# Virtual Element Method for Parabolic Integro-differential Equations

**THESIS**

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# CERTIFICATE

This is to certify that the thesis entitled, **"Virtual Element Method for Parabolic Integro-differential Equations"** and submitted by **Ms. Meghana** ID No. **2019PHXF0039P** for the award of Ph.D. degree of the institute embodies original work done by her under my supervision.

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Date: 12/09/2024

***Dedicated to***

***My Beloved Sister***
***Shruti Suthar***

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# ABSTRACT

This thesis develops and analyzes the virtual element method (VEM) and mixed VEM for parabolic integro-differential equations (PIDEs), focusing on both the semi-discrete and fully discrete cases. The fully discrete scheme employs the left rectangular rule for integral term discretization and the backward Euler method for time derivative approximation.

To handle the integral term within the VEM framework, the Ritz-Volterra (R.V.) projection is introduced, and its estimates are derived. With the help of R.V. projection, optimal error estimates are derived for both the semi-discrete and fully-discrete cases. Numerical experiments are conducted to confirm the convergence rates for both singular and weakly singular kernels. Additionally, experiments are performed with local mesh refinements to demonstrate the practical utility of VEMs. These refinements are essential for reducing the overall computational cost, a capability often limited in conforming finite element methods (FEMs). Furthermore, the results illustrate VEM's ability to handle hanging nodes, which eventually helps in the local refining of the mesh, i.e., one can have finer mesh around the singular point and coarser mesh in the rest of the domain.

As far as mixed VEM for PIDE is concerned, there are two formulations, each offering unique advantages. One formulation includes a resolvent kernel, while the other does not. A new mixed projection is introduced for each formulation to handle integral terms effectively. This approach results in optimal *a priori* error estimates of order $O(h^{k+1})$ for the velocity and pressure, where $h$ is the mesh size and $k$ is the degree of the polynomial. Furthermore, a step-by-step analysis is proposed for the super convergence of the discrete solution of order $O(h^{k+2})$. The fully discrete case is also analyzed to achieve $O(\tau)$ in time. Several computational experiments are discussed to validate the computational efficiency of the proposed schemes and to support the theoretical conclusions. Using various numerical experiments, we demonstrate the advantages of each formulation.

An analysis is presented for VEM with non-smooth initial data. Through the repeated application of integration by parts and using regularity results, we establish estimates of the intermediate projection solely in terms of the initial data in $L^2$. Moreover, with the help

of the estimates of the intermediate projection, optimal error estimates were established for the semi-discreet case.

For the mixed VEM with non-smooth initial data, analysis has been presented for the two formulations. By using a new projection that includes a memory component, applying energy arguments, and employing an integral operator iteratively, this research establishes optimal $L^2$-error estimates for both pressure and velocity. These findings provide a thorough analysis of the VEM, covering both formulations.

Finally, the possible extensions with scope for future investigations are discussed in the concluding Chapter.

# Contents

# List of Abbreviations

| Sr. No. | Abbrevation | Stands for |
|---|---|---|
| 1 | PDE | Partial Differential Equation |
| 2 | PIDE | Parabolic Integro-differential Equation |
| 3 | FEM | Finite Element Method |
| 4 | FVM | Finite Volume Method |
| 5 | FDM | Finite Difference Method |
| 6 | BEM | Boundary Element Method |
| 7 | DG | Discontinuous Galerkin |
| 8 | HDG | Hybridized Discontinuous Galrekin |
| 9 | VEM | Virtual Element Method |
| 10 | VCFEM | Voronoi Cell Finite Element Method |
| 11 | PFEM | Polygonal Finite Element Method |
| 12 | HPE | Hybrid Polygonal Element |
| 13 | nSFEM | n-Sided Polygonal Smoothed Finite Element Method |
| 14 | PSBFEM | Polygonal Scaled Boundary Finite Element Method |
| 15 | MFD | Mimetic Finite Difference |
| 16 | dof | Degrees of Freedom |
| 17 | R.V. | Ritz- Volterra |

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# Chapter 1

# Introduction

*"Solving partial differential equations is like navigating the intricate paths of reality"*

- Vladimir Arnold

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We see partial differential equations (PDEs) everywhere; more or less, every physical, chemical, or even biological phenomenon can be represented in terms of PDEs. They are the core topics in multi-variable calculus and are used to describe the evolution of gases in fluid dynamics [1], the formation of galaxies [2], the nature of quantum mechanics (Schrodinger's Equations) [3], gravitation [4], heat transfer [5] etc. Unlike traditional PDEs, which solely involve derivatives of a function with respect to time and space, parabolic integro-differential equations (PIDEs) extend the framework by including integral operators. These integral terms account for memory effects or history-dependent behavior in the system. They generally occur in demonstrating specific physical processes in which memory effects are considered. For instance, these equations appeared in solving the electrical circuit problems that govern the Kirchhoff voltage laws [6], for a disease spread by the dispersal of infectious individuals [7], heat flow in material with memory [8] and many more. There are several significant methods for finding the solution of PDE; some of the analytical techniques include separation of variables, method of characteristic, variation of parameters, etc., whereas some numerical methods are: finite element method

(FEM), finite volume method (FVM), finite difference method (FDM), boundary element method (BEM), discontinuous Galerkin (DG) method, hybridized discontinuous Galerkin (HDG) method, virtual element method (VEM), etc.

## 1.1 Model Problem

The focus of this thesis is on developing VEMs for the linear PIDE (1.1.1) [9] defined on a bounded polygon domain $\mathcal{D} \subset \mathbb{R}^2$, having $\partial\mathcal{D}$ as the boundary; furthermore, the interval $(0, T]$ represents a finite time span. Find $u(\boldsymbol{x}, t)$ such that

$$u_t(\boldsymbol{x}, t) + \mathbb{A}u(\boldsymbol{x}, t) - \int_0^t \mathbb{B}(t, s)u(\boldsymbol{x}, s)ds = f(\boldsymbol{x}, t) \quad (\boldsymbol{x}, t) \in \mathcal{D} \times (0, T], \tag{1.1.1}$$

subjects to the homogenous boundary conditions:

$$u(\boldsymbol{x}, t) = 0 \quad (\boldsymbol{x}, t) \in \partial\mathcal{D} \times [0, T],$$

and initial data:

$$u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}) \quad \boldsymbol{x} \in \mathcal{D},$$

where $u_t = \frac{\partial u}{\partial t}$; $\mathbb{A}$ and $\mathbb{B}(t, s)$ are second-order elliptic operators of the form:

$$\mathbb{A} = -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_j} \left( a_{ij}(\boldsymbol{x}) \frac{\partial}{\partial x_i} \right) + a_0(\boldsymbol{x})I,$$

and

$$\mathbb{B}(t, s) = -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_j} \left( b_{ij}(\boldsymbol{x}; t, s) \frac{\partial}{\partial x_i} \right) + \sum_{j=1}^{2} \boldsymbol{b}_j(\boldsymbol{x}; t, s) \frac{\partial}{\partial x_j} + b_0(\boldsymbol{x}; t, s)I.$$

For our analysis, we require the following assumptions on the operators $\mathbb{A}$, $\mathbb{B}(t, s)$ and the function $f$:

- the operator $\mathbb{A}$ is positive-definite,

- all the coefficients of $\mathbb{A}$ are real-valued, bounded, and smooth, along with $a(\boldsymbol{x}) \geq \alpha_0 > 0$ and $a_0(\boldsymbol{x}) \geq 0$ for all $\boldsymbol{x} \in \mathcal{D}$,

- the coefficients of $\mathbb{B}(t, s)$ and their derivative with respect to $t$ and $s$ are real-valued, bounded and smooth,

- The function $f$ is real-valued and smooth enough.

Under these assumptions, the well-posedness of (1.1.1) and the continuous solution $u$ regularity is reported in [10].

## 1.2 Background and Motivation

Addressing widespread PDEs across various disciplines enables us to analyze and understand a diverse array of physical and mathematical phenomena. Discovering exact or analytical solutions to PDEs poses a significant challenge in many real-world scenarios, which is why pursuing solutions on a consistent discretized mesh remains a highly active field of study. FEM is highly favored as one of the widely adopted numerical methods for addressing PDE-related challenges due to its manifold benefits. The core idea behind FEM is to approximate the behavior of a continuous system by discretizing it into a finite number of elements, typically triangles or quadrilaterals in two dimensions and tetrahedra or hexahedra in three dimensions. It comes out to be a system of linear or non-linear equations. After applying the boundary conditions, the resulting system of equations is solved numerically, often using iterative techniques. Some of the advantages of FEM are:

- FEM can provide highly accurate solutions when the mesh is refined. By increasing the mesh density, the accuracy of the solution can be improved, making it suitable for high-precision simulations.

- Localized approximations in discretized problems result in sparse equation systems, leading to efficient storage and faster computation.

- It is also possible to implement higher-order elements in the model.

- As various finite elements are available for the discretizing domain, it becomes easier to model complex geometries and irregular shapes.
- Boundary conditions can be easily incorporated in FEM.
- FEM allows for adaptive mesh refinement, meaning that more elements can be concentrated in areas of interest while reducing the mesh density in less critical regions. This adaptability enhances the efficiency and accuracy of the analysis.
- Availability of a large number of computer software packages and literature makes FEM a versatile and powerful numerical method.

In recent years, there has been an increasing demand for the use of meshes containing general polygonal elements. Employing polygonal meshes provides several advantages over relying solely on triangular or quadrilateral meshes. These advantages include simplifying the partitioning of domains with complex geometries and reducing the complexity of adaptive mesh refinement. Various methods have been proposed over the years to form polygonal/polyhedral finite elements, such as the Voronoi cell finite element method (VCFEM), polygonal finite element method (PFEM), hybrid polygonal element method (HPEM), $n$-sided polygonal smoothed finite element method (nSFEM), polygonal scaled boundary FEM (PSBFEM), etc., for more details see [11] and reference within.

One of the challenges encountered when constructing polygonal finite elements lies in creating interpolation functions, which extend into the element's interior. In the article [12], the mimetic finite difference (MFD) method for polygonal meshes is introduced, which demonstrates the efficiency of MFD in solving problems involving polygonal meshes as it relies solely on the surface representation of discrete unknowns. Notably, this approach was effective even for meshes containing degenerate and non-convex polygonal elements. Since no extension of basis functions inside the mesh elements is required, practical implementation of the MFD method is simple for polygonal meshes. It has been established that incorporating degrees of freedom (dof) into trial/test functions located within the elements would significantly enhance the simplicity of the method, even if the functions are not always polynomials. Considering this, the MFD method was extended and reintroduced under the name 'virtual element method' (VEM) [13]. The finite element spaces considered in this work are virtual in the sense that basis functions are not explicitly required to implement the method. The scientific community's interest in VEMs has grown due to the fact that the convergence analysis of these methods can be incorporated into the well-established framework of finite element methods documented in the literature. Other interesting features of these methods are admitting hanging nodes in the mesh generation, avoiding explicit constructions of nodal basis functions, handling complicated domains, and allowing higher-order polynomials, which in turn improve the accuracy and suitability to work with convex and non-convex polygonal meshes. The local virtual element spaces are defined over an element or polygon consisting of polynomial and non-polynomial functions. One of the most appealing characteristics of these methods is that the discrete bilinear forms that appeared in the discrete formulations can be computed directly with the help of the degrees of freedom only (without using the basis functions as done in the case of finite element methods). In contrast with finite element formulation, virtual element discretization requires two projection operators: One is $L^2$- projection ($\Pi_k^0$), and the other is energy projection ($\Pi_k^\nabla$), which is defined in the next section. The involvement of these operators makes the convergence analysis more challenging. The advantages of the virtual element method can be summarized as follows:

- VEM can be interpreted as a generalization of the FEM that allows the use of general polygonal and polyhedral meshes.
- By carefully choosing dof and introducing a novel formulation corresponding to the stiffness and mass matrices, VEM avoids the need for explicit integration of the basis function.
- The trial and test functions on each element contain the polynomials plus other functions that generally are not polynomials.
- VEM provides more freedom in local refinement (hanging nodes are manageable).

We stress that the applications of VEMs have not yet been explored in terms of the numerical solution of PIDEs. In view of the computational advantages mentioned herein, we intend to employ VEM for a class of PIDEs and rigorously study their convergence behavior. The mathematical ideas developed here to obtain the optimal convergence analysis results can be used while dealing with the virtual discretization of more applicable problems, such as to solve the electrical circuit problems that govern the Kirchhoff voltage laws [6], for a disease spread by the dispersal of infectious individuals [7], heat flow in material with memory [8]. Furthermore, for the two formulations of mixed FEM [14, 15], only the semi-discrete case is discussed, but the fully-discrete case has not been explored yet to the best of our knowledge. So, we attempt to develop and analyze the mixed VEM scheme for these formulations and verify the theoretical results with the help of several numerical experiments. When dealing with non-smooth initial data, the analysis takes a different and more intricate path compared to the one for smooth initial data; here, we attempt to develop and analyze VEM for PIDE with the non-smooth initial data. We believe the present study can be considered a Bridgestone for studying mathematical/physical models governed by integral-differential equations.

## 1.3 Literature Review

VEM was first introduced for elliptic problems to explain the essential features of this method and show it as the ultimate evolution of MFD [13]. The challenge of formulating MFD arising from the absence of trial functions within the element's interior has led to its generalization and subsequent reintroduction as the VEM. The method is designed in such a way that it enables the construction of high-order approximation spaces, which may include an arbitrary degree of global regularity [16] on meshes consisting of very general polygonal (or polyhedral) elements.

Since its beginning, the VEM has been applied to a variety of problems, such as elliptic problem [17–19], Stokes equation [20, 21], plate bending problems [22], linear parabolic and hyperbolic problems [5, 23], convection dominated diffusion equation [24, 25], Navier-Stokes equation [26] and so on. A posteriori error estimates are derived and employed for adaptive analysis [27, 28]. The VEM has also been extended to semilinear and non-linear problems; see [26, 29–31].

In the context of the mixed VEM, its first introduction can be traced back to the work presented in [32]. The primary objective of this article was to provide a straightforward and introductory presentation of extending the VEM framework to discretize $H(\text{div})$-conforming vector fields. The mixed VEM has been developed for many problems; for example, for elliptic problem [32, 33], for the pseudo-stress-velocity formulation of the Stokes problem [34], for quasi-Newtonian Stokes flows [35]. The mixed VEMs for Brinkman and non-linear Brinkman problems were studied in [36, 37], respectively. Moreover, the mixed VEMs for the buckling problem of Kirchhoff plate [38], the Laplacian eigenvalue problem [39], and for the wormhole propagation, arising in petroleum engineering [40] were proposed.

As far as PIDEs are concerned, a range of methods has been developed to acquire numerical solutions for (1.1.1), such as FEM [14, 15, 41–45], FDM [46, 47], mixed FEM [14, 15, 48], FVM [49], least-square Galerkin method [50], $hp$-local discontinuous Galerkin method [51], and He's variational iteration method [52]. Further, by extending these ideas in [9, 45, 53–55], fully-discrete schemes were proposed in which discretization of time is implemented via implicit finite difference schemes. To simplify the problems with integral terms, R.V. projection was introduced in [45, 56]. The maximum norm estimates for R.V. projections to some time-dependent problems were presented in [57]. To establish connections with existing literature on PIDEs, we initially review the conventional FEM [58] to provide a meaningful context for our results:

*Classical Finite Element Method:*

Consider the following PIDE (1.3.1):

$$u_t(\boldsymbol{x},t) - \nabla\cdot\left(a(\boldsymbol{x})\nabla u(\boldsymbol{x},t) - \int_0^t b(\boldsymbol{x};t,s)\nabla u(\boldsymbol{x},s)ds\right)$$
$$= f(\boldsymbol{x},t) \quad (\boldsymbol{x},t)\in\mathcal{D}\times(0,T], \tag{1.3.1}$$

with the boundary condition:

$$u(\boldsymbol{x},t)=0 \quad (\boldsymbol{x},t)\in\partial\mathcal{D}\times(0,T],$$

and initial condition:

$$u(\boldsymbol{x},0)=u_0(\boldsymbol{x}) \quad \boldsymbol{x}\in\mathcal{D}.$$

To derive a variational formulation of (1.3.1) we multiply(1.3.1) by a test function $\chi$, which is assumed to vanish on the boundary $\partial\mathcal{D}$, see [59] and by integrate using Green's formula (i.e., integration by parts) to arrive at:

$$(u_t,\chi)+(a(\boldsymbol{x})\nabla u(\boldsymbol{x},t),\nabla\chi)-\int_0^t (b(\boldsymbol{x};t,s)\nabla u(\boldsymbol{x},s),\nabla\chi)ds = (f,\chi) \quad \forall\chi\in H_0^1(\mathcal{D}), \tag{1.3.2a}$$

along with the initial condition $u(\boldsymbol{x},0)=u_0(\boldsymbol{x})$. For numerical solution purposes, let's assume that we are provided with a family $S_h$ of finite-dimensional subspaces of $H_0^1(\mathcal{D})$. We require that the following inequality holds for all $v$ in $H_0^1\cap H^s$, where $1\le s\le r$ ($r\ge 2$ is a predetermined integer):

$$\min_{\chi\in S_h}\{\|v-\chi\|+h\|\nabla v-\nabla\chi\|\}\le Ch^s\|v\|_s.$$

The semi-discrete FEM that we shall study is: Find $u_h:(0,T]\to S_h$ such that:

$$(u_{h,t},\chi_h)+(a(\boldsymbol{x})\nabla u_h(\boldsymbol{x},t),\nabla\chi_h)-\int_0^t (b(\boldsymbol{x};t,s)\nabla u_h(\boldsymbol{x},s),\nabla\chi_h)ds = (f,\chi_h) \quad \forall\chi_h\in S_h, \tag{1.3.3}$$

with initial condition $u_h(0)=u_{h,0}$. Now, (1.3.3) gives a system of initial value problem, and by solving that, we can find the numerical solution. For the mixed FEM applied to PIDE, see [14, 15, 42, 43, 48]. The mixed formulation described in [14] is as follows:

Introduce $\boldsymbol{\sigma}(\boldsymbol{x}, t)$ as:

$$\boldsymbol{\sigma}(\boldsymbol{x}, t) = a(x)\nabla u(\boldsymbol{x}, t) - \int_0^t b(x; t, s)\nabla u(\boldsymbol{x}, s)ds, \tag{1.3.4}$$

and rewrite (1.3.1) as:

$$u_t(\boldsymbol{x}, t) - \nabla \cdot \boldsymbol{\sigma}(\boldsymbol{x}, t) = f(\boldsymbol{x}, t). \tag{1.3.5}$$

Consider the space $\mathcal{V} = H(\text{div}; \mathcal{D})$ and $\mathcal{Q} = L^2(\mathcal{D})$ and the corresponding discrete spaces $\mathcal{V}_h$ and $\mathcal{Q}_h$ having property $\nabla \cdot \mathcal{V}_h \subset \mathcal{Q}_h$. Since $a(x) > 0$, assuming $\mu(x) = a^{-1}(x)$, (1.3.4) becomes:

$$\nabla u(\boldsymbol{x}, t) = \mu(x)\boldsymbol{\sigma}(\boldsymbol{x}, t) + \int_0^t \mathcal{R}(x; t, s)\mu(x)\boldsymbol{\sigma}(\boldsymbol{x}, s)ds, \tag{1.3.6}$$

where $\mathcal{R}(x; t, s)$ is the resolvent kernel of $\mu(x)b(x; t, s)$, see [14, 60, 61] and satisfy the following:

$$\mathcal{R}(x; t, s) = \mu(x)b(x; t, s) + \int_s^t \mu(x)b(x; t, z)\mathcal{R}(x; z, s)dz \quad t > s \geq 0. \tag{1.3.7}$$

By denoting $\mathcal{K}(x; t, s) = \mathcal{R}(x; t, s)\mu(x)$, define variational formulation as: Find $(u, \boldsymbol{\sigma}) \in \mathcal{Q} \times \mathcal{V}$ such that:

$$\begin{aligned} (u_t, \phi) - (\nabla \cdot \boldsymbol{\sigma}, \phi) &= (f, \phi) \quad \forall \phi \in \mathcal{Q}, \\ (\mu\boldsymbol{\sigma}, \boldsymbol{\chi}) + \int_0^t (\mathcal{K}(t, s)\boldsymbol{\sigma}(s), \boldsymbol{\chi})ds + (\nabla \cdot \boldsymbol{\chi}, u) &= 0 \quad \forall \boldsymbol{\chi} \in \mathcal{V}. \end{aligned} \tag{1.3.8}$$

Now, the mixed discrete formulation reads as: Find $(u_h, \boldsymbol{\sigma}_h) \in \mathcal{Q}_h \times \mathcal{V}_h$ such that:

$$\begin{aligned} (u_{h,t}, \phi_h) - (\nabla \cdot \boldsymbol{\sigma}_h, \phi_h) &= (f, \phi_h) \quad \forall \phi_h \in \mathcal{Q}_h, \\ (\mu\boldsymbol{\sigma}_h, \boldsymbol{\chi}_h) + \int_0^t (\mathcal{K}_h(t, s)\boldsymbol{\sigma}_h(s), \boldsymbol{\chi}_h)ds + (\nabla \cdot \boldsymbol{\chi}_h, u_h) &= 0 \quad \forall \boldsymbol{\chi}_h \in \mathcal{V}_h. \end{aligned} \tag{1.3.9}$$

The other formulation corresponding to mixed FEM for PIDE as defined in [15], which avoided the use of the resolvent kernel and is characterized by:

Assuming $\mu(x) = a^{-1}(x)$, (1.3.4) becomes:

$$\nabla u(\boldsymbol{x}, t) = \mu(x)\boldsymbol{\sigma}(\boldsymbol{x}, t) + \int_0^t b_0(x; t, s)\nabla u(\boldsymbol{x}, s)ds,$$

where $\mu(x)b(x; t, s) = b_0(x; t, s)$, then the variational formulation reads: Find $(u, \boldsymbol{\sigma}) \in \mathcal{Q} \times \mathcal{V}$ such that:

$$(u_t, \phi) - (\nabla \cdot \boldsymbol{\sigma}, \phi) = (f, \phi) \quad \forall \phi \in \mathcal{Q}, \tag{1.3.10a}$$

$$(\mu\boldsymbol{\sigma}, \boldsymbol{\chi}) + (u, \nabla \cdot \boldsymbol{\chi}) - \int_0^t ((b_0(x; t, s)u(s), \nabla \cdot \boldsymbol{\chi}) + (\nabla b_0(x; t, s)u(s), \boldsymbol{\chi}))ds$$
$$= 0 \quad \forall \boldsymbol{\chi} \in \mathcal{V}. \tag{1.3.10b}$$

The discrete formulation corresponding to (1.3.10a) is: Find $(u_h, \boldsymbol{\sigma}_h) \in \mathcal{Q}_h \times \mathcal{V}_h$ such that:

$$(u_{h,t}, \phi_h) - (\nabla \cdot \boldsymbol{\sigma}_h, \phi_h) = (f, \phi_h) \quad \forall \phi_h \in \mathcal{Q}_h,$$

$$(\mu\boldsymbol{\sigma}_h, \boldsymbol{\chi}_h) + (u_h, \nabla \cdot \boldsymbol{\chi}_h)$$
$$- \int_0^t ((b_0(x; t, s)u_h(s), \nabla \cdot \boldsymbol{\chi}_h) + (\nabla b_0(x; t, s)u_h(s), \boldsymbol{\chi}_h))ds = 0 \quad \forall \boldsymbol{\chi}_h \in \mathcal{V}_h.$$

For the PIDEs with non-smooth initial data, there have been very few articles: Semi-discrete FEM for PIDEs has been discussed in [62] for the case of a homogenous equation with non-smooth initial data. An alternate approach to a priori error estimates for the semi-discrete Galerkin approximation to a PIDE with non-smooth initial data was proposed and analyzed [63]. Energy type arguments and the duality technique were used to obtain an $L^2$ error estimate of order $O(\frac{h^2}{t})$ when the given initial data is only in $L^2$ [64]. Fully-discrete FEM scheme with a backward Euler method for discretization in time has been proposed and analyzed when the initial data is in $H_0^1$ [65] and in $L^2$ [66]. A new mixed FEM for PIDE with non-smooth initial data has been discussed in [67] with three field formulations. Using the resolvent kernel, the semi-discrete case has been discussed in [14, 15] for the smooth and non-smooth initial data. As we can see from the literature, VEM and mixed

VEM in the context of PIDE are still unexplored. Here, we're trying to develop VEM and mixed VEM approaches that can handle PIDEs, whether the initial data is smooth or not.

## 1.4 Objectives of the Thesis

Based on the existing literature and the gap in the research direction, the following objectives are proposed and accomplished:

- To develop and analyze the semi-discrete scheme of the VEM and mixed VEM for PIDE with smooth initial data.
- To develop and analyze the fully-discrete scheme of the VEM and mixed VEM for PIDE with smooth initial data.
- To analyze the super convergence behavior of the discrete solution in mixed VEM for the smooth initial data.
- To validate the theoretical findings of the above-mentioned objective with the numerical experiments.
- To develop and analyze VEM for PIDE with non-smooth initial data.
- To develop and analyze mixed VEM for PIDE with non-smooth initial data.

## 1.5 Preliminaries

This section introduces preliminaries that will be frequently used throughout the thesis.

### 1.5.1 Function Spaces

We shall make use of the following spaces frequently, see [68, 69]:

1. **Polynomial Spaces**: $\mathbb{P}_k(\mathcal{D})$ is the set consisting of polynomials of degree $\leq k$ in $\mathcal{D}$.
2. **$L^p$ Spaces**: $L^p$ spaces, also known as Lebesgue spaces, are mathematical function spaces used to assess the behavior and characteristics of functions in terms of their

integrability. For $1 \leq p < \infty$; $L^p(\mathcal{D})$ contains measurable functions $\varphi : \mathcal{D} \to \mathbb{R}$ such that $\int_{\mathcal{D}} |\varphi(\boldsymbol{x})|^p d\boldsymbol{x} < \infty$, whereas the $L^p$-norm is given by:

$$\|\varphi\|_{L^p(\mathcal{D})} = \left( \int_{\mathcal{D}} |\varphi(\boldsymbol{x})|^p d\boldsymbol{x} \right)^{\frac{1}{p}}.$$

The following inclusion is valid for a bounded domain $\mathcal{D}$ and $1 \leq p < q \leq \infty$:

$$L^q(\mathcal{D}) \subset L^p(\mathcal{D}).$$

For $\frac{1}{q} = \frac{1}{p} + \frac{1}{r}$, where $r > 0$, we have:

$$\|\phi\|_{L^p(\mathcal{D})} \leq \text{ meas } (\mathcal{D})^{\frac{1}{r}} \|\phi\|_{L^q(\mathcal{D})}.$$

The proof follows from Hölder inequality.

3. $C_c^1(\mathcal{D})$ consists of continuous functions having compact support and possesses continuous first-order derivatives.

4. $C^0(\mathcal{D})$ consists of continuous functions.

5. **Sobolev Spaces**: Sobolev spaces are function spaces that contain functions with certain degrees of weak derivatives. For natural number $m \geq 1$ and a real number $n$; $1 \leq n \leq \infty$, we establish the space $W^{m,n}(\mathcal{D})$ using an iterative approach as:

   - The space $W^{m,n}(\mathcal{D})$ is defined as:

   $$W^{m,n}(\mathcal{D}) = \{\phi \in L^n(\mathcal{D}), D^\alpha \phi \in L^n(\mathcal{D}) \text{ for all } |\alpha| \leq m\}$$

   A multi-index $\alpha$ is an $N$-tuple of non-negative integers. Thus, $\alpha = (\alpha_1, \cdots, \alpha_N)$ where the $\alpha_i$ are all non-negative integers. We define

   $$|\alpha| = \sum_{i=1}^{N} \alpha_i$$

$$x^{\alpha} = x_1^{\alpha_1} \cdots x_N^{\alpha_N} \qquad \text{for } x = (x_1, x_2, \cdots, x_N) \in \mathbb{R}^N$$
$$D^{\alpha} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}}$$

- The space $W_0^{1,n}$ referred as the space which contains all the functions of $W^{1,n}$, with homogenous boundary condition.
- Moreover, when $n = 2$, we denote $W^{m,2}(\mathcal{D}) = H^m(\mathcal{D})$.

We write commonly $(\cdot, \cdot)$ and $\|\cdot\|$ to indicate the $L^2$ inner product and norm. As usual, $|\cdot|_{s,\mathcal{D}}$ and $\|\cdot\|_{s,\mathcal{D}}$ denote the $H^s(\mathcal{D})$ semi-norm and norm respectively.

6. $H(\text{div}; \mathcal{D}) = \{\boldsymbol{\chi} \in (L^2(\mathcal{D}))^2 : \text{div}\, \boldsymbol{\chi} \in L^2(\mathcal{D})\}$.

7. $H(\text{rot}; \mathcal{D}) = \{\boldsymbol{\chi} \in (L^2(\mathcal{D}))^2 : \text{rot}\, \boldsymbol{\chi} \in L^2(\mathcal{D})\}$, where rot $\boldsymbol{\chi}$ for $\boldsymbol{\chi} = (\boldsymbol{\chi}_1, \boldsymbol{\chi}_2)$ is defined as:
$$\text{rot}\, \boldsymbol{\chi} = \frac{\partial \boldsymbol{\chi}_2}{\partial x} - \frac{\partial \boldsymbol{\chi}_1}{\partial y}.$$

8. $L^n(0, T, H^s(\mathcal{D}))$ for $n \geq 1$ and $s \geq 1$ with the standard modification for $n = \infty$ is defined as:

$$L^n(0, T, H^s(\mathcal{D})) = \{u(\boldsymbol{x}, t) \in H^s(\mathcal{D}) \text{ for a.e. } t \in (0, T] \text{ and } \left(\int_0^T \|u(\cdot, t)\|_{s,\mathcal{D}}^n dt\right)^{1/n} < \infty\},$$

$$L^n(0, T, L^2(\mathcal{D})) = \{u(\boldsymbol{x}, t) \in L^2(\mathcal{D}) \text{ for a.e. } t \in (0, T] \text{ and } \left(\int_0^T \|u(\cdot, t)\|_{L^2(\mathcal{D})}^n dt\right)^{1/n} < \infty\},$$

with the norms

$$\|u\|_{L^n(0,T,H^s(\mathcal{D}))} = \left(\int_0^T \|u(\cdot, t)\|_{s,\mathcal{D}}^n dt\right)^{1/n},$$

$$\|u\|_{L^\infty(0,T,H^s(\mathcal{D}))} = \sup_{0\le t\le T} \|u(\cdot,t)\|_{s,\mathcal{D}},$$

$$\|u\|_{L^n(0,T,L^2(\mathcal{D}))} = \left(\int_0^T \|u(\cdot,t)\|^n_{L^2(\mathcal{D})} dt\right)^{1/n}, \quad \|u\|_{L^\infty(0,T,L^2(\mathcal{D}))} = \sup_{0\le t\le T} \|u(\cdot,t)\|_{L^2(\mathcal{D})}.$$

Moreover, $\mathcal{V} = H(\text{div};\mathcal{D})$ and $\mathcal{Q} = L^2(\mathcal{D})$. When no confusion can occur, the indication of the domain $\mathcal{D}$ will be omitted.

### 1.5.2 Standard Inequalities

Throughout this study, we shall make use of the following inequalities frequently: [70]

1. **Cauchy–Schwarz inequality**:

   For $\nu$ and $\mu$ as the two real-valued functions within an inner product space, then:

$$|\langle \nu, \mu\rangle| \le \|\nu\|\|\mu\|.$$

2. **Young's inequality**:

   For $a, b \ge 0$ and $\epsilon > 0$, we have:

$$ab \le \frac{a^2}{2\epsilon} + \frac{\epsilon b^2}{2}.$$

3. **Hölder's inequality**:

   For $i, j \ge 1$, such that $\frac{1}{i} + \frac{1}{j} = 1$ then for all measurable real valued functions $\omega$ and $\psi$, we have:

$$\|\omega\psi\|_{L^1(\mathcal{D})} \le \|\omega\|_{L^i(\mathcal{D})}\|\psi\|_{L^j(\mathcal{D})}.$$

4. **Poincaré inequality**:

   For $p \ge 1$, and $\varphi \in W_0^{1,p}(\mathcal{D})$, it holds:

$$\|\varphi\|_{L^p(\mathcal{D})} \le C\|\nabla\varphi\|_{L^p(\mathcal{D})}.$$

### 1.5.3 Grönwall's Lemma

Now, we present Grönwall's lemmas, which will be used frequently in the analysis:

**Lemma 1.5.1.** *Let $g(t)$, $f(t)$ be continuous function and $h(t)$ be non-negative continuous function on $t_0 \leq t \leq T$ and satisfy:*

$$f(t) \leq g(t) + \int_{t_0}^{t} f(s)h(s)ds \quad \text{for } t \in [t_0, T].$$

*Then, we have:*

$$f(t) \leq g(t) + \int_{t_0}^{t} g(s)h(s)\text{exp}\left(\int_{s}^{t} h(\tau)d\tau\right) ds \quad \text{for } t \in [t_0, T].$$

For more details, see [71].

**Lemma 1.5.2. (Discrete version of Grönwall's lemma)**: *If $\{\zeta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are nonnegative sequences and satisfy:*

$$\zeta_n \leq \gamma_n + \sum_{j=0}^{n-1} \delta_j \zeta_j, \quad \text{for } n \geq 0.$$

*Then, we have:*

$$\zeta_n \leq \gamma_n + \sum_{j=0}^{n-1} \gamma_j \delta_j \text{exp}\left(\sum_{k=j+1}^{n-1} \delta_k\right) \quad \text{for } n \geq 0.$$

For more details about discrete Grönwall's lemma, see [72].

### 1.5.4 Virtual Element Subdivision

We assume $\{\mathcal{I}_h\}_h$ to be the sequence of decomposition of $\mathcal{D}$ into star-shaped sub-polygons $\mathbb{K}$, whereas $\mathbb{K}_h$ is the set of edges $e$ of $\mathcal{I}_h$. The vertices of each polygon $\mathbb{K}$ are designated by $v_i(i = 1, \cdots, N_{\mathbb{K}})$ with $N_{\mathbb{K}}$ as the number of vertices in polygon $\mathbb{K}$. Along with this, we postulate that for each element $\mathbb{K}$, there exists a $\delta_{\mathbb{K}} > 0$ such that $\mathbb{K}$ is star-shaped from every point of the disc $D_{\delta_{\mathbb{K}}}$ with radius $\delta_{\mathbb{K}} h_{\mathbb{K}}$ ($h_{\mathbb{K}}$ represents the element $\mathbb{K}$'s diameter), while $h_e$ represents the edge $e$'s length, of the element $\mathbb{K}$ and fulfills $h_e \geq \delta_{\mathbb{K}} h_{\mathbb{K}}$. While considering this decomposition $\{\mathcal{I}_h\}_h$, we assume $\delta_{\mathbb{K}} \geq \delta_0 > 0$ for some $\delta_0$ independent of $\mathbb{K}$ and $\mathcal{I}_h$. The largest diameter of $\mathcal{I}_h$'s elements is $h$, as is customary.

**Degrees of Freedom**:

Here, we introduce dof, which we shall use in our subsequent Chapters.

1. **Degrees of Freedom for Confirming Virtual Element Method**

   We present the well-defined collection of operators $(D_i)_{i=1}^3$ from $\tilde{V}_{k|\mathbb{K}}$ (1.5.1) into $\mathbb{R}$, $\forall\, v \in \tilde{V}_{k|\mathbb{K}}$:

   $D_1$ : the value of $v$ at the $N_{\mathbb{K}}$ vertices of $\mathbb{K}$;

   $D_2$ : for $k > 1$, value of $v$ at the $k$-1 distinct internal points on each of the edges $e$ of $\partial\mathbb{K}$;

   $D_3$ : for $k > 1$, all internal moments $\dfrac{1}{|\mathbb{K}|}\displaystyle\int_{\mathbb{K}} v(\boldsymbol{x}) m_{k-2}(\boldsymbol{x})\, d\boldsymbol{x}, \quad \forall m_{k-2}(\boldsymbol{x})$, where $m_{k-2}(\boldsymbol{x})$ are scaled monomials upto degree $k-2$.

   The first two sets of dof are the ones that define the boundary. They determine the unique value of $v$ on the boundary of polygon $\mathbb{K}$, and the third one is the internal dof.

2. **Degrees of Freedom for Mixed Virtual Element Method** The discrete bilinear forms in spaces $V_h^k(\mathbb{K})$(1.5.4) and $Q_h^k(\mathbb{K})$ can be computed via dof. For the space $Q_h^k$, we are considering the scaled monomials on each element $\mathbb{K}$ as dof:

   $$\left(1, \left(\frac{x - x_{c_{\mathbb{K}}}}{h_{\mathbb{K}}}\right), \left(\frac{y - y_{c_{\mathbb{K}}}}{h_{\mathbb{K}}}\right)\right) \quad \text{for } k = 1;$$

   $$\left(1, \left(\frac{x - x_{c_{\mathbb{K}}}}{h_{\mathbb{K}}}\right), \left(\frac{y - y_{c_{\mathbb{K}}}}{h_{\mathbb{K}}}\right), \left(\frac{x - x_{c_{\mathbb{K}}}}{h_{\mathbb{K}}}\right)^2, \left(\frac{x - x_{c_{\mathbb{K}}}}{h_{\mathbb{K}}}\right)\left(\frac{y - y_{c_{\mathbb{K}}}}{h_{\mathbb{K}}}\right), \left(\frac{y - y_{c_{\mathbb{K}}}}{h_{\mathbb{K}}}\right)^2\right)$$

   for $k = 2$ and in a similar way for the higher value of $k$. For the local space $V_h^k(\mathbb{K})$, we present the well-defined collection of operators as dof, see [17]:

- $\displaystyle\int_e \boldsymbol{\chi} \cdot \boldsymbol{n} q_k d\gamma$ for each edge $e$ of the element $\mathbb{K}$, $\forall q_k \in \mathbb{P}_k(e)$,
- $\displaystyle\int_{\mathbb{K}} \boldsymbol{\chi} \cdot \boldsymbol{g}_{k-1} d\boldsymbol{x}, \quad \forall \boldsymbol{g}_{k-1} \in \nabla \mathbb{P}_k(\mathbb{K})$,
- $\displaystyle\int_{\mathbb{K}} \boldsymbol{\chi} \cdot \boldsymbol{g}_k^{\perp} d\boldsymbol{x}, \quad \forall \boldsymbol{g}_k^{\perp} \in L^2(\mathbb{K})$ orthogonal of $\nabla \mathbb{P}_{k+1}(\mathbb{K})$ in $(\mathbb{P}_k(\mathbb{K}))^2$.

**Scaled Monomials**:

We denote $M_k(\mathbb{K})$ as the set of scaled monomials of degree less than or equal to $k$ such that:

$$M_k(\mathbb{K}) := \{m_{\boldsymbol{\beta}} : 0 \leq |\boldsymbol{\beta}| \leq k\},$$

where $\boldsymbol{\beta}$ indicates a multi-index in two dimensions; $\boldsymbol{\beta} = (\beta_1, \beta_2)$ with the usual notation $|\boldsymbol{\beta}| = \beta_1 + \beta_2$. If $\boldsymbol{x} = (x_1, x_2)$, then $\boldsymbol{x}^{\beta} = x_1^{\beta_1} x_2^{\beta_2}$,

$$m_{\boldsymbol{\beta}} = \left( \frac{\boldsymbol{x} - \boldsymbol{x}_{c_{\mathbb{K}}}}{h_{\mathbb{K}}} \right)^{\boldsymbol{\beta}}.$$

where $\boldsymbol{x}_{c_{\mathbb{K}}} = (x_{c_{\mathbb{K}}}, y_{c_{\mathbb{K}}})$ is the centroid and $h_{\mathbb{K}}$ be the diameter of $\mathbb{K}$. For more details, see [19].

### 1.5.5 Virtual Element Spaces and Local Projection

Here, we introduce discrete spaces and local projections that we shall use in our subsequent Chapters.

1. **Virtual Element Space and Projections**:

   We describe the augmented local space $\tilde{V}_{k|\mathbb{K}}$ [17], for all $\mathbb{K}$ as:

$$\tilde{V}_{k|\mathbb{K}} = \{v \in H^1(\mathbb{K}) : \Delta v \in \mathbb{P}_k(\mathbb{K}), \quad v \in C^0(\partial K) : v|_e \in \mathbb{P}_k(e) \, \forall e \subset \partial \mathbb{K}\}, \tag{1.5.1}$$

   where, $\{e\}_{e \in \partial \mathbb{K}}$ represents the collection of polygon $\mathbb{K}'$s edges.

   Now, consider the operator $\Pi_k^{\nabla} : \tilde{V}_{k|\mathbb{K}} \to \mathbb{P}_k(\mathbb{K})$, as defined by Da Veiga [19], for every $v \in \tilde{V}_{k|\mathbb{K}}$ (1.5.1):

$$\left(\nabla p_k, \nabla\left(\Pi_k^{\nabla} v - v\right)\right)_{0,\mathbb{K}} = 0 \quad \forall p_k \in \mathbb{P}_k(\mathbb{K}). \tag{1.5.2}$$

To deal with constant, we need to add one more term, i.e. $P_0 : \tilde{V}_{k|\mathbb{K}} \to \mathbb{P}_0(\mathbb{K})$ defined for $v \in \tilde{V}_{k|\mathbb{K}}$:

$$P_0\left(\Pi_k^{\nabla} v - v\right) = 0.$$

Here, we are choosing

$$P_0 v := \begin{cases} \frac{1}{N_{\mathbb{K}}} \sum_{i=1}^{N_{\mathbb{K}}} v(v_i) & k = 1, \\ \frac{1}{|\mathbb{K}|} \int_{\mathbb{K}} v dx & k > 1. \end{cases}$$

Define $L^2$-operators $\Pi_k^0$ and $\mathbf{\Pi}_k^0$, for every $v \in \tilde{V}_{k|\mathbb{K}}$ and $\boldsymbol{v} \in (\tilde{V}_{k|\mathbb{K}})^2$ respectively, with the following orthogonality condition:

$$(p_k, \Pi_k^0 v - v)_{0,\mathbb{K}} = 0, \quad \forall p_k \in \mathbb{P}_k(\mathbb{K}), \tag{1.5.3a}$$

$$(\boldsymbol{p_k}, \mathbf{\Pi}_k^0 \boldsymbol{v} - \boldsymbol{v})_{0,\mathbb{K}} = 0, \quad \forall \boldsymbol{p_k} \in (\mathbb{P}_k(\mathbb{K}))^2. \tag{1.5.3b}$$

On $\tilde{V}_{k|\mathbb{K}}$, the polynomial $\Pi_k^{\nabla} v$ can be calculated solely by utilizing the values of the operators $(D_i)_{i=1}^3$ calculated on $v$, see 1.5.4. Now, we are all set to create our virtual local space:

$$W_{k,\mathbb{K}} = \{w \in \tilde{V}_{k|\mathbb{K}} : \int_{\mathbb{K}} (\Pi_k^{\nabla} w) q d\boldsymbol{x} = \int_{\mathbb{K}} w q d\boldsymbol{x} \quad \forall q \in \mathbb{P}_k(\mathbb{K})/\mathbb{P}_{k-2}(\mathbb{K})\},$$

where $\mathbb{P}_k(\mathbb{K})/\mathbb{P}_{k-2}(\mathbb{K})$ denotes the polynomials of degree $k$ living on $\mathbb{K}$ that are $L^2$-orthogonal to all polynomials of degree $k$-2 on $\mathbb{K}$. Moreover, the space $W_{k,\mathbb{K}}$ satisfies following properties:

(i) $\mathbb{P}_k(\mathbb{K}) \subset W_{k,\mathbb{K}}$.

(ii) The function $\Pi_k^{\nabla} v$ can be simply calculated with the help of dof ($D1$-$D3$) of $v$, $\forall\, v \in W_{k,\mathbb{K}}$.

(iii) The function $\Pi_k^0 v$ can be simply calculated from the dof $(D1\text{-}D3)$ of $v$, $\forall$ $v \in W_{k,\mathbb{K}}$.

(iv) The function $\Pi_{k-1}^0 v$ can be simply calculated from the dof $(D1\text{-}D3)$ of $v$, $\forall$ $v \in W_{k,\mathbb{K}}$.

