The State-of-the-art Economic Analysis of the Fault-tolerant Redundant Repairable System with Different Types of Failures and Repair Mechanism

THESIS

Submitted in partial fulfilment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

by

MAHENDRA DEVANDA

Under the Supervision of

PROF. CHANDRA SHEKHAR

BIRLA INSTITUTE OF TECHNOLOGY AND SCIENCE, PILANI

2024

"*In a gentle way, you can shake the world.*" — *by Mahatma Gandhi*

BIRLA INSTITUTE OF TECHNOLOGY & SCIENCE, PILANI

CERTIFICATE

This is certify that the thesis titled " **The State-of-the-art Economic Analysis of the Fault-tolerant Redundant Repairable System with Different Types of Failures and Repair Mechanism**" submitted by **Mr. Mahendra Devanda**, ID No. **2019PHXF0402P** for the award of Ph.D. of the institute embodies original work done by him under my supervision.

Shekhar 04/09/24

Signature of the Supervisor

Name: **PROF. CHANDRA SHEKHAR**

Designation: **Professor**

Date: **September 2024**

Dedicated to my respected Parents, my cherished Wife,
and my beloved Children

Acknowledgements

I dedicate my thesis, first and foremost, to my parents, whose guidance and encouragement led me to pursue a Ph.D.

Firstly, I would like to extend my heartfelt gratitude to my supervisor, Prof. Chandra Shekhar, for his mentorship, insightful guidance, and invaluable feedback throughout the entire research process. His expertise and dedication have been instrumental in shaping the direction of this thesis. He has not only been a guide in academia but also a source of motivation during the challenging phases. His belief in my potential has encouraged me to strive for higher standards. The collaborative atmosphere he fostered in our interactions has been instrumental in broadening my perspectives.

I also extend my heartfelt thanks to Mrs. Shikha Gupta for her moral support and motivating encouragement to work hard and strive for success. Whenever I felt lonely, her remarkable guidance stood out even more. I am also thankful for her children Tejasv and Pranjal, who always loved and respected me like an elder brother.

I am profoundly grateful to Professor Ram Niwas Jat for his unwavering guidance, support, and encouragement throughout my Ph.D. journey and every step of my life.

With great reverence, I express my gratitude to Prof. V. Ramgopal Rao (Vice Chancellor), Prof. Sudhirkumar Barai (Director), Prof. M.B. Srinivas (Dean, AGSRD), and Prof. Shamik Chakraborty (Associate Dean, AGSRD) for providing necessary facilities.

I am grateful to Prof. Bhupendra Kumar Sharma, Prof. Devendra Kumar (Ex.HoDs), and Prof. Ashish Tiwari, HoD, Department of Mathematics, for their unwavering sup-port. I extend my acknowledgment to all the faculty members and research scholars for their valuable assistance and cooperation throughout my research journey. I am also thankful to all the professors of the Department of Mathematics of BITS Pilani, Pilani Campus, for their guidance and valuable suggestions in my research work.

I feel fortunate to have my Doctoral Advisory Committee (DAC) members, Prof. Rajesh Prasad Mishra and Prof. Shivi Agarwal, for their insightful discussions and valuable suggestions that have significantly enriched the quality of this work.

I am grateful to my all seniors (Shreekant, Amit, Sandeep, Deepak, Pallav, Komal, Barkha, Yogendra, Ankit, Kapil, Sarita, Chandan) for their wisdom and mentorship. A heartfelt thanks to my friends (HP, Kamlesh, Deepak, Kamalkant, Kailash, Satta, Himanshu, Amit, Gourav, Sanjiv, Umesh, Parveen, Sitaram, Manoj, Deva, Arun, Sonu, Anshu, Suman, Pooja, Shipra) for their constant companionship and support. I would also like to acknowledge the encouragement and cooperation of my juniors Vijay, Yogesh, Jakhar(Ajay), Anish, Saddam, Praveen, Anup, Rohit, Ravindra, Anil, Rahul, Shariq, Vikram, Ashish, Ankit, Ankur, Vijender, Prashant, Ravi, Sakshi, Shiksha, Shweta, Little Pooja, Sonu, and Jyoti whose energy and enthusiasm have been motivating throughout this journey.

I am deeply thankful to my grandparents (Shri. Gheesaram, Smt. Lada Devi), my beloved parents (Mr. Harlal Singh Devanda and Mrs. Santra Devi), and my uncles (Suresh, Rajendra, Veer Bahadur, and Subhas) for their boundless love and encouragement, which have always inspired me. A special thanks to my elder sister Saroj, brother-in-law Sunil Kumar, younger sister Dr. Anita, and her fiancé Dr. Shushil Kudi. I also appreciate the support of my cousins Sunita, Pinky, Himanshu, Kapil, Sakshi, Anshu, and Nihar. Above all, I am grateful to my wife, Mrs. Maina, and my children, Paridhi and Taksh, for their unwavering love, patience, and support throughout this journey.

I must acknowledge the financial support provided by CSIR/UGC, which enabled me to carry out this research effectively and efficiently. I am also thankful to the office staff members who helped me with all the formalities. In conclusion, this journey has been a collective effort, and I am humbled by the support I have received. My deepest gratitude goes to each and every individual who has played a role, no matter how small, in helping me reach this milestone.

Place: BITS Pilani, Pilani Campus

Date: __/__/____ **Mahendra Devanda**

Abstract

In every industrial application, operating units tend to experience random failure. With progress, people are becoming more dependent on machining systems, and machines have become a part of human's daily lives, providing comfort.

In chapter 2, the reliability analysis of complex systems is crucial for ensuring their dependable operation. This study focuses on a dual-active single-standby storage unit system, which plays a critical role in various industrial and technological applications. The study investigates the reliability metrics of this system in detail, addressing challenges posed by unreliable repairs and standby switching failures. The application of Bayesian inference with Gamma and Beta distribution priors, coupled with Monte Carlo simulations, provides a robust methodology for estimating unknown parameters and deriving posterior distributions. This approach assumes exponential distributions for time-to-failures and time-to-repair. Additionally, time-to-inspections follow exponential distributions for perfect and imperfect rejuvenations, and the probability of unsuccessful standby switching, denoted as 'q', is also considered. The findings, presented through comprehensive tables and graphs, offer valuable insights into the system's reliability and the effectiveness of the statistical inferences employed.

In the contemporary world, the prevalence of machine repair problems (MRPs) in real-time machining models has increased with the development of technology for socioeconomic advancement, mobility, security, and safety. The uninterrupted operations of critical appliances, monitoring controllers, advanced devices, and data exchange systems are expected whenever needed prompt. When active units fail, the results may be catastrophic, injury, or loss, leading to critical reliability challenges that must be resolved. The critical reliability challenges are leading to when active units fail; the results may be catastrophic, injury, or loss that must be reconciled. Chapter 3 aims to provide a comprehensive, state-of-the-art study for failure/repair/operation uncertainties and impreciseness in optimistic and pessimistic conditions. We consider the fault-tolerant machining system consisting of two-active units, a single-warm standby unit, and a single-repair facility in a fuzzy environment governing the involved imperfectness, vagueness, and uncertainty. Switching the standby unit to the failed active unit is also subject to failure. The notion of imperfect repair makes the proposed model more insightful. A membership grade function of the reliability characteristics, mean time-to-failure, and system availability are constructed to study uncertainties in-depth for the fault-tolerant redundant repairable system with switching failure and imperfect repair for well to poor design. The nonlinear parametric program

technique converts the studied problem into a set of conventional problems. It is employed to compute the upper and lower bounds of the reliability characteristic based on the γ-cut approach and Zadeh's extension principle for extreme design-constrained limits. Extensive numerical simulations are also performed for the governing parameters ranging from well-conditioned to ill-conditioned. The concluding remarks and future scopes are also included.

In today's highly automated world, human interaction with machines is constant, aided by technologies like IoT and cloud manufacturing. Chapter 4 focuses on a novel multi-unit machine repair issue involving M active units, S standby units, and a reliable repairer with deteriorating standby units. The decision to retain or reject deteriorated standby units is crucial, guided by cost analysis and performance evaluation. Failures in active and standby units pose risks. We activate standby units upon active unit failure. Deteriorating standby units degrade over time while ready for replacement. This concept applies to engineering, maintenance, and reliability analysis. Our research introduces distinct failure rates between active and standby units due to deterioration. Rejecting deteriorating standbys isn't cost-effective, as our analysis shows. Using Teaching-Learning based Optimization, we determine optimal decision parameters through a recursive matrix approach for steady-state probabilities. Our numerical assessment evaluates various system performance metrics under optimal conditions.

Chapter 5 delves into the complex interplay between failure characteristics, repair capabilities, and system performance metrics within a multi-unit redundant repairable system, focusing specifically on the deterioration of standby units. This examination shows how different distributions in repair times-namely Exponential, Erlang, and Hyperexponential affect the system's overall performance. By employing this more accurate repair time distribution, the system for repairing machines can effectively represent and assess different performance metrics such as system availability and mean time to repair.

The multi-unit system is the basis of many efficiency evaluations in operational machining management. It is indispensable to assess the critical parameter. Chapter 6 evaluates a Markovian MRP with a controllable strategy and imperfect repair. The imperfect repair means that the repairer provides the service to failed machines, but the service may be imperfect. We opt for the controllable threshold-based strategy, which deals with the issue of allowing the failed units when the number of failed units in the reparable queue reaches up to the system's capacity to avoid largely expected waiting. The repairers stop allowing new failed units to wait until the queue size decreases to a pre-fixed level. Failed units that are not allowed may be repaired by the external facility at additional costs. The number of failed units in the waiting line and the expected number of service-gained units play an essential role in a well-designed control policy. Using the Laplace transform method, we obtain the transient-state distribution of the failed units in the system, establish different performance measures, and compute the system's reliability function and mean-time-to-first failure. Extensive numerical simulation and sensitivity analysis have also been performed to analyze the studied system deeply.

In Chapter 7, the prospective study analyzes the reliability characteristics of multi-unit systems with standby provisioning, accounting for failures, degradation, random delays, and probabilistic imperfections. The investigation sheds light on how these unreliable attributes hinder the performance and availability of machining systems. Given the frequent occurrence of these negative attributes within machining systems, their impact on production flow, performance, and resource utilization is significant. Moreover, these unfavorable attributes hinder the adoption of advanced technologies that rely on the continuous availability of machining systems. Thorough research on operational traits serves as a foundation for developing solutions to enhance machining system efficiency and elucidates the underlying causes of machining system failures throughout their performance lifecycle. The reliability analysis employs a state-of-the-art queue-theoretic approach. In order to model the stochastic behavior of the investigated machine repair problem systematically, we consider various statistically independent failure modes, including active/standby unit failure, degraded failure, switching failure, and common-cause failure. The unreliable characteristics of machining systems are further compounded by factors such as imperfect fault coverage, reboot delay, and imperfect repair, necessitating a comprehensive examination to implement preventive, corrective, and predictive measures strategically. The seamless operation of multi-unit machining systems is essential for successfully integrating advanced technologies such as cloud computing, industry 4.0, and IoT. Failures, delays, degradation, and imperfections within machining systems have detrimental effects on their efficiency and availability, demanding in-depth investigation. To facilitate numerical experimentation and sensitivity analysis of the reliability aspects of the proposed machining system, we develop performance indices such as system reliability, mean-time-to-failure, and failure frequency. These metrics provide valuable insights for decision-makers seeking to implement measures that ensure the uninterrupted availability of the machining system.

Chapter 8 addresses the challenge of repairing a complex machine system with warm spares and multiple operational units. When a unit malfunctions, prompt repair is crucial. Our two-stage hierarchical repair facility comprises primary repairers responsible for the preparatory stage, handling routine maintenance and low-skilled repairs. In contrast, the execution stage involves a highly skilled secondary repairer dedicated to addressing critical issues. We focus on efficient work sequential allocation between these stages, optimizing work distribution for quicker resolution of critical problems. We use the Matrix Recursive Method to analyze the system to determine steady-state probabilities. Our mathematical model, solved with a recursive method, allows us to calculate performance indices based on probability distributions. Additionally, we create a cost function and fine-tune decision parameters to minimize expected costs per unit of time. We employ Particle Swarm Optimization, a meta-heuristic technique, to optimize these parameters for a cost-effective service system. This approach streamlines the repair process, enhancing overall efficiency. Our research aims to provide practical insights into managing complex machinery repairs, ensuring optimal resource utilization and cost-effectiveness.

At the end of the whole work, the main findings of the thesis work are summarized in Chapter 9. The potential exploration and future scope of this work is also outlined in Chapter 9. Additionally, the bibliography section offers a compilation of references used throughout the thesis.

Contents

Certificate **v**

Acknowledgements **ix**

Abstract **xi**

1 General Introduction **1**

1.1 Motivation 1
1.2 Machining System 3
1.3 Fundamental Configuration of Machining Setup 5
1.4 Classification by Machine Type or Industry Sector 10
1.5 Machine Repair Problem 13
1.6 Machining System with Redundancy 14
1.7 Fundamental components of Queueing systems 16
1.8 Distinct Important Processes 18
1.8.1 Stochastic Process: 18
1.8.2 Counting Process: 18
1.8.3 Markov Process: 19
1.8.4 Markov Chain: 19
1.8.5 Birth-Death Process: 20
1.8.6 Poisson Process: 20
1.8.7 Renewal Process: 20
1.8.8 Chapmann-Kolmogorov Equation: 21
1.9 Some Important Distribution 21
1.10 Machining System with Various Failure 26
1.11 Problem-solving Methods 28
1.12 Performance Metrics of the Machining System 33
1.12.1 Queueing Metrics: 33
1.12.2 Reliability Metrics: 35
1.13 Machine repair system with different kinds 37
1.13.1 Series Systems: 37

1.13.2 Parallel Systems: 38
1.13.3 Standby Redundant Systems: 39
1.13.4 K-out-of-M:G Systems: 39
1.13.5 Maintained Systems: 39
1.13.6 (K,m) Systems: 40
1.13.7 Series-parallel Systems: 40
1.13.8 Parallel-series Systems: 40
1.13.9 Complex Systems: 40
1.13.10 Controlled Policies 40
1.14 Fuzzy Redundant Machine Repair Problem 42
1.14.1 Normal Fuzzy Set: 42
1.14.2 Support of a Fuzzy Set: 42
1.14.3 α-level Set: 43
1.14.4 Convex of a Fuzzy Set: 43
1.14.5 Scalar Cardinality of a Fuzzy Set: 43
1.14.6 Union of two Fuzzy Set: 43
1.14.7 Intersection of two Fuzzy Set: 43
1.14.8 Complement of a normalized Fuzzy Set: 44
1.14.9 Algebraic Sum of two Fuzzy Set: 44
1.14.10 Bounded Sum of two Fuzzy Set: 44
1.14.11 Bounded Difference of two Fuzzy Set: 44
1.14.12 Algebraic Product of two Fuzzy Set: 44
1.15 Bayesian Analysis 48
1.16 Review of Literature 49
1.16.1 Literature on Queueing Theory and its Historical Significance 49
1.16.2 Machine Repair Problem with Spare Parts 51
1.16.3 Contributions in MRP with Varied failures 54
1.16.4 Machine Repair Problem with Vacations and Controlled policy 61
1.16.5 MRP with Bayesian point of view 64
1.17 Gaps in the Existing Literature 69
1.18 Thesis Objective 69

2 Reliability Analysis of Imperfect Repair and Switching Failures: A Bayesian Inference and Monte Carlo Simulation Approach 71

2.1 Introduction 71
2.2 Literature Survey 73
2.3 Model Description 76
2.3.1 Assumption 76

2.4 The system reliability characteristics 79
2.4.1 Mean time-to-failure 79
2.4.2 Availability of the system 81
2.5 Bayesian Estimation of reliability characteristics 83
2.5.1 Likelihood Function 85
2.5.2 Two-parameter gamma prior 85
2.5.3 Beta distribution of second kind 87
2.6 Numerical illustrations and simulation study 90
2.6.1 Comparison of Bayesian and asymptotic estimates 92
2.7 Conclusion 98

3 Fuzzified Imperfect Repair Redundant Machine Repair Problem 99
3.1 Introduction 99
3.2 Model Description 102
3.3 The system reliability characteristics 105
3.3.1 Mean time-to-failure 105
3.3.2 Availability of the system 108
3.4 The fuzzy redundant repairable system 110
3.5 Parametric nonlinear programs 112
3.6 Numerical illustration and results 115
3.6.1 The fuzzy mean time-to-failure 116
3.6.2 The fuzzy system's availability 118
3.6.3 Numerical simulation 119
3.6.4 Discussion 125
3.7 Conclusion 126

4 Performance Analysis of a Repairable Machining System with Standby Deterioration 127
4.1 Introduction 127
4.2 Model Description 130
4.2.1 Steady-state equations 132
4.3 Matrix Analytic Method 134
4.4 System characteristics 137
4.4.1 Cost Analysis 139
4.5 Teaching-Learning based Optimization(TLBO) 139
4.5.1 Mathematical Model and Algorithm 140
4.6 Numerical illustration 142
4.7 Conclusion 151

5 Performance Analysis of Machine Repair Systems with Deteriorating Standby Using Phase-Type Distributions: Comparative Study of Erlang, Exponential, and Hyper-Exponential Distributions 153
5.1 Introduction . . . 153
5.2 Model Description . . . 158
5.3 Steady-State Analysis . . . 161
5.4 System characteristics . . . 165
5.5 Numerical Analysis . . . 166
5.6 Conclusion . . . 169

6 The State of The Art Methodologies For Reliability Analysis of Imperfect Repair and Thresholed-Based Measures 171
6.1 Introduction . . . 171
6.2 Literature Review . . . 173
6.3 Model Description . . . 174
6.4 The governing equation . . . 179
6.4.1 Laplace transform . . . 180
6.5 Performance measures . . . 184
6.5.1 Reliability Measures . . . 184
6.5.2 Queueing characteristics . . . 185
6.6 Cost function . . . 187
6.7 Sensitivity analysis . . . 188
6.8 Special Cases . . . 189
6.9 Numerical Results . . . 190
6.10 Conclusion . . . 200

7 Reliability analysis of Standby Provision Multi-Unit Machining Systems with Varied Failures, Degradations, Imperfections, and Delays 203
7.1 Introduction . . . 203
7.2 Model Description . . . 206
7.2.1 Transient Equations . . . 210
7.2.2 Laplace transform . . . 212
7.3 Reliability of the system . . . 217
7.3.1 Sensitivity Analysis of $R_X(t)$ . . . 220
7.4 Mean time-to-failure . . . 223
7.4.1 Sensitivity Analysis of $MTTF$ . . . 224
7.5 Failure Frequency of the System . . . 224
7.6 Special Cases . . . 226
7.7 Discussion, Conclusion, Future scope . . . 226

8 Exploring Hierarchical Repair Strategies for Multi-Unit Redundant Machining Systems **229**
8.1 Introduction . . . 229
8.2 Literature Review . . . 231
8.2.1 Machine Repair Problems . . . 231
8.2.2 Machine Repair Problems with Redundancy . . . 232
8.2.3 Machine Repair Systems with Repair Facilities . . . 232
8.3 Model Description . . . 233
8.3.1 Work division policy . . . 236
8.3.2 Steady-state equations . . . 238
8.4 Matrix-recursive method . . . 239
8.4.1 The State Probabilities . . . 242
8.5 Optimal Analysis . . . 243
8.5.1 Performance measures . . . 243
8.5.2 Cost Function . . . 246
8.5.3 Paticle swarm optimization . . . 246
8.6 Numerical Analysis . . . 248
8.7 Conclusion . . . 258

9 Conclusions and Future Work **261**
9.1 Summary and Conclusions . . . 261
9.2 Contributions Through this Research . . . 263
9.3 Future Scope of the Present Research Work . . . 264

Publications **297**

Presented Works **300**

Brief Biography of the Candidate **301**

Brief Biography of the Supervisor **302**

List of Figures

1.1 Breakdown of each states of the machining system . . . 3
1.2 Maintenance policy . . . 4
1.3 Single-Line,Single-Phase . . . 5
1.4 Single-Line, Multi-Phase . . . 6
1.5 Multi-Line, Single-Phase . . . 7
1.6 Multi-Line, Multi-Phase . . . 7
1.7 Priority queue . . . 8
1.8 State transition diagram of basic Machine repair system . . . 13
1.9 Queueing system . . . 16
1.10 MRP with common-cause failure . . . 26
1.11 MRP with Switching failure . . . 27
1.12 State transition diagram of Machine repair problem . . . 33
1.13 State transition diagram for system failure . . . 36
1.14 Multi-unit series system . . . 37
1.15 Multi-unit parallel system . . . 38
1.16 K-out-of-$M : G$ system . . . 39
1.17 MRP with N-policy . . . 41
1.18 MRP with F-policy . . . 41

2.1 MTTF for Gamma Prior . . . 84
2.2 $A(\infty)$ for Gamma Prior . . . 89
2.3 MTTF for Beta Prior . . . 96
2.4 $A(\infty)$ for Beta Prior . . . 97

3.1 Transition diagram for reliability analysis . . . 103
3.2 Transition diagram for availability analysis . . . 108
3.3 $\eta_{\widetilde{T}}$ and $\eta_{\widetilde{A}}$ for numerical illustration . . . 117
3.4 $\eta_{\widetilde{T}}$ and $\eta_{\widetilde{A_v}}$ for fuzzified $\widetilde{\lambda}_i$ . . . 120
3.5 $\eta_{\widetilde{T}}$ and $\eta_{\widetilde{A_v}}$ for fuzzified $\widetilde{\nu}_i$ . . . 121
3.6 $\eta_{\widetilde{T}}$ and $\eta_{\widetilde{A_v}}$ for fuzzified $\widetilde{\mu}_i$ . . . 122
3.7 $\eta_{\widetilde{T}}$ and $\eta_{\widetilde{A_v}}$ for fuzzified $\widetilde{\beta_1}_i$ . . . 123
3.8 $\eta_{\widetilde{T}}$ and $\eta_{\widetilde{A_v}}$ for fuzzified $\widetilde{\beta_2}_i$ . . . 124

3.9 $\eta_{\widetilde{T}}$ and $\eta_{\widetilde{A_v}}$ for fuzzified $\widetilde{q}_i$ 125

4.1 Expected number of failed units in the system (E_N) for different parameters, (For $M = 20, S = 8, \lambda = 4, \nu = 1.8, \lambda_1 = 3, \mu = 75$) 143
4.2 Expected number of operating standby units in the system (E_{OS}) for different parameters, (For $M = 20$, $S = 8$, $\lambda = 4$, $\nu = 1.8$, $\lambda_1 = 3$, $\mu = 75$) 144
4.3 Throughput of the system (Th) for the different parameters, (For $M = 20$, $S = 8$, $\lambda = 4$, $\nu = 1.8$, $\lambda_1 = 3$, $\mu = 75$) 144
4.4 Expected waiting time of the system (W_S) for the different parameters, (For $M = 20$, $S = 8$, $\lambda = 4$, $\nu = 1.8$, $\lambda_1 = 3$, $\mu = 75$) 145
4.5 Comaprision of Expected Machining availability of the system for proposed model(left side) and general machine repair model (right-side) (MA) for the different parameters, (For $M = 20$, $S = 8$, $\lambda = 4$, $\nu = 1.8$, $\lambda_1 = 3$, $\mu = 75$) 146
4.6 Expected total cost of the system (TC) for the different parameters, (For $M = 20$, $S = 8$, $\lambda = 4$, $\nu = 1.8$, $\lambda_1 = 3$, $\mu = 75$, $C_1 = 295$, $C_2 = 30$, $C_3 = 100$, $C_4 = 220$, $C_5 = 90$) 147
4.7 Convergence of iteration of Teaching-Learning based Optimization . 148

5.1 Expected number of active units in the system 167
5.2 Expected number of deteriorate units in the system 167
5.3 Expected number of the failed units in the system 168
5.4 Expected failure rate of the system 168
5.5 Expected number of the standby units in the system 168
5.6 Expected waiting time of the system 169
5.7 Machine availability of the system 169
5.8 Throughput of the system 169

6.1 Transition diagram 177
6.2 Reliability of the system ($R_X(t)$) wrt. t for different parameters. For ($M = 12$, $S = 5$, $m = 2$, $\lambda = 0.4$, $\nu = 0.35$, $\lambda_d = 0.8$, $\mu = 16$, $\beta_1 = 8$, $\beta_2 = 1$, $\gamma = 4$) 191
6.3 Sensitivity and relative sensitivity of the reliability of the system, For ($M = 12, S = 5, m = 2, \lambda = 0.4, \nu = 0.35, \lambda_d = 0.8, \mu = 16, \beta_1 = 8, \beta_2 = 1$, $\gamma = 4, C_H = 90, C_S = 60, C_M = 10, C_1 = 13, C_2 = 2, C_3 = 2$) 192
6.4 Mean time to failure of the system ($MTTF$) wrt. t for different parameters, For ($M = 12, S = 5, m = 2, \lambda = 0.4, \nu = 0.35, \lambda_d = 0.8$, $\mu = 16, \beta_1 = 8, \beta_2 = 1, \gamma = 4$) 193

6.5 Expected number of failed units in the system ($E_N(t)$) wrt. t for different parameters.For ($M=12$, $S=5$, $m=2$, $\lambda=0.4$, $\nu=0.35$, $\lambda_d=0.8$, $\mu=16$, $\beta_1=8$, $\beta_2=1$, $\gamma=4$) 195
6.6 Throughput of the system ($Th(t)$) wrt. t for different parameters. For ($M=12$, $S=5$, $m=2$, $\lambda=0.4$, $\nu=0.35$, $\lambda_d=0.8$, $\mu=16$, $\beta_1=8$, $\beta_2=1$, $\gamma=4$) 196
6.7 Expected total cost ($E_{TC}(t)$) by varying parameters, For ($M=12$, $S=5$, $m=2$, $\lambda=0.4$, $\nu=0.35$, $\lambda_d=0.8$, $\mu=16$, $\beta_1=8$, $\beta_2=1$, $\gamma=4$) 199

7.1 State transition rate diagram $\Lambda_i=[M\lambda\bar{q}+(S-i)\nu]c$, $0\leq i\leq K-1$, $\Upsilon_{j,i}=M\lambda\bar{q}q^{j-i-1}c$ $\phi_i=M\lambda q^{S-i}c$ 207
7.2 Reliability $R_X(t)$ of the system w.r.t. design parameter, (For $\lambda=0.5$, $\lambda_d=1$, $\lambda_c=0.01$, $\nu=0.1$, $\alpha=0.05$, $\mu=5$, $\beta_1=15$, $\beta_2=0.1$, $\beta=10$, $\gamma=60$, $c=0.9$, $q=0.1$) 217
7.3 Reliability $R_X(t)$ of the system w.r.t. different types of failures, (For $M=10$, $S=5$, $m=2$, $\mu=5$, $\beta_1=15$, $\beta=10$, $\gamma=60$, $c=0.9$, $q=0.1$) 218
7.4 Reliability $R_X(t)$ of the system w.r.t. repair facility, (For $M=10$, $S=5$, $m=2$, $\lambda=0.5$, $\lambda_d=1$, $\lambda_c=0.01$, $\nu=0.1$, $\alpha=0.05$, $c=0.9$, $q=0.1$) 219
7.5 Reliability $R_X(t)$ of the system w.r.t. probability of coverage and switching failure, (For $M=10$, $S=5$, $m=2$, $\lambda=0.5$, $\lambda_d=1$, $\lambda_c=0.01$, $\nu=0.1$, $\alpha=0.05$, $\mu=5$, $\beta_1=15$, $\beta_2=0.1$, $\beta=10$, $\gamma=60$) 220
7.6 Sensitivity analysis Δ_Θ of $R_X(t)$ for different parameters, (For $M=10$, $S=5$, $m=2$, $\lambda=0.5$, $\lambda_d=1$, $\lambda_c=0.01$, $\nu=0.1$, $\alpha=0.05$, $\mu=5$, $\beta_1=15$, $\beta_2=0.1$, $\beta=10$, $\gamma=60$, $c=0.9$, $q=0.1$) 221
7.7 Relative sensitivity analysis Γ_Θ of $R_X(t)$ for different parameters, (For $M=10$, $S=5$, $m=2$, $\lambda=0.5$, $\lambda_d=1$, $\lambda_c=0.01$, $\nu=0.1$, $\alpha=0.05$, $\mu=5$, $\beta_1=15$, $\beta_2=0.1$, $\beta=10$, $\gamma=60$, $c=0.9$, $q=0.1$) 222
7.8 Mean time-to-failure $MTTF$ of the system by varying parameters, (For $M=10$, $S=5$, $m=2$, $\lambda=0.5$, $\lambda_d=1$, $\lambda_c=0.01$, $\nu=0.1$, $\alpha=0.05$, $\mu=5$, $\beta_1=15$, $\beta_2=0.1$, $\beta=10$, $\gamma=60$, $c=0.9$, $q=0.1$) 223
7.9 Failure frequency FF for different parameters, (For $M=10$, $S=5$, $m=2$, $\lambda=0.5$, $\lambda_d=1$, $\lambda_c=0.01$, $\nu=0.1$, $\alpha=0.05$, $\mu=5$, $\beta_1=15$, $\beta_2=0.1$, $\beta=10$, $\gamma=60$, $c=0.9$, $q=0.1$) 225

8.1 Flow chart of the repair system 233
8.2 Steady-state Rate Diagram 234

8.3 Effects of the varying parameters on the system's expected number of failed units (E_S).(for $M=25; S=7; m=2; R=5; \lambda=2; \nu=1.2; \lambda_d=3; \mu=12; \zeta=10; b=200; \omega=1.3; \delta=1$) . . . 249
8.4 Effects of the distinct parameters on the system's expected number of primary repairers in F-stage (E_F), (for $M=25; S=7; m=2; R=5; \lambda=2; \nu=1.2; \lambda_d=3; \mu=12; \zeta=10; b=200; \omega=1.3; \delta=1$) . . 249
8.5 Effects of the distinct parameters on the system's expected waiting time (W_S), (for $M=25; S=7; m=2; R=5; \lambda=2; \nu=1.2; \lambda_d=3; \mu=12; \zeta=10; b=200; \omega=1.3; \delta=1$) . . . 250
8.6 Effects of the distinct parameters on the system's expected waiting time of primary repairers in F-stage (W_F), (for $M=25; S=7; m=2; R=5; \lambda=2; \nu=1.2; \lambda_d=3; \mu=12; \zeta=10; b=200; \omega=1.3; \delta=1$) 251
8.7 Effects of the distinct parameters on the system's Failure frequency (FF), (for $M=25; S=7; m=2; R=5; \lambda=2; \nu=1.2; \lambda_d=3; \mu=12; \zeta=10; b=200; \omega=1.3; \delta=1$) . . . 251
8.8 Effects of the distinct parameters on the system's Quality difference ($Q_d(\mu)$), (for $M=25; S=7; m=2; R=5; \lambda=2; \nu=1.2; \lambda_d=3; \mu=12; \zeta=10; b=200; \omega=1.3; \delta=1$) . . . 252
8.9 Effects of the distinct parameters on the system's Throughput (Th_S), (for $M=25; S=7; m=2; R=5; \lambda=2; \nu=1.2; \lambda_d=3; \mu=12; \zeta=10; b=200; \omega=1.3; \delta=1$) . . . 252
8.10 Effects of the distinct parameters on the Throughput or effective service rate of primary repairer in F-stage (Th_F), (for $M=25; S=7; m=2; R=5; \lambda=2; \nu=1.2; \lambda_d=3; \mu=12; \zeta=10; b=200; \omega=1.3; \delta=1$) 253
8.11 Effects of the distinct parameters on the Expected total cost of the system (TC), (for $M=25; S=7; m=2; R=5; \lambda=2; \nu=1.2; \lambda_d=3; \mu=12; \zeta=10; b=200; \omega=1.3; \delta=1; C_1=50; C_2=250; C_3=80; C_4=12; C_5=25; C_6=5$) . . . 253
8.12 Convergence of iteration of Partical swarm optimization, (for $M=25; S=7; m=2; R=5; \lambda=2; \nu=1.2; \lambda_d=3; \mu=12; \zeta=10; b=200; \omega=1.3; \delta=1; C_1=50; C_2=250; C_3=80; C_4=12; C_5=25; C_6=5$) . . . 256

List of Tables

1.1 Table to Contribution of MRP with different cold, warm, hot, and mixed standby units . . . 51
1.4 Table to Contribution of MRP with . . . 55
1.5 Table to Contribution of MRP varied failures . . . 58
1.6 Table to MRP with Vacations and Controlled Policy . . . 62
1.7 Table to Contribution of MRP solved by Bayesian approach . . . 64
1.8 Table to Contribution of Machine Repair problem in Fuzzy environment 66

2.1 Estimates for MTTF . . . 90
2.2 Estimates for $A(\infty)$ . . . 90
2.3 Estimates for MTTF . . . 91
2.4 Estimates for A(∞) . . . 91
2.5 PM, HPD intervals and ACI of MTTF . . . 93
2.6 PM, HPD intervals and ACI of A(∞) . . . 93

3.1 Trapezoidal fuzzified system parameters . . . 120
3.2 Properties of $\widetilde{T}$ and $\widetilde{A}_v$ for fuzzified $\widetilde{\lambda}_i$ . . . 121
3.3 Properties of $\widetilde{T}$ and $\widetilde{A}_v$ for fuzzified $\widetilde{v}_i$ . . . 122
3.4 Properties of $\widetilde{T}$ and $\widetilde{A}_v$ for fuzzified $\widetilde{\mu}_i$ . . . 123
3.5 Properties of $\widetilde{T}$ and $\widetilde{A}_v$ for fuzzified $\widetilde{\beta}_{1_i}$ . . . 124
3.6 Properties of $\widetilde{T}$ and $\widetilde{A}_v$ for fuzzified $\widetilde{\beta}_{2_i}$ . . . 124
3.7 Properties of $\widetilde{T}$ and $\widetilde{A}_v$ for fuzzified $\widetilde{q}_i$ . . . 125

4.1 Optimal expected total cost of the system $(TC(S^*,\mu^*))$ for different parameters using TLBO algorithm, For $C_1 = 295, C_2 = 30, C_3 = 100, C_4 = 220, C_5 = 90$ . . . 147
4.2 Optimal expected total cost of the system $(TC(S^*,\mu^*))$ for different parameters using TLBO algorithm, For $M = 20, \lambda = 4, v = 1.8, \lambda_1 = 3$ 149
4.3 Optimal expected total cost of the system $(TC(S^*,\mu^*))$ for different parameters using PSO algorithm, For $C_1 = 295, C_2 = 30, C_3 = 100, C_4 = 220, C_5 = 90$ . . . 149

4.4 Optimal expected total cost of the system $(TC(S^*,\mu^*))$ for different parameters using PSO algorithm, For $M = 20, \lambda = 4, \nu = 1.8, \lambda_1 = 3$. 151

6.1 Sensitivity and relative sensitivity of the mean time to failure of the system for ($M = 12, S = 5, m = 2, \lambda = 0.4, \lambda_d = 0.8, \nu = 0.35, \mu = 16, \beta_1 = 8, \beta_2 = 1, \gamma = 4, C_H = 90, C_S = 60, C_M = 10, C_1 = 13, C_2 = 2, C_3 = 2$) . . . 194

6.2 Performance evaluations for different parameters M, λ and t with $S = 5$, $m = 2$, $\lambda_d = 0.8$, $\nu = 0.35$, $\mu = 16$, $\beta_1 = 8$, $\beta_2 = 1$, $\gamma = 4$ . . . 197

6.3 Performance evaluations for different parameters S, ν and t with $M = 12$, $m = 2$, $\lambda = 0.4$, $\lambda_d = 0.8$, $\mu = 16$, $\beta_1 = 8$, $\beta_2 = 1$, $\gamma = 4$ . . . 198

6.4 Performance evaluations for different parameters m, λ_d and t with $M = 12$, $S = 5$, $\lambda = 0.4$, $\nu = 0.35$, $\mu = 16$, $\beta_1 = 8$, $\beta_2 = 1$, $\gamma = 4$ . . . 200

6.5 Performance evaluations for different parameters μ, β_1 and t with $M = 12$, $S = 5$,$m = 2$, $\lambda = 0.4$, $\lambda_d = 8$, $\nu = 0.35$, $\beta_2 = 1$, $\gamma = 4$ . . . 201

6.6 Performance evaluations for different parameters β_1, β_2 and t with $M = 12$, $S = 5$,$m = 2$, $\lambda = 0.4$, $\lambda_d = 8$, $\nu = 0.35$, $\mu = 16$, $\gamma = 4$ . . . 202

7.1 Sensitivity analysis Φ_Θ of the $MTTF$ of the system, (For $\lambda = 0.5$, $\lambda_d = 1$, $\lambda_c = 0.01$, $\nu = 0.1$, $\alpha = 0.05$, $\mu = 5$, $\beta_1 = 15$, $\beta_2 = 0.1$, $\beta = 10$, $\gamma = 60$, $c = 0.9$, $q = 0.1$) . . . 224

7.2 Relative sensitivity analysis Ω_Θ of the $MTTF$ of the system,(For $\lambda = 0.5$, $\lambda_d = 1$, $\lambda_c = 0.01$, $\nu = 0.1$, $\alpha = 0.05$, $\mu = 5$, $\beta_1 = 15$, $\beta_2 = 0.1$, $\beta = 10$, $\gamma = 60$, $c = 0.9$, $q = 0.1$) . . . 225

8.1 Optimal expected total cost of the system $TC(R^*,\mu^*)$ for different parameters, for ($C_1 = 50, C_2 = 250, C_3 = 80, C_4 = 12, C_5 = 25, C_6 = 5$) 257

8.2 Optimal expected total cost of the system $TC(R^*,\mu^*)$ for the different parameters, for ($M = 25; S = 7; m = 2; R = 5; \lambda = 2; \nu = 1.2; \lambda_d = 3; \mu = 12; \zeta = 10; b = 200; \omega = 1.3; \delta = 1$) . . . 258

Chapter 1

General Introduction

"Efficiency is doing better what is already being done".

Peter Drucker

1.1 Motivation

The modern world heavily relies on machines. Without them, it would be impossible for the world to function correctly and faster. Over time, the progress in science and technology has resulted in highly sophisticated, complex, and robust engineering systems that consist of interconnected hardware and software components that are more powerful and efficient than ever before. This machining system consists of several operating units, each with its own unique features and structures. These operating units are prone to random failures, directly impacting the system's efficiency and indirectly affecting the economy and society. The constant need for repairs is having a significant impact on the overall productivity of the system. The fault is due to degraded performance, deterioration, or entire failure of the units or system. We understand the importance of maintaining system integrity to avoid the negative consequences of reduced reliability, mismanagement, and loss of time, money, or even life. Random failures can lead to a loss in production, longer waiting times, increased costs, manufacturing delays, reduced reliability, and longer runtimes. To mitigate these issues, repair and spare parts are crucial in improving the system's reliability and production, and reducing the expected total cost. The thesis aims to improve the reliability and availability of the machining system while optimizing the total cost over time.

In machine repair, Reliability refers to the ability of a system or unit to perform its intended functions consistently and without failure over a specified period and under specific operating conditions. It has critical importance in machine repair problems for various reasons, such as its direct impact on system performance, efficiency, and cost-effectiveness. Reliability is especially crucial in industries where continuous operation is essential for productivity; it helps to reduce machining downtime and contributes to stable and predictable production processes. Random faults can result in significant costs associated with emergency repairs, replacement parts, and idle labor. Adopting a reliable machine repair strategy that combines preventive and predictive maintenance is necessary to minimize these costs. Reliability is crucial in machining to enhance equipment lifespan, safety, customer satisfaction, operational efficiency, and predict maintenance costs. Statistical and stochastic models provide a robust framework to analyze various system performance measures such as unit indices, optimal system design, and expected cost analyses. Analyzing the models in-depth is necessary for accurate predictions. The study of such models addresses reliability prediction for life testing, structural reliability, machine maintenance, and replacement.

Reliability, availability, maintainability, and safety (RAMS) are indispensable in modern technological systems. Significant investments are made in research to develop sophisticated techniques, quality-of-service (QoS) standards, state-of-the-art designs, and just-in-time (JIT) maintenance for machining systems. This thesis focuses on developing reliability models for unpredictable machining systems prone to random and discriminate failures, emphasizing modeling methodologies, analysis techniques, and real-time problem-solving approaches. Each model is thoroughly discussed, covering model descriptions, analytical or numerical solutions, reliability characteristics, optimal parameters, and future applications. However, obtaining optimal solutions for such complex problems is often challenging or impossible.

The provision of spare units is necessary when the consequences of unit failures are unacceptable. Machining systems require various spare units, including hardware and software, homogeneous or heterogeneous, and in-house or third-party, to ensure continuity. Safety, security, financial considerations, and data protection drive provisioning spare units. Safety concerns arise in systems where human life or health is at risk, such as aviation, medical equipment, or nuclear plants. Security is essential to protect against vandalism or espionage in critical areas like communication or military installations. Financial losses due to industrial disruptions necessitate the manufacturing or production of spare units. Data loss prevention is crucial in systems reliant on data storage, such as computer-based data processing or IoT (Internet of Things) devices utilized in long-term laboratory experiments.

This thesis presents state-of-the-art reliability models for future research, providing insights for researchers, system designers, analysts, and engineers. It covers various system structures mirroring real-life problems and addresses reliability evaluation, optimal design, algorithms, mathematical tools, and references. Key features include complex analysis, Markov chain embeddable structures, tools for optimal design development, and application in predictive, preventive, or corrective measures.

1.2 Machining System

A machining system refers to a collection of operating units, tools, processes, and technology used in manufacturing. Each operating unit has a specific function that works in synchronization to produce the desired amount of products of the necessary quality within the required timeframe and most efficiently and economically. Various techniques, such as turning, milling, drilling, and grinding, shape raw materials into precise and intricate forms. The essential components of any machining system are processes, tools, machinery, materials, cutting fluids and coolants, computer-aided design (CAD) and computer-aided manufacturing (CAM), and quality control. The machining process involves well-defined stages that ensure an efficient and systematic transformation of raw materials into finished products, including system planning, operation, and control.

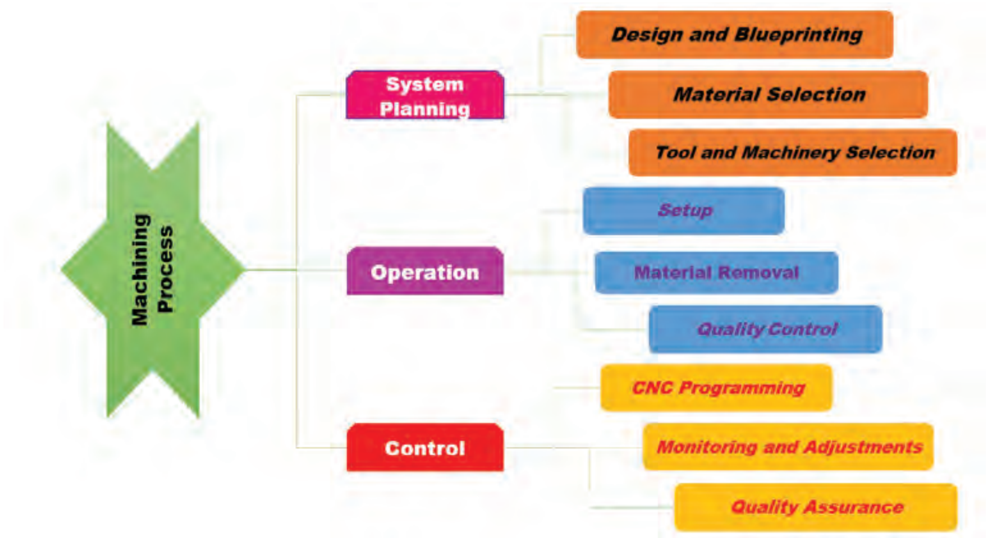


Figure 1.1: Breakdown of each states of the machining system

The machining system requires critical research to ensure quality, quantity, cost-effectiveness, timeliness, and reliability.

The proper functioning of the machining system depends on various factors, and a thorough understanding of the repair system is necessary for smooth operation. Effective and on-time repair processes reduce machine downtime, enhance productivity, and promote equipment longevity. It is beneficial for manufacturing economics. Cost optimization and reliability are the compelling reasons for delving into machine repair modeling. The first step in performance modeling of the machining system is to develop a functional design that satisfies the customers' needs and is suitable for production. Before establishing any machining system, the system analyst must know the cost of job flow, in-process inventories, maintenance, spare parts, etc.

The reliability and availability of the machining system can be enhanced by including spare parts, redundancy, proactive maintenance strategies, replacement policy, quality control, repairers' training and involvement, optimized maintenance scheduling, etc. The faults in operating units are random, causing severe effects on production and downtime of the machining system. A state-of-the-art repair facility design is necessary to ensure efficient production planning and control.

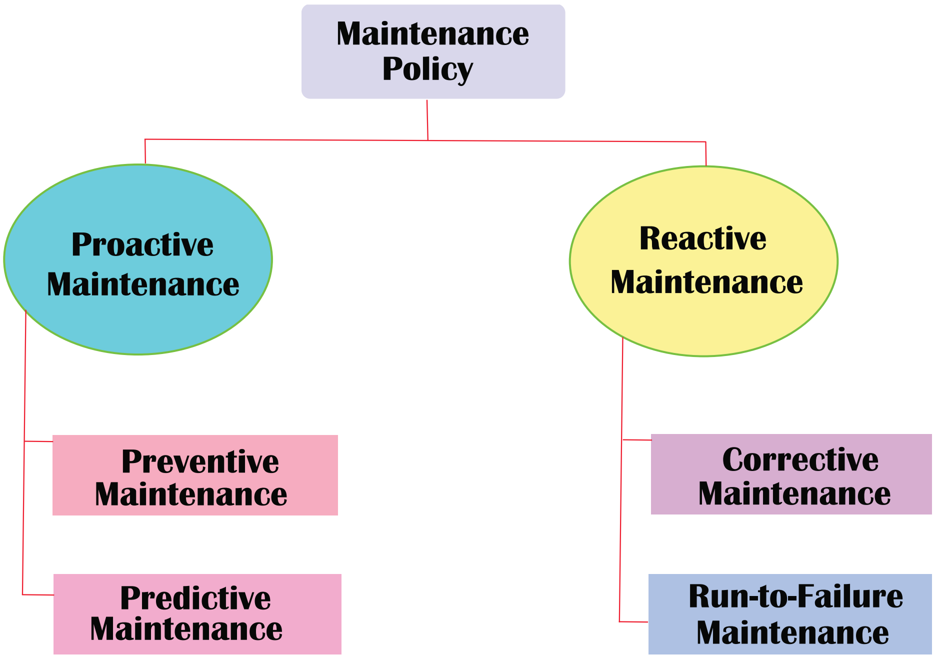


Figure 1.2: Maintenance policy

The Industrial Revolution is typically linked with the late 1700s and the mid-1800s. After World War II, Queueing Theory and Operations Research gained importance in manufacturing systems. Operations Research (OR) is a field of study offering various analytical tools and models. These tools and models can be applied to various machine repair problems. For instance, OR can help develop optimal maintenance schedules, allocate resources, diagnose faults, make informed decisions, predict maintenance needs, optimize spare parts inventory levels, and streamline machine flow through maintenance service systems.

The proper functioning of operating units is critical to any machining system. Therefore, maintaining these units through just-in-time maintenance is essential. The timely replacement and maintenance of these units can improve their availability. To achieve this, it is essential to have a well-planned maintenance policy in place. Generally, maintenance policies are classified into two categories as shown in Fig. 1.2.

1.3 Fundamental Configuration of Machining Setup

Queueing theory is a branch of probability theory and operations research that deals with the time disparity between customer arrivals and service provision. Queueing Theory provides a systematic and mathematical approach to analyzing and optimizing machine repair processes. Before establishing any machining systems, the exclusion of Queueing Theory affects economic consequences such as enhanced downtime costs, inefficient resource allocation, improper allocation of resources for repairs, and impacts on customer satisfaction. Understanding the fundamental structures of queueing systems can be incredibly valuable for modeling and analyzing waiting lines in various industries and settings. In general, some fundamental structures of the waiting line problem related to the machining systems can be categorized into the following types:

Single-Line, Single-Phase

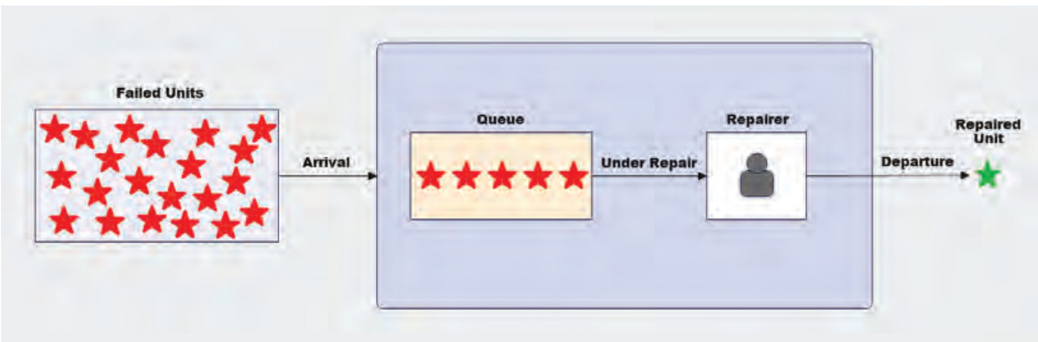


Figure 1.3: Single-Line,Single-Phase

A Single-Channel, Single-Phase waiting line (Fig. 1.3) is a queuing system in which customers form a single line and are served singly at a time by a single server. This approach guarantees efficient and fair service in the order of arrival. Real-world examples of this type of waiting line include checkout at supermarkets, bank tellers, hospital triage, and machine repairs in manufacturing facilities.

Single-Line, Multi-Phase

In a Single-Line, Multi-Phase waiting line (Fig. 1.4), customers wait in a single line but can access multiple service points or channels to receive service. This structure optimizes the use of resources and allows customers to be served according to their unique needs or priorities. Examples of real-world applications of this type of waiting line include airport check-in counters, bank branches, fast-food drive-thrus, automotive repair shops, computer repair centers, and appliance repair services.

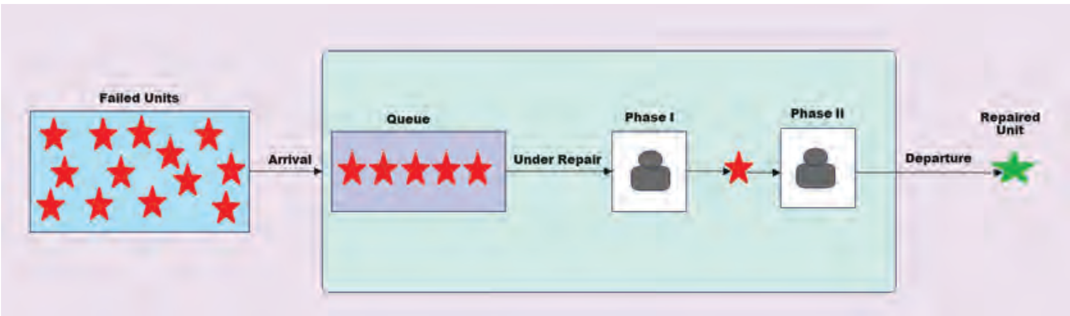


Figure 1.4: Single-Line, Multi-Phase

Multi-Line, Single-Phase

A Multi-Line, Single-Phase waiting line system (Fig. 1.5) is a queuing arrangement where multiple parallel queues lead to a single service point. Customers can choose a queue based on length, convenience, or service type. However, the service is provided on a first-come, first-served basis, irrespective of the queue chosen by the customer. Examples of this type of waiting line are supermarkets with multiple checkout lines, ticket counters at train stations, banks with multiple ATMs, commercial HVAC (Heating, Ventilation, and Air Conditioning) repair companies, and equipment repair service centers.

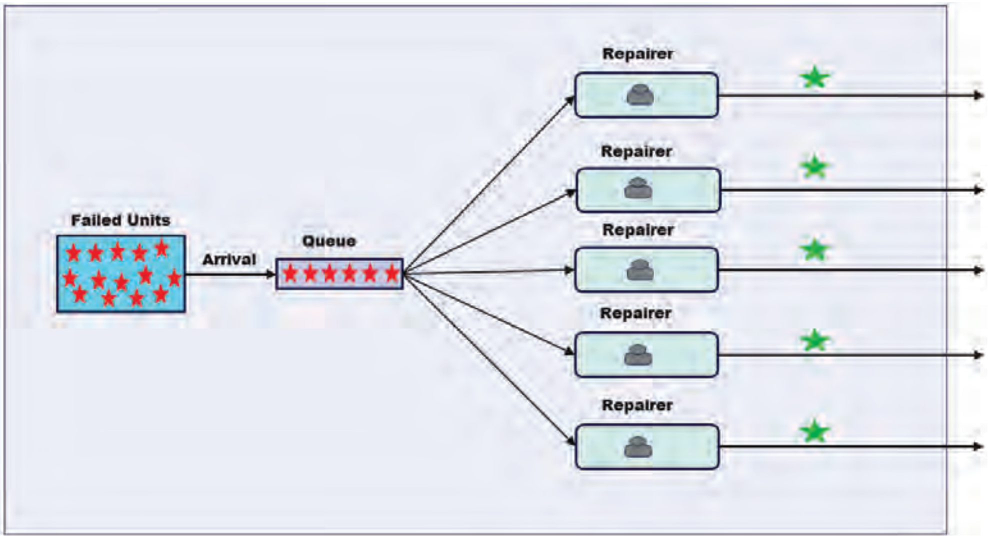


Figure 1.5: Multi-Line, Single-Phase

Multi-Line, Multi-Phase

A Multi-Line, Multi-Phase waiting line system (Fig. 1.6) is a queuing mechanism that comprises several parallel queues leading to multiple service points or channels. The system directs customers or entities to specific service points based on predefined criteria and serves them simultaneously or in a coordinated manner across different phases of the waiting line. Examples of this system in real-world scenarios include Airport Security Checkpoints, Supermarket Deli Counters, Customer Service Call Centers, and Industrial Maintenance Departments.

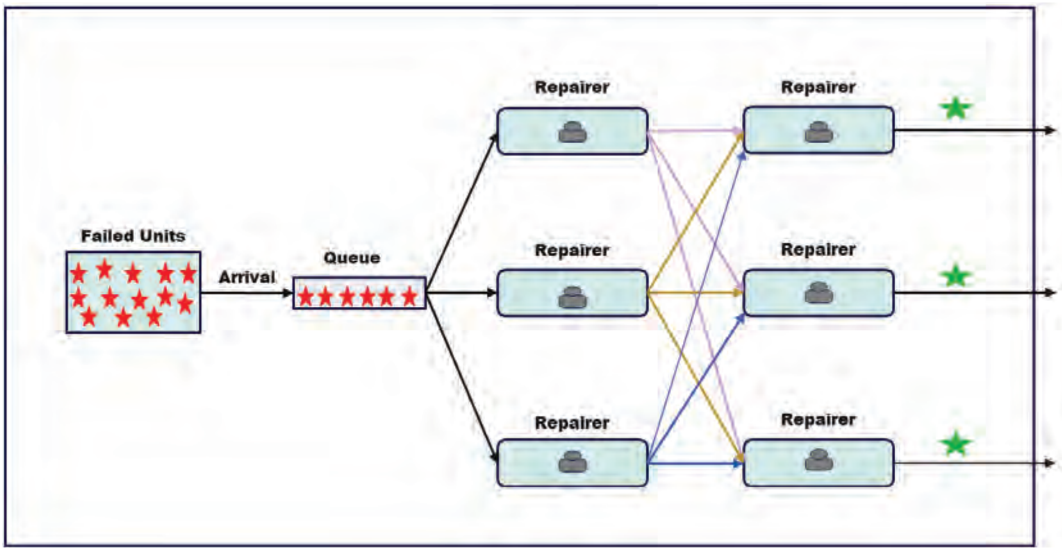


Figure 1.6: Multi-Line, Multi-Phase

Some additional queueing configurations in machining systems or different sectors can be categorized as follows:

Priority queue

A Priority Queue waiting line system (Fig. 1.7) prioritizes tasks based on predefined criteria such as urgency, importance, or customer status. Tasks with higher priority are processed first, regardless of their arrival order. This type of system is used in various real-world scenarios, such as in a hospital emergency room, an airline boarding process, equipment maintenance in manufacturing, and software development bug fixes.

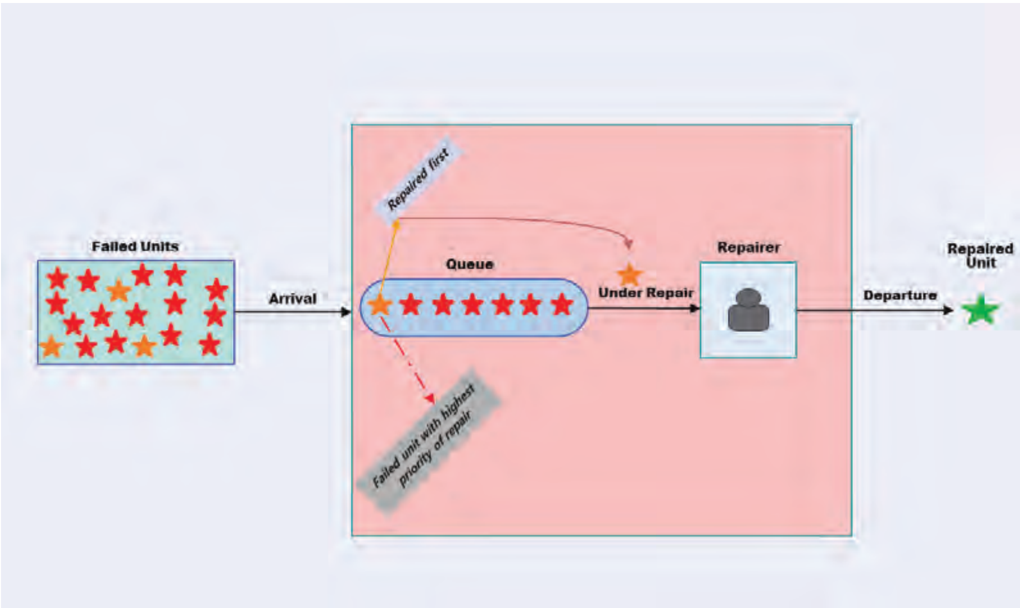


Figure 1.7: Priority queue

Virtual Waiting Lines

A Virtual Waiting Line is a queueing system where customers or entities are placed in a queue without physically waiting in line. Instead, they receive notifications or updates about their queue position and estimated wait time, allowing them to engage in other activities until their turn for service. This system has applications in various real-world scenarios, such as online appointment scheduling, retail store pickup services, customer service call centers, restaurant waitlists, remote troubleshooting and support, and online repair ticketing systems.

Self-Service Queues

Self-service queues are systems where customers can join a queue and request a service without requiring direct assistance from the staff. This means that customers have autonomy over their waiting experience by managing their entry into the queue and using self-service kiosks or digital interfaces to initiate service. The system has various applications in real-world scenarios, such as self-checkout lanes in supermarkets, online appointment scheduling, mobile check-in at airports and hotels, self-service ordering kiosks in restaurants, and mobile maintenance apps.

Balking

Balking in waiting lines occurs when customers or entities refuse to join a waiting line due to long wait times, overcrowding, or poor service conditions. It leads to losing business opportunities for the service provider as these customers leave the queue area without receiving service. In machine repairs, balking occurs when users/operators choose not to join a repair queue or request maintenance for various reasons. In supermarkets, customers may balk at long checkout lines, Customer Service Call Centers, Restaurant Waiting Lists, Ticket Lines at Events, and Public Transportation customers may balk at long checkout lines.

Reneging

Reneging is a phenomenon that occurs when customers or entities abandon the queue or waiting line before receiving service despite joining the queue in the first place. This usually happens due to long wait times, dissatisfaction with the quality of service, or changing priorities.

Batch Arrivals

When multiple customers or entities arrive at a service facility simultaneously, it is called a batch arrival. This is different from individuals arriving over time. Batch arrival models are characterized by arrivals in groups or clusters, which can impact queue dynamics and service efficiency. Some examples of batch arrival waiting lines in real life include bus arrivals at a bus stop, customer arrivals at a retail store, patient arrivals at a medical clinic, and package deliveries to a distribution.

1.4 Classification by Machine Type or Industry Sector

Classification by machine types or industry sections provides a framework for organizing and understanding the diverse machinery used in various sectors. This classification aids in identifying specific needs, challenges, and opportunities within each industry, leading to the development of tailored solutions and technological advancements.

Industry 4.0

Industry 4.0, also known as the Fourth Industrial Revolution, is the continuous transformation of conventional manufacturing and industrial practices through digitalization, automation, and the integration of advanced technologies. It signifies a significant change in how products are designed, manufactured, and delivered, utilizing cutting-edge technologies to create more intelligent, efficient, and interconnected production systems.

Cyber-Physical Systems (CPS)

Cyber-physical systems (CPS) are the amalgamation of physical components such as sensors, actuators, machinery, and production systems with digital technologies like computing devices, communication networks, and software platforms. The concept of Industry 4.0 involves the integration of physical systems with digital technologies to create CPS, which enables real-time monitoring, control, and optimization of manufacturing processes.

Internet of Things (IoT)

The Internet of Things (IoT) has transformed how we interact with the physical world. It enables various healthcare, transportation, manufacturing, agriculture, and innovative city applications. The Internet of Things (IoT) is a network of connected devices and sensors integrated into machines and equipment used for repair. These devices gather and exchange data in real-time to monitor the machines' performance, condition, and health. IoT facilitates predictive maintenance, remote monitoring, and automation of repair processes, resulting in improved efficiency, reduced downtime, and cost savings.

Additive Manufacturing(3D)

Additive Manufacturing, also known as 3D Printing, is a technology that can transform various industries, such as aerospace, automotive, healthcare, and consumer goods. It offers many benefits, such as rapid prototyping, customization, and the ability to fabricate complex geometries. This revolutionary technology has changed how products are designed, manufactured, and distributed, making it a game-changer for the manufacturing industry.

Robotics and Automation

Robotics is a field that focuses on creating mechanical devices called robots. These robots are equipped with sensors, actuators, and control systems, enabling them to interact with their environment, manipulate objects, and perform pre-programmed tasks. Robotics is a diverse field encompassing several subfields, such as industrial, mobile, humanoid, and soft robotics. Each subfield addresses specific application domains and challenges. Automation utilizes technology and software to automate tasks, procedures, and processes that humans conventionally execute. The process involves using intricate algorithms, sensors, actuators, and control systems to regulate and optimize the operation of machinery, equipment, and systems. The spectrum of automation can vary from simple programmable logic controllers (PLCs) used in industrial settings to multifaceted autonomous systems that operate with minimal human intervention.

Flexible Manufacturing Systems (FMS)

Flexible Manufacturing Systems (FMS) are computer-controlled production systems that integrate various manufacturing processes such as material handling, machining, assembly, and testing into a single automated system. FMSs are designed to be adaptable to changing production requirements and can produce a diverse range of products efficiently and with high quality standards. They are particularly useful for companies that have a wide range of products, experience fluctuating demand, and require quick response times to market changes. FMSs are crucial in enabling swift and competitive manufacturing operations in today's dynamic business environment.

Supply Chain Digitization

Supply Chain Digitization is a process that leverages digital technologies to streamline operations and improve responsiveness across the supply chain. This approach helps organizations optimize processes, enhance visibility, and improve stakeholder collaboration, improving efficiency, profitability, and customer satisfaction. By digitizing the supply chain, businesses can gain a competitive advantage by driving innovation, reducing costs, and delivering higher value to their customers.

Cloud computing

Cloud computing is a revolutionary approach to delivering services via the Internet, such as servers, storage, databases, networking, software, and analytics. This innovative technology empowers users to access and utilize computing resources and services provided by cloud service providers (CSPs) without requiring any on-premises infrastructure. With cloud computing, users only pay for their services, resulting in a cost-effective and flexible solution to various business challenges and opportunities in today's digital economy. This transformative technology empowers industries to innovate, compete, and thrive in an increasingly interconnected and data-driven world.

Cloud manufacturing

Cloud manufacturing is a cutting-edge production method that utilizes cloud computing technologies, state-of-the-art resources, and data-centric processes to achieve efficient, high-quality, and environmentally friendly manufacturing. Using cloud-based platforms over the Internet enables collaborative and dispersed manufacturing, where various organizations, locations, and supply chain partners can share resources such as production equipment, software tools, and expertise. Cloud manufacturing enhances agility, efficiency, and innovation in production processes by integrating manufacturing capabilities and services. It allows manufacturers to access, leverage, and share manufacturing resources and services flexibly, efficiently, and collaboratively, driving innovation and competitiveness in the digital era.

Sustainable Manufacturing

Sustainable Manufacturing is a comprehensive approach incorporating eco-friendly practices and social responsibility into manufacturing. Its foremost goal is to reduce harmful effects on the environment, society, and the economy while enhancing economic feasibility and social welfare. The methodology employs innovative technologies, materials, and processes to curtail resource consumption, minimize waste production, and promote fairness across the product cycle. It strikes a balance between economic expansion and environmental preservation, guaranteeing that manufacturing activities meet current requirements without jeopardizing the ability of future generations to meet their own.

1.5 Machine Repair Problem

A comprehensive machine repair system includes a collection of protocols, methodologies, and assets utilized to maintain, repair, and sustain industrial or organizational machinery and equipment. Its main goal is to minimize downtime, increase equipment uptime, improve reliability, and enhance the performance and dependability of machines to ensure seamless operations and higher production efficiency. Machine repairing can be modeled as a finite queueing system, where the number of failed units represents the population of potential customers, each unit breakdown is considered a new customer, and the repair team serves as the server. We illustrate a Machine repair model having K identical operating units with R repairers. The time to failure of each operating unit follows an exponential distribution with the mean time to failure $1/\lambda$, and the time-to-repair follows an exponential distribution with a mean time of $1/\mu$. Initially, at $t = 0$, there are no failed units; after time t, if there are n failed units, then P_n represents the steady-state probability of n failed units. Fig. (1.8) denotes the state transition diagram for the continuous-time Markov chain (CTMC) used in the machine repair model.

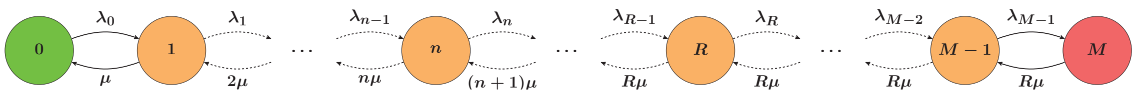


Figure 1.8: State transition diagram of basic Machine repair system

The failure rate and repair rate, which depend on the system's state, can be denoted by λ_n and μ_n, respectively, and are defined as

$$\lambda_n = \begin{cases} (M-n)\lambda; & if \quad 0 \le n < M \\ 0; & otherwise \end{cases} \tag{1.1}$$

and

$$\mu_n = \begin{cases} n\mu; & 0 \le n < R \\ R\mu; & R \le n \le M \end{cases} \tag{1.2}$$

The steady-state probability P_n is derived for this model by solving the Chapman-Kolmogorov equations, which dictate the model utilizing the transition failure and repair rates as described in Eqns. 1.1 and 1.2. Hence, the distribution of queue sizes can be determined by employing the product-type solution as outlined below.

$$P_n = \begin{cases} \binom{M}{n} \left(\frac{\lambda}{\mu}\right)^n P_0; & \text{when} \quad 1 \leq n < R \\ \\ \binom{M}{n} \frac{n!}{R^{n-R}R!} \left(\frac{\lambda}{\mu}\right)^n P_0; & \text{when} \quad R \leq n \leq M \end{cases} \tag{1.3}$$

By applying the normalizing conditions, the initial probability P_0 can be defined as below.

$$P_0 = \left(\sum_{n=1}^{R} \binom{M}{n} \left(\frac{\lambda}{\mu}\right)^n + \sum_{n=R+1}^{M} \frac{n!}{R^{n-R}R!} \left(\frac{\lambda}{\mu}\right)^n \right)^{-1} \tag{1.4}$$

Efficient and prompt repair of failed units ensures the system's smooth operation. Consequently, machine repair constitutes a crucial aspect of all machining systems. Occasionally, overcrowding occurs, resulting in prolonged inspection times for the last unit in line. Delays in repair can lead to extended queues or waiting lines, hindering service delivery. However, accurately predicting failed units' arrival and repair duration is challenging, as these decisions are often unpredictable. To address these challenges, system designers may increase service facilities at a higher cost. Conversely, reducing service facilities can lower costs but result in longer waiting times. Inadequate repair services can lead to prolonged waiting periods and increased downtime for failed units, incurring significant costs such as social costs, loss of customers, or production delays. Hence, the primary objective is to strike a balance between service costs and waiting times.

1.6 Machining System with Redundancy

A redundant machine repair system is a proactive approach to mitigating the impact of machine failures and maintenance activities on production operations. These systems help minimize downtime, improve reliability, and ensure continuous production in manufacturing environments by providing backup machines and automated failover mechanisms. Redundancy can be categorized as either active or passive. In active redundancy, redundant units actively participate in the production, and if primary operating units fail, they are ready to take their place immediately. Passive redundancy involves keeping redundant units or components in a standby or idle state until required. Typically, redundant units are activated only when primary units fail or when repair activities are scheduled. There are three types of passive redundancy: cold, warm, and hot.

- Cold Redundancy: In this redundancy, the failure characteristics of the standby units are zero, meaning they are reliable and can take over production tasks seamlessly whenever required.
- Warm Redundancy: In this redundancy, the failure characteristics of the standby units are both nonzero and less than that of the operating unit.
- Hot Redundancy: In this redundancy, the failure characteristics of the redundant units are equal to primary operating unit in inactive state.

If we construct a machine repair system incorporating M active units with S_1 cold standby units, the at any time t the failure rate is given as

$$\lambda_n = \begin{cases} M\lambda; & 0 \le n < S_1 \\ (M+S_1-n)\lambda; & S_1 \le n \le M+S_1 \\ 0; & \text{otherwise} \end{cases} \tag{1.5}$$

if we construct a machine repair system incorporating M active units with S_2 warm standby units, and when warm standby unit is in inactive state then the failure rate of warm standby unit S_2 is given by $\nu, (0 < \nu < \lambda)$

$$\lambda_n = \begin{cases} M\lambda + S_2\nu; & 0 \le n < S_2 \\ (M+S_2-n)\lambda; & S_2 \le n \le M+S_2 \\ 0; & \text{otherwise} \end{cases} \tag{1.6}$$

similarly if there are S_3 hot standby units with same failure rate (λ) as an active unit in the system then at any instant the failure rate is given by

$$\lambda_n = \begin{cases} (M+S_3-n)\lambda; & 0 \le n < S_2 \\ 0; & \text{otherwise} \end{cases} \tag{1.7}$$

If we formulate a machine repair scenario involving M active units, S_1 cold standby units, S_2 warm standby units, and S_3 hot standby units managed by R repairers, the failure and repair rates are defined as follows:

$$\lambda_n = \begin{cases} (M+S_3)\lambda + S_2\nu; & 0 \le n < S_1 \\ (M+S_3)\lambda + (S_1+S_2-n)\nu; & S_1 \le n < S_1+S_2 \\ (M+S_1+S_2+S_3-n)\lambda; & S_1+S_2 \le n < M+S_1+S_2+S_3 \end{cases} \tag{1.8}$$

and the state-dependent repair rate μ_n is

$$\mu_n = \begin{cases} n\mu; & 1 \leq n < R \\ R\mu; & R \leq n \leq M + S_1 + S_2 + S_3 \end{cases} \tag{1.9}$$

1.7 Fundamental components of Queueing systems

A queueing system is a mathematical model that studies and analyzes waiting lines (queues) behavior and associated service processes. This model deals with customers who arrive individually or in batches and must wait for service if the server is busy; otherwise, they receive service only when the server is available. The queueing system consists of an arrival process, service process, queue discipline, queue capacity, service channels, population source, and exit mechanism. The service facility consists of one or more servers that are organized in various specifications or designs. The servers select the waiting customers based on pre-specified service policies or discipline and provide them with the necessary service. The queueing system also includes customers who are currently receiving service.

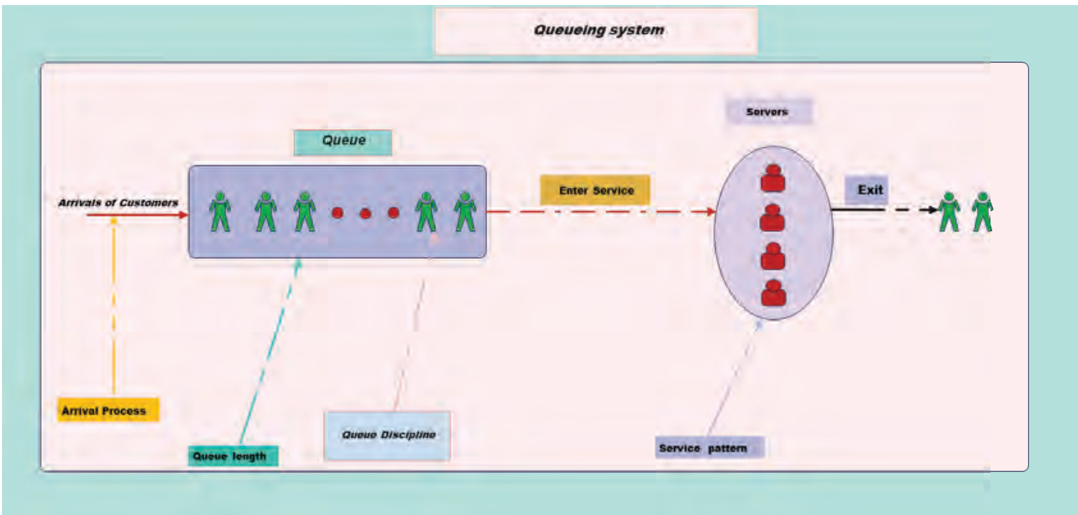


Figure 1.9: Queueing system

Figure 1.9 illustrates the direction of customer flow in the queuing system. Primary factors mainly categorize different types of queuing systems.

Arrival Process:

In a queueing system, the arrival process deals with how the customers and entities enter the system at time $t_1, t_2, .., t_m$ where $t_{n+1} > t_n$ and $n = 1, 2, 3, ..., m$. The inter-arrival time of $(m+1)^{\text{th}}$ and m^{th} customers are denoted by τ_m and defined as $\tau_m = t_{m+1} - t_m$. It describes the pattern or distribution of arrivals and is a fundamental aspect of queueing theory. The arrival process helps characterize the queue dynamics and influences factors such as queue length, waiting times, and system utilization. It specifies the pattern and rate at which entities arrive, often modeled using probability distributions such as Poisson or exponential distributions and deterministic and non-Poisson arrival processes. If at any time t, there are N customers in the system, then it is denoted by $N(t)$.

Service Mechanism:

The mechanism for serving customers or entities within the system is defined as the service mechanism. It specifies the distribution of service times or how long it takes to complete a service for each entity.

Queue Discipline:

Queue discipline refers to the policies or rules that dictate how customers are selected from a waiting line or queue when a service opportunity arises. This system plays a crucial role in managing the flow of customers, optimizing system performance, and ensuring fairness. The primary queue discipline is first-in, first-out (FIFO). However, sometimes, the queues are served by the discipline of last-come, first-served (LCFS), priority-based queueing, shortest processing time (SPT), random, and round robin (RR).

Customer behavior:

It's essential to understand customer behavior when designing service facilities. To ensure optimal service delivery and customer satisfaction, service providers should consider incorporating controlled or uncontrolled arrivals, addressing impatience behavior, and implementing threshold policies. If the customer finds long waiting lines, they either join the queue and wait for service or decide not to join the queue, i.e., balking. After joining the queue, if customers perceive the wait time to be longer than expected, they may leave the queue; such behavior of customers is known as reneging. Jockeying involves customers switching between lines to find the shortest queue, based on observation of queue progress.

Service Channels:

In a queuing system, service channels refer to the points where customers receive service. The system may have one or more service channels to provide service according to the arrival of customers. It is known as a homogeneous service system if all the service channels identically provide the same types of services. In a heterogeneous system, the service channels may differ regarding service type or rate. Additionally, a queuing system may have a finite or infinite capacity to hold the waiting customers.

1.8 Distinct Important Processes

In this section, we present a concise overview of several crucial discrete-time and continuous-time processes that hold significance for in-depth studies within queueing theory.

1.8.1 Stochastic Process:

A stochastic process $\{X(t) : t \in T\}$ is the collection of the random variables. Where the set T is the index set of the process. The index t is interpreted as time, and we refer to $X(t)$ as the state of the process at time t.

State Space:

The state space of a stochastic process is the set of all possible values that the random variables $X(t)$ can take on.

Discrete-time stochastic process:

If the index set T is a countable set., i.e., $X_n, n = 0, 1, 2, 3, ...,$

Continuous-time stochastic process:

If the index set T is an interval of real line or infinite set, the stochastic process is said to be continuous-time stochastic process; i.e., $T = \{X(t), t \geq 0\}$

1.8.2 Counting Process:

A counting process is a stochastic process $\{N(t), t > 0\}$ that represents the number of events that have occurred up to time t. The counting process satisfies the following properties.

(i) $N(0) = 0$, i.e.,no event have occured before time $t = 0$.
(ii) $N(t) \leq N(t+1)$, i.e. $N(t)$ is a non-decreasing function.
(iii) $N(t)$ counts discrete events.

- **Transition Probability Function:** This function expresses the likelihood of transitioning from one state to another during a specific time interval. For any states i, j in the state space S and time parameters t, s such that $s \leq t$, then the transition probability function can be expressed as:

$$P(X_{t+s} = j | X_s = i, X_u = x_u, 0 \leq u < s)$$

where (i) X_t denotes the state of the process at time t.
(ii) $X_s = i$ denotes indicates that the process is in state i at time s.
(iii) $X_u = x_u$ represents the sequence of states from time 0 to s.

1.8.3 Markov Process:

A stochastic process is known as the Markov process if it satisfies the Markovian property, which states that the future state of the process depends only on the present state and not on the sequence of events that preceded it. Mathematically, it can be expressed as

$$P(X_{t+s} = j | X_s = i, X_u = x_u, 0 \leq u < s) = P(X_{t+s} = j | X_s = i)$$

1.8.4 Markov Chain:

A Markov process with a countable or countably infinite state space, i.e., as S is discrete and an index set T that takes values from $T = \{0, 1, 2, 3, \dots\}$ is referred to as a Markov chain. Markov chain is categorized in two types namely:

Discrete-Time Markov Chain (DTMC):

A stochastic process $\{N(t), t \geq 0\}$ on a state space S is said to be a discrete-time Markov chain if for all i, j in S

$$\begin{aligned} P(N(t+1) = j | N(t) = i, N(t-1) = i_{i-1}, \dots, N(2) = i_2, N(1) = i_1, N(0) = i_0) \\ = P(N(t+1) = j | N(t) = i) \end{aligned}$$

Continuous-Time Markov Chain (CTMC):

A stochastic process $\{N(t), t \geq 0\}$ on a state space S is said to be a continuous-time Markov chain if for all i, j in S and $t, s \geq 0$,

$$P(N(s+t) = j|N(s) = i, N(u), 0 \leq u \leq s) = P(N(s+t) = j|N(s) = i)$$

1.8.5 Birth-Death Process:

A continuous-time Markov chain $\{N(t), t \geq 0\}$ with countable or countably infinite state space $\{0, 1, 2, 3, \dots\}$ for which transitions from state n may go only to either state $n-1$ or state $n+1$ and homogeneous transition rate matrix V_{ij} is known a birth and death process if $V_{ij} = 0$ for all i and j such that $|i - j| > 1$.

1.8.6 Poisson Process:

A counting process $\{N(t), t \geq 0\}$ is said to be a Poisson process with parameter λ, $\lambda > 0$ if it satisfies the following conditions:
(i) $N(0) = 0$. i.e., the counting of events begins at time $t = 0$.
(ii) The process exhibits independent increments. i.e., if $t < s$ then the number of events that occur by time t $(N(t))$ must be independent of the number of events that occur between times t and s $(N(s) - N(t))$.
(iii) $P(N(t + \Delta t) - N(t) = 1) = \lambda \Delta t + O(\Delta t)$
(iv) $P(N(t + \Delta t) - N(t) > 1) = O(\Delta t)$
If the number of events in any interval of length Δt follows the Poisson distribution with parameter $\lambda \Delta t$, i.e., for all $t \geq 0$ and $\Delta t \geq 0$.

$$P\{N(t + \Delta t) - N(t) = n\} = P(N(\Delta t) = n) = \frac{(\lambda \Delta t)^n e^{-\lambda \Delta t}}{n!}; n = 0, 1, 2, \dots \tag{1.10}$$

1.8.7 Renewal Process:

Let $\{X_n\}$ be the time between the $(n-1)^{\text{th}}$ and n^{th} event, $n \geq 1$. If the sequence of non-negative random variables $\{X_1, X_2, \dots\}$ is identically independent distributed, then the counting process $\{N(t), t \geq 0\}$ is said to be renewal process. For a renewal process having inter-arrival times $X_1, X_2, X_3, \dots$, the time of the

n^{th} renewal is denoted by Z_n and defined as below

$$Z_n = \sum_{i=1}^{n} X_i$$

1.8.8 Chapmann-Kolmogorov Equation:

The Chapman-Kolmogorov equation helps compute the multi-step transition probabilities. The n-step transition probabilities from state i to state j over all possible k values and are expressed as

$$P_{i,j}(t+s) = \sum_{k=0}^{\infty} P_{ik}(t)P_{kj}(s) \tag{1.11}$$

The above equation states that in order to transition from state i to state j in time t, $X(t)$ moves to state k in time t and then from k to j in the remaining time s.

1.9 Some Important Distribution

An experiment with uncertain or random outcomes is called a random experiment (E). Let S be the sample space (set of all possible outcomes) associated with the experiment. A random variable X is a variable taking real values corresponding to each outcome (or element) of a random experiment or a sample space S. Random variables are categorized into two types, namely Discrete and Continuous. Discrete random variables can only take on a countable number of distinct values, such as the number of goals scored in a soccer match. In contrast, continuous random variables can take any value within a specific range, such as a person's height. Both these random variables are also categorized into probability density function $f(x)$ and commutative distribution function $F(x)$. One of the key concepts associated with random variables is the moment-generating function $m_X(t)$, mean $E(X)$, and variance $V(X)$.

Point Mass Distribution:

A discrete random variable X is said to have a point mass distribution if its probability density function is given by

$$f(x) = \begin{cases} 1; & x = k, \quad k \in (-\infty, \infty) \\ 0; & x \neq k \end{cases}$$

the mean and variance for point mass distribution is given as

$$E(X) = k \quad \text{and} \quad V(X) = 0$$

Geometric Distribution:

A discrete random variable X in which a random experiment consists of a series of independent trials is said to have a geometric distribution if its probability density function is given by

$$f(x) = \begin{cases} p(1-p)^{x-1}; & x = 1,2,3,\dots \quad \& \quad 0 < p < 1 \\ 0; & otherwise \end{cases}$$

the mean and variance for geometric distribution is given as

$$E(X) = \frac{1}{p} \quad \text{and} \quad V(X) = \frac{(1-p)}{p^2}$$

Binomial Distribution:

A discrete random variable X in which a random experiment consists of a finite number (n) of independent trials is said to have binomial distribution if its probability density function is given by

$$f(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}; & x = 0,1,2,3,\dots,n \\ 0; & otherwise \end{cases}$$

the mean and variance for binomial distribution is given as

$$E(X) = np \quad \text{and} \quad V(X) = npq, \quad q = 1-p$$

Poisson Distribution:

Let X be the discrete random variable representing the number of events occurring during a given period, and λ be the average number of events. If discrete random variable X follows a Poisson distribution, then the probability of observing x events over the period is

$$f(x) = P(X = x) = \begin{cases} \frac{\lambda^x e^{-\lambda}}{x!}; & x = 0,1,2,3,\dots \\ 0; & \text{otherwise} \end{cases}$$

Uniform Discrete Distribution:

A discrete random variable X is said to have uniform distribution if it assumes a finite number of values, all with the same chance of occurrence or equal probabilities.

$$f(x) = \frac{1}{k+1}; \quad x = x_0, x_1, x_2, \ldots, x_k$$

Uniform or Rectangular Distribution:

A continuous random variable X is said to have uniform distribution if its density function $f(x)$ takes constant value for all values of x

$$f(x) = \begin{cases} k; & x_1 \leq x \leq x_2 \\ 0; & \text{otherwise} \end{cases}$$

where $k = \frac{1}{x_2 - x_1}$ and the mean and variance for uniform distribution over interval $[x_1, x_2]$ is given as

$$E(X) = \frac{x_1 + x_2}{2} \quad \text{and} \quad V(X) = \frac{(x_2 - x_1)^2}{12}$$

Gamma Distribution:

A continuous random variable X is said to have gamma distribution with parameters $\alpha > 0$ and $\beta > 0$ if its density function $f(x)$

$$f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)} \frac{x^{\alpha-1} e^{\frac{-x}{\beta}}}{\beta^{\alpha}}; & x > 0 \\ 0 \qquad ; & \text{otherwise} \end{cases}$$

the mean and variance for gamma distribution is given as

$$E(X) = \alpha\beta \quad \text{and} \quad V(X) = \alpha\beta^2$$

Exponential Distribution:

It is also a special case of gamma distribution, a gamma distribution with $\alpha = 1$ is called exponential distribution if its probability density function $f(x)$ is given as

$$f(x) = \begin{cases} \frac{1}{\beta} e^{\frac{-x}{\beta}}; & x > 0 \;\&\; \beta > 0 \\ 0; & \text{otherwise} \end{cases}$$

the mean and variance for exponential distribution is given as

$$E(X) = \beta \quad \text{and} \quad V(X) = \beta^2$$

Chi-Squared (χ^2) Distribution:

It is also special case of gamma distribution, a gamma distribution with $\beta = 2$ and $\alpha = \frac{\nu}{2}$, where $\nu > 0$ is known as a degree of freedom, is known as chi-squared χ^2 distribution if its probability density function $f(x)$ is

$$f(\chi^2) = \begin{cases} \frac{1}{\Gamma(\frac{\nu}{2})2^{\frac{\nu}{2}}}(\chi^2)^{\frac{\nu}{2}-1}e^{\frac{-\chi^2}{2}}; & x > 0 \\ 0 \qquad ; & \text{otherwise} \end{cases}$$

the mean and variance for chi-squared χ^2 distribution is given as

$$E(X) = \nu \quad \text{and} \quad V(X) = 2\nu$$

Normal Distribution (Gaussian distribution)

A continuous random variable X is said to have Normal distribution with parameters μ(mean) and σ (standard deviation) if its probability density function is

$$f(x) = \frac{1}{2\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}; \quad -\infty < x < \infty,\ -\infty < \mu < \infty,\ \sigma > 0$$

Standard Normal Distribution

Let X be normally distributed random variable with mean 0 and variance 1 is said to have standard normal distribution.

Erlang Distribution

A continuous random variable X is said to have Erlang distribution if its density function $f(x)$

$$f(x) = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{\Gamma(x)}; \quad x > 0$$

where, k is a shape parameter (representing the number of exponential distributions being summed) and λ $(\lambda > 0)$ is a rate parameter. the mean and variance for erlang distribution is given as

$$E(X) = \frac{k}{\lambda} \quad \text{and} \quad V(X) = \frac{k}{\lambda^2}$$

Hyperexponential Distribution

A continuous random variable X is said to have hyperexponential distribution if its probability density function $f(x)$ is

$$f(x)=\sum_{i=1}^{k} b_i\lambda_i e^{-\lambda_i x}; \quad x\geq 0;\ 0\leq b_i\leq 1, \text{such that} \sum_{i=1}^{k} b_i=1$$

where, $\lambda_i>0\ \ \forall i$. the mean and variance for hyperexponential distribution is given as

$$E(X)=\sum_{i=1}^{k}\frac{b_i}{\lambda_i} \quad \text{and} \quad V(X)=2\sum_{i=1}^{k}\frac{b_i}{\lambda_i{}^2}-\left[\sum_{i=1}^{k}\frac{b_i}{\lambda_i}\right]^2$$

Phase-type distribution

A phase-type distribution is a probability distribution formed through the convolution or combination of exponential distributions. It arises from a system where one or more interconnected Poisson processes occur sequentially or in various phases.

Let us contemplate a continuous-time Markov process comprising $k+1$ states, where $k\geq 1$. Among these states, $1\ldots k$ are transient, while state 0 is absorbing. Moreover, suppose the process is initialized with a probability distribution across the $m+1$ phases, denoted by the probability vector (α_0,α), where α_0 is a scalar, and α is a $1\times m$ vector. The continuous phase-type distribution describes the time duration from initiating the abovementioned process until it is absorbed into the absorbing state. The transition rate matrix of this process is given as:

$$Q=\begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{T}^0 & T \end{bmatrix}$$

Here T is $m\times m$ matrix represents the transition between transient states, and $\mathbf{T}^0$ = $1-Te$, where e is the $m\times 1$ column vector with each element is 1.

Hypoexponential or Generalized Erlang Distribution

Hypoexponential distribution is a particular case of phase-type distribution. Like phase-type distribution, hypo exponential distribution also has first k transient states, and the state $k+1$ is an absorbing state. In hypo exponential distribution, we start the process always from state 1 and move skip-free from state j to $j+1$ with rate λ_j until we reach the state k then absorb in state $k+1$ with rate λ_k. So for hypo exponential distribution α=[1, 0, 0, ..., 0] and matrix T has the following structure:

$$T = \begin{bmatrix} -\lambda_1 & \lambda_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & -\lambda_2 & \lambda_2 & 0 & \dots & 0 & 0 \\ 0 & 0 & -\lambda_3 & \lambda_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -\lambda_{k-1} & \lambda_{k-1} \\ 0 & 0 & 0 & 0 & \dots & 0 & -\lambda_k \end{bmatrix} \tag{1.12}$$

1.10 Machining System with Various Failure

Machine repair problems encompass a range of failures that can occur in industrial machinery, leading to downtime, production disruptions, and reduced productivity. Common types of failures in machine repair can include:

Common Cause Failure:

In machine repair, a common cause failure (CCF) occurs when multiple components or subsystems within a unit or system fail simultaneously because they are affected by the exact root cause or external factor. Common cause failures can manifest in various ways, such as environmental factors (e.g., extreme temperatures, humidity, vibration, or chemical exposure, etc.), design flaws, manufacturing defects (e.g., material impurities, incorrect assembly, or inadequate quality control, etc.), maintenance errors (e.g., incorrect procedures, insufficient training, or inadequate inspection), and external events like power surges, natural disasters, accidents, or sabotage. Improving system reliability, reducing downtime, and enhancing overall performance requires addressing common cause failures in machine repair problems.

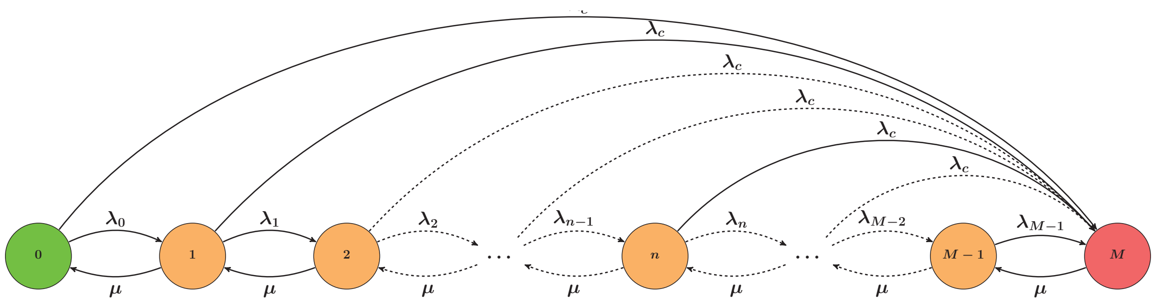


Figure 1.10: MRP with common-cause failure

Degraded Failure:

When all the standby units fail, the load is shared with existing units, then the active unit's time-to-failure is exponentially distributed at an increasing rate.

Reboot Delay:

The term "reboot delay" refers to the time a system, device, or machine takes to shut down, undergo any necessary maintenance or repair tasks, and then restart or reboot before it can resume regular operation. This includes the entire process, from initiating the shutdown procedure to completing the reboot sequence.

Switching Failure:

In the machining system, spare units are crucial for ensuring uninterrupted operation. In advanced control systems, on the failure of an active unit, the redundancy automatically takes its place in negligible switchover time. Its working/failure characteristics are also changed to active on successful switchover. The supplantation of the standby unit instead of the failed active unit may not be successful due to some random switchover hitch with probability q, known as switching failure of the spare unit. If a standby unit miscarries to switch to a failed active unit, the next existing standby unit randomly attempts to switch geometrically. This switching process continues until switching is successful or available standby units have been exhausted.

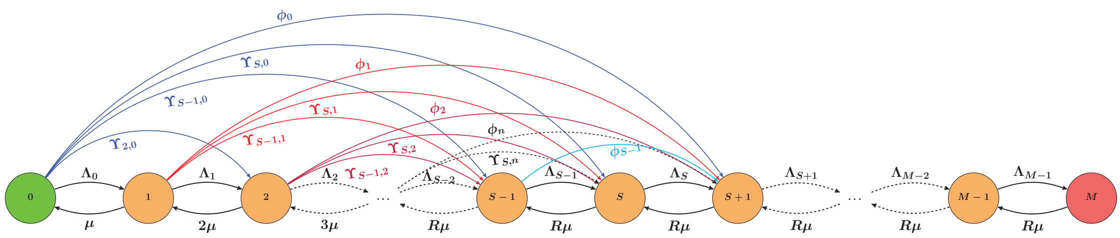


Figure 1.11: MRP with Switching failure

Deterioration:

In machine repair problems, the term "deterioration" refers to the gradual decline in the condition or performance of a machine over time. This decline can be caused by wear and tear, aging, environmental conditions, usage patterns, and inadequate maintenance.

Imperfect Coverage:

On failure of active/standby units or their components, they may not be successfully fully addressed, detected, located, and covered. Then, it is referred to as imperfect coverage.

1.11 Problem-solving Methods

The machining system's queueing models can be solved using analytical and numerical techniques to obtain stationary or non-stationary queue-size distribution. The outcomes are significant in evaluating the sensitivity and optimum performance of the machining system. This section outlines techniques used to solve queueing models investigated in this thesis. A finite state continuous time Markov with state space t is denoted by $\{X(t), t \geq 0\}$. The transition rate to state n from state m is represented by r_{mn}, where the block diagonal elements $r_{mn} = -r_{mn} = \sum_{m=n} r_{mn}; m \neq n$, and $\mathbf{Q} = r_{mn}$ is a generator matrix. Let us assume that $\mathbf{Q}$ has k number of nonzero entries and $P_m(t)$ is the unconditional probability of the continuous-time Markov chain at time t in state m. $\mathbf{P}(t)$ is associated with a probability row vector in the context of Markov chains. The Chapman-Kolmogorov difference equation $\mathbf{P}(t)$ describes the behavior of a continuous-time Markov chain is

$$\frac{d\mathbf{P}(t)}{dt} = \mathbf{Q}(t)\mathbf{P}(t); \quad \mathbf{P}(0) = \mathbf{P}_0 \tag{1.13}$$

where, at initial state $t = 0$, the probability vector of the CTMC is represented by $\mathbf{P}_0$, and the transient-state probability vector is denoted by $\mathbf{P}(t)$. For the steady-state solution, i.e., as $t \to \infty$, the above Eqn. 1.13 is deduced to the system of linear equation

$$\mathbf{QP} = 0 \tag{1.14}$$

The steady-state solution also satisfies the normalizing condition of the probability

$$e^T\mathbf{P} = 1 \tag{1.15}$$

Transient Solution Method:

The Transient Solution Method is a powerful tool for analyzing the machining systems and predicting their behavior over time. Eqn. 1.13 can be solved as

$$\frac{d\mathbf{P}(t)}{\mathbf{P}(t)} = \mathbf{Q}(t)dt$$

By integrating the above equation, we have

$$\mathbf{P}(t) = \mathbf{P}(0)e^{\int \mathbf{Q}(t)dt} = \mathbf{P}(0)e^{\int \mathbf{P}t} \tag{1.16}$$

where, the exponential matrix $e^{\mathbf{P}t}$ is defined by the Taylor series

$$e^{\mathbf{P}t} = \sum_{i=0}^{\infty} \frac{(\mathbf{P}t)^i}{i!}$$

This method is significantly advantageous over the implicit ODE method.

Laplace transform

The Laplace transformation is the mathematical operation used to transform the given derivative function with real variable t to convert it into a complex function with variable u. The Laplace transform of a function $f(t)$ is denoted by $L\{f(t)\}$ or $F(u)$ and defined as below

$$L\{f(t)\} = F(u) = \int_0^{\infty} e^{-ut} f(t)dt; \quad \mathbb{R}(u) \geq 0 \tag{1.17}$$

Applying the Laplace transform is essential for resolving complex systems of differential equations. It converts the system of differential equations with initial conditions into the system of linear equations.

Quasi-Newton Method:

The quasi-Newton method helps find zeros or local maxima/minima of functions, and it is also an alternative to Newton's method. The Quasi-Newton method is employed to determine the global values of continuous decision variables $\{x_1, x_2\}$ by minimizing the objective function. Consider $F(x_1, x_2)$, a non-linear convex function that is also continuously differentiable to the second order. This method operates iteratively and terminates based on specific stopping criteria a tolerance limit determines. One of its notable advantages lies in its rapid convergence and affine invariance. The primary theoretical iterative step is defined as follows:

$$x^{i+1} = x^i - t\nabla^2 f(x)^{-1} \nabla f(x)$$

The steps for implementing the Quasi-Newton method to attain the minimum value of $F^*(x_1^*, x_2^*)$ and the corresponding decision variables x_1^* and x_2^* are as follows:

(i) Initialize the decision variable with an initial value $\vec{\Omega}_0 = [x_{1_0}, x_{2_0}]^T$, where $i = 0$, and set the tolerance $\varepsilon = 10^{-8}$.

(ii) Set the initial trial solution for $\vec{\Omega}_0$ and compute $F(\vec{\Omega}_0)$.

(iii) Compute the objective gradient $\nabla F(\vec{\Omega}_i) = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right]_{\Omega=\Omega_i}$ and the objective Hessian matrix

$$H(\vec{\Omega}_i) = \begin{bmatrix} \frac{\partial^2 F}{\partial^2 x_1} & \frac{\partial^2 F}{\partial x_1 x_2} \\ \frac{\partial^2 F}{\partial x_2 x_1} & \frac{\partial^2 F}{\partial^2 x_2} \end{bmatrix}_{\vec{\Omega}=\vec{\Omega}_i}$$

(iv) Find the new trial solution

$$\vec{\Omega_{i+1}} = \vec{\Omega}_i - \left[H(\vec{\Omega}_i)\right]^{-1} \nabla F(\vec{\Omega}_i).$$

(v) Set $i = i+1$ and repeat steps (iii) and (iv) until max

$$\left(\left|\frac{\partial F}{\partial x_1}\right|, \left|\frac{\partial F}{\partial x_2}\right|\right) < \varepsilon$$

(vi) Find the global minimum value

$$F^*(x_1^*, x_2^*) = F^*(\vec{\Omega}_i{}^*)$$

Eigenvalue and Eigenvector:

Let $\mathbf{Q}$ be any square matrix and λ be any scalar. Then the scalar value λ is known as eigenvalue of matrix $\mathbf{Q}$ if $\exists$ a column vector $\mathbf{P}$; $\mathbf{P} \neq 0$ such that

$$\mathbf{PQ} = \lambda \mathbf{P} \tag{1.18}$$

Any vector which satisfies the above condition Eqn. 1.18 is known as an eigen vector of $\mathbf{Q}$ corresponding the eigen value of λ.

Matrix Analytical Method:

The matrix analytical method [215] & [216] is a mathematical approach used in queueing theory to analyze the performance of queueing systems. The method of matrix analytics is a technique that can be used to calculate the probability distribution of a Markov chain once it has reached a stationary state. This method is applicable when the chain has a repeating structure after a certain point and an unbounded state space in one dimension or less. These models are commonly labeled as $M/G/1$ type Markov chains as they can detect transitions within an $M/G/1$ queueing model.

This method is a classic solution technique for $M/G/1$ chains and is a more complex version of the matrix geometric method.

One possible form of a stochastic matrix for an $M/G/1$ type is as follows.

$$Q = \begin{bmatrix} \mathbf{Y}_0 & \mathbf{Y}_1 & \mathbf{Y}_2 & \mathbf{Y}_3 & \dots \\ \mathbf{X}_0 & \mathbf{X}_1 & \mathbf{X}_2 & \mathbf{X}_3 & \dots \\ \mathbf{0} & \mathbf{X}_0 & \mathbf{X}_1 & \mathbf{X}_2 & \dots \\ \mathbf{0} & \mathbf{0} & \mathbf{X}_0 & \mathbf{X}_1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Where the matrix $\mathbf{X}_i$ and $\mathbf{Y}_i$ are the square matrix of order K, Now if the block tridiagonal matrix $\mathbf{Q}$ is irreducible and positive recursive then, the stationary queue-size distribution is specified by the solution to the equations

$$\mathbf{QP} = \mathbf{P} \quad \text{and} \quad \mathbf{e}^T\mathbf{P} = 1 \tag{1.19}$$

Where $\mathbf{e}$ is a unit vector of suitable dimension. The probability vector $\mathbf{P}$ is partitioned to $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_4, \dots,$. To compute these probabilities, the column stochastic matrix $\mathbf{H}$ is calculated as

$$\mathbf{H} = \sum_{i=0}^{\infty} \mathbf{H}^i \mathbf{X}_i$$

here matrix $\mathbf{H}$ is known as auxiliary matrix and can be defined as below

$$\bar{\mathbf{X}}_{i+1} = \sum_{j=i+1}^{\infty} \mathbf{H}^{j-i-1}\mathbf{X}_j \tag{1.20}$$

$$\bar{\mathbf{Y}}_i = \sum_{j=i}^{\infty} \mathbf{H}^{j-i}\mathbf{Y}_j \tag{1.21}$$

then the initial probability vector $\mathbf{P}_0$ can be found by solving

$$\bar{\mathbf{Y}}_0\mathbf{P}_0 = \mathbf{P}_0$$

$$\left(\mathbf{e}^T + \mathbf{e}^T\left(\mathbf{I} - \sum_{i=1}^{\infty}\bar{\mathbf{X}}_i\right)^{-1}\sum_{i=1}^{\infty}\bar{\mathbf{Y}}_i\right)\mathbf{P}_0 = 1$$

and hence,

$$\mathbf{P}_i = \left(\mathbf{I} - \bar{\mathbf{X}}_i\right)^{-1}\left[\bar{\mathbf{Y}}_{i+1}\mathbf{P}_0 + \sum_{j=1}^{i-1}\bar{\mathbf{X}}_{i-j+1}\mathbf{P}_j\right], \quad i \geq 1$$

Teaching-Learning Based Optimization

The concept of the global optimization method known as the Teaching-Learning based Optimization (TLBO) algorithm was initially proposed by Rao et al. ([236], [237]) to optimize highly nonlinear functions. This algorithm follows a population-based approach, where the influence of a teacher or class on learners' performance in a classroom is utilized. The TLBO algorithm employs a population of solutions to search for the global optimum. In this context, a group of students in a classroom represents the TLBO population. Similar to other population-based optimization methods, TLBO involves several design variables associated with different subjects taught to the learners, and the learners' fitness is determined based on their performance. The best solution obtained through TLBO corresponds to the teacher, who is considered an intellectual member of society. The algorithm operates in two phases: the Teacher Phase (T-Phase) and the Learner Phase (L-Phase). During the T-Phase, learners acquire knowledge from the teacher, while the L-Phase focuses on learning through interactions among the learners. The T phase and L phase of Teaching-Learning Based Optimization (TLBO) algorithm are responsible for conducting exploration and exploitation, respectively, in the context of meta-heuristic optimization. For the detailed study of the TLBO algorithm, refer the Chapter 4, Section 4.5.

Particle Swarm Optimization

The Particle Swarm Optimization (PSO) algorithm is a stochastic global optimization technique inspired by swarm intelligence observed in social behaviors, such as birds flocking and fish schooling. This algorithm was originally introduced by Kennedy and Eberhart in 1995 [142]. PSO operates on the principle of exploring and refining a population of entities, referred to as a "swarm," comprising individuals known as "particles." These particles traverse the solution space with fixed velocities, evolving across generations to converge towards optimal global positions. In problem-solving scenarios, the solution space is represented as a search space, where each point signifies a potential solution to the problem. The fitness value of each particle within the swarm is determined by the optimization objective, which involves both maximization and minimization tasks. Particles possess awareness of the coordinates of decision variables and maintain connections to the best solutions or fitness values they have achieved. For the detailed study of the PSO algorithm, refer the Chapter 8, Section 8.5.3.

1.12 Performance Metrics of the Machining System

Analyzing the performance characteristics of machine repair problems is crucial for organizations to monitor, evaluate, and improve their performance efficiently. This ultimately leads to increased efficiency, effectiveness, and competitiveness in today's constantly evolving business environment. The performance metrics of machine repair problems are broadly categorized into two types: (i) Queueing characteristics and (ii) Reliability characteristics. The performance of machine repair problems is evaluated based on various factors such as repair time, downtime, repair costs, resource utilization, and quality of repair. Additionally, rates such as repair, failure, and downtime rate and states such as operating and ideal are also considered. The machining system's thresholds are also considered while deriving the performance metrics.

Let us assume a Machining system incorporating M active units and S spare units have state-dependent failure and repair rates are λ_n and μ_n, respectively. The state transition diagram of the MRP is depicted in Fig. (1.12)

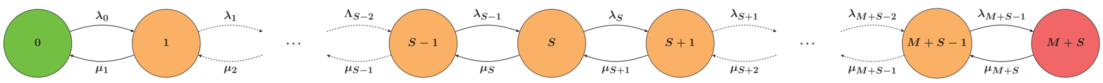


Figure 1.12: State transition diagram of Machine repair problem

1.12.1 Queueing Metrics:

Optimizing machine repair systems requires understanding and managing queueing characteristics. This includes allocating resources effectively, meeting service level agreements, enhancing customer satisfaction, managing costs, and monitoring performance.

Expected number of failed units:

A system's expected number of failed units represents the average number of units in a failed state at a given time. It is a statistical measure that considers the probability distribution of the system's failed units. Expected number of failed units $(E_N(t))$ in the system at a time t

$$E_N(t) = \sum_{i=0}^{M+S} iP_i(t) \tag{1.22}$$

Throughput of the System:

The throughput of a system $(TP(t))$ refers to the rate at which the system processes failed units. This is typically measured as the number of failed units processed per unit of time. Below is the expression for the system's throughput at a time t

$$TP(t) = \sum_{i=0}^{M+S} \mu_i P_i(t) \tag{1.23}$$

Expected number of Operating units:

A system's mean number of active (operational) units is the average number of operational units present over a specified period. The mean number of active units in the system at time t

$$E_O(t) = \sum_{i=0}^{M} (M-i) P_i(t) \tag{1.24}$$

Expected number of Standby units:

The expected number of standby units $E_S(t)$ in a system represents the average number of standby units that are present in the system over a specific period of time. The expected number of standby units in the system at a time t is given as

$$E_S(t) = \sum_{i=0}^{S} (S-i) P_i(t) \tag{1.25}$$

Machine availability:

Machine availability of the system at time t is defined as the ratio of the expected number of functioning units in the system at time t out of the total number of units available in the system initially

$$MA(t) = 1 - \frac{E_N(t)}{M+S} \tag{1.26}$$

Expected delay time:

The delay time of the failed unit $(E_D(t))$ represents the waiting time experienced by a failed unit before being attended to by the repair process. The delay time is a crucial metric in assessing the system's performance and reliability, providing insights into how quickly the system can address failures. The delay time of a failed unit at a time

t

$$E_D(t) = \frac{E_N(t)}{TP(t)} \tag{1.27}$$

Effective failure rate:

The effective failure rate in the context of a system refers to a composite or overall failure rate that considers various factors, such as the failure rates of individual components, redundancy, repair processes, and other relevant aspects. It provides a more comprehensive measure of the system's reliability by considering the occurrence of failures and the effectiveness of repair processes. The effective failure rate at time t

$$\lambda_{eff} = \sum_{i=0}^{M+S-1} \lambda_i P_i(t) \tag{1.28}$$

Expected waiting time:

The expected waiting time of failed units $E_W(t)$ in a system refers to the average time a failed unit spends in a waiting state before it undergoes repair and receives perfect service. This metric is essential for evaluating the efficiency and performance of the repair process within the system. Expected waiting time of the failed units in the system at time t

$$E_W(t) = \frac{E_N(t)}{\lambda_{eff}} \tag{1.29}$$

Probability of the server being Idle:

Probability when there is no failed unit in the system at time t and is expressed as

$$P_I(t) = P(E_N(t) = 0) = P_0(t) \tag{1.30}$$

Probability of the server being Busy:

Probability when there is at least one failed unit in the system at time t and is expressed as

$$P_B(t) = 1 - P_I(t) \tag{1.31}$$

1.12.2 Reliability Metrics:

We are analyzing the reliability of a machining system using a strategy that controls failed unit arrival based on a threshold and includes imperfect repair. The RAMS

(Reliability, Availability, Maintainability, and Safety) analysis is a fundamental framework for the efficient and timely utilization of machining systems. It is an interdisciplinary methodology integrating design elements to achieve a machining system's operational goals. In machining systems engineering, RAMS ensures that the inherent design characteristics align with the required standards for optimal performance. Reliability, the foremost factor in RAMS analysis, highlights the system's ability to operate consistently without failure. Let us assume a Machining system incorporating M active units and S spare units (Fig. 1.13) have state-dependent failure and repair rates are λ_n and μ_n, respectively.

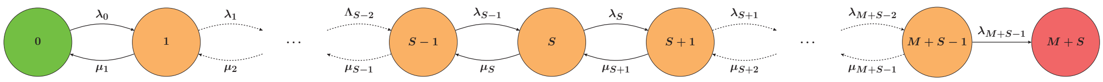


Figure 1.13: State transition diagram for system failure

Reliability:

Let X denote the continuous random variable representing the time to failure of the system characterized by probability density function $f(x)$ and cumulative distribution function $F(x)$. The reliability of a system or unit is determined by the probability of its intended function being performed without failure over a specific period of time $(0,t)$, assuming certain stated operating conditions. It denotes the probability of a nonfailure over time. Mathematically,

$$R_X(t) = Pr\{X \geq t\} = \int_t^{\infty} f(x)dx = 1 - F(t) \tag{1.32}$$

at the initial state $t = 0$, $R_X(0) = 1$ and at the steady-state, i.e., $\lim_{t\to\infty} R_X(t) = 0$ The reliability $R_X(t)$ is a non-decreasing of time t satisfies

$$f(t) = -\frac{dR_X(t)}{dt}$$

The system's unreliability is denoted by $F(t)$.

Failure frequency:

The failure frequency refers to the rate at which machines or equipment experience malfunctions or breakdowns. It measures how often a machine fails and requires maintenance or repair. The failure frequency of the system at time t is denoted by $FF(t)$ and defined by

$$FF(t) = \lambda_{M+S-1} P_{M+S-1} \tag{1.33}$$

Mean time-to-failure (MTTF):

In the field of machine repairs, Mean Time to Failure (MTTF) is the amount of time, on average, a unit or equipment operates before experiencing a failure or breakdown. i.e., Mathematically, MTTF can be derived as

$$MTTF = \int_0^\infty xf(x)dx = \int_0^\infty R_X(t)dt \tag{1.34}$$

System availability:

System availability is a metric that quantifies the time during a specified period that a system remains functional and capable of carrying out its designated tasks. At any time t, it can be expressed as

$$Av = 1 - P_{M+S}(t) \tag{1.35}$$

1.13 Machine repair system with different kinds

We need to examine alternative policies to achieve high reliability thoroughly, better performance of the machining system, less total cost, and different unit arrangements. To accomplish this, we have compiled a summary of joint arrangements.

1.13.1 Series Systems:

In Machine repair problems, a configuration where multiple units are arranged sequentially, and each unit must operate successfully for the system to function correctly, is known as a series systems. The functionality of the series system depends on the successful operation of each unit; if any unit experiences a failure or requires repair, it can disrupt the entire production process.

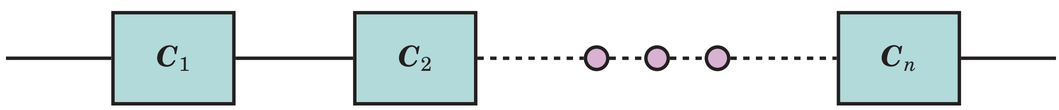


Figure 1.14: Multi-unit series system

Let $R_i(t)$ be the reliability of the i^{th} unit at any time t, where $i = 1, 2, \cdots, n-1, n$, and X_i & Y_S denotes the time-to-failure of the i^{th} unit and the time-to failure of the system having n units, respectively. Therefor Y_S can be defined as

$$Y_S = \min\{X_1, X_2, \cdots, X_{n-1}, X_n\}$$

hence, the reliability of the series system at any time t can be represented as

$$R_S(t) = R_1(t)R_2(t)\cdots R_{n-1}(t)R_n(t)$$

1.13.2 Parallel Systems:

In machine repair, a parallel system refers to a configuration of multiple identical units (say n) or subsystems that operate independently of each other. The system can operate as long as at least one of the n units is functional.

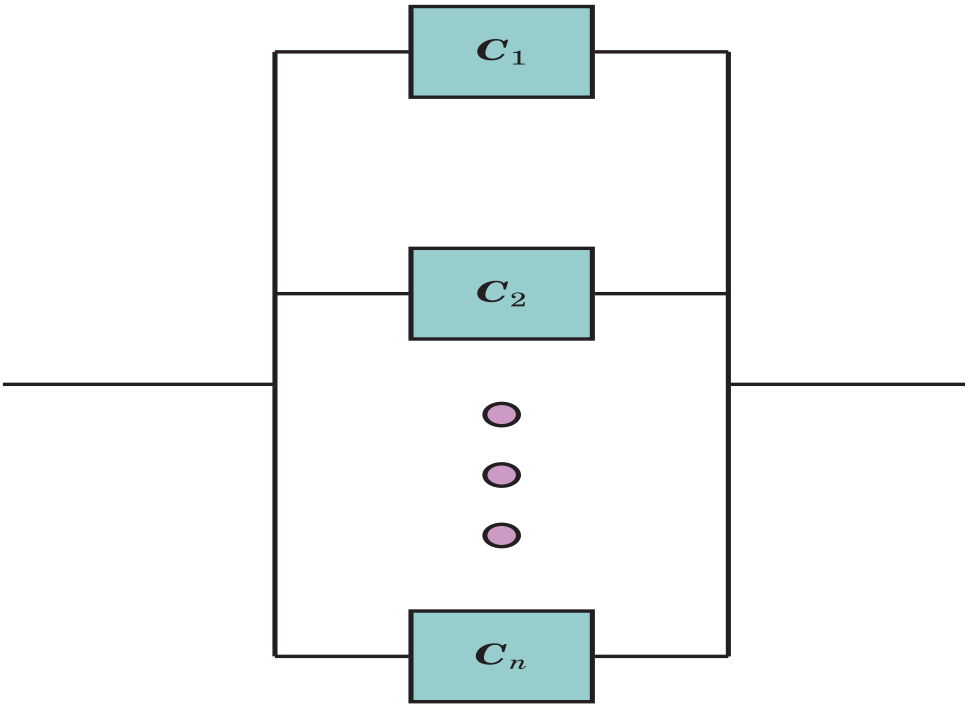


Figure 1.15: Multi-unit parallel system

Let $R_i(t)$ be the reliability of the i^{th} unit at any time t, where $i = 1, 2, \cdots, n-1, n$, and X_i & Y_S denotes the time-to-failure of the i^{th} unit and the time-to failure of the system having n units, respectively. Therefor Y_S can be defined as

$$Y_S = \max\{X_1, X_2, \cdots, X_{n-1}, X_n\}$$

hence, the reliability of the parallel system at any time t can be represented as

$$R_S(t) = 1 - (1 - R_1(t))(1 - R_2(t))\cdots(1 - R_{n-1}(t))(1 - R_n(t))$$

1.13.3 Standby Redundant Systems:

In a Machine repair problem, a configuration where backup or standby units or components are kept in reserve to replace a failed operating unit. i.e., On failure of an active unit, a standby unit is switched online, and the failed unit is taken off-line.

1.13.4 K-out-of-M:G Systems:

When analyzing systems with multiple redundant components, a reliability model ensures successful operation if at least K out of M units are operational. If the number of failed units exceeds $M - K$, the system will fail.

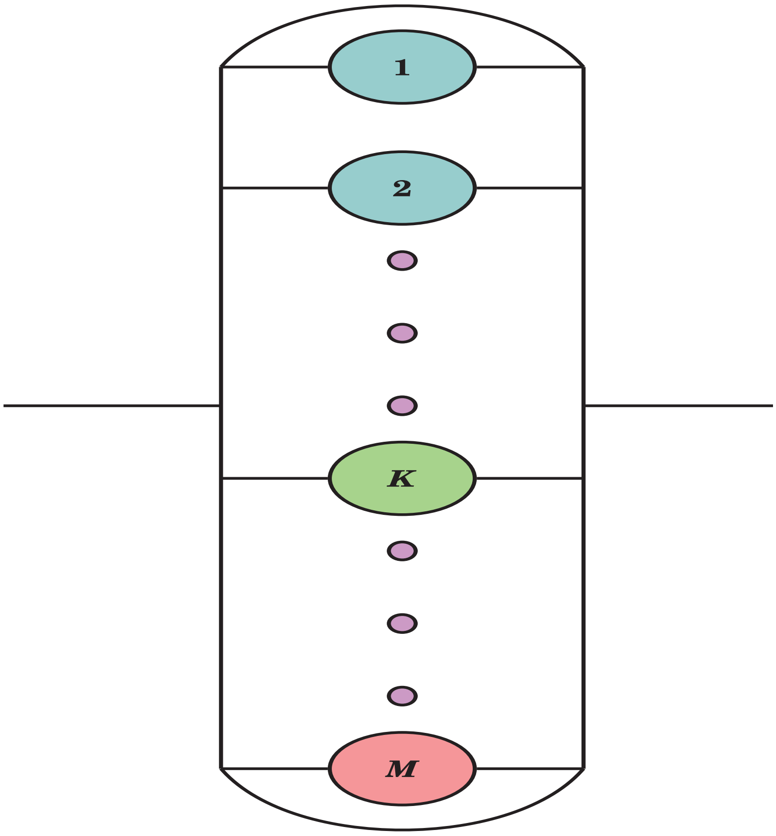


Figure 1.16: K-out-of-M : G system

1.13.5 Maintained Systems:

Maintainable systems are designed and engineered to allow maintenance actions to be carried out during specific intervals of time without causing extensive disruption to operations.

1.13.6 (K,m) Systems:

In machine repair, a configuration with m independent units, and for the system's successful operation, at least K out of those m components must be operational, is known as a (K,m) system.

1.13.7 Series-parallel Systems:

In reliability engineering, a Series-parallel system of order (m,n) is a configuration that consists of m identical parallel systems, each of order n, arranged in series. This configuration is frequently used to model complex systems requiring high reliability and redundancy.

1.13.8 Parallel-series Systems:

In reliability engineering, a Parallel-series system of order (m,n) is a configuration that consists of m identical series systems, each of order n, arranged in parallel. This configuration is frequently used to model complex systems requiring high reliability and redundancy.

1.13.9 Complex Systems:

Any combination of above defined systems or different from above categories having complex structures are called complex systems.

1.13.10 Controlled Policies

N-Policy:

In Machine repair problems, the N policy is a type of repair-controlled policy used to repair a system. Under this policy, the repairer initiates the repair process when N failed units accumulate in the system. The process continues until all the failed units get repaired. Once the number of failed units in the system reaches or exceeds N, repair is initiated.

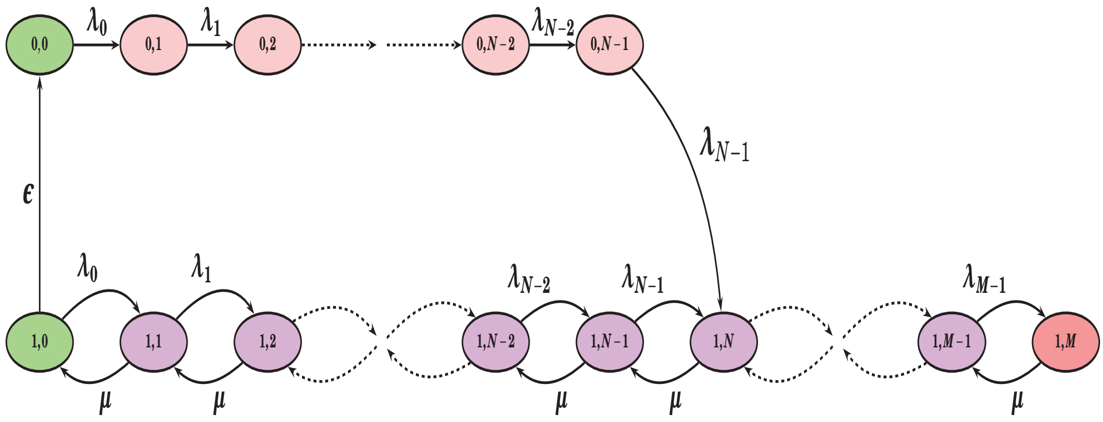


Figure 1.17: MRP with N-policy

F-Policy (Fixed-size Policy):

For admission control to avoid long queues in waiting, if the queue size of failed units reaches a threshold K, then no prospective failed unit is permitted to join the repair queue until the queue size of failed units reduces a predefined threshold value $F(1 \le F \le K-1)$.

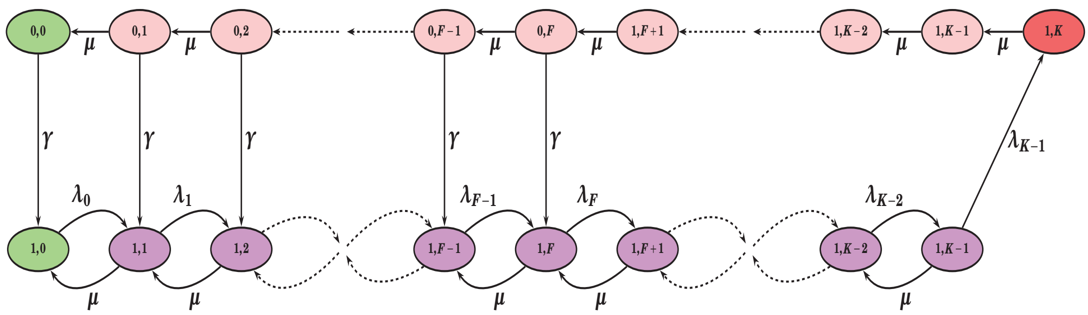


Figure 1.18: MRP with F-policy

T-Policy (Time-limited Policy):

The T-policy is a control policy that determines the maximum waiting time for entities before they are served. This policy is beneficial when excessive waiting times must be avoided, and timely service is crucial. From the repairer's perspective, the T-policy means that they take a break from the system for a specified period after completing each busy period. Maintenance request handling in a manufacturing plant is the real-world scenario example of T-policy.

D-Policy (Deadline-based Policy):

D-policy is a control policy that prioritizes repair tasks based on their deadlines or due dates. Repair tasks with earlier deadlines are assigned a higher priority for repair to ensure timely completion and minimize lateness or delays. Handling customer support tickets in a tech company and automotive manufacturing plant exemplifies the importance of a *D* policy.

S-Policy (Service-time Policy):

It is a control policy that prioritizes repair tasks based on expected repair times. The *S*-policy prefers shorter repair tasks than those with longer ones. This policy aims to minimize waiting time for unit repair, improve system throughput, and resolve maintenance issues quickly. In fast-food restaurants, IT help-desk support, vehicle servicing, manufacturing maintenance, and medical triage are the best examples of this policy.

1.14 Fuzzy Redundant Machine Repair Problem

If U is a collection of objects denoted normally by u, then a fuzzy set A in U is a set of ordered pairs

$$\tilde{A} = \{(u, \mu_{\tilde{A}}(u)) | \; u \in U\}$$

where $\mu_{\tilde{A}}(u)$ is known as the membership function (degree of truth) of u in $\tilde{A}$. that maps U to the membership space M.

$$\mu_{\tilde{A}}(u) \quad : \quad U \quad \longrightarrow \quad [0,1]$$

1.14.1 Normal Fuzzy Set:

A fuzzy set $\tilde{A}$ defined on a universe of discourse U is said to be normal fuzzy set if $\exists$ $u_i \in U, \quad \mu_{\tilde{A}}(u_i) = 1$.

1.14.2 Support of a Fuzzy Set:

The support of a fuzzy set $\tilde{A}$ defined on a universe of discourse U is denoted by $Supp(\tilde{A})$ and defined as below

$$Supp(\tilde{A}) = \{u \in U | \mu_{\tilde{A}} > 0\}$$

1.14.3 α-level Set:

The α-level set of a fuzzy set $\tilde{A}$ defined on a universe of discourse U is denoted by $\tilde{A}_\alpha$ and defined as below

$$\tilde{A}_\alpha = \{u \in U | \mu_{\tilde{A}}(u) \geq \alpha\}$$

1.14.4 Convex of a Fuzzy Set:

A fuzzy set $\tilde{A}$ on a universe of discourse U is convex if

$$\mu_{\tilde{A}}(\lambda u_1 + (1-\lambda)u_2) \geq min\{\mu_{\tilde{A}}(u_1), \mu_{\tilde{A}}(u_2)\}, \quad u_1, u_2 \in U, \lambda \in [0,1]$$

1.14.5 Scalar Cardinality of a Fuzzy Set:

The Cardinality of a fuzzy set $\tilde{A}$ defined on a universe of discourse U is denoted by $|\tilde{A}|$ and defined as below

$$|\tilde{A}| = \sum_{u \in U} \mu_{\tilde{A}}(u)$$

the relative cardinality of $\tilde{A}$ is define as

$$\|\tilde{A}\| = \frac{|\tilde{A}|}{u}$$

1.14.6 Union of two Fuzzy Set:

Let $\tilde{A}_1$ and $\tilde{A}_2$ are two fuzzy set and $\mu_{\tilde{A}_1}(u)$ & $\mu_{\tilde{A}_2}(u)$ are the membership grade function of $\tilde{A}_1$ and $\tilde{A}_2$ respectively, then the membership function $\mu_{\tilde{A}_1 \cup \tilde{A}_2}$ of the union $\tilde{A}_1 \cup \tilde{A}_2$ is pointwise defined by

$$\mu_{\tilde{A}_1 \cup \tilde{A}_2}(u) = max\{\mu_{\tilde{A}_1}(u), \mu_{\tilde{A}_2}(u)\}, \quad \forall\, u \in U$$

1.14.7 Intersection of two Fuzzy Set:

Let $\tilde{A}_1$ and $\tilde{A}_2$ are two fuzzy set and $\mu_{\tilde{A}_1}(u)$ & $\mu_{\tilde{A}_2}(u)$ are the membership grade function of $\tilde{A}_1$ and $\tilde{A}_2$ respectively, then the membership function $\mu_{\tilde{A}_1 \cap \tilde{A}_2}$ of the intersection $\tilde{A}_1 \cap \tilde{A}_2$ is pointwise defined by

$$\mu_{\tilde{A}_1 \cap \tilde{A}_2}(u) = min\{\mu_{\tilde{A}_1}(u), \mu_{\tilde{A}_2}(u)\}, \quad \forall\, u \in U$$

1.14.8 Complement of a normalized Fuzzy Set:

Let $\tilde{A}$ be the normalized fuzzy set and its membership function is denoted by $\mu_{\complement\tilde{A}}(u)$ and defined by

$$\mu_{\complement\tilde{A}}(u) = 1 - \mu_{\tilde{A}}(u); \quad \forall\, u \in U$$

1.14.9 Algebraic Sum of two Fuzzy Set:

Let $\tilde{A}_1$ and $\tilde{A}_2$ are two fuzzy set, then its algebraic sum is denoted by $\tilde{A}_3 = \tilde{A}_1 + \tilde{A}_2$ is defined as

$$\tilde{A}_3 = \{(u, \mu_{\tilde{A}_1+\tilde{A}_2}(u))|u \in U\}$$

where $\quad \mu_{\tilde{A}_1+\tilde{A}_2}(u) = \mu_{\tilde{A}_1}(u) + \mu_{\tilde{A}_2}(u) - \mu_{\tilde{A}_1}(u)\mu_{\tilde{A}_2}(u)$

1.14.10 Bounded Sum of two Fuzzy Set:

Let $\tilde{A}_1$ and $\tilde{A}_2$ are two fuzzy set, then its bounded sum is denoted by $\tilde{A}_3 = \tilde{A}_1 \oplus \tilde{A}_2$ is defined as

$$\tilde{A}_3 = \{(u, \mu_{\tilde{A}_1\oplus\tilde{A}_2}(u))|u \in U\}$$

where $\quad \mu_{\tilde{A}_1\oplus\tilde{A}_2}(u) = min\{1,\ \mu_{\tilde{A}_1}(u) + \mu_{\tilde{A}_2}(u)\}$

1.14.11 Bounded Difference of two Fuzzy Set:

Let $\tilde{A}_1$ and $\tilde{A}_2$ are two fuzzy set, then its bounded difference is denoted by $\tilde{A}_3 = \tilde{A}_1 \ominus \tilde{A}_2$ is defined as

$$\tilde{A}_3 = \{(u, \mu_{\tilde{A}_1\ominus\tilde{A}_2}(u))|u \in U\}$$

where $\quad \mu_{\tilde{A}_1\ominus\tilde{A}_2}(u) = max\{0,\ \mu_{\tilde{A}_1}(u) + \mu_{\tilde{A}_2}(u) - 1\}$

1.14.12 Algebraic Product of two Fuzzy Set:

Let $\tilde{A}_1$ and $\tilde{A}_2$ are two fuzzy set, then its algebraic product is denoted by $\tilde{A}_3 = \tilde{A}_1.\tilde{A}_2$ is defined as

$$\tilde{A}_3 = \{(u, \mu_{\tilde{A}_1}(u).\mu_{\tilde{A}_2}(u))|u \in U\}$$

A fuzzy number, denoted as $\tilde{A}$, established on the universe of discourse U, can be identified by a trapezoidal distribution function described by the parameters (a_1, a_2, a_3, a_4).

$$\mu_{\tilde{A}}(u) = \begin{cases} 0, & u < a_1 \\ \frac{u-a_1}{a_2-a_1}; & a_1 \leq u \leq a_2 \\ 1, & a_2 \leq u \leq a_3 \\ \frac{a_4-u}{a_4-a_3}; & a_3 \leq u \leq a_4 \\ 0, & a_4 \leq u \end{cases} \tag{1.36}$$

Let $\tilde{A}$ and $\tilde{B}$ represent two trapezoidal fuzzy numbers, characterized by the quadruplet (a_1, a_2, a_3, a_4) and (b_1, b_2, b_3, b_4) respectively, where $a_1 \leq b_1, a_2 \leq b_2, a_3 \leq b_3$ and $a_4 \leq b_4$. The $\alpha-$cut of corresponding trapezoidal fuzzy numbers is given by

$$\begin{aligned} \tilde{A_\alpha} &= [a_1 + (a_2 - a_1)\alpha, a_4 - (a_4 - a_3)\alpha] \\ \tilde{B_\alpha} &= [b_1 + (b_2 - b_1)\alpha, b_4 - (b_4 - b_3)\alpha] \end{aligned} \tag{1.37}$$

where, $\alpha \in [0,1]$ For example, let $\tilde{A} = [1,5,6,9]$ and $\tilde{B} = [2,6,7,10]$ are two trapezoidal fuzzy numbers. The $\alpha-$cut of $\tilde{A}$ can be given as $\tilde{A_\alpha}$=$[1+4\alpha, 9-3\alpha]$, $\tilde{B_\alpha}$=$[2+4\alpha, 10-3\alpha], \alpha \in [0,1]$. Hence we have

$$\tilde{A_\alpha} \oplus \tilde{B_\alpha} = [1+4\alpha, 9-3\alpha] \oplus [2+4\alpha, 10-3\alpha] = [3+8\alpha, 19-6\alpha]$$

$$\tilde{A_\alpha} \ominus \tilde{B_\alpha} = [1+4\alpha, 9-3\alpha] \ominus [2+4\alpha, 10-3\alpha] = [-9+7\alpha, 7-7\alpha]$$

$$\begin{aligned} \tilde{A_\alpha} \otimes \tilde{B_\alpha} &= [1+4\alpha, 9-3\alpha] \otimes [2+4\alpha, 10-3\alpha] \\ &= [\min\{(1+4\alpha)(2+4\alpha), (1+4\alpha)(10-3\alpha), (9-3\alpha)(2+4\alpha), (9-3\alpha)(10-3\alpha)\}, \\ &\quad \max\{(1+4\alpha)(2+4\alpha), (1+4\alpha)(10-3\alpha), (9-3\alpha)(2+4\alpha), (9-3\alpha)(10-3\alpha)\}] \\ &= [1+4\alpha)(2+4\alpha), (9-3\alpha)(10-3\alpha)] = [16\alpha^2 + 12\alpha + 2, 90 - 57\alpha + 9\alpha^2] \end{aligned}$$

$$\begin{aligned} \tilde{A_\alpha} \oslash \tilde{B_\alpha} &= [1+4\alpha, 9-3\alpha] \oslash [2+4\alpha, 10-3\alpha] \\ &= \left[\min\left\{\frac{1+4\alpha}{2+4\alpha}, \frac{9-3\alpha}{2+4\alpha}, \frac{1+4\alpha}{10-3\alpha}, \frac{9-3\alpha}{10-3\alpha}\right\}, \right. \\ &\quad \left. \max\left\{\frac{1+4\alpha}{2+4\alpha}, \frac{9-3\alpha}{2+4\alpha}, \frac{1+4\alpha}{10-3\alpha}, \frac{9-3\alpha}{10-3\alpha}\right\}\right] \\ &= \left[\frac{1+4\alpha}{10-3\alpha}, \frac{9-3\alpha}{2+4\alpha}\right] \end{aligned}$$

hence the bounded sum of $\tilde{A}$ and $\tilde{B}$ is

$$\tilde{A_\alpha} \oplus \tilde{B_\alpha} = [3+8\alpha, 19-6\alpha], \alpha \in [0,1]$$

For $\alpha = 0, \tilde{A_\alpha} \oplus \tilde{B_\alpha}$=$[3,19]$ and for $\alpha = 1, \tilde{A_\alpha} \oplus \tilde{B_\alpha}$=$[11,13]$. So the membership grade function of $\tilde{A_\alpha} \oplus \tilde{B_\alpha}$ is given by

$$\mu_{\tilde{A_\alpha} \oplus \tilde{B_\alpha}}(u) = \begin{cases} 0, & u < 3 \\ \frac{u-3}{8}; & 3 \leq u \leq 11 \\ 1, & 11 \leq u \leq 13 \\ \frac{17-u}{5}, & 13 \leq u \leq 19 \\ 0, & u > 19 \end{cases} = (3, 11, 13, 19)$$

The bounded difference or subtraction of two fuzzy number $\tilde{A_\alpha} \ominus \tilde{B_\alpha}$ is

$$\tilde{A_\alpha} \ominus \tilde{B_\alpha} = [-9+7\alpha, 7-7\alpha]$$

For $\alpha = 0, \tilde{A_\alpha} \ominus \tilde{B_\alpha} = [-9,7]$ and for $\alpha = 1, \tilde{A_\alpha} \ominus \tilde{B_\alpha} = [-2,0]$. So the membership grade function of $\tilde{A_\alpha} \ominus \tilde{B_\alpha}$ is given by

$$\mu_{\tilde{A_\alpha} \ominus \tilde{B_\alpha}}(u) = \begin{cases} 0, & u < -9 \\ \frac{u+9}{7}; & -9 \leq u \leq -2 \\ 1, & -2 \leq u \leq 0 \\ \frac{7-u}{7}, & 0 \leq u \leq 7 \\ 0, & u > 7 \end{cases} = (-9, -2, 0, 7)$$

The multiplication of two fuzzy number $\tilde{A_\alpha} \otimes \tilde{B_\alpha}$ is

$$\tilde{A_\alpha} \otimes \tilde{B_\alpha} = [16\alpha^2 + 12\alpha + 2, 90 - 57\alpha + 9\alpha^2]$$

For $\alpha = 0, \tilde{A_\alpha} \otimes \tilde{B_\alpha} = [2,90]$ and for $\alpha = 1, \tilde{A_\alpha} \otimes \tilde{B_\alpha} = [30,42]$ so

$$\begin{aligned}
&\Longrightarrow 16\alpha^2 + 12\alpha + 2 = u \\
&\Longrightarrow 16\alpha^2 + 12\alpha + (2-u) = 0 \\
&\Longrightarrow \alpha = \frac{-12 \pm \sqrt{144 - 64(2-u)}}{32} \\
&\Longrightarrow \alpha = \frac{-12 \pm \sqrt{64u+16}}{32} \\
&\Longrightarrow \alpha = \frac{-3 \pm \sqrt{4u+1}}{8}
\end{aligned}$$

and

$$\begin{aligned}
&90 - 57\alpha + 9\alpha^2 = u \\
&\Longrightarrow 9\alpha^2 - 57\alpha + (90-u) = 0 \\
&\Longrightarrow \alpha = \frac{57 \pm \sqrt{(57)^2 - 36(90-u)}}{18}
\end{aligned}$$

$$\Longrightarrow \alpha = \frac{21 \pm \sqrt{4u+1}}{6}$$

so the membership grade function of $\tilde{A_\alpha} \otimes \tilde{B_\alpha}$ is given as

$$\mu_{\tilde{A_\alpha} \otimes \tilde{B_\alpha}}(u) = \begin{cases} 0, & u < 2 \\ \frac{-3+\sqrt{4u+1}}{8}; & 2 \leq u \leq 30 \\ 1, & 30 \leq u \leq 42 \\ \frac{19-\sqrt{4u+1}}{6}, & 42 \leq u \leq 90 \\ 0, & u > 90 \end{cases} = (2,30,42,90)$$

The division of two fuzzy numbers $\tilde{A_\alpha} \oslash \tilde{B_\alpha}$ is

$$\tilde{A_\alpha} \oslash \tilde{B_\alpha} = \left[\frac{1+4\alpha}{10-3\alpha}, \frac{9-3\alpha}{2+4\alpha}\right]$$

For $\alpha = 0, \tilde{A_\alpha} \oslash \tilde{B_\alpha} = \left[\frac{1}{10}, \frac{9}{2}\right]$ and for $\alpha = 1, \tilde{A_\alpha} \oslash \tilde{B_\alpha} = [\frac{5}{7}, \frac{6}{6}]$ *i.e.* $[\frac{1}{10}, \frac{5}{7}, 1, \frac{9}{2}]$

Let

$$\frac{1+4\alpha}{10-3\alpha} = u$$
$$\implies 1+4\alpha = u(10-3\alpha)$$
$$\implies \alpha(4+3u) = 10u-1$$
$$\implies \alpha = \frac{10u-1}{4+3u}$$

and

$$\frac{9-3\alpha}{2+4\alpha} = u$$
$$\implies 9-3\alpha = u(2+4\alpha)$$
$$\implies 9-2u = (3+4u)\alpha$$
$$\implies \alpha = \frac{9-2u}{3+4u}$$

The membership grade function of $\tilde{A_\alpha} \oslash \tilde{B_\alpha}$ is given by

$$\mu_{\tilde{A_\alpha} \oslash \tilde{B_\alpha}}(u) = \begin{cases} 0, & u < \frac{1}{10} \\ \frac{10u-1}{4+3u}; & \frac{1}{10} \le u \le \frac{5}{7} \\ 1, & \frac{5}{7} \le u \le 1 \\ \frac{9-2u}{3+4u}, & 1 \le u \le \frac{9}{2} \\ 0, & u > \frac{9}{2} \end{cases}$$

1.15 Bayesian Analysis

Bayesian Analysis is a fundamental aspect of contemporary statistical inference, notable for its rigorous and systematic method of combining prior knowledge with empirical data. Originating from the pioneering work of Reverend Thomas Bayes, this statistical framework employs Bayes' Theorem to revise the probability of a hypothesis in light of new evidence. This introduction covers the theoretical underpinnings, historical evolution, and various applications of Bayesian Analysis, offering a detailed overview for professionals in the field.

Theoretical Foundations:

At its core, Bayesian Analysis is governed by Bayes' Theorem, a fundamental principle expressed as:

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

where

- $P(A \mid B)$ denotes the posterior probability, representing the updated probability of hypothesis A given the evidence B.
- $P(B \mid A)$ is the likelihood, indicating the probability of observing B under the assumption that A is true.
- $P(A)$ signifies the prior probability, encapsulating prior beliefs about A before observing B.
- $P(B)$ is the marginal likelihood, providing the normalizing constant that ensures the posterior distribution sums to one over all hypotheses.

For the detailed study of the Bayesian Analysis, refer the Chapter 2, Section 2.5.

1.16 Review of Literature

Machining systems are vital to numerous industries, and any issues can lead to notable negative consequences. In light of this, the provisioning of spare units and the implementation of corrective maintenance procedures by repair facilities play crucial roles in restoring the optimal functioning of these systems. Thus, it is imperative to implement efficient maintenance practices that minimize downtime, boost productivity, and protect investments. Proactive measures such as predictive maintenance, data analysis, and AI algorithms can be utilized to anticipate and prevent potential malfunctions. By adopting this approach, companies can optimize their operational performance, foster sustainable growth, and stay ahead of the competition. We wholeheartedly recommend this strategy to ensure uninterrupted workflow and long -term success.

1.16.1 Literature on Queueing Theory and its Historical Significance

The queueing theory field, initiated by the pioneering work of Danish Engineer A.K. Erlang in telecommunications, has gained immense popularity since its inception. Erlang developed mathematical models to analyze congestion and waiting times in telephone networks, introducing queueing theory, which continue to thrive today. As a tribute to him, the system addressed through M/M/1 notation and the measure of communication activity are named after Erlang. The evolution of queueing theory gained significant momentum with the advent of Operations Research in the late 1940s and early 1950s. This interdisciplinary field provided a platform for rigorous technical analysis of queueing systems, leading to a deeper understanding of their behavior and performance characteristics. The literature on queueing theory has grown exponentially over the years, reflecting the growing interest and relevance of the subject across diverse domains.

Queueing models have been applied in various sectors such as transportation, healthcare, manufacturing, and more, transcending traditional telecommunication contexts. Queueing theory has witnessed a remarkable evolution thanks to the dynamic interplay between theoretical frameworks and practical applications. The symbiotic relationship between theory and practice has facilitated continuous innovation, developing sophisticated methodologies, tools, and algorithms for managing queues, enhancing system performance, and optimizing resource utilization. The cross-fertilization of ideas and best practices between academia, industry, and government agencies has enriched and refined theoretical models and practical solutions. Queueing theory has proven to be a powerful tool for effectively addressing contemporary challenges and shaping the future of complex systems engineering and management. Its continued relevance and impact underscore its importance in advancing state-of-the-art queueing analysis and optimization.

Various researchers in the field of queueing theory have done a vast amount of research work. Besides Erlang, Kendall [141] thoroughly examined stochastic processes within queueing theory, establishing the foundation for subsequent advancements in the discipline. In 1975, Kendall [149] compiled his research of queueing theory into a book *Queueing systems: theory*. Marshall [196] queueing models with exponential service times, a fundamental aspect of many single server queueing systems. Conway et al. wrote a book *Theory of Scheduling* that introduced the concept of scheduling theory, which is closely related to queueing theory and deals with the efficient allocation of resources over time. Gross [75] offers a thorough investigation into queueing theory, encompassing a broad spectrum of subjects ranging from fundamental principles to complex mathematical models and practical applications in the book *Fundamentals of QUEUEING THEORY*. The literature concerning queues is extensive. Here, we focus on developing queueing models, which are closely aligned with our investigation into predicting machining system performance. Numerous noteworthy studies have examined performance modeling in machine repair problems (MRP). Our attention now turns to concisely reviewing past literature in this domain.

Naor [212] studied the machine interference model and developed a probability distribution function to describe the distinct states of the system. Mack [190], Koenigsberg [150], Phipps Jr [228], Morse [207], Wilson et al. [330], Bunday and Scraton [23], and Wolff [331], and Palm [222] conducted a comprehensive examination of machine repair problems, proposing multiple metrics of interest from both theoretical and practical perspectives.

1.16.2 Machine Repair Problem with Spare Parts

In machine repair, spare parts improve reliability, availability, and maintainability. Spare parts help to minimize downtime and cost, ensure safety and flexibility, and prevent production delays. Taylor and Jackson [293] first introduced the concept of redundancy (cold standby) in MRP. Srinivasan [282] implemented the r spare parts in two cases: (i) constant working and (ii) intermittently. Some significant work has been done by various researchers in the field of machine repair problems, like Natarajan [213], Hastings [89], Thomas [296], Fortuin [62], and Schouten et al. [302]. Some notable work on MRP with cold standby units is done by Gopalan and Naidu [72], Sivazlian and Wang [272] and Gupta [77]. MRP with warm standby units has been conceptualized by Dhillon and Yang [50], Wang and Kuo [317], and Wang and Wu [323]. Numerous researchers have also introduced machine repair problems with mixed kinds of spare, like Wang [310], Cho and Parlar [40], and Wang [311]. The pioneering efforts in modeling spare units within machining systems are evident in the contributions of the researchers listed in Table 1.1 .

Table 1.1: Table to Contribution of MRP with different cold, warm, hot, and mixed standby units

Author	**Year**	**Keywords**	**Methodology**
Reetu Malhotra and Gulshan Taneja [194]	2014	Two-Unit Cold standby unit, Availability, Mean Time to System Failure	Semi-Markov Processes, Regenerative Point Techniques
R. Jamshidi and Mir Mehdi Seyyed Esfahani [119]	2015	Reliability, Maintenance, Cold standby, Bi-objective model, Cost-optimization	Non-dominated sorting genetic algorithm II
Xiang Jia, Hao Chen, Zhijun Cheng and Bo Guo [122]	2016	Two-unit standby system, Switching policy, Mean time to failure, Imperfect switching	Weibull distribution, Sensitivity analysis

Madhu Jain, Chandra Shekhar and Shalini Shukla [116]	2016	Markov chain, Machine repair, Mixed spares, Time sharing, Threshold policy, Additional repairman, Queue length	Recursive Method
Madhu Jain and Rakesh Kumar Meena [104]	2017	Fault-tolerant MRP, Unreliable server, Working vacation, Standbys, F-policy	Neuro-fuzzy technique
Chandra Shekhar, Madhu Jain, Ather Aziz Raina and Javid Iqbal [261]	2017	Reliability, Fault tolerant system, Active redundancy, Reboot, MTTF, Common cause failure	Spectral method
Wu-Lin Chen and Kuo-Hsiung Wang [39]	2018	Retrial machine repair system, Controllable server, Reliability analysis, Working breakdown, Sensitivity analysis, MTTF	Crammer's rule, Laplace transform
Jau-Chuan Ke, Tzu-Hsin Liua and Dong-Yuh Yang [135]	2018	Cost analysis, Imperfect switchover, Machine interference problem, Optimization, Availability	Supplementary variable method
Kamlesh Kumar, Madhu Jain and Chandra Shekhar [156]	2019	MRP, Threshold, F-policy, Warm standbys, Two heterogeneous servers, Queue size	Matrix Method
Ching-Chang Kuo and Jau-Chuan Ke [163]	2019	Unreliable server, Spare system, General repair	Supplementary variable technique, Laplace–Stieltje's transform

Chandra Shekhar, Amit Kumar and Shreekant Varshney [262]	2020	Redundant repairable system, Switching failure, Common-cause failure,	Newton-Quasi method, Successive over-relaxation (SOR) method
Chia-Huang Wu and Dong-Yuh Yang [332]	2020	Dynamic control, MRP, Sensitivity analysis, Switching failure, Vacation	Matrix method
Hui Wua, Yan-Fu Lia and Christophe Bérenguer [336]	2020	Multiphase Markov process, Periodic inspection, Repair, Warm standby system, Cost analysis	Monte Carlo simulation
Shan Gao and Jinting Wang [65]	2021	Availability, MTTF, Reliability, Retrial, Mixed standby, Preventive maintenance	Crammer's rule, Laplace transform method
Ritu Gupta and Divya Agarwal [80]	2021	MRP, Warm spares, N-policy, First essential repair, Second optional repair, Reliability, Vacation, Cost function, Markov process	Runge-Kutta method
Jasdev Bhatti and Mohit K. Kakkar [15]	2021	Cold standby units, MTTF, Availability, Busy period	Regenerative techniques, Geometric distribution
Kuo-Hsiung Wang, Tseng-Chang Yen and Chia Huang Wu [327]	2022	Availability, MTTF, Reliability, MRP ,Sensitivity analysis, Switching failure	Laplace transforms, Matrix-analytical method
Rakesh Kumar Meena, Madhu Jain, Assif Assad, Rachita Sethi and Deepika Garg [199]	2022	Machine repair problem, Optimal policy, M/G/1 queue, N-policy	Supplementary variable approach, Particle swarm optimization, Recursive method

Jia Kanga, Linmin Hu, Rui Peng,Yan Li and Ruiling Tian [127]	2023	Warm standby, Retrial, Availability, Cost-benefit ratio	Matrix-analytic method, Bayesian approach
Farjam Kayedpour, Maghsoud Amiri, Mahmoud Rafizadeh, Arash Shahryai Nia and Mani Sharifi [128]	2023	Redundancy allocation problem (RAP), Warm standby, MTTF, Imperfect switching, Reliability	Genetic algorithm, Non-dominated Sorting Genetic algorithm (NSGA-II)
Shalini Sharma and Kamlesh Kumar [254]	2024	MRP, Differentiated vacation, Threshold control policies, Discouragement, Cost optimization	Fibonacci search algorithm, Artificial bee colony
Taha Tetik, Gulesin Sena Das and Burak Birgoren [294]	2024	Multi-objective optimization, Satellite design, Satellite reliability optimization, RAP	Simulated Annealing algorithm
Qi Shao, Linmin Hu and Fan Xu [251]	2024	k-out-of-n: G retrial system, Mixed standby, Mean time to first failure, Reliability, Availability	Genetic algorithm, Monte Carlo method
Ibrahim Yusuf, Muhammad Sagir Aliyu and Mus'abu Musa [356]	2024	MTTF, Mixed standby, Redundancy, Series-parallel	Linear algebra

1.16.3 Contributions in MRP with Varied failures

Resolving machine repair challenges necessitates a structured method comprising root cause diagnosis, corrective action implementation, and preventive measures to mitigate future occurrences. Effective troubleshooting and resolution of machine issues heavily rely on the collaboration between maintenance personnel, technicians, engineers, and stakeholders.

MRP with switching failure

The spare unit is an essential component of the machining system, which activates automatically in case of a primary unit failure. A successful and efficient switching process is crucial for seamless operation. However, poor automation and mishandling may lead to switching failures, denoted by probability q. Therefore, prompt switching is crucial for uninterrupted functioning, and standby units in the pool repeat this process until all units successfully switch or exhaust. Kumar and Agarwal [155] introduced the concept of imperfect switching within a two-unit standby redundant system. Moreover, extensive research on machine repair problems with imperfect switching has been conducted by other scholars, such as Gopalan and Waghmare [73], Goel et al. [71], Goyal and Murari [74], Gupta and Chaudhary [78], and Subramanian and Sarma [286].

MRP with common-cause failure

Common Cause Failures (CCFs) often complicate machine repair, where multiple components or subsystems fail simultaneously due to shared root causes or external factors. CCFs can be caused by environmental, design, manufacturing, and maintenance issues. These factors can influence component integrity and system functionality, leading to machine failures. Additionally, we discuss strategies for mitigating CCFs and enhancing system resilience, emphasizing the importance of a comprehensive approach to machine repair that addresses underlying vulnerabilities and promotes sustainable operational practices. A state transition diagram Fig. 1.10 is depicted for a machine repair model where each unit has an individual failure rate denoted by λ, and the system can also fail due to common cause failure with a rate denoted by $\lambda_C, (\lambda_C << \lambda)$. McGranaghan et al. [198] introduced the concept of common cause failure within voltage sags in industrial plants, Stamatis [283], and Birolini [16] conceptualized the common cause failure in distinct machine repair models in their research work. Numerous researchers have conducted extensive research across various time frames on Switching failure and common cause failure. The significant contributions regarding switching failure and common cause failure have been consolidated in Table 1.4.

Table 1.4: Table to Contribution of MRP with

Authors	Year	Key Feature	Methodology
Madhu Jain, G.C. Sharma and Varsha Rani [109]	2015	MRP, Two mode failure, Warm spares, Cost analysis	Runge–Kutta technique

Mangey Ram and Monika Manglik [233]	2016	Availability, Reliability, MTTF, mean time to repair, CCF	Supplementary variable technique, Laplace transformation
Huan Yu, Jun Yang, Jing Lin and Yu Zhao [351]	2017	Phased-mission common bus (PMCB) system, CCF, Reliability	Recursive algorithm, Genetic algorithm
Ruiying Li, Qiong Li, Ning Huang and Rui Kang [173]	2017	Cloud computing systems ,CCF, Reliability,	Monte Carlo simulation, State enumeration
Amit Kumar, Mangey Ram, Sangeeta Pant and Anuj Kumar [152]	2018	Industrial-based complex system, Reliability, Availability, MTTF,	Supplementary variable technique, Laplace transform
Madhu Jain and Ritu Gupta [100]	2018	Redundant repairable system, N-policy, Switching failure, MTSF, Availability	Neuro fuzzy technique
Neetu Singh [269]	2019	Machining system, CCF, Cost analysis, Mixed standby, N-Policy, Queue size	Recursive method
Chandra Shekhar, Neeraj Kumar, Amit Gupta, Amit Kumar and Shreekant Varshney [264]	2020	CCF, Reliability, Transient queue-size distribution, Vacation interruption	Laplace transform method, Spectral method
Chandra Shekhar, Neeraj Kumar, Madhu Jain and Amit Gupta [265]	2020	Redundant Fault-tolerant Computing Network, CCF, Switching failure, MTTF, Reliability, Availability, Failure Frequency	Spectral method
Xiao-Jian Yi, Chen-Hao Xu, Shu-Lin Liu, Man-Xi Xing and Hui-na Mu [350]	2021	Reliability analysis, CCF, Maintenance correlation	Goal oriented method

Yan-Feng Li, Hong-Zhong Huang, Jinhua Mi, Weiwen Peng and Xiaomeng Han [175]	2022	Multi-state systems, CCF, Fuzzy probability	Defuzzification
Jinhua Mi, Ning Lu, Yan-Feng Li, Hong-Zhong Huang and Lib-ing Bai [203]	2022	CCF, Complex systems, Sensitivities analysis, Reliability	Network-based hierarchical method
Kuo-Hsiung Wang, Chia-Huang Wu and Tseng-Chang Yen [322]	2023	Redundant retrial machining system, Warm standbys, Reliability, MTSF,	Laplace transform technique, Matrix-analytical method
Jing Li, Linmin Hu, Yuyu Wang and Jia Kang [172]	2023	Reliability, Retrial, Two failure mode, Preventive maintenance	Markov process theory, Laplace transform method,
Qinglai Dong, Pin Liu and Xujie Jia [52]	2023	k-out-of-n: G system, CCF, Three-level repair strategy, Mean time to first failure, Availability	Laplace transform

Contributions in MRP with Varied failures, Degradations Imperfect Coverage, and Reboot Delay

In systems with backup units, it is crucial to have mechanisms for detecting, isolating, and reconfiguring them in case of failure. This process is called redundancy management with perfect coverage. However, achieving complete certainty in redundancy management tasks is rare in real-world situations, resulting in system obstacles and the need for practical corrective actions. This situation is an example of imperfect coverage. Corrective actions may include initiating a reboot or recovery process, which may prolong the delay time. Rebooting refers to intentionally restarting a system that has been hindered due to unpredictable faults. This process can be either hard or soft. Hard rebooting involves physically cycling the system's power to facilitate the initial unit boot-up, while soft rebooting allows for system restart without power interruption. Rebooting quickly removes the faulty unit and reconfigures the system to restore functionality as soon as possible. Table 1.5 summarizes notable past research on varied failure, degradations, imperfect coverage and reboot delay.

Table 1.5: Table to Contribution of MRP varied failures

Authors	Year	Key Feature	Methodology
Tseng-Chang Yen, Haitao Wu, Kuo-Hsiung Wang and Wen-Kuang Chou [349]	2015	MRP, Warm standbys, Working breakdown, Removable repairman	Matrix-analytic method
Jau-Chuan Ke, Tzu-Hsin Liu and Dong-Yuh Yang [134]	2015	Machine repair system, Standby switching failure, Weibull distributions	Recursive procedure, Lausanne method, Supplementary variable technique
Madhu Jain [99]	2016	Redundant repairable system, Availability, Failure Frequency, Imperfect Switching	Runge–Kutta fourth-order technique
Ching-Chang Kuo and Jau-Chuan Ke [162]	2016	Switching Failure, Optimal Availability, Unreliable server	Supplementary variable method
Chandra Shekhar, Madhu Jain, Ather Aziz Raina and Rajesh Prasad Mishra [260]	2017	MRP, Spare, Geometric reneging, Switching failure, Reliability, MTTF	Runge-Kutta method of fourth order
Madhu Jain, Chandra Shekhar and Rakesh Kumar Meena [112]	2017	MRP. Controlled policy, Maintenance, Start-up time, Server breakdown	Successive over-relaxation, Quasi-Newton method, Direct search method
Madhu Jain and Rakesh Kumar Meena [103]	2017	FTMS, Vacation, Queue length, Imperfect coverage	Runge–Kutta method
Meisam Sadeghi and Emad Roghanian [243]	2017	Markov Process, Switching mechanisms, Reliability, MTTF, Availability	Laplace transforms

Jau-Chuan Ke, Tzu-Hsin Liu and Dong-Yuh Yang [135]	2017	Cost analysis, Imperfect switchover, Machine interference problem, Optimization	Supplementary variable method
Dong-Yuh Yang and Ya-Dun Chang [341]	2017	Busy period, Cost optimization, Retrial machine, Availability	Supplementary variable technique
Wu-Lin Chen and Kuo-Hsiung Wang [39]	2018	Warm standby, N-policy, Retrial MRP, MTTF, Reliability	Laplace transform
Jau-Chuan Ke, Tzu-Hsin Liu, Ying-Lin Hsu and Hui-Tzu Ku [133]	2018	MRP, Cost analysis, Partial breakdowns, Delayed repairs, Warm standby	Matrix-analytic method, Lausanne method
Dong-Yuh Yang and Chih-Lung Tsao [342]	2019	Reliability, Warm spare, MTTF, FTMS, Working Vacation	Matrix-analytic method, Laplace transform technique
Chandra Shekhar, Amit Kumar, Shreekant Varshney, Sherif I. Ammar [263]	2019	Fault-tolerant redundant repairable system,	Laplace–Stieltjes transform, Supplementary variable technique
Chandra Shekhar, Amit Kumar and Shreekant Varshney [262]	2020	FTMS, CCF, Standby unit, Reboot delay	Newton-quasi method
Chia-Huang Wu and Dong-Yuh Yang [332]	2020	MRP, Switching failure, Standby, Unreliable repairmen	Matrix analytic method
Pankaj Kumar and Madhu Jain [158]	2020	Fault-tolerant system (FTS), Imperfect switching, Reboot, MTTF, RELiability and Sensitivity analysis	Spectral method
Madhu Jain, Rakesh Kumar Meena and Pankaj Kumar [106]	2021	Imperfect fault detection, Reboot delay, Imperfect Switching, Maintainability	Recursive approach, Supplementary variable technique

Dong-Yuh Yang and Chia-Huang Wu [343]	2021	Availability, Reliability, MRP, Warm standbys, Imperfect switchovers	Runge–Kutta Method, Laplace transform method
Baoliang Liu, Yanqing Wen, Qingan Qiu, Haiyan Shi and Jianhui Chen [178]	2022	Unreliable switching failure, Phase-type(Ph) distribution, Redundancy, MRP	Matrix-analytic method
Chia-Huang Wu, Tseng-Chang Yen, Kuo-Hsiung Wang [335]	2021	Four retrial systems, Availability, Imperfect coverage, MTTF	Supplementary variable technique
Madhu Jain, Pankaj Kumar and Sudeep Singh Sanga [101]	2021	FTMS, Availability, MTTF, Imperfect coverage, Reboot	Non-linear programming, Yager's approach
Amit Kumar, Mohamed Boualem, Amina Angelika Bouchentouf and Savita [159]	2022	Markovian machine interference problem, Synchronized Reneging, Vacation Interruption, Random Switching Failure	Successive over-Relaxation, Quasi-Newton optimisation method
Shan Gao, Jinting Wang and Jie Zhang [66]	2022	Redundant series system, CCF, Delayed vacation, Cost benefit analysis	Laplace-transform, Markov renewal technique, Sequential Least Squares Programming (SLSQP) algorithm
Kuo-Hsiung Wang, Chia-Huang Wu and Tseng-Chan [322]	2023	Redundant retrial machining system, Switching failure, Warm standbys,Relibility, MTSF	Laplace transform technique, Matrix-analytical method
Chandra Shekhar Mahendra Devanda and Suman Kaswan [258]	2023	Multi-unit machining system, Common-cause failure, Reboot delay, Switching failure, Unreliable repairer, Failure frequency	Laplace transform, Cramer's method

Sudeep Kumar and Ritu Gupta [161]	2023	Reliability, Standby switching failure, N-policy, Server breakdown, Optional service	Laplace transform, Cramer's rule

1.16.4 Machine Repair Problem with Vacations and Controlled policy

When the system is free from failed units in any real-time framework, implementing server leave is a prudent strategy to minimize service costs. Including vacations in machine repair schedules enhances workforce well-being and promotes system efficiency, performance optimization, cost management, and service continuity. This ultimately contributes to the success and sustainability of repair operations. Various vacation strategies exist for this purpose, among which some common and important ones include:

- **Multiple vacation**: Upon repairing all failed units, the repairer takes a hiatus for a random period. At the end of the vacation, if no waiting failed units are present, the repairer takes another random vacation; otherwise, it resumes serving the failed units.

- **N-policy**: The repairer initiates service only when there are N failed units in the system and continues until all failed units are repaired.

- **Single vacation**: A repairer takes a single vacation after fixing all failed units, during which the repairer is unavailable until the vacation period ends.

- **Working vacation**: While on break, the repairer can operate remotely within the system, albeit at a reduced repair rate, a practice commonly called a working vacation. The concept of a working vacation decreases waiting times for failed units or systems and enhances overall system efficiency. These working vacations can be categorized as multiple or single, depending on whether the repairer takes more than one break or just one break during the vacation period.

- **Flexible Working Vacation**: Repairers can select their vacation periods within predetermined guidelines in a flexible working vacation arrangement. This vacation policy accommodates individual preferences and personal circumstances, ensuring that repair or service coverage remains effective throughout the year.

- **Vacation interruption**: At times, failed units encounter prolonged waiting periods during scheduled vacations. To mitigate these delays, the system employs a distinctive approach known as vacation interruption, wherein the server is summoned to conduct repairs before the scheduled conclusion of the vacation period. The policy governing vacation interruption may be based on the waiting time of failed units or the number of failed units in the system.

A survey on the queueing system with vacation studied by doshi [54]. Some researchers like Gupta [83], Lee et al. [168], and Li et al. [174] implemented the concept of working vacations, single vacation, breakdown and vacation in machine repair problems. Table 1.6 summarizes significant contributions to machine repair problems involving vacations, breakdowns, and controlled policies.

Table 1.6: Table to MRP with Vacations and Controlled Policy

Authors	Year	Key Feature	Methodology
Jau-Chuan Ke, and Kuo-Hsiung Wang[139]	2007	MRP, Two types spare, Multiple vacation, Single vacation	Matrix geometric method
Kuo-Hsiung Wang, Wei-Lun Chen, and Dong-Yuh Yang [312]	2009	MRP, Cost Optimization, Working vacation	Newton's method, Matrix-geometric method
Kuo-Hsiung Wang, Cheng-Dar Liou, and Ya-Lin Wang [319]	2014	MRP,Multiple vacation, warm-standby, Unreliable repairman	Matrix-analytic method
Cheng-Dar Liou [177]	2015	MRP, Multiple vacation, Working breakdown	Matrix-analytic method
Madhu Jain, Chandra Shekhar, and Shalini Shukla [117]	2015	MRP, Working vacation, Startup time F-policy	Matrix method
Madhu Jain and Rakesh Kumar Meena [105]	2018	Unreliable Heterogeneous servers, Mixed standby, Bi-level threshold policy	Runge–Kutta method
Gang He, Wenqing Wu, and Yuanyuan Zhang [90]	2018	MRP, Single working vacation, Phase type distribution	Matrix analytical method,

Chandra Shekhar, Shreekant Varshney, and Amit Kumar [265]	2019	MRP, Vacation policy, Reliability, MTTF, N-policy	Matrix method
Madhu Jain, Chandra Shekhar and Rakesh Kumar Meena [113]	2019	FTMS, Working vacation, Warm standby F-Policy, Maintainability	Runge–Kutta method
Madhu Jain, Rakesh Kumar Meena, and Pankaj Kumar [106]	2020	N-policy, Balking, Imperfect recovery, Reboot delay	Recursive method, Quasi-Newton method, ANFIS
Chandra Shekhar, Shreekant Varshney, and Amit Kumar [267]	2021	Preventive, Corrective, Predictive maintenance policies, Vacation interruption	Laplace transformation
Rakesh Kumar Meena, Madhu Jain, Sudeep Singh Sanga, and Assif Assad [200]	2021	$M/G/1/K$, $FM/FG/1/K$. MRP, Standby, Vacation	Laplace–Stieltjes transform, Recursive Method, Supplementary variable approache
Teketel Ketema, Seleshi Demie, and Melisew Tefera Belachew [143]	2021	MRP, Multiple working vacation, triadic$(0,Q,N,M)$ policy	Grid search algorithm
Praveen Deora, Umesh Kumari and D. C. Sharma [46]	2021	MRP, Feedback F-policy, Cost analysis, Vacation	Particle swarm optimization
Rakesh Kumar Meena, Madhu Jain, Assif Assad, Rachita Sethi, and Deepika Garg [199]	2022	MRP, Optimal policy, Vacation policy, N-policy	Recursive method, Supplementary variable techniques
Chia-Huang Wu, Dong-Yuh Yang, and Ting-En He [333]	2023	MRP, Unreliable repairman, Working breakdowns, Multiple vacations	Matrix-augmentation approach
Pankaj Kumar, Madhu Jain, and Rakesh Kumar Meena [160]	2023	FTS, Working vacation,Double retrial orbits, Admission control policy	Spectral expansion method

Chia-Huang Wu, Dong-Yuh Yang, and Min-Hui Ko [334]	2024	MRP, Two failure mode, Working vacation	Matrix analysis method, Matrix decomposition
Sudeep Singh Sanga and Khushbu S. Antala [247]	2024	FTMS, F-policy, Multiple vacation,Trapezoidal fuzzy number	Laplace-Stieltjes transform, Recursive method
Qi Shao, Linmin Hu, and Fan Xu [251]	2024	k-out-of-n : G, J-vacation policy, MTTF, Redundant dependency, Reliability	Monte Carlo method, Runge-Kutta method
Parmeet Kaur Chahal and Kamlesh Kumar [28]	2024	MRP, Generalized triadic control policy, Multiple working vacations, Redundant system	Recursive approach

1.16.5 MRP with Bayesian point of view

Utilizing Bayesian analysis offers a significant avenue for tackling machine repair challenges, providing valuable insights and methodologies to enhance maintenance strategies, operational efficiency, and reliability within industrial systems. Embracing Bayesian principles in machine repair has the potential to foster innovation, streamline resource allocation, and ultimately propel the evolution of maintenance practices across a variety of industries. The table 1.7 summarizes significant contributions to machine repair models solved by bayesian analysis.

Table 1.7: Table to Contribution of MRP solved by Bayesian approach

Authors	**Year**	**Key Feature**	**Methodology**
Venkata SS Yadavalli, Andriétte Bekker, and J Pauw [338]	2005	Two-unit system, CCF, Individual failures	Monte Carlo simulation, Bayes estimation, γ-prior, β-prior
Gregory Levitin and Min Xie [170]	2007	FTS, CCF, Reliability	$N-$version programming method

Jau-Chuan Ke, Ssu-Lang Lee, and Ying-Lin Hsu[132]	2008	Availability, Coverage, Detection, Reboot, Mean time to system failure	Monte Carlo simulation
Ying-Lin Hsu, Ssu-Lang Lee, and Jau-Chuan Ke [95]	2009	Mean time to system failure, Simulation, Availability	Monte Carlo simulation, Bayesian estimation
Jau-Chuan Ke, Ssu-Lang Lee, and Ming-Yang Ko [120]	2011	Availability, Confidence limits, Detection delay, Imperfect coverage, MTTF	Confidence Interval Estimation
Jau-Chuan Ke, Zheng-Long Su, Kuo-Hsiung Wang, and Ying-Lin Hsu [136]	2010	Availability, Distribution-free, Imperfect coverage, Power function, Standby	Simulation, Interval estimation, Hypothesis test
Ram Kishan and Divya Jain [148]	2014	MTSF, Highest posterior density (HPD) intervals, Fisher information matrix	Regenerative point technique
P. Chandrasekhar, V. S. Vaidyanathan, V. S. S. Yadavalli, and S. Xavier [32]	2013	Two-unit standby system, Steady state availability	Multivariate central limit theorem,Slutsky theorem
Tzu-Hsin Liu, Jau-Chuan Ke, Meng-Feng Yen, and Ying-Lin Hsu [184]	2017	Availability, Power function, Reboot delay, Standby switching failure	Hypothesis test, Logit transformation, Simulation
Rohit Patawa, Pramendra Singh Pundir, Alok Kumar Sigh, and Abhinav Singh [224]	2022	MTSF, Availability, Maintenance, Lindley distribution, Maximum likelihood	Metropolis-hastings algorithm, Bayesian inference
S. A. Thiagarajan and S. Thobias[295]	2023	Cold standby system, Point availability, Slutsky theorem, Steadystate availability	Bayes estimator, Quadrivariate exponential distribution

Machine repair problem in Fuzzy enviornment

Fuzzy sets and fuzzy logic are crucial in addressing numerous real-time challenges encountered in the research and advancement of technological, social, and economic processes. These include but are not limited to engineering problems spanning mechanical, civil, chemical, electrical, aerospace, biomedical, agricultural, computer, environmental, industrial, geological, and mechatronics fields. Additionally, they are instrumental in computer software development, natural sciences such as Mathematics, Biology, Chemistry, Physics, medical science, and social science encompassing economics, management, political science, psychology, and public policy. Fuzzy sets and fuzzy logic are apt tools for mitigating the vagueness or uncertainty stemming from linguistic errors, experimental errors, and similar sources. They offer a well-defined framework characterized by membership grade functions, facilitating the representation and manipulation of imprecise or uncertain information in a structured manner.

Fuzzy logic has proven highly effective across various domains, including image processing, control systems engineering, power engineering, industrial automation, robotics, consumer electronics, and optimization. Fuzzy queues, characterized by their ability to handle vague and uncertain information, offer more excellent value and realism than conventional crisp queues. Consequently, fuzzy queues are recognized as more practical and valuable, given their capacity to accommodate imprecise data and uncertainties inherent in real-world scenarios.

Firstly, Zadeh introduced the concept of vagueness information in decision-making in the research article [358]. Buckley [22] implemented the concept of uncertainty in queueing theory. Jain and Agogino [230] suggested incorporating Bayesian fuzzy probabilities into an influence diagram computational scheme to analyze sensitivity while solving decision and probabilistic inference problems. Various researchers have made a vast contribution to fuzzy machiner repair problems by [274, 181, 176]. The table 1.8 summarizes significant contributions to fuzzified machine repair problems involving varied failure and repair strategies.

Table 1.8: Table to Contribution of Machine Repair problem in Fuzzy environment

Author	Year	Keywords	Methodology
Shih-Pin Chen[36]	2006	Membership function, Crisp value, Lower and upper bounds, MIP	Mathematical programming approach

Shih-Pin Chen [37]	2007	MRP, MIP, Fuzzy sets, Non-linear programming	α-cut approach
Jau-Chuan Ke, Hsin-I Huang, and Chuen-Horng Lin [130]	2007	Zadeh's extension principle, Availability, Membership function, Fuzzy sets	α-cut approach
Lilly Robert and W. Ritha [240]	2010	MIP, Fuzzy trapezoidal numbers	Function principle
Harish Garg [67]	2013	Reliability, Availability optimization, PSO, CIBFLT, Fuzzy methodology	Fuzzy and statistical methodology, Fuzzy confidence interval
Komal and S.P. Sharma [255]	2014	Artificial neural networks, Fuzzy reliability, Availability,	FLT technique, GABLT technique, NGABLT technique
G.A Rastorguev and F.A Elerian [238]	2014	Inventory management, Machine tool , Spare parts, Fuzzy logic	Fuzzification, Defuzzification
Ali Azadeh and Saeed Abdolhossein Zadeh [11]	2016	Condition-based maintenance policies, Triangular fuzzy numbers, crisp numbers,	Analytic hierarchy process, Fuzzy MCDM approach
Sayed Javad Aghili and Hamze Hajian-Hoseinabadi [1]	2017	Markov Processes, Fuzzy reliability, Fuzzy transformation, Repairable systems, Substation automation	Standard Fuzzy Arithmetic, Fuzzy Transformation Method (FTM)
Yanli Meng, Xiaodong Liu, and Muyan Zhou [201]	2017	M/M/c/m/m queueing system, Utilization rate of servers, Expected degree of acceptability, System cost, Optimal number of servers	Decision-making index

R. Sivaraman and Dr. Sonal Bharti [271]	2017	MIP, Graded MeanIntegration Representation, Trapezoidal fuzzy numbers, Fuzzy queues	Function principle
Rakesh Kumar Meena, Madhu Jain, Sudeep Singh Sanga, Assif Assad [200]	2019	MRP, Standby, Fuzzy environmen	Laplace–Stieltjes transform, Supplementary variable approaches
Madhu Jain, Sudeep Singh Sanga [108]	2020	F-policy, Fuzzy environmen, MIP, Retrial	α-cut approach, Supplementary variable Method
Madhu Jain, Pankaj Kumar, and Rakesh Kumar Meena [108]	2020	Non-markov fuzzy model, FTS, Harmony search, Server breakdown, Imperfect recovery	Supplementary variable, Parametric non-linear programming approach
Madhu Jain, Pankaj Kumar, and Sudeep Singh Sanga [101]	2020	FTS, Redundancy, RAM, Reboot provisioning	α-cut, Parametric non-linear programming method
H. Merlyn Margaret and P. Thirunavukarasu [202]	2021	MRP, Parametric non-linear programming, Fuzzy sets	α-cut approach
Pankaj Kumar, Madhu Jain, and Rakesh Kumar Meena [159]	2022	FTMS, Reboot, Vacation, Fuzzy environments, N-policy	Parametric non-linear programming
Abbas Al-Refaie and Ahmad Al-Hawadi [6]	2022	Scheduling, Sequencing, Optimization, Maintenance planning, Multi-skill	Fuzzy arithmatic
Ananda Prasad Panta, Ram Prasad Ghimire, Dinesh Panthi, and Shankar Raj Pant [223]	2023	Fuzzy Environment, Poisson, Optimal, Reneging	Recursive method
Sibasish Dhibar, Madhu Jain [49]	2024	Cloud storage, Retrial queue, Working vacation, Catastrophes	Fuzzy PSO, Qausi-Newton Method, α-cut approach

1.17 Gaps in the Existing Literature

- The notion of working vacation, working breakdown, and vacation interruption needs to be implemented to enhance the reliability measures.
- The existing MRP's can be extended for general failure and repair time distributions.
- The different architectural arrangments like series, parallel, K-out of M, and other combinations can be compared for optimal topology.
- Retrial attempts, restricted attempts, point abandonment of failed units are real issues that need research for the performance evaluation of fault-tolerant redundant repairable machining system.
- Imperfect coverage, threshold-based repair policy, pressure coefficient, and feedback are strategies that help predict the redundant machining system's performance and require in-depth analysis.

1.18 Thesis Objective

- To focus on reliability and optimization issues related to the fault-tolerant redundant repairable system.
- Computation of transient-state and steady-state measures of probabilistic events involved in the machining systems.
- Developing numerical techniques to focus on statistical and optimal analysis for studying MRP's redundancy and RAM characteristics.
- Sensitivity analysis and optimal analysis to classify the critical parameter(s) for the fault-tolerant machining system.
- To simulate the mathematical model and use the results to design machining system optimally

Chapter 2

Reliability Analysis of Imperfect Repair and Switching Failures: A Bayesian Inference and Monte Carlo Simulation Approach

"In machine repair problems, statistics unveils insights into failure patterns and guides efficient maintenance strategies".

Prof. Jane Smith

2.1 Introduction

In today's data-driven technological advancement, where data is crucial for operations, storage systems play a pivotal role in information management. Redundancy is essential in engineering, ensuring uninterrupted functionality and enhancing infrastructure resilience, reliability, and availability across various domains. It shields against vulnerability attacks, unplanned downtimes, errors, and hardware failures, ensuring data integrity and system stability. Additionally, redundancy aids in load balancing and system optimization, improving overall performance and efficiency. Operational excellence (OE) is a priority for enterprise software companies, directly impacting performance, cost-efficiency, and customer satisfaction. Poor OE can lead to financial losses, increased repair costs, customer churn, and damage to reputation. Key metrics for assessing storage systems' OE include Mean Time-to-Failure (MTTF) and availability, driving decision-makers to invest in maximizing these metrics. Given the critical role of redundancy and these metrics, further research is needed to comprehensively study repairable redundant storage systems and analyze MTTF and availability.

Cutting-edge storage and infrastructure systems rely on Triple-Modular Redundancy (TMR), known for its role in fault-tolerant operation and improved availability. TMR employs three identical units, with one or two active units and the rest in standby. It excels in quickly detecting anomalies among units, aiding in rapid fault identification. This study examines a dual-active, one-standby redundant storage framework to minimize system downtime by promptly replacing a malfunctioning active unit with its standby counterpart. However, the switchover process is vulnerable to various issues, such as human errors, manufacturing defects, and software glitches, resulting in imperfect switching. In such cases, the switchover operation is repeated until successful, introducing complexities that can impact system performance and reliability. Given the intricacies and uncertainties surrounding this process, comprehensive research is needed to study redundant storage systems with unreliable switchover mechanisms.

With modern technological advancements in software infrastructures, storage units, and computing systems, the complexity of these entities is increasing. These systems integrate interconnected units, creating intricate layers of software and hardware. Prompt dispatch to repair facilities is crucial in case of component faults, but the inherent complexities introduce challenges in system rejuvenation. Repair may be unreliable due to factors like component interdependencies, hardware variability, compatibility issues, bad sectors, misconfigurations, and incorrect root cause analysis. Technicians often face iterative repair cycles until the unit is perfectly repaired, each iteration adding uncertainties and complexities. This process is time-consuming and resource-intensive. Within repairable redundant storage systems, imperfect repair is a relatively unexplored area. As system complexities grow, there is a need for a deeper understanding of imperfect repair processes in uncertain environments.

To measure the Mean Time-to-Failure (MTTF) and availability of storage units, parameters like time to failure, repair, and inspection are assumed to follow specific probability distributions with known parameters. Real-world uncertainties often render these distribution parameters unknown or imprecise. Therefore, choosing an appropriate estimation method is crucial for accurately computing failure, repair, and inspection time distributions. The Bayesian approach, which incorporates prior knowledge from analogous reliability data, offers robustness by translating this information into a flexible prior density. While the Bayesian approach is flexible in handling uncertainties, asymptotic confidence intervals, reliant on large-sample properties, provide computational efficiency for complex system characteristics but may be less accurate with small sample sizes. Given the historical efficacy of Bayesian analysis in accurately estimating reliability characteristics, a comparison with asymptotic estimators is necessary to highlight its effectiveness. This investigation is essential for demonstrating the robustness of Bayesian analysis in achieving precise estimations for MTTF and availability, particularly in the presence of uncertainties in distribution parameters.

The research study focuses on three primary highlights to advance the understanding of reliability metrics in storage systems. Firstly, it aims to develop robust methods capable of handling uncertainties in distribution parameters, particularly for failure, repair, and inspection times. These methods will enhance the accuracy of reliability assessments, ensuring more precise evaluations of system performance. Secondly, the research seeks to compare the effectiveness of Bayesian analysis and asymptotic estimators in scenarios with limited data. This comparative analysis will shed light on the strengths and limitations of each approach, providing valuable insights into their applicability in practical settings. Finally, the study aims to evaluate the resilience of Bayesian analysis in the face of uncertainties. By assessing the performance of Bayesian methods in real-world scenarios, the research will contribute to a deeper understanding of their reliability and effectiveness.

The chapter is organized as follows. Section 2.2 presents a comprehensive literature survey aimed at identifying research gaps. In Section 2.3, we provide a detailed description of the model, focusing on the key stages in the life cycle of a storage unit. This section introduces mathematical notations and assumptions, laying the groundwork for deriving the system's reliability characteristics, including Mean Time-to-Failure (MTTF) and availability (Section 2.4). Section 2.5 incorporates Bayesian inferences to estimate crucial unknown parameters necessary for computing availability. We adopt different prior distributions, specifically the gamma and beta probability distributions with varying hyperparameters. To validate the accuracy of the posterior analysis, we conduct numerical simulations and compare the results with asymptotic confidence intervals (Section 2.6). Additionally, we provide graphical representations of the Monte Carlo simulations to illustrate the performance of Bayesian analysis. Finally, in Section 2.7, we discuss the conclusions drawn from our study and suggest potential avenues for future research.

2.2 Literature Survey

In the field of repair and reliability engineering, availability stands as a critical performance metric, ensuring operational efficiency, cost-effectiveness, and optimized maintenance strategies for units. Redundancy, a technique employed in engineering systems, plays a pivotal role in enhancing system reliability and availability. Osaki and Nakagawa [220], Osaki [219], and Nakagawa [209, 210] introduced the concept of mean-time failure for a two-unit standby redundant system, laying the foundation for subsequent research. Subramanian and Anantharaman [285] delved into the reliability characteristics and optimization of comprehensive cost functions for complex standby redundant systems. Huang and Ke [96] conducted a comparative cost/benefit analysis of repairable redundant systems with distinct configurations. Tan [290] analyzed the reliability and availability of two-unit warm standby microcomputer systems with self-reset functionality and repair facilities. A considerable body of research on two-unit standby systems with redundancy in computer and industrial applications has been explored by [195, 239, 355, 51, 289, 81].

Kumar et al. [154] developed a stochastic model for a data processing system comprising two identical units, one operational while the other in standby mode, introducing a novel approach for hardware fault detection in computer systems. Wang et al. [325] emphasized that the coverage factor of active unit failure differs from standby unit failure, developing steady-state availability using recursive methods and supplementary variable techniques for the two systems. Yen and Wang [347] analyzed three configurations with imperfect coverage and standby switching failures, evaluating system reliability and availability based on their findings. Catelani et al. [26] examined standby redundancy for improving wireless sensor network reliability. Other researchers (cf. [193], [340], [144]) have presented the concept of a two-unit standby redundant system with distinct failures, ensuring high reliability and availability.

Repair facilities play a vital role in the long-term sustainability of systems by ensuring their maintenance and proper functioning. Previous research by Mokaddis and Tawfek [204] evaluated the reliability metrics of redundant systems operating in warm standby mode with different types of repair facilities. Lam [166] studied the maintenance model of a two-unit redundant system with various replacement policies, while Sridharan and Mohanavadivu [279] introduced the concept of two types of repairers and patience time in evaluating the reliability metrics of a unit standby system. Chandrasekhar et al. [31] investigated a two-unit standby system using Erlangen repair time, and Wang et al. [320] examined the cost-effectiveness of three distinct series systems featuring redundancy and general repair time. These studies offer valuable insights into the performance and optimization of complex systems considering repair and redundancy. Saini et al. [244] examined two-unit redundant systems with various repair strategies. Other researchers [79, 337, 313, 63, 262] have also contributed to this area, evaluating mean time to failure, availability, and reliability of systems with different repair strategies. Gao and Wang [65] proposed the use of an unreliable repair facility to evaluate the reliability and availability of mixed standby retrial systems. In a recent study, Kamal et al. [126] analyzed the behavior of two non-identical units with redundancy subjected to arbitrary repair facilities and replacements, focusing on cost benefits and providing insights into the reliability metrics of such systems.

Chang and Thompson [33] proposed a technique to obtain Bayes confidence limits for the reliability of a series-parallel system with failure-independent modules, aiming to accurately estimate the system's reliability. Vaurio [303] introduced estimation methods for multiple failure rates and other model parameters. Hsu et al. [92] studied the reliability characteristics of a two-unit redundant system using a Bayesian approach to derive the posterior distribution for the Mean Time-to-Failure (MTTF) and steady-state availability of the system. Several researchers (*cf.* [131], [132], [169], [93], [292]) have investigated the Mean Time-to-Failure and availability of repairable redundant systems with imperfect coverage, switching failure, reboot delay, and unreliable repair facilities from a statistical perspective. Kishan and Jain [148] proposed a model where the failure and repair time distributions of each unit are considered Weibull with typical shape parameters but different scale parameters. Jia and Guo [123] analyzed the reliability metrics of the k-out-of-n non-repairable cold-standby system using Bayes theory. Cüran and Kizilaslan [45] constructed asymptotic confidence intervals and the highest probability density credible intervals for a series system with cold standby redundancy. Thiagarajan and Thobias [295] comprehensively analyzed a dependent structure three-unit cold standby system from both classical and Bayesian inference perspectives, demonstrating the system's capability for exhaustive analysis.

Several studies have explored repairable redundant systems with failures and rejuvenation in probabilistic environments, as highlighted in our literature review. However, critical research gaps have been identified: (i) limited exploration of imperfect repair in redundant multi-unit storage systems, and (ii) the absence of Bayesian analysis integration, particularly in the context of imperfect repair. In contrast, our proposed model stands out due to its comprehensive investigation into the imperfect repair facility within the context of redundant storage system, estimation of reliability metrics derived from inherently vague, ambiguous, imprecise, and uncertain failure and imperfect repair environments, and the application of the Bayesian approach to estimate unknown parameters, comparing it to asymptotic confidence intervals. The primary objective of this chapter is to address impreciseness, vagueness, and uncertainties that may arise at any stage in the life cycle of a system of storage units, with a focus on studying imperfect repair and switching failure in a repairable redundant storage system with a warm standby unit, and exploring these aspects within a vague and imprecise environment. This study can be further extended to any redundant and repairable system with an imperfect rejuvenation environment and an imprecise switchover process. Despite advancements in reliability analysis, several research gaps persist in the study of storage system metrics. Firstly, there is a need for more sophisticated methods that can effectively handle uncertainties in distribution parameters. Current approaches often struggle to account for these uncertainties, leading to inaccuracies in reliability estimations. Secondly, while Bayesian analysis has shown promise in addressing uncertainties, its comparative effectiveness against asymptotic estimators needs further exploration, especially in scenarios with limited data availability. Finally, the robustness of Bayesian analysis in practical applications remains uncertain, particularly in complex storage system environments. Bridging these gaps will not only improve the accuracy of reliability assessments but also enhance the overall understanding of storage system performance in real-world settings.

2.3 Model Description

This chapter investigates a redundant storage system with two identical storage units operating in parallel, supported by a redundant standby unit overseen by a single processor. This setup is relevant in fault-tolerant systems such as enterprise servers, networking equipment, data storage devices, data centers, cloud services, telecommunications, and distributed databases. The system is akin to failover load balancing mechanisms in network systems, ensuring uninterrupted service by redirecting traffic to a standby server during primary server failures. The study presents mathematical models describing storage device behavior, malfunction likelihood, and the impact of standby redundancy on system reliability.

2.3.1 Assumption

This chapter investigates a redundant storage configuration consisting of two active storage units and one standby storage unit, along with an unreliable storage maintenance approach. The mathematical modeling of storage unit interactions and interferences within the system is guided by assumptions and notational conventions. These are delineated across three distinct phases of the storage system's lifecycle.

Malfunction Stage:

- The active storage units and the warm standby storage unit follow independent exponential distributions for their time-to-malfunction, with rates λ and $\nu(0 < \nu < \lambda)$, respectively.
- In the event of an active storage unit malfunction, an available warm standby storage unit promptly takes over, inheriting the operational characteristics of the active storage unit.

Switching Stage:

- The elapsed time for a malfunctioned unit to switch with an available standby unit is considered negligible.
- Successful switchover of the warm standby unit in place of the malfunctioned unit may be impeded by factors such as mishandling, unit defects, or issues in the automation process, quantified by the probability q. Consequently, the probability of a successful switchover is denoted by $\bar{q}$.

Rejuvenation Stage:

- Malfunctioning units are sent for rejuvenation immediately. An idle processor initiates rejuvenation instantly, while a busy processor leads to queued rejuvenation.
- Time-to-rejuvenation follows an exponential distribution with a mean time of $\frac{1}{\mu}$.
- The rejuvenation facility may be perfect or imperfect. Inspection times during perfect and imperfect rejuvenation follow exponential distributions with mean rates β_1 and β_2, respectively.
- After perfect rejuvenation, units are restored to operational states like new units subsequently assigned to active or standby roles as per system requirements.

The proposed repairable redundant storage system's state at any time 't' is described below to develop the forward Chapman-Kolmogorov equations. To establish a repairable redundant storage system, we correlate the system's condition with both the active unit count denoted by $I_1(t)$ and the available standby unit count denoted by $I_2(t)$. At any given time t, the storage system comprises two types of units: active units $I_1(t)$ and standby units $I_2(t)$. Thus, the state notations, along with the pre-defined state of the repairable redundant system, the process $(I_1(t), I_2(t)); t > 0$ form a continuous-time Markov chain in the state space $\Pi = \{(I_1(t) = i, I_2(t) = j) | i = 0, 1, 2$ and $j = 0, 1\}$. This analysis operates on the premise that a system experiences failure when there are no active units. The probabilities of the system's states are introduced in the following manner:

- $P_{21}(t)$= Prob $\{$ At any given time t, the storage system comprises two active units and one warm standby unit.$\}$i.e. $P_{21}(t)$=Prob$[I_1(t) = 2, I_2(t) = 1]$
- $P_{20}(t)$=Prob$\{$At any given time t, the storage system comprises two active units and no warm standby unit.$\}$ i.e. $P_{20}(t)$=Prob$[I_1(t) = 2, I_2(t) = 0]$

- $P_{10}(t)$= Prob{At any given time t, the storage system comprises one active unit and no warm standby unit.} i.e.$P_{10}(t)$=Prob$[I_1(t) = 1, I_2(t) = 0]$

- $P_F(t)$=Prob{There are no active units in the storage system at any instant t.} i.e. $P_F(t)$=Prob$[I_1(t) = 0, I_2(t) = 0]$

- $Q_{20}(t)$=Prob{At any instant t, there are two active units and the repaired unit is under inspection.} i.e. $Q_{20}(t)$=Prob$[I_1(t) = 2, I_2(t) = 0]$

- $Q_{10}(t)$= Prob{At any instant t, there is one active unit and the repaired unit is under inspection.} i.e. $Q_{10}(t)$=Prob$[I_1(t) = 1, I_2(t) = 0]$

Therefore, the forward Chapman-Kolmogorov differential-difference equations, developed in terms of $\lambda, \nu, \mu, \beta_1, \beta_2, q$, and $\bar{q}$, which balance the inflow-outflow rate for the reliability analysis of the repairable redundant storage system is defined as follows:

$$\frac{dP_{21}(t)}{dt} = -(2\lambda + \nu)P_{21}(t) + \beta_1 Q_{20}(t) \tag{2.1}$$
$$\frac{dP_{20}(t)}{dt} = -(2\lambda + \mu)P_{20}(t) + (2\lambda\bar{q} + \nu)P_{21}(t) + \beta_2 Q_{20}(t) + \beta_1 Q_{10}(t) \tag{2.2}$$
$$\frac{dQ_{20}(t)}{dt} = -(\beta_1 + \beta_2)Q_{20}(t) + \mu P_{20}(t) \tag{2.3}$$
$$\frac{dP_{10}(t)}{dt} = -(\lambda + \mu)P_{10}(t) + 2\lambda P_{20}(t) + 2\lambda q P_{21}(t) + \beta_2 Q_{10}(t) \tag{2.4}$$
$$\frac{dQ_{10}(t)}{dt} = -(\beta_1 + \beta_2)Q_{10}(t) + \mu P_{10}(t) \tag{2.5}$$
$$\frac{dP_F(t)}{dt} = \lambda P_{10}(t) \tag{2.6}$$

The above Eqns 2.1- 2.6 , can be defined in matrix form as shown below.

$$\begin{bmatrix} -(2\lambda+\nu) & 0 & \beta_1 & 0 & 0 & 0 \\ (2\lambda\bar{q}+\nu) & -(2\lambda+\mu) & \beta_2 & 0 & \beta_1 & 0 \\ 0 & \mu & -(\beta_1+\beta_2) & 0 & 0 & 0 \\ 2\lambda q & 2\lambda & 0 & -(\lambda+\mu) & \beta_2 & 0 \\ 0 & 0 & 0 & \mu & -(\beta_1+\beta_2) & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 \end{bmatrix} \begin{bmatrix} P_{21}(t) \\ P_{20}(t) \\ Q_{20}(t) \\ P_{10}(t) \\ Q_{10}(t) \\ P_F(t) \end{bmatrix} = \begin{bmatrix} \frac{dP_{21}(t)}{dt} \\ \frac{dP_{20}(t)}{dt} \\ \frac{dQ_{20}(t)}{dt} \\ \frac{dP_{10}(t)}{dt} \\ \frac{dQ_{10}(t)}{dt} \\ \frac{dP_F(t)}{dt} \end{bmatrix} \tag{2.7}$$

To define the initial conditions, it is assumed that at the initial time $t = 0$, there are no failed units, the system is functioning correctly with two active units and one warm standby unit. Therefore $P_{21}(0) = 1$ and $P_{20}(0) = Q_{20}(0) = P_{10}(0) = Q_{10}(0) = P_F(0) = 0$ are the initial conditions.

2.4 The system reliability characteristics

We utilize the mathematical theory of Laplace transformation to analytically derive the expression of transient state probabilities from the system of first-order differential equations in matrix form Eqn. 2.7 with initial conditions mentioned earlier in this section. The Laplace transformation of state probability and its derivative can be defined as below.

$$L(P_{i,j}(t)) = \ddot{P}_{i,j}(s) = \int_0^{\infty} e^{-st} P_{i,j}(t)dt \quad \forall\, i,j$$

$$L\left(\frac{d}{dt}P_{ij}(t)\right) = s\ddot{P}_{i,j}(s) - P_{i,j}(0) \quad \forall\, i,j$$

2.4.1 Mean time-to-failure

After applying the Laplace transformation, the system of linear equations in matrix form is obtained as follows.

$$\begin{bmatrix} -(s+2\lambda+\nu) & 0 & \beta_1 & 0 & 0 & 0 \\ (2\lambda\bar{q}+\nu) & -(s+2\lambda+\mu) & \beta_2 & 0 & \beta_1 & 0 \\ 0 & \mu & -(s+\beta_1+\beta_2) & 0 & 0 & 0 \\ 2\lambda q & 2\lambda & 0 & -(s+\lambda+\mu) & \beta_2 & 0 \\ 0 & 0 & 0 & \mu & -(s+\beta_1+\beta_2) & 0 \\ 0 & 0 & 0 & \lambda & 0 & -s \end{bmatrix} \begin{bmatrix} \ddot{P}_{21}(s) \\ \ddot{P}_{20}(s) \\ \ddot{Q}_{20}(s) \\ \ddot{P}_{10}(s) \\ \ddot{Q}_{10}(s) \\ \ddot{P}_F(s) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.8}$$

After solving Eqn. 2.8, the state probabilities are obtained in their transformed form.

$$\ddot{P}_{21}(s) = \left[\frac{\begin{array}{c}(s+\beta_1+\beta_2)^2(\lambda+s)(2\lambda+s)+\mu(\lambda+s)(3s+\beta_1)(s+\beta_1+\beta_2) \\ -\mu(s+\beta_1+\beta_2)s(s-\beta_1)+\mu^2(s+\beta_1)^2\end{array}}{\begin{array}{c}(s+\beta_1+\beta_2)^2(\lambda+s)(2\lambda+s)(2\lambda+\nu+s)+(s+\beta_1+\beta_2) \\ \big((2\lambda+\nu+s)s(3\lambda+2s+\beta_1)\mu+\beta_1(\lambda+s)(2\lambda q+s)\mu\big) \\ +(2\lambda+\nu+s)(s+\beta_1)\mu^2 s+\beta_1(2\lambda q+s+\beta_1)\mu^2 s\end{array}} \right] \tag{2.9}$$

$$\ddot{P}_{20}(s)=\left[\frac{\begin{array}{c}-\left(s+\beta_1+\beta_2\right)^2(\lambda+s)(2\lambda q-2\lambda-\nu)+2\left(s+\beta_1+\beta_2\right)\\(1-q)\,\mu\,\lambda\,s+\left(\beta_1(2\lambda+\nu)+\nu s\right)\left(s+\beta_1+\beta_2\right)\mu\end{array}}{\begin{array}{c}\left(s+\beta_1+\beta_2\right)^2(\lambda+s)(2\lambda+s)(2\lambda+\nu+s)+\left(s+\beta_1+\beta_2\right)\\\left((2\lambda+\nu+s)\,s\,(3\lambda+2s+\beta_1)\,\mu+\beta_1(\lambda+s)(2\lambda q+s)\,\mu\right)\\+(2\lambda+\nu+s)\left(s+\beta_1\right)\mu^2 s+\beta_1\left(2\lambda q+s+\beta_1\right)\mu^2 s\end{array}}\right] \tag{2.10}$$

$$\ddot{Q}_{20}(s)=\left[\frac{\begin{array}{c}-\left(s+\beta_1+\beta_2\right)(\lambda+s)(2\lambda q-2\lambda-\nu)\,\mu+(2\lambda+\nu)\left(s+\beta_1\right)\mu^2\\-2\lambda\, qs\mu^2\end{array}}{\begin{array}{c}\left(s+\beta_1+\beta_2\right)^2(\lambda+s)(2\lambda+s)(2\lambda+\nu+s)+\left(s+\beta_1+\beta_2\right)\\\left((2\lambda+\nu+s)\,s\,(3\lambda+2s+\beta_1)\,\mu+\beta_1(\lambda+s)(2\lambda q+s)\,\mu\right)\\+(2\lambda+\nu+s)\left(s+\beta_1\right)\mu^2 s+\beta_1\left(2\lambda q+s+\beta_1\right)\mu^2 s\end{array}}\right] \tag{2.11}$$

$$\ddot{P}_{10}(s)=\left[\frac{2\left(s+\beta_1+\beta_2\right)^2(qs+2\lambda+\nu)\,\lambda+2\,\mu\, q\lambda\left(s+\beta_1\right)\left(s+\beta_1+\beta_2\right)}{\begin{array}{c}\left(s+\beta_1+\beta_2\right)^2(\lambda+s)(2\lambda+s)(2\lambda+\nu+s)+\left(s+\beta_1+\beta_2\right)\\\left((2\lambda+\nu+s)\,s\,(3\lambda+2s+\beta_1)\,\mu+\beta_1(\lambda+s)(2\lambda q+s)\,\mu\right)\\+(2\lambda+\nu+s)\left(s+\beta_1\right)\mu^2 s+\beta_1\left(2\lambda q+s+\beta_1\right)\mu^2 s\end{array}}\right] \tag{2.12}$$

$$\ddot{Q}_{10}(s)=\left[\frac{2\left(s+\beta_1+\beta_2\right)(qs+2\lambda+\nu)\,\lambda\,\mu+2\mu^2 q\left(s+\beta_1\right)\lambda}{\begin{array}{c}\left(s+\beta_1+\beta_2\right)^2(\lambda+s)(2\lambda+s)(2\lambda+\nu+s)+\left(s+\beta_1+\beta_2\right)\\\left((2\lambda+\nu+s)\,s\,(3\lambda+2s+\beta_1)\,\mu+\beta_1(\lambda+s)(2\lambda q+s)\,\mu\right)\\+(2\lambda+\nu+s)\left(s+\beta_1\right)\mu^2 s+\beta_1\left(2\lambda q+s+\beta_1\right)\mu^2 s\end{array}}\right] \tag{2.13}$$

$$\ddot{P}_{F}(s)=\left[\frac{2\left(s+\beta_1+\beta_2\right)^2(qs+2\lambda+\nu)\,\lambda^2+2\,\mu\, q\lambda^2\left(s+\beta_1\right)\left(s+\beta_1+\beta_2\right)}{s\left(\begin{array}{c}\left(s+\beta_1+\beta_2\right)^2(\lambda+s)(2\lambda+s)(2\lambda+\nu+s)+\left(s+\beta_1+\beta_2\right)\\\left((2\lambda+\nu+s)\,s\,(3\lambda+2s+\beta_1)\,\mu+\beta_1(\lambda+s)(2\lambda q+s)\,\mu\right)\\+(2\lambda+\nu+s)\left(s+\beta_1\right)\mu^2 s+\beta_1\left(2\lambda q+s+\beta_1\right)\mu^2 s\end{array}\right)}\right] \tag{2.14}$$

To calculate the transient state probabilities $P_{21}(t), P_{20}(t), Q_{20}(t), P_{10}(t), Q_{10}(t)$, and $P_F(t)$ of a system at any moment t, we can derive them by taking the inverse Laplace transform for $\ddot{P}_{21}(s), \ddot{P}_{20}(s), \ddot{Q}_{20}(s), \ddot{P}_{10}(s), \ddot{Q}_{10}(s)$, and $\ddot{P}_F(s)$, respectively. Here, $P_F(t)$ represents the probability of the storage system's complete failure when all active and standby units fail. Let X be a continuous time random variable representing the system's time-to-failure. The function $R_X(t)$ defines the system's reliability at any time t

and is defined as

$R_X(t) = Prob$(the storage system operates satisfactorily within a defined time frame)

$$R_X(t) = 1 - P_F(t), \quad t > 0 \tag{2.15}$$

The failure density function $X(t)$ can be defined in reliability theory as

$$X(t) = -\frac{d}{dt}R_X(t) = -\frac{d}{dt}(1 - P_F(t)) = \frac{d}{dt}P_F(t) \tag{2.16}$$

By applying the Laplace transform, we can express the failure density function as $\ddot{X}(s) = s\ddot{P}_F(s) - P_F(s)$. This enables us to calculate the mean time-to-failure of the storage system.

$$MTTF = -\frac{d}{dt}\ddot{X}(u) = \left[\frac{(\beta_1+\beta_2)\Big((\beta_1+\beta_2)\left((-2q+8)\lambda^2+3\lambda\nu\right)+(2\lambda+\nu)(3\lambda+\beta_1)\mu - \lambda\,(2\lambda q-2q\beta_1-\beta_1)\mu\Big)+\mu^2\beta_1(2\lambda q+2\lambda+\nu+\beta_1)}{2(2\lambda+\nu)(\beta_1+\beta_2)^2\lambda^2+2\mu q\beta_1(\beta_1+\beta_2)\lambda^2} \right] \tag{2.17}$$

2.4.2 Availability of the system

Accurately predicting the performance of a storage system necessitates a comprehensive analysis of its reliability measures. This section focuses on the availability of the storage system, a crucial factor in system evaluation. The forward Chapman-Kolmogorov equations are represented as a system of linear equations in matrix form. Upon applying the normalizing condition of probability, we derive the system's steady-state probabilities. The system's availability is then determined based on these steady-state probabilities.

$$\begin{bmatrix} -(2\lambda+\nu) & 0 & \beta_1 & 0 & 0 & 0 & 0 \\ (2\lambda\bar{q}+\nu) & -(2\lambda+\mu) & \beta_2 & 0 & \beta_1 & 0 & 0 \\ 0 & \mu & -(\beta_1+\beta_2) & 0 & 0 & 0 & 0 \\ 2\lambda q & 2\lambda & 0 & -(\lambda+\mu) & \beta_2 & 0 & \beta_1 \\ 0 & 0 & 0 & \mu & -(\beta_1+\beta_2) & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & -\mu & \beta_2 \\ 0 & 0 & 0 & 0 & 0 & \mu & -(\beta_1+\beta_2) \end{bmatrix} \begin{bmatrix} P_{21}(s) \\ P_{20}(s) \\ Q_{20}(s) \\ P_{10}(s) \\ Q_{10}(s) \\ P_F(s) \\ Q_F(s) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.18}$$

The following is the mathematical expression for the normalizing condition that arises from the total probability rule.

$$P_{21} + P_{20} + Q_{20} + P_{10} + Q_{10} + P_F + Q_F = 1 \tag{2.19}$$

The steady-state probabilities $P_{21}, P_{20}, Q_{20}, P_{10}, Q_{10}, P_F, Q_F$ can be obtained by solving the system of linear equations (Eqn. 2.18) using the normalizing condition 2.19. This process ensures that the probabilities sum up to 1, reflecting the system's steady-state behavior.

$$P_{21} = \left[\frac{\mu^3 \beta_1{}^3}{\begin{array}{c} 2\,(\beta_1+\beta_2)^3\,\lambda^2\,(2\,\lambda+\nu)+(2\,\lambda+\nu)\,\left((\beta_1+\beta_2)\left(2\,\lambda^2\beta_1+2\,\lambda^2\beta_2\right.\right. \\ \left.\left.+2\,\lambda\,\mu\,\beta_1+2\,\lambda\,\beta_1{}^2+2\,\lambda\,\beta_1\beta_2+\mu\,\beta_1{}^2\right)\mu+\mu^3\beta_1{}^2\right)+2\,(\beta_1+\beta_2) \\ \lambda\,((\mu+\beta_1+\beta_2)\,\lambda+\mu\,\beta_1)\,q\beta_1\mu+\mu^3\beta_1{}^2\,(2\,\lambda\,q+\beta_1) \end{array}} \right] \tag{2.20}$$

$$P_{20} = \left[\frac{(\beta_1+\beta_2)(2\,\lambda+\nu)\mu^2\beta_1{}^2}{\begin{array}{c} 2\,(\beta_1+\beta_2)^3\,\lambda^2\,(2\,\lambda+\nu)+(2\,\lambda+\nu)\,\left((\beta_1+\beta_2)\left(2\,\lambda^2\beta_1+2\,\lambda^2\beta_2\right.\right. \\ \left.\left.+2\,\lambda\,\mu\,\beta_1+2\,\lambda\,\beta_1{}^2+2\,\lambda\,\beta_1\beta_2+\mu\,\beta_1{}^2\right)\mu+\mu^3\beta_1{}^2\right)+2\,(\beta_1+\beta_2) \\ \lambda\,((\mu+\beta_1+\beta_2)\,\lambda+\mu\,\beta_1)\,q\beta_1\mu+\mu^3\beta_1{}^2\,(2\,\lambda\,q+\beta_1) \end{array}} \right] \tag{2.21}$$

$$Q_{20} = \left[\frac{(2\,\lambda+\nu)\mu^3\beta_1{}^2}{\begin{array}{c} 2\,(\beta_1+\beta_2)^3\,\lambda^2\,(2\,\lambda+\nu)+(2\,\lambda+\nu)\,\left((\beta_1+\beta_2)\left(2\,\lambda^2\beta_1+2\,\lambda^2\beta_2\right.\right. \\ \left.\left.+2\,\lambda\,\mu\,\beta_1+2\,\lambda\,\beta_1{}^2+2\,\lambda\,\beta_1\beta_2+\mu\,\beta_1{}^2\right)\mu+\mu^3\beta_1{}^2\right)+2\,(\beta_1+\beta_2) \\ \lambda\,((\mu+\beta_1+\beta_2)\,\lambda+\mu\,\beta_1)\,q\beta_1\mu+\mu^3\beta_1{}^2\,(2\,\lambda\,q+\beta_1) \end{array}} \right] \tag{2.22}$$

$$P_{10} = \left[\frac{2\,(\beta_1+\beta_2)^2(2\,\lambda+\nu)\lambda\,\mu\,\beta_1+2\,\mu^2q\beta_1{}^2\lambda\,(\beta_1+\beta_2)}{\begin{array}{c} 2\,(\beta_1+\beta_2)^3\,\lambda^2\,(2\,\lambda+\nu)+(2\,\lambda+\nu)\,\left((\beta_1+\beta_2)\left(2\,\lambda^2\beta_1+2\,\lambda^2\beta_2\right.\right. \\ \left.\left.+2\,\lambda\,\mu\,\beta_1+2\,\lambda\,\beta_1{}^2+2\,\lambda\,\beta_1\beta_2+\mu\,\beta_1{}^2\right)\mu+\mu^3\beta_1{}^2\right)+2\,(\beta_1+\beta_2) \\ \lambda\,((\mu+\beta_1+\beta_2)\,\lambda+\mu\,\beta_1)\,q\beta_1\mu+\mu^3\beta_1{}^2\,(2\,\lambda\,q+\beta_1) \end{array}} \right] \tag{2.23}$$

$$Q_{10} = \left[\frac{2(\beta_1+\beta_2)\lambda\beta_1\mu^2(2\lambda+\nu)+2\mu^3 q\beta_1{}^2\lambda}{\begin{array}{c} 2(\beta_1+\beta_2)^3\lambda^2(2\lambda+\nu)+(2\lambda+\nu)\Big((\beta_1+\beta_2)\Big(2\lambda^2\beta_1+2\lambda^2\beta_2 \\ +2\lambda\mu\beta_1+2\lambda\beta_1{}^2+2\lambda\beta_1\beta_2+\mu\beta_1{}^2\Big)\mu+\mu^3\beta_1{}^2\Big)+2(\beta_1+\beta_2) \\ \lambda((\mu+\beta_1+\beta_2)\lambda+\mu\beta_1)q\beta_1\mu+\mu^3\beta_1{}^2(2\lambda q+\beta_1) \end{array}} \right] \tag{2.24}$$

$$P_F = \left[\frac{2(\beta_1+\beta_2)^3\lambda^2(2\lambda+\nu)+2\lambda^2\mu q\beta_1(\beta_1+\beta_2)^2}{\begin{array}{c} 2(\beta_1+\beta_2)^3\lambda^2(2\lambda+\nu)+(2\lambda+\nu)\Big((\beta_1+\beta_2)\Big(2\lambda^2\beta_1+2\lambda^2\beta_2 \\ +2\lambda\mu\beta_1+2\lambda\beta_1{}^2+2\lambda\beta_1\beta_2+\mu\beta_1{}^2\Big)\mu+\mu^3\beta_1{}^2\Big)+2(\beta_1+\beta_2) \\ \lambda((\mu+\beta_1+\beta_2)\lambda+\mu\beta_1)q\beta_1\mu\mu^3\beta_1{}^2(2\lambda q+\beta_1) \end{array}} \right] \tag{2.25}$$

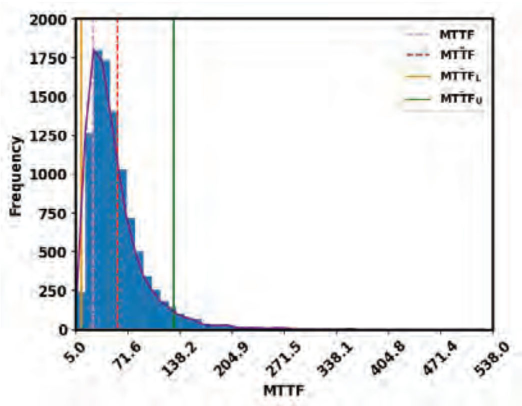
$$Q_F = \left[\frac{2(\beta_1+\beta_2)^2(2\lambda+\nu)\lambda^2\mu+2\mu^2 q\beta_1\lambda^2(\beta_1+\beta_2)}{\begin{array}{c} 2(\beta_1+\beta_2)^3\lambda^2(2\lambda+\nu)+(2\lambda+\nu)\Big((\beta_1+\beta_2)\Big(2\lambda^2\beta_1+2\lambda^2\beta_2 \\ +2\lambda\mu\beta_1+2\lambda\beta_1^2+2\lambda\beta_1\beta_2+\mu\beta_1^2\Big)\mu+\mu^3\beta_1^2\Big)+2(\beta_1+\beta_2) \\ \lambda((\mu+\beta_1+\beta_2)\lambda+\mu\beta_1)q\beta_1\mu+\mu^3\beta_1^2(2\lambda q+\beta_1) \end{array}} \right] \tag{2.26}$$

Therefore, the expression for the availability of the storage system is derived as follows:

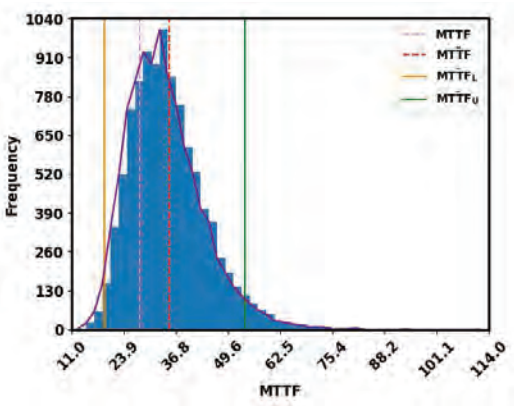
$$A(\infty) = \left[\frac{\begin{array}{c} 2(\beta_1+\beta_2)^2\lambda(\lambda+\beta_1)(2\lambda+\nu)\mu+(\beta_1+\beta_2)\mu^2\beta_1\Big(2\lambda^2 q+2\lambda q\beta_1 \\ +4\lambda^2+2\lambda\nu+2\lambda\beta_1+\nu\beta_1\Big)+\mu^3\beta_1^2(2\lambda q+2\lambda+\nu+\beta_1) \end{array}}{\begin{array}{c} 2(\beta_1+\beta_2)^3\lambda^2(2\lambda+\nu)+(2\lambda+\nu)\Big((\beta_1+\beta_2)\Big(2\lambda^2\beta_1+2\lambda^2\beta_2 \\ +2\lambda\beta_1\mu+2\beta_1^2\lambda+2\lambda\beta_2\beta_1+\beta_1^2\mu\Big)\mu+\mu^3\beta_1^2\Big)+2(\beta_1+\beta_2) \\ \lambda((\mu+\beta_1+\beta_2)\lambda+\mu\beta_1)q\beta_1\mu+\mu^3\beta_1^2(2\lambda q+\beta_1) \end{array}} \right] \tag{2.27}$$

2.5 Bayesian Estimation of reliability characteristics

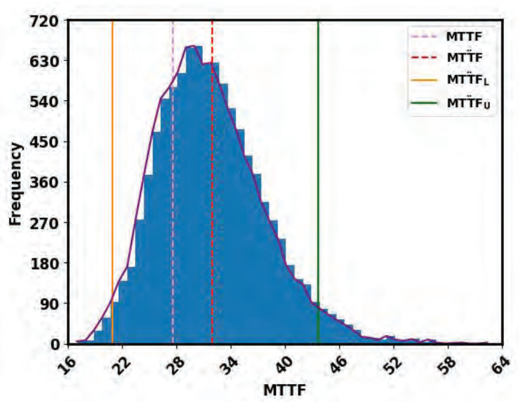
In this section, we present a Bayesian approach for estimating system parameters, including λ, ν, μ, β_1, and β_2. These parameters are initially unknown and require estimation based on suitable prior distributions and empirical data. We start by defining the likelihood function for λ, ν, μ, β_1, and β_2. Next, we consider two appropriate prior distributions: the two-parameter gamma distribution and the beta distribution of the second kind.



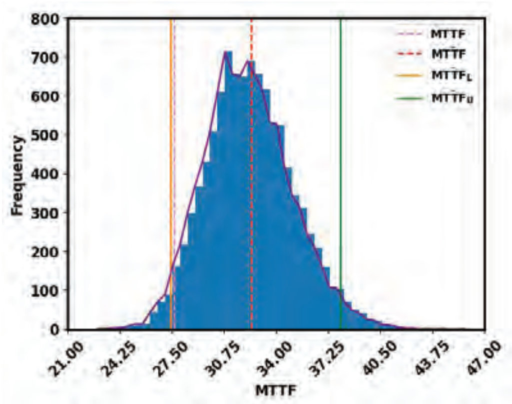
2-Parameter Gamma Prior with $w_1 = 4$, $v_1 = 8$, and $n = 10$



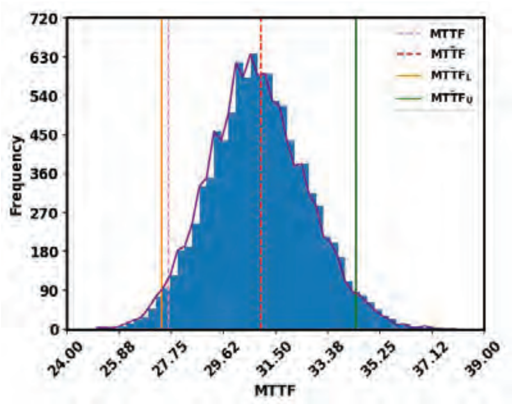
2-Parameter Gamma Prior with $w_1 = 4$, $v_1 = 8$, and $n = 50$



2-Parameter Gamma Prior with $w_1 = 4$, $v_1 = 8$, and $n = 100$



2-Parameter Gamma Prior with $w_1 = 4$, $v_1 = 8$, and $n = 500$



2-Parameter Gamma Prior with $w_1 = 4$, $v_1 = 8$, and $n = 1000$

Figure 2.1: MTTF for Gamma Prior

2.5.1 Likelihood Function

The durations of various events within the system are governed by independent exponential distributed random variables. These events include the time-to-failure of an active storage unit, the time-to-failure of a standby storage unit, and the time-to-rejuvenate, time-to-inspection for both perfect and imperfect rejuvenations. We consider random samples $U_1 = (U_{11}, U_{12}, \ldots U_{1n_1})$ and $U_2 = (U_{21}, U_{22}, \ldots, U_{2n_2})$ of sizes n_1 and n_2, respectively, representing the time between failures of active and standby storage units. Additionally, let $U_3 = (U_{31}, U_{32}, \ldots, U_{3n_3})$ be a collection of random samples of size n_3 denoting the time required to rejuvenate a failed storage unit. Furthermore, let $U_4 = (U_{41}, U_{42}, \ldots, U_{4n_4})$ and $U_5 = (U_{51}, U_{52}, \ldots, U_{5n_5})$ be the random samples of size n_4 and n_5, respectively, representing inspection times during perfect and imperfect rejuvenations. Each sample is drawn from independent exponential distributions. The likelihood function for λ, ν, μ, β_1, and β_2 can be computed using the following formula:

$$L(\lambda, \nu, \mu, \beta_1, \beta_2 | T_1, T_2, T_3, T_4, T_5) = \lambda^{n_1} \nu^{n_2} \mu^{n_3} {\beta_1}^{n_4} {\beta_2}^{n_5} e^{-(\lambda T_1 + \nu T_2 + \mu T_3 + \beta_1 T_4 + \beta_2 T_5)} \tag{2.28}$$

The expression for T_i is given by $T_i = \sum_{j=1}^{n_i} U_{ij}$ for $i = 1,2,3,4,5$. The vector $(T_1, T_2, T_3, T_4, T_5)$ is sufficient for estimating λ, ν, μ, β_1, and β_2.

2.5.2 Two-parameter gamma prior

A suitable choice for the prior distribution for λ is a gamma distribution $G(w_1, v_1)$, which is defined by its probability density function.

$$p(\lambda) = \frac{v_1^{w_1} \lambda^{w_1 - 1} e^{-v_1 \lambda}}{\Gamma(w_1)}, \qquad \text{for } \lambda > 0 \tag{2.29}$$

The prior distribution for λ is denoted by $\lambda \sim G(w_1, v_1)$, where $\Gamma(x)$ represents the gamma function defined as $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}, dt$, and $w_1 > 0$ and $v_1 > 0$ are specified parameters. Here, $E(\lambda) = w_1/v_1$ and $Var(\lambda) = w_1/v_1^2$. According to Bayesian theory, the posterior density of λ given T_1 can be expressed as:

$$h(\lambda | T_1) = \frac{(T_1 + v_1)^{n_1 + w_1} \lambda^{n_1 + w_1 - 1} e^{-\lambda (T_1 + v_1)}}{\Gamma(n_1 + w_1)} \tag{2.30}$$

The posterior density of λ given T_1 corresponds to the probability density of a gamma distribution with parameters $n_1 + w_1, T_1 + v_1$. The Bayesian estimate for λ, denoted by $\widehat{\lambda}_B$, is given by $\frac{n_1+w_1}{T_1+v_1}$, which determines the mean of the posterior distribution. Notably, $\widehat{\lambda}_B$ can be seen as a weighted average of the maximum likelihood estimate, $\frac{n_1}{T_1}$, for λ, and the prior mean, $\frac{w_1}{v_1}$, with corresponding weights $\frac{T_1}{T_1+v_1}$ and $\frac{v_1}{T_1+v_1}$. Consequently, as the sample size n_1 increases, $\widehat{\lambda}_B$ tends to align more closely with the maximum likelihood estimate of λ. It is also observed that the influence of the prior mean diminishes with the increasing sample size n_1, as the weight assigned to the prior mean decreases. This indicates that the effect of hyperparameters on the posterior mean decreases as n_1 increases.

To conduct Bayesian inference, the Metropolis-Hastings algorithm is used to generate random samples from the posterior density function $h(\lambda|T_1)$. The following steps outline the procedure of the algorithm are as below 1:
Let Y be distributed as $h(\lambda|T_1)$, and V be distributed as $G(n_1+w_1, T_1+v_1)$ with the probability density function f_V. Our objective is to generate a sequence of random numbers $Z =< Z_0, Z_1, Z_2, \ldots, Z_N >$ that will serve as a simulation of the posterior distribution as described by Eqn. 2.30.

The pseudo-code for the algorithm is as follows:

1. Generate V from $G(n_1+w_1, T_1+v_1)$. Set $Z_0 = V$.

2. For $i = 1, 2, \ldots, N$

   a. Generate U_i and V_i from uniform $(0,1)$ and $G(n_1+w_1, T_1+v_1)$, respectively, and calculate

$$\rho_i = min\left\{\frac{h(V_i|T_1)}{f_V(V_i)}\frac{f_V(Z_{i-1})}{h(Z_{i-1}|T_1)}, 1\right\}. \tag{2.31}$$

   b. Set

$$Z_i = \begin{cases} V_i, & \text{if } U_i \leq \rho_i \\ Z_{i-1}, & \text{otherwise} \end{cases} \tag{2.32}$$

After a sufficient number of burn-in iterations, the remaining samples can be used to estimate the desired parameters. Similarly, for other parameters such as $\nu \sim G(w_2, v_2), \mu \sim G(w_3, v_3), \beta_1 \sim G(w_4, v_4)$, and $\beta_2 \sim G(w_5, v_5)$, prior distributions are assumed. It is considered that all system parameters have independent prior distributions. Hence, the joint distribution of $\lambda, \nu, \mu, \beta_1$, and β_2 is obtained by multiplying the individual prior distributions for each parameter. Following a similar approach, we derive the joint posterior distribution as follows:

$$\lambda, \nu, \mu, \beta_1, \beta_2 | T_1, T_2, T_3, T_4, T_5 \sim \prod_{i=1}^{5} G(n_i+w_i, T_i+v_i) \tag{2.33}$$

or

$$\{\lambda,\nu,\mu,\beta_1,\beta_2|T_1,T_2,T_3,T_4,T_5\} \propto$$
$$\lambda^{n_1+w_1}\nu^{n_2+w_2}\mu^{n_3+w_3}{\beta_1}^{n_4+w_4}{\beta_2}^{n_5+w_5}e^{-[\lambda(T_1+v_1)+\nu(T_2+v_2)+\mu(T_3+v_3)+\beta_1(T_4+v_4)+\beta_2(T_5+v_5)]} \tag{2.34}$$

and use the Metropolis-Hastings algorithm to obtain random samples of ν,μ,β_1 and β_2 as well.

2.5.3 Beta distribution of second kind

Another suitable prior distribution for λ is a beta distribution of the second kind $BP(m_1,r_1)$ with probability density function

$$g(\lambda)=\frac{\lambda^{m_1-1}}{B(m_1,r_1)(1+\lambda)^{m_1+r_1}}, \qquad \text{for } \lambda>0, m_1>0, r_1>0 \tag{2.35}$$

where $B(a,b)=\dfrac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ denotes the beta function and the suitable values for the hyperparameters m_1 and r_1 are

$$m_1=\frac{\mu_1(\mu_1+\mu_1^2+\sigma_1^2)}{\sigma_1^2} \quad \text{and } r_1=\frac{\mu_1(\mu_1+\mu_1^2+2\sigma_1^2)}{\sigma_1^2} \tag{2.36}$$

The mean and variance of λ in the prior distribution are as follows:

$$\mu_1=E(\lambda)=\frac{m_1}{r_1-1}, \quad \sigma_1^2=\mathrm{Var}(\lambda)=\frac{m_1}{(r_1-1)^2}\cdot\frac{m_1+r_1-1}{r_1-2} \tag{2.37}$$

According to the Bayesian theory, the posterior probability density of λ given T_1 is calculated as follows:

$$h(\lambda|T_1)=\frac{\lambda^{n_1+m_1-1}e^{-\lambda T_1}}{\Gamma(n_1+m_1)U(n_1+m_1,n_1-r_1+1,T_1)(1+\lambda)^{m_1+r_1}} \tag{2.38}$$

The confluent hypergeometric function $U(a,b,z)$ is defined as follows:

$$U(a,b,z)=\frac{1}{\Gamma(a)}\int_0^\infty x^{a-1}(1+x)^{b-a-1}e^{-zx}\,dx \tag{2.39}$$

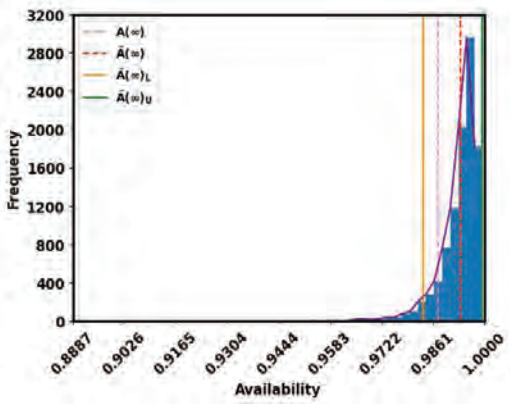
We will employ the Metropolis-Hastings algorithm described in Section 2.5.2 to generate random samples from the posterior density function $h(\lambda|T_1)$. For this prior, we will generate random samples following the distribution $G(n_1+m_1,T_1)$ as outlined as below 1:

Consider Y, which follows the distribution $h(\lambda|T_1)$, and W, distributed as $G(n_1+m_1,T_1)$ with the probability density function f_W. Our objective is to generate a sequence of random numbers $Z =< Z_0,Z_1,Z_2,\ldots,Z_N >$ that will serve as a simulation of the posterior distribution as described by Eqn. 2.38.

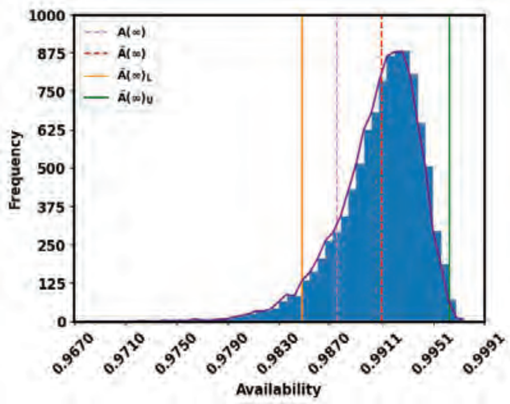
1. Generate W from $G(n_1+m_1,T_1)$. Set $Z_0 = W$.
2. For $i = 1,2,\ldots,N$
   a. Generate U_i and W_i from uniform $(0,1)$ and $G(n_1+m_1,T_1)$, respectively, and calculate
   $$\rho_i = min\left\{\frac{h(W_i|T_1)}{f_W(W_i)}\frac{f_W(Z_{i-1})}{h(Z_{i-1}|T_1)},1\right\}. \quad (2.40)$$
   b. Set
   $$Z_i = \begin{cases} W_i, & \text{if } U_i \le \rho_i \\ Z_{i-1}, & \text{otherwise} \end{cases} \quad (2.41)$$

After a sufficient number of burn-in iterations, the remaining samples can be used to estimate the desired parameters. Similarly, for other parameters such as $\nu \sim BP(m_2,r_2), \mu \sim BP(m_3,r_3), \beta_1 \sim BP(m_4,r_4)$, and $\beta_2 \sim BP(m_5,r_5)$, prior distributions are assumed. It is considered that all the system parameters have independent prior distributions. Hence, the joint distribution of λ,ν,μ,β_1, and β_2 is obtained by multiplying the individual prior distributions for each parameter. Following a similar approach, we derive the joint posterior distribution as follows:

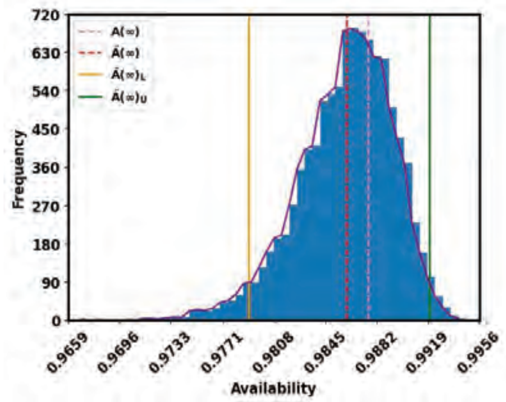
$$\{\lambda,\nu,\mu,\beta_1,\beta_2|T_1,T_2,T_3,T_4,T_5\} \propto \frac{\lambda^{n_1+m_1-1}\nu^{n_2+m_2-1}\mu^{n_3+m_3-1}{\beta_1}^{n_4+m_4-1}{\beta_2}^{n_5+m_5-1}e^{-(\lambda T_1+\nu T_2+\mu T_3+\beta_1 T_4+\beta_2 T_5)}}{(1+\lambda)^{m_1+r_1}(1+\nu)^{m_2+r_2}(1+\mu)^{m_3+r_3}(1+\beta_1)^{m_4+r_4}(1+\beta_2)^{m_5+r_5}} \quad (2.42)$$



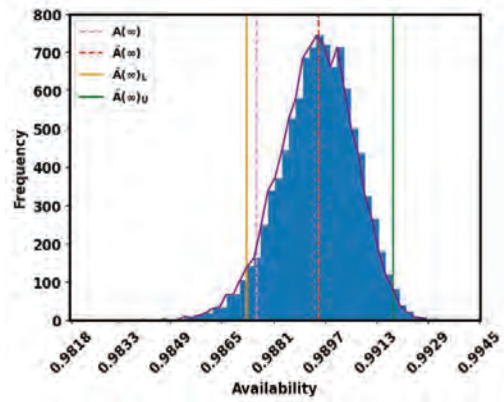
2-Parameter Gamma Prior
with $w_1 = 4, v_1 = 8,$ and $n = 10$



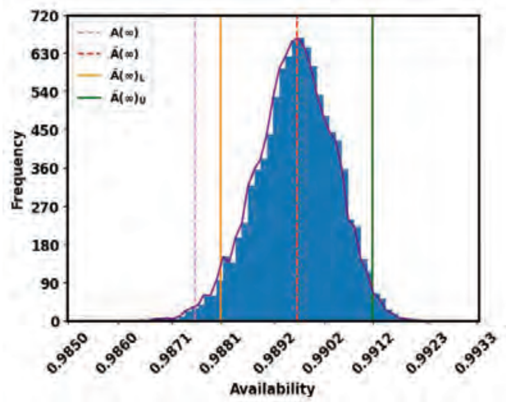
2-Parameter Gamma Prior
with $w_1 = 4, v_1 = 8,$ and, $n = 50$



2-Parameter Gamma Prior
with $w_1 = 4, v_1 = 8,$ and, $n = 100$



2-Parameter Gamma Prior
with $w_1 = 4, v_1 = 8,$ and, $n = 500$



2-Parameter Gamma Prior
with $w_1 = 4, v_1 = 8,$ and, $n = 1000$

Figure 2.2: $A(\infty)$ for Gamma Prior

Monte Carlo simulation methods can be employed to generate values for $\lambda, \nu, \mu, \beta_1$, and β_2 from the joint posterior distributions (2.34) and (2.42). By substituting these values into Eqns. 2.17 and 2.27, we can obtain the MTTF and $A(\infty)$, respectively. Drawing N pairs of (MTTF, $A(\infty)$) allows us to deduce the posterior probability distributions for MTTF and $A(\infty)$ in the redundant storage system. This enables us to compute the posterior mean (PM) and establish the highest posterior density (HPD) intervals. Finally, we conduct numerical comparisons between the posterior and asymptotic performance assessments of MTTF and $A(\infty)$.

Table 2.1: Estimates for MTTF

n	Two-parameter gamma prior								
	(w_2,v_2)=(0.2, 2); (w_3,v_3)=(0.25, 0.05); (w_4,v_4)=(0.2, 0.01); (w_5,v_5)=(0.2, 0.02)								
	$(w_1,v_1)=(1, 2)$			$(w_1,v_1)=(2, 4)$			$(w_1,v_1)=(4, 8)$		
	PM	95% HPD		PM	95% HPD		PM	95% HPD	
10	38.77513	8.79880	88.41238	20.37647	4.61062	44.26413	58.03652	11.93471	129.89640
20	28.37646	9.86603	52.69028	27.33942	9.56065	51.03924	34.92172	12.66506	64.41127
30	17.10610	7.84124	28.16865	22.06975	9.93284	37.22638	34.76448	15.38369	58.32739
40	23.66663	12.07125	37.73457	19.42212	9.73381	30.88027	21.62716	11.05049	33.70226
50	33.31886	18.07983	50.61657	38.63704	21.05113	59.22056	34.90206	19.11225	53.64189
100	39.62346	26.23843	55.16201	30.03778	19.87146	40.96811	32.00108	20.94992	43.64067
300	24.94264	19.85874	29.96692	24.22402	19.46838	29.26775	33.46380	26.63250	40.32767
500	24.81710	21.08440	28.74482	28.00345	23.76649	32.55987	32.45963	27.45427	38.04614
1000	29.27615	26.02347	32.60847	27.33950	24.37346	30.45248	30.97007	27.42766	34.39956

True value of MTTF=27.6434. $\lambda = 0.5, \nu = 0.1, \mu = 5, \beta_1 = 20, \beta_2 = 10$ and $q = 0.1$

Table 2.2: Estimates for $A(\infty)$

n	Two-parameter gamma prior								
	(w_2,v_2)=(0.2, 2); (w_3,v_3)=(0.25, 0.05); (w_4,v_4)=(0.2, 0.01); (w_5,v_5)=(0.2, 0.02)								
	$(w_1,v_1)=(1, 2)$			$(w_1,v_1)=(2, 4)$			$(w_1,v_1)=(4, 8)$		
	PM	95% HPD		PM	95% HPD		PM	95% HPD	
10	0.97862	0.94282	0.99855	0.97500	0.93826	0.99792	0.99354	0.98354	0.99951
20	0.98355	0.96499	0.99680	0.98679	0.97171	0.99775	0.98730	0.97284	0.99770
30	0.97344	0.95010	0.99186	0.97974	0.96087	0.99335	0.98811	0.97778	0.99671
40	0.98355	0.97079	0.99426	0.98527	0.97382	0.99442	0.98082	0.96632	0.99241
50	0.98874	0.98090	0.99539	0.98999	0.98292	0.99606	0.99104	0.98481	0.99636
100	0.99191	0.98790	0.99563	0.98754	0.98156	0.99329	0.98606	0.97899	0.99206
300	0.98573	0.98164	0.98952	0.98467	0.98038	0.98865	0.99026	0.98750	0.99277
500	0.98441	0.98111	0.98763	0.98642	0.98348	0.98917	0.98951	0.98726	0.99182
1000	0.98840	0.98660	0.99010	0.98748	0.98553	0.98931	0.98966	0.98810	0.99119

True value of $A(\infty) = 0.9875808$. $\lambda = 0.5, \nu = 0.1, \mu = 5, \beta_1 = 20, \beta_2 = 10$ and $q = 0.1$

2.6 Numerical illustrations and simulation study

This section analyzes the posterior performances of MTTF and $A(\infty)$ for the repairable redundant storage system through simulation results. To achieve this, we set $n_1 = n_2 = n_3 = n_4 = n_5 = n$ and conduct $10,000$ simulations alongside $1,000$ burn-in

Table 2.3: Estimates for MTTF

n	Beta distribution of the second kind prior								
	(m_2, r_2)=(0.21, 3.1); (m_3, r_3)=(7.5, 2.5); (m_4, r_4)=(15.4, 1.77); (m_5, r_5)=(10.6, 2.06)								
	(m_1, r_1) = (0.875, 2.75)			(m_1, r_1) = (1.4, 3.8)			(m_1, r_1) = (1.85, 4.7)		
	PM	**95% HPD**		**PM**	**95% HPD**		**PM**	**95% HPD**	
10	48.68384	8.89792	108.49190	51.68776	10.31633	115.76100	67.67263	12.53421	152.30040
20	37.13349	12.62753	67.56152	51.59278	16.97870	96.05151	60.89101	21.08990	113.51930
30	37.06706	15.76362	62.31617	33.27675	15.22303	55.96245	49.99936	22.81621	85.86473
40	22.06475	11.80836	34.68400	38.63405	18.78184	62.23411	26.47469	13.28312	41.77789
50	27.25532	14.69239	40.60095	42.49344	22.40990	64.26993	32.89291	17.98850	50.02012
100	30.65912	20.44666	42.02218	36.75366	24.51721	50.48484	20.71058	14.08770	27.99056
300	26.91254	21.67018	32.55875	28.93372	23.27796	34.98927	30.36855	24.08519	36.49637
500	29.15087	24.45674	33.78723	26.48571	22.37118	30.92120	29.61144	24.95037	34.30941
1000	26.40221	23.32730	29.30190	26.37681	23.53415	29.43929	25.40335	22.71340	28.24662

True value of MTTF=27.6434. $\lambda = 0.5, \nu = 0.1, \mu = 5, \beta_1 = 20, \beta_2 = 10$ and $q = 0.1$

Table 2.4: Estimates for A(∞)

n	Beta distribution of the second kind prior								
	(m_2, r_2)=(0.21, 3.1); (m_3, r_3)=(7.5, 2.5); (m_4, r_4)=(15.4, 1.77); (m_5, r_5)=(10.6, 2.06)								
	(m_1, r_1) = (0.875, 2.75)			(m_1, r_1) = (1.4, 3.8)			(m_1, r_1) = (1.85, 4.7)		
	PM	**95% HPD**		**PM**	**95% HPD**		**PM**	**95% HPD**	
10	0.99431	0.98581	0.99957	0.99304	0.98232	0.99948	0.99741	0.99351	0.99975
20	0.99114	0.98147	0.99808	0.99316	0.98563	0.99867	0.99523	0.98988	0.99884
30	0.99048	0.98208	0.99738	0.98756	0.97609	0.99634	0.99401	0.98861	0.99811
40	0.98165	0.96861	0.99310	0.99155	0.98504	0.99723	0.98780	0.97826	0.99511
50	0.98478	0.97468	0.99351	0.99254	0.98760	0.99699	0.98978	0.98297	0.99582
100	0.98971	0.98494	0.99449	0.98991	0.98494	0.99431	0.98276	0.97459	0.99012
300	0.98598	0.98242	0.98999	0.98857	0.98539	0.99152	0.98822	0.98501	0.99138
500	0.98792	0.98531	0.99051	0.98724	0.98448	0.98987	0.98848	0.98594	0.99083
1000	0.98708	0.98507	0.98891	0.98698	0.98503	0.98891	0.98596	0.98388	0.98797

True value of $A(\infty) = 0.9875808$. $\lambda = 0.5, \nu = 0.1, \mu = 5, \beta_1 = 20, \beta_2 = 10$ and $q = 0.1$

simulations for each prior distribution discussed in Section 2.5. During each simulation run, values are generated from the assumed prior distributions. These simulated values serve as parameter values for various time variables such as the time between failures of the active and standby storage units, the time to rejuvenate a failed storage unit, and inspection times during perfect and imperfect rejuvenations. A sample of size n is subsequently drawn from each of the five time-related variables, following which the posterior mean (PM) and the associated highest posterior density (HPD) intervals are computed. The tables provided herein present the mean, alongside the $10,000$ PM values and their estimated standard deviation ($s/\sqrt{10000}$). The samples were generated using appropriate functions of Python 3.10. Subsequently, we compare the PM and HPD results with the asymptotic estimates and confidence intervals derived from the samples generated from the prior distributions. Finally, we use graphical illustrations to demonstrate how MTTF and $A(\infty)$ perform according to the Bayesian approach discussed in this chapter.

Table 2.1 presents the posterior mean (PM) and highest posterior density (HPD) intervals for the MTTF across various sample sizes. The scenarios involve fixed values of $\lambda = 0.5$, $\nu = 0.1$, $\mu = 5$, $\beta_1 = 20$, $\beta_2 = 10$, and $q = 0.1$, with different choices for the hyperparameters under the assumption of a two-parameter gamma prior. These results are compared against the true MTTF value of 27.6434 and $A(\infty)$ value of 0.98758, derived by substituting $(\lambda, \nu, \mu, \beta_1, \beta_2, q) = (0.5, 0.1, 5, 20, 10, 0.1)$ into the respective Eqns. 2.17 and 2.27. Analyzing Table 2.1, it becomes evident that the PM estimates are more stable and accurate, and the HPD intervals are significantly narrower as the sample size increases. This trend is consistent with similar observations made in Table 2.2, which pertains to the steady-state availability of the storage system.

Table 2.3 and 2.4 further extend the analysis by employing the Beta distribution of the second kind as a prior with varying hyperparameter values and sample sizes to obtain Bayesian estimates for MTTF and $A(\infty)$. These tables yield results that align with the patterns observed in Table 2.1 and 2.2. Collectively, across Table 2.1 - 2.4, it becomes clear that larger sample sizes lead to PM estimates that closely approximate the true values, regardless of the specific hyperparameter configurations selected when choosing either the two-parameter gamma prior or the beta distribution of the second kind as a prior.

2.6.1 Comparison of Bayesian and asymptotic estimates

Next, we assess the performance of the PM and HPD intervals in comparison to the asymptotic estimate and asymptotic confidence intervals (ACI) under the application

Table 2.5: PM, HPD intervals and ACI of MTTF

n	PM	St. dev	95% HPD		$\widehat{\textbf{MTTF}}$	s.e.($\widehat{\textbf{MTTF}}$)	95% ACI	
10	20.37647	12.67709	4.61062	44.26413	45.12256	22.44159	1.13704	89.10808
20	27.33942	11.92200	9.56065	51.03924	35.79044	15.86860	4.68798	66.89290
30	22.06975	7.52530	9.93284	37.22638	31.87973	12.95666	6.48468	57.27479
40	19.42212	5.74477	9.73381	30.88027	28.20393	11.22079	6.21117	50.19669
50	38.63704	10.36802	21.05113	59.22056	33.19165	10.03618	13.52072	52.86257
100	30.03778	5.55201	19.87146	40.96811	26.95992	7.09666	13.05047	40.86936
300	24.22402	2.51342	19.46838	29.26775	22.73905	4.09726	14.70843	30.76967
500	28.00345	2.24284	23.76649	32.55987	25.73041	3.17372	19.50992	31.95091
1000	27.33950	1.55393	24.37346	30.45248	29.91742	2.24416	25.51887	34.31597

True value of MTTF=27.6434. $\lambda = 0.5, \nu = 0.1, \mu = 5, \beta_1 = 20, \beta_2 = 10$ and $q = 0.1$;
Two-parameter gamma prior is assumed.

Table 2.6: PM, HPD intervals and ACI of A(∞)

n	PM	St. dev	95% HPD		$\widehat{\textbf{A}}(\infty)$	s.e.($\widehat{\textbf{A}}(\infty)$)	95% ACI	
10	0.97500	0.01857	0.93826	0.99792	0.97325	0.01354	0.94671	0.99979
20	0.98679	0.00773	0.97171	0.99775	0.98073	0.00957	0.96196	0.99949
30	0.97974	0.00920	0.96087	0.99335	0.98215	0.00782	0.96683	0.99747
40	0.98527	0.00576	0.97382	0.99442	0.98660	0.00677	0.97333	0.99987
50	0.98999	0.00357	0.98292	0.99606	0.98668	0.00606	0.97481	0.99854
100	0.98754	0.00313	0.98156	0.99329	0.98609	0.00428	0.97770	0.99448
300	0.98467	0.00212	0.98038	0.98865	0.98419	0.00247	0.97935	0.98904
500	0.98642	0.00147	0.98348	0.98917	0.98652	0.00191	0.98277	0.99028
1000	0.98748	0.00096	0.98553	0.98931	0.98870	0.00135	0.98604	0.99135

True value of $A(\infty) = 0.9875808$. $\lambda = 0.5, \nu = 0.1, \mu = 5, \beta_1 = 20, \beta_2 = 10$ and $q = 0.1$;
Two-parameter gamma prior is assumed.

of the standard gamma prior. The asymptotic estimate, denoted as $\widehat{MTTF}$, is defined as follows:

$$\widehat{MTTF} = \left[\frac{\begin{array}{c}(\overline{U_4}+\overline{U_5})\left((\overline{U_4}+\overline{U_5})\left((-2q+8)\overline{U_1}^2+3\overline{U_1}\,\overline{U_2}\right)+(2\overline{U_1}+\overline{U_2})(3\overline{U_1}+\overline{U_4})\overline{U_3} \\ -\overline{U_1}\,(2\overline{U_1}q-2q\overline{U_4}-\overline{U_4})\overline{U_3}\right)+\overline{U_3}^2\overline{U_4}\,(2\overline{U_1}q+2\overline{U_1}+\overline{U_2}+\overline{U_4})\end{array}}{2\,(2\overline{U_1}+\overline{U_2})\,(\overline{U_4}+\overline{U_5})^2\,\overline{U_1}^2+2\overline{U_3}\,q\overline{U_4}\,(\overline{U_4}+\overline{U_5})\,\overline{U_1}^2} \right] \tag{2.43}$$

Likewise, the asymptotic estimate $\widehat{A}(\infty)$ is defined as follows:

$$\widehat{A}(\infty) = \left[\frac{\begin{array}{c}2\,(\overline{U_4}+\overline{U_5})^2\,\overline{U_1}\,(\overline{U_1}+\overline{U_4})\,(2\overline{U_1}+\overline{U_2})\,\overline{U_3}+(\overline{U_4}+\overline{U_5})\,\overline{U_3}^2\overline{U_4}\Big(2\overline{U_1}^2q+2\overline{U_1}\,q\overline{U_4} \\ +4\overline{U_1}^2+2\overline{U_1}\,\overline{U_2}+2\overline{U_1}\,\overline{U_4}+\overline{U_2}\,\overline{U_4}\Big)+\overline{U_3}^3\overline{U_4}^2\,(2\overline{U_1}q+2\overline{U_1}+\overline{U_2}+\overline{U_4})\end{array}}{\begin{array}{c}2\,(\overline{U_4}+\overline{U_5})^3\,\overline{U_1}^2\,(2\overline{U_1}+\overline{U_2})+(2\overline{U_1}+\overline{U_2})\Big((\overline{U_4}+\overline{U_5})\Big(2\overline{U_1}^2\overline{U_4}+2\overline{U_1}^2\overline{U_5} \\ +2\overline{U_1}\,\overline{U_4U_3}+2\overline{U_4}^2\overline{U_1}+2\overline{U_1}\,\overline{U_5U_4}+\overline{U_4}^2\overline{U_3}\Big)\overline{U_3}+\overline{U_3}^3\overline{U_4}^2\Big)+2\,(\overline{U_4}+\overline{U_5})\,\overline{U_1} \\ \Big((\overline{U_3}+\overline{U_4}+\overline{U_5})\,\overline{U_1}+\overline{U_3}\,\overline{U_4}\Big)q\overline{U_4U_3}+\overline{U_3}^3\overline{U_4}^2\,(2\overline{U_1}q+\overline{U_4})\end{array}} \right] \tag{2.44}$$

where $\overline{U_i} = \dfrac{n_i}{\sum_{j=1}^{n_i} U_{ij}}$, $i = 1,2,3,4$ and 5.

Furthermore, the ACI for MTTF is calculated as $\widehat{MTTF} \pm z_{\alpha/2}\dfrac{\sigma(\widehat{\Theta})}{\sqrt{n}}$, and for $A(\infty)$, it is computed as $\widehat{A}(\infty) \pm z_{\alpha/2}\dfrac{\Phi(\widehat{\Theta})}{\sqrt{n}}$, where $\sigma^2(\widehat{\Theta})$ and $\Phi^2(\widehat{\Theta})$ serve as consistent estimators.

$$\sigma^2(\widehat{\Theta}) = \sum_{i=1}^{5}\left[\frac{\partial MTTF}{\partial \vartheta_i}\right]^2 \vartheta_{ii}^2 \text{ and } \Phi^2(\widehat{\Theta}) = \sum_{i=1}^{5}\left[\frac{\partial A(\infty)}{\partial \vartheta_i}\right]^2 \vartheta_{ii}^2 \text{ with } \Theta = (\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5) = \left(\frac{1}{\lambda}, \frac{1}{\nu}, \frac{1}{\mu}, \frac{1}{\beta_1}, \frac{1}{\beta_2}\right) \tag{2.45}$$

Table 2.5 presents the PM, HPD intervals, and the ACI for $\widehat{MTTF}$ across various sample sizes. These estimates are derived under the standard two-parameter gamma prior with hyperparameters $\omega_1 = 0.5,\ \omega_2 = 0.1,\ \omega_3 = 5,\ \omega_4 = 20,\ \omega_5 = 10$ and fixed values of $\lambda = 0.5,\ \nu = 0.1,\ \mu = 5,\ \beta_1 = 20,\ \beta_2 = 10$ and $q = 0.1$. Comparing these estimates with the true value of 27.6434, the results indicate that as the sample size increases, both the PM and the asymptotic estimate $\widehat{MTTF}$ tend to converge closer to the true value. As anticipated, larger sample sizes lead to narrower spreads in the posterior and asymptotic distributions.

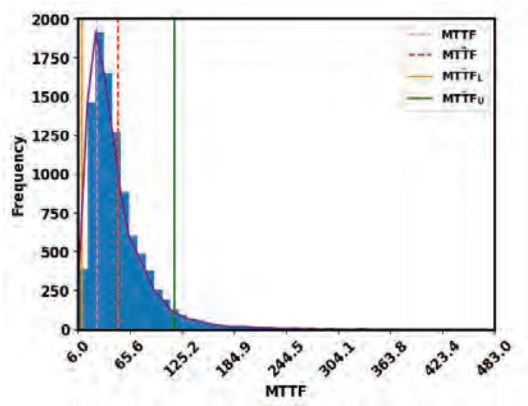
Table 2.6 yields similar insights, with the true value being 0.9875808 for $A(\infty)$. Additionally, it is noteworthy that the HPD intervals provide estimates that closely align with the expected MTTF values, outperforming the asymptotic confidence intervals. Overall, these results emphasize the effectiveness of Bayesian approaches developed in this chapter, particularly when dealing with smaller sample sizes, in constructing more reliable HPD intervals. With larger sample sizes, both Bayesian and asymptotic approaches yield comparable results.

Finally, we present graphical representations of the posterior distributions to visually illustrate the performance of MTTF and $A(\infty)$. For the sake of consistency, Figures 2.1 - 2.4 are based on the parameters $\lambda = 0.5, \nu = 0.1, \mu = 5, \beta_1 = 20, \beta_2 = 10$ and $q = 0.1$. Figures 2.1 depict the posterior distributions for MTTF across sample sizes ranging from 10 to 1000, assuming the two-parameter gamma prior with hyper-parameter values of $w_1 = 4$ and $v_1 = 8$. Similarly, Figures 2.2 display the posterior distribution for $A(\infty)$ within the same sample size range. Furthermore, Figures 2.3 and Fig. 2.4 illustrate the posterior distributions for MTTF and $A(\infty)$ for the beta distribution of the second kind prior.

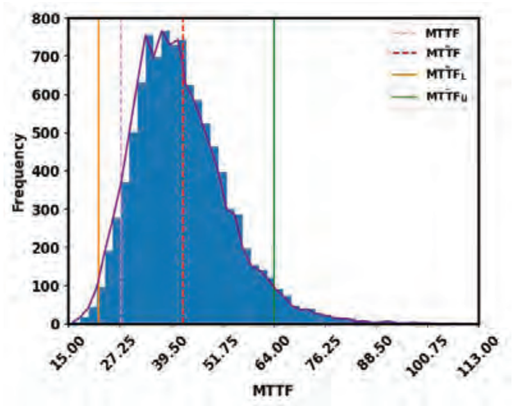
In all these graphs, the true value of the system characteristic is represented in purple, the posterior mean is shown in red, and the lower and upper HPD intervals are presented in orange and green, respectively. It is noteworthy that regardless of the chosen prior, the spreads of the posterior distributions for MTTF and $A(\infty)$ decrease as the sample size increases. Additionally, the histograms for the posterior distributions of MTTF exhibit a slight leftward skew, while those of $A(\infty)$ demonstrate a rightward skew.

In summary:

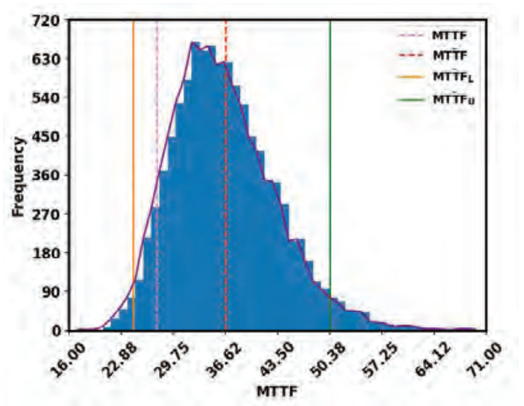
- The study used simulation results to analyze the posterior performance of MTTF and $A(\infty)$ for a repairable redundant storage system.
- Increasing the sample size led to more stable and accurate posterior mean estimates (PM) with narrower highest posterior density (HPD) intervals.
- Both the two-parameter gamma prior and the Beta distribution of the second kind as priors showed consistent trends in PM estimates and HPD intervals.
- Comparison of Bayesian and asymptotic estimates revealed that larger sample sizes resulted in PM estimates closely approximating the true values.
- Graphical illustrations demonstrated that regardless of the prior chosen, the spreads of the posterior distributions for MTTF and $A(\infty)$ decreased as the sample size increased, with slight leftward skew in MTTF histograms and rightward skew in $A(\infty)$ histograms.



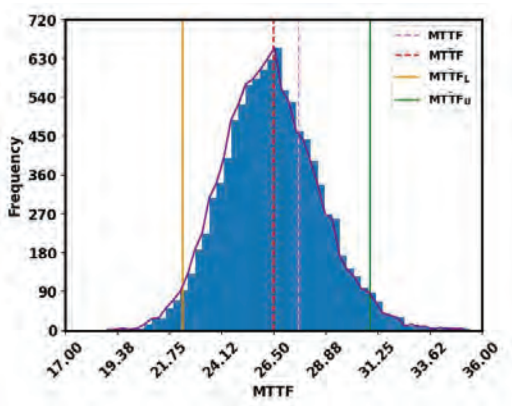
Beta distribution of the second kind
with $m_1 = 1.4, r_1 = 3.8,$ and, $n = 10$



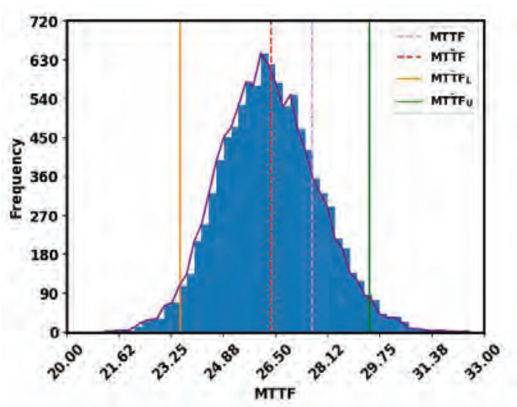
Beta distribution of the second kind
with $m_1 = 1.4, r_1 = 3.8,$ and, $n = 50$



Beta distribution of the second kind
with $m_1 = 1.4, r_1 = 3.8,$ and, $n = 100$

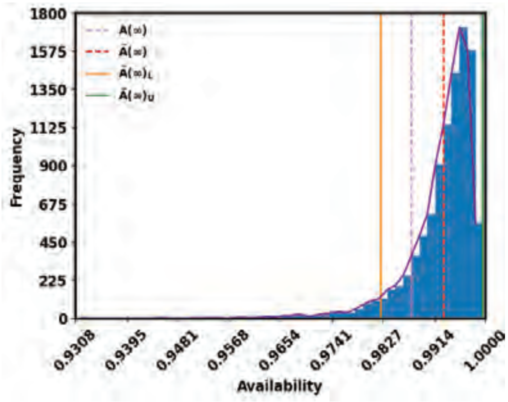


Beta distribution of the second kind
with $m_1 = 1.4, r_1 = 3.8,$ and, $n = 500$

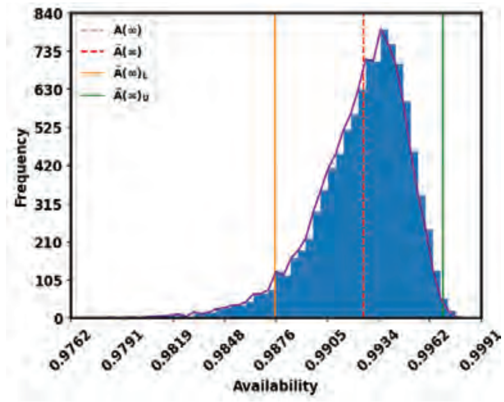


Beta distribution of the second kind
with $m_1 = 1.4, r_1 = 3.8,$ and $n = 1000$

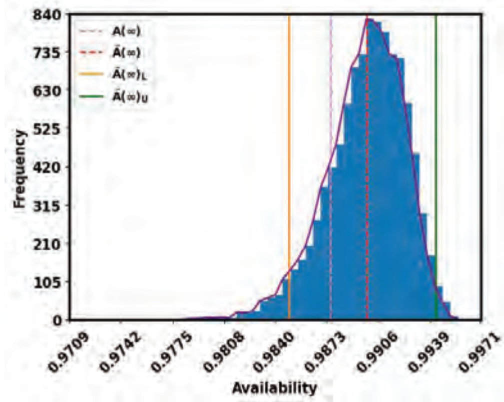
Figure 2.3: MTTF for Beta Prior



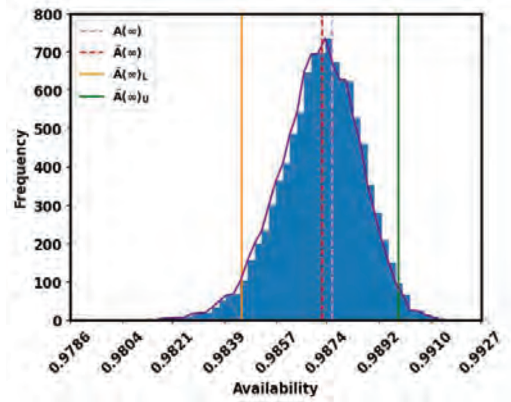
Beta distribution of the second kind
with $m_1 = 1.4, r_1 = 3.8,$ and $n = 10$



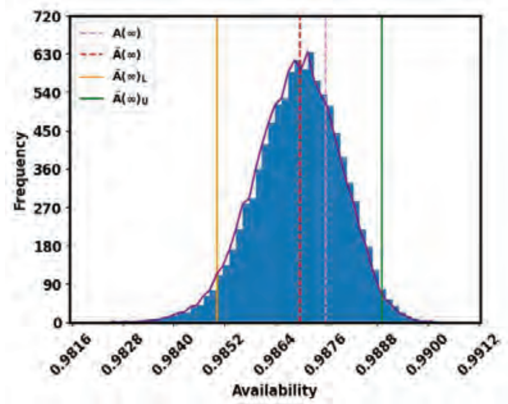
Beta distribution of the second kind
with $m_1 = 1.4, r_1 = 3.8,$ and, $n = 50$



Beta distribution of the second kind
with $m_1 = 1.4, r_1 = 3.8,$ and, $n = 100$



Beta distribution of the second kind
with $m_1 = 1.4, r_1 = 3.8,$ and, $n = 500$



Beta distribution of the second kind
with $m_1 = 1.4, r_1 = 3.8,$ and, $n = 1000$

Figure 2.4: $A(\infty)$ for Beta Prior

2.7 Conclusion

In this chapter, we have explored the realm of redundant storage systems, employing Bayesian inferences and carefully selected prior distributions, particularly when dealing with unpredictable repair services and switching failure. We have evaluated two crucial reliability metrics: mean time-to-failure (MTTF) and availability ($A(\infty)$). Through a combination of simulations and inference techniques, we have provided insights into the intricate workings of these systems, offering valuable insights into their performance. Under the two-parameter gamma and the beta distribution of the second kind, with various hyperparameters, Bayesian methodology has emerged as a reliable tool for estimating MTTF and $A(\infty)$ with remarkably narrow intervals, even with smaller sample sizes. Moreover, as the sample size increases, our Bayesian posterior mean (PM) estimates converge closer to the true values, regardless of the chosen prior. In our final sections, we compared the performance of our PM-HPD intervals against asymptotic estimates, finding that Bayesian methods outperform asymptotic confidence intervals, providing superior and computationally efficient estimates. This is particularly true when analyzing the complex dynamics of repairable redundant storage systems, where unreliable repair and switching failure add a layer of complexity.

In the rapidly evolving landscape of computer technology, where system reliability is paramount, our findings underscore the usefulness of Bayesian methods for engineers and practitioners. They offer a way to gain deeper insights into the performance of repairable redundant storage systems, enabling informed decision-making and robust system design. As computational capabilities continue to advance, the Bayesian paradigm is poised to play a pivotal role in strengthening the resilience and dependability of modern storage infrastructures. With engineering systems becoming increasingly complex and redundancy becoming the norm, our study on the incorporation of unreliable service stations can be extended to any number of active and standby units, each characterized by distinct failure probability distributions. Additionally, the concept can be applied to general storage systems with caching, recovery protocols, rebooting, susceptibility to soft failures, and an unreliable rejuvenation process. As the number of units increases, a promising area for further exploration is the consideration of a multi-unit multi-repair facility. Such a facility could introduce variability through differential repair times and varying levels of certainty associated with achieving a perfect repair outcome. Given the critical role of operational excellence and uninterrupted uptime in enterprise software, further exploration of the intricacies of MTTF and availability, especially when faced with additional unknowns and probabilistic challenges, warrants further study.

Chapter 3

Fuzzified Imperfect Repair Redundant Machine Repair Problem

"Just as every individual is unique and multifaceted, fuzzified machine repair problems reflect the complexity and diversity of human existence, offering opportunities for innovation and growth amidst uncertainty."

Prof. Michael Nguyen

3.1 Introduction

The crucial factor prior to instituting any machining system is the reliability analysis. The uncertainty, vagueness, and impreciseness can be seen in failure, operation, data, and repair facilities. Failure of units hinders the functioning and hence, impacts the cost of operation and quality of the service facility. Reliability and performance can be enhanced by using a the-state-of-the-art redundant and repairable system. The operating capacity of any industry, power plant, manufacturing system, and engineering sector can be improved by managing preventive and corrective measures. The present study is centered on a two-active one-standby unit redundant repairable system with unreliable attributes switching failure and imperfect repair. Various researchers *cf.* [44], Albright [8], Ke and Wang [137], Jain et al. [114] have studied fault-tolerant systems for better system performance and engineering management. The survey and necessity of the machining system for advancement entail exploring the machine repair problems in depth in a vague environment.

Redundancy is crucial for uninterrupted functioning and improving the machining system' reliability and availability. Redundancy is classified as active and passive and requires detailed research due to its importance and cost involvement. According to units' failure rates in inactive states, passive redundancy is categorized into three types: cold, warm, and hot. The concept of a standby unit was introduced by Srinivasan [281]. The comprehensive research (Srinivasan and Subramanian [280], Jain et al. [116], Shekhar et al. [266], Ke et al. [135]) were explored for new methods and techniques to increase the system's reliability. Chen [38] discussed reliability analysis of retrial machine repair systems having standbys and single repairer with recovery policy. Jain et al. [106] studied the maintainability of redundant machining systems having imperfect fault detection and reboot delay. Recently, Yeh et al. [346] analyzed a redundant repair model to maximize its reliability and diminish cost and weight limitations. Due to its great importance, more extensive research needs to study the critical issues of the two-active unit-one-standby redundant machining system in imprecise environments.

When an active unit fails, an available warm standby unit replaces the failed unit in negligible time to prevent interruption in the operation of the machining system. The switchover for a failed unit with an available warm standby unit may be perfect or imperfect. When the switchover process is imperfect, switchover repeats until it becomes successful or available standby units are exhausted. Wang et al. [316] studied a machine repair problem with R repairers and switching failure. Lately, significant work on switching mechanisms has been done by various researchers (Wang et al. [315], Jain et al. [115], Ke et al. [134], Shekhar et al. [260]). More recently, Shekhar et al. [262] investigated the optimal design of fault-tolerant machining systems with various machining hindrances. Due to critical issues involved in switching with the advancement of technological mechanisms, more insightful investigations are required for the uncertain environment.

When an active/standby unit fails, it is sent to a repair facility for repair without losing time so that the system can avoid hindrance to the service facility. In real-life practice, the repair is only sometimes perfect or precise. It may be imperfect due to insufficient faults data, more load on the repairer, vagueness in detecting fault rightly, etc. A repairer provides unsuccessful efforts as many times before successfully repairing. This notion of repair facility is referred to as imperfect repair in queueing literature that was first conceptualized by Patterson and Korzeniowski [226]. Again, Patterson and Korzeniowski [227] extended their previous research for a server working vacation and derived the probability-generating function of the stationary queue length and Laplace Stieltjes transform of the stationary waiting time for the Markovian queue. Due to prevalent imperfect repair issues, it is also conceptualized in machine repair problems. Recently Shekhar et al. [262] presented the mathematical study of the stochastic model of a machine repair problem of standbys provisioning in Markovian conditions with unreliable service and vacation interruption. The imperfect repair in MRP with a two-unit-one-standby machine system still needs to be investigated in the literature. It needs in-depth study in a vague, imperfect, imprecise, and uncertain environment.

The past study of machine repair problems was based on the inter-failure times and times-to-repair required to follow a specific distribution with certain parameters. However, due to a vague, imprecise, uncertain functioning environment, the failure and repair patterns are seldom known. The failure and repair pattern more suitably designate subjectively in the etymological terms such as slow, fast, moderate. Fuzzy set theory, attributing a degree to which a specific event belongs to a set, is appropriate to deal with such vagueness, imprecise, and uncertainties. The multilateral approach of defining the imprecision in language, vague pieces of information, uncertainty in nature, an approximate estimation of governing parameters, and possibilities instead of probabilistic expostulates us to develop an alternative method (Zadeh, [358], Dubois et al.,[55], Dubois et al., [56], Buckley [22], Buckley et al., [21]. Chen [36] investigated mathematical programming techniques to develop the membership grade function of the performance measure of the machine interference system in a fuzzy occurrence. Huang et al. [97] examined the reliability analysis of a two-unit machine repair problem with a single warm standby and demonstrated the practicality of the proposed technique. Liu et al. [187] employed parametric programming techniques to obtain a fuzzy Markov model's reliability and performance measures where the system's parameters have linguistic impreciseness. Shekhar et al. [259] analyzed availability characteristics of multi-active and standby units with switching failure and reboot delay employing parametric nonlinear programming with γ-cut. Kumar et al. [159] studied the fault tolerance system in which a repair facility is based on optimal N-policy in a fuzzy environment. The fuzzy analysis of a two-unit-one-standby fault-tolerant machining system with imperfect repair and switching failure is essential to study the reliability behavior in a vague, imprecise, and uncertain environment.

The studied model has a lot of potential applications in real-time technology-based machining systems. The importance of redundancy and imperfection in switching and repair is of a great deal in all application sectors like cloud movement prediction in satellite images (Son et al. [278]), wind turbine blade structure (Sarkar and Gunturi [249], Singh and Rizwan [270]), electric vehicle operation (Kim et al. [146], Suhail et al. [287], Venugopal et al. [305], Tang and You [291]), load forecasting (Sengar and Liu [250], Tian [297]), battery charge monitoring (Ganguly et al. [64]). For the analysis of the application sector, soft computing-based methodologies are used to deal with the high level of complexity due to involved technological advancement.

Many researchers have investigated a wide range of work on the redundant two-unit repairable system with different failure/service notions in fuzzy environments. A few of them have been discussed in our literature review above. The underlying research gaps are identified as (i) no research on imperfect repair in redundant multi-unit machining systems, (ii) the real-time vagueness, ambiguity, impreciseness, uncertainty in failures, and imperfect repairs are not proposed earlier. The proposed model is different from the above discussed and other models in literature because it endows the following characteristic: (i) a study of the imperfect repair facility in machine repair problem (MRP), (ii) a convenient estimation value from the vague environment, (iii) a correlation between fuzzy theory and the conventional method. Looking at high-grade applications of the studied model and their sensitivity issues, the prime objective is to deal with the impreciseness, vagueness, and uncertainties involved. The underlying contribution of the present study is to investigate imperfect repair and switch failure in redundant machining systems with active and passive redundancy in a vague and imprecise environment. The study is essential for preventive measures, corrective measures, and predictive measures for the machining systems.

We categorized our work in multiple sections: The notations, assumptions, and mathematical formulation of the proposed machining model have been presented in section 3.2 along with associated Chapman-Kolmogorov differential-difference equations in transient-state. Further, in section 3.3, we employ the mathematical concept of Laplace transform, linear algebra, and the hazard failure rate to derive the reliability characteristics, viz. mean time-to-failure and availability for the redundant repairable system with an imperfect repair and switching failure. In section 3.4, we extend the applicable rate and probabilities in fuzzy natures and briefly introduce the methodology for obtaining the corresponding membership grade functions for the MTTF and the system's availability. In section 3.5, a parametric nonlinear programming technique is discussed to derive the membership grade functions for the reliability characteristics. Further, we present a numerical illustration to develop the validity of our proposed concept and methodology in section 3.6. We also present numerical illustrations for in-depth tractability. In the last section 3.7, the article's conclusion and future scope are discussed.

3.2 Model Description

We investigate a machine repair problem with two identical active units functioning simultaneously in parallel with the redundancy of a single standby unit under the supervision of a single reliable repairer. We delineate the following assumptions and notations utilized for the detailed illustrations of the

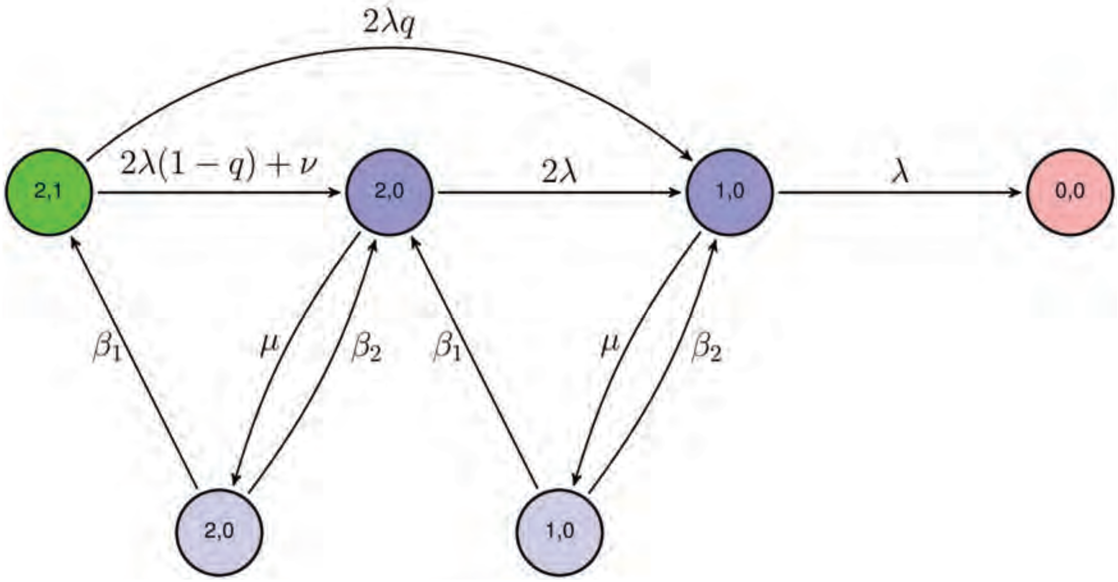


Figure 3.1: Transition diagram for reliability analysis

Failure Process

- The time-to-failure of an active and a warm standby unit follow an exponential distribution with the rate λ and ν $(0 < \nu < \lambda)$, respectively. The failure of active units and a warm standby unit are independent.
- When an active unit fails, it is immediately supplanted by an available warm standby unit. After the switchover, the standby unit's operational and failure characteristics are as same as the active unit.

Switching Process

- The switchover time of a failed unit with an available standby unit is assumed to be negligible.
- The switchover of the warm standby unit to the failed active unit may not always be successful due to mishandling, defects in the unit, or unreliable automation. The probability of unsuccessful switching is q, whereas its complementary $\bar{q}$ denotes the probability of successful switching.

Repair Process

- A failed unit is instantly sent to the repair facility; if the repairer is idle, it instantly begins its repair; otherwise, the failed unit joins the queue and waits for its turn.

- The time-to-repair follows an exponential distribution with the meantime $\frac{1}{\mu}$.
- The repair may be perfect or imperfect. The time-to-inspect the perfect and imperfect repair follows an exponential distribution with the mean rate β_1 and β_2, respectively.
- The repaired unit will function as same as the new unit. After recovering, it is instantly installed as the active unit or a warm standby unit as per the system's requirements.

The system's state at instant t to develop the forward Chapman-Kolmogorov equations of the proposed redundant repairable system is described as follows.

$I_1(t) \equiv$ The number of the active units in the system at instant t.
$I_2(t) \equiv$ The number of the standby units in the system at instant t.

With the support of the pre-defined state of the machining system, the process $\{I_1(t), I_2(t)\}$, stochastic in nature, describes the continuous-time Markov chain (*CTMC*) in the state space $\Pi = \{(I_1(t) = i, I_2(t) = j \mid i = 0, 1, 2 \text{ and } j = 0, 1\}$. The present analysis is based on the assumption that the system fails when there is no active unit. The state probabilities are introduced as follows.

$P_{21}(t) = \text{Prob}[I_1(t) = 2, I_2(t) = 1] =$ likelihood that there are two active units and a single warm standby unit in the machining system at any moment t

$P_{20}(t) = \text{Prob}[I_1(t) = 2, I_2(t) = 0] =$ likelihood that there are two active units in the machining system at any moment t.

$Q_{20}(t) = \text{Prob}[I_1(t) = 2, I_2(t) = 0] =$ likelihood that there are two active units and repaired unit is under inspection at any moment t.

$P_{10}(t) = \text{Prob}[I_1(t) = 1, I_2(t) = 0] =$ likelihood that there is one active unit in the machining system at any moment t.

$Q_{10}(t) = \text{Prob}[I_1(t) = 1, I_2(t) = 0] =$ likelihood that there is one active unit and re- -paired unit is under inspection at any moment t.

$P_{00}(t) = \text{Prob}[I_1(t) = 0, I_2(t) = 0] =$ likelihood that there is no active unit in the machining system at any moment t.

Hence, from the state transition diagram for reliability analysis in Fig.3.1, the following governing set of forward Chapman-Kolmogorov differential-difference equations are developed in terms of λ, ν, μ, β_1, β_2, q, and $\bar{q}$ by balancing input-output rate flow as follows.

$$\left(\frac{d}{dt}+2\lambda+\nu\right)P_{21}(t)=\beta_1 Q_{20}(t) \tag{3.1}$$

$$\left(\frac{d}{dt}+2\lambda+\mu\right)P_{20}(t)=(2\lambda\bar{q}+\nu)P_{21}(t)+\beta_2 Q_{20}(t)+\beta_1 Q_{10}(t) \tag{3.2}$$

$$\left(\frac{d}{dt}+\lambda+\mu\right)P_{10}(t)=2\lambda P_{20}(t)+2\lambda q P_{21}(t)+\beta_2 Q_{10}(t) \tag{3.3}$$

$$\left(\frac{d}{dt}\right)P_{00}(t)=\lambda P_{10}(t) \tag{3.4}$$

$$\left(\frac{d}{dt}+\beta_1+\beta_2\right)Q_{20}(t)=\mu P_{20}(t) \tag{3.5}$$

$$\left(\frac{d}{dt}+\beta_1+\beta_2\right)Q_{10}(t)=\mu P_{10}(t) \tag{3.6}$$

At the initial instant $t=0$, there is no failed unit in the system, i.e., the system has two active units with a single warm standby unit functioning properly. Hence, $P_{21}(0)=1$ and $P_{20}(0)=Q_{20}(0)=P_{10}(0)=Q_{10}(0)=P_{00}(0)=0$ are the initial conditions.

3.3 The system reliability characteristics

3.3.1 Mean time-to-failure

We employ the mathematical theory of Laplace transformation to derive the expression of transient state probabilities analytically from the set of the earlier section's differential-difference equations (3.1)-(3.6) with initial conditions. The Laplace transformation of state probability and its derivative can be defined as.

$$L\left(P_{ij}(t)\right)=\ddot{P}_{ij}(u)=\int_0^\infty e^{-ut}P_{ij}(t)dt \qquad \forall \quad i,j$$

$$L\left(\frac{d}{dt}P_{ij}(t)\right)=u\ddot{P}_{ij}(u)-P_{ij}(0) \qquad \forall \quad i,j$$

We get the set of linear equations (3.7)-(3.12) after employing Laplace transform on the set of governing differential-difference equations (3.1)-(3.6) in the following manner.

$$(u+2\lambda+\nu)\ddot{P}_{21}(u)-1=\beta_1\ddot{Q}_{20}(u) \tag{3.7}$$

$$(u+2\lambda+\mu)\ddot{P}_{20}(u)=(2\lambda\bar{q}+\nu)\ddot{P}_{21}(u)+\beta_2\ddot{Q}_{20}(u)+\beta_1\ddot{Q}_{10}(u)=0 \tag{3.8}$$

$$(u+\lambda+\mu)\ddot{P}_{10}(u)=2\lambda\ddot{P}_{20}(u)+2\lambda q\ddot{P}_{21}(u)+\beta_1\ddot{Q}_{10}(u)=0 \tag{3.9}$$

$$u\ddot{P}_{00}(u)=\lambda\ddot{P}_{10}(u)=0 \tag{3.10}$$

$$(u+\beta_1+\beta_2)\ddot{Q}_{20}(u)=\mu\ddot{P}_{20}(u)=0 \tag{3.11}$$

$$(u+\beta_1+\beta_2)\ddot{Q}_{10}(u)=\mu\ddot{P}_{10}(u)=0 \tag{3.12}$$

The state probabilities in the transformed form are obtained by solving the above linear equations system (3.7)-(3.12) as follows.

$$\ddot{P}_{21}(u)=\left[\frac{\begin{array}{c}(u+\beta_1+\beta_2)^2(u+2\lambda+\mu)(u+\lambda+\mu)-\Big((2u+3\lambda+\mu)\beta_2{}^2+(2u^2+\\(3\lambda+2\mu+2\beta_1)u+\beta_1(5\lambda+2\mu))\beta_2+2\lambda\beta_1(u+\beta_1)\Big)\mu\end{array}}{\begin{array}{c}(u+\beta_1+\beta_2)\Big((u+2\lambda+\nu)\Big((u+\lambda+\mu)\Big(u^2+(2\lambda+\mu+\beta_1+\beta_2)u\\+(2\lambda-\mu)\beta_2+2\lambda\beta_1\Big)-\lambda\mu(2\beta_1+\beta_2)\Big)+\mu\Big((u+\lambda+\mu)\\(u+2\lambda q)\beta_1+\beta_2{}^2\mu\Big)\Big)+(-\beta_2{}^2+\beta_2(2\lambda+\nu)-2\lambda q\beta_1)(\beta_1+\beta_2)\mu^2\end{array}}\right] \tag{3.13}$$

$$\ddot{P}_{20}(u)=\left[\frac{\begin{array}{c}(u+\beta_1+\beta_2)^2(2\lambda\bar{q}+\nu)(u+\lambda+\mu)-((2\lambda\bar{q}+\nu)\beta_2-2\lambda\beta_1q)\\\mu(u+\beta_1+\beta_2)\end{array}}{\begin{array}{c}(u+\beta_1+\beta_2)\Big((u+2\lambda+\nu)\Big((u+\lambda+\mu)\Big(u^2+(2\lambda+\mu+\beta_1+\beta_2)u\\+(2\lambda-\mu)\beta_2+2\lambda\beta_1\Big)-\lambda\mu(2\beta_1+\beta_2)\Big)+\mu\Big((u+\lambda+\mu)\\(u+2\lambda q)\beta_1+\beta_2{}^2\mu\Big)\Big)+(-\beta_2{}^2+\beta_2(2\lambda+\nu)-2\lambda q\beta_1)(\beta_1+\beta_2)\mu^2\end{array}}\right] \tag{3.14}$$

$$\ddot{Q}_{20}(u)=\left[\frac{\begin{array}{c}2(u+\beta_1+\beta_2)(2\lambda\bar{q}+\nu)\mu(u+\lambda+\mu)-\Big((2\lambda\bar{q}+\nu)\beta_2\\-2\lambda\beta_1q\Big)\mu^2\end{array}}{\begin{array}{c}(u+\beta_1+\beta_2)\Big((u+2\lambda+\nu)\Big((u+\lambda+\mu)\Big(u^2+(2\lambda+\mu+\beta_1+\beta_2)u\\+(2\lambda-\mu)\beta_2+2\lambda\beta_1\Big)-\lambda\mu(2\beta_1+\beta_2)\Big)+\mu\Big((u+\lambda+\mu)\\(u+2\lambda q)\beta_1+\beta_2{}^2\mu\Big)\Big)+(-\beta_2{}^2+\beta_2(2\lambda+\nu)-2\lambda q\beta_1)(\beta_1+\beta_2)\mu^2\end{array}}\right] \tag{3.15}$$

$$\ddot{P}_{10}(u) = \left[\frac{\lambda\,(u+\beta_1+\beta_2)\,\big((u+\beta_1+\beta_2)\,\big((u+\mu)\,q+2\lambda+\nu\big)-\mu\,\beta_2 q\big)}{\begin{array}{c}(u+\beta_1+\beta_2)\Big((u+2\lambda+\nu)\Big((u+\lambda+\mu)\Big(u^2+(2\lambda+\mu+\beta_1+\beta_2)\,u\\ +(2\lambda-\mu)\,\beta_2+2\lambda\,\beta_1\Big)-\lambda\,\mu\,(2\beta_1+\beta_2)\Big)+\mu\Big((u+\lambda+\mu)\\ (u+2\lambda\,q)\,\beta_1+{\beta_2}^2\mu\Big)\Big)+\big(-{\beta_2}^2+\beta_2\,(2\lambda+\nu)-2\lambda\,q\beta_1\big)\,(\beta_1+\beta_2)\,\mu^2\end{array}} \right] \tag{3.16}$$

$$\ddot{Q}_{10}(u) = \left[\frac{2\lambda\,\mu\,\Big(qu^2+\big((\mu+\beta_1+\beta_2)\,q+\nu+2\lambda\big)u+\mu\,\beta_1 q+(\beta_1+\beta_2)\,(2\lambda+\nu)\Big)}{\begin{array}{c}(u+\beta_1+\beta_2)\Big((u+2\lambda+\nu)\Big((u+\lambda+\mu)\Big(u^2+(2\lambda+\mu+\beta_1+\beta_2)\,u\\ +(2\lambda-\mu)\,\beta_2+2\lambda\,\beta_1\Big)-\lambda\,\mu\,(2\beta_1+\beta_2)\Big)+\mu\Big((u+\lambda+\mu)\\ (u+2\lambda\,q)\,\beta_1+{\beta_2}^2\mu\Big)\Big)+\big(-{\beta_2}^2+\beta_2\,(2\lambda+\nu)-2\lambda\,q\beta_1\big)\,(\beta_1+\beta_2)\,\mu^2\end{array}} \right] \tag{3.17}$$

$$\ddot{P}_{00}(u) = \left[\frac{\begin{array}{c}\lambda^2\,(u+\beta_1+\beta_2)\Big(qu^2+(q\beta_1+(\mu+\beta_2)\,q+\nu+2\lambda)\,u+(\mu\,q+2\lambda+\nu)\,\beta_1\\ +\beta_2\,(2\lambda+\nu)\Big)\end{array}}{\begin{array}{c}u\Big((u+\beta_1+\beta_2)\Big((u+2\lambda+\nu)\Big((u+\lambda+\mu)\Big(u^2+(2\lambda+\mu+\beta_1+\beta_2)\,u\\ +(2\lambda-\mu)\,\beta_2+2\lambda\,\beta_1\Big)-\lambda\,\mu\,(2\beta_1+\beta_2)\Big)+\mu\Big((\lambda+\mu+u)\\ (2\lambda\,q+u)\,\beta_1+{\beta_2}^2\mu\Big)\Big)+\big(-{\beta_2}^2+\beta_2\,(2\lambda+\nu)-2\lambda\,\beta_1 q\big)\,(\beta_1+\beta_2)\,\mu^2\Big)\end{array}} \right] \tag{3.18}$$

The transient state probabilities $P_{21}(t)$, $P_{20}(t)$, $Q_{20}(t)$, $P_{10}(t)$, $Q_{10}(t)$ and $P_{00}(t)$ of the system at any instant t can be derived on taking inverse Laplace transform for $\ddot{P}_{21}(u)$, $\ddot{P}_{20}(u)$, $\ddot{Q}_{20}(u)$, $\ddot{P}_{10}(u)$, $\ddot{Q}_{10}(u)$ and $\ddot{P}_{00}(u)$ respectively. The state (0,0) represents that there is no active units in the system, i.e., $P_{00}(t)$ denotes the probability that the system completely fails at instant t. Let X be the continuous random variable denoting the time-to-failure of the system, then $R_X(t)$ represents the reliability of the system and defined as

$$R_X(t) = \text{Prob (the machining system function adequately for a specified period of time)} \tag{3.19}$$

$$= 1 - P_{00}(t), \quad t \geq 0 \tag{3.20}$$

From the reliability theory, the failure density function $X(t)$ is defined as:

$$X(t) = -\frac{d}{dt}R_X(t) = -\frac{d}{dt}(1-P_{00}(t)) = \frac{d}{dt}P_{00}(t) \tag{3.21}$$

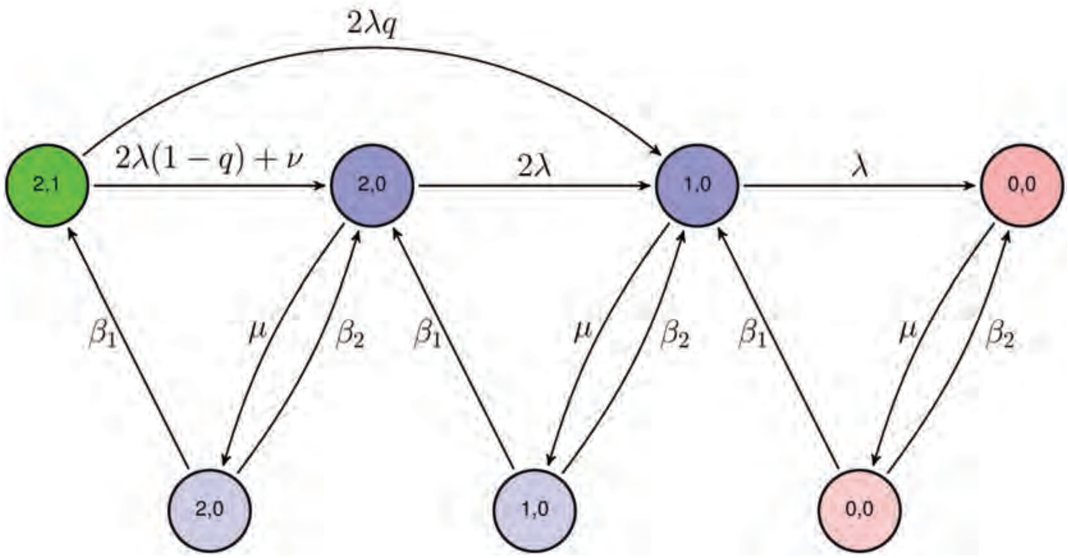


Figure 3.2: Transition diagram for availability analysis

The Laplace transform of failure density is deduced as $\ddot{X}(u) = u\ddot{P}_{00}(u) - P_{00}(0)$. Hence, the system's mean time-to-failure is obtained as follows.

$$\begin{aligned} T =& MTTF = -\frac{d}{du}\ddot{X}(u)\bigg|_{u=0} \\ =& \frac{1}{2}\left[\frac{\begin{array}{c}(2\lambda+\nu)\left((3\lambda+\mu)\beta_1+3\lambda\beta_2\right)(\mu+\beta_1+\beta_2)+2\left(-\mu q+(\beta_1+\beta_2)\bar{q}\right)\\ \lambda^2(\beta_1+\beta_2)+\mu\left((2q+1)\beta_1+2\mu q+\beta_2(2q+1)\right)\beta_1\lambda+\mu^2{\beta_1}^2\end{array}}{\left((\mu q+2\lambda+\nu)\beta_1+\beta_2(2\lambda+\nu)\right)(\beta_1+\beta_2)\lambda^2}\right] \end{aligned} \tag{3.22}$$

3.3.2 Availability of the system

This subsection is enriched with another reliability characteristic, the availability of the redundant repairable system with switching failure and imperfect repair. The forward Chapman Kolmogorov equations are derived in the form of the system of the linear equations from Fig. 3.2. We quest the steady-state probabilities of the system using normalizing condition of probability. The system's availability is derived after getting the steady-state probabilities.

In this sequel, the steady-state forward Chapman Kolmogorov equations are developed by balancing inflow-outflow rates for each state as follows.

$$(2\lambda+\nu)P_{21} = \beta_1 Q_{20} \tag{3.23}$$

$$(2\lambda+\mu)P_{20} = (2\lambda\bar{q}+\nu)P_{21}+\beta_2 Q_{20}+\beta_1 Q_{10} \tag{3.24}$$

$$(\lambda+\mu)P_{10} = 2\lambda P_{20}+2\lambda q P_{21}+\beta_2 Q_{10}+\beta_1 Q_{00} \tag{3.25}$$

$$\mu P_{00} = \lambda P_{10} + \beta_2 Q_{00} \tag{3.26}$$

$$(\beta_1 + \beta_2) Q_{20} = \mu P_{20} \tag{3.27}$$

$$(\beta_1 + \beta_2) Q_{10} = \mu P_{10} \tag{3.28}$$

$$(\beta_1 + \beta_2) Q_{00} = \mu P_{00} \tag{3.29}$$

The normalizing condition from the total probability rule for steady-state probabilities is written below as

$$P_{21} + P_{20} + Q_{20} + P_{10} + Q_{10} + P_{00} + Q_{00} = 1 \tag{3.30}$$

We obtain the steady-state transition probabilities P_{21}, P_{20}, Q_{20}, P_{10}, Q_{10}, P_{00}, and, Q_{00} by solving the system of linear equations (3.23)-(3.29) using the normalizing condition (3.30).

$$P_{21} = \left[\frac{\mu^3 {\beta_1}^3}{2\,(2\lambda + \nu + \mu\, q)\left(\beta_1\,(\lambda + \mu) + \lambda\,\beta_2\right)\lambda\beta_1\,(\mu + \beta_1 + \beta_2) + (2\lambda + \nu)\left(\mu^2 {\beta_1}^2 + 2\lambda\,\beta_2\,(\lambda + \mu)\,\beta_1 + 2\lambda^2 {\beta_2}^2\right)(\mu + \beta_1 + \beta_2) + \mu^3 {\beta_1}^3} \right] \tag{3.31}$$

$$P_{20} = \left[\frac{(\beta_1 + \beta_2)(2\lambda + \nu)\mu^2 {\beta_1}^2}{2\,(2\lambda + \nu + \mu\, q)\left(\beta_1\,(\lambda + \mu) + \lambda\,\beta_2\right)\lambda\beta_1\,(\mu + \beta_1 + \beta_2) + (2\lambda + \nu)\left(\mu^2 {\beta_1}^2 + 2\lambda\,\beta_2\,(\lambda + \mu)\,\beta_1 + 2\lambda^2 {\beta_2}^2\right)(\mu + \beta_1 + \beta_2) + \mu^3 {\beta_1}^3} \right] \tag{3.32}$$

$$Q_{20} = \left[\frac{(2\lambda + \nu)\mu^3 {\beta_1}^2}{2\,(2\lambda + \nu + \mu\, q)\left(\beta_1\,(\lambda + \mu) + \lambda\,\beta_2\right)\lambda\beta_1\,(\mu + \beta_1 + \beta_2) + (2\lambda + \nu)\left(\mu^2 {\beta_1}^2 + 2\lambda\,\beta_2\,(\lambda + \mu)\,\beta_1 + 2\lambda^2 {\beta_2}^2\right)(\mu + \beta_1 + \beta_2) + \mu^3 {\beta_1}^3} \right] \tag{3.33}$$

$$P_{10} = \left[\frac{2((\mu\, q + 2\lambda + \nu)\beta_1 + \beta_2(2\lambda + \nu))\beta_1(\beta_1 + \beta_2)\lambda\,\mu}{2\,(2\lambda + \nu + \mu\, q)\left(\beta_1\,(\lambda + \mu) + \lambda\,\beta_2\right)\lambda\beta_1\,(\mu + \beta_1 + \beta_2) + (2\lambda + \nu)\left(\mu^2 {\beta_1}^2 + 2\lambda\,\beta_2\,(\lambda + \mu)\,\beta_1 + 2\lambda^2 {\beta_2}^2\right)(\mu + \beta_1 + \beta_2) + \mu^3 {\beta_1}^3} \right] \tag{3.34}$$

$$Q_{10} = \left[\frac{2((2\lambda + \mu\, q + \nu)\beta_1 + \beta_2(2\lambda + \nu))\beta_1\lambda\,\mu^2}{2\,(2\lambda + \nu + \mu\, q)\left(\beta_1\,(\lambda + \mu) + \lambda\,\beta_2\right)\lambda\beta_1\,(\mu + \beta_1 + \beta_2) + (2\lambda + \nu)\left(\mu^2 {\beta_1}^2 + 2\lambda\,\beta_2\,(\lambda + \mu)\,\beta_1 + 2\lambda^2 {\beta_2}^2\right)(\mu + \beta_1 + \beta_2) + \mu^3 {\beta_1}^3} \right] \tag{3.35}$$

$$P_{00}=\left[\frac{2((\mu q+2\lambda+\nu)\beta_1+\beta_2(2\lambda+\nu))(\beta_1+\beta_2)^2\lambda^2}{\begin{array}{c}2\,(2\lambda+\nu+\mu q)\left(\beta_1\,(\lambda+\mu)+\lambda\,\beta_2\right)\lambda\beta_1\,(\mu+\beta_1+\beta_2)+(2\lambda+\nu)\left(\mu^2{\beta_1}^2+\right.\\ \left.2\lambda\,\beta_2\,(\lambda+\mu)\,\beta_1+2\lambda^2{\beta_2}^2\right)(\mu+\beta_1+\beta_2)+\mu^3{\beta_1}^3\end{array}}\right] \tag{3.36}$$

$$Q_{00}=\left[\frac{2((\mu q+2\lambda+\nu)\beta_1+\beta_2(2\lambda+\nu))(\beta_1+\beta_2)\lambda^2\mu}{\begin{array}{c}2\,(2\lambda+\nu+\mu q)\left(\beta_1\,(\lambda+\mu)+\lambda\,\beta_2\right)\lambda\beta_1\,(\mu+\beta_1+\beta_2)+(2\lambda+\nu)\left(\mu^2{\beta_1}^2+\right.\\ \left.2\lambda\,\beta_2\,(\lambda+\mu)\,\beta_1+2\lambda^2{\beta_2}^2\right)(\mu+\beta_1+\beta_2)+\mu^3{\beta_1}^3\end{array}}\right] \tag{3.37}$$

As a result, the manifest expression of the system's availability is derived as

$$A_\nu=\left[\frac{\begin{array}{c}2\,(\mu q+2\lambda+\nu)\left({\beta_1}^2+(\mu+\lambda+\beta_2)\,\beta_1+\lambda\,\beta_2\right)\beta_1\mu\,\lambda+(2\lambda+\nu)\left(\mu\,{\beta_1}^3\right.\\ \left.+\left((2\lambda+\mu)\,\beta_2+\mu^2\right){\beta_1}^2+2\lambda\,\beta_2\,(\mu+\lambda+\beta_2)\,\beta_1+2\lambda^2{\beta_2}^2\right)\mu+\mu^3{\beta_1}^3\end{array}}{\begin{array}{c}2\,(2\lambda+\nu+\mu q)\left(\beta_1\,(\lambda+\mu)+\lambda\,\beta_2\right)\lambda\beta_1\,(\mu+\beta_1+\beta_2)+(2\lambda+\nu)\left(\mu^2{\beta_1}^2+\right.\\ \left.2\lambda\,\beta_2\,(\lambda+\mu)\,\beta_1+2\lambda^2{\beta_2}^2\right)(\mu+\beta_1+\beta_2)+\mu^3{\beta_1}^3\end{array}}\right] \tag{3.38}$$

3.4 The fuzzy redundant repairable system

The compendium of the extant perusal is to comprise the efficacy of redundant repairable systems with switching failure and imperfect repair facilities by language ambiguity, uncertainty in data, and vagueness in system parameters. The system parameters are extended to follow fuzzy specifications to expand the applicability of two-active single-standby unit redundant systems with switching failure and imperfect repair. The adjacency and lack of certainty of the failure rate of an active unit (λ), the failure rate of a standby unit (ν), the repair rate (μ), the perfect inspection rate (β_1), the imperfect inspection rate (β_2), and the probability of switching failure (q) can be represented by fuzzy numbers $\widetilde{\lambda}$, $\widetilde{\nu}$, $\widetilde{\mu}$, $\widetilde{\beta}_1$, $\widetilde{\beta}_2$, and $\widetilde{q}$, respectively. Let the membership grade function for fuzzy numbers $\widetilde{\lambda}$, $\widetilde{\nu}$, $\widetilde{\mu}$, $\widetilde{\beta}_1$, $\widetilde{\beta}_2$, and $\widetilde{q}$, are denoted by $\eta_{\widetilde{\lambda}}(u_1)$, $\eta_{\widetilde{\nu}}(u_2)$, $\eta_{\widetilde{\mu}}(u_3)$, $\eta_{\widetilde{\beta}_1}(u_4)$, $\eta_{\widetilde{\beta}_2}(u_5)$, and $\eta_{\widetilde{q}}(u_6)$, respectively. Then, we have the following fuzzy representations

$$\widetilde{\lambda}=\{(u_1,\eta_{\widetilde{\lambda}}(u_1))\mid u_1\in U_1\} \tag{3.39a}$$

$$\widetilde{\nu}=\{(u_2,\eta_{\widetilde{\nu}}(u_2))\mid u_2\in U_2\} \tag{3.39b}$$

$$\widetilde{\mu}=\{(u_3,\eta_{\widetilde{\mu}}(u_3))\mid u_3\in U_3\} \tag{3.39c}$$

$$\widetilde{\beta}_1=\{(u_4,\eta_{\widetilde{\beta}_1}(u_4))\mid u_4\in U_4\} \tag{3.39d}$$

$$\widetilde{\beta}_2=\{(u_5,\eta_{\widetilde{\beta}_2}(u_5))\mid u_5\in U_5\} \tag{3.39e}$$

$$\widetilde{q}=\{(u_6,\eta_{\widetilde{q}}(u_6))\mid u_6\in U_6\} \tag{3.39f}$$

where $U_1, U_2, U_3, U_4, U_5,$ and U_6 denote the universe of discourse in crisp nature for the system parameters, namely the failure rate of an active unit/a warm standby unit, the repair rate, the perfect inspection rate, the imperfect inspection rate, and the probability of switching failure, respectively.

The fuzzy number, a normal and convex fuzzy set, is an extension of a real number in the vagueness, ambiguity, uncertainty, and approximation environment. In the fuzzy number, the element value ranges in possible intervals with the membership grade between 0 and 1 instead of a specific value. The algebra of fuzzy numbers covers the uncertainty involved in initial conditions, design, operations, parameters, observations, etc. For the study of the system characteristics, suppose that $k(u_1, u_2, u_3, u_4, u_5, u_6)$ represents the reliability characteristic of the proposed repairable system. Since $\widetilde{\lambda}$, $\widetilde{\nu}$, $\widetilde{\mu}$, $\widetilde{\beta}_1$, $\widetilde{\beta}_2$, and $\widetilde{q}$ are convex-normalized fuzzy set having piecewise continuous membership grade function defined on the real number, $\widetilde{k}\left(\widetilde{\lambda}, \widetilde{\nu}, \widetilde{\mu}, \widetilde{\beta}_1, \widetilde{\beta}_2, \widetilde{q}\right)$ is also the convex-normalized fuzzy set having piecewise continuous membership grade function defined on a real number. The membership grade function of the intended characteristic $\widetilde{k}\left(\widetilde{\lambda}, \widetilde{\nu}, \widetilde{\mu}, \widetilde{\beta}_1, \widetilde{\beta}_2, \widetilde{q}\right)$ of studied machining system is derived for reliability analysis following Zadeh's extension principle as

$$\eta_{\widetilde{k}\left(\widetilde{\lambda},\widetilde{\nu},\widetilde{\mu},\widetilde{\beta}_1,\widetilde{\beta}_2,\widetilde{q}\right)}(z) = \sup_{\Psi} \min\{\eta_{\widetilde{\lambda}}(u_1), \eta_{\widetilde{\nu}}(u_2), \eta_{\widetilde{\mu}}(u_3), \eta_{\widetilde{\beta}_1}(u_4), \eta_{\widetilde{\beta}_2}(u_5), \eta_{\widetilde{q}}(u_6)\} \tag{3.40}$$

$$\text{st} \quad z = k(u_1, u_2, u_3, u_4, u_5, u_6)$$

We obtain the reliability characteristics k as mean time-to-failure T in Eqn. (3.22) and availability of the system A_v in Eqn. (3.38), for the proposed model. Since pertinent parameters are fuzzy, $\widetilde{T}$ and $\widetilde{A_v}$ are also fuzzy. So, the membership grade function for the MTTF and availability is denoted by $\eta_{\widetilde{T}}(z)$ and $\eta_{\widetilde{A_v}}(z)$, respectively, and define as follow

$$\eta_{\widetilde{T}}(z) = \sup_{\Psi} \min\{\eta_{\widetilde{\lambda}}(u_1), \eta_{\widetilde{\nu}}(u_2), \eta_{\widetilde{\mu}}(u_3), \eta_{\widetilde{\beta}_1}(u_4), \eta_{\widetilde{\beta}_2}(u_5), \eta_{\widetilde{q}}(u_6) \mid z = k_T(u_1, u_2, u_3, u_4, u_5, u_6)\} \tag{3.41}$$

and

$$\eta_{\widetilde{A_v}}(z) = \sup_{\Psi} \min\{\eta_{\widetilde{\lambda}}(u_1), \eta_{\widetilde{\nu}}(u_2), \eta_{\widetilde{\mu}}(u_3), \eta_{\widetilde{\beta}_1}(u_4), \eta_{\widetilde{\beta}_2}(u_5), \eta_{\widetilde{q}}(u_6) \mid z = k_{A_v}(u_1, u_2, u_3, u_4, u_5, u_6)\} \tag{3.42}$$

respectively where

$$k_T(u_1,u_2,u_3,u_4,u_5,u_6) = \frac{1}{2}\left[\frac{\begin{array}{c}(2u_1+u_2)\left((3u_1+u_3)u_4+3u_1u_5\right)(u_3+u_4+u_5)\\ +2\left(-u_3u_6+(u_4+u_5)(1-u_6)\right)u_1^2(u_4+u_5)+u_3\left((2u_6\right.\\ \left.+1)u_4+2u_3u_6+u_5(2u_6+1)\right)u_1u_4+u_3^2u_4^2\end{array}}{\left((u_3u_6+2u_1+u_2)u_4+u_5(2u_1+u_2)\right)(u_4+u_5)u_1^2}\right]$$

$$k_{A_v}(u_1,u_2,u_3,u_4,u_5,u_6) = \left[\frac{\begin{array}{c}2(2u_1+u_2+u_3u_6)\left(u_4^2+(u_1+u_3+u_5)u_4+u_1u_5\right)\\ u_1u_3u_4+(2u_1+u_2)\left(u_3u_4^3+((2u_1+u_3)u_5+u_3^2)u_4^2\right.\\ \left.+2u_1u_5(u_1+u_3+u_5)u_4+2u_1^2u_5^2\right)u_3+u_3^3u_4^3\end{array}}{\begin{array}{c}2(2u_1+u_2+u_3u_6)\left(u_4(u_1+u_3)+u_1u_5\right)u_1u_4\\ (u_3+u_4+u_5)+(2u_1+u_2)\left(u_3^2u_4^2+2u_1u_5(u_1+u_3)u_4\right.\\ \left.+2u_1^2u_5^2\right)(u_3+u_4+u_4)+u_3^3u_4^3\end{array}}\right]$$

with

$$\Psi = \{u_1\in U_1, u_2\in U_2, u_3\in U_3, u_4\in U_4, u_5\in U_5, u_6\in U_6 \mid u_i>0; \quad i=1,2,3,4,5,6\}$$

Since the proposed redundant repairable system is more complex with switching failure and imperfect repair, associated reliability characteristics are also more tedious under a fuzzy environment to infer the results precisely. The obtained membership grade function for fuzzified reliability characteristics needs to be more explicit, and usability is impractical. Even the shape of the membership grade function can not be depicted, and the conclusion is hardly inferential. The following section employs parametric nonlinear programs based on the extension principle to overcome the implications of derived membership grade functions and their practical applicability.

3.5 Parametric nonlinear programs

For the defuzzification, we quest the γ-cut of fuzzy number $\widetilde{k}\left(\widetilde{\lambda},\widetilde{\nu},\widetilde{\mu},\widetilde{\beta}_1,\widetilde{\beta}_2,\widetilde{q}\right)$ following extension principal discussed by Zadeh's ([358]). For that purpose, parametric nonlinear programs (*NLPs*), mathematical programming techniques, are executed.

For the fuzzy reliability characteristic $\widetilde{k}$, we construct the membership grade function $\eta_{\widetilde{k}\left(\widetilde{\lambda},\widetilde{\nu},\widetilde{\mu},\widetilde{\beta}_1,\widetilde{\beta}_2,\widetilde{q}\right)}$ in the comprehensible and customary form in terms of γ-cut extending from γ-cuts of pertinent parameters. In the following manner, the γ-cuts of $\widetilde{\lambda},\widetilde{\nu},\widetilde{\mu},\widetilde{\beta}_1,\widetilde{\beta}_2$, and $\widetilde{q}$ are determined that represent different levels of crisp confidence intervals.

$$\lambda(\gamma)=[\underline{u_{1\gamma}},\overline{u_{1\gamma}}]=\left[\min_{u_1\in U_1}\{u_1\mid\eta_{\widetilde{\lambda}}(u_1)\geq\gamma\},\max_{u_1\in U_1}\{u_1\mid\eta_{\widetilde{\lambda}}(u_1)\geq\gamma\}\right]\tag{3.43a}$$

$$\nu(\gamma)=[\underline{u_{2\gamma}},\overline{u_{2\gamma}}]=\left[\min_{u_2\in U_2}\{u_2\mid\eta_{\widetilde{\nu}}(u_2)\geq\gamma\},\max_{u_2\in U_2}\{u_2\mid\eta_{\widetilde{\nu}}(u_2)\geq\gamma\}\right]\tag{3.43b}$$

$$\mu(\gamma)=[\underline{u_{3\gamma}},\overline{u_{3\gamma}}]=\left[\min_{u_3\in U_3}\{u_3\mid\eta_{\widetilde{\mu}}(u_3)\geq\gamma\},\max_{u_3\in U_3}\{u_3\mid\eta_{\widetilde{\mu}}(u_3)\geq\gamma\}\right]\tag{3.43c}$$

$$\beta_1(\gamma)=[\underline{u_{4\gamma}},\overline{u_{4\gamma}}]=\left[\min_{u_4\in U_4}\{u_4\mid\eta_{\widetilde{\beta}_1}(u_4)\geq\gamma\},\max_{u_4\in U_4}\{u_4\mid\eta_{\widetilde{\beta}_1}(u_4)\geq\gamma\}\right]\tag{3.43d}$$

$$\beta_2(\gamma)=[\underline{u_{5\gamma}},\overline{u_{5\gamma}}]=\left[\min_{u_5\in U_5}\{u_5\mid\eta_{\widetilde{\beta}_2}(u_5)\geq\gamma\},\max_{u_5\in U_5}\{u_5\mid\eta_{\widetilde{\beta}_2}(u_5)\geq\gamma\}\right]\tag{3.43e}$$

$$q(\gamma)=[\underline{u_{6\gamma}},\overline{u_{6\gamma}}]=\left[\min_{u_6\in U_6}\{u_6\mid\eta_{\widetilde{q}}(u_6)\geq\gamma\},\max_{u_6\in U_6}\{u_6\mid\eta_{\widetilde{q}}(u_6)\geq\gamma\}\right]\tag{3.43f}$$

Therefore, the upper and lower bounds of above defined crisp intervals given by 3.43(a-f) can be expressed as a function of γ as $\underline{u_{1\gamma}}=\min\eta_{\widetilde{\lambda}}^{-1}(\gamma)$, $\overline{u_{1\gamma}}=\max\eta_{\widetilde{\lambda}}^{-1}(\gamma)$, $\underline{u_{2\gamma}}=\min\eta_{\widetilde{\nu}}^{-1}(\gamma)$, $\overline{u_{2\gamma}}=\max\eta_{\widetilde{\nu}}^{-1}(\gamma)$, $\underline{u_{3\gamma}}=\min\eta_{\widetilde{\mu}}^{-1}(\gamma)$, $\overline{u_{3\gamma}}=\max\eta_{\widetilde{\mu}}^{-1}(\gamma)$, $\underline{u_{4\gamma}}=\min\eta_{\widetilde{\beta}_1}^{-1}(\gamma)$, $\overline{u_{4\gamma}}=\max\eta_{\widetilde{\beta}_1}^{-1}(\gamma)$, $\underline{u_{5\gamma}}=\min\eta_{\widetilde{\beta}_2}^{-1}(\gamma)$, $\overline{u_{5\gamma}}=\max\eta_{\widetilde{\beta}_2}^{-1}(\gamma)$, $\underline{u_{6\gamma}}=\min\eta_{\widetilde{q}}^{-1}(\gamma)$, and $\overline{u_{6\gamma}}=\max\eta_{\widetilde{q}}^{-1}(\gamma)$. Hence, we have $u_1\in\lambda(\gamma)$ or $u_1\in[\underline{u_{1\gamma}},\overline{u_{1\gamma}}]$, $u_2\in\nu(\gamma)$ or $u_2\in[\underline{u_{2\gamma}},\overline{u_{2\gamma}}]$, $u_3\in\mu(\gamma)$ or $u_3\in[\underline{u_{3\gamma}},\overline{u_{3\gamma}}]$, $u_4\in\beta_1(\gamma)$ or $u_4\in[\underline{u_{4\gamma}},\overline{u_{4\gamma}}]$, $u_5\in\beta_2(\gamma)$ or $u_5\in[\underline{u_{5\gamma}},\overline{u_{5\gamma}}]$, and $u_6\in q(\gamma)$ or $u_6\in[\underline{u_{6\gamma}},\overline{u_{6\gamma}}]$. Thus, a family of traditional MRPs with different γ-cut sets expresses the studied fuzzified MRP.

The membership grade function of reliability characteristics $\eta_{\widetilde{k}}(z)$ is obtained by employing Zadeh's extension principle for which it is indispensable that at least one of the following cases to be hold such that $z=k(u_1,u_2,u_3,u_4,u_5,u_6)$ satisfies $\eta_{\widetilde{k}}(z)=\gamma$

$$\text{Case (i)}:\left(\eta_{\widetilde{\lambda}}(u_1)=\gamma,\eta_{\widetilde{\nu}}(u_2)\geq\gamma,\eta_{\widetilde{\mu}}(u_3)\geq\gamma,\eta_{\widetilde{\beta}_1}(u_4)\geq\gamma,\eta_{\widetilde{\beta}_2}(u_5)\geq\gamma,\eta_{\widetilde{q}}(u_6)\geq\gamma\right)$$

$$\text{Case (ii)}:\left(\eta_{\widetilde{\lambda}}(u_1)\geq\gamma,\eta_{\widetilde{\nu}}(u_2)=\gamma,\eta_{\widetilde{\mu}}(u_3)\geq\gamma,\eta_{\widetilde{\beta}_1}(u_4)\geq\gamma,\eta_{\widetilde{\beta}_2}(u_5)\geq\gamma,\eta_{\widetilde{q}}(u_6)\geq\gamma\right)$$

$$\text{Case (iii)}:\left(\eta_{\widetilde{\lambda}}(u_1)\geq\gamma,\eta_{\widetilde{\nu}}(u_2)\geq\gamma,\eta_{\widetilde{\mu}}(u_3)=\gamma,\eta_{\widetilde{\beta}_1}(u_4)\geq\gamma,\eta_{\widetilde{\beta}_2}(u_5)\geq\gamma,\eta_{\widetilde{q}}(u_6)\geq\gamma\right)$$

$$\text{Case (iv)}:\left(\eta_{\widetilde{\lambda}}(u_1)\geq\gamma,\eta_{\widetilde{\nu}}(u_2)\geq\gamma,\eta_{\widetilde{\mu}}(u_3)\geq\gamma,\eta_{\widetilde{\beta}_1}(u_4)=\gamma,\eta_{\widetilde{\beta}_2}(u_5)\geq\gamma,\eta_{\widetilde{q}}(u_6)\geq\gamma\right)$$

$$\text{Case (v)}:\left(\eta_{\widetilde{\lambda}}(u_1)\geq\gamma,\eta_{\widetilde{\nu}}(u_2)\geq\gamma,\eta_{\widetilde{\mu}}(u_3)\geq\gamma,\eta_{\widetilde{\beta}_1}(u_4)\geq\gamma,\eta_{\widetilde{\beta}_2}(u_5)=\gamma,\eta_{\widetilde{q}}(u_6)\geq\gamma\right)$$

$$\text{Case (vi)}:\left(\eta_{\widetilde{\lambda}}(u_1)\geq\gamma,\eta_{\widetilde{\nu}}(u_2)\geq\gamma,\eta_{\widetilde{\mu}}(u_3)\geq\gamma,\eta_{\widetilde{\beta}_1}(u_4)\geq\gamma,\eta_{\widetilde{\beta}_2}(u_5)\geq\gamma,\eta_{\widetilde{q}}(u_6)=\gamma\right)$$

Parametric nonlinear programming techniques help for that purpose. Hence, the NLPs for computing the lower and upper bounds of the γ-cut of $\eta_{\widetilde{k}}(z)$ for the cases (i)-(vi) discussed above are as follows.

$$\underline{(k)_\gamma} = \min_{\Psi} k(u_1,u_2,u_3,u_4,u_5,u_6)\,;\forall \quad \text{cases (i)-(vi)} \tag{3.44a}$$

$$\overline{(k)_\gamma} = \max_{\Psi} k(u_1,u_2,u_3,u_4,u_5,u_6)\,;\forall \quad \text{cases (i)-(vi)} \tag{3.44b}$$

Let γ_1 and γ_2 be the two γ-cuts such that $0<\gamma_2<\gamma_1\leq 1$. so, we have the following inclusion relations $[\underline{u_{i\gamma_1}},\overline{u_{i\gamma_1}}]\subseteq[\underline{u_{i\gamma_2}},\overline{u_{i\gamma_2}}]; i=1,2,3,4,5,6$. The lower bounds Eqn.(3.44a) and the upper bounds Eqn.(3.44b) have the same smallest and largest element respectively because these γ-cuts structure a nested interval with respect to γ. To depict the membership grade function $\eta_{\widetilde{k}}(z)$ graphically for the $\widetilde{k}$, we retrace the left and right shape function of $\eta_{\widetilde{k}}(z)$ which is compeer to quest γ-cuts' lower bound $\underline{(k)_\gamma}$ and upper bound $\overline{(k)_\gamma}$. The bounds are expressed as

$$\begin{aligned}\underline{(k)_\gamma} &= \min_{\Psi} k(u_1,u_2,u_3,u_4,u_5,u_6)\\ & st\ \underline{u_{i\gamma}}\leq u_i\leq\overline{u_{i\gamma}}; i=1,2,3,4,5,6\end{aligned} \tag{3.45a}$$

$$\begin{aligned}\overline{(k)_\gamma} &= \max_{\Psi} k(u_1,u_2,u_3,u_4,u_5,u_6)\\ & st\ \underline{u_{i\gamma}}\leq u_i\leq\overline{u_{i\gamma}}; i=1,2,3,4,5,6\end{aligned} \tag{3.45b}$$

At least one of $u_i; i=1,2,3,4,5,6$ must trace the limits of their γ-cuts to calculate the membership grade function $\eta_{\widetilde{k}}(z)$. In the ensuing section, parametric nonlinear programs based on the extension principle are employed to overcome the implications of derived membership grade functions and their practical applicability. This is a traditional mathematical model with boundary restrictions and makes progress itself to the logical study of whence the optimum solutions vary with $\underline{u_{i\gamma}},\overline{u_{i\gamma}}; i=1,2,3,4,5,6$ as γ differs over $(0,1]$. It is a distinctive case of parametric nonlinear programming. The crisp intervals $[\underline{(k)_\gamma},\overline{(k)_\gamma}]$ obtained in the Eqn.(3.45)(a-b) represents the γ-cut of $\widetilde{k}$. Therefore, for $0<\gamma_2<\gamma_1\leq 1$, we have $\underline{(k)_{\gamma_1}}\geq\underline{(k)_{\gamma_2}}$ and $\overline{(k)_{\gamma_1}}\leq\overline{(k)_{\gamma_2}}$ since $\widetilde{k}$ is convex in nature. It implies that on increasing the value of γ the value of $\underline{(k)_\gamma}$ increases and $\overline{(k)_\gamma}$ decreases. Hence, the membership grade function $\eta_{\widetilde{k}}(z)$ can be derived from Eqn.(3.45)(a-b) precisely.

If both the $\underline{(k)_\gamma}$ and $\overline{(k)_\gamma}$ are invertible with respect to γ, the explicit expression of a left function $L(z)=[\underline{(k)_\gamma}]^{-1}$ and a right function $R(z)=[\underline{(k)_\gamma}]^{-1}$ can be manifested, otherwise, shape function $L(z)$ and $R(z)$ can be depicted as a graph. Therefore the membership grade function $\eta_{\widetilde{k}}(z)$ is given as

$$\eta_{\widetilde{k}}(z) = \begin{cases} L(z), & \underline{(k)_{\gamma=0}} \leq z \leq \underline{(k)_{\gamma=1}} \\ 1, & \underline{(k)_{\gamma=1}} \leq z \leq \overline{(k)_{\gamma=1}} \\ R(z), & \overline{(k)_{\gamma=1}} \leq z \leq \overline{(k)_{\gamma=0}} \end{cases} \tag{3.46}$$

For the intricate models including realistic notions and involved complex process, it is not easy to solve for the value of $\underline{(k)_{\gamma}}$ and $\overline{(k)_{\gamma}}$ analytically. Hence, an explicit form of membership grade function for $\widetilde{k}$ is also not facile to formulate. To overcome this problem we depict the approximate shape graphically. To delineate the approximate shape of $L(z)$ and $R(z)$, the numerical values for $\underline{(k)_{\gamma}}$ and $\overline{(k)_{\gamma}}$ at distinct possibility level γ can be summed up. To derive the membership grade function for system's reliability characteristics of proposed model in a similar manner, consider k_T and k_{A_v} as a $k(u_1,u_2,u_3,u_4,u_5,u_6)$ for MTTF and availability of the system respectively. A membership grade function of the reliability characteristics of the studied redundant repairable system conserves the properties such as ambiguity, fuzziness, vagueness, uncertainty, approximation, etc., involved in the pertinent parameters. Nevertheless, in practice, the analysts favor a specific or exact single crisp value rather than a fuzzy set for action as a measure. The Yager,s ranking index method (Yager,[339]) is used to defuzzify the fuzzy values of the reliability characteristics for questing the required crisp value. Since Yager,s method holds area reimbursement properties, the proposed approach is embraced to modify imprecise fuzzified reliability attributes into precise, crisp values. Proper estimations of system attributes are determined as

$$O(D[\omega]) = \int_0^1 \frac{\underline{(D[\omega])_{\gamma}} + \overline{(D[\omega])_{\gamma}}}{2} d\gamma \tag{3.47}$$

where $D[\omega]$ is a convex fuzzy number and $\left(\underline{(D[\omega])_{\gamma}}, \overline{(D[\omega])_{\gamma}}\right)$ is the corresponding γ-cut. The proposed technique is a vigorous ranking approach that holds reimbursement, linearity, and additivity properties (Fortemps and Roubens [61]).

3.6 Numerical illustration and results

In this section, we contemplate a real-time instance to exhibit the practical applicability of a redundant repairable system in power generation. The power generating system consists of two identical electrical power generator units with a standby generator redundancy to increase the system's usability under the vigilance of a single operator. The automation system monitors the faults in generator units and fixes them for repair. The activation of the standby generator unit may also be unpredictable at the instant of requirement. Both the units (active and standby) are repairable with an imperfect repair due to inappropriate rendering, faulty fixing, defective parts, lack of knowledge, etc. The efficiency and potency of the power system can be accessed by its reliability characteristics (MTTF and availability of the system). Assume the failure rate of an active or standby unit, repair rate, and rate of other involved processes are fuzzy to characterize the entailed vagueness, uncertainty, and approximation. The affine rate parameters are trapezoidal fuzzy numbers and are as follows.

$$\widetilde{\lambda} = [0.6, 0.8, 1.0, 1.2],\ \widetilde{\nu} = [0.1, 0.2, 0.3, 0.4],\ \widetilde{\mu} = [3, 4, 5, 6],\ \widetilde{\beta_1} = [10, 15, 20, 25],$$
$$\widetilde{\beta_2} = [0.5, 1.0, 1.5, 2.0],\ \widetilde{q} = [0.06, 0.08, 0.10, 0.12]$$

For pre-specified $\gamma \in (0, 1]$, a crisp confidence interval is obtained for defuzzification. The crisp γ-cuts corresponding to given fuzzy numbers associated with applicable rates are as follows.

$$\left[\underline{u_{1\gamma}}, \overline{u_{1\gamma}}\right] = [0.6 + 0.2\gamma, 1.2 - 0.2\gamma],\ \left[\underline{u_{2\gamma}}, \overline{u_{2\gamma}}\right] = [0.1 + 0.1\gamma, 0.4 - 0.1\gamma],$$
$$\left[\underline{u_{3\gamma}}, \overline{u_{3\gamma}}\right] = [3 + \gamma, 6 - \gamma],\ \left[\underline{u_{4\gamma}}, \overline{u_{4\gamma}}\right] = [10 + 5\gamma, 25 - 5\gamma],$$
$$\left[\underline{u_{5\gamma}}, \overline{u_{5\gamma}}\right] = [0.5 + 0.5\gamma, 2 - 0.5\gamma], \text{and } \left[\underline{u_{6\gamma}}, \overline{u_{6\gamma}}\right] = [0.06 + 0.02\gamma, 0.12 - 0.02\gamma].$$

It is noticeable that the reliability characteristics accomplish their minimum value for the lower value of failure and upper value of repair facility and vice versa.

3.6.1 The fuzzy mean time-to-failure

The fuzzy mean time-to-failure $(\widetilde{T})$ can be obtained using the Eqns. 3.45(a) and 3.45(b) for $k = k_T$. The left shape of MTTF is obtained from minimum value which can be attained for $u_1 = \overline{u_{1\gamma}}, u_2 = \overline{u_{2\gamma}}, u_3 = \underline{u_{3\gamma}}, u_4 = \overline{u_{4\gamma}}, u_5 = \overline{u_{5\gamma}}$, and $u_6 = \overline{u_{6\gamma}}$ and its right shape corresponding to maximum value when $u_1 = \underline{u_{1\gamma}}, u_2 = \underline{u_{2\gamma}}, u_3 = \overline{u_{3\gamma}}, u_4 = \underline{u_{4\gamma}}, u_5 = \underline{u_{5\gamma}}$, and $u_6 = \underline{u_{6\gamma}}$. The left and right limits of crisp interval corresponding to γ-cut are as follows.

$$\underline{(T)_\gamma} = \frac{1}{2} \frac{\left(\frac{3123423}{125} - \frac{4722336\gamma}{625} + \frac{1068403\gamma^2}{2500} - \frac{6832\gamma^3}{125} + \frac{66019\gamma^4}{5000} + \frac{549\gamma^5}{2500}\right)}{\left(27 - \frac{11\gamma}{2}\right)\left(\frac{\gamma}{5} - \frac{6}{5}\right)^2 \left(\frac{423}{5} - \frac{146\gamma}{5} + \frac{39\gamma^2}{20} + \frac{\gamma^3}{10}\right)} \tag{3.48a}$$

$$\overline{(T)_\gamma} = \frac{1}{2}\,\frac{\left(\frac{4075320}{701} + \frac{2476167\,\gamma}{500} + \frac{884693\,\gamma^2}{1250} - \frac{77221\,\gamma^3}{625} + \frac{82489\,\gamma^4}{5000} - \frac{549\,\gamma^5}{2500}\right)}{\left(\frac{21}{2} + \frac{11\gamma}{2}\right)\left(\frac{3}{5} + \frac{\gamma}{5}\right)^2\left(\frac{69}{4} + \frac{74\,\gamma}{5} + \frac{57\,\gamma^2}{20} - \frac{\gamma^3}{10}\right)} \tag{3.48b}$$

The function $\overline{(T)_\gamma}$ and $\underline{(T)_\gamma}$ are invertible, which yields the membership grade function as

$$\eta_{\widetilde{T}}(z) = \begin{cases} L(z), & \frac{347047}{91368} \leq z \leq \frac{712711}{98814} \\ 1, & \frac{712711}{98814} \leq z \leq \frac{7967841913}{499605504} \\ R(z), & \frac{7967841913}{499605504} \leq z \leq \frac{135844000}{3047247} \end{cases} \tag{3.49}$$

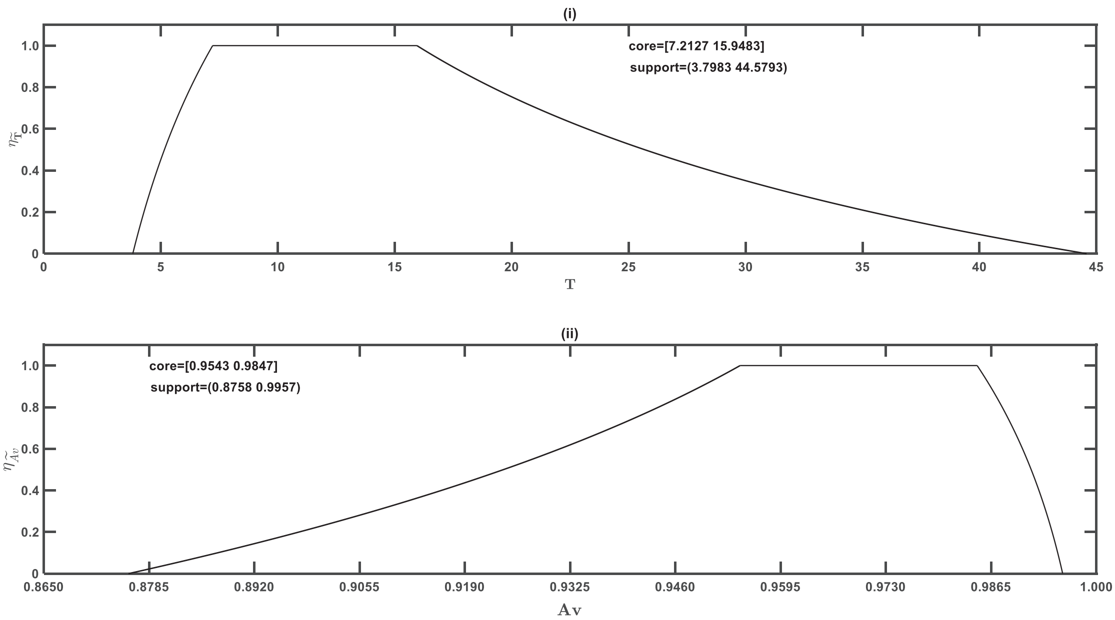


Figure 3.3: $\eta_{\widetilde{T}}$ and $\eta_{\widetilde{A}}$ for numerical illustration

It is not so convenient to express the explicit form of the membership left/right shape function $L(z)$ and $R(z)$; Fig. 3.3(i) depicts the variational shape of the membership grade function of fuzzified reliability characteristics. From 3.3(i), we can make inferential for reliability characteristics under uncertain, vague, or approximate environments.

From Fig. 3.3(i), two significant crisp quantities associated with the fuzzy mean time-to-failure $\widetilde{T}$ are notable. Firstly, at the possible level $\gamma = 0$, the support of $\widetilde{T}$

varies from 3.7983 to 44.5793; which depicts that in the pessimistic condition, the mean time-to-failure can not be beyond 3.7983 or in the optimistic condition it can not be more than 44.5793. Second, the γ-cut at plausibility level $\gamma = 1$ holds the values 7.2127 to 15.9483, which are the most comprehensible values for the MTTF under normal conditions.

3.6.2 The fuzzy system's availability

Similar to previous subsection, the γ-cut of the system availability $\widetilde{A_v}$ are coined from Eqns. 3.45(a) and 3.45(b) for $k = k_{A_v}$. The left shape of system availability is obtained from minimum value which is attained for $u_1 = \overline{u_{1\gamma}}, u_2 = \overline{u_{2\gamma}}, u_3 = \underline{u_{3\gamma}}, u_4 = \underline{u_{4\gamma}}, u_5 = \overline{u_{5\gamma}}$, and $u_6 = \overline{u_{6\gamma}}$ and its right shape corresponding to maximum value when $u_1 = \underline{u_{1\gamma}}, u_2 = \underline{u_{2\gamma}}, u_3 = \overline{u_{3\gamma}}, u_4 = \overline{u_{4\gamma}}, u_5 = \underline{u_{5\gamma}}$, and $u_6 = \underline{u_{6\gamma}}$. The γ-cuts are as follows.

$$\underline{(A_v)_\gamma} = \left[\frac{(3+\gamma)\left(\frac{4534704}{125} + \frac{6456126\,\gamma}{125} + \frac{3532933\,\gamma^2}{125} + \frac{75731\,\gamma^3}{10} + \frac{107227\,\gamma^4}{100} + \frac{39031\,\gamma^5}{500} + \frac{133\,\gamma^6}{125}\right)}{\left(\frac{3106512}{25} + \frac{25173162\,\gamma}{125} + \frac{16980513\,\gamma^2}{125} + \frac{12465863\,\gamma^3}{250} + \frac{216169\,\gamma^4}{20} + \frac{33691\,\gamma^5}{25} + \frac{78851\,\gamma^6}{1000} + \frac{451\,\gamma^7}{500}\right)}\right] \tag{3.50a}$$

$$\overline{(A_v)_\gamma} = \left[\frac{(6-\gamma)\left(\frac{91556843}{121} - \frac{15516227\,\gamma}{27} + \frac{44173361\,\gamma^2}{250} - \frac{701012\,\gamma^3}{25} + \frac{59671\,\gamma^4}{25} - \frac{48607\,\gamma^5}{500} + \frac{133\,\gamma^6}{125}\right)}{\left(\frac{68396138}{15} - \frac{71322439\,\gamma}{17} + \frac{11427788\,\gamma^2}{7} - \frac{12801599\,\gamma^3}{37} + \frac{1700813\,\gamma^4}{40} - \frac{734359\,\gamma^5}{250} + \frac{97793\,\gamma^6}{1000} - \frac{451\,\gamma^7}{500}\right)}\right] \tag{3.50b}$$

The functions $\underline{(A_v)_\gamma}$ and $\overline{(A_v)_\gamma}$ are invertible due to the involved complexity in the proposed model, which makes the function rational with high degree polynomial, so the membership grade function in shape form is

$$\eta_{\widetilde{A_v}}(z) = \begin{cases} L(z), & \frac{3499}{3995} \le z \le \frac{2220704}{2327063} \\ 1, & \frac{2220704}{2327063} \le z \le \frac{2394580821877027}{2431769598868698} \\ R(z), & \frac{2394580821877027}{2431769598868698} \le z \le \frac{4120057935}{4137966349} \end{cases} \tag{3.51}$$

Fig. 3.3(ii) depicts the shape of the membership grade function of the fuzzy availability of the system to reveal the variation wrt to the possibility level. The core of availability is approximately [0.9543 0.9847] from Fig. 3.3(ii) for the possibility level $\gamma = 1$, which exhibits the falling behavior of availability in the interval, although it is vague. The support of availability of the system is approximately (0.8758 0.9957) for the likelihood level $\gamma = 0$. This interval indicates that the system's availability cannot outdo 0.9957 and fall beneath 0.8758 in optimistic and pessimistic conditions, respectively.

3.6.3 Numerical simulation

The flow chart for simulation

INPUT $< \tilde{\lambda}, \tilde{\nu}, \tilde{\mu}, \tilde{\beta}_1, \tilde{\beta}_2, \tilde{q}, \text{and } \gamma >$

Compute γ-cut $\lambda(\gamma), \nu(\gamma), \mu(\gamma), \beta_1(\gamma), \beta_2(\gamma), q(\gamma)$

Put γ-cut in Eqn.3.22 for MTTF and Eqn.3.38 for availability

Compute $\underline{(T)}_\gamma, \overline{(T)}_\gamma, \underline{(A_v)}_\gamma, \overline{(A_v)}_\gamma$

Compute Support for $\underline{(T)}_{\gamma=0}$ $\underline{(A_v)}_{\gamma=0}$

and

Core for $\overline{(T)}_{\gamma=1}$ $\overline{(A_v)}_{\gamma=1}$

Derive $\eta_{\tilde{T}}(z)$ and $\eta_{\tilde{A}_v}(z)$ by inverting $\underline{(T)}_\gamma, \overline{(T)}_\gamma, \underline{(A_v)}_\gamma, \overline{(A_v)}_\gamma$

i.e.

$L(z)$ for $\eta_{\tilde{T}}(z)$ by equating $\underline{(T)}_\gamma$=z

$R(z)$ for $\eta_{\tilde{T}}(z)$ by equating $\overline{(T)}_\gamma$=z

$L(z)$ for $\eta_{\tilde{A}_v}(z)$ by equating $\underline{(A_v)}_\gamma$=z

$R(z)$ for $\eta_{\tilde{A}_v}(z)$ by equating $\overline{(A_v)}_\gamma$=z

OUTPUT $< \eta_{\tilde{T}}(z) \text{ and } \eta_{\tilde{A}_v}(z) >$

Plot $\eta_{\tilde{T}}(z)$ and $\eta_{\tilde{A}_v}(z)$

We illustrate numerical simulation using the flow chart present herewith for the different conglomerations of fuzzy numbers. The fuzzy numbers represent intrinsic impreciseness in governing system parameters. The numerical simulations are useful for the tractability of a nonlinear program introduced for finding the membership grade function of fuzzified MTTF and the system's availability for the studied two-active unit single-warm standby repairable system with imperfect repair and switching failure. Apart from the assumed trapezoidal fuzzy number for numerical illustration in the previous subsection, we also consider diverse congregations of trapezoidal fuzzy numbers. The trapezoidal fuzzy numbers, tabulated in Table 3.1, are commensurate to governing parameters for experimenting numerically to represent a more comprehensive range of uncertainties, approximation, vagueness, impreciseness. Figs. 3.4-3.9

show the varied shape of the membership grade functions, and Tables 3.2-3.7 illustrate the corresponding support and core. Membership functions can be defined as a technique to solve practical problems by experience rather than knowledge.

Table 3.1: Trapezoidal fuzzified system parameters

i	1	2	3	4
$\widetilde{\lambda}_i$	[0.6, 0.8, 1.0, 1.2]	[0.6, 0.7, 0.8, 0.9]	[0.4, 0.5, 0.6, 0.7]	[0.6, 0.9, 1.2, 1.5]
$\widetilde{\nu}_i$	[0.1, 0.2, 0.3, 0.4]	[0.10, 0.15, 0.20, 0.25]	[0.05, 0.15, 0.25, 0.35]	[0.10, 0.25, 0.40, 0.55]
$\widetilde{\mu}_i$	[3.0, 4.0, 5.0, 6.0]	[3.0, 3.5, 4.0, 4.5]	[1.0, 2.0, 3.0, 4.0]	[3.0, 5.0, 7.0, 9.0]
$\widetilde{\beta}_{1i}$	[10, 15, 20, 25]	[10, 12, 14, 16]	[5, 10, 15, 20]	[10, 20, 30, 40]
$\widetilde{\beta}_{2i}$	[0.5, 1.0, 1.5, 2.0]	[0.6, 0.8, 1.0, 1.2]	[0.25, 0.50, 0.75, 1.0]	[0.5, 1.5, 2.5, 3.5]
$\widetilde{q}_i$	[0.06, 0.08, 0.10, 0.12]	[0.06, 0.10, 0.14, 0.18]	[0.02, 0.04, 0.06, 0.08]	[0.1, 0.2, 0.3, 0.4]

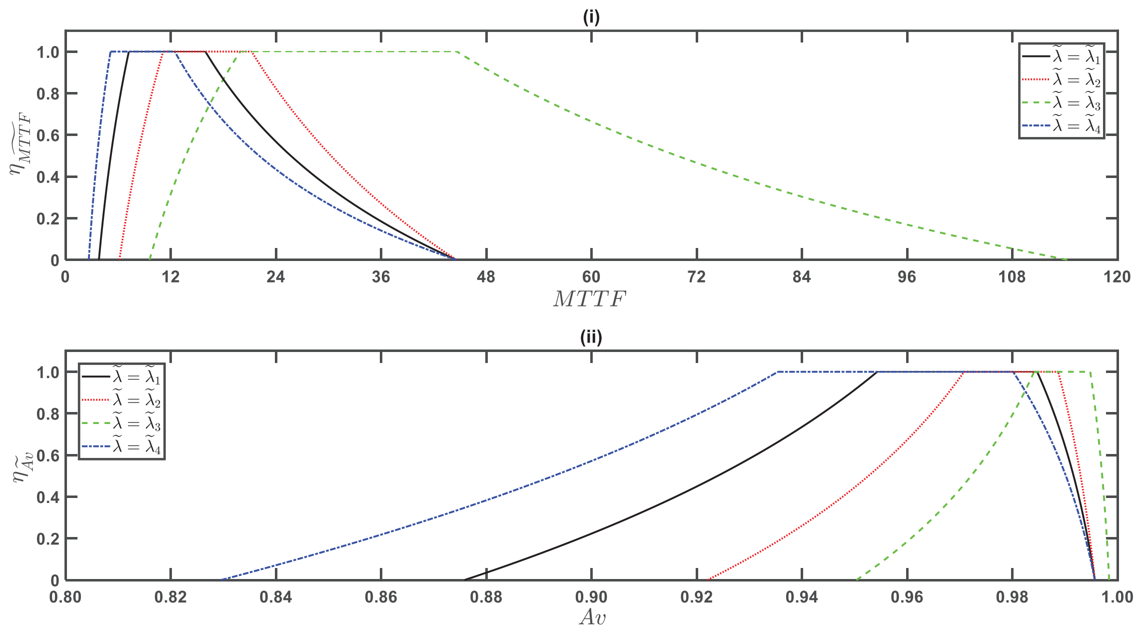


Figure 3.4: $\eta_{\widetilde{T}}$ and $\eta_{\widetilde{A_v}}$ for fuzzified $\widetilde{\lambda}_i$

In Fig. 3.4, the shape of the membership grade function of the system's MTTF and the availability are depicted as relevant to the active unit fuzzy failure rate. Table 3.2 represents the associated support and core values for the reliability characteristics for various $\widetilde{\lambda}_i$. The wide range of support and narrow range of core for MTTF corresponding to $\widetilde{\lambda}_3$ sustenance the fact that the failure rate must be lower. The higher uncertainties in the failure of active units are oppositely symmetrical to the possibility of MTTF and the system's availability, which is an endorsement of our modeling. The system's availability increases with the preventive failure. Preventive failure may help enhance the system's availability, and preventive failure may be obtained with the proper protective and prophesy maintenance.

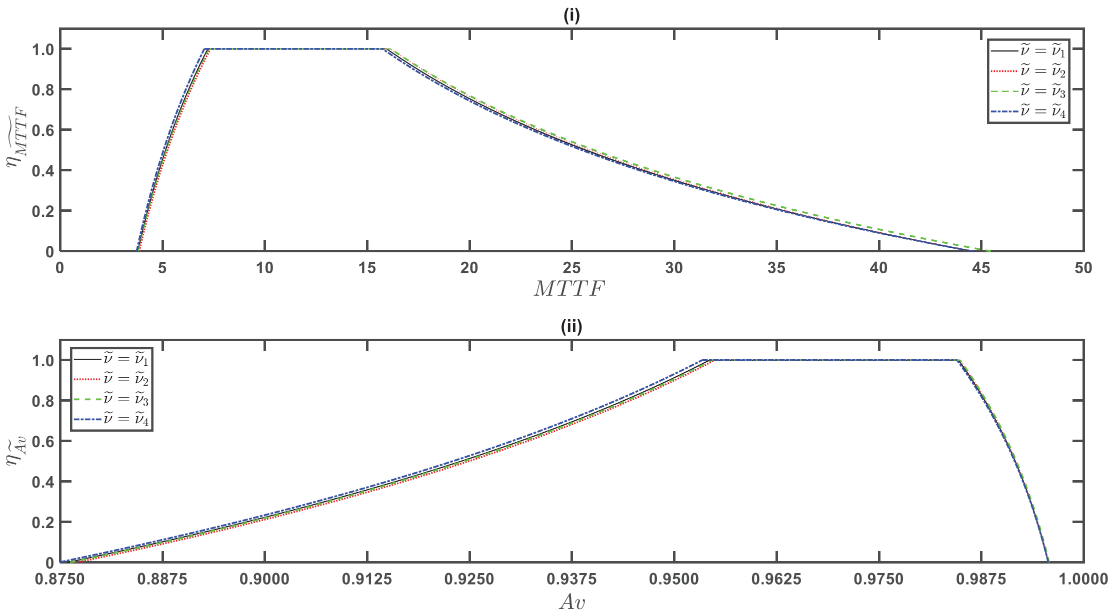


Figure 3.5: $\eta_{\widetilde{T}}$ and $\eta_{\widetilde{A_v}}$ for fuzzified $\widetilde{\nu}_i$

Table 3.2: Properties of $\widetilde{T}$ and $\widetilde{A}_v$ for fuzzified $\widetilde{\lambda}_i$

$\widetilde{\lambda}_i$	[0.6, 0.8, 1.0, 1.2]	[0.6, 0.7, 0.8, 0.9]	[0.4, 0.5, 0.6, 0.7]	[0.6, 0.9, 1.2, 1.5]
$(T)_{\gamma>0}$	(3.79834,44.57925)	(6.13630,44.57925)	(9.57629,114.24885)	(2.64212, 44.57925)
$(T)_{\gamma=1}$	[7.21265,15.94827]	[11.12202,21.22130]	[19.90512,44.63414]	[5.12590,12.45469]
$(A_v)_{\gamma>0}$	(0.87584,0.99567)	(0.92202,0.99567)	(0.95035, 0.99839)	(0.82956,0.99567)
$(A_v)_{\gamma=1}$	[0.95429,0.98471]	[0.97081, 0.98869]	[0.98412,0.99480]	[0.93542, 0.98014]

The system's reliability features (MTTF and availability) for distinct sets of the fuzzy failure rate of the standby unit ($\widetilde{\nu}_i$) are delineated in Fig.3.5 with the shape of the corresponding membership grade function. Table 3.3 represents the corresponding support and core of the fuzzy set of the MTTF and the system's availability for different ν_i. The shape, support, and core for varied ranges of ν_i are nearly the same. The results show that the standby unit's failure rate ($\widetilde{\nu}$) in the inactive state does not modify much in reliability characteristics. practically, the standby needs to be perfect for uninterrupted functioning at the time of switching. The preventive measures provisioning upkeep the working of standby units.

Table 3.3: Properties of $\widetilde{T}$ and $\widetilde{A_v}$ for fuzzified $\widetilde{v}_i$

$\widetilde{v}_i$	[0.1, 0.2, 0.3, 0.4]	[0.10, 0.15, 0.20, 0.25]	[0.05, 0.15, 0.25, 0.35]	[0.10, 0.25, 0.40, 0.55]
$(T)_{\gamma>0}$	(3.79834,44.57925)	(3.86114,44.57925)	(3.82004, 45.46380)	(3.73723,44.57925)
$(T)_{\gamma=1}$	[7.21265,15.94827]	[7.33811,16.15252]	[7.27419,16.15252]	[7.04113,15.75319]
$(A_v)_{\gamma>0}$	(0.87584,0.99567)	(0.87700,0.99567)	(0.87622 ,0.99576)	(0.87478,0.99567)
$(A_v)_{\gamma=1}$	[0.95429,0.98471]	[0.95493,0.98490]	[0.95461,0.98490]	[0.95340 ,0.98452]

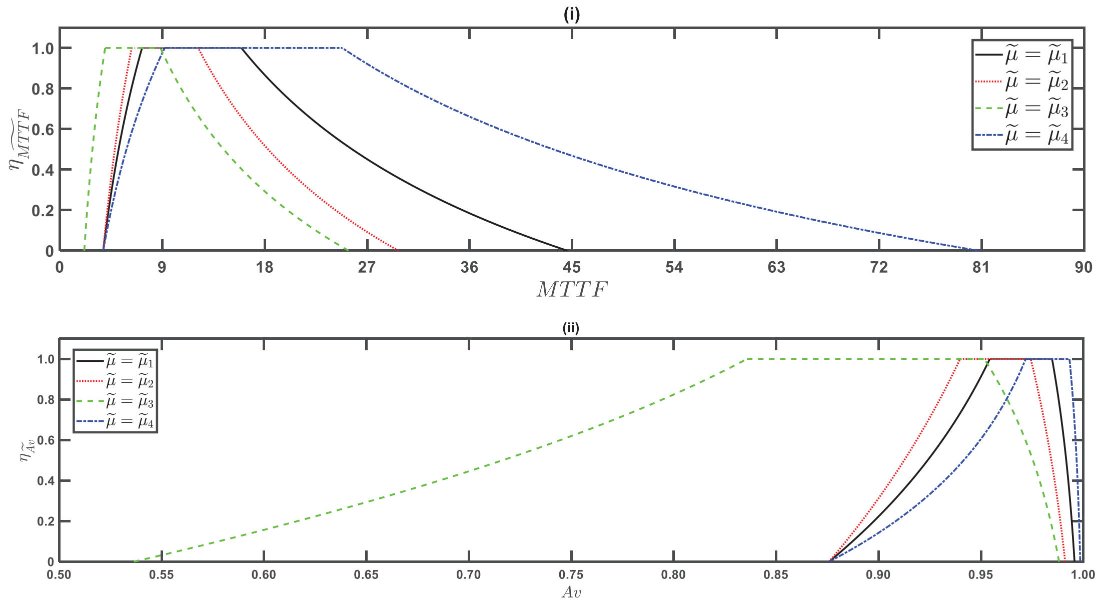


Figure 3.6: $\eta_{\widetilde{T}}$ and $\eta_{\widetilde{A_v}}$ for fuzzified $\widetilde{\mu}_i$

The shape of the membership grade function of fuzzified MTTF and the system's availability commensurate to the specific fuzzy numbers, which represent the vagueness of the repair rate of the repairer, is portrayed in Fig. 3.6. Table 3.4 denotes the most likely range of possible values of the MTTF and availability of the system for various $\widetilde{\mu}_i$ as support and core, respectively. The shape permits the gradual assessment of the membership of reliability characteristics. The wide range of repair rate $\widetilde{\mu}_4$ shows the wide range of availability of the system. It is an apparent fact. The result indicates that the increments in a higher possible range of repair rates are directly proportional to the system's availability and MTTF. The simulation outcomes prompt the corrective measures to opt for the machining systems' long-run functioning.

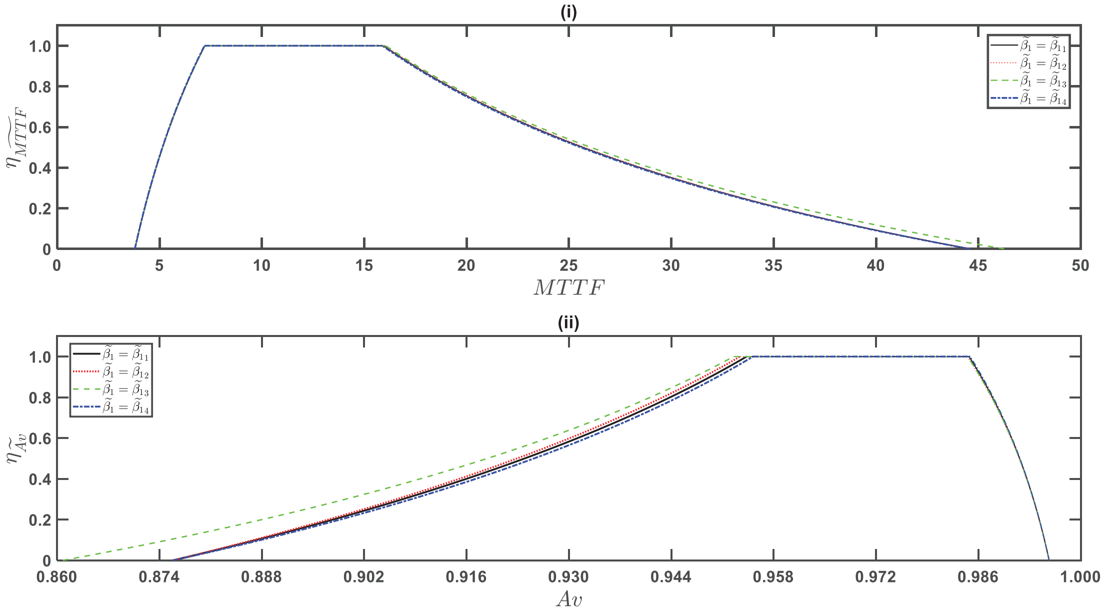


Figure 3.7: $\eta_{\widetilde{T}}$ and $\eta_{\widetilde{A_v}}$ for fuzzified $\widetilde{\beta}_{1_i}$

In Fig. 3.7, the shape of the membership grade function of the fuzzified system's characteristics wrt the different trapezoidal fuzzy numbers is shown, representing the vagueness of the perfect inspection of the repairer. The majority range of possible values of the MTTF and availability of the system for diverse $\widetilde{\beta}_{1_i}$ as support and core, respectively, collected in Table 3.5. The possibility of a perfect inspection rate also increases the system's characteristics. Whereas its converse, the increasing behavior of imperfect inspection diminishes the MTTF and system's availability shown in Fig 3.8. The majority range of possible values of the MTTF and availability of the system for diverse $\widetilde{\beta}_{2_i}$ as support and core, respectively, collected in Table 3.6. The result signifies how predictive measures are important for judging the services. The support and core corresponding to $\widetilde{\beta}_{24}$ characterize the likelihood of the availability of the system. Knowing the most likely range is important for management to prioritize decision-making and resource allocation.

Table 3.4: Properties of $\widetilde{T}$ and $\widetilde{A}_v$ for fuzzified $\widetilde{\mu}_i$

$\widetilde{\mu}_i$	[3.0, 4.0, 5.0, 6.0]	[3.0, 3.5, 4.0, 4.5]	[1.0, 2.0, 3.0, 4.0]	[3.0, 5.0, 7.0, 9.0]
$(T)_{\gamma>0}$	(3.79834,44.57925)	(3.79834,29.73744)	(2.13604,25.35750)	(3.79834,80.71567)
$(T)_{\gamma=1}$	[7.21265 ,15.94827]	[6.30554 ,12.14732]	[3.96792 ,8.83016]	[9.19938 ,24.81424]
$(A_v)_{\gamma>0}$	(0.87584,0.99567)	(0.87584,0.99111)	(0.53643,0.98812)	(0.87584,0.99845)
$(A_v)_{\gamma=1}$	[0.95429,0.98471]	[0.93980,0.97431]	[0.83575,0.95163]	[0.97184,0.99321]

Table 3.5: Properties of $\widetilde{T}$ and $\widetilde{A_v}$ for fuzzified $\widetilde{\beta}_{1i}$

$\widetilde{\beta}_{1i}$	[10, 15, 20, 25]	[10, 12, 14, 16]	[5, 10, 15, 20]	[10, 20, 30, 40]
$(T)_{\gamma>0}$	(3.79834,44.57925)	(3.80108,44.57925)	(3.79378,46.25565)	(3.79549,44.57925)
$(T)_{\gamma=1}$	[7.21265,15.94827]	[7.22547,16.00322]	[7.22293,16.05346]	[7.20005,15.88825]
$(A_v)_{\gamma>0}$	(0.87584,0.99567)	(0.87584,0.99567)	(0.86079,0.99567)	(0.87584,0.99567)
$(A_v)_{\gamma=1}$	[0.95429,0.98471]	[0.95341,0.98448]	[0.95253,0.98453]	[0.95517,0.98488]

Table 3.6: Properties of $\widetilde{T}$ and $\widetilde{A_v}$ for fuzzified $\widetilde{\beta}_{2i}$

$\widetilde{\beta}_{2i}$	[0.5, 1.0, 1.5, 2.0]	[0.6, 0.8, 1.0, 1.2]	[0.25, 0.50, 0.75, 1.0]	[0.5, 1.5, 2.5, 3.5]
$(T)_{\gamma>0}$	(3.79834,44.57925)	(3.88924,43.97597)	(3.91303,46.15377)	(3.48389,44.57925)
$(T)_{\gamma=1}$	[7.21265,15.94827]	[7.39216,16.20396]	[7.48598,16.60327]	[6.80054,15.42492]
$(A_v)_{\gamma>0}$	(0.87584,0.99567)	(0.88973,0.99563)	(0.89263,0.99592)	(0.84943,0.99567)
$(A_v)_{\gamma=1}$	[0.95429,0.98471]	[0.95718,0.98505]	[0.95843,0.98565]	[0.94832,0.98383]

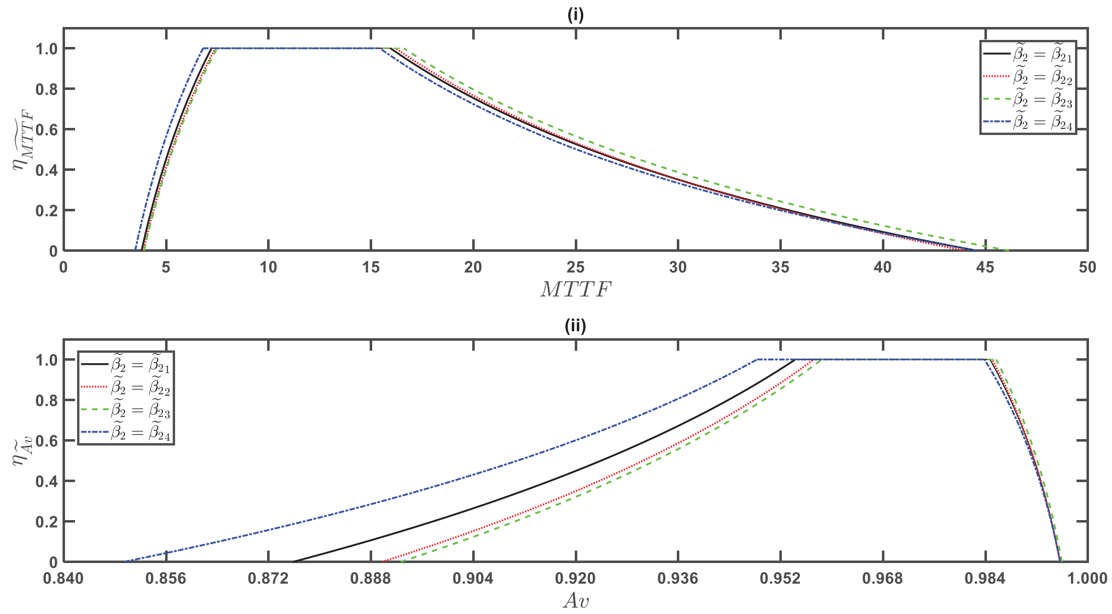


Figure 3.8: $\eta_{\widetilde{T}}$ and $\eta_{\widetilde{A_v}}$ for fuzzified $\widetilde{\beta}_{2i}$

Fig. 3.9 exhibits fuzzified MTTF and availability' membership grade function shape wrt distinct trapezoidal fuzzy numbers, which represents the implications of the switching failure probability $\widetilde{q}_i$. Table 3.7 summarizes the most likely range and most generality value of MTTF and the system's availability. The difference in the likelihood of the availability of the system for $\widetilde{q}_1$ and $\widetilde{q}_4$ backing for provisioning of the corrective measures.

Table 3.7: Properties of $\widetilde{T}$ and $\widetilde{A_v}$ for fuzzified $\widetilde{q}_i$

$\widetilde{q}_i$	[0.06, 0.08, 0.10, 0.12]	[0.06, 0.10, 0.14,0.18]	[0.02, 0.04, 0.06, 0.08]	[0.1, 0.2, 0.3, 0.4]
$(T)_{\gamma>0}$	(3.79834,44.57925)	(3.63307,44.57925)	(3.91111,51.40515)	(3.14758,39.42076)
$(T)_{\gamma=1}$	[7.21265,15.94827]	[6.72337,15.34547]	[7.58694,17.32446]	[5.83041,12.95081]
$(A_v)_{\gamma>0}$	(0.87584,0.99567)	(0.87278,0.99567)	(0.87797,0.99626)	(0.86265,0.99511)
$(A_v)_{\gamma=1}$	[0.95429,0.98471]	[0.95159,0.98413]	[0.95616,0.98588]	[0.94572,0.98134]

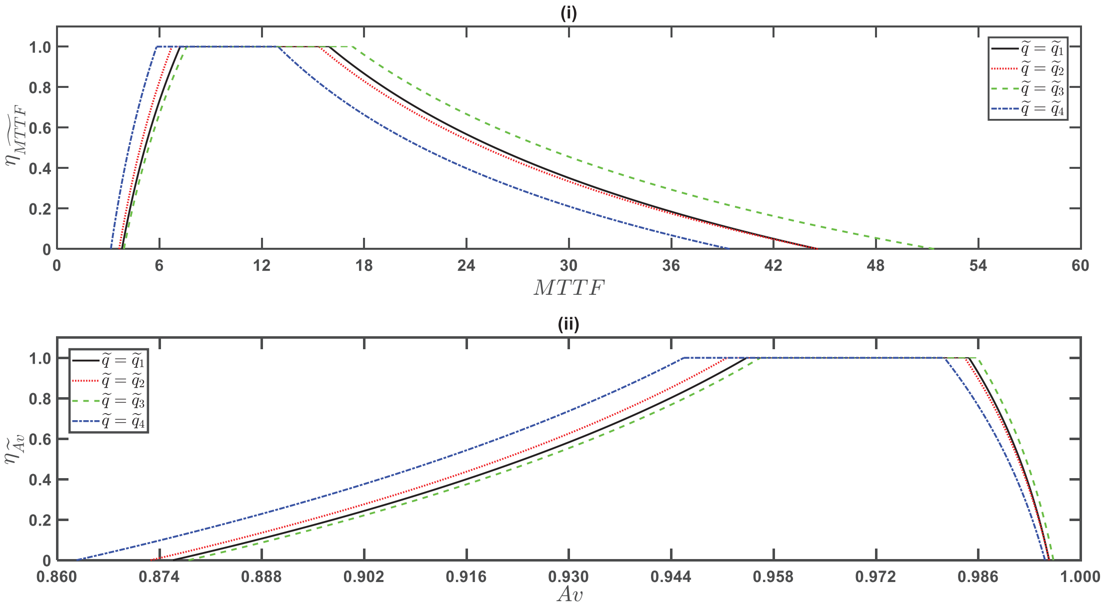


Figure 3.9: $\eta_{\widetilde{T}}$ and $\eta_{\widetilde{A_v}}$ for fuzzified $\widetilde{q}_i$

These results correlate with the ambiguity, unpredictability, assumptions, and error in estimating the system's parameters to reliability characteristics. The sensitive range of core values and support helps broaden the MTTF and system's availability. The support and core directly prospect the enormous feasible scale and most likely range of desired characteristics. As a system's complexity increases, the decision-maker's ability to make precise and yet significant. This research article helps form and analyze any real-time service delivery system. The figure-of-merits of the present study are (i) it deals with a complex system, (ii) covenant with a human-free system, (iii) it signifies with approximations, errors, uncertainties, vagueness, and ambiguities in inputs and operations.

3.6.4 Discussion

In the previous section, we present various real-time practical applications computing to service, computer to communication system, energy sector to the electric vehicle, etc., where the proposed model is valid. The machining system with active and passive redundancy can be analyzed in the studied frameworks. Imperfect repair is a common imprecise, and the vague regime applicable in machining systems is a crucial term of our investigation. We present highly measurable performance indices for the analysis of the studied model. The intensive simulation qualifies our study a lot for multi-scenario benefits. The membership of performance indices indicates the optimistic and pessimistic range under the extremely well and poor design. The investigative results infer the following points

- The preventive measures should opt to avoid frequent failure of active and standby units and avoid switching failure.
- The corrective measures should opt for timely and perfect repair and perfect switching.
- The predictive measures should opt to avoid the delay in repair.

3.7 Conclusion

In the present chapter, the fuzzy doctrines γ-cut and Zadeh's extension principle are employed to develop the fuzzified MTTF and the system availability associated membership grade function by using parametric nonlinear programming from fuzzified pertinent system parameters. We evaluate the γ-cut of the membership grade function, and their equate interval limits are inverted to obtain machining system performance explicit expressions. The proposed two-active single standby unit system is more practical and applicable in power systems, communication systems, service systems, etc. The adapted Imperfect repair and switching failure are also realistic. The shortcoming of the present study is to determine the expression of characteristics to be analyzed for which a high-grade computing system is required. Otherwise, we cannot handle multi-unit systems within complex proximity.

In the future, this research work can be extended to derive the membership grade function for the system's reliability and queueing measures and apply this concept to the generalized MRP with any number of standby units with different types of failure. The studied concept can also be employed on a generalized MRP with reboot, recovery, and imperfect repairer with perfect and imperfect repair facilities. The failure, delay, imperfection, and unreliable can be dealt with in the future in designing the machining system for real-time technology-based sectors. The vacation, working vacation, distinguished service regime, and varied design can also be implemented in the future.

Chapter 4

Performance Analysis of a Repairable Machining System with Standby Deterioration

"In the journey of research, much like the operation of machines, the passage of time reveals both strengths and vulnerabilities. Yet, it is through perseverance and innovation that both machines and researchers evolve and endure."

Dr. Emily Chen

4.1 Introduction

In today's evolving technologies in socio-economic constraints, the assimilation of machines into human life has emerged as profoundly advantageous. Diverse industrial domains including but not limited to the internet of thing, cloud manufacturing, power plants, assembly lines, security systems, and laboratories have become profoundly reliant upon the performance metrics of machining systems. However, machines are inherently prone to failure or degrade over time, leading to increased production costs, disruption in production flow, and logistical supply delays. To mitigate these challenges and ensure uninterrupted operations, the development of robust modern industries or machining systems with improved cost-effectiveness assumes paramount.

The contemporary world confronts both internal and external security challenges. The pervasive deployment of CCTV cameras or costly security gadgets has become ubiquitous in cities, academic/research institutions, service sectors, and industries, with each location playing a instrumental role in security measures. Moreover, internal and external factors, including environmental issues, can disrupt the functioning of security systems that rely on a steady power supply. To address challenges effectively, the provisioning of standby generator systems is pivotal for security agencies. By provisioning standby systems, the reliability and sustainability of security systems are significantly augmented, with judicious operational upkeep. Standby systems are commonly categorized into three types: cold, warm, and hot. Warm standby units can be inherently integrated with machining systems to enhance system capacity, thereby offering a comprehensive enhancement to security system.

This research article focuses on a multi-unit machining system comprising active redundancy, involving M active units supervised by one repairer and passive redundancy encompassing an high-priced assemblage of S warm standby units, which experience degradation over time while in a state of readiness. The strategic deployment of standby units play a vital role in ensuring uninterrupted production flow (UPF) for machining systems. When an active unit fails, it is immediately replaced by an available standby unit to maintain the balance of demand and supply within the industry. Standby units are commonly used in systems to provide redundancy and improve system reliability. Numerous scholarly studies have investigated machining systems with standbys provisioning since the pioneering work of Srinivasan [281], who laid foundational insights that redundancy increases the mean time to system failure. The concept of standby failure in the spare state was first proposed by Osaki and Nakagawa [220], and subsequent studies have continuum exploration. Sivazlian and Wang [273] & Gupta and Chaudhary [78] analyzed a machining system with R repairers to maximize the expected profit function derived from optimal values of standby units and repairers.

In traditional research, standby units are often assumed to remain unaffected by degradation over time in machine repair scenarios and completely reject the deteriorated standby unit, if any. However, real-world machining systems show that both active and standby units degrade. Recognizing big-budgeted standby unit deterioration is vital for accurate reliability analysis and efficient system design. This study develops mathematical models to predict exorbitant standby unit behavior as they degrade, aiding in maintenance decisions and system performance optimization. Ignoring extortionate standby unit deterioration can lead to increased downtime and reduced system reliability. By considering degradation patterns, optimal maintenance strategies can be devised, managing costs while ensuring system dependability. Queueing theory, a mathematical framework, proves useful for studying high-cost standby unit deterioration by modeling interactions between operational and standby units, offering insights into system performance. In essence, understanding standby unit deterioration is crucial for cost-effective system design, analysis, and maintenance, improving redundancy and overall system effectiveness.

Yeh [345] developed an optimal repairable-replacement model for deteriorating machining systems. Tsang [299] introduced preventive and corrective maintenance notions within a condition-based framework, considering discrete deterioration states with transitions modeled as semi-Markov or Poisson processes. Wang [307] comprehensively examined maintenance policies including age, random age replacement, repair cost, and time policies. Lai and Chen [165] analyzed a two-unit system with inter-unit failure rate interaction. Montoro-Cazorla and Perez-Ocon [205] explored a cold standby system's degradation phases using phase-type distributions. Others (*cf.* Kamaitis [125], Sarje et al. [248], Ahmad and Kamaruddin [2], Yusuf and Bala [357], Ahmadi [3], Alaswad and Xiang [7]) investigated condition-based maintenance, multi-component corrosion protection, and unit deteriorating rates, culminating in reliability attribute evaluations.

Dong et al. [53] analyzed an optimal preventive replacement strategy for a single-unit system with two maintenance policies and optimal cost considerations. Liu et al. [179] proposed a multi-dimensional degradation process-aware maintenance policy with reliability and cost analyses. Recent work by Wang et al. [328], Al-Jararha et al. [5], and Hsieh [91] introduced novel criteria for component reallocation in spare parts systems with condition-based maintenance, enhancing production lines. Salmasnia and Shabani [246] explored bottleneck-centered opportunistic maintenance using particle swarm optimization, determining optimal preventive intervals and reliability thresholds. Strategic cost function design aids in judicious maintenance decisions.

Wang and Chiu [314] discussed cost analysis of warm standby system availability using supplementary variable techniques. Haque and Armstrong [88] conducted a comprehensive survey on machine interference problems. Complex warm standby systems with contingent failures were explored (*cf.* Yun & Cha [354], Shekhar et al. [266], [263]). Bai et al. [12] proposed a novel approach for warm standby optimization with genetic algorithms. Liu et al. [183] assessed cost-benefit of a standby retrial system with switching failure. Wang et al. [321] analyzed retrial systems with preventive maintenance and unreliable repairer, focusing on cost-benefit analysis using Laplace-Stieltjes transforms.

It is evident from the literature survey that researchers have categorized standby failures into three types in machining systems. However, no research has been conducted on the theory that the failure rate of a standby unit in the operational state differs from that of its active unit due to deterioration, i.e. degraded failure rate or degraded working efficiency of highly-priced standby. This research article explores a novel notion in which the failure rate of the standby unit in the operative state differs from that of the active unit instead of completely rejecting the deteriorated standby unit. The current study holds paramount significance as it extends its focus beyond one or two unit systems to encompass generalized machining systems with arbitrary number of active and standby units. Notably, this investigation uniquely considers the degradation of high-priced standby units while in the spare state, presenting a novel and comprehensive approach. This study has practical significance in various cost efficient domains, including batteries, security systems, and industrial units that use plastic and rubber components, as these materials deteriorate over time.

The primary objectives of this study are as follows: (i) develop a Markovian model that incorporates realistic qualities of machining systems, such as failures, standby failures in the spare state, and failures of operating standby units; (ii) employ mathematical concepts like linear algebra and matrix method to simplify the inflow and outflow balance equations derived from the Markovian model; and (iii) demonstrate the computational feasibility of cost optimization and system availability.

The remainder of this chapter is structured as follows: Section 4.2 provides a concise model description, including assumptions, mathematical notations, and steady-state equations of the governing model. Section 4.3 solves the steady-state probability equations using the matrix analytic method and recursively obtains the state probabilities. Section 4.4 presents the system's queueing characteristics and the expected cost function. Section 4.5 introduces the teaching-learning based optimization technique, a nature-inspired metaheuristic technique. Section 4.6 presents numerical analysis and cost optimization. Finally, Section 4.7 concludes the inquiry by summarizing the significant aspects and relevance of the current study.

4.2 Model Description

Our analysis demonstrates that outright rejecting the deteriorating standby unit does not lead to favorable cost outcomes. Mitigating highly-priced standby unit deterioration requires a strategic blend of measures, including adept maintenance strategies, vigilant condition monitoring, and informed decisions on replacements and repairs. Essential considerations encompass regular inspection and monitoring, predictive maintenance, optimal storage and environmental conditions, rotational deployment of standby units, maintenance optimization, adaptive replacement policies, life-cycle cost analysis, advanced material adoption, rigorous research and development, skillful training, and comprehensive documentation. Employing these strategies synergistically enables organizations to effectively curtail standby unit deterioration, optimizing the dependability and efficacy of systems reliant on such units. The underpinning of these endeavors is underscored by the imperative of robust mathematical modeling.

This section presents the Markov model customized for a multi-unit machining system incorporating active redundancy of M identical active units and exorbitant passive redundancy of S standby units with a single repairer. The model accounts for the two types of failure of standby units based on their states. The research article rely on the realistic premise of deterioration of high-priced standby unit overtime that the failure rate of an active unit differs from that of a replaced standby unit. Both active and standby units are susceptible to failures, and their time-to-failure is assumed to follow exponential distributions with parameters λ and ν, respectively. The following subsections elaborate on the underlying assumptions and system parameters.

Unit Failure Process

The machining system comprises active units, standby units, and deteriorating standby units, all of which are susceptible to failure under the following assumptions.

- The active units are subject to failures, and the inter-failure time of active units follows an exponential distribution with a mean of $\frac{1}{\lambda}$.
- It is assumed that standby units may fail before entering the operating state. The time-to-failure of warm standby units follows an exponential distribution with a mean of ν^{-1} $(0 < \nu < \lambda)$ in the spare state. The advantage of incorporating warm standby units in this assumption lies in its potential for generalization to all types of standby units.
- When an active unit fails, it is immediately replaced by an available standby unit. Once a standby unit transitions to the operating state, its failure characteristics may no longer identical to those of an active unit due to deterioration in spare state.
- The switching of standby units is assumed to be flawless. The hypothesis of switching failures could be considered in a broader scope of the study.
- The inter-failure duration of replaced units (the standby unit in operating state) is assumed to follow an exponential distribution with a mean of ${\lambda_1}^{-1}$ $(\nu < \lambda_1 \; \& \; \lambda \leq \lambda_1)$.

Unit Repair Facility

Upon failure, all types of units undergo repair, guided by the subsequent assumptions.

- A repair facility is maintained by a reliable repairer to ensure uninterrupted operation of the system. The current study's scope could be expanded to encompass scenarios involving unreliable and/or multiple repairers.
- When an operating unit fails, it is sent for repair. If the repairer is idle, the failed unit is repaired immediately; otherwise, it joins a queue, awaits repair, and is serviced on a first-come, first-served (FCFS) basis.
- The repair time of a failed unit follows an exponential distribution with a mean of μ^{-1}.
- The repair process is assumed to be perfect, resulting in a restored unit functioning equivalently to new. Its subsequent role as an active or standby unit is contingent upon the system's state. The current research concept could potentially be extended to investigate scenarios involving unreliable service postulates.

There is no statistical correlation between the failure and repair processes of the machining system. The comprehensive study could potentially be extended to encompass non-Markovian assumptions as well.

Let $I_1(t)$ be the number of active units in the system at time t, $I_2(t)$ be the number of standby units in the system at time t, and $I_3(t)$ be the number of standby units in the operating state at time t. Thus, the system state at time t can be described as a three-dimensional Markov process $\Xi = (I_1(t) = i, I_2(t) = j, I_3(t) = k; t \geq 0)$ with a state space defined as $\Pi = \{(i,j,k) | i = 0,1,2,\cdots,M-1,M \ \& \ j = 0,1,2,\cdots,S-1,S \ \& \ k = 0,1,2,\cdots,M-i-1,M-i\}$. In the steady state, the steady-state probabilities can be represented by the equations:

$$P_{i,j,k} = \lim_{t\to\infty} \Pr(I_1(t) = i, I_2(t) = j, I_3(t) = k); \quad \{i = 0,1,2,\ldots,M-1,M \ ; \ j = 0,1,2, \ldots, S-1,S \ ; \ k = 0,1,2,\ldots,M-i\}$$

To determine the steady-state probabilities of the multi-unit machining system with standby deterioration, we derive the Chapman-Kolmogorov difference equations based on the inflow and outflow rates within the proposed model.

4.2.1 Steady-state equations

In this section, we establish the governing system of equations by equating the rates of inflow and outflow for each state, ensuring a balanced representation.

When system is in intial state

At the outset, the system consists of M active units and S standby units, all operating in a satisfactory manner. There are no standby units in a deteriorating state within the system.

$$-(M\lambda+S\nu)P_{M,S,0}+\mu P_{M,S-1,0}=0 \tag{4.1}$$

When system is in operating mode

Certain active or standby units within the system are non-functional.

$$-(M\lambda_1+S\nu)P_{0,S,M}+\mu P_{0,S-1,M}=0 \tag{4.2}$$

$$-(M\lambda_1+(S-j)\nu+\mu)P_{0,S-j,M}+\lambda P_{1,S-j+1,M-1}+ (M\lambda_1+(S-j+1)\nu)P_{0,S-j+1,M}+\mu P_{0,S-j-1,M}=0\,; \quad 1\le j\le S-1 \tag{4.3}$$

$$-(M\lambda_1+\mu)P_{0,0,M}+\lambda P_{1,1,M-1}+(M\lambda_1+\nu)P_{0,1,M}=0 \tag{4.4}$$

$$-((M-j)\lambda_1+\mu)P_{0,0,M-j}+(M-j+1)\lambda_1P_{0,0,M-j+1}+\lambda P_{1,0,M-j}=0 \;;\; 1\le j\le M-1 \tag{4.5}$$

$$-(i\lambda+k\lambda_1+S\nu)P_{i,S,k}+\mu P_{i,S-1,k}=0\,; \quad 1\le i\le M-1 \quad \&\, i+k=M \tag{4.6}$$

$$-(M\lambda+(S-j)\nu+\mu)P_{M,S-j,0}+\mu P_{M,S-j-1,0}+(S-j+1)\nu P_{M,S-j+1,0}=0 \;;1\le j\le S-1 \tag{4.7}$$

$$-(M\lambda+\mu)P_{M,0,0}+\mu P_{M-1,0,0}+\nu P_{M,1,0}=0 \tag{4.8}$$

$$-(i\lambda+\mu)P_{i,0,0}+(i+1)\lambda P_{i+1,0,0}+\lambda_1P_{i,0,1}+\mu P_{i-1,0,0}=0; \quad 1\le i\le M-1 \tag{4.9}$$

$$-(i\lambda+(S-j)\nu+k\lambda_1+\mu)P_{i,S-j,k}+(i+1)\lambda P_{i+1,S-j+1,k-1}+\mu P_{i,S-j-1,k}+ (k\lambda_1+(S-j+1)\nu)P_{i,S-j+1,k}=0\,;\; 1\le i\le M-1,\; 1\le j\le S-1 \;\&\; i+k=M \tag{4.10}$$

$$-(i\lambda+k\lambda_1+\mu)P_{i,0,k}+(i+1)\lambda P_{i+1,1,k-1}+(k\lambda_1+\nu)P_{i,1,k}+\mu P_{i-1,0,k}=0 \;;1\le i\le M-1 \;\&\; i+k=M \tag{4.11}$$

$$-(i\lambda+(k-j)\lambda_1+\mu)P_{i,0,k-j}+(i+1)\lambda P_{i+1,0,k-j}+(k-j+1)\lambda_1P_{i,0,k-j+1}+ \mu P_{i-1,0,k-j}=0\,;\; 1\le i\le M-2\,,\; 1\le j\le k-1 \;\&\; i+k=M \tag{4.12}$$

When system is in failed state

At this state, there are no operational active, standby, or deteriorating units within the system.

$$-\mu P_{0,0,0}+\lambda P_{1,0,0}+\lambda_1P_{0,0,1}=0 \tag{4.13}$$

By employing the principle of total probabilities, we can establish the normalization condition for the probabilities in the steady state as follows:

$$\sum_{i=0}^{M}\sum_{j=0}^{S}\sum_{k=0}^{M-i} P_{i,j,k} = 1 \tag{4.14}$$

In this section, we have formulated the steady-state probability equations. To compute these probabilities, an appropriate methodology must be applied. In the subsequent section, the matrix analytical method is employed to derive numerical values for state probabilities. The intricate nature of the problem precludes the derivation of explicit expressions for state probabilities.

4.3 Matrix Analytic Method

The systematic approach and techniques employed involve gathering, analyzing, and interpreting the proposed model to investigate and understand the system's behavior. This encompasses an overarching framework and procedures to determine state probabilities and subsequently evaluate performance metrics, ensuring the reliability, validity, and rigor of the study's findings. In this section, we leverage the concept of standby units deterioration to analyze the steady-state problem of machine repair. The steady-state probabilities of a Markov process can be effectively computed using the recursive matrix method. This technique capitalizes on the idea that a complex system of equations can be simplified by decomposing it into smaller subsystems that exhibit the same underlying structure. By formulating the block tridiagonal matrix **Q** through Eqns. 4.1 to 4.13, we establish a transition rate matrix **Q** that facilitates the calculation of steady-state probabilities for the multiunit redundant machine repair system. The transition rate matrix **Q** is a square matrix of order $(M+1)$.

$$\mathbf{Q} = \begin{pmatrix}
\mathbf{X}_0 & \mathbf{Y}_0 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\
\mathbf{Z}_0 & \mathbf{X}_1 & \mathbf{Y}_1 & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{Z}_1 & \mathbf{X}_2 & \mathbf{Y}_2 & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Z}_{i-1} & \mathbf{X}_i & \mathbf{Y}_i & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Z}_i & \mathbf{X}_{i+1} & \mathbf{Y}_{i+1} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Z}_{i+1} & \mathbf{X}_{i+2} & \mathbf{Y}_{i+2} & \mathbf{0} & \dots & \mathbf{0} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{Z}_{M-3} & \mathbf{X}_{M-2} & \mathbf{Y}_{M-2} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Z}_{M-2} & \mathbf{X}_{M-1} & \mathbf{Y}_{M-1} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Z}_{M-1} & \mathbf{X}_M
\end{pmatrix}$$

The main diagonal block entries, denoted as $\mathbf{X}i$, represent square matrices of size $(S+1+i)$, where $0 \le i \le M$. The first block diagonal below, denoted as $\mathbf{Z}_i$, corresponds to square matrices of size $(S+1+i)$, with $0 \le i \le M-1$. Similarly, the first block diagonal above, represented by $\mathbf{Y}_i$, is a matrix of size $(S+1+i) \times (S+2+i)$, where $0 \le i \le M-1$. These block entries collectively constitute the transition matrix $\mathbf{Q}$, which characterizes the dynamics of the system.

$$\mathbf{X}_0 = \begin{bmatrix} -(M\lambda+S\nu) & S\nu & 0 & 0 & \dots & 0 \\ \mu & -(M\lambda+(S-1)\nu+\mu) & (S-1)\nu & 0 & \dots & 0 \\ 0 & \mu & -(M\lambda+(S-2)\nu+\mu) & (S-2)\nu & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \mu & -(M\lambda+2\nu+\mu) & 2\nu & 0 \\ 0 & 0 & 0 & \mu & -(M\lambda+\nu+\mu) & \nu \\ 0 & 0 & 0 & 0 & \mu & -(M\lambda+\mu) \end{bmatrix}$$

$$\mathbf{X}_i = \begin{pmatrix} A_{i,0} & B_{i,0} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \mu & A_{i,1} & B_{i,1} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \mu & A_{i,2} & B_{i,2} & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \mu & A_{i,S-1} & B_{i,S-1} & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \mu & A_{i,S} & B_{i,S} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu & C_{i,0} & D_{i,0} & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_{i,1} & D_{i,1} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & C_{i,i-2} & D_{i,i-2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & C_{i,i-1} & D_{i,i-1} \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & C_{i,i} \end{pmatrix}; \; 1 \le i \le M$$

where

$$A_{i,0} = -[(M-i)\lambda + S\nu + i\lambda_1], \quad A_{i,j} = -[(M-i)\lambda + (S-j)\nu + i\lambda_1 + \mu], \quad 1 \le j \le S$$
$$C_{i,j} = -[(M-i)\lambda + (i-j)\lambda_1 + \mu], \quad B_{i,j} = -[(S-j)\nu + i\lambda_1], \quad 0 \le j \le S,$$
$$D_{i,j} = -(i-j)\lambda_1, \; 1 \le j \le i$$

$$\mathbf{Y}_i = \begin{bmatrix} 0 & (M-i)\lambda & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & (M-i)\lambda & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & (M-i)\lambda & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & (M-i)\lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & (M-i)\lambda \end{bmatrix} \;\&\; \mathbf{Z}_i = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \mu \end{bmatrix}$$

where, $0 \leq i \leq M-1$

Let $\mathbf{P}$ represent the stationary probability vector corresponding to $\mathbf{Q}$, which can be partitioned as $\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2, \cdots, \mathbf{P}_{M-1}, \mathbf{P}_M$. The subvector $\mathbf{P}_i = [P_{M-i,S,i}, P_{M-i,S-1,i}, P_{M-i,S-2,i}, \cdots, P_{M-i,1,i}, P_{M-i,0,i}, P_{M-i,0,i-1}, P_{M-i,0,i-2}, \cdots, P_{M-i,0,1}, P_{M-i,0,0}]$; where $0 \leq i \leq M$, has a dimension of order $1 \times (S+1+i)$. The balance equations can be expressed in matrix form, known as the matrix form of the stationary probability equations.

$$\mathbf{PQ} = \mathbf{0} \tag{4.15}$$

where, $\mathbf{0}$ represents the zero vector of dimension $(M+1)$, and $\mathbf{Q}$ denotes the transition rate matrix associated with the Markov process Ξ. It is important to note that the initial condition for the system is as follows:

$$\mathbf{Pe} = 1 \tag{4.16}$$

Based on Equation 4.15, the computation of the stationary probability vector $\mathbf{P}$ involves solving the following set of equations.

$$\mathbf{P}_0\mathbf{X}_0 + \mathbf{P}_1\mathbf{Y}_0 = \mathbf{0} \tag{4.17}$$
$$\mathbf{P}_0\mathbf{Z}_0 + \mathbf{P}_1\mathbf{X}_1 + \mathbf{P}_2\mathbf{Y}_1 = \mathbf{0} \tag{4.18}$$
$$\mathbf{P}_1\mathbf{Z}_1 + \mathbf{P}_2\mathbf{X}_2 + \mathbf{P}_3\mathbf{Y}_2 = \mathbf{0} \tag{4.19}$$
$$\mathbf{P}_{i-1}\mathbf{Z}_{i-1} + \mathbf{P}_i\mathbf{X}_i + \mathbf{P}_{i+1}\mathbf{Y}_i = \mathbf{0};\ 2 \leq i \leq M-2 \tag{4.20}$$
$$\mathbf{P}_{M-2}\mathbf{Z}_{M-2} + \mathbf{P}_{M-1}\mathbf{X}_{M-1} + \mathbf{P}_M\mathbf{Y}_{M-1} = \mathbf{0} \tag{4.21}$$
$$\mathbf{P}_{M-1}\mathbf{Z}_{M-1} + \mathbf{P}_M\mathbf{X}_M = \mathbf{0} \tag{4.22}$$

The solution is obtained by employing matrix manipulation techniques to derive the state probabilities in vector form. As the matrix $\mathbf{X}_0$ is invertible, Eqn. 4.17 yields the following result..

$$\mathbf{P}_0 = \mathbf{P}_1\mathbf{V}_0 \quad \text{where} \quad \mathbf{V}_0 = -\mathbf{X}_0^{-1}\mathbf{Y}_0 \tag{4.23}$$

From Eqns. 4.18 and 4.23, we have the following results

$$\mathbf{P}_1 = \mathbf{P}_2\mathbf{V}_1 \quad \text{where} \quad \mathbf{V}_1 = -(\mathbf{V}_0\mathbf{Z}_0 + \mathbf{X}_1)^{-1}\mathbf{Y}_1 \tag{4.24}$$

Similarly, from Eqns. 4.19 and 4.24, we obtain:

$$\mathbf{P}_2 = \mathbf{P}_3\mathbf{V}_2 \quad \text{where} \quad \mathbf{V}_2 = -(\mathbf{V}_1\mathbf{Z}_1 + \mathbf{X}_2)^{-1}\mathbf{Y}_2 \tag{4.25}$$

By further applying Eqns. 4.20 and 4.25, we can iteratively derive the following results:

$$\mathbf{P}_i = \mathbf{P}_{i+1}\mathbf{V}_i \quad \text{where} \quad \mathbf{V}_i = -(\mathbf{V}_{i-1}\mathbf{Z}_{i-1} + \mathbf{X}_i)^{-1}\mathbf{Y}_i;\ 2 \leq i \leq M-2 \tag{4.26}$$

Furthermore, from Eqns. 4.21 and 4.26, we find:

$$\mathbf{P}_{M-1} = \mathbf{P}_M\mathbf{V}_{M-1} \quad \text{where} \quad \mathbf{V}_{M-1} = -(\mathbf{V}_{M-2}\mathbf{Z}_{M-2} + \mathbf{X}_{M-1})^{-1}\mathbf{Y}_{M-1} \tag{4.27}$$

The state probabilities $\mathbf{P}_i$ for $0 \leq i \leq M$ can be expressed in terms of the state probability $\mathbf{P}_M$ by recursively computing Eqns. 4.23 - 4.27:.

$$\mathbf{P}_i = \mathbf{P}_{i+1}\mathbf{V}_i = \mathbf{P}_{i+2}\mathbf{V}_i\mathbf{V}_{i+1} = \cdots = \mathbf{P}_M \prod_{\xi=1}^{M-i} \mathbf{V}_{M-\xi} = \mathbf{P}_M \tau_i^* \tag{4.28}$$

Here, $\tau_i^* = \prod_{\xi=1}^{M-i} \mathbf{V}_{M-\xi}$ and $\mathbf{V}_i$ for $0 \leq i \leq M$ are given in the aforementioned equations. Combining Eqns. 4.17 and 4.28, we obtain:

$$\begin{aligned}
\sum_{i=0}^{M} \mathbf{P}_i\mathbf{e}_i &= \mathbf{P}_0\mathbf{e}_0 + \mathbf{P}_1\mathbf{e}_1 + \cdots + \mathbf{P}_{M-1}\mathbf{e}_{M-1} + \mathbf{P}_M\mathbf{e}_M \\
&= \mathbf{P}_M\tau_0^*\mathbf{e}_0 + \mathbf{P}_M\tau_1^*\mathbf{e}_1 + \cdots + \mathbf{P}_{M-1}\tau_{M-1}^*\mathbf{e}_{M-1} + \mathbf{P}_M\tau_M^*\mathbf{e}_M \\
&= \mathbf{P}_M\left[\sum_{i=0}^{M} \tau_i^*\mathbf{e}_i\right] = 1
\end{aligned} \tag{4.29}$$

Here, $\mathbf{e}_i = [1, 1, \cdots, 1, 1]^T_{1\times(S+1+i)}$ for $0 \leq i \leq M$. Therefore, Eqn. 4.22 can be expressed as:

$$\mathbf{P}_M\left[\mathbf{V}_{M-1}\mathbf{Z}_{M-1} + \mathbf{X}_M\right] = \mathbf{0} \tag{4.30}$$

By solving Eqns. 4.29 - 4.30, we can determine the state probability $\mathbf{P}_M$. Hence, we can obtain the steady-state probabilities for $\mathbf{P}_i$; $i = 0, 1, 2, ..., M$. The forthcoming section focuses on the queueing characteristics of the system, and the cost function is also constructed.

4.4 System characteristics

The fundamental objective of modeling the studied multi-unit redundant machining system is to assess various queueing measures that can effectively capture and characterize the system's behavior, thereby offering valuable insights for improving its overall performance. This section presents range of measures that assess the efficiency, effectiveness, and behavior of a queueing system of failed units in the studied machining system with standby deterioration. These metrics provide insights into various aspects of system performance and help in evaluating and optimizing the system's functioning. Some key performance metrics in the context of queueing theory include expected queue length, throughput, etc. ,as performance indicators to evaluate the system's efficiency quantitatively. By employing these performance indices, we comprehensively understand the system's operational dynamics and its impact on overall performance. The details along with the corresponding mathematical expressions are presented in this section.

- The expected number of failed units E_N in the system E_N is given as

$$E_N = \sum_{i=0}^{M} \left[\sum_{j=1}^{S} (S-j) P_{i,j,M-i} + \sum_{k=0}^{M-i-1} (M-i+S-k) P_{i,0,k} \right] \tag{4.31}$$

- The throughput of the system T_P is expressed as

$$Th = \sum_{i=0}^{M} \sum_{j=0}^{S-1} \mu P_{i,j,M-i} + \sum_{i=0}^{M-1} \sum_{k=0}^{M-i-1} \mu P_{i,0,k} \tag{4.32}$$

- The expected number of standby units in the system E_S is given by

$$E_S = \sum_{i=0}^{M} \sum_{j=0}^{S-1} (S-j) P_{i,S-j,M-i} \tag{4.33}$$

- The mean number of active units in the system E_O is defined as

$$E_O = \sum_{i=0}^{M} i \left[\sum_{j=1}^{S} P_{i,j,M-i} + \sum_{k=0}^{M-i} P_{i,0,k} \right] \tag{4.34}$$

- The mean number of operating standby units in the system E_{OS} is given by

$$E_{OS} = \sum_{k=1}^{M} k \left[\sum_{j=1}^{S} P_{M-k,j,k} + \sum_{i=0}^{M-k} P_{i,0,k} \right] \tag{4.35}$$

- The effective failure rate $E_{fr}$is defined as

$$E_{fr} = \left[\sum_{i=0}^{M} \sum_{j=1}^{S} (i\lambda + (M-i)\lambda_1 + j\nu) P_{i,j,M-i} + \sum_{i=0}^{M-1} \sum_{k=0}^{M-i} (i\lambda + k\lambda_1) P_{i,0,k} \right] \tag{4.36}$$

- The Expected waiting time of the failed units in the system E_W is given by

$$E_W = \frac{E_N}{E_{fr}} \tag{4.37}$$

- The delay time of failed units E_D is expressed as

$$E_D = \frac{E_N}{Th} \tag{4.38}$$

- The availability of the system MA is given by

$$MA = 1 - \frac{E_N}{M+S} \tag{4.39}$$

4.4.1 Cost Analysis

The primary objective of conducting cost analysis for machining systems is to minimize the overall cost of the system while ensuring its availability over a specified period. To achieve this, we have formulated the steady-state expected cost function per unit of time, considering various cost elements associated with the different states of the system. The cost elements associated with the different states of the system are defined as follows:

$C_1 \equiv$ Cost per unit time for a failed unit present in the system.

$C_2 \equiv$ Cost per unit time for an operating standby unit present in the system.

$C_3 \equiv$ Cost per unit time for a standby unit available in the system.

$C_4 \equiv$ Cost per unit time for a repaired unit present in the system.

$C_5 \equiv$ Cost per unit time for repairing a failed unit.

The expected cost function is defined as follows, expressing the cost per unit of time in terms of the relevant cost elements.

$$TC = C_1 \times E_N + C_2 \times E_{OS} + C_3 \times E_S + C_4 \times E_O + C_5 \times \mu \tag{4.40}$$

Therefore, the optimization problem can be formulated as follows:

$$TC^*(S^*, \mu^*) = \min_{S,\mu} TC \tag{4.41}$$

4.5 Teaching-Learning based Optimization(TLBO)

The concept of the global optimization method known as the Teaching-Learning based Optimization (TLBO) algorithm was initially proposed by Rao et al. ([236], [237]) to optimize highly nonlinear functions. This algorithm follows a population-based approach, where the influence of a teacher or class on learners' performance in a classroom is utilized. The TLBO algorithm employs a population of solutions to search for the global optimum. In this context, a group of students in a classroom represents the TLBO population. Similar to other population-based optimization methods, TLBO involves several design variables associated with different subjects taught to the learners, and the learners' fitness is determined based on their performance. The best solution obtained through TLBO corresponds to the teacher, who is considered an intellectual member of society. The algorithm operates in two phases: the Teacher Phase (T-Phase) and the Learner Phase (L-Phase). During the T-Phase, learners acquire knowledge from the teacher, while the L-Phase focuses on learning through interactions among the learners. The T phase and L phase of Teaching-Learning Based Optimization (TLBO) algorithm are responsible for conducting exploration and exploitation, respectively, in the context of meta-heuristic optimization. In the following subsection, we provide a concise overview of the execution of this optimization method.

4.5.1 Mathematical Model and Algorithm

We use the following notations to describe TLBO:

- L : Total number of learners in the class i.e. "Class Size"
- H : Total number of courses offered to the learners
- I_T : Maximum number of iterations

The TLBO algorithm starts by defining a search space represented by a matrix with L rows and H columns, which initializes the population X. The j^{th} parameter of the i^{th} learner is randomly generated using the following equation:

$$X_{(i,j)}^{0} = X_j^{min} + r(X_j^{max} - X_j^{min}) \tag{4.42}$$

Here, r is a random number uniformly distributed in the range $[0,1]$, and $X^{min} j$ and $X^{max} j$ represent the minimum and maximum values of the j^{th} parameter, respectively. For generation G, the parameters of the i^{th} learner are represented as:

$$\chi_{(i)}^{G} = \left[X_{(i,1)}^{G}, X_{(i,2)}^{G}, X_{(i,3)}^{G}, \cdots, X_{(i,j)}^{G}, \cdots, X_{(i,H)}^{G}\right] \tag{4.43}$$

Teacher Phase ($T-$Phase)

At generation G, the mean parameters M^G of the learners for each subject in the class are calculated as:

$$M^{G} = \left[m_1^{G}, m_2^{G}, m_3^{G}, \cdots, m_j^{G}, \cdots, m_H^{G}\right] \tag{4.44}$$

The learner with the minimum objective function value for the current iteration is considered as the teacher $\chi^G Teacher$. The algorithm aims to improve the positions of other individuals $(\chi^G i)$ by moving them closer to the position of $\chi^G Teacher$, while considering the current mean value of the individuals $(\chi mean)$. This is achieved by using the mean values of each parameter within the problem space and constructing attributes for all learners in the current generation. Two random parameters, T_f and r, are used, where T_f can be either 0 or 1, and r is in the range $[0,1]$. A random weighted difference vector is created based on the actual and desired mean parameters to generate a set of optimal learners, which are then added to the existing population.

$$\chi_{new}{}^{G}_{(i)} = \chi_i^G + r(\chi_{Teacher}^G - T_f M^G) \tag{4.45}$$

The value of T_f is determined heuristically and randomly using the following equation:

$$T_f = round\,[1 + r(0,1)\{2-1\}] \tag{4.46}$$

If $\chi_{new}{}^G(i)$ in generation G is found to be a better learner than $\chi^G(i)$, the inferior learner is replaced by the better one in the matrix.

Learner Phase ($L-$Phase)

In this phase, learners interact with each other to enhance their knowledge. Two types of interactions occur: learners receive input from the teacher and learners interact with each other. During the interactions, if a learner has less knowledge compared to others, they improve their knowledge by learning from the better-performing learner. A learner $\chi^G(r)$ is randomly selected for a given learner $\chi^G(i)$ $(i \neq r)$. In the learner phase, the j^{th} parameter of the matrix χ_{new} is updated as follows:

For $i = 1 : N$
  Randomly select two learners χ_i^G and χ_r^G, where $(i \neq r)$
  if $f(\chi_{(i)}^G) \leq f(\chi_{(r)}^G)$
    $\chi_{new}{}^{G}_{(i)} = \chi_{(i)}^G + r\left(\chi_{(i)}^G - \chi_{(r)}^G\right)$
  else
    $\chi_{new}{}^{G}_{(i)} = \chi_{(i)}^G + r\left(\chi_{(r)}^G - \chi_{(i)}^G\right)$
  end if
end for

Here, N represents the population size.

The updated $\chi_{new}{}^G(i)$ is accepted if it provides a better solution, and the algorithm terminates after a maximum number of iterations (IT).

The pseudo-code for Teaching-Learning based optimization algorithm is as follows:

1 Input Objective function $\chi^{LB}, \chi^{UB}, N, I_T$
2 Initialize a random population $[P]$
3 Evaluate Objective function F of $[P]$
4 for $t = 1$ to T
    for $i = 1$ to N_P
        Select $\chi^G_{Teacher}$
        Determine χ^G_{mean}
            $\chi_{new}{}^G_{(i)} = \chi^G_{(i)} + R * (\chi^G_{Teacher} - T_f \chi^G_{mean})$
        Bound $\chi_{new}{}^G_{(i)}$ and evaluate the objective function $F(\chi_{new}{}^G_{(i)})$
        Accept $\chi_{new}{}^G_{(i)}$ if it is better than $\chi^G_{(i)}$
        Chose any solution randomly, $\chi^G_{(r)}$ $(r \neq i)$
        Determine $\chi_{new}{}^G_{(i)}$ as :
            If $F(\chi^G_{(i)}) < F(\chi^G_{(r)})$
                $\chi_{new}{}^G_{(i)} = \chi^G_{(i)} + R * \left(\chi^G_{(i)} - \chi^G_{(r)}\right)$
            else
                $\chi_{new}{}^G_{(i)} = \chi^G_{(i)} + R * \left(\chi^G_{(r)} - \chi^G_{(i)}\right)$
        Bound $\chi_{new}{}^G_{(i)}$ and ecaluate the objective function $F(\chi_{new}{}^G_{(i)})$
        Accept $\chi_{new}{}^G_{(i)}$ if it is better than $\chi^G_{(i)}$ to update (P)
            end
    end
end

4.6 Numerical illustration

In the following section, we provide a comprehensive numerical illustration to demonstrate the practical application and effectiveness of the proposed notion of advantages of detriorating standby units in the multi-unit machining system. Completely discarding the deteriorating big-budget standby unit does not lead to favorable cost outcomes, as demonstrated by our comprehensive analysis. Through this illustrative example, we aim to highlight the real-world implications of our research findings and showcase the potential benefits of incorporating the proposed notion. By applying the developed mathematical models to a relevant scenario, we seek to provide valuable insights into the behavior, performance, and optimization of the studied multi-unit machining system with deteriorating standby units. This numerical illustration serves as a practical validation of our theoretical framework and contributes to the practicality and relevance of our research in the field.

This section presents a numerical exploration to analyze the influence of failure characteristics of both active units and standby units (in spare and operating states) on various performance measures. Additionally, we exhibit sensitivity of a cost function and employ nature-inspired techniques to determine the optimal cost of the system. We generate plots for performance metrics including E_N, E_{OS}, Th, W_S, T_C, and MA by varying the failure characteristics and repair facility. For illustrative purpose, the system parameters are fixed at $M = 20$, $S = 8$, $\lambda = 4$, $\nu = 1.8$, $\lambda_1 = 3$, and $\mu = 75$, which are given in the published articles mentioned in the introduction. We implement a MATLAB R2020b program to simulate the performance indices for different input parameters.

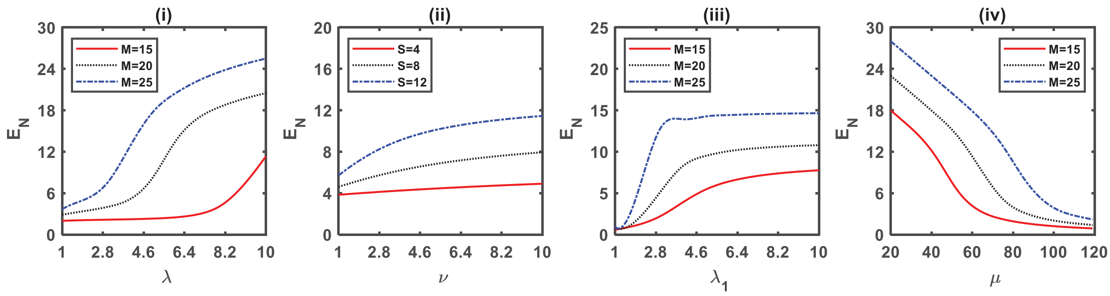


Figure 4.1: Expected number of failed units in the system (E_N) for different parameters, (For $M = 20, S = 8, \lambda = 4, \nu = 1.8, \lambda_1 = 3, \mu = 75$)

Figure 4.1(i & ii) illustrates that the mean number of failed units initially increases with the failure rate of active units and standby units. The results prompt for routine preventive maintenance to delay the failure of units. After reaching a certain threshold of failure rate, the mean number of failed units stabilizes. It demonstrates the steady-state behavior of the system that validates our modelling. Similarly, Figure 4.1(iii) demonstrates a rapid increase in the mean number of failed units with the failure rate of operating standby units, which then becomes constant after a certain rate. The results recommend to ensure that highly-priced standby units are stored in suitable conditions to prevent unnecessary deterioration. Factors such as temperature, humidity, and exposure to corrosive substances can accelerate deterioration. Proper storage can extend the lifespan of standby units. Figure 4.1(iv) shows an inverse relationship between the repair rate and the system's expected number of failed units. The findings provoke for just-in-time corrective measures for maintenance. The observed graphical patterns in all the illustrations align with the principles of machining systems and reliability theory.

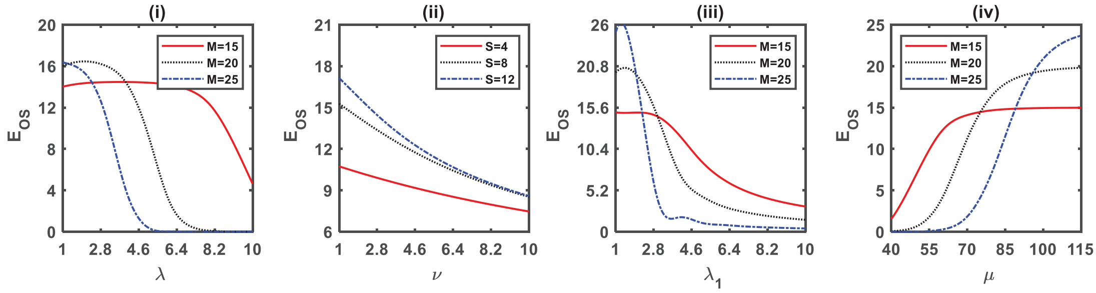


Figure 4.2: Expected number of operating standby units in the system (E_{OS}) for different parameters, (For $M = 20$, $S = 8$, $\lambda = 4$, $\nu = 1.8$, $\lambda_1 = 3$, $\mu = 75$)

Figure 4.2(iv) depicts the increase in the mean number of operating standby units with the effective repair facility. Conversely, the mean number of operating standby units decreases as the failure characteristics of active units, standby units, and operating standby units increase. Figure 4.2(i) demonstrates a rapid decrease in the system's expected number of operating standby units with an increase in the failure rate of active units. After a certain rate, the mean number of operating standby units tends to be low. This pattern is discernible and consistent. Incorporating advanced attributes characterized by enhanced resistance to corrosion, wear, and various forms of degradation can significantly prolong the operational lifespan of standby units, thereby substantially enhancing the overall applicability and utility of the machining system. The results also prompt, if feasible, periodically rotate the use of standby units to distribute wear and tear more evenly. This can help prevent individual units from deteriorating more rapidly than others. The extended degradation of units further emphasizes that choosing to outright dismiss the deteriorating standby unit does not lead to favorable cost-related outcomes, as underscored by our comprehensive analysis.

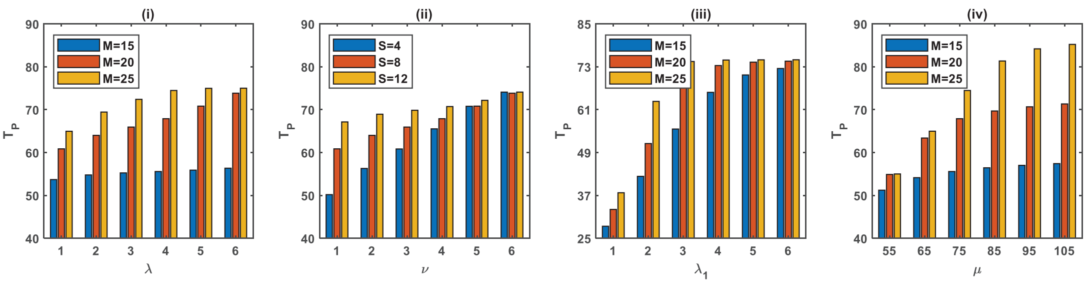


Figure 4.3: Throughput of the system (Th) for the different parameters, (For $M = 20$, $S = 8$, $\lambda = 4$, $\nu = 1.8$, $\lambda_1 = 3$, $\mu = 75$)

Figure 4.3(ii & iii) shows that the system's throughput increases rapidly with the failure rate of standby and operating standby units, and stabilizes after a certain failure rate. The upward trajectory in system throughput reflects an increased demand for repairs as the number of failed units within the system rises. These findings emphasize the importance of establishing a consistent regimen for inspecting and monitoring the state of standby units. Early detection of deterioration enables timely interventions and proactive measures. Moreover, the outcomes aid in predicting the occurrence of deterioration and thus facilitate the scheduling of appropriate maintenance actions. Figure 4.3(i & iv) presents the system's throughput enhancement with the failure rate of active units and the efficiency of repair facilities. As the failure characteristics of units increase, the repair facility also needs to improve to sustain the system in an operational state. The analysis prompts to develop maintenance optimization models that consider the costs and benefits of different maintenance actions, such as preventive maintenance, corrective maintenance, and replacement. These models can help determine the optimal timing for maintenance based on the degradation profile of standby units.

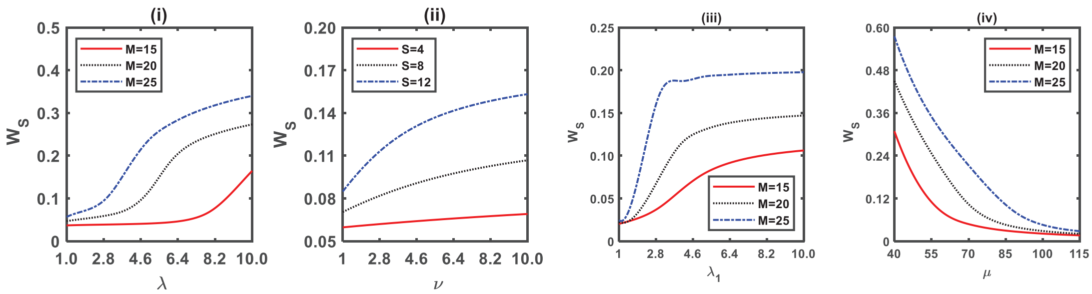


Figure 4.4: Expected waiting time of the system (W_S) for the different parameters, (For $M = 20$, $S = 8$, $\lambda = 4$, $\nu = 1.8$, $\lambda_1 = 3$, $\mu = 75$)

Figure 4.4(i, ii, iii) reveals the proportional relationship between failure characteristics and the system's waiting time, which is obvious. Conversely, Figure 4.4(iv) demonstrates an inverse relationship between the system's waiting time and the repair rate. The study stimulates the consideration of lifecycle cost analyses to compare the expenses associated with maintaining deteriorating exorbitant standby units versus replacing them. This facilitates informed decision-making regarding investment in either maintenance or replacement, taking into account the comprehensive long-term costs and benefits.

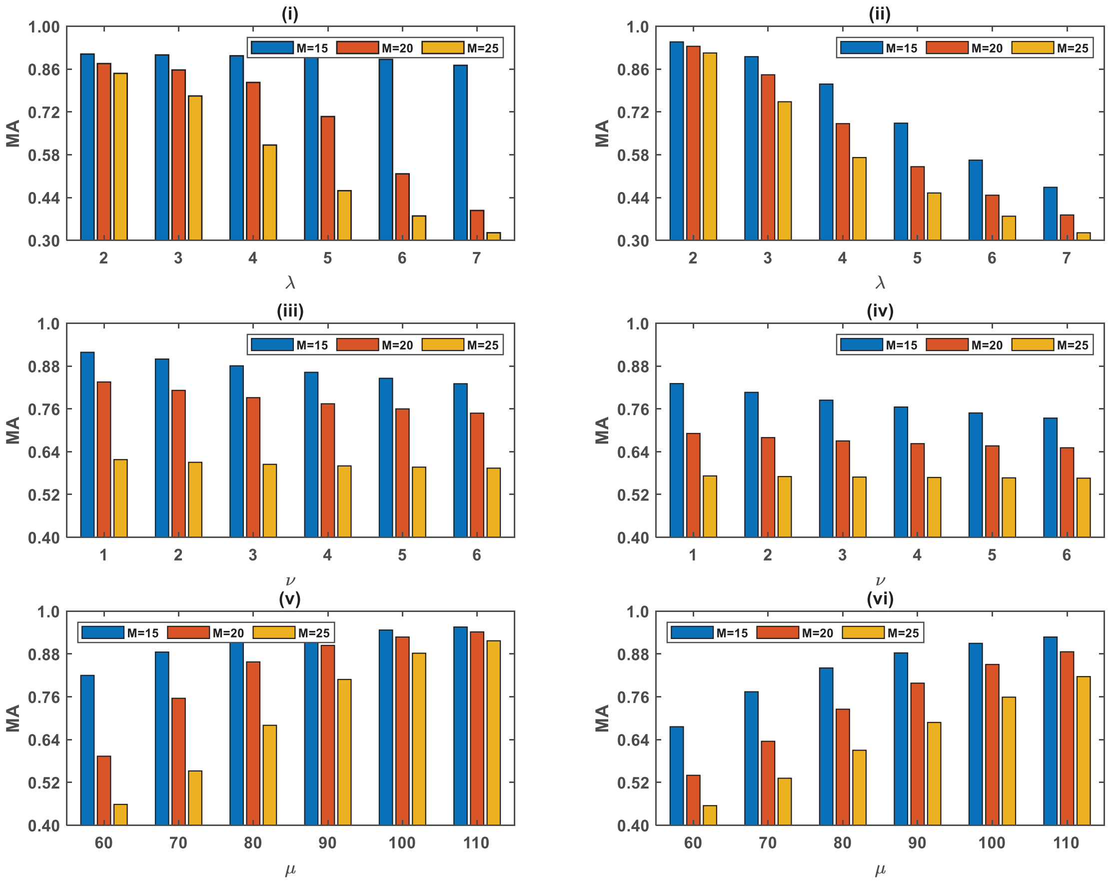


Figure 4.5: Comaprision of Expected Machining availability of the system for proposed model(left side) and general machine repair model (right-side) (MA) for the different parameters, (For $M = 20$, $S = 8$, $\lambda = 4$, $\nu = 1.8$, $\lambda_1 = 3$, $\mu = 75$)

Figure 4.5 provides a comparative analysis of machining system availability considering highly-priced standby deterioration in the context of machine repair problems with warm standbys provisioning, presented as two distinct cases: Case (i) involving a multi-unit machining system with standby units and their deterioration, and Case (ii) involving a multi-unit machining system with standby units but no allowed deterioration. This comparison underscores the crucial importance of our study in informing decision-making processes, emphasizing that the decision not to entirely reject the deteriorating standby unit does not lead to favorable cost-related outcomes. In Figure 4.5 (i, iii, v), the graphical behavior showcases the result for case (i), i.e., the availability of the machining system with standby provisioning and their deterioration. Whereas, Figure 4.5 (ii, iv, vi) represents the result of case (ii), i.e., the availability of the machining system with standby provisioning but without considering their deterioration. Across both cases, the findings illustrate a decline in the machine availability of the system with increasing failure rates. Additionally, the results demonstrate an initial increase in machine availability with the repair rate, followed by stabilization after reaching a certain repair rate. These observed trends validate the efficacy of our proposed model. A direct comparison between the two cases clearly elucidates that the system's availability in case (i) significantly surpasses that of case (ii). This outcome underscores the essential contribution of considering standby deterioration in enhancing the overall availability and reliability of the machining system. The study recommends for not to reject the deteriorating unit outright and minimizing standby unit deterioration involves effective maintenance, predictive strategies, proper storage, rotation, optimized maintenance models, replacement policies, lifecycle cost analysis, advanced materials, research, training, and documentation. Implementing these strategies enhances standby unit reliability and optimizes system performance.

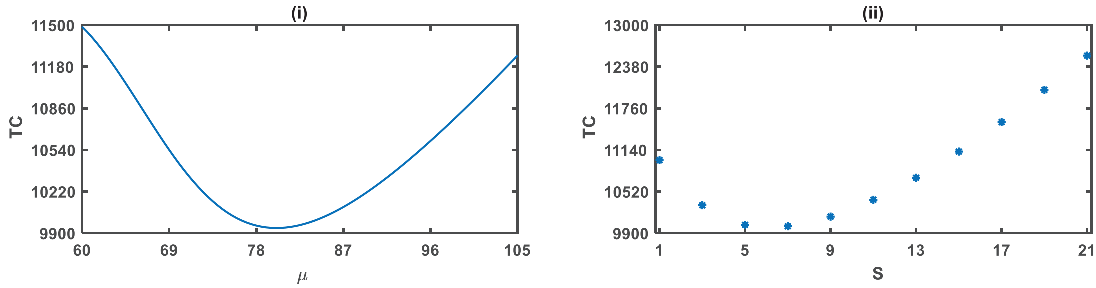


Figure 4.6: Expected total cost of the system (TC) for the different parameters, (For $M = 20$, $S = 8$, $\lambda = 4$, $\nu = 1.8$, $\lambda_1 = 3$, $\mu = 75$, $C_1 = 295$, $C_2 = 30$, $C_3 = 100$, $C_4 = 220$, $C_5 = 90$)

Table 4.1: Optimal expected total cost of the system $(TC(S^*, \mu^*))$ for different parameters using TLBO algorithm, For $C_1 = 295, C_2 = 30, C_3 = 100, C_4 = 220, C_5 = 90$

$M, \lambda, \nu, \lambda_1$	S^*	μ^*	$TC(S^*, \mu^*)$	Mean	Maximum	Time Elapsed
15, 4, 1.8, 3	5	69.00000000	8617.11189180	1.00000041196	1.00000060516	899.54982130
20, 4, 1.8, 3	6	77.16146310	10720.89963510	1.00000015151	1.00000031927	899.74991970
25, 4, 1.8, 3	7	97.04682170	13128.04331680	1.00000090934	1.00000080059	951.31288070
20, 2, 1.8, 3	4	69.00000000	9768.63993000	1.00000024471	1.00000083000	934.18288370
20, 4, 1.8, 3	6	77.16146310	10720.89963510	1.00000020271	1.00000012045	899.74991970
20, 6, 1.8, 3	8	92.25556110	11651.25054900	1.00000077918	1.00000090956	934.43656590
20, 4, 0.9, 3	7	76.93986360	10561.93927290	1.00000029389	1.00000066319	983.95043780
20, 4, 1.8, 3	6	77.16146310	10720.89963510	1.00000010906	1.00000027286	899.7499197
20, 4, 2.7, 3	5	76.35569960	10829.93348960	1.00000045143	1.00000082179	987.6567021
20, 4, 1.8, 3	6	77.16146310	10720.89963510	1.00000081059	1.00000088578	899.7499197
20, 4, 1.8, 5	4	69.00000000	11620.79567780	1.00000020031	1.00000011409	870.3748059
20, 4, 1.8, 7	4	69.00000000	11797.56884690	1.00000031419	1.00000098442	863.6995901

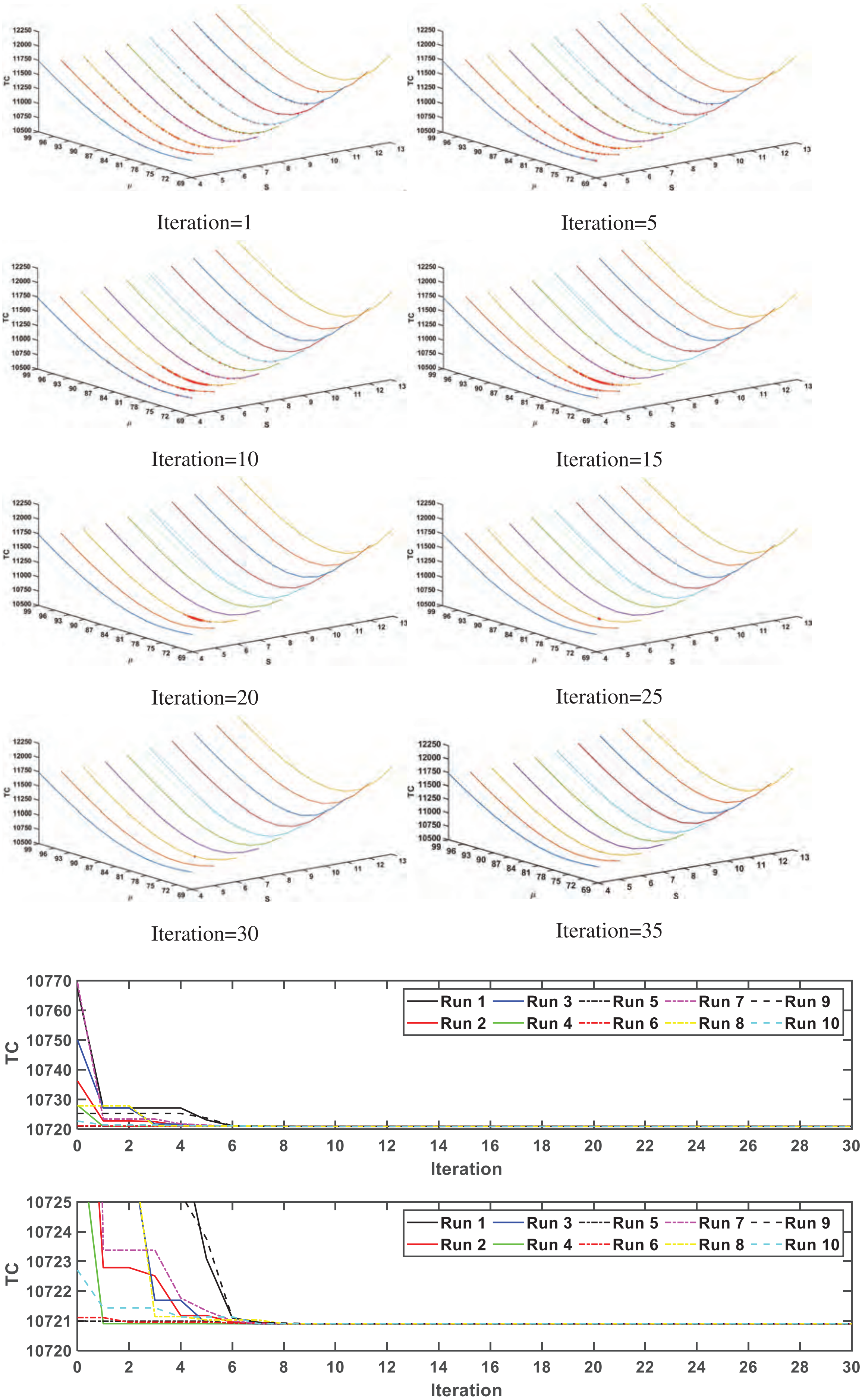


Figure 4.7: Convergence of iteration of Teaching-Learning based Optimization

Table 4.2: Optimal expected total cost of the system $(TC(S^*,\mu^*))$ for different parameters using TLBO algorithm, For $M=20,\lambda=4,\nu=1.8,\lambda_1=3$

C_1,C_2,C_3,C_4,C_5	S^*	μ^*	$TC(S^*,\mu^*)$	Mean	Maximum	Time Elapsed
240, 100, 30, 220, 90	6	74.96138010	10489.22798980	1.00000009600	1.00000052141	923.36916370
295, 100, 30, 220, 90	6	77.16146340	10720.89963510	1.00000088594	1.00000067988	899.74991970
350, 100, 30, 220, 90	6	78.86097810	10930.62964790	1.00000054321	1.00000040023	890.64309570
295, 70, 30, 220, 90	7	79.44917240	10284.53704230	1.00000029123	1.00000069461	945.52531740
295, 100, 30, 220, 90	6	77.16146340	10720.89963510	1.00000058289	1.00000064603	899.74991970
295, 130, 30, 220, 90	6	69.00000000	11017.24555610	1.00000052998	1.00000087073	853.47022050
295, 100, 10, 220, 90	6	77.55162070	10670.62058440	1.00000007797	1.00000073474	715.60339200
295, 100, 30, 220, 90	6	77.16146340	10720.89963510	1.00000033682	1.00000060641	899.74991970
295, 100, 50, 220, 90	5	75.83910930	10763.25880380	1.00000096670	1.00000010818	935.95150790
295, 100, 30, 180, 90	4	69.00000000	10392.27824060	1.00000016773	1.00000082179	858.12666990
295, 100, 30, 220, 90	6	77.16146340	10720.89963510	1.00000031325	1.00000021260	899.74991970
295, 100, 30, 260, 90	7	79.57288920	10903.35585380	1.00000054367	1.00000078785	844.06782680
295, 100, 30, 220, 60	6	82.82309870	8325.28862860	1.00000056897	1.00000048454	887.68834430
295, 100, 30, 220, 90	6	77.16146340	10720.89963510	1.00000055697	1.00000048061	899.74991970
295, 100, 30, 220, 120	4	69.00000000	12869.51517820	1.00000060602	1.00000048204	897.79616760

Table 4.3: Optimal expected total cost of the system $(TC(S^*,\mu^*))$ for different parameters using PSO algorithm, For $C_1=295,C_2=30,C_3=100,C_4=220,C_5=90$

M,λ,ν,λ_1	S^*	μ^*	$TC(S^*,\mu^*)$	Mean	Maximum	Time Elapsed
15, 4, 1.8, 3	5	69.00000000	8617.11189180	1.00000013611	1.00000030554	1348.36518690
20, 4, 1.8, 3	6	77.16146330	10720.89963510	1.00000097010	1.00000052101	1349.45679990
25, 4, 1.8, 3	7	97.04682100	13128.04331680	1.00000011807	1.00000091120	1493.34594530
20, 2, 1.8, 3	4	69.00000000	9768.63993000	1.00000082264	1.00000016694	1329.95894890
20, 4, 1.8, 3	6	77.16146270	10720.89963510	1.00000090634	1.00000011189	1349.52871830
20, 6, 1.8, 3	8	92.25556050	11651.25054900	1.00000059529	1.00000072096	1489.02060450
20, 4, 0.9, 3	7	76.93986400	10561.93927290	1.00000077633	1.00000080469	1581.85008890
20, 4, 1.8, 3	6	77.16146270	10720.89963510	1.00000091759	1.00000041987	1349.52871830
20, 4, 2.7, 3	5	76.35569980	10829.93348960	1.00000079057	1.00000029589	1522.50358090
20, 4, 1.8, 3	6	77.16146270	10720.89963510	1.00000090012	1.00000018322	1349.52871830
20, 4, 1.8, 5	4	69.00000000	11620.79567780	1.00000060249	1.00000082903	1298.49546720
20, 4, 1.8, 7	3	60.00000000	11797.56884690	1.00000078416	1.00000051414	1280.77642290

The following content delineates a cost optimization analysis, evaluating the cost-effectiveness of distinct strategies, while considering factors such as maintenance, replacement, and system performance. For optimal analysis of the cost-optimization function, we set $M=20$, $S=8$, $\lambda=4$, $\nu=1.8$, $\lambda_1=3$, $\mu=75$, $C_1=295$, $C_2=30$, $C_3=100$, $C_4=220$, and $C_5=90$. Figure 4.6(i & ii) depicts both graphs as convex, indicating the existence of an optimal solution within the convex region for the decision variables S and μ. Due to S being discrete and μ continuous, the optimization problem becomes inherently more changeableness. Traditional optimization techniques are not well-suited to address this challenge since total cost function reliance on system performances expressed in terms of state probabilities. We utilize the teaching-learning based optimization (TLBO) technique to obtain the optimal total cost and the corresponding values of the decision variables. For employing TLBO, we set the population size to 300 and performed 10 multiple runs to obtain the optimal decision parameters and total cost values. The collection of graphs in figure 4.7 illustrates the convergence of solution points to optimal point through different intermediate solution points during iterations and demonstrates the convergence to optimal point across multiple runs, considering a range of initial random solutions. Figure 4.7 illustrates the convergence behavior of the optimal total cost TC for different iterations of multiple runs, thereby validating the effectiveness of teaching-learning based optimization for optimal analysis. Tables 4.1 and 4.2 present the optimal values of the expected total cost function and decision variables for each range of input parameters employing TLBO techniques where as Tables 4.3 and 4.4 present the simillar results from PSO technique. The evident outcome underscores that sustaining the operation of a substantial number of active units necessitates a higher-tier repair facility and redundancy, consequently leading to higher costs. In addressing the recurrent failure of active units, the demand for additional standbys, coupled with an efficient repair facility, introduces a higher budgetary allocation. When the failure rate of standby units in the spare state is pronounced, the findings advocate for a reduced inventory of standby units in the spare state. The tabulated data also forecasts the reduction in operational costs within a machining system through the incorporation of deteriorating units. These outcomes align with the hypothesis that the decision to reject deteriorating standby units demands a nuanced assessment, encompassing not only immediate cost savings but also the enduring operational repercussions. To provide statistical justification for the utilization of the meta-heuristic optimization technique, we calculate the mean and maximum of ratios between the overall optimum value of the total cost, denoted as $TC(S^*, \mu^*)$, across all runs and the optimal total cost $TC_i(S^*, \mu^*)$ obtained from the i^{th} run. The results presented in the numerical illustration demonstrate that the variations across different runs are of negligible magnitude.

Graphs and tables validate the effectiveness of the optimization approach and present the optimal values of the cost function and decision variables. These findings provide insights for improving the machining system's performance and optimizing its cost by understanding the impact of failure characteristics and making informed decisions regarding maintenance and standby provisioning.

In a nutshell,

Table 4.4: Optimal expected total cost of the system $(TC(S^*,\mu^*))$ for different parameters using PSO algorithm, For $M = 20, \lambda = 4, \nu = 1.8, \lambda_1 = 3$

C_1,C_2,C_3,C_4,C_5	S^*	μ^*	$TC(S^*,\mu^*)$	**Mean**	**Maximum**	**Time Elapsed**
240, 100, 30, 220, 90	6	74.96137990	10489.22798980	1.00000010081	1.00000018530	1456.05380520
295, 100, 30, 220, 90	6	77.16146300	10720.89963510	1.00000087843	1.00000043689	1349.78670430
350, 100, 30, 220, 90	6	78.86097790	10930.62964790	1.00000019232	1.00000014231	1346.46043420
295, 70, 30, 220, 90	7	79.44917220	10284.53704230	1.00000025341	1.00000089116	1481.95494260
295, 100, 30, 220, 90	6	77.16146300	10720.89963510	1.00000091181	1.00000010012	1349.78670430
295, 130, 30, 220, 90	4	69.00000000	11017.24555610	1.00000014176	1.00000086129	1390.72374590
295, 100, 10, 220, 90	6	77.55161990	10670.62058440	1.00000076899	1.00000077285	1276.43717580
295, 100, 30, 220, 90	6	77.16146300	10720.89963510	1.00000088905	1.00000080518	1349.78670430
295, 100, 50, 220, 90	5	75.83910970	10763.25880380	1.00000041991	1.00000095490	1274.74491670
295, 100, 30, 180, 90	4	69.00000000	10392.27824060	1.00000075788	1.00000014494	1299.82566120
295, 100, 30, 220, 90	6	77.16146300	10720.89963510	1.00000013312	1.00000062117	1349.78670430
295, 100, 30, 260, 90	7	79.57289010	10903.35585380	1.00000098286	1.00000099830	1342.16140430
295, 100, 30, 220, 60	6	82.82309890	8325.28862860	1.00000013331	1.00000056390	1297.53929790
295, 100, 30, 220, 90	6	77.16146300	10720.89963510	1.00000078736	1.00000018269	1349.78670430
295, 100, 30, 220, 120	4	69.00000000	12869.51517820	1.00000012788	1.00000091102	1345.36541600

- In a systematic approach, the installation of an optimal number of standby units proves cost-effective by avoiding the maintenance of excessive standbys or encountering shortages.
- To prevent failures, it is essential to implement preventive measures consistently. Instead of replacing standbys with new units, utilizing deteriorating standbys can be a viable option.
- Just-in-time corrective measures should be implemented promptly to mitigate the risk of significant shortages or downtime.
- Regular observations and data collection are crucial for conducting predictive analysis and anticipating unit failures.

By leveraging mathematical models and cost-benefit analyses, the study demonstrates that a balanced approach to standby unit maintenance is crucial for optimizing system performance, reliability, and economic efficiency. By following these practices, organizations can optimize their operational efficiency, minimize costs, and enhance overall system reliability.

4.7 Conclusion

In conclusion, this research presents a comprehensive exploration of standby unit deterioration dynamics within multi-unit machining systems, revealing the intricate interplay between failure characteristics, repair facilities, and system performance metrics. Our model, incorporating degradation effects, failure rates, and repair dynamics for active and standby units, serves as a valuable tool to analyze and optimize system behavior in realistic operational scenarios. The study emphasizes the critical importance of considering standby unit deterioration, offering insights into the delicate balance between cost considerations and system reliability. Through advanced methodologies encompassing queueing theory, mathematical modeling, and metaheuristic optimization techniques, our research contributes to both theoretical understanding and practical applications. Results highlight the strategic implications of managing deteriorating standby units, advocating for nuanced evaluations that encompass short-term savings and long-term operational sustainability. These findings are particularly pertinent for industries reliant on standby units, providing actionable guidelines to enhance reliability, efficiency, and cost-effectiveness. In summary, our study provides a robust foundation for analyzing and optimizing multi-unit machining systems under the influence of standby unit deterioration. By delving into practical complexities and leveraging diverse analytical tools, this research extends its relevance to a range of real-world scenarios. As we move forward, the potential for further investigations and informed decision-making processes is undeniable, opening avenues for continued advancements in reliability engineering and machining system optimization. The current research can be extended to encompass scenarios involving an unreliable repairer, multi-repairer setups, multi-type standbys, and the consideration of switching failures in standby units. Furthermore, the comprehensive analysis presented here holds potential for extension into non-Markovian frameworks, thereby offering a broader understanding of complex industrial systems and their dynamics.

Chapter 5

Performance Analysis of Machine Repair Systems with Deteriorating Standby Using Phase-Type Distributions: Comparative Study of Erlang, Exponential, and Hyper-Exponential Distributions

"Life is like a phase-type distribution; it transitions between different states, each with its own challenges and opportunities."

John Doe

5.1 Introduction

In the contemporary world, machines have made our lives easier, faster, and more convenient in numerous ways; therefore, they have become an integral part of our modern-day lives. The world is highly dependent on machines; the day-to-day life of a human being is also affected by machines. Machines play an important role in such diverse fields as manufacturing, transportation, healthcare, education, logistics, computer networking, telecommunications, entertainment, and many daily-life service operations. The challenges faced during machine repair can be multifaceted and demanding, often requiring a deep understanding of technical concepts, access to specialized resources, and a skillful approach to problem-solving. The successful restoration of machines to their optimal state depends on effectively applying these elements. Machine repair problems (MRP), like Industry 4.0, cloud computing, and the internet of things(IOTs), are advantageous in the current industrial scenario. Machine rapir problems can predict machine failures, automate troubleshooting and maintenance processes, remote monitoring and repair, automate diagnostics, detect mechanical issues, and automate reporting of machine performance. It helps to dwindle downtime and costs associated with the breakdown of machines. Machine repair problems enhance production productivity, efficiency, and product quality. However, the machine is also prone to failure like any other artificial object, wear and tear, and other problems that can significantly impact its performance; the supply chain management, the service facility, and many other factors are also interrupted. Due to various combined impacts of loads, external factors like the environment, the quality of materials used in production, and the internal factor of materials, the performances of the machining systems deteriorate with time. If the machine fails, it might disrupt many different sectors and enterprises, resulting in costly downtime and damage to reputation. To overcome this, the maintenance and monitoring of the machines to prevent or minimize the chances of failure and prolong their lifespan.

Various researchers have done a vast amount of research on the machine repair problem. Wang [309] developed the profit model of the machining system using different types of breakdowns to determine the optimal values of repairers. The machining system with different types of failure and maintenance has been studied in detail by various researchers (*cf.* [273], [40], [114], [35], [318], [115]). Recently, Shekhar et al. [264] analyzed the Markovian warm-spare node provisioning computing network with different failures. The Laplace transform technique is used to determine the steady-state probabilities and analyzed the proposed model's reliability characteristics. Yen et al.[348] discussed a retrial machine repair problem using F-policy with working breakdown. The Laplace transform technique and matrix analytic method are applied to solve the differential-difference equations of the proposed model. More recently, a finite capacity Markovian multi-server machining system with working vacation and customer impatience was discussed by Bouchentouf et al. [18]. The optimal values of decision variables are found using the direct search and Quasi-Newton methods. A non-Markovian machining system with spare parts and operating units with working vacations under N-policy was discussed by Meena et al [199].

When an active unit fails in the multi-unit machining system, the industry faces losses like loss of production, revenue, data, time, profit, customers, and reputation. Spare parts are beneficial for machining systems to prevent such kinds of losses. When an operating unit fails, it is instantly replaced with available spare parts, and the failed unit is sent to the repairer. After repair, the repaired unit joins as an active or spare unit as per system requirements. Spare parts and well-planned maintenance and repair schedules are essential for the decline in unit failure. The spare parts are categorized into three types according to their failure characteristics: cold, warm, and hot. A standby unit is known as cold standby unit if its failure rate is zero. Suppose the failure rate of a standby unit is the same as that of an active unit, then it is known as a hot standby unit. A standby unit is known as a warm standby unit if its failure rate is neither zero nor equal to an active unit. Various researchers have done a vast amount of research on multi-unit machining systems incorporating standby units.

The self-testing and repairing system with dynamic redundancy has been studied by Avizienis et al. [10]. The fault-tolerant machine repair system with mixed standby units investigated by Wang and Kuo [317]. A direct search method and the steepest descent method are imposed to find the optimal value of decision parameters that help to reduce the total expected cost function. A Markovian multi-unit machine repair system has warm standby with R repairer and vacation policy developed by researchers *(cf.* [316], [139], [275], [234]). These machining systems' reliability, availability, maintainability, and profit analysis have been studied. The redundancy helps to enhance the system's reliability and mean time to failure (MTTF). The MTTF and the system's availability significantly enhance by incorporating redundancy with the system. The availability of the fault-tolerant system with standby units and a waiting strategy can be enhanced if it is free from any human error and has an instant repair facility. We investigate preventative maintenance optimization for parallel k-of-n multi-unit systems where production might be lowered while some units remain operating. The classification for high availability cloud solution (HACS) based on multilayer discussed by *(cf.* [17], [57]). The Markovian K-out-of-$M+Y+S$ redundant repairable machining system has mixed standbys with various types of failure analyzed in the literature [262]. Recently, Wang et al. [327] studied a redundant retrial machining system subject to standby switching failure. In addition, they used the matrix analytical method and the Laplace transform technique to determine the system's performance indices. Kumar et al. [159] developed a non-Markovian $M/G/1$ fault-tolerant machine repair system with the vacationing server. The fuzzy performance indices are computed using the non-linear parametric technique, and the cost function is defined using harmony search to get the best descriptors at the lowest cost. More recently, Gao [66] developed a redundant machining system having N-active, W-warm, and C-cold standby units with two dependent failures. Performance matrices, reliability, and availability in the transient are obtained by applying the Markov and Markov renewal theories. Rani et al. [235] used particle swarm optimization and harmony search optimization to find the optimal decision parameters and minimum total cost of the fault-tolerant redundant system.

In the prosaic research of the machine repair model, it is considered that the standby units are often intact and unaffected by degradation over time, Often, if a unit's performance deteriorates slightly, it is better to fix it rather than discard it entirely. Recognizing the deterioration of the standby units in a highly financially estimated set-up is helpful to preemptive management, maintenance decisions, performance optimization, system downtime and production reduction, redundancy, and system effectiveness, and optimizing system design. Therefore, this study creates a mathematical model to predict extortionate standby unit practices as they degrade, assisting in the system's maintenance decisions and performance optimization. It is important to pay attention to the deterioration of standby units in a machine repair system, both for reliability and cost-effectiveness. Ignoring this issue can increase system downtime and reduce production. To balance system dependability and cost management, it is necessary to observe degradation patterns and implement optimal maintenance strategies. Mathematical modeling using queueing theory can be a useful approach to analyzing the degradation of high-priced standby units. Analyzing the correlations between operational and standby units is crucial for obtaining significant inputs into system performances. Understanding the deterioration of standby units plays a vital role in the development, assessment, and maintenance of efficient systems, thereby improving redundancy and overall system effectiveness.

A standby unit can degrade in its spare state without any operation. Bearing, seals, gaskets, hoses, belts, fans, batteries, filters, and rubber items are a few examples of machine components that wear out over time. This deterioration depends on the quality of the parts, the environmental factors such as temperature, humidity, dust, etc. Pierskalla and Voelker [229] conducted a comprehensive review of maintenance models for systems that are prone to deterioration. In his investigation, Wang [307] examined various maintenance policies, such as age replacement, block replacement, periodic preventive maintenance, failure limit, and sequential preventive maintenance policies for deteriorating systems. The study delved into technical details related to these policies and their effectiveness in maintaining such systems. Yuang and Meng [352] utilized the Laplace transform technique to examine the reliability aspects of a two-unit warm standby repairable system with precedence in use and a fickle switch. Meanwhile, Ghaleb et al. [68] studied a single deteriorating machine that experiences degradation-based failures and scrutinized the impact of deterioration, failures, and maintenance policies. The study of condition-based maintenance, deteriorating machining systems in the energy field, degraded manufacturing with imperfect repair, various effects of deterioration on machining systems, and deteriorating maintenance activity are analyzed by numerous researchers *(cf.* [329], [221], [69], [288], [308], [124]). In a phase-type service distribution, the random variable representing the time between machine failures (or repairable system failures) is modeled as a Markov process. The Markov process is divided into a finite number of phases. Each phase is linked to an exponential time distribution. The system moves from one phase to another based on specific criteria, such as the occurrence of failures or the execution of repairs. It is applicable in reliability engineering, queueing theory, and mathematical finance to model stochastic processes. A MRP can precisely predict repair durations for various machines using the phase type service distribution. It improves future repair time forecasts, helping the repair system schedule repairs and assess service availability. Neuts [217] studied the reliability modelling of systems with two components assuming the time-to-failure and repair time distributions are of phase type. Brière and Chaudhry [19] discussed the bulk-arrival queueing model $M^x/G/1$ with four service-time distribution cases: hyper exponential, deterministic, uniform, and Erlang. Numerous researchers have studied machine repair systems with exponentially distributed machine failure, Markovian arrival process, phase type breakdown of machines, and the service time of the repairer is assumed to be a phase type. The supplementary variable technique and Laplace-Stieltjes transform are applied to derive the steady-state probabilities and perform the machine availability sensitivity analysis. Phase type distributions enhance the systems' flexibility and practicality. The time-dependent reliability of nonrepairable systems is designed and analyzed using realistic stochastic models. The phase type distribution in the machine repair system with different characteristics is studied in the literature *(cf.*[29], [312], [242], [341], [30], [145], [321], [197]).

When an active unit fails, it is replaced with an available standby unit, and the failed unit is sent back to the repairer. A standby unit may fail in the spare state before operation mode. A standby unit may deteriorate with time due to internal and external factors, and the failure rate of a deteriorated standby unit is much higher than a standby unit in a spare state and an active unit. It is assumed that a standby unit may fail in its spare state without any operation. So the failure characteristics of a standby unit in a spare state will be different from the failure characteristics of the operating standby unit. To the best of our knowledge, it is found that no research has yet to be done on this theory. Hence our best knowledge, it is a novel approach in which the failure rate of the standby unit in spare state is different from that of its operating standby unit. Hence, standby unit failure characteristics in the spare and operating states should be studied. This study might optimize standby unit maintenance and replacement, boosting system reliability and efficiency.

Motivation Behind the study

Standby provisioning is standard in many industries and systems to ensure uninterrupted operation. However, even standby units can deteriorate with time, affecting their reliability and performance. In the case of electric vehicles, batteries are a critical component, and their deterioration over time can affect the performance and range of the vehicle. It is a challenge that the industry is actively working to address by improving battery technology and implementing maintenance and replacement strategies to ensure optimal battery performance. Similarly, in security services, equipment such as vehicles, communication devices, and surveillance systems must be maintained and replaced regularly to ensure reliable operation. Rubber and plastic parts used in industrial machinery and machining systems can also deteriorate over time due to wear and tear exposure to heat, chemicals, and environmental factors. Regular maintenance and replacement of these components can prevent equipment failures and downtime. Overall, it is vital for industries and systems to consider the potential deterioration of standby units and components and to implement strategies to ensure their ongoing reliability and performance.

The remainder of the research article's contents are as follows: The rest of this work is split into six sections: Section 5.2 provides a model description, Section 5.3 goes over matrix analytic method, Section 5.4 describes the system's characteristics, Section 5.5 presents a numerical analysis, and Section 5.6 provides a conclusion and ideas for future studies.

5.2 Model Description

From the above literature survey, it is evident that the outright rejection of the deteriorating standby unit does not usher in beneficial cost repercussions. The cost of standby units varies depending on the type of industrial setup, with some being highly expensive and others being average or low-priced. In order to mitigate the degradation of costly standby units, it is essential to implement a practical approach that includes appropriate repair, vigilant monitoring, timely replacement, and the expertise of skilled technicians. To ensure optimal performance and longevity of equipment, systematic inspection and monitoring should be prioritized, along with the implementation of predictive maintenance approaches, ideal storage, and environmental conditions, standby unit rotation as needed, optimization of maintenance procedures, utilization of adaptive replacement policies, consideration of lifecycle cost analysis, adoption of advanced materials, thorough research and development, skillful training, and comprehensive documentation maintenance. This research article presents a multi-unit machine repair model having M identical active units, S standby units, and a repairer. The active units are prone to failure, and whenever a unit fails, it is immediately replaced with an available standby unit, while the failed unit is sent to the repair pool. Since the standby units may fail in their standby state, they can fail without being in operation. The standby units may deteriorate with time, leading to higher failure rates than active and standby units.

Unit failure process

- When an active unit fails in the operating state, then the inter-failure time of the active units follows an exponential distribution with the parameter λ.
- It is presumed that the warm standby units may fail in the spare state before their operational state. The inter-failure time of the warm standby unit is exponentially distributed with the mean ν^{-1}, $(0 < \nu < \lambda)$, where ν is the failure rate of the warm standby unit.
- When the standby unit deteriorates over time for various factors, its failure characteristics will differ from those of the active and standby units.
- When an active unit fails, it is replaced with an available warm standby unit without losing time. The inter-failure time between two successive operating standby units follows an exponential distribution with the mean $1/\lambda$, $(\nu < \lambda_1 \ \& \ \lambda \leq \lambda_1)$.

Unit repair facility

- Developing a repair facility to ensure an uninterrupted operating system (UOS) will be beneficial.
- When an operating unit fails, it is sent to the repair facility without loss of time and will be repaired instantly if the repairer is idle; otherwise, join the queue wait for its turn and repair under a first come, first serve policy (FCFS).
- The time to repair the failed units is exponentially distributed with the mean $1/\mu$, $\mu > 0$.

Phase type Distribution

A phase-type distribution (Pht) is defined as the distribution of the life time, i.e., the time to enter an absorbing state from the set of transient states of an absorbing continuous time Markov process. In this article, repairing of the unit is done in various phases and to model this type of repair process we considered the phase type distribution. Let there are $m+1$ repair stages of the units and all the states are transient states except the $(m+1)^{\text{th}}$ stage. Then the time until absorption is defined by the phase type distribution (Pht disribution). A. K. Erlang [58] initially proposed the idea of a phase type distribution, and Neuts [214] later formalized this idea. The distribution $F(.)$ on $\mathbb{R}^+$ is a phase-type distribution with the representation $(\beta, \mathbf{V})$ if it is the distribution of the absorption time in the $(m+1)^{\text{th}}$ state of a Markov process defined on the states $\{1, 2, 3, \cdots, m, m+1\ \}$, given initial probability distribution $\beta = (\beta_1, \beta_2, \beta_3, \cdots, \beta_m, \beta_{m+1})$. where the states $1, 2, 3, \cdots, m, m+1$ are transient states and the state $(m+1)^{\text{th}}$ absorbing. Consider a finite continuous time Markov chain (CTMC) with $m+1$ transient states and one absorbing state (no transition from this state) with the infinitesimal generator T partitioned as.

$$\mathbf{T} = \begin{bmatrix} \mathbf{V} & \mathbf{V^0} \\ \mathbf{0} & 0 \end{bmatrix}$$

where $\mathbf{V}$ is a $m \times m$ square matrix, $\mathbf{V}^0$ is a column vector of order m such that $\mathbf{Ve} + \mathbf{V}^0 = \mathbf{0}$. Matrix $\mathbf{V}$ should be nonsingular for absorption into the absorbing state from any initial state, and assume the matrix $\mathbf{V} + \mathbf{V}^0\beta$ is irreducible. Let's assume the initial state of the Markov chain is obtained using the (β, β_{m+1}) probability vector. Our objective is to figure out how much time the system requires for the system to enter the absorbing stage. Let Y denote the transition interval from state m to state $m+1$. Thus, Y is a continuous random variable on $[0, \infty]$, and its probability density and cumulative probability distribution function are provided by

$$f(t) = \beta e^{Vt} V^0, t \geq 0, \quad F(t) = P(Y \leq t) = 1 - \beta e^{Vt}\mathbf{e},\ t \geq 0, \text{ respectively.}$$

Random variable Y follows a PH-distribution with representation $(\beta, \mathbf{V})$ of order m and denoted by $Y \equiv PH(\beta, V)$ of order m. The mean and variance of Y are denoted by μ_Y and ${\sigma_Y}^2$ and defined as

$$\mu_Y = \beta(-\mathbf{V})^{-1}\mathbf{e} \quad \text{and} \quad {\sigma_Y}^2 = 2\beta\mathbf{V}^{-2}\mathbf{e} - {\mu_Y}^2.$$

In order to define the exponential matrix, we have

$$e^{\mathbf{Q}} = \mathbf{I} + \mathbf{Q} + \mathbf{Q}^2/2 + \cdots$$

Some special cases of PH-distribution are given as

Exponential Distribution

If $m = 1, \beta = 1$, and $\mathbf{V}$=$(-\lambda)$ then we obtain exponential distribution with parameter λ.

Erlang Distribution

The erlang distribution of order m with parameter λ is a PH-distribution with representation (β,**V**) of order m given by

$$\beta = (1,0,0,\cdots,0) \quad \text{and} \quad \mathbf{V} = \begin{bmatrix} -\lambda_1 & \lambda_1 & 0 & \cdots & 0 \\ 0 & -\lambda_2 & \lambda_2 & \cdots & 0 \\ 0 & 0 & -\lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda_m \end{bmatrix}.$$

Hyperexponential Distribution

A hyperexponential is a mixture of m exponential with parameters $\lambda_1,\lambda_2,\lambda_3,\cdots,\lambda_m$ and the mixing probabilities are $P_1,P_2,P_3,\cdots,P_m$. This is a PH-distribution with representation (β, **V**) of order m given by

$$\beta = (P_1,P_2,P_3,\cdots,P_m) \quad \text{and} \quad \mathbf{V} = \begin{bmatrix} -\lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & -\lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & -\lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda_m \end{bmatrix}.$$

5.3 Steady-State Analysis

There is no statistical correlation between machining system failure and its repair. Let P(t) be the phase of service, if any, $I_1(t)$ be the number of active units in the system at some time t, $I_2(t)$ be the number of standby units in the system at any instant t, and $I_3(t)$ be the number of standby units acting in an operating state in the system at any time t. Thus, the system state at time t can be described as a four-dimensional Markov process $\Xi = (P(t)=p, I_1(t)=i, I_2(t)=j, I_3(t)=k; t \geq 0)$ with state space is defined as $\Omega = \{(p,i,j,k)| \quad p=1,2,\cdots,m-1,m \quad \& \quad i=0,1,2,\cdots,M-1,M \quad \& \quad j=0,1,2,\cdots,S-1,S \quad \& \quad k=0,1,2,\cdots,M-i-1,M-i\}$. In the steady state, let us represent the steady-state probabilities equations as follows

$$P_{p,i,j,k} = \lim_{t\to 0}\{P(t)=p, I_1(t)=i, I_2(t)=j, I_3(t)=k\};\ p=1,2,\cdots,m-1,m \ \&\ i=0,1,2$$
$$\cdots,m-1,m \ \&\ j=0,1,2,\cdots,S-1,S \ \&\ k=0,1,2,\cdots,M-i$$

To find the steady-state probabilities of the multi-unit machining system, we derive the Chapman Kolmogrove differential-difference equation in terms of inflow and outflow rates in of the proposed model.

$$\mathbf{Q} = \begin{pmatrix} \mathbf{E}_0 & \mathbf{F}_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{G}_1 & \mathbf{E}_1 & \mathbf{F}_1 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{G}_2 & \mathbf{E}_2 & \mathbf{F}_2 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{G}_3 & \mathbf{E}_3 & \mathbf{F}_3 & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{G}_{M-2} & \mathbf{E}_{M-2} & \mathbf{F}_{M-2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{G}_{M-1} & \mathbf{E}_{M-1} & \mathbf{F}_{M-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{G}_M & \mathbf{E}_M \end{pmatrix},$$

The transition rate matrix $\mathbf{Q}$ is a square matrix of order $M+1$. The main block diagonal matrix, denoted as $\mathbf{E}_i$, represents a square matrix of order $(S+i)m+1 \times (S+i)m+1$; where $0 \leq i \leq M$. The first diagonal above matrix denoted as $\mathbf{F}_i$, represents a matrix of order $(S+(i+1))m+1 \times (S+(i+1))m+1$; where $0 \leq i \leq M-1$. The first diagonal below matrices, denoted as $\mathbf{G}_i$, represent a matrix of order $(S+i)m+1 \times (S+i)m+1$; where $1 \leq i \leq M$.

$$\mathbf{E}_0 = \begin{pmatrix} b_{0,0} & \beta \otimes S\nu & 0 & 0 & \cdots & 0 & 0 \\ V^0 & B_{0,1}+V & C_{0,1} & 0 & \cdots & 0 & 0 \\ 0 & V^0\beta & B_{0,2}+V & C_{0,2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & V^0\beta & B_{0,S-1}+V & C_{0,S-1} \\ 0 & 0 & 0 & 0 & 0 & V^0\beta & B_{0,S}+V \end{pmatrix}$$

$$\mathbf{F}_0 = \begin{pmatrix} 0 & M\lambda I & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & M\lambda I & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & M\lambda I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & M\lambda I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & M\lambda I \end{pmatrix}$$

$$\mathbf{E}_i = \begin{pmatrix} b_{i,0} & \beta\otimes S\nu & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ V^0 & B_{i,1}+V & C_{i,1} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & V^0\beta & B_{i,2}+V & C_{i,2} & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & V^0\beta & B_{i,S-1}+V & C_{i,S-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & V^0\beta & B_{i,S}+V & C_{i,S} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_{i,1}+V & H_{i,1} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_{i,i}+V \end{pmatrix}$$

$$\mathbf{F}_i = \begin{pmatrix} 0 & (M-i)\lambda I & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & (M-i)\lambda I & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & (M-i)\lambda I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & (M-i)\lambda I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & (M-i)\lambda I \end{pmatrix}$$

$$\mathbf{G}_i = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & V^0\beta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & V^0\beta & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & V^0\beta & 0 & 0 \end{pmatrix}$$

The elements $B_{j,i}, b_{j,i}, C_{j,i}, D_{j,i},$ and$H_{j,i}$ the above of component matrix are defined belows

$B_{j,i} = -((M-j)\lambda + j\lambda_1 + (S-i)\nu)I$, $b_{j,i} = -((M-j)\lambda + j\lambda_1 + (S-i)\nu)$, $\forall i \le S$

$C_{j,i} = (j\lambda_1 + (S-i)\nu)I$, $D_{j,i} = -((M-j)\lambda + (j-i)\lambda_1)I$, $H_{j,i} = (j-i)\lambda I$.

Let $\mathbf{P}$ denote the stationary probability column vector conforms to $\mathbf{Q}$, which can be divide up as $\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2, \cdots, \mathbf{P}_{M-1}, \mathbf{P}_M$. The subvector $\mathbf{P}_i = [\mathbf{P}_{M-i,S,i}, \mathbf{P}_{M-i,S-1,i}, \mathbf{P}_{M-i,S-2,i}, \cdots, \mathbf{P}_{M-i,1,i}, \mathbf{P}_{M-i,0,i}, \mathbf{P}_{M-i,0,i-1}, \mathbf{P}_{M-i,0,i-2}, \cdots, \mathbf{P}_{M-i,0,1}, \mathbf{P}_{M-i,0,0}]$; $0 \le i \le M$, has dimension of order $1 \times (S+i)m + 1$. Then, we have

$$\mathbf{QP} = \mathbf{0} \tag{5.1}$$

where $\mathbf{0}$ is zero vector of dimension $(M+1)(Sm+1)+m(M(M+1)/2)$, and $\mathbf{Q}$ is the generator matrix of the Markov process Ξ. with the normalizing condition

$$\mathbf{P}^{\mathbf{T}}\mathbf{e} = 1 \tag{5.2}$$

From Eqn. 5.1, the stationary probability vector $\mathbf{P}$ can be calculated by solving the following equations.

$$\mathbf{E}_0\mathbf{P}_0 + \mathbf{F}_0\mathbf{P}_1 = \mathbf{0} \tag{5.3}$$

$$\mathbf{G}_1\mathbf{P}_0 + \mathbf{E}_1\mathbf{P}_1 + \mathbf{F}_1\mathbf{P}_2 = \mathbf{0} \tag{5.4}$$

$$\mathbf{G}_i\mathbf{P}_{i-1} + \mathbf{E}_i\mathbf{P}_i + \mathbf{F}_i\mathbf{P}_{i+1} = \mathbf{0}\,; \quad 2 \le i \le M-2 \tag{5.5}$$

$$\mathbf{G}_{M-1}\mathbf{P}_{M-2} + \mathbf{E}_{M-1}\mathbf{P}_{M-1} + \mathbf{F}_{M-1}\mathbf{P}_M = \mathbf{0} \tag{5.6}$$

$$\mathbf{G}_M\mathbf{P}_{M-1} + \mathbf{E}_M\mathbf{P}_M = \mathbf{0} \tag{5.7}$$

We get the solution now from Eqn. 5.3, we have

$$\mathbf{P}_0 = \mathbf{P}_1\mathbf{X}_0, \quad \text{where} \quad \mathbf{X}_0 = -\mathbf{E}_0^{-1}\mathbf{F}_0 \tag{5.8}$$

From Eqns. 5.4 and 5.8, we get

$$\mathbf{P}_1 = \mathbf{P}_2\mathbf{X}_1, \quad \text{where} \quad \mathbf{X}_1 = -\left(\mathbf{X}_0\mathbf{G}_1 + \mathbf{E}_1\right)^{-1}\mathbf{F}_1 \tag{5.9}$$

From Eqns. 5.5 and 5.9, we determine

$$\mathbf{P}_i = \mathbf{P}_{i+1}\mathbf{X}_i, \quad \text{where} \quad \mathbf{X}_i = -\left(\mathbf{X}_{i-1}\mathbf{G}_i + \mathbf{E}_i\right)^{-1}\mathbf{F}_i, \quad 2 \le i \le M-2 \tag{5.10}$$

From Eqns. 5.6 and 5.10, we obtain

$$\mathbf{P}_{M-1} = \mathbf{P}_M\mathbf{X}_{M-1}, \quad \text{where} \quad \mathbf{X}_{M-1} = -(\mathbf{X}_{M-2}\mathbf{G}_{M-1} + \mathbf{E}_{M-1})^{-1}\mathbf{F}_{M-1} \tag{5.11}$$

By recursively evaluating Eqns. 5.8-5.11, we can express the state probabilities $\mathbf{P}_i$; $0 \le i \le M$ in terms of state probabilities $\mathbf{P}_M$

$$\mathbf{P}_i = \mathbf{P}_{i+1}\mathbf{X}_i = \mathbf{P}_{i+2}\mathbf{X}_i\mathbf{X}_{i+1} = \cdots = \mathbf{P}_M\prod_{\xi=1}^{M-i}\mathbf{X}_{M-\xi} = \mathbf{P}_M\tau_i^* \tag{5.12}$$

where $\tau_i^* = \prod_{\xi=1}^{M-i} \mathbf{X}_{M-\xi}$ and $\mathbf{X}_i;\quad 0 \leq i \leq M$ are given above Eqns. 5.8 - 5.11. By Eqns. 5.7 and 5.12 we get

$$\sum_{i=0}^{M} \mathbf{P}_i \mathbf{e}_i = \mathbf{P}_0\mathbf{e}_0 + \mathbf{P}_1\mathbf{e}_1 + \cdots + \mathbf{P}_{M-1}\mathbf{e}_{M-1} + \mathbf{P}_M\mathbf{e}_M$$

$$= \mathbf{P}_M\tau_0^*\mathbf{e}_0 + \mathbf{P}_M\tau_1^*\mathbf{e}_1 + \cdots + \mathbf{P}_{M-1}\tau_{M-1}^*\mathbf{e}_{M-1} + \mathbf{P}_M\tau_M^*\mathbf{e}_M = \mathbf{P}_M\left[\sum_{i=0}^{M} \tau_i^*\mathbf{e}_i\right] = 1 \tag{5.13}$$

where $\mathbf{e}_i = [1,1,\cdots,1,1]^T_{1\times(S+i)m+1}$; where $0 \leq i \leq M$. hence Eqn. 5.7 can be written as

$$\mathbf{P}_M\left[\mathbf{X}_{M-1}\mathbf{G}_M + \mathbf{E}_M\right] = \mathbf{0} \tag{5.14}$$

therefore, on solving Eqns. 5.13 - 5.14, we can obtain state probability $\mathbf{P}_M$. So, we can determine the steady-state probabilities for $\mathbf{P}_M$

5.4 System characteristics

Modeling this multi-unit redundant machining system is done primarily to assess various queueing methods to define the system's behavior, which may improve the system's performance. We have used indices such as steady-state probabilities to characterize the system's effectiveness. The system's behavior may be significantly interpreted with the help of these performance indicators.

- Expected number of failed units in the system is given as

$$E_N = \sum_{i=0}^{M} e\left[\sum_{j=1}^{S}(S-j)P_{i,j,M-i} + \sum_{k=0}^{M-i-1}(M-i+S-k)P_{i,0,k}\right] \tag{5.15}$$

- The throughput of the system is given as

$$Th = \sum_{i=0}^{M}\sum_{j=0}^{S-1} e\mu P_{i,j,M-i} + \sum_{i=0}^{M-1}\sum_{k=0}^{M-i-1} e\mu P_{i,0,k} \tag{5.16}$$

- Expected number of standby units in the system is given as

$$E_S = \sum_{i=0}^{M} SP_{i,S,M-i} + \sum_{i=0}^{M}\sum_{j=1}^{S-1} e(S-j)P_{i,S-j,M-i} \tag{5.17}$$

- Mean number of active units in the system is given as

$$E_O = \sum_{i=1}^{M} iP_{i,S,M-i} + \sum_{i=2}^{M}\sum_{j=1}^{S-1} ieP_{i,j,M-i} + \sum_{i=1}^{M}\sum_{k=0}^{M-i} P_{i,0,k} \tag{5.18}$$

- Mean number of operating standby units in the system is given as

$$E_{OS} = \sum_{k=1}^{M} kP_{M-k,S,k} + \sum_{k=1}^{M} ke\left[\sum_{j=1}^{S-1} P_{M-k,j,k} + \sum_{i=0}^{M-k} P_{i,0,k}\right] \tag{5.19}$$

- Effective failure rate is given as

$$\begin{aligned} E_{fr} =& \sum_{i=0}^{M} (i\lambda + (M-i)\lambda_1 + S\nu)\, P_{i,S,M-i} \\ &+ \left[\sum_{i=0}^{M}\sum_{j=1}^{S-1} (i\lambda + (M-i)\lambda_1 + j\nu)\, eP_{i,j,M-i} + \sum_{i=0}^{M-1}\sum_{k=0}^{M-i} (i\lambda + k\lambda_1) eP_{i,0,k}\right] \end{aligned} \tag{5.20}$$

- Expected waiting time of the failed units in the system is given as

$$E_W = \frac{E_N}{E_{fr}} \tag{5.21}$$

- Delay time of failed units is given as below

$$E_D = \frac{E_N}{Th} \tag{5.22}$$

- Availability of the system

$$MA = 1 - \frac{E_N}{M+S} \tag{5.23}$$

5.5 Numerical Analysis

In the subsequent section, we present an extensive numerical demonstration to exemplify the practical application and efficacy of the proposed concept regarding the advantages of integrating deteriorating standby units within the multi-unit machining system. Our thorough analysis reveals that entirely disregarding the deteriorating, albeit costly, standby unit does not yield favorable cost outcomes. Through this illustrative example, we endeavor to underscore the tangible real-world implications of our research findings and underscore the potential advantages of embracing the proposed concept. By employing the developed mathematical models in a pertinent scenario, we aim to offer valuable insights into the behavior, performance, and optimization of the examined multi-unit machining system with deteriorating standby units. This numerical demonstration validates our theoretical framework and enhances the practicality and relevance of our research in the field.

In this section, we undertake a numerical investigation to evaluate the impact of failure characteristics on performance metrics for both active and standby units (in both spare and operating states). We utilize MATLAB R2020b, running on a 12th Gen Intel(R) Core(TM) i7 − 12700 processor with a clock speed of 2100 MHz, 12 cores, and 20 logical processors, to simulate the performance indices for various input parameters. Specifically, we set the system parameters as follows: $M = 20$, $S = 8$, $\lambda = 4$, $\nu = 1.8$, $\lambda_1 = 3$, and $\mu = 75$, as per the references cited in the introduction.

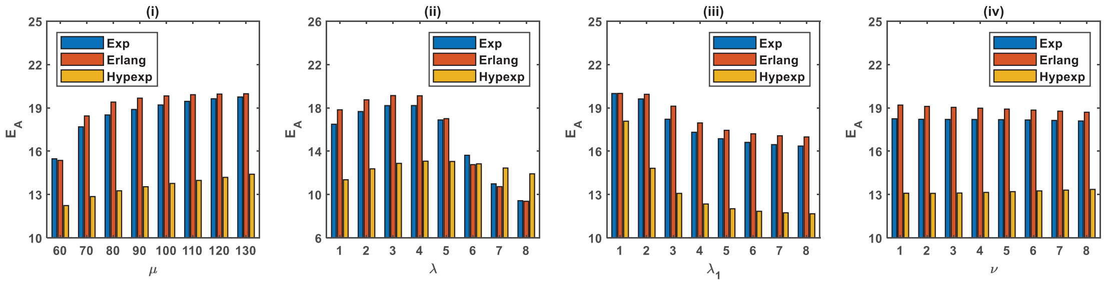


Figure 5.1: Expected number of active units in the system

Figure 5.1 deploys how the system's expected number of active units changes when different parameters are changed. In Figure 5.1(i) the expected number of active units with different distributions enhances as we improve the system's service rate. In contrast, in Figure 5.1(ii-iv), the expected number of active units decreases as we increase the failure rate of active units, operating standby units, and standby units.

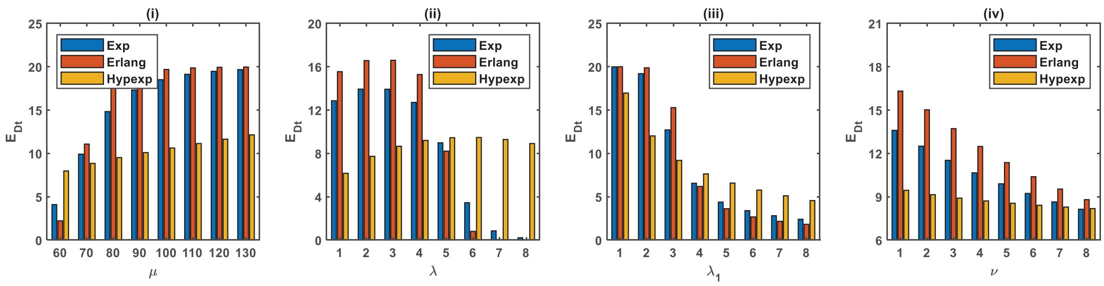


Figure 5.2: Expected number of deteriorate units in the system

The expected number of active units is more in the erlang distribution than exponential and hyper exponential. Figure 5.2(i) represents the proportional behavior of the mean number of deteriorated units and service rate. Figure 5.2(ii-iv) shows the inverse relationship between the mean number of deteriorated units and the failure rate of active units, operating standby units, and standby units. Enhancing the failure rates of the operating standby units and the standby units results in rapid decreases in the system's deteriorated units.

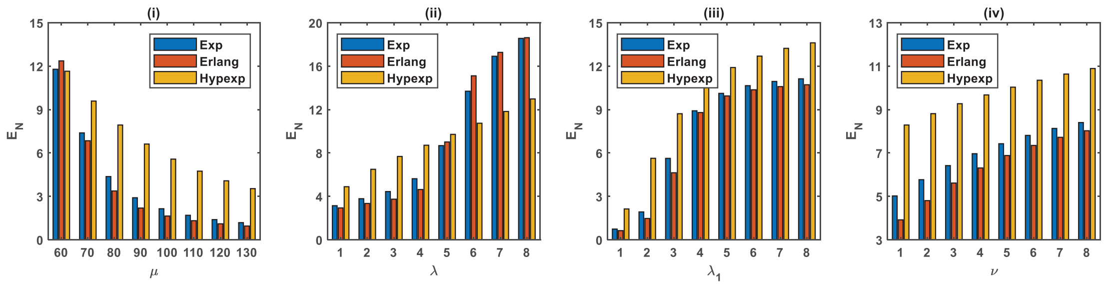


Figure 5.3: Expected number of the failed units in the system

Figure 5.3(i) demonstrates increasing the service rate of the system results in the expected number of failed units in the system decreasing. In contrast, Figure 5.3(ii-iv) displays that the system's failure rates behave proportionally to the expected number of failed units in the system.

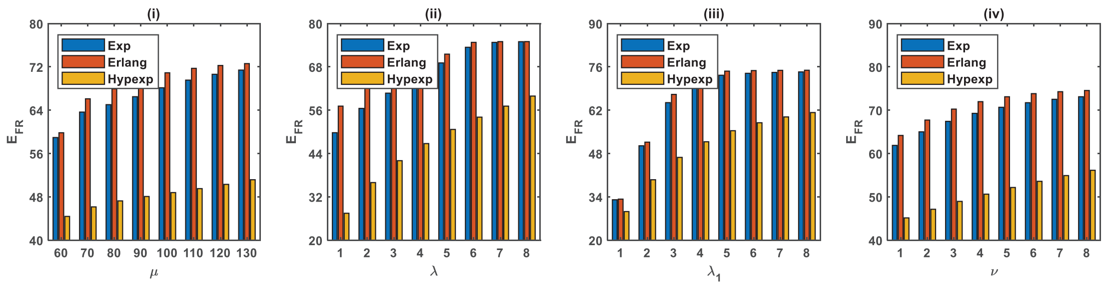


Figure 5.4: Expected failure rate of the system

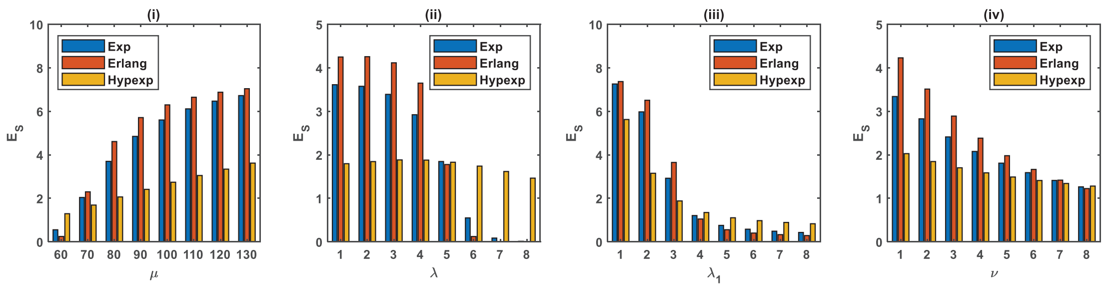


Figure 5.5: Expected number of the standby units in the system

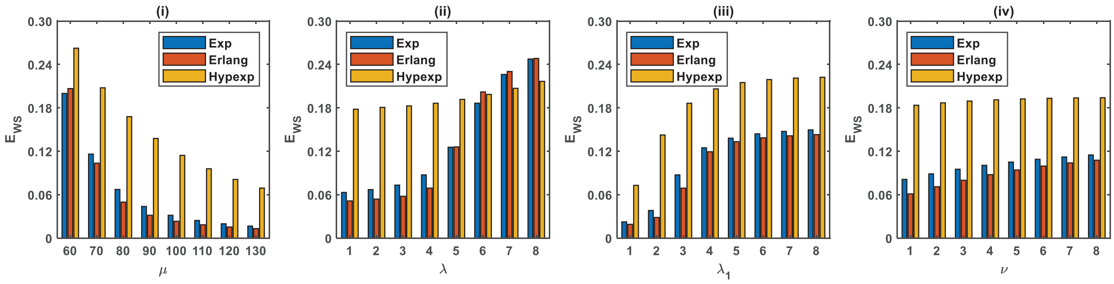


Figure 5.6: Expected waiting time of the system

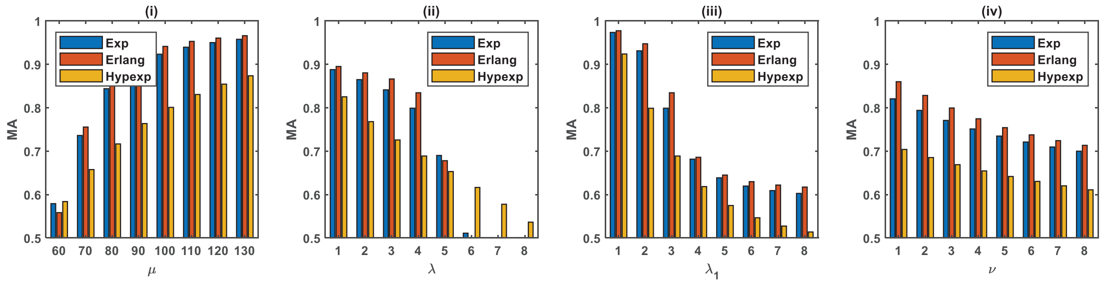


Figure 5.7: Machine availability of the system

Figure 5.6 also represents the same behavior as figure 5.3. Figures 5.5 & 5.7(i) depict how increasing the service rate increases the mean number of standby units and system availability. Figures 5.5 & 5.7(ii-iv) demonstrate improving the failure rates of the system reduces the system's availability and the mean number of standby units. Figures 5.4 & 5.8 (i) exhibit the system's effective failure rate and throughput. As the system's service rate increases, the system's effective failure rate, and throughput are also enhanced. Figures 5.4 & 5.8 (ii-iv) demonstrate that the increasing failure rate of the system results in enhancing the effective failure rate and throughput of the system.

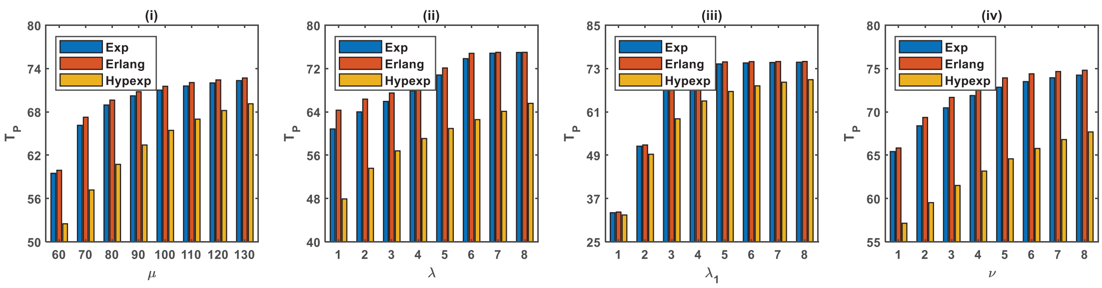


Figure 5.8: Throughput of the system

5.6 Conclusion

The phase-type distribution can be useful in modeling machine repair systems because it allows for a more accurate representation of service time distributions than simpler distributions, such as the exponential distribution. By using a more accurate service time distribution, the machine repair system can more effectively model and analyze the system's performance, including measures such as mean time to repair and system availability.

Chapter 6

The State of The Art Methodologies For Reliability Analysis of Imperfect Repair and Thresholed-Based Measures

> "The imperfect repair of the machine mirrors the human condition-a perpetual striving towards perfection amidst the inevitability of imperfection. In this dance, the researcher becomes both artisan and philosopher, shaping progress with each adjustment."
>
> Plato

6.1 Introduction

In today's world, machines and devices are deeply intertwined with daily life, making it nearly impossible to imagine human existence without them. Machines have become an integral part of human life, seamlessly integrating into society and playing a crucial role in meeting the growing demands for products and services. Fault-tolerant systems (FTSs) are especially important in ensuring uninterrupted operations within socio-techno-economic constraints. These systems find wide applications in various industries like textiles, automobiles, and Fast-moving consumer goods. Machine interference occurs when there's a mismatch between the units and the repairer. This research addresses the novel concept of imperfect repair in FTSs with finite active/standby units. It focuses on strategic control to improve maintenance and enhance redundancy for increased utility and reliability. In machining systems, unexpected unit failures due to wear and tear lead to increased costs, delays, and operational inefficiencies. Implementing a preventive maintenance strategy with standby units can help promptly replace failed units, improving system reliability despite additional costs.

Previous research has focused on managing failed units in systems, often through threshold-based policies. These policies control the arrival of failed units to minimize expected downtime. For instance, under the commonly studied F-policy, failed units are not allowed to enter the system when the number of waiting units reaches capacity. They are only permitted to enter after the queue size drops to a specified level.

Service systems have often been studied assuming consistently successful service provision. However, real-world situations may involve instances of unsuccessful service attempts before achieving success. For example, the COVID-19 pandemic has forced academic institutions to shift to untested online teaching methods, leading to network glitches hindering effective knowledge transfer. Cloud computing has emerged as a significant enhancer of educational efficiency, dynamically allocating computing and storage resources for teaching materials and addressing network errors during online sessions. The term "unreliable service" describes this interplay of unsuccessful and successful service instances, where customers experiencing unsuccessful service rejoin the queue until they receive successful service.

This chapter comprehensively explores the novel service regime of imperfect repair in FTSs, aiming to bridge gaps in existing literature by introducing controlled arrival processes for failed units. Its objectives are to formulate a stochastic model of the machining system considering imperfect repair and controlled arrival, propose a computationally efficient numerical scheme for calculating transient-state probabilities, and establish the system's reliability and queueing characteristics. The study also addresses sensitivity and relative sensitivity analysis, offering valuable insights for decision-makers. Additionally, it discusses potential applications of the FTS in various domains. The methodology involves a systematic approach, including an extensive literature survey, the introduction of a novel model, the formulation of a stochastic model, and the development of numerical schemes and exploration of system characteristics using mathematical theories. The chapter is organized into sections covering literature review, model description, Chapman-Kolmogorov differential-difference equations, transient-state probabilities derivation, system characteristics, cost function formulation, sensitivity analysis, identification of standard models, numerical results, and conclusion.

6.2 Literature Review

The exploration of machine interference has been a prominent focus in the literature, with foundational reviews (*cf.* Stecke and Aronson [284], Valdez-Flores and Feldman [301], Haque and Armstrong [88]). The articles on machine interference problems (MIPs) highlight significant applications across various sectors, including telecommunications (Kryvinska [151]), cloud computing (Luo et al. [188]), computer networking (Bunday et al. [24]), artificial intelligence (Luo et al. [182]), ambulance fleet management (Firooze et al. [59]), and more. Zagia et al. [359] introduced a hybrid model optimizing facility maintenance scheduling for efficiency and cost savings.

From the conceptualization of standby units (Taylor and Jackson [293]), the literature has been extensively enriched (*cf.* Yun [353], Cho and Parlar [40]). The fault-tolerant systems with multi-active and standby units were analyzed for availability with imperfect coverage (Ke et al. [136]), reliability in a fuzzy environment (Shekhar et al. [259]), reliability in a probabilistic environment (Shekhar et al. [261]), reliability with switching failure and reboot delay (Shekhar et al. [262]), reliability characterizing the temperature deviation procedure via a two-stage Wiener process (Ma et al. [189]), and availability and mean time to failure (MTTF) in a fuzzy environment (Devanda et al. [47]).

The total cost of a mathematical model was scrutinized for coordinating production scheduling and Condition-Based Maintenance (CBM) planning in a manufacturing system with a single machine (Sharifi and Taghipour [253]) and parallel-machine (Sharifi et al.[252]) experiencing multiple failures and discrete stages of deterioration. Maintenance planning and production scheduling were jointly optimized in intelligent manufacturing systems (*cf.* Ghaleb et al. [68], Ghaleb et al. [70], Ghaleb et al. [69]), addressing factors such as new job arrivals, due date changes, stochastic deterioration-based failures, minimal repairs, and CBM. Recent research explores applications for multiunit systems, including studies on bipropellant rocket engines with electric pump-fed systems (Bai et al. [13]) and hydrogen-air-steam mixture gas behavior under steam condensation (Liu et al. [180]). Structural reliability and design analysis for complex systems have been conducted (*cf.* She et al. [256], Qi et al. [231]), alongside the development of an iterative threshold algorithm of Log-Sum Regularization for sparse problems (Zhou et al. [360]).

After the inception of the F-policy (Gupta [82]), subsequent research extensively explored the concept of controllable arrival in MRPs for reliability characteristics analysis. Steady-state results were derived for a single removable and unreliable server using the matrix analytical method (Wang and Yang [324]), while transient results were investigated for a retrial system with working breakdowns and randomized setup time using the Laplace transform technique (Yen et al. [348]).

The concept of unreliable service was introduced in a Markovian queue with a single server, both without (Patterson and Korzeniowski [225]) and with (Patterson and Korzeniowski [226]) working vacations. Recognizing its practicality, unreliable service as imperfect repair in MRPs with standby provisioning, a strategic threshold-based F-policy, and vacation interruption was examined (Shekhar et al. [267]), highlighting its importance in reducing power consumption and preventing thermal trip errors through discouragement and feedback strategies (Shekhar et al. [257]). The reliability of multi-unit systems with standby provisioning was explored, considering failures, degradation, random delays, and probabilistic imperfections (Shekhar et al. [258]).

Although FTSs have been widely studied, the incorporation of imperfect repair mechanisms, where failed units are subject to controlled arrival processes, remains underexplored in the literature. Addressing this gap, our research introduces a more realistic model that:

- Pioneers the incorporation of imperfect repair mechanisms, presenting a novel service regime beyond traditional fault-tolerance approaches.
- Explores controlled arrival processes for failed units, offering in-depth understanding of system dynamics.
- Extends the understanding of service scenarios to include unsuccessful attempts before achieving success, relevant in contexts like online teaching during the COVID-19 pandemic.

These findings collectively advance the understanding of system reliability and performance in real-world scenarios.

6.3 Model Description

In this section, we present a detailed model description outlining the key components and fundamental characteristics of the proposed fault-tolerant machining system. The model intricately captures the dynamic interactions between active and standby units, the repair process, and the controlled arrival of failed units. Each aspect is systematically defined and formulated, incorporating essential parameters such as failure rates, repair rates, inspection rates, and strategic thresholds. The model's design is grounded in established principles of reliability theory, queueing theory, and controlled processes, ensuring a comprehensive. The main assumptions of the current research are as follows:

- The proposed machine repair problem focuses on the reliability characteristics in the context of a service regime involving imperfect repair by a single repairer for failed units, controlled by the F-policy.

- The machining system comprises M identical active units and S warm standby units, aiming to enhance reliability and availability. Standby units promptly replace failed active units with negligible delay.

- During normal mode (operation), all M active units operate simultaneously, and the system continues in short mode until there are at least $m; (m; 1 \leq m \leq M-1)$ active units operational.

- The proposed preventive strategy regulates the influx of failed units, with an excess allowed up to a specified threshold $K = M + S - m + 1$.

Future research may extend the study to incorporate switching failures and significant switching delays. Notations and key definitions are provided for clarity, including parameters such as failure rates, repair rates, inspection rates, and system reliability indicators.

Notations:

M	:	Number of active units in the system
S	:	Number of warm standby units in the system
m	:	The least number of units required for working of the system
K	:	System capacity
λ	:	Failure rate of active units in normal state
ν	:	Failure rate of warm standby units
λ_d	:	Degraded failure rate of active units in short mode
μ	:	Repair rate for failed units
β_1	:	Inspection rate for perfect repair
β_2	:	Inspection rate for imperfect repair
γ	:	Setup rate for allowing failed units to join the queue
$R_Y(t)$	:	Reliability of the system at the time t
$MTTF$	:	Mean time to failure of the system
$E_N(t)$	:	Expected number of failed units in the system at the time t
$T_P(t)$	:	Throughput of the system at the time t

$E_S(t)$: Expected number of standby units in the system at the time t

$E_O(t)$: Mean number of operating units in the system at the time t

$E_{fr}(t)$: Expected carrying load of failed machines in the system at the time t

$E_W(t)$: Expected waiting time of the failed operating units in the system at time t

$E_D(t)$: Expected delay time of failed machine at time t

$FF(t)$: Failure frequency of the system at the time t

Failure process: The failure process involves independent exponential failures for active and standby units, with degradation upon exhausting standby units.

- Each of the active and standby units experiences independent failures, and the time-to-failure for each active and standby unit follows an exponential distribution with mean time to failure $\frac{1}{\lambda}$ and $\frac{1}{\nu}$ (where $0 < \nu < \lambda$), respectively.
- Upon switching to the on-the-go state on the active unit's failure, the standby unit inherits the same failure and working characteristics as those of an active unit.
- In the event that all available standby units are exhausted, the time to failure for each active unit is degraded with a mean time to failure of $\frac{1}{\lambda_d}$ (where $0 < \lambda < \lambda_d$).

Repair process: The repair process assumes immediate repair without delay with inspection for perfect and imperfect repair.

- When a unit becomes futile, immediate repair is essential without any delay. If the repairer is available, the failed unit undergoes instant repair; otherwise, it waits in the queue.
- The queue discipline of this repairable system is $FCFS$ (first-come, first-served).
- The time-to-repair follows an exponential distribution with a mean time of $\frac{1}{\mu}$.
- After repair, the unit undergoes inspection. Generally, perfect repair is assumed, but in practice, it may be imperfect.
- The inter-time-to-inspect for both perfect and imperfect repair follows an exponential distribution with mean rates β_1 and β_2, respectively.
- The unit with imperfect repair rejoins the queue until it undergoes perfect repair.
- The fixed unit is considered as good as a new active or standby unit and is returned to the pool of active units or standby units when the system is operating in short or normal mode, respectively.

Controlled process: A controlled process prevents additional failed units from joining the queue mitigating prolonged waiting times.

- When the number of failed units reaches the system capacity K, the system prevents additional failed units from joining the queue until the system reverts to normal mode. Specifically, the value of F in the F-policy influences the system's ability to admit failed units, preventing significant expected waiting times and affecting the overall efficiency of the control strategy.This strategy aims to mitigate prolonged waiting times. The excluded failed units may undergo repair at an external facility, incurring an additional cost.
- Upon resuming, the system experience a random setup time, which follows an exponential distribution with a parameter of γ. The setup rate refers to the rate at which failed units, which were initially excluded from the system due to capacity constraints, are permitted to enter the repair queue after the system transitions back to normal mode. This setup rate is a parameter in the model that influences the controlled admission of failed units based on the system's capacity strategy. It represents the speed or frequency at which the system allows previously excluded failed units to rejoin the repair process. The setup rate is a crucial factor in managing the repair queue and mitigating prolonged waiting times for failed units.

The failure, degraded failure, perfect repair, imperfect repair, setup, etc., of each unit are independent events. The analyzed model is illustrated in the transition diagram in Fig. 6.1. Here, the blue nodes represent the state when there are a printed number of failed units in the system, and the red node with the print F represents the system failure state. The transition arrow signifies the transition between states with marked rates. The j-th row, where $j = 0,1,2,3$, of blue nodes represents the j-th state of the system as defined above. Each single node in the j-th row represents the state of the system (j,i) in a two-tuple form, where i is the marked number in the node, representing the number of failed units in the system or failure state F.

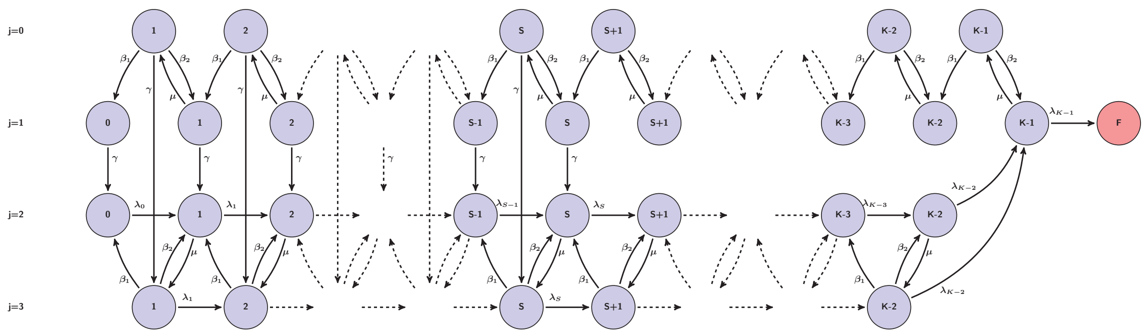


Figure 6.1: Transition diagram

All the active and standby units are operational in the initial stage, i.e., there is no failed unit in the system at time $t = 0$. If there are i failed units in the system at time t, the state-dependent effective failure rate of units is expressed as:

$$\lambda_i = \begin{cases} M\lambda + (S-i)\nu; & \text{if } i = 0,1,2,3,...,S-1 \\ (M+S-i)\lambda_d; & \text{if } i = S,S+1,...,K-1,K \\ 0; & \text{otherwise} \end{cases}$$

For mathematical modeling, the following notations are defined. The state of the failed units $J(t)$ in the system at time t is defined as follows:

$$J(t) \equiv \begin{cases} \text{0; The newly failed unit cannot be allowed to ingress into the system for repair,} \\ \text{while the existing failed unit is promptly repaired and reinstated back into the} \\ \text{system.} \\ \text{1; Access to the system to repair the newly failed unit has been prohibited, and} \\ \text{the repairer is currently occupied.} \\ \text{2; The newly failed unit is granted access to the system for repair, while the} \\ \text{repairer is engaged.} \\ \text{3; The newly failed unit is allowed to ingress into the system for repair, and} \\ \text{the existing failed unit immediately after the repair for inspection.} \end{cases}$$

and

$I_1(t) \equiv$ Number of failed units in the system at the time t

$F(t) \equiv$ The state when system has been failed at the time t.

With the above definition of the system states, the system states form a continuous-time Markov chain $(J(t) = j, I_1(t) = i) \cup F(t);\ t \geq 0$ in the state space, which can be represented as:

$$\begin{aligned} \Pi = &\{(j,i) \mid j = 0; i = 1,2,...,K-2,K-1\} \cup \{(j,i) \mid j = 1; i = 0,1,2,...,K-1\} \\ &\cup \{(j,i) \mid j = 2; i = 0,1,2,...,K-2\} \cup \{(j,i) \mid j = 3; i = 1,2,...,K-2\} \cup F \end{aligned}$$

The probabilities of the different states in the system at any time t are defined as follows:

$P_{j,i}(t) \equiv$ Probability that at time t there are n failed units in the system and the system is in the state j, where $(j,i) \in \Pi$

$P_F(t) \equiv$ Probability that at time t the system is in the failed state.

6.4 The governing equation

Referring to the transition diagram shown in Fig. 6.1 for the threshold-based failed unit arrival controlled strategy with imperfect repair and an exponential setup time, we have developed the forward Chapman-Kolmogorov differential-difference equations in terms of state probabilities and governing parameters. The differential-difference equation for a particular node or state is expressed as a balance between the inflow and outflow rates. For example, for Eqn. (6.1), the governing differential-difference equation for state $(0,i)$ is derived by balancing the outflow rates γ, β_1, and β_2 from the $(0,i)$ state with the inflow rate μ from the $(1,i)$ state.

$$\frac{dP_{0,i}(t)}{dt} = -(\gamma+\beta_1+\beta_2)P_{0,i}(t)+\mu P_{1,i}(t)\ ;\ 1\le i\le S \tag{6.1}$$

$$\frac{dP_{0,i}(t)}{dt} = -(\beta_1+\beta_2)P_{0,i}(t)+\mu P_{1,i}(t)\ ;\ S+1\le i\le K-1 \tag{6.2}$$

$$\frac{dP_{1,0}(t)}{dt} = -\gamma P_{1,0}(t)+\beta_1 P_{0,1}(t) \tag{6.3}$$

$$\frac{dP_{1,i}(t)}{dt} = -(\mu+\gamma)P_{1,i}(t)+\beta_2 P_{0,i}(t)+\beta_1 P_{0,i+1}(t)\ ;\ 1\le i\le S \tag{6.4}$$

$$\frac{dP_{1,i}(t)}{dt} = -\mu P_{1,i}(t)+\beta_2 P_{0,i}(t)+\beta_1 P_{0,i+1}(t)\ ;\ S+1\le i\le K-2 \tag{6.5}$$

$$\frac{dP_{1,K-1}(t)}{dt} = -(\mu+\lambda_{K-1})P_{1,K-1}(t)+\beta_2 P_{0,K-1}(t)+\lambda_{K-2}P_{2,K-2}(t)$$
$$+\lambda_{K-2}P_{3,K-2}(t) \tag{6.6}$$

$$\frac{dP_{2,0}(t)}{dt} = -\lambda_0 P_{2,0}(t)+\gamma P_{1,0}(t)+\beta_1 P_{3,1}(t) \tag{6.7}$$

$$\frac{dP_{2,i}(t)}{dt} = -(\lambda_i+\mu)P_{2,i}(t)+\gamma P_{1,i}(t)+\lambda_{i-1}P_{2,i-1}(t)+\beta_2 P_{3,i}(t)+\beta_1 P_{3,i+1}(t)$$
$$;\ 1\le i\le S \tag{6.8}$$

$$\frac{dP_{2,i}(t)}{dt} = -(\lambda_i+\mu)P_{2,i}(t)+\lambda_{i-1}P_{2,i-1}(t)+\beta_2 P_{3,i}(t)+\beta_1 P_{3,i+1}(t)$$
$$;\ S+1\le i\le K-3 \tag{6.9}$$

$$\frac{dP_{2,K-2}(t)}{dt} = -(\lambda_{K-2}+\mu)P_{2,K-2}(t)+\lambda_{K-3}P_{2,K-3}(t)+\beta_2 P_{3,K-2}(t) \tag{6.10}$$

$$\frac{dP_{3,1}(t)}{dt} = -(\lambda_1+\beta_1+\beta_2)P_{3,1}(t)+\gamma P_{0,1}(t)+\mu P_{2,1}(t) \tag{6.11}$$

$$\frac{dP_{3,i}(t)}{dt} = -(\lambda_i+\beta_1+\beta_2)P_{3,i}(t)+\gamma P_{0,i}(t)+\mu P_{2,i}(t)+\lambda_{i-1}P_{3,i-1}(t)$$
$$;\ 2\le i\le S \tag{6.12}$$

$$\frac{dP_{3,i}(t)}{dt} = -(\lambda_i + \beta_1 + \beta_2)P_{3,i}(t) + \mu P_{2,i}(t) + \lambda_{i-1}P_{3,i-1}(t)$$
$$;\ S+1 \leq i \leq K-2 \quad (6.13)$$
$$\frac{dP_F(t)}{dt} = \lambda_{K-1}P_{1,K-1}(t) \quad (6.14)$$

At $t = 0$, the initial conditions are defined as

$$P(0,i) = 0; \quad i = 1,2,...,K-1 \quad (6.15)$$
$$P(1,i) = 0; \quad i = 0,1,...,K-1 \quad (6.16)$$
$$P(2,0) = 1 \quad (6.17)$$
$$P(2,i) = 0; \quad i = 0,1,...,K-2 \quad (6.18)$$
$$P(3,i) = 0; \quad i = 1,2,...,K-2 \quad (6.19)$$
$$P(F) = 0 \quad (6.20)$$

6.4.1 Laplace transform

For solving the developed system of equations (Eqns.6.1-6.14) under initial conditions for computing transient-state probabilities, we employ the Laplace transform of state probabilities, and their derivatives as

$$\ddot{P}_{j,i}(u) = L(P_{j,i}(t)) = \int_0^\infty e^{-ut}P_{j,i}(t)dt\ ; \mathbb{R}(u) \geq 0,\ \forall\ (j,i) \in \Pi \quad (6.21)$$
$$\ddot{P}_F(u) = L(P_F(t)) = \int_0^\infty e^{-ut}P_F(t)dt;\ \mathbb{R}(u) \geq 0 \quad (6.22)$$
$$L\left(\frac{dP_{j,i}(t)}{dt}\right) = u\ddot{P}_{j,i}(u) - P_{j,i}(0);\ \mathbb{R}(u) \geq 0\ \forall\ (j,i) \in \Pi \quad (6.23)$$
$$L\left(\frac{dP_F(t)}{dt}\right) = u\ddot{P}_F(u) - P_F(0);\ \mathbb{R}(u) \geq 0 \quad (6.24)$$

After applying the defined Laplace transform over the governing differential-difference equations with initial conditions, we get the following system of linear equations

$$u\ddot{P}_{0,i}(u) = -(\gamma + \beta_1 + \beta_2)\ddot{P}_{0,i}(u) + \mu\ddot{P}_{1,i}(u)\ ;\ 1 \leq i \leq S \quad (6.25)$$
$$u\ddot{P}_{0,i}(u) = -(\beta_1 + \beta_2)\ddot{P}_{0,i}(u) + \mu\ddot{P}_{1,i}(u)\ ;\ S+1 \leq i \leq K-1 \quad (6.26)$$
$$u\ddot{P}_{1,0}(u) = -\gamma\ddot{P}_{1,0}(u) + \beta_1\ddot{P}_{0,1}(u) \quad (6.27)$$
$$u\ddot{P}_{1,i}(u) = -(\mu + \gamma)\ddot{P}_{1,i}(u) + \beta_2\ddot{P}_{0,i}(u) + \beta_1\ddot{P}_{0,i+1}(u)\ ;\ 1 \leq i \leq S \quad (6.28)$$
$$u\ddot{P}_{1,i}(u) = -\mu\ddot{P}_{1,i}(u) + \beta_2\ddot{P}_{0,i}(u) + \beta_1\ddot{P}_{0,i+1}(u)\ ;\ S+1 \leq i \leq K-2 \quad (6.29)$$

$$u\ddot{P}_{1,K-1}(u) = -(\mu + \lambda_{K-1})\ddot{P}_{1,K-1}(u) + \beta_2\ddot{P}_{0,K-1}(u) + \lambda_{K-2}\ddot{P}_{2,K-2}(u) + \lambda_{K-2}\ddot{P}_{3,K-2}(u) \tag{6.30}$$

$$u\ddot{P}_{2,0}(u) - 1 = -\lambda_0\ddot{P}_{2,0}(u) + \gamma\ddot{P}_{1,0}(u) + \beta_1\ddot{P}_{3,1}(u) \tag{6.31}$$

$$u\ddot{P}_{2,i}(u) = -(\lambda_i + \mu)\ddot{P}_{2,i}(u) + \gamma\ddot{P}_{1,i}(u) + \lambda_{i-1}\ddot{P}_{2,i-1}(u) + \beta_2\ddot{P}_{3,i}(u) + \beta_1\ddot{P}_{3,i+1}(u);\ 1 \le i \le S \tag{6.32}$$

$$u\ddot{P}_{2,i}(u) = -(\lambda_i + \mu)\ddot{P}_{2,i}(u) + \lambda_{i-1}\ddot{P}_{2,i-1}(u) + \beta_2\ddot{P}_{3,i}(u) + \beta_1\ddot{P}_{3,i+1}(u);\ S+1 \le i \le K-3 \tag{6.33}$$

$$u\ddot{P}_{2,K-2}(u) = -(\lambda_{K-2} + \mu)\ddot{P}_{2,K-2}(u) + \lambda_{K-3}\ddot{P}_{2,K-3}(u) + \beta_2\ddot{P}_{3,K-2}(u) \tag{6.34}$$

$$u\ddot{P}_{3,1}(u) = -(\lambda_1 + \beta_1 + \beta_2)\ddot{P}_{3,1}(u) + \gamma\ddot{P}_{0,1}(u) + \mu\ddot{P}_{2,1}(u) \tag{6.35}$$

$$u\ddot{P}_{3,i}(u) = -(\lambda_i + \beta_1 + \beta_2)\ddot{P}_{3,i}(u) + \gamma\ddot{P}_{0,i}(u) + \mu\ddot{P}_{2,i}(u) + \lambda_{i-1}\ddot{P}_{3,i-1}(u)\ ;\ 2 \le i \le S \tag{6.36}$$

$$u\ddot{P}_{3,i}(u) = -(\lambda_i + \beta_1 + \beta_2)\ddot{P}_{3,i}(u) + \mu\ddot{P}_{2,i}(u) + \lambda_{i-1}\ddot{P}_{3,i-1}(u)\ ;\ S+1 \le i \le K-2 \tag{6.37}$$

$$u\ddot{P}_F(u) = \lambda_{K-1}\ddot{P}_{1,K-1}(u) \tag{6.38}$$

We represent the transient-state probabilities subscript in a unary code to facilitate the solution procedure, as shown below.

$$[P_{0,1}(t), P_{0,2}(t), ..., P_{0,K-1}(t)]^T \equiv [\pi_1(t), \pi_2(t), ..., \pi_{K-1}(t)]^T$$
$$[P_{1,0}(t), P_{1,1}(t), ..., P_{1,K-1}(t)]^T \equiv [\pi_K(t), \pi_{K+1}(t), ..., \pi_{2K-1}(t)]^T$$
$$[P_{2,0}(t), P_{2,1}(t), ..., P_{2,K-2}(t)]^T \equiv [\pi_{2K}(t), \pi_{2K+1}(t), ..., \pi_{3K-2}(t)]^T$$
$$[P_{3,1}(t), P_{3,2}(t), ..., P_{3,K-2}(t)]^T \equiv [\pi_{3K-1}(t), \pi_{3K}(t), ..., \pi_{4K-4}(t)]^T$$
$$P_F(t) \equiv \pi_{4K-3}(t)$$

The Laplace transform probabilities relevant to the problem can be calculated using the following equation.

$$\ddot{\pi}_r(u) = L\{\pi_r(t)\};\ 1 \le r \le 4K-3$$

Delimitate the subsequent column vectors of order $4K-3$.

$$\ddot{\Xi}(u) = [\ddot{\pi}_1(u), \ddot{\pi}_2(u), \ddot{\pi}_3(u), ..., \ddot{\pi}_{4K-4}(u), \ddot{\pi}_{4K-3}(u)]^T, \tag{6.39}$$

$$\Xi(0) = [\pi_1(0), \pi_2(0), \pi_3(0), ..., \pi_{4K-4}(0), \pi_{4K-3}(0)]^T \tag{6.40}$$

We represent the system of linear equations (Equations 6.25-6.38) in matrix form using the aforementioned column vectors as follows.

$$\mathcal{F}(u)\ddot{\Xi}(u) = \Xi(0) \tag{6.41}$$

Here, $\mathcal{F}(u)$ denotes the coefficient square matrix of order $4K-3$. Applying Cramer's rule to the matrix equation 6.41, we explicitly express $\ddot{\pi}_r(u)$ as follows.

$$\ddot{\pi}_r(u) = \frac{|\mathcal{F}_r(u)|}{|\mathcal{F}(u)|};\ 1 \leq r \leq 4K-3 \tag{6.42}$$

Here, $\mathcal{F}_r(u)$ represents a square matrix of order $4K-3$. The matrix $\mathcal{F}_r(u)$ is derived from $\mathcal{F}(u)$ by replacing its r^{th} column with the right-hand side column vector $\Xi(0)$. To determine $\ddot{\pi}_r(u)$ from Eqn. 6.42, we initially calculate the denominator $|\mathcal{F}(u)|$. Notably, $|\mathcal{F}(u)|$ becomes singular due to the balanced inflow and outflow rates inherent in its nature, resulting in $u=0$ as one latent root. Additionally, we identify $u=-\xi$ as another nonzero latent root of $|\mathcal{F}(u)|=0$.

$$\mathcal{F}(-\xi) = \mathbf{A} - \xi\mathbf{I} \tag{6.43}$$

where $\mathbf{A} = \mathcal{F}(0)$ and $\mathbf{I}$ is an identity matrix of order $4K-3$. Now expression is also written as

$$\mathcal{F}(-\xi)\ddot{\Xi}(u) = (\mathbf{A} - \xi\mathbf{I})\ddot{\Xi}(u) \tag{6.44}$$

Let $\xi_h(\neq 0)$, for $h = 1,2,3,\ldots,4K-5,4K-4$, denote $4K-4$ distinct latent roots of $|\mathbf{A}-\xi\mathbf{I}| = 0$, which may be real or complex numbers. Consider $\xi_1,\xi_2,\xi_3,\ldots,\xi_{i_1}$ as i_1 real latent roots, and $\xi_{i_1+1},\bar{\xi}_{i_1+1},\xi_{i_2+1},\bar{\xi}_{i_2+1},\ldots,\xi_{i_1+i_2},\bar{\xi}_{i_1+i_2}$ as $2i_2$ complex latent roots, existing in conjugate pairs such that $i_1+2i_2 = 4K-4$. Therefore,

$$|\mathcal{F}(u)| = u\prod_{h=1}^{i_1}(u+\xi_h)\prod_{h=1}^{i_2}(u^2+(\xi_{i_1+h}+\bar{\xi}_{i_1+h})u+\xi_{i_1+h}\bar{\xi}_{i_1+h}) \tag{6.45}$$

Hence, Eqn. 6.42 reduces to

$$\ddot{\pi}_r(u) = \frac{|\mathcal{F}_r(u)|}{\mathcal{F}(u)} = \frac{|\mathcal{F}_r(u)|}{u\prod_{h=1}^{i_1}(u+\xi_h)\prod_{h=1}^{i_2}(u^2+(\xi_{i_1+h}+\bar{\xi}_{i_1+h})u+\xi_{i_1+h}\bar{\xi}_{i_1+h})};$$
$$1 \leq r \leq 4K-3 \tag{6.46}$$

The expression in Equation 6.46 for $\ddot{\pi}_r(u)$ can be represented in partial fraction form as follows:

$$\ddot{\pi}_r(u)=\frac{a_{0,r}}{u}+\sum_{h=1}^{i_1}\frac{a_{h,r}}{(u+\xi_h)}+\sum_{h=1}^{i_2}\frac{b_{h,r}(u)+c_{h,r}}{(u^2+(\xi_{i_1+h}+\bar{\xi}_{i_1+h})u+\xi_{i_1+h}\bar{\xi}_{i_1+h})};1\leq r\leq 4K-3 \tag{6.47}$$

The coefficients in the partial fraction representation are computed as follows:

$$a_{0,r}=\frac{|F_r(0)|}{\prod_{h=1}^{i_1}(\xi_h)\prod_{h=1}^{i_2}(\xi_{i_1+h}\bar{\xi}_{i_2+h})} \tag{6.48}$$

$$a_{h,r}=\frac{|F_r(-\xi_h)|}{(-\xi_h)\prod_{g=1,g\neq h}^{i_1}(\xi_g-\xi_h)\prod_{g=1}^{i_2}({\xi_h}^2+(\xi_{i_1+g}+\bar{\xi}_{i_1+g})(-\xi_h)+\xi_{i_1+g}\bar{\xi}_{i_1+g})};\quad h=1,2,3,...,i_1 \tag{6.49}$$

and

$$b_{h,r}(-\xi_{i_1+h})+c_{h,r}=\frac{|F_r(-\xi_{i_1+h})|}{(-\xi_{i_1+h})\prod_{g=1}^{i_1}(\xi_g-\xi_{i_1+h})\prod_{g=1,g\neq h}^{i_2}\left((-\xi_{i_1+h})^2+\left(\xi_{i_1+g}+\bar{\xi}_{i_1+g}\right)(-\xi_{i_1+h})+\xi_{i_1+h}\bar{\xi}_{i_1+h}\right)};\quad h=1,2,3,...,i_2 \tag{6.50}$$

The explicit expression of the transient-state probabilities $\pi_r(t);1\leq r\leq 4K-3$ is obtained by taking the inverse Laplace transform of Eqn. 6.50. Hence, we have the following expressions for each $\pi_r(t)$:

$$\pi_r(t)=a_{0,r}+\sum_{h=1}^{i_1}a_{h,r}e^{-\xi_h t}+\sum_{h=1}^{i_2}\left[b_{h,r}e^{-x_h t}\cos y_h t+\frac{c_{h,r}-b_{h,r}x_h}{y_h}e^{-x_h t}\sin y_h t\right];1\leq r\leq 4K-3 \tag{6.51}$$

The arbitrary constants $a_{0,r},a_{h,r},b_{h,r}$, and $c_{h,r}$ are computed in the above equations (Eqns. 6.48-6.50), and x_h and y_h represent the real and imaginary parts of the respective complex latent root ξ_{i_1+h}.

6.5 Performance measures

The primary objective of the current research is to investigate imperfect repair within the context of the Markovian threshold-based arrival controlled strategy for machine repair problems. To achieve this goal, we delineate the reliability and performance indices using the transient-state probabilities obtained in the preceding section, taking into account the governing parameters. These assessments contribute to the improvement of the system's reliability and queueing characteristics.

6.5.1 Reliability Measures

In this subsection, we delve into the reliability analysis of the machining system employing a threshold-based failed unit arrival controlled strategy coupled with imperfect repair. The core of Reliability, Availability, Maintainability, and Safety (RAMS) analysis establishes a fundamental framework for the efficient and timely utilization of machining systems. RAMS, an interdisciplinary methodology, encompasses the integration of design elements aimed at facilitating the operational goals of a machining system. In machining systems engineering, RAMS assumes a pivotal role, guaranteeing that the inherent design characteristics align with the requisite standards for optimal performance. Reliability, positioned as the foremost factor in RAMS analysis, underscores the system's ability to consistently operate without failure.

Let X denote the continuous random variable representing the time to failure of the system. The reliability of a machining system is the probability that the system will execute its intended functions under specified conditions for the stipulated duration without encountering failure. Denoting $P_F(t)$ as the probability that the system will fail at or before time t, the machining system's reliability is defined as follows:

$$R_X(t) = 1 - P_F(t);\ t \geq 0 \tag{6.52}$$

Another vital reliability index is the mean time to failure ($MTTF$) of the system, a key metric in assessing the system's performance and durability. In the context of the present study, the focus is on optimizing $MTTF$ through the recommendation of appropriate preventive, corrective, and predictive maintenance strategies.

The Mean Time To Failure ($MTTF$) is a reliability metric that represents the average time a system or component is expected to operate before experiencing a failure.

The mean time to failure ($MTTF$) of the machining system is defined as

$$\begin{aligned} MTTF &= \int_{t=0}^{\infty} R_X(t)dt = \int_{t=0}^{\infty}(1-P_F(t))dt = \int_{t=0}^{\infty}(1-\pi_{4K-3}(t))dt \\ &= \lim_{u\to 0}\left[\frac{1-a_{0,4K-3}}{u} - \sum_{h=1}^{i_1}\frac{a_{h,4k-3}}{u+\xi_h} - \sum_{h=1}^{i_2}\frac{b_{h,4k-3}u + c_{h,4K-3}}{u^2+(\xi_{i_1+h}+\bar{\xi}_{i_1+h})u+\xi_{i_1+h}\bar{\xi}_{i_1+h}}\right] \\ &= -\sum_{h=1}^{i_1}\frac{a_{h,4K-3}}{\xi_h} - \sum_{h=1}^{i_2}\frac{c_{h,4K-3}}{\xi_{i_1+h}\bar{\xi}_{i_1+h}} \end{aligned} \tag{6.53}$$

The failure frequency refers to the rate at which failures occur in a system over a given time period. It is a measure of the frequency or likelihood of failures happening within a specified duration. In mathematical terms, the failure frequency (often denoted as ($FF(t)$) can be expressed as the derivative of the reliability function with respect to time. The failure frequency of the machining system at a given time t is denoted by $FF(t)$ and is defined as:

$$FF(t) = \lambda_{K-1}P_{K-1}(t) \tag{6.54}$$

6.5.2 Queueing characteristics

The queueing attributes characterize the congestion and waiting induced by unit failures and strategic imperfection. These metrics are crucial for refining maintenance strategies and informing the future design of the machining system. We investigate various queueing indices, including expected queue length, throughput, expected number of available active/standby units, waiting time, and delay time, among others.

- The expected number of failed units in a system represents the average number of units that are in a failed state at a given point in time. It is a statistical measure that takes into account the probability distribution of the number of failed units in the system. Expected number of failed units in the system at a time t

$$E_N(t) = \sum_{j=0}^{3}\sum_{i=1}^{K-2} iP_{j,i}(t) + \left(\sum_{j=0}^{1}(K-1)P_{j,K-1}(t)\right) + KP_F(t) \tag{6.55}$$

- The throughput of a system refers to the rate at which the system processes failed units, usually measured as the number of failed units processed per unit of time. The throughput of the system at the time t is given as below

$$Th(t) = \sum_{i=1}^{K-2}\beta_1\left[P_{0,i}(t)+P_{3,i}(t)\right] + \beta_1 P_{0,K-1}(t) \tag{6.56}$$

- The expected number of standby units in a system represents the average number of standby units that are present in the system over a specific period of time. Expected number of standby units in the system at a time t is given as

$$E_S(t) = \sum_{j=1}^{2} SP_{j,0}(t) + \sum_{j=0}^{3} \sum_{i=1}^{S-1} (S-i)\, P_{j,i}(t) \tag{6.57}$$

- The mean number of active units in a system represents the average number of active (operational) units present in the system over a specified period of time. Mean number of active units in the system at time t

$$\begin{aligned} E_O(t) =& M\left[\sum_{j=1}^{2} P_{j,0}(t) + \sum_{j=0}^{3}\sum_{i=1}^{S} P_{j,i}(t)\right] + \sum_{j=0}^{3}\sum_{i=S+1}^{K-2} (M+S-i)\, P_{j,i}(t) \\ &+ \sum_{j=0}^{1} mP_{j,K-1}(t) \end{aligned} \tag{6.58}$$

- The effective failure rate in the context of a system refers to a composite or overall failure rate that takes into account various factors, such as the failure rates of individual components, redundancy, repair processes, and other relevant aspects. It provides a more comprehensive measure of the system's reliability by considering both the occurrence of failures and the effectiveness of repair processes. Effective failure rate at time t

$$\begin{aligned} E_{fr}(t) =& \sum_{j=0}^{3}\sum_{i=1}^{S} (M\lambda + (S-i)\nu)\, P_{j,i}(t) + \sum_{j=0}^{3}\sum_{i=S+1}^{K-2} (M+S-i)\, \lambda_d P_{j,i}(t) \\ &+ \sum_{j=0}^{1} m\lambda_d P_{j,K-1}(t) \end{aligned} \tag{6.59}$$

- The expected waiting time of failed units in a system refers to the average amount of time that a failed unit spends in a waiting state before it undergoes repair and receives perfect service. This metric is essential for evaluating the efficiency and performance of the repair process within the system. Expected waiting time of the failed units in the system at time t

$$E_W(t) = \frac{E_N(t)}{E_{fr}(t)} \tag{6.60}$$

- Delay time of the failed unit represents the waiting time experienced by a failed unit before being attended to by the repair process. The delay time is a crucial metric in assessing the system's performance and reliability, providing insights into how quickly the system can address failures. Delay time of failed unit at a time t

$$E_D(t) = \frac{E_N(t)}{Th(t)} \tag{6.61}$$

6.6 Cost function

Ensuring the uninterrupted and effective operation of a machining system is essential in various domains, including manufacturing, production, communication, computations, and service. Mathematical modeling and cost analysis play a crucial role in ensuring that any machining system implements an optimal maintenance strategy at the lowest possible cost. Cost analysis is significant for the success of industrial or commercial endeavors. To address this, the expected total cost function is formulated in terms of pertinent variables, such as the expected number of failed units, standby units, mean service rate, mean inspection rate for perfect/imperfect repair, and mean setup time. For the formulation of the total cost, we define the following incurred costs in designing the machining system.

- Holding costs are associated with the expenses incurred while maintaining and managing the system to ensure its continued operation and reliability. These costs may include various elements relevant to fault tolerance, such as redundancy costs, monitoring and maintenance costs, training costs, system upgrades, energy and cooling costs, downtime costs, etc.

- The repair cost refers to the total expenditure associated with restoring a damaged or malfunctioning unit to its operational state. This cost encompasses expenses such as labor, materials, parts replacement, and any other resources utilized in the repair process.

- Inspection costs refer to the expenditures associated with examining and evaluating products, processes, or services to ensure compliance with specified standards, quality requirements, or regulations. These costs are incurred as part of quality control measures to identify defects, errors, or deviations from established criteria during different stages of production or service delivery.

- Setup costs, also known as preparation costs, refer to the expenses associated with preparing a repair facility, machine, or process for new repair services. These costs are incurred when transitioning from one state to another.

It's important for decision-makers to carefully consider incurred costs when designing and implementing fault-tolerant systems. Balancing the level of fault tolerance with associated costs is crucial to achieving the desired level of system reliability without excessive financial burden.The unit costs associated with distinct system states are defined and denoted for the cost evaluation as follows.

$C_H \equiv$ Holding cost per unit time of each failed unit present in the system

$C_S \equiv$ Cost per component per unit time of each standby unit present in the system

$C_M \equiv$ Fixed cost per unit time for providing a service with the rate μ to the failed unit

$C_1 \equiv$ Fixed cost per unit time for inspecting an perfect repair with the rate β_1 for the failed unit

$C_2 \equiv$ Fixed cost per unit time for inspecting an imperfect repair with the rate β_2 for the failed unit

$C_3 \equiv$ Fixed cost per unit time for taking a setup time by a system with the rate γ for allowing the failed unit

Hence, the anticipated total cost function accrued for the machining system at a given time t is expressed as:

$$E_{TC}(t) = C_H E_N(t) + C_S E_S(t) + C_M \mu + C_1 \beta_1 + C_2 \beta_2 + C_3 \gamma \tag{6.62}$$

The assumption is made that all unit costs employed in the assessment of the cost function in Eqn. 6.62 exhibit linearity, being directly proportional to the respective governing parameters and derived performance indices.

6.7 Sensitivity analysis

The differential calculus theory knowledge for maxima or minima delves into the sensitivity of the reliability function and $MTTF$ concerning the governing parameters. The comprehensive variability pattern of the studied performance function can be elucidated by computing the first derivatives of the function with respect to the decision variable Θ, where Θ is a general variable representing the governing parameters involved in system design.

Now, by computing the first derivatives of Eqn. 6.41, we obtain:

$$\frac{\partial \digamma(u)}{\partial \Theta} \ddot{\Xi}(u) + \digamma(u) \frac{\partial \ddot{\Xi}(u)}{\partial \Theta} = 0 \tag{6.63}$$

$$\frac{\partial \ddot{\Xi}(u)}{\partial \Theta} = -(\digamma(u))^{-1} \frac{\partial \digamma(u)}{\partial \Theta} \ddot{\Xi}(u) \tag{6.64}$$

Now, deriving the first derivative of the reliability function from Eqn. 6.52, we obtain:

$$\Delta_\Theta(t) = \frac{\partial R_X(t)}{\partial \Theta} = 0 - \frac{\partial P_F(t)}{\partial \Theta} = L^{-1}\left(-\frac{\partial \ddot{P}_F(u)}{\partial \Theta}\right) = L^{-1}\left(\frac{\partial \ddot{\pi}_{4K-3}(u)}{\partial \Theta}\right) \tag{6.65}$$

The chain rule is employed to calculate $\frac{\partial \ddot{P}_F(u)}{\partial \Theta}$. Additionally, the subsequent ratio is utilized to assess the relative sensitivity analysis of the reliability function.

$$\Gamma_\Theta(t) = \frac{\partial R_X(t)/R_X(t)}{\partial \Theta/\Theta} = \Delta_\Theta(t).\frac{\Theta}{R_X(t)} \tag{6.66}$$

Similarly, for the sensitivity analysis of the Mean Time To Failure ($MTTF$), we obtain the first derivative of $MTTF$ with respect to Θ from Equation 6.53 as follows:

$$\Phi_\Theta = \frac{\partial\,(MTTF)}{\partial \Theta} = \frac{\partial\,(\int_{t=0}^{\infty} R_X(t)dt)}{\partial \Theta} = \lim_{s\to 0}\left[\int_{t=0}^{\infty} \frac{\partial R_X(t)}{\partial \Theta} e^{-ut} dt\right] = \lim_{u\to 0}\left[-\frac{\partial \ddot{P}_F(u)}{\partial \Theta}\right]$$
$$= \lim_{u\to 0}\left[\frac{\partial \ddot{\pi}_{4K-3}(u)}{\partial \Theta}\right] \tag{6.67}$$

Again, we calculate the following ratio to determine the relative sensitivity of the Mean Time To Failure ($MTTF$):

$$\Omega_\Theta = \frac{\frac{\partial (MTTF)}{MTTF}}{\frac{\partial \Theta}{\Theta}} = \Phi_\Theta \frac{\Theta}{MTTF} \tag{6.68}$$

The detailed examination of the sensitivity of reliability and Mean Time To Failure ($MTTF$) is discussed with numerical illustrations in the next section.

6.8 Special Cases

The studied model represents an extension of previously published research works, and these works serve to validate our modeling approach.

Case 1: In the limit as β_1 approaches infinity, the model converges to a Markovian single repairer machining system consisting of active/standby units with a threshold-based corrective strategy [114].

Case 2: When the threshold value is set to $K-1$ with β_1 approaching infinity, the current model exhibits behavior akin to a classical machine interference problem with standby provisioning [312], [82], [83] .

Case 3: For $0 < \beta_1 < \infty$, $\beta_2 = 0$, $\gamma \to \infty$, setting the threshold value to $K-1$, $S = 0$, and $\mu = \beta_1$, the model simulates a finite population queueing model with Erlangian service

Case 4: In case 3 with $\mu > \beta_1$, the model transitions to a finite population single-server queueing model with hyper-exponential service time distribution [29].

6.9 Numerical Results

In this section, we conduct a comprehensive numerical analysis to explore the reliability measures, queueing characteristics, and sensitivity of various parameters. The objective is to delve deeper into the impact of these parameters, providing additional insights to complement the theoretical analysis. Given the intricate nature of continuous-time Markov chains representing machining systems with real-time constraints in equilibrium, the stochastic process and probability distribution are intricate, as discussed in earlier sections. Consequently, despite analytical derivation of sensitivity, the investigation heavily relies on numerical experiments.

Our research focuses on the following analyses. Firstly, we perform a graphical analysis to investigate how various parameters influence system reliability. Subsequently, we examine the impact of multiple factors on the Mean Time To Failure ($MTTF$). In the proposed model, consider a scenario analogous to a data center's server infrastructure. The active units represent operational servers, while the warm standby units are additional servers available for immediate use in case of a server failure. Failures and repairs, such as hardware malfunctions or crashes, follow exponential distributions with constant rates. The system employs a preventive strategy to control the influx of failed servers, aiming to maintain optimal performance. This model can be applied to enhance the reliability of the server infrastructure in data centers, ensuring uninterrupted service and minimizing downtime through strategic standby provisioning and controlled arrival of failed units. The numerical experiments of performance indices are conducted using the MATLAB R2020b program, considering distinct input parameters that govern the system. For this purpose, we hypothetically set default parameters as follows: $M = 12$, $S = 5$, $m = 2$, $\lambda = 0.4$, $\nu = 0.35$, $\lambda_d = 0.8$, $\mu = 16$, $\beta_1 = 8$, $\beta_2 = 1$, and $\gamma = 4$.

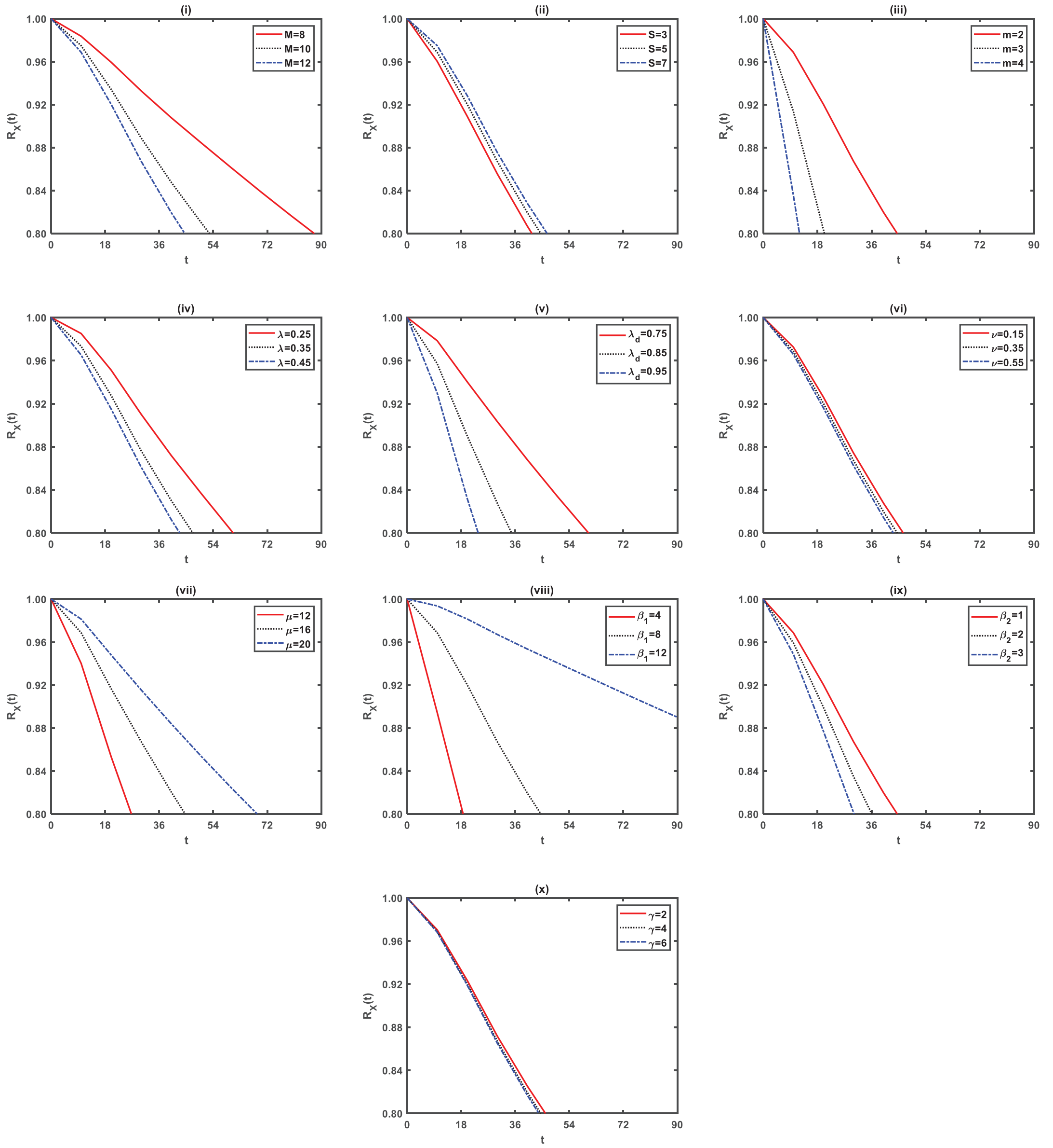


Figure 6.2: Reliability of the system ($R_X(t)$) wrt. t for different parameters. For ($M = 12$, $S = 5$, $m = 2$, $\lambda = 0.4$, $\nu = 0.35$, $\lambda_d = 0.8$, $\mu = 16$, $\beta_1 = 8$, $\beta_2 = 1$, $\gamma = 4$)

The variability of the system's reliability, denoted as $R_X(t)$ and computed using Eqn. 6.52, is illustrated in Fig. 6.2 for varying time t and other system parameters. Fig. 6.2 reveals a decreasing trend in the reliability level as time progresses in all its subgraphs (i)-(x). The evident outcomes for the system's reliability indicate a decrease with an increase in the failure rate of active/standby units and the inspection rate of imperfect repair. Conversely, the system's reliability shows improvement with an increase in the repair rate of failed units and the inspection rate of perfect repair, as depicted in Fig. 6.2 (vii)-(viii). Additionally, the system's reliability enhances with an increase in the number of standby units and a decrease in setup time. These results provide support for the preventive maintenance policy, emphasizing the provisioning of standbys and controlled arrivals of failed units.

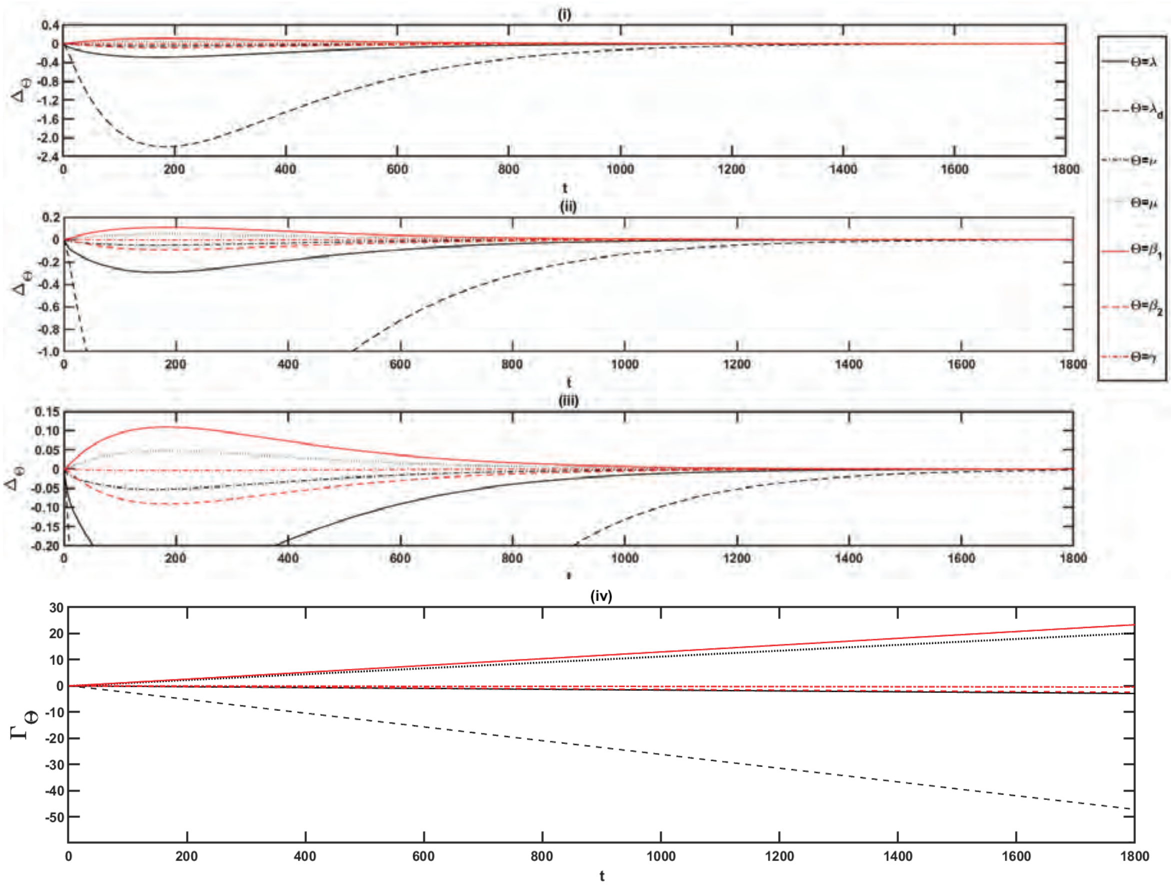


Figure 6.3: Sensitivity and relative sensitivity of the reliability of the system, For ($M = 12, S = 5, m = 2, \lambda = 0.4, \nu = 0.35, \lambda_d = 0.8, \mu = 16, \beta_1 = 8, \beta_2 = 1, \gamma = 4, C_H = 90, C_S = 60, C_M = 10, C_1 = 13, C_2 = 2, C_3 = 2$)

The sensitivities of the system's reliability are depicted in Fig. 6.3, offering a comparative view of the sensitivities of different system parameters. In this figure, it is evident that the sensitivities of λ_d, β_1, and μ are notably higher than those of other parameters, namely λ, ν, β_2, and γ. Consequently, parameters λ_d, β_1, and μ emerge as crucial and highly sensitive factors for enhancing reliability while maintaining cost-effectiveness. Upon closer examination of the sensitivity of parameters for reliability, it is observed that, at any given time t, the order of sensitive parameters is as follows: $\lambda_d > \lambda > \beta_1 > \beta_2 > \nu > \mu > \gamma$. These findings suggest that implementing appropriate preventive and corrective measures is essential to mitigate degradation and improve system reliability.

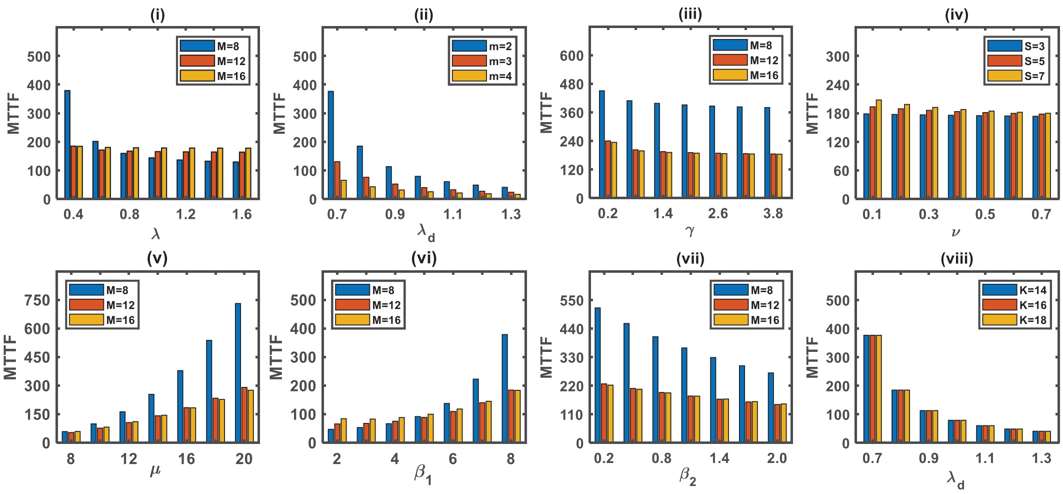


Figure 6.4: Mean time to failure of the system ($MTTF$) wrt. t for different parameters, For ($M = 12, S = 5, m = 2, \lambda = 0.4, \nu = 0.35, \lambda_d = 0.8,\ \mu = 16, \beta_1 = 8, \beta_2 = 1, \gamma = 4$)

Reliability stands as a critical determinant for the efficacy of any machining system, and a pivotal metric in forecasting reliability is the mean time to failure $(MTTF)$, as computed in Eqn. 6.53. The variability of the system's $MTTF$ is visually represented in Figure 6.4 through a bar plot. Upon scrutiny, it is discerned that $MTTF$ experiences a rapid decline for smaller values of M, exhibiting a more pronounced sensitivity. In contrast, the sensitivity diminishes for larger values of active units when confronted with an escalation in the failure rate of any kind. Conversely, an augmentation in the inspection rate for perfect repair and repair rates correlates with a notable increase in $MTTF$, especially in scenarios involving fewer active units. The quantity of active and standby units emerges as a pivotal consideration in the design of a reliable system within the constraints of cost-effectiveness.

Table 6.1: Sensitivity and relative sensitivity of the mean time to failure of the system for ($M = 12, S = 5, m = 2, \lambda = 0.4, \lambda_d = 0.8, \nu = 0.35, \mu = 16, \beta_1 = 8, \beta_2 = 1, \gamma = 4,$ $C_H = 90, C_S = 60, C_M = 10, C_1 = 13, C_2 = 2, C_3 = 2$)

	M,S,m	Θ						
		λ	λ_d	ν	μ	β_1	β_2	γ
Φ_Θ	10, 5, 2	-501.53683	-1412.45511	-109.38124	33.27704	86.35596	-63.59000	-2.59526
	12, 5, 2	-142.75361	-1077.72249	-25.66233	22.98489	53.53459	-44.55887	-1.92651
	14, 5, 2	-61.45667	-1019.16402	-9.50638	20.99768	46.71424	-41.16135	-1.55964
	12, 3, 2	-64.87038	-1036.59368	-8.63333	21.25930	48.58546	-41.55342	-1.23844
	12, 5, 2	-142.75361	-1077.72249	-25.66233	22.98489	53.53459	-44.55887	-1.92651
	12, 7, 2	-203.79150	-1098.62437	-44.13549	24.13662	57.02288	-46.53853	-2.38493
	12, 5, 1	-385.97641	-7082.77422	-67.59066	139.34465	377.53003	-287.92860	-3.85303
	12, 5, 2	-142.75361	-1077.72249	-25.66233	22.98489	53.53459	-44.55887	-1.92651
	12, 5, 3	-90.11140	-335.08105	-16.46149	7.89276	15.98151	-14.65419	-1.28434
Ω_Θ	10, 5, 2	-0.91374	-5.14666	-0.17437	2.42508	3.14661	-0.28963	-0.04728
	12, 5, 2	-0.30951	-4.67327	-0.04868	1.99336	2.32139	-0.24152	-0.04177
	14, 5, 2	-0.13584	-4.50548	-0.01839	1.85652	2.06513	-0.22746	-0.03447
	12, 3, 2	-0.14750	-4.71394	-0.01718	1.93355	2.20944	-0.23621	-0.02816
	12, 5, 2	-0.30951	-4.67327	-0.04868	1.99336	2.32139	-0.24152	-0.04177
	12, 7, 2	-0.43000	-4.63621	-0.08149	2.03714	2.40637	-0.24549	-0.05032
	12, 5, 1	-0.17196	-6.31086	-0.02635	2.48316	3.36385	-0.32069	-0.01717
	12, 5, 2	-0.30951	-4.67327	-0.04868	1.99336	2.32139	-0.24152	-0.04177
	12, 5, 3	-0.47725	-3.54930	-0.07629	1.67206	1.69282	-0.19403	-0.06802

Table 6.1 provides an overview of the sensitivity and relative sensitivity of the $MTTF$ with respect to various parameters, as determined by the respective equations 6.67 and 6.68. Across distinct configurations of governing parameters (M, S, m), the sensitivity order is observed to be $\lambda_d > \lambda > \nu > \beta_1 > \beta_2 > \mu > \gamma$. This insight implies a strategic emphasis on preventive measures over corrective measures as an effective approach to mitigate degradation in system performance.

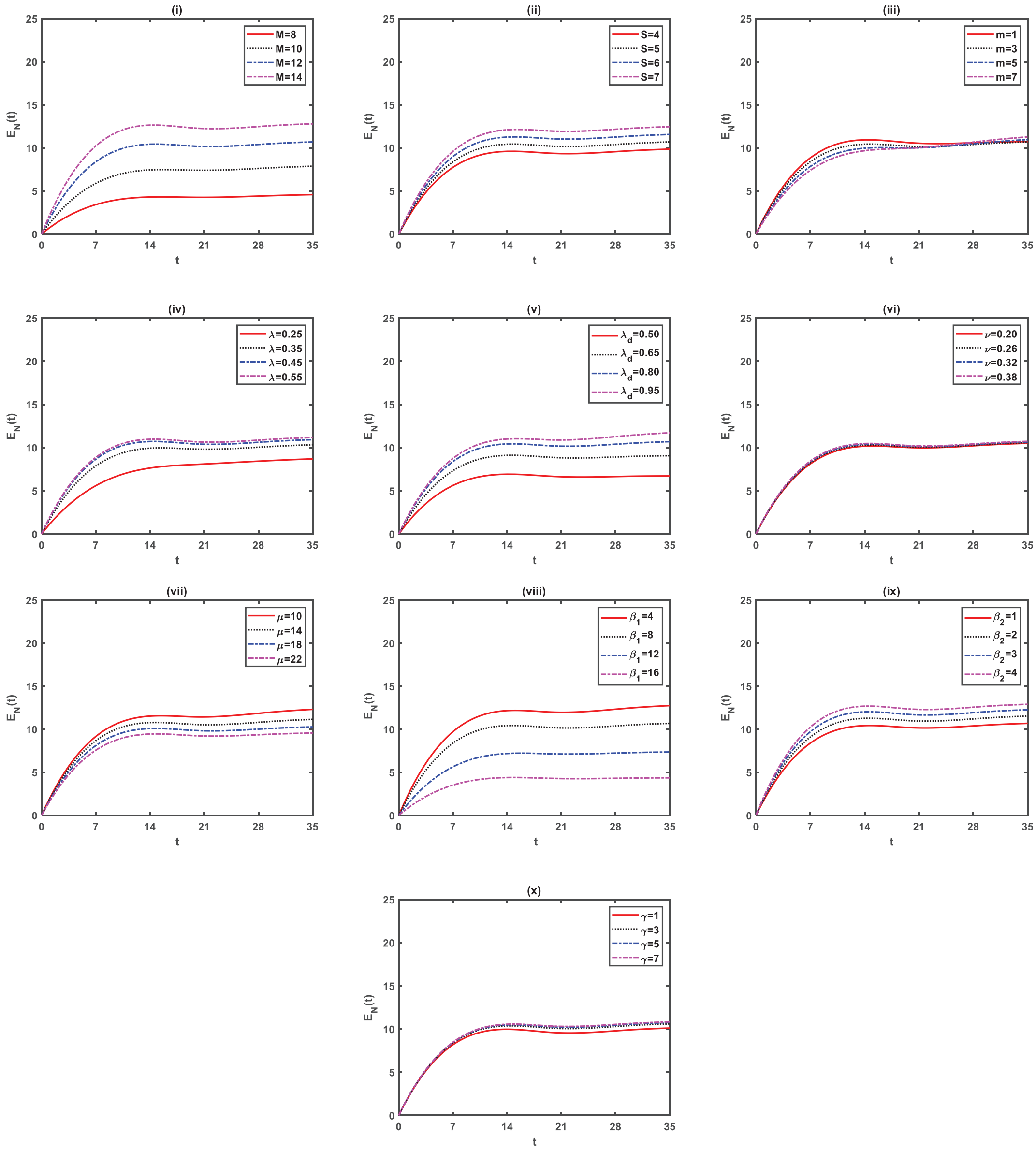


Figure 6.5: Expected number of failed units in the system ($E_N(t)$) wrt. t for different parameters.For ($M = 12$, $S = 5$, $m = 2$, $\lambda = 0.4$, $\nu = 0.35$, $\lambda_d = 0.8$, $\mu = 16$, $\beta_1 = 8$, $\beta_2 = 1$, $\gamma = 4$)

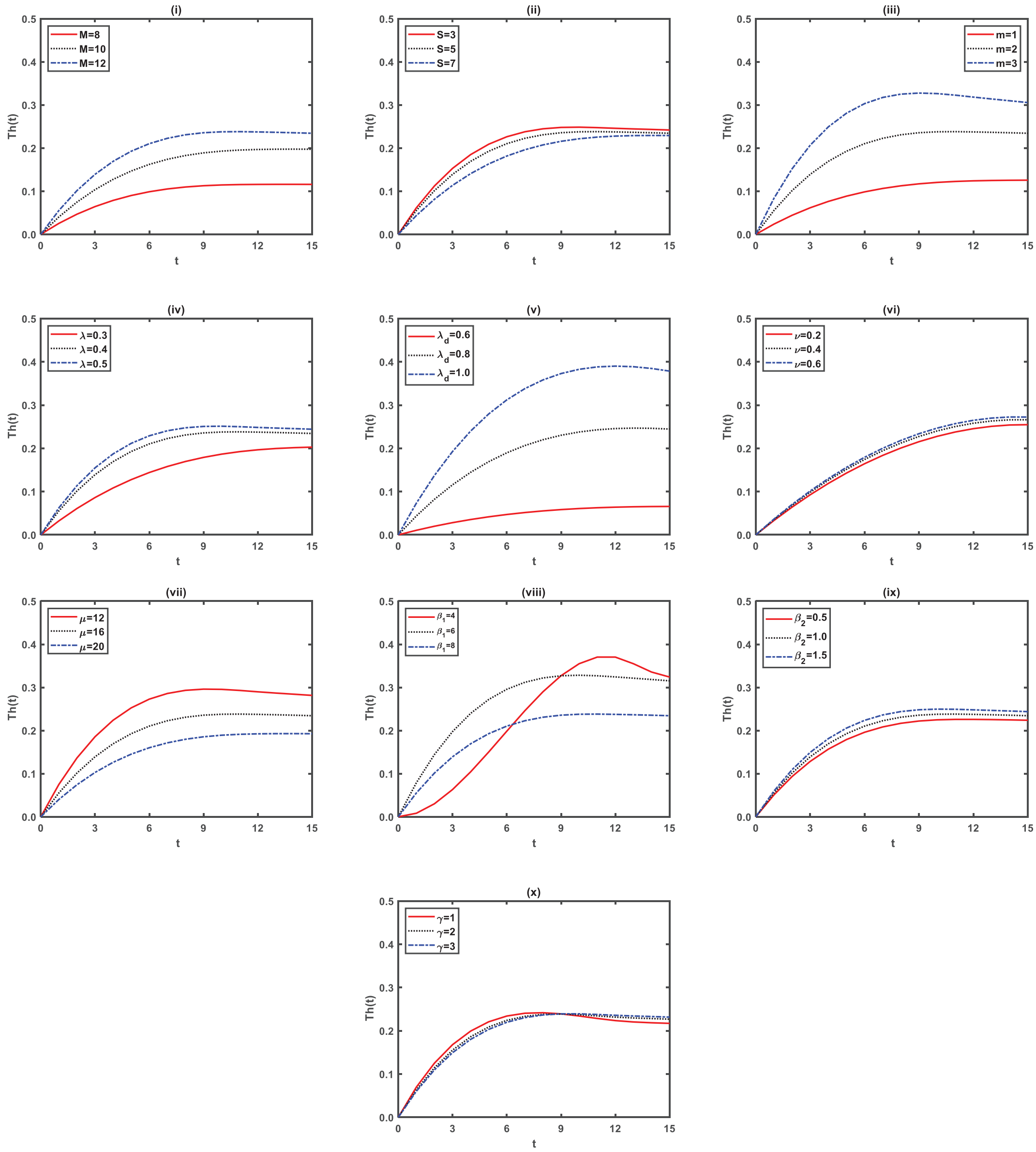


Figure 6.6: Throughput of the system ($Th(t)$) wrt. t for different parameters. For ($M = 12$, $S = 5$, $m = 2$, $\lambda = 0.4$, $\nu = 0.35$, $\lambda_d = 0.8$, $\mu = 16$, $\beta_1 = 8$, $\beta_2 = 1$, $\gamma = 4$)

Figure 6.5 visually depicts the temporal evolution of the expected number of failed units in the system, denoted as $E_N(t)$ and computed using Eqns. 6.55. The legend associated with each sub-graph (i)-(x) denotes diverse system parameters. Initially, $E_N(t)$ exhibits a rising trend over time for each sub-graph. After a specific time instant t, it tends to stabilize, experiencing a modest decline, and eventually maintains a constant level beyond a fixed time t. Notably, the influence of the threshold-based failed units arrival controlled policy is evident in these dynamics. The expected number of failed units in the system, $E_N(t)$, correlates positively with an augmented count

Table 6.2: Performance evaluations for different parameters M, λ and t with $S = 5$, $m = 2$, $\lambda_d = 0.8$, $\nu = 0.35$, $\mu = 16$, $\beta_1 = 8$, $\beta_2 = 1$, $\gamma = 4$

M	λ	t	$R_X(t)$	$MTTF$	$E_N(t)$	$Th(t)$	$E_S(t)$	$E_O(t)$	$E_{fr}(t)$	$E_W(t)$	$FF(t)$	$E_D(t)$
8	0.4	10	0.98846	541.82368	3.18858	0.09225	2.44401	7.40995	4.27817	1.34172	0.01847	0.02893
		25	0.96140	541.82368	3.45954	0.09210	2.35823	7.19634	4.15894	1.20217	0.06175	0.02662
		40	0.93499	541.82368	3.69422	0.08958	2.29342	6.99862	4.04468	1.09487	0.10401	0.02425
	0.55	10	0.97732	298.57274	4.69611	0.16912	1.43222	6.90909	4.82406	1.02725	0.03629	0.03601
		25	0.92902	298.57274	5.07057	0.16206	1.35351	6.56211	4.58256	0.90376	0.11356	0.03196
		40	0.88308	298.57274	5.41328	0.15404	1.28656	6.23757	4.35592	0.80467	0.18707	0.02846
	0.7	10	0.96772	225.37488	5.65095	0.22361	0.83744	6.54157	5.14780	0.91096	0.05165	0.03957
		25	0.90478	225.37488	6.06542	0.20926	0.78214	6.11541	4.81234	0.79341	0.15235	0.03450
		40	0.84594	225.37488	6.45138	0.19565	0.73128	5.71769	4.49937	0.69743	0.24649	0.03033
12	0.4	10	0.97626	233.30656	8.96966	0.21370	0.37828	7.69183	5.48997	0.61206	0.03798	0.02382
		25	0.91461	233.30656	9.51588	0.20591	0.32145	7.13690	5.12877	0.53897	0.13662	0.02164
		40	0.85666	233.30656	9.92679	0.19287	0.30106	6.68464	4.80378	0.48392	0.22935	0.01943
	0.55	10	0.96916	216.69154	9.61277	0.23370	0.14669	7.27359	5.58515	0.58101	0.04935	0.02431
		25	0.90348	216.69154	10.04924	0.21808	0.13599	6.77782	5.20524	0.51797	0.15443	0.02170
		40	0.84225	216.69154	10.45255	0.20330	0.12678	6.31846	4.85246	0.46424	0.25240	0.01945
	0.7	10	0.96601	211.05855	9.79541	0.23827	0.08628	7.14815	5.66004	0.57783	0.05439	0.02432
		25	0.89900	211.05855	10.23436	0.22234	0.07896	6.64509	5.26216	0.51417	0.16160	0.02172
		40	0.83662	211.05855	10.63441	0.20691	0.07348	6.18403	4.89705	0.46049	0.26140	0.01946
16	0.4	10	0.97205	227.77787	13.08031	0.22078	0.09419	7.86172	5.99480	0.45831	0.04472	0.01688
		25	0.90932	227.77787	13.59297	0.20899	0.08115	7.29524	5.58012	0.41051	0.14508	0.01537
		40	0.85054	227.77787	14.00721	0.19548	0.07590	6.82359	5.21938	0.37262	0.23913	0.01396
	0.55	10	0.96889	223.88611	13.22960	0.22105	0.05316	7.75019	6.06845	0.45870	0.04977	0.01671
		25	0.90550	223.88611	13.76412	0.21122	0.04387	7.15739	5.61510	0.40795	0.15121	0.01535
		40	0.84612	223.88611	14.17300	0.19737	0.04099	6.68809	5.24692	0.37021	0.24621	0.01393
	0.7	10	0.96758	222.37667	13.31460	0.22225	0.03697	7.68001	6.10746	0.45870	0.05188	0.01669
		25	0.90389	222.37667	13.82931	0.21205	0.03130	7.10308	5.65094	0.40862	0.15378	0.01533
		40	0.84429	222.37667	14.23615	0.19807	0.02924	6.63476	5.27836	0.37077	0.24914	0.01391

of active/standby units and their respective failure rates. In contrast, the system's expected number of failed units diminishes with increased rates of repair and inspection for perfect repair.

Figure 6.6 illustrates the temporal variations in the system's throughput, calculated using Eqn. 6.56, over time while varying several system parameters. As depicted in Fig. 6.6, the throughput is higher when there are more active units in the system, increased failure rates of any kind, and higher inspection rates for imperfect repair. This trend is expected, considering that a higher number of failed units in the system contributes to increased throughput. Parameters that result in higher throughput are crucial for enhancing existing systems. Conversely, throughput decreases with higher service rates, increased standby units, and longer setup times. A noteworthy trend is observed in Fig. 6.6(viii) for the inspection rate for perfect repair, β_1. The throughput decreases when the inspection rate for perfect repair is significantly large. In contrast, a lower inspection rate for perfect repair initially yields lower throughput, gradually becoming more prominent over time.

Table 6.3: Performance evaluations for different parameters S, ν and t with $M = 12$, $m = 2$, $\lambda = 0.4$, $\lambda_d = 0.8$, $\mu = 16$, $\beta_1 = 8$, $\beta_2 = 1$, $\gamma = 4$

S	ν	t	$R_X(t)$	$MTTF$	$E_N(t)$	$Th(t)$	$E_S(t)$	$E_O(t)$	$E_{fr}(t)$	$E_W(t)$	$FF(t)$	$E_D(t)$
3	0.1	10	0.97044	224.63368	7.41111	0.22645	0.19273	7.42878	5.36248	0.72357	0.04730	0.03055
		25	0.90693	224.63368	7.83340	0.21104	0.18191	6.94973	5.01274	0.63992	0.14891	0.02694
		40	0.84760	224.63368	8.23681	0.19723	0.17001	6.49508	4.68481	0.56876	0.24384	0.02395
	0.2	10	0.96982	223.13436	7.45321	0.22754	0.17929	7.39968	5.38028	0.72187	0.04829	0.03053
		25	0.90597	223.13436	7.87832	0.21217	0.16864	6.91725	5.02700	0.63808	0.15045	0.02693
		40	0.84633	223.13436	8.28128	0.19820	0.15754	6.46192	4.69610	0.56707	0.24587	0.02393
	0.3	10	0.96926	221.81462	7.49025	0.22850	0.16751	7.37400	5.39580	0.72038	0.04918	0.03051
		25	0.90511	221.81462	7.91811	0.21317	0.15700	6.88837	5.03921	0.63642	0.15183	0.02692
		40	0.84520	221.81462	8.32064	0.19906	0.14661	6.43245	4.70568	0.56554	0.24768	0.02392
5	0.1	10	0.98036	245.83784	8.29273	0.19199	0.65103	8.09883	5.40617	0.65192	0.03142	0.02315
		25	0.92212	245.83784	9.11859	0.19745	0.47926	7.38341	5.07529	0.55659	0.12462	0.02165
		40	0.86643	245.83784	9.53650	0.18563	0.44947	6.93604	4.76874	0.50005	0.21371	0.01947
	0.2	10	0.97857	239.85848	8.62274	0.20239	0.51536	7.90340	5.45424	0.63254	0.03428	0.02347
		25	0.91867	239.85848	9.30627	0.20146	0.40368	7.26813	5.10330	0.54837	0.13012	0.02165
		40	0.86192	239.85848	9.72041	0.18904	0.37853	6.81872	4.78798	0.49257	0.22093	0.01945
	0.3	10	0.97698	235.23638	8.87041	0.21043	0.41686	7.75308	5.48133	0.61793	0.03682	0.02372
		25	0.91584	235.23638	9.45369	0.20459	0.34557	7.17615	5.12191	0.54179	0.13465	0.02164
		40	0.85824	235.23638	9.86552	0.19173	0.32379	6.72475	4.79977	0.48652	0.22681	0.01943
7	0.1	10	0.98773	267.92154	8.43309	0.14452	1.61272	9.00390	5.43958	0.64503	0.01963	0.01714
		25	0.93697	267.92154	10.25991	0.18304	0.91557	7.82173	5.15178	0.50213	0.10086	0.01784
		40	0.88473	267.92154	10.76116	0.17524	0.83246	7.34807	4.86412	0.45201	0.18444	0.01628
	0.2	10	0.98506	253.70672	9.36436	0.16795	1.13378	8.55029	5.55201	0.59289	0.02390	0.01794
		25	0.92975	253.70672	10.76844	0.19320	0.67068	7.55119	5.18720	0.48170	0.11239	0.01794
		40	0.87506	253.70672	11.21593	0.18267	0.62164	7.09450	4.88077	0.43516	0.19991	0.01629
	0.3	10	0.98259	244.32256	10.04759	0.18664	0.80206	8.19701	5.59375	0.55673	0.02785	0.01858
		25	0.92421	244.32256	11.10095	0.19965	0.51674	7.36713	5.19938	0.46837	0.12126	0.01798
		40	0.86789	244.32256	11.52706	0.18772	0.48292	6.91483	4.88193	0.42352	0.21137	0.01629

Figure 6.7 presents a surface plot depicting the correlation between the expected total cost, as computed in Eqn. 6.57, time, and system parameters such as μ, β_1, λ, and λ_d. The plot reveals that an increase in either the service rate or the inspection rate for the perfect repair by the repairer results in the repair of more failed units, ultimately reducing the overall cost incurred by the system. Conversely, the failure rate of active units and those in a deteriorating state contributes to an increase in the expected total cost of the system, as illustrated in Fig. 6.7(iii)-6.7(iv). To minimize the expected cost associated with machine repair problems, proactive preventive measures should be implemented to prevent deterioration and mitigate delays.

Tables 6.2-6.6 provide a comprehensive overview of the system's characteristics as parameters vary over time. In the numerical experiments, default parameter values are set to $M = 12$, $S = 5$, $m = 2$, $\lambda = 0.4$, $\lambda_d = 8$, $\nu = 0.35$, $\mu = 16$, $\beta_2 = 1$, and $\gamma = 4$. The results observed in Tables 6.2-6.6 indicate that the reliability $R_X(t)$ decreases, while $E_N(t)$ increases with time, regardless of parameter variations. The mean time to failure ($MTTF$) exhibits a monotonically decreasing trend concerning

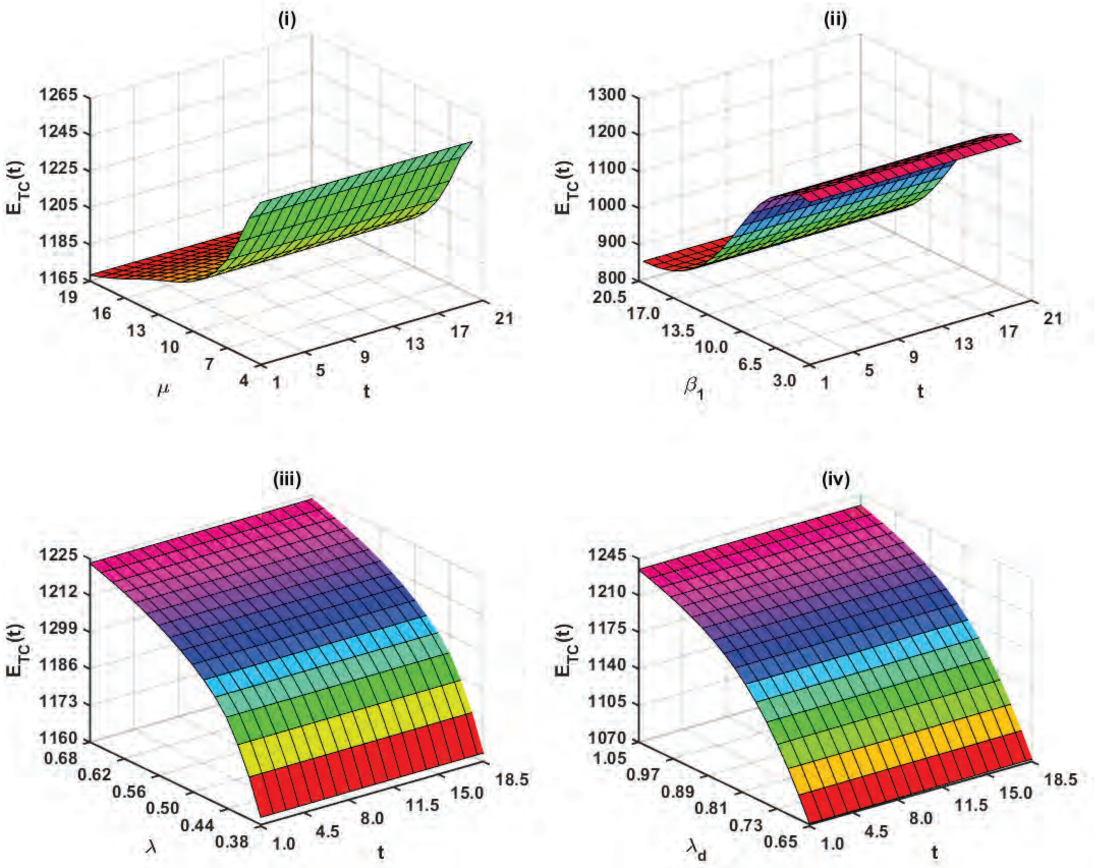


Figure 6.7: Expected total cost ($E_{TC}(t)$) by varying parameters, For ($M = 12$, $S = 5$, $m = 2$, $\lambda = 0.4$, $\nu = 0.35$, $\lambda_d = 0.8$, $\mu = 16$, $\beta_1 = 8$, $\beta_2 = 1$, $\gamma = 4$)

all parameters except μ. Additional insights and observations from the summarized results can be utilized to inform design decisions. These concise summaries provide valuable information for system designers to make informed decisions.

In summary, based on the numerical illustrations, we draw the following concluding notes:

- In implementing preventive measures, the strategy involves delaying the failure of both active and standby units. Switching must be instantaneous and perfect to ensure the uninterrupted functioning of the system.
- As a corrective measure, repairs should be executed with excellence, emphasizing a high inspection rate. Optimal thresholds and capacities for the system are essential to prevent a large number of lost failed units.
- The number of standby units should be optimized within cost constraints to prevent degradation during short-mode operation.

Table 6.4: Performance evaluations for different parameters m, λ_d and t with $M = 12$, $S = 5$, $\lambda = 0.4$, $\nu = 0.35$, $\mu = 16$, $\beta_1 = 8$, $\beta_2 = 1$, $\gamma = 4$

m	λ_d	t	$R_X(t)$	$MTTF$	$E_N(t)$	$Th(t)$	$E_S(t)$	$E_O(t)$	$E_{fr}(t)$	$E_W(t)$	$FF(t)$	$E_D(t)$
1	0.80	10	0.99615	1225.10702	9.41232	0.10018	0.28664	7.36624	5.39825	0.57353	0.00308	0.01064
		25	0.98413	1225.10702	9.78649	0.10662	0.20365	7.07454	5.27795	0.53931	0.01270	0.01089
		40	0.97210	1225.10702	9.87540	0.10534	0.20094	6.98755	5.21330	0.52791	0.02232	0.01067
	0.95	10	0.98777	443.28894	10.29402	0.21668	0.23428	6.53640	5.64243	0.54813	0.01162	0.02105
		25	0.95470	443.28894	10.63120	0.21497	0.19444	6.23702	5.43416	0.51115	0.04304	0.02022
		40	0.92263	443.28894	10.84516	0.20775	0.18790	6.02749	5.25162	0.48424	0.07350	0.01916
	1.10	10	0.97333	229.55458	10.58896	0.33263	0.27209	6.20249	5.96139	0.56298	0.02934	0.03141
		25	0.91086	229.55458	11.02444	0.31270	0.24685	5.78819	5.57756	0.50593	0.09805	0.02836
		40	0.85238	229.55458	11.40811	0.29262	0.23100	5.41655	5.21944	0.45752	0.16238	0.02565
2	0.80	10	0.97626	233.30656	8.96966	0.21370	0.37828	7.69183	5.48997	0.61206	0.03798	0.02382
		25	0.91461	233.30656	9.51588	0.20591	0.32145	7.13690	5.12877	0.53897	0.13662	0.02164
		40	0.85666	233.30656	9.92679	0.19287	0.30106	6.68464	4.80378	0.48392	0.22935	0.01943
	0.95	10	0.94314	111.23438	9.54672	0.34211	0.38025	7.07733	5.78594	0.60607	0.10804	0.03584
		25	0.82081	111.23438	10.40262	0.29909	0.32414	6.14732	5.03486	0.48400	0.34045	0.02875
		40	0.71431	111.23438	11.12889	0.26028	0.28208	5.34969	4.38158	0.39371	0.54281	0.02339
	1.10	10	0.90141	69.64193	9.77973	0.42181	0.43162	6.74820	6.05819	0.61946	0.21689	0.04313
		25	0.72055	69.64193	11.03544	0.33786	0.34198	5.38962	4.84347	0.43890	0.61478	0.03062
		40	0.57597	69.64193	12.03160	0.27006	0.27336	4.30816	3.87160	0.32179	0.93286	0.02245
3	0.80	10	0.93055	91.18102	8.43583	0.31082	0.51005	7.97519	5.48570	0.65029	0.16668	0.03685
		25	0.78465	91.18102	9.49022	0.26414	0.42043	6.70925	4.62400	0.48724	0.51685	0.02783
		40	0.66153	91.18102	10.35473	0.22269	0.35446	5.65653	3.89847	0.37649	0.81233	0.02151
	0.95	10	0.86766	51.99721	8.98637	0.40114	0.51491	7.28968	5.66269	0.63014	0.37717	0.04464
		25	0.64045	51.99721	10.56711	0.29666	0.37759	5.37729	4.18026	0.39559	1.02471	0.02807
		40	0.47272	51.99721	11.72805	0.21897	0.27870	3.96902	3.08549	0.26309	1.50274	0.01867
	1.10	10	0.80274	36.17703	9.36952	0.43603	0.53177	6.75540	5.75924	0.61468	0.65095	0.04654
		25	0.51670	36.17703	11.37737	0.28075	0.34154	4.34746	3.70749	0.32587	1.59488	0.02468
		40	0.33258	36.17703	12.66829	0.18071	0.21983	2.79825	2.38634	0.18837	2.20250	0.01426

6.10 Conclusion

This chapter explores a Markovian model of a redundant repairable machining system, integrating real-time paradigms into the modeling process for increased practicality. To the best of our knowledge, this research represents one of the initial attempts to quantitatively assess the reliability of a machining system, incorporating controlled failed unit arrival policies and imperfect repair. The study employs an efficient numerical computation technique based on Laplace transform, eigenvalue, and linear algebra to calculate transient-state probabilities, reliability measures, and queueing characteristics. Additionally, sensitivity analysis is conducted to pinpoint critical parameters for the machining system. However, there are some limitations to this research. For instance, the proposed model could be extended by integrating differentiated working vacations, working breakdowns, common-cause failures, switching failures, and switching delays. Furthermore, further exploration into the analysis, design, and optimization of the system to derive optimal values for decision parameters based on

Table 6.5: Performance evaluations for different parameters μ, β_1 and t with $M = 12$, $S = 5, m = 2$, $\lambda = 0.4$, $\lambda_d = 8$, $\nu = 0.35$, $\beta_2 = 1$, $\gamma = 4$

μ	β_1	t	$R_X(t)$	$MTTF$	$E_N(t)$	$Th(t)$	$E_S(t)$	$E_O(t)$	$E_{fr}(t)$	$E_W(t)$	$FF(t)$	$E_D(t)$
10	4	10	0.86035	57.65581	11.06680	0.34604	0.07488	5.77903	4.30223	0.38875	0.22343	0.03127
		25	0.65964	57.65581	12.20387	0.26341	0.05796	4.44409	3.30621	0.27091	0.54457	0.02158
		40	0.50580	57.65581	13.09365	0.20259	0.04426	3.40337	2.53267	0.19343	0.79073	0.01547
	8	10	0.93082	93.72755	10.15188	0.31010	0.18463	6.64514	4.88416	0.48111	0.11069	0.03055
		25	0.78887	93.72755	11.04994	0.26306	0.15727	5.62474	4.13631	0.37433	0.33781	0.02381
		40	0.66850	93.72755	11.80525	0.22292	0.13327	4.76649	3.50517	0.29692	0.53040	0.01888
	12	10	0.96939	172.67989	8.87112	0.19255	0.40356	7.73761	5.50311	0.62034	0.04898	0.02171
		25	0.88720	172.67989	9.63133	0.18414	0.31893	6.97634	5.01571	0.52077	0.18047	0.01912
		40	0.81152	172.67989	10.17482	0.16844	0.29166	6.38108	4.58781	0.45090	0.30157	0.01656
16	4	10	0.91458	92.87546	10.63334	0.36521	0.10978	6.24365	4.61269	0.43380	0.13667	0.03435
		25	0.77610	92.87546	11.44500	0.30979	0.09321	5.29915	3.91474	0.34205	0.35824	0.02707
		40	0.65860	92.87546	12.13602	0.26306	0.07902	4.49554	3.32133	0.27368	0.54624	0.02168
	8	10	0.97626	233.30656	8.96966	0.21370	0.37828	7.69183	5.48997	0.61206	0.03798	0.02382
		25	0.91461	233.30656	9.51588	0.20591	0.32145	7.13690	5.12877	0.53897	0.13662	0.02164
		40	0.85666	233.30656	9.92679	0.19287	0.30106	6.68464	4.80378	0.48392	0.22935	0.01943
	12	10	0.99564	1045.22855	5.69163	0.05326	1.42399	9.93129	6.13204	1.07738	0.00698	0.00936
		25	0.98180	1045.22855	6.19719	0.05989	1.24753	9.58830	6.04287	0.97510	0.02912	0.00966
		40	0.96774	1045.22855	6.34353	0.05915	1.22708	9.44764	5.95629	0.93896	0.05161	0.00932
22	4	10	0.93925	129.25142	10.38952	0.36735	0.13399	6.49430	4.77457	0.45956	0.09720	0.03536
		25	0.83509	129.25142	11.02459	0.32820	0.11831	5.76204	4.23881	0.38449	0.26386	0.02977
		40	0.74245	129.25142	11.57724	0.29188	0.10514	5.12217	3.76825	0.32549	0.41207	0.02521
	8	10	0.98869	454.02483	7.92094	0.14931	0.64943	8.48790	5.79481	0.73158	0.01810	0.01885
		25	0.95658	454.02483	8.42134	0.15420	0.54356	8.05938	5.58439	0.66312	0.06947	0.01831
		40	0.92518	454.02483	8.67100	0.14918	0.52541	7.79425	5.40097	0.62288	0.11972	0.01720
	12	10	0.99892	4664.74338	3.80928	0.01801	2.28720	10.95531	6.21547	1.63167	0.00172	0.00473
		25	0.99577	4664.74338	3.99093	0.01967	2.20845	10.84950	6.20045	1.55363	0.00676	0.00493
		40	0.99257	4664.74338	4.03091	0.01963	2.20066	10.81394	6.18057	1.53329	0.01188	0.00487

evaluation results would be valuable.

Table 6.6: Performance evaluations for different parameters β_1, β_2 and t with $M = 12$, $S = 5, m = 2$, $\lambda = 0.4$, $\lambda_d = 8$, $\nu = 0.35$, $\mu = 16$, $\gamma = 4$

β_1	β_2	t	$R_X(t)$	$MTTF$	$E_N(t)$	$Th(t)$	$E_S(t)$	$E_O(t)$	$E_{fr}(t)$	$E_W(t)$	$FF(t)$	$E_D(t)$
6	0.1	10	0.94987	134.15670	9.98867	0.32047	0.20131	6.82790	4.99127	0.49969	0.08020	0.03208
		25	0.84738	134.15670	10.64981	0.28674	0.17980	6.07843	4.44717	0.41758	0.24419	0.02692
		40	0.75587	134.15670	11.22758	0.25577	0.16039	5.42201	3.96692	0.35332	0.39061	0.02278
	0.6	10	0.94288	119.45581	10.39791	0.32523	0.19368	6.74753	4.93299	0.47442	0.09139	0.03128
		25	0.82923	119.45581	11.09036	0.28735	0.16996	5.91734	4.33103	0.39052	0.27323	0.02591
		40	0.72918	119.45581	11.68273	0.25268	0.14946	5.20340	3.80848	0.32599	0.43331	0.02163
	1.1	10	0.93566	107.43943	10.78592	0.32889	0.18643	6.66884	4.87617	0.45209	0.10295	0.03049
		25	0.81096	107.43943	11.50211	0.28682	0.16069	5.75961	4.21727	0.36665	0.30247	0.02494
		40	0.70276	107.43943	12.10223	0.24855	0.13925	4.99115	3.65458	0.30198	0.47559	0.02054
10	0.1	10	0.98977	466.90210	7.35916	0.11236	0.81641	8.87194	5.91591	0.80388	0.01637	0.01527
		25	0.95875	466.90210	7.98018	0.12028	0.65914	8.37588	5.70739	0.71519	0.06600	0.01507
		40	0.92811	466.90210	8.23951	0.11653	0.63694	8.10632	5.52479	0.67052	0.11502	0.01414
	0.6	10	0.98764	394.65560	7.89991	0.12545	0.73788	8.69098	5.85972	0.74175	0.01977	0.01588
		25	0.95096	394.65560	8.51787	0.13196	0.59483	8.16931	5.61786	0.65954	0.07846	0.01549
		40	0.91505	394.65560	8.80252	0.12706	0.57160	7.85950	5.40556	0.61409	0.13591	0.01443
	1.1	10	0.98528	338.31595	8.41338	0.13853	0.66849	8.51874	5.80162	0.68957	0.02354	0.01647
		25	0.94254	338.31595	9.02417	0.14319	0.53967	7.97129	5.52561	0.61231	0.09194	0.01587
		40	0.90108	338.31595	9.33242	0.13694	0.51543	7.61978	5.28244	0.56603	0.15827	0.01467
14	0.1	10	0.99806	2448.48951	4.33914	0.02482	2.01960	10.68440	6.20550	1.43012	0.00310	0.00572
		25	0.99211	2448.48951	4.63162	0.02770	1.90059	10.50518	6.17371	1.33295	0.01262	0.00598
		40	0.98604	2448.48951	4.70436	0.02758	1.88744	10.43926	6.13601	1.30433	0.02234	0.00586
	0.6	10	0.99756	1926.13960	4.82465	0.02970	1.89839	10.54398	6.19447	1.28392	0.00391	0.00616
		25	0.99000	1926.13960	5.15955	0.03319	1.76753	10.33331	6.15160	1.19228	0.01600	0.00643
		40	0.98230	1926.13960	5.24752	0.03299	1.75207	10.25104	6.10383	1.16318	0.02832	0.00629
	1.1	10	0.99696	1541.70141	5.30705	0.03508	1.78134	10.40038	6.18059	1.16460	0.00486	0.00661
		25	0.98753	1541.70141	5.68336	0.03919	1.64005	10.15689	6.12462	1.07764	0.01994	0.00690
		40	0.97794	1541.70141	5.78764	0.03887	1.62228	10.05609	6.06517	1.04795	0.03529	0.00672

Chapter 7

Reliability analysis of Standby Provision Multi-Unit Machining Systems with Varied Failures, Degradations, Imperfections, and Delays

"In the face of every malfunction lies an opportunity for enlightenment. Embrace each challenge as a philosopher does, for within the labyrinth of failure resides the path to discovery."

Immanuel Kant

7.1 Introduction

The fault-tolerant multi-unit machining system (FMMS) has become a prerequisite of contemporary technology in the fast-growing industrial environment to support uninterrupted operation, especially in computer-driven automated machining systems using artificial intelligence and remote access. The FMMSs are the primary need for developing smart industry 4.0, communication networks, power plants, assembly lines, distributed setups, etc. As preventive and corrective measures, redundancy and strategic maintenance, respectively are primarily operated to reduce the risk of system failure and service disruption. The queueing-theoretic approach based on stochastic modeling facilitates a robust methodology for predicting reliability/availability metrics in FMMSs such as internet/communication networks, distributed service/data/cloud computing centers, manufacturing systems/production lines, hydraulic control systems, etc. The unexpected failure of embedded units is undesirable; it increases downtime, which can sometimes be the primary cause of a system-wide essential shutdown. The unexpected failure interrupts the operation, degrades system performance and service quality, and increases economic losses. The expert machining system is desirable to propose the prevention of unforeseen faults/failures of units and related unpleasant scenarios. The reliability models on machine repair problem (MRP) have been broadly investigated in the literature (*cf.*[88], [242], [266], [4], [159], [14]) for the better design of the FMMSs.

Besides active redundancy, standby redundancy can strategically enhance system reliability in machining systems. Most studies on the prediction of fault-tolerant system (FTS) reliability and queueing characteristics have expected that the transition from standby to active units is perfect. Although it is not practically conceivable in many machining systems, the perfect switching paradigm simplifies problem analysis. Lewis [171] conceptualized the notion of standby switching failures in the reliability analysis of standby provisioning systems. Switching failure of standby support in a machining system is a challenge in real-time scenarios, such as cloud computations, internet systems, call centers, safety devices, etc. Some distinguished researchers have also investigated the reliability analysis and optimal analysis by integrating the concept of unsuccessful standby switchover in stochastic redundant multi-unit repairable machining systems (*cf.* [260], [100]).

In short mode, once available standby units are exhausted, the active-redundant system starts functioning in degraded mode since the likelihood of active unit failure increases due to load sharing. In short mode, the active units fail at a higher rate. It is worthwhile to refer to some notable references in this direction (*cf.* [118], [147], [34], [185], [172]).

The expert design has been extensively used in multi-unit systems that consider the common cause of failure with numerous redundant mechanisms. Under common cause failure, all existing units fail simultaneously owing to some randomly generated causes like natural catastrophes or artificial errors. The occurrence of common cause failures (CCFs) has an adverse impact on multi-unit machining system reliability and contributes significantly to system risk. There have been attempts to analyze the CCFs phenomenon in the multi-unit system (*cf.* [208], [304], [218], [60], [76], [232]). Recently, Soga et al. [277] proposed a systematic approach to approximate an inter-unit common cause failure probability.

Imperfect coverage or unrecovered faults owing to a lack of fault detection or switching failure are prevalent problems in redundant machining systems. It significantly impacts the standby unit's operations and the system's availability. The coverage factor is the likelihood of successful coverage, which takes account of the chances of efficacious detection, location, and recovery from a failure (*cf.* [129], [164], [20]).

The faulty units under inadequate coverage can be removed with the reboot process. Trivedi [298] detailed the notion of reboot delay and examined its effect on the availability or reliability of a repairable multi-unit machining system. Significant investigations on unreliable attributes like imperfect coverage and reboot delay have been done in the past (*cf.* [315], [132], [107], [326]) in different frameworks using the distinguished methodology.

Most conventional investigations of repairable redundant multi-unit machining systems are centered on the assumption of a reliable server and perfect repair. However, these unrealistic presumption appears to be speculative for predicting the factual performance of the repair facility since the repair facility or repairer may fail while performing the repair task or may repair imperfectly. The repairer may be imperfect any number of times before it gives a perfect repair. Some researchers (*cf.* [138], [135], [121]) explored the performance indices of machine repairable system with warm standby provisioning and a failure-prone repair facility. For more comprehensive theories and excellent reviews on the imperfect repair facility, the researcher may refer to the following investigations (*cf.* [225], [226], [267], [47],[157]). The unreliable repair facility with the imperfect repair is a more realistic representation of the machining system and never has been studied in literature before.

In a power distribution system, transformers play a crucial role as active units by converting high voltage electricity into a lower voltage suitable for consumer use. However, in the event of a transformer failure, a standby transformer is available for immediate replacement, ensuring uninterrupted power supply to consumers. The failed unit is promptly sent for repair. However, if the repair facility relies on an unreliable service that fails to effectively resolve the underlying issues, the repaired transformer may experience recurring failures or suboptimal performance due to imperfect repairs. This can lead to increased maintenance costs and a reduction in the overall reliability of the power distribution system. A switching failure occurs when the automation responsible for maintaining the power supply malfunctions or fails to operate correctly. Such failures can disrupt the uninterrupted power supply, causing downtime issues and impacting the reliability of the infrastructure. In power stations, the installation of multiple transformers is essential to ensure redundancy and enhance uninterrupted power supply. However, a common-cause failure scenario could arise if all transformers are affected by a single event, such as high temperature or connection problems. This type of failure can result in a loss of power and pose a significant threat to the reliability of the power distribution system. Over time, transformer systems may deteriorate due to factors such as carbon deficiency, rust on coils, problems in the magnetic chamber, or general wear and tear. As a result, their performance gradually degrades, leading to failures and a decrease in energy output. This degradation negatively impacts the overall efficiency and reliability of the transformer system. In a fault detection and suppression system, coverage refers to the system's ability to effectively detect and suppress faults. If the coverage is imperfect, certain areas may not be adequately monitored, leading to undetected faults and compromised reliability, particularly in terms of early fire detection and suppression. Occasional reboots may be necessary for maintenance or updates in various systems. However, if the reboot process is slow or inefficient, it can result in extended downtime, thereby impacting the availability and reliability of the services hosted on the system.

The foremost objective of the projected repairable redundant multi-unit machining system with standby units provisioning and unreliable attributes is threefold: (i) to formulate the Markovian model for fault-tolerant-system in strategic exploration by integrating the realistic unreliable attributes of machining system such as failures, imperfections, delays, degradation, viz. unreliable server, imperfect repair, imperfect switching process, degraded failure, common-cause failure, imperfect coverage, and reboot delay, (ii) to simplify the governing differential-difference forward equations transient solution procedure by employing mathematical notions like Laplace transform, linear algebra, etc., and (iii) to endorse the computational submissiveness of reliability characteristics along with sensitivity and relative sensitivity analysis.

The contents of the present article are structured section-wise as follows: In the next Section 7.2, the detailed description of the proposed model with state notations and transient equations is outlined. Successively, we present an efficient methodology to compute the transient solution. In the next Section 7.3, we define the reliability function and present its sensitivity analysis. The mean time-to-failure and failure frequency are defined and studied in detail in Section 7.4 and 7.5 respectively. To justify the research gap, special cases are discussed in Section 7.6. The significance of the investigation is outlined, and the conclusion and future scope are highlighted in Section 7.7.

7.2 Model Description

This section presents the mathematical model for a Markovian multi-unit machining system of M operating units with S warm standby units and a single repairer. The operating/standby units and repairer, all of which are unreliable but repairable. In the event of a failure, the active/standby units and repairer are immediately sent to restore. The failed unit is replaced with the available warm standby unit in insignificant switchover time. When all M operating units are in working mode, any system can function normally. The system is degraded if less than M but more than m operating units are working. A system failure is defined when less than m; $m = 1,2,...,M$ active units remain working. Therefore, the machining system's strategic policy is known as the (M,m) policy. It is worth mentioning that the system can tolerate up to $K = M+S-m+1$ unit failures.

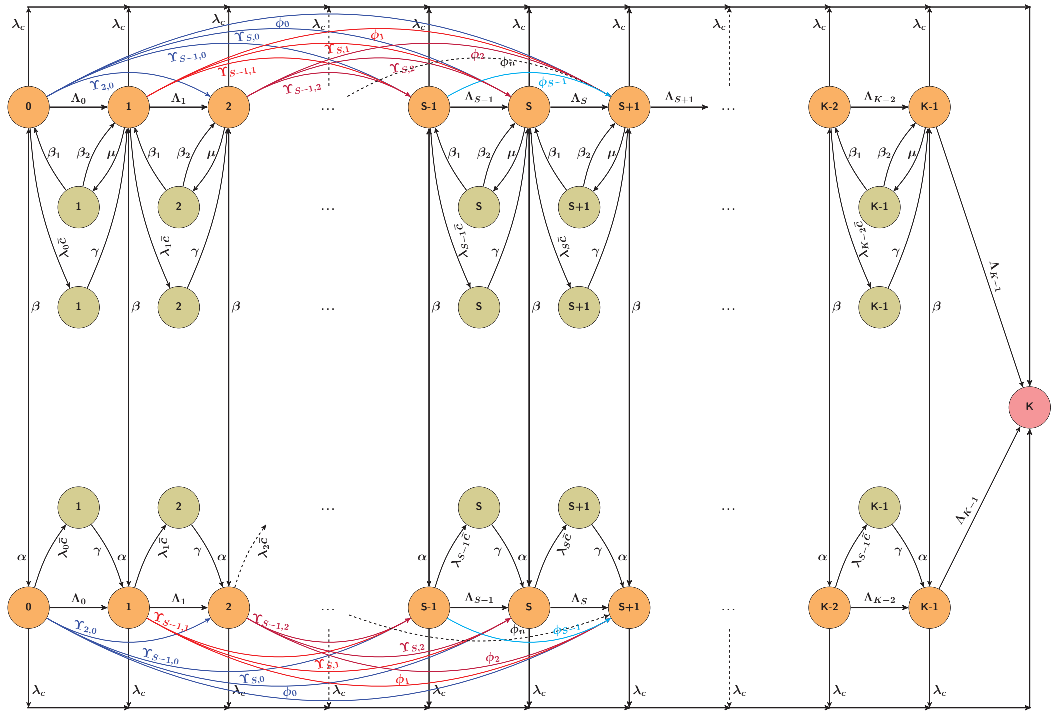


Figure 7.1: State transition rate diagram $\Lambda_i = [M\lambda\bar{q}+(S-i)\nu]c, \quad 0 \leq i \leq K-1,$ $\Upsilon_{j,i} = M\lambda\bar{q}q^{j-i-1}c \qquad \phi_i = M\lambda q^{S-i}c$

We set forth the following assumptions and notations employed for the detailed illustrations of the studied model.

Units failure process

- Since the operating units may fail unpredictably. The inter-time between successive failures of an operating unit is assumed to follow an exponential distribution with a mean failure rate of λ.
- In the context of standby units, it is common for the warm standby units in the spare state to fail before being put into the whole operation. The inter-failure time of the warm standby units in the spare state is also modeled by an exponential distribution with a parameter ν, where $0 < \nu < \lambda$.

- Independent of the state of the active or standby units, the active/standby unit fails randomly.
- In a degraded state, the time-to-failure of the active unit is distributed exponentially with an increasing rate $\lambda_d; \lambda_d > \lambda$ since the load is shared with the existing unit.

Switching process

- Since the operating units are subject to failures, on the failure of any operating unit, the operating unit is ejected with an accessible, warm standby unit in trifling switchover time. After a successful switchover, switched units' failure/working characteristics are also changed to operating.
- Replacing standby units instead of failed operating units may be unsuccessful due to some random switchover hitch. The probability of successful switchover is $1-q$, whereas the switching failure probability is q.
- In a system with standby units, if a standby unit fails to take over from a failed operating unit, the next one in line will randomly attempt to take over. This process of attempting to switch between standby and operating units continues until a successful switch occurs or all the available standby units have been exhausted.

Catastrophe process

- The machining system may cause catastrophe wherein the simultaneous failures of multiple units due to some random common causes. The cause may be external, such as natural calamities or extreme environmental conditions, or internal, such as failures propagated from certain system elements.
- Since a catastrophe in the machining system is a rare event, the random catastrophe cause arrives in a Poisson fashion with parameter $\lambda_c(\lambda_c << \lambda, \nu)$.

Unit recovery process

- When an operating unit is futile, it queued up for the recovery facility of one unreliable repairer without losing time. If the repairer is idle, the failed unit gets repaired; otherwise, the futile unit waits in the queue and is repaired following the first-come, first-served ($FCFS$) service discipline.
- The repair time of the failed units, a continuous random variable, follows an exponential distribution with meantime $1/\mu$.

- The repair may be perfect or imperfect. The time-to-inspect perfect and imperfect repair follows an exponential distribution with parameters β_1 and β_2, respectively.
- The recovered units are as good as the new ones and experience no deterioration in performance with age. They are sent back to the pool of operating or standby units depending on the system's state: degraded or normal.

Reboot process

- When the operating/standby unit fails, it may be instantly successfully detected, located, and recovered with a perfect coverage probability c. As any one of the breakdowns that are not covered successfully is referred to as the unsafe failure state of the system.
- A reboot process refreshes the unsafe state. The reboot follows a Poisson process with a parameter γ.
- The mean reboot delay is too slight to have any other events like failure/repair of operating/standby units, i.e., $\gamma >> \mu, \lambda$.

Unreliable repair facility

- The recovery facility may become unavailable when the repairer fails following an exponentially distributed time-to-breakdown with the rate α.
- The failure of units continues when the recovery facility is unavailable.
- The recovery facility's time-to-restore also follows memoryless property holding distribution with parameter β.

The events of failure/repair of operating/standby units, switching failure, catastrophe, coverage, reboot delay, and breakdown/recovery of the repair facility are statistically independent of each other.

Let $I_1(t)$, $J(t)$, and $F(t)$ denote the number of failed units in the multi-unit system, the state of the repairer, and the inoperative state of the machining system at an arbitrary time t, respectively. The state of the repairer at any time t is defined as follows.

$$J(t) = \begin{cases} 0; & \text{when the repairer is in normal busy state} \\ 1; & \text{when the repairer is in breakdown state} \end{cases}$$

Hence, the state notations defined above form a continuous-time Markov chain $\{(J(t)$ $,I_1(t)) \cup F(t); t \geq 0\}$, in the following solution space.

$$\Pi = \{\{(J(t)=j,\ I_1(t)=i)\,|\,j=0;\ i=0,1,2,\ldots,K-1\}\cup\{(j,i)\,|\,j=0;\ i=1,2,3,\ \ldots,K-2,K-1\}\cup\{(j,i)\,|\,j=0;\ i=1,2,3,\ldots,K-2,K-1\}\cup\{(j,i)\,|\,j=1;\ i=0,1,2,\ldots,K-2,K-1\}\cup\{(j,i)\,|\,j=1;\ i=1,2,3,\ldots,K-2,K-1\}\cup F\}.$$

The governing state probabilities of the system at arbitrary time t are defined as follows.

$P_{j,i}(t) \equiv Pr\left(J(t)=j, I_1(t)=i\right), i.e.,$ Probability of the machining system having i failed units and the repairer is in the state j, where $(j,i)\in\Pi$

$Q_{j,i}(t) \equiv Pr\left(J(t)=j, I_1(t)=i\right)$ and serviced unit is in state of decision for perfect service, i.e., Probability of the machining system having i failed units after service is rendered and the repairer is in the state j, where $(j,i)\in\Pi$

$R_{j,i}(t) \equiv Pr\left(J(t)=j, I_1(t)=i\right)$, and failed unit's fault is not perfectly detected i.e., Probability of the machining system having i failed units with imperfect coverage and the repaireris in the state j, where $(j,i)\in\Pi$

$P_F(t) \equiv$ Probability that the machining system is in inoperative state

At each point of time, number of working units are fixed and changed with event of failure and repair. Depending on the state of the system, failure rate changes. Following the above-defined pertinent notations and assumptions, the state-dependent failure rate of the active/standby unit is summed as

$$\lambda_i = \begin{cases} M\lambda + (S-i)\nu; & \text{if } i = 0,1,\ldots,S-1 \\ (M+S-i)\lambda_d; & \text{if } i = S,S+1,\ldots,K-1 \\ 0; & \text{otherwise} \end{cases}$$

7.2.1 Transient Equations

Following the above-defined notations, assumptions, states, and states probabilities, the associated forward Chapman-Kolmogrove equations are formulated by balancing the inflow-outflow rates in the state-transition diagram in Fig. 7.1. The developed equations are differential-difference equations regarding time and the number of failed units in the system, respectively. For this purpose, we define following operator

$$D = \frac{d}{dt}$$

When the repairer is in normal busy state $\mathbf{J(t) = 0}$

$$DP_{0,0}(t) = -(\lambda_0 + \alpha + \lambda_c)P_{0,0}(t) + \beta_1 Q_{0,1}(t) + \beta P_{1,0}(t) \tag{7.1}$$

$$\begin{aligned} DP_{0,1}(t) = &-(\lambda_1 + \alpha + \lambda_c + \mu)P_{0,1}(t) + (M\lambda\bar{q} + S\nu)cP_{0,0}(t) + \beta_2 Q_{0,1}(t) \\ &+ \beta_1 Q_{0,2}(t) + \gamma R_{0,1}(t) + \beta P_{1,1}(t) \end{aligned} \tag{7.2}$$

$$\begin{aligned} DP_{0,i}(t) = &-(\lambda_i + \alpha + \lambda_c + \mu)P_{0,i}(t) + (M\lambda\bar{q} + (S - i + 1)\nu)cP_{0,i-1}(t) \\ &+ \beta_2 Q_{0,i}(t) + \beta_1 Q_{0,i+1}(t) + \gamma R_{0,i}(t) + \sum_{l=0}^{i-2} M\lambda\bar{q}q^{i-l-1}cP_{0,l}(t) \\ &+ \beta P_{1,i}(t); \quad 2 \leq i \leq S - 1 \end{aligned} \tag{7.3}$$

$$\begin{aligned} DP_{0,S}(t) = &-(\lambda_S + \alpha + \lambda_c + \mu)P_{0,S}(t) + (M\lambda\bar{q} + \nu)cP_{0,S-1}(t) + \beta_2 Q_{0,S}(t) \\ &+ \beta_1 Q_{0,S+1}(t) + \gamma R_{0,S}(t) + \beta P_{1,S}(t) + \sum_{l=0}^{S-2} M\lambda\bar{q}q^{S-l-1}cP_{0,l}(t) \end{aligned} \tag{7.4}$$

$$\begin{aligned} DP_{0,S+1}(t) = &-(\lambda_{S+1} + \alpha + \lambda_c + \mu)P_{0,S+1}(t) + \lambda_S cP_{0,S}(t) + \beta_2 Q_{0,S+1}(t) \\ &+ \beta_1 Q_{0,S+2}(t) + \gamma R_{0,S+1}(t) + \beta P_{1,S+1}(t) + \sum_{l=0}^{S-1} M\lambda q^{S-l}cP_{0,l}(t) \end{aligned} \tag{7.5}$$

$$\begin{aligned} DP_{0,i}(t) = &-(\lambda_i + \alpha + \lambda_c + \mu)P_{0,i}(t) + \lambda_{i-1}cP_{0,i-1}(t) + \beta_2 Q_{0,i}(t) \\ &+ \beta_1 Q_{0,i+1}(t) + \gamma R_{0,i}(t) + \beta P_{1,i}(t); \quad S + 2 \leq i \leq K - 2 \end{aligned} \tag{7.6}$$

$$\begin{aligned} DP_{0,K-1}(t) = &-(\lambda_{K-1} + \alpha + \lambda_c + \mu)P_{0,K-1}(t) + \lambda_{K-2}cP_{0,K-2}(t) \\ &+ \beta_2 Q_{0,K-1}(t) + \gamma R_{0,K-1}(t) + \beta P_{1,K-1}(t) \end{aligned} \tag{7.7}$$

$$DQ_{0,i}(t) = -(\beta_1 + \beta_2)Q_{0,i}(t) + \mu P_{0,i}(t); \quad 1 \leq i \leq K - 1 \tag{7.8}$$

$$DR_{0,i}(t) = -\gamma R_{0,i}(t) + \lambda_{i-1}\bar{c}P_{0,i-1}(t); \quad 1 \leq i \leq S \tag{7.9}$$

$$DR_{0,i}(t) = -\gamma R_{0,i}(t) + \lambda_{i-1}\bar{c}P_{0,i-1}(t); \quad S + 1 \leq i \leq K - 1 \tag{7.10}$$

When the repairer is in breakdown state $\mathbf{J(t) = 1}$

$$DP_{1,0}(t) = -(\lambda_0 + \beta + \lambda_c)P_{1,0}(t) + \alpha P_{0,0}(t) \tag{7.11}$$

$$\begin{aligned} DP_{1,1}(t) = &-(\lambda_1 + \beta + \lambda_c)P_{1,1}(t) + (M\lambda\bar{q} + S\nu)cP_{1,0}(t) + \alpha P_{0,1}(t) \\ &+ \gamma R_{1,1}(t) \end{aligned} \tag{7.12}$$

$$\begin{aligned} DP_{1,i}(t) = &-(\lambda_i + \beta + \lambda_c)P_{1,i}(t) + (M\lambda\bar{q} + (S - i + 1)\nu)cP_{1,i-1}(t) \\ &+ \alpha P_{0,i}(t) + \gamma R_{1,i}(t) + \sum_{l=0}^{i-2} M\lambda\bar{q}q^{i-l-1}cP_{1,l}(t); \quad 2 \leq i \leq S - 1 \end{aligned} \tag{7.13}$$

$$\begin{aligned} DP_{1,S}(t) = &-(\lambda_S + \beta + \lambda_c)P_{1,S}(t) + (M\lambda\bar{q} + \nu)cP_{1,S-1}(t) + \alpha P_{0,S}(t) \\ &+ \gamma R_{1,S}(t) + \sum_{l=0}^{S-2} M\lambda\bar{q}q^{S-l-1}cP_{1,l}(t) \end{aligned} \tag{7.14}$$

$$\begin{aligned} \mathrm{D}P_{1,S+1}(t) = &-(\lambda_{S+1}+\beta+\lambda_c)P_{1,S+1}(t)+\lambda_S cP_{1,S}(t)+\alpha P_{0,S+1}(t) \\ &+\gamma R_{1,S+1}(t)+\sum_{l=0}^{S-1} M\lambda q^{S-l}cP_{1,l}(t) \end{aligned} \tag{7.15}$$

$$\begin{aligned} \mathrm{D}P_{1,i}(t) = &-(\lambda_i+\beta+\lambda_c)P_{1,i}(t)+\lambda_{i-1}cP_{1,i-1}(t)+\alpha P_{0,i}(t)+\gamma R_{1,i}(t); \\ &S+2\leq i\leq K-2 \end{aligned} \tag{7.16}$$

$$\begin{aligned} \mathrm{D}P_{1,K-1}(t) = &-(\lambda_{K-1}+\beta+\lambda_c)P_{1,K-1}(t)+\lambda_{K-2}cP_{1,K-2}(t) \\ &+\alpha P_{0,K-1}(t)+\gamma R_{1,K-1}(t) \end{aligned} \tag{7.17}$$

$$\mathrm{D}R_{1,i}(t) = -\gamma R_{1,i}(t)+\lambda_{i-1}\bar{c}P_{1,i-1}(t); \quad 1\leq i\leq S \tag{7.18}$$

$$\mathrm{D}R_{1,i}(t) = -\gamma R_{1,i}(t)+\lambda_{i-1}\bar{c}P_{1,i-1}(t); \quad S+1\leq i\leq K-1 \tag{7.19}$$

When the system is in failed state

$$\begin{aligned} \mathrm{D}P_F(t) =&(\lambda_{K-1}+\lambda_c)P_{0,K-1}(t)+(\lambda_{K-1}+\lambda_c)P_{1,K-1}(t)+\sum_{l=0}^{K-2}\lambda_c P_{0,l}(t) \\ &+\sum_{l=0}^{K-2}\lambda_c P_{1,l}(t) \end{aligned} \tag{7.20}$$

In the preliminary stage, at $t=0$, all the operating and standby units are functioning, i.e., there is no failed unit at the initial state. Hence the initial conditions are

$$\begin{cases} P_{j,i}(0)=1 & ; j=0, i=0 \\ P_{j,i}(0)=0 & ;\text{otherwise} \end{cases} \tag{7.21}$$

7.2.2 Laplace transform

Under initial conditions, the Laplace transform of state probabilities and their derivatives are defined as

$$\mathrm{L}\left(P_{j,i}(t)\right) = \int_0^\infty e^{-ut}P_{j,i}(t)dt = \ddot{P}_{j,i}(u);\ \mathbb{R}(u)\geq 0, \quad \forall \quad (j,i)\in\Pi \tag{7.22}$$

$$\mathrm{L}\left(Q_{j,i}(t)\right) = \int_0^\infty e^{-ut}Q_{j,i}(t)dt = \ddot{Q}_{j,i}(u);\ \mathbb{R}(u)\geq 0, \quad \forall \quad (j,i)\in\Pi \tag{7.23}$$

$$\mathrm{L}\left(R_{j,i}(t)\right) = \int_0^\infty e^{-ut}R_{j,i}(t)dt = \ddot{R}_{j,i}(u);\ \mathbb{R}(u)\geq 0, \quad \forall \quad (j,i)\in\Pi \tag{7.24}$$

$$\mathrm{L}\left(P_F(t)\right) = \int_0^\infty e^{-ut}P_F(t)dt = \ddot{P}_F(u);\ \mathbb{R}(u)\geq 0 \tag{7.25}$$

$$\mathrm{L}\left(\mathrm{D}P_{j,i}(t)\right) = u\ddot{P}_{j,i}(u)-P_{j,i}(0);\ \mathbb{R}(u)\geq 0, \quad \forall \quad (j,i)\in\Pi \tag{7.26}$$

$$\mathrm{L}\left(\mathrm{D}Q_{j,i}(t)\right) = u\ddot{Q}_{j,i}(u) - Q_{j,i}(0) \quad ; \quad \forall \quad (j,i) \in \Pi \tag{7.27}$$

$$\mathrm{L}\left(\mathrm{D}R_{j,n}(t)\right) = u\ddot{R}_{j,i}(u) - R_{j,i}(0) \quad ; \quad \forall \quad (j,i) \in \Pi \tag{7.28}$$

$$\mathrm{L}\left(\mathrm{D}P_F(t)\right) = u\ddot{P}_F(u) - P_F(0) \tag{7.29}$$

The developed system of differential-difference equations 7.1-7.20 in previous subsection can be transformed as system of linear equations 7.30-7.49 following above-defined Laplace transform as follows.

When the repairer is in normal state J(t) = 0

$$-(u+\lambda_0+\alpha+\lambda_c)\ddot{P}_{0,0}(u)+\beta_1\ddot{Q}_{0,1}(u)+\beta\ddot{P}_{1,0}(u)+1=0 \tag{7.30}$$

$$-(u+\lambda_1+\alpha+\lambda_c+\mu)\ddot{P}_{0,1}(u)+(M\lambda\bar{q}+S\nu)c\ddot{P}_{0,0}(u)+\beta_2\ddot{Q}_{0,1}(u)+ \\ \beta_1\ddot{Q}_{0,2}(u)+\gamma\ddot{R}_{0,1}(u)+\beta\ddot{P}_{1,1}(u)=0 \tag{7.31}$$

$$-(u+\lambda_i+\alpha+\lambda_c+\mu)\ddot{P}_{0,i}(u)+(M\lambda\bar{q}+(S-i+1)\nu)c\ddot{P}_{0,i-1}(u)+ \\ \beta_2\ddot{Q}_{0,i}(u)+\beta_1\ddot{Q}_{0,i+1}(u)+\gamma\ddot{R}_{0,i}(u)+\sum_{l=0}^{i-2}M\lambda\bar{q}q^{i-l-1}c\ddot{P}_{0,l}(u)+ \\ \beta\ddot{P}_{1,i}(u)=0;\ 2\le i\le S-1 \tag{7.32}$$

$$-(u+\lambda_S+\alpha+\lambda_c+\mu)\ddot{P}_{0,S}(u)+(M\lambda\bar{q}+\nu)c\ddot{P}_{0,S-1}(u)+\beta_2\ddot{Q}_{0,S}(u)+ \\ \beta_1\ddot{Q}_{0,S+1}(u)+\gamma\ddot{R}_{0,S}(u)+\beta\ddot{P}_{1,S}(u)+\sum_{l=0}^{S-2}M\lambda\bar{q}q^{S-l-1}c\ddot{P}_{0,l}(u)=0 \tag{7.33}$$

$$-(u+\lambda_{S+1}+\alpha+\lambda_c+\mu)\ddot{P}_{0,S+1}(s)+\lambda_S c\ddot{P}_{0,S}(u)+\beta_2\ddot{Q}_{0,S+1}(u)+ \\ \beta_1\ddot{Q}_{0,S+2}(u)+\gamma\ddot{R}_{0,S+1}(u)+\beta\ddot{P}_{1,S+1}(u)+\sum_{l=0}^{S-1}M\lambda q^{S-l}c\ddot{P}_{0,l}(u)=0 \tag{7.34}$$

$$-(u+\lambda_i+\alpha+\lambda_c+\mu)\ddot{P}_{0,i}(u)+\lambda_{i-1}c\ddot{P}_{0,i-1}(u)+\beta_2\ddot{Q}_{0,i}(u)+\beta_1\ddot{Q}_{0,i+1}(u) \\ +\gamma\ddot{R}_{0,i}(u)+\beta\ddot{P}_{1,i}(u)=0 \quad ;S+2\le i\le K-2 \tag{7.35}$$

$$-(u+\lambda_{K-1}+\alpha+\lambda_c+\mu)\ddot{P}_{0,K-1}(u)+\lambda_{K-2}c\ddot{P}_{0,K-2}(u)+\beta_2\ddot{Q}_{0,K-1}(u) \\ +\gamma\ddot{R}_{0,K-1}(u)+\beta\ddot{P}_{1,K-1}(u)=0 \tag{7.36}$$

$$-(u+\beta_1+\beta_2)\ddot{Q}_{0,i}(u)+\mu\ddot{P}_{0,i}(u)=0; \quad 1\le i\le K-1 \tag{7.37}$$

$$-(u+\gamma)\ddot{R}_{0,i}(u)+\lambda_{i-1}\bar{c}\ddot{P}_{0,i-1}(u)=0; \quad 1\le i\le S \tag{7.38}$$

$$-(u+\gamma)\ddot{R}_{0,i}(u)+\lambda_{i-1}\bar{c}\ddot{P}_{0,i-1}(u)=0; \quad S+1\le i\le K-1 \tag{7.39}$$

When the repairer is in breakdown state J(t) = 1

$$-(u+\lambda_0+\beta+\lambda_c)\ddot{P}_{1,0}(u)+\alpha\ddot{P}_{0,0}(u)=0 \tag{7.40}$$

$$-(u+\lambda_1+\beta+\lambda_c)\ddot{P}_{1,1}(u)+(M\lambda\bar{q}+S\nu)c\ddot{P}_{1,0}(u)+\alpha\ddot{P}_{0,1}(u)+ \gamma\ddot{R}_{1,1}(u)=0 \tag{7.41}$$

$$-(u+\lambda_i+\beta+\lambda_c)\ddot{P}_{1,i}(u)+(M\lambda\bar{q}+(S-i+1)\nu)c\ddot{P}_{1,i-1}(u)+\alpha\ddot{P}_{0,i}(u)+ \gamma\ddot{R}_{1,i}(u)+\sum_{l=0}^{i-2}M\lambda\bar{q}q^{i-l-1}c\ddot{P}_{1,l}(u)=0; \quad 2\leq i\leq S-1 \tag{7.42}$$

$$-(u+\lambda_S+\beta+\lambda_c)\ddot{P}_{1,S}(u)+(M\lambda\bar{q}+\nu)c\ddot{P}_{1,S-1}(u)+\alpha\ddot{P}_{0,S}(u)+\gamma\ddot{R}_{1,S}(u)+ \sum_{l=0}^{S-2}M\lambda\bar{q}q^{S-l-1}c\ddot{P}_{1,l}(u)=0 \tag{7.43}$$

$$-(u+\lambda_{S+1}+\beta+\lambda_c)\ddot{P}_{1,S+1}(u)+\lambda_S c\ddot{P}_{1,S}(u)+\alpha\ddot{P}_{0,S+1}(u)+\gamma\ddot{R}_{1,S+1}(u)+ \sum_{l=0}^{S-1}M\lambda q^{S-l}c\ddot{P}_{1,l}(u)=0 \tag{7.44}$$

$$-(u+\lambda_i+\beta+\lambda_c)\ddot{P}_{1,i}(u)+\lambda_{i-1}c\ddot{P}_{1,i-1}(u)+\alpha\ddot{P}_{0,i}(u)+\gamma\ddot{R}_{1,i}(u)=0; \quad S+2\leq i\leq K-2 \tag{7.45}$$

$$-(u+\lambda_{K-1}+\beta+\lambda_c)\ddot{P}_{1,K-1}(u)+\lambda_{K-2}c\ddot{P}_{1,K-2}(u)+\alpha\ddot{P}_{0,K-1}(u)+ \gamma\ddot{R}_{1,K-1}(u)=0 \tag{7.46}$$

$$-(u+\gamma)\ddot{R}_{1,i}(u)+\lambda_{i-1}\bar{c}\ddot{P}_{1,i-1}(u)=0; \quad 1\leq i\leq S \tag{7.47}$$

$$-(u+\gamma)\ddot{R}_{1,i}(u)+\lambda_{i-1}\bar{c}\ddot{P}_{1,i-1}(u)=0; \quad S+1\leq i\leq K-1 \tag{7.48}$$

When the system is in failed state

$$-u\ddot{P}_F(u)+(\lambda_{K-1}+\lambda_c)\ddot{P}_{0,K-1}(u)+(\lambda_{K-1}+\lambda_c)\ddot{P}_{1,K-1}(u)+ \sum_{l=0}^{K-2}\lambda_c\ddot{P}_{0,l}(u)+\sum_{l=0}^{K-2}\lambda_c\ddot{P}_{1,l}(u)=0 \tag{7.49}$$

For employing the theory of linear algebra to solve the system of linear equations 7.30-7.49, the transient-state probabilities are depicted in a single subscript in the following manner.

$$[P_{0,0}(t),P_{0,1}(t),\ldots,P_{0,K-1}(t)]^T\equiv[\pi_0(t),\pi_1(t),\ldots,\pi_{K-1}(t)]^T$$
$$[Q_{0,1}(t),Q_{0,2}(t),\ldots,Q_{0,K-1}(t)]^T\equiv[\pi_K(t),\pi_{K+1}(t),\ldots,\pi_{2K-1}(t)]^T$$
$$[R_{0,1}(t),R_{0,2}(t),\ldots,R_{0,K-1}(t)]^T\equiv[\pi_{2K}(t),\pi_{2K+1}(t),\ldots,\pi_{3K-2}(t)]^T$$
$$[P_{1,0}(t),P_{1,1}(t),\ldots,P_{1,K-1}(t)]^T\equiv[\pi_{3K-1}(t),\pi_{3K}(t),\ldots,\pi_{4K-2}(t)]^T$$
$$[R_{1,1}(t),R_{1,2}(t),\ldots,R_{1,K-1}(t)]^T\equiv[\pi_{4K-1}(t),\pi_{4K}(t),\ldots,\pi_{5K-3}(t)]^T$$

$$[P_F(t)]^T \equiv [\pi_{5K-2}(t)]^T$$

For matrix-representation, we define the column vectors of order $5K-1$ as

$$\ddot{\Xi}(u) = [\ddot{\pi}_0(u), \ddot{\pi}_1(u), \ddot{\pi}_2(u), \ldots, \ddot{\pi}_{5K-3}(u), \ddot{\pi}_{5K-2}(u)]^T \quad (7.50)$$

$$\Xi(0) = [\pi_0(0), \pi_1(0), \pi_2(0), \ldots, \pi_{5K-3}(0), \pi_{5K-2}(0)]^T \quad (7.51)$$

The matrix representation of the system of linear equations 7.30-7.49 in pre-defined vectors is as follows

$$\mathcal{F}(u)\ddot{\Xi}(u) = \Xi(0) \quad (7.52)$$

where $\mathcal{F}(u)$ is the square matrix with coefficients as elements of order $5K-1$.

Using Cramer's method for solving matrix equation, we express $\ddot{\pi}_k(u)$ explicitly as follows

$$\ddot{\pi}_k(u) = \frac{|\mathcal{F}_k(u)|}{|\mathcal{F}(u)|};\ 0 \leq k \leq 5K-2 \quad (7.53)$$

where $\mathcal{F}_k(u)$ defined from $\mathcal{F}(u)$ by substituting the right-hand side column vector $\Xi(0)$ in the k^{th} column. For obtaining expression of $\ddot{\pi}_k(u)$ explicitly from Eqn. 7.53, we first determine the value of denominator $|\mathcal{F}(u)|$. $u=0$ is one latent characteristics of $|\mathcal{F}(u)|=0$ since $\mathcal{F}(u)$ is derived from singular coefficient matrix obtained from balanced outflow and inflow rates. Suppose that $u=-\xi$ be any other unknown nonzero latent characteristics of $|\mathcal{F}(u)|=0$, then we have

$$\mathcal{F}(-\xi) = \mathbf{C} - \xi\mathbf{I} \quad (7.54)$$

where $\mathbf{C} = \mathcal{F}(0)$ and $\mathbf{I}$ is an unit matrix of order $5K-1$. The expression in Eqn. 7.52 can also be represented as

$$\mathcal{F}(-\xi)\ddot{\Xi}(u) = (\mathbf{C} - \xi\mathbf{I})\ddot{\Xi}(u) \quad (7.55)$$

Let $\xi_l\,(\neq 0)$, $l = 1,2,3,\ldots,5K-3,5K-2$ are distinct real or complex latent characteristics of the characteristics equation $|\mathbf{C}-\xi\mathbf{I}| = 0$. Consider $\xi_1, \xi_2, \xi_3, \ldots, \xi_{i_1-1}, \xi_{i_1}$ are i_1 real latent characteristics. Let $\xi_{i_1+1}, \bar{\xi}_{i_1+1}, \xi_{i_1+1}, \bar{\xi}_{i_1+1}, \ldots, \xi_{i_1+i_2-1}$, $\bar{\xi}_{i_1+i_2-1}$, $\xi_{i_1+i_2}, \bar{\xi}_{i_1+i_2}$ be i_2 conjugate pair of complex latent characteristics. Hence, $i_1 + 2i_2 =$

$5K-2$. Therefore, $|F(u)|$ can be expressed as

$$|F(u)| = u\prod_{l=1}^{i_1}(u+\xi_l)\prod_{l=1}^{i_2}\left(u^2+(\xi_{i_1+l}+\bar{\xi}_{i_1+l})u+\xi_{i_1+l}\bar{\xi}_{i_1+l}\right) \tag{7.56}$$

Hence, Eqn. 7.53 reduces to

$$\ddot{\pi}_k(u) = \frac{|F_k(u)|}{|F(u)|} = \frac{|F_k(u)|}{u\prod_{l=1}^{i_1}(u+\xi_l)\prod_{l=1}^{i_2}\left(u^2+(\xi_{i_1+l}+\bar{\xi}_{i_1+l})u+\xi_{i_1+l}\bar{\xi}_{i_1+l}\right)};$$
$$0 \le k \le 5K-2 \tag{7.57}$$

The Eqn. 7.57 can be expressed in the partial fraction form as

$$\ddot{\pi}_k(u) = \frac{a_{0,k}}{u} + \sum_{l=1}^{i_1}\frac{a_{l,k}}{(u+\xi_l)} + \sum_{l=1}^{i_2}\frac{b_{l,k}u+c_{l,k}}{(u^2+(\xi_{i_1+l}+\bar{\xi}_{i_1+l})u+\xi_{i_1+l}\bar{\xi}_{i_1+l})};$$
$$0 \le k \le 5K-2 \tag{7.58}$$

The partial fraction constants are evaluated as

$$a_{0,k} = \frac{|F_k(0)|}{\prod_{l=1}^{i_1}(\xi_l)\prod_{l=1}^{i_2}(\xi_{i_1+l}\bar{\xi}_{i_1+l})} \tag{7.59}$$

$$a_{l,k} = \frac{|F_k(-\xi_l)|}{(-\xi_l)\prod_{m=1,m\neq l}^{i_1}(\xi_m-\xi_l)\prod_{m=1}^{i_2}({\xi_l}^2+(\xi_{i_1+m}+\bar{\xi}_{i_1+m})(-\xi_l)+\xi_{i_1+m}\bar{\xi}_{i_1+m})};$$
$$l = 1,2,3,\ldots,i_1 \tag{7.60}$$

and

$$b_{l,k}(-\xi_{i_1+l})+c_{l,k}$$
$$= \frac{|F_k(-\xi_{i_1+l})|}{(-\xi_{i_1+l})\prod_{m=1}^{i_1}(\xi_m-\xi_{i_1+l})\prod_{m=1,m\neq l}^{i_2}\left((-\xi_{i_1+l})^2+(\xi_{i_1+m}+\bar{\xi}_{i_1+m})(-\xi_{i_1+l})+\xi_{i_1+m}\bar{\xi}_{i_1+m}\right)};$$
$$l = 1,2,3,\ldots,i_2 \tag{7.61}$$

On taking inverse Laplace transform of Eqn. 7.61, the explicit solution of the governing system of differential-difference equation can be obtained. Hence, the explicit expression of transient-state probabilities is deduced as

$$\pi_k(t) = a_{0,k} + \sum_{l=1}^{i_1} a_{l,k} e^{-\xi_l t} + \sum_{l=1}^{i_2} \left[b_{l,k} e^{-x_l t} \cos y_l t + \frac{c_{l,k} - b_{l,k} x_l}{y_l} e^{-x_l t} \sin y_l t \right]; \quad 0 \le k \le 5K-2 \tag{7.62}$$

The arbitrary constant $a_{0,k}, a_{l,k}, b_{l,k}$ and $c_{l,k}$ are computed in the above Eqns. 7.59-7.61. The real numbers x_l and y_l are respectively real and imaginary components of the complex latent characteristics ξ_{i_1+l}.

Using the derived state probabilities, we develop the reliability characteristics of a fault-tolerant multi-unit machining system with standbys provisioning under different unreliable attributes. In the following sections, we derived the expression of reliability of the system, mean time-to-failure, and failure frequency of the system and studied the parametric behavior.

7.3 Reliability of the system

This section discusses the reliability and parametric behavior of the repairable fault-tolerant redundant machining system. Let X, a continuous random variable, represent the time-to-failure of the system. The state F represents the inoperative state of the system. The state probability $P_F(t)$ is the likelihood that the system is inoperative on or before time t. Employing a probability theory, the system reliability $R_X(t)$ at time t is the probability that the system will operate without failure over the interval $(0,t]$, a specific period, under certain specified operating conditions. Mathematically, $R_X(t)$ is given as

$$R_X(t) = 1 - P_F(t); \quad t \ge 0 \tag{7.63}$$

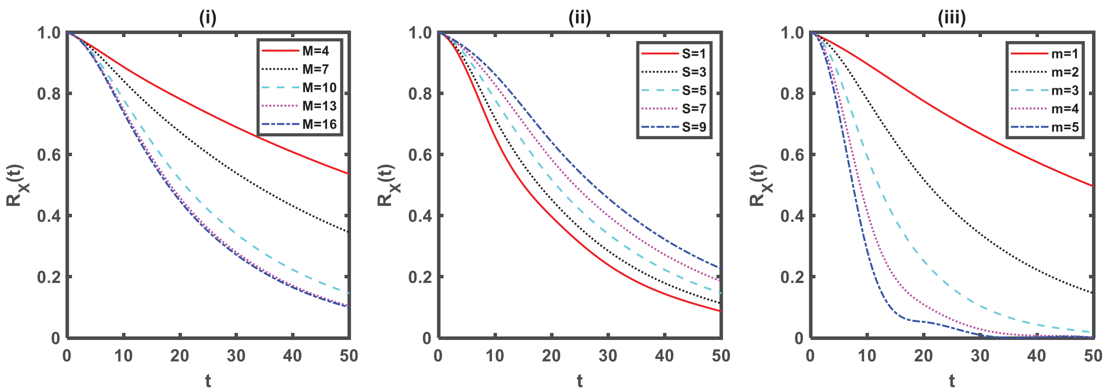


Figure 7.2: Reliability $R_X(t)$ of the system w.r.t. design parameter, (For $\lambda = 0.5$, $\lambda_d = 1$, $\lambda_c = 0.01$, $\nu = 0.1$, $\alpha = 0.05$, $\mu = 5$, $\beta_1 = 15$, $\beta_2 = 0.1$, $\beta = 10$, $\gamma = 60$, $c = 0.9$, $q = 0.1$)

The reliability function $R_X(t)$ is a non-increasing function of t and ranges from 1 to 0. The results can be observed in the following numerical results. For the numerical experiments, we set the default parameters of the studied repairable fault-tolerant redundant machining system as follows $M = 10$, $S = 5$, $m = 2$, $\lambda = 0.5$, $\lambda_d = 1$, $\lambda_c = 0.01$, $\nu = 0.1$, $\alpha = 0.05$, $\mu = 5$, $\beta_1 = 15$, $\beta_2 = 0.1$, $\beta = 10$, $\gamma = 60$, $c = 0.9$, $q = 0.1$. From the Figs. 7.2-7.5, the obvious nonincreasing trends of the system reliability with time t are observed. The depiction supports the modeling, methodology used for the studied model.

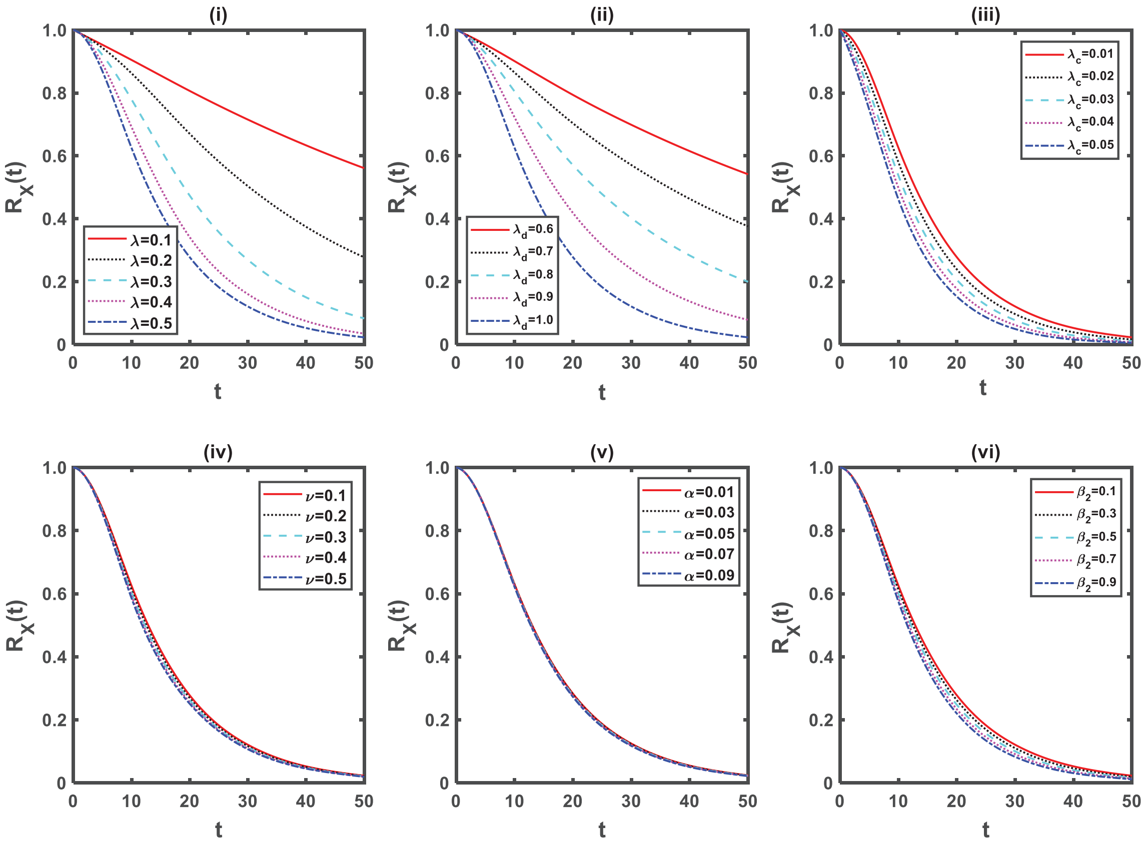


Figure 7.3: Reliability $R_X(t)$ of the system w.r.t. different types of failures, (For $M = 10$, $S = 5$, $m = 2$, $\mu = 5$, $\beta_1 = 15$, $\beta = 10$, $\gamma = 60$, $c = 0.9$, $q = 0.1$)

Fig. 7.2 depicts the variability of the system reliability for different design parameters. Fig. 7.2(i) depicts how the reliability of the system depends on the number of operating units. It is important to decision-making for developing the system. An increase in standby units prevents the system from failing for longer. The apparent result is observed in Fig 7.2(ii), where the reliability enhances with an increase in standby units. In Fig 7.2(iii), for increased m, an interesting trend is observed for variability in $R_X(t)$ after some time interval. Fig 7.3(i-vi) shows that the reliability of the system decreases due to unreliable attributes. With the increased failure rate of active units in normal state λ, the failure rate of active units in degraded state λ_d, common-cause failure λ_c, the failure rate of warm standby units ν, inspection rate for imperfect repair β_2, and repairer breakdown rate α, the reliability of the system decreases. The decreasing trends prompt a scheduled preventive maintenance strategy for the uninterrupted long run and high efficiency.

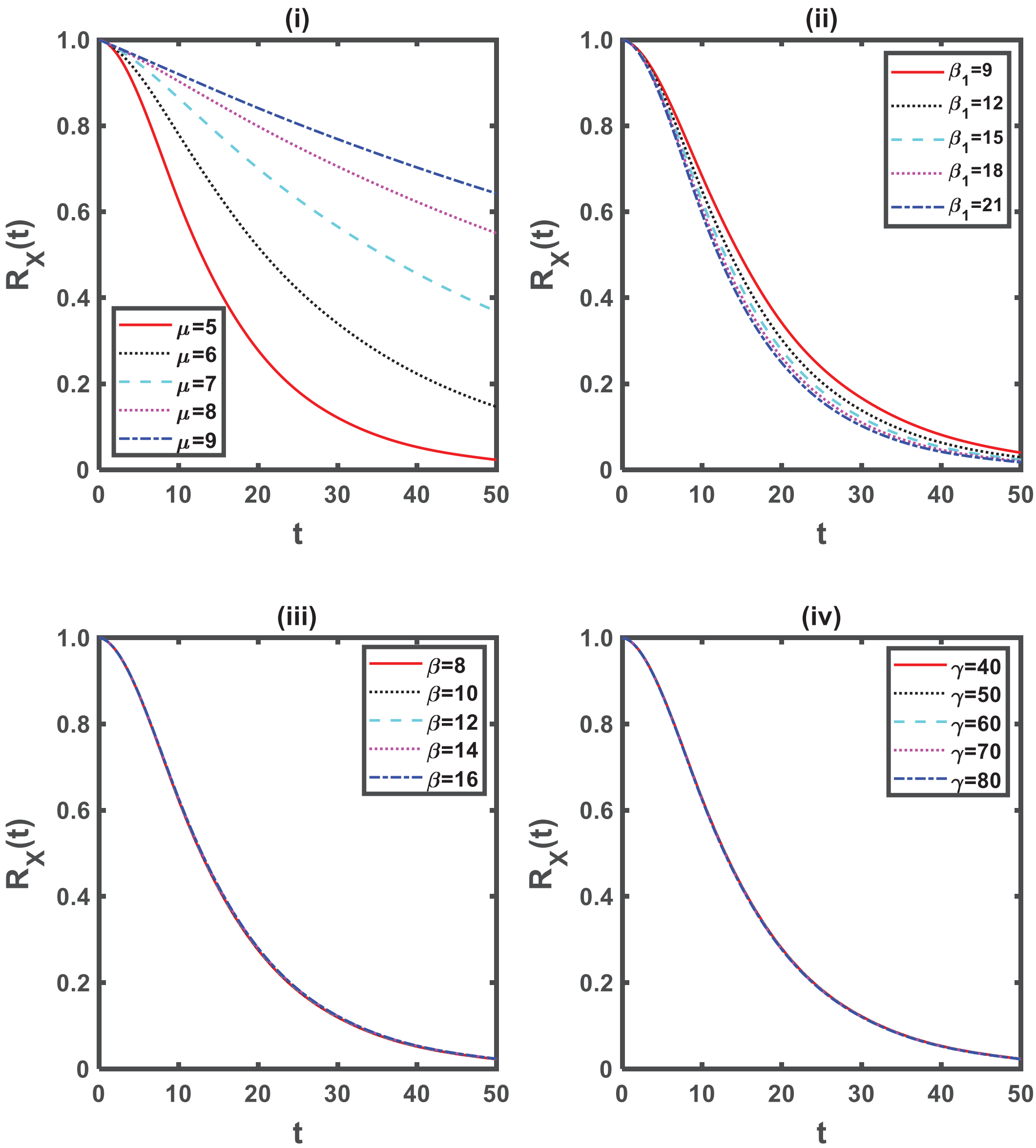


Figure 7.4: Reliability $R_X(t)$ of the system w.r.t. repair facility, (For $M = 10$, $S = 5$, $m = 2$, $\lambda = 0.5$, $\lambda_d = 1$, $\lambda_c = 0.01$, $\nu = 0.1$, $\alpha = 0.05$, $c = 0.9$, $q = 0.1$)

Fig. 7.4(i) gives an apparent increasing trend of the system's reliability when service improves. The results support the prompt corrective measures. From Fig 7.4(ii) and Fig 7.3(vi), it is observed that whether the repair is perfect or imperfect, the inspection of units after repair should be carried out as early as possible for maintaining system reliability higher. The repairer breakdown happens rarely, and if it happens, the recovery rate should be high enough to maintain the system's reliability (Fig 7.4(iii)) better. The reliability of the system mainly depend on number of active unit present and breakdown/recovery is rare event. It is prompted from the graphical result. The reboot process is carried out in a negligible amount of time. Thus the reboot delay rate has the most insignificant impact on the variability of the system's reliability, as shown in Fig 7.4(iv).

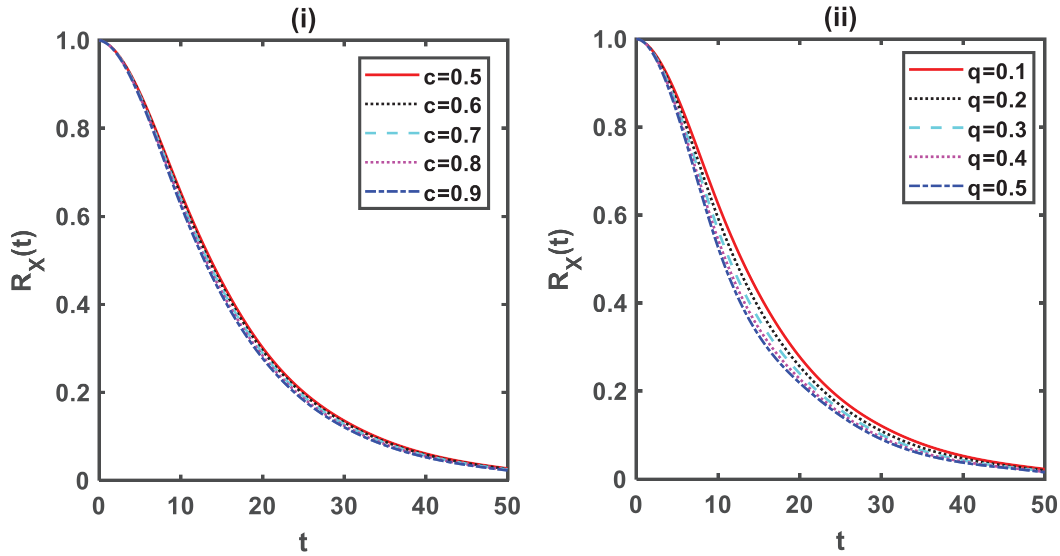


Figure 7.5: Reliability $R_X(t)$ of the system w.r.t. probability of coverage and switching failure, (For $M = 10$, $S = 5$, $m = 2$, $\lambda = 0.5$, $\lambda_d = 1$, $\lambda_c = 0.01$, $\nu = 0.1$, $\alpha = 0.05$, $\mu = 5$, $\beta_1 = 15$, $\beta_2 = 0.1$, $\beta = 10$, $\gamma = 60$)

When a fault in either an active unit or a standby unit is successfully detected in a redundant machining system, it goes under repair. The system then remains with fewer units, decreasing reliability as coverage probability c increases, as shown in Fig 7.5(i). The switchover of standby to replace failed units is not always successful, and these attempts are made until successful or all standbys get exhausted. It is vital to successfully switch standby in place of failed units for the system to be more reliable. Notably, our predictions in Fig 7.5(ii) are shown to agree ideally with the practical results. Predictive and corrective measures on time are necessary to seek high reliability.

7.3.1 Sensitivity Analysis of $R_X(t)$

Regular calibration is a necessary measure for the machining system. The governing parameter's sensitivity for different performance indices should be known. The theory of differentiation establishes the platform for sensitivity analysis. This section presents the sensitivity analysis of the reliability function and mean time-to-failure for

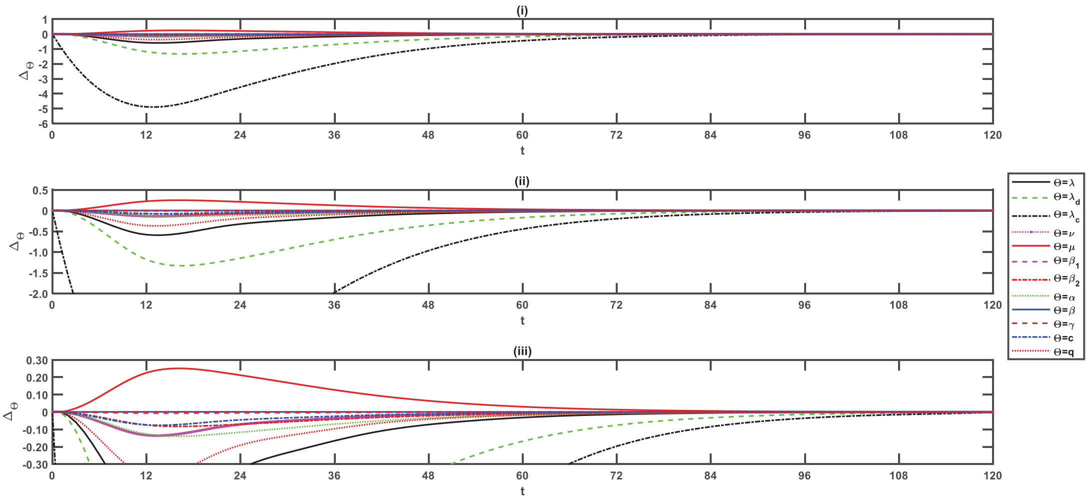


Figure 7.6: Sensitivity analysis Δ_Θ of $R_X(t)$ for different parameters, (For $M=10$, $S=5$, $m=2$, $\lambda=0.5$, $\lambda_d=1$, $\lambda_c=0.01$, $\nu=0.1$, $\alpha=0.05$, $\mu=5$, $\beta_1=15$, $\beta_2=0.1$, $\beta=10$, $\gamma=60$, $c=0.9$, $q=0.1$)

the governing parameters. The wide variability of the function with respect to governing decision variables can be described by taking the first derivatives.

On taking the first derivative of the Eqn.7.52 partially with respect to the decision variable using product rule, we have

$$\mathrm{D}_\Theta \digamma(u)\ddot{\Xi}(u) + \digamma(u)\mathrm{D}_\Theta \ddot{\Xi}(u) = 0 \tag{7.64}$$

$$\mathrm{D}_\Theta \ddot{\Xi}(u) = -(\digamma(u))^{-1}\mathrm{D}_\Theta \digamma(u)\ddot{\Xi}(u) \tag{7.65}$$

We have the first derivative of the reliability function from Eqn.7.63 as

$$\Delta_\Theta(t) = \mathrm{D}_\Theta R_X(t) = -\mathrm{D}_\Theta P_F(t) = \mathrm{L}^{-1}\left(-\mathrm{D}_\Theta \ddot{P}_F(u)\right) = \mathrm{L}^{-1}\left(-\mathrm{D}_\Theta \ddot{\pi}_{5K-2}(u)\right) \tag{7.66}$$

The numerical scheme is used to compute $\frac{\partial \ddot{P}_F(u)}{\partial \Theta}$ since state probabilities are derived from a system of differential equations which are developed in terms of various governing parameters and decision variables. The pictorial trend is depicted in Fig. 7.6.

We infer from Fig 7.6 that (i) the positive sensitivity of parameters μ implies a slight increment of parameter value will improve $R_X(t)$; (ii) the negative sign sensitivity of parameters λ_c, λ_d, λ, q, α, ν, β_2, c, and γ mean slightly increments of parameter's value will reduce $R_X(t)$); and (iii) the parameters β_1 and β have a very less affect $R_X(t)$. From Fig. 7.6, we also rank the order of magnitude of the effect as

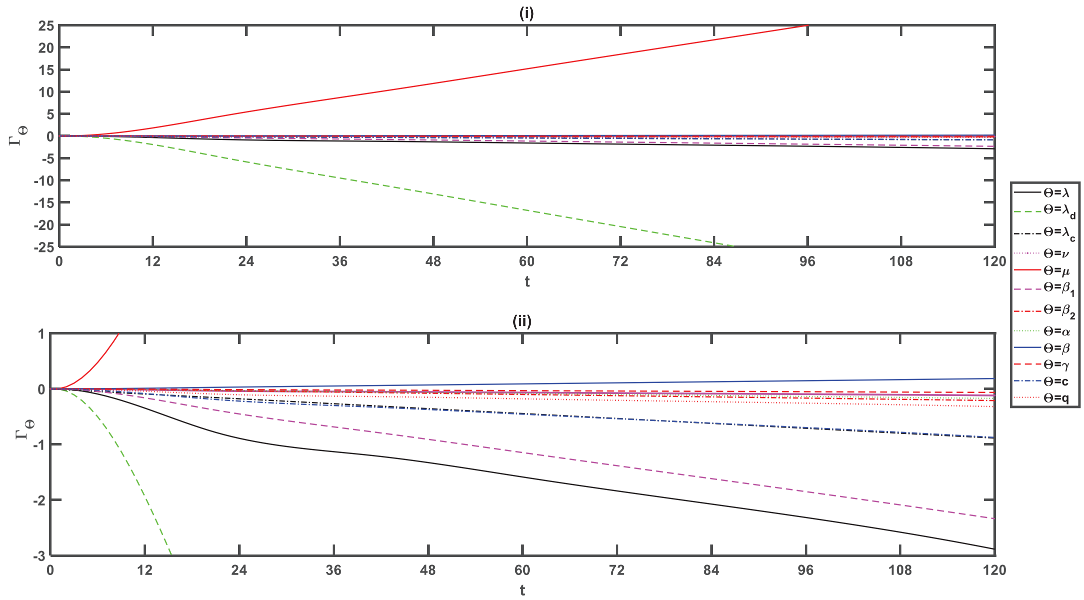


Figure 7.7: Relative sensitivity analysis Γ_Θ of $R_X(t)$ for different parameters, (For $M = 10$, $S = 5$, $m = 2$, $\lambda = 0.5$, $\lambda_d = 1$, $\lambda_c = 0.01$, $\nu = 0.1$, $\alpha = 0.05$, $\mu = 5$, $\beta_1 = 15$, $\beta_2 = 0.1$, $\beta = 10$, $\gamma = 60$, $c = 0.9$, $q = 0.1$)

$\lambda_c > \lambda_d > \lambda > q > \mu > \alpha > \nu > \beta_2 > c > \gamma > \beta_1 \approx \beta$.

For the in-depth analysis, we determine the relative change parameter-wise. We compute the following ratio for the relative sensitivity analysis of the reliability function

$$\Gamma_\Theta(t) = \frac{\frac{\partial R_X(t)}{R_X(t)}}{\frac{\partial \Theta}{\Theta}} = \Delta_\Theta \frac{\Theta}{R_X(t)} \tag{7.67}$$

From Fig 7.7, the order of relative sensitivity of $R_X(t)$ for distinguished parameters can be ranked as $\lambda_d > \mu > \lambda > \beta_1 > c > \lambda_c > q > \beta_2 > \beta > \alpha > \nu > \gamma$. We notice negative sign from Fig 7.7 of relative sensitivity for parameters λ, λ_d, λ_c, ν, β_1, β_2, α, γ, c, q. The result prompts a diminishing value of $R_X(t)$ on increasing these parameters, which we expect in the real-time fault-tolerant system too. The positive sign of the relative sensitivity of $R_X(t)$ for μ and β are observed, which indicate that by surplussing the service facility, there is an improvement in the system reliability. From numerical experiments, the order of magnitude of all parameters affecting $R_X(t)$ can be ordered as $\lambda_d > \mu > \lambda > \beta_1 > c > \lambda_c > q > \beta_2 > \beta > \alpha > \nu > \gamma$. This order in magnitude defines the critical parameters in well manner and give direct insight to decision maker for predictive, preventive, and corrective measures.

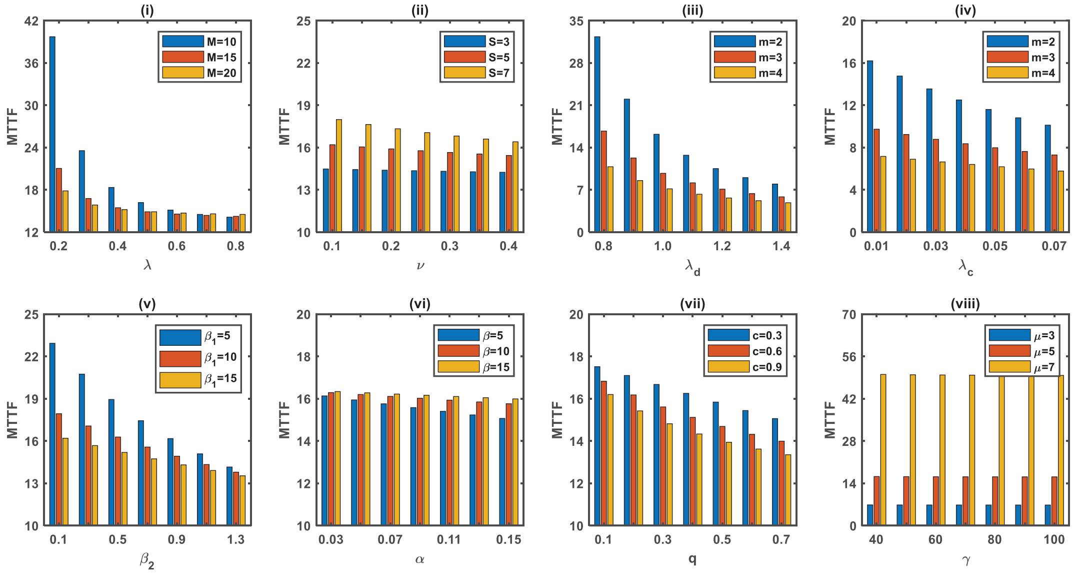


Figure 7.8: Mean time-to-failure $MTTF$ of the system by varying parameters, (For $M = 10$, $S = 5$, $m = 2$, $\lambda = 0.5$, $\lambda_d = 1$, $\lambda_c = 0.01$, $\nu = 0.1$, $\alpha = 0.05$, $\mu = 5$, $\beta_1 = 15$, $\beta_2 = 0.1$, $\beta = 10$, $\gamma = 60$, $c = 0.9$, $q = 0.1$)

7.4 Mean time-to-failure

The mean time-to-failure ($MTTF$) of the system is defined as

$$\begin{aligned} MTTF &= \int_0^\infty R_X(t)dt = \int_0^\infty (1 - P_F(t))\,dt = \int_0^\infty (1 - \pi_{5K-2}(t))\,dt \\ &= \lim_{u \to 0} \left[\frac{1 - a_{0,5K-2}}{u} - \sum_{l=1}^{i_1} \frac{a_{l,5K-2}}{u + \xi_l} - \sum_{l=1}^{i_2} \frac{b_{l,5K-2}u + c_{l,5K-2}}{u^2 + \left(\xi_{i_1+l} + \bar{\xi}_{i_1+l}\right) u + \xi_{i_1+l}\bar{\xi}_{i_1+l}} \right] \\ &= -\sum_{l=1}^{i_1} \frac{a_{l,5K-2}}{\xi_l} - \sum_{l=1}^{i_2} \frac{c_{l,5K-2}}{\xi_{i_1+l}\bar{\xi}_{i_1+l}} \end{aligned} \tag{7.68}$$

An increased failure rate λ results in more units failing, and thus mean time-to-failure ($MTTF$) decreases significantly till a specific failure rate λ. Later on, the trend reverses for a further increase in failure rate λ since, in this case, units fail more rapidly, for which more active units are required to increase $MTTF$ (Fig 7.8(i)). A similar trend is observed in Fig 7.8(iii-v, vii). With the discussion of Fig 7.2(ii) and Fig 7.3(iv), it can be concluded that it is beneficial to have more standbys for the system to work efficiently and longer. In our model, even standby can be unreliable, so $MTTF$ reduces with an increase in the failure rate of standby units (Fig 7.8(ii)). Fig 7.8(vi, viii) gives a similar trend, which is self-explanatory from the previous discussion.

7.4.1 Sensitivity Analysis of $MTTF$

Following the similar approach for sensitivity analysis of the $MTTF$, we derive the first derivative of $MTTF$ with respect to decision parameter Θ from Eqn.7.68 as follow

$$\begin{aligned}\Phi_\Theta &= \mathrm{D}_\Theta MTTF = \mathrm{D}_\Theta\left(\int_0^\infty R_X(t)dt\right) = \lim_{u\to 0}\left[\int_{t=0}^\infty \mathrm{D}_\Theta R_X(t)e^{-ut}dt\right] \\ &= \lim_{u\to 0}\left[-\mathrm{D}_\Theta \ddot{P}_F(u)\right] = \lim_{u\to 0}\left[-\mathrm{D}_\Theta \ddot{\pi}_{5K-2}(u)\right]\end{aligned} \tag{7.69}$$

The relativity of $MTTF$ also can be derived using the following ratio

$$\Omega_\Theta = \frac{\frac{\partial(MTTF)}{MTTF}}{\frac{\partial\Theta}{\Theta}} = \Phi_\Theta\frac{\Theta}{MTTF} \tag{7.70}$$

Table 7.1: Sensitivity analysis Φ_Θ of the $MTTF$ of the system, (For $\lambda = 0.5$, $\lambda_d = 1$, $\lambda_c = 0.01$, $\nu = 0.1$, $\alpha = 0.05$, $\mu = 5$, $\beta_1 = 15$, $\beta_2 = 0.1$, $\beta = 10$, $\gamma = 60$, $c = 0.9$, $q = 0.1$)

	M,S,m	Θ											
		λ	λ_d	λ_c	ν	α	μ	β_1	β_2	β	γ	c	q
Φ_Θ	10,5,2	-14.43066	-43.93492	-156.79707	-3.21775	8.14282	-0.23423	-2.69630	-4.50278	0.02460	-0.00187	-2.00685	-8.77218
	14,5,2	-4.80466	-41.37748	-131.38707	-0.82197	6.78573	-0.22452	-2.24693	-3.82317	0.02116	-0.00193	-1.43322	-2.74012
	18,5,2	-2.77013	-41.80451	-129.58830	-0.37734	6.67390	-0.22544	-2.20990	-3.76385	0.02086	-0.00205	-1.38133	-1.51297
	10,2,2	-2.89051	-41.39255	-120.14940	-0.34229	6.73070	-0.19912	-2.22871	-3.79910	0.02106	-0.00159	-1.14521	-1.81959
	10,5,2	-14.43066	-43.93492	-156.79707	-3.21775	8.14282	-0.23423	-2.69630	-4.50278	0.02460	-0.00187	-2.00685	-8.77218
	10,8,2	-30.83339	-44.77156	-201.59107	-10.26737	9.78371	-0.27427	-3.23964	-5.32680	0.02876	-0.00219	-3.09453	-17.79475
	10,5,2	-14.43066	-43.93492	-156.79707	-3.21775	8.14282	-0.23423	-2.69630	-4.50278	0.02460	-0.00187	-2.00685	-8.77218
	10,5,3	-13.82419	-19.53307	-52.20377	-3.14180	3.96284	-0.13794	-1.31220	-2.11703	0.01113	-0.00120	-1.57233	-8.50364
	10,5,4	-13.47699	-10.85909	-27.88477	-3.09121	2.52887	-0.09925	-0.83737	-1.33247	0.00688	-0.00091	-1.38451	-8.34144

Table 7.1 and 7.2 summarizes the sensitivity and relative sensitivity of the mean time-to-failure of the system, respectively. The observation prompts the order of sensitivity as follows $\lambda_c > \lambda_d > \lambda > q > \alpha > \beta_2 > \nu > \beta_1 > c > \mu > \beta > \gamma$. The other trends are observed from the tables for distinguished parameters for quick decision-making insight. The results highlight that preventive measures are more critical than corrective and predictive measures. Significant measures need to diminish the chance of failure of active/standby units like frequent inspection, cooling, greasing, periodical maintenance, etc.

7.5 Failure Frequency of the System

Failure frequency of the system is denoted by FF and defined as the number of times the fault-tolerant repairable machining system fails in a unit of time. Mathematically, it is defined as

$$FF = (m\lambda_d + \lambda_c)(P_{0,K-1}(t) + P_{1,K-1}(t)) + \sum_{i=0}^{K-2} \lambda_c (P_{0,i}(t) + P_{1,i}(t)) \tag{7.71}$$

Table 7.2: Relative sensitivity analysis Ω_Θ of the $MTTF$ of the system,(For $\lambda = 0.5$, $\lambda_d = 1$, $\lambda_c = 0.01$, $\nu = 0.1$, $\alpha = 0.05$, $\mu = 5$, $\beta_1 = 15$, $\beta_2 = 0.1$, $\beta = 10$, $\gamma = 60$, $c = 0.9$, $q = 0.1$)

	M,S,m	Θ											
		λ	λ_d	λ_c	ν	α	μ	β_1	β_2	β	γ	c	q
Ω_Θ	10,5,2	-0.44538	-2.71195	-0.09679	-0.01986	2.51314	-0.21687	-0.01664	-0.01390	0.01518	-0.00694	-0.11149	-0.05415
	14,5,2	-0.16085	-2.77054	-0.08797	-0.00550	2.27178	-0.22550	-0.01504	-0.01280	0.01417	-0.00774	-0.08637	-0.01835
	18,5,2	-0.09322	-2.81347	-0.08721	-0.00254	2.24579	-0.22758	-0.01487	-0.01267	0.01404	-0.00826	-0.08367	-0.01018
	10,2,2	-0.10546	-3.02041	-0.08767	-0.00250	2.45569	-0.21795	-0.01626	-0.01386	0.01537	-0.00695	-0.07521	-0.01328
	10,5,2	-0.44538	-2.71195	-0.09679	-0.01986	2.51314	-0.21687	-0.01664	-0.01390	0.01518	-0.00694	-0.11149	-0.05415
	10,8,2	-0.81741	-2.37383	-0.10689	-0.05444	2.59371	-0.21813	-0.01718	-0.01412	0.01525	-0.00697	-0.14767	-0.09435
	10,5,2	-14.43066	-43.93492	-156.79707	-3.21775	8.14282	-0.23423	-2.69630	-4.50278	0.02460	-0.00187	-2.00685	-8.77218
	10,5,3	-0.71010	-2.00670	-0.05363	-0.03228	2.03558	-0.21257	-0.01348	-0.01087	0.01143	-0.00738	-0.14538	-0.08736
	10,5,4	-0.94031	-1.51531	-0.03891	-0.04314	1.76443	-0.20775	-0.01168	-0.00930	0.00961	-0.00764	-0.17388	-0.11640

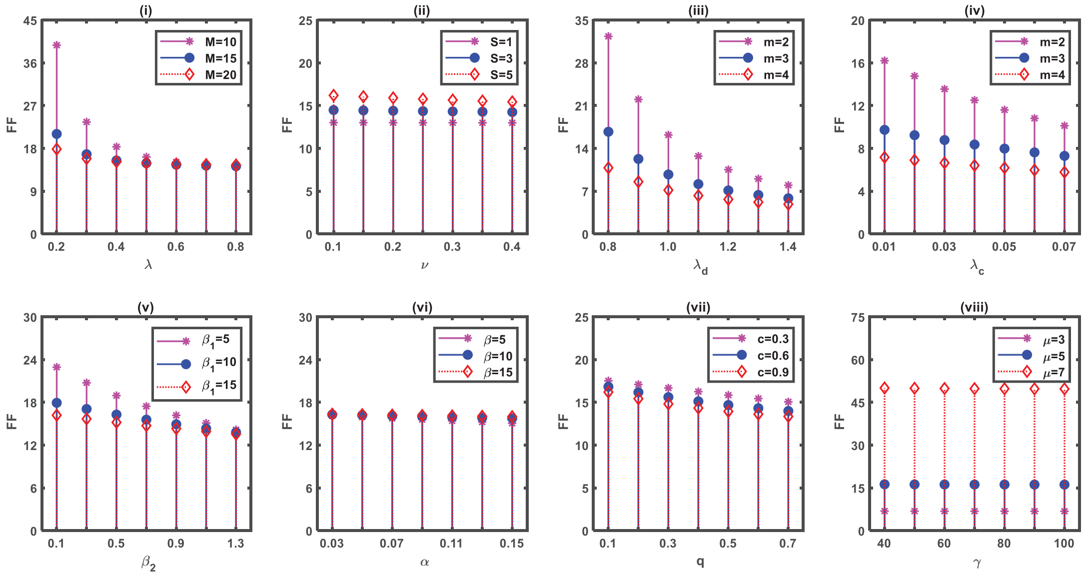


Figure 7.9: Failure frequency FF for different parameters, (For $M = 10$, $S = 5$, $m = 2$, $\lambda = 0.5$, $\lambda_d = 1$, $\lambda_c = 0.01$, $\nu = 0.1$, $\alpha = 0.05$, $\mu = 5$, $\beta_1 = 15$, $\beta_2 = 0.1$, $\beta = 10$, $\gamma = 60$, $c = 0.9$, $q = 0.1$)

The variability of the failure frequency of the system is depicted in Fig.7.9. The figure depicts how unreliable attributes increase the chance of system failure or increase the frequency of the system failure. The risk of hindrance in operations due to unreliable attributes: failures, imperfections, degradations, and delays increases, and a strategic balance between preventive, corrective, and predictive measures needs to opt.

7.6 Special Cases

We address some problems as special cases which are already discussed in literature. These special cases require a tailored approach to validate the modelling, computation, and analysis. By recognizing the distinct nature of governing parameters and implementing targeted strategies, it becomes possible to overcome challenges and achieve the desired objectives.

- Case 1: When $\lambda_c = 0$, $\lambda_d = \lambda$, $\beta_1 = \mu$ and $\beta_2 = 0$, the studied module reduces as modeling of multi-unit repair problem with switching failure and reboot delay (*cf.* [94]).
- Case 2: If we set $\beta_1 = \mu$, $\beta_2 = 0$, $\lambda_d = \lambda$, $\lambda_c = 0$, and $q = 0$, the model approaches to reliability analysis of a multi-component machining system with service interruption, imperfect coverage,and reboot (*cf.* [158]).
- Case 3: For $\lambda_c = 0$, $\lambda_d = \lambda$, $c = 1$, $\beta_1 = \mu$, $\beta_2 = 0$, $\alpha = 0, \beta \to \infty$, and $\gamma \to \infty$, the proposed model resembles with unit repairing systems with standby switching failure (*cf.* [134]).
- Case 4: When $S = 0$, $q = 0$, $\lambda_c = 0$, $\lambda_d = \lambda$, $c = 1, \beta_1 = \mu, \beta_2 = 0, \alpha = 0, \beta \to \infty$, and $\gamma \to \infty$, the model approaches to $M/M/1$ model with unreliable service (*cf.* [225])

7.7 Discussion, Conclusion, Future scope

Based on the findings, the following recommendations are suggested:

- Increase the number of standby units to enhance system reliability.
- Prioritize reliable repair services to minimize recurring failures and improve reliability.
- Invest in robust automation to mitigate switching failures and ensure uninterrupted power supply.
- Implement preventive maintenance strategies to detect and address issues proactively.
- Emphasize regular calibration and inspection to maintain system reliability.
- Optimize parameters based on sensitivity analysis to improve system performance.
- Continuously monitor and improve system reliability through feedback mechanisms.

By implementing these recommendations, system reliability can be enhanced, minimizing downtime and ensuring uninterrupted operation.

The technologies such as digital twins, digital threads, augmented reality/virtual reality (AR/VR), cloud and edge computing, artificial intelligence (AI), and the Industrial Internet of Things (IIoT) are the next generation of advancement. Innovative technologies need multi-unit systems, which require cutting-edge strategies for uninterrupted availability under varied unreliable attributes. The unreliable attributes in fault-tolerant multi-unit machining systems (FTMS) significantly affect reliability, availability, maintainability, and safety. In the present investigation, we present the reliability and sensitivity analysis using the queueing theoretic approach for an FTMS with multi-standby support. The unreliable attributes are abdicate characteristics involved in the machining system, such as failures, imperfections, delays, degradation, etc., that directly decrease the working efficiency and system availability which indirectly affect continuous implementations of the technologies. The unreliable attributes studied herein are the active/standby unit failure, switching failure, imperfect repairer, imperfect repair, imperfect fault coverage, common-cause failure, degraded failure, and reboot delay which are incorporated into the modeling to make the machining system more realistic. We have presented how reliability, $MTTF$, and failure frequency are affected with discussed unreliable attributes. The sensitivity analysis identifies the sensitive parameters involved in studied systems and reveals to the decision-makers how reliability and $MTTF$ can be improved by adjudging the sensitive parameters. The results prompt (i) regular predictive measures are to be taken to avoid the failures, (ii) strategical corrective measures need to be taken to avoid long downtime or hindrance, and (iii) preventive measures are systematically implemented to diminish failures, delays, and imperfections. The reliability prediction of standby provisioning FTMS has many real-time applications, including the electronic industry, service industry, safety systems, power plants, etc. wherein unreliable attributes are not preferable.

The present research work can be further extended by relaxing the exponential distribution for different failures and repairs to explore the impact of generally distributed failure and repair time. Furthermore, the application of optimization techniques can be employed to optimize reliability-costs-value, with the objective of either maximizing or minimizing them. In addition, the utilization of cutting-edge meta-heuristic optimization techniques holds promise for determining a suitable compromise solution.

Chapter 8

Exploring Hierarchical Repair Strategies for Multi-Unit Redundant Machining Systems

"As the philosopher stone turns base metal into gold, so does the professor transmute ignorance into wisdom, through the alchemy of teaching and the artistry of perfect repair."

Isaac Newton

8.1 Introduction

In today's technology-driven era, machines are the backbone of various industries, assuming a pivotal role in ensuring seamless operations. The Machine Repair Problem (MRP) has emerged as a matter of paramount importance, spanning a wide array of sectors, including assembly lines, automotive, information technology, manufacturing, medical equipment, construction, healthcare, home automation, transportation, security systems, and beyond. The relentless progress of technological evolution has bestowed upon us remarkable gains in terms of speed, precision, and overall operational efficiency. This transformation is particularly pronounced within the Industry 4.0 framework, characterized by the convergence of the Internet of Things (IoT), artificial intelligence (AI), and production automation.

In this rapidly evolving socio-econo-techno-landscape, MRP stands out as a strategic imperative for cost reduction. It champions a proactive approach through predictive and preventive maintenance, charting a course that proactively mitigates breakdowns. By adopting this approach, businesses can avert the specter of costly equipment failures, substantially reduce maintenance expenditures, and significantly extend the operational lifespan of their machinery.

However, within the realm of MRP, the challenge of component failures looms large, presenting substantial hurdles to industries across the board. These failures encompass a multifaceted spectrum, ranging from the breakdown of vital components to electrical malfunctions, software glitches, and vexing mechanical failures. The ramifications of these occurrences reverberate throughout industries, disrupting operational flows, tarnishing a company's reputation, and inflicting a heavy toll in terms of increased repair costs and more. To counteract these formidable challenges, redundancy emerges as a linchpin strategy. Redundancy encompasses the provision of backup components, serving as a resilient bulwark against system failures. It augments fault tolerance, bolsters system reliability, ensures continuous availability, and achieves the overarching goal of minimizing downtime. Within this backdrop, the concept of standby units assumes a pivotal role, with these units classified into three distinct categories based on their failure characteristics: cold, warm, and hot standby units. A hot standby unit corresponds to a scenario where its failure rate aligns with that of the operational unit it backs up. Conversely, a cold standby unit exhibits a failure rate of zero, signifying its readiness to spring into action. In contrast, a warm standby unit has a nonzero but lower failure rate compared to the operational unit, positioning it as an intermediate solution.

Establishing high-quality repair facilities emerges as a critical piece of the puzzle. The formulation of robust strategies encompasses a multifaceted approach. It involves meticulous needs assessment, judicious budget allocation, the recruitment of highly skilled repair personnel, investment in cutting-edge equipment, and the development of standardized operating procedures. In scenarios where operational unit failures inevitably transpire, the seamless transition to an accessible warm standby unit becomes imperative. This swift replacement minimizes system downtime, thereby enhancing overall operational efficiency. This proactive approach minimizes delays in system operations, thus extending the runtime and ensuring continuity in critical processes.

Our research introduces a pioneering approach within the realm of machine repair-a multi-unit redundant machining system distinguished by two categories of repairers: Primary and Secondary. This intricate system architecture comprises M operating units, S warm standby units, R primary repairers, and one secondary repairer. This multifaceted repair facility operates across two distinct stages: the preparatory stage, where primary repairers take charge of addressing repairs, and the execution stage, marked by a collaborative effort with the secondary repairer. When the secondary repairer is unavailable, a meticulously organized queue system comes into play, adhering diligently to the first-come, first-repaired policy.

Our study aims to unveil the impact of dividing repair tasks between primary and secondary repairers on the repair facility. Additionally, we craft a cost function and system performance metrics to identify optimal costs, the ideal number of primary repairers, and the optimal repair rate. We scrutinize the work division policy to enhance repair quality. Importantly, our research explores uncharted territory-a multi-unit redundant machine repair system with two types of repairers and a two-stage repair process.

The structure of this article unfolds as follows: Section 8.2 provides a thorough literature review. Section 8.3 lays the foundation, describing the model, introducing notations, and unveiling steady-state equations. Section 8.3.1 dives into the core theme—the work division policy-and its profound implications for repair quality. Section 8.4 employs the matrix recursive method to decode steady-state probabilities. Section 8.5.2 elucidates the intricacies of the cost function and system performance metrics. Section 8.6 invites readers on a numerical journey, dissecting system performance measures. Finally, Section 8.7 brings this voyage to a close, offering panoramic insights and charting potential avenues for future research. In summary, our research aims to contribute significantly to understanding and optimizing complex machine repair systems, propelling us towards a future of efficient and effective machine repair.

8.2 Literature Review

8.2.1 Machine Repair Problems

Machine repair problems have garnered significant attention from various researchers, who have employed diverse approaches and studied different cases. Sivazlian and Wang [273] conducted an economic analysis of a Markovian model of machine repair problems. Lee et al. [167] explored continuous approximations of machine repair systems, introducing the concepts of two boundary policies: the elementary return boundary (ERB) and the instantaneous return boundary (IRB). Gupta [83] delved into machine interference problems involving spares, different types of server vacations, and exhaustive service scenarios. Jain et al. [111] tackled the finite queue-dependent heterogeneous multiprocessor service system, where processors are shared among multiple jobs, and developed a cost function to determine the optimal threshold level for active processors. Kumar et al. [153] delved into reliability measures for systems with subsystems in series configuration. Shekhar et al. [266] and Liu et al. [186] conducted surveys on machine repair problems and real-time machine management systems in the context of the Internet of Things (IoT). More recently, Devanda et al. [47] and Meena et al. [200] conducted cost and reliability analyses of fault-tolerant machining systems in fuzzy environments, considering server vacations, general repair, and imperfect repair.

8.2.2 Machine Repair Problems with Redundancy

In fault-tolerant machining systems, when an operating unit fails, an accessible warm standby unit takes its place. Redundant machine repair systems are categorized as non-repairable and repairable. Nakagawa and Osaki [211] analyzed the cost and reliability of a single-unit system with no repairable spare unit and explored its optimization applications. Several researchers have explored machine repair problems with warm standby, common cause failure, balking, reneging, reboot delay, switching failure, geometric-reneging, and threshold-based recovery policies (*cf.* [300], [268], [138], [344], [102], [107], [262], [258]). Ke et al. [140] conducted reliability and sensitivity analyses of redundant machine repair systems with common cause failures and utilized Laplace transform techniques to solve transient state probabilities. Wang et al. [325] employed the recursive method and supplementary variable technique to develop steady-state availability for systems with warm standby units and different imperfect coverage. Ruiz et al. [241] explored a complex warm standby system with preventive maintenance, an indeterminate number of repairers, and unit loss using a Markovian Arrival Process with Marked arrivals. Cha et al. [27] and Huang et al. [98] analyzed the reliability model for redundant systems and the performance evaluation of general standby units. Srinivasan and Subramanian [280] investigated a three-unit warm standby redundant system with repairs involving general lifespan and repair time distributions. Yun and Cha [354] delved into the redundant machine repair system with an optimally designed general warm standby system featuring different switching types. Saini et al. [245] conducted RAM (reliability, availability, and maintainability) analysis of a hot standby system. Devi et al. [48] provided a detailed review of redundancy allocation problems spanning two decades. More recently, Kumar et al. [157] investigated queueing measures for machine repair problems with a threshold recovery strategy, unreliable repairers, and K-types of stage repairs. Shekhar et al. [258] conducted a comprehensive study on the reliability characteristics of multi-unit systems with standby provisioning, accounting for failures, degradation, random delays, and probabilistic imperfections.

8.2.3 Machine Repair Systems with Repair Facilities

Machine repair systems commonly include two types of repairers and two repair stages: the preparatory (P-stage) and execution (E-stage) stages. Primary repairers handle the preparatory stage, while the execution stage involves collaboration between primary and secondary repairers. Several researchers have explored this area (*cf.* [191], [192], [110], [43], [9], [42], [41], [85], [86], [87], [306]). Hanukov [84] conducted an in-depth analysis of server utilization time, splitting service into two stages: preliminary service and service reserved for subsequent customers. This article, too, divides the repair facility into initial and final stages, with primary repairers handling the preparatory repairs and secondary repairers taking over in the execution stage. An intriguing question arises: how should the workload be allocated between these stages? The division of tasks among repairers has long been recognized as a means to enhance productivity and effectiveness in producing goods and services, echoing the principles laid out by Adam Smith [276]. A detailed analysis of the concept of work division is presented in Subsection 8.3.1.

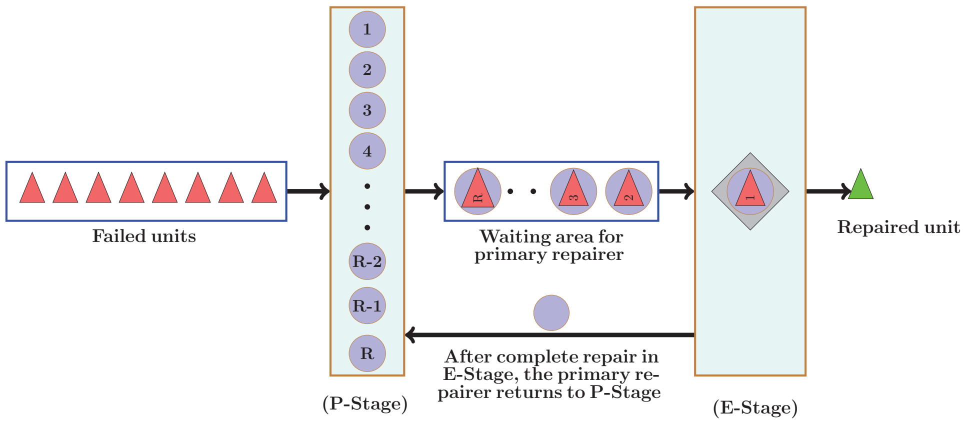


Figure 8.1: Flow chart of the repair system

8.3 Model Description

In this section, we present a finite-population Markovian machine repair model that encompasses two types of repairers: primary and secondary. The model comprises M operating units and S warm standby units, alongside R primary repairers and one secondary repairer. A standby unit is categorized as a warm standby unit when its failure rate is both non-zero and lower than that of an operating unit. The underlying assumptions of this model are as follows:

Failure process

- The failure of both operating and warm standby units follows a Poisson process with mean rate, λ and ν $(0 \leq \nu \leq \lambda)$, respectively.
- When an operating unit fails, an available standby unit replaces it almost instantaneously.
- Upon successful replacement, a warm standby unit transitions to the role of an operating unit, with all its characteristics identical to those of an operating unit.

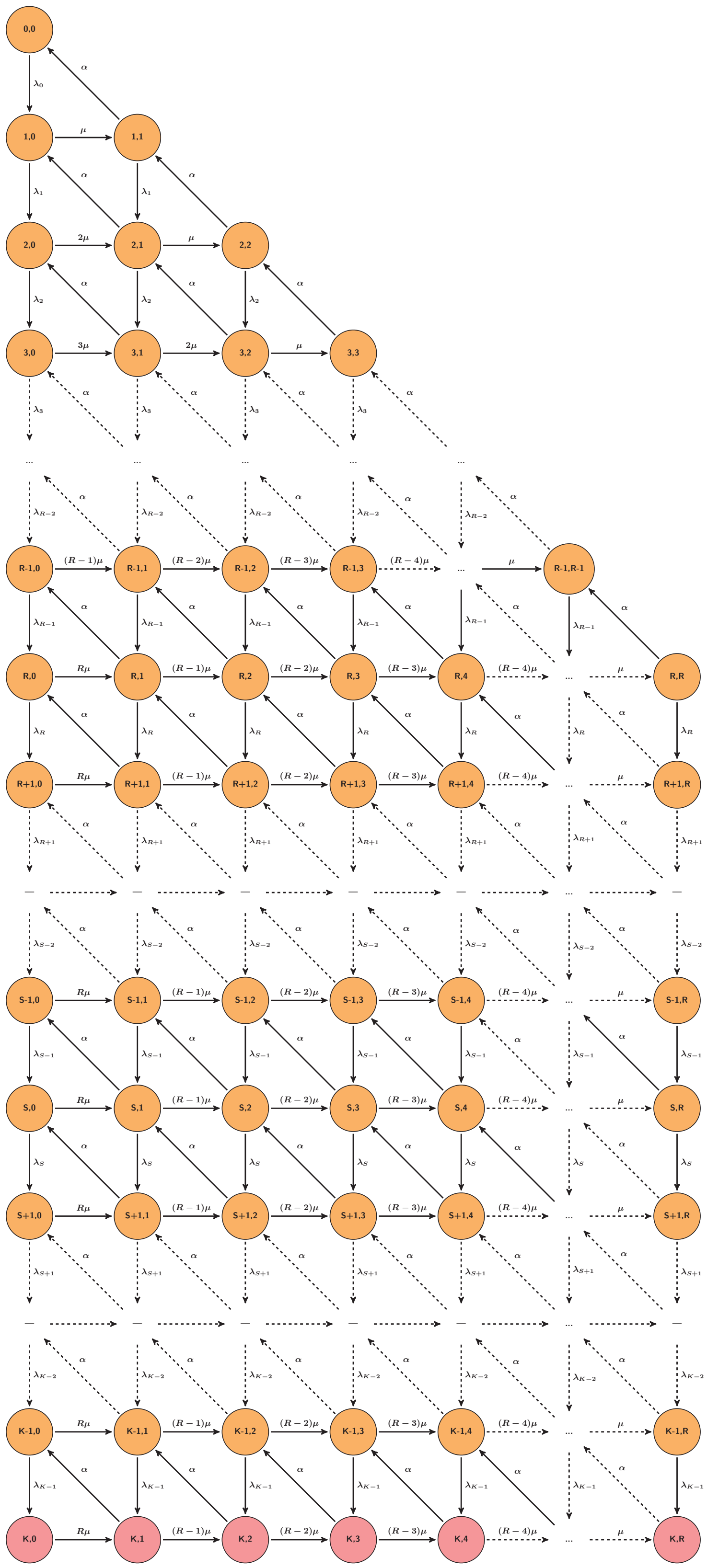


Figure 8.2: Steady-state Rate Diagram

- If the availability of standby units in the system is depleted, the inter-failure time of each operating unit deteriorates and follows an exponential distribution with a mean rate of λ_d ($\lambda < \lambda_d$).

At the initial instant, no unit has failed, meaning all operating and standby units ($K = M + S$) are operational. As time progresses, the system experiences n failed units; therefore, the instantaneous failure rate is calculated as follows:

$$\lambda_n = \begin{cases} M\lambda + (S-n)\nu & ; \quad n = 0,1,2,\cdots,S-1, \\ (M+S-n)\lambda_d & ; \quad n = S, S+1, S+2, \cdots, K-1, K \\ 0 & ; \quad \text{Otherwise} \end{cases}$$

Repair process

- A failed unit undergoes hierarchical two stages of repair: the P-stage, managed by any available primary repairer, and the E-stage, overseen by the secondary repairer.
- A failed unit cannot directly proceed to the E-stage for secondary repair. Instead, it first goes to the primary repairer in the P-stage, where an available primary repairer works on the failed unit. After completing this preparatory repair, the unit requires approval or advanced repair from the secondary repairer.
- To facilitate the comprehensive repair process, the primary repairer confers with the secondary repairer, detailing the unit's faults and the work performed during the P-stage.
- According to the work division policy, the secondary repairer collaborates with the primary repairer to complete the remaining portion of the service that necessitates the presence of a secondary repairer. Ultimately, the secondary repairer approves the unit for reintegration into the regular operating or standby unit after repairs.
- If a primary repairer finds that the secondary repairer is occupied with another primary repairer, the primary repairer joins a queue. In the approval stage, the primary repairer assumes the role of a customer in the E-stage and departs this stage after approval.
- The service facility adheres to the First-In-First-Out (FIFO) policy in both P-stage and E-stage.
- Both primary and secondary repairers have exponentially distributed repair times with rates μ and α, respectively.

Hence, the instantaneous repair rate μ_i is defined as follows:

$$\mu_i = \begin{cases} (i-j)\mu & ; \quad 0 \le i \le R, \ \& \ 0 \le j \le i \\ (R-j)\mu & ; \quad R \le i \le K, \ \& \ 0 \le j \le R-1 \end{cases}$$

All events, such as the failure of operating/standby units, primary repair, and secondary repair, are statistically independent.

This model constitutes an $M/M/R+1/K$ machine repair system comprising M identical operating units, S standby units, and is serviced by R primary repairers and one secondary repairer. We begin by deriving the steady-state probabilities using the Chapman-Kolmogorov difference equations, followed by employing the matrix recursive method and an efficient MATLAB program.

8.3.1 Work division policy

This section provides insight into the state-of-the-art allocation of repair work between the primary and secondary repairers in both stages of the repair process. As previously mentioned, the repair facility is divided into two distinct stages: the preparatory (P) and exccution (E) stages. During the preparatory (P) stage, one of the available primary repairers undertakes the primary repair task, while in the execution (E) stage, the secondary repairer collaborates with the primary repairer to complete the repair process. The predetermined work division policy governs the mean duration of repair in the P-stage, denoted as $(\frac{1}{\mu})$, and in the E-stage, denoted as $(\frac{1}{\alpha})$. To formulate this policy, we first establish a function involving α and μ. We then fix the value of μ to define the policy, subsequently determining the value of α as a function of μ. Let us denote $\alpha(\mu)$ as the repair rate in the E-stage, which varies inversely with the repair rate in the P-stage, μ. The repair rate, represented as ζ, is applicable when a single repairer (either primary or secondary) handles continuous repairs within a single stage. Both μ and $\alpha(\mu)$ are bounded below by ζ within the range $(0 < \zeta < \alpha(\mu), \mu)$. We assume a convex decreasing relationship between the difference in repair rates between the execution stage and the overall repair rate and the difference in repair rates between the preparatory stage and the overall repair rate, expressed as $\alpha(\mu) - \zeta = \frac{b}{(\mu-\zeta)}$. This convex function bears resemblance to a line used by other researchers to develop inventory control strategies ([206], [25]). Upon solving for the value of $\alpha(\mu)$, we obtain:

$$\alpha(\mu) = \zeta + \frac{b}{(\mu - \zeta)} \tag{8.1}$$

- Case I: In the event that the primary repairer does not conduct repairs during the preparatory stage ($\mu \to \infty$), meaning that the secondary repairer exclusively handles the entire repair process, the value of $\alpha(\mu)$ converges to ζ. This scenario reflects a transition in which the secondary repairer assumes full responsibility for the repair task, resulting in $\alpha(\mu)$ reaching its limiting value of ζ.
- Case II: In the event that the entirety of the repair process is handled exclusively by the primary repairer during the preparatory stage ($\mu \to \zeta^+$), it follows that in this scenario, the secondary repairer becomes ideal in the execution stage ($\alpha(\mu) \to \infty$). As ($\mu \to \zeta^+$), the expression $\frac{b}{(\mu-\zeta)}$ tends towards zero, indicating that the complete repair rate approaches the value of ζ. This behavior aligns with the concept of the complete repair rate converging to ζ as the primary repairer takes on the entire repair process during the preparatory stage.

Following the analysis of the aforementioned scenarios, we have arrived at the conclusion that when the entire repair process is consistently managed by a single repairer, whether primary or secondary, within a single stage, the repair time approaches $\frac{1}{\zeta}$. This finding underscores the relationship between repair duration and the singular responsibility of a repairer, highlighting the convergence of repair time to the reciprocal of ζ. This scenario bears a resemblance to standard Markovian single-server machine interference queueing models that incorporate redundancy.

- Case III:The requirement for the combined duration of the two repair stages to be identical to the duration when a single repairer continuously manages the repair within a single stage is not obligatory, as denoted by $\frac{1}{\alpha} + \frac{1}{\mu} \lesseqgtr \frac{1}{\zeta}$. This discrepancy arises due to the additional tasks associated with dividing the repair process, such as the transition to the secondary repairer, waiting time in the execution stage, and consultations with the secondary server to address the issues encountered, among other factors. These intricacies introduce a level of variability in the overall repair duration.

The proficiency and technical acumen of a secondary repairer contribute to a notably higher standard of work compared to that of a primary repairer. Consequently, the efficacy of the repair process is contingent on the proportion of repairs conducted by the secondary repairer. As a result, it becomes imperative to establish the value of μ in order to regulate the quality standards of the repair facility. We can formulate a function, denoted as $Q(\mu)$, to quantify the quality of the repair, defined as follows:

$$Q(\mu) = 1 - \omega^{(\zeta-\mu-\delta)} \tag{8.2}$$

where $0 \leq Q(\mu) \leq 1$. When the parameter ω exceeds unity ($\omega > 1$), it indicates that the contributions of the primary repairer significantly influence the overall service quality. A higher value of ω implies a reduced disparity in the skill levels between the primary and secondary repairers. This may suggest that the secondary repairer possesses exceptional abilities to rectify any errors or discrepancies introduced during the preparatory repair stage. As observed in Equation 8.2, the quality of repair, denoted as $Q(\mu)$, approaches 1 as μ approaches infinity, signifying that the repair process is entirely carried out by the secondary repairer. Conversely, if the repair process is exclusively handled by the primary repairer during the preparatory stage, the quality of repair reaches its lowest value, i.e., $Q(\mu) \to 1 - \omega^{-\delta}$ as μ approaches ζ^+. Hence, it becomes evident that enhancing the involvement of the secondary repairer can effectively augment the quality of the repair. However, it is essential to strike a balance, as allocating an excessively high workload to the secondary repairer can have detrimental effects on the repair facility:

- Increased Component Waiting Time: A greater workload for the secondary repairer leads to longer waiting times for failed components, as the secondary repairer is responsible for the entire repair process.
- Prolonged Waiting Times for Primary Repairers: The extended duration required by the secondary repairer affects the waiting times of primary repairers in the execution stage.
- Impact on Total Cost: The heightened idle time of primary repairers due to extended waiting periods in the final stage can impact the overall cost of the system.

Hence, achieving an optimal balance in workload allocation to the secondary repairer is crucial to maintain both service quality and operational efficiency in the repair facility.

8.3.2 Steady-state equations

The system's state can be effectively represented by pairs denoted as $Z = (I, F)$, where I represents the number of failed units (referred to as customers) in the system within a state-space, while F signifies the count of customers in the E-stage. The E-stage count corresponds to the number of primary repairers in the queue, awaiting their turn to interact with the secondary repairer. Let $P_{i,j}$ denote the joint probability distribution function associated with the system state, defined as follows:

$$P_{i,j} = Prob\,(I = i, F = j)$$

where $i = 0, 1, \cdots, R, \cdots, S, \cdots, K-1, K$, and $j = 0, 1, \cdots, R-1, R$. Figure 8.2 provides a visual representation of the steady-state transition rate diagram for the multi-unit machine repair model, featuring two stages hierarchical of repairers. The mathematical framework expressing the steady-state conditions of the system is presented below:

$$-\lambda_0 P_{0,0} + \alpha P_{1,1} = 0 \tag{8.3}$$

$$-(\lambda_i + i\mu)P_{i,0} + \lambda_{i-1}P_{i-1,0} + \alpha P_{i+1,1} = 0;\ 1 \le i \le R-1 \tag{8.4}$$

$$-(\lambda_i + R\mu)P_{i,0} + \lambda_{i-1}P_{i-1,0} + \alpha P_{i+1,1} = 0;\ R+1 \le i \le K-1 \tag{8.5}$$

$$-R\mu P_{K,0} + \lambda_{K-1}P_{K-1,0} = 0 \tag{8.6}$$

$$-(\lambda_i + \alpha)P_{i,j} + \mu P_{i,j-1} + \alpha P_{i+1,j+1} = 0;\ 1 \le i \le R-1,\ 1 \le j \le i \tag{8.7}$$

$$-(\lambda_R + \alpha)P_{R,R} + \mu P_{R,R-1} = 0 \tag{8.8}$$

$$-(\lambda_i + \alpha)P_{i,R} + \lambda_{i-1}P_{i-1,R} + \mu P_{i,R-1} = 0;\ R+1 \le i \le K-1 \tag{8.9}$$

$$-\alpha P_{K,R} + \lambda_{K-1}P_{K-1,R} + \mu P_{K-1,R-1} = 0 \tag{8.10}$$

$$-(\lambda_i + (i-j)\mu + \alpha)P_{i,j} + (i-j+1)\mu P_{i,j-1} + \lambda_{i-1}P_{i-1,j} + \alpha P_{i+1,j+1} = 0$$
$$;\quad 2 \le i \le R-1,\ 1 \le j \le i-1 \tag{8.11}$$

$$-(\lambda_i + (R-j)\mu + \alpha)P_{i,j} + (R-j+1)\mu P_{i,j-1} + \lambda_{i-1}P_{i-1,j} + \alpha P_{i+1,j+1} = 0$$
$$;R \le i \le K-1,\ 1 \le j \le R-1 \tag{8.12}$$

$$-((R-j)\mu + \alpha)P_{K,j} + (R-j+1)\mu P_{K,j-1} + \lambda_{K-1}P_{K-1,j} = 0; 1 \le j \le R-1 \tag{8.13}$$

To determine the steady-state probabilities for the machine repair model, considering exponentially distributed lifespans and repair times for the operating/standby units and repairers, we derive the Chapman-Kolmogorov forward equations (Eq. 8.3 - 8.13) by balancing inflow and outflow rates. To effectively calculate these probabilities, an appropriate method is required. In the subsequent section, we employ the matrix recursive method to achieve this.

8.4 Matrix-recursive method

This section is dedicated to the computation of steady-state probabilities using a matrix recursive approach. To facilitate this, we construct a block-tridiagonal matrix based on the transition rate matrix **Q** of the underlying Markov chain. The block-tridiagonal matrix **Q** is defined as follows:

$$\mathbf{Q} = \begin{bmatrix} \mathbf{A}_0 & \mathbf{C}_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_0 & \mathbf{A}_1 & \mathbf{C}_1 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_1 & \mathbf{A}_2 & \mathbf{C}_2 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_2 & \mathbf{A}_3 & \mathbf{C}_3 & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{R-2} & \mathbf{A}_{R-1} & \mathbf{C}_{R-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{R-1} & \mathbf{D}_R & \mathbf{C}_R & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{E}_R & \mathbf{D}_{R+1} & \mathbf{C}_R & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{E}_{R+1} & \mathbf{D}_{R+2} & \mathbf{C}_R & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{E}_{K-2} & \mathbf{D}_{K-1} & \mathbf{C}_R \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{E}_{K-1} & \mathbf{D}_K \end{bmatrix}_{K+1\times K+1}$$

The block tridiagonal matrix $\mathbf{Q}$ is a square matrix of order $K+1$. Its principal diagonal block entries, denoted as $\mathbf{A}_i$ for $1 \le i \le R-1$, $\mathbf{D}_i$ for $R \le i \le K-1$, and $\mathbf{D}_K$, are square matrices of order $i+1$, $R+1$, and $R+1$, respectively. The first diagonal above block entries, represented as $\mathbf{C}_i$ and $\mathbf{C}_R$, correspond to matrices of order $(i+1)\times(i+2)$ and $(R+1)\times(R+1)$, respectively, where $0 \le i \le R-1$. Conversely, the first diagonal below block entries, denoted as $\mathbf{B}_i$ for $0 \le i \le R-1$ and $\mathbf{E}_i$ for $R \le i \le K-1$, are matrices of order $(i+2)\times(i+1)$ and $(R+1)\times(R+1)$, respectively. In matrix form, the entries of the transition matrix $\mathbf{Q}$ are defined as follows:

$$\mathbf{A}_0 = \begin{bmatrix} -\lambda_0 \end{bmatrix}_{1\times 1}$$

$$\mathbf{A}_i = \begin{bmatrix} -(\lambda_i + (i-0)\mu) & 0 & 0 & \cdots & 0 & 0 & 0 \\ \Theta_{i,0} & -\Lambda_{i,1} & 0 & \cdots & 0 & 0 & 0 \\ 0 & \Theta_{i,1} & -\Lambda_{i,2} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \Theta_{i,j-1} & -\Lambda_{i,j} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \Theta_{i,j} & -(\lambda_i+\alpha) \end{bmatrix}_{(i+1)\times(i+1)}$$

where $\Theta_{i,j} = (i-j)\mu$; $\quad \Lambda_{i,j} = [\lambda_i + \alpha + (i-j)\mu]$; $1 \le i \le R-1$ & $0 \le j \le i-1$

$$\mathbf{B}_i = \begin{bmatrix} \lambda_i & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_i & 0 & \cdots & 0 & 0 \\ 0 & 0 & \lambda_i & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i & 0 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_i \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{(i+2)\times(i+1)} \qquad \mathbf{C}_i = \begin{bmatrix} 0 & \alpha & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \alpha & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \alpha & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \alpha & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \alpha \end{bmatrix}_{(i+1)\times(i+2)}$$

where $1 \leq i \leq R-1$

$$\mathbf{C}_R = \begin{bmatrix} 0 & \alpha & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \alpha & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \alpha & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \alpha & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \alpha \end{bmatrix}_{(R+1)\times(R+1)} \qquad \mathbf{E}_i = \begin{bmatrix} \lambda_i & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_i & 0 & \cdots & 0 & 0 \\ 0 & 0 & \lambda_i & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i & 0 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_i \end{bmatrix}_{(R+1)\times(R+1)}$$

where $R \leq i \leq K-1$

$$\mathbf{D}_i = \begin{bmatrix} -(\lambda_i + (R-0)\mu) & 0 & 0 & \cdots & 0 & 0 & 0 \\ \Theta_{R,0} & -\kappa_{i,1} & 0 & \cdots & 0 & 0 & 0 \\ 0 & \Theta_{R,1} & -\kappa_{i,2} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \Theta_{R,j-1} & -\kappa_{i,j} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \Theta_{R,j} & -(\lambda_i + \alpha) \end{bmatrix}_{(R+1)\times(R+1)}$$

where $\Theta_{R,j} = (R-j)\mu; \quad \kappa_{i,j} = [\lambda_i + \alpha + (R-j)\mu] \quad R \leq i \leq K-1 \;\&\; 0 \leq j \leq R-1$

$$\mathbf{D}_K = \begin{bmatrix} -R\mu & 0 & 0 & \cdots & 0 & 0 & 0 \\ R\mu & -((R-1)\mu + \alpha) & 0 & \cdots & 0 & 0 & 0 \\ 0 & (R-1)\mu & -((R-2)\mu + \alpha) & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2\mu & -(\mu + \alpha) & 0 \\ 0 & 0 & 0 & \cdots & 0 & \mu & -\alpha \end{bmatrix}_{(R+1)\times(R+1)}$$

Now, consider $\mathbf{P}_i = (P_{i,0}, P_{i,1}, \cdots, P_{i,i})$ to represent the probability vector of the i^{th} row, which has a dimension of $(i+1)$, where $i = 0, 1, \cdots, R-1$. Additionally, let $\mathbf{P}_i = (P_{i,0}, P_{i,1}, \cdots, P_{i,R})$ be the probability vector of dimension $(R+1)$ for the i^{th} row, where $R \leq i \leq K$. We can construct the steady-state probability vector $\mathbf{P}$ as follows:

$$\mathbf{P} = [\mathbf{P_0}, \mathbf{P_1}, \mathbf{P_2}, \cdots, \mathbf{P_{K-1}}, \mathbf{P_K}]$$

This vector $\mathbf{P}$ represents the steady-state probabilities corresponding to the block-tridiagonal matrix $\mathbf{Q}$. To compute the numerical values of these state probabilities, we need to solve the following problem:

$$\mathbf{PQ} = \mathbf{0} \tag{8.14}$$

Alternatively, we can refer to this as a system of block-matrix equations.

$$\mathbf{P}_0\mathbf{A}_0 + \mathbf{P}_1\mathbf{B}_0 = \mathbf{0} \tag{8.15}$$

$$\mathbf{P}_{i-1}\mathbf{C}_{i-1} + \mathbf{P}_i\mathbf{A}_i + \mathbf{P}_{i+1}\mathbf{B}_i = \mathbf{0} \quad ; \quad \mathbf{1} \leq \mathbf{i} \leq \mathbf{R-1} \tag{8.16}$$

$$\mathbf{P}_{R-1}\mathbf{C}_{R-1} + \mathbf{P}_R\mathbf{D}_R + \mathbf{P}_{R+1}\mathbf{E}_R = \mathbf{0} \tag{8.17}$$

$$\mathbf{P}_{i-1}\mathbf{C}_R + \mathbf{P}_i\mathbf{D}_i + \mathbf{P}_{i+1}\mathbf{E}_i = \mathbf{0} \quad ; \quad \mathbf{R+1} \leq \mathbf{i} \leq \mathbf{K-1} \tag{8.18}$$

$$\mathbf{P}_{K-1}\mathbf{C}_R + \mathbf{P}_K\mathbf{D}_K = \mathbf{0} \tag{8.19}$$

8.4.1 The State Probabilities

Utilizing fundamental matrix manipulation techniques to deduce the state probabilities in vector form, we present the solution as follows. Consequently, it can be observed that matrix $\mathbf{A}_0$ is non-singular as indicated by Eqn. 8.15.

$$\mathbf{P}_0 = \mathbf{P}_1\mathbf{X}_0 \quad \text{where} \quad \mathbf{X}_0 = -\mathbf{B}_0\mathbf{A}_0^{-1} \tag{8.20}$$

Based on Eqn. 8.20 and Eqn. 8.16, we can derive the following result.

$$\mathbf{P}_i = \mathbf{P}_{i+1}\mathbf{X}_i \quad \text{where} \quad \mathbf{X}_i = -\mathbf{B}_i\left[\mathbf{X}_{i-1}\mathbf{C}_{i-1} + \mathbf{A}_i\right]^{-1} \; ; \quad 1 \leq i \leq R-1 \tag{8.21}$$

After solving Eqn. 8.21 and Eqn. 8.17, the following result is obtained.

$$\mathbf{P}_R = \mathbf{P}_{R+1}\mathbf{X}_R \quad \text{where} \quad \mathbf{X}_R = -\mathbf{E}_R\left[\mathbf{X}_{R-1}\mathbf{C}_R + \mathbf{D}_R\right]^{-1} \tag{8.22}$$

After computing Eqn. 8.22 and Eqn. 8.18, the following result is obtained.

$$\mathbf{P}_i = \mathbf{P}_{i+1}\mathbf{X}_i \quad \text{where} \quad \mathbf{X}_i = -\mathbf{E}_i\left[\mathbf{X}_{i-1}\mathbf{C}_R + \mathbf{D}_i\right]^{-1} \quad R+1 \leq i \leq K-1 \tag{8.23}$$

from Eqn.8.23 & Eqn.8.19, we have the following result Hence, by solving Eqns. 8.20 through 8.23 recursively, the state probabilities $\mathbf{P}_i$ for $0 \leq i \leq K-1$ can be expressed in terms of state probabilities $\mathbf{P}_K$ as follows.

$$\mathbf{P}_i = \mathbf{P}_{i+1}\mathbf{X}_i = \mathbf{P}_{i+2}\mathbf{X}_i\mathbf{X}_{i+1} = \cdots = \mathbf{P}_K\prod_{\xi=1}^{K-i}\mathbf{X}_{K-\xi} = \mathbf{P}_K\tau_i^* \quad \tau_i^* = \prod_{\xi=1}^{K-i}\mathbf{X}_{K-\xi} \tag{8.24}$$

and $\mathbf{X}_i$ for $0 \leq i \leq K-1$ are presented in Eqns. 8.20 through 8.23. By applying the normalization condition $\mathbf{Pe} = 1$, as derived from the total probability theorem, and

Eqn. 8.24, we derive the following result.

$$\begin{aligned}
\sum_{i=0}^{R-1} \mathbf{P}_i \mathbf{e}_i + \sum_{i=R}^{K} \mathbf{P}_i \mathbf{e}_R &= \mathbf{P}_0 \mathbf{e}_0 + \mathbf{P}_1 \mathbf{e}_1 + \cdots + \mathbf{P}_{R-1} \mathbf{e}_{R-1} + [\mathbf{P}_R + \mathbf{P}_{R+1} + \cdots + \mathbf{P}_K] \, \mathbf{e}_R \\
&= \mathbf{P}_K \tau_0^* \mathbf{e}_0 + \mathbf{P}_K \tau_1^* \mathbf{e}_1 + \cdots + \mathbf{P}_K \tau_{K-1}^* \mathbf{e}_{K-1} + \\
&\left[\mathbf{P}_K \tau_R^* + \mathbf{P}_K \tau_{R+1}^* + \cdots + \mathbf{P}_K \tau_K^*\right] \mathbf{e}_R \\
&= \mathbf{P}_K \left[\sum_{i=0}^{R-1} \tau_i^* \mathbf{e}_i + \sum_{i=R}^{K} \tau_i^* \mathbf{e}_R \right] = 1
\end{aligned} \tag{8.25}$$

where $\mathbf{e}_i$=$[1,1,1,\cdots,1]^T$ and $\mathbf{e}_R$=$[1,1,1,\cdots,1]^T$ are the column vectors of ones of dimension $(i+1)$ and $(R+1)$, respectively. Therefore, Eqn. 8.19 can be expressed as:

$$\mathbf{P}_K \left[\mathbf{X}_{K-1} \mathbf{C}_R + \mathbf{D}_K\right] = \mathbf{0} \tag{8.26}$$

Hence, by solving Eqns. 8.25 and 8.26, we can determine the state probabilities $\mathbf{P}_K$. Consequently, we are able to calculate the steady-state probabilities for $\mathbf{P}_i$; $0 \leq i \leq K-1$ using Eqn. 8.24. Furthermore, we have developed a MATLAB program to numerically compute these steady-state probabilities.

8.5 Optimal Analysis

In research, performance measures, cost functions, and optimization techniques are pivotal. They help evaluate efficiency, balance objectives, and fine-tune parameters to achieve optimal outcomes. This exploration delves into their significance and synergy in research and problem-solving. This section is dedicated to the comprehensive examination of the system's performance measures and cost analysis for the proposed Markovian model of multi-unit machine repair, featuring both primary and secondary repairers.

8.5.1 Performance measures

The ensuing performance metrics provide a deeper understanding of the system's behavior and efficiency:

- Mean number of failed units in the system

$$E_S = \sum_{i=0}^{R-1} i P_i e_{i+1} + \sum_{i=R}^{K} i P_i e_{R+1} \tag{8.27}$$

This metric provides valuable insights into the system's reliability and operational efficiency, shedding light on the average number of units experiencing failures over a specific period.

- Mean number of primary repairer in the E-stage

$$E_F = \sum_{i=0}^{R-1} P_i a_i + \sum_{i=R}^{K} P_i a_R \tag{8.28}$$

It represents the average count of primary repairers involved in the initial repair stage (E-Stage) across various operational scenarios. This metric offers insights into the utilization and workload distribution among primary repairers, contributing to the system's efficiency assessment.

- Throughput of the system

$$Th_S = \sum_{i=0}^{R-1} (\alpha + i\mu) P_i + \sum_{i=R}^{K} \sum_{j=0}^{R-1} (\alpha + (R-j)\mu) P_i \tag{8.29}$$

It quantifies the rate at which units, both operating and standby, move through the repair process and return to operational status. This metric is vital for evaluating the system's overall efficiency, capacity utilization, and its ability to meet demands in a timely manner.

- Throughput of the primary repairer in the E-stage

$$Th_F = \sum_{i=0}^{R-1} i\mu P_i + \sum_{i=R}^{K} \sum_{j=0}^{R-1} (\alpha + (R-j)\mu) P_i \tag{8.30}$$

This metric assesses the rate at which primary repairers complete repairs during the initial stage (E-stage) of the repair process. It provides insights into the efficiency and workload of primary repairers, which is essential for optimizing the allocation of resources and ensuring timely repairs.

- Mean number of failed operating units in the system

$$E_{fr} = \sum_{i=0}^{S} (M\lambda_i + (S-i)\nu) P_i + \sum_{i=S+1}^{K-1} (M+S-i)\lambda_d P_i + m\lambda_d P_K \tag{8.31}$$

This metric provides valuable insights into the system's reliability and maintenance requirements, playing a crucial role in optimizing resource allocation and overall system efficiency.

- Effective service rate of primary repairer in E-stage

$$\mu_{eff} = \sum_{i=0}^{R-1} P_i b_i + \sum_{i=R}^{K} P_i b_R \tag{8.32}$$

This metric quantifies the rate at which the primary repairer completes repairs in the final stage (E-stage) of the repair process. It takes into account various factors, including the primary repairer's individual service rate and the system's dynamics, offering insights into the efficiency of the repair operations. The effective service rate of the primary repairer in the F-stage plays a pivotal role in optimizing system performance, resource allocation, and overall cost-effectiveness, making it a critical parameter for decision-making in maintenance and repair strategies.

- Mean waiting time of failed operating units in the system

$$W_S = \frac{E_S}{E_{fr}} \tag{8.33}$$

It's crucial for system efficiency, reliability, and decision-making.

- Mean waiting time of primary repairer in E-stage

$$W_F = \frac{E_F}{\mu_{eff}} \tag{8.34}$$

It's a vital indicator of system performance, affecting resource utilization and repair efficiency. Calculating and optimizing this metric is crucial for enhancing system reliability and cost-effectiveness in multi-unit machine repair scenarios.

- Failure Frequency of the system

$$FF = \lambda_{K-1} P_{K-1} \tag{8.35}$$

This metric quantifies the rate at which failures occur within the system, providing valuable insights into its reliability and robustness. Analyzing and optimizing the failure frequency is essential for improving the performance and sustainability of multi-unit machine repair models with primary and secondary repairers.

- Quality Difference $Q_d(\mu)$ of the system

$$Q_d(\mu) = Q(\mu+1) - Q(\mu) \tag{8.36}$$

This metric is instrumental in assessing the efficiency and effectiveness of repair services provided by the two types of repair personnel. Analyzing $Q_d(\mu)$ is vital for optimizing the allocation of repair tasks and resources to enhance overall system performance.

The a_i and b_i are the $(i+1)$-dimensional column vectors, where $a_i = (0,1,2,...,i)^T$ &$b_i = \mu a_i J_i$ for $i = 0,1,2,\ldots,R$. Here, J_i represents an exchange matrix of order i. e_{i+1} and e_R are column vectors with all entries equal to 1 and of order $(i+1)$ and R, respectively. Specifically, $e_{i+1} = (1,1,1,\ldots,1)^T$ and $e_R = (1,1,1,\ldots,1)^T$.

8.5.2 Cost Function

We formulate a steady-state expected cost function for the $M/M/R+1/K$ machine repair problem featuring two stages of repair: Preparatory and Execution. This cost function encompasses two decision parameters, namely, R and μ. The discrete parameter R represents a natural number, signifying the count of primary repairers in the system. In contrast, the continuous variable μ is a non-negative real number. Our ultimate objective is to determine the optimal number of primary repairers in the system, denoted as R^*, and the optimal values of the repair rate μ, denoted as μ^*. These optimizations aim to minimize the system's cost while maximizing its profitability. The governing cost function is defined as:

$$TC = \begin{bmatrix} C_1 \times (1 - I_T) + C_2 \times E_S + C_3 \times E_F + C_4 \times E_{fr} \times (1 - Q(\mu)) \\ +C_5 \times R + C_6 \times \mu \end{bmatrix} \tag{8.37}$$

where

C_1 ≡ The unit time cost for the amount of the secondary server's time that is spent in a utilization time

C_2 ≡ The unit time cost for each failed unit in the system

C_3 ≡ The unit time cost for the expected number of primary repairers in E-stage

C_4 ≡ The unit time cost for each failed operating unit in the system

C_5 ≡ The unit time cost for one primary repairer

C_6 ≡ The unit time cost for repair

8.5.3 Paticle swarm optimization

The Particle Swarm Optimization (PSO) algorithm is a stochastic global optimization technique inspired by swarm intelligence observed in social behaviors, such as birds flocking and fish schooling. This algorithm was originally introduced by Kennedy and Eberhart in 1995 [142]. PSO operates on the principle of exploring and refining a population of entities, referred to as a "swarm," comprising individuals known as "particles." These particles traverse the solution space with fixed velocities, evolving across generations to converge towards optimal global positions. In problem-solving scenarios, the solution space is represented as a search space, where each point signifies a potential solution to the problem. The fitness value of each particle within the swarm is determined by the optimization objective, which involves both maximization and minimization tasks. Particles possess awareness of the coordinates of decision variables and maintain connections to the best solutions or fitness values they have achieved. Particle movement in the PSO algorithm is influenced by various factors, including the particle's inertia and its best-known position (pbest), denoted as $\mathbf{pb}_i^l$, as well as the global best-known position (gbest), represented as $\mathbf{gb}^l$. The particle's velocity during the most recent iteration is regulated using an inertia weight. Inertia serves as a restraining force, preventing particles from immediately returning to their previous positions. The best solution attained by each individual particle is referred to as $\mathbf{pb}_i^l$, while the best solution achieved by the entire swarm family is termed gbest, denoted as $\mathbf{gb}^l$, representing the optimal position among all particles. This algorithm leverages natural communication principles to enhance optimization results.

In the context of PSO algorithms, each particle serves as a representation of a potential solution within the search space. The movement of each particle is determined by its velocity, which is influenced by both its individual best-known position (pbest) and the global best-known position (gbest). Let $\mathbf{S}_i^l$ and $\mathbf{V}_i^l$ represent the position vector and velocity vector of the i^{th} particle in dimension d, respectively in l^{th} iteration. These vectors can be expressed as $\mathbf{S}_i^l = (S_{i,1}^l, S_{i,2}^l, \ldots, S_{i,d}^l)$ and $\mathbf{V}_i^l = (V_{i,1}^l, V_{i,2}^l, \ldots, V_{i,d}^l)$, respectively. Following each iteration, the velocity vector undergoes updates based on the following formula:

$$\mathbf{V}_i^{l+1} = w \otimes \mathbf{V}_i^l + \kappa_1 \otimes \mathbf{r}_1 \otimes \left(\mathbf{pb}_i^l - \mathbf{S}_i^l\right) + \kappa_2 \otimes \mathbf{r}_2 \otimes \left(\mathbf{gb}^l - \mathbf{S}_i^l\right) \tag{8.38}$$

$$\mathbf{S}_i^{l+1} = \mathbf{S}_i^l + \mathbf{V}_i^{l+1} \tag{8.39}$$

where

w represents the inertia weight, governing the impact of previous velocity on the current one.

κ_1 and κ_2 denote the cognitive and social parameters, respectively, influencing the particle's response to particle-best and global-best.

$\mathbf{r}_1$ and $\mathbf{r}_2$ are random values between 0 and 1, contributing to stochasticity in the particle's movement.

The pseudo-code for Particle Swarm Optimization algorithm is as follows:

Pseudo-code for Particle Swarm Optimization

Initialize population, $\mathbf{S}_{\max}$, $\mathbf{S}_{\min}$, $\mathbf{V}_{\max}$, $\mathbf{V}_{\min}$
Find $\mathbf{pb}_i^1$ and $\mathbf{gb}^1$
for l=1:maximum generation
  for i=1:population size
    if $f(\mathbf{S}_i^l) < f(\mathbf{pb}_i^l)$ **then** $\mathbf{pb}_i^l = \mathbf{S}_i^l$
      $f(\mathbf{gb}^l) = \min_i \left(f(\mathbf{pb}_i^l)\right)$
    end
    for j=1:dimension(d)
      $\mathbf{V}_i^{l+1} = w \otimes \mathbf{V}_i^l + \kappa_1 \otimes \mathbf{r}_1 \otimes \left(\mathbf{pb}_i^l - \mathbf{S}_i^l\right) + \kappa_2 \otimes \mathbf{r}_2 \otimes \left(\mathbf{gb}^l - \mathbf{S}_i^l\right)$
      $\mathbf{S}_i^{l+1} = \mathbf{S}_i^l + \mathbf{V}_i^{l+1}$
      if $\mathbf{V}_i^{l+1} > \mathbf{V}_{\max}$ **then** $\mathbf{V}_i^{l+1} = \mathbf{V}_{\max}$
      else if $\mathbf{V}_i^{l+1} < \mathbf{V}_{\min}$ **then** $\mathbf{V}_i^{l+1} = \mathbf{V}_{\min}$
      end
      if $\mathbf{S}_i^{l+1} > \mathbf{S}_{\max}$ **then** $\mathbf{S}_i^{l+1} = \mathbf{S}_{\max}$
      else if $\mathbf{S}_i^{l+1} < \mathbf{S}_{\min}$ **then** $\mathbf{S}_i^{l+1} = \mathbf{S}_{\min}$
      end
    end
  end
end

8.6 Numerical Analysis

Numerical evaluations are conducted to assess the computational feasibility of the Markovian multi-unit machine repair problem, encompassing two distinct repair stages and involving two categories of repairers. These analyses are carried out utilizing the MATLAB software environment (license number 925317), employing a computational system equipped with an Intel(R) Core(TM) i5-1035G1 processor operating at a CPU speed of 1.19 GHz and an 8.0 GB RAM capacity. These investigations encompass a comprehensive exploration of diverse system parameters.

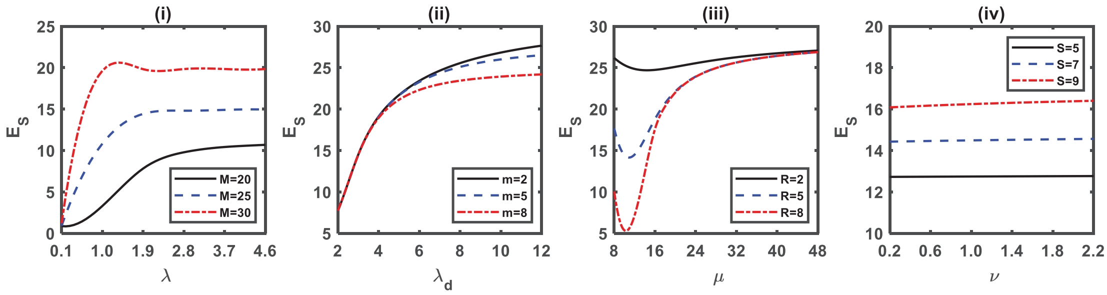


Figure 8.3: Effects of the varying parameters on the system's expected number of failed units (E_S).(for $M = 25; S = 7; m = 2; R = 5; \lambda = 2; \nu = 1.2; \lambda_d = 3; \mu = 12; \zeta = 10; b = 200; \omega = 1.3; \delta = 1$)

Figure 8.3(i-iv) illustrates the behavior of the expected number of failed units, denoted as E_S, within the system. In Fig. 8.3(i), it is discernible that in the initial state, the expected number of failed units corresponds to the failure rate of the operating units in the system. However, beyond a certain threshold of the failure rate, the expected number of failed units stabilizes. This same trend is replicated in Fig. 8.3(ii) when considering the degraded failure rate alongside the expected number of failed units in the system. In Fig. 8.3(iii), a converse relationship is evident between the repair rate μ of the primary repairer and the expected number of failed units. Initially, as the repair rate μ increases, the mean number of failed units within the system decreases, reaching a minimum value at a specific μ. Subsequently, it begins to rise again and eventually reaches a constant level as μ continues to increase. It is worth noting that the failure rate of standby units, denoted as ν, has a relatively minor impact on the expected number of failed units within the system.

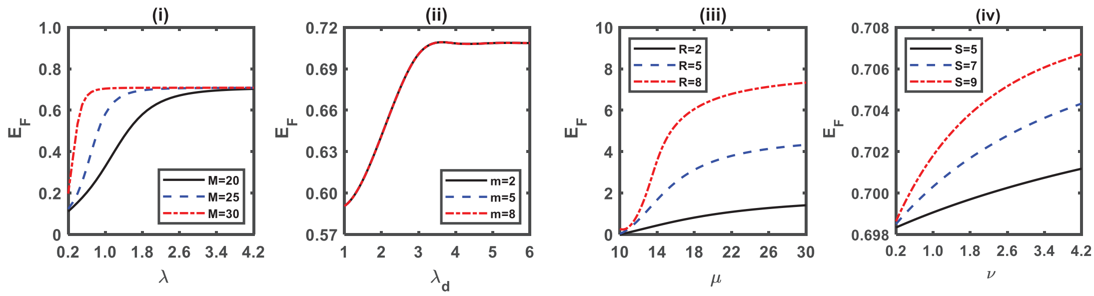


Figure 8.4: Effects of the distinct parameters on the system's expected number of primary repairers in F-stage (E_F), (for $M = 25; S = 7; m = 2; R = 5; \lambda = 2; \nu = 1.2; \lambda_d = 3; \mu = 12; \zeta = 10; b = 200; \omega = 1.3; \delta = 1$)

Figure 8.4(i) illustrates that as the failure rate of operating units escalates, the expected number of primary repairers in the E-stage increases. At a constant failure rate λ, a steady-state level is achieved. A similar pattern is observed for the degraded failure rate λ_d, as depicted in Figure 8.4(ii). In Figure 8.4(iii), an increase in the service rate of primary repairers leads to a higher number of customers in the E-stage. This indicates a quicker transition of primary repairers to the secondary repairer for complete repair. However, after a certain threshold value of μ, a steady-state behavior is observed. Figure 8.4(iv) shows that the expected number of primary repairers in the E-stage is influenced by the failure rate (ν) of warm standby units.

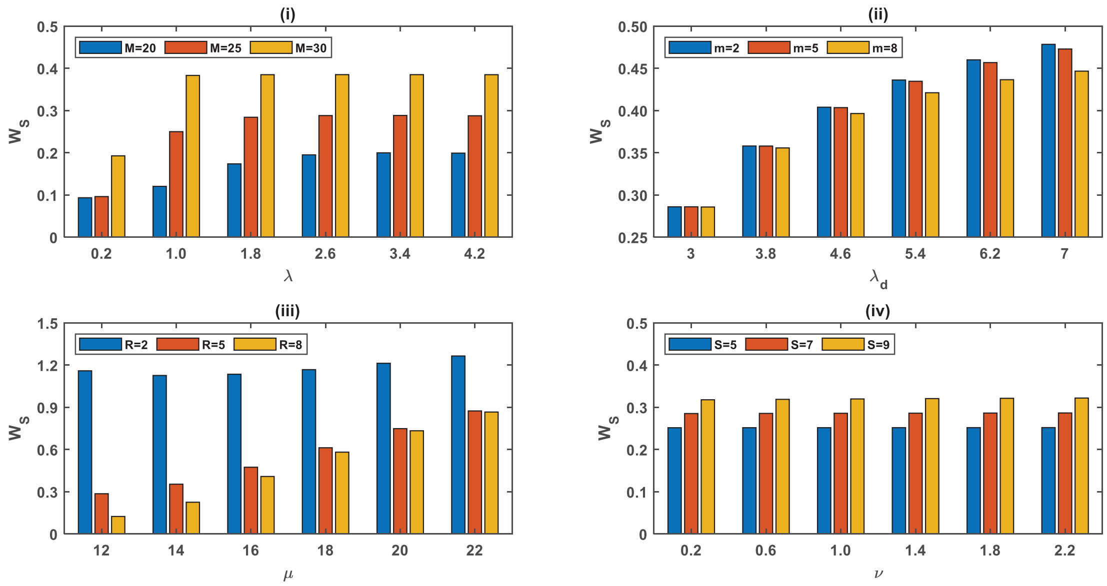


Figure 8.5: Effects of the distinct parameters on the system's expected waiting time (W_S), (for $M = 25; S = 7; m = 2; R = 5; \lambda = 2; \nu = 1.2; \lambda_d = 3; \mu = 12; \zeta = 10; b = 200; \omega = 1.3; \delta = 1$)

Figure 8.5(iii) reveals that an increased number of primary repairers reduces the expected waiting time (W_S) for failed units in the system, indicating improved service efficiency. However, Figures 8.5(i, ii, and iv) show that an increment in the failure rate of units (λ and λ_d) and the presence of warm standby units (ν) contribute to an extended expected waiting time for failed units in the system, reflecting increased repair demand and delays.

Figure 8.6(i) illustrates that an increased failure rate of operating units (λ) leads to an extended expected waiting time for primary repairers in the E-stage. This is attributed to primary repairers accumulating at the E-stage to collaborate with the secondary repairer. On the other hand, Figures 8.6(ii) and (iv) demonstrate a relatively lower dependence on the degraded failure and standby failure rates with respect to the waiting time of primary repairers in the E-stage. However, Figure 8.6(iii) reveals that an increase in the number of primary repairers (R) and their service rate (μ) reduces the expected waiting time of primary repairers in the F-stage, reaching a steady state after a certain service rate threshold.

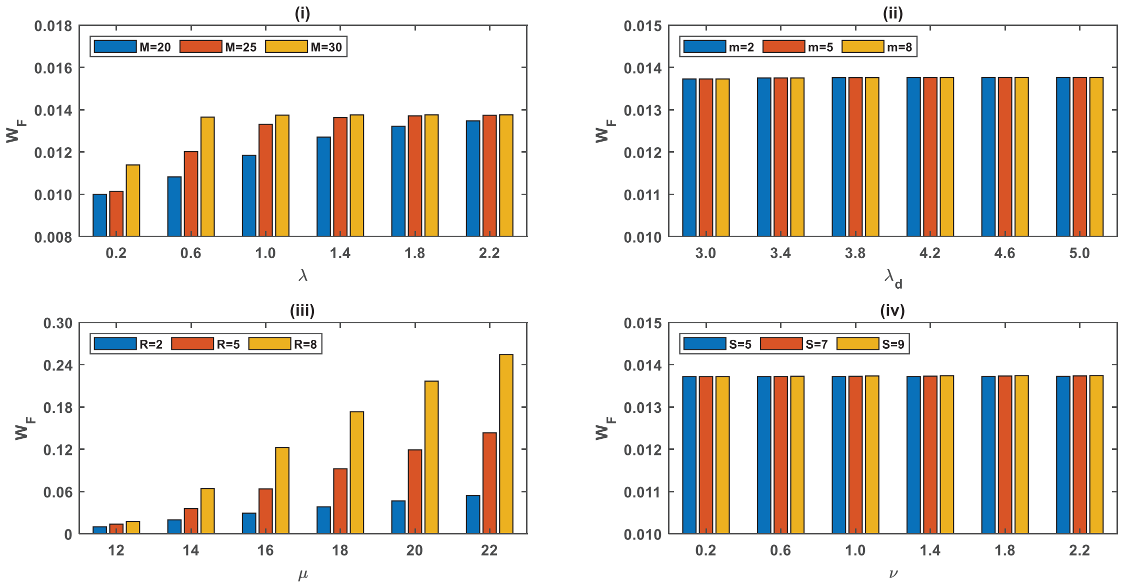


Figure 8.6: Effects of the distinct parameters on the system's expected waiting time of primary repairers in F-stage (W_F), (for $M = 25; S = 7; m = 2; R = 5; \lambda = 2; \nu = 1.2; \lambda_d = 3; \mu = 12; \zeta = 10; b = 200; \omega = 1.3; \delta = 1$)

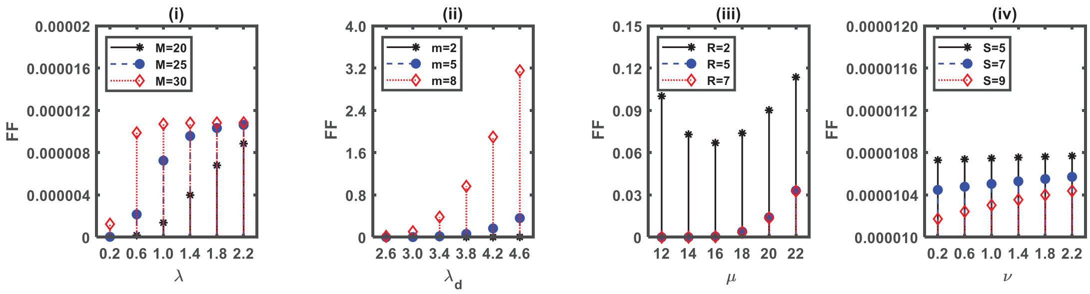


Figure 8.7: Effects of the distinct parameters on the system's Failure frequency (FF), (for $M = 25; S = 7; m = 2; R = 5; \lambda = 2; \nu = 1.2; \lambda_d = 3; \mu = 12; \zeta = 10; b = 200; \omega = 1.3; \delta = 1$)

Figure 8.7(i, ii, and iv) illustrates that an increase in the failure rate of operating units (λ), warm standby units (ν), and degraded failure rate (λ_d) leads to an escalation in the system's failure frequency (FF). This indicates that these conditions contribute to an enhanced failure propensity within the system. However, Figure 8.7(iii) portrays the repair-related dynamics affecting the failure frequency. It demonstrates that the repair facility plays a crucial role in mitigating system failures, thereby reducing the failure frequency.

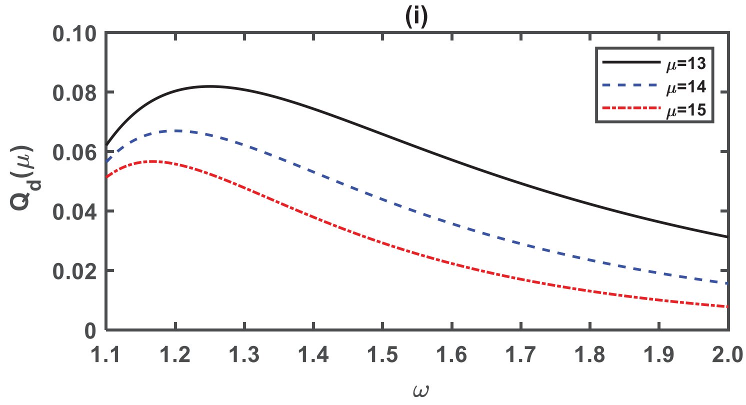


Figure 8.8: Effects of the distinct parameters on the system's Quality difference ($Q_d(\mu)$), (for $M = 25; S = 7; m = 2; R = 5; \lambda = 2; \nu = 1.2; \lambda_d = 3; \mu = 12; \zeta = 10; b = 200; \omega = 1.3; \delta = 1$)

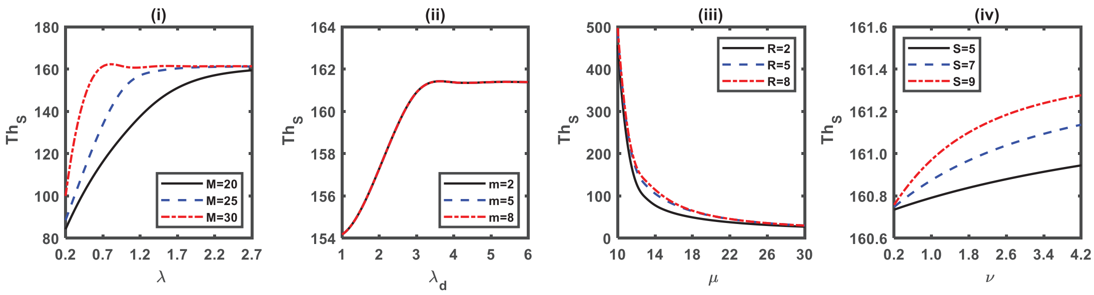


Figure 8.9: Effects of the distinct parameters on the system's Throughput (Th_S), (for $M = 25; S = 7; m = 2; R = 5; \lambda = 2; \nu = 1.2; \lambda_d = 3; \mu = 12; \zeta = 10; b = 200; \omega = 1.3; \delta = 1$)

Figure 8.8 provides insights into the relationship between the workload allocated to the primary repairer and the quality of service delivered. The findings indicate that, even when primary repairers exert a substantial influence on overall quality, their incremental impact on quality remains marginal. When distributing tasks between primary and secondary repairers, it is observed that reducing the workload assigned to the primary repairer results in only a slight improvement in repair quality. This trend holds true for a wide range of ω values, making it economically unfavorable to burden the secondary repairer with additional tasks. Consequently, it is advisable to allocate a greater workload to the primary repairer, regardless of whether ω is small or large, as opposed to intermediate values. In scenarios where the contribution of primary repairers to overall repair quality is negligible ($\omega > 2$), opting for complete repair execution by the primary repairer may be a more prudent choice.

Figures 8.9 and 8.10 depict the behavior of the primary repairers' effective service rate and the system throughput under various parameter settings. In Fig. 8.9(i, ii, and iv), the system's throughput (Th_S) experiences an initial increase with rising failure rates of operating units, degraded failures (λ and λ_d), and warm standby (ν). However, once a certain threshold of failure rate is reached, the system's throughput stabilizes. Conversely, an increase in the service rate of primary repairers leads to a reduction in system throughput. A similar trend is observed in the throughput of primary repairers, as shown in Fig. 8.10. In Figures (8.9 and 8.10)(ii), it is evident that an escalation in the repair rate of primary repairers results in the accumulation of more primary repairers in the E-stage. Consequently, the waiting time for failed units increases, leading to a decrease in the system's throughput.

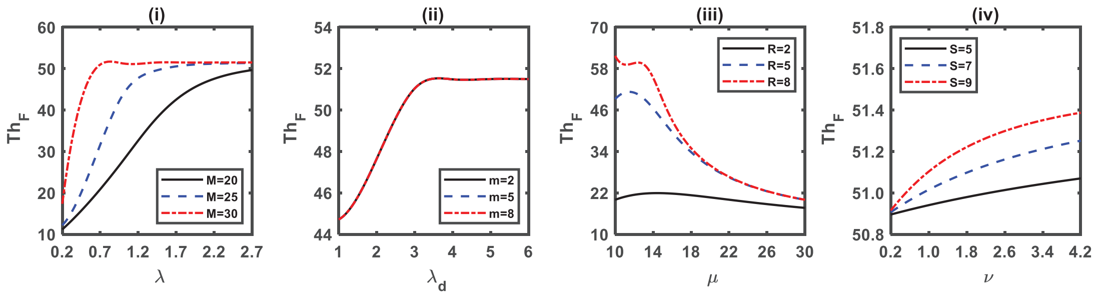


Figure 8.10: Effects of the distinct parameters on the Throughput or effective service rate of primary repairer in F-stage (Th_F), (for $M = 25; S = 7; m = 2; R = 5; \lambda = 2; \nu = 1.2; \lambda_d = 3; \mu = 12; \zeta = 10; b = 200; \omega = 1.3; \delta = 1$)

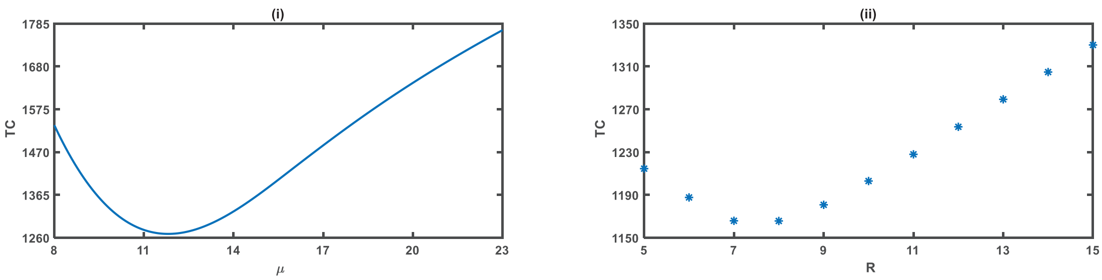


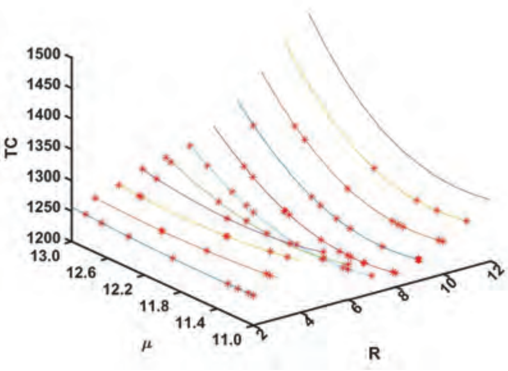
Figure 8.11: Effects of the distinct parameters on the Expected total cost of the system (TC), (for $M = 25; S = 7; m = 2; R = 5; \lambda = 2; \nu = 1.2; \lambda_d = 3; \mu = 12; \zeta = 10; b = 200; \omega = 1.3; \delta = 1; C_1 = 50; C_2 = 250; C_3 = 80; C_4 = 12; C_5 = 25; C_6 = 5$)

Based on the numerical results of the present research, the following recommendations are offered for decision-makers:

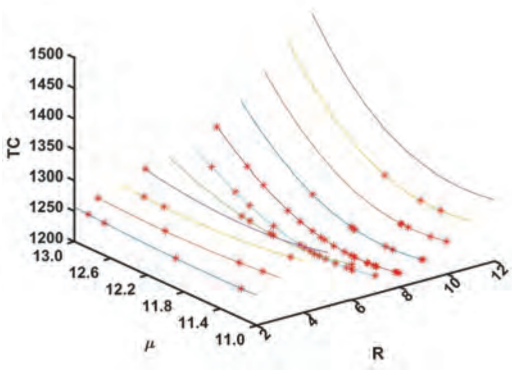
- Optimal Allocation of Repair Work: The research highlights that the allocation of repair work between primary and secondary repairers significantly impacts the overall quality of service. Decision-makers should consider the trade-off between the workload assigned to primary and secondary repairers. For situations where the impact of primary repairers on quality is minimal (e.g., $\omega > 2$), it may not be advantageous to divide the repair workload. Instead, allocating the entire repair to a primary repairer could be a more cost-effective choice.

- System Throughput Management: The study shows that the system's throughput is influenced by various parameters, including failure rates, repair rates, and the number of repairers. Decision-makers should carefully manage these parameters to optimize system throughput. For instance, increasing the repair rate of primary repairers may lead to congestion in the repair process, resulting in longer waiting times for failed units and reduced system throughput.
- Failure Frequency Mitigation: To minimize system failure frequency, decision-makers should focus on strategies that reduce failure rates of operating units, degraded failures, and warm standby units. Implementing preventive maintenance measures or improving the reliability of components can contribute to a more reliable system.
- Quality Improvement: When quality is a critical factor, decision-makers should assess the impact of primary repairers on repair quality (indicated by ω). Depending on the specific context and the desired level of quality, they can adjust the allocation of repair work to optimize repair outcomes.

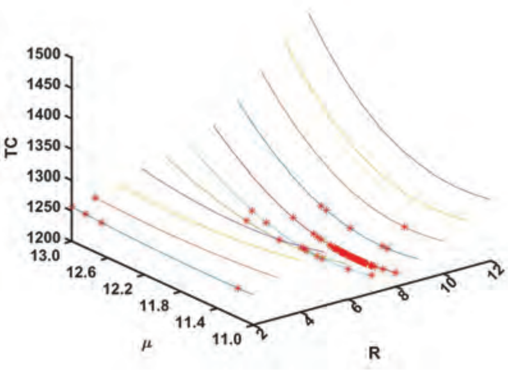
These recommendations provide valuable insights for decision-makers in managing and optimizing multi-unit machine repair systems with primary and secondary repairers, contributing to improved system performance and cost-effectiveness. Building on the results obtained in this study, we propose a state-of-the-art optimal analysis utilizing metaheuristic techniques for a diverse range of parameter configurations. This approach leverages the insights gained from our research to explore optimization strategies that can enhance the performance and efficiency of the system under varying conditions. By employing metaheuristic methods, we aim to identify optimal solutions and decision-making strategies that account for complex interactions between parameters, ultimately advancing the state-of-the-art in system optimization and resource allocation within multi-unit machine repair systems.



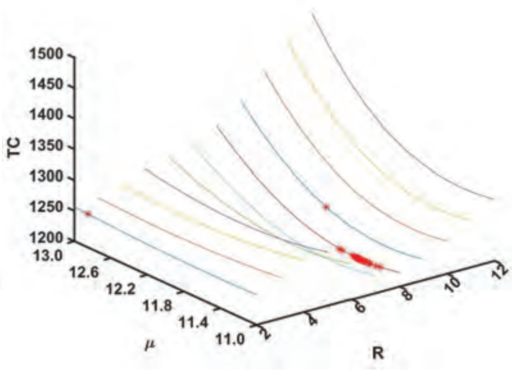
Iteration=1



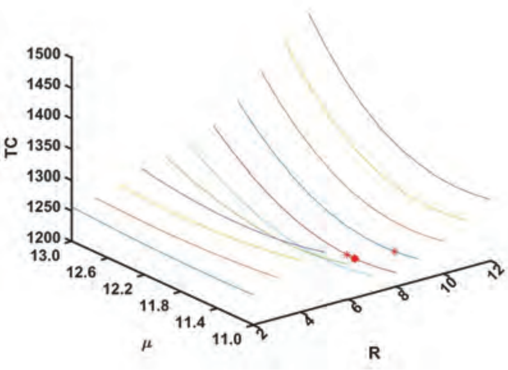
Iteration=5



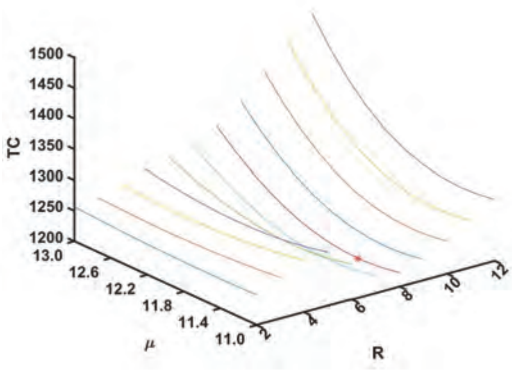
Iteration=10



Iteration=15



Iteration=20



Iteration=25

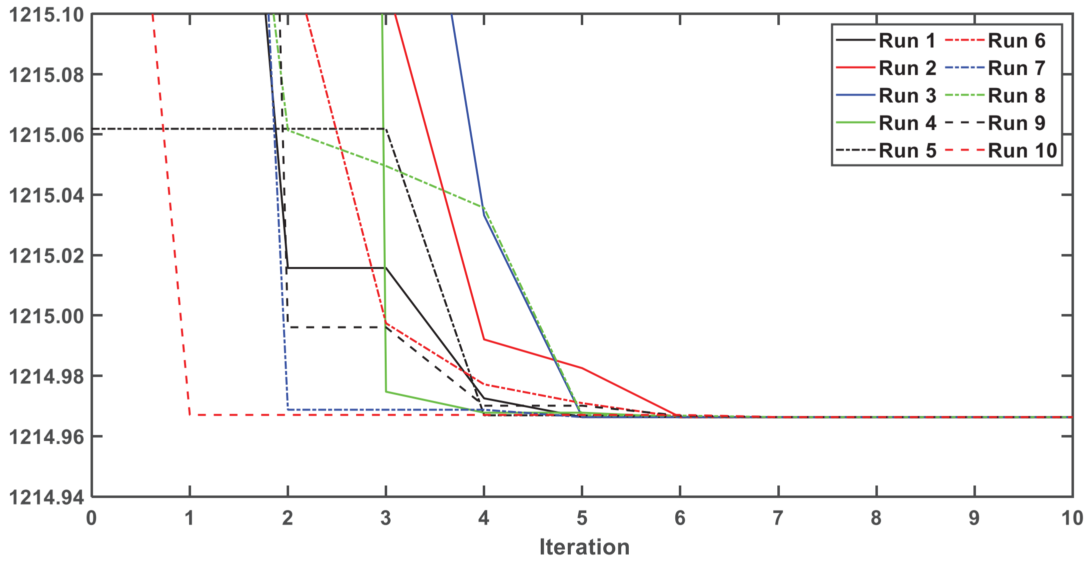


Figure 8.12: Convergence of iteration of Partical swarm optimization, (for $M = 25; S = 7; m = 2; R = 5; \lambda = 2; \nu = 1.2; \lambda_d = 3; \mu = 12; \zeta = 10; b = 200;$ $\omega = 1.3; \delta = 1; C_1 = 50; C_2 = 250; C_3 = 80; C_4 = 12; C_5 = 25; C_6 = 5$)

To optimize the cost function effectively, we conducted experiments with specific parameter values, including $M = 25$, $S = 7$, $m = 2$, $R = 5$, $\lambda = 2$, $\nu = 1.2$, $\lambda_d = 3$, $\mu = 12$, $\zeta = 10$, $b = 200$, $\omega = 1.3$, and $\delta = 1$. Figure 8.11 (i and ii) illustrates the convex nature of the total cost function concerning the primary repairer count (R) and the repair rate (μ). This convexity indicates an optimal solution within a desirable convex region for the decision variables μ and R. The optimization problem involves both discrete and continuous decision variables (R and μ, respectively) when evaluating the expected total cost function. Given the distinct nature of these variables, conventional optimization techniques are inadequate. Hence, we employ the Particle Swarm Optimization (PSO) method, a metaheuristic approach, to determine optimal total cost and corresponding decision variable values. The cost elements considered for optimization include $C_1 = 50$, $C_2 = 250$, $C_3 = 80$, $C_4 = 12$, $C_5 = 25$, and $C_6 = 5$. A population size of 300 is fixed for the PSO algorithm, and we performed 10 multiple runs to obtain optimal values for the decision variables and expected total cost.

Figure 8.12 provides a visual representation of the convergence process, illustrating the transition from initial random solutions to optimal points during iterative computations. This figure also effectively demonstrates the convergence behavior of optimal total cost values across multiple runs. Notably, Figure 8.12 offers a clear visualization of how the initial random solution on lines corresponding to different values of R at different μ gradually converge towards the optimal solution as iterations progress. This convergence is achieved within a remarkably short span, typically requiring only 15-20 iterations and maintaining precision to the level of 10^{-7}.

The presented graphical evidence strongly supports the utilization of metaheuristic optimization techniques for addressing complex applied problems of this nature. It highlights the efficiency of these methods in consistently guiding diverse initial solutions towards the optimal solution, thereby facilitating the decision-making process in complex systems.

Table 8.1: Optimal expected total cost of the system $TC(R^*, \mu^*)$ for different parameters, for $(C_1 = 50, C_2 = 250, C_3 = 80, C_4 = 12, C_5 = 25, C_6 = 5)$

Parameter	$M,S,m,\lambda,\nu,\lambda_d,\zeta$	R^*	μ^*	$TC(R^*,\mu^*)$	Mean Fitness	Worst Fitness	Time elapsed
λ	25, 7, 2, 1.5, 1.2, 3, 10	10	11.2148	1960.2229	1.0000009809	1.0000004914	233.2427
	25, 7, 2, 2.0, 1.2, 3, 10	10	11.1431	2371.7661	1.0000002483	1.0000007196	262.8704
	25, 7, 2, 2.5, 1.2, 3, 10	11	11.1285	2658.0494	1.0000004524	1.0000002830	229.2556
ν	25, 7, 2, 2, 0.6, 3, 10	10	11.1479	2326.3320	1.0000002840	1.0000006986	228.9490
	25, 7, 2, 2, 0.9, 3, 10	10	11.1454	2349.6173	1.0000004500	1.0000000260	223.8101
	25, 7, 2, 2, 1.2, 3, 10	10	11.1431	2371.7661	1.0000000890	1.0000007485	227.0956
λ_d	25, 7, 2, 2, 1.2, 2.0, 10	8	11.5165	2087.2420	1.0000003909	1.0000002054	123.1258
	25, 7, 2, 2, 1.2, 2.5, 10	9	11.3096	2210.9562	1.0000007341	1.0000000039	139.6072
	25, 7, 2, 2, 1.2, 3.0, 10	10	11.1431	2371.7661	1.0000003544	1.0000005958	146.7713
ζ	25, 7, 2, 2, 1.2, 3, 9.5	11	10.7016	2456.8478	1.0000003188	1.0000001863	234.7805
	25, 7, 2, 2, 1.2, 3, 10.0	10	11.1431	2371.7661	1.0000003319	1.0000003254	218.1552
	25, 7, 2, 2, 1.2, 3, 10.5	10	11.5874	2292.6716	1.0000001623	1.0000009815	236.9302

Tables 8.1 and 8.2 provide detailed information on the optimal values of decision variables and total cost (TC) for various parameter settings. Notably, the results reveal that an increase in the failure rate of operating units, standby units, and degraded failure rates leads to an enhancement in either the optimal number of primary repairers or the repairer rate of the system, validating the efficacy of our models. This optimization approach enables decision-makers to make informed choices while considering complex interactions between parameters in multi-unit machine repair systems.

The statistical analysis presented in the tables provides robust evidence supporting the efficacy of employing metaheuristic optimization techniques in tackling intricate decision-oriented problems. To establish this, we initially calculate the ratio of optimal solutions obtained in various runs to the overall optimal solution. Subsequently, we compute the mean and maximum values of these ratios. The findings presented in the tables demonstrate a high degree of consistency and precision in the results. The mean and maximum ratios consistently align with the overall optimal solution, affirming the reliability and effectiveness of employing metaheuristic optimization methods for addressing complex decision-making challenges.

In summary, our research study leveraged advanced numerical analyses to optimize decision variables in a complex system involving multi-unit machine repair.

Utilizing metaheuristic optimization techniques, we successfully determined optimal solutions, demonstrating the efficiency and effectiveness of our approach. These results contribute valuable insights for decision-makers in similar challenging scenarios, showcasing the potential for improved cost-efficiency and system performance.

Table 8.2: Optimal expected total cost of the system $TC(R^*, \mu^*)$ for the different parameters, for ($M = 25; S = 7; m = 2; R = 5; \lambda = 2; \nu = 1.2; \lambda_d = 3; \mu = 12; \zeta = 10; b = 200; \omega = 1.3; \delta = 1$)

Parameters	$C_1, C_2, C_3, C_4, C_5, C_6$	R^*	μ^*	$TC(R^*, \mu^*)$	Mean Fitness	Worst Fitness	Time elapsed
C_1	30, 250, 80, 12, 25, 5	10	11.1662	2365.3093	1.00000010097	1.00000098520	123.0082
	50, 250, 80, 12, 25, 5	10	11.1431	2371.7661	1.00000032533	1.00000017767	175.7418
	60, 250, 80, 12, 25, 5	10	11.1196	2378.0993	1.00000076520	1.00000014905	175.6347
C_2	50, 220, 80, 12, 25, 5	10	11.1638	2193.6693	1.00000013586	1.00000068607	201.9567
	50, 250, 80, 12, 25, 5	10	11.1431	2371.7661	1.00000034673	1.00000022109	179.9409
	50, 280, 80, 12, 25, 5	10	11.1256	2549.7801	1.00000054876	1.00000040558	185.1130
C_3	50, 250, 60, 12, 25, 5	10	11.1942	2362.0490	1.00000080959	1.00000060970	176.0793
	50, 250, 80, 12, 25, 5	10	11.1431	2371.7661	1.00000009117	1.00000093433	145.6521
	50, 250, 100, 12, 25, 5	10	11.0938	2380.9083	1.00000039292	1.00000050500	156.9453
C_4	50, 250, 80, 10, 25, 5	10	11.0664	2306.3636	1.00000074945	1.00000073998	158.0788
	50, 250, 80, 12, 25, 5	10	11.1431	2371.7661	1.00000037542	1.00000011805	142.1406
	50, 250, 80, 14, 25, 5	10	11.2132	2435.9444	1.00000004805	1.00000005431	140.6111
C_5	50, 250, 80, 12, 15, 5	11	11.1424	2264.5258	1.00000064894	1.00000039808	148.0741
	50, 250, 80, 12, 25, 5	10	11.1431	2371.7661	1.00000074296	1.00000027732	188.6907
	50, 250, 80, 12, 35, 5	11	11.1431	2471.7661	1.00000024805	1.00000050725	141.6968
C_6	50, 250, 80, 12, 15, 3	10	11.1518	2349.4711	1.00000024037	1.00000068248	204.0200
	50, 250, 80, 12, 25, 5	10	11.1431	2371.7661	1.00000020636	1.00000029405	137.3643
	50, 250, 80, 12, 35, 7	10	11.1343	2394.0436	1.00000010402	1.00000024201	192.3328

8.7 Conclusion

The proposed model encompasses essential characteristics of practical machine repair problems, including operating units, warm standby units, failure rates, system throughput, two types of repairers, and repair rates. This Markovian model focuses on the state-of-the-art analysis of a multi-unit machine repair system with spare units and two-stage repair services. The introduction of a work-division policy within the repair facility offers potential advantages to industrial stakeholders, enabling cost optimization and supply-demand equilibrium. The matrix recursive method is employed to determine state probabilities, queue properties, and system performance metrics. This study's examination of queueing measures' sensitivity empowers industrial decision-makers to enhance system uptime and reduce operational costs. The numerical analysis section underscores the feasibility of leveraging optimal decision variable values to boost system availability and minimize total costs, delivering tangible benefits for industrial operations. Furthermore, the analysis of queueing measures under varying parameters highlights how optimal decision variables contribute to enhanced system efficiency and cost-effectiveness.

The present study opens doors to several promising avenues for future research in the field of multi-unit machine repair systems. (a) Further exploration of cutting-edge optimization algorithms, beyond Particle Swarm Optimization, may yield even more efficient solutions for decision variable selection. (b) Investigating the feasibility of implementing the proposed work-division policy in real-world industrial settings, along with the development of practical decision support systems. (c) Incorporating machine learning techniques to predict system failures and repair requirements, enhancing predictive maintenance strategies. (d) Assessing the environmental impact of machine repair policies and their alignment with sustainability goals. (e) Extending the research to encompass multi-objective optimization, considering not only cost but also environmental impact and system robustness. (f) Conducting case studies across different industries to validate the applicability and benefits of the proposed model. These future endeavors can contribute to the advancement of decision-making processes in the domain of machine repair systems, ultimately benefiting industries in terms of cost-efficiency, system reliability, and sustainability.

Chapter 9

Conclusions and Future Work

"Success is not final, failure is not fatal: It is the courage to continue that counts".

Winston Churchill

This chapter serves as a summary of the primary findings of the thesis. Additionally, it offers insights into potential research avenues that could be explored in future studies.

9.1 Summary and Conclusions

In this thesis, diverse queueing models are formulated within the domain of machine repair problems to forecast the performance of machining systems providing different kinds of spare unit support, prompt repair facility, and enhancing the reliability and availability of the systems.

In Chapter 2, we have explored the realm of redundant storage systems, employing Bayesian inferences and carefully selected prior distributions, particularly when dealing with unpredictable repair services and switching failure. We have evaluated two crucial reliability metrics: mean time-to-failure (MTTF) and availability ($A(\infty)$). Through a combination of simulations and inference techniques, we have provided insights into the intricate workings of these systems, offering valuable insights into their performance.

In the chapter 3, the fuzzy doctrines γ-cut and Zadeh's extension principle are employed to develop the fuzzified MTTF and the system availability associated membership grade function by using parametric nonlinear programming from fuzzified pertinent system parameters. The adapted Imperfect repair and switching failure are also realistic. The shortcoming of the present study is to determine the expression of characteristics to be analyzed for which a high-grade computing system is required. Otherwise, we cannot handle multi-unit systems within complex proximity.

Chapter 4 of our study extensively investigates the dynamics of standby unit degradation in multi-unit machining systems, uncovering the complex relationships among failure traits, repair capacities, and system performance indicators. Our model, which integrates degradation impacts, failure probabilities, and repair dynamics for both active and standby units, proves instrumental in analyzing and optimizing system conduct in practical operational scenarios. This research underscores the crucial necessity of accounting for standby unit deterioration, providing insights into the intricate balance between cost considerations and system reliability. By employing advanced methodologies such as queueing theory, mathematical modeling, and metaheuristic optimization techniques, our study contributes significantly to both theoretical comprehension and real-world applications in this domain.

Chapter 5 of our research rigorously examines the degradation dynamics of standby units in multi-unit machining systems, revealing the intricate connections between failure characteristics, repair capabilities, and system performance metrics. Utilizing phase-type distribution proves advantageous in modeling machine repair systems, as it enables a more precise depiction of service time distributions compared to simpler models like the exponential distribution. By employing this more accurate service time distribution, the machine repair system can better model and evaluate various performance measures, including mean time to repair and system availability.

Chapter 6 explores a Markovian model of a redundant repairable machining system, integrating real-time paradigms into the modeling process for increased practicality. To the best of our knowledge, this research represents one of the initial attempts to quantitatively assess the reliability of a machining system, incorporating controlled failed unit arrival policies and imperfect repair. The study employs an efficient numerical computation technique based on Laplace transform, eigenvalue, and linear algebra to calculate transient-state probabilities, reliability measures, and queueing characteristics. Additionally, sensitivity analysis is conducted to pinpoint critical parameters for the machining system.

In the Chpater 7 investigation, we present the reliability and sensitivity analysis using the queueing theoretic approach for an FTMS with multi-standby support. The unreliable attributes are abdicate characteristics involved in the machining system, such as failures, imperfections, delays, degradation, etc., that directly decrease the working efficiency and system availability which indirectly affect continuous implementations of the technologies. The unreliable attributes studied herein are the active/standby unit failure, switching failure, imperfect repairer, imperfect repair, imperfect fault coverage, common-cause failure, degraded failure, and reboot delay which are incorporated into the modeling to make the machining system more realistic. We have presented how reliability, $MTTF$, and failure frequency are affected with discussed unreliable attributes. The sensitivity analysis identifies the sensitive parameters involved in studied systems and reveals to the decision-makers how reliability and $MTTF$ can be improved by adjudging the sensitive parameters. The results prompt (i) regular predictive measures are to be taken to avoid the failures, (ii) strategical corrective measures need to be taken to avoid long downtime or hindrance, and (iii) preventive measures are systematically implemented to diminish failures, delays, and imperfections. The reliability prediction of standby provisioning FTMS has many real-time applications, including the electronic industry, service industry, safety systems, power plants, etc. wherein unreliable attributes are not preferable.

The proposed model of Chapter 8 encompasses essential characteristics of practical machine repair problems, including operating units, warm standby units, failure rates, system throughput, two types of repairers, and repair rates. This Markovian model focuses on the state-of-the-art analysis of a multi-unit machine repair system with spare units and two-stage repair services. Introducing a work-division policy within the repair facility offers potential advantages to industrial stakeholders, enabling cost optimization and supply-demand equilibrium. The matrix recursive method determines state probabilities, queue properties, and system performance metrics. This study's examination of queueing measures' sensitivity empowers industrial decision-makers to enhance system uptime and reduce operational costs. The numerical analysis section underscores the feasibility of leveraging optimal decision variable values to boost system availability and minimize total costs, delivering tangible benefits for industrial operations. Furthermore, analyzing queueing measures under varying parameters highlights how optimal decision variables enhance system efficiency and cost-effectiveness.

9.2 Contributions Through this Research

Below are the key discoveries from the current investigation. Based on the findings, the following recommendations are suggested:

- Increase the number of standby units to enhance system reliability.
- Prioritize reliable repair services to minimize recurring failures and improve reliability.
- Invest in robust automation to mitigate switching failures and ensure uninterrupted power supply.

- Implement preventive maintenance strategies to detect and address issues proactively.
- Emphasize regular calibration and inspection to maintain system reliability.
- Optimize parameters based on sensitivity analysis to improve system performance.
- Continuously monitor and improve system reliability through feedback mechanisms.

By implementing these recommendations, system reliability can be enhanced, minimizing downtime and ensuring uninterrupted operation.

9.3 Future Scope of the Present Research Work

- This study can be expanded to include random processes, such as failures or repair times, that follow a general distribution rather than being limited to exponential distributions. This extension would better accommodate practical systems and enhance the applicability of the findings.
- An extension could involve increasing the number of repair phases beyond two for a more realistic machine repair system. In this scenario, repairs would be conducted sequentially, with repairers from the previous phase becoming customers for the subsequent phase. This concept has been explored in the existing research literature, where a model has been studied to illustrate this approach.
- MRPs with imperfect coverage, imperfect repair, types of abandonment, repair in phases, retention, and restoration are many real-time machining variants that can be extended further.

Bibliography

[1] S. J. Aghili and H. Hajian Hoseinabadi, “Reliability evaluation of repairable systems using various fuzzy-based methods–a substation automation case study,” *International Journal of Electrical Power & Energy Systems*, vol. 85, pp. 130–142, 2017.

[2] R. Ahmad and S. Kamaruddin, “An overview of time-based and condition-based maintenance in industrial application,” *Computers & Industrial Engineering*, vol. 63, no. 1, pp. 135–149, 2012.

[3] R. Ahmadi, “Scheduling preventive maintenance for a nonperiodically inspected deteriorating system,” *International Journal of Reliability, Quality and Safety Engineering*, vol. 22, no. 06, p. 1 550 029, 2015.

[4] Q. Ahmed, S. A. Raza, and D. M. Al-Anazi, “Reliability-based fault analysis models with industrial applications: A systematic literature review,” *Quality and Reliability Engineering International*, vol. 37, no. 4, pp. 1307–1333, 2021.

[5] A. Al Hanbali, H. H. Saleh, O. G. Alsawafy, A. M. Attia, A. M. Ghaithan, and A. Mohammed, “Spare parts supply with incoming quality control and inspection errors in condition based maintenance,” *Computers & Industrial Engineering*, vol. 172, p. 108 534, 2022.

[6] A. Al-Refaie and A. Al-Hawadi, “Optimal fuzzy repairs scheduling and sequencing of failure types over multiple periods,” *Journal of Ambient Intelligence and Humanized Computing*, vol. 13, no. 1, pp. 201–217, 2022.

[7] S. Alaswad and Y. Xiang, “A review on condition-based maintenance optimization models for stochastically deteriorating system,” *Reliability Engineering & System Safety*, vol. 157, pp. 54–63, 2017.

[8] S. C. Albright, “Optimal maintenance-repair policies for the machine repair problem,” *Naval Research Logistics Quarterly*, vol. 27, no. 1, pp. 17–27, 1980.

[9] J. Artalejo and G Choudhury, “Steady state analysis of an $M/G/1$ queue with repeated attempts and two-phase service,” *Quality Technology & Quantitative Management*, vol. 1, no. 2, pp. 189–199, 2004.

[10] A. Avizienis, G. C. Gilley, F. P. Mathur, D. A. Rennels, J. A. Rohr, and D. K. Rubin, “The star (self-testing and repairing) computer: An investigation of the theory and practice of fault-tolerant computer design,” *IEEE Transactions on Computers*, vol. 100, no. 11, pp. 1312–1321, 1971.

[11] A. Azadeh and S. Abdolhossein Zadeh, “An integrated fuzzy analytic hierarchy process and fuzzy multiple-criteria decision-making simulation approach for maintenance policy selection,” *Simulation*, vol. 92, no. 1, pp. 3–18, 2016.

[12] S. Bai, X. Jia, Z. Cheng, B. Guo, Q. Zhao, and X. Zhang, “Operation optimization model for warm standby system based on nonperiodic and imperfect multiple active switching policy,” *Computers & Industrial Engineering*, vol. 167, p. 108 001, 2022.

[13] X. Bai, M. Xu, Q. Li, and L. Yu, “Trajectory-battery integrated design and its application to orbital maneuvers with electric pump-fed engines,” *Advances in Space Research*, vol. 70, no. 3, pp. 825–841, 2022.

[14] A. S. Bhandari, A. Kumar, and M. Ram, “Grey wolf optimizer and hybrid pso-gwo for reliability optimization and redundancy allocation problem,” *Quality and Reliability Engineering International*, vol. 39, no. 3, pp. 905–921, 2023.

[15] J. Bhatti and M. K. Kakkar, “Reliability analysis of cold standby parallel system possessing failure and repair rate under geometric distribution,” *Recent Advances in Computer Science and Communications (Formerly: Recent Patents on Computer Science)*, vol. 14, no. 3, pp. 962–968, 2021.

[16] A. Birolini, *Reliability engineering*. Springer, 2007, vol. 5.

[17] M. Bohlin and M. Wärja, “Maintenance optimization with duration-dependent costs,” *Annals of Operations Research*, vol. 224, pp. 1–23, 2015.

[18] A. A. Bouchentouf, M. Boualem, L. Yahiaoui, and H. Ahmad, “A multi-station unreliable machine model with working vacation policy and customers’ impatience,” *Quality Technology & Quantitative Management*, vol. 19, no. 6, pp. 766–796, 2022.

[19] G Briere and M. L. Chaudhry, “Computational analysis of single-server bulk-arrival queues $M^X/G/1$,” *Computers & Operations Research*, vol. 15, no. 3, pp. 283–292, 1988.

[20] L. C. Brito, G. A. Susto, J. N. Brito, and M. A. Duarte, "An explainable artificial intelligence approach for unsupervised fault detection and diagnosis in rotating machinery," *Mechanical Systems and Signal Processing*, vol. 163, p. 108 105, 2022.

[21] J. J. Buckley, T. Feuring, and Y. Hayashi, "Fuzzy queueing theory revisited," *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, vol. 9, no. 05, pp. 527–537, 2001.

[22] J. Buckley, "Elementary queueing theory based on possibility theory," *Fuzzy Sets and Systems*, vol. 37, no. 1, pp. 43–52, 1990.

[23] B. Bunday and R. Scraton, "The $G/M/r$ machine interference model," *European Journal of Operational Research*, vol. 4, no. 6, pp. 399–402, 1980.

[24] B. D. Bunday, H. M. Bokhari, and E. Khorram, "The efficiency of two groups of heterogeneous stations cared for by r operatives with a priority group," *Applied Mathematical Modelling*, vol. 21, no. 1, pp. 42–47, 1997.

[25] L. E. Cárdenas-Barrón and S. S. Sana, "A production-inventory model for a two-echelon supply chain when demand is dependent on sales teams initiatives," *International Journal of Production Economics*, vol. 155, pp. 249–258, 2014.

[26] M. Catelani, L. Ciani, A. Bartolini, G. Guidi, and G. Patrizi, "Standby redundancy for reliability improvement of wireless sensor network," in *2019 IEEE 5th International Forum on Research and Technology for Society and Industry (RTSI)*, IEEE, 2019, pp. 364–369.

[27] J. H. Cha, J. Mi, and W. Y. Yun, "Modelling a general standby system and evaluation of its performance," *Applied Stochastic Models in Business and Industry*, vol. 24, no. 2, pp. 159–169, 2008.

[28] P. K. Chahal and K. Kumar, "A genetic algorithm for cost optimization in queueing models of machining systems with multiple working vacation and generalized triadic policy," *Decision Analytics Journal*, vol. 10, p. 100 395, 2024.

[29] S. R. Chakravarthy and A. Agarwal, "Analysis of a machine repair problem with an unreliable server and phase type repairs and services," *Naval Research Logistics (NRL)*, vol. 50, no. 5, pp. 462–480, 2003.

[30] S. R. Chakravarthy, R. K. Meena, and A. Choudhary, “Queues with markovian arrivals, phase type services, breakdowns, and repairs,” in *Distributed Computer and Communication Networks: 23rd International Conference, DCCN 2020, Moscow, Russia, September 14–18, 2020, Revised Selected Papers 23*, Springer, 2020, pp. 259–281.

[31] P Chandrasekhar, R Natarajan, and V. S. Yadavalli, “A study on a two unit standby system with erlangian repair time,” *Asia-Pacific Journal of Operational Research*, vol. 21, no. 03, pp. 271–277, 2004.

[32] P Chandrasekhar, V. Vaidyanathan, V. Yadavalli, and S Xavier, “Statistical inference for a two unit warm standby system with dependent structure,” *Opsearch*, vol. 50, pp. 372–382, 2013.

[33] E. Y. Chang and W. E. Thompson, “Bayes analysis of reliability for complex systems,” *Operations Research*, vol. 24, no. 1, pp. 156–168, 1976.

[34] M. Chang, X. Huang, F. P. Coolen, and T. Coolen Maturi, “Reliability analysis for systems based on degradation rates and hard failure thresholds changing with degradation levels,” *Reliability Engineering & System Safety*, vol. 216, p. 108 007, 2021.

[35] B Chen, J Wan, L Shu, P Li, M Mukherjee, and B Yin, “Smart factory of industry 4.0: Key technologies, application case, and challenges,” *IEEE Access*, vol. 6, pp. 6505–6519, 2017.

[36] S. P. Chen, “A mathematical programming approach to the machine interference problem with fuzzy parameters,” *Applied Mathematics and Computation*, vol. 174, no. 1, pp. 374–387, 2006.

[37] S. P. Chen, “Non-linear programming for the optimization of machine repair problems in fuzzy environments,” *Engineering Optimization*, vol. 38, no. 7, pp. 789–799, 2006.

[38] W. L. Chen, “System reliability analysis of retrial machine repair systems with warm standbys and a single server of working breakdown and recovery policy,” *Systems Engineering*, vol. 21, no. 1, pp. 59–69, 2018.

[39] W. L. Chen and K. H. Wang, “Reliability analysis of a retrial machine repair problem with warm standbys and a single server with N-policy,” *Reliability Engineering & System Safety*, vol. 180, pp. 476–486, 2018.

[40] D. I. Cho and M. Parlar, “A survey of maintenance models for multi-unit systems,” *European Journal of Operational Research*, vol. 51, no. 1, pp. 1–23, 1991.

[41] G. Choudhury, “Steady state analysis of an $M/G/1$ queue with linear retrial policy and two phase service under bernoulli vacation schedule,” *Applied Mathematical Modelling*, vol. 32, no. 12, pp. 2480–2489, 2008.

[42] G. Choudhury and M. Deka, “A batch arrival unreliable server delaying repair queue with two phases of service and bernoulli vacation under multiple vacation policy,” *Quality Technology & Quantitative Management*, vol. 15, no. 2, pp. 157–186, 2018.

[43] G. Choudhury and L. Tadj, “An $M/G/1$ queue with two phases of service subject to the server breakdown and delayed repair,” *Applied Mathematical Modelling*, vol. 33, no. 6, pp. 2699–2709, 2009.

[44] L. Coleman Goheen, “On the optimal operating policy for the machine repair problem when failure and repair times have erlang distribution,” *Operations Research*, vol. 25, no. 3, pp. 484–492, 1977.

[45] G. Cúran and F. Kizilaslan, “Statistical inference of the stress-strength reliability and mean remaining strength of series system with cold standby redundancy at system and component levels,” *Hacettepe Journal of Mathematics and Statistics*, vol. 50, no. 6, pp. 1793–1821, 2021.

[46] P. Deora, U. Kumari, and D. Sharma, “Cost analysis and optimization of machine repair model with working vacation and feedback-policy,” *International Journal of Applied and Computational Mathematics*, vol. 7, pp. 1–14, 2021.

[47] M. Devanda, C. Shekhar, and S. Kaswan, “Fuzzified imperfect repair redundant machine repair problems,” *International Journal of System Assurance Engineering and Management*, pp. 1–20, 2023.

[48] S. Devi, H. Garg, and D. Garg, “A review of redundancy allocation problem for two decades: Bibliometrics and future directions,” *Artificial Intelligence Review*, vol. 56, no. 8, pp. 7457–7548, 2023.

[49] S. Dhibar and M. Jain, “Particle swarm optimization and $FM/FM/1/WV$ retrial queues with catastrophes: Application to cloud storage,” *The Journal of Supercomputing*, pp. 1–35, 2024.

[50] B. Dhillon and N Yang, “Reliability and availability analysis of warm standby systems with common-cause failures and human errors,” *Microelectronics Reliability*, vol. 32, no. 4, pp. 561–575, 1992.

[51] S. Distefano, “Standby redundant systems: A quantitative reliability perspective,” *International Journal of Reliability and Safety*, vol. 8, no. 1, pp. 70–93, 2014.

[52] Q. Dong, P. Liu, and X. Jia, "Reliability analysis of k-out-of-n: G Repairable systems considering common cause failure and multi-level maintenance strategy," *Proceedings of the Institution of Mechanical Engineers, Part O: Journal of Risk and Reliability*, vol. 238, no. 1, pp. 44–59, 2024.

[53] W. Dong, S. Liu, Y. Cao, and S. A. Javed, "Scheduling optimal replacement policies for a stochastically deteriorating system subject to two types of shocks," *ISA Transactions*, vol. 112, pp. 292–301, 2021.

[54] B. T. Doshi, "Queueing systems with vacations—a survey," *Queueing Systems*, vol. 1, pp. 29–66, 1986.

[55] D. Dubois, H. T. Nguyen, and H. Prade, "Possibility theory, probability and fuzzy sets misunderstandings, bridges and gaps: Misunderstandings, bridges and gaps," in *Fundamentals of Fuzzy Sets*, Springer, 2000, pp. 343–438.

[56] D. Dubois, W. Ostasiewicz, and H. Prade, "Fuzzy sets: History and basic notions," in *Fundamentals of Fuzzy Sets*, Springer, 2000, pp. 21–124.

[57] P. T. Endo, M. Rodrigues, G. E. Gonçalves, J. Kelner, D. H. Sadok, and C. Curescu, "High availability in clouds: Systematic review and research challenges," *Journal of Cloud Computing*, vol. 5, no. 1, pp. 1–15, 2016.

[58] A. K. Erlang, "The theory of probabilities and telephone conversations," *Nyt. Tidsskr. Mat. Ser. B*, vol. 20, pp. 33–39, 1909.

[59] S. Firooze, M. Rafiee, and S. M. Zenouzzadeh, "An optimization model for emergency vehicle location and relocation with consideration of unavailability time," *Scientia Iranica*, vol. 25, no. 6, pp. 3685–3699, 2018.

[60] K. N. Fleming, "Reliability model for common mode failures in redundant safety systems," General Atomics, San Diego, CA (United States), Tech. Rep., 1974.

[61] P. Fortemps and M. Roubens, "Ranking and defuzzification methods based on area compensation," *Fuzzy Sets and Systems*, vol. 82, no. 3, pp. 319–330, 1996.

[62] L. Fortuin, "The all-time requirement of spare parts for service after sales—theoretical analysis and practical results," *International Journal of Operations & Production Management*, vol. 1, no. 1, pp. 59–70, 1980.

[63] M. Gahlot, V. Singh, H. I. Ayagi, and C. Goel, "Performance assessment of repairable system in series configuration under different types of failure and repair policies using copula linguistics," *International Journal of Reliability and Safety*, vol. 12, no. 4, pp. 348–363, 2018.

[64] P. Ganguly, S. Chattopadhyay, and B. Biswas, “An adaptive algorithm for battery charge monitoring based on frequency domain analysis,” *IETE Journal of Research*, vol. 69, no. 9, pp. 6398–6408, 2023.

[65] S. Gao and J. Wang, “Reliability and availability analysis of a retrial system with mixed standbys and an unreliable repair facility,” *Reliability Engineering & System Safety*, vol. 205, p. 107 240, 2021.

[66] S. Gao, J. Wang, and J. Zhang, “Reliability analysis of a redundant series system with common cause failures and delayed vacation,” *Reliability Engineering & System Safety*, vol. 239, p. 109 467, 2023.

[67] H. Garg, “Performance analysis of complex repairable industrial systems using pso and fuzzy confidence interval based methodology,” *ISA Transactions*, vol. 52, no. 2, pp. 171–183, 2013.

[68] M. Ghaleb, S. Taghipour, M. Sharifi, and H. Zolfagharinia, “Integrated production and maintenance scheduling for a single degrading machine with deterioration-based failures,” *Computers & Industrial Engineering*, vol. 143, p. 106 432, 2020.

[69] M. Ghaleb, S. Taghipour, and H. Zolfagharinia, “Real-time integrated production-scheduling and maintenance-planning in a flexible job shop with machine deterioration and condition-based maintenance,” *Journal of Manufacturing Systems*, vol. 61, pp. 423–449, 2021.

[70] M. Ghaleb, H. Zolfagharinia, and S. Taghipour, “Real-time production scheduling in the industry-4.0 context: Addressing uncertainties in job arrivals and machine breakdowns,” *Computers & Operations Research*, vol. 123, p. 105 031, 2020.

[71] L. Goel, R. Agnihotri, and R. Gupta, “Stochastic analysis of a two unit warm standby system with fault detection and inspection,” *Microelectronics Reliability*, vol. 30, no. 1, pp. 61–65, 1990.

[72] M. Gopalan and R. S. Naidu, “Cost—benefit analysis of a one-server two-unit cold standby system subject to inspection,” *Microelectronics Reliability*, vol. 22, no. 4, pp. 699–705, 1982.

[73] M. Gopalan and S. Waghmare, “Cost-benefit analysis of single server N-unit imperfect switch system with delayed repair,” *Microelectronics Reliability*, vol. 32, no. 1-2, pp. 5–9, 1992.

[74] V. Goyal and K Murari, “Cost analysis in a two-unit standby system with a regular repairman and patience time,” *Microelectronics Reliability*, vol. 24, no. 3, pp. 453–459, 1984.

[75] D. Gross, *Fundamentals of queueing theory*. John wiley & sons, 2008.

[76] Y. Guo, M. Zhong, C. Gao, H. Wang, X. Liang, and H. Yi, "A discrete-time bayesian network approach for reliability analysis of dynamic systems with common cause failures," *Reliability Engineering & System Safety*, vol. 216, p. 108 028, 2021.

[77] R. Gupta, "Probabilistic analysis of a two-unit cold standby system with two-phase repair and preventive maintenance," *Microelectronics Reliability*, vol. 26, no. 1, pp. 13–18, 1986.

[78] R. Gupta and A. Chaudhary, "Profit analysis of a system with two-units having guarantee periods and delayed operation of standby," *Microelectronics Reliability*, vol. 34, no. 8, pp. 1387–1390, 1994.

[79] R. Gupta, M. Mahi, and V. Sharma, "A two component two unit standby system with correlated failure and repair times," *Journal of Statistics and Management Systems*, vol. 11, no. 1, pp. 77–90, 2008.

[80] R. Gupta and D. Agarwal, "Cost analysis of N-policy vacation machine repair problem with optional repair," *International Journal of Mathematics in Operational Research*, vol. 19, no. 3, pp. 354–374, 2021.

[81] S. Gupta, N. Jaiswal, and L. Goel, "Reliability analysis of a two-unit cold standby redundant system with two operating modes," *Microelectronics Reliability*, vol. 22, no. 4, pp. 747–758, 1982.

[82] S. M. Gupta, "Interrelationship between controlling arrival and service in queueing systems," *Computers & operations research*, vol. 22, no. 10, pp. 1005–1014, 1995.

[83] S. M. Gupta, "Machine interference problem with warm spares, server vacations and exhaustive service," *Performance Evaluation*, vol. 29, no. 3, pp. 195–211, 1997.

[84] G. Hanukov, T. Avinadav, T. Chernonog, U. Spiegel, and U. Yechiali, "Improving efficiency in queueing systems by utilizing the server's idle time," 2016.

[85] G. Hanukov, T. Avinadav, T. Chernonog, U. Spiegel, and U. Yechiali, "A queueing system with decomposed service and inventoried preliminary services," *Applied Mathematical Modelling*, vol. 47, pp. 276–293, 2017.

[86] G. Hanukov, T. Avinadav, T. Chernonog, and U. Yechiali, "A service system with perishable products where customers are either fastidious or strategic," *International Journal of Production Economics*, vol. 228, p. 107 696, 2020.

[87] G. Hanukov, T. Avinadav, T. Chernonog, and U. Yechiali, “A multi-server system with inventory of preliminary services and stock-dependent demand,” *International Journal of Production Research*, vol. 59, no. 14, pp. 4384–4402, 2021.

[88] L. Haque and M. J. Armstrong, “A survey of the machine interference problem,” *European Journal of Operational Research*, vol. 179, no. 2, pp. 469–482, 2007.

[89] N. Hastings, “The repair limit replacement method,” *Journal of the Operational Research Society*, vol. 20, pp. 337–349, 1969.

[90] G. He, W. Wu, and Y. Zhang, “Performance analysis of machine repair system with single working vacation,” *Communications in Statistics-Theory and Methods*, vol. 48, no. 22, pp. 5602–5620, 2019.

[91] T. J. Hsieh, “Performance indicator-based multi-objective reliability optimization for multi-type production systems with heterogeneous machines,” *Reliability Engineering & System Safety*, vol. 230, p. 108 970, 2023.

[92] Y. L. Hsu, J. C. Ke, and S. L. Lee, “On a redundant repairable system with switching failure: Bayesian approach,” *Journal of Statistical Computation and Simulation*, vol. 78, no. 12, pp. 1163–1180, 2008.

[93] Y. L. Hsu, J. C. Ke, and T. H. Liu, “Standby system with general repair, reboot delay, switching failure and unreliable repair facility a statistical standpoint,” *Mathematics and Computers in Simulation*, vol. 81, no. 11, pp. 2400–2413, 2011.

[94] Y. L. Hsu, J. C. Ke, T. H. Liu, and C. H. Wu, “Modeling of multi-server repair problem with switching failure and reboot delay and related profit analysis,” *Computers & Industrial Engineering*, vol. 69, pp. 21–28, 2014.

[95] Y. L. Hsu, S. L. Lee, and J. C. Ke, “A repairable system with imperfect coverage and reboot: Bayesian and asymptotic estimation,” *Mathematics and Computers in Simulation*, vol. 79, no. 7, pp. 2227–2239, 2009.

[96] H. I. Huang and J. C. Ke, “Comparative analysis on a redundant repairable system with different configurations,” *Engineering Computations*, vol. 26, no. 4, pp. 422–439, 2009.

[97] H. I. Huang, C. H. Lin, and J. C. Ke, “Parametric nonlinear programming approach for a repairable system with switching failure and fuzzy parameters,” *Applied Mathematics and Computation*, vol. 183, no. 1, pp. 508–517, 2006.

[98] W. Huang, J. Loman, and T. Song, "A reliability model of a warm standby configuration with two identical sets of units," *Reliability Engineering & System Safety*, vol. 133, pp. 237–245, 2015.

[99] M. Jain, "Reliability prediction of repairable redundant system with imperfect switching and repair," *Arabian Journal for Science and Engineering*, vol. 41, pp. 3717–3725, 2016.

[100] M. Jain and R. Gupta, "N-policy for redundant repairable system with multiple types of warm standbys with switching failure and vacation," *International Journal of Mathematics in Operational Research*, vol. 13, no. 4, pp. 419–449, 2018.

[101] M. Jain, P. Kumar, and S. S. Sanga, "Fuzzy markovian modeling of machining system with imperfect coverage, spare provisioning and reboot," *Journal of Ambient Intelligence and Humanized Computing*, vol. 12, no. 7, pp. 7935–7947, 2021.

[102] M. Jain, S. Maheshwari, *et al.*, "N-policy for a machine repair system with spares and reneging," *Applied Mathematical Modelling*, vol. 28, no. 6, pp. 513–531, 2004.

[103] M. Jain and R. K. Meena, "Fault tolerant system with imperfect coverage, reboot and server vacation," *Journal of Industrial Engineering International*, vol. 13, pp. 171–180, 2017.

[104] M. Jain and R. K. Meena, "Markovian analysis of unreliable multi-components redundant fault tolerant system with working vacation and F-policy," *Cogent Mathematics*, vol. 4, no. 1, p. 1 306 961, 2017.

[105] M. Jain and R. K. Meena, "Vacation model for markov machine repair problem with two heterogeneous unreliable servers and threshold recovery," *Journal of Industrial Engineering International*, vol. 14, pp. 143–152, 2018.

[106] M. Jain, R. K. Meena, and P. Kumar, "Maintainability of redundant machining system with vacation, imperfect recovery and reboot delay," *Arabian Journal for Science and Engineering*, vol. 45, pp. 2145–2161, 2020.

[107] M. Jain and S. Rani, "Availability analysis for repairable system with warm standby, switching failure and reboot delay," *International Journal of Mathematics in Operational Research*, vol. 5, no. 1, pp. 19–39, 2013.

[108] M. Jain and S. S. Sanga, "Fuzzy cost optimization and admission control for machine interference problem with general retrial," *Journal of Testing and Evaluation*, vol. 48, no. 6, pp. 4488–4502, 2020.

[109] M. Jain, G. Sharma, and V. Rani, "Performance prediction of machine repair problem with spares and two modes of failure," *International Journal of Operational Research*, vol. 22, no. 2, pp. 167–193, 2015.

[110] M. Jain, G. Sharma, and R. Sharma, "Optimal control of (N,F) policy for unreliable server queue with multi-optional phase repair and start-up," *International Journal of Mathematics in Operational Research*, vol. 4, no. 2, pp. 152–174, 2012.

[111] M. Jain, G. Sharma, and C. Shekhar, "Processor-shared service systems with queue-dependent processors," *Computers & Operations Research*, vol. 32, no. 3, pp. 629–645, 2005.

[112] M. Jain, C. Shekhar, and R. K. Meena, "Admission control policy of maintenance for unreliable server machining system with working vacation," *Arabian Journal for Science and Engineering*, vol. 42, pp. 2993–3005, 2017.

[113] M. Jain, C. Shekhar, and R. K. Meena, "Performance analysis and control F-policy for fault-tolerant system with working vacation," *Opsearch*, vol. 56, pp. 409–431, 2019.

[114] M. Jain, C. Shekhar, and S. Shukla, "Machine repair problem with an unreliable server and controlled arrival of failed machines," *Opsearch*, vol. 51, pp. 416–433, 2014.

[115] M. Jain, C. Shekhar, and S. Shukla, "Markov model for switching failure of warm spares in machine repair system," *Journal of Reliability and Statistical Studies*, vol. 7, no. 1, pp. 57–68, 2014.

[116] M. Jain, C. Shekhar, and S. Shukla, "A time-shared machine repair problem with mixed spares under *N*-policy," *Journal of Industrial Engineering International*, vol. 12, pp. 145–157, 2016.

[117] M. Jain, C. Shekhar, and S. Shukla, "Queueing analysis of machine repair problem with controlled rates and working vacation under F-policy," *Proceedings of the National Academy of Sciences, India Section A: Physical Sciences*, vol. 86, pp. 21–31, 2016.

[118] M. Jain and S. Upadhyaya, "Threshold N-policy for degraded machining system with multiple type spares and multiple vacations," *Quality Technology & Quantitative Management*, vol. 6, no. 2, pp. 185–203, 2009.

[119] R Jamshidi and M. M. S. Esfahani, "Maintenance policy determination for a complex system consisting of series and cold standby system with multiple levels of maintenance action," *The International Journal of Advanced Manufacturing Technology*, vol. 78, pp. 1337–1346, 2015.

[120] K. Jau Chuan, L. Ssu Lang, and K. Ming Yang, "Two-unit redundant system with detection delay and imperfect coverage: Confidence interval estimation," *Quality Technology & Quantitative Management*, vol. 8, no. 1, pp. 1–14, 2011.

[121] H. Jia, R. Peng, L. Yang, T. Wu, D. Liu, and Y. Li, "Reliability evaluation of demand-based warm standby systems with capacity storage," *Reliability Engineering & System Safety*, vol. 218, p. 108 132, 2022.

[122] X. Jia, H. Chen, Z. Cheng, and B. Guo, "A comparison between two switching policies for two-unit standby system," *Reliability engineering & system safety*, vol. 148, pp. 109–118, 2016.

[123] X. Jia and B. Guo, "Analysis of non-repairable cold-standby systems in bayes theory," *Journal of Statistical Computation and Simulation*, vol. 86, no. 11, pp. 2089–2112, 2016.

[124] X. Jia, D. Y. Lv, Y. Hu, J. B. Wang, Z. Wang, and E. Wang, "Slack due-window assignment scheduling problem with deterioration effects and a deteriorating maintenance activity," *Asia-Pacific Journal of Operational Research*, vol. 39, no. 06, p. 2 250 005, 2022. DOI: 10.1142/S0217595922500051.

[125] Z. Kamaitis, "Modelling of corrosion protection as standby system for coated reinforced concrete structures," *Journal of Civil Engineering and Management*, vol. 15, no. 4, pp. 387–394, 2009.

[126] A. Kamal, M. Mahmoud, R. EL Sagheer, and M. Moaz, "Reliability and cost-benefit efficient in a two-dissimilar-unit with warm unit standby case subject to arbitrary repair and replacement," *Inf. Sci. Lett*, vol. 12, no. 3, pp. 1117–1129, 2023.

[127] J. Kang, L. Hu, R. Peng, Y. Li, and R. Tian, "Availability and cost-benefit evaluation for a repairable retrial system with warm standbys and priority," *Statistical Theory and Related Fields*, vol. 7, no. 2, pp. 164–175, 2023.

[128] F. Kayedpour, M. Amiri, M. Rafizadeh, A. S. Nia, and M. Sharifi, "A markov chain-based genetic algorithm for solving a redundancy allocation problem for a system with repairable warm standby components," *Proceedings of the Institution of Mechanical Engineers, Part O: Journal of Risk and Reliability*, p. 1748006X231164848, 2023.

[129] J. C. Ke and T. H. Liu, "A repairable system with imperfect coverage and reboot," *Applied Mathematics and Computation*, vol. 246, pp. 148–158, 2014.

[130] J. C. Ke, H. I. Huang, and C. H. Lin, "Fuzzy analysis for steady-state availability: A mathematical programming approach," *Engineering Optimization*, vol. 38, no. 8, pp. 909–921, 2006.

[131] J. C. Ke, S. L. Lee, and Y. L. Hsu, "Bayesian analysis for a redundant repairable system with imperfect coverage," *Communications in Statistics-Simulation and Computation®*, vol. 37, no. 5, pp. 993–1004, 2008.

[132] J. C. Ke, S. L. Lee, and Y. L. Hsu, "On a repairable system with detection, imperfect coverage and reboot: Bayesian approach," *Simulation Modelling Practice and Theory*, vol. 16, no. 3, pp. 353–367, 2008.

[133] J. C. Ke, T. H. Liu, Y. L. Hsu, and H. T. Ku, "Analysis of machine-repairing systems with server breakdown and delayed repairs," *International Journal of Industrial and Systems Engineering*, vol. 31, no. 4, pp. 517–534, 2019.

[134] J. C. Ke, T. H. Liu, and D. Y. Yang, "Machine repairing systems with standby switching failure," *Computers & Industrial Engineering*, vol. 99, pp. 223–228, 2016.

[135] J. C. Ke, T. H. Liu, and D. Y. Yang, "Modeling of machine interference problem with unreliable repairman and standbys imperfect switchover," *Reliability Engineering & System Safety*, vol. 174, pp. 12–18, 2018.

[136] J. C. Ke, Z. L. Su, K. H. Wang, and Y. L. Hsu, "Simulation inferences for an availability system with general repair distribution and imperfect fault coverage," *Simulation Modelling Practice and Theory*, vol. 18, no. 3, pp. 338–347, 2010.

[137] J. C. Ke and K. H. Wang, "Cost analysis of the $M/M/R$ machine repair problem with balking, reneging, and server breakdowns," *Journal of the Operational Research Society*, vol. 50, no. 3, pp. 275–282, 1999.

[138] J.-C. Ke and K.-H. Wang, "The reliability analysis of balking and reneging in a repairable system with warm standbys," *Quality and Reliability Engineering International*, vol. 18, no. 6, pp. 467–478, 2002.

[139] J. C. Ke and K. H. Wang, "Vacation policies for machine repair problem with two type spares," *Applied Mathematical Modelling*, vol. 31, no. 5, pp. 880–894, 2007.

[140] J. B. Ke, W. C. Lee, and J. C. Ke, "Reliability-based measure for a system with standbys subjected to switching failures," *Engineering Computations*, vol. 25, no. 7, pp. 694–706, 2008.

[141] D. G. Kendall, "Stochastic processes occurring in the theory of queues and their analysis by the method of the imbedded markov chain," *The Annals of Mathematical Statistics*, pp. 338–354, 1953.

[142] J. Kennedy and R. Eberhart, “Particle swarm optimization,” in *Proceedings of ICNN’95-international conference on neural networks*, ieee, vol. 4, 1995, pp. 1942–1948.

[143] T. Ketema, S. Demie, and M. T. Belachew, “Controllable $M/M/2$ machine repair problem with multiple working vacations and triadic $(0,Q,N,M)$ policy,” *International Journal of Management Science and Engineering Management*, vol. 16, no. 3, pp. 184–196, 2021.

[144] A. Khati and S. Singh, “Interval-valued availability of a standby system with imperfect switch incorporating uncertainty & multiple failure modes,” *Quality Technology & Quantitative Management*, vol. 21, no. 1, pp. 72–87, 2024.

[145] H. Kim and P. Kim, “Reliability models for a nonrepairable system with heterogeneous components having a phase-type time-to-failure distribution,” *Reliability Engineering & System Safety*, vol. 159, pp. 37–46, 2017.

[146] J. Kim, J. Jeong, H. Kim, and J.-S. Park, “Cloud-based battery replacement scheme for smart electric bus system,” *IETE Journal of Research*, vol. 66, no. 3, pp. 341–352, 2020.

[147] S. H. Kim and S. I. Sung, “Modeling and analysis of the catastrophic failure and degradation data,” *Microelectronics Reliability*, vol. 114, p. 113 764, 2020.

[148] R. Kishan and D. Jain, “Classical and bayesian analysis of reliability characteristics of a two-unit parallel system with weibull failure and repair laws,” *International Journal of System Assurance Engineering and Management*, vol. 5, pp. 252–261, 2014.

[149] L. Kleinrock, *Queueing systems: theory*. John Wiley, 1975.

[150] E. Koenigsberg, “Finite queues and cyclic queues,” *Operations Research*, vol. 8, no. 2, pp. 246–253, 1960.

[151] N. Kryvinska, “Intelligent network analysis by closed queuing models,” *Telecommunication Systems*, vol. 27, pp. 85–98, 2004.

[152] A. Kumar, M. Ram, S. Pant, and A. Kumar, “Industrial system performance under multistate failures with standby mode,” *Modeling and Simulation in Industrial Engineering*, pp. 85–100, 2018.

[153] A. Kumar, S. Pant, and S. Singh, “Availability and cost analysis of an engineering system involving subsystems in series configuration,” *International Journal of Quality & Reliability Management*, vol. 34, no. 6, pp. 879–894, 2017.

[154] A. Kumar, M. Saini, and S. Malik, "Performance analysis of a computer system with imperfect fault detection of hardware," *Procedia Computer Science*, vol. 45, pp. 602–610, 2015.

[155] A. Kumar and M. Agarwal, "Behaviour of a two-unit standby redundant system with imperfect switching and delayed repair," *Microelectronics Reliability*, vol. 20, no. 3, pp. 315–321, 1980.

[156] K. Kumar, M. Jain, and C. Shekhar, "Machine repair system with F-policy, two unreliable servers, and warm standbys," *Journal of Testing and Evaluation*, vol. 47, no. 1, pp. 361–383, 2019.

[157] K. Kumar, M. Jain, and C. Shekhar, "Machine repair system with threshold recovery policy, unreliable servers and phase repairs," *Quality Technology & Quantitative Management*, pp. 1–24, 2023.

[158] P. Kumar and M. Jain, "Reliability analysis of a multi-component machining system with service interruption, imperfect coverage, and reboot," *Reliability Engineering & System Safety*, vol. 202, p. 106 991, 2020.

[159] P. Kumar, M. Jain, and R. K. Meena, "Optimal control of fault tolerant machining system with reboot and recovery in fuzzy environment using harmony search algorithm," *ISA Transactions*, vol. 119, pp. 52–64, 2022.

[160] P. Kumar, M. Jain, and R. K. Meena, "Transient analysis and reliability modeling of fault-tolerant system operating under admission control policy with double retrial features and working vacation," *ISA Transactions*, vol. 134, pp. 183–199, 2023.

[161] S. Kumar and R. Gupta, "Reliability analysis of N-policy vacation-based ftc system subject to standby switching failures," *Operations Research and Decisions*, vol. 33, 2023.

[162] C. C. Kuo and J. C. Ke, "Comparative analysis of standby systems with unreliable server and switching failure," *Reliability Engineering & System Safety*, vol. 145, pp. 74–82, 2016.

[163] C. C. Kuo and J. C. Ke, "Availability and comparison of spare systems with a repairable server," *International Journal of Reliability, Quality and Safety Engineering*, vol. 26, no. 02, p. 1 950 008, 2019.

[164] C. C. Kuo and J. C. Ke, "Modeling and comparison of the series systems with imperfect coverage for an unreliable server," *Soft Computing*, vol. 23, no. 6, pp. 2073–2082, 2019.

[165] Y. C. Lai Min Tsai & Chen, "Optimal periodic replacement policy for a two-unit system with failure rate interaction," *The International Journal of Advanced Manufacturing Technology*, vol. 29, pp. 367–371, 2006.

[166] Y. Lam, "A maintenance model for two-unit redundant system," *Microelectronics Reliability*, vol. 37, no. 3, pp. 497–504, 1997.

[167] H. W. Lee, S. H. Yoon, and S. S. Lee, "Continuous approximations of the machine repair system," *Applied Mathematical Modelling*, vol. 19, no. 9, pp. 550–559, 1995.

[168] S. S. Lee, H. W. Lee, S. H. Yoon, and K. C. Chae, "Batch arrival queue with *N*-policy and single vacation," *Computers & Operations Research*, vol. 22, no. 2, pp. 173–189, 1995.

[169] S. L. Lee, J. C. Ke, and Y. L. Hsu, "Bayesian assessing for a repairable system with standby imperfect switching and reboot delay," *International Journal of Systems Science*, vol. 40, no. 11, pp. 1149–1159, 2009.

[170] G. Levitin and M. Xie, "Performance distribution of a fault-tolerant system in the presence of failure correlation," *IIE Transactions*, vol. 38, no. 6, pp. 499–509, 2006.

[171] E. E. Lewis, *Introduction to reliability engineering*. John Wiley & Sons, 1987.

[172] J. Li, L. Hu, Y. Wang, and J. Kang, "Reliability analysis and optimization design of a repairable k-out-of-n retrial system with two failure modes and preventive maintenance," *Communications in Statistics-Theory and Methods*, pp. 1–29, 2023.

[173] R. Li, Q. Li, N. Huang, and R. Kang, "A simplified reliability analysis method for cloud computing systems considering common-cause failures," *Proceedings of the Institution of Mechanical Engineers, Part O: Journal of Risk and Reliability*, vol. 231, no. 3, pp. 324–333, 2017.

[174] W. Li, D. Shi, and X. Chao, "Reliability analysis of $M/G/1$ queueing systems with server breakdowns and vacations," *Journal of Applied Probability*, vol. 34, no. 2, pp. 546–555, 1997.

[175] Y. F. Li, H. Z. Huang, J. Mi, W. Peng, and X. Han, "Reliability analysis of multi-state systems with common cause failures based on bayesian network and fuzzy probability," *Annals of Operations Research*, pp. 1–15, 2022.

[176] Y. Li, P. B. Luh, and X. Guan, "Fuzzy optimization-based scheduling of identical machines with possible breakdown," in *Proceedings of the 1994 IEEE International Conference on Robotics and Automation*, IEEE, 1994, pp. 3447–3452.

[177] C. D. Liou, "Optimization analysis of the machine repair problem with multiple vacations and working breakdowns," *Journal of Industrial and Management Optimization*, vol. 11, no. 1, pp. 83–104, 2015.

[178] B. Liu, Y. Wen, Q. Qiu, H. Shi, and J. Chen, "Reliability analysis for multi-state systems under *K*-mixed redundancy strategy considering switching failure," *Reliability Engineering & System Safety*, vol. 228, p. 108 814, 2022.

[179] B. Liu, X. Zhao, Y. Liu, and P. Do, "Maintenance optimisation for systems with multi-dimensional degradation and imperfect inspections," *International Journal of Production Research*, vol. 59, no. 24, pp. 7537–7559, 2021.

[180] F. Liu, Z. Sun, H. Bian, M. Ding, and X. Meng, "Identification and classification of the flow pattern of hydrogen-air-steam mixture gas under steam condensation," *International Journal of Thermal Sciences*, vol. 183, p. 107 854, 2023.

[181] J. N. Liu and K. Sin, "Fuzzy neural networks for machine maintenance in mass transit railway system," *IEEE Transactions on Neural Networks*, vol. 8, no. 4, pp. 932–941, 1997.

[182] R. Liu, B. Yang, E. Zio, and X. Chen, "Artificial intelligence for fault diagnosis of rotating machinery: A review," *Mechanical Systems and Signal Processing*, vol. 108, pp. 33–47, 2018.

[183] T. H. Liu, Y. L. Huang, Y. B. Lin, and F. M. Chang, "Cost-benefit analysis of a standby retrial system with an unreliable server and switching failure," *Computation*, vol. 10, no. 4, p. 48, 2022.

[184] T. H. Liu, J. C. Ke, M. F. Yen, and Y. L. Hsu, "Statistical inference for the availability of repairable system with general repair, reboot delay, and switching railure," *Communications in Statistics-Simulation and Computation*, vol. 46, no. 1, pp. 579–592, 2017.

[185] T. Liu, F. M. Chang, J. C. Ke, and S. H. Sheu, "Optimization of retrial queue with unreliable servers subject to imperfect coverage and reboot delay," *Quality Technology & Quantitative Management*, vol. 19, no. 4, pp. 428–453, 2022.

[186] Y. Liu, H. Zhang, and R. Y. Zhong, "A survey on internet of things-enabled real-time machine management system in new zealand," *Procedia CIRP*, vol. 72, pp. 868–873, 2018.

[187] Y. Liu, H. Z. Huang, and G. Levitin, "Reliability and performance assessment for fuzzy multi-state elements," *Proceedings of the Institution of Mechanical Engineers, Part O: Journal of Risk and Reliability*, vol. 222, no. 4, pp. 675–686, 2008.

[188] L. Luo, S. Meng, X. Qiu, and Y. Dai, “Improving failure tolerance in large-scale cloud computing systems,” *IEEE Transactions on Reliability*, vol. 68, no. 2, pp. 620–632, 2019.

[189] X. Ma, B. Liu, L. Yang, R. Peng, and X. Zhang, “Reliability analysis and condition-based maintenance optimization for a warm standby cooling system,” *Reliability Engineering & System Safety*, vol. 193, p. 106 588, 2020.

[190] C Mack, “The efficiency of N machines uni-directionally patrolled by one operative when walking time is constant and repair times are variable,” *Journal of the Royal Statistical Society Series B: Statistical Methodology*, vol. 19, no. 1, pp. 173–178, 1957.

[191] K. C. Madan, “On a single server queue with two stage general heterogeneous service and binomial schedule server vacations,” *The Egyptian Statistical Journal*, vol. 44, no. 1, pp. 39–55, 2000.

[192] K. C. Madan, “On a single server queue with two-stage heterogeneous service and deterministic server vacations,” *International Journal of Systems Science*, vol. 32, no. 7, pp. 837–844, 2001.

[193] R. Malhotra, F. S. Alamri, and H. A. E. W. Khalifa, “Novel analysis between two-unit hot and cold standby redundant systems with varied demand,” *Symmetry*, vol. 15, no. 6, p. 1220, 2023.

[194] R. Malhotra and G. Taneja, “Stochastic analysis of a two-unit cold standby system wherein both units may become operative depending upon the demand,” *Journal of Quality and Reliability Engineering*, vol. 2014, 2014.

[195] S. Malik and V. Munday, “Stochastic modelling of a computer system with hardware redundancy,” *International Journal of Computer Applications*, vol. 89, no. 7, 2014.

[196] K. T. Marshall, “Some inequalities for single server queues.,” Ph.D. dissertation, Univ. of California, Berkeley, 1966.

[197] J. de Mast, S. H. Steiner, W. P. Nuijten, and D. Kapitan, “Analytical problem solving based on causal, correlational and deductive models,” *The American Statistician*, vol. 77, no. 1, pp. 51–61, 2023.

[198] M. F. McGranaghan, D. R. Mueller, and M. J. Samotyj, “Voltage sags in industrial systems,” *IEEE Transactions on Industry Applications*, vol. 29, no. 2, pp. 397–403, 1993.

[199] R. K. Meena, M. Jain, A. Assad, R. Sethi, and D. Garg, "Performance and cost comparative analysis for $M/G/1$ repairable machining system with N-policy vacation," *Mathematics and Computers in Simulation*, vol. 200, pp. 315–328, 2022.

[200] R. K. Meena, M. Jain, S. S. Sanga, and A. Assad, "Fuzzy modeling and harmony search optimization for machining system with general repair, standby support and vacation," *Applied Mathematics and Computation*, vol. 361, pp. 858–873, 2019.

[201] Y. Meng, X. Liu, and M. Zhou, "A novel model to determine the optimal number of servers in finite input source fuzzy queueing system," in *2017 36th Chinese Control Conference (CCC)*, IEEE, 2017, pp. 4175–4180.

[202] H Merlyn Margaret and P Thirunavukarasu, "Analysis of the fuzzy machine repair problem with spares using parametric nonlinear programming," *Journal of Interdisciplinary Mathematics*, vol. 25, no. 1, pp. 63–77, 2022.

[203] J. Mi, N. Lu, Y.-F. Li, H.-Z. Huang, and L. Bai, "An evidential network-based hierarchical method for system reliability analysis with common cause failures and mixed uncertainties," *Reliability Engineering & System Safety*, vol. 220, p. 108 295, 2022.

[204] G. Mokaddis and M. Tawfek, "Stochastic analysis of a two-dissimilar unit warm standby redundant system with two types of repair facilities," *Microelectronics Reliability*, vol. 35, no. 12, pp. 1467–1472, 1995.

[205] R. Montoro-Cazorla Delia & Pérez-Ocón, "A deteriorating two-system with two repair modes and sojourn times phase-type distributed," *Reliability Engineering & System Safety*, vol. 91, no. 1, pp. 1–9, 2006.

[206] I. Moon and S. Choi, "The distribution free continuous review inventory system with a service level constraint," *Computers & Industrial Engineering*, vol. 27, no. 1-4, pp. 209–212, 1994.

[207] P. M. Morse, "Stochastic properties of waiting lines," *Journal of the Operations Research Society of America*, vol. 3, no. 3, pp. 255–261, 1955.

[208] A. Mosleh, "Common cause failures: An analysis methodology and examples," *Reliability Engineering & System Safety*, vol. 34, no. 3, pp. 249–292, 1991.

[209] T. Nakagawa, "Two-unit redundant models," *Stochastic Models in Reliability and Maintenance*, pp. 165–191, 2002.

[210] T. Nakagawa and S. Osaki, “A summary of optimum preventive maintenance policies for a two-unit standby redundant system,” *Zeitschrift fúr Operations Research*, vol. 20, pp. 171–187, 1976.

[211] T. Nakagawa and S. Osaki, “Reliability analysis of a one-unit system with unrepairable spare units and its optimization applications,” *Journal of the Operational Research Society*, vol. 27, no. 1, pp. 101–110, 1976.

[212] P Naor, “On machine interference,” *Journal of the Royal Statistical Society Series B: Statistical Methodology*, vol. 18, no. 2, pp. 280–287, 1956.

[213] R Natarajan, “A reliability problem with spares and multiple repair facilities,” *Operations Research*, vol. 16, no. 5, pp. 1041–1057, 1968.

[214] M. F. Neuts, “A versatile markovian point process,” *Journal of Applied Probability*, vol. 16, no. 4, pp. 764–779, 1979.

[215] M. F. Neuts, “Matrix-analytic methods in queuing theory,” *European Journal of Operational Research*, vol. 15, no. 1, pp. 2–12, 1984.

[216] M. F. Neuts, *Matrix-geometric solutions in stochastic models: an algorithmic approach*. Courier Corporation, 1994.

[217] M. F. Neuts and K. S. Meier, “On the use of phase type distributions in reliability modelling of systems with two components,” *Operations-Research-Spektrum*, vol. 2, no. 4, pp. 227–234, 1981.

[218] A. N. OĆonnor, *A general cause based methodology for analysis of dependent failures in system risk and reliability assessments*. University of Maryland, College Park, 2013.

[219] S. Osaki, “Reliability analysis of a two-unit standby-redundant system with preventive maintenance,” *IEEE Transactions on Reliability*, vol. 21, no. 1, pp. 24–29, 1972.

[220] S. Osaki and T. Nakagawa, “On a two-unit standby redundant system with standby failure,” *Operations Research*, vol. 19, no. 2, pp. 510–523, 1971.

[221] S. Ouaret, J. P. Kenné, and A. Gharbi, “Production and replacement planning of a deteriorating remanufacturing system in a closed-loop configuration,” *Journal of Manufacturing Systems*, vol. 53, pp. 234–248, 2019.

[222] C. Palm, “Intensitatsschwankungen im fernsprechverker,” *Ericsson Technics*, 1943.

[223] A. P. Panta, R. P. Ghimire, D. Panthi, S. R. Pant, *et al.*, “Optimization of $M/M/s/N$ queueing model with reneging in a fuzzy environment,” *American Journal of Operations Research*, vol. 11, no. 03, p. 121, 2021.

[224] R. Patawa, P. S. Pundir, A. K. Sigh, and A. Singh, “Some inferences on reliability measures of two-non-identical units cold standby system waiting for repair,” *International Journal of System Assurance Engineering and Management*, pp. 1–17, 2022.

[225] J. Patterson and A. Korzeniowski, “$M/M/1$ Model with unreliable service,” *International Journal of Statistics and Probability*, vol. 7, no. 1, pp. 125–136, 2018.

[226] J. Patterson and A. Korzeniowski, “$M/M/1$ Model with unreliable service and a working vacation,” *International Journal of Statistics and Probability*, vol. 8, no. 2, pp. 1–10, 2019.

[227] J. Patterson and A. Korzeniowski, “Decomposition of $M/M/1$ with unreliable service and a working vacation,” *International Journal of Statistics and Probability*, vol. 9, no. 1, pp. 1–63, 2020.

[228] T. E. Phipps Jr, “Machine repair as a priority waiting-line problem,” *Operations Research*, vol. 4, no. 1, pp. 76–85, 1956.

[229] W. P. Pierskalla and J. A. Voelker, “A survey of maintenance models: The control and surveillance of deteriorating systems,” *Naval Research Logistics Quarterly*, vol. 23, no. 3, pp. 353–388, 1976.

[230] J. Pramod and A. M. Agogino, “Stochastic sensitivity analysis using fuzzy influence diagrams,” in *Machine Intelligence and Pattern Recognition*, vol. 9, Elsevier, 1990, pp. 79–92.

[231] X. Qi, F. Yu, Z. Meng, Z. Sun, N. Zhang, and Z. Guo, “Preliminary design of the suppressive containment system based on hpr1000,” *Nuclear Engineering and Design*, vol. 415, p. 112 743, 2023.

[232] S. Qiu and X. Ming, “Explicit and implicit valuation-based system methods for the risk assessment of systems subject to common-cause failures under uncertainty,” *Knowledge-Based Systems*, vol. 214, p. 106 665, 2021.

[233] M. Ram and M. Manglik, “An analysis to multi-state manufacturing system with common cause failure and waiting repair strategy,” *Cogent Engineering*, vol. 3, no. 1, p. 1 266 185, 2016.

[234] M. Ram, S. B. Singh, and V. V. Singh, “Stochastic analysis of a standby system with waiting repair strategy,” *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 43, no. 3, pp. 698–707, 2013.

[235] S. Rani, M. Jain, and R. K. Meena, “Queueing modeling and optimization of a fault-tolerant system with reboot, recovery, and vacationing server operating under admission control policy,” *Mathematics and Computers in Simulation*, vol. 209, pp. 408–425, 2023.

[236] R. V. Rao, V. J. Savsani, and D. Vakharia, “Teaching–learning-based optimization: A novel method for constrained mechanical design optimization problems,” *Computer-Aided Design*, vol. 43, no. 3, pp. 303–315, 2011.

[237] R. Rao, V. Savsani, and D. Vakharia, “Teaching–learning-based optimization: An optimization method for continuous non-linear large scale problems,” *Information Sciences*, vol. 183, no. 1, pp. 1–15, 2012.

[238] G. Rastorguev and F. Elerian, “Spare parts management for the repair of machine tools using fuzzy logic approach.,” *MEJ-Mansoura Engineering Journal*, vol. 39, no. 2, pp. 14–21, 2020.

[239] S. Rizwan, V. Khurana, and G. Taneja, “Reliability analysis of a hot standby industrial system,” *International Journal of Modelling and Simulation*, vol. 30, no. 3, pp. 315–322, 2010.

[240] L. Robert and W Ritha, “Machine interference problem with fuzzy environment,” *Int. J. Contemp. Math. Sciences*, vol. 5, no. 39, pp. 1905–1912, 2010.

[241] J. E. Ruiz-Castro and M. Dawabsha, “A multi-state warm standby system with preventive maintenance, loss of units and an indeterminate multiple number of repairpersons,” *Computers & Industrial Engineering*, vol. 142, p. 106 348, 2020.

[242] R S Pundhir, G. Sharma, and M Jain, “Some perspectives of machine repair problems,” *International Journal of Engineering*, vol. 23, no. 3, pp. 253–268, 2010.

[243] M. Sadeghi and E. Roghanian, “Reliability analysis of a warm standby repairable system with two cases of imperfect switching mechanism,” *Scientia Iranica*, vol. 24, no. 2, pp. 808–822, 2017.

[244] M. Saini, A. Kumar, and K. Devi, “Stochastic investigation of two-unit redundant system with provision of different repair schemes,” in *Innovations in Computational Intelligence and Computer Vision: Proceedings of ICICV 2020*, Springer, 2021, pp. 3–9.

[245] M. Saini, J. Yadav, and A. Kumar, “Reliability, availability and maintainability analysis of hot standby database systems,” *International Journal of System Assurance Engineering and Management*, vol. 13, no. 5, pp. 2458–2471, 2022.

[246] A. Salmasnia and A. Shabani, "Opportunistic maintenance modeling for series production systems based on bottleneck by considering energy consumption and market demand," *Journal of Industrial and Production Engineering*, vol. 40, no. 6, pp. 506–518, 2023.

[247] S. S. Sanga and K. S. Antala, "Cost optimization and machine learning based prediction of waiting time for fault-tolerance machining system with general vacation and F-policy," *Applied Soft Computing*, p. 111 404, 2024.

[248] L. G. B. S. Sarje SH, "Cbm policy for an online continuously monitored deteriorating system with random change of mode," *Journal of The Institution of Engineers (India): Series C*, vol. 93, pp. 27–32, 2012.

[249] D. Sarkar and S. K. Gunturi, "Wind turbine blade structural state evaluation by hybrid object detector relying on deep learning models," *Journal of Ambient Intelligence and Humanized Computing*, vol. 12, no. 8, pp. 8535–8548, 2021.

[250] S. Sengar and X. Liu, "Ensemble approach for short term load forecasting in wind energy system using hybrid algorithm," *Journal of Ambient Intelligence and Humanized Computing*, vol. 11, pp. 5297–5314, 2020.

[251] Q. Shao, L. Hu, and F. Xu, "Reliability and optimization for k-out-of-n: G Mixed standby retrial system with dependency and J-vacation," *Methodology and Computing in Applied Probability*, vol. 26, no. 1, p. 8, 2024.

[252] M. Sharifi, M. Ghaleb, and S. Taghipour, "Joint parallel-machine scheduling and maintenance planning optimisation with deterioration, unexpected breakdowns, and condition-based maintenance," *International Journal of Systems Science: Operations & Logistics*, vol. 10, no. 1, p. 2 200 888, 2023.

[253] M. Sharifi and S. Taghipour, "Optimal production and maintenance scheduling for a degrading multi-failure modes single-machine production environment," *Applied Soft Computing*, vol. 106, p. 107 312, 2021.

[254] S. Sharma and K. Kumar, "Cost optimal analysis for a differentiated vacation machining system with discouragement under the two threshold control policies," *Results in Control and Optimization*, p. 100 409, 2024.

[255] S. Sharma *et al.*, "Fuzzy reliability analysis of repairable industrial systems using soft-computing based hybridized techniques," *Applied Soft Computing*, vol. 24, pp. 264–276, 2014.

[256] A. She, L. Wang, Y. Peng, and J. Li, "Structural reliability analysis based on improved wolf pack algorithm ak-ss," in *Structures*, Elsevier, vol. 57, 2023, p. 105 289.

[257] C Shekhar, A Gupta, N Kumar, A Kumar, and S Varshney, "Transient solution of multiple vacation queue with discouragement and feedback," *Scientia Iranica*, vol. 29, no. 5, pp. 2567–2577, 2022.

[258] C. Shekhar, M. Devanda, and S. Kaswan, "Reliability analysis of standby provision multi-unit machining systems with varied failures, degradations, imperfections, and delays," *Quality and Reliability Engineering International*, vol. 39, no. 7, pp. 3119–3139, 2023.

[259] C. Shekhar, M. Jain, and S. Bhatia, "Fuzzy analysis of machine repair problem with switching failure and reboot," *Journal of Reliability and Statistical Studies*, pp. 41–55, 2014.

[260] C. Shekhar, M. Jain, A Raina, and R Mishra, "Sensitivity analysis of repairable redundant system with switching failure and geometric reneging," *Decision Science Letters*, vol. 6, no. 4, pp. 337–350, 2017.

[261] C. Shekhar, M. Jain, A. A. Raina, and J. Iqbal, "Reliability prediction of fault tolerant machining system with reboot and recovery delay," *International Journal of System Assurance Engineering and Management*, vol. 9, pp. 377–400, 2018.

[262] C. Shekhar, A. Kumar, and S. Varshney, "Load sharing redundant repairable systems with switching and reboot delay," *Reliability Engineering & System Safety*, vol. 193, p. 106 656, 2020.

[263] C. Shekhar, A. Kumar, S. Varshney, and S. I. Ammar, "Fault-tolerant redundant repairable system with different failures and delays," *Engineering Computations*, vol. 37, no. 3, pp. 1043–1071, 2019.

[264] C. Shekhar, N. Kumar, A. Gupta, A. Kumar, and S. Varshney, "Warm-spare provisioning computing network with switching failure, common cause failure, vacation interruption, and synchronized reneging," *Reliability Engineering & System Safety*, vol. 199, p. 106 910, 2020.

[265] C. Shekhar, N. Kumar, M. Jain, and A. Gupta, "Reliability prediction of computing network with software and hardware failures," *International Journal of Reliability, Quality and Safety Engineering*, vol. 27, no. 02, p. 2 040 006, 2020.

[266] C. Shekhar, A. Raina, A Kumar, and J Iqbal, "A survey on queues in machining system: Progress from 2010 to 2017," *Yugoslav Journal of Operations Research*, vol. 27, no. 4, pp. 391–413, 2017.

[267] C. Shekhar, S. Varshney, and A. Kumar, “Standbys provisioning in machine repair problem with unreliable service and vacation interruption,” in *The Handbook of Reliability, Maintenance, and System Safety through Mathematical Modeling*, Elsevier, 2021, pp. 101–133.

[268] J. Singh, “A warm standby redundant system with common cause failures,” *Reliability Engineering & System Safety*, vol. 26, no. 2, pp. 135–141, 1989.

[269] N. Singh, “Cost analysis for N-policy machining system with spares, service halt, set up and common cause failure,” *International Journal of System & Software Engineering*, vol. 7, no. 2, 2019.

[270] U. Singh and M Rizwan, “Analysis of wind turbine dataset and machine learning based forecasting in scada-system,” *Journal of Ambient Intelligence and Humanized Computing*, vol. 14, no. 6, pp. 8035–8044, 2023.

[271] R Sivaraman and S. Bharti, “Interference problem of machines embedded with fuzziness,” *International Journal of Mathematics Trends and Technology-IJMTT*, vol. 43, 2017.

[272] B. Sivazlian and K. Wang, “Diffusion approximation to the $G/G/R$ machine repair problem with warm standby spares,” *Naval Research Logistics (NRL)*, vol. 37, no. 5, pp. 753–772, 1990.

[273] K. Sivazlian BD & Wang, “Economic analysis of the M/M/R machine repair problem with warm standbys,” *Microelectronics Reliability*, vol. 29, no. 1, pp. 25–35, 1989.

[274] W. Slany, “Scheduling as a fuzzy multiple criteria optimization problem,” *Fuzzy Sets and Systems*, vol. 78, no. 2, pp. 197–222, 1996.

[275] K. S. de Smidt-Destombes, M. C. van der Heijden, and A. van Harten, “Joint optimisation of spare part inventory, maintenance frequency and repair capacity for k-out-of-N systems,” *International Journal of Production Economics*, vol. 118, no. 1, pp. 260–268, 2009.

[276] A. Smith, *On the causes of improvement in the productive powers. on labour, and on the order according to which its’ produce is naturally distributed among the different ranks of the people*, 1776.

[277] S. Soga, E. Higo, and H. Miura, “A systematic approach to estimate an inter-unit common-cause failure probability,” *Reliability Engineering & System Safety*, vol. 215, p. 107 802, 2021.

[278] Y. Son, X. Zhang, Y. Yoon, J. Cho, and S. Choi, “Lstm–gan based cloud movement prediction in satellite images for pv forecast,” *Journal of Ambient Intelligence and Humanized Computing*, vol. 14, no. 9, pp. 12 373–12 386, 2023.

[279] V Sridharan and P Mohanavadivu, "Stochastic behaviour of a two-unit standby system with two types of repairmen and patience time," *Mathematical and Computer Modelling*, vol. 28, no. 9, pp. 63–71, 1998.

[280] S. K. Srinivasan and R. Subramanian, "Reliability analysis of a three unit warm standby redundant system with repair," *Annals of Operations Research*, vol. 143, no. 1, pp. 227–235, 2006.

[281] V. Srinivasan, "The effect of standby redundancy in system's failure with repair maintenance," *Operations Research*, vol. 14, no. 6, pp. 1024–1036, 1966.

[282] V. Srinivasan, "First emptiness in the spare parts problem for reparable components," *Operations Research*, vol. 16, no. 2, pp. 407–415, 1968.

[283] D. H. Stamatis, *Failure mode and effect analysis*. Quality Press, 2003.

[284] K. E. Stecke and J. E. Aronson, "Review of operator/machine interference models," *International Journal of Production Research*, vol. 23, no. 1, pp. 129–151, 1985.

[285] R Subramanian and V Anantharaman, "Reliability analysis of a complex standby redundant systems," *Reliability Engineering & System Safety*, vol. 48, no. 1, pp. 57–70, 1995.

[286] R Subramanian and Y. Sarma, "Two-unit standby systems with non-instantaneous switchover and imperfect switch," *Microelectronics Reliability*, vol. 27, no. 2, pp. 237–247, 1987.

[287] M. Suhail, I. Akhtar, and S. Kirmani, "Objective functions and infrastructure for optimal placement of electrical vehicle charging station: A comprehensive survey," *IETE Journal of Research*, vol. 69, no. 8, pp. 5250–5260, 2023.

[288] X. Sun and X. N. Geng, "Single-machine scheduling with deteriorating effects and machine maintenance," *International Journal of Production Research*, vol. 57, no. 10, pp. 3186–3199, 2019.

[289] Y Takemoto and I Arizono, "A study of mttf in two-unit standby redundant system with priority under limited information about failure and repair times," *Proceedings of the Institution of Mechanical Engineers, Part O: Journal of Risk and Reliability*, vol. 230, no. 1, pp. 67–74, 2016.

[290] Z. B. Tan, "Reliability and availability analysis of two-unit warm standby microcomputer systems with self-reset function and repair facility," *Microelectronics Reliability*, vol. 37, no. 8, pp. 1251–1253, 1997.

[291] M. Tang and Z. You, "Design and research of electric bicycle networking system based on nb-iot technology," *IETE Journal of Research*, pp. 1–8, 2021.

[292] O. Tannous, L. Xing, R. Peng, and M. Xie, "Reliability of warm-standby systems subject to imperfect fault coverage," *Proceedings of the Institution of Mechanical Engineers, Part O: Journal of Risk and Reliability*, vol. 228, no. 6, pp. 606–620, 2014.

[293] J Taylor and R. Jackson, "An application of the birth and death process to the provision of spare machines," *Journal of the Operational Research Society*, vol. 5, no. 4, pp. 95–108, 1954.

[294] T. Tetik, G. S. Daş, and B. Birgoren, "A multi-objective perspective to satellite design and reliability optimization," *Expert Systems with Applications*, vol. 246, p. 123 178, 2024.

[295] S. Thiagarajan and S Thobias, "Classical and bayesian inference procedures of a three unit cold standby system with dependent structure," *Communications in Statistics-Theory and Methods*, vol. 52, no. 12, pp. 4247–4258, 2023.

[296] L. Thomas, "A survey of maintenance and replacement models for maintainability and reliability of multi-item systems," *Reliability Engineering*, vol. 16, no. 4, pp. 297–309, 1986.

[297] Z. Tian, "Analysis and research on chaotic dynamics behaviour of wind power time series at different time scales," *Journal of Ambient Intelligence and Humanized Computing*, vol. 14, no. 2, pp. 897–921, 2023.

[298] K. S. Trivedi, *Probability & Statistics with Reliability, Queuing and Computer Science Applications*. John Wiley & Sons, 2008.

[299] A. H. Tsang, "Condition-based maintenance: Tools and decision making," *Journal of Quality in Maintenance Engineering*, vol. 1, no. 3, pp. 3–17, 1995.

[300] R. Tuteja and G. Taneja, "Cost-benefit analysis of a two-server, two-unit, warm standby system with different types of failure," *Microelectronics Reliability*, vol. 32, no. 10, pp. 1353–1359, 1992.

[301] C. Valdez-Flores and R. M. Feldman, "A survey of preventive maintenance models for stochastically deteriorating single-unit systems," *Naval Research Logistics (NRL)*, vol. 36, no. 4, pp. 419–446, 1989.

[302] F. Van Der Duyn Schouten and P Wartenhorst, "A two-machine repair model with variable repair rate," *Naval Research Logistics (NRL)*, vol. 40, no. 4, pp. 495–523, 1993.

[303] J. K. Vaurio, "The theory and quantification of common cause shock events for redundant standby systems," *Reliability Engineering & System Safety*, vol. 43, no. 3, pp. 289–305, 1994.

[304] J. K. Vaurio, "Common cause failure probabilities in standby safety system fault tree analysis with testing—scheme and timing dependencies," *Reliability Engineering & system safety*, vol. 79, no. 1, pp. 43–57, 2003.

[305] P. Venugopal and S. Reka S, "State of charge estimation of lithium batteries in electric vehicles using indrnn," *IETE Journal of Research*, vol. 69, no. 5, pp. 2886–2896, 2023.

[306] V. Vijayan, S. K. Chaturvedi, and R. Chandra, "A failure interaction model for multicomponent repairable systems," *Proceedings of the Institution of Mechanical Engineers, Part O: Journal of Risk and Reliability*, vol. 234, no. 3, pp. 470–486, 2020.

[307] H. Wang, "A survey of maintenance policies of deteriorating systems," *European Journal of Operational Research*, vol. 139, no. 3, pp. 469–489, 2002.

[308] J. B. Wang, J. X. Xu, F. Guo, and M. Liu, "Single-machine scheduling problems with job rejection, deterioration effects and past-sequence-dependent setup times," *Engineering Optimization*, vol. 54, no. 3, pp. 471–486, 2022.

[309] K. H. Wang, "Profit analysis of the machine-repair problem with a single service station subject to breakdowns," *Journal of the Operational Research Society*, vol. 41, pp. 1153–1160, 1990.

[310] K. H. Wang, "Cost analysis of the $M/M/R$ machine-repair problem with mixed standby spares," *Microelectronics Reliability*, vol. 33, no. 9, pp. 1293–1301, 1993.

[311] K. H. Wang, "An approach to cost analysis of the machine repair problem with two types of spares and service rates," *Microelectronics Reliability*, vol. 35, no. 11, pp. 1433–1436, 1995.

[312] K. H. Wang, W. L. Chen, and D. Y. Yang, "Optimal management of the machine repair problem with working vacation: Newton's method," *Journal of Computational and Applied Mathematics*, vol. 233, no. 2, pp. 449–458, 2009.

[313] K. H. Wang and Y. J. Chen, "Comparative analysis of availability between three systems with general repair times, reboot delay and switching failures," *Applied Mathematics and Computation*, vol. 215, no. 1, pp. 384–394, 2009.

[314] K. H. Wang and L. W. Chiu, "Cost benefit analysis of availability systems with warm standby units and imperfect coverage," *Applied Mathematics and Computation*, vol. 172, no. 2, pp. 1239–1256, 2006.

[315] K. H. Wang, W. L. Dong, and J. B. Ke, "Comparison of reliability and the availability between four systems with warm standby components and standby switching failures," *Applied Mathematics and Computation*, vol. 183, no. 2, pp. 1310–1322, 2006.

[316] K. H. Wang, J. B. Ke, and J. C. Ke, "Profit analysis of the $M/M/R$ machine repair problem with balking, reneging, and standby switching failures," *Computers & Operations Research*, vol. 34, no. 3, pp. 835–847, 2007.

[317] K. H. Wang and C. C. Kuo, "Cost and probabilistic analysis of series systems with mixed standby components," *Applied Mathematical Modelling*, vol. 24, no. 12, pp. 957–967, 2000.

[318] K. H. Wang, C. D. Liou, and Y. H. Lin, "Comparative analysis of the machine repair problem with imperfect coverage and service pressure condition," *Applied Mathematical Modelling*, vol. 37, no. 5, pp. 2870–2880, 2013.

[319] K. H. Wang, C. D. Liou, and Y. L. Wang, "Profit optimisation of the multiple-vacation machine repair problem using particle swarm optimisation," *International Journal of Systems Science*, vol. 45, no. 8, pp. 1769–1780, 2014.

[320] K. H. Wang, Y. C. Liu, and W. L. Pearn, "Cost benefit analysis of series systems with warm standby components and general repair time," *Mathematical Methods of Operations Research*, vol. 61, pp. 329–343, 2005.

[321] K. H. Wang, C. H. Wu, and T. C. Yen, "Comparative cost-benefit analysis of four retrial systems with preventive maintenance and unreliable service station," *Reliability Engineering & System Safety*, vol. 221, p. 108 342, 2022.

[322] K. H. Wang, C. H. Wu, and T. C. Yen, "Reliability analysis of redundant retrial machining system subject to standby switching failure," *Quality Technology & Quantitative Management*, vol. 20, no. 5, pp. 561–576, 2023.

[323] K. H. Wang and J. D. Wu, "Cost analysis of the $M/M/R$ machine repair problem with spares and two modes of failure," *Journal of the Operational Research Society*, vol. 46, pp. 783–790, 1995.

[324] K. H. Wang and D. Y. Yang, "Controlling arrivals for a queueing system with an unreliable server: Newton-quasi method," *Applied Mathematics and Computation*, vol. 213, no. 1, pp. 92–101, 2009.

[325] K. H. Wang, T. C. Yen, and Y. C. Fang, "Comparison of availability between two systems with warm standby units and different imperfect coverage," *Quality Technology & Quantitative Management*, vol. 9, no. 3, pp. 265–282, 2012.

[326] K. H. Wang, T. C. Yen, and J. J. Jian, "Reliability and sensitivity analysis of a repairable system with imperfect coverage under service pressure condition," *Journal of Manufacturing Systems*, vol. 32, no. 2, pp. 357–363, 2013.

[327] K. H. Wang, T. C. Yen, and C. H. Wu, "Reliability and sensitivity analyses of an unreliable retrial machine repair system with warm standby switching failure," *International Journal of Reliability, Quality and Safety Engineering*, vol. 29, no. 04, p. 2 141 008, 2022.

[328] L. Wang, Y. Song, W. Zhang, and X. Ling, "Condition-based inspection, component reallocation and replacement optimization of two-component interchangeable series system," *Reliability Engineering & System Safety*, vol. 230, p. 108 907, 2023.

[329] Y. Wang, A. Che, and J. Feng, "Energy-efficient unrelated parallel machine scheduling with general position-based deterioration," *International Journal of Production Research*, vol. 61, no. 17, pp. 5886–5900, 2023.

[330] J. R. Wilson and A. A. B. Pritsker, "Evaluation of startup policies in simulation experiments," *Simulation*, vol. 31, no. 3, pp. 79–89, 1978.

[331] R. W. Wolff, "Problems of statistical inference for birth and death queuing models," *Operations Research*, vol. 13, no. 3, pp. 343–357, 1965.

[332] C. H. Wu and D. Y. Yang, "Dynamic control of a machine repair problem with switching failure and unreliable repairmen," *Arabian Journal for Science and Engineering*, vol. 45, pp. 2219–2234, 2020.

[333] C. H. Wu, D. Y. Yang, and T. E. He, "Matrix-augmentation approach for machine repair problem with generally distributed repair times during working breakdown periods," *Mathematics and Computers in Simulation*, 2023.

[334] C. H. Wu, D. Y. Yang, and M. H. Ko, "Performance sensitivity analysis for machine repair problem with two failure modes and working vacation," *International Journal of Reliability, Quality and Safety Engineering*, p. 2 350 040, 2024.

[335] C. H. Wu, T. C. Yen, and K. H. Wang, "Availability and comparison of four retrial systems with imperfect coverage and general repair times," *Reliability Engineering & System Safety*, vol. 212, p. 107 642, 2021.

[336] H. Wu, Y. F. Li, and C. Bérenguer, "Optimal inspection and maintenance for a repairable k-out-of-n: G Warm standby system," *Reliability Engineering & System Safety*, vol. 193, p. 106 669, 2020.

[337] Q. Wu and S. Wu, "Reliability analysis of two-unit cold standby repairable systems under poisson shocks," *Applied Mathematics and Computation*, vol. 218, no. 1, pp. 171–182, 2011.

[338] V. S. Yadavalli, A. Bekker, and J Pauw, "Bayesian study of a two-component system with common-cause shock failures," *Asia-Pacific Journal of Operational Research*, vol. 22, no. 01, pp. 105–119, 2005.

[339] R. R. Yager, "A procedure for ordering fuzzy subsets of the unit interval," *Information Sciences*, vol. 24, no. 2, pp. 143–161, 1981.

[340] A. Yaghoubi and S. T. A. Niaki, "Reliability analysis of a two-unit standby system under marshall–olkin dependency," *International Journal of Quality & Reliability Management*, vol. 40, no. 6, pp. 1587–1596, 2023.

[341] D. Y. Yang and Y. D. Chang, "Sensitivity analysis of the machine repair problem with general repeated attempts," *International Journal of Computer Mathematics*, vol. 95, no. 9, pp. 1761–1774, 2018.

[342] D. Y. Yang and C. L. Tsao, "Reliability and availability analysis of standby systems with working vacations and retrial of failed components," *Reliability Engineering & System Safety*, vol. 182, pp. 46–55, 2019.

[343] D. Y. Yang and C. H. Wu, "Evaluation of the availability and reliability of a standby repairable system incorporating imperfect switchovers and working breakdowns," *Reliability Engineering & System Safety*, vol. 207, p. 107 366, 2021.

[344] D. Y. Yang, Z. R. Wu, and C. S. Tsou, "Reliability analysis of a repairable system with geometric reneging and threshold-based recovery policy," *Proceedings of the Institution of Mechanical Engineers, Part B: Journal of Engineering Manufacture*, vol. 229, no. 11, pp. 2047–2062, 2015.

[345] L. Yeh, "An optimal repairable replacement model for deteriorating systems," *Journal of Applied Probability*, vol. 28, no. 4, pp. 843–851, 1991.

[346] W. C. Yeh, W. Zhu, S. Y. Tan, G. G. Wang, and Y. H. Yeh, "Novel general active reliability redundancy allocation problems and algorithm," *Reliability Engineering & System Safety*, vol. 218, p. 108 167, 2022.

[347] T. C. Yen and K. H. Wang, "Cost benefit analysis of three systems with imperfect coverage and standby switching failures," *International Journal of Mathematics in Operational Research*, vol. 12, no. 2, pp. 253–272, 2018.

[348] T. C. Yen, K. H. Wang, and C. H. Wu, "Reliability-based measure of a retrial machine repair problem with working breakdowns under the F-policy," *Computers & Industrial Engineering*, vol. 150, p. 106 885, 2020.

[349] T.-C. Yen, H. Wu, K.-H. Wang, and W.-K. Chou, "Optimal control of the machine repair problem with removable repairman subject to working breakdowns," *Journal of Testing and Evaluation*, vol. 43, no. 6, pp. 1487–1496, 2015.

[350] X. J. Yi, C. H. Xu, S. L. Liu, M. X. Xing, and H. n. Mu, "Failure analysis of repairable systems with k-out-of-m configuration considering with common cause failure and maintenance correlation based on goal oriented method," *Engineering Failure Analysis*, vol. 140, p. 106 615, 2022.

[351] H. Yu, J. Yang, J. Lin, and Y. Zhao, "Reliability evaluation of non-repairable phased-mission common bus systems with common cause failures," *Computers & Industrial Engineering*, vol. 111, pp. 445–457, 2017.

[352] L. Yuan and X. Y. Meng, "Reliability analysis of a warm standby repairable system with priority in use," *Applied Mathematical Modelling*, vol. 35, no. 9, pp. 4295–4303, 2011.

[353] W. Y. Yun, "Optimal number of redundant units for a standby system," *Reliability Engineering & System Safety*, vol. 25, no. 4, pp. 365–369, 1989.

[354] W. Y. Yun and J. H. Cha, "Optimal design of a general warm standby system," *Reliability Engineering & System Safety*, vol. 95, no. 8, pp. 880–886, 2010.

[355] I. Yusuf, "Comparison of some reliability characteristics between redundant systems requiring supporting units for their operations," *J. Math. Comput. Sci.*, vol. 3, no. 1, pp. 216–232, 2013.

[356] I. Yusuf and M. S. Aliyu, "Mean time to failure modelling and sensitivity analysis of mixed standby serial systems," *International Journal of Mathematics in Operational Research*, vol. 27, no. 2, pp. 267–289, 2024.

[357] I. Yusuf and S. I. Bala, "Stochastic modeling of a two unit parallel system under two types of failures," *International Journal of Latest Trends in Mathematics*, vol. 2, no. 1, pp. 44–53, 2012.

[358] L. A. Zadeh, "Fuzzy sets as a basis for a theory of possibility," *Fuzzy Sets and Systems*, vol. 1, no. 1, pp. 3–28, 1978.

[359] F. Zagia, O. Motamedi Sedeh, and B. Ostadi, "A three-stage optimization model for scheduling the facilities maintenance considering random failure rate," *Scientia Iranica*, 2021.

[360] X. Zhou, X. Liu, G. Zhang, L. Jia, X. Wang, and Z. Zhao, "An iterative threshold algorithm of log-sum regularization for sparse problem," *IEEE Transactions on Circuits and Systems for Video Technology*, 2023.

Publications

This thesis comprises the following works as a chapter that have been published/under revision/ communicated in distinct journals.

1. **Devanda M.**, Shekhar, C., and Kaswan, S. (2023): "Fuzzifed imperfect repair redundant machine repair problems", *International Journal of System Assurance Engineering and Management* (Springer) (Scopus), doi. 10.1007/s13198-023-01922-3
   https://link.springer.com/article/10.1007/s13198-023-01922-3
2. Shekhar, C., **Devanda M.**, and Kaswan, S. (2023): "Reliability analysis of standby provision multi-unit machining systems with varied failures, degradations, imperfections, and delays", *Quality and Reliability Engineering International*, (Wiley Online Library) (SCIE), Vol. 39, no. 7, pp. 3119–3139.
   https://doi.org/10.1002/qre.3421
3. Shekhar, C., **Devanda M.**, and Kaswan, S. (2024): "The state-of-the-art methodologies for reliability analysis of imperfect repair and threshold-based measures", *Scientia Iranica*, (Sharif University of Technology) (SCIE)**(Accepted)**.
4. Shekhar, C., **Devanda M.**, and Kaswan, S.: "Exploring Hierarchical Repair Strategies for Multi-Unit Redundant Machining Systems", **(Revision Submitted)**.
5. Shekhar, C., **Devanda M.**, and Kaswan, S.: "Performance Analysis of a Repairable Machining System with Standby Deterioration",*Opsearch*, (ESCI) **(Accepted)**.
6. Shekhar, C., **Devanda M.**, and Sharma, K.: "Reliability Analysis of Imperfect Repair and Switching Failures: A Bayesian Inference and Monte Carlo Simulation Approach", **(Revision Submitted)**.
7. Shekhar, C., **Devanda M.**, and Choudhary, A.: " Performance Analysis of Machine Repair Systems with Deteriorating Standby Using Phase-Type Distributions: Comparative Study of Erlang, Exponential, and Hyper-Exponential Distributions", **(Communicated)**.

List of research article(s) in conference (published)

1. **Devanda M.**, Kaswan, S., and Shekhar, C. (2023): "Optimizing Costs for the Multi-unit Machine Repair Problem with Primary and Secondary Repairer in

the M/M/R+1 Configuration", *Proceedings of the* 1st *International Conference on Mathematical and Statistical Sciences(ICMSS-1).*

Furthermore, several research works attributed to my profile have been published/under revision/communicated in the following journals:

1. **Devanda M.**, Kaswan, S., and Shekhar, C. (2024): "Quasi and metaheuristic optimization approach for service system with strategic policy and unreliable service", *Journal of Ambient Intelligence and Humanized Computing*, (Springer) (Scopus), Vol. 15, pp. 2295–2315.
https://doi.org/10.1007/s12652-024-04756-4

2. Kumar, A., Kaswan, S., **Devanda M.**, and Shekhar, C. (2023): "Transient Analysis of Queueing-Based Congestion with Differentiated Vacations and Customer's Impatience Attributes", *Arabian Journal for Science and Engineering*, (Springer) (SCIE), Vol. 48, no. 11, pp.15655–15665.
https://doi.org/10.1007/s13369-023-08020-3

3. Varshney, S., Kaswan, S., **Devanda M.**, and Shekhar, C. (2024): "Finite Capacity Service System with Partial Server Breakdown and Recovery Policy: An Economic Perspective", *Journal of Systems Science and Systems Engineering*, (Springer) (SCIE).
https://doi.org/10.1007/s11518-024-5612-1

4. Kaswan, S., **Devanda M.**, and Shekhar, C. (2023): "Economic Analysis of a Retrial Queueing System with Balking, Orbit Search, Multiple Vacation, and Unreliable Server", *Proceedings of the* 1st *International Conference on Mathematical and Statistical Sciences(ICMSS-1).*

5. Kaswan, K., **Devanda M.**, and Shekhar, C.: "Admission Control Policy on Online and Impatience Attributes of Offline Customers in Multi-phase Queueing Systems", **(Under review)**.

6. Kaswan, S., **Devanda, M.**, and Shekhar, C.: "Economic analysis of a service system with unreliable service of two types of servers", **(Under Review)**.

7. Kumar K., Chahal PK, **Devanda M.**, and Shekhar, C.: "Reliability analysis for a production system with threshold recovery policy, servers'unreliability, vacation and phase repairs", **(Under Review)**.

8. Kaswan, K., **Devanda M.**, and Shekhar, C.: "Cost Analysis of Customer's Impatience Attributes in the Service System", **(Communicated)**.

9. Kaswan, K., **Devanda M.**, and Shekhar, C.: "Cost optimization of a retrial queueing system with an unreliable server incorporating an orbital search mechanism, multiple vacation policies, and the balking phenomenon", **(Communicated)**.

Conferences / Workshops Attended

- Presented paper entitled 'Exploring Hierarchical Repair Strategies for Multi-Unit Redundant Machining Systems' in International Conference on "**Operations Research and Game Theoretic Approach in Decision Making (ICORGTDM 2024)**" organized by Department of Mathematics (Indian Statistical Institute, Delhi Center) on January 17-19', 2024.

- Presented paper entitled 'Optimizing Costs for the Multi-unit Machine Repair Problem with Primary and Secondary Repairer in the M/M/R+1 Configuration' in International Conference on "**Proceedings of the** 1^{st} **International Conference on Mathematical and Statistical Sciences(ICMSS-1)**" organized by Department of Mathematics, Statistics and Actuarial Science, (Namibia University of Science and Technology.) on July 3-4', 2023.

- Presented paper entitled 'Machine repair problem with unpredictable failures, delays, and imperfection' in International Conference on "**Advances in Mechanics, Modelling, Computing and Statistics (ICAMMCS-2022)**" organized by Department of Mathematics (BITS Pilani - Pilani Campus) on March 19-21', 2022.

- Five days workshop in "**National Workshop on Python**" organized by Department of Mathematics (Indira Gandhi University Meerpur, Rewari, Haryana) on October 18-22', 2021.

- Five days workshop in "**Online Short-Term Course on Mathematics with Computational Learning for Engineering and Technological Applications (MCET 2023)**" organized by Department of Mathematics (Dr B. R. Ambedkar National Institute of Technology Jalandhar, Punjab, India) on November 20-24', 2023.

- Ten days workshop in "**Asian and European schools in Mathematics (AESIM-2023)**" organized by Department of Mathematics (BITS Pilani, Pilani Campus, India) on December 28 - January 06', 2024.

Brief Biography of the Candidate

Mahendra Devanda is a Research Scholar in the Department of Mathematics at Birla Institute of Technology and Science, Pilani. He achieved an all-India rank of 122 (1081/CSIR-UGC NET DEC. 2018) in the NET-JRF exam and successfully qualified the GATE exam in 2019. He holds a Master of Science in Mathematics from the University of Rajasthan, completed in 2017. With five research articles published in peer-reviewed journals and commendable research contributions in international publications, his primary areas of interest include the development of queueing models, stochastic modeling, and reliability theory. He is currently working as an Assistant Professor in the Department of Mathematics at Maharaja Surajmal Brij University, Bharatpur.

Brief Biography of the Supervisor

Prof. Chandra Shekhar, the Professor and ex-HoD in the Department of Mathematics in BITS Pilani, India, is actively involved in research and teaching in the area of queueing theory, computer and communication systems, machine repair problems, reliability and maintainability, stochastic process, evolutionary computation, statistical analysis,fuzzy set, and logic. Besides attending, presenting scientific papers, and delivering invited talks in national/international conferences and FDPs, he has organized a number of conferences, workshops, and symposiums as convener and organizing secretary. The best research paper award has been bestowed at the international conference. He has more than 50 research articles in these fields in journals of high repute and has supervised three Ph.D. theses. Besides some book chapters in an edited book published by the publisher of international reputation, authorship of the textbook entitled Differential Equations, Calculus of Variations and Special Functions and the edited book entitled Mathematical Modeling and Computation of Real-Time Problems: An Interdisciplinary Approach is also to his credit. He is also a member of the editorial board and reviewer of many reputed journals, academic societies and Doctoral Research Committee, advisory board, faculty selection committee, the examination board of many governments and private universities, institutions, or research labs. As a professional, he has visited IIRS (ISRO), CSIR-IIP, NIH, WIHG, CPWD, Bank of Maharashtra, APS Lifetech.