1}{16}$	7.0692e-05	2.9778	1.0849e-04	2.9931	1.2275e-06	4.1156
3	$\frac{1}{2}$	7.5423e-03		1.7558e-02		1.2973e-03	
	$\frac{1}{4}$	4.8000e-04	3.9739	1.0136e-03	4.1145	3.4910e-05	5.2158
	$\frac{1}{8}$	3.0353e-05	3.9831	6.2655e-05	4.0160	1.0729e-06	5.0240
	$\frac{1}{16}$	1.9087e-06	3.9912	3.9054e-06	4.0039	3.3399e-08	5.0056

Table 4.3: Computed convergence rates and L^2 error estimates in the context of $Example\ 2$

are of the order $O(h^{k+1})$, $O(h^{k+1})$ and $O(h^{k+2})$, respectively. Table 4.2 gives the time convergence for u for the example for different time steps.

4.8 Conclusions

This chapter have introduced and analyzed the HDG method applied to a hyperbolic integro-differential equation. The derivation of error estimates employs both the HDG and Ritz-Volterra projections. Additionally, element-by-element post-processing of the numerical solution is achieved through the dual of the problem. The results clearly indicate that all three variables, namely, u, $\boldsymbol{q}$, and $\boldsymbol{\sigma}$, exhibit convergence of order $k+1$, for non-negative k in terms of h, which represents the discretization parameter of the spatial domain. In contrast, the post-processed solution has attained super-convergence; that is, it converges with order $k + 2$, for $k \geq 1$. The analysis of this chapter have provided better accuracy results compared to [79]. Finally, numerical results were reviewed. This study may be carried over to the three-dimensional domain by making the appropriate adjustments.

Chapter 5

HDG Method for Nonlinear Hyperbolic Integro-Differential Equations

5.1 Introduction

In this chapter, we have considered the following nonlinear hyperbolic integro-differential equation with nonlinear kernel:

$$u_{tt} - \nabla \cdot \left(a(u)\nabla u + \int_0^t b(u(s))\nabla u(s)ds \right) = f(u) \qquad \text{in } \Omega \times (0,T], \tag{5.1a}$$

$$u(x,t) = 0 \qquad \text{on } \partial\Omega \times (0,T], \tag{5.1b}$$

$$u(x,0) = u_0(x) \qquad \text{for } x \in \Omega, \tag{5.1c}$$

$$u(x,0) = u_1(x) \qquad \text{for } x \in \Omega, \tag{5.1d}$$

where $u : \Omega \times (0,T] \to \mathbb{R}$. We assume that there are positive constants a_* and M such that $0 < a_* \leq a(u) \leq M$ and $|b(u)| \leq M$ for the functions $a : \mathbb{R} \to \mathbb{R}$ and $b : \mathbb{R} \to \mathbb{R}$. We also assume that the functions $a(u)$, $b(u)$, their derivatives up to and including order 2, and $f(u)$ meet the Lipschitz continuity condition near u. For the existence and uniqueness of the solution of (5.1), we refer to [24].

In literature, Pani *et al.* [114] have analyzed fully discrete schemes for time-dependent partial integro-differential equations, using energy methods, paying attention to the storage required during time-stepping. Further, errors are estimated in L^2 and H^1 norms. In the work by Saedpanah in [123], a continuous space-time Finite Element Method (FEM) of degree one is formulated for a hyperbolic integro-differential equation featuring mixed boundary conditions. The study has also established a set of a posteriori estimates. In a separate study, presented in [26], Chen *et al.* have proposed and investigated a numerical technique centered on a two-grid finite element discretization approach designed to solve semi-linear hyperbolic integro-differential equations. This method utilizes piece-wise continuous finite element approximation and strategically employs a two-grid strategy to address the semi-linearity within the model. Importantly, it is proven that this approach achieves a level of accuracy comparable to that of the traditional FEM. Similarly, in the research outlined in [134], Tan *et al.* have introduced and analyzed a two-grid finite element discretization approach for (5.1). The study also includes numerical examples to empirically support the theoretical findings.

In this chapter, the HDG method is implemented on (5.1). The major contributions of this chapter are as follows:

- For the error analysis, only derivatives of order up to two of the nonlinear variables a and b, along with their Lipschitz continuity condition, has been considered.
- To deal with the integral term, Ritz-Volterra projection of extended type is introduced and analyzed. This helps to achieve optimal estimates of order $O(h^{k+1})$ when using functions from the space of polynomials of degree $k \geq 0$ for approximating both the function 'u' and its gradient '∇u'.
- Dual problem is used for element-by-element post-processing to attain super-convergence outcomes for the post-processed solution.
- Using central difference scheme for time derivative, a completely discrete method is proposed, and corresponding estimations of error are calculated.

- With the help of different numerical examples, it has been verified that the unknown variable and its approximate gradient achieve convergence of optimum order and the post-processed solution attains the super-convergence.

We have used central difference scheme for the time derivative, but higher-order methods can also be applied to derive higher-order convergence in the temporal direction. For the sake of simplicity, C is used to denote an inclusive, positive constant independent of discretizing parameter h as well as the degree of polynomial k. Also, argument x of functions will not be written explicitly, whereas t and s will be written as and when required.

The chapter's structure is as follows: Section 5.2 defines the HDG method for hyperbolic integro-differential equation (5.1). It also introduces an intermediate projection, along with its estimates. Section 5.3 analyses the error for the semi-discrete problem. In Section 5.4, the post-processed solution is introduced, along with its estimates. Section 5.5 deals with the fully discrete scheme. Section 5.6 validates the theoretical results with the help of a few numerical examples. Finally, Section 5.7 gives some concluding remarks.

5.2 HDG Method

To define the technique for equation (5.1), we make use of the following auxiliary variables:

$$\boldsymbol{q} = -\nabla u, \;\; \boldsymbol{\sigma} = a(u)\boldsymbol{q} + \int_0^t b(u(s))\boldsymbol{q}(s)ds,$$

using these variables, equation (5.1) is rewritten as:

$$\boldsymbol{q} = -\nabla u, \qquad \text{in } \Omega, \tag{5.2a}$$

$$\boldsymbol{\sigma} = a(u)\boldsymbol{q} + \int_0^t b(u(s))\boldsymbol{q}(s)ds, \qquad \text{in } \Omega, \tag{5.2b}$$

$$u_{tt} + \nabla\cdot\boldsymbol{\sigma} = f(u), \qquad \text{in } \Omega. \tag{5.2c}$$

At each time t within the interval $(0, T]$, the method provides an approximation $u_h(t)$ of the scalar function $u(t)$, an approximation $\boldsymbol{q}_h(t)$ and $\boldsymbol{\sigma}_h(t)$ of the vector function $\boldsymbol{q}(t)$ and

$\boldsymbol{\sigma}(t)$, respectively, and an approximation $\hat{u}_h(t)$ of the trace of $u(t)$ on the boundaries of the elements. These approximations are computed in the function spaces V_h, $\boldsymbol{W_h}$, $\boldsymbol{W_h}$, and M_h, respectively.

With these spaces, the HDG formulation seeks approximation $(u_h, \boldsymbol{q}_h, \boldsymbol{\sigma}_h, \hat{u}_h)(t) \in (V_h \times \boldsymbol{W}_h \times \boldsymbol{W}_h \times M_h)$, for $t \in (0, T]$, for any $(v_h, \boldsymbol{w}_h, \boldsymbol{\tau}_h, \mu_h, m_h) \in (V_h \times \boldsymbol{W}_h \times \boldsymbol{W}_h \times M_h \times M_h)$, satisfying

$$(\boldsymbol{q}_h, \boldsymbol{w}_h) - (u_h, \nabla \cdot \boldsymbol{w}_h) + \langle \hat{u}_h, \boldsymbol{w}_h \cdot \boldsymbol{\nu} \rangle_{\partial \mathcal{T}_h} = 0, \tag{5.3a}$$

$$(a(u_h)\boldsymbol{q}_h, \boldsymbol{\tau}_h) - (\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + \int_0^t (b(u_h(s))\boldsymbol{q}_h(s), \boldsymbol{\tau}_h) ds = 0, \tag{5.3b}$$

$$(u_{h_{tt}}, v_h) - (\boldsymbol{\sigma}_h, \nabla v_h) + \langle \hat{\boldsymbol{\sigma}}_h \cdot \boldsymbol{\nu}, v_h \rangle_{\partial \mathcal{T}_h} = (f(u_h), v_h), \tag{5.3c}$$

$$\langle \hat{u}_h, \mu_h \rangle_{\partial \Omega} = 0, \tag{5.3d}$$

$$\langle \hat{\boldsymbol{\sigma}}_h \cdot \boldsymbol{\nu}, m_h \rangle_{\partial \mathcal{T}_h \setminus \partial \Omega} = 0, \tag{5.3e}$$

$$u_h(0) = \Pi_V u_0, \tag{5.3f}$$

$$u_{h_t}(0) = \Pi_V u_1, \tag{5.3g}$$

where the numerical trace for flux is defined by

$$\hat{\boldsymbol{\sigma}}_h \cdot \boldsymbol{\nu} = \boldsymbol{\sigma}_h \cdot \boldsymbol{\nu} + \tau(u_h - \hat{u}_h) \ \text{ on } \partial \mathcal{T}_h,$$

for a non-negative stabilisation parameter τ specified on Γ, it is assumed that τ is piece-wise constant on the faces. It is seen that the precise solution u, $\boldsymbol{q}$, and $\boldsymbol{\sigma}$ adhere to equation (5.3). Therefore, the error equations may be derived as follows, considering any $(v_h, \boldsymbol{w}_h, \boldsymbol{\tau}_h, \mu_h, m_h) \in (V_h \times \boldsymbol{W}_h \times \boldsymbol{W}_h \times M_h \times M_h)$.

$$(\boldsymbol{q} - \boldsymbol{q}_h, \boldsymbol{w}_h) - (u - u_h, \nabla \cdot \boldsymbol{w}_h) + \langle u - \hat{u}_h, \boldsymbol{w}_h \cdot \boldsymbol{\nu} \rangle_{\partial \mathcal{T}_h} = 0, \tag{5.4a}$$

$$(a(u)\boldsymbol{q} - a(u_h)\boldsymbol{q}_h, \boldsymbol{\tau}_h) - (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + \int_0^t ((b(u(s))\boldsymbol{q} - b(u_h(s))\boldsymbol{q}_h)(s), \boldsymbol{\tau}_h) ds = 0, \tag{5.4b}$$

$$(u_{tt} - u_{h_{tt}}, v_h) - (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \nabla v_h) + \langle (\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}_h) \cdot \boldsymbol{\nu}, v_h \rangle_{\partial \mathcal{T}_h} = (f(u) - f(u_h), v_h), \tag{5.4c}$$

$$\langle u - \hat{u}_h, \mu_h \rangle_{\partial\Omega} = 0, \tag{5.4d}$$

$$\langle (\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}_h) \cdot \boldsymbol{\nu}, m_h \rangle_{\partial\mathcal{T}_h \setminus \partial\Omega} = 0. \tag{5.4e}$$

For the further analysis, we add and subtract $a(u)\boldsymbol{q}_h + \int_0^t b(u(s))\boldsymbol{q}_h(s)ds$ in (5.4b), to get the error equation in the following form, for any $(v_h, \boldsymbol{w}_h, \boldsymbol{\tau}_h, \mu_h, m_h) \in (V_h \times \boldsymbol{W}_h \times \boldsymbol{W}_h \times M_h \times M_h)$:

$$(\boldsymbol{q} - \boldsymbol{q}_h, \boldsymbol{w}_h) - (u - u_h, \nabla \cdot \boldsymbol{w}_h) + \langle u - \hat{u}_h, \boldsymbol{w}_h \cdot \boldsymbol{\nu} \rangle_{\partial\mathcal{T}_h} = 0, \tag{5.5a}$$

$$\begin{aligned} (a(u)(\boldsymbol{q} - \boldsymbol{q}_h), \boldsymbol{\tau}_h) - (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + \int_0^t (b(u(s))(\boldsymbol{q} - \boldsymbol{q}_h)(s), \boldsymbol{\tau}_h)\, ds \\ = ((a(u_h) - a(u))\boldsymbol{q}_h, \boldsymbol{\tau}_h) + \int_0^t ((b(u_h(s)) - b(u(s)))\boldsymbol{q}_h(s), \boldsymbol{\tau}_h) ds, \end{aligned} \tag{5.5b}$$

$$(u_{tt} - u_{h_{tt}}, v_h) - (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \nabla v_h) + \langle (\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}_h) \cdot \boldsymbol{\nu}, v_h \rangle_{\partial\mathcal{T}_h} = (f(u) - f(u_h), v_h), \tag{5.5c}$$

$$\langle u - \hat{u}_h, \mu_h \rangle_{\partial\Omega} = 0, \tag{5.5d}$$

$$\langle (\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}_h) \cdot \boldsymbol{\nu}, m_h \rangle_{\partial\mathcal{T}_h \setminus \partial\Omega} = 0. \tag{5.5e}$$

The main outcome of this chapter is presented in the form of the following theorem:

Theorem 5.2.1. *Let* $(u, \boldsymbol{q}, \boldsymbol{\sigma})$ *be the solution of* (5.2) *with* $u, u_t, u_{tt} \in L^\infty(H^{k+2}(\mathcal{T}_h))$ *and* $u_0, u_1 \in H^{k+2}(\mathcal{T}_h)$ *for* $k \geq 0$*. Additionally, let* $(u_h, \boldsymbol{q}_h, \boldsymbol{\sigma}_h, \hat{u}_h) \in V_h \times \boldsymbol{W}_h \times \boldsymbol{W}_h \times M_h$ *be the solution of* (5.3) *along with* $u_h(0) = \Pi_V u_0$, $u_{h_t}(0) = \Pi_V u_1$, $\boldsymbol{q}_h(0) = -\boldsymbol{I}_h^k \nabla u_0$ *and* $\boldsymbol{\sigma}_h(0) = \boldsymbol{\Pi}_{\boldsymbol{W}}(a\nabla u_0)$*. Consequently, the following estimations hold true:*

$$\|(u_t - u_{h_t})(t)\| \leq Ch^{k+1},$$

$$\|(u - u_h)(t)\| + \|(\boldsymbol{q} - \boldsymbol{q}_h)(t)\| + \|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)(t)\| + \|(u - \hat{u}_h)(t)\|_\tau \leq Ch^{k+1}.$$