It is now possible to create the global discrete space as in traditional finite element methods:

$$W_h = \{w \in H_0^1(\mathcal{D}) : w_{|\mathbb{K}} \in W_{k,\mathbb{K}} \,\forall\, \mathbb{K} \in \mathcal{I}_h\}.$$

2. **Mixed Virtual Element Spaces and Projections**:

We define the local space [33]:

$$\begin{aligned} V_h^k(\mathbb{K}) = \{\boldsymbol{\chi} \in H(\text{div};\mathbb{K}) \cap H(\text{rot};\mathbb{K}) : \boldsymbol{\chi} \cdot \boldsymbol{n}_{|e} \in \mathbb{P}_k(e) \forall e \in \partial\mathbb{K}, \\ \nabla \cdot \boldsymbol{\chi} \in \mathbb{P}_k(\mathbb{K}) \text{ and, rot } \boldsymbol{\chi} \in \mathbb{P}_{k-1}(\mathbb{K}) \text{ for } k \geq 0\}, \end{aligned} \tag{1.5.4}$$

where $\mathbb{P}_{-1}(\mathbb{K}) = \{0\}$. For our analysis, we define the global discrete spaces as:

$$V_h^k := \{\boldsymbol{\chi} \in \mathcal{V} : \ \boldsymbol{\chi}_{|\mathbb{K}} \in V_h^k(\mathbb{K}) \quad \forall\, \mathbb{K} \text{ in } \mathcal{I}_h\}, \tag{1.5.5}$$

$$Q_h^k := \{q \in L^2(\mathcal{D}) : \ q_{|\mathbb{K}} \in \mathbb{P}_k(\mathbb{K}) \quad \forall\, \mathbb{K} \text{ in } \mathcal{I}_h\}. \tag{1.5.6}$$

To define discrete variational formulation, we make use of the $L^2$-projection operators denoted by $\Pi_k^0 : \mathcal{Q} \to Q_h^k$(1.5.6) and $\boldsymbol{\Pi}_k^0 : \mathcal{V} \to V_h^k$(1.5.5), and defined for $q \in \mathcal{Q}$ and $\boldsymbol{\chi} \in \mathcal{V}$ as:

$$\begin{aligned} \int_{\mathbb{K}} (q - \Pi_k^0 q) p_k d\boldsymbol{x} = 0 \quad \forall p_k \in \mathbb{P}_k(\mathbb{K}), \quad \forall \mathbb{K} \in \mathcal{I}_h, \\ \int_{\mathbb{K}} (\boldsymbol{\chi} - \boldsymbol{\Pi}_k^0 \boldsymbol{\chi}) \boldsymbol{p_k} d\boldsymbol{x} = 0 \quad \forall \boldsymbol{p_k} \in (\mathbb{P}_k(\mathbb{K}))^2, \quad \forall \mathbb{K} \in \mathcal{I}_h. \end{aligned}$$

$\Pi_k^0$ and $\boldsymbol{\Pi}_k^0$ satisfies the following estimates, see [17]:

$$||q - \Pi_k^0 q||_0 \leq Ch^r |q|_r, \qquad ||\boldsymbol{\chi} - \boldsymbol{\Pi}_k^0 \boldsymbol{\chi}||_0 \leq Ch^r |\boldsymbol{\chi}|_r \quad 0 \leq r \leq k+1. \tag{1.5.7}$$

Now, define "Fortin" operator $\mathbf{\Pi}_h^F$: $(H^1(\mathcal{D}))^2 \to V_h^k$ through the dof of the space $V_h^k$ as:

- $\displaystyle\int_e (\boldsymbol{\chi} - \mathbf{\Pi}_h^F \boldsymbol{\chi}) \cdot \boldsymbol{n} q_k d\gamma = 0$ for each edge $e$, $\forall q_k \in \mathbb{P}_k(\mathbb{K})$,
- $\displaystyle\int_{\mathbb{K}} (\boldsymbol{\chi} - \mathbf{\Pi}_h^F \boldsymbol{\chi}) \cdot \boldsymbol{g}_{k-1} d\boldsymbol{x} = 0$ for each element $\mathbb{K}$, $\forall \boldsymbol{g}_{k-1} \in \nabla\mathbb{P}_k(\mathbb{K})$,
- $\displaystyle\int_{\mathbb{K}} (\boldsymbol{\chi} - \mathbf{\Pi}_h^F \boldsymbol{\chi}) \cdot \boldsymbol{g}_k^{\perp} d\boldsymbol{x} = 0$ for each element $\mathbb{K}$, $\forall \boldsymbol{g}_k^{\perp} \in L^2(\mathbb{K})$ orthogonal of $\nabla\mathbb{P}_{k+1}(\mathbb{K})$ in $(\mathbb{P}_k(\mathbb{K}))^2$.

$\mathbf{\Pi}_h^F$ satisfy the following properties and estimates:

$$\nabla \cdot \mathbf{\Pi}_h^F \boldsymbol{\chi} = \Pi_k^0 \nabla \cdot \boldsymbol{\chi}, \tag{1.5.8}$$

$$||\boldsymbol{\chi} - \mathbf{\Pi}_h^F \boldsymbol{\chi}||_0 \le C h^r |\boldsymbol{\chi}|_r, \quad \|\nabla \cdot (\boldsymbol{\chi} - \mathbf{\Pi}_h^F \boldsymbol{\chi})\|_0 \le C h^r |\nabla \cdot \boldsymbol{\chi}|_r \quad 0 \le r \le k+1. \tag{1.5.9}$$

For more details about $\mathbf{\Pi}_h^F$, we refer to [73].

## 1.6 Organization of the Thesis

Following the clarification of the main objective and contributions of the thesis, this section provides a brief summary of the chapter-wise roadmap. The thesis comprises a total of seven chapters. Chapter 1 includes the motivation behind our work, important preliminaries, a literature survey, and objectives of the thesis.

Chapter 2 develops and analyzes a virtual element scheme for the spatial discretization of PIDEs combined with backward Euler's scheme for temporal discretization. We derive optimal a priori error estimates for both the semi-discrete and fully-discrete cases using R.V. and $L^2$ projection operators. Several numerical experiments are being presented to confirm the computational efficiency of the proposed scheme and validate the theoretical findings. To demonstrate the real application of VEMs, we conduct numerical experiments with local mesh refinements and show that errors can be reduced efficiently by using hanging nodes. The results of this chapter have been published in [74].

Chapter 3 presents mixed VEM for semi-discrete and fully-discrete cases. The formulation in this chapter uses the concept of a resolvent kernel. By defining the mixed R.V. projection, optimal error estimates are established for the two unknowns. Along with that, the super convergence of the discrete solution is analyzed. Several numerical experiments are presented to support the theoretical findings. Along with that, we also show that if we don't know the explicit formulation of the resolvent kernel, we can get the optimal convergence by truncating the resolvent kernel series after a few steps.

Chapter 4 develops and analyzes the new mixed VEM for the semi-discrete and fully-discrete cases without using a resolvent kernel. We define the new mixed intermediate projection and establish the optimal convergence for the two unknowns. Super convergence of the discrete solution is also proposed. With the help of numerical experiments, we show that this formulation is more generalized in the sense that it can be applied to a wider range of applications.

Chapter 5 deals with the confirming VEM applied to PIDEs for the semi-discrete case when the initial data is non-smooth. We define the intermediate projection and find out the intermediate projection's estimates in terms of non-smooth initial data. By the repetitive use of the integral operator, error estimates are established in the case when the initial data is non-smooth.

Chapter 6 presents and analyzes two distinctive approaches to the mixed VEM applied to PIDEs with non-smooth initial data. In the first part of the chapter, we introduce and analyze a mixed virtual element scheme for PIDE that eliminates the need for the resolvent operator. Through the introduction of a novel projection involving a memory term, coupled with the application of energy arguments and the repeated use of an integral operator, this study establishes optimal $L^2$-error estimates for the two unknowns $p$ and $\boldsymbol{\sigma}$. Furthermore, optimal error estimates are derived for the standard mixed formulation with a resolvent kernel. The chapter offers a comprehensive analysis of the VEM, encompassing both formulations.

Finally, in Chapter 7, we present some critical assessments of our results and discuss possible extensions and also scope for future problems.

# Chapter 2

# Virtual Element Method for Parabolic Integro-Differential Equations[1]

In this chapter, we develop and analyze a conforming virtual element scheme for the spatial discretization of PIDEs combined with backward Euler's scheme for temporal discretization. For deriving optimal error estimates, we introduce a projection operator (in literature, known as R.V. projection) that contains the memory term. In contrast with finite element formulation, virtual element discretization requires two projection operators: One is $L^2-$ projection ($\Pi_k^0$) and energy projection ($\Pi_k^\nabla$), defined in (1.5.2) and (1.5.3a). The involvement of these operators makes the convergence analysis more challenging. In this chapter, a sophisticated analysis is carried out to establish the optimal convergence rates for the proposed fully and semi-discrete schemes in the $L^2$ and $H^1$ norms. The mathematical ideas developed here to obtain the optimal convergence analysis results can be used while dealing with the virtual discretization of more applicable problems. Moreover, in order to show the real application of VEMs, numerical experiments are conducted with local mesh refinements, which are necessary to reduce the overall computational cost but may not be

[1]The substantial part of this chapter has been published in the following publication:
S Yadav, **M Suthar**, and S Kumar "A Conforming Virtual Element Method for Parabolic Integro-Differential Equations", *Computational Methods in Applied Mathematics* https://doi.org/10.1515/cmam-2023-0061 (2023).

possible in the context of conforming FEMs. In this chapter, we focus on the following linear PIDE (1.1.1) defined on $\mathcal{D} \subset \mathbb{R}^2$. Find $u(\boldsymbol{x}, t)$ such that:

$$u_t(\boldsymbol{x}, t) + \mathbb{A}u(\boldsymbol{x}, t) - \int_0^t \mathbb{B}(t, s)u(\boldsymbol{x}, s)ds = f(\boldsymbol{x}, t) \quad (\boldsymbol{x}, t) \in \mathcal{D} \times (0, T], \tag{2.0.1}$$

along with the homogenous boundary condition and initial data $u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x})$, where

$$\mathbb{B}(t, s)u(\boldsymbol{x}, t) := -\nabla \cdot (b(\boldsymbol{x}; t, s)\nabla u(\boldsymbol{x}, t)) + \mathbf{b}_1(\boldsymbol{x}; t, s) \cdot \nabla u(\boldsymbol{x}, t) + b_0(\boldsymbol{x}; t, s)u(\boldsymbol{x}, t),$$
$$\mathbb{A}u(\boldsymbol{x}, t) := -\nabla \cdot (a(\boldsymbol{x})\nabla u(\boldsymbol{x}, t)) + a_0(\boldsymbol{x})u(\boldsymbol{x}, t).$$

Now, by multiplying the suitable test function, the variational form corresponds to (2.0.1) read as follows. Find $u \in L^2(0, T; H_0^1(\mathcal{D}))$ such that,

$$\langle u_t, v\rangle + \mathcal{A}(u, v) - \int_0^t \mathcal{B}(t, s; u(s), v)ds = \langle f, v\rangle \quad t \in (0, T],\ \forall v \in H_0^1(\mathcal{D}), \tag{2.0.2}$$

with $u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x})$ and

$$\mathcal{A}(u, v) := \int_{\mathcal{D}} [a(\boldsymbol{x})\nabla u(\boldsymbol{x}, t) \cdot \nabla v(\boldsymbol{x}) + a_0(\boldsymbol{x})u(\boldsymbol{x}, t)v(\boldsymbol{x})]\, d\boldsymbol{x}, \tag{2.0.3a}$$
$$\mathcal{B}(t, s; u, v) := \int_{\mathcal{D}} [b(\boldsymbol{x}; t, s)\nabla u(\boldsymbol{x}, s) \cdot \nabla v(\boldsymbol{x}) + \boldsymbol{b}_1(\boldsymbol{x}; t, s) \cdot \nabla u(\boldsymbol{x}, s)v(\boldsymbol{x})$$
$$+ b_0(\boldsymbol{x}; t, s)u(\boldsymbol{x}, s)v(\boldsymbol{x})]\, d\boldsymbol{x}. \tag{2.0.3b}$$

## 2.1 VEM Semi-discrete Formulation

Defining local counterparts of bilinear forms in (2.0.3a)-(2.0.3b) as:

$$\mathcal{A}(u, v) := \sum_{\mathbb{K} \in \mathcal{T}_h} \mathcal{A}^{\mathbb{K}}(u, v) \quad \text{for all } u, v \in V,$$

$$\mathcal{B}(t, s; u, v) := \sum_{\mathbb{K} \in \mathcal{T}_h} \mathcal{B}^{\mathbb{K}}(t, s; u, v) \quad \text{for all } u, v \in V,$$

where

$$\mathcal{A}^{\mathbb{K}}(u,v) := \int_{\mathbb{K}} (a(\boldsymbol{x})\nabla u(\boldsymbol{x},t)\cdot\nabla v(\boldsymbol{x}) + a_0(\boldsymbol{x})u(\boldsymbol{x},t)v(\boldsymbol{x}))d\boldsymbol{x},$$

$$\mathcal{B}^{\mathbb{K}}(t,s;u,v) := \int_{\mathbb{K}} (b(\boldsymbol{x};t,s)\nabla u(\boldsymbol{x},s)\cdot\nabla v(\boldsymbol{x}) + \boldsymbol{b}_1(\boldsymbol{x};t,s)\cdot\nabla u(\boldsymbol{x},s)v(\boldsymbol{x}) + b_0(\boldsymbol{x};t,s)u(\boldsymbol{x},s)v(\boldsymbol{x}))d\boldsymbol{x}.$$

The semi-discrete approximation to the problem (2.0.1) can be constructed as:

$$m_h(u_{h,t},v_h) + \mathcal{A}_h(u_h,v_h) - \int_0^t \mathcal{B}_h(t,s;u_h(s),v_h)ds = \langle f_h(t), v_h\rangle \ \forall v_h\in W_h, \quad (2.1.1)$$

along with the initial data $u_h(0)=u_{h,0}$, where $u_{h,0}$ will be defined later in the proof of Theorem-2.2.3. The above discrete bilinear forms are computable and defined for all $p_h, q_h \in W_h$ as:

$$m_h(p_h,q_h) := \sum_{\mathbb{K}\in\mathcal{T}_h} m_h^{\mathbb{K}}(p_h,q_h), \quad \mathcal{A}_h(p_h,q_h) := \sum_{\mathbb{K}\in\mathcal{T}_h} \mathcal{A}_h^{\mathbb{K}}(p_h,q_h),$$

$$\mathcal{B}_h(t,s;p_h,q_h) := \sum_{\mathbb{K}\in\mathcal{T}_h} \mathcal{B}_h^{\mathbb{K}}(t,s;p_h,q_h).$$

The local discrete bilinear forms on every element $\mathbb{K}\in\mathcal{T}_h$ are defined for any $v_h, w_h \in W_{k,\mathbb{K}}$ (see [17]) as below:

$$\mathcal{A}_h^{\mathbb{K}}(\cdot,\cdot): W_{k,\mathbb{K}}\times W_{k,\mathbb{K}}\to\mathbb{R}, \quad \mathcal{B}_h^{\mathbb{K}}(\cdot,\cdot): W_{k,\mathbb{K}}\times W_{k,\mathbb{K}}\to\mathbb{R},$$

$$m_h^{\mathbb{K}}(\cdot,\cdot): W_{k,\mathbb{K}}\times W_{k,\mathbb{K}}\to\mathbb{R},$$

$$m_h^{\mathbb{K}}(v_h,w_h) := (\Pi_k^0 v_h, \Pi_k^0 w_h)_{0,\mathbb{K}} + S_1^{\mathbb{K}}((I-\Pi_k^0)v_h,(I-\Pi_k^0)w_h),$$

$$\mathcal{A}_h^{\mathbb{K}}(v_h,w_h) := \int_{\mathbb{K}} (a(\boldsymbol{x})\boldsymbol{\Pi}_{k-1}^0\nabla v_h\cdot\boldsymbol{\Pi}_{k-1}^0\nabla w_h d\boldsymbol{x} + a_0(\boldsymbol{x})\Pi_{k-1}^0 v_h \Pi_{k-1}^0 w_h)d\boldsymbol{x} + S_0^{\mathbb{K}}((I-\Pi_k^{\nabla})v_h,(I-\Pi_k^{\nabla})w_h,),$$

$$\mathcal{B}_h^{\mathbb{K}}(t,s;v_h,w_h) := \int_{\mathbb{K}} (b(\boldsymbol{x};t,s)\boldsymbol{\Pi}_{k-1}^0\nabla v_h\cdot\boldsymbol{\Pi}_{k-1}^0\nabla w_h + \mathbf{b_1}(\boldsymbol{x};t,s)\cdot\boldsymbol{\Pi}_{k-1}^0\nabla v_h\Pi_k^0 w_h + b_0(\boldsymbol{x};t,s)\Pi_{k-1}^0 v_h\Pi_{k-1}^0 w_h)d\boldsymbol{x}.$$

The stability term $S_0^{\mathbb{K}} : W_{k,\mathbb{K}} \times W_{k,\mathbb{K}} \to \mathbb{R}$ should be build in such a way that, $\exists\, \alpha_*, \alpha^*$, independent of $h$ with $0 < \alpha_* \leq \alpha^*$ and satisfy the following:

$$\alpha_* a^{\mathbb{K}}(v_h, v_h) \leq S_0^{\mathbb{K}}(v_h, v_h) \leq \alpha^* a^{\mathbb{K}}(v_h, v_h) \quad \forall v_h \in \ker \Pi_k^{\nabla}. \tag{2.1.2}$$

where

$$a^{\mathbb{K}}(v_h, w_h) = \int_{\mathbb{K}} (a(\boldsymbol{x})\boldsymbol{\Pi}_{k-1}^0 \nabla v_h \cdot \boldsymbol{\Pi}_{k-1}^0 \nabla w_h d\boldsymbol{x}$$

One of the possible choices of $S_0^{\mathbb{K}}(\cdot,\cdot)$ and $S_1^{\mathbb{K}}(\cdot,\cdot)$ are:

$$S_1^{\mathbb{K}}(v_h, w_h) := h_K^2 \sum_{i=1}^{N_{\text{dof}}} \text{dof}_i(v_h)\text{dof}_i(w_h), \tag{2.1.3}$$

$$S_0^{\mathbb{K}}(v_h, w_h) := \bar{a} \sum_{i=1}^{N_{\text{dof}}} \text{dof}_i(v_h)\text{dof}_i(w_h), \tag{2.1.4}$$

or more precisely

$$S_0^K((I - \Pi_k^{\nabla})v_h, (I - \Pi_k^{\nabla})w_h) := \bar{a}((I - \Pi_k^{\nabla})v_h, (I - \Pi_k^{\nabla})w_h).$$

where $N_{\text{dof}}$ is the number of degrees of freedom (dof), $\text{dof}_i$ is the operator that selects the $i$th dof, and $\bar{a}$ is some positive constant approximation of the coefficients $a(\mathbf{x})$ (e.g., local averages), for more details, we refer to [24, 27].

First, we note that $\forall w_h \in W_{k,\mathbb{K}}$, we have:

$$\|\nabla(v_h - \Pi_k^{\nabla} v_h)\|_{0,\mathbb{K}}^2 \geq \|\nabla v_h - \boldsymbol{\Pi}_{k-1}^0 \nabla v_h\|_{0,\mathbb{K}}^2. \tag{2.1.5}$$

Now, we proceed to establish the coercivity $\mathcal{A}_h^{\mathbb{K}}(\cdot,\cdot)$. Under the assumption that $a_0(x) \geq 0$ $\forall x \in \mathcal{D}$ and in view of (2.1.2) and (2.1.5), the following holds

$$\begin{aligned}\mathcal{A}_h^{\mathbb{K}}(v_h, v_h) \geq{}& (a(\mathbf{x})\boldsymbol{\Pi}_{k-1}^0 \nabla v_h, \boldsymbol{\Pi}_{k-1}^0 \nabla v_h)_{0,\mathbb{K}} + \alpha_* a^{\mathbb{K}}((I - \Pi_k^{\nabla})v_h, (I - \Pi_k^{\nabla})v_h) \\ &+ (a_0(\boldsymbol{x})\Pi_{k-1}^0 v_h, \Pi_{k-1}^0 v_h)\end{aligned}$$

$$\geq \min(1,\alpha_*)c_0(\|\mathbf{\Pi}^0_{k-1}\nabla v_h\|^2_{0,\mathbb{K}} + \|\nabla v_h - \nabla\Pi^\nabla_k v_h\|^2_{0,\mathbb{K}})$$
$$= c_{\alpha_*}\|\nabla v_h\|^2_{0,\mathbb{K}}. \tag{2.1.6}$$

With the help of discrete inner product $m_h(v_h, w_h)$, we define the following induced $L^2$ norm on $W_h$.

$$\|p_h\|^2_h = m_h(p_h, p_h) \quad \forall p_h \in W_h.$$

For the right-hand side, we assume $f \in L^2(0,T;L^2(\mathcal{D}))$ and define: $f_h(\cdot,t) = \Pi^0_k f(\cdot,t)$ for a.e. $t \in (0,T)$. The discrete forms $m^{\mathbb{K}}_h(\cdot,\cdot)$ and $\mathcal{A}^{\mathbb{K}}_h(\cdot,\cdot)$ satisfy the following two conditions:

- $k$**-consistency**

  $\forall\ p_k \in \mathbb{P}_k(\mathbb{K})$ and $\forall\ w_h \in W_{k,\mathbb{K}}$

  $$m^{\mathbb{K}}_h(p_k, w_h) = (p_k, w_h)_{0,\mathbb{K}};$$

- **stability**

  there exist positive constants $m_*, m^*$ independent of $h$, such that $\forall\ v_h \in W_{k,\mathbb{K}}$

  $$m_*(v_h, v_h)_{0,\mathbb{K}} \leq m^{\mathbb{K}}_h(v_h, v_h) \leq m^*(v_h, v_h)_{0,\mathbb{K}}.$$

**Lemma 2.1.1.** *Let $\mathbb{K} \in \mathcal{T}_h$, for smooth vector-valued function $\mathbf{b}_1$ and smooth scalar-valued functions $p$ and $q$, the following holds true:*

$$(\mathbf{b_1}\cdot\nabla p, q)_{0,\mathbb{K}} - (\mathbf{b_1}\cdot\mathbf{\Pi}^0_{k-1}\nabla p, \Pi^0_{k-1}q)_{0,\mathbb{K}} \leq Ch|p|_{1,\mathbb{K}}|q|_{1,\mathbb{K}}$$

*with $C$ to be a constant depending on $\mathbf{b_1}$.*

*Proof.* By the definition of $\Pi^0_{k-1}$ and $\mathbf{\Pi}^0_{k-1}$, we observe

$$(\mathbf{b_1}\cdot\nabla p, q)_{0,\mathbb{K}} - (\mathbf{b_1}\cdot\mathbf{\Pi}^0_{k-1}\nabla p, \Pi^0_{k-1}q)_{0,\mathbb{K}}$$

$$
\begin{aligned}
&= (\mathbf{b_1} \cdot \nabla p - \Pi^0_{k-1}(\mathbf{b_1} \cdot \nabla p), q - \Pi^0_{k-1} q)_{0,\mathbb{K}} + (\nabla p - \mathbf{\Pi}^0_{k-1} \nabla p, \mathbf{b_1}(q - \Pi^0_{k-1} q))_{0,\mathbb{K}} \\
&\quad - (\nabla p - \mathbf{\Pi}^0_{k-1} \nabla p, \mathbf{b_1} q - \mathbf{\Pi}^0_{k-1}(\mathbf{b_1} q))_{0,\mathbb{K}} \\
&\leq Ch \|\mathbf{b_1} \cdot \nabla p\|_{0,\mathbb{K}} |q|_{1,\mathbb{K}} + Ch \|\nabla p\|_{0,\mathbb{K}} |q|_{1,\mathbb{K}} + Ch \|\nabla p\|_{0,\mathbb{K}} |q \mathbf{b_1}|_{1,\mathbb{K}} \\
&\leq Ch |p|_{1,\mathbb{K}} |q|_{1,\mathbb{K}}.
\end{aligned}
$$

□

Below, we state two lemmas, proof of which follows from [17].

**Lemma 2.1.2.** *For $\mathbb{K} \in \mathcal{T}_h$, let all the coefficients and $p, q$ be smooth scalar or vector-valued functions on $\mathbb{K}$. Then,*

$$
\begin{aligned}
\mathcal{A}^{\mathbb{K}}_h(p, q) - \mathcal{A}^{\mathbb{K}}(p, q) &\leq C_{a_0,a} |p|_{1,\mathbb{K}} |q|_{1,\mathbb{K}}, \\
\mathcal{B}^{\mathbb{K}}_h(t, s; p, q) - \mathcal{B}^{\mathbb{K}}(t, s; p, q) &\leq C_{b,b_0,\mathbf{b_1}} |p|_{1,\mathbb{K}} |q|_{1,\mathbb{K}}.
\end{aligned}
$$

**Lemma 2.1.3.** *(Consistency) For $\mathbb{K} \in \mathcal{T}_h$, $p$ to be sufficiently regular and for all $q_h \in W_h$, it holds*

$$
\begin{aligned}
\mathcal{A}^{\mathbb{K}}_h(\Pi^0_k p, q_h) - \mathcal{A}^{\mathbb{K}}(\Pi^0_k p, q_h) &\leq C_{a,a_0} h^k \|p\|_{k+1,\mathbb{K}} |q_h|_{1,\mathbb{K}}, \\
\mathcal{B}^{\mathbb{K}}_h(t, s; \Pi^0_k p, q_h) - \mathcal{B}^{\mathbb{K}}(t, s; \Pi^0_k p, q_h) &\leq C_{b,b_0,\mathbf{b_1}} h^k \|p\|_{k+1,\mathbb{K}} |q_h|_{1,\mathbb{K}}.
\end{aligned}
$$

**Lemma 2.1.4.** *For $\mathbb{K} \in \mathcal{T}_h$, let $p_I$ be the interpolant of $p \in H^2(\mathbb{K})$ then for all $q_h \in W_h$, it holds*

$$
\mathcal{A}^{\mathbb{K}}_h(q_h, p_I) - \mathcal{A}^{\mathbb{K}}(q_h, p_I) \leq C_{a,a_0} h |q_h|_{1,\mathbb{K}} \|p\|_{2,\mathbb{K}}.
$$

*Proof.* By using the properties of $\Pi^0_k$ projection, we can write:

$$
\begin{aligned}
&(a \Pi^0_{k-1} \nabla q_h, \Pi^0_{k-1} \nabla p_I)_{0,\mathbb{K}} - (a \nabla q_h, \nabla p_I)_{0,\mathbb{K}} \\
&= (a \nabla q_h - \Pi^0_{k-1}(a \nabla q_h), \nabla p_I - \Pi^0_{k-1} \nabla p_I)_{0,\mathbb{K}} - (\nabla q_h - \Pi^0_{k-1} \nabla q_h, a(\nabla p_I - \Pi^0_{k-1} \nabla p_I))_{0,\mathbb{K}} \\
&\quad + (\nabla q_h - \Pi^0_{k-1} \nabla q_h, a \nabla p_I - \Pi^0_{k-1}(a \nabla p_I))_{0,\mathbb{K}}
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\|\nabla q_h\|_{0,\mathbb{K}}(\|\nabla p_I - \Pi^0_{k-1}\nabla p_I\|_{0,\mathbb{K}} + \|a\nabla p_I - \Pi^0_{k-1}(a\nabla p_I)\|_{0,\mathbb{K}})\\
&\leq C|q_h|_{1,\mathbb{K}}(\|\nabla p_I - \Pi^0_{k-1}\nabla p\|_{0,\mathbb{K}} + \|a\nabla p_I - \Pi^0_{k-1}(a\nabla p)\|_{0,\mathbb{K}})\\
&\leq C|q_h|_{1,\mathbb{K}}(\|\nabla p_I - \nabla p + \nabla p - \Pi^0_{k-1}\nabla p\|_{0,\mathbb{K}} + \|a\nabla p_I - a\nabla p + a\nabla p - \Pi^0_{k-1}(a\nabla p)\|_{0,\mathbb{K}})\\
&\leq Ch|q_h|_{1,\mathbb{K}}\|p\|_{2,\mathbb{K}}. \qquad (2.1.7)
\end{aligned}
$$

Similarly, we can write

$$
(a_0\Pi^0_{k-1}q_h, \Pi^0_{k-1}p_I) - (a_0 q_h, p_I) \leq Ch|q_h|_{1,\mathbb{K}}\|p\|_{2,\mathbb{K}}. \qquad (2.1.8)
$$

Using (2.1.7) and (2.1.8) we arrive at:

$$
\begin{aligned}
\mathcal{A}^{\mathbb{K}}_h(q_h, p_I) - \mathcal{A}^{\mathbb{K}}(q_h, p_I) &\leq Ch|q_h|_{1,\mathbb{K}}\|p\|_{2,\mathbb{K}} + S^{\mathbb{K}}_0((I - \Pi^\nabla_k)q_h, (I - \Pi^\nabla_k)p_I)\\
&\leq Ch|q_h|_{1,\mathbb{K}}\|p\|_{2,\mathbb{K}}.
\end{aligned} \qquad (2.1.9)
$$

□

## 2.2 Ritz-Volterra Projection

For the optimal error analysis, we need to define a new projection with a memory term called the R.V. projection, $R^h : H^1_0(\mathcal{D}) \to W_h$, for $t \in \bar{J}$, where $J = (0, T]$ as the solution $\forall v_h \in W_h$:

$$
\mathcal{A}_h(R^h u, v_h) - \int_0^t \mathcal{B}_h(t, s; R^h u(s), v_h)ds = \mathcal{A}(u, v_h) - \int_0^t \mathcal{B}(t, s; u(s), v_h)ds. \quad (2.2.1)
$$

Below, we present the estimates for the R.V. projection.

**Theorem 2.2.1.** *If for each $t$, $u(t) \in H^{k+1}(\mathcal{D}) \cap L^1(0, T, H^{k+1}(\mathcal{D}))$,then there is a unique function $R^h u(t) \in W_h$, for $t \in \bar{J}$ satisfying (2.2.1) and the following estimate:*

$$
\|R^h u(t) - u(t)\| + h|R^h u(t) - u(t)|_1 \leq Ch^{k+1}\left(\|u(t)\|_{k+1} + \int_0^t \|u(s)\|_{k+1}ds\right), \quad (2.2.2)
$$

*where $C$ depends on the coefficients.*

*Proof.* We use the result explained in [75] for the existence and uniqueness of $R^h u$. Considering a function $u_I$ in $W_h$ which is interpolant of $u$, and satisfies

$$\text{dof}_n(u_I) = \text{dof}_n(u) \quad n = 1, \cdots, N_{dof},$$

where $\text{dof}_n(\cdot)$ indicates the operator which relate the $n$th degree of freedom to each smooth enough function. $N_{dof}$ stands for the number of degrees of freedom. The following approximation is satisfied by the interpolant function $u_I$:

$$\|u - u_I\|_{0,\mathbb{K}} + h|u - u_I|_{1,\mathbb{K}} \leq Ch^{k+1}|u|_{k+1} \quad \forall \mathbb{K} \in \mathcal{T}_h, \quad \text{for details see [13].}$$

Let $\mathcal{D}_h = R^h u - u_I$. By using coercivity of $\mathcal{A}_h(\cdot,\cdot)$, we arrive at:

$$\begin{aligned} c_{\alpha_*}|\mathcal{D}_h|_1^2 &\leq \mathcal{A}_h(\mathcal{D}_h, \mathcal{D}_h) \\ &= \mathcal{A}_h(R^h u, \mathcal{D}_h) - \mathcal{A}_h(u_I, \mathcal{D}_h). \end{aligned}$$

Using the definition of R.V. Projection (2.2.1):

$$\begin{aligned} c_{\alpha_*}|\mathcal{D}_h|_1^2 \leq{}& [\mathcal{A}(u, \mathcal{D}_h) - \mathcal{A}_h(u_I, \mathcal{D}_h)] \\ &- \left[\int_0^t \mathcal{B}(t, s; u(s), \mathcal{D}_h)ds - \int_0^t \mathcal{B}_h(t, s; R^h u(s), \mathcal{D}_h)ds\right]. \end{aligned} \tag{2.2.3}$$

Let's take a closer look at the first term on the right-hand side of (2.2.3):

$$\begin{aligned} \mathcal{A}(u, \mathcal{D}_h) - \mathcal{A}_h(u_I, \mathcal{D}_h) ={}& \mathcal{A}(u - u_I, \mathcal{D}_h) + \mathcal{A}(u_I - \Pi_k^0 u, \mathcal{D}_h) - \mathcal{A}_h(u_I - \Pi_k^0 u, \mathcal{D}_h) \\ &+ \mathcal{A}(\Pi_k^0 u, \mathcal{D}_h) - \mathcal{A}_h(\Pi_k^0 u, \mathcal{D}_h). \end{aligned} \tag{2.2.4}$$

Using Lemma 2.1.2 and Lemma 2.1.3, we can bound right-hand side of (2.2.4) as below:

$$\begin{aligned} \mathcal{A}(u, \mathcal{D}_h) - \mathcal{A}_h(u_I, \mathcal{D}_h) &\leq C(|u - u_I|_1 + |u - \Pi_k^0 u|_1 + h^k\|u\|_{k+1})|\mathcal{D}_h|_1 \\ &\leq Ch^k\|u\|_{k+1}|\mathcal{D}_h|_1. \end{aligned}$$

For the second term in the sum on the right-hand side of (2.2.3), we proceed as:

$$\begin{aligned}
&\int_0^t \mathcal{B}_h(t,s;R^h u(s),\mathcal{D}_h)ds - \int_0^t \mathcal{B}(t,s;u(s),\mathcal{D}_h)ds \\
&= \int_0^t [\mathcal{B}_h(t,s;R^h u(s)-u_I,\mathcal{D}_h) - \mathcal{B}(t,s;(u-u_I)(s),\mathcal{D}_h) + \mathcal{B}_h(t,s;(u_I-\Pi_k^0 u)(s),\mathcal{D}_h) \\
&\quad - \mathcal{B}(t,s;(u_I-\Pi_k^0 u)(s),\mathcal{D}_h) + \mathcal{B}_h(t,s;\Pi_k^0 u(s),\mathcal{D}_h) - \mathcal{B}(t,s;\Pi_k^0 u(s),\mathcal{D}_h)]ds.
\end{aligned}$$

Again, using Lemma 2.1.2 and Lemma 2.1.3

$$\begin{aligned}
&\int_0^t \mathcal{B}_h(t,s;R^h u(s),\mathcal{D}_h)ds - \int_0^t \mathcal{B}(t,s;u(s),\mathcal{D}_h)ds \\
&\le C\int_0^t \big(|(R^h u-u_I)(s)|_1|\mathcal{D}_h|_1 + |(u-u_I)(s)|_1|\mathcal{D}_h|_1 + |(u_I-\Pi_{k-1}^0 u)(s)|_1|\mathcal{D}_h|_1 \\
&\quad + h^k\|u(s)\|_{k+1}|\mathcal{D}_h|_1\big)\, ds \\
&\le C\int_0^t (|\mathcal{D}_h(s)|_1 + h^k\|u(s)\|_{k+1})|\mathcal{D}_h|_1 ds.
\end{aligned}$$

Combining all these terms, we obtain:

$$|\mathcal{D}_h|_1 \le C\left(h^k\|u\|_{k+1} + \int_0^t (h^k\|u(s)\|_{k+1} + |\mathcal{D}_h(s)|_1)ds\right).$$

Use of Grönwall's lemma, followed by a triangle inequality, yields:

$$|R^h u - u|_1 \le Ch^k\left(\|u\|_{k+1} + \int_0^t \|u(s)\|_{k+1}ds\right).$$

The duality approach will be used to demonstrate the $L^2$ error estimate. Let $\phi \in H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})$, with $\mathcal{D}$ to be convex and bounded, be the solution of

$$-\nabla\cdot(a\nabla\phi) + a_0\phi = \rho; \quad \text{in } \mathcal{D} \quad \phi = 0 \quad \text{on } \partial\mathcal{D}, \tag{2.2.5}$$

where $\rho = R^h u - u$ and it satisfies the following estimate:

$$\|\phi\|_2 \leq C\|\rho\|.$$

Using (2.2.5), we obtain:

$$\|\rho\|^2 = \mathcal{A}(\rho, \phi - \phi_I) + \mathcal{A}(\rho, \phi_I). \tag{2.2.6}$$

The first term in the right-hand side of (2.2.6) can be simplified as:

$$\begin{aligned}
\mathcal{A}(\rho, \phi - \phi_I) &= \mathcal{A}(R^h u - u, \phi - \phi_I) \\
&\leq |R^h u - u|_1 h |\phi|_2 \\
&\leq Ch^{k+1} \left( \|u\|_{k+1} + \int_0^t \|u(s)\|_{k+1} ds \right) \|\rho\|.
\end{aligned}$$

The second term on the right-hand side of (2.2.6) can be simplified as:

$$\mathcal{A}(\rho, \phi_I) = \mathcal{A}(R^h u - u, \phi_I) = \mathcal{A}(R^h u, \phi_I) - \mathcal{A}(u, \phi_I).$$

Using the definition of R.V. projection (2.2.1):

$$\begin{aligned}
\mathcal{A}(\rho, \phi_I) = &\left(\mathcal{A}(R^h u, \phi_I) - \mathcal{A}_h(R^h u, \phi_I)\right) \\
&+ \left( \int_0^t (\mathcal{B}_h(t, s; R^h u(s), \phi_I) - \mathcal{B}(t, s; u(s), \phi_I)) ds \right).
\end{aligned} \tag{2.2.7}$$

The first term on the right-hand side of (2.2.7) can be rewritten as:

$$\begin{aligned}
&\mathcal{A}(R^h u, \phi_I) - \mathcal{A}_h(R^h u, \phi_I) \\
&= \mathcal{A}(R^h u - u_I, \phi_I) - \mathcal{A}_h(R^h u - u_I, \phi_I) + \mathcal{A}(u_I - \Pi_k^0 u, \phi_I) - \mathcal{A}_h(u_I - \Pi_k^0 u, \phi_I) \\
&\quad + \mathcal{A}(\Pi_k^0 u, \phi_I) - \mathcal{A}_h(\Pi_k^0 u, \phi_I),
\end{aligned} \tag{2.2.8}$$

where all the terms on the right-hand side of (2.2.8) can be simplified with the help of

Lemma 2.1.4. So,

$$\begin{aligned}\mathcal{A}(R^h u, \phi_I) - \mathcal{A}_h(R^h u, \phi_I) &\leq Ch|\mathcal{D}_h|_1\|\rho\| + Ch|u_I - \Pi_k^0 u|_1\|\rho\| + Ch^{k+1}\|u\|_{k+1}\|\rho\| \\ &\leq Ch^{k+1}\left(\|u\|_{k+1} + \int_0^t \|u(s)\|_{k+1} ds\right)\|\rho\|.\end{aligned}$$

The second term on the right-hand side of (2.2.7) can be estimated as:

$$\begin{aligned}&\int_0^t (\mathcal{B}_h(t,s;R^h u(s),\phi_I) - \mathcal{B}(t,s;u(s),\phi_I))ds \\ &\quad= \int_0^t [\mathcal{B}_h(t,s;R^h u(s),\phi_I) - \mathcal{B}(t,s;R^h u(s),\phi_I) - \mathcal{B}(t,s;(u-R^h u)(s),\phi_I)]ds \\ &\quad= \int_0^t [\mathcal{B}_h(t,s;R^h u(s),\phi_I) - (\mathcal{B}(t,s;R^h u(s),\phi_I) - \mathcal{B}(t,s;(u-R^h u)(s),\phi_I - \phi) \\ &\quad\quad - \mathcal{B}(t,s;(u-R^h u)(s),\phi)]ds \\ &\quad\leq C\left(h^{k+1}\int_0^t \|u(s)\|_{k+1}ds + \int_0^t (h|\rho(s)|_1 + \|\rho(s)\|)ds\right)\|\rho\|.\end{aligned}$$

Combining all these terms, applying Grönwall's inequality, and using $s \leq t$, we obtain:

$$\|\rho(t)\| \leq Ch^{k+1}\left(\|u\|_{k+1} + \int_0^t \|u(s)\|_{k+1}ds\right).$$

□

**Theorem 2.2.2.** *Under all the assumptions of Theorem 2.2.1 and $u_t \in H^{k+1}(\mathcal{D})$, where $u_t$ and $(R^h u)_t$ be the time derivative of u and $R^h u$, we have the following estimates:*

$$\begin{aligned}&\|(R^h u(t))_t - u_t(t)\| + h|(R^h u(t))_t - u_t(t)|_1 \\ &\quad\quad \leq Ch^{k+1}\left(\|u_t(t)\|_{k+1} + \|u(t)\|_{k+1} + \int_0^t \|u(s)\|_{k+1}ds\right).\end{aligned}$$

*where the constant $C$ depends on coefficients.*

*Proof.* Differentiating (2.2.1) with respect to $t$ we get:

$$\mathcal{A}_h((R^h u)_t, v_h) - \mathcal{B}_h(t,t;R^h u, v_h) + \int_0^t \mathcal{B}_{h,t}(t,s;R^h u, v_h)ds$$
$$= \mathcal{A}(u_t, v_h) - \mathcal{B}(t,t;u,v_h) + \int_0^t \mathcal{B}_t(t,s;u,v_h)ds,$$

whereas we define the bilinear form $\mathcal{B}_t(t,s;v,w)$ as:

$$\mathcal{B}_t(t,s;v,w) := \sum_{\mathbb{K}\in\mathcal{T}_h} \mathcal{B}_t^{\mathbb{K}}(t,s;v,w),$$
$$\mathcal{B}_t^{\mathbb{K}}(t,s;v,w) := \left(\frac{\partial b(\boldsymbol{x},t,s)}{\partial t}\nabla v, \nabla w\right)_{0,\mathbb{K}} + \left(\frac{\partial \mathbf{b}_1(\boldsymbol{x},t,s)}{\partial t}\cdot\nabla v, w\right)_{0,\mathbb{K}} + \left(\frac{\partial b_0(\boldsymbol{x},t,s)}{\partial t} v, w\right)_{0,\mathbb{K}}.$$

The bilinear form $\mathcal{B}_{h,t}(t,s;p_h,q_h)$ is defined as:

$$\mathcal{B}_{h,t}(t,s;p_h,q_h) = \sum_{\mathbb{K}\in\mathcal{T}_h} \mathcal{B}_{h,t}^{\mathbb{K}}(t,s;p_h,q_h) \quad \forall p_h, q_h \in W_h, \tag{2.2.9}$$

where,

$$\mathcal{B}_{h,t}^{\mathbb{K}}(t,s;v_h,w_h) := \int_{\mathbb{K}} \left(\frac{\partial b(\boldsymbol{x},t,s)}{\partial t}\mathbf{\Pi}_{k-1}^0\nabla v_h \cdot \mathbf{\Pi}_{k-1}^0 \nabla w_h + \frac{\partial \mathbf{b}_1(\boldsymbol{x},t,s)}{\partial t}\cdot \mathbf{\Pi}_{k-1}^0\nabla v_h \Pi_k^0 w_h \right.$$
$$\left. + \frac{\partial b_0(\boldsymbol{x},t,s)}{\partial t}\Pi_{k-1}^0 v_h \Pi_{k-1}^0 w_h\right) d\boldsymbol{x} \quad \forall v_h, w_h \in W_{k,\mathbb{K}}.$$

The proof is similar to the proof of the previous theorem. □

**Theorem 2.2.3.** *Let $u$ and $u_h$ be the solution of continuous problem (2.0.2) and semi-discrete formulation (2.1.1), respectively. Assuming $f, u, u_t \in L^\infty(0,T,H^{k+1}(\mathcal{D})) \cap L^2(0,T,H^{k+1}(\mathcal{D}))$, $u_0 \in H^{k+1}(\mathcal{D})$ with $u_{h,0} = u_0^I$, then:*

$$\|u(\cdot,t) - u_h(\cdot,t)\|^2 \leq Ch^{2(k+1)}\left(\|u_0\|_{k+1}^2 + \|u(\cdot,t)\|_{k+1}^2 + \int_0^T \psi(s)ds\right),$$

*where $\psi(s) = |f(\cdot,s)|_{k+1}^2 + \|u(\cdot,s)\|_{k+1}^2 + \|u_s(\cdot,s)\|_{k+1}^2$.*

*Proof.* Let us set

$$u_h(\cdot,t) - u(\cdot,t) = (u_h(\cdot,t) - R^h u(\cdot,t)) + (R^h u(\cdot,t) - u(\cdot,t)) =: \theta(\cdot,t) + \rho(\cdot,t).$$

We set $u_{h,0} = u_0^I$, as an interpolate of the beginning data $u_0$, we get:

$$\begin{aligned}\|\theta(\cdot,0)\| &\le \|u_h(\cdot,0) - u_0\| + \|R_h u(\cdot,0) - u_0\| \\ &\le \|u_0^I - u_0\| + \|R_h u_0 - u_0\| \\ &\le Ch^{k+1}\|u_0\|_{k+1}. \end{aligned} \tag{2.2.10}$$

We already have the estimates for $\rho(\cdot,t)$. Now to deal with $\theta(\cdot,t)$, we use (2.1.1) and (2.2.1) as:

$$\begin{aligned} & m_h(\theta_t(\cdot,t), v_h) + \mathcal{A}_h(\theta(\cdot,t), v_h) - \int_0^t \mathcal{B}_h(t,s;\theta(\cdot,s),v_h)ds \\ &= \langle f_h(\cdot,t), v_h\rangle - \left(\mathcal{A}(u(\cdot,t),v_h) - \int_0^t \mathcal{B}(t,s;u(\cdot,s),v_h)ds\right) - m_h((R^h u)_t(\cdot,t),v_h). \\ &= \langle f_h(\cdot,t), v_h\rangle - m_h((R^h u)_t(\cdot,t),v_h) - (\langle f(\cdot,t),v_h\rangle - (u_t(\cdot,t),v_h)) \\ &= \langle f_h(\cdot,t), v_h\rangle - \langle f(\cdot,t),v_h\rangle - (m_h((R^h u)_t(\cdot,t),v_h) - (u_t(\cdot,t),v_h)) \\ &\le Ch^{k+1}|f(\cdot,t)|_{k+1}\|v_h\| + \sum_{\mathbb{K}\in\mathcal{T}_h} \left((u_t(\cdot,t),v_h)_{0,\mathbb{K}} - m_h^{\mathbb{K}}((R^h u)_t(\cdot,t),v_h)\right) \\ &= Ch^{k+1}|f(\cdot,t)|_{k+1}\|v_h\| \\ &\quad + \sum_{\mathbb{K}\in\mathcal{T}_h} ((u_t(\cdot,t) - \Pi_k^0 u_t(\cdot,t), v_h)_{0,\mathbb{K}} - m_h^{\mathbb{K}}((R^h u)_t(\cdot,t) - \Pi_k^0 u_t(\cdot,t), v_h)) \\ &\le Ch^{k+1}(|f(\cdot,t)|_{k+1}\|v_h\| + \|u_t(\cdot,t)\|_{k+1}\|v_h\|) + \|\Pi_k^0 u_t(\cdot,t) - (R^h u)_t(\cdot,t)\|\|v_h\| \\ &\le Ch^{k+1}\left(|f(\cdot,t)|_{k+1} + \|u_t(\cdot,t)\|_{k+1} + \|u(\cdot,t)\|_{k+1} + \int_0^t \|u(\cdot,s)\|_{k+1}ds\right)\|v_h\|. \end{aligned}$$

So, we have:

$$\begin{aligned} & m_h(\theta_t(\cdot,t), v_h) + \mathcal{A}_h(\theta(\cdot,t), v_h) - \int_0^t \mathcal{B}_h(t,s;\theta(\cdot,s),v_h)ds \\ &\le Ch^{k+1}\left(|f(\cdot,t)|_{k+1} + \|u_t(\cdot,t)\|_{k+1} + \|u(\cdot,t)\|_{k+1} + \int_0^t \|u(\cdot,s)\|_{k+1}ds\right)\|v_h\|. \end{aligned} \tag{2.2.11}$$

Since, $\theta(\cdot,t) \in W_h$. Put $v_h = \theta(\cdot,t)$ in (2.2.11), we get:

$$\begin{aligned} m_h(\theta_t(\cdot,t),\theta(\cdot,t)) &+ \mathcal{A}_h(\theta(\cdot,t),\theta(\cdot,t)) - \int_0^t \mathcal{B}_h(t,s;\theta(\cdot,s),\theta(\cdot,t))ds \\ &\leq Ch^{k+1}\left(|f(\cdot,t)|_{k+1} + \|u_t(\cdot,t)\|_{k+1} + \|u(\cdot,t)\|_{k+1} + \int_0^t \|u(\cdot,s)\|_{k+1}ds\right)\|\theta(\cdot,t)\| \\ &= I(\cdot,t)\|\theta\|, \end{aligned}$$

where

$$I(\cdot,t) = Ch^{k+1}\left(|f(\cdot,t)|_{k+1} + \|u_t(\cdot,t)\|_{k+1} + \|u(\cdot,t)\|_{k+1} + \int_0^t \|u(\cdot,s)\|_{k+1}ds\right).$$

We can write

$$m_h(\theta_t(\cdot,t),\theta(\cdot,t)) = \frac{1}{2}\frac{d}{dt}m_h(\theta(\cdot,t),\theta(\cdot,t)) = \frac{1}{2}\frac{d}{dt}\|\theta(\cdot,t)\|_h^2.$$

From the coercivity of $\mathcal{A}_h(\cdot,\cdot)$, we get:

$$\|\theta(\cdot,t)\|_h\frac{d}{dt}\|\theta(\cdot,t)\|_h + c_{\alpha_*}|\theta(\cdot,t)|_1^2 \leq I(\cdot,t)\|\theta(\cdot,t)\| + \int_0^t B_h(t,s;\theta(\cdot,s),\theta(\cdot,t))ds.$$

From the $L^2$ norm $\|\cdot\|$ equivalence with the norm $\|\cdot\|_h$, we apply Young's inequality for suitable $\epsilon, \epsilon_1 > 0$, and then integrate the above equation to get:

$$\begin{aligned} \|\theta(\cdot,t)\|^2 &+ c_{\epsilon_1'}\int_0^t |\theta(\cdot,s)|_1^2 ds \\ &\leq \|\theta(\cdot,0)\|^2 + \frac{\epsilon}{2}\int_0^t (I(\cdot,s))^2 ds + \int_0^t \frac{\|\theta(\cdot,s)\|^2}{2\epsilon}ds + c_{\epsilon_1}\int_0^t\int_0^z \|\theta(\cdot,s)\|_1^2 ds dz. \end{aligned}$$

Now,

$$\begin{aligned} &\min(1,c_{\epsilon_1'})\left(\|\theta(\cdot,t)\|^2 + \int_0^t |\theta(\cdot,s)|_1^2 ds\right) \\ &\leq \|\theta(\cdot,0)\|^2 + \frac{\epsilon}{2}\int_0^t (I(\cdot,s))^2 ds + \max\left(\frac{1}{2\epsilon}, c_{\epsilon_1}\right)\left(\int_0^t \|\theta(\cdot,s)\|^2 ds + \int_0^t\int_0^z \|\theta(\cdot,s)\|_1^2 ds dz\right). \end{aligned}$$

Applying Grönwall's lemma, we obtain:

$$\|\theta(\cdot,t)\|^2 + \int_0^t |\theta(\cdot,s)|_1^2 \le C\left(\|\theta(\cdot,0)\|^2 + \int_0^t (I(\cdot,s))^2 ds\right),$$

$$\|\theta(\cdot,t)\|^2 \le C\big(\|\theta(\cdot,0)\|^2 \\ + h^{2(k+1)} \int_0^t \left(|f(\cdot,s)|_{k+1}^2 + \|u_s(\cdot,s)\|_{k+1}^2 + \|u(\cdot,s)\|_{k+1}^2 + T\|u(\cdot,s)\|_{k+1}^2\right) ds\bigg).$$

Using a triangle inequality, (2.2.10) and $t \le T$ we arrive at:

$$\|u(\cdot,t) - u_h(\cdot,t)\|^2 \le Ch^{2(k+1)}\left(\|u_0\|_{k+1}^2 + \|u(\cdot,t)\|_{k+1}^2 + \int_0^T \psi(s)ds\right),$$

where $\psi(s) = |f(\cdot,s)|_{k+1}^2 + \|u(\cdot,s)\|_{k+1}^2 + \|u_s(\cdot,s)\|_{k+1}^2$. □

**Theorem 2.2.4.** *Under the assumption of the previous theorem, the following estimates hold true:*

$$|u_h(\cdot,t) - u(\cdot,t)|_1^2 \le Ch^{2k}\left(\|u_0\|_{k+1}^2 + \|u(\cdot,t)\|_{k+1}^2 + \int_0^T \psi(s)ds\right).$$

*Proof.* Put $v_h = \theta_t$ in (2.2.11), and use Young's inequality to arrive at:

$$m_h(\theta_t(\cdot,t),\theta_t(\cdot,t)) + \mathcal{A}_h(\theta(\cdot,t),\theta_t(\cdot,t)) - \int_0^t \mathcal{B}_h(t,s;\theta(\cdot,s),\theta_t(\cdot,t))ds = I(\cdot,t)\|\theta_t(\cdot,t)\|,$$

$$m_*\|\theta_t(\cdot,t)\|^2 + \frac{c_{\alpha_*}}{2}\frac{d|\theta(\cdot,t)|_1^2}{dt} \le \int_0^t \mathcal{B}_h(t,s;\theta(\cdot,s),\theta_t(\cdot,t))ds + I(\cdot,t)\|\theta_t(\cdot,t)\|. \quad (2.2.12)$$

Use Young's inequality in (2.2.12) and then integrate from 0 to '$t$',

$$\int_0^t \frac{m_*}{2}\|\theta_s(\cdot,s)\|^2 + \frac{c_{\alpha_*}|\theta(\cdot,t)|_1^2}{2} \\ \le \frac{c_{\alpha_*}|\theta(\cdot,0)|_1^2}{2} + \int_0^t \frac{(I(\cdot,s))^2}{2m_*}ds + \int_0^t\int_0^s \mathcal{B}_h(s,z;\theta(\cdot,z),\theta_s(\cdot,s))dzds. \quad (2.2.13)$$

Now, by changing the order of integration in (2.2.13), we get:

$$\int_0^t \frac{m_*}{2}\|\theta_s(\cdot,s)\|^2 + \frac{c_{\alpha_*}|\theta(\cdot,t)|_1^2}{2}$$
$$\leq \frac{c_{\alpha_*}|\theta(\cdot,0)|_1^2}{2} + \int_0^t \frac{(I(\cdot,s))^2}{2m_*}ds - \int_0^t\int_z^t \mathcal{B}_{h,s}(s,z;\theta(\cdot,z),\theta(\cdot,s))dsdz$$
$$+ \int_0^t (\mathcal{B}_h(t,z;\theta(\cdot,z),\theta(\cdot,t)) - \mathcal{B}_h(z,z;\theta(\cdot,z),\theta(\cdot,z)))dz,$$

where $\mathcal{B}_{h,s}$ is defined as in (2.2.9). Since we have assumed the coefficients of $\mathcal{B}$ and $\mathcal{B}_s$ are smooth and bounded, the terms on the right-hand side of the above equation are bounded. Now, by using Young's inequality, we arrive at:

$$\int_0^t \frac{m_*}{2}\|\theta_s(\cdot,s)\|^2 + \frac{c_{\alpha_*}|\theta(\cdot,t)|_1^2}{2}$$
$$\leq \frac{c_{\alpha_*}|\theta(\cdot,0)|_1^2}{2} + \int_0^t \frac{(I(\cdot,s))^2}{2m_*}ds + C_\epsilon\|\theta(\cdot,t)\|_1^2 + C_{\epsilon'}\int_0^t \|\theta(\cdot,s)\|_1^2 ds,$$

$$C'_\epsilon|\theta(\cdot,t)|_1^2 \leq \frac{c_{\alpha_*}|\theta(\cdot,0)|_1^2}{2} + \int_0^t \frac{(I(\cdot,s))^2}{2m_*}ds + C_{\epsilon'}\int_0^t \|\theta(\cdot,s)\|_1^2 ds.$$

Applying Grönwall's lemma, followed by a triangle inequality, $t \leq T$ and the fact that $|\theta(\cdot,0)|_1 \leq Ch^k\|u_0\|_{k+1}$, it follows:

$$|u_h(\cdot,t) - u(\cdot,t)|_1^2 \leq Ch^{2k}\left(\|u_0\|_{k+1}^2 + \|u(\cdot,t)\|_{k+1}^2 + \int_0^T \psi(s)ds\right).$$

□

## 2.3 Fully-discrete Scheme

The error produced by a fully-discrete scheme has two ingredients in theory: the error caused by spatial discretization, which is dependent on $h$, and the error caused by time discretization, which is dependent on time phase size $\tau$.

Now, we discretize our problem in time. To discretize in time, we use the Euler backward process. We divide our time interval into $N$ distinct points that are evenly spaced, let $t_n = n\tau$ and the sequence $\{U_n\}$ be generated as:

$$U_n \approx u_h(\cdot, t_n), \quad n = 0, 1, 2, ..., N,$$

$$\tau = T/N.$$

We use the left rectangular rule for the partitioning of the integral term and any function $g(t)$, this rule is defined as:

$$\int_0^{t_n} g(s)ds \approx \tau \sum_{j=0}^{n-1} g(t_j).$$

Therefore, the fully-discrete scheme is defined as:

$$m_h\left(\frac{U_n - U_{n-1}}{\tau}, v_h\right) + \mathcal{A}_h(U_n, v_h) - \sum_{j=0}^{n-1} \tau \mathcal{B}_h(t_n, t_j; U_j, v_h) = \langle f_h(t_n), v_h\rangle. \quad (2.3.1)$$

**Theorem 2.3.1.** *Let $u(\cdot, t_n)$ and $U_n$ be the solution of continuous problem (2.0.2) and fully-discrete formulation (2.3.1) at time $t = t_n$ respectively. Assuming $f \in L^\infty(0, T, H^{k+1}(\mathcal{D}))$, $u_{tt} \in L^\infty(0, T, L^2(\mathcal{D})) \cap L^1(0, T, L^2(\mathcal{D}))$, $u, u_t \in L^\infty(0, T, H^{k+1}(\mathcal{D})) \cap L^1(0, T, H^{k+1}(\mathcal{D}))$ and $u_0 \in H^{k+1}(\mathcal{D})$ with $U_0 = u_0^I$, then $\forall n = 1, 2, \cdots, N$, we have:*

$$\|U_n - u(\cdot, t_n)\| \leq O(h^{k+1} + \tau).$$

*Proof.* As in the previous theorem:

$$U_n - u(\cdot, t_n) = (U_n - R^h u(\cdot, t_n)) + (R^h u(\cdot, t_n) - u(\cdot, t_n)) =: \theta^n + \rho^n.$$

The term $\rho^n$ can be restricted by earlier arguments. By using eq. (2.0.2), (2.2.1) and

(2.3.1), we get for all $v_h \in W_h$:

$$
\begin{aligned}
& m_h\left(\frac{\theta^n-\theta^{n-1}}{\tau}, v_h\right)+\mathcal{A}_h(\theta^n, v_h)-\tau \sum_{j=0}^{n-1} \mathcal{B}_h(t_n, t_j ; \theta^j, v_h) \\
& =[\langle f_h(\cdot, t_n), v_h\rangle-\langle f(\cdot, t_n), v_h\rangle]+\left[(u_t(\cdot, t_n), v_h)-m_h\left(\frac{R^h u(\cdot, t_n)-R^h u(\cdot, t_{n-1})}{\tau}, v_h\right)\right] \\
& +\left[\int_0^{t_n} \mathcal{B}_h(t_n, s ; R^h u(\cdot, s), v_h) d s-\tau \sum_{j=0}^{n-1} \mathcal{B}_h(t_n, t_j ; R^h u(\cdot, j \tau), v_h)\right], \qquad (2.3.2)
\end{aligned}
$$

where the first term in right-hand side of (2.3.2) can be simplified as:

$$
\begin{aligned}
\langle f_h(\cdot, t_n), v_h\rangle-\langle f(\cdot, t_n), v_h\rangle & \leq C h^{k+1}|f(\cdot, t_n)|_{k+1}\|v_h\|, \\
& =I_1^n\|v_h\|,
\end{aligned}
$$

where $I_1^n=C h^{k+1}|f(\cdot, t_n)|_{k+1}$. The second term in the right-hand side of (2.3.2) can be simplified as:

$$
\begin{aligned}
& (u_t(\cdot, t_n), v_h)-m_h\left(\frac{R^h u(\cdot, t_n)-R^h u(\cdot, t_{n-1})}{\tau}, v_h\right) \\
& =\sum_{\mathbb{K} \in \mathcal{T}_h}\left((u_t(\cdot, t_n), v_h)_{0, \mathbb{K}}-m_h^{\mathbb{K}}\left(\frac{R^h u(\cdot, t_n)-R^h u(\cdot, t_{n-1})}{\tau}, v_h\right)\right) \\
& =\sum_{\mathbb{K} \in \mathcal{T}_h}\left(\left(u_t(\cdot, t_n)-\frac{u(\cdot, t_n)-u(\cdot, t_{n-1})}{\tau}, v_h\right)_{0, \mathbb{K}}\right. \\
& \quad+\left(\frac{u(\cdot, t_n)-u(\cdot, t_{n-1})}{\tau}-\frac{\Pi_k^0(u(\cdot, t_n)-u(\cdot, t_{n-1}))}{\tau}, v_h\right)_{0, \mathbb{K}} \\
& \quad\left.+m_h^{\mathbb{K}}\left(\frac{\Pi_k^0(u(\cdot, t_n)-u(\cdot, t_{n-1}))}{\tau}-\frac{R^h u(\cdot, t_n)-R^h u(\cdot, t_{n-1})}{\tau}, v_h\right)\right) \\
& \leq C\left(\left\|u_t(\cdot, t_n)-\left(\frac{u(\cdot, t_n)-u(\cdot, t_{n-1})}{\tau}\right)\right\|\right. \\
& \quad+\frac{1}{\tau}\|u(\cdot, t_n)-u(\cdot, t_{n-1}))-\Pi_k^0(u(\cdot, t_n)-u(\cdot, t_{n-1}))\| \\
& \quad\left.+\frac{1}{\tau}\|\Pi_k^0(u(\cdot, t_n)-u(\cdot, t_{n-1}))-(R^h u(\cdot, t_n)-R^h u(\cdot, t_{n-1}))\|\right)\|v_h\| \\
& =I_2^n\|v_h\|,
\end{aligned}
$$

where

$$|I_2^n| \leq \frac{C_1}{\tau} \left( \|\tau u_t(\cdot, t_n) - (u(\cdot, t_n) - u(\cdot, t_{n-1}))\| \right.$$
$$\left. + h^{k+1} \int_{t_{n-1}}^{t_n} \left( \|u_s(\cdot, s)\|_{k+1} + \|u(\cdot, s)\|_{k+1} + \int_0^s \|u(\cdot, z)\|_{k+1} dz \right) ds \right).$$

We can deal with the third term in the right-hand side of (2.3.2) as see [76]:

$$\int_0^{t_n} \mathcal{B}_h(t_n, s; R^h u(\cdot, s), v_h) ds - \tau \sum_{j=0}^{n-1} \mathcal{B}_h(t_n, t_j; R^h u(\cdot, j\tau), v_h) \leq I_3^n |v_h|_1,$$

where $I_3^n = C_2 \tau \left( \int_0^{t_n} (|\rho(\cdot, s)|_1 + |\rho_s(\cdot, s)|_1 + |u(\cdot, s)|_1 + |u_s(\cdot, s)|_1) ds \right)$.
Now, put $v_h = \theta^n$ in (2.3.2) to arrive at:

$$\frac{1}{2} \left( \frac{\|\theta^n\|^2 - \|\theta^{n-1}\|^2}{\tau} + \frac{\|\theta^n - \theta^{n-1}\|^2}{\tau} \right) + c_{\alpha_*} |\theta^n|_1^2$$
$$\leq C_3 \left( \tau \sum_{j=0}^{n-1} |\theta^j|_1 |\theta^n|_1 + (I_1^n + I_2^n) \|\theta^n\| + I_3^n |\theta^n|_1 \right).$$

Using Young's inequality on the first term in the right-hand side of the above equation, followed by the kickback argument, gives us the following:

$$\left( \frac{\|\theta^n\|^2 - \|\theta^{n-1}\|^2}{\tau} \right) + |\theta^n|_1^2 \leq C \left( \tau \sum_{j=0}^{n-1} |\theta^j|_1^2 + (I_1^n + I_2^n) \|\theta^n\| + I_3^n |\theta^n|_1 \right).$$

Multiplying by $\tau$ and summing over $n$ from 1 to $m$, with $1 \leq m \leq N$ yields:

$$\|\theta^m\|^2 + \tau \sum_{n=1}^{m} |\theta^n|_1^2$$
$$\leq \|\theta(\cdot, 0)\|^2 + C \left[ \tau \sum_{n=1}^{m} (I_1^n + I_2^n) \|\theta^n\| + \tau \sum_{n=1}^{m} I_3^n |\theta^n|_1 + \tau \sum_{n=1}^{m} \tau \sum_{j=0}^{n-1} |\theta^j|_1^2 \right]. \quad (2.3.3)$$

Define the left-hand side as $E_m$ and $\delta_{\mathcal{M}} = \max_{0 \leq i \leq \mathcal{M}} E_i$ for $\mathcal{M} \leq N$, and using the Holder's inequality for the third term on the right-hand side of the above equation, then:

$$\delta_{\mathcal{M}} \leq C' \left[ \|\theta(\cdot, 0)\| \delta_{\mathcal{M}}^{\frac{1}{2}} + \tau \sum_{n=1}^{\mathcal{M}} (I_1^n + I_2^n) \delta_{\mathcal{M}}^{\frac{1}{2}} + \left( \tau \sum_{n=1}^{\mathcal{M}} (I_3^n)^2 \right)^{\frac{1}{2}} \delta_{\mathcal{M}}^{\frac{1}{2}} + \tau \sum_{n=1}^{\mathcal{M}} \delta_{i-1}^{\frac{1}{2}} \delta_{\mathcal{M}}^{\frac{1}{2}} \right].$$

Dividing both sides by $\delta_{\mathcal{M}}^{\frac{1}{2}}$, we may replace $\mathcal{M}$ by $m$, and then using discrete Grönwall's inequality, gives us the required result. All these terms $\tau\sum_{i=1}^{m} I_1^i,\ \tau\sum_{i=1}^{m} I_2^i,\ \tau\sum_{i=1}^{m}(I_3^i)^2$ can be simplified as:

$$\left|\tau\sum_{i=1}^{m} I_1^i\right| \leq Ch^{k+1}\|f\|_{X_\infty},$$

$$\left|\tau\sum_{i=1}^{m} I_2^i\right| \leq C\left(\tau\|u_{tt}\|_{L^1(0,T,L^2(\mathcal{D}))} + h^{k+1}\int_0^T(\|u_s(\cdot,s)\|_{k+1} + \|u(\cdot,s)\|_{k+1})ds\right),$$

$$\left|\left(\tau\sum_{i=1}^{m}(I_3^i)^2\right)^{\frac{1}{2}}\right| \leq C\tau\left(\int_0^T(h^{2(k+1)}(\|u_t(\cdot,s)\|_{k+1}^2 + \|u(\cdot,s)\|_{k+1}^2) + |u(\cdot,s)|_1^2 + |u_s(\cdot,s)|_1^2)ds\right)^{\frac{1}{2}}.$$

Applying a triangle inequality along with Theorem 2.2.1 completes the rest of the proof.

□

Below, we state the theorem which provides the optimal estimates for $|U_n - u(\cdot,t_n)|_1$. The proof is similar to the proof of Theorem 2.3.1.

**Theorem 2.3.2.** *Under the assumption of the previous theorem, the following estimates hold true:*

$$|U_n - u(\cdot,t_n)|_1 \leq O(h^k + \tau). \tag{2.3.4}$$

## 2.4 Numerical Experiments

In this section, we carry out different numerical examples to justify the performance of the proposed virtual element scheme for the linear integro-differential problem (2.0.1). For simplicity, in all experiments, we consider the domain $\mathcal{D}$ to be the unit square in $\mathbb{R}^2$. To support our theoretical findings, we verified the convergence rates for both singular and weakly singular kernels. It is well known that hanging nodes can be easily managed in the context of VEM. Hence, this particular feature would allow us local refinement, i.e., one can have finer mesh around the singular point and coarser mesh in the rest of the domain. Considering this point, we present our numerical experiments with adaptive mesh refinement and obtain accurate numerical solutions for problems that have sharp changes at some particular points. We have also compared numerical solutions for uniform and adaptive mesh refinement. We stress that local refinement would not be an easy task while working with classical finite element schemes. Therefore, VEMs are more suitable when local mesh refinement is mandatory, for instance, at singular points. We use the backward

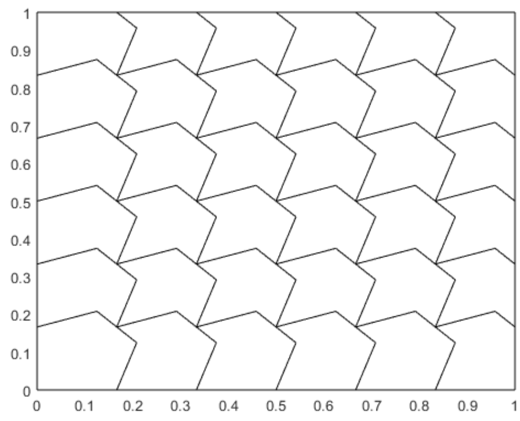


Figure 2.1: The representation of mesh employed in this study.

Euler approach for time discretization coupled with VEM discretization to tackle the fully-discrete problem for the polygonal mesh sequences introduced in Figure.2.1.

### 2.4.1 Uniform Mesh Refinement

**Example 2.4.1.** *Consider the linear parabolic integro-differential equation* (2.0.1)*, with variable coefficients* $a(x)$*,* $a_0(x)$*,* $b(x;t,s)$*,* $\mathbf{b}_1(x;t,s)$*,* $b_0(x;t,s)$*, in which the load term* $f$*, the boundary data, and the beginning data* $u_0$ *are all determined based on the exact solution.*

$$u(x,t) = t(x - x^2)(y - y^2). \tag{2.4.1}$$

**Example 2.4.2.** ***(Weakly-Singular Kernel)*** *Consider the linear parabolic integro-differential equation* (2.0.1)*,with coefficients* $a(x) = a_0(x) = 1$ *and* $b(x;t,s) = (t-s)^{-0.01}e^s$ *and* $\mathbf{b}_1(x;t,s) = b_0(x;t,s) = 0$*; in which the load term* $f$*, the boundary data, and the beginning data* $u_0$ *are all determined based on the exact solution.*

$$u(x,t) = e^{-t}(x - x^2)(y - y^2). \tag{2.4.2}$$

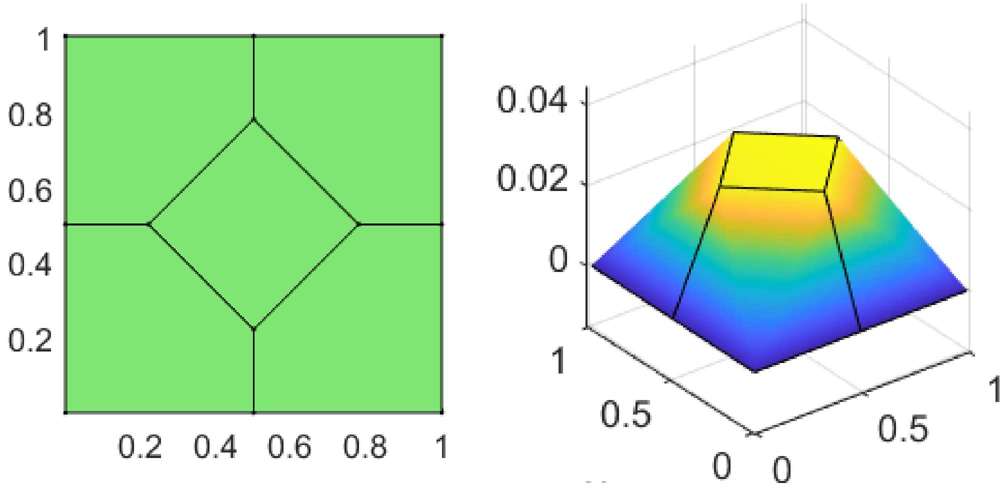


Figure 2.2: The left panel shows the initial domain with only five elements; the right panel shows the approximate solution at this domain.

### 2.4.2 Adaptive Mesh Refinement

Here, we show that for some problems that have sharp changes at some particular point, it is advisable to use an adaptive mesh rather than a uniform mesh; it gives better accuracy and saves run time for code. Here we have considered the solution as

$$u(x,t) = te^{-100((x-1/2)^2+(y-1/2)^2)}, \tag{2.4.3}$$

and this solution has a sharp peak at $(\frac{1}{2}, \frac{1}{2})$. So, rather than doing the uniform refinement, we take advantage of the fact that hanging nodes can be dealt with easily in VEM, so we go for non-uniform refinement around $(\frac{1}{2}, \frac{1}{2})$, and by comparing the results in uniform and adaptive mesh, we can see that adaptive mesh refinement has less $H^1$-error as compared to the uniform refinement for a particular $\tau$ and $h$. Figure2.2 shows the approximate solution at the initial mesh with $5$ elements, whereas Figure2.3 shows the solution after $25$ iterations. Figure2.4 depicts the solution for uniform refinement and the comparison of $H^1$-error for adaptive and uniform mesh refinement. **Convergence Curves**

Figure 2.5 and Figure 2.6 depict the errors for Example 2.4.1 and Example 2.4.2, respectively. It is evident that our theoretical estimates are well according to our numerical

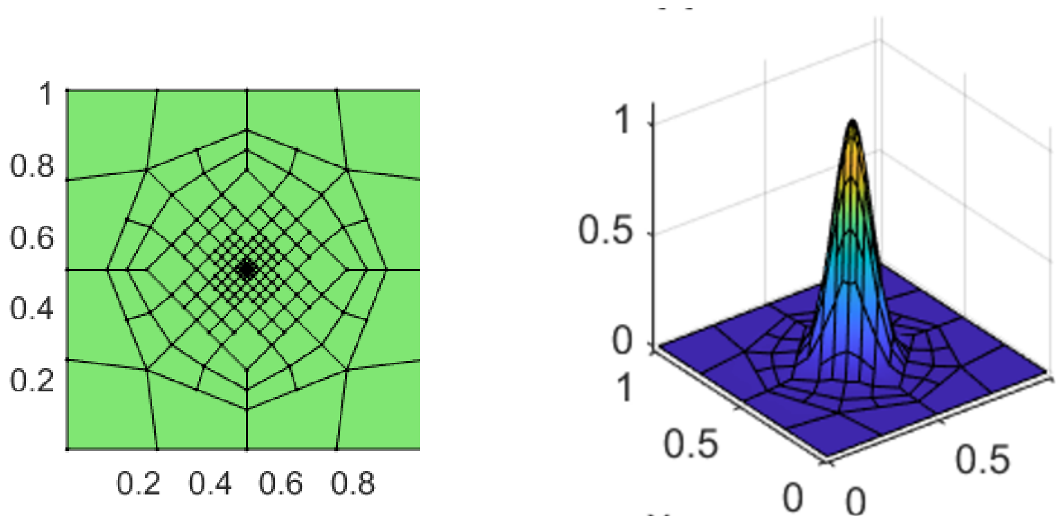


Figure 2.3: The left panel shows the adaptive domain (finer mesh at $(\frac{1}{2}, \frac{1}{2})$); the right panel shows the approximate solution at this domain.

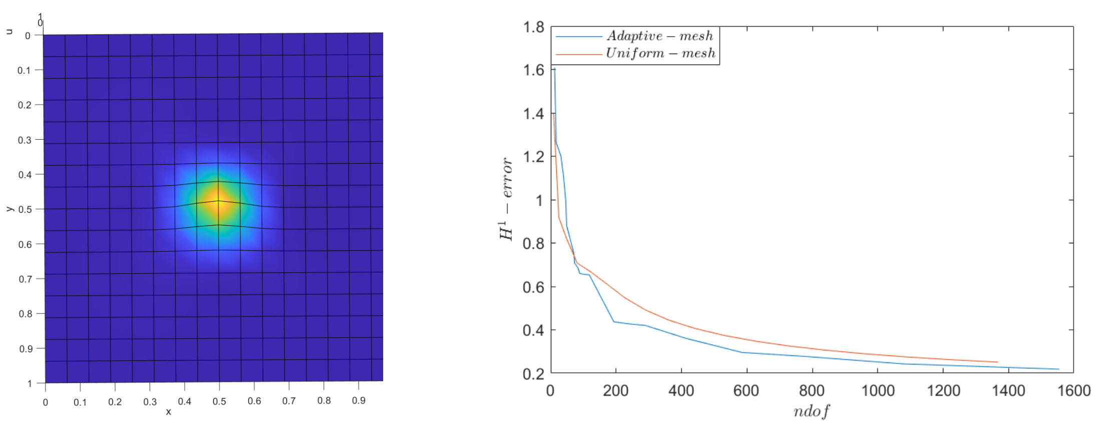


Figure 2.4: The left panel shows the approximate solution at uniform mesh; the right panel shows the comparison of the error in adaptive and uniform mesh

results.

## 2.5 Conclusion

In this Chapter, we develop and analyze the confirming VEM for PIDE on polygonal meshes. We establish the $L^2$ and $H^1$-error estimates of order $O(h^{k+1})$ and $O(h^k)$ respectively, for the semi-discrete case. Using the left-rectangular rule for partitioning the integral term and the backward Euler's method for the time derivative approximation, the

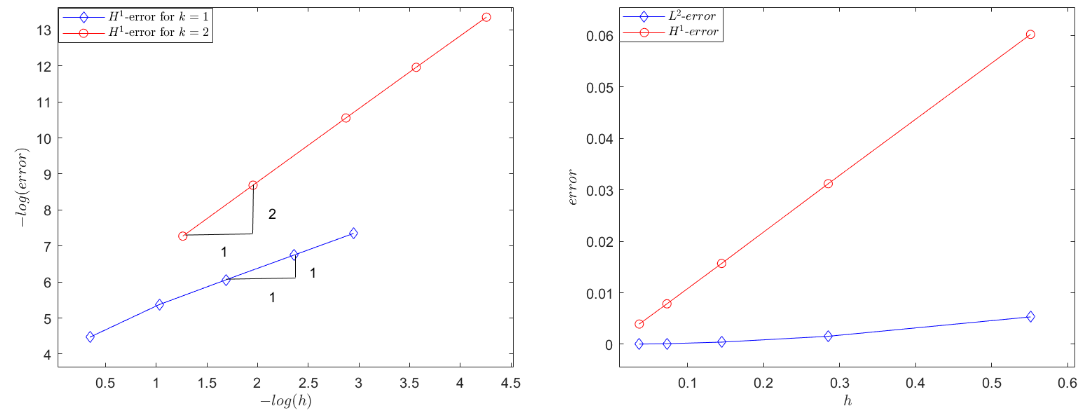


Figure 2.5: The left panel shows the order of convergence for $k = 1$ and $k = 2$; the right panel shows the $L^2$-error and $H^1$-error for $k = 1$, in case of Example 2.4.1.

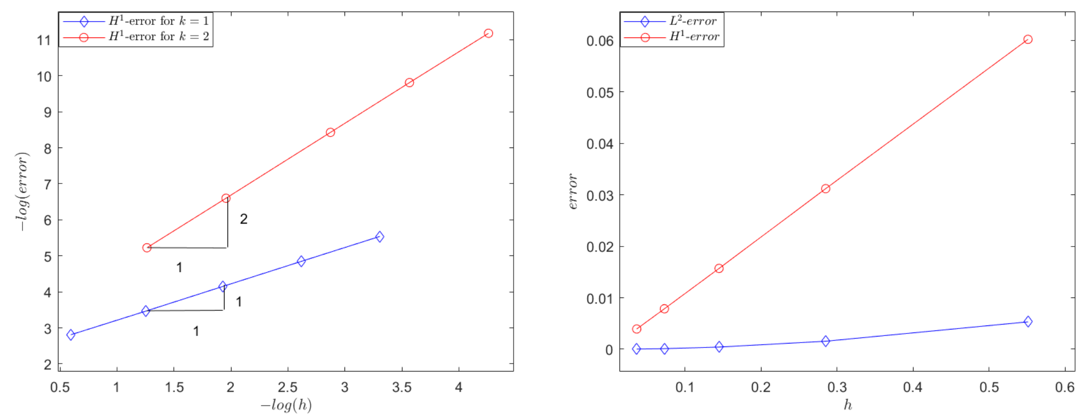


Figure 2.6: The left panel shows the order of convergence for $k = 1$ and $k = 2$; the right panel shows the $L^2$-error and $H^1$-error for $k = 1$, in case of Example 2.4.2.

error of order $O(h^{k+1} + \tau)$ and $O(h^k + \tau)$ are established. Furthermore, to demonstrate the practical application of VEMs, numerical experiments are carried out involving local mesh refinements with hanging nodes, a technique essential for minimizing computational expenses but one that may not be feasible within the framework of conforming FEMs.

# Chapter 3

# Mixed Virtual Element Method for Linear Parabolic Integro-Differential Equations [1]

In this chapter, we develop and analyze a mixed virtual element scheme for the spatial discretization of parabolic integro-differential equations combined with backward Euler's scheme for temporal discretization. Our focus lies in the exploration of PIDEs of the form given by (1.3.1).

One of our concerns in (1.3.1) is to determine the flux or velocity in addition to the pressure; the typical Galerkin method yields a loss of precision because it is estimated from the approximated solution via post-processing. The mixed methods, on the other hand, provide a direct estimate of this physical quantity and lead to locally conservative solutions. Another advantage of using a mixed technique here is the ability to introduce one more unknown of physical importance, which may be computed directly without adding any new sources of error. Here, we introduce $\boldsymbol{\sigma}(\boldsymbol{x}, t)$, defined by (1.3.4) then, (1.3.1) can be

[1]The substantial part of this chapter has been communicated as follows: **M Suthar**, and S Yadav, "Mixed Virtual Element Method for Linear Parabolic Integro-Differential Equations", *International Journal of Numerical Analysis and Modeling* (**Accepted**).

written as in the form of (1.3.5). The meaning of this independent variable '$\boldsymbol{\sigma}$' is velocity field while discussing flow in porous media, whereas (1.3.5) expresses a mass balance in any subdomain of $\mathcal{D}$, see [14]. So, the mixed formulation for this setting simultaneously approximates the pressure and the velocity field while maintaining the underlying local mass conservation. Since there is an integral term in (1.3.4) which involves $\nabla u$, we introduce a new kernel known as the resolvent kernel to deal with this integral term. Although determining the resolvent kernel for a given kernel may be challenging, but computationally, this approach proves significantly more efficient. This is evident when comparing it to the formulation outlined in [15], which involves two terms under the integral sign and hence requires $N \times N$ times more computation of a matrix. Additionally, in the 3-field formulation discussed in [48], the system of equations is considerably larger than the system arising from this formulation. Hence, whenever the resolvent kernel is available, utilizing this formulation yields computational cost cutting. Furthermore, if the resolvent kernel turns out to be a series, we can truncate the series and get the desired result. This chapter implements the mixed VEM method on (1.3.1) and presents several significant contributions, which are outlined as follows:

- To tackle the integral term, an approach involving a novel projection with a memory term (referred to as mixed R.V. projection) is introduced, which helps in achieving the optimal convergence of order $O(h^{k+1})$ for both the unknowns.

- A fully discretized scheme is put forth, utilizing the backward Euler's method for temporal derivative and the left rectangular rule for the discretization of the integral term.

- The analysis is performed to show the super convergence of the discrete solution, which has been verified with the different numerical experiments.

- Theoretical results have been validated through the implementation of numerical experiments.

## 3.1 The Continuous and Mixed VEM Semi-discrete Formulation

One of the possible ways of finding a resolvent kernel $R(t,s)$ defined by (1.3.6) and (1.3.7), for any kernel $K(t,s)$ is:

$$R(t,s)=\sum_{m=1}^{\infty}K_m(t,s),$$

where $K_m(t,s)$ is given by:

$$K_1(t,s)=K(t,s);\quad K_m(t,s)=\int_s^t K(t,z)K_{m-1}(z,s)dz. \tag{3.1.1}$$

The smoothness and boundedness of the resolvent kernel are derived from the smoothness and boundedness of $a^{-1}(x)b(x;t,s)$. For further details, please see [61]. The mixed VEM formulation corresponding to (1.3.8) reads as: Find $(u_h,\boldsymbol{\sigma}_h)\in Q_h^k\times V_h^k$ such that:

$$\begin{aligned}(u_{h,t},\phi_h)-(\nabla\cdot\boldsymbol{\sigma}_h,\phi_h)&=(f,\phi_h)\quad\forall\phi_h\in Q_h^k,\\ a_h(\boldsymbol{\sigma}_h,\boldsymbol{\chi}_h)+\int_0^t\mathcal{K}_h(t,s;\boldsymbol{\sigma}_h(s),\boldsymbol{\chi}_h)ds+(\nabla\cdot\boldsymbol{\chi}_h,u_h)&=0\quad\forall\boldsymbol{\chi}_h\in V_h^k,\end{aligned} \tag{3.1.2}$$

The discrete bilinear forms in (3.1.2) are defined $\forall\boldsymbol{z}_h,\boldsymbol{\chi}_h\in V_h^k$ as:

$$a_h(\boldsymbol{z}_h,\boldsymbol{\chi}_h):=\sum_{\mathbb{K}\in\mathcal{I}_h}a_h^{\mathbb{K}}(\boldsymbol{z}_h,\boldsymbol{\chi}_h),\qquad\mathcal{K}_h(t,s;\boldsymbol{z}_h,\boldsymbol{\chi}_h):=\sum_{\mathbb{K}\in\mathcal{I}_h}\mathcal{K}_h^{\mathbb{K}}(t,s;\boldsymbol{z}_h,\boldsymbol{\chi}_h),$$

whereas the bilinear forms $a_h^{\mathbb{K}}(\cdot,\cdot):V_h^k(\mathbb{K})\times V_h^k(\mathbb{K})\to\mathbb{R}$ and $\mathcal{K}_h^{\mathbb{K}}(\cdot,\cdot):V_h^k(\mathbb{K})\times V_h^k(\mathbb{K})\to\mathbb{R}$, on every element $\mathbb{K}\in\mathcal{I}_h$ are defined as:

$$\begin{aligned}&a_h^{\mathbb{K}}(\boldsymbol{l}_h,\boldsymbol{q}_h):=(\mu\boldsymbol{\Pi}_k^0\boldsymbol{l}_h,\boldsymbol{\Pi}_k^0\boldsymbol{q}_h)_{0,\mathbb{K}}+S_0^{\mathbb{K}}((I-\boldsymbol{\Pi}_k^0)\boldsymbol{l}_h,(I-\boldsymbol{\Pi}_k^0)\boldsymbol{q}_h)\quad\forall\boldsymbol{l}_h,\boldsymbol{q}_h\in V_h^k(\mathbb{K}),\\ &\mathcal{K}_h^{\mathbb{K}}(t,s;\boldsymbol{l}_h,\boldsymbol{q}_h):=(\mathcal{K}(t,s)\boldsymbol{\Pi}_k^0\boldsymbol{l}_h,\boldsymbol{\Pi}_k^0\boldsymbol{q}_h)_{0,\mathbb{K}}\quad\forall\boldsymbol{l}_h,\boldsymbol{q}_h\in V_h^k(\mathbb{K}).\end{aligned}$$

The stability term $S_0^{\mathbb{K}} : V_h^k(\mathbb{K}) \times V_h^k(\mathbb{K}) \to \mathbb{R}$ should be constructed in such a way that, $\exists\, \mu_*, \mu^*$ independent of $h$ with $0 < \mu_* \leq \mu^*$ and satisfies the following:

$$\mu_* a^{\mathbb{K}}(\boldsymbol{l}_h, \boldsymbol{l}_h) \leq S_0^{\mathbb{K}}(\boldsymbol{l}_h, \boldsymbol{l}_h) \leq \mu^* a^{\mathbb{K}}(\boldsymbol{l}_h, \boldsymbol{l}_h) \quad \forall \boldsymbol{l}_h \in V_h^k(\mathbb{K}).$$

One of the possible choices of $S_0^{\mathbb{K}}(\cdot,\cdot)$ is:

$$S_0^K((I - \Pi_k^0)\boldsymbol{l}_h, (I - \Pi_k^0)\boldsymbol{m}_h) := \bar{\mu}|E| \sum_{i=1}^{N_{\text{dof}}} \text{dof}_i(\boldsymbol{l}_h - \boldsymbol{\Pi}_k^0 \boldsymbol{l}_h)\text{dof}_i(\boldsymbol{m}_h - \boldsymbol{\Pi}_k^0 \boldsymbol{m}_h),$$

where $\bar{\mu}$ is some positive constant approximation of the coefficients $\mu(x)$ [77]. Moreover, $\exists\, \mu_1, \mu_2 > 0$, such that:

$$a_h(\boldsymbol{\chi}_h, \boldsymbol{\chi}_h) \geq \mu_1 \|\boldsymbol{\chi}_h\|^2 \text{ and } |a_h(\boldsymbol{z}_h, \boldsymbol{\chi}_h)| \leq \mu_2 \|\boldsymbol{z}_h\| \|\boldsymbol{\chi}_h\| \quad \forall \boldsymbol{\chi}_h \in V_h^k. \tag{3.1.3}$$

For more details about $S_0^{\mathbb{K}}$ and its properties, see [13, 77].

## 3.2 Error Analysis for the Semi-discrete Case

**Lemma 3.2.1.** *For $\mathbb{K} \in \mathcal{I}_h$, let the coefficients $\mu(x)$ and $\mathcal{K}(x; t, s)$ be smooth scaler-valued functions in $\mathcal{D}$ and $\boldsymbol{p}$ be smooth vector-valued function and $\boldsymbol{\chi}_h \in V_h^k(\mathbb{K})$. Then,*

$$\begin{aligned} a_h^{\mathbb{K}}(\boldsymbol{\Pi}_k^0 \boldsymbol{p}, \boldsymbol{\chi}_h) - (\mu \boldsymbol{\Pi}_k^0 \boldsymbol{p}, \boldsymbol{\chi}_h)_{0,\mathbb{K}} &\leq C_\mu h^{k+1} |\boldsymbol{p}|_{k+1,\mathbb{K}} |\boldsymbol{\chi}_h|_{0,\mathbb{K}}, \\ \mathcal{K}_h(t, s; \boldsymbol{\Pi}_k^0 \boldsymbol{p}, \boldsymbol{\chi}_h) - (\mathcal{K}(t,s) \boldsymbol{\Pi}_k^0 \boldsymbol{p}, \boldsymbol{\chi}_h)_{0,\mathbb{K}} &\leq C_{\mathcal{K}} h^{k+1} |\boldsymbol{p}|_{k+1,\mathbb{K}} |\boldsymbol{\chi}_h|_{0,\mathbb{K}}. \end{aligned} \tag{3.2.1}$$

*Proof.* Let $\boldsymbol{l}_h, \boldsymbol{\chi}_h$ be vector-valued functions in $V_h^k(\mathbb{K})$. Then, by using the properties of $\boldsymbol{\Pi}_k^0$, we arrive at:

$$\begin{aligned} &a_h^{\mathbb{K}}(\boldsymbol{l}_h, \boldsymbol{\chi}_h) - (\mu \boldsymbol{l}_h, \boldsymbol{\chi}_h)_{0,\mathbb{K}} \\ &= (\mu \boldsymbol{l}_h - \boldsymbol{\Pi}_k^0(\mu \boldsymbol{l}_h), \boldsymbol{\chi}_h - \boldsymbol{\Pi}_k^0 \boldsymbol{\chi}_h)_{0,\mathbb{K}} + (\boldsymbol{l}_h - \boldsymbol{\Pi}_k^0 \boldsymbol{l}_h, \mu \boldsymbol{\chi}_h - \boldsymbol{\Pi}_k^0(\mu \boldsymbol{\chi}_h))_{0,\mathbb{K}} \\ &\quad - (\boldsymbol{l}_h - \boldsymbol{\Pi}_k^0 \boldsymbol{l}_h, \mu(\boldsymbol{\chi}_h - \boldsymbol{\Pi}_k^0 \boldsymbol{\chi}_h))_{0,\mathbb{K}} + S_0^{\mathbb{K}}((I - \boldsymbol{\Pi}_k^0)\boldsymbol{l}_h, (I - \boldsymbol{\Pi}_k^0)\boldsymbol{\chi}_h). \end{aligned} \tag{3.2.2}$$

Now, put $\boldsymbol{l}_h = \boldsymbol{\Pi}^0_k \boldsymbol{p}$ in (3.2.2) and using the properties of $\boldsymbol{\Pi}^0_k$, last three terms becomes zero and we arrive at:

$$a_h^{\mathbb{K}}(\boldsymbol{\Pi}^0_k \boldsymbol{p}, \boldsymbol{\chi}_h) - (\mu \boldsymbol{\Pi}^0_k \boldsymbol{p}, \boldsymbol{\chi}_h)_{0,\mathbb{K}} \leq C_\mu h^{k+1} |\boldsymbol{p}|_{k+1,\mathbb{K}} |\boldsymbol{\chi}_h|_{0,\mathbb{K}}.$$

Using the similar arguments, we can also prove (3.2.1). □

### 3.2.1 Mixed Ritz Volterra Projection

For the formulation described in (1.3.8) and (3.1.2), we can now derive the optimal error estimates for both the semi-discrete and fully-discrete cases, and for that, we need to deal with the memory term. Therefore, we introduce a new projection here with the memory term known as mixed R.V. projection. Given $(p(t), \boldsymbol{\sigma}(t)) \in Q \times V$ for $t \in (0, T]$, define mixed R.V. projection $(\tilde{p}(t), \tilde{\boldsymbol{\sigma}}(t)) \in Q_h^k \times V_h^k$, as:

$$\begin{aligned} a_h(\tilde{\boldsymbol{\sigma}}, \boldsymbol{\chi}_h) &+ \int_0^t \mathcal{K}_h(t, s; \tilde{\boldsymbol{\sigma}}(s), \boldsymbol{\chi}_h) ds + (\nabla \cdot \boldsymbol{\chi}_h, \tilde{u}) \\ &= (\mu \boldsymbol{\sigma}, \boldsymbol{\chi}_h) + \int_0^t (\mathcal{K}(t, s)\boldsymbol{\sigma}(s), \boldsymbol{\chi}_h) ds + (\nabla \cdot \boldsymbol{\chi}_h, u) \quad \forall \boldsymbol{\chi}_h \in V_h^k, \\ (\nabla \cdot (\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}), \phi_h) &= 0 \quad \forall \phi_h \in Q_h^k. \end{aligned} \tag{3.2.3}$$

Since (3.2.3) is a linear system. To prove the existence and uniqueness of the mixed R.V. projection, it is sufficient to prove that the associated homogenous system (3.2.4a)-(3.2.4b) has only a trivial solution.

$$a_h(\tilde{\boldsymbol{\sigma}}, \boldsymbol{\chi}_h) + \int_0^t \mathcal{K}_h(t, s; \tilde{\boldsymbol{\sigma}}(s), \boldsymbol{\chi}_h) ds + (\nabla \cdot \boldsymbol{\chi}_h, \tilde{p}) = 0 \quad \forall \boldsymbol{\chi}_h \in V_h^k, \tag{3.2.4a}$$

$$(\nabla \cdot \tilde{\boldsymbol{\sigma}}, \phi_h) = 0 \quad \forall \phi_h \in Q_h^k. \tag{3.2.4b}$$

Put $\phi_h = \nabla \cdot \tilde{\boldsymbol{\sigma}}$ in (3.2.4b) to arrive at $\nabla \cdot \tilde{\boldsymbol{\sigma}} = 0$. Substitute $\boldsymbol{\chi}_h = \tilde{\boldsymbol{\sigma}}$ in (3.2.4a) and by using (3.1.3), we arrive at the following:

$$\mu_1 \|\tilde{\boldsymbol{\sigma}}\|^2 \leq -\int_0^t \mathcal{K}_h(t, s; \tilde{\boldsymbol{\sigma}}(s), \tilde{\boldsymbol{\sigma}}) ds,$$

$$\|\tilde{\boldsymbol{\sigma}}\| \leq C \int_0^t \|\tilde{\boldsymbol{\sigma}}(s)\| ds.$$

Using Grönwall's lemma, we have $\|\tilde{\boldsymbol{\sigma}}\| = 0$. Now, we use the $inf$-$sup$ condition mentioned in [33], [32], which is:

$$\exists\, \beta > 0 \text{ such that } \inf_{q \in Q_h^k} \sup_{v \in V_h^k} \frac{(\nabla \cdot \boldsymbol{v}, q)}{\|q\|_Q \|\boldsymbol{v}\|_V} \geq \beta > 0.$$

So,

$$\|\tilde{p}\| \leq c \sup_{v \in V_h^k} \frac{(\nabla \cdot \boldsymbol{\chi}_h, \tilde{p})}{\|\boldsymbol{\chi}_h\|_V} \leq C \left( \|\tilde{\boldsymbol{\sigma}}\| + \int_0^t \|\tilde{\boldsymbol{\sigma}}(s)\| ds \right).$$

We arrive at $\|\tilde{p}\| = 0$. Hence, $\tilde{p} = 0$ and $\tilde{\boldsymbol{\sigma}} = 0$. Below, we present the estimates for R.V. projection.

**Theorem 3.2.1.** *Under the assumptions* H.1-H.3, $\boldsymbol{\sigma}$ *and* $u \in L^\infty(0, T, H^{k+1}(\mathcal{D}))$, $\exists$ *a unique solution* $(\tilde{u}, \tilde{\boldsymbol{\sigma}}) \in Q_h^k \times V_h^k$, *which satisfies* (3.2.3). *Furthermore, the following estimates hold true:*

$$\|\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}\| \leq C h^{k+1} \left( |\boldsymbol{\sigma}|_{k+1} + \int_0^t |\boldsymbol{\sigma}(s)|_{k+1} ds \right), \tag{3.2.5}$$

$$\|u - \tilde{u}\| \leq C h^{k+1} \left( |u|_{k+1} + |\boldsymbol{\sigma}|_{k+1} + \int_0^t |\boldsymbol{\sigma}(s)|_{k+1} ds \right). \tag{3.2.6}$$

*Proof.* In order to prove (3.2.5) and (3.2.6), we proceed by considering $\boldsymbol{\vartheta} = \boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}$, $\boldsymbol{\psi}_h = \boldsymbol{\Pi_h^F \sigma} - \tilde{\boldsymbol{\sigma}}$, $\varrho = u - \tilde{u}$, and $\tau_h = \Pi_k^0 u - \tilde{u} \in Q_h^k$ . Now, By the definition of the mixed R.V. projection (3.2.3):

$$\begin{aligned} a_h(\boldsymbol{\psi_h}, \boldsymbol{\chi}_h) &+ \int_0^t \mathcal{K}_h(t, s; \boldsymbol{\psi_h}(s), \boldsymbol{\chi}_h) ds \\ &= (\nabla \cdot \boldsymbol{\chi}_h, \tilde{u} - u) + \left( \int_0^t \mathcal{K}_h(t, s; \boldsymbol{\Pi_h^F \sigma}(s), \boldsymbol{\chi}_h) ds - \int_0^t (\mathcal{K}(t, s)\boldsymbol{\sigma}(s), \boldsymbol{\chi}_h) ds \right) \\ &\quad + \left( a_h(\boldsymbol{\Pi_h^F \sigma}, \boldsymbol{\chi}_h) - (\mu \boldsymbol{\sigma}, \boldsymbol{\chi}_h) \right). \end{aligned} \tag{3.2.7}$$

For solving the third term in the right-hand side of (3.2.7), we use Lemma 3.2.1, (3.1.3) and Cauchy-Schwarz inequality as:

$$\begin{aligned}
&a_h(\boldsymbol{\Pi_h^F}\boldsymbol{\sigma}, \boldsymbol{\chi}_h) - (\mu\boldsymbol{\sigma}, \boldsymbol{\chi}_h) \\
&\quad = a_h(\boldsymbol{\Pi_h^F}\boldsymbol{\sigma} - \boldsymbol{\Pi}_k^0\boldsymbol{\sigma}, \boldsymbol{\chi}_h) + a_h(\boldsymbol{\Pi}_k^0\boldsymbol{\sigma}, \boldsymbol{\chi}_h) - (\mu(\boldsymbol{\sigma} - \boldsymbol{\Pi}_k^0\boldsymbol{\sigma}), \boldsymbol{\chi}_h) - (\mu\boldsymbol{\Pi}_k^0\boldsymbol{\sigma}, \boldsymbol{\chi}_h) \\
&\quad \leq C(\|\boldsymbol{\sigma} - \boldsymbol{\Pi}_k^0\boldsymbol{\sigma}\| + \|\boldsymbol{\Pi_h^F}\boldsymbol{\sigma} - \boldsymbol{\Pi}_k^0\boldsymbol{\sigma}\| + h^{k+1}|\boldsymbol{\sigma}|_{k+1})\|\boldsymbol{\chi}_h\| \\
&\qquad\quad [\text{ By using (1.5.7) and (1.5.9) }] \\
&\quad \leq Ch^{k+1}|\boldsymbol{\sigma}|_{k+1}\|\boldsymbol{\chi}_h\|. \qquad (3.2.8)
\end{aligned}$$

In a similar manner, one can address the solution of the second term on the right-hand side of equation (3.2.7) as:

$$\int_0^t \mathcal{K}_h(t,s;\boldsymbol{\Pi_h^F}\boldsymbol{\sigma}(s), \boldsymbol{\chi}_h)ds - \int_0^t (\mathcal{K}(t,s)\boldsymbol{\sigma}(s), \boldsymbol{\chi}_h)ds \leq Ch^{k+1}\|\boldsymbol{\chi}_h\| \int_0^t |\boldsymbol{\sigma}(s)|_{k+1}ds. \qquad (3.2.9)$$

By using (3.2.3), (3.2.8), (3.2.9), $\boldsymbol{\Pi}_h^F$, $\|\nabla\cdot\boldsymbol{\psi}_h\| = 0$ and by considering $\boldsymbol{\chi}_h = \boldsymbol{\psi}_h$ in (3.2.7), we arrive at the following:

$$a_h(\boldsymbol{\psi}_h, \boldsymbol{\psi}_h) \leq Ch^{k+1}\left(|\boldsymbol{\sigma}|_{k+1} + \int_0^t |\boldsymbol{\sigma}(s)|_{k+1}ds\right)\|\boldsymbol{\psi}_h\| - \int_0^t \mathcal{K}_h(t,s;\boldsymbol{\psi}_h(s), \boldsymbol{\psi}_h)ds.$$

Use of boundedness of $\mathcal{K}_h(t,s;\cdot,\cdot)$ see [61], coercivity of $a_h(\boldsymbol{\psi}_h, \boldsymbol{\psi}_h)$ and (3.1.3), followed by Grönwall's lemma, yields:

$$\|\boldsymbol{\psi}_h\| \leq Ch^{k+1}\left(|\boldsymbol{\sigma}|_{k+1} + \int_0^t |\boldsymbol{\sigma}(s)|_{k+1}ds\right).$$

Now, the use of triangle inequality completes the proof of (3.2.5):

$$\|\boldsymbol{\vartheta}\| \leq Ch^{k+1}\left(|\boldsymbol{\sigma}|_{k+1} + \int_0^t |\boldsymbol{\sigma}(s)|_{k+1}ds\right). \qquad (3.2.10)$$

To prove (3.2.6), we proceed by using the definition of mixed R.V. projection (3.2.3) as:

$$\begin{aligned}
(\mu\boldsymbol{\vartheta}, \boldsymbol{\chi}_h) + (\nabla\cdot\boldsymbol{\chi}_h, \tau_h) &= \mathcal{F}(\boldsymbol{\chi}_h) \quad \forall\boldsymbol{\chi}_h \in V_h^k, \\
(\nabla\cdot\boldsymbol{\vartheta}, \phi_h) &= 0 \quad \forall\phi_h \in Q_h^k,
\end{aligned} \qquad (3.2.11)$$

where

$$\mathcal{F}(\boldsymbol{\chi}_h) = a_h(\tilde{\boldsymbol{\sigma}}, \boldsymbol{\chi}_h) - (\mu\tilde{\boldsymbol{\sigma}}, \boldsymbol{\chi}_h) + \int_0^t \mathcal{K}_h(t, s; \tilde{\boldsymbol{\sigma}}(s), \boldsymbol{\chi}_h)ds - \int_0^t (\mathcal{K}(t, s)\boldsymbol{\sigma}(s), \boldsymbol{\chi}_h)ds. \tag{3.2.12}$$

Let $\xi \in H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})$, with $\mathcal{D}$ to be convex and bounded, be the solution of the dual problem:

$$-\nabla \cdot (a\nabla\xi) = \tau_h, \quad \xi = 0 \quad \text{on } \partial\mathcal{D}. \tag{3.2.13}$$

and satisfy the following regularity condition:

$$\|\xi\|_2 \leq \|\tau_h\|. \tag{3.2.14}$$

Consider $\boldsymbol{\Phi} = a\nabla\xi$, then (3.2.13) satisfies:

$$\begin{aligned} (\mu\boldsymbol{\Phi}, \boldsymbol{\chi}) + (\nabla \cdot \boldsymbol{\chi}, \xi) = 0 \quad \forall \boldsymbol{\chi} \in \mathcal{V} \\ -(\nabla \cdot \boldsymbol{\Phi}, \phi) = (\tau_h, \phi) \quad \forall \phi \in \mathcal{Q}. \end{aligned} \tag{3.2.15}$$

Now, put $\phi = \tau_h$ in (3.2.15) to get:

$$\begin{aligned} \|\tau_h\|^2 &= (\tau_h, -\nabla \cdot (\boldsymbol{\Pi}_h^F a\nabla\xi)) \\ &= (\mu\boldsymbol{\vartheta}, \boldsymbol{\Pi}_h^F(a\nabla\xi)) - \mathcal{F}(\boldsymbol{\Pi}_h^F(a\nabla\xi)) \quad [\text{ By using (3.2.11) }]. \end{aligned} \tag{3.2.16}$$

Now, from (3.2.12), we can rewrite $\mathcal{F}(\boldsymbol{\Pi}_h^F(a\nabla\xi))$ as:

$$\begin{aligned} &\mathcal{F}(\boldsymbol{\Pi}_h^F(a\nabla\xi)) \\ &= (a_h(\tilde{\boldsymbol{\sigma}} - \boldsymbol{\Pi}_k^0\boldsymbol{\sigma}, \boldsymbol{\Pi}_h^F(a\nabla\xi)) - (\mu(\tilde{\boldsymbol{\sigma}} - \boldsymbol{\Pi}_k^0\boldsymbol{\sigma}), \boldsymbol{\Pi}_h^F(a\nabla\xi))) \\ &\quad + (a_h(\boldsymbol{\Pi}_k^0\boldsymbol{\sigma}, \boldsymbol{\Pi}_h^F(a\nabla\xi)) - (\mu\boldsymbol{\Pi}_k^0\boldsymbol{\sigma}, \boldsymbol{\Pi}_h^F(a\nabla\xi))) - \left(\int_0^t (\mathcal{K}(t, s)(\boldsymbol{\vartheta})(s), \boldsymbol{\Pi}_h^F(a\nabla\xi))\right) \\ &\quad + \left(\int_0^t \mathcal{K}_h(t, s; (\tilde{\boldsymbol{\sigma}} - \boldsymbol{\Pi}_k^0\boldsymbol{\sigma})(s), \boldsymbol{\Pi}_h^F(a\nabla\xi))ds - \int_0^t (\mathcal{K}(t, s)(\tilde{\boldsymbol{\sigma}} - \boldsymbol{\Pi}_k^0\boldsymbol{\sigma})(s), \boldsymbol{\Pi}_h^F(a\nabla\xi))\right) \\ &\quad + \left(\int_0^t \mathcal{K}_h(t, s; \boldsymbol{\Pi}_k^0\boldsymbol{\sigma}(s), \boldsymbol{\Pi}_h^F(a\nabla\xi))ds - \int_0^t (\mathcal{K}(t, s)\boldsymbol{\Pi}_k^0\boldsymbol{\sigma}(s), \boldsymbol{\Pi}_h^F(a\nabla\xi))ds\right). \end{aligned} \tag{3.2.17}$$

The first and fourth terms on the right-hand side of (3.2.17) can be simplified in a similar way as (3.2.2). We apply Lemma 3.2.1 to deal with the second and last terms, whereas we use Cauchy-Schwarz inequality and (3.2.10) for the third term. Therefore, we arrive at the following:

$$\mathcal{F}(\mathbf{\Pi}_h^F(a\nabla\xi)) \leq Ch^{k+1}\left(|\boldsymbol{\sigma}|_{k+1} + \int_0^t |\boldsymbol{\sigma}(s)|_{k+1}ds\right)\|\xi\|_1, \tag{3.2.18}$$

whereas

$$(\mu\boldsymbol{\vartheta}, \mathbf{\Pi}_h^F(a\nabla\xi)) \leq C\|\boldsymbol{\vartheta}\|\|\xi\|_1. \tag{3.2.19}$$

Using (3.2.18), (3.2.19), (3.2.10) and (3.2.14) in (3.2.16), we arrive at:

$$\|\tau_h\| \leq Ch^{k+1}\left(|\boldsymbol{\sigma}|_{k+1} + \int_0^t |\boldsymbol{\sigma}(s)|_{k+1}ds\right). \tag{3.2.20}$$

We get our desired estimates using a triangle inequality and (1.5.7). □

### 3.2.2 Super Convergence Property of Mixed Ritz Volterra Projection

As evident from the equation (3.2.20), it is clear that $\tau_h$ exhibits convergence of order $O(h^{k+1})$. We can enhance the convergence order of $\tau_h$ by utilizing the dual norm approach, resulting in an order of $O(h^{k+2})$. This can be shown by rewriting (3.2.16) as:

$$\|\tau_h\|^2 = (\mu\boldsymbol{\vartheta}, \mathbf{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi) - a\nabla\xi) + (\nabla\cdot\boldsymbol{\vartheta}, \Pi_k^0\xi - \xi) - \mathcal{F}(\mathbf{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi)). \tag{3.2.21}$$

Now, by using (1.5.7) and (1.5.9) , we arrive at:

$$(\mu\boldsymbol{\vartheta}, \mathbf{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi) - a\nabla\xi) \leq Ch\|\boldsymbol{\vartheta}\|\|\xi\|_2, , \tag{3.2.22}$$

$$(\nabla\cdot\boldsymbol{\vartheta}, \Pi_k^0\xi - \xi) \leq Ch^2\|\nabla\cdot\boldsymbol{\vartheta}\|\|\xi\|_2. \tag{3.2.23}$$

Now, (3.2.17), can be rewritten as:

$$\begin{aligned}
&\mathcal{F}(\mathbf{\Pi}_h^F(a\nabla\xi)) \\
&= (a_h(\tilde{\boldsymbol{\sigma}} - \mathbf{\Pi}_k^0\boldsymbol{\sigma}, \mathbf{\Pi}_h^F(a\nabla\xi)) - (\mu(\tilde{\boldsymbol{\sigma}} - \mathbf{\Pi}_k^0\boldsymbol{\sigma}), \mathbf{\Pi}_h^F(a\nabla\xi))) \\
&\quad + (a_h(\mathbf{\Pi}_k^0\boldsymbol{\sigma}, \mathbf{\Pi}_h^F(a\nabla\xi)) - (\mu\mathbf{\Pi}_k^0\boldsymbol{\sigma}, \mathbf{\Pi}_h^F(a\nabla\xi)))
\end{aligned}$$

$$+\left(\int_0^t \mathcal{K}_h(t,s;(\tilde{\boldsymbol{\sigma}}-\mathbf{\Pi}_k^0\boldsymbol{\sigma})(s),\mathbf{\Pi}_h^F(a\nabla\xi))ds-\int_0^t(\mathcal{K}(t,s)(\tilde{\boldsymbol{\sigma}}-\mathbf{\Pi}_k^0\boldsymbol{\sigma})(s),\mathbf{\Pi}_h^F(a\nabla\xi))\right)$$
$$+\left(\int_0^t \mathcal{K}_h(t,s;\mathbf{\Pi}_k^0\boldsymbol{\sigma}(s),\mathbf{\Pi}_h^F(a\nabla\xi))ds-\int_0^t(\mathcal{K}(t,s)\mathbf{\Pi}_k^0\boldsymbol{\sigma}(s),\mathbf{\Pi}_h^F(a\nabla\xi))ds\right)$$
$$-\int_0^t(\mathcal{K}(t,s)(\boldsymbol{\vartheta})(s),\mathbf{\Pi}_h^F(a\nabla\xi)-a\nabla\xi)-\int_0^t(\mathcal{K}(t,s)(\boldsymbol{\vartheta})(s),a\nabla\xi).$$

For the last term in the right-hand side of (3.2.24), we use the dual norm approach, whereas all the remaining terms can be solved similarly to (3.2.17) by considering the higher regularity of $\xi$, i.e., $\|\xi\|_2$ as:

$$\mathcal{F}((\mathbf{\Pi}_h^F(a\nabla\xi))$$
$$\leq Ch^{k+2}\left(|\boldsymbol{\sigma}|_{k+1}+\int_0^t|\boldsymbol{\sigma}(s)|_{k+1}ds\right)\|\xi\|_2+\int_0^t\|\boldsymbol{\vartheta}(s)\|_{-1}\|a\nabla\xi\|_1. \tag{3.2.24}$$

The term in the (3.2.22) can be bounded by using (3.2.10) and for (3.2.23), we proceed as:

$$\begin{aligned}\|\nabla\cdot\boldsymbol{\vartheta}\|^2&=(\nabla\cdot\boldsymbol{\vartheta},\nabla\cdot(\boldsymbol{\sigma}-\boldsymbol{\Pi}_h^F\boldsymbol{\sigma}))\\&\leq\|\nabla\cdot\boldsymbol{\vartheta}\|\|\nabla\cdot(\boldsymbol{\sigma}-\boldsymbol{\Pi}_h^F\boldsymbol{\sigma}).\end{aligned}$$

Now,

$$\|\nabla\cdot\boldsymbol{\vartheta}\|\leq Ch^k|\nabla\cdot\boldsymbol{\sigma}|_k. \tag{3.2.25}$$

For the estimate of $\|\boldsymbol{\vartheta}\|_{-1}$ in (3.2.24), let $\boldsymbol{\varkappa}\in(H^1(\mathcal{D}))^2$, then:

$$(\mu\boldsymbol{\vartheta},\boldsymbol{\varkappa})=a_h(\tilde{\boldsymbol{\sigma}},\boldsymbol{\Pi}_k^0\boldsymbol{\varkappa})-(\mu\tilde{\boldsymbol{\sigma}},\boldsymbol{\Pi}_k^0\boldsymbol{\varkappa})+\int_0^t\mathcal{K}_h(\tilde{\boldsymbol{\sigma}}(s),\boldsymbol{\Pi}_k^0\boldsymbol{\varkappa})ds-\int_0^t(\mathcal{K}\tilde{\boldsymbol{\sigma}}(s),\boldsymbol{\Pi}_k^0\boldsymbol{\varkappa})ds$$
$$+(\mu\boldsymbol{\vartheta},\boldsymbol{\varkappa}-\boldsymbol{\Pi}_k^0\boldsymbol{\varkappa})-\int_0^t(\mathcal{K}\boldsymbol{\vartheta}(s),\boldsymbol{\Pi}_k^0\boldsymbol{\varkappa})ds-(\nabla\cdot\boldsymbol{\Pi}_k^0\boldsymbol{\varkappa},\tau_h).$$

Solving all these terms and using Grönwall's lemma, we get the following:

$$\|\boldsymbol{\vartheta}\|_{-1}\leq Ch^{k+2}\left(|\boldsymbol{\sigma}|_{k+1}+\int_0^t|\boldsymbol{\sigma}(s)|_{k+1}ds\right)+\|\tau_h\|. \tag{3.2.26}$$

Using (3.2.26), (3.2.24), (3.2.23) and (3.2.25) in (3.2.21) followed by Grönwall's lemma:

$$\|\Pi_k^0 u - \tilde{u}\| \le Ch^{k+2}\left(|\boldsymbol{\sigma}|_{k+1} + |\nabla\cdot\boldsymbol{\sigma}|_k + \int_0^t |\boldsymbol{\sigma}(s)|_{k+1} ds\right).$$

To prove the super convergence of $\Pi_k^0 u - u_h$, we must estimate $\Pi_k^0 u_t - \tilde{u}_t$. First, we differentiate (3.2.3) and then follow the similar steps as above; we get the following:

$$\|\tau_{h,t}\| \le Ch^{k+2}\left(|\boldsymbol{\sigma}_t|_{k+1} + |\nabla\cdot\boldsymbol{\sigma}_t|_k + |\boldsymbol{\sigma}|_{k+1} + \int_0^t (|\boldsymbol{\sigma}(s)|_{k+1} + |\nabla\cdot\boldsymbol{\sigma}(s)|_k) ds\right).$$

**Lemma 3.2.2.** *Under all the assumptions of Theorem 3.2.1and $\boldsymbol{\sigma}_t, u_t \in L^\infty(0,T,H^{k+1}(\mathcal{D}))$, where $u_t$, $\boldsymbol{\sigma}_t$ and $\tilde{u}_t$, $\tilde{\boldsymbol{\sigma}}_t$ be the time derivative of $u$, $\boldsymbol{\sigma}$ and $\tilde{u}$, $\tilde{\boldsymbol{\sigma}}$ respectively, the following estimates hold true:*

$$\|\boldsymbol{\sigma}_t - \tilde{\boldsymbol{\sigma}}_t\| \le Ch^{k+1}\left(|\boldsymbol{\sigma}_t|_{k+1} + |\boldsymbol{\sigma}|_{k+1} + \int_0^t |\boldsymbol{\sigma}(s)|_{k+1} ds\right), \tag{3.2.27}$$

$$\|u_t - \tilde{u}_t\| \le Ch^{k+1}\left(|u_t|_{k+1} + |\boldsymbol{\sigma}_t|_{k+1} + |\boldsymbol{\sigma}|_{k+1} + \int_0^t |\boldsymbol{\sigma}(s)|_{k+1} ds\right). \tag{3.2.28}$$

**Theorem 3.2.2.** *Let $u$, $\boldsymbol{\sigma}$ and $u_h$, $\boldsymbol{\sigma}_h$ be the solution of continuous problem (1.3.8) and semi-discrete formulation (3.1.2), respectively. Under all the assumptions of Lemma 3.2.2, the following estimates hold true:*

$$\|u - u_h\|^2 \le C\left(\|\varrho_h(\cdot,0)\|^2 + h^{2(k+1)}\left(|u|_{k+1}^2 + |\boldsymbol{\sigma}|_{k+1}^2 + \int_0^T g(s) ds\right)\right), \tag{3.2.29}$$

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|^2 \le C\left(\|\boldsymbol{\vartheta}_h(\cdot,0)\|^2 + Ch^{2(k+1)}\left(|\boldsymbol{\sigma}|_{k+1}^2 + \int_0^T g(s) ds\right)\right), \tag{3.2.30}$$

*where $g(s) = |u_t(s)|_{k+1}^2 + |\boldsymbol{\sigma}_t(s)|_{k+1}^2 + |\boldsymbol{\sigma}(s)|_{k+1}^2$.*

*Proof.* Writing $u - u_h = \varrho + \varrho_h$ and $\boldsymbol{\sigma} - \boldsymbol{\sigma}_h = \boldsymbol{\vartheta} + \boldsymbol{\vartheta}_h$ where $\varrho_h = (\tilde{u} - u_h)$ and $\boldsymbol{\vartheta}_h = (\tilde{\boldsymbol{\sigma}} - \boldsymbol{\sigma}_h)$. Since, we already have the estimates of $\|\varrho\|$ and $\|\boldsymbol{\vartheta}\|$, we need to find $\|\varrho_h\|$ and $\|\boldsymbol{\vartheta}_h\|$. Use (1.3.8) and (3.1.2) to have the error equation as:

$$(u_t, \phi_h) - (u_{h,t}, \phi_h) - (\nabla\cdot(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h), \phi_h) = 0 \quad \forall \phi_h \in Q_h^k, \tag{3.2.31a}$$

$$(\mu\boldsymbol{\sigma},\boldsymbol{\chi}_h) - a_h(\boldsymbol{\sigma}_h,\boldsymbol{\chi}_h) + \int_0^t [(\mathcal{K}(t,s)\boldsymbol{\sigma}(s),\boldsymbol{\chi}_h) - \mathcal{K}_h(t,s;\boldsymbol{\sigma}_h(s),\boldsymbol{\chi}_h)]ds$$
$$= (\nabla\cdot\boldsymbol{\chi}_h, u_h - u) \quad \forall \boldsymbol{\chi}_h \in V_h^k. \qquad (3.2.31b)$$

Again rewrite (3.2.31a) and (3.2.31b) as:

$$(\varrho_{h,t},\phi_h) - (\nabla\cdot\boldsymbol{\vartheta}_h,\phi_h) = -(\varrho_t,\phi_h), \qquad (3.2.32a)$$

$$a_h(\boldsymbol{\vartheta}_h,\boldsymbol{\chi}_h) + \int_0^t \mathcal{K}_h(t,s;\boldsymbol{\vartheta}_h(s),\boldsymbol{\chi}_h)ds + (\nabla\cdot\boldsymbol{\chi}_h,\varrho_h) = 0. \qquad (3.2.32b)$$

Putting $\phi_h = \varrho_h$ in (3.2.32a) and $\boldsymbol{\chi}_h = \boldsymbol{\vartheta}_h$ in (3.2.32b), then adding these equations, we get:

$$(\varrho_{h,t},\varrho_h) + a_h(\boldsymbol{\vartheta}_h,\boldsymbol{\vartheta}_h) = -(\varrho_t,\varrho_h) - \int_0^t \mathcal{K}_h(t,s;\boldsymbol{\vartheta}_h(s),\boldsymbol{\vartheta}_h)ds.$$

By utilizing (3.1.3), along with the Cauchy-Schwarz inequality, Young's inequality, and employing the Kickback argument, we reach at the following result:

$$\frac{1}{2}\frac{d}{dt}\|\varrho_h\|^2 + C_{\mu_1,\mathcal{K}'}\|\boldsymbol{\vartheta}_h\|^2 \le C_\epsilon\|\varrho_t\|^2 + C_{\epsilon'}\|\varrho_h\|^2 + C_{\mathcal{K}}\int_0^t \|\boldsymbol{\vartheta}_h(s)\|^2 ds. \qquad (3.2.33)$$

Integrating (3.2.33) from 0 to $t$, and then using Grönwall's lemma, we get:

$$\|\varrho_h\|^2 + \int_0^t \|\boldsymbol{\vartheta}_h(s)\|^2 ds \le C\left(\|\varrho_h(\cdot,0\|^2 + \int_0^t \|\varrho_t(s)\|^2 ds\right),$$

$$\|\varrho_h\|^2 \le C\left(\|\varrho_h(\cdot,0)\|^2 + h^{2(k+1)}\int_0^t g(s)ds\right).$$

Now, using a triangle inequality and (3.2.6):

$$\|u-u_h\|^2 \le C\left(\|\varrho_h(\cdot,0)\|^2 + h^{2(k+1)}\left(|u|_{k+1}^2 + |\boldsymbol{\sigma}|_{k+1}^2 + \int_0^t g(s)ds\right)\right).$$

For the proof of (3.2.30), differentiate (3.2.32b), and then put $\boldsymbol{\chi}_h = \boldsymbol{\vartheta}_h$ and, $\phi_h = \varrho_{h,t}$ in

(3.2.32a) to get:

$$
\begin{aligned}
&(\varrho_{h,t}, \varrho_{h,t}) + a_h(\boldsymbol{\vartheta}_{h,t}, \boldsymbol{\vartheta}_h) + \mathcal{K}_h(t,t;\boldsymbol{\vartheta}_h, \boldsymbol{\vartheta}_h) - \int_0^t k_{h,t}(t,s;\boldsymbol{\vartheta}_h(s), \boldsymbol{\vartheta}_h)ds = -(\varrho_t, \varrho_{h,t}) \\
&\quad \|\varrho_{h,t}\|^2 + \frac{\mu_1}{2}\frac{d}{dt}\|\boldsymbol{\vartheta}_h\|^2 \le C_{\mathcal{K}}\|\boldsymbol{\vartheta}_h\|^2 + C_{\mathcal{K}_t}\|\boldsymbol{\vartheta}_h\| \int_0^t \|\boldsymbol{\vartheta}_h(s)\|ds + C_\epsilon\|\varrho_t\|^2 + C_{\epsilon'}\|\varrho_{h,t}\|^2.
\end{aligned}
$$

Using the Kickback argument followed by the integration from 0 to $t$, we arrive at the following:

$$
\int_0^t \|\varrho_{h,s}(s)\|^2 ds + \|\boldsymbol{\vartheta}_h\|^2 \le C_1 \left( \|\boldsymbol{\vartheta}_h(\cdot,0)\|^2 + \int_0^t \left(\|\boldsymbol{\vartheta}_h(s)\|^2 + \|\varrho_t(s)\|^2\right) ds \right).
$$

Now, using Grönwall's lemma:

$$
\|\boldsymbol{\vartheta}_h\|^2 \le C \left( \|\boldsymbol{\vartheta}_h(\cdot,0)\|^2 + \int_0^t \|\varrho_t(s)\|^2 ds \right).
$$

Using a triangle inequality and (3.2.5), we arrive at:

$$
\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|^2 \le C \left( \|\boldsymbol{\vartheta}_h(\cdot,0)\|^2 + C h^{2(k+1)} \left( |\boldsymbol{\sigma}|^2_{k+1} + \int_0^t g(s) ds \right) \right).
$$

□

**Remark 3.2.1.** *The estimate* (3.2.29) *and* (3.2.30) *involve the term* $\varrho_h(\cdot,0)$ *and* $\boldsymbol{\vartheta}_h(\cdot,0)$ *respectively. We need to choose* $u_h(\cdot,0)$ *and* $\boldsymbol{\sigma}_h(\cdot,0)$ *in such a way that* $\varrho_h(\cdot,0)$ *and* $\boldsymbol{\vartheta}_h(\cdot,0)$ *is of* $O(h^{k+1})$.

### 3.2.3 Super Convergence Analysis of the Discrete Solution

**Theorem 3.2.3.** *Let* $u$ *and* $u_h$ *be the solution of continuous problem* (1.3.8) *and semi-discrete formulation* (3.1.2)*, respectively. In accordance with all the presumptions outlined in Theorem* 3.2.2*, the following assertion remains valid:*

$$
\|\Pi_k^0 u - u_h\| \le O(h^{k+2}).
$$

*Proof.* As shown in Section 3.2.2, the convergence of $\tau_h$ can be extended to $O(h^{k+2})$. Here, we analyze the super convergence properties of $\Pi_k^0 u - u_h$ by considering

$$\Pi_k^0 u - u_h = \Pi_k^0 u - \tilde{u} + \tilde{u} - u_h =: \tau_h + \varrho_h.$$

Since we know the estimate of $\tau_h$ (3.2.21) so, our aim is to find the estimate for $\varrho_h$ and for that, we proceed by using the fact that $\phi_h \in Q_h^k$ and $\nabla \cdot \boldsymbol{\chi}_h \in \mathbb{P}_k(\mathbb{K})$ and use the definition of $\Pi_k^0$ projection, we write (1.3.8) and (3.1.2)as:

$$(\Pi_k^0 u_t, \phi_h) - (u_{h,t}, \phi_h) - (\nabla \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h), \phi_h) = 0 \quad \forall \phi_h \in Q_h^k, \tag{3.2.34}$$

$$\begin{aligned}(\mu \boldsymbol{\sigma}, \boldsymbol{\chi}_h) - a_h(\boldsymbol{\sigma}_h, \boldsymbol{\chi}_h) + \int_0^t [(\mathcal{K}(t,s)\boldsymbol{\sigma}(s), \boldsymbol{\chi}_h) - \mathcal{K}_h(t,s;\boldsymbol{\sigma}_h(s), \boldsymbol{\chi}_h)]ds \\ = (u_h - \Pi_k^0 u, \nabla \cdot \boldsymbol{\chi}_h) \quad \forall \boldsymbol{\chi}_h \in V_h^k.\end{aligned} \tag{3.2.35}$$

Rewriting (3.2.34) and (3.2.35) as:

$$(\delta_{h,t}, \phi_h) - (\nabla \cdot \boldsymbol{\vartheta}_h, \phi_h) = -(\Psi_{h,t}, \phi_h), \tag{3.2.36a}$$

$$a_h(\boldsymbol{\vartheta}_h, \boldsymbol{\chi}_h) + \int_0^t \mathcal{K}_h(\boldsymbol{\vartheta}_h(s), \boldsymbol{\chi}_h)ds + (\nabla \cdot \boldsymbol{\chi}_h, \varrho_h) = 0. \tag{3.2.36b}$$

Put $\phi_h = \varrho_h$ in (3.2.36a) and $\boldsymbol{\chi}_h = \boldsymbol{\vartheta}_h$ in (3.2.36b), followed by the similar steps as in Theorem 3.2.2, we get:

$$\begin{aligned}\|\varrho_h\|^2 \leq & C\|\varrho_h(\cdot,0)\|^2 \\ & + Ch^{2(k+2)} \int_0^t (|\boldsymbol{\sigma}_t(s)|_{k+1}^2 + |\nabla \cdot \boldsymbol{\sigma}_t(s)|_k^2 + |\nabla \cdot \boldsymbol{\sigma}(s)|_k^2 + |\boldsymbol{\sigma}(s)|_{k+1}^2)ds.\end{aligned}$$

□

## 3.3 Fully-discrete Scheme

The error produced by a fully-discrete scheme has two ingredients in theory: the error caused by spatial discretization, which is dependent on $h$, and the error caused by the time discretization, which is dependent on time step size $\tau$.

Now, we discretize our problem in time. To discretize in time, we use the Euler backward process. Divide the time interval into $N$ distinct points that are evenly spaced, let $t_n = n\tau$ and the sequence $\{U_n\}$ and $\{\boldsymbol{\sigma}_n\}$ be generated as:

$$U_n \approx u_h(\cdot, t_n), \quad \boldsymbol{\sigma}_n \approx \boldsymbol{\sigma}_h(\cdot, t_n), \;\; n = 0, 1, 2, ..., N,$$

$$\tau = T/N.$$

Define $\bar{\partial}_t \Phi^n = \frac{\Phi(t_n)-\Phi(t_{n-1})}{\tau}$ and the left rectangular rule for the partitioning of the integral term for any function $\Phi(t)$ as:

$$\int_0^{t_n} \Phi(s)ds \approx \tau \sum_{j=0}^{n-1} \Phi(t_j).$$

Therefore, the fully-discrete scheme is defined as:

$$\begin{aligned} \left(\frac{U_n - U_{n-1}}{\tau}, \phi_h\right) - (\nabla \cdot \boldsymbol{\sigma}_n, \phi_h) = (f(t_n), \phi_h), \\ a_h(\boldsymbol{\sigma}_n, \boldsymbol{\chi}_h) + \tau \sum_{j=0}^{n-1} \mathcal{K}_h(t_n, t_j; \boldsymbol{\sigma}_j, \boldsymbol{\chi}_h) + (\nabla \cdot \boldsymbol{\chi}_h, U_n) = 0. \end{aligned} \tag{3.3.1}$$

**Theorem 3.3.1.** *Let $u(\cdot, t_n)$ and $U_n$ be the solution of continuous problem (1.3.8) and fully-discrete formulation (3.3.1) at time $t = t_n$ respectively. In accordance with all the presumptions outlined in Theorem 3.2.2 and $u_{tt} \in L^2(0, T, L^2(\mathcal{D}))$, the following assertion remains valid:*

$$\|U_n - u(\cdot, t_n)\| \leq O(h^{k+1} + \tau), \quad \forall n = 1, 2, \cdots, N,$$
$$\|\boldsymbol{\sigma}_n - \boldsymbol{\sigma}(\cdot, t_n)\| \leq O(h^{k+1} + \tau) \quad \forall n = 1, 2, \cdots, N.$$

*Proof.* Let us write,

$$U_n - u(\cdot, t_n) = U_n - \tilde{u}(\cdot, t_n) + \tilde{u}(\cdot, t_n) - u(\cdot, t_n) =: \varrho_h^n + \varrho^n,$$

$$\boldsymbol{\sigma}_n - \boldsymbol{\sigma}(\cdot, t_n) = \boldsymbol{\sigma}_n - \tilde{\boldsymbol{\sigma}}(\cdot, t_n) + \tilde{\boldsymbol{\sigma}}(\cdot, t_n) - \boldsymbol{\sigma}(\cdot, t_n) =: \boldsymbol{\vartheta}_h^n + \boldsymbol{\vartheta}^n.$$

Since, we know the estimates for $\varrho^n$ and $\boldsymbol{\vartheta}^n$, we need to find $\|\varrho_h^n\|$ and $\|\boldsymbol{\vartheta}_h^n\|$ and for that, we proceed by rewriting (3.3.1) and using (1.3.8) as:

$$\left(\bar{\partial}_t \varrho_h^n, \phi_h\right) - (\nabla \cdot \boldsymbol{\vartheta}_h^n, \phi_h) = \left(u_t(\cdot, t_n) - \bar{\partial}_t \tilde{u}^n, \phi_h\right), \tag{3.3.2a}$$

$$a_h(\boldsymbol{\vartheta}_h^n, \boldsymbol{\chi}_h) + \tau \sum_{j=0}^{n-1} \mathcal{K}_h(t_n, t_j; \boldsymbol{\vartheta}_h^j, \boldsymbol{\chi}_h) + (\nabla \cdot \boldsymbol{\chi}_h, \varrho_h^n)$$
$$= (\nabla \cdot \boldsymbol{\chi}_h, u(\cdot, t_n) - \tilde{u}(\cdot, t_n)) + \int_0^{t_n} (\mathcal{K}(t, s)\boldsymbol{\sigma}(s), \boldsymbol{\chi}_h) ds - \tau \sum_{j=0}^{n-1} \mathcal{K}_h(t_n, t_j; \tilde{\boldsymbol{\sigma}}(\cdot, t_j), \boldsymbol{\chi}_h)$$
$$+ (\mu \boldsymbol{\sigma}(\cdot, t_n), \boldsymbol{\chi}_h) - a_h(\tilde{\boldsymbol{\sigma}}(\cdot, t_n), \boldsymbol{\chi}_h). \tag{3.3.2b}$$

Add (3.3.2a) and (3.3.2b), after putting $\phi_h = \varrho_h^n$ and $\boldsymbol{\chi}_h = \boldsymbol{\vartheta}_h^n$, and then use the definition of mixed R.V. projection to obtain:

$$\left(\bar{\partial}_t \varrho_h^n, \varrho_h^n\right) + a_h(\boldsymbol{\vartheta}_h^n, \boldsymbol{\vartheta}_h^n) + \tau \sum_{j=0}^{n-1} \mathcal{K}_h(t_n, t_j; \boldsymbol{\vartheta}_h^j, \boldsymbol{\vartheta}_h^n)$$
$$= \int_0^{t_n} \mathcal{K}_h(t_n, s; \tilde{\boldsymbol{\sigma}}(s), \boldsymbol{\vartheta}_h^n) ds - \tau \sum_{j=0}^{n-1} \mathcal{K}_h(t_n, t_j; \tilde{\boldsymbol{\sigma}}(\cdot, t_j), \boldsymbol{\vartheta}_h^n) + \left(u_t(\cdot, t_n) - \bar{\partial}_t \tilde{u}^n, \varrho_h^n\right).$$

Using (3.1.3) and boundedness of $\mathcal{K}_h(t_n, t_j; \cdot, \cdot)$, we arrive at:

$$\frac{1}{2}\left(\frac{\|\varrho_h^n\|^2 - \|\varrho_h^{n-1}\|^2}{\tau} + \frac{\|\varrho_h^n - \varrho_h^{n-1}\|^2}{\tau}\right) + \mu_1 \|\boldsymbol{\vartheta}_h^n\|^2$$
$$\leq (I_1^n, \varrho_h^n) + (I_2^n, \boldsymbol{\vartheta}_h^n) + C\tau \sum_{j=0}^{n-1} \|\boldsymbol{\vartheta}_h^j\| \|\boldsymbol{\vartheta}_h^n\|, \tag{3.3.3}$$

where

$$(I_1^n, \varrho_h^n) = \left(u_t(\cdot, t_n) - \bar{\partial}_t \tilde{u}^n, \varrho_h^n\right)$$
$$\leq (\left\|u_t(\cdot, t_n) - \bar{\partial}_t u^n\right\| + \left\|\bar{\partial}_t u^n - \bar{\partial}_t \tilde{u}^n\right\|) \|\varrho_h^n\|.$$

So,

$$(I_1^n, \varrho_h^n) \leq C \left( \int_{t_{n-1}}^{t_n} \|u_{tt}(s)\| ds + \frac{h^{k+1}}{\tau} \int_{t_{n-1}}^{t_n} (|u_t(s)|_{k+1} + |\boldsymbol{\sigma}_t(s)|_{k+1} + |\boldsymbol{\sigma}(s)|_{k+1} \right.$$
$$\left. + \int_0^s |\boldsymbol{\sigma}(z)|_{k+1} dz) ds \right) \|\varrho_h^n\|, \tag{3.3.4}$$

where the term $(I_2^n, \boldsymbol{\vartheta}_h^n)$ can be solved as:

$$\begin{aligned}
(I_2^n, \boldsymbol{\vartheta}_h^n) &= \int_0^{t_n} \mathcal{K}_h(t_n, s; \tilde{\boldsymbol{\sigma}}(s), \boldsymbol{\vartheta}_h^n) ds - \tau \sum_{j=0}^{n-1} \mathcal{K}_h(t_n, t_j; \tilde{\boldsymbol{\sigma}}(\cdot, t_j), \boldsymbol{\vartheta}_h^n) \\
&\leq C\tau \int_0^{t_n} \left| \frac{\partial}{\partial s} (\mathcal{K}_h(t_n, s; \tilde{\boldsymbol{\sigma}}(s), \boldsymbol{\vartheta}_h^n) ds \right| \\
&\leq C\tau \int_0^{t_n} (\|\tilde{\boldsymbol{\sigma}}(s) - \boldsymbol{\sigma}(s)\| + \|\boldsymbol{\sigma}(s)\| + \|\tilde{\boldsymbol{\sigma}}_t(s) - \boldsymbol{\sigma}_t(s)\| + \|\boldsymbol{\sigma}_t(s)\|) ds \|\boldsymbol{\vartheta}_h^n\|, \\
&\leq C\tau \int_0^{t_n} (\|\boldsymbol{\sigma}(s)\| + \|\boldsymbol{\sigma}_t(s)\| + h^{k+1}(|\boldsymbol{\sigma}(s)|_{k+1} + |\boldsymbol{\sigma}_t(s)|_{k+1} \\
&\quad + \int_0^s |\boldsymbol{\sigma}(z)|_{k+1} dz)) ds \|\boldsymbol{\vartheta}_h^n\|.
\end{aligned} \tag{3.3.5}$$

Using (3.3.4) and (3.3.5) in (3.3.3), we get the following:

$$\begin{aligned}
&\frac{1}{2} \left( \frac{\|\varrho_h^n\|^2 - \|\varrho_h^{n-1}\|^2}{\tau} \right) + \mu_1 \|\boldsymbol{\vartheta}_h^n\|^2 \\
&\leq C \left( \left( \tau \sum_{j=0}^{n-1} \|\boldsymbol{\vartheta}_h^j\| + \tau \int_0^{t_n} (\|\boldsymbol{\sigma}(s)\| + \|\boldsymbol{\sigma}_t(s)\| + h^{k+1}(|\boldsymbol{\sigma}(s)|_{k+1} + |\boldsymbol{\sigma}_t(s)|_{k+1})) ds \right) \|\boldsymbol{\vartheta}_h^n\| \right. \\
&\quad + \left( \int_{t_{n-1}}^{t_n} \|u_{tt}(s)\| ds + \frac{h^{k+1}}{\tau} \int_{t_{n-1}}^{t_n} (|u_t(s)|_{k+1} + |\boldsymbol{\sigma}_t(s)|_{k+1} + |\boldsymbol{\sigma}(s)|_{k+1} \right. \\
&\quad \left.\left. + \int_0^s |\boldsymbol{\sigma}(z)|_{k+1} dz) ds \right) \|\varrho_h^n\| \right).
\end{aligned}$$

Applying Young's inequality and subsequently employing the Kickback argument leads us to the following:

$$\frac{1}{2} \left( \frac{\|\varrho_h^n\|^2 - \|\varrho_h^{n-1}\|^2}{\tau} \right) + C_1 \|\boldsymbol{\vartheta}_h^n\|^2$$

$$\leq C_2 \left( \tau \sum_{j=0}^{n-1} \|\boldsymbol{\vartheta}_h^j\|^2 + \|\varrho_h^n\|^2 + \tau \int_{t_{n-1}}^{t_n} \|u_{tt}(s)\|^2 ds. + \frac{h^{2(k+1)}}{\tau} \int_{t_{n-1}}^{t_n} \left( g(s) + \int_0^s |\boldsymbol{\sigma}(z)|_{k+1}^2 dz \right) ds \right.$$
$$\left. + \tau^2 \int_0^{t_n} (\|\boldsymbol{\sigma}(s)\|^2 + \|\boldsymbol{\sigma}_t(s)\|^2 + h^{2(k+1)}(|\boldsymbol{\sigma}(s)|_{k+1}^2 + |\boldsymbol{\sigma}_t(s)|_{k+1}^2))ds \right). \quad (3.3.6)$$

Multiplying (3.3.6) by $2\tau$ and summing from 1 to $m$, with $1 \leq m \leq N$ gives:

$$\|\varrho_h^m\|^2 + 2\tau C_1 \sum_{n=1}^{m} \|\boldsymbol{\vartheta}_h^n\|^2$$
$$\leq \|\varrho(\cdot, 0)\|^2 + 2C_2 \left( \tau^2 \sum_{n=1}^{m} \sum_{j=0}^{n-1} \|\boldsymbol{\vartheta}_h^j\|^2 + \tau \sum_{n=1}^{m} \|\varrho_h^n\|^2 + \tau^2 \int_0^T (\|\boldsymbol{\sigma}(s)\|^2 + \|\boldsymbol{\sigma}_t(s)\|^2) ds \right.$$
$$\left. + \tau^2 \int_0^T \|u_{tt}(s)\|^2 ds + h^{2(k+1)} \int_0^T g(s) ds + \tau^2 h^{2(k+1)} \int_0^T (|\boldsymbol{\sigma}(s)|_{k+1}^2 + |\boldsymbol{\sigma}_t(s)|_{k+1}^2) ds \right).$$

Using Grönwall's lemma and replacing $m$ by $n$, we get our desired result:

$$\|\varrho_h^n\| \leq O(h^{k+1} + \tau).$$

For the estimate of $\boldsymbol{\vartheta}_h^n$, we proceed by rewriting (3.3.2b) as:

$$a_h(\boldsymbol{\vartheta}_h^n, \boldsymbol{\chi}_h) + \tau \sum_{j=0}^{n-1} \mathcal{K}_h(t_n, t_j; \boldsymbol{\vartheta}_h^j, \boldsymbol{\chi}_h) + (\nabla \cdot \boldsymbol{\chi}_h, \varrho_h^n)$$
$$= \int_0^{t_n} \mathcal{K}_h(t_n, s; \tilde{\boldsymbol{\sigma}}(s), \boldsymbol{\chi}_h) ds - \tau \sum_{j=0}^{n-1} \mathcal{K}_h(t_n, t_j; \tilde{\boldsymbol{\sigma}}(t_j), \boldsymbol{\chi}_h). \quad (3.3.7)$$

Again considering (3.3.7) at time step $t = t_{n-1}$, we obtain:

$$a_h(\boldsymbol{\vartheta}_h^{n-1}, \boldsymbol{\chi}_h) + \tau \sum_{j=0}^{n-2} \mathcal{K}_h(t_{n-1}, t_j; \boldsymbol{\vartheta}_h^j, \boldsymbol{\chi}_h) + (\nabla \cdot \boldsymbol{\chi}_h, \varrho_h^{n-1})$$
$$= \int_0^{t_{n-1}} \mathcal{K}_h(t_{n-1}, s; \tilde{\boldsymbol{\sigma}}(s), \boldsymbol{\chi}_h) ds - \tau \sum_{j=0}^{n-2} \mathcal{K}_h(t_{n-1}, t_j; \tilde{\boldsymbol{\sigma}}(t_j), \boldsymbol{\chi}_h). \quad (3.3.8)$$

Now, subtracting (3.3.8) from(3.3.7), and then dividing by $\tau$, we arrive at:

$$
\begin{aligned}
&a_h\left(\bar{\partial}_t\boldsymbol{\vartheta}_h^n,\boldsymbol{\chi}_h\right)+\sum_{j=0}^{n-1}\mathcal{K}_h(t_n,t_j;\boldsymbol{\vartheta}_h^j,\boldsymbol{\chi}_h)-\sum_{j=0}^{n-2}\mathcal{K}_h(t_{n-1},t_j;\boldsymbol{\vartheta}_h^j,\boldsymbol{\chi}_h)+\left(\nabla\cdot\boldsymbol{\chi}_h,\bar{\partial}_t\varrho_h^n\right)\\
&=\frac{1}{\tau}\left(\int_0^{t_n}\mathcal{K}_h(t_n,s;\tilde{\boldsymbol{\sigma}}(s),\boldsymbol{\chi}_h)ds-\tau\sum_{j=0}^{n-1}\mathcal{K}_h(t_n,t_j;\tilde{\boldsymbol{\sigma}}(t_j),\boldsymbol{\chi}_h)\right)\\
&\quad-\frac{1}{\tau}\left(\int_0^{t_{n-1}}\mathcal{K}_h(t_{n-1},s;\tilde{\boldsymbol{\sigma}}(s),\boldsymbol{\chi}_h)ds-\tau\sum_{j=0}^{n-2}\mathcal{K}_h(t_{n-1},t_j;\tilde{\boldsymbol{\sigma}}(t_j),\boldsymbol{\chi}_h)\right). \qquad (3.3.9)
\end{aligned}
$$

Put $\phi_h=\bar{\partial}_t\varrho_h^n$ in (3.3.2a) and $\boldsymbol{\chi}_h=\boldsymbol{\vartheta}_h^n$ in (3.3.9) and then add, we obtain:

$$
\begin{aligned}
&\|\bar{\partial}_t\varrho_h^n\|^2+a_h\left(\bar{\partial}_t\boldsymbol{\vartheta}_h^n,\boldsymbol{\vartheta}_h^n\right)+\sum_{j=0}^{n-1}\mathcal{K}_h(t_n,t_j;\boldsymbol{\vartheta}_h^j,\boldsymbol{\vartheta}_h^n)-\sum_{j=0}^{n-2}\mathcal{K}_h(t_{n-1},t_j;\boldsymbol{\vartheta}_h^j,\boldsymbol{\vartheta}_h^n)\\
&=\left(I_1^n,\bar{\partial}_t\varrho_h^n\right)+\frac{1}{\tau}\left(\int_0^{t_n}\mathcal{K}_h(t_n,s;\tilde{\boldsymbol{\sigma}}(s),\boldsymbol{\vartheta}_h^n)ds-\tau\sum_{j=0}^{n-1}\mathcal{K}_h(t_n,t_j;\tilde{\boldsymbol{\sigma}}(t_j),\boldsymbol{\vartheta}_h^n)\right)\\
&\quad-\frac{1}{\tau}\left(\int_0^{t_{n-1}}\mathcal{K}_h(t_{n-1},s;\tilde{\boldsymbol{\sigma}}(s),\boldsymbol{\vartheta}_h^n)ds-\tau\sum_{j=0}^{n-2}\mathcal{K}_h(t_{n-1},t_j;\tilde{\boldsymbol{\sigma}}(t_j),\boldsymbol{\vartheta}_h^n)\right)\\
&=(I_1^n,\bar{\partial}_t\varrho_h^n)+\frac{1}{\tau}(I_3^n,\boldsymbol{\vartheta}_h^n), \qquad (3.3.10)
\end{aligned}
$$

where

$$
\begin{aligned}
(I_1^n,\bar{\partial}_t\varrho_h^n)\le C_\epsilon&\left(\tau\int_{t_{n-1}}^{t_n}\|u_{tt}(s)\|^2ds+\frac{h^{2k+2}}{\tau}\int_{t_{n-1}}^{t_n}\left(g(s)+\int_0^s|\boldsymbol{\sigma}(z)|_{k+1}^2dz\right)ds\right)\\
&+C_{\epsilon'}\|\bar{\partial}_t\varrho_h^n\|^2. \qquad (3.3.11)
\end{aligned}
$$

and

$$
\begin{aligned}
(I_3^n,\boldsymbol{\vartheta}_h^n)=&\left(\int_0^{t_n}\mathcal{K}_h(t_n,s;\tilde{\boldsymbol{\sigma}}(s),\boldsymbol{\vartheta}_h^n)ds-\tau\sum_{j=0}^{n-1}\mathcal{K}_h(t_n,t_j;\tilde{\boldsymbol{\sigma}}(t_j),\boldsymbol{\vartheta}_h^n)\right)\\
&-\left(\int_0^{t_{n-1}}\mathcal{K}_h(t_{n-1},s;\tilde{\boldsymbol{\sigma}}(s),\boldsymbol{\vartheta}_h^n)ds-\tau\sum_{j=0}^{n-2}\mathcal{K}_h(t_{n-1},t_j;\tilde{\boldsymbol{\sigma}}(t_j),\boldsymbol{\vartheta}_h^n)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^{t_{n-1}} (\mathcal{K}_h(t_n, s; \tilde{\boldsymbol{\sigma}}(s), \boldsymbol{\vartheta}_h^n) - \mathcal{K}_h(t_{n-1}, s; \tilde{\boldsymbol{\sigma}}(s), \boldsymbol{\vartheta}_h^n))\, ds \\
&\quad - \tau \sum_{j=0}^{n-2} (\mathcal{K}_h(t_n, t_j; \tilde{\boldsymbol{\sigma}}(t_j), \boldsymbol{\vartheta}_h^n) - \mathcal{K}_h(t_{n-1}, t_j; \tilde{\boldsymbol{\sigma}}(t_j), \boldsymbol{\vartheta}_h^n)) \\
&\quad + \int_{t_{n-1}}^{t_n} \mathcal{K}_h(t_n, s; \tilde{\boldsymbol{\sigma}}(s), \boldsymbol{\vartheta}_h^n) ds - \tau \mathcal{K}_h(t_n, t_{n-1}, \tilde{\boldsymbol{\sigma}}(t_{n-1}), \boldsymbol{\vartheta}_h^n), \\
&\le \tau^2 \int_0^{t_{n-1}} \left| \frac{\partial}{\partial s} (\mathcal{K}_{h,t}(t_{n^*}, s; \tilde{\boldsymbol{\sigma}}(s), \boldsymbol{\vartheta}_h^n)) \right| ds + \tau \int_{t_{n-1}}^{t_n} \left| \frac{\partial}{\partial s} (\mathcal{K}_h(t, s; \tilde{\boldsymbol{\sigma}}(s), \boldsymbol{\vartheta}_h^n)) \right| ds,
\end{aligned}
$$

where $t_{n^*} \in (t_{n-1}, t_n)$.

$$
\begin{aligned}
&\frac{1}{\tau}(I_3^n, \boldsymbol{\vartheta}_h^n) \\
&\le \tau \int_0^{t_{n-1}} \left| \frac{\partial}{ds} (\mathcal{K}_{h,t}(t_{n^*}, s; \tilde{\boldsymbol{\sigma}}(s), \boldsymbol{\vartheta}_h^n)) \right| ds + \int_{t_{n-1}}^{t_n} \left| \frac{\partial}{ds} (\mathcal{K}_h(t, s; \tilde{\boldsymbol{\sigma}}(s), \boldsymbol{\vartheta}_h^n)) \right| ds \\
&\le C \left( \tau^2 \int_0^{t_{n-1}} (\|\boldsymbol{\sigma}(s)\|^2 + \|\boldsymbol{\sigma}_t(s)\|^2) ds + \tau \int_{t_{n-1}}^{t_n} (\|\boldsymbol{\sigma}(s)\|^2 + \|\boldsymbol{\sigma}_t(s)\|^2) ds \right. \\
&\quad + \tau^2 h^{2(k+1)} \int_0^{t_{n-1}} \left( |\boldsymbol{\sigma}(s)|_{k+1}^2 + |\boldsymbol{\sigma}_t(s)|_{k+1}^2 + \int_0^s |\boldsymbol{\sigma}(z)|_{k+1}^2 dz \right) ds \\
&\quad \left. + \tau h^{2(k+1)} \int_{t_{n-1}}^{t_n} \left( |\boldsymbol{\sigma}(s)|_{k+1}^2 + |\boldsymbol{\sigma}_t(s)|_{k+1}^2 + \int_0^s |\boldsymbol{\sigma}(z)|_{k+1}^2 dz \right) ds + \|\boldsymbol{\vartheta}_h^n\|^2 \right).
\end{aligned}
\tag{3.3.12}
$$

Put (3.3.11) and (3.3.12) in (3.3.10), to arrive at:

$$
\begin{aligned}
&\|\bar{\partial}_t \varrho_h^n\|^2 + \mu_1 \left( \frac{\|\boldsymbol{\vartheta}_h^n\|^2 - \|\boldsymbol{\vartheta}_h^{n-1}\|^2}{2\tau} \right) \\
&\le -\tau \sum_{j=0}^{n-2} \mathcal{K}_{h,t}(t_{n^*}, t_j; \boldsymbol{\vartheta}_h^j, \boldsymbol{\vartheta}_h^n) - \mathcal{K}_h(t_n, t_{n-1}; \boldsymbol{\vartheta}_h^{n-1}, \boldsymbol{\vartheta}_h^n) + C \left( \|\bar{\partial}_t \varrho_h^n\|^2 + \|\boldsymbol{\vartheta}_h^n\|^2 \right. \\
&\quad + \frac{h^{2k+2}}{\tau} \int_{t_{n-1}}^{t_n} \left( g(s) + \int_0^s \|\boldsymbol{\sigma}(z)\|_{k+1}^2 dz \right) ds \\
&\quad + \tau^2 \int_0^{t_{n-1}} (\|\boldsymbol{\sigma}(s)\|^2 + \|\boldsymbol{\sigma}_t(s)\|^2) ds + \tau \int_{t_{n-1}}^{t_n} (\|\boldsymbol{\sigma}(s)\|^2 + \|\boldsymbol{\sigma}_t(s)\|^2) ds \\
&\quad + \tau^2 h^{2(k+1)} \int_0^{t_{n-1}} \left( |\boldsymbol{\sigma}(s)|_{k+1}^2 + |\boldsymbol{\sigma}_t(s)|_{k+1}^2 + \int_0^s |\boldsymbol{\sigma}(z)|_{k+1}^2 dz \right) ds
\end{aligned}
$$

$$+\tau \int_{t_{n-1}}^{t_n} \left( \|u_{tt}(s)\|^2 + h^{2(k+1)} \left( |\boldsymbol{\sigma}(s)|_{k+1}^2 + |\boldsymbol{\sigma}_t(s)|_{k+1}^2 + \int_0^s |\boldsymbol{\sigma}(z)|_{k+1}^2 dz \right) \right) ds \Bigg). \tag{3.3.13}$$

Multiplying (3.3.13) by $2\tau$, using kickback argument and then summing from $n = 1$ to $m$, we obtain:

$$\begin{aligned}\|\boldsymbol{\vartheta}_h^m\|^2 \leq & C_1 \tau \sum_{n=1}^{m} \|\boldsymbol{\vartheta}_h^n\|^2 + C_2 \left( \|\vartheta_h(\cdot, 0)\|^2 + \tau^2 \int_0^T (\|u_{tt}(s)\|^2 + \|\boldsymbol{\sigma}(s)\|^2 + \|\boldsymbol{\sigma}_t(s)\|^2) ds \right. \\ & \left. + h^{2k+2} \int_0^T g(s) ds \right).\end{aligned}$$

By using Grönwall's lemma, we may replace $m$ by $n$ to get our desired estimate:

$$\|\boldsymbol{\vartheta}_h^n\| \leq O(h^{k+1} + \tau).$$

□

## 3.4 Numerical Results

Within this section, we are set to conduct numerical experiments aimed at validating the effectiveness of the introduced mixed virtual element scheme for the PIDE (1.3.1). Our investigation encompasses two distinct mesh types: a quadrilateral mesh and a hexagonal mesh, as illustrated in Figure 3.1. Here, we consider the domain $\mathcal{D}$ the unit square in $\mathbb{R}^2$. Before presenting the numerical results, it is important for us to have a better understanding of both the spaces, dofs, and how bilinear forms can be computed on these spaces. All the discrete forms are already explained in Section-3.1. In (3.3.1), the bilinear forms $(U_n, \phi_h)$ and $(f(t_n), \phi_h)$ involves multiplication with polynomials so, can be done by using any quadrature rule of appropriate order, whereas for the discrete forms $a_h^E(\cdot,\cdot)$, $\mathcal{K}_h^E(\cdot,\cdot)$ and $(\nabla \cdot \boldsymbol{\sigma}_n, \phi_h)_E$, dofs of $V_h^k(E)$ will be needed (as defined in section-3.1). For more details about the implementation, we refer to [77]. We use the backward Euler approach for time discretization coupled with mixed VEM discretization to tackle the fully-discrete problem for the polygonal mesh sequences.

**Remark 3.4.1.** *From Remark 6.3 of [32], we can see that the lowest order Raviart Thomas element can be constructed for* $k = 0$ *with the usual convention* $\mathbb{P}_{-1}(E) = 0$.

**Example 3.4.1.** *Consider the linear PIDE* (1.3.1) *, with coefficients* $a(x) = 1 + x$, $b(x;t,s) = (1+x)e^{(t-s)}$, *exact solution* $u(x,t) = t\sin(2\pi x)\sin(2\pi y)$ *whereas* $\mathcal{K}(x;t,s) = \frac{e^{2(t-s)}}{1+x}$. *Notably, the load term* $f$, *boundary data, and initial data* $u_0$ *are all determined using the exact solution as a reference point.*

**Example 3.4.2.** *Consider the linear PIDE* (1.3.1) *, with cofficients* $a(x) = 1 + x$, $b(x;t,s) = (1+x)\left(\frac{2+\cos(s)}{2+\cos(t)}\right)$, *exact solution* $u(x,t) = te^{x+t}(x - x^2)\sin(2\pi y)$ *whereas* $\mathcal{K}(x;t,s) = \left(\frac{2+\cos(s)}{(1+x)(2+\cos(t))}\right)e^{(t-s)}$. *Notably, the load term* $f$, *boundary data, and initial data* $u_0$ *are all determined using the exact solution as a reference point.*

Now, we present a numerical example showing that even if we don't know the explicit form of a resolvent kernel, our formulation still works by truncating the series of resolvent kernel after a few steps. In Example 3.4.3, the resolvent kernel comes out to be in a series, so here, we have considered the first five terms of the resolvent kernel and find out that numerical results are still in accordance with the theoretical results.

**Example 3.4.3.** *Consider the linear PIDE* (1.3.1) *, with coefficients* $a(x) = 1$, $b(x;t,s) = ts$, *exact solution* $u(x,t) = t(x - x^2)(y - y^2)$. *Here* $\mathcal{K}(x;t,s)$ *is approximated by the first five terms of the series generated using* (3.1.1). *Notably, the load term* $f$, *boundary data, and initial data* $u_0$ *are all determined using the exact solution as a reference point.*

Fig. 3.3 and 3.4 depicts the order of convergence for both $u_h$ and $\boldsymbol{\sigma}_h$ for Example 3.4.1 in case of $k = 1, 2$ and 3 on quadrilateral and hexagonal mesh respectively. Both the figures show that these orders of convergence are accomplished in perfect accordance with theory, while Fig. 3.5 shows the super convergence results for both Example 3.4.1 and Example 3.4.2 in the case of $k$= 1, 2, and 3 on the quadrilateral mesh whereas in Fig 3.6 shows the order of convergence for Example 3.4.2 on the quadrilateral mesh. Fig 3.7 shows the convergence corresponding to Example 3.4.3. From all the figures, we can see that our theory is well according to our numerical results.

**Remark 3.4.2.** *Whenever we are unable to find the explicit form of the resolvent kernel, we can use the first few terms of the series to achieve the optimal order of convergence, as shown in Example 3.4.3.*

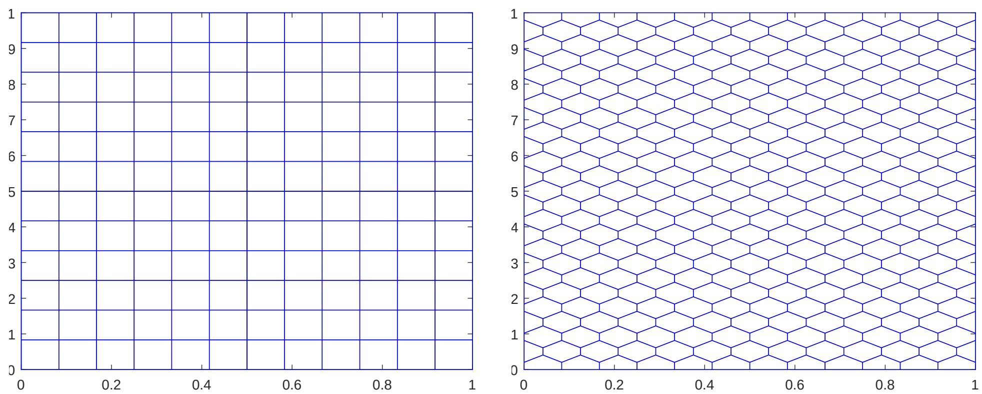


Figure 3.1: An illustration of polygonal meshes: on the left $\mathcal{Q}_{1/12}$, and on the right, $\mathcal{H}_{1/12}$.

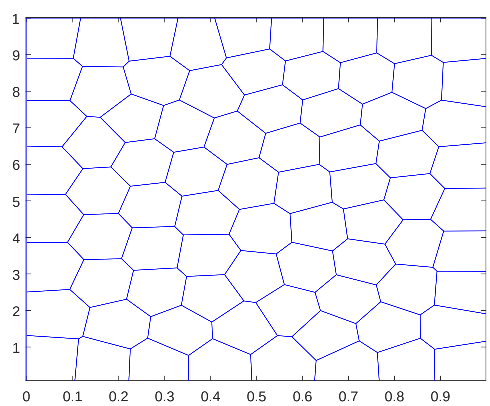


Figure 3.2: An illustration of Voronoi mesh $\mathcal{V}_{1/6}$.

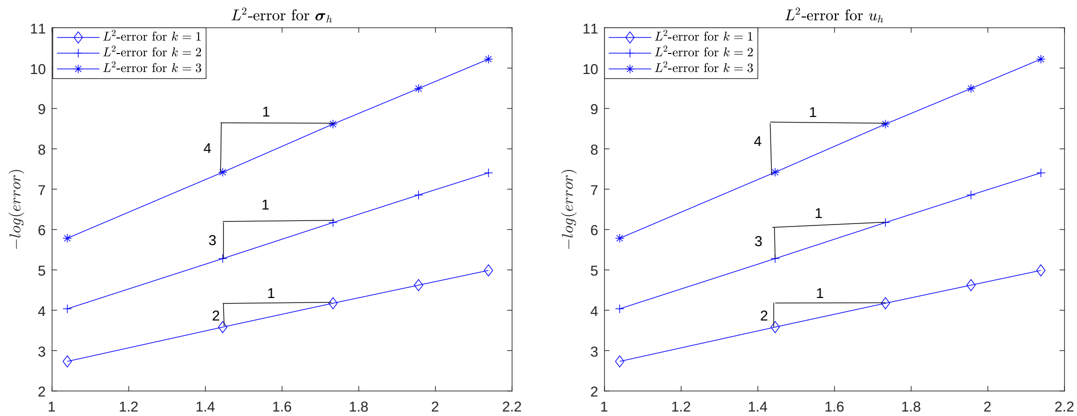


Figure 3.3: Order of convergence for Example 3.4.1 on the quadrilateral mesh.

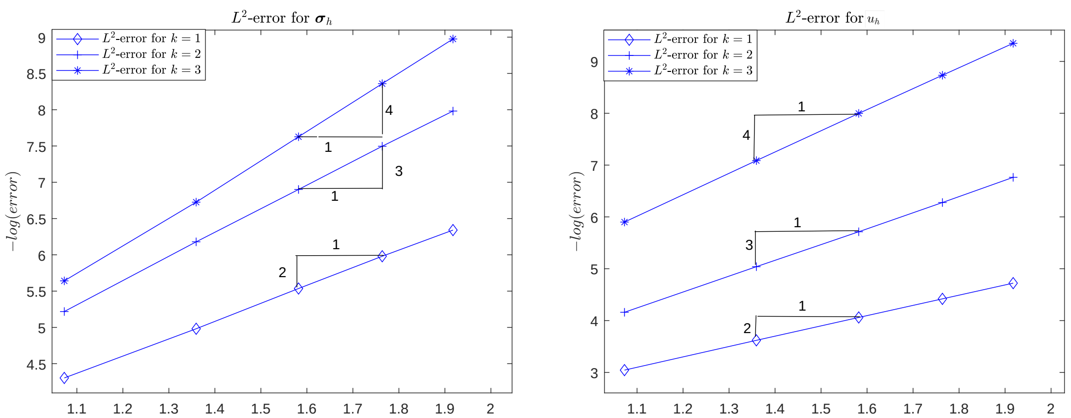


Figure 3.4: Order of convergence for Example 3.4.1 on the hexagonal mesh.

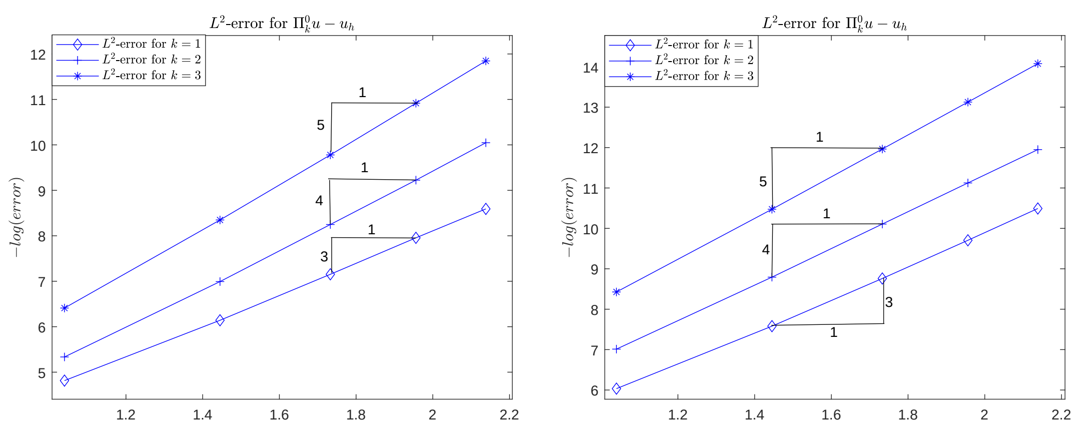


Figure 3.5: Order of convergence for $\Pi_k^0 u - u_h$ on the quadrilateral mesh. The left-hand panel pertains to Example 3.4.1, while the right-hand panel pertains to Example 3.4.2.

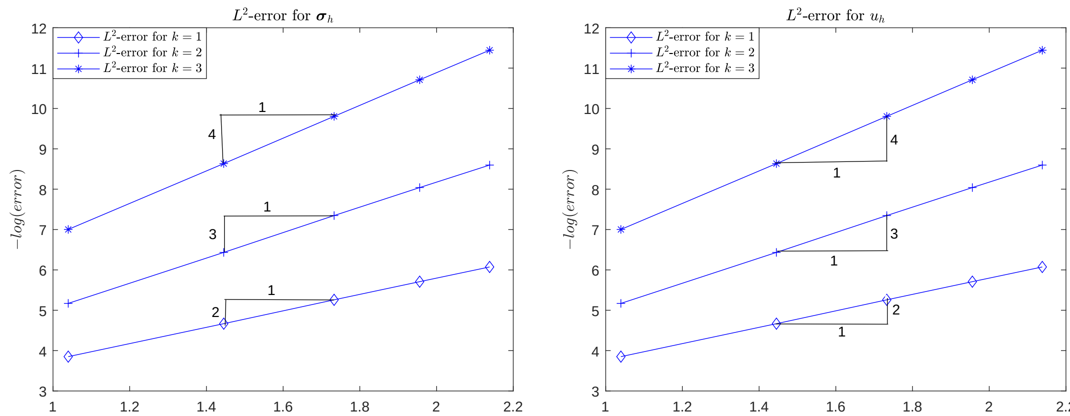


Figure 3.6: Order of convergence for Example 3.4.2 on the quadrilateral mesh.

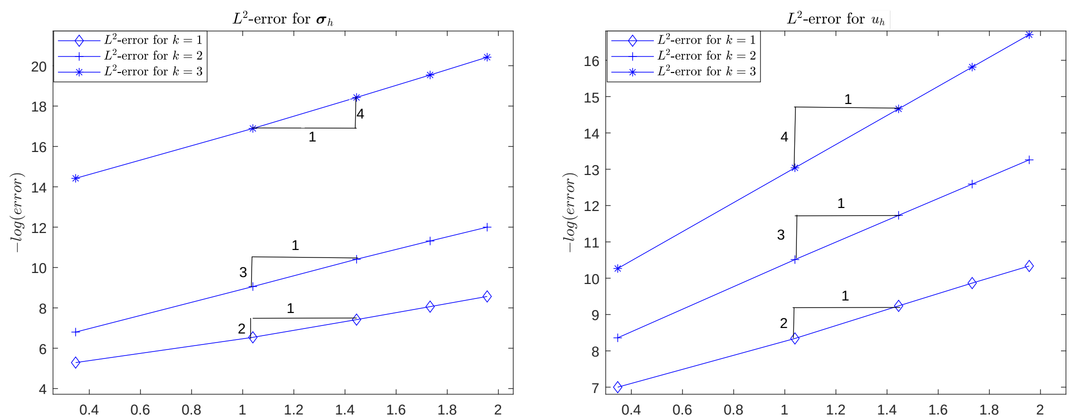


Figure 3.7: Order of convergence for Example 3.4.3 on the quadrilateral mesh in case of $k$=1, 2 and 3.

## 3.5 Conclusions

Considering the advantages of VEM and mixed methods, we applied a mixed VEM approach to address both the semi-discrete and fully-discrete schemes to solve the PIDE (1.3.1). In this chapter, we have introduced a novel projection known as mixed R.V. projection, which helps in handling the integral term. The semi-discrete scheme and error estimates presented in this work align with those obtained in the previous study [78]. This research marks a significant contribution to the literature [14, 61, 78] only with semi-discrete formulation, while here, we represent the first comprehensive examination of the fully-discrete scheme within this formulation. Furthermore, a step-by-step analysis

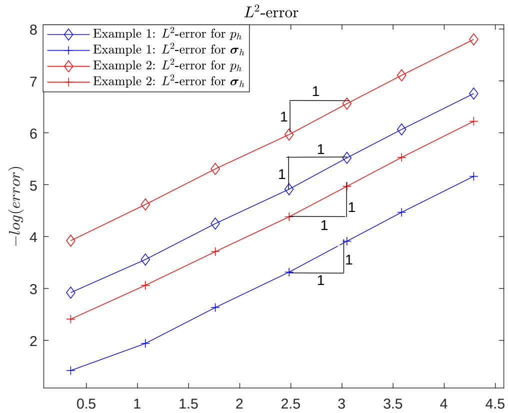


Figure 3.8: Order of convergence for Example 1 and Example 2 on the Voronoi mesh in case of $k$=0.

is proposed for the super convergence of the discrete solution of order $O(h^{k+2})$. Several computational experiments on different polygonal meshes are discussed to validate the proposed scheme's computational efficiency and support the theoretical conclusions.

# Chapter 4

# Mixed Virtual Element Method for Integro-Differential Equations of Parabolic type without Resolvent Kernel[1]

---

This chapter presents and analyzes a new mixed virtual element approach for discretizing PIDEs (1.3.1) in a bounded domain of $\mathbb{R}^2$, complemented by the backward Euler scheme for temporal discretization. It focuses on the variational formulation that avoids the use of a resolvent kernel. As discussed in the previous chapter, the integral term in equation (1.3.4) relies on $\nabla u$; we adopt an approach in our formulation that prevents the necessity of having $u$ in $H^1$. One possible strategy involves the introduction of a resolvent kernel, although it should be noted that deriving a resolvent kernel for a specific problem may not be an easy task. Since this is the case, we use the formulation described by [15]. Ziwen Jiang [15] introduced this formulation (1.3.10a) so that we can avoid the use of a

[1]The substantial part of this chapter has been published as follows:
**M Suthar**, S Yadav, and S Kumar, "Mixed Virtual Element Method for Integro-Differential Equations of Parabolic Type." *Journal of Applied Mathematics and Computing*, https://doi.org/10.1007/s12190-024-02066-8.

resolvent kernel. This article explains the existence and uniqueness of the semi-discrete solution corresponding to the (4.1.1) and mixed intermediate projection. By finding out the estimates of mixed intermediate projection, it establishes optimal error estimates for not only the variables $u$ and $\boldsymbol{\sigma}$ but also with the temporal derivative $u_t$ and $\nabla \cdot \boldsymbol{\sigma}$.

While it's worth noting that, because of the one extra term under the integral sign, this formulation may have a longer computation time compared to the one described in [14], but this is more generalized in the sense that it has a wide range of applications, even when it is tough to find the resolvent kernel, we can go with this formulation and can achieve the required convergence (demonstrated with the help of numerical experiment). This work presents several significant contributions, which are outlined as follows:

- To tackle the integral term, an approach involving a novel projection with a memory term referred to as mixed intermediate projection is introduced, which helps in achieving the optimal convergence of order $O(h^{k+1})$ for both the unknowns.
- A fully discretized scheme is put forth, utilizing the backward Euler's method for temporal derivative and the left rectangular rule for the discretization of the integral term.
- The analysis is performed to show the super convergence of the discrete solution, which has been verified with the different numerical experiments.
- Several different numerical experiments on different meshes have been conducted to validate our theoretical findings.

## 4.1 The Continuous and Semi-discrete Formulation

The mixed VEM discrete formulation corresponding to (1.3.10a) reads as: Find $(\boldsymbol{\sigma}_h, u_h) \in V_h^k \times Q_h^k$ such that:

$$(u_{h,t}, \phi_h) - (\nabla \cdot \boldsymbol{\sigma}_h, \phi_h) = (f, \phi_h) \quad \forall \phi_h \in Q_h^k,$$

$$a_h(\boldsymbol{\sigma}_h, \boldsymbol{\chi}_h) + (u_h, \nabla \cdot \boldsymbol{\chi}_h) - \int_0^t (b_0(\boldsymbol{x}; t, s) u_h(s), \nabla \cdot \boldsymbol{\chi}_h) ds$$

$$-\int_0^t (\nabla b_0(x;t,s)u_h(s), \mathbf{\Pi}_k^0 \boldsymbol{\chi}_h)ds = 0 \quad \forall \boldsymbol{\chi}_h \in V_h^k. \quad (4.1.1)$$

The bilinear form $a_h(\cdot,\cdot)$ and its properties are discussed in Section 3.1. Now, we provide a lemma regarding the consistency of the bilinear form $a_h^{\mathbb{K}}(\cdot,\cdot)$.

**Lemma 4.1.1.** *For $\mathbb{K} \in \mathcal{I}_h$, let the coefficients $\mu(\boldsymbol{x})$ be smooth scaler-valued function in $\mathcal{D}$ and $\boldsymbol{p}$ be any smooth vector-valued function and $\boldsymbol{\chi}_h \in V_h^k(\mathbb{K})$. Then,*

$$a_h^{\mathbb{K}}(\mathbf{\Pi}_k^0 \boldsymbol{p}, \boldsymbol{\chi}_h) - (\mu \mathbf{\Pi}_k^0 \boldsymbol{p}, \boldsymbol{\chi}_h)_{0,\mathbb{K}} \leq C_\mu h^{k+1} |\boldsymbol{p}|_{k+1,\mathbb{K}} \|\boldsymbol{\chi}_h\|_{0,\mathbb{K}}.$$

*Proof.* Let $\boldsymbol{l}_h, \boldsymbol{\chi}_h$ be vector valued functions in $V_h^k(\mathbb{K})$. By the definition of $a_h^{\mathbb{K}}(\cdot,\cdot)$ and $\Pi_k^0$ projection, it is easy to note that:

$$\begin{aligned} &a_h^{\mathbb{K}}(\boldsymbol{l}_h, \boldsymbol{\chi}_h) - (\mu \boldsymbol{l}_h, \boldsymbol{\chi}_h)_{0,\mathbb{K}} \\ &= (\mu \boldsymbol{l}_h - \mathbf{\Pi}_k^0(\mu \boldsymbol{l}_h), \boldsymbol{\chi}_h - \mathbf{\Pi}_k^0 \boldsymbol{\chi}_h)_{0,\mathbb{K}} + (\boldsymbol{l}_h - \mathbf{\Pi}_k^0 \boldsymbol{l}_h, \mu \boldsymbol{\chi}_h - \mathbf{\Pi}_k^0(\mu \boldsymbol{\chi}_h))_{0,\mathbb{K}} \\ &\quad - (\boldsymbol{l}_h - \mathbf{\Pi}_k^0 \boldsymbol{l}_h, \mu(\boldsymbol{\chi}_h - \mathbf{\Pi}_k^0 \boldsymbol{\chi}_h))_{0,\mathbb{K}} + S_0^{\mathbb{K}}((I - \mathbf{\Pi}_k^0)\boldsymbol{l}_h, (I - \mathbf{\Pi}_k^0)\boldsymbol{\chi}_h). \end{aligned} \quad (4.1.2)$$

Now, put $\boldsymbol{l}_h = \mathbf{\Pi}_k^0 \boldsymbol{p}$ in (4.1.2) and using the properties of $\mathbf{\Pi}_k^0$, last three terms becomes zero, then by using (1.5.7), we get:

$$a_h^{\mathbb{K}}(\mathbf{\Pi}_k^0 \boldsymbol{p}, \boldsymbol{\chi}_h) - (\mu \mathbf{\Pi}_k^0 \boldsymbol{p}, \boldsymbol{\chi}_h)_{0,\mathbb{K}} \leq C_\mu h^{k+1} |\boldsymbol{p}|_{k+1,\mathbb{K}} \|\boldsymbol{\chi}_h\|_{0,\mathbb{K}}.$$

□

## 4.2 Error Analysis for the Semi-discrete Case

For the formulation described in (1.3.10a) and (4.1.1), we now derive the optimal error estimates for both the semi-discrete and fully-discrete cases. So, to deal with the memory term, we define a new projection here with the memory term known as mixed intermediate projection.

### 4.2.1 Mixed Intermediate Projection

Mixed intermediate projection is defined as: $(\mathscr{I}^V \boldsymbol{\sigma}, \mathscr{I}^Q u) : (0,T] \to V_h^k \times Q_h^k$ such that:

$$\begin{aligned}
&a_h(\mathscr{I}^V \boldsymbol{\sigma}, \boldsymbol{\chi}_h) - (\mu\boldsymbol{\sigma}, \boldsymbol{\chi}_h) + (\mathscr{I}^Q u - u, \nabla\cdot\boldsymbol{\chi}_h) \\
&- \int_0^t ((b_0(\boldsymbol{x}; t,s)\mathscr{I}^Q u(s), \nabla\cdot\boldsymbol{\chi}_h) + (\nabla b_0(x;t,s)\mathscr{I}^Q u(s), \boldsymbol{\Pi}_k^0\boldsymbol{\chi}_h))ds \\
&+ \int_0^t ((b_0(\boldsymbol{x}; t,s)u(s), \nabla\cdot\boldsymbol{\chi}_h) + (\nabla b_0(x;t,s)u(s), \boldsymbol{\chi}_h))ds = 0 \quad \forall \boldsymbol{\chi}_h \in V_h^k, \qquad \text{(4.2.1a)} \\
&(\nabla\cdot(\boldsymbol{\sigma} - \mathscr{I}^V\boldsymbol{\sigma}), \phi_h) = 0 \;\; \forall \phi_h \in Q_h^k, \qquad \text{(4.2.1b)}
\end{aligned}$$

along with the suitable choice of $(\mathscr{I}^V\boldsymbol{\sigma}(0), \mathscr{I}^Q u(0)) \in V_h^k \times Q_h^k$.

**Theorem 4.2.1.** *If $\boldsymbol{\sigma}$ and $u \in L^\infty(0,T,H^{k+1}(\mathcal{D}))$, then $\exists$ a unique pair $(\mathscr{I}^V\boldsymbol{\sigma}, \mathscr{I}^Q u) \in V_h^k \times Q_h^k$ satisfying (4.2.1b). Also, the following estimates hold true:*

$$\|\boldsymbol{\sigma} - \mathscr{I}^V\boldsymbol{\sigma}\| \le Ch^{k+1}\left(|\boldsymbol{\sigma}(t)|_{k+1} + \int_0^t (|u(s)|_{k+1} + |\boldsymbol{\sigma}(s)|_{k+1})ds\right), \qquad (4.2.2)$$

$$\|u - \mathscr{I}^Q u\| \le Ch^{k+1}\left(|u(t)|_{k+1} + |\boldsymbol{\sigma}|_{k+1} + \int_0^t |u(s)|_{k+1}ds\right). \qquad (4.2.3)$$

*Proof.* To establish the proofs of (4.2.2) and (4.2.3), we utilize the definition of the mixed intermediate projection given by (4.2.1b). By considering $\boldsymbol{\Psi}_h = \boldsymbol{\Pi_h^F}\boldsymbol{\sigma} - \mathscr{I}^V\boldsymbol{\sigma} \in V_h^k$, $\eta = u - \mathscr{I}^Q u$ and $\nu_h = \Pi_k^0 u - \mathscr{I}^Q u \in Q_h^k$, we proceed as:

$$\boldsymbol{\Theta} = \boldsymbol{\sigma} - \mathscr{I}^V\boldsymbol{\sigma} = \boldsymbol{\sigma} - \boldsymbol{\Pi_h^F}\boldsymbol{\sigma} + \boldsymbol{\Psi}_h.$$

From the definition of mixed intermediate projection (4.2.1b), we observe:

$$\begin{aligned}
&a_h(\boldsymbol{\Psi}_h, \boldsymbol{\chi}_h) \\
&= [a_h(\boldsymbol{\Pi_h^F}\boldsymbol{\sigma}, \boldsymbol{\chi}_h) - (\mu\boldsymbol{\sigma}, \boldsymbol{\chi}_h)] + (\mathscr{I}^Q u - u, \nabla\cdot\boldsymbol{\chi}_h) \\
&\quad + \int_0^t ((b_0(\boldsymbol{x}; t,s)(u - \mathscr{I}^Q u)(s), \nabla\cdot\boldsymbol{\chi}_h)ds
\end{aligned}$$

$$-\int_0^t ((\nabla b_0(x;t,s)\mathscr{I}^Q u(s), \mathbf{\Pi}_k^0 \boldsymbol{\chi}_h) - (\nabla b_0(x;t,s)u(s), \boldsymbol{\chi}_h))ds. \tag{4.2.4}$$

To deal with the first expression on the right-hand side of (4.2.4), we proceed as follows:

$$\begin{aligned}
& a_h(\mathbf{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}\boldsymbol{\sigma}, \boldsymbol{\chi}_h) - (\mu\boldsymbol{\sigma}, \boldsymbol{\chi}_h) \\
&= a_h(\mathbf{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}\boldsymbol{\sigma} - \mathbf{\Pi}_k^0\boldsymbol{\sigma}, \boldsymbol{\chi}_h) + a_h(\mathbf{\Pi}_k^0\boldsymbol{\sigma}, \boldsymbol{\chi}_h) - (\mu(\boldsymbol{\sigma} - \mathbf{\Pi}_k^0\boldsymbol{\sigma}), \boldsymbol{\chi}_h) - (\mu\mathbf{\Pi}_k^0\boldsymbol{\sigma}, \boldsymbol{\chi}_h) \\
&\leq C(\|\boldsymbol{\sigma} - \mathbf{\Pi}_k^0\boldsymbol{\sigma}\| + \|\mathbf{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}\boldsymbol{\sigma} - \mathbf{\Pi}_k^0\boldsymbol{\sigma}\| + h^{k+1}|\boldsymbol{\sigma}|_{k+1})\|\boldsymbol{\chi}_h\| \\
&\leq Ch^{k+1}|\boldsymbol{\sigma}|_{k+1}\|\boldsymbol{\chi}_h\|.
\end{aligned}$$

While the last term on the right-hand side of equation (4.2.4) can be resolved as follows:

$$\begin{aligned}
&\int_0^t ((\nabla b_0(\boldsymbol{x};t,s)\mathscr{I}^Q u(s), \mathbf{\Pi}_k^0\boldsymbol{\chi}_h) - (\nabla b_0(\boldsymbol{x};t,s)u(s), \boldsymbol{\chi}_h))ds \\
&\quad\leq \int_0^t (\nabla b_0(\boldsymbol{x};t,s)(\mathscr{I}^Q u - u)(s), \mathbf{\Pi}_k^0\boldsymbol{\chi}_h)ds + \int_0^t (\nabla b_0(\boldsymbol{x};t,s)u(s), \mathbf{\Pi}_k^0\boldsymbol{\chi}_h - \boldsymbol{\chi}_h)ds \\
&\quad\leq \int_0^t (\nabla b_0(\boldsymbol{x};t,s)(\mathscr{I}^Q u - u)(s), \mathbf{\Pi}_k^0\boldsymbol{\chi}_h)ds \\
&\qquad + \int_0^t (\nabla b_0(\boldsymbol{x};t,s)u(s) - \mathbf{\Pi}_k^0(\nabla b_0(\boldsymbol{x};t,s)u(s)), \mathbf{\Pi}_k^0\boldsymbol{\chi}_h - \boldsymbol{\chi}_h)ds \\
&\quad\leq C_{\nabla b_0}\int_0^t (\|\nu_h(s)\| + h^{k+1}|u(s)|_{k+1})ds\|\boldsymbol{\chi}_h\|.
\end{aligned} \tag{4.2.5}$$

Put $\boldsymbol{\chi}_h = \mathbf{\Psi}_h$ in (4.2.4) and using (3.1.3), boundedness of $b_0(\boldsymbol{x};t,s)$ and the fact that $\|\nabla \cdot \mathbf{\Psi}_h\| = 0$, to arrive at the following:

$$\mu_1\|\mathbf{\Psi}_h\|^2 \leq \left(Ch^{k+1}\left(|\boldsymbol{\sigma}|_{k+1} + \int_0^t |u(s)|_{k+1}ds\right) + \int_0^t \|\nu_h(s)\|ds\right)\|\mathbf{\Psi}_h\|. \tag{4.2.6}$$

By using the triangle inequality, we obtain the following:

$$\|\mathbf{\Theta}\| \leq Ch^{k+1}\left(|\boldsymbol{\sigma}|_{k+1} + \int_0^t |u(s)|_{k+1}ds\right) + \int_0^t \|\nu_h(s)\|ds. \tag{4.2.7}$$

Now, for the proof of (4.2.3), we use the duality argument along with the definition of mixed intermediate projection (4.2.1b) as:

$$(\mu\boldsymbol{\Theta}, \boldsymbol{\chi}_h) + (\nabla \cdot \boldsymbol{\chi}_h, \nu_h) = \mathcal{F}(\boldsymbol{\chi}_h) \quad \forall \boldsymbol{\chi}_h \in V_h^k,$$
$$(\nabla \cdot \boldsymbol{\Theta}, \phi_h) = 0 \quad \forall q_h \in Q_h^k,$$

where

$$\mathcal{F}(\boldsymbol{\chi}_h) = a_h(\mathscr{I}^V\boldsymbol{\sigma}, \boldsymbol{\chi}_h) - (\mu\mathscr{I}^V\boldsymbol{\sigma}, \boldsymbol{\chi}_h) - \int_0^t (b_0(\boldsymbol{x}; t, s)(\mathscr{I}^Q u - u)(s), \nabla \cdot \boldsymbol{\chi}_h) ds$$
$$- \int_0^t ((\nabla b_0(\boldsymbol{x}; t, s)\mathscr{I}^Q u(s), \boldsymbol{\Pi}_k^0 \boldsymbol{\chi}_h) - (\nabla b_0(\boldsymbol{x}; t, s) u(s), \boldsymbol{\chi}_h)) ds.$$

Let $\omega \in H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})$, with $\mathcal{D}$ to be convex and bounded, be the solution of the following dual problem:

$$-\nabla \cdot (a\nabla\omega) = \nu_h \quad \text{in } \mathcal{D}; \quad \omega = 0 \quad \text{on } \partial\mathcal{D}. \tag{4.2.8}$$

and satisfy the following regularity condition $\|\omega\|_2 \leq \|\nu_h\|$. Consider $\boldsymbol{\Phi} = a\nabla\omega$, then the mixed variational formulation corresponding to (4.2.8) is; Find $(\boldsymbol{\Phi}, \omega) \in \mathcal{V} \times \mathcal{Q}$ such that:

$$\begin{aligned} (\mu\boldsymbol{\Phi}, \boldsymbol{\chi}) + (\omega, \nabla \cdot \boldsymbol{\chi}) &= 0 \quad \forall \boldsymbol{\chi} \in \mathcal{V}, \\ -(\nabla \cdot \boldsymbol{\Phi}, \phi) &= (\nu_h, \phi) \quad \forall \phi \in \mathcal{Q}. \end{aligned} \tag{4.2.9}$$

Now, put $\phi = \nu_h$ in (4.2.9), to arrive at:

$$\begin{aligned} \|\nu_h\|^2 &= (\nu_h, -\nabla \cdot (\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}} a\nabla\omega)) \\ &= (\mu\boldsymbol{\Theta}, \boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\omega)) - \mathcal{F}(\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\omega)). \end{aligned} \tag{4.2.10}$$

Now, we rewrite $\mathcal{F}(\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\omega))$ as:

$$\mathcal{F}(\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\omega))$$
$$= a_h(\mathscr{I}^V\boldsymbol{\sigma} - \boldsymbol{\Pi}_k^0\boldsymbol{\sigma}, \boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\omega)) - (\mu(\mathscr{I}^V\boldsymbol{\sigma} - \boldsymbol{\Pi}_k^0\boldsymbol{\sigma}), \boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\omega)) + a_h(\boldsymbol{\Pi}_k^0\boldsymbol{\sigma}, \boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\omega))$$

$$
\begin{aligned}
&- (\mu \boldsymbol{\Pi}_k^0 \boldsymbol{\sigma}, \boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\omega)) - \int_0^t (b_0(\boldsymbol{x};t,s)(\mathscr{I}^Q u - \Pi_k^0 u + \Pi_k^0 u - u)(s), \nabla \cdot \boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\omega))ds \\
&- \int_0^t (\nabla b_0(\boldsymbol{x};t,s)(\mathscr{I}^Q u - u)(s), \boldsymbol{\Pi}_k^0(\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\omega)))ds \\
&- \int_0^t (\nabla b_0(\boldsymbol{x};t,s)u(s) - \boldsymbol{\Pi}_k^0(\nabla b_0(\boldsymbol{x};t,s)u(s)), \boldsymbol{\Pi}_k^0(\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\omega)) - \boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\omega))ds.
\end{aligned}
$$

Using Lemma 4.1.1, the estimates of $\Pi_k^0$ (1.5.7), and boundedness of $b_0(\boldsymbol{x};t,s)$, we arrive at the following:

$$
\begin{aligned}
&|\mathcal{F}(\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\omega))| \\
&\quad \leq C\left(\|\boldsymbol{\Psi}_h\| + h^{k+1}\left(|\boldsymbol{\sigma}|_{k+1} + \int_0^t |u(s)|_{k+1}ds\right) + \int_0^t \|\nu_h(s)\|)ds\right)\|\nu_h\|. \quad (4.2.11)
\end{aligned}
$$

Now, using (4.2.6) and (4.2.11) in (4.2.10) followed by Grönwall's lemma, we have:

$$
\|\nu_h\| \leq Ch^{k+1}\left(|\boldsymbol{\sigma}|_{k+1} + \int_0^t |u(s)|ds\right). \tag{4.2.12}
$$

Using (4.2.12), the estimates of $\Pi_k^0$ (1.5.7) and the triangle inequality, we achieve (4.2.3). Now, substitute (4.2.12) in (4.2.7) to achieve (4.2.2). □

**Theorem 4.2.2.** *Under all the assumptions of Theorem 4.2.1 and $\boldsymbol{\sigma}_t, u_t \in L^\infty(0,T,H^{k+1}(\mathcal{D}))$, where $u_t$, $\boldsymbol{\sigma}_t$ and $\mathscr{I}^Q u_t$, $\mathscr{I}^V \boldsymbol{\sigma}_t$ be the time derivative of $u$, $\boldsymbol{\sigma}$ and $\mathscr{I}^Q u$, $\mathscr{I}^V \boldsymbol{\sigma}$ respectively, the following estimates hold true:*

$$
\begin{aligned}
\|\boldsymbol{\sigma}_t - \mathscr{I}^V \boldsymbol{\sigma}_t\| &\leq Ch^{k+1}\left(|\boldsymbol{\sigma}_t|_{k+1} + g(t)\right), \\
\|u_t - \mathscr{I}^Q u_t\| &\leq Ch^{k+1}\left(|\boldsymbol{\sigma}_t|_{k+1} + |u_t|_{k+1} + g(t)\right),
\end{aligned} \tag{4.2.13}
$$

*where $g(t) = |\boldsymbol{\sigma}|_{k+1} + |u(t)|_{k+1} + \int_0^t (|\boldsymbol{\sigma}(s)|_{k+1} + |u(s)|_{k+1})ds$.*

**Theorem 4.2.3.** *Let $u$, $\boldsymbol{\sigma}$ and $u_h$, $\boldsymbol{\sigma}_h$ be the solution of continuous problem (1.3.10a) and semi-discrete formulation (4.1.1) , respectively. Under all the assumptions of Theorem 4.2.2, we have the following:*

$$\|u - u_h\|^2 \leq C(\|\eta_h(\cdot, 0)\|^2 + h^{2(k+1)} \int_0^t (|u_t(s)|_{k+1}^2 + |\boldsymbol{\sigma}_t(s)|_{k+1}^2 + |u(s)|_{k+1}^2 + |\boldsymbol{\sigma}(s)|_{k+1}^2) ds), \tag{4.2.14}$$

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|^2 \leq C(\|\boldsymbol{\Theta}_h(\cdot, 0)\|^2 + T\|\eta_h(\cdot, 0)\|^2 + h^{2(k+1)} \int_0^t (|u_t(s)|_{k+1}^2 + |\boldsymbol{\sigma}_t(s)|_{k+1}^2 + |u(s)|_{k+1}^2 + |\boldsymbol{\sigma}(s)|_{k+1}^2) ds). \tag{4.2.15}$$

*Proof.* Writing $u - u_h = u - \mathscr{I}^Q u + \mathscr{I}^Q u - u_h = \eta + \eta_h$ and $\boldsymbol{\sigma} - \boldsymbol{\sigma}_h = \boldsymbol{\sigma} - \mathscr{I}^V \boldsymbol{\sigma} + \mathscr{I}^V \boldsymbol{\sigma} - \boldsymbol{\sigma}_h = \boldsymbol{\Theta} + \boldsymbol{\Theta}_h$. Since, we already have the estimates for $\|\eta\|$ and $\|\boldsymbol{\Theta}\|$, our goal is to determine $\|\eta_h\|$ and $\|\boldsymbol{\Theta}_h\|$. To proceed, we use (1.3.10a) and (4.1.1) to obtain:

$$(u_t - u_{h,t}, \phi_h) - (\nabla \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h), \phi_h) = 0 \quad \forall \phi_h \in Q_h^k, \tag{4.2.16}$$

$$(\mu \boldsymbol{\sigma}, \boldsymbol{\chi}_h) - a_h(\boldsymbol{\sigma}_h, \boldsymbol{\chi}_h) + (u - u_h, \nabla \cdot \boldsymbol{\chi}_h) - \int_0^t (b_0(\boldsymbol{x}; t, s)(u - u_h)(s), \nabla \cdot \boldsymbol{\chi}_h) ds - \int_0^t ((\nabla b_0(\boldsymbol{x}; t, s) u(s), \boldsymbol{\chi}_h) - (\nabla b_0(\boldsymbol{x}; t, s) u_h(s), \boldsymbol{\Pi}_k^0 \boldsymbol{\chi}_h)) ds = 0 \quad \forall \boldsymbol{\chi}_h \in V_h^k. \tag{4.2.17}$$

Rewriting (4.2.16) and (4.2.17) as:

$$(\eta_{h,t}, \phi_h) - (\nabla \cdot \boldsymbol{\Theta}_h, \phi_h) = -(\eta_t, \phi_h), \tag{4.2.18}$$

$$a_h(\boldsymbol{\Theta}_h, \boldsymbol{\chi}_h) + (\eta_h, \nabla \cdot \boldsymbol{\chi}_h) - \int_0^t (b_0(\boldsymbol{x}; t, s)\eta_h(s), \nabla \cdot \boldsymbol{\chi}_h) ds - \int_0^t (\nabla b_0(\boldsymbol{x}; t, s)\eta_h(s), \boldsymbol{\Pi}_k^0 \boldsymbol{\chi}_h) ds = 0. \tag{4.2.19}$$

Putting $\phi_h = \eta_h - \int_0^t \Pi_k^0(b_0(\boldsymbol{x}; t, s)\eta_h(s)) ds$ in (4.2.18) and $\boldsymbol{\chi}_h = \boldsymbol{\Theta}_h$ in (4.2.19), then adding the equations, we get:

$$a_h(\boldsymbol{\Theta}_h, \boldsymbol{\Theta}_h) + (\eta_{h,t}, \eta_h) - \int_0^t (\nabla b_0(\boldsymbol{x}; t, s)\eta_h(s), \boldsymbol{\Pi}_k^0 \boldsymbol{\Theta}_h) ds$$
$$= -\left(\eta_t, \eta_h - \int_0^t \Pi_k^0(b(b(\boldsymbol{x}; t, s)\eta_h(s))_h(s)) ds\right) + \left(\eta_{h,t}, \int_0^t \Pi_k^0(b\Pi_k^0(\boldsymbol{x}; t, s)\eta_h(s)) ds\right).$$

Using (1.5.7) and differentiation by parts, we arrive at:

$$\begin{aligned}
&\alpha_1\|\mathbf{\Theta}_h\|^2 + \frac{1}{2}\frac{d\|\eta_h\|^2}{dt} \\
&= -\left[\eta_h, \Pi_k^0(b_0(\boldsymbol{x};t,t)\eta_h) + \int_0^t \Pi_k^0(b_{0,t}(\boldsymbol{x};t,s)\eta_h(s))ds\right] + \int_0^t (\nabla b_0(\boldsymbol{x};t,s)\eta_h(s), \mathbf{\Pi_k^0}\mathbf{\Theta}_h)ds \\
&\quad + \frac{d}{dt}\left(\eta_h, \int_0^t \Pi_k^0(b_0(\boldsymbol{x};t,s)\eta_h(s))ds\right) - \left(\eta_t, \eta_h - \int_0^t \Pi_k^0(b_0(\boldsymbol{x};t,s)\eta_h(s))ds\right).
\end{aligned}$$

Using Young's inequality followed by a kickback argument, we arrive at the following:

$$\begin{aligned}
&c_{\alpha_1,\epsilon}\|\mathbf{\Theta}_h\|^2 + \frac{1}{2}\frac{d\|\eta_h\|^2}{dt} \\
&\quad = C\left(\|\eta_h\|^2 + \int_0^t \|\eta_h(s)\|^2 ds + \|\eta_t\|^2\right) + \frac{d}{dt}\left(\eta_h, \int_0^t \Pi_k^0(b(\boldsymbol{x};t,s)\eta_h(s))ds\right).
\end{aligned}$$

By integrating from 0 to $t$ coupled with Young's inequality and kickback argument:

$$\|\eta_h(\cdot,t)\|^2 \leq C\left(\|\eta_h(\cdot,0)\|^2 + \int_0^t (\|\eta_h(s)\|^2 + \|\eta_t(s)\|^2)ds\right).$$

Applying Grönwall's lemma, the resulting expression obtained is as follows:

$$\begin{aligned}
&\|\eta_h(\cdot,t)\|^2 \leq C(\|\eta_h(\cdot,0)\|^2 \\
&\qquad\qquad + h^{2(k+1)}\int_0^t (|u_t(s)|_{k+1}^2 + |\boldsymbol{\sigma}_t(s)|_{k+1}^2 + |u(s)|_{k+1}^2 + |\boldsymbol{\sigma}(s)|_{k+1}^2)ds).
\end{aligned}$$

Using the triangle inequality, we complete the proof of (4.2.14). For the proof of (4.2.15), differentiate (4.2.19), and then put $\boldsymbol{\chi}_h = \mathbf{\Theta}_h$ and, $\phi_h = \eta_{h,t} - \Pi_k^0(b_0(\boldsymbol{x};t,t)\eta_h) - \int_0^t \Pi_k^0(b_{0,t}(\boldsymbol{x};t,s)\eta_h(s))ds$ in (4.2.18) and then add, to achieve:

$$\begin{aligned}
&(\eta_{h,t}, \eta_{h,t}) - (\nabla b_0(\boldsymbol{x};t,t)\eta_h(t), \mathbf{\Pi_k^0}\mathbf{\Theta}_h) + a_h(\mathbf{\Theta}_{h,t}, \mathbf{\Theta}_h) \\
&= \left(\eta_{h,t}, \Pi_k^0(b_0(\boldsymbol{x};t,t)\eta_h(t)) + \int_0^t \Pi_k^0(b_{0,t}(\boldsymbol{x};t,s)\eta_h(s))ds\right) \\
&\quad + \int_0^t (\nabla b_{0t}(\boldsymbol{x};t,s)\eta_h(s), \mathbf{\Pi_k^0}\mathbf{\Theta}_h)ds
\end{aligned}$$

$$-\left(\eta_t, \eta_{h,t} - \Pi_k^0(b_0(\boldsymbol{x};t,t)\eta_h) - \int_0^t \Pi_k^0(b_{0,t}(\boldsymbol{x};t,s)\rho_h)ds\right).$$

Using Young's inequality followed by a Kickback argument, we arrive at:

$$c_\epsilon\|\eta_{h,t}\|^2 + \frac{\alpha_1}{2}\frac{d}{dt}\|\boldsymbol{\Theta}_h\|^2 \leq C\left(\|\eta_h\|^2 + \int_0^t \|\eta_h(s)\|^2 ds + \|\boldsymbol{\Theta}_h\|^2 + \|\eta_t\|^2\right).$$

By integration form 0 to $t$ and using Grönwall's lemma:

$$\|\boldsymbol{\Theta}_h(\cdot,t)\|^2 \leq C\left(\|\boldsymbol{\Theta}_h(\cdot,0)\|^2 + \int_0^t (\|\eta_t(s)\|^2 + \|\eta_h(s)\|^2)ds\right).$$

Using a triangle inequality and (1.5.7), we arrive at the following:

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|^2 \leq C(\|\boldsymbol{\Theta}_h(\cdot,0)\|^2 + T\|\eta_h(\cdot,0)\|^2 \\ + h^{2(k+1)}\int_0^t (|u_t(s)|_{k+1}^2 + |\boldsymbol{\sigma}_t(s)|_{k+1}^2 + |u(s)|_{k+1}^2 + |\boldsymbol{\sigma}(s)|_{k+1}^2)ds).$$

Hence, it completes the proof. □

**Remark 4.2.1.** *The estimate* (4.2.14) *and* (4.2.15) *involve the term* $\eta_h(\cdot,0)$ *and* $\boldsymbol{\Theta}_h(\cdot,0)$ *respectively. To achieve the optimal convergence, we need to choose* $u_h(\cdot,0)$ *and* $\boldsymbol{\sigma}_h(\cdot,0)$ *in such a way that* $\eta_h(\cdot,0)$ *and* $\boldsymbol{\Theta}_h(\cdot,0)$ *is of* $O(h^{k+1})$.

### 4.2.2 Super Convergence Analysis of the Discrete Solution

As evident from the equation (4.2.12), it is clear that $\nu_h$ exhibits convergence of order $O(h^{k+1})$. We can potentially enhance the convergence order of $\nu_h$ by utilizing the dual norm approach, resulting in an order of $O(h^{k+2})$.

**Theorem 4.2.4.** *Let* $u$ *and* $u_h$ *be the solution of continuous problem* (1.3.10a) *and semi-discrete formulation* (4.1.1) *, respectively. Under all the assumptions of Theorem 4.2.3, the following holds:*

$$\|\Pi_k^0 u - u_h\| \leq O(h^{k+2}). \tag{4.2.20}$$

*Proof.* Rewriting (4.2.10) as:

$$\begin{aligned}\|\nu_h\|^2 =&(\mu\boldsymbol{\Theta}, \boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\omega)) - \mathcal{F}(\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\omega))\\ =&(\mu\boldsymbol{\Theta}, \boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\omega) - a\nabla\omega) - \mathcal{F}(\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\omega)) + (\nabla\cdot\boldsymbol{\Theta}, \Pi_k^0\omega - \omega)\\ \leq& C(h\|\boldsymbol{\Theta}\|\|\omega\|_2 + h^2\|\nabla\cdot\boldsymbol{\Theta}\|\|\omega\|_2 + \mathcal{F}(\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\omega))), \end{aligned} \tag{4.2.21}$$

where $\mathcal{F}(\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\omega))$ can be solved as:

$$\begin{aligned}&\mathcal{F}(\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\omega))\\ &= (a_h(\mathscr{I}^V\boldsymbol{\sigma} - \boldsymbol{\Pi}_k^0\boldsymbol{\sigma}, \boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\omega)) - (\mu(\mathscr{I}^V\boldsymbol{\sigma} - \boldsymbol{\Pi}_k^0\boldsymbol{\sigma}), \boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\omega))\\ &\quad + a_h(\boldsymbol{\Pi}_k^0\boldsymbol{\sigma}, \boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\omega)) - (\mu\boldsymbol{\Pi}_k^0\boldsymbol{\sigma}, \boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\omega)))\\ &\quad - \left(\int_0^t (b_0(\boldsymbol{x};t,s)(\mathscr{I}^Q u - u)(s), \nabla\cdot\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\omega))ds\right)\\ &\quad - \left(\int_0^t ((\nabla b_0(\boldsymbol{x};t,s)\mathscr{I}^Q u(s), \boldsymbol{\Pi}_k^0(\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\omega)) - (\nabla b_0(\boldsymbol{x};t,s)u(s), \boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\omega)))ds\right)\\ &= I + II + III. \end{aligned} \tag{4.2.22}$$

The initial expression on the right-hand side of equation (4.2.22) can be addressed by using (4.1.2) in the following manner:

$$\begin{aligned}|I| &\leq C(\|\mathscr{I}^V\boldsymbol{\sigma} - \boldsymbol{\Pi}_k^0\boldsymbol{\sigma}\| + h^{k+1}|\boldsymbol{\sigma}|_{k+1})\|\boldsymbol{\Pi}_k^0(\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\omega)) - \boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\omega)\|\\ &\leq Ch^{k+2}\left(|\boldsymbol{\sigma}|_{k+1} + \int_0^t (|u(s)|_{k+1} + |\boldsymbol{\sigma}(s)|_{k+1})ds\right)\|\omega\|_2. \end{aligned} \tag{4.2.23}$$

The second term on the right-hand side of (4.2.22) can be solved by using $\Pi^0$ projection of $b_0(\boldsymbol{x};t,s)$ onto space of piecewise constant functions; for more details, see [79]:

$$\begin{aligned}|II| \leq& \int_0^t (b_0(\boldsymbol{x};t,s) - \Pi^0 b(\boldsymbol{x};t,s)(\mathscr{I}^Q u - u)(s), \nabla\cdot\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\omega))\\ &+ (\Pi^0 b(\boldsymbol{x};t,s)(\mathscr{I}^Q u - \Pi_k^0 u)(s), \nabla\cdot\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\omega)))ds\\ \leq& C\int_0^t (h|b_0(s)|_{1,\infty}\|\mathscr{I}^Q u - u\| + \|\nu_h\|)ds\|\omega\|_2. \end{aligned} \tag{4.2.24}$$

The third expression on the right-hand side of equation (4.2.22) can be solved in a similar way as (4.2.5) as:

$$\begin{aligned} |III| &= \int_0^t ((\nabla b_0(\boldsymbol{x};t,s)\mathscr{I}^Q u(s), \boldsymbol{\Pi_h^F}(a\nabla\omega)) - (\nabla b_0(\boldsymbol{x};t,s)u(s), \boldsymbol{\Pi_h^F}(a\nabla\omega)))ds \\ &\leq C\int_0^t (h^{k+2}|u(s)|_{k+1} + \|\nu_h\|)\|\omega\|_2. \end{aligned} \tag{4.2.25}$$

Putting (4.2.23), (4.2.24), (4.2.25) in (4.2.22), we arrive at:

$$\begin{aligned} \mathcal{F}(\boldsymbol{\Pi_h^F}(a\nabla\omega)) \leq C(h^{k+2}|\boldsymbol{\sigma}|_{k+1} + h^{k+2}\int_0^t (|u(s)|_{k+1} + |\boldsymbol{\sigma}(s)|_{k+1})ds \\ + \int_0^t \|\nu_h(s)\|ds)\|\omega\|_2. \end{aligned} \tag{4.2.26}$$

To derive an estimate for $\nabla\cdot\boldsymbol{\Theta}$, we follow the following procedure:

$$\begin{aligned} \|\nabla\cdot\boldsymbol{\Theta}\|^2 &= (\nabla\cdot\boldsymbol{\Theta}, \nabla\cdot(\boldsymbol{\sigma} - \boldsymbol{\Pi_h^F}\boldsymbol{\sigma})) \\ &\leq \|\nabla\cdot\boldsymbol{\Theta}\|\|\nabla\cdot(\boldsymbol{\sigma} - \boldsymbol{\Pi_h^F}\boldsymbol{\sigma})\| \\ \|\nabla\cdot\boldsymbol{\Theta}\| &\leq Ch^k|\nabla\cdot\boldsymbol{\sigma}|_k. \end{aligned} \tag{4.2.27}$$

Now, using (4.2.2), (4.2.26) and (4.2.27) in (4.2.21) followed by Grönwall's lemma, we arrive at:

$$\|\nu_h\| \leq Ch^{k+2}\left(|\boldsymbol{\sigma}|_{k+1} + |\nabla\cdot\boldsymbol{\sigma}|_k + \int_0^t (|u(s)|_{k+1} + |\boldsymbol{\sigma}(s)|_{k+1})ds\right). \tag{4.2.28}$$

To prove the super convergence, we must estimate $\Pi_k^0 u_t - \mathscr{I}^Q u_t$. By following the similar steps as above, we get the following:

$$\begin{aligned} \|\nu_{h,t}\| \leq Ch^{k+2}(|\boldsymbol{\sigma}_t|_{k+1} + |\nabla\cdot\boldsymbol{\sigma}_t|_k + |\boldsymbol{\sigma}|_{k+1} + |u|_{k+1} \\ + \int_0^t (|\boldsymbol{\sigma}(s)|_{k+1} + |u(s)|_{k+1} + |\nabla\cdot\boldsymbol{\sigma}(s)|_k)ds). \end{aligned} \tag{4.2.29}$$

Now, we need to estimate $\Pi_k^0 u - u_h = \Pi_k^0 u - \mathscr{I}^Q u + \mathscr{I}^Q u - u_h = \nu_h + \eta_h$. Since, we have the estimate of $\nu_h$ from(4.2.28), we need to estimate $\eta_h$ in terms of $\nu_h$. For that, we use the properties of $\Pi_k^0$ operator to rewrite (4.2.18) and (4.2.19) as:

$$(\eta_{h,t}, \phi_h) - (\nabla \cdot \boldsymbol{\Theta}_h, \phi_h) = -(\nu_{h,t}, \phi_h), \quad (4.2.30)$$

$$a_h(\boldsymbol{\Theta}_h, \boldsymbol{\chi}_h) + (\eta_h, \nabla \cdot \boldsymbol{\chi}_h) - \int_0^t (b_0(\boldsymbol{x}; t, s)\eta_h(s), \nabla \cdot \boldsymbol{\chi}_h) ds$$
$$- \int_0^t (\nabla b_0(\boldsymbol{x}; t, s)\eta_h(s), \boldsymbol{\Pi}_k^0 \boldsymbol{\chi}_h) ds = 0. \quad (4.2.31)$$

Put $\phi_h = \eta_h - \int_0^t \Pi_k^0(b_0(\boldsymbol{x}; t, s)\eta_h(s))ds$ in (4.2.30) and $\boldsymbol{\chi}_h = \boldsymbol{\Theta}_h$ in (4.2.31) and then adding the two equations to arrive at:

$$a_h(\boldsymbol{\Theta}_h, \boldsymbol{\Theta}_h) + (\eta_{h,t}, \eta_h) - \int_0^t (\nabla b_0(\boldsymbol{x}; t, s)\eta_h(s), \boldsymbol{\Pi}_k^0 \boldsymbol{\Theta}_h) ds$$
$$= -\left(\nu_{h,t}, \eta_h - \int_0^t \Pi_k^0(b(\boldsymbol{x}; t, s)\eta_h(s))ds\right) + \left(\eta_{h,t}, \int_0^t \Pi_k^0(b(\boldsymbol{x}; t, s)\eta_h(s))ds\right).$$

Now, using the similar steps as in Theorem 4.2.3, we arrive at:

$$\|\eta_h(\cdot, t)\|^2 \leq C\left(\|\eta_h(\cdot, 0)\|^2 + \int_0^t (\|\eta_h(s)\|^2 + \|\nu_{h,t}(s)\|^2) ds\right).$$

Applying Grönwall's lemma and using (4.2.29), the resulting expression obtained is as follows:

$$\|\eta_h(\cdot, t)\|^2 \leq C(\|\eta_h(\cdot, 0)\|^2 + h^{2(k+1)} \int_0^t (|\boldsymbol{\sigma}(s)|_{k+1}^2 + |u(s)|_{k+1}^2 + |\boldsymbol{\sigma}_t(s)|_{k+1}^2$$
$$+ |\nabla \cdot \boldsymbol{\sigma}(s)|_{k+1}^2 + |\nabla \cdot \boldsymbol{\sigma}_t(s)|_{k+1}^2) ds).$$

Hence, we get our desired estimate. □

## 4.3 Fully-discrete Scheme

The error produced by a fully-discrete scheme has two ingredients in theory: the error caused by spatial discretization, which is dependent on $h$, and the error caused by the time discretization, which is dependent on time phase size $\tau$.

Now, we discretize our problem in time. To discretize in time, we use the Euler backward process. Divide the time interval into $N$ distinct points that are evenly spaced, let $t_n = n\tau$ and the sequence $\{U_n\}$ and $\{\boldsymbol{\sigma}_n\}$ be generated as:

$$U_n \approx u_h(\cdot, t_n), \quad \boldsymbol{\sigma}_n \approx \boldsymbol{\sigma}_h(\cdot, t_n), \;\; n = 0, 1, 2, ..., N,$$

$$\tau = T/N.$$

The fully-discrete scheme is defined as:

$$\begin{aligned}
&(\partial_t U^n, \phi_h) - (\nabla \cdot \boldsymbol{\sigma}_n, \phi_h) = (f(\cdot, t_n), \phi_h),\\
&a_h(\boldsymbol{\sigma}_n, \boldsymbol{\chi}_h) + (U_n, \nabla \cdot \boldsymbol{\chi}_h) - \tau \sum_{j=0}^{n-1} ((b_0(t_n, t_j)U_j, \nabla \cdot \boldsymbol{\chi}_h) + (\nabla b_0(t_n, t_j)U_j, \boldsymbol{\Pi}_k^0 \boldsymbol{\chi}_h)) = 0.
\end{aligned} \tag{4.3.1}$$

Define $\Xi(\gamma(t_n, s), \varphi(s))$, for any function $\gamma(t, s)\varphi(s)$ as:

$$\begin{aligned}
\Xi(\gamma(t_n, s), \varphi(s)) = &\left( \int_0^{t_n} \gamma(t_n, s)\varphi(s) ds - \tau \sum_{j=0}^{n-1} \gamma(t_n, t_j)\varphi(t_j) \right)\\
&- \left( \int_0^{t_{n-1}} \gamma(t_{n-1}, s)\varphi(s) ds - \tau \sum_{j=0}^{n-2} \gamma(t_{n-}, t_j)\varphi(t_j) \right).
\end{aligned}$$

Now, by using Taylor's remainder theorem, we arrive at the following:

$$\begin{aligned}
&\Xi(\gamma(t_n, s), \varphi(s))\\
&= \left( \tau \int_0^{t_n-1} \gamma_t(t_{n^*}, s)\varphi(s) ds - \tau^2 \sum_{j=0}^{n-2} \gamma_t(t_{n^*}, t_j)\varphi(t_j) \right)\\
&\quad + \left( \int_{t_{n-1}}^{t_n} \gamma(t_n, s)\varphi(s) ds - \tau\gamma(t_n, t_{n-1})\varphi(t_{n-1}) \right) \quad \text{for some } t_{n^*} \in (t_{n-1}, t_n)\\
&\leq \left( \tau^2 \int_0^{t_{n-1}} \left| \frac{\partial}{\partial s}(\gamma_t(t_{n^*}, s)\varphi(s)) \right| ds \right) + \left( \tau \int_{t_{n-1}}^{t_n} \left| \frac{\partial}{\partial s}(\gamma(t_n, s)\varphi(s)) \right| ds \right).
\end{aligned} \tag{4.3.2}$$

**Theorem 4.3.1.** *Let $u(\cdot, t_n)$ and $U_n$ be the solution of continuous problem (1.3.10a) and fully-discrete formulation (4.3.1) at time $t = t_n$, respectively. In accordance with all the* *presumptions outlined in Theorem 4.2.3 and $u_{tt} \in L^2(0,T;L^2(\mathcal{D}))$, the following remains valid:*

$$\|U_n - u(\cdot,t_n)\| \le O(h^{k+1}+\tau), \quad \forall n = 1,2,\cdots,N,$$
$$\|\boldsymbol{\sigma}_n - \boldsymbol{\sigma}(\cdot,t_n)\| \le O(h^{k+1}+\tau) \quad \forall n = 1,2,\cdots,N.$$

*Proof.* For the proof,

$$U_n - u(\cdot,t_n) = U_n - \mathscr{I}^Q u(\cdot,t_n) + \mathscr{I}^Q u(\cdot,t_n) - u(\cdot,t_n) =: \eta_h^n + \eta^n,$$
$$\boldsymbol{\sigma}_n - \boldsymbol{\sigma}(\cdot,t_n) = \boldsymbol{\sigma}_n - \mathscr{I}^V \boldsymbol{\sigma}(\cdot,t_n) + \mathscr{I}^V \boldsymbol{\sigma}(\cdot,t_n) - \boldsymbol{\sigma}(\cdot,t_n) =: \boldsymbol{\Theta}_h^n + \boldsymbol{\Theta}^n.$$

Since we know the estimates for $\eta^n$ and $\boldsymbol{\Theta}^n$, we need to find $\|\eta_h^n\|$ and $\|\boldsymbol{\Theta}_h^n\|$ and for that, we proceed by rewriting (4.3.1) and using the definition of mixed intermediate projection, to obtain:

$$(\partial_t \eta_h^n, \phi_h) - (\nabla\cdot\boldsymbol{\Theta}_h^n, \phi_h) = \left(u_t(\cdot,t_n) - \partial_t \mathscr{I}^Q \overline{u}^n, \phi_h\right), \tag{4.3.3}$$

$$\begin{aligned} &a_h(\boldsymbol{\Theta}_h^n, \boldsymbol{\chi}_h) + (\nabla b_0(t_n,t_j)\eta_h^j, \boldsymbol{\Pi}_k^0\boldsymbol{\chi}_h)) \\ &+ \left(\eta_h^n + \int_0^{t_n} (b_0(t_n,s)\mathscr{I}^Q u(s))ds - \tau\sum_{j=0}^{n-1}(b_0(t_n,t_j)\eta_h^j + b_0(t_n,t_j)\mathscr{I}^Q u(t_j)), \nabla\cdot\boldsymbol{\chi}_h\right) \\ &= \tau\sum_{j=0}^{n-1}((\nabla b_0(t_n,t_j)\mathscr{I}^Q u, \boldsymbol{\Pi}_k^0\boldsymbol{\chi}_h)) - \int_0^t (\nabla b_0(t_n,s)\mathscr{I}^Q u(s), \boldsymbol{\Pi}_k^0\boldsymbol{\chi}_h)ds. \end{aligned} \tag{4.3.4}$$

Put $\boldsymbol{\chi}_h = \boldsymbol{\Theta}_h^n$ and $\phi_h = \eta_h^n + \int_0^{t_n} \Pi_k^0(b_0(t_n,s)\mathscr{I}^Q u(s))ds - \tau\sum_{j=0}^{n-1}(\Pi_k^0(b_0(t_n,t_j)\eta_h^j) + \Pi_k^0(b_0(t_n,t_j)\mathscr{I}^Q u(t_j)))$ in (4.3.4) and (4.3.3) respectively and then add, to arrive at:

$$\begin{aligned} &(\partial_t \eta_h^n, \eta_h^n) + a_h(\boldsymbol{\Theta}_h^n, \boldsymbol{\Theta}_h^n) - \tau\sum_{j=0}^{n-1}(\nabla b_0(t_n,t_j)\eta_h^j, \boldsymbol{\Pi}_k^0\boldsymbol{\Theta}_h^n) \\ &= \left(u_t(\cdot,t_n) - \partial_t\mathscr{I}^Q \overline{u}^n, \eta_h^n - \tau\sum_{j=0}^{n-1}\Pi_k^0(b_0(t_n,t_j)\eta_h^j) + \int_0^{t_n}\Pi_k^0(b_0(t_n,s)\mathscr{I}^Q u(s))ds\right) \end{aligned}$$

$$
\begin{aligned}
&-\left(u_t(\cdot,t_n)-\partial_t \mathscr{I}^Q u^n, \tau\sum_{j=0}^{n-1}\Pi_k^0(b_0(t_n,t_j)\mathscr{I}^Q u(t_j))\right)\\
&-\int_0^{t_n}(\nabla b_0(t_n,s)\mathscr{I}^Q u(s),\mathbf{\Pi}_k^0\mathbf{\Theta}_h^n)-\tau\sum_{j=0}^{n-1}(\nabla b_0(t_n,t_j)\mathscr{I}^Q u,\mathbf{\Pi}_k^0\mathbf{\Theta}_h^n)\\
&+\left(\partial_t\eta_h^n,\tau\sum_{j=0}^{n-1}\Pi_k^0(b_0(t_n,t_j)\eta_h^j)-\int_0^{t_n}\Pi_k^0(b_0(t_n,s)\mathscr{I}^Q u(s))ds\right)\\
&+\left(\partial_t\eta_h^n,\tau\sum_{j=0}^{n-1}\Pi_k^0(b_0(t_n,t_j)\mathscr{I}^Q u(t_j))\right). \qquad (4.3.5)
\end{aligned}
$$

The last term in the right-hand side of (4.3.5) can be solved by writing $R_1^n=\tau\sum_{j=0}^{n-1}\Pi_k^0(b_0(t_n,t_j)\eta_h^j)-\int_0^{t_n}\Pi_k^0(b_0(t_n,s)\mathscr{I}^Q u(s))ds+\tau\sum_{j=0}^{n-1}\Pi_k^0(b_0(t_n,t_j)\mathscr{I}^Q u(t_j))$, as:

$$
(\partial_t\eta_h^n,R_1^n)=\left(\frac{(R_1^n,\eta_h^n)-(R_1^{n-1},\eta_h^{n-1})}{\tau}\right)-\left(\partial_t R_1^n,\eta_h^{n-1}\right), \qquad (4.3.6)
$$

where the term $(\partial_t R_1^n,\eta_h^{n-1})$ can be solved by using (4.3.2) and Taylor remainder theorem as:

$$
\begin{aligned}
&(\partial_t R_1^n,\eta_h^{n-1})\\
&=\left(\tau\sum_{j=0}^{n-2}\Pi_k^0(b_{0,t}(t_{n_1^*},t_j)\eta_h^j)+\Pi_k^0(b_0(t_n,t_{n-1})\eta_h^{n-1}),\eta_h^{n-1}\right)\\
&\quad+(\Xi(\Pi_k^0(b_0(t_n,s)\mathscr{I}^Q u(s))),\eta_h^{n-1})\\
&\le C\left(\tau\sum_{j=0}^{n-2}\Pi_k^0(b_{0,t}(t_{n_1^*},t_j)\eta_h^j,\eta_h^{n-1})+\tau\int_0^{t_n}\left|\frac{\partial}{\partial s}(\Pi_k^0(b_{0,t}(t_{n^*},s)\mathscr{I}^Q u(s))\right|ds\|\eta_h^{n-1}\|\right.\\
&\quad\left.+\|\eta_h^{n-1}\|^2+\int_{t_{n-1}}^{t_n}\left|\frac{\partial}{\partial s}(\Pi_k^0(b_0(t_n,s)\mathscr{I}^Q u(s))\right|ds\|\eta_h^{n-1}\|\right)\ \text{ for } t_{n^*},t_{n_1^*}\in(t_{n-1},t_n).
\end{aligned}
$$

So, (4.3.6) can be written as:

$$
(\partial_t\eta_h^n,R_1^n)
$$

$$
\begin{aligned}
&\leq \left(\frac{(R_1^n,\eta_h^n)-(R_1^{n-1},\eta_h^{n-1})}{\tau}\right)+C\left(\tau\sum_{j=0}^{n-2}(\Pi_k^0(b_{0,t}(t_{n_1^*},t_j)\eta_h^j),\eta_h^{n-1})+\|\eta_h^{n-1}\|^2\right.\\
&\quad+\tau\int_0^{t_n}\left|\frac{\partial}{\partial s}(\Pi_k^0(b_{0,t}(t_{n^*},s)\mathscr{I}^Qu(s)))\right|ds\|\eta_h^{n-1}\|\\
&\quad\left.+\int_{t_{n-1}}^{t_n}\left|\frac{\partial}{\partial s}(\Pi_k^0(b_0(t_n,s)\mathscr{I}^Qu(s)))\right|ds\|\eta_h^{n-1}\|\right). \qquad (4.3.7)
\end{aligned}
$$

Using (4.3.7), (3.1.3) in (4.3.5) and boundedness of $\nabla b_0$, we arrive at:

$$
\begin{aligned}
&\frac{1}{2}\left(\frac{\|\eta_h^n\|^2-\|\eta_h^{n-1}\|^2}{\tau}\right)+\alpha_1\|\boldsymbol{\Theta}_h^n\|^2\\
&\leq C\left(\|I_1^n\|\left(\|\eta_h^n\|+\tau\sum_{j=0}^{n-1}\|\eta_h^j\|+\tau\int_0^{t_n}\left|\frac{\partial}{\partial s}(\Pi_k^0(b_0(t_n,s)\mathscr{I}^Qu(s)))\right|ds\right)\right.\\
&\quad+\tau\int_0^{t_n}\left|\frac{\partial}{\partial s}(\nabla b_0(t_n,s)\mathscr{I}^Qu(s))\right|\|\boldsymbol{\Theta}_h^n\|ds+\left(\frac{(R_1^n,\eta_h^n)-(R_1^{n-1},\eta_h^{n-1})}{\tau}\right)\\
&\quad+\tau\sum_{j=0}^{n-1}\|\eta_h^j\|\|\boldsymbol{\Theta}_h^n\|+\tau\sum_{j=0}^{n-2}(\Pi_k^0(b_{0,t}(t_{n_1^*},t_j)\eta_h^j),\eta_h^{n-1})+\|\eta_h^{n-1}\|^2\\
&\quad+\left(\tau\int_0^{t_n}\left|\frac{\partial}{\partial s}(\Pi_k^0(b_{0,t}(t_{n^*},s)\mathscr{I}^Qu(s)))\right|ds\right.\\
&\quad\left.\left.+\int_{t_{n-1}}^{t_n}\left|\frac{\partial}{\partial s}(\Pi_k^0(b_0(t_n,s)\mathscr{I}^Qu(s)))\right|ds,\eta_h^{n-1}\right)\right),
\end{aligned}
$$

where $I_1^n=\left(u_t-\partial_t\mathscr{I}^Qu^n\right)$.

Using Young's inequality followed by the kickback argument, we arrive at:

$$
\begin{aligned}
&\frac{1}{2}\left(\frac{\|\eta_h^n\|^2-\|\eta_h^{n-1}\|^2}{\tau}\right)+c_{\alpha_1}\|\boldsymbol{\Theta}_h^n\|^2\\
&\leq C\left(\tau\sum_{j=0}^{n-1}\|\eta_h^j\|^2+\|\eta_h^n\|^2+\|\eta_h^{n-1}\|^2+\|I_1^n\|^2+\tau^2\int_0^{t_n}\left|\frac{\partial}{\partial s}(\Pi_k^0(b_0(t_n,s)\mathscr{I}^Qu(s)))\right|^2ds\right.\\
&\quad+\left(\frac{(R_1^n,\eta_h^n)-(R_1^{n-1},\eta_h^{n-1})}{\tau}\right)+\tau^2\int_0^{t_n}\left|\frac{\partial}{\partial s}(\Pi_k^0(b_{0,t}(t_{n^*},s)\mathscr{I}^Qu(s)))\right|^2ds\\
&\quad\left.+\tau\int_{t_{n-1}}^{t_n}\left|\frac{\partial}{\partial s}(\Pi_k^0(b_0(t_n,s)\mathscr{I}^Qu(s)))\right|^2ds+\tau^2\int_0^{t_n}\left|\frac{\partial}{\partial s}(\nabla b_0(t_n,s)\mathscr{I}^Qu(s))\right|^2ds\right).
\end{aligned}
$$

Multiplying the above equation by $2\tau$ and sum from $n = 1 \cdots N$, we arrive at the following:

$$
\begin{aligned}
&\|\eta_h^N\|^2 + 2\tau c_{\alpha_1} \sum_{n=1}^{N} \|\mathbf{\Theta}_h^n\| \\
&\leq C \Bigg( \|\eta_h(\cdot,0)\|^2 + \tau \sum_{n=1}^{N} \|\eta_h^n\|^2 + \tau \sum_{n=1}^{N} \|I_1^n\|^2 + C_{\epsilon_1}\|R_1^N\|^2 + C_{\epsilon_1'}\|\eta_h^N\|^2 \\
&\quad + \tau^3 \sum_{n=1}^{N} \int_0^{t_n} \left( \left| \frac{\partial}{\partial s}(\Pi_k^0(b_0(t_n,s)\mathscr{I}^Q u(s))) \right|^2 + \left| \frac{\partial}{\partial s}(\nabla b_0(t_n,s)\mathscr{I}^Q u(s)) \right|^2 \right) ds \\
&\quad + \tau^3 \sum_{n=1}^{N} \int_0^{t_n} \left( \left| \frac{\partial}{\partial s}\Pi_k^0((b_{0,t}(t_{n^*},s)\mathscr{I}^Q u(s))) \right|^2 \right) ds \\
&\quad + \tau^2 \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \left| \frac{\partial}{\partial s}(\Pi_k^0(b_0(t_n,s)\mathscr{I}^Q u(s))) \right|^2 ds \Bigg).
\end{aligned}
$$

Using the kickback argument followed by Grönwall's lemma, we have:

$$
\begin{aligned}
\|\eta_h^N\|^2 \leq C \Bigg( &\|\eta_h(\cdot,0)\|^2 + \tau \sum_{n=1}^{N} \|I_1^n\|^2 + \tau^2 \int_0^T \left| \frac{\partial}{\partial s}(\Pi_k^0(b_0(t_n,s)\mathscr{I}^Q u(s))) \right|^2 ds \\
&+ \tau^2 \int_0^T \left| \frac{\partial}{\partial s}(\Pi_k^0(b_{0,t}(t_{n^*},s)\mathscr{I}^Q u(s))) \right|^2 ds + \tau^2 \int_0^T \left| \frac{\partial}{\partial s}(\nabla b_0(t_n,s)\mathscr{I}^Q u(s)) \right|^2 ds \Bigg),
\end{aligned} \tag{4.3.8}
$$

where

$$
\begin{aligned}
\tau \sum_{n=1}^{N} |I_1^n|^2 &\leq C\tau \sum_{n=1}^{N} (\|u_t(\cdot,t_n) - \partial_t u^n\|^2 + \|\partial_t u^n - \partial_t \mathscr{I}^Q u^n\|^2) \\
&= \frac{C}{\tau} \sum_{n=1}^{N} \left( \left( \int_{t_{n-1}}^{t_n} (s - t_{n-1}) u_{tt} ds \right)^2 + \left( \int_{t_{n-1}}^{t_n} (u_t - \mathscr{I}^Q u_t)(s) ds \right)^2 \right) \\
&\leq C \int_0^{t_N} (\tau^2 \|u_{tt}(s)\|^2 ds + O(h^{2(k+1)}) \qquad (\text{ by using } (4.2.13)),
\end{aligned} \tag{4.3.9}
$$

and

$$
\tau^2 \int_0^T \left| \frac{\partial}{\partial s}(b_0(t_n,s)\mathscr{I}^Q u(s)) \right|^2 ds
$$

$$\leq C\tau^2 \int_0^T (\|\mathscr{I}^Q u(s)\| + \|\mathscr{I}^Q u_s(s)\|)ds$$
$$\leq C\tau^2 \int_0^T (\|(\mathscr{I}^Q u - u)(s)\| + \|u(s)\| + \|(\mathscr{I}^Q u_s - u(s))(s)\| + \|p_s(s)\|)ds.$$

Similarly, we can bound the remaining terms $\tau^2 \int_0^T \left|\frac{\partial}{\partial s}(b_{0,t}(t_{n^*}, s)\mathscr{I}^Q u(s))\right|^2 ds$ and $\tau^2 \int_0^T \left|\frac{\partial}{\partial s}(\nabla b_0(t_n, s)\mathscr{I}^Q u(s))\right|^2 ds$ of (4.3.8), and then using (4.2.3), we arrive at:

$$\|\eta_h^N\|^2 \leq (h^{2(k+1)} + \tau^2).$$

Now, the use of triangle inequality completes the proof of $\|\eta_h^N\|$.

To obtain the estimate of $\boldsymbol{\Theta}_h^n$, we begin by examining equation (4.3.4) at the time step $t = t_{n-1}$. Then, by subtracting and dividing by $\tau$, we arrive at the following expression:

$$\begin{aligned}
&a_h(\bar{\partial}_t \boldsymbol{\Theta}_h^n, \boldsymbol{\chi}_h) \\
&+ \left(\bar{\partial}_t \eta_h^n - \sum_{j=0}^{n-1} b_0(t_n, t_j)\eta_h^j + \sum_{j=0}^{n-2} b_0(t_{n-1}, t_j)\eta_h^j + \frac{1}{\tau}\Xi(b_0(t_n, s)\mathscr{I}^Q u(s)), \nabla\cdot\boldsymbol{\chi}_h\right) \\
&= \sum_{j=0}^{n-1}(\nabla b_0(t_n, t_j)\eta_h^j, \boldsymbol{\Pi}_k^0 \chi_h) - \sum_{j=0}^{n-2}(\nabla b_0(t_{n-1}, t_j)\eta_h^j, \boldsymbol{\Pi}_k^0 \chi_h) \\
&+ \frac{1}{\tau}(\Xi(\nabla b_0(t_n, s)\mathscr{I}^Q u(s)), \boldsymbol{\Pi}_k^0 \boldsymbol{\chi}_h).
\end{aligned} \tag{4.3.10}$$

Put $\boldsymbol{\chi}_h = \boldsymbol{\Theta}_h^n$ and $\phi_h = \partial_t \eta_h^n - \sum_{j=0}^{n-1}\Pi_k^0(b_0(t_n, t_j)\eta_h^j) + \sum_{j=0}^{n-2}\Pi_k^0(b_0(t_{n-1}, t_j)\eta_h^j)$ $+ \frac{1}{\tau}\Xi(\Pi_k^0(b_0(t_n, s)\mathscr{I}^Q u(s)))$ in (4.3.10) and (4.3.3) respectively and then add to arrive at the following:

$$\begin{aligned}
&a_h\left(\bar{\partial}_t \boldsymbol{\Theta}_h^n, \boldsymbol{\Theta}_h^n\right) + \left(\bar{\partial}_t \eta_h^n, \bar{\partial}_t \eta_h^n\right) \\
&\leq C\Bigg(\left(\nabla b_0(t_n, t_{n-1})\eta_h^{n-1}, \boldsymbol{\Pi}_{\boldsymbol{k}}^{\mathbf{0}} \boldsymbol{\Theta}_h^n\right) + \tau\sum_{j=0}^{n-2}\left(\nabla b_0(t_{n_1^*}, s)\eta_h^j, \boldsymbol{\Pi}_{\boldsymbol{k}}^{\mathbf{0}} \boldsymbol{\Theta}_h^n\right)
\end{aligned}$$

$$
\begin{aligned}
&\quad + \left(\bar{\partial}_t \eta_h^n, \sum_{j=0}^{n-1} \Pi_k^0 (b_0(t_n, t_j)\eta_h^j)\right) \\
&+ \tau \int_0^{t_{n-1}} \left| \frac{\partial}{\partial s} (\nabla b_{0,t}(t_{n^*}, s) \mathscr{I}^Q u(s)) \right| ds \|\mathbf{\Pi}_k^0 \mathbf{\Theta}_h^n\| - \left(\bar{\partial}_t \eta_h^n, \sum_{j=0}^{n-2} \Pi_k^0 (b_0(t_{n-1}, t_j)\eta_h^j)\right) \\
&+ \int_{t_{n-1}}^{t_n} \left| \frac{\partial}{\partial s} (\nabla b_0(t_n, s) \mathscr{I}^Q u(s)) \right| ds \|\mathbf{\Pi}_k^0 \mathbf{\Theta}_h^n\| + \left(\bar{\partial}_t \eta_h^n, \frac{1}{\tau} \Xi(\Pi_k^0 (b_0(t_n, s) \mathscr{I}^Q u(s)))\right) \\
&+ \left(I_1^n, \partial_t \eta_h^n - \sum_{j=0}^{n-1} (\Pi_k^0 (b_0(t_n, t_j)\eta_h^j)) + \sum_{j=0}^{n-2} (\Pi_k^0 (b_0(t_{n-1}, t_j)\eta_h^j)) - \frac{1}{\tau} \Xi(\Pi_k^0 (b_0(t_n, s) \mathscr{I}^Q u(s))\right)\Bigg) \\
&\qquad\qquad \text{for some } t_{n^*}, t_{n_1^*} \in (t_{n-1}, t_n).
\end{aligned}
$$

Now, using Young's inequality followed by the kickback argument, we have:

$$
\begin{aligned}
&C_{\alpha_1,\epsilon} \left( \frac{\|\mathbf{\Theta}_h^n\|^2 - \|\mathbf{\Theta}_h^{n-1}\|^2}{2\tau} \right) + C_{\epsilon_1'} \|\partial_t \eta_h^n\|^2 \\
&\leq C \left( \tau \sum_{j=0}^{n-2} \|\eta_h^j\|^2 + \|\eta_h^{n-1}\|^2 + \|\mathbf{\Theta}_h^n\|^2 + \|I_1^n\|^2 + \tau^2 \int_0^{t_{n-1}} \left| \frac{\partial}{\partial s} (\nabla b_{0t}(t_{n^*}, s) \mathscr{I}^Q u(s) \right|^2 ds \right. \\
&\quad + \tau^2 \int_0^{t_{n-1}} \left| \frac{\partial}{\partial s} (\Pi_k^0 (b_{0t}(t_{n_1^*}, s) \mathscr{I}^Q u(s)) \right|^2 ds \\
&\quad \left. + \tau \int_{t_{n-1}}^{t_n} \left| \frac{\partial}{\partial s} (\nabla b_0(t_n, s) \mathscr{I}^Q u(s) \right|^2 ds + \tau \int_{t_{n-1}}^{t_n} \left| \frac{\partial}{\partial s} (\Pi_k^0 (b_0(t_n, s) \mathscr{I}^Q u(s))) \right|^2 ds \right).
\end{aligned}
$$

Multiplying the above equation by $2\tau$ and sum from $n = 1$ to $N$, we get:

$$
\begin{aligned}
\|\mathbf{\Theta}_h^N\|^2 \leq C &\left( \tau^2 \sum_{n=1}^{N} \sum_{j=0}^{n-2} \|\eta_h^j\|^2 + \tau \sum_{n=1}^{N} \|\mathbf{\Theta}_h^n\|^2 + \tau \sum_{n=1}^{N} \|I_1^n\|^2 \right. \\
&+ \tau^3 \sum_{n=1}^{N} \int_0^{t_n} \left( \left| \frac{\partial}{\partial s} (\nabla b_{0t}(t_{n^*}, s) \mathscr{I}^Q u(s) \right|^2 + \left| \frac{\partial}{\partial s} (\Pi_k^0 (b_{0t}(t_{n_1^*}, s) \mathscr{I}^Q u(s))) \right|^2 \right) ds \\
&\left. + \tau^2 \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \left( \left| \frac{\partial}{\partial s} (\nabla b_0(t_n, s) \mathscr{I}^Q u(s)) \right|^2 + \left| \frac{\partial}{\partial s} (\Pi_k^0 (b_0(t_n, s) \mathscr{I}^Q u(s))) \right|^2 \right) ds \right).
\end{aligned}
$$

Use of (4.3.9) and (4.3.8) followed by Grönwall's lemma completes the proof of $\|\theta_h^N\|$. □

## 4.4 Numerical Results

In this section, we carry out a numerical example to justify the performance of the proposed mixed virtual element scheme for the linear parabolic integro-differential problem (1.1.1) on the square mesh in Figure 3.1. Here, we consider the domain $\mathcal{D}$ the unit square in $\mathbb{R}^2$. We use the backward Euler approach for time discretization coupled with Mixed VEM discretization to tackle the fully-discrete problem for the mesh sequences introduced in Figure 3.1.

**Example 4.4.1.** *Consider the linear PIDE (1.1.1), with coefficients* $a(\boldsymbol{x}) = 1 + x$, $b(\boldsymbol{x}; t, s) = (1 + x)e^{t-s}$. *Notably, the load term* $f$*, boundary data, and initial data* $u_0$ *are all determined using the exact solution as a reference point:*

$$u(\boldsymbol{x}, t) = te^t \sin 2\pi x \sin 2\pi y.$$

**Example 4.4.2.** *Consider the linear PIDE (1.1.1), with coefficients* $a(\boldsymbol{x}) = 1 + x$, $b(\boldsymbol{x}; t, s) = (1 - x^2)ts$. *Notably, the load term* $f$*, boundary data, and initial data* $u_0$ *are all determined using the exact solution as a reference point:*

$$u(\boldsymbol{x}, t) = t(x - x^2)(y - y^2).$$

**Example 4.4.3.** *Consider the linear PIDE (1.1.1), with coefficients* $a(\boldsymbol{x}) = 1$, $b(\boldsymbol{x}; t, s) = 10e^{-1000t^2s^2}$. *Notably, the load term* $f$*, boundary data, and initial data* $u_0$ *are all determined using the exact solution as a reference point:*

$$u(\boldsymbol{x}, t) = t(x - x^2)(y - y^2).$$

With the help of this example, we can show that this formulation is more generalized in the sense that it is applicable to a wider range of problems satisfying the assumptions outlined in the Introduction. Here, in this example, the challenge lies in determining a resolvent kernel because manual integration of each function is not possible due to the unavailability of anti-derivatives. In this example, we can compute the first two terms of the series of the resolvent kernel, and if we terminate the series at this point, even after further discretization, the error comes out to be constant without further reduction, as shown in Figure 4.5. But comparatively, if we go with the formulation (1.3.10a), we are getting an appropriate order of convergence. Although this formulation takes more computation time but it is more effective for a vast range of examples. Fig. 4.1 and 4.2 depict the order of convergence for both $u_h$ and $\boldsymbol{\sigma}_h$ for Example 4.4.1 in case of $k = 1, 2$ and 3 on quadrilateral and hexagonal mesh respectively. Both the figures show that these orders of convergence are accomplished in perfect accordance with theory, while Fig. 4.3 shows the superconvergence results for both Example 4.4.1 and Example 2 in the case of $k$= 1, 2, and 3 on the quadrilateral mesh whereas in Fig 4.4 shows the order of convergence for Example 4.4.2 on the quadrilateral mesh. From all the figures, we can see that our theory matches our numerical results well.

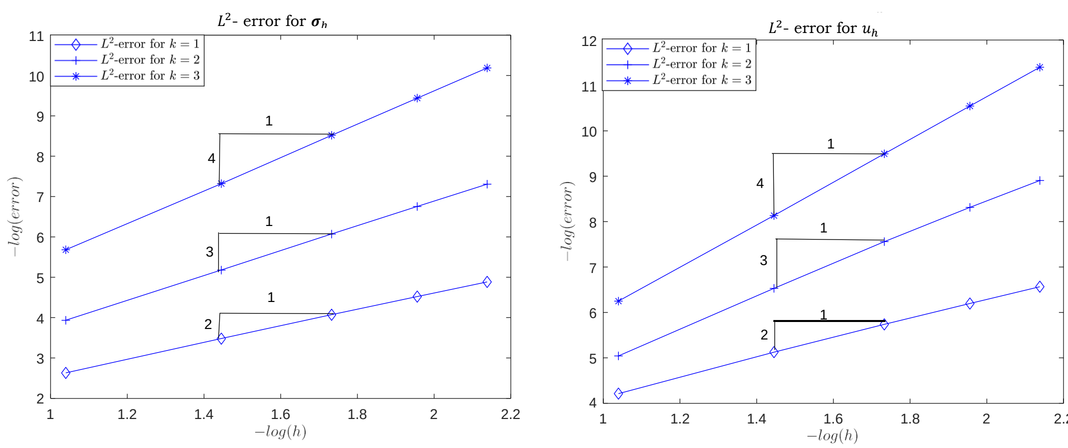


Figure 4.1: Order of convergence for Example 4.4.1 on the quadrilateral mesh in case of $k$=1, 2 and 3 with $\tau = 1.1e-04$.

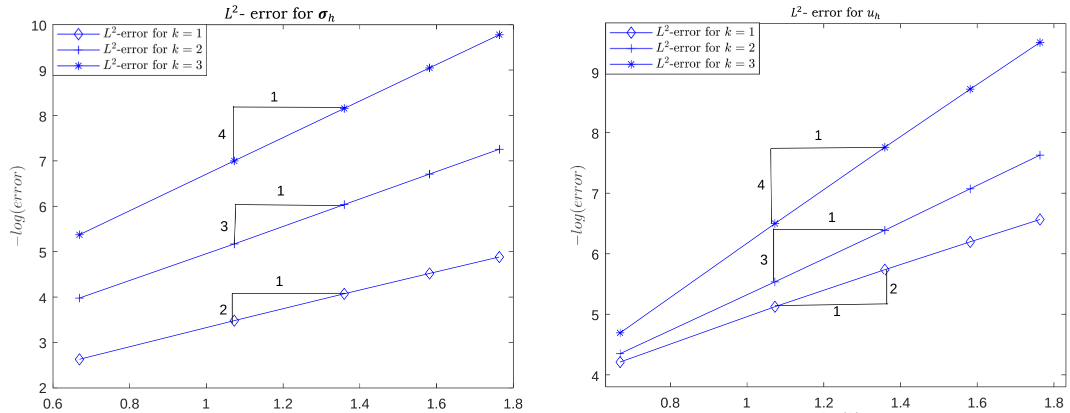


Figure 4.2: Order of convergence for Example 4.4.1 on the hexagonal mesh in case of $k = 1$, 2 and 3 with $\tau = 1.6e - 04$.

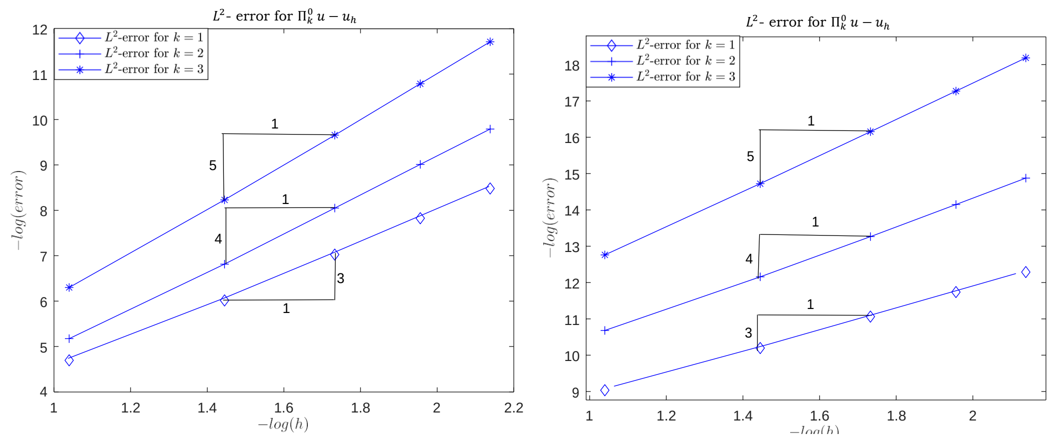


Figure 4.3: Order of convergence for $\Pi_k^0 u - u_h$ in case of $k$=1, 2 and 3 on the quadrilateral mesh with $\tau = 1.1e - 04$.. The left panel corresponds to Example 4.4.1 , and the right panel corresponds to Example 4.4.2 .

## 4.5 Conclusions

Considering the advantages of VEM and mixed methods, we applied a mixed VEM approach to address both the semi-discrete and fully-discrete cases of the PIDE (1.1.1). In this chapter, we have introduced a novel projection, the mixed intermediate projection, which helps in handling the integral term. Significantly, this work represents the first instance in the literature [15] where the fully-discrete case has been thoroughly examined for this formulation. Furthermore, a step-by-step analysis is proposed for the supercon-

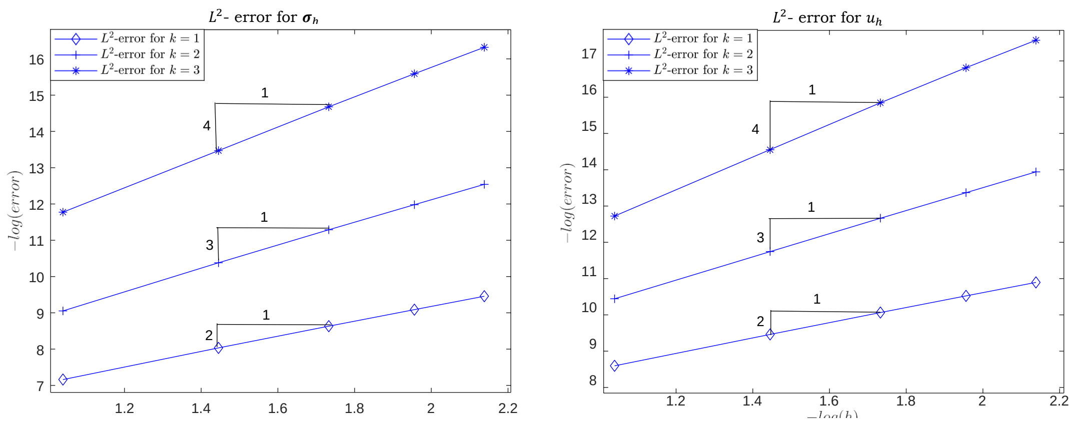


Figure 4.4: Order of convergence for Example 4.4.2 on the quadrilateral mesh in case of $k$=1, 2 and 3 with $\tau = 1.1e-04$..

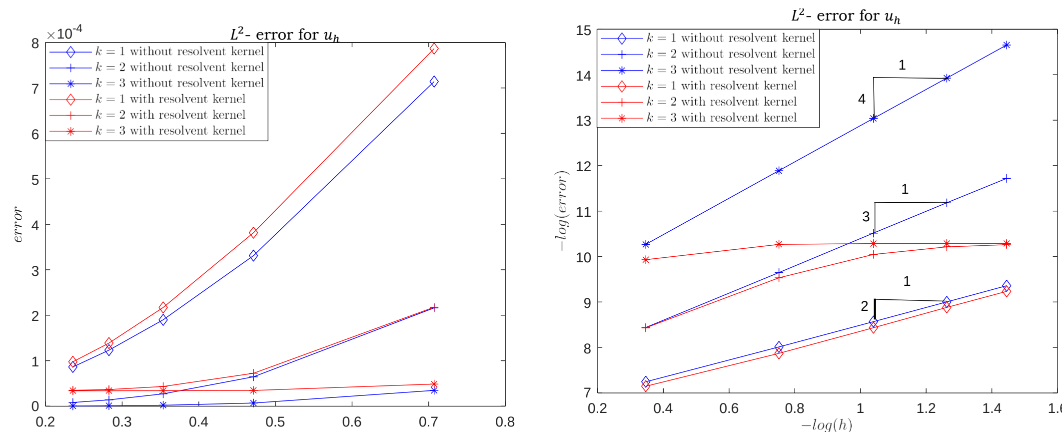


Figure 4.5: Order of convergence for Example 4.4.2 on the quadrilateral mesh in case of $k$=1, 2 and 3 with $\tau = 1.1e-04$..

vergence of the discrete solution of order $O(h^{k+2})$. Several computational experiments are discussed to validate the proposed scheme's computational efficiency and support the theoretical conclusions.

# Chapter 5

# Virtual Element Method for Parabolic Integro-Differential Equations with Non-smooth Initial Data

In Chapter 2, we analyzed the PIDE with smooth initial data, specifically when $u_0 \in H^{k+1}(\mathcal{D})$. The focus of the current chapter is to extend this analysis to cases where the initial data is less smooth, specifically when $u_0 \in L^2(\mathcal{D})$ but not in $H^1(\mathcal{D})$. In this Chapter, we focus on developing virtual element methods for the following linear parabolic integro-differential equation defined on $\Omega \subset \mathbb{R}^2$. Find $u(\boldsymbol{x}, t)$ such that

$$\begin{aligned} u_t(\boldsymbol{x}, t) - \nabla \cdot \left( a(\boldsymbol{x}) \nabla u(\boldsymbol{x}, t)) - \int_0^t (b(\boldsymbol{x}; t, s) \nabla u(\boldsymbol{x}, s) \right) ds = 0 \quad (\boldsymbol{x}, t) \in \Omega \times (0, T], \\ u(\boldsymbol{x}, t) = 0 \quad (\boldsymbol{x}, t) \in \partial\Omega \times (0, T], \\ u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}) \quad \boldsymbol{x} \in \Omega. \end{aligned} \tag{5.0.1}$$

Now, by multiplying the suitable test function, the variational form corresponds to (5.0.1) read as follows. Find $u \in L^2(0,T;H_0^1(\Omega))$ such that:

$$\langle u_t, v\rangle + \mathscr{A}(u,v) - \int_0^t \mathscr{B}(t,s;u(s),v)ds = 0 \quad t \in (0,T], \quad \forall v \in H_0^1(\Omega), \qquad (5.0.2)$$
$$u(\cdot,0) = u_0,$$

where

$$\mathscr{A}(u,v) = (a(\boldsymbol{x})\nabla u(\boldsymbol{x},t), \nabla v(\boldsymbol{x})), \quad \mathscr{B}(t,s;u,v) = (b(\boldsymbol{x};t,s)\nabla u(\boldsymbol{x},t), \nabla v(\boldsymbol{x})).$$

Defining local counterparts of bilinear forms in (5.0.2):

$$\mathscr{A}(u,v) := \sum_{\mathbb{K}\in\mathcal{I}_h} \mathscr{A}^{\mathbb{K}}(u,v) \quad \forall u,v \in V,$$

$$\mathscr{B}(t,s;u,v) := \sum_{\mathbb{K}\in\mathcal{I}_h} \mathscr{B}^{\mathbb{K}}(t,s;u,v) \quad \forall u,v \in V,$$

where

$$\mathscr{A}^{\mathbb{K}}(u,v) := \int_{\mathbb{K}} a(\boldsymbol{x})\nabla u(\boldsymbol{x},t)\cdot\nabla v(\boldsymbol{x})d\boldsymbol{x}, \quad \mathscr{B}^{\mathbb{K}}(t,s;u,v) := \int_{\mathbb{K}} b(\boldsymbol{x};t,s)\nabla u(\boldsymbol{x},s)\cdot\nabla v(\boldsymbol{x})d\boldsymbol{x}.$$

For deriving optimal error estimates, we introduce a new projection known as intermediate projection ($I^h u$) that contains the memory term. We employ an iterative process wherein we repeatedly apply the integral operator to obtain estimations for the integration of the intermediate projection denoted as $\widehat{I^h u}$. Using these estimations of $\widehat{I^h u}$, we further derive estimations for $I^h u$. By combining regularity outcomes and intermediate projection estimations, we establish precise error estimates at an optimal order of $O(h^2 t^{-1})$.

## 5.1 The Discrete Formulation

After introducing the global virtual space $W_h$, the semi-discrete approximation to the problem (5.0.1) can be constructed as:

$$m_h(u_{h,t}, v_h) + \mathscr{A}_h(u_h, v_h) - \int_0^t \mathscr{B}_h(t,s;u_h(s),v_h)ds = 0 \quad \forall v_h \in W_h, \qquad (5.1.1)$$

$$u_h(0) = u_{h,0},$$

where $u_{h,0}$ will be defined later in the proof of Theorem 2.2.3 and the above discrete bilinear forms are computable and defined for all $p_h, q_h \in W_h$ as

$$m_h(p_h, q_h) := \sum_{\mathbb{K}\in\mathcal{I}_h} m_h^{\mathbb{K}}(p_h, q_h), \quad \mathscr{A}_h(p_h, q_h) := \sum_{\mathbb{K}\in\mathcal{I}_h} \mathscr{A}_h^{\mathbb{K}}(p_h, q_h),$$
$$\mathscr{B}_h(t, s; p_h, q_h) := \sum_{\mathbb{K}\in\mathcal{I}_h} \mathscr{B}_h^{\mathbb{K}}(t, s; p_h, q_h).$$

The bilinear forms on every element $\mathbb{K} \in \mathcal{I}_h$ are defined for any $v_h, w_h \in W_{k,\mathbb{K}}$ (see [17]) as below:

$$\mathscr{A}_h^{\mathbb{K}}(\cdot,\cdot) : W_{k,\mathbb{K}}\times W_{k,\mathbb{K}} \to \mathbb{R}, \quad \mathscr{B}_h^{\mathbb{K}}(\cdot,\cdot) : W_{k,\mathbb{K}}\times W_{k,\mathbb{K}} \to \mathbb{R}, \quad m_h^{\mathbb{K}}(\cdot,\cdot) : W_{k,\mathbb{K}}\times W_{k,\mathbb{K}} \to \mathbb{R}$$

$$m_h^{\mathbb{K}}(v_h, w_h) := (\Pi_k^0 v_h, \Pi_k^0 w_h)_{0,\mathbb{K}} + S_1^{\mathbb{K}}((I - \Pi_k^0)v_h, (I - \Pi_k^0)w_h),$$
$$\mathscr{A}_h^{\mathbb{K}}(v_h, w_h) := \int_K (a(\boldsymbol{x})\boldsymbol{\Pi}_{k-1}^0 \nabla v_h \cdot \boldsymbol{\Pi}_{k-1}^0 \nabla w_h) d\boldsymbol{x} + S_0^{\mathbb{K}}((I - \Pi_k^{\nabla})v_h, (I - \Pi_k^{\nabla})w_h, ),$$
$$\mathscr{B}_h^{\mathbb{K}}(t, s; v_h, w_h) := \int_K (b(\boldsymbol{x}; t, s)\boldsymbol{\Pi}_{k-1}^0 \nabla v_h \cdot \boldsymbol{\Pi}_{k-1}^0 \nabla w_h d\boldsymbol{x}.$$

The stability term $S_0^{\mathbb{K}} : W_{k,\mathbb{K}} \times W_{k,\mathbb{K}} \to \mathbb{R}$ should be build in such a way that, $\exists\, \alpha_{**}, \alpha^*$, independent of $h$ with $0 < \alpha_{**} \le \alpha^*$ and satisfy the following:

$$\alpha_{**} a^{\mathbb{K}}(v_h, v_h) \le S_0^{\mathbb{K}}(v_h, v_h) \le \alpha^* a^{\mathbb{K}}(v_h, v_h) \quad \forall v_h \in \text{ker}\, \Pi_k^{\nabla}.$$

where $S_0^{\mathbb{K}}(\cdot,\cdot)$ and $S_1^{\mathbb{K}}(\cdot,\cdot)$ is defined in (2.1.3) and (2.1.4). Now, we proceed to establish the coercivity $\mathscr{A}_h^{\mathbb{K}}(\cdot,\cdot)$:

$$\begin{aligned}\mathscr{A}_h^{\mathbb{K}}(v_h, v_h) &\ge (a(\boldsymbol{x})\boldsymbol{\Pi}_{k-1}^0 \nabla v_h, \boldsymbol{\Pi}_{k-1}^0 \nabla v_h)_{0,\mathbb{K}} + \alpha_{**} a^{\mathbb{K}}((I - \Pi_k^{\nabla})v_h, (I - \Pi_k^{\nabla})v_h)\\ &\ge \min(1, \alpha_{**}) c_0 (\|\boldsymbol{\Pi}_{k-1}^0 \nabla v_h\|_{0,\mathbb{K}}^2 + \|\nabla v_h - \nabla \Pi_k^{\nabla} v_h\|_{0,\mathbb{K}}^2)\\ &= c_{\alpha_{**}} \|\nabla v_h\|_{0,\mathbb{K}}^2. \end{aligned} \tag{5.1.2}$$

Below, we state two lemmas, proof of which follows from [17].

**Lemma 5.1.1.** *Let $u$ be a solution of the PIDE (5.0.1) and $u_0 \in L^2$. Then the following estimates hold for $t \in (0, T]$ and $j \in \{1, 2\}$:*

*(i)* $t\|u(t)\|_1^2 + \int_0^t s\|u_s(s)\|^2 ds \leq C\|u_0\|^2$,

*(ii)* $t^2\|u_t(t)\|^2 + \int_0^t s^2\|u_s(s)\|_1^2 ds \leq C\|u_0\|^2$,

*(iii)* $\|\hat{u}(t)\|_2 + t\|u(t)\|_2 \leq C\|u_0\|$,

*(iv)* $t\|u(t)\|_2 \leq C\|u_0\|$,

*(v)* $\|u_t(t)\|_j \leq Ct^{-(1+\frac{j}{2})}\|u_0\|$.

**Lemma 5.1.2.** *For $\mathbb{K} \in \mathcal{I}_h$, let all the coefficients and $p, q$ be smooth scalar or vector-valued functions on $\mathbb{K}$. Then,*

$$\mathscr{A}_h^{\mathbb{K}}(p, q) - \mathscr{A}^{\mathbb{K}}(p, q) \leq C_{a_0,a}|p|_{1,\mathbb{K}}|q|_{1,\mathbb{K}},$$
$$\mathscr{B}_h^{\mathbb{K}}(t, s; p, q) - \mathscr{B}^{\mathbb{K}}(t, s; p, q) \leq C_{b,b_0,\mathbf{b}_1}|p|_{1,\mathbb{K}}|q|_{1,\mathbb{K}}.$$

*Proof.* Proof of this lemma follows from [74]. □

**Lemma 5.1.3.** *For $\mathbb{K} \in \mathcal{I}_h$, let the coefficients $a(\boldsymbol{x})$ and $b(\boldsymbol{x}; t, s)$ be smooth scalar-valued function in $\mathcal{D}$ and $p$ be any smooth scaler-valued function. Then,*

$$\mathscr{A}_h^{\mathbb{K}}(\Pi_k^0 p, q_h) - \mathscr{A}^{\mathbb{K}}(\Pi_k^0 p, q_h) \leq C_a h^k\|p\|_{k+1}|q_h|_{1,\mathbb{K}},$$
$$\mathscr{B}_h^{\mathbb{K}}(t, s; \Pi_k^0 p, q_h) - \mathscr{B}^{\mathbb{K}}(t, s; \Pi_k^0 p, q_h) \leq C_b h^k\|p\|_{k+1}|q_h|_{1,\mathbb{K}}.$$

*Proof.* Proof of this lemma follows from [74]. □

Our approach relies on an energy-based argument coupled with the repeated application of a time-integral operator defined for any function $g(t)$:

$$\hat{g}(t) = \int_0^t g(s)ds.$$

To derive the optimal error estimates, first, we need to define a projection new projection with a memory term called the intermediate projection (5.2.1). Since we need the estimates of intermediate projection for $u_0 \in L^2$. For that, we first find out the estimates for the $\hat{\rho}$, where $\hat{\rho} = \widehat{I^h u} - \hat{u}$

## 5.2 Intermediate Projection

Define intermediate projection $I^h : H_0^1(\Omega) \to W_h$, for $t \in \bar{J}$, where $J = (0, T]$, as the solution:

$$\mathscr{A}_h(I^h u, v_h) - \int_0^t \mathscr{B}_h(t, s; I^h u(s), v_h) ds = \mathscr{A}(u, v_h) - \int_0^t \mathscr{B}(t, s; u(s), v_h) ds \quad \forall v_h \in W_h. \tag{5.2.1}$$

By employing integration by parts, we rewrite (5.2.1) as:

$$\begin{aligned} &\mathscr{A}(u, v_h) - \left( \mathscr{B}(t, t; \hat{u}(t), v_h) - \int_0^t \mathscr{B}_s(t, s; \hat{u}(s), v_h) ds \right) \\ &= \mathscr{A}_h(I^h u, v_h) - \left( \mathscr{B}_h(t, t; \widehat{I^h u}(t), v_h) - \int_0^t \mathscr{B}_{h,s}(t, s; \widehat{I^h u}(s), v_h) ds \right). \end{aligned} \tag{5.2.2}$$

The bilinear form $\mathscr{B}_s(t, s; v, w)$ is defined as:

$$\mathscr{B}_s(t, s; v, w) := \left( \frac{\partial b(\boldsymbol{x}, t, s)}{\partial s} \nabla v, \nabla w \right) \quad \forall v, w \in H_0^1(\Omega),$$

and the discrete bilinear form $\mathscr{B}_{h,t}(t, s; p_h, q_h)$ is defined as:

$$\mathscr{B}_{h,s}(t, s; p_h, q_h) := \sum_{\mathbb{K} \in \mathcal{I}_h} \mathscr{B}_{h,s}^{\mathbb{K}}(t, s; p_h, q_h) \quad \forall p_h, q_h \in W_h,$$

where,

$$\mathscr{B}_{h,s}^{\mathbb{K}}(t, s; v_h, w_h) := \int_K \left( \frac{\partial b(\boldsymbol{x}; t, s)}{\partial s} \mathbf{\Pi}_{k-1}^0 \nabla v_h \cdot \mathbf{\Pi}_{k-1}^0 \nabla w_h \right) d\boldsymbol{x} \quad \forall v_h, w_h \in W_{k,\mathbb{K}}.$$

**Theorem 5.2.1.** *For $u(t) \in H_0^1 \cap H^2$, $t > 0$, with an initial condition $u_0 \in L^2$, then there* *exists a positive constant $C$ that is not dependent on the parameter $h$, and under these conditions, the following estimates are valid:*

$$\|\widehat{I^h u}(t) - \hat{u}(t)\| + h|\widehat{I^h u}(t) - \hat{u}(t)|_1 \le Ch^2\|u_0\|. \tag{5.2.3}$$

*Proof.* Write $\hat{\rho} = \widehat{I^h u} - \Pi_k^0 \hat{u} + \Pi_k^0 \hat{u} - \hat{u} = \hat{\delta}_h + \Pi_k^0 \hat{u} - \hat{u}$, where $\Pi_k^0$ is defined by using (1.5.3a) and satisfies (1.5.7). By integrating (5.2.2) from 0 to $t$, we arrive at:

$$\begin{aligned}
&\mathscr{A}(\hat{u}, v_h) - \int_0^t \left( \mathscr{B}(s, s; \hat{u}(s), v_h) - \int_0^s \mathscr{B}_z(s, z; \hat{u}(z), v_h) dz \right) ds \\
&= \mathscr{A}_h(\widehat{I^h u}, v_h) - \int_0^t \left( \mathscr{B}_h(s, s; \widehat{I^h u}(s), v_h) - \int_0^s \mathscr{B}_{h,z}(s, z; \widehat{I^h u}(z), v_h) dz \right) ds.
\end{aligned} \tag{5.2.4}$$

Now, we proceed by using coricivty of $\mathscr{A}_h(\cdot, \cdot)$ (5.1.2) as:

$$\begin{aligned}
&c_{\alpha_{**}} |\hat{\delta}_h|_1^2 \\
&\quad \le \mathscr{A}_h(\hat{\delta}_h, \hat{\delta}_h) = \mathscr{A}_h(\widehat{I^h u}, \hat{\delta}_h) - \mathscr{A}_h(\Pi_k^0 \hat{u}, \hat{\delta}_h) \\
&\quad = \left[ \mathscr{A}(\hat{u}, \hat{\delta}_h) - \mathscr{A}_h(\Pi_k^0 \hat{u}, \hat{\delta}_h) \right] - \left[ \int_0^t \left( \mathscr{B}(s, s; \hat{u}(s), \hat{\delta}_h) - \mathscr{B}_h(s, s; \widehat{I^h u}(s), \hat{\delta}_h) \right) ds \right] \\
&\qquad + \int_0^t \int_0^s \left( \mathscr{B}_z(s, z; \hat{u}(z), \hat{\delta}_h) - \mathscr{B}_{h,z}(s, z; \widehat{I^h u}(z), \hat{\delta}_h) \right) dz ds.
\end{aligned} \tag{5.2.5}$$

The first term on the right-hand side of (5.2.5) can be solved as:

$$\begin{aligned}
\mathscr{A}(\hat{u}, \hat{\delta}_h) - \mathscr{A}_h(\Pi_k^0 \hat{u}, \hat{\delta}_h) &= \mathscr{A}(\hat{u} - \Pi_k^0 \hat{u}, \hat{\delta}_h) + \mathscr{A}(\Pi_k^0 \hat{u}, \hat{\delta}_h) - \mathscr{A}_h(\Pi_k^0 \hat{u}, \hat{\delta}_h) \\
&\le C(|\hat{u} - \Pi_k^0 \hat{u}|_1 + h^2 \|\hat{u}\|_2) |\hat{\delta}_h|_1 \quad \text{(By using Lemma 5.1.3)} \\
&\le Ch \|u_0\| |\hat{\delta}_h|_1 \quad \text{(By using Lemma 5.1.1)}.
\end{aligned} \tag{5.2.6}$$

The second term on the right-hand side of (5.2.5) can be solved as:

$$\begin{aligned}
&\int_0^t \left( \mathscr{B}(s, s; \hat{u}(s), \hat{\delta}_h) - \mathscr{B}_h(s, s; \widehat{I^h u}(s), \hat{\delta}_h) \right) ds \\
&= \int_0^t \Big( \mathscr{B}(s, s; (\hat{u} - \Pi_k^0 \hat{u})(s), \hat{\delta}_h) - \mathscr{B}_h(s, s; (\widehat{I^h u} - \Pi_k^0 \hat{u})(s), \hat{\delta}_h)
\end{aligned}$$

$$+\mathscr{B}(s,s;\Pi_k^0\hat{u}(s),\hat{\delta}_h)-\mathscr{B}_h(s,s;\Pi_k^0\hat{u}(s),\hat{\delta}_h)\Big)\,ds$$

$$\leq C\int_0^t\Big(|(\hat{u}-\Pi_k^0\hat{u})(s)|_1+|\hat{\delta}_h(s)|_1+h^2\|\hat{u}(s)\|_2\Big)\,ds|\hat{\delta}_h|_1$$

(By using the boundedness of $b(\boldsymbol{x};t,s)$ and Lemma 5.1.3)

$$\leq C\int_0^t\Big(h\|u_0\|+|\hat{\delta}_h(s)|_1\Big)\,ds|\hat{\delta}_h|_1 \quad \text{(By using Lemma 5.1.1)}. \tag{5.2.7}$$

The third term on the right-hand side of (5.2.5) can be solved by considering the boundedness of $b_s(\boldsymbol{x};t,s)$ and by following similar arguments as above:

$$\int_0^t\int_0^s\Big(\mathscr{B}_z(s,z;\hat{u}(z),\hat{\delta}_h)-\mathscr{B}_{h,z}(s,z;\widehat{I^hu}(z),\hat{\delta}_h)\Big)\,dzds$$
$$\leq C\int_0^t\int_0^s\Big(h\|u_0\|+|\hat{\delta}_h(z)|_1\Big)\,dzds|\hat{\delta}_h|_1. \tag{5.2.8}$$

By using (5.2.6),(5.2.7),(5.2.8) in (5.2.5) and the fact that $t\leq T$, we arrive at:

$$c_{\alpha_{**}}|\hat{\delta}_h|_1^2\leq Ch\left(\|u_0\|+\int_0^t\|u_0\|ds+\int_0^t\int_0^s\|u_0\|dzds+\int_0^t|\hat{\delta}(s)|_1\right)|\hat{\delta}_h|_1.$$

Now, by using the boundedness of $t$ and applying Grönwall's inequality, we arrive at:

$$c_{\alpha_{**}}|\hat{\delta}_h|_1\leq C_1h\|u_0\|.$$

The application of the triangle's inequality completes the proof:

$$|\widehat{I^hu}-\hat{u}|_1\leq Ch\|u_0\|. \tag{5.2.9}$$

For the $L^2$ estimate, we follow the duality argument. Let $\phi\in H^2(\Omega)\cap H_0^1(\Omega)$ be the solution of

$$-\nabla\cdot(a\nabla\phi)=\hat{\rho};\quad \text{in }\Omega\quad \phi=0\quad \text{on }\partial\Omega, \tag{5.2.10}$$

where $\hat{\rho}=\widehat{I^hu}-\hat{u}$ and, satisfies the estimate: $\|\phi\|_2\leq C\|\hat{\rho}\|$. Using (5.2.10), we arrive at:

$$\|\hat{\rho}\|^2=\mathscr{A}(\hat{\rho},\phi-\phi_I)+\mathscr{A}(\hat{\rho},\phi_I). \tag{5.2.11}$$

The first term in the right-hand side of (5.2.11) can be solved as:

$$
\begin{aligned}
\mathscr{A}(\hat{\rho}, \phi - \phi_I) &= \mathscr{A}(\widehat{I^h u} - \hat{u}, \phi - \phi_I) \\
&\leq Ch|\widehat{I^h u} - \hat{u}|_1 \|\phi\|_2 \\
&\leq Ch^2 \|u_0\| \|\hat{\rho}\|.
\end{aligned}
$$

The second term in the right-hand side of (5.2.11) can be solved by using (5.2.4), as:

$$
\begin{aligned}
&\mathscr{A}(\hat{\rho}, \phi_I) \\
&= \left[\mathscr{A}(\widehat{I^h u}, \phi_I) - \mathscr{A}_h(\widehat{I^h u}, \phi_I)\right] + \left[\mathscr{B}_h(s, s; \widehat{I^h u}(s), \phi_I) - \int_0^t [\mathscr{B}(s, s; \hat{u}(s), \phi_I)]\, ds\right] \\
&\quad + \int_0^t \int_0^s \left(\mathscr{B}_z(s, z; \hat{u}(z), \phi_I) - \mathscr{B}_{h,z}(s, z; \widehat{I^h u}(z), \phi_I) dz\right) dz ds. \qquad (5.2.12)
\end{aligned}
$$

Now, to solve the first term on the right-hand side of the (5.2.12), we proceed as:

$$
\begin{aligned}
\mathscr{A}(\widehat{I^h u}, \phi_I) - \mathscr{A}_h(\widehat{I^h u}, \phi_I) &= \mathscr{A}(\widehat{I^h u} - \Pi_k^0 \hat{u}, \phi_I) - \mathscr{A}_h(\widehat{I^h u} - \Pi_k^0 \hat{u}, \phi_I) \\
&\quad + \mathscr{A}(\Pi_k^0 \hat{u}, \phi_I) - \mathscr{A}_h(\Pi_k^0 \hat{u}, \phi_I) \\
&\leq C(h|\hat{\delta}_h|_1 \|\rho\| + h^2 \|u_0\| \|\hat{\rho}\|) \quad \text{(By using Lemma 5.1.3)} \\
&\leq Ch^2 \|u_0\| \|\hat{\rho}\| \quad \text{(By using (5.2.9))}. \qquad (5.2.13)
\end{aligned}
$$

The second term on the right-hand side of (5.2.12) can be estimated as:

$$
\begin{aligned}
&\int_0^t \left(\mathscr{B}_h(s, s; \widehat{I^h u}(s), \phi_I) - \mathscr{B}(s, s; \hat{u}(s), \phi_I)\right) ds \\
&\quad = \int_0^t [\mathscr{B}_h(s, s; \widehat{I^h u}(s), \phi_I) - \mathscr{B}(s, s; \widehat{I^h u}(s), \phi_I) - \mathscr{B}(s, s; (\hat{u} - \widehat{I^h u})(s), \phi_I)] ds \\
&\quad = \int_0^t [\mathscr{B}_h(s, s; \widehat{I^h u}(s), \phi_I) - (\mathscr{B}(s, s; \widehat{I^h u}(s), \phi_I) - \mathscr{B}(s, s; (\hat{u} - \widehat{I^h u})(s), \phi_I - \phi) \\
&\qquad - \mathscr{B}(s, s; (\hat{u} - \widehat{I^h u})(s), \phi)] ds \\
&\quad \leq C \int_0^t \left(h^2 \|u_0\| + h|\hat{\rho}(s)|_1 + \|\hat{\rho}(s)\|\right) ds \|\hat{\rho}\|. \qquad (5.2.14)
\end{aligned}
$$

By similar arguments, we can solve the third term on the right-hand side of (5.2.12) and arrive at:

$$\int_0^t \int_0^s \Big( \mathscr{B}_z(s,z;\hat{u}(z),\phi_I) - \mathscr{B}_{h,z}(s,z;\widehat{I^h u}(z),\phi_I) \Big)\, dzds \\ \leq C \int_0^t \int_0^s \left( h^2 \|u_0\| + h|\hat{\rho}(z)|_1 + \|\hat{\rho}(z)\| \right) dzds \|\hat{\rho}\|. \tag{5.2.15}$$

Using (5.2.13), (5.2.14), (5.2.15), in (5.2.12), and then put (5.2.12) in (5.2.11), followed by the use of Grönwall's lemma, we arrive at:

$$\|\hat{\rho}\| \leq Ch^2 \left( \|u_0\| + \int_0^t \|u_0\| ds + \int_0^t \int_0^s \|u_0\| dzds \right).$$

Now, by using the fact that $t \leq T$, we arrive at:

$$\|\hat{\rho}\| \leq Ch^2 \|u_0\|.$$

□

**Theorem 5.2.2.** *For $u(t) \in H_0^1 \cap H^2$, $t > 0$, with an initial condition $u_0 \in L^2$, then there exists a positive constant $C$ that is not dependent on the parameter $h$, and under these conditions, the following estimates are valid:*

$$\|I^h u(t) - u(t)\| + h|I^h u(t) - u(t)|_1 \leq Ch^2 t^{-1} \|u_0\|. \tag{5.2.16}$$

*Proof.* We write $\rho = I^h u - u = \delta_h + \Pi_k^0 u - u$, where $\delta_h = I^h u - \Pi_k^0 u$. and now proceed by using the coercivity of $\mathscr{A}(\cdot,\cdot)$ as:

$$c_{\alpha_{**}} |\delta_h|_1^2 \leq \mathscr{A}_h(\delta_h, \delta_h) \\ = \mathscr{A}_h(I^h u, \delta_h) - \mathscr{A}_h(\Pi_k^0 u, \delta_h).$$

Now, by using (5.2.2), we arrive at:

$$c_{\alpha_{**}} |\delta_h|_1^2 \leq \left[ \mathscr{A}(u,\delta_h) - \mathscr{A}_h(\Pi_k^0 u, \delta_h) \right] - \Big[ \mathscr{B}(t,t;\hat{u},\delta_h) - \mathscr{B}_h(t,t;\widehat{I^h u},\delta_h) \Big]$$

$$+\int_0^t \Big(\mathscr{B}_s(t,s;\hat{u}(s),\delta_h) - \mathscr{B}_{h,s}(t,s;\widehat{I^h u}(s),\delta_h)\Big)\,ds. \tag{5.2.17}$$

The first term on the right-hand side of (5.2.17) can be solved as:

$$\begin{aligned}\mathscr{A}(u,\delta_h) - \mathscr{A}_h(\Pi_k^0 u,\delta_h) &= \mathscr{A}(u-\Pi_k^0 u,\delta_h) + \mathscr{A}(\Pi_k^0 u,\delta_h) - \mathscr{A}_h(\Pi_k^0 u,\delta_h)\\ &\le C(|u-\Pi_k^0 u|_1 + h^2\|u\|_2)|\delta_h|_1 \quad \text{(By using Lemma 5.1.3)}\\ &\le Cht^{-1}\|u_0\||\hat{\delta}_h|_1 \quad \text{(By using Lemma 5.1.1)},\end{aligned} \tag{5.2.18}$$

whereas the second and third terms on the right-hand side of (5.2.17) can be solved by following the similar arguments as in (5.2.7):

$$\mathscr{B}_h(t,t;\widehat{I^h u},\delta_h) - \mathscr{B}(t,t;\hat{u},\delta_h) \le Ch\|u_0\||\delta_h|_1, \tag{5.2.19}$$

$$\int_0^t \Big(\mathscr{B}_s(t,s;\hat{u}(s),\delta_h) - \mathscr{B}_{h,s}(t,s;\widehat{I^h u}(s),\delta_h)\Big)\,ds \le Ch\|u_0\||\delta_h|_1. \tag{5.2.20}$$

Putting (5.2.18), (5.2.19) and (5.2.20) in (5.2.17) followed by the triangle inequality, we get our required estimate:

$$|\rho|_1 \le Cht^{-1}\|u_0\|.$$

The duality approach will be used to demonstrate the $L^2$ error estimate. Let $\phi \in H^2(\Omega)\cap H_0^1(\Omega)$ be the solution of

$$-\nabla\cdot(a\nabla\phi) = \rho; \quad \text{in } \Omega \quad \phi = 0 \quad \text{on } \partial\Omega, \tag{5.2.21}$$

and it satisfies the estimate: $\|\phi\|_2 \le C\|\rho\|$. Using (5.2.21), we arrive at:

$$\|\rho\|^2 = \mathscr{A}(\rho,\phi-\phi_I) + \mathscr{A}(\rho,\phi_I). \tag{5.2.22}$$

The first term in the right-hand side of (5.2.22) can be solved as:

$$\begin{aligned}\mathscr{A}(\rho,\phi-\phi_I) &= \mathscr{A}(I^h u - u,\phi-\phi_I)\\ &\le Ch\|I^h u - u\|_1\|\phi\|_2\\ &\le Ch^2 t^{-1}\|u_0\|\|\rho\|.\end{aligned} \tag{5.2.23}$$

The second term on the right-hand side of (5.2.22) can be solved by using (5.2.2) as:

$$\begin{aligned}\mathscr{A}(\rho,\phi_I) &= \mathscr{A}(I^h u,\phi_I) - \mathscr{A}(u,\phi_I)\\ &= \left(\mathscr{A}(I^h u,\phi_I) - \mathscr{A}_h(I^h u,\phi_I)\right) - \left[\mathscr{B}(t,t;\hat{u},\phi_I) - \mathscr{B}_h(t,t;\widehat{I^h u},\phi_I)\right]\\ &\quad + \int_0^t \left(\mathscr{B}_s(t,s;\hat{u}(s),\phi_I) - \mathscr{B}_{h,s}(t,s;\widehat{I^h u}(s),\phi_I)\right) ds, \qquad (5.2.24)\end{aligned}$$

where the first term on the right-hand side of (5.2.24) can be solved as:

$$\begin{aligned}&\mathscr{A}(I^h u,\phi_I) - \mathscr{A}_h(I^h u,\phi_I)\\ &= \mathscr{A}(I^h u - \Pi_k^0 u,\phi_I) - \mathscr{A}_h(I^h u - \Pi_k^0 u,\phi_I) + \mathscr{A}(\Pi_k^0 u,\phi_I) - \mathscr{A}_h(\Pi_k^0 u,\phi_I)\\ &\leq Ch^2 t^{-1}\|u_0\|\|\rho\| \quad (\text{ By using Lemma 5.1.1}). \qquad (5.2.25)\end{aligned}$$

The second and third terms on the right-hand side of (5.2.24) can be solved by following the similar arguments as in (5.2.19) and (5.2.20):

$$\mathscr{B}(t,t;\hat{u},\phi_I) - \mathscr{B}_h(t,t;\widehat{I^h u},\phi_I) \leq Ch^2\|u_0\|\|\rho\|, \qquad (5.2.26)$$

$$\int_0^t \left(\mathscr{B}_s(t,s;\hat{u}(s),\phi_I) - \mathscr{B}_{h,s}(t,s;\widehat{I^h u}(s),\phi_I)\right) ds \leq Ch^2\|u_0\|\|\rho\|. \qquad (5.2.27)$$

Now, using (5.2.23) , (5.2.24), (5.2.25) , (5.2.26), (5.2.27) in (5.2.22), we get our desired estimate:

$$\|\rho\| \leq Ch^2 t^{-1}\|u_0\|.$$

□

**Theorem 5.2.3.** *For $u(t) \in H_0^1 \cap H^2$, $t > 0$, with an initial condition $u_0 \in L^2$, and $u_t$ be the time derivative of $u$, then there exists a positive constant $C$ that is not dependent on the parameter $h$, and under these conditions, the following estimates are valid:*

$$\|(I^h u)_t(t) - u_t(t)\| + h|(I^h u)_t(t) - u_t(t)|_1 \leq Ch^2 t^{-2}\|u_0\|. \qquad (5.2.28)$$

*Proof.* Differentiating (5.2.1) with respect to $t$ we get:

$$\begin{aligned}\mathscr{A}_h((I^hu)_t, v_h) - \mathscr{B}_h(t,t;I^hu,v_h) + \int_0^t \mathscr{B}_{h,t}(t,s;I^hu(s),v_h)ds \\ = \mathscr{A}(u_t,v_h) - \mathscr{B}(t,t;u,v_h) + \int_0^t \mathscr{B}_t(t,s;u(s),v_h)ds,\end{aligned}$$

where $\mathscr{B}(t,t;v_h,w_h)$ and $\mathscr{B}_t(t,s;v_h,w_h)$ are defined as:

$$\mathscr{B}(t,t;v_h,w_h) = (b(\boldsymbol{x},t,t)\nabla v_h, \nabla w_h), \quad \mathscr{B}_t(t,s;v_h,w_h) = \left(\frac{\partial b(\boldsymbol{x},t,s)}{\partial t}\nabla v_h, \nabla w_h\right).$$

$\mathscr{B}_{h,t}(t,s;u_h,v_h)$ and $\mathscr{B}_h(t,t;u_h,v_h)$ are the discrete bilinear forms corresponding to $\mathscr{B}_t(t,s;u_h,v_h)$ and $\mathscr{B}(t,t;u_h,v_h)$ respectively defined using $\Pi^0_{k-1}$ projection. The proof is similar to the previous theorem arguments. □

## 5.3 Error Estimates

Now, we prove some lemmas, and for that, we proceed by integrating (5.3.15) as:

$$\begin{aligned}&m_h(\theta(\cdot,t),v_h) + \mathscr{A}_h(\hat{\theta}(\cdot,t),v_h) \\ &\quad = ((u(\cdot,t),v_h) - m_h((I^hu)(\cdot,t),v_h)) + \int_0^t \mathscr{B}_h(s,s;\hat{\theta}(s),v_h)ds \\ &\quad - \int_0^t\int_0^s \mathscr{B}_{h,z}(s,z;\hat{\theta}(z),v_h)dzds. \qquad (5.3.1)\end{aligned}$$

Again integrating (5.3.1) form 0 to $t$, we get:

$$\begin{aligned}&m_h(\hat{\theta}(\cdot,t),v_h) + \mathscr{A}_h(\hat{\hat{\theta}}(\cdot,t),v_h) \\ &= ((\hat{u}(\cdot,t),v_h) - m_h(\widehat{I^hu}(\cdot,t),v_h)) + \int_0^t \mathscr{B}_h(s,s;\hat{\hat{\theta}}(s),v_h)ds \\ &- 2\int_0^t\int_0^s \mathscr{B}_{h,z}(z,z;\hat{\hat{\theta}}(z),v_h)dzds + \int_0^t\int_0^s\int_0^z \mathscr{B}_{h,z\tau'}(z,\tau';\hat{\hat{\theta}}(z),v_h)d\tau'dzds. \qquad (5.3.2)\end{aligned}$$

**Lemma 5.3.1.** *Let $\hat{\theta}$ and $\hat{\hat{\theta}}$ satisfies (5.3.1) and (5.3.2) respectively, then $\exists$ a positive constant $C$, such that the following estimates hold true:*

$$\|\hat{\hat{\theta}}\|^2 + \int_0^t |\hat{\hat{\theta}}(s)|_1^2 ds \leq Ch^4 t \|u_0\|^2, \tag{5.3.3}$$

$$\int_0^t \|\hat{\theta}(s)\|^2 ds + |\hat{\hat{\theta}}(t)|_1^2 \leq Ch^4 t \|u_0\|^2. \tag{5.3.4}$$

*Proof.* By putting $v_h = \hat{\hat{\theta}}(t)$ in (5.3.2), we arrive at:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\hat{\hat{\theta}}(t)\|^2 + c_{\alpha_{**}}|\hat{\hat{\theta}}(t)|_1^2 \\
&\leq \left(\left(\hat{u}(\cdot,t), \hat{\hat{\theta}}(t)\right) - m_h\left(\widehat{I^h u}(\cdot,t), \hat{\hat{\theta}}(t)\right)\right) + \int_0^t \mathscr{B}_h(s,s;\hat{\hat{\theta}}(s), \hat{\hat{\theta}}(t)) ds \\
&\quad - 2\int_0^t \int_0^s \mathscr{B}_{h,z}(z,z;\hat{\hat{\theta}}(z), \hat{\hat{\theta}}(t)) dz ds + \int_0^t \int_0^s \int_0^z \mathscr{B}_{h,z\tau'}(z,\tau';\hat{\hat{\theta}}(z), \hat{\hat{\theta}}(t)) d\tau' dz ds.
\end{aligned} \tag{5.3.5}$$

The first term on the right-hand side of the (5.3.5) can be solved as:

$$\begin{aligned}
&\left(\left(\hat{u}(\cdot,t), \hat{\hat{\theta}}(t)\right) - m_h\left(\widehat{I^h u}(\cdot,t), \hat{\hat{\theta}}(t)\right)\right) \\
&\quad = \sum_{\mathbb{K}\in\mathcal{I}_h}\left(\left(\hat{u}(\cdot,t), \hat{\hat{\theta}}\right)_{0,\mathbb{K}} - m_h^{\mathbb{K}}\left(\widehat{I^h u}(\cdot,t), \hat{\hat{\theta}}\right)\right) \\
&\quad = \sum_{\mathbb{K}\in\mathcal{I}_h}\left(\left(\hat{u}(\cdot,t) - \Pi_k^0\hat{u}(\cdot,t), \hat{\hat{\theta}}\right)_{0,\mathbb{K}} - m_h^{\mathbb{K}}\left(\widehat{I^h u}(\cdot,t) - \Pi_k^0\hat{u}(\cdot,t), \hat{\hat{\theta}}\right)\right) \\
&\quad \leq Ch^2\left(\|\hat{u}\|_2 + \|u_0\|\right)\|\hat{\hat{\theta}}\| \quad \text{By using (5.2.3)} \\
&\quad \leq Ch^2\|u_0\|\|\hat{\hat{\theta}}\|.
\end{aligned} \tag{5.3.6}$$

By the use of (5.3.6), Cauchy Schwarz inequality in (5.3.5) followed by the use of kickback argument, we can rewrite (5.3.5) as:

$$\frac{1}{2}\frac{d}{dt}\|\hat{\hat{\theta}}(t)\|^2 + |\hat{\hat{\theta}}(t)|_1^2 \leq C\left(h^4\|u_0\|^2 ds + \|\hat{\hat{\theta}}(t)\|^2 + \int_0^t |\hat{\hat{\theta}}(s)|_1^2 ds\right). \tag{5.3.7}$$

Now, integrating (5.3.7) from 0 to $t$, we arrive at:

$$\|\hat{\hat{\theta}}(t)\|^2 + \int_0^t |\hat{\hat{\theta}}(s)|_1^2 ds \leq C\left(h^4 \int_0^t \|u_0\|^2 ds + \int_0^t \|\hat{\hat{\theta}}(s)\|^2 + \int_0^t \int_0^s |\hat{\hat{\theta}}(z)|_1^2 dz ds\right).$$

The use of Grönwall's inequality completes the proof:

$$\|\hat{\hat{\theta}}(t)\|^2 + \int_0^t |\hat{\hat{\theta}}(s)|_1^2 ds \leq Ch^4 t \|u_0\|^2.$$

For the proof of (5.3.4), we proceed by putting $v_h = \hat{\theta}(t)$ in (5.3.2) as:

$$\begin{aligned}
&\|\hat{\theta}(\cdot,t)\|^2 + \frac{c_{\alpha_{**}}}{2}\frac{d}{dt}|\hat{\hat{\theta}}(\cdot,t)|^2 \\
&= \left(\left(\hat{u}(\cdot,t), \hat{\theta}(t)\right) - m_h\left(\widehat{I^h u}(\cdot,t), \hat{\theta}(t)\right)\right) + \mathscr{B}_h(t,t;\hat{\hat{\theta}}(t), \hat{\hat{\theta}}(t)) \\
&\quad + \frac{d}{dt}\left(-\int_0^t \mathscr{B}_h(s,s;\hat{\hat{\theta}}(s), \hat{\hat{\theta}}(t) ds + 2\int_0^t \int_0^s \mathscr{B}_{h,z}(z,z;\hat{\hat{\theta}}(z), \hat{\hat{\theta}}(t)) dz ds \right. \\
&\quad \left. + \int_0^t \int_0^s \int_0^z \mathscr{B}_{h,z\tau'}(z,\tau';\hat{\hat{\theta}}(\tau'), \hat{\hat{\theta}}(t)) d\tau' dz ds\right) \\
&\quad - 2\int_0^t \mathscr{B}_{h,s}(s,s;\hat{\hat{\theta}}(s), \hat{\hat{\theta}}(t)(t)) ds - \int_0^t \int_0^s \mathscr{B}_{h,sz}(s,z;\hat{\hat{\theta}}(z), \hat{\hat{\theta}}(t)) dz ds. \qquad (5.3.8)
\end{aligned}$$

Following the similar step as in (5.3.6) for the first term on the right-hand side of (5.3.8) and then integrating (5.3.8) from 0 to $t$, followed by an application of Cauchy-Schwarz inequality, we arrive at:

$$\int_0^t \|\hat{\theta}(s)\|^2 ds + |\hat{\hat{\theta}}(t)|_1^2 \leq C\left(h^4 \int_0^t \|u_0\|^2 ds + \int_0^t |\hat{\hat{\theta}}(s)|_1^2\right).$$

Use of (5.3.3) completes the proof:

$$\int_0^t \|\hat{\theta}(s)\|^2 ds + |\hat{\hat{\theta}}(t)|_1^2 \leq Ch^4 t \|u_0\|^2.$$

□

**Lemma 5.3.2.** *Let* $\hat{\theta}$ *and* $\hat{\hat{\theta}}$ *satisfies* (5.3.1) *and* (5.3.2) *respectively, then* $\exists$ *a positive* *constant $C$, such that the following estimates hold true:*

$$t\|\hat{\theta}\|^2 + \int_0^t s|\hat{\theta}(s)|_1^2 ds \le Ch^4 t\|u_0\|^2, \tag{5.3.9}$$

$$\int_0^t s^2\|\theta(s)\|^2 ds + t^2|\hat{\theta}(t)|_1^2 \le Ch^4 t\|u_0\|^2. \tag{5.3.10}$$

*Proof.* Put $v_h = t\hat{\theta}(t)$ in (5.3.1), we arrive at:

$$\begin{aligned} &tm_h(\theta(\cdot,t),\hat{\theta}(t)) + t\mathscr{A}_h(\hat{\theta}(\cdot,t),\hat{\theta}(t)) \\ &= t\Big((u(\cdot,t),\hat{\theta}(t)) - m_h((I^h u)(\cdot,t),\hat{\theta}(t))\Big) + t\int_0^t \mathscr{B}_h(s,s;\hat{\theta}(s),\hat{\theta}(t))ds \\ &\quad - t\int_0^t\int_0^s \mathscr{B}_{h,z}(s,z;\hat{\theta}(z),\hat{\theta}(t))dzds. \end{aligned} \tag{5.3.11}$$

Now, rewriting (5.3.11) as:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(t\|\hat{\theta}\|^2) + c_{\alpha_{**}}|\hat{\theta}|_1^2 \\ &= t\Big((u(\cdot,t),\hat{\theta}(t)) - m_h((I^h u)(\cdot,t),\hat{\theta}(t))\Big) + \frac{1}{2}\|\hat{\theta}\|^2 + tB_h(t,t;\hat{\hat{\theta}}(t),\hat{\theta}) \\ &\quad - 2t\int_0^t \mathscr{B}_{h,s}(s,s;\hat{\hat{\theta}}(s),\hat{\theta})ds + t\int_0^t\int_0^s \mathscr{B}_{h,zz}(s,z;\hat{\hat{\theta}}(z),\hat{\theta})dzds. \end{aligned} \tag{5.3.12}$$

The first term on the right-hand side of the above equation follows as:

$$\begin{aligned} &t((u(\cdot,t),\hat{\theta}) - m_h((I^h u)(\cdot,t),\hat{\theta})) \\ &\qquad = \sum_{\mathbb{K}\in\mathcal{I}_h} t\Big((u(\cdot,t),\hat{\theta})_{0,\mathbb{K}} - m_h^{\mathbb{K}}(I^h u(\cdot,t),\hat{\theta})\Big) \\ &\qquad = \sum_{\mathbb{K}\in\mathcal{I}_h} t((u(\cdot,t) - \Pi_k^0 u(\cdot,t),\hat{\theta})_{0,\mathbb{K}} - m_h^{\mathbb{K}}(I^h u(\cdot,t) - \Pi_k^0 u(\cdot,t),\hat{\theta})) \\ &\qquad \le Ch^2 t(\|u\|_2 + t^{-1}\|u_0\|)\|\hat{\theta}\| \quad \text{By using (5.2.16)} \\ &\qquad \le Ch^2\|u_0\|\|\hat{\theta}\| \quad \text{By using (5.1.1).} \end{aligned} \tag{5.3.13}$$

Now, integrating (5.3.12) from 0 to $t$, and then the using (5.3.13) and Cauchy Schwarz inequality completes the proof:

$$t\|\hat{\theta}\|^2 + \int_0^t s|\hat{\theta}(s)|_1^2 ds \leq Ch^4 t\|u_0\|^2.$$

Proof of the other follows from [63]. □

**Theorem 5.3.1.** *Let $u$ and $u_h$ be the solution of (5.0.2) and (5.1.1) respectively, Then there exists a positive constant $C$, independent of $h$ such that the following estimate holds true:*

$$\|u - u_h\| \leq Ch^2 t^{-1}\|u_0\|. \tag{5.3.14}$$

*Proof.* Write $u - u_h = u - I^h u + I^h u - u_h =: \rho + \theta$. We already have the $\rho(\cdot, t)$ estimates. Now to deal with $\theta(\cdot, t)$, we use (5.0.2) and (5.1.1), along with the intermediate projection (5.2.1) to arrive at:

$$\begin{aligned} m_h(\theta_t(\cdot,t), v_h) + \mathscr{A}_h(\theta(\cdot,t), v_h) - \int_0^t \mathscr{B}_h(t,s;\theta(\cdot,s), v_h) ds \\ = ((u_t(\cdot,t), v_h) - m_h((I^h u)_t(\cdot,t), v_h)). \end{aligned} \tag{5.3.15}$$

Put $v_h = t^3\theta$ in (5.3.15), we get:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(t^3\|\theta\|^2) + c_{\alpha_{**}} t^3|\theta|_1^2 \\ &\leq \frac{3}{2}t^2\|\theta\|^2 + t^3((u_t(\cdot,t),\theta) - m_h((I^h u)_t(\cdot,t),\theta)) + t^3\int_0^t \mathscr{B}_h(t,s;\theta(\cdot,s),\theta) ds. \end{aligned}$$

Integrating the above equation with respect to time form 0 to $t$, we have:

$$\begin{aligned} t^3\|\theta\|^2 &+ 2c_{\alpha_{**}}\int_0^t s^3|\theta(\cdot,s)|_1^2 ds \\ &\leq \int_0^t 3s^2\|\theta(\cdot,s)\|^2 ds + \int_0^t\int_0^s s^3\mathscr{B}_h(s,z;\theta(\cdot,z),\theta(\cdot,s)) dz ds \\ &\quad + \left(\int_0^t s^3((u_s(\cdot,s),\theta(\cdot,s)) - m_h((I^h u)_s(\cdot,s),\theta(\cdot,s))) ds\right). \end{aligned} \tag{5.3.16}$$

To solve $\left(\int_0^t s^3((u_s(\cdot,s),\theta(\cdot,s)) - m_h((I^h u)_s(\cdot,s),\theta(\cdot,s)))ds\right)$, we proceed as:

$$
\begin{aligned}
&(u_t(\cdot,t),\theta(\cdot,t)) - m_h((I^h u)_t(\cdot,t),\theta(\cdot,t)) \\
&= \sum_{\mathbb{K}\in\mathcal{I}_h} \big((u_t(\cdot,t),\theta(\cdot,t))_{0,\mathbb{K}} - m_h^{\mathbb{K}}((I^h u)_t(\cdot,t),\theta(\cdot,t))\big) \\
&= \sum_{\mathbb{K}\in\mathcal{I}_h} \big((u_t(\cdot,t) - \Pi_k^0 u_t(\cdot,t),\theta(t))_{0,\mathbb{K}} - m_h^{\mathbb{K}}((I^h u)_t(\cdot,t) - \Pi_k^0 u_t(\cdot,t),\theta(\cdot,t))\big) \\
&\leq C(h^2\|u_t(\cdot,t)\|_2 + \|\Pi_k^0 u_t(\cdot,t) - (I^h u)_t(\cdot,t)\|)\|\theta(\cdot,t)\| \\
&\leq Ch^2 t^{-2}\|u_0\|\|\theta(\cdot,t)\| \quad \text{(By using Lemma 5.1.1).}
\end{aligned}
$$

Now,

$$
\begin{aligned}
&\int_0^t s^3((u_t(\cdot,s),\theta(\cdot,\cdot,s)) - m_h((I^h u)_t(\cdot,s),\theta(\cdot,s)))ds \\
&\quad\leq Ch^2\int_0^t s\|u_0\|\|\theta(\cdot,s)\|ds \\
&\quad\leq C\left(h^4 t\|u_0\|^2 + \int_0^t s^2\|\theta(\cdot,s)\|^2 ds\right) \quad \text{(By using Young's inequality).} \qquad (5.3.17)
\end{aligned}
$$

So, (5.3.16) can be rewritten by using (5.3.17) as:

$$
\begin{aligned}
&t^3\|\theta\|^2 + 2c_{\alpha_{**}}\int_0^t s^3|\theta(\cdot,s)|_1^2 ds \\
&\quad\leq C\left(h^4 t\|u_0\|^2 + \int_0^t s^2\|\theta(\cdot,s)\|^2 ds + \int_0^t\int_0^s s^3\mathscr{B}_h(s,z;\theta(\cdot,z),\theta(\cdot,s))dzds\right) \\
&\quad\leq C\Bigg(h^4 t\|u_0\|^2 + \int_0^t s^2\|\theta(\cdot,s)\|^2 ds + \int_0^t s^3\mathscr{B}_h(s,s;\hat{\theta}(\cdot,s),\theta(\cdot,s))ds \\
&\qquad - \int_0^t s^3\mathscr{B}_{h,s}(s,s;\hat{\hat{\theta}}(\cdot,s),\theta(\cdot,s))ds + \int_0^t\int_0^s s^3\mathscr{B}_{h,zz}(s,z;\hat{\hat{\theta}}(\cdot,z),\theta(\cdot,s))dzds.
\end{aligned}
$$

Now, using Young's inequality and kickback argument:

$$
t^3\|\theta\|^2 + \int_0^t s^3|\theta(\cdot,s)|_1^2 ds \leq C\left(h^4 t\|u_0\|^2 + \int_0^t (s^2\|\theta(\cdot,s)\|^2 + s|\hat{\theta}(s)|_1^2 + |\theta(\hat{\hat{s}})|_1^2 ds\right).
$$

By using (5.3.3), (5.3.9) and (5.3.10), we arrive at:

$$t^3\|\theta\|^2 + \int_0^t s^3|\theta(\cdot,s)|_1^2 ds \leq Ch^4 t\|u_0\|^2.$$

So,

$$\|\theta(\cdot,t)\| \leq Ch^2 t^{-1}\|u_0\|.$$

Use of the triangle inequality and (5.2.16) completes the proof of (5.3.14):

$$\|u - u_h\| \leq Ch^2 t^{-1}\|u_0\|.$$

□

## 5.4 Conclusion

In this Chapter, we develop and analyze the confirming VEM for PIDE with non-smooth initial data. Our approach involves iteratively applying the integral operator to derive estimates for the integration of the intermediate projection denoted as $\widehat{I^h u}$. With the help of estimates of $\widehat{I^h u}$, we derived the estimates of $I^h u$. Through a combination of regularity results and intermediate projection estimates, we establish optimal error estimates on the order of $O(h^2 t^{-1})$.

# Chapter 6

# Two Mixed Virtual Element Formulations for Parabolic Integro-Differential Equations with Nonsmooth Initial Data[1]

## 6.1 Introduction

This chapter presents and analyzes the two mixed virtual element formulations applied to linear PIDEs (1.1.1) with $f = 0$, with non-smooth initial data i.e. $u_0 \in L^2(\mathcal{D})$ and not in $H^1(\mathcal{D})$. Along with the assumptions defined in Chapter 1, we consider the coefficient $b(\boldsymbol{x}; t, s)$ and its higher-order derivatives up to the second order, with respect to the variables $t$ and $s$, are real-valued, bounded and, and smooth. In cases where the initial data exhibits sufficient smoothness, the corresponding solution possesses certain regularities. However, when dealing with non-smooth initial data, such as $u_0 \in L^2(\mathcal{D})$, the solution to

[1]The substantial part of this chapter has been communicated as follows:
**M Suthar**, and S Yadav, "Two Mixed Virtual Element Formulations for Parabolic Integro-Differential Equations with Nonsmooth Initial Data" (**Communicated**).

(1.1.1) will no longer have the regularity observed in its parabolic counterpart, as detailed in [62], because of this higher value of $k$ will not help us in higher convergence, so we consider $k$=1 only for the non-smooth initial data. This means that the memory term plays a dominant role in these systems since it memorizes the singularities developed at $t$ = 0. Consequently, in such instances, the smoothness results presented in [74] become inapplicable. Hence, we require a different approach for the non-smooth analysis. As we know, we can proceed with two different formulations, one without a resolvent kernel (6.1.1a)-(6.1.1b) and one with a resolvent kernel (6.1.2) given by Find $(\boldsymbol{\sigma}, p) : (0, T] \to \mathcal{V} \times \mathcal{Q}$ such that:

$$(u_t, \phi) - (\nabla \cdot \boldsymbol{\sigma}, \phi) = 0 \;\; \forall \phi \in \mathcal{Q}, \tag{6.1.1a}$$

$$(\mu\boldsymbol{\sigma}, \boldsymbol{\chi}) + (u, \nabla \cdot \boldsymbol{\chi}) - \int_0^t ((b_0(\boldsymbol{x}; t, s)u(s), \nabla \cdot \boldsymbol{\chi}) + (\nabla b_0(x; t, s)u(s), \boldsymbol{\chi}))ds = 0 \;\; \forall \boldsymbol{\chi} \in \mathcal{V}, \tag{6.1.1b}$$

and

$$\begin{aligned} (u_t, \phi) - (\nabla \cdot \boldsymbol{\sigma}, \phi) &= 0 \quad \forall \phi \in \mathcal{Q}, \\ (\mu\boldsymbol{\sigma}, \boldsymbol{\chi}) + \int_0^t (\mathcal{K}(t, s)\boldsymbol{\sigma}(s), \boldsymbol{\chi})ds + (\nabla \cdot \boldsymbol{\chi}, u) &= 0 \quad \forall \boldsymbol{\chi} \in \mathcal{V}. \end{aligned} \tag{6.1.2}$$

The discrete formulation corresponding to (6.1.1a)-(6.1.1b) and (6.1.2) is given by Find $(u_h, \boldsymbol{\sigma}_h) \in Q_h^k \times V_h^k$ such that:

$$\begin{aligned} (u_{h,t}, \phi_h) - (\nabla \cdot \boldsymbol{\sigma}_h, \phi_h) &= 0 \quad \forall \phi_h \in Q_h^k, \\ a_h(\boldsymbol{\sigma}_h, \boldsymbol{\chi}_h) + (u_h, \nabla \cdot \boldsymbol{\chi}_h) - \int_0^t (b_0(\boldsymbol{x}; t, s)u_h(s), \nabla \cdot \boldsymbol{\chi}_h)ds & \\ \int_0^t (\nabla b_0(x; t, s)u_h(s), \mathbf{\Pi}_k^0 \boldsymbol{\chi}_h)ds &= 0 \quad \forall \boldsymbol{\chi}_h \in V_h^k. \end{aligned} \tag{6.1.3}$$

and

$$\begin{aligned} (u_{h,t}, \phi_h) - (\nabla \cdot \boldsymbol{\sigma}_h, \phi_h) &= 0 \quad \forall \phi_h \in Q_h^k, \\ a_h(\boldsymbol{\sigma}_h, \boldsymbol{\chi}_h) + \int_0^t \mathcal{K}_h(t, s; \boldsymbol{\sigma}_h(s), \boldsymbol{\chi}_h) ds + (\nabla \cdot \boldsymbol{\chi}_h, u_h) &= 0 \quad \forall \boldsymbol{\chi}_h \in V_h^k, \end{aligned} \tag{6.1.4}$$

The treatment of PIDEs with non-smooth initial data remains unexplored within the context of the VEM. While the literature has addressed the non-smooth analysis of the formulation (6.1.2) for the FEM [14], the discussion on non-smooth analysis for the formulation (6.1.1a)-(6.1.1b) is absent, even in the context of FEM.

1. For the formulation (6.1.1a)-(6.1.1b), our contributions are as follows:

   - To tackle the integral term, a projection with the memory term (referred to as mixed intermediate projection) is introduced, which helps in achieving the optimal convergence of order $O(h^2 t^{-1})$ for the unknown $u$.
   - By the repeated use of the integral operator and the properties of $L^2$- projection and Fortin operator, optimal error estimate is obtained for the unknown $\boldsymbol{\sigma}$ of order $O(ht^{-1})$.
   - Our primary contribution in this paper is to provide a comprehensive analysis of the VEM, and that holds true for FEM also, for the formulation presented in (6.1.1a). Significantly, this analysis addresses a void in the current literature, as there is a notable absence of any chapter, including those related to the FEM.

2. For the formulation (6.1.2), our contributions are as follows:

   - To tackle the integral term, a projection with the memory term (referred to as mixed Ritz-Volterra projection) is introduced, which helps in achieving the optimal convergence of order $O(h^2 t^{-1})$ for the unknown $u$.
   - By the repeated use of the integral operator and the properties of $L^2$- projection and Fortin operator, the optimal error estimate