5.2.1 An Extended Mixed Ritz-Volterra Projection and Associated Estimates

The subsequent Ritz-Volterra projection is now defined as follows: For each t and given $(u, \boldsymbol{q}, \boldsymbol{\sigma})$, find $(\tilde{u}_h, \tilde{\boldsymbol{q}}_h, \tilde{\boldsymbol{\sigma}}_h, \hat{\tilde{u}}_h) \in (V_h \times \boldsymbol{W}_h \times \boldsymbol{W}_h \times M_h)$ satisfying

$$(\boldsymbol{q} - \tilde{\boldsymbol{q}}_h, \boldsymbol{w}_h) - (u - \tilde{u}_h, \nabla \cdot \boldsymbol{w}_h) + \langle u - \hat{\tilde{u}}_h, \boldsymbol{w}_h \cdot \boldsymbol{\nu} \rangle_{\partial\mathcal{T}_h} = 0, \tag{5.6a}$$

$$(a(u)(\boldsymbol{q}-\tilde{\boldsymbol{q}}_h),\boldsymbol{\tau}_h)-(\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}}_h,\boldsymbol{\tau}_h)+\int_0^t(b(u(s))(\boldsymbol{q}-\tilde{\boldsymbol{q}}_h)(s),\boldsymbol{\tau}_h)ds=0, \tag{5.6b}$$

$$-(\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}}_h,\nabla v_h)+\langle(\boldsymbol{\sigma}-\hat{\tilde{\boldsymbol{\sigma}}}_h)\cdot\boldsymbol{\nu},v_h\rangle_{\partial\mathcal{T}_h}=0, \tag{5.6c}$$

$$\langle u-\hat{\tilde{u}}_h,\mu_h\rangle_{\partial\Omega}=0, \tag{5.6d}$$

$$\langle(\boldsymbol{\sigma}-\hat{\tilde{\boldsymbol{\sigma}}}_h)\cdot\boldsymbol{\nu},m_h\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega}=0, \tag{5.6e}$$

for all $(v_h,\boldsymbol{w}_h,\boldsymbol{\tau}_h,\mu_h,m_h)\in(V_h\times\boldsymbol{W}_h\times\boldsymbol{W}_h\times M_h\times M_h)$, where

$$\hat{\tilde{\boldsymbol{\sigma}}}_h\cdot\boldsymbol{\nu}=\tilde{\boldsymbol{\sigma}}_h\cdot\boldsymbol{\nu}+\tau(\tilde{u}_h-\hat{\tilde{u}}_h)\ \text{ on }\partial\mathcal{T}_h.$$

We decompose the errors in terms θ's and ρ's with the help of the projection $\boldsymbol{I}_h^k$ (L^2-projection onto $\boldsymbol{W}_h$) and P_M (L^2-projection onto M_h) as done in Chapter 2.
Now, the system of equations (5.6) become

$$(\boldsymbol{\rho_q},\boldsymbol{w}_h)-(\rho_u,\nabla\cdot\boldsymbol{w}_h)+\langle\hat{\rho}_u,\boldsymbol{w}_h\cdot\boldsymbol{\nu}\rangle_{\partial\mathcal{T}_h}=0, \tag{5.7a}$$

$$\begin{aligned}(a(u)\boldsymbol{\rho_q},\boldsymbol{\tau_h})-(\boldsymbol{\rho_\sigma},\boldsymbol{\tau}_h)+\int_0^t(b(u(s))\boldsymbol{\rho_q}(s),\boldsymbol{\tau}_h)ds&=(a(u)\boldsymbol{\theta_q},\boldsymbol{\tau}_h)-(\boldsymbol{\theta_\sigma},\boldsymbol{\tau}_h)\\&\quad+\int_0^t(b(u(s))\boldsymbol{\theta_q}(s),\boldsymbol{\tau}_h)ds,\end{aligned} \tag{5.7b}$$

$$-(\boldsymbol{\rho_\sigma},\nabla v_h)+\langle\hat{\boldsymbol{\rho}}_{\boldsymbol{\sigma}}\cdot\boldsymbol{\nu},v_h\rangle_{\partial\mathcal{T}_h}=0, \tag{5.7c}$$

$$\langle\hat{\rho}_u,\mu_h\rangle_{\partial\Omega}=0, \tag{5.7d}$$

$$\langle\hat{\boldsymbol{\rho}}_{\boldsymbol{\sigma}}\cdot\boldsymbol{\nu},m_h\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega}=0, \tag{5.7e}$$

for all $(v_h,\boldsymbol{w}_h,\boldsymbol{\tau}_h,\mu_h,m_h)\in(V_h\times\boldsymbol{W}_h\times\boldsymbol{W}_h\times M_h\times M_h)$.

Note that, $\langle\hat{\boldsymbol{\rho}}_{\boldsymbol{\sigma}}\cdot\boldsymbol{\nu},\mu\rangle_{\partial\mathcal{T}_h}=\langle\boldsymbol{\rho_\sigma}\cdot\boldsymbol{\nu}+\tau(\rho_u-\hat{\rho}_u),\mu\rangle_{\partial\mathcal{T}_h}$ for all $\mu\in P_k(F)$.

We now present the estimates of ρ's in accordance with the subsequent lemma:

Lemma 5.2.2. *There is a positive constant C that does not rely on h and k such that $\forall\, t\in(0,T]$, the inequality below is valid for $l=0,1,2$*

$$\left\|\frac{\partial^l\rho_u}{\partial t^l}\right\|+\left\|\frac{\partial^l\boldsymbol{\rho_q}}{\partial t^l}\right\|+\left\|\frac{\partial^l\boldsymbol{\rho_\sigma}}{\partial t^l}\right\|\le Ch^{k+1}. \tag{5.8}$$

Proof. **STEP I: Estimates of $\|\boldsymbol{\rho_\sigma}(t)\|$ and $\|\boldsymbol{\rho_q}(t)\|$.**

Use $\boldsymbol{\tau}_h = \boldsymbol{\rho_\sigma}$ in (5.7b). Combining Cauchy-Schwarz inequality with $0 < a_* \leq a \leq M$ and $|b(u)| \leq M$ yield

$$\begin{aligned}\|\boldsymbol{\rho_\sigma}\|^2 &= (a(u)\boldsymbol{\rho_q}, \boldsymbol{\rho_\sigma}) + \int_0^t (b(u(s))\boldsymbol{\rho_q}(s), \boldsymbol{\rho_\sigma})ds - (a(u)\boldsymbol{\theta_q}, \boldsymbol{\rho_\sigma}) + (\boldsymbol{\theta_\sigma}, \boldsymbol{\rho_\sigma}) \\ &\quad - \int_0^t (b(u(s))\boldsymbol{\theta_q}(s), \boldsymbol{\rho_\sigma})ds, \\ &\leq C\Big[\|\boldsymbol{\rho_q}\| + \|\boldsymbol{\theta_q}\| + \|\theta_\sigma\| + \int_0^t (\|\boldsymbol{\rho_q}(s)\| + \|\boldsymbol{\theta_q}(s)\|)ds\Big]\|\boldsymbol{\rho_\sigma}\|,\end{aligned}$$

and hence,

$$\|\boldsymbol{\rho_\sigma}\| \leq C\Big[\|\boldsymbol{\rho_q}\| + \|\boldsymbol{\theta_q}\| + \|\theta_\sigma\| + \int_0^t (\|\boldsymbol{\rho_q}(s)\| + \|\boldsymbol{\theta_q}(s)\|)ds\Big]. \tag{5.9}$$

Next, take $\boldsymbol{w}_h = \boldsymbol{\rho_\sigma}$, $\boldsymbol{\tau}_h = \boldsymbol{\rho_q}$, $v_h = \rho_u$, $\mu_h = -\hat{\boldsymbol{\rho}}_{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu}$ and $m_h = -\hat{\rho}_u$ in (5.7a), (5.7b), (5.7c), (5.7d) and (5.7e), respectively. Then, by combining the resultant equations, we get

$$\begin{aligned}(a(u)\boldsymbol{\rho_q}, \boldsymbol{\rho_q}) + \|\hat{\rho}_u - \rho_u\|_\tau^2 = (a(u)\boldsymbol{\theta_q}, \boldsymbol{\rho_q}) - (\boldsymbol{\theta_\sigma}, \boldsymbol{\rho_q}) + \int_0^t \big[(b(u(s))\boldsymbol{\theta_q}(s), \boldsymbol{\rho_q}) \\ - (b(u(s))\boldsymbol{\rho_q}(s), \boldsymbol{\rho_q})\big]ds.\end{aligned}$$

Further, use of the boundedness of a and b yield

$$\|\hat{\rho}_u - \rho_u\|_\tau + \|\boldsymbol{\rho_q}\| \leq C\Big[\|\boldsymbol{\theta_q}\| + \|\boldsymbol{\theta_\sigma}\| + \int_0^t (\|\boldsymbol{\theta_q}(s)\| + \|\boldsymbol{\rho_q}(s)\|)ds\Big].$$

Finally, use of Gronwall's lemma along with (5.9) give the following desired estimates:

$$\|\hat{\rho}_u - \rho_u\|_\tau + \|\boldsymbol{\rho_\sigma}(t)\| + \|\boldsymbol{\rho_q}(t)\| \leq C\Big[\|\boldsymbol{\theta_q}(t)\| + \|\boldsymbol{\theta_\sigma}(t)\| + \int_0^t \|\boldsymbol{\theta_q}(s)\|ds\Big].$$

STEP II: Estimates of $\|\rho_u(t)\|$.

To derive this estimate, we use the following dual problem:

$$\begin{aligned}-\nabla \cdot (a(u)\nabla\psi) &= \Lambda \qquad \text{in } \Omega, \\ \psi &= 0 \qquad \text{on } \partial\Omega,\end{aligned}$$

with the following regularity

$$\|\psi\|_{H^2(\Omega)} \leq \|\Lambda\|.$$

The following is a mixed form of the dual problem, which will be used ahead:

$$\boldsymbol{\phi} = -\nabla\psi \qquad \text{in } \Omega, \tag{5.10a}$$

$$\boldsymbol{p} = a(u)\boldsymbol{\phi} \qquad \text{in } \Omega, \tag{5.10b}$$

$$\nabla\cdot\boldsymbol{p} = \Lambda \qquad \text{in } \Omega. \tag{5.10c}$$

Then, taking $\Lambda = \rho_u$ and using L^2 inner product between (5.10c) and ρ_u, there holds

$$\|\rho_u\|^2 = (\rho_u, \rho_u) = (\rho_u, \nabla\cdot\boldsymbol{p}),$$

using Lemma 1.4.6, we obtain

$$\begin{aligned}\|\rho_u\|^2 &= (\rho_u, \nabla\cdot\boldsymbol{\Pi_W p}) + \langle\rho_u, \tau(\Pi_V\psi - \psi)\rangle_{\partial\mathcal{T}_h}\\ &= (\boldsymbol{\rho_q}, \boldsymbol{\Pi_W p}) + \langle\hat{\rho}_u, \boldsymbol{\Pi_W p}\cdot\boldsymbol{\nu}\rangle_{\partial\mathcal{T}_h} + \langle\rho_u, \tau(\Pi_V\psi - \psi)\rangle_{\partial\mathcal{T}_h}. \qquad \text{by(5.7a)}\end{aligned}$$

By continuity of $\boldsymbol{p}\cdot\boldsymbol{\nu}$ and (5.7d), we arrive at

$$\begin{aligned}\|\rho_u\|^2 &= (\boldsymbol{\rho_q}, \boldsymbol{\Pi_W p}) + \langle\hat{\rho}_u, (\boldsymbol{\Pi_W p} - \boldsymbol{p})\cdot\boldsymbol{\nu}\rangle_{\partial\mathcal{T}_h} + \langle\rho_u, \tau(\Pi_V\psi - \psi)\rangle_{\partial\mathcal{T}_h}\\ &= (\boldsymbol{\rho_q}, \boldsymbol{\Pi_W p}) + \langle\tau(\rho_u - \hat{\rho}_u), \Pi_V\psi\rangle_{\partial\mathcal{T}_h} + \langle\boldsymbol{\rho_\sigma}\cdot\boldsymbol{\nu}, P_M\psi\rangle_{\partial\mathcal{T}_h} \qquad \text{by (1.1c), (5.7e)}\\ &= (\boldsymbol{\rho_q}, \boldsymbol{\Pi_W p}) + \langle\tau(\rho_u - \hat{\rho}_u) - \boldsymbol{\rho_\sigma}\cdot\boldsymbol{\nu}, \Pi_V\psi\rangle_{\partial\mathcal{T}_h} + \langle\boldsymbol{\rho_\sigma}\cdot\boldsymbol{\nu}, \psi\rangle_{\partial\mathcal{T}_h}\\ &\quad + (\boldsymbol{\rho_\sigma}, \nabla\Pi_V\psi) \qquad \text{by (5.7c)}\\ &= (\boldsymbol{\rho_q}, \boldsymbol{\Pi_W p}) + (\boldsymbol{\rho_\sigma}, \nabla\psi) \qquad \text{by (1.1a)}\\ &= (\boldsymbol{\rho_q}, \boldsymbol{\Pi_W p} - \boldsymbol{p}) + (\boldsymbol{\rho_q}, \boldsymbol{p}) - (\boldsymbol{\rho_\sigma}, \boldsymbol{\phi}) \qquad \text{by (5.10a)}\\ &= (\boldsymbol{\rho_q}, \boldsymbol{\Pi_W p} - \boldsymbol{p}) + (\boldsymbol{\rho_q}, \boldsymbol{p}) - (\boldsymbol{\rho_\sigma}, \boldsymbol{\phi} - \boldsymbol{I}_h^k\boldsymbol{\phi}) - (\boldsymbol{\rho_\sigma}, \boldsymbol{I}_h^k\boldsymbol{\phi}).\end{aligned}$$

Now, use (5.7b) with $\boldsymbol{\tau}_h = \boldsymbol{I}_h^k\boldsymbol{\phi}$, to obtain

$$\begin{aligned}\|\rho_u\|^2 &= (\boldsymbol{\rho_q}, \boldsymbol{\Pi_W p} - \boldsymbol{p}) + (\boldsymbol{\rho_q}, \boldsymbol{p}) - (a(u)\boldsymbol{\rho_q}, \boldsymbol{I}_h^k\boldsymbol{\phi}) + \int_0^t (b(u(s))(\boldsymbol{\theta_q}(s) - \boldsymbol{\rho_q}(s)), \boldsymbol{I}_h^k\boldsymbol{\phi})ds\\ &\quad + (a(u)\boldsymbol{\theta_q}, \boldsymbol{I}_h^k\boldsymbol{\phi}) - (\boldsymbol{\theta_\sigma}, \boldsymbol{I}_h^k\boldsymbol{\phi})\end{aligned}$$

$$= (\boldsymbol{\rho_q}, \boldsymbol{\Pi_W p} - \boldsymbol{p}) + (a(u)\boldsymbol{\rho_q}, \boldsymbol{\phi}) - (a(u)\boldsymbol{\rho_q}, \boldsymbol{I}_h^k\boldsymbol{\phi}) + \int_0^t (b(u(s))(\boldsymbol{\theta_q}(s)$$

$$- \boldsymbol{\rho_q}(s)), \boldsymbol{I}_h^k\boldsymbol{\phi})ds + (a(u)\boldsymbol{\theta_q}, \boldsymbol{I}_h^k\boldsymbol{\phi}) - (\boldsymbol{\theta_\sigma}, \boldsymbol{I}_h^k\boldsymbol{\phi}) \qquad \text{by (5.10b)}$$

$$= (\boldsymbol{\rho_q}, \boldsymbol{\Pi_W p} - \boldsymbol{p}) + (a(u)\boldsymbol{\rho_q}, \boldsymbol{\phi} - \boldsymbol{I}_h^k\boldsymbol{\phi}) + (a(u)\boldsymbol{\theta_q}, \boldsymbol{I}_h^k\boldsymbol{\phi}) - (\boldsymbol{\theta_\sigma}, \boldsymbol{I}_h^k\boldsymbol{\phi})$$

$$+ \int_0^t (b(u(s))(\boldsymbol{\theta_q}(s) - \boldsymbol{\rho_q}(s)), \boldsymbol{I}_h^k\boldsymbol{\phi})ds.$$

The following inequality is then obtained by applying the Cauchy-Schwarz inequality.

$$\|\rho_u\|^2 \leq \|\boldsymbol{\rho_q}\|\|\boldsymbol{\Pi_W p} - \boldsymbol{p}\| + C\|\boldsymbol{\rho_q}\|\|\boldsymbol{\phi} - \boldsymbol{I}_h^k\boldsymbol{\phi}\| + C\|\boldsymbol{\theta_q}\|\|\boldsymbol{I}_h^k\boldsymbol{\phi}\|_{\boldsymbol{H}^1(\Omega)} + \|\boldsymbol{\theta_\sigma}\|\|\boldsymbol{I}_h^k\boldsymbol{\phi}\|_{\boldsymbol{H}^1(\Omega)}$$

$$+ C\int_0^t \Big(\|\boldsymbol{\theta_q}(s)\| + \|\boldsymbol{\rho_q}(s)\|\Big)\|\boldsymbol{I}_h^k\boldsymbol{\phi}\|_{\boldsymbol{H}^1(\Omega)}.$$

Now, use of (1.2), estimates of $\|\boldsymbol{\rho_q}\|$, Lemma 1.4.1, elliptic regularity, $\|\boldsymbol{\phi}\|_{\boldsymbol{H}^1(\Omega)} \leq M\|\boldsymbol{p}\|_{\boldsymbol{H}^1(\Omega)}$ and $\|\boldsymbol{p}\|_{\boldsymbol{H}^1(\Omega)} \leq \|\psi\|_{H^2(\Omega)}$, yield the desired result.

STEP III: $\left\|\dfrac{\partial \rho_u}{\partial t}\right\|$**,** $\left\|\dfrac{\partial \boldsymbol{\rho_q}}{\partial t}\right\|$ **and** $\left\|\dfrac{\partial \boldsymbol{\rho_\sigma}}{\partial t}\right\|$**.**

We will begin by differentiating (5.7a)-(5.7e) w.r.t. t, to obtain

$$(\boldsymbol{\rho_{q_t}}, \boldsymbol{w_h}) - (\rho_{u_t}, \nabla\cdot\boldsymbol{w_h}) + \langle\hat{\rho}_{u_t}, \boldsymbol{w_h}\cdot\boldsymbol{\nu}\rangle_{\partial\mathcal{T}_h} = 0, \tag{5.11a}$$

$$\begin{aligned}(a_u(u)u_t\boldsymbol{\rho_q} + a(u)\boldsymbol{\rho_{q_t}}, \boldsymbol{\tau_h}) - (\boldsymbol{\rho_{\sigma_t}}, \boldsymbol{\tau}_h) + (b(u)\boldsymbol{\rho_q}(t), \boldsymbol{\tau_h}) &= (a_u(u)u_t\boldsymbol{\theta_q} + a(u)\boldsymbol{\theta_{q_t}}, \boldsymbol{\tau}_h)\\ &\quad -(\boldsymbol{\theta_{\sigma_t}}, \boldsymbol{\tau}_h) + (b(u)\boldsymbol{\theta_q}(t), \boldsymbol{\tau_h}),\end{aligned} \tag{5.11b}$$

$$-(\boldsymbol{\rho_{\sigma_t}}, \nabla v_h) + \langle\boldsymbol{\rho_{\sigma_t}}\cdot\boldsymbol{\nu}, v_h\rangle_{\partial\mathcal{T}_h} = 0, \tag{5.11c}$$

$$\langle\hat{\rho}_{u_t}, \mu_h\rangle_{\partial\Omega} = 0, \tag{5.11d}$$

$$\langle\hat{\boldsymbol{\rho}}_{\boldsymbol{\sigma_t}}\cdot\boldsymbol{\nu}, m_h\rangle_{\partial\mathcal{T}_h\setminus\partial\mathcal{T}_h} = 0, \tag{5.11e}$$

for all $(v_h, \boldsymbol{w}_h, \boldsymbol{\tau}_h, \mu_h, m_h) \in V_h \times \boldsymbol{W}_h \times \boldsymbol{W}_h \times M_h \times M_h$. Now, adding (5.11) after taking $\boldsymbol{w}_h = \boldsymbol{\rho_{\sigma_t}}$, $\boldsymbol{\tau}_h = \boldsymbol{\rho_{q_t}}$, $v_h = \rho_{u_t}$, $\mu_h = -\hat{\boldsymbol{\rho}}_{\boldsymbol{\sigma}_t}\cdot\boldsymbol{\nu}$ and $m_h = -\hat{\rho}_{u_t}$ and simplifying using Cauchy Schwarz inequality and the Gronwall's lemma, will give the estimate of $\|\boldsymbol{\rho_{q_t}}\|$, whereas taking $\boldsymbol{\tau}_h = \boldsymbol{\rho_{\sigma_t}}$ in (5.11b) gives the estimate of $\|\boldsymbol{\rho_{\sigma_t}}\|$.

For the estimate of $\|\rho_{u_t}\|$, we begin by taking into account the dual problem (5.10) with $\Lambda = \rho_{u_t}$ along with:

$$\|\psi\|_{H^2(\Omega)} \leq \|\rho_{u_t}\|. \tag{5.12}$$

Consider,

$$\|\rho_{u_t}\|^2 = (\rho_{u_t}, \nabla \cdot \boldsymbol{p}).$$

Lastly, we make use of (5.11), (1.2) and (5.12) along with the estimates of $\|\boldsymbol{\rho}_{\boldsymbol{q_t}}\|$ and $\|\boldsymbol{\rho}_{\boldsymbol{\sigma_t}}\|$ to yield the desired estimates.

STEP IV: Estimates of $\left\|\frac{\partial^2 \rho_u}{\partial t^2}\right\|$**,** $\left\|\frac{\partial^2 \boldsymbol{\rho}_{\boldsymbol{q}}}{\partial t^2}\right\|$ **and** $\left\|\frac{\partial^2 \boldsymbol{\rho}_{\boldsymbol{\sigma}}}{\partial t^2}\right\|$**.**

We will differentiate (5.11) again, w.r.t. t, and proceed in a similar fashion to obtain the desired estimates. □

Theorem 5.2.3. *For $t \in (0, T]$, if $u, u_t, u_{tt} \in L^\infty(H^{k+2}(\mathcal{T}_h))$ and $l = 0, 1, 2$ then irrespective of the values of h and k, there is a positive constant C such that*

$$\left\|\frac{\partial^l \eta_u}{\partial t^l}\right\| + \left\|\frac{\partial^l \boldsymbol{\eta}_{\boldsymbol{q}}}{\partial t^l}\right\| + \left\|\frac{\partial^l \boldsymbol{\eta}_{\boldsymbol{\sigma}}}{\partial t^l}\right\| \leq Ch^{k+1} \tag{5.13}$$

$$\left\|I_h^{k-1}\left(\frac{\partial^l \eta_u}{\partial t^l}\right)\right\| \leq Ch^{k+2}. \tag{5.14}$$

Proof. The inequality (5.13) can be obtained with the help of (1.2), Lemma 5.2.2 and the triangle inequality.

For the estimates of $\|I_h^{k-1}\eta_u\|$, the following dual problem is considered in $\Omega \times (0, T]$

$$\begin{aligned} \boldsymbol{\phi} &= -\nabla \psi, \\ \boldsymbol{p} &= a\boldsymbol{\phi}, \\ \nabla \cdot \boldsymbol{p} &= \theta, \end{aligned}$$

which satisfies the elliptic regularity

$$\|\psi\|_{H^2(\Omega)} \leq \|\theta\|.$$

Now, using (5.6a) and proceeding as in [39], conclude the proof.

$$\begin{aligned} (I_h^{k-1}\eta_u, \theta) &= (I_h^{k-1}\eta_u, \nabla \cdot \boldsymbol{p}) \\ &= (\eta_u, \nabla \cdot \mathbf{\Pi}_{k-1}^{RT}\boldsymbol{p}) \end{aligned}$$

$$= (\boldsymbol{\eta_q}, \nabla\cdot\boldsymbol{\Pi}_{k-1}^{RT}\boldsymbol{p} - \boldsymbol{p}) - (a\boldsymbol{\eta_q}, \nabla\psi)$$
$$\leq Ch^{k+2}\|\theta\|.$$

Similar procedure can be followed for $l = 1, 2$. □

5.3 A Priori Error Estimates for Semidiscrete Scheme

To prove Theorem 5.2.1, we decompose the error in terms of $\tilde{u}_h$, $\tilde{\boldsymbol{q}}_h$, $\tilde{\boldsymbol{\sigma}}_h$, $\hat{\tilde{u}}_h$and $\hat{\tilde{\boldsymbol{\sigma}}}_h$ as below

$$e_u = u - u_h = (u - \tilde{u}_h) - (u_h - \tilde{u}_h) = \eta_u - \xi_u,$$

similarly, we will decompose $\boldsymbol{e_q}$, $\boldsymbol{e_\sigma}$, $\hat{e}_u$ and $\hat{\boldsymbol{e}}_{\boldsymbol{\sigma}}$ in terms of η's and ξ's. With the help of this decomposition, (5.5) can be rewritten as

$$(\boldsymbol{\xi_q}, \boldsymbol{w_h}) - (\xi_u, \nabla\cdot\boldsymbol{w_h}) + \langle\hat{\xi}_u, \boldsymbol{w_h}\cdot\boldsymbol{\nu}\rangle_{\partial\mathcal{T}_h} = 0, \quad (5.15a)$$

$$(a(u_h)\boldsymbol{\xi_q}, \boldsymbol{\tau_h}) - (\boldsymbol{\xi_\sigma}, \boldsymbol{\tau}_h) + \int_0^t (b(u_h(s))\boldsymbol{\xi_q}(s), \boldsymbol{\tau}_h)ds = -((a(u) - a(u_h))\tilde{\boldsymbol{q}}_h, \boldsymbol{\tau}_h) - \int_0^t ((b(u(s)) - b(u_h(s)))\tilde{\boldsymbol{q}}_h, \boldsymbol{\tau}_h), \quad (5.15b)$$

$$(\xi_{u_{tt}}, v_h) - (\boldsymbol{\xi_\sigma}, \nabla v_h) + \langle\hat{\boldsymbol{\xi}}_{\boldsymbol{\sigma}}\cdot\boldsymbol{\nu}, v_h\rangle_{\partial\mathcal{T}_h} = (f(u) - f(u_h), v_h) - (\eta_{u_{tt}}, v_h), \quad (5.15c)$$

$$\langle\hat{\xi}_u, \mu_h\rangle_{\partial\Omega} = 0, \quad (5.15d)$$

$$\langle\hat{\boldsymbol{\xi}}_{\boldsymbol{\sigma}}\cdot\boldsymbol{\nu}, m_h\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega} = 0, \quad (5.15e)$$

for all $(v_h, \boldsymbol{w}_h, \boldsymbol{\tau}_h, \mu_h, m_h) \in V_h \times \boldsymbol{W}_h \times \boldsymbol{W}_h \times M_h \times M_h$.

For any w in $[0, t]$, let us define $\bar{w}$ as:

$$\bar{w}(t) = \int_0^t w(s)ds.$$

Clearly, $\bar{w}_t = w$ and $\bar{w}(0) = 0$.

Now, we present a series of lemma's which will help to prove Theorem 5.2.1.

Lemma 5.3.1. *([134]) If $u \in L^\infty(H^2)$ and $\boldsymbol{q} \in L^\infty(\boldsymbol{H}^1)$, then there exists $C = C(\boldsymbol{q})$, such that*

$$\|\tilde{\boldsymbol{q}}_h\|_\infty + \|\tilde{\boldsymbol{q}}_{h_t}\|_\infty + \|\tilde{\boldsymbol{q}}_{h_{tt}}\|_\infty \leq C(\boldsymbol{q}).$$

Proof. From the elliptic projection with memory, Lemma 1.4.5 and Lemma 1.4.10, we have the following

$$\begin{aligned}
\|\tilde{\boldsymbol{q}}_{h_{tt}}\|_\infty &\leq \|\tilde{\boldsymbol{q}}_{h_{tt}} - \boldsymbol{R}\boldsymbol{q}_{tt}\|_\infty + \|\boldsymbol{R}\boldsymbol{q}_{tt}\|_\infty \\
&\leq Ch^{-1}\|\tilde{\boldsymbol{q}}_{h_{tt}} - \boldsymbol{R}\boldsymbol{q}_{tt}\| + \|\boldsymbol{q}_{tt}\|_\infty \\
&\leq Ch^{-1}\left(\|\tilde{\boldsymbol{q}}_{h_{tt}} - \boldsymbol{q}_{tt}\| + \|\boldsymbol{q}_{tt} - \boldsymbol{R}\boldsymbol{q}_{tt}\|\right) + C \\
&\leq Ch^{-1}\left(h + h\right) + C \\
&\leq C.
\end{aligned}$$

Similarly, the estimates of $\|\tilde{\boldsymbol{q}}_h\|_\infty$ and $\|\tilde{\boldsymbol{q}}_{h_t}\|_\infty$ can be derived, see Lemma 3.3.2. □

Lemma 5.3.2. *There is a positive constant C independent of h and k such that*

$$\|\xi_u(t)\|^2 + \|\bar{\boldsymbol{\xi}}_q(t)\|^2 + \|(\bar{\hat{\xi}}_u - \bar{\xi}_u)(t)\|_\tau^2 \leq C\Bigg(\|\xi_u(0)\|^2 + \|a^{1/2}\bar{\boldsymbol{\xi}}_q(0)\|^2 + \|\bar{\hat{\xi}}_u(0) - \bar{\xi}_u(0)\|_\tau^2 + \int_0^T (\|\eta_u(t)\|^2 + \|\eta_{u_t}(t)\|^2)dt\Bigg).$$

Proof. We integrate (5.15b), (5.15c) and (5.15e) from 0 to t and then, choose $\boldsymbol{w}_h = \bar{\boldsymbol{\xi}}_{\boldsymbol{\sigma}}$, $\boldsymbol{\tau}_h = \boldsymbol{\xi}_{\boldsymbol{q}}$, $v_h = \xi_u$, $\mu_h = -\bar{\hat{\boldsymbol{\xi}}}_{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu}$ and $m_h = -\hat{\xi}_u$ in (5.15) and add them, to obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left(a_*\|\bar{\boldsymbol{\xi}}_q\|^2 + \|\xi_u\|^2 + \|\bar{\hat{\xi}}_u - \bar{\xi}_u\|_\tau^2\right) &= (\eta_{u_t}, \xi_u) - \int_0^t ((a(u(s)) - a(u_h(s))\tilde{\boldsymbol{q}}_h(s), \boldsymbol{\xi}_{\boldsymbol{q}}(s))ds \\
&+ \int_0^t (f(u(s)) - f(u_h(s)), \xi_u(s))ds - \int_0^t \int_0^s (b(u_h(\gamma))\boldsymbol{\xi}_{\boldsymbol{q}}(\gamma), \boldsymbol{\xi}_{\boldsymbol{q}}(s))d\gamma ds \\
&+ \int_0^t \int_0^s ((b(u(\gamma)) - b(u_h(\gamma)))\tilde{\boldsymbol{q}}_h(s), \boldsymbol{\xi}_{\boldsymbol{q}}(s))d\gamma ds.
\end{aligned}$$

It follows from integrating aforementioned inequality that

$$\begin{aligned}
a_*\|\bar{\boldsymbol{\xi}}_q\|^2 + \|\xi_u\|^2 + \|\bar{\hat{\xi}}_u - \bar{\xi}_u\|_\tau^2 &\leq \|\xi_u(0)\|^2 + \|a^{1/2}\bar{\boldsymbol{\xi}}_q(0)\|^2 + \|\bar{\hat{\xi}}_u(0) - \bar{\xi}_u(0)\|_\tau^2 \\
&+ 2\Bigg[\int_0^t (\eta_{u_s}, \xi_u)ds + \int_0^t \int_0^s (f(u(\gamma)) - f(u_h(\gamma)), \xi_u(\gamma))d\gamma ds
\end{aligned}$$

$$- \int_0^t \int_0^s \int_0^\gamma (b(u_h(\gamma^*))\boldsymbol{\xi_q}(\gamma^*), \boldsymbol{\xi_q}(\gamma))d\gamma^* d\gamma ds - \int_0^t \int_0^s (a(u(\gamma)) - a(u_h(\gamma))\tilde{\boldsymbol{q}}_h(\gamma),$$

$$\boldsymbol{\xi_q}(\gamma))d\gamma ds + \int_0^t \int_0^s \int_0^\gamma (b(u(\gamma^*)) - b(u_h(\gamma^*))\tilde{\boldsymbol{q}}_h(\gamma), \boldsymbol{\xi_q}(\gamma))d\gamma^* d\gamma ds\Big],$$

which can be written as

$$\begin{aligned} a_* \|\bar{\boldsymbol{\xi}}_q\|^2 + \|\xi_u\|^2 + \|\bar{\hat{\xi}}_u - \bar{\xi}_u\|_\tau^2 &\le \|\xi_u(0)\|^2 + \|a^{1/2}\bar{\boldsymbol{\xi}}_q(0)\|^2 + \|\bar{\hat{\xi}}_u(0) - \bar{\xi}_u(0)\|_\tau^2 \\ &+2\Big[\int_0^t (\eta_{u_s}, \xi_u)ds + \int_0^t \int_0^s (f(u(\gamma)) - f(u_h(\gamma)), \xi_u(\gamma))d\gamma ds - I_1 - I_2 + I_3\Big], \end{aligned}$$

where,

$$\begin{aligned} I_2 &= \int_0^t \int_0^s (a(u(\gamma)) - a(u_h(\gamma))\tilde{\boldsymbol{q}}_h(\gamma), \boldsymbol{\xi_q}(\gamma))d\gamma ds \\ &= \int_0^t (a(u(s)) - a(u_h(s)))\tilde{\boldsymbol{q}}_h(s), \bar{\boldsymbol{\xi}}_q(s))ds - \int_0^t \int_0^s \frac{d}{d\gamma}\left((a(u(\gamma)) - a(u_h(\gamma))\tilde{\boldsymbol{q}}_h(\gamma)\right)\bar{\boldsymbol{\xi}}_q(\gamma)d\gamma ds \\ &\le \int_0^t \|\eta_u(s) + \xi_u(s)\| \|\bar{\boldsymbol{\xi}}_{\boldsymbol{q}}(s)\| ds - \int_0^t \int_0^s \|\eta_u(\gamma) + \xi_u(\gamma)\| \|\bar{\boldsymbol{\xi}}_{\boldsymbol{q}}(\gamma)\| d\gamma ds. \end{aligned}$$

Simplifying the other terms in a similar fashion and applying the Gronwall's lemma will finish the proof. □

Lemma 5.3.3. *There is a positive constant C independent of h and k such that*

$$\begin{aligned} \|\xi_{u_t}(t)\|^2 + \|\boldsymbol{\xi_\sigma}(t)\|^2 + \|\boldsymbol{\xi_q}(t)\|^2 + \|\hat{\xi}_u - \xi_u\|_\tau^2 &\le C\bigg(\|\xi_{u_t}(0)\|^2 + \|\boldsymbol{\xi_q}(0)\|^2 \\ &+ \|\xi_u(0)\|^2 + \|(\hat{\xi}_u - \xi_u)(0)\|_\tau^2 + \int_0^T \left(\|\eta_u(t)\|^2 + \|\eta_{u_{tt}}(t)\|^2\right) dt\bigg). \end{aligned}$$

Proof. To begin with, we differentiate (5.15a) with respect to t and then choose $\boldsymbol{w_h} = \boldsymbol{\xi_\sigma}$, $\boldsymbol{\tau_h} = \boldsymbol{\xi_{q_t}}$, $v_h = \xi_{u_t}$ in (5.15a), (5.15b), (5.15c) respectively. Now, differentiate (5.15d) with respect to t and choose $\mu = -\hat{\boldsymbol{\xi}}_{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu}$ and $\mu = -\hat{\xi}_{u_t}$ in (5.15d) and (5.15e), respectively. Then adding the resulting equations, we obtain

$$\begin{aligned} (a(u_h)\boldsymbol{\xi_q}, \boldsymbol{\xi_{q_t}}) + \frac{1}{2}\frac{d}{dt}\left(\|\hat{\xi}_u - \xi_u\|_\tau^2 + \|\xi_{u_t}\|^2\right) + \int_0^t (b(u_h(s))\boldsymbol{\xi_q}(s), \boldsymbol{\xi_{q_t}})ds = -(\eta_{u_{tt}}, \xi_{u_t}) \\ - ((a(u) - a(u_h))\tilde{\boldsymbol{q}}_h, \boldsymbol{\xi_{q_t}}) - \int_0^t ((b(u(s)) - b(u_h(s)))\tilde{\boldsymbol{q}}_h(s), \boldsymbol{\xi_{q_t}})ds + (f(u) - f(u_h), \xi_{u_t}), \end{aligned}$$

which yields the following inequality

$$C\frac{d}{dt}\left(\|\boldsymbol{\xi_q}\|^2+\|\hat{\xi}_u-\xi_u\|_\tau^2+\|\xi_{u_t}\|^2\right)\leq -\frac{d}{dt}((a(u)-a(u_h))\tilde{\boldsymbol{q}}_h,\boldsymbol{\xi_q})-(\eta_{u_{tt}},\xi_{u_t})$$
$$+((a_u(u)u_t-a_u(u_h)u_{h_t})\tilde{\boldsymbol{q}}_h,\boldsymbol{\xi_q})+((a(u)-a(u_h))\tilde{\boldsymbol{q}}_{h_t},\boldsymbol{\xi_q})-\int_0^t\frac{d}{dt}(b(u_h(s))\boldsymbol{\xi_q}(s),\boldsymbol{\xi_q})ds$$
$$-\int_0^t\frac{d}{dt}((b(u(s))-b(u_h(s)))\tilde{\boldsymbol{q}}_h(s),\boldsymbol{\xi_q})ds+(f(u)-f(u_h),\xi_{u_t}).$$

Next, we will use the Cauchy Schwarz inequality, Leibnitz's Theorem and Lemma 5.3.1 along with the fact that a, a_u, b, b_u and f are Lipschitz continuous with respect to u. Then, integrating the resulting equations from 0 to t, yields the following inequality

$$\|\boldsymbol{\xi_q}\|^2+\|\hat{\xi}_u-\xi_u\|_\tau^2+\|\xi_{u_t}\|^2\leq C(\|\boldsymbol{\xi_q}(0)\|^2+\|\xi_u(0)\|^2+\|\eta_u\|^2+\|\xi_u\|^2)$$
$$+C\int_0^t\left(\|\eta_u\|^2+\|\xi_u\|^2+\|\boldsymbol{\xi_q}\|^2+\|\eta_{u_{tt}}\|^2+\|\xi_{u_t}\|^2\right)ds.$$

Finally, a use of the Gronwall's lemma yields the following result

$$\|\xi_{u_t}\|^2+\|\boldsymbol{\xi_q}\|^2+\|\hat{\xi}_u-\xi_u\|_\tau^2\leq C\left(\|\boldsymbol{\xi_q}(0)\|^2+\|\xi_u(0)\|^2+\int_0^T\left(\|\eta_u(t)\|^2+\|\eta_{u_{tt}}(t)\|^2\right)dt\right).$$

Now, choosing $\boldsymbol{\tau_h}=\boldsymbol{\xi_\sigma}$ in (5.15b) and then proceeding as above will give

$$\|\boldsymbol{\xi_\sigma}\|\leq C\left(\|\eta_u\|+\|\xi_u\|\right).$$

Combining the last two inequalities yields the desired result. □

Proof of Theorem 5.2.1: To prove the theorem, we use triangle's inequality, Theorem 5.2.3, Lemma 5.3.2 and Lemma 5.3.3.

□

5.4 Post-processing

To begin with, we define the function $\psi(s)\in H^2(\Omega)\cap H_0^1(\Omega),\ s\leq t$ to be the solution of the following problem:

$$\psi_{ss}-\nabla\cdot\left(a(u)\nabla\psi+\int_s^t b(u(\gamma))\nabla\psi(\gamma)d\gamma\right)+a_u(u)\boldsymbol{q}\cdot\nabla\psi+\int_s^t b_u(u(\gamma))\boldsymbol{q}(\gamma)\cdot\nabla\psi(\gamma)d\gamma$$

$$+ f_u(u)\psi = 0, \qquad (5.16)$$

with the following conditions:

$$\begin{aligned} \psi(x,s) &= 0 && \text{on } \partial\Omega, s \le t, \\ \psi(x,t) &= 0 && \text{in } x \in \Omega, \\ \psi_s(x,t) &= \lambda(x) && \text{in } x \in \Omega. \end{aligned}$$

Lemma 5.4.1. *(Regularity Results) There exists a constant C dependent on the data of the above problem, such that it satisfies the following inequality:*

$$\|\psi(s)\|_{L^\infty(H^1)} + \|\psi_s(s)\|_{L^\infty(L^2)} \le C\|\lambda\|, \qquad (5.17a)$$

$$\|\underline{\psi}(s)\|_2 \le C\|\lambda\|, \qquad (5.17b)$$

where, $\underline{\psi}(s) = \int_s^t \psi(\gamma)d\gamma$.

Proof. The first inequality can be proved using a simple kickback argument [42]. To prove the second inequality, we begin by integrating (5.16) from s to t, noting that $-\psi_s(s) = \underline{\psi}_{ss}(s)$ and using the boundary condition, to obtain

$$\begin{aligned} &\underline{\psi}_{ss}(s) - \nabla\cdot\left(a(u)\nabla\underline{\psi} - \int_s^t a_u(u(\gamma))u_\gamma(\gamma)\nabla\underline{\psi}(\gamma)d\gamma + \int_s^t\int_\gamma^t b(u(\gamma^*))\nabla\psi(\gamma^*)d\gamma^*d\gamma\right) \\ &= \int_s^t a_u(u(\gamma))\boldsymbol{q}(\gamma)\cdot\nabla\psi(\gamma)d\gamma + \int_s^t\int_\gamma^t b_u(u(\gamma^*))\boldsymbol{q}(\gamma^*)\cdot\nabla\psi(\gamma^*)d\gamma^*d\gamma \\ &\quad + \int_s^t f_u(u(\gamma))\psi(\gamma)d\gamma - \lambda. \end{aligned}$$

Next, we assume the following elliptic regularity on $\underline{\psi}$ [42], and use (5.17a) to get

$$\begin{aligned} &\|\underline{\psi}\|_2 \le C\left\|\nabla\cdot\left(a(u)\nabla\underline{\psi}\right)\right\| \\ &\le C\Bigg(\|\psi_s(s)\| + \|\lambda\| + \left\|\int_s^t a_u(u(\gamma))u_\gamma(\gamma)\nabla\underline{\psi}(\gamma)d\gamma\right\| + \left\|\int_s^t\int_\gamma^t b(u(\gamma^*))\nabla\psi(\gamma^*)d\gamma^*d\gamma\right\| \\ &\quad + \left\|\int_s^t a_u(u(\gamma))\boldsymbol{q}(\gamma)\cdot\nabla\psi(\gamma)d\gamma\right\| + \left\|\int_s^t\int_\gamma^t b_u(u(\gamma^*))\boldsymbol{q}(\gamma^*)\cdot\nabla\psi(\gamma^*)d\gamma^*d\gamma\right\| \\ &\quad + \left\|\int_s^t f_u(u(\gamma))\psi(\gamma)d\gamma\right\|\Bigg) \end{aligned}$$

$$\leq C\,\|\lambda\|\,.$$

This concludes the proof. □

Lemma 5.4.2. *For the method of the form (5.3), there exists a positive constant C which does not rely on h and k such that $\forall t \in (0,T]$, the inequality below is valid*

$$\|I_h^{k-1}e_u\|_{L^2(K)} \leq Ch^{k+2}, \tag{5.18}$$

where, I_h^{k-1} is L^2-projection onto the space of polynomial for degree at most $k-1$.

Proof. Since, $e_u = \eta_u - \xi_u$, therefore, $\|I_h^{k-1}e_u\| \leq \|I_h^{k-1}\eta_u\| + \|I_h^{k-1}\xi_u\|$.
For the estimates of $\|I_h^{k-1}\xi_u\|$, we start by rewriting (5.16) in the following mixed form:

$$\boldsymbol{\phi}(s) = \nabla\psi(s) \quad \text{in } \Omega,\ s \leq t, \tag{5.19a}$$

$$\boldsymbol{p}(s) = a(u)\boldsymbol{\phi}(s) + \int_s^t b(u(\gamma))\boldsymbol{\phi}(\gamma)d\gamma \quad \text{in } \Omega,\ s \leq t, \tag{5.19b}$$

$$\psi_{ss}(s) + \nabla\cdot\boldsymbol{p}(s) = a_u(u)\boldsymbol{q}\cdot\boldsymbol{\phi} + \int_s^t b_u(u(\gamma))\boldsymbol{q}(\gamma)\cdot\boldsymbol{\phi}(\gamma)d\gamma + f_u(u)\psi \quad \text{in } \Omega,\ s \leq t, \tag{5.19c}$$

$$\psi(s) = 0 \quad \text{on } \partial\Omega,\ s \leq t, \tag{5.19d}$$

$$\psi(t) = 0 \quad \text{in } \Omega, \tag{5.19e}$$

$$\psi_s(t) = I_h^{k-1}\xi_u(t) \quad \text{in } \Omega. \tag{5.19f}$$

We begin by taking the inner product of (5.19c) with $I_h^{k-1}\xi_u(s)$, to obtain

$$\begin{aligned}
&(\psi_{ss}(s), I_h^{k-1}\xi_u(s)) - (\nabla\cdot\boldsymbol{p}(s), I_h^{k-1}\xi_u(s)) = (a_u(u)\boldsymbol{q}\cdot\boldsymbol{\phi}, I_h^{k-1}\xi_u(s)) \\
&+ (f_u(u)\psi, I_h^{k-1}\xi_u(s)) + \left(\int_s^t b_u(u(\gamma))\boldsymbol{q}(\gamma)\cdot\boldsymbol{\phi}(\gamma)d\gamma, I_h^{k-1}\xi_u(s)\right).
\end{aligned}$$

Now,

$$\begin{aligned}
&\frac{d}{ds}\left[(\psi_s(s), I_h^{k-1}\xi_u(s)) - (\psi(s), I_h^{k-1}\xi_{u_s}(s))\right] \\
&= (\psi_{ss}(s), I_h^{k-1}\xi_u(s)) - (\psi(s), I_h^{k-1}\xi_{u_{ss}}(s)) \\
&= -(\psi(s), I_h^{k-1}\xi_{u_{ss}}(s)) + (\nabla\cdot\boldsymbol{p}(s), I_h^{k-1}\xi_u(s)) + (a_u(u)\boldsymbol{q}\cdot\boldsymbol{\phi}, I_h^{k-1}\xi_u(s))
\end{aligned}$$

$$+ (f_u(u)\psi, I_h^{k-1}\xi_u(s)) + \left(\int_s^t b_u(u(\gamma))\boldsymbol{q}(\gamma) \cdot \boldsymbol{\phi}(\gamma) d\gamma, I_h^{k-1}\xi_u(s) \right).$$

Use of (5.15) and intermediate projections, see [139], yield the following equality

$$\frac{d}{ds}\left[(\psi_s(s), I_h^{k-1}\xi_u(s)) - (\psi(s), I_h^{k-1}\xi_{u_s}(s))\right] = (\xi_{u_{ss}}(s), I_h^k\psi(s) - \psi(s))$$
$$- (\xi_{u_{ss}}(s), I_h^{k-1}\psi(s) - \psi(s)) + (\boldsymbol{\xi_q}(s), \boldsymbol{\Pi}_{k-1}^{RT}\boldsymbol{p}(s) - \boldsymbol{p}(s)) + (a(u)\boldsymbol{\xi_q}(s), \boldsymbol{\phi}(s) - \boldsymbol{I}_h^k\boldsymbol{\phi}(s))$$
$$+ (\boldsymbol{\xi_\sigma}(s), \boldsymbol{I}_h^k\boldsymbol{\phi}(s) - \boldsymbol{\phi}(s)) + (\boldsymbol{\xi_\sigma}(s), \nabla(\psi - I_h^k\psi)(s)) + \langle \hat{\boldsymbol{\xi}}_{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu}, I_h^k\psi\rangle - (\eta_{u_{ss}}, I_h^k\psi)$$
$$- ((a(u_h) - a(u))\boldsymbol{q}_h, \boldsymbol{I}_h^k\boldsymbol{\phi}(s)) + (a_u(u)\boldsymbol{q} \cdot \boldsymbol{\phi}, I_h^{k-1}\xi_u(s))$$
$$- (f(u_h) - f(u), I_h^k\psi) + (f_u(u)\psi, I_h^{k-1}\xi_u) - \int_0^s ((b(u_h(\gamma)) - b(u(\gamma)))\boldsymbol{q}_h(\gamma), \boldsymbol{I}_h^k\boldsymbol{\phi}(s))d\gamma$$
$$+ \int_s^t (b_u(u(\gamma))\boldsymbol{q}(\gamma) \cdot \boldsymbol{\phi}(\gamma), I_h^{k-1}\xi_u(s))d\gamma + \int_0^s (b(u(\gamma))\boldsymbol{\xi_q}(\gamma), \boldsymbol{I}_h^k\boldsymbol{\phi}(s))d\gamma$$
$$- \int_s^t (b(u(\gamma))\boldsymbol{\phi}(\gamma), \boldsymbol{\xi_q}(s))d\gamma.$$

Taking $\xi_u(0) = \xi_{u_s}(0) = 0$ and integrating the equation from 0 to t followed by a change of order of integration of the last term, we obtain

$$\|I_h^{k-1}\xi_u\|^2 = \int_0^t \Big[(\xi_{u_{ss}}(s), I_h^k\psi(s) - \psi(s)) - (\xi_{u_{ss}}(s), I_h^{k-1}\psi(s) - \psi(s))$$
$$+ (\boldsymbol{\xi_q}(s), \boldsymbol{\Pi}_{k-1}^{RT}\boldsymbol{p}(s) - \boldsymbol{p}(s)) + (a(u)\boldsymbol{\xi_q}(s), \boldsymbol{\phi}(s) - \boldsymbol{I}_h^k\boldsymbol{\phi}(s)) + (\boldsymbol{\xi_\sigma}(s), \boldsymbol{I}_h^k\boldsymbol{\phi}(s) - \boldsymbol{\phi}(s))$$
$$+ (\boldsymbol{\xi_\sigma}(s), \nabla(\psi - I_h^k\psi)(s)) + \langle \hat{\boldsymbol{\xi}}_{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu}, I_h^k\psi\rangle - (\eta_{u_{ss}}, I_h^k\psi)$$
$$+ \big[- ((a(u_h) - a(u))\boldsymbol{q}_h, \boldsymbol{I}_h^k\boldsymbol{\phi}(s)) + (a_u(u)\boldsymbol{q} \cdot \boldsymbol{\phi}, I_h^{k-1}\xi_u(s))\big] + \big[- (f(u_h) - f(u), I_h^k\psi)$$
$$+ (f_u(u)\psi, I_h^{k-1}\xi_u)\big]\Big]ds - \int_0^t \int_s^t (b(u(\gamma))\boldsymbol{\xi_q}(s), \boldsymbol{\phi}(\gamma) - \boldsymbol{I}_h^k\boldsymbol{\phi}(\gamma))d\gamma ds$$
$$+ \Big[\int_0^t \int_s^t (b_u(u(\gamma))\boldsymbol{q}(\gamma) \cdot \boldsymbol{\phi}(\gamma), P^{k-1}e_u(s))d\gamma ds$$
$$- \int_0^t \int_0^s ((b(u_h(\gamma)) - b(u(\gamma)))\boldsymbol{q}_h(\gamma), \boldsymbol{P}^k\boldsymbol{\phi}(s))d\gamma ds\Big]$$
$$= \int_0^t [E_1 + E_2 + E_3 + E_4 + E_5 + E_6 + E_7 + E_8 + E_9 + E_{10}]ds + E_{11} + E_{12}. \tag{5.20}$$

Cauchy Schwarz's inequality and (5.17a) show

$$E_1 + E_2 \leq Ch^{k+2}\|I_h^{k-1}\xi_u(s)\|.$$

Next, a use of identity $\int_0^t f(z)g(z)dz = f(0)\bar{g}(0) + \int_0^t f_z(z)\bar{g}(z)dz$ along with (5.17b), yield

$$|E_3 + E_4 + E_5 + E_6| \leq Ch^{k+2}\|I_h^{k-1}\xi_u\|.$$

Use of (5.5e), properties of the projection I_h and (5.17a) give

$$|E_7| \leq \|\hat{\boldsymbol{\xi}}_{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu}\|_{\partial K}\|I_h^k\psi - \psi\|_{\partial K} \leq Ch^{k+2}\|I_h^{k-1}\xi_u\|.$$

We rewrite E_8 as follows

$$\begin{aligned}(\eta_{u_{ss}}, I_h^k\psi) &= (\eta_{u_{ss}}, I_h^k\psi - I_h^{k-1}\psi) + (\eta_{u_{ss}}, I_h^{k-1}\psi)\\ &= (\eta_{u_{ss}}, I_h^k\psi - I_h^{k-1}\psi) + (I_h^{k-1}\eta_{u_{ss}}, I_h^{k-1}\psi)\\ &\leq \|\eta_{u_{ss}}\|\|I_h^k\psi - I_h^{k-1}\psi\| + \|I_h^{k-1}\eta_{u_{ss}}\|\|I_h^{k-1}\psi\|\\ &\leq Ch^{k+2}\|I_h^{k-1}\xi_u\|.\end{aligned}$$

Next, we have

$$\begin{aligned}E_9 &= -((a(u_h) - a(u))\boldsymbol{q}_h, \boldsymbol{P}^k\boldsymbol{\phi}(s)) + (a_u(u)\boldsymbol{q}\cdot\boldsymbol{\phi}, P^{k-1}e_u(s))\\ &= -((a(u_h) - a(u))(\boldsymbol{q}_h - \boldsymbol{q}), \boldsymbol{P}^k\boldsymbol{\phi}(s)) - ((a(u_h) - a(u))\boldsymbol{q}, \boldsymbol{P}^k\boldsymbol{\phi}(s) - \boldsymbol{\phi}(s))\\ &\quad + (a_u(u)\boldsymbol{q}\cdot\boldsymbol{\phi}, P^{k-1}e_u(s) - e_u(s)) - ((a(u_h) - a(u)), \boldsymbol{q}\cdot\boldsymbol{\phi}(s)) + (a_u(u)\boldsymbol{q}\cdot\boldsymbol{\phi}, e_u(s)).\end{aligned}$$

Use of Taylor's series expansion yields

$$\begin{aligned}E_9 &= -((a(u_h) - a(u))(\boldsymbol{q}_h - \boldsymbol{q}), \boldsymbol{P}^k\boldsymbol{\phi}(s)) - ((a(u_h) - a(u))\boldsymbol{q}, \boldsymbol{P}^k\boldsymbol{\phi}(s) - \boldsymbol{\phi}(s))\\ &\quad + (a_u(u)\boldsymbol{q}\cdot\boldsymbol{\phi}, P^{k-1}e_u(s) - e_u(s)) + (a_u(u) - a_u(u_h + \lambda(u_h - u))e_u(s), \boldsymbol{q}\cdot\boldsymbol{\phi}).\end{aligned}$$

Use of generalized Holder's inequality yields

$$|E_9| \leq Ch^{k+2}\|I_h^{k-1}\xi_u\|.$$

For E_{10}, a similar procedure can be followed, whereas for E_{11}, a change of order of integration followed by a similar procedure as for E_9 will give the estimates. Finally, use of boundedness of b shows

$$|E_{12}| \leq M\left|\int_0^t \left(\int_s^t \boldsymbol{\phi}(\gamma) - \boldsymbol{I}_h^k\boldsymbol{\phi}(\gamma)\right)d\gamma, \boldsymbol{\xi}_{\boldsymbol{q}}(s)ds\right|$$

$$= M \left| \int_0^t \bar{\boldsymbol{\phi}}(\gamma) - \boldsymbol{I}_h^k \bar{\boldsymbol{\phi}}(\gamma), \boldsymbol{\xi}_{\boldsymbol{q}}(s) ds \right|$$
$$\leq Ch^{k+2} \|\bar{\psi}(s)\|_2$$
$$\leq Ch^{k+2} \|I_h^{k-1} \xi_u\| \qquad \text{(by(5.17b))}.$$

Substituting in (5.20), we get

$$\|I_h^{k-1} \xi_u(t)\|^2 \leq Ch^{k+2} \int_0^t \|I_h^{k-1} \xi_u(s)\| ds.$$

Use of Young's inequality and Gronwall's lemma yield the following estimate

$$\|I_h^{k-1} \xi_u(t)\| \leq Ch^{k+2}. \tag{5.21}$$

Finally, (5.21) and (5.14) conclude the proof of the theorem. □

5.5 Fully Discrete Scheme

In this section, a completely discrete scheme is derived for the problem (5.3), based on the central difference scheme, along with the mid-point rule to approximate the integral term. We first divide the interval $[0, T]$ into M equally spaced sub-intervals by the following points

$$0 = t_0 < t_1 < ... < t_M = T,$$

with $t_n = n\Delta t$, $\Delta t = T/M$, be the time step. We begin by defining the following notations,

$$\Upsilon U^n = \frac{U^{n+1} + U^n}{2}, \quad \Phi U^n = \frac{U^{n+1} + 2U^n + U^{n-1}}{4} = \frac{\Upsilon U^n + \Upsilon U^{n-1}}{2},$$
$$\partial_t \Upsilon U^n = \frac{U^{n+1} - U^n}{\Delta t}, \ \partial_t^2 U^n = \frac{U^{n+1} - 2U^n + U^{n-1}}{\Delta t^2}, \ \delta_t U^n = \frac{\partial_t \Upsilon U^n + \partial_t \Upsilon U^{n-1}}{2},$$
$$E_h^n(\boldsymbol{Q}) = \Delta t \sum_{j=0}^{n-1} b(U^{j+1/2}) \Upsilon \boldsymbol{Q}^j, \quad \Upsilon E_h^n(\boldsymbol{Q}) = \frac{E_h^{n+1}(\boldsymbol{Q}) + E_h^n(\boldsymbol{Q})}{2}.$$

For $1 \leq n \leq M$, find $(U^n, \boldsymbol{Q}^n, \boldsymbol{S}^n, \hat{U}^n) \in (V_h \times \boldsymbol{W}_h \times \boldsymbol{W}_h \times M_h)$, such that, for any $(v_h, \boldsymbol{w}_h, \boldsymbol{\tau}_h, \mu_h, m_h) \in (V_h \times \boldsymbol{W}_h \times \boldsymbol{W}_h \times M_h \times M_h)$, we require

$$\frac{2}{\Delta t}(\partial_t \Upsilon U^0, v_h) - (\Upsilon \boldsymbol{S}^0, \nabla v_h) + \langle \Upsilon \hat{\boldsymbol{S}}^0 \cdot \boldsymbol{\nu}, v_h \rangle_{\partial \mathcal{T}_h} = (\Upsilon f(U^0) + \frac{2}{\Delta t} u_1, v_h), \tag{5.22a}$$

$$\langle \Upsilon \hat{\boldsymbol{S}}^0 \cdot \boldsymbol{\nu}, \mu \rangle_{\partial \mathcal{T}_h \setminus \partial\Omega} = 0, \quad (5.22b)$$

$$(\Upsilon \boldsymbol{Q}^n, \boldsymbol{w}_h) - (\Upsilon U^n, \nabla \cdot \boldsymbol{w}_h) + \langle \Upsilon \hat{U}^n, \boldsymbol{w}_h \cdot \boldsymbol{\nu} \rangle_{\partial \mathcal{T}_h} = 0, \quad (5.22c)$$

$$(a(\Upsilon U^n)\Upsilon \boldsymbol{Q}^n, \boldsymbol{\tau}_h) - (\Upsilon \boldsymbol{S}^n, \boldsymbol{\tau}_h) + (\Upsilon E_h^n(\boldsymbol{Q}), \boldsymbol{\tau}_h) = 0, \quad (5.22d)$$

$$(\partial_t^2 U^n, v_h) - (\Phi \boldsymbol{S}^n, \nabla v_h) + \langle \Phi \hat{\boldsymbol{S}}^n \cdot \boldsymbol{\nu}, v_h \rangle_{\partial \mathcal{T}_h} = (\Phi f(U^n), v_h), \quad (5.22e)$$

$$\langle \Upsilon \hat{U}^n, \mu_h \rangle_{\partial\Omega} = 0, \quad (5.22f)$$

$$\langle \Phi \hat{\boldsymbol{S}}^n \cdot \boldsymbol{\nu}, m_h \rangle_{\partial \mathcal{T}_h \setminus \partial\Omega} = 0, \quad (5.22g)$$

where, (5.22c), (5.22d), (5.22f) are defined for $n \geq 0$, and (5.22e), (5.22g) are defined for $n \geq 1$.

Theorem 5.5.1. *Let* u *be the solution of* (5.1), $u, u_t, u_{tt} \in L^\infty(H^{k+2}(\mathcal{T}_h))$, $u_h(0) = U^0 = \Pi_V u_0$ *and* $\boldsymbol{q}_h(0) = Q^0 = -I_h \nabla u_0$, *then for all* $1 \leq n \leq M$,

$$\|\partial_t \Upsilon \zeta_u^n\| + \|\Upsilon \boldsymbol{\zeta}_{\boldsymbol{q}}^n\| + \|\Upsilon \boldsymbol{\zeta}_{\boldsymbol{\sigma}}^n\| + \|\Upsilon \hat{\zeta}_u^n\|_\tau \leq O(h^{k+1} + \Delta t^2), \quad (5.23a)$$

$$\|\zeta_u^{n+1}\| \leq O(h^{k+1} + \Delta t^2). \quad (5.23b)$$

Proof. We will write $\|u(t_n) - U^n\| \leq \|u(t_n) - u_h(t_n)\| + \|u_h(t_n) - U^n\|$. We only need to derive the estimates of $\|u_h(t_n) - U^n\|$, which will be denoted by $\|\zeta_u^n\|$. Similarly, $\boldsymbol{\zeta}_{\boldsymbol{q}}^n$, $\boldsymbol{\zeta}_{\boldsymbol{\sigma}}^n$ and $\hat{\zeta}_u^n$.

Now, using (5.3) and (5.22), we have the following

$$\frac{2}{\Delta t}(\partial_t \Upsilon \zeta_u^0, v_h) - (\Upsilon \boldsymbol{\zeta}_{\boldsymbol{\sigma}}^0, \nabla v_h) + \langle \Upsilon \hat{\boldsymbol{\zeta}}_{\boldsymbol{\sigma}}^0 \cdot \boldsymbol{\nu}, v_h \rangle_{\partial \mathcal{T}_h} = \Bigg(\frac{2}{\Delta t}\left(\partial_t \Upsilon u_h^0 - u_1\right) - \Upsilon u_{h_{tt}}^0 + \Upsilon f(u_h^0) - \Upsilon f(U^0), v_h \Bigg), \quad (5.24a)$$

$$\langle \Upsilon \hat{\boldsymbol{\zeta}}_{\boldsymbol{\sigma}}^0 \cdot \boldsymbol{\nu}, \mu \rangle_{\partial \mathcal{T}_h \setminus \partial\Omega} = 0, \quad (5.24b)$$

$$(\Upsilon \boldsymbol{\zeta}_{\boldsymbol{q}}^n, \boldsymbol{w}_h) - (\Upsilon \zeta_u^n, \nabla \cdot \boldsymbol{w}_h) + \langle \Upsilon \hat{\zeta}_u^n, \boldsymbol{w}_h \cdot \boldsymbol{\nu} \rangle_{\partial \mathcal{T}_h} = 0, \quad (5.24c)$$

$$(A^n, \boldsymbol{\tau}_h) - (\Upsilon \boldsymbol{\zeta}_{\boldsymbol{\sigma}}^n, \boldsymbol{\tau}_h) + (\Upsilon I^n(\boldsymbol{q_h}), \boldsymbol{\tau}_h) = (\Upsilon E_h^n(\boldsymbol{Q}), \boldsymbol{\tau}_h), \quad (5.24d)$$

$$(\partial_t^2 \zeta_u^n, v_h) - (\Phi \boldsymbol{\zeta}_{\boldsymbol{\sigma}}^n, \nabla v_h) + \langle \Phi \hat{\boldsymbol{\zeta}}_{\boldsymbol{\sigma}}^n \cdot \boldsymbol{\nu}, v_h \rangle_{\partial \mathcal{T}_h} = \left(\partial_t^2 u_h^n - \Phi u_{h_{tt}}^n, v_h\right) + \left(\Phi f(u_h^n) - \Phi f(U^n), v_h\right), \quad (5.24e)$$

$$\langle \Upsilon \hat{\zeta}_u^n, \mu_h \rangle_{\partial\Omega} = 0, \quad (5.24f)$$

$$\langle \Phi \hat{\boldsymbol{\zeta}}^n_{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu}, m_h \rangle_{\partial \mathcal{T}_h \setminus \partial \Omega} = 0, \tag{5.24g}$$

for all $(v_h, \boldsymbol{w}_h, \boldsymbol{\tau}_h, \mu_h, m_h) \in (V_h \times \boldsymbol{W}_h \times \boldsymbol{W}_h \times M_h \times M_h)$. Here,

$$I^n(\boldsymbol{q_h}) = \int_0^{t_n} b(u_h(s))\boldsymbol{q}_h(s)ds$$

and

$$A^n = (a(\Upsilon u_h^n)\Upsilon \boldsymbol{q}_h^n - a(\Upsilon U^n)\Upsilon \boldsymbol{Q}^n).$$

We begin with the proof of (5.23a). Let $n \geq 1$; then, we start by subtracting (5.24c) from itself after replacing n by $n-1$ and then, dividing the resulting equation by $2\Delta t$. Secondly, we will perform the same operations in (5.24f). Next, in (5.24d), we will replace n by $n-1$ and take the average of the resulting equation with itself. Now, take $\boldsymbol{w}_h = \Phi \boldsymbol{\zeta}^n_{\boldsymbol{\sigma}}$, $\boldsymbol{\tau}_h = \delta_t \boldsymbol{\zeta}^n_{\boldsymbol{q}}$, $v_h = \delta_t \zeta^n_u$, $\mu_h = -\Phi \hat{\boldsymbol{\zeta}}^n_{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu}$ and $m_h = -\delta_t \hat{\zeta}^n_u$ in (5.24c), (5.24d), (5.24e), (5.24f) and (5.24g), respectively and then, add (5.24c)-(5.24e), (5.24f) and (5.24g) to obtain

$$\alpha_* \left(\Phi \boldsymbol{\zeta}^n_{\boldsymbol{q}}, \delta_t \boldsymbol{\zeta}^n_{\boldsymbol{q}}\right) + \left(\partial_t^2 \zeta^n_u, \delta_t \zeta^n_u\right) + \left\langle \Phi \hat{\zeta}^n_u - \Phi \zeta^n_u, \tau(\delta_t \hat{\zeta}^n_u - \delta_t \zeta^n_u) \right\rangle \leq \left(\Phi E^n_h(\boldsymbol{Q}), \delta_t \boldsymbol{\zeta}^n_{\boldsymbol{q}}\right)$$
$$- \left(\Phi I^n(\boldsymbol{q_h}), \delta_t \boldsymbol{\zeta}^n_{\boldsymbol{q}}\right) + \left(\Phi u^n_{h_{tt}} - \partial_t^2 u^n_h, \delta_t \boldsymbol{\zeta}^n_{\boldsymbol{q}}\right) + \left(\Phi f(u^n_h) - \Phi f(U^n), \delta_t \zeta^n_u\right).$$

The equation can be further written as

$$\frac{1}{2\Delta t}\Big[\Upsilon \|\partial_t \Upsilon \zeta^n_u\|^2 - \|\partial_t \Upsilon \zeta^{n-1}_u\|^2 + \|\Upsilon \boldsymbol{\zeta}^n_{\boldsymbol{q}}\|^2 - \|\Upsilon \boldsymbol{\zeta}^{n-1}_{\boldsymbol{q}}\|^2 + \|\Upsilon \hat{\zeta}^n_u - \Upsilon \zeta^n_u\|^2_\tau$$
$$- \|\Upsilon \hat{\zeta}^{n-1}_u - \Upsilon \zeta^{n-1}_u\|^2_\tau\Big] \leq \left(\Phi E^n_h(\boldsymbol{q_h}), \delta_t \boldsymbol{\zeta}^n_{\boldsymbol{q}}\right) - \left(\Phi I^n(\boldsymbol{q_h}), \delta_t \boldsymbol{\zeta}^n_{\boldsymbol{q}}\right) + \left(\Phi E^n_h(\boldsymbol{\zeta_q}), \delta_t \boldsymbol{\zeta}^n_{\boldsymbol{q}}\right)$$
$$+ \left(\Phi u^n_{h_{tt}} - \partial_t^2 u^n_h, \delta_t \boldsymbol{\zeta}^n_{\boldsymbol{q}}\right) + \left(\Phi f(u^n_h) - \Phi f(U^n), \delta_t \zeta^n_u\right).$$

Now, multiplying the equation by $2\Delta t$ and adding from $n = 1$ to $n = m$, we obtain the following inequality

$$\|\partial_t \Upsilon \zeta^m_u\|^2 + \|\Upsilon \boldsymbol{\zeta}^m_{\boldsymbol{q}}\|^2 + \|\Upsilon \hat{\zeta}^m_u - \Upsilon \zeta^m_u\|^2_\tau \leq \|\partial_t \Upsilon \zeta^0_u\|^2 + \|\Upsilon \boldsymbol{\zeta}^0_{\boldsymbol{q}}\|^2 + \|\Upsilon \hat{\zeta}^0_u - \Upsilon \zeta^0_u\|^2_\tau$$
$$+ 2\Delta t \sum_{n=1}^{m} (J^n_1 + J^n_2 + J^n_3 + J^n_4), \tag{5.25}$$

where

$$J_1^n = \left(\Phi E_h^n(\boldsymbol{q_h}), \delta_t \boldsymbol{\zeta_q^n}\right) - (\Phi I^n(\boldsymbol{q_h}), \delta_t \boldsymbol{\zeta_q^n}),\ J_2^n = \left(\Phi E_h^n(\boldsymbol{\zeta_q}), \delta_t \boldsymbol{\zeta_q^n}\right),$$
$$J_3^n = \left(\Phi u_{h_{tt}}^n - \partial_t^2 u_h^n, \delta_t \boldsymbol{\zeta_q^n}\right)\ J_4^n = (\Phi f(u_h^n) - \Phi f(U^n), \delta_t \zeta_u^n).$$

For the estimates of $\|\partial_t \Upsilon \zeta_u^0\|^2 + \|\Upsilon \boldsymbol{\zeta_q^0}\|^2 + \|\Upsilon \hat{\zeta}_u^0 - \Upsilon \zeta_u^0\|_\tau^2$, we consider the following equations

$$\frac{2}{\Delta t}(\partial_t \Upsilon \zeta_u^0, v_h) - (\Upsilon \boldsymbol{\zeta_\sigma^0}, \nabla v_h) + \langle \Upsilon \hat{\boldsymbol{\zeta}}_{\boldsymbol{\sigma}}^0 \cdot \boldsymbol{\nu}, v_h \rangle_{\partial \mathcal{T}_h} = \left(\frac{2}{\Delta t}\left(\partial_t \Upsilon u_h^0 - u_1\right) - \Upsilon u_{h_{tt}}^0, v_h\right), \tag{5.26a}$$

$$(\Upsilon \boldsymbol{\zeta_q^0}, \boldsymbol{w}_h) - (\Upsilon \zeta_u^0, \nabla \cdot \boldsymbol{w}_h) + \langle \Upsilon \hat{\zeta}_u^0, \boldsymbol{w}_h \cdot \boldsymbol{\nu} \rangle_{\partial \mathcal{T}_h} = 0, \tag{5.26b}$$
$$(A^0, \boldsymbol{\tau}_h) - (\Upsilon \boldsymbol{\zeta_\sigma^0}, \boldsymbol{\tau}_h) + \left(I_1^0, \boldsymbol{\tau}_h\right) ds = \left(\Upsilon E_h^0(\boldsymbol{Q}), \boldsymbol{\tau}_h\right), \tag{5.26c}$$
$$\langle \Upsilon \hat{\zeta}_u^0, \mu_h \rangle_{\partial \Omega} = 0, \tag{5.26d}$$
$$\langle \Upsilon \hat{\boldsymbol{\zeta}}_{\boldsymbol{\sigma}}^0 \cdot \boldsymbol{\nu}, m_h \rangle_{\partial \mathcal{T}_h \setminus \partial \Omega} = 0, \tag{5.26e}$$

for all $(v_h, \boldsymbol{w}_h, \boldsymbol{\tau}_h, \mu_h, m_h) \in (V_h \times \boldsymbol{W}_h \times \boldsymbol{W}_h \times M_h \times M_h)$. We take $v_h = \Upsilon \zeta_u^0$, $\boldsymbol{w}_h = \Upsilon \boldsymbol{\sigma}^0$, $\boldsymbol{\tau}_h = \Upsilon \boldsymbol{\zeta_q^0}$, $\mu_h = -\Upsilon \hat{\boldsymbol{\zeta}}_{\boldsymbol{\sigma}}^0 \cdot \boldsymbol{\nu}$ and $m_h = -\Upsilon \delta_t \hat{\zeta}_u^0$ in (5.26a), (5.26b), (5.26c), (5.26d) and (5.26e), respectively and add the resulting equations, to get the following inequality

$$\|\partial_t \Upsilon \zeta_u^0\|^2 + \|\Upsilon \boldsymbol{\zeta_q^0}\|^2 + \|\Upsilon \hat{\zeta}_u^0 - \Upsilon \zeta_u^0\|_\tau^2 \leq \frac{1}{2}\left(\Upsilon E_h^0(\boldsymbol{Q}), \Upsilon \boldsymbol{\zeta_q^0}\right) - \frac{1}{2}\int_0^{t_1} \left(b(t_1, s)\boldsymbol{q}_h(s), \Upsilon \boldsymbol{\zeta_q^0}\right) ds$$
$$+ \left(\frac{2}{\Delta t}\left(\partial_t \Upsilon u_h^0 - u_1\right) - \Upsilon u_{h_{tt}}^0, \Upsilon \zeta_u^0\right).$$

Now, proceeding in the similar way as to obtain (5.25) will prove that

$$\|\partial_t \Upsilon \zeta_u^0\|^2 + \|\Upsilon \boldsymbol{\zeta_q^0}\|^2 + \|\Upsilon \hat{\zeta}_u^0 - \Upsilon \zeta_u^0\|_\tau^2 \leq C(h^{2(k+1)} + \Delta t^4).$$

Next, for J_1^n, use of Theorem 5.2.1 along with quadrature error yield

$$\|\Phi E_h^n(\boldsymbol{q_h}) - \Phi I^n(\boldsymbol{q_h})\| \leq \|\Phi E_h^n(\boldsymbol{q}) - \Phi I^n(\boldsymbol{q}) - \Phi E_h^n(\boldsymbol{e_q}) + \Phi I^n(\boldsymbol{e_q})\|$$
$$\leq C(h^{k+1} + \Delta t^2).$$

Further, use of Young's inequality yields

$$\Delta t \sum_{n=1}^{m} |J_1^n| \leq C\left(h^{2(k+1)} + \Delta t^4\right) + \frac{\Delta t}{2} \sum_{n=1}^{m} \left\| \frac{\Upsilon \boldsymbol{\zeta}_{\boldsymbol{q}}^n - \Upsilon \boldsymbol{\zeta}_{\boldsymbol{q}}^{n-1}}{\Delta t} \right\|^2. \tag{5.27}$$

Use of Taylor's series expansion, along with Young's inequality, yield

$$\Delta t \sum_{n=1}^{m} |J_3^n| \leq C\left(h^{2(k+1)} + \Delta t^4\right) + \frac{1}{2} \sum_{n=1}^{m} \left\| \frac{\partial_t \Upsilon \zeta_u^n + \partial_t \Upsilon \zeta_u^{n-1}}{2} \right\|^2. \tag{5.28}$$

Use of (5.27) and (5.28) in (5.25) along with discrete Gronwall's lemma yield

$$\|\partial_t \Upsilon \zeta_u^m\|^2 + \|\Upsilon \boldsymbol{\zeta}_{\boldsymbol{q}}^m\|^2 + \|\Upsilon \hat{\zeta}_u^m - \Upsilon \zeta_u^m\|_\tau^2 \leq C\left(h^{2(k+1)} + \Delta t^4\right).$$

Finally, use of triangle inequality and Theorem 5.2.1, finish the proof of (5.23a).

Now, for the proof of (5.23b), we introduce the following notations:

$$\underset{\sim}{\phi}^0 = 0, \quad \underset{\sim}{\phi}^n = \Delta t \sum_{j=0}^{n-1} \Upsilon \phi^j, \quad \partial_t \Upsilon \underset{\sim}{\phi}^n = \Upsilon \phi^n, \quad \Delta t \sum_{j=0}^{n} \Phi \phi^j = \Upsilon \underset{\sim}{\phi}^n - \frac{\Delta t}{2} \Upsilon \phi^0.$$

Next, we multiply (5.24d), (5.24e) and (5.24g) by k, take summation over n and use (5.24a) and (5.24b) to get the following system of equations

$$(\Upsilon \boldsymbol{\zeta}_{\boldsymbol{q}}^n, \boldsymbol{w}_h) - (\Upsilon \zeta_u^n, \nabla \cdot \boldsymbol{w}_h) + \langle \Upsilon \hat{\zeta}_u^n, \boldsymbol{w}_h \cdot \boldsymbol{\nu} \rangle_{\partial \mathcal{T}_h} = 0, \tag{5.29a}$$

$$(\Delta t \sum_{j=0}^{n} A^j, \boldsymbol{\tau}_h) - (\Upsilon \underset{\sim}{\boldsymbol{\zeta}_{\boldsymbol{\sigma}}}^n, \boldsymbol{\tau}_h) + \left(\Upsilon \underset{\sim}{E_h}^n(\boldsymbol{\zeta}_{\boldsymbol{q}}^n), \boldsymbol{\tau}_h\right) = \left(\Upsilon \underset{\sim}{F_h}^n(\boldsymbol{q}_{\boldsymbol{h}}), \boldsymbol{\tau}_h\right), \tag{5.29b}$$

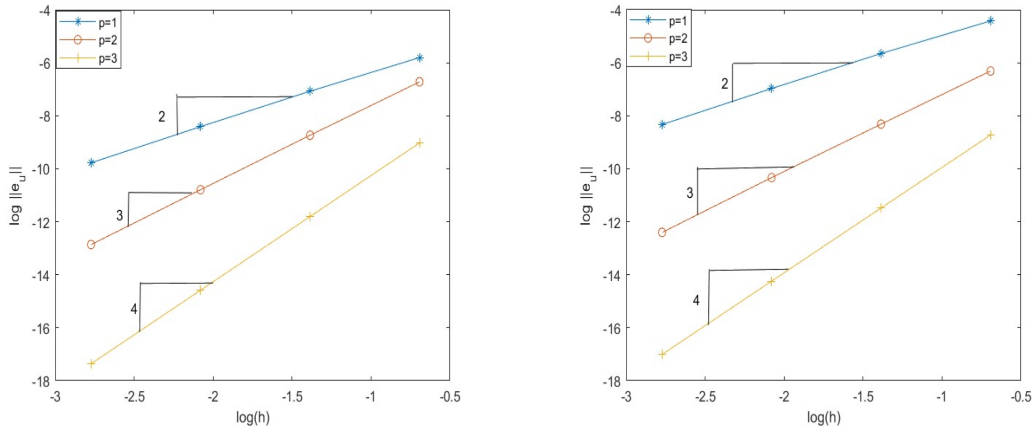
$$\begin{aligned}(\partial_t \Upsilon \zeta_u^n, v_h) - (\Upsilon \underset{\sim}{\boldsymbol{\zeta}_{\boldsymbol{\sigma}}}^n, \nabla v_h) + \langle \Upsilon \underset{\sim}{\hat{\boldsymbol{\zeta}}_{\boldsymbol{\sigma}}}^n \cdot \boldsymbol{\nu}, v_h \rangle_{\partial \mathcal{T}_h} &= \left(\Delta t \sum_{j=0}^{n} \left(\partial_t^2 u_h^j - \Phi u_{h_{tt}}^j\right), v_h\right) \\ &+ \left(\Delta t \sum_{j=0}^{n} (\Phi f(u_h^j) - \Phi f(U^j)), v_h\right),\end{aligned} \tag{5.29c}$$

$$\langle \Upsilon \hat{\zeta}_u^n, \mu_h \rangle_{\partial \Omega} = 0, \tag{5.29d}$$

$$\langle \Upsilon \underset{\sim}{\hat{\boldsymbol{\zeta}}_{\boldsymbol{\sigma}}}^n \cdot \boldsymbol{\nu}, m_h \rangle_{\partial \mathcal{T}_h \setminus \partial \Omega} = 0. \tag{5.29e}$$

Choose $\boldsymbol{w}_h = \Upsilon \underset{\sim}{\hat{\boldsymbol{\zeta}}_{\boldsymbol{\sigma}}}^n$, $\boldsymbol{\tau}_h = \Upsilon \boldsymbol{\zeta}_{\boldsymbol{q}}^n$, $v_h = \Upsilon \zeta_u^n$, $\mu_h = -\Upsilon \underset{\sim}{\hat{\boldsymbol{\zeta}}_{\boldsymbol{\sigma}}}^n \cdot \boldsymbol{\nu}$ and $m_h = -\Upsilon \hat{\zeta}_u^n$ in (5.29a), (5.29b), (5.29c), (5.29d) and (5.29e), respectively, and add the resulting equations. After simplifying as above, we attain the desired estimate.

□



(a) Example 1

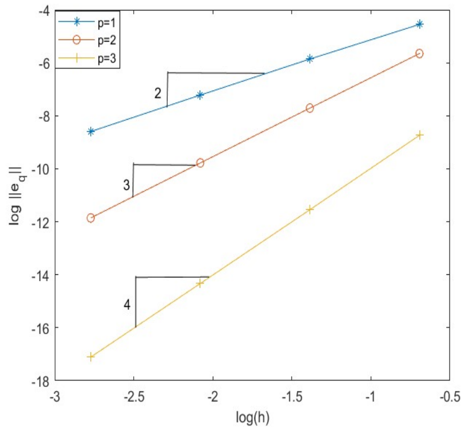
(b) Example 2

Figure 5.1: Convergence behaviour of $\|e_u\|$ at $t = 1$

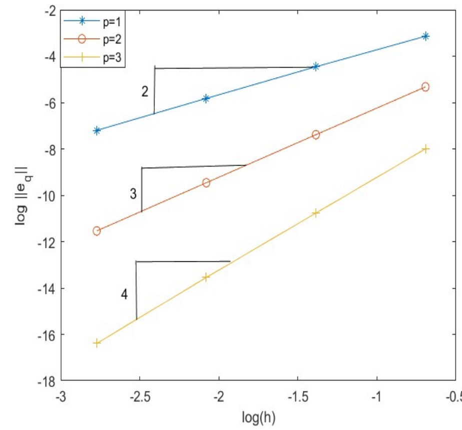
5.6 Numerical Results

This section consists of two numerical examples which are used to verify the theoretical results that are proved in the chapter. The examples consists of (5.1a)-(5.1c) for $\Omega = (0, 1)\times(0, 1)$ and $T = 1$. We have used central difference scheme to completely discretize the problem, along with the mid-point rule to approximate the integral term. Newton's method is used for the implementation of nonlinear terms. We note that the optimal order of convergence in the case of u and $\boldsymbol{q}$ and the super-convergence in the case of u_h^* predicted by our theory is achieved.

Example 1. Let $u(x, y, t) = t^2e^tx(1-x)y(1-y)$ and the coefficients be $a(u) = 1+u^2$, $b(u) = u$ and $f(u) = u - u^3 + g(x, y, t)$, where $g(x, y, t)$ is decided by the exact solution u. We compute the order of convergence for $e_u = u - u_h$, $\boldsymbol{e_q} = \boldsymbol{q} - \boldsymbol{q}_h$ and $e_u^* = u - u_h^*$ for the cases $k = 1$, $k = 2$ and $k = 3$ with different choices of h. In Figures 5.1a, 5.2a and 5.3a, we plot the computed error with the mesh sizes for different degrees of polynomials. We observe that the convergence rates for $\|e_u\|$, $\|\boldsymbol{e_q}\|$ and $\|e_u^*\|$ at $t = 1$ are of the order of $O(h^{k+1})$, $O(h^{k+1})$ and $O(h^{k+2})$, respectively. Table 5.1 gives the time convergence for u for the example for different time steps.



(a) Example 1



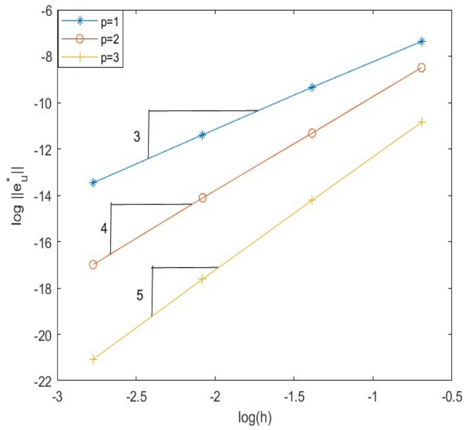
(b) Example 2

Figure 5.2: Convergence behaviour of $\|\boldsymbol{e_q}\|$ at $t = 1$

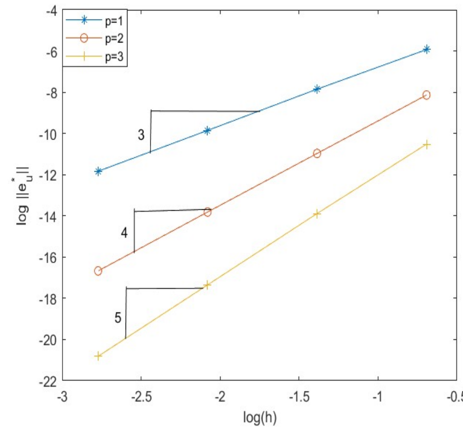
$\boldsymbol{\Delta t}$ $(h = 1/4)$	**Order** (Ex. 1)	**Order** (Ex. 2)
0.25	1.7945	1.8743
0.125	1.9023	1.9986
0.0625	2.0567	2.1789
0.03125	2.1842	2.2014

Table 5.1: Order of convergence for time

Example 2. Let $u(x, y, t) = t\sin(\pi t)\sin(\pi x)\sin(\pi y)$ and the coefficients be $a(u) = 1 + u^2$, $b(u) = u$ and $f(u) = u - u^3 + g(x, y, t)$, where $g(x, y, t)$ is decided by the exact solution u. We compute the order of convergence for $e_u = u - u_h$, $\boldsymbol{e_q} = \boldsymbol{q} - \boldsymbol{q}_h$ and $e_u^* = u - u_h^*$ for the cases $k = 1$, $k = 2$ and $k = 3$ with different choices of h. In Figures 5.1b, 5.2b and 5.3b, we plot the computed error with the mesh sizes for different degrees of polynomials. We observe that the convergence rates for $\|e_u\|$, $\|\boldsymbol{e_q}\|$ and $\|e_u^*\|$ at $t = 1$ are of the order of $O(h^{k+1})$, $O(h^{k+1})$ and $O(h^{k+2})$, respectively. Table 5.1 gives the time convergence for u for the example for different time steps.



(a) Example 1



(b) Example 2

Figure 5.3: Convergence behaviour of $\|e_u^*\|$ at $t = 1$

5.7 Conclusions

Due to various theoretical and computational benefits of the HDG method, it has been proposed and analyzed for nonlinear hyperbolic integro-differential equation (5.1). The nonlinear functions have been considered to be Lipschitz continuous to prove the a priori error estimates. Throughout this chapter, HDG and Ritz-Volterra projections have been used to derive the error estimates. Further, element-by-element post-processing has been proposed. It has been shown that the solution and its gradient achieved the optimal rate of convergence, that is, of order $k + 1$, $k \geq 0$ in the discretizing parameter h, whereas super-convergence has been achieved, that is, of order $k + 2$, $k \geq 1$, for the post-processed solution, when the function f was differentiable and its derivative was Lipschitz continuous. A fully discrete scheme has also been discussed, which is of order $O(h^{k+1} + \Delta t^2)$. Higher order fully discrete scheme can be achieved by using higher order difference scheme for the derivative term and higher order quadrature rule for the integral term. Finally, numerical results have been discussed. This analysis can be extended to a 3-dimensional domain by incorporating the changes accordingly.

Chapter 6

Conclusions and Future Directions

6.1 Concluding Remarks and Critical Analysis of the Results

This dissertation examines the HDG method for linear and nonlinear PIDEs and linear and non-linear hyperbolic integro-differential equations. Significant attention has been directed into the error analysis of the methodology used for the model problems. Optimum rate of convergence has been achieved for the scalar variable and its approximate gradient. A post-processing technique has been used on an element-by-element basis in order to enhance the rate of convergence. The theoretical findings have also been verified by performing numerical experiments for each model problem.

In Chapter 2, we have discussed the HDG method for approximating the solution of linear PIDE. We have used the HDG projection and the Ritz-Volterra projection as intermediate projection for the semi-discrete error analysis. The estimates of the Ritz-Volterra projection were derived by taking particular values of the test functions in the discrete formulation, whereas, the estimates of the scalar variable u, were derived by taking into account an elliptic dual problem that satisfies the elliptic regularity condition. Then, for a priori error estimates, the estimates were derived by taking particular values of the test functions. Finally, it was proved that all the unknown variables achieved the order of convergence $O(h^{k+1})$. The convergence rates there, were improved compared to [115].

For the estimates of the post-processed solution, a dual of linear parabolic-integro differential equation was taken into account and required regularity results were derived. This helped in achieving super-convergence. For the complete discretization of the scheme, the backward Euler method and the left rectangle rule are used to approximate the derivative and integral term, respectively. This helped in achieving convergence of order 1 in the temporal direction.

Finally, the theoretical findings were verified by a series of numerical examples in the 2-dimensional domain. It was verified that for the examples in consideration, the HDG approximation of the scalar and flux variable exhibited optimum order of convergence. Additionally, the post-processed approximation of the scalar variable demonstrated super-convergence. It was noted that the results can be extended in 3-dimensional domain.

In Chapter 3, we have discussed the HDG method to approximate the solution for a class of quasi-linear PIDE. In this case, for the semi-discrete error analysis, only the first order derivative of the nonlinear variables a and b, along with the Lipschitz continuity condition, has been considered, without taking their second order derivative. Then, to deal with the integral term, Ritz-Volterra projection of extended type was introduced and analyzed. This helped to achieve optimal estimates of order $O(h^{k+1})$ when polynomials of degree $k \geq 0$ were used to approximate both 'u' and '∇u'. Dual problem was used for element-by-element post-processing to achieve super-convergence results for the post-processed solution. The super-convergence was achieved by considering the derivative of order only up to one of the nonlinear variables f, a, and b. To derive a completely discrete scheme and corresponding error estimates, we have used the backward Euler's method and left rectangle rule to approximate the time derivative and integral, respectively. Further, with the help of two different numerical examples, it has been verified that the unknown variable and the flux achieved optimal order of convergence, whereas the post-processed solution attained the super-convergence.

In Chapter 4, we have discussed the HDG method to approximate the solution of non-linear hyperbolic integro-differential equation. Error estimates have been derived using HDG and Ritz-Volterra projections. In addition, the element-by-element post-processing of the numerical solution was accomplished by utilizing the dual of the problem. The findings demonstrated that all the three variables, namely, u, $\boldsymbol{q}$ and $\boldsymbol{\sigma}$ attain convergence of order $k+1$, for non-negative k in h, which was the discretizing parameter of the space domain. In contrast, the post-processed solution attained super-convergence; that is, it converged with order $k+2$, for $k \geq 1$. The analysis of the chapter provided better accuracy results compared to [79]. Finally, numerical results were reviewed.

In Chapter 5, we build upon the groundwork laid in Chapter 4 by extending our analysis to address nonlinear hyperbolic integro-differential equations. In this context, we introduce and rigorously analyze the HDG method tailored specifically for handling these nonlinear equations. To facilitate our analysis, we assume the nonlinear functions to be Lipschitz continuous, a crucial assumption for establishing a priori error estimates. Throughout this chapter, we leverage both HDG and Ritz-Volterra projections as analytical tools to derive error estimates. Additionally, we introduce an innovative element-by-element post-processing technique. Our investigations reveal that both the solution and its gradient achieve the optimal convergence rate of order $k+1$, where $k \geq 0$, concerning the discretization parameter h. Of particular interest is the observation that super-convergence, characterized by an order of $k+2$ (where $k \geq 1$), is attainable for the post-processed solution. This remarkable achievement is contingent upon the function f being differentiable and its derivative being Lipschitz continuous. Furthermore, we delve into a comprehensive discussion of a fully discrete scheme. This scheme exhibits a convergence rate of $O(h^{k+1} + \Delta t^2)$. Within this scheme, we employ the central difference rule to approximate derivatives and the mid-point rule to handle integral terms. In the concluding section of this chapter, we present and scrutinize numerical results that serve to validate the robustness of our theoretical findings and their practical applicability.

As stated in Chapter 1, the HDG method has various advantages over other finite element methods. To conclude that, table 6.1 gives a comparison of convergence results of continuous, discontinuous and mixed FEM with those derived for HDG method with the relative advantages and complexities.

Method	Order of Convergence for $\boldsymbol{q}$	Relative Advantages	Complexities
Continuous FEM [114]	k	Simplicity in formulation	Global assembly and solution
DG Method [115]	k	Flexibility in handling irregular meshes	Element-wise computations and numerical fluxes
Mixed FEM [111]	k	Natural handling of mixed formulations	Construction of suitable mixed spaces
HDG Method	$k+1$	Reduced global unknowns, Compatibility with existing tools	Implementation of hybrid variables and numerical fluxes

Table 6.1: Comparison of Finite Element Methods and HDG

6.2 Possible Extensions and Future Problems

The findings of this dissertation may be extrapolated to the three-dimensional domain by the implementation of suitable modifications. Similarly, for all the problems that has been discussed, the order of convergence in the temporal direction, for the fully-discrete case can be improved by approximating the derivative and the integral term by formulas with higher rate of convergence.

In Chapter 2, the HDG method had been discussed to a linear PIDE, with smooth kernel. Hence, in future, it can be developed for the following equation with weakly singular kernel:

$$u_t - \nabla \cdot \left(a(x)\nabla u + \int_0^t (t-s)^{-\alpha} b(t,s)\nabla u(s)ds \right) = f(x,t) \quad \text{in } \Omega \times (0,T], \tag{6.1a}$$

$$u(x,t) = 0 \qquad \text{on } \partial\Omega \times (0,T], \tag{6.1b}$$

$$u(x,0) = u_0(x) \qquad \text{for } x \in \Omega, \tag{6.1c}$$

where $0 < \alpha < 1$ and $u : \Omega \times (0,T] \to \mathbb{R}$. The coefficients $a : \Omega \to \mathbb{R}$, $b : \Omega \times (0,T] \to \mathbb{R}$ and $f : \Omega \times (0,T] \to \mathbb{R}$ are smooth with bounded derivatives and there exist positive constants α_0 and M such that $0 < \alpha_0 \leq a \leq M$ and $|b| \leq M$.

PIDEs with weakly singular kernel are often encountered in various fields, for instance, heat conduction, non-Fickian diffusion and image-processing, see, [25]. In the literature, Chen $et\ al.$ [25], have analyzed finite element approximation of a PIDE with weakly singular kernel. They have shown that optimal order estimates are achieved for both spatially semi-discrete and completely discrete schemes. Zhou $et\ al.$ [142] have developed a weak Galerkin FEM for the PIDE with a weakly singular kernel. They have derived the scheme's stability and optimal convergence order estimates in the L^2 norm and established numerical experiments to verify the theory. In [28], Chen $et\ al.$ have developed and analyzed the FEM for various types of integro-differential equations, along with the problem of the above type. They have developed semi-discrete and fully discrete schemes for the method, along with the error estimates. In [46], Da has considered backward Euler method for a PIDE with a memory term containing a weakly singular kernel. The have treated the integral term through a convolution quadrature, whereas the stability and convergence properties of the time discretizations were derived and applied to the semi-discrete equations built by the Galerkin FEMs in the space variables.

To define the method for equation (6.1), we make use of the following auxiliary variables:

$$\boldsymbol{q} = -\nabla u, \ \ \boldsymbol{\sigma} = a\boldsymbol{q} + \int_0^t (t-s)^{-\alpha} b(s)\ \boldsymbol{q}(s) ds,$$

and then, rewrite it as the following system of equations:

$$\boldsymbol{q} = -\nabla u \qquad \text{in } \Omega, \qquad (6.2a)$$

$$\boldsymbol{\sigma} = a\boldsymbol{q} + \int_0^t (t-s)^{-\alpha} b(s)\ \boldsymbol{q}(s) ds \qquad \text{in } \Omega, \qquad (6.2b)$$

$$u_t + \nabla \cdot \boldsymbol{\sigma} = f \qquad \text{in } \Omega. \qquad (6.2c)$$

Defining the Ritz-Volterra projection as in Chapter 2, we can get the following relation:

$$(\boldsymbol{\rho_q}, \boldsymbol{w}_h) - (\rho_u, \nabla \cdot \boldsymbol{w}_h) + \langle \hat{\rho}_u, \boldsymbol{w}_h \cdot \boldsymbol{\nu} \rangle_{\partial \mathcal{T}_h} = 0, \qquad (6.3a)$$

$$(a\boldsymbol{\rho_q}, \boldsymbol{z}_h) - (\boldsymbol{\rho_\sigma}, \boldsymbol{z}_h) + \int_0^t ((t-s)^{-\alpha} b(s)\ \boldsymbol{\rho_q}, \boldsymbol{z}_h) ds = (a\boldsymbol{\theta_q}, \boldsymbol{z}_h) - (\boldsymbol{\theta_\sigma}, \boldsymbol{z}_h) + \int_0^t ((t-s)^{-\alpha} b(s)\ \boldsymbol{\theta_q}, \boldsymbol{z}_h) ds, \qquad (6.3b)$$

$$-(\boldsymbol{\rho_\sigma}, \nabla v_h) + \langle \hat{\boldsymbol{\rho}}_{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu}, v_h \rangle_{\partial \mathcal{T}_h} = 0, \qquad (6.3c)$$

$$\langle \hat{\rho}_u, \mu_h \rangle_{\partial\Omega} = 0, \qquad (6.3d)$$

$$\langle \hat{\boldsymbol{\rho}}_{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu}, m_h \rangle_{\partial \mathcal{T}_h \backslash \partial\Omega} = 0, \qquad (6.3e)$$

for all $(v_h, \boldsymbol{w}_h, \boldsymbol{z}_h, \mu_h, m_h) \in (V_h \times \boldsymbol{W}_h \times \boldsymbol{W}_h \times M_h \times M_h)$.

We expect to have the following results.

Theorem 6.2.1. *There is a constant C that does not rely on h and k such that*

$$\|(u - \tilde{u}_h)(t)\| + \|(\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h)(t)\| + \|(\boldsymbol{q} - \tilde{\boldsymbol{q}}_h)(t)\| \leq C h^{k+1} \sup_{t\in(0,T]} \|u(t)\|_{H^{k+2}(\mathcal{T}_h)}.$$

And, with the similar meanings of ξ's as in Chapter 2, we have the following estimates:

Lemma 6.2.2. *For $t \in (0, T]$, there exists a positive constant C independent of h and k such that*

$$\|\xi_u\|^2 + \int_0^t \|\boldsymbol{\xi_\sigma}(s)\|^2 ds \leq C\Big(\|\xi_u(0)\|^2 + \int_0^T \|\eta_{u_t}\|^2 dt \Big).$$

So, with the help of Theorem 6.2.1 and Lemma 6.2.2, it can be proved that for the PIDE with weakly singular kernel (6.1), the HDG approximation of the scalar variables achieves optimal order of convergence, however, the analysis for the order of convergence of variables $\boldsymbol{q}_h$ and $\boldsymbol{\sigma}_h$ is due as our future work.

Our future work entails an extension of the research conducted by Chen *et al.* [26] and Tan *et al.* [134], focusing on nonlinear parabolic and hyperbolic integro-differential equations, respectively. Their pioneering work introduced a two-grid finite element method to address these complex problems. Our objective is to expand upon their contributions by advancing to a two-grid HDG method. Our research agenda also encompasses the development and in-depth analysis of the HDG method applied to Kirchhoff's equations, which encompass both elliptic and parabolic types. Furthermore, we plan on deriving a posteriori error estimates of linear and non-linear PIDEs. This extension forms a vital component of our future endeavors in the field.

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List of Research Publications

Journal Publications

1. R. Jain, A.K. Pani, and S. Yadav, HDG method for linear parabolic integro-Differential equations, Applied Mathematics and Computation, vol. 450, pp. 127987, 2023.

2. R. Jain, and S. Yadav, HDG Method for Nonlinear Parabolic Integro-Differential Equations, Computation Methods in Applied Mathematics, (Under Review).

3. R. Jain, and S. Yadav, Hybridizable Discontinuous Galerkin Method for Linear Hyperbolic Integro-Differential Equations, International Journal of Numerical Analysis and Modeling, (Communicated).

Paper Presentations in Conferences and Workshops

1. HDG Method for Linear Hyperbolic Integro-Differential Equations, **International Conference on Differential Equations and Control Problems**, School of Mathematical and Statistical Sciences, Indian Institute of Technology, Mandi (Jun, 2023).

2. HDG Method for Linear PIDE with Weakly Singular Kernel, **International Conference on Dynamical Systems, Control and their Applications**, Indian Institute of Technology, Roorkee (Jun, 2022).

3. HDG Method for Linear Parabolic Integro- Differential Equations, **4th International Conference on Frontiers in Industrial and Applied Mathematics**, Department of Mathematics SLIET, Longowal (Dec, 2021).

4. Weak Galerkin Method for Quasi Linear Elliptic Problem, **International Conference and 22nd Annual Convention of Vijnana Parishad of India on Advances in Operations Research Statistics and Mathematics**, BITS Pilani, Pilani Campus (Dec, 2019).

Biography of the Candidate

Ms. Riya Jain has been a full-time research scholar in the Department of Mathematics, BITS Pilani, Pilani Campus, since Jan 2019. She received her Bachelor in Science with honours (B.Sc.(H)) in Mathematics from Kamala Nehru College, University of Delhi, Delhi, in 2016. She completed her Master's in Applied Mathematics from South Asian University, New Delhi, in 2018. She then joined the Department of Mathematics at BITS Pilani, Pilani Campus, to pursue her Doctorate in Philosophy (Ph.D.) under the guidance of Prof. Sangita Yadav. She has published a research paper in a reputed international journal and has presented her work at various international conferences.

Biography of the Supervisor

Prof. Sangita Yadav is an Associate Professor in the Department of Mathematics at Birla Institute of Technology and Science Pilani, Pilani campus. She earned her Ph.D. degree from the Indian Institute of Technology, Bombay. Her research interests include numerical analysis, scientific computing, finite element methods, discontinuous Galerkin methods, multi-grid methods and adaptive methods. She has several research publications in reputed international journals. She is a member of the Indian Mathematical Society (IMS), the Indian Society of Industrial and Applied Mathematics (ISIAM) and the Ramanujan Mathematical Society (RMS). She has also attended several international conferences and symposiums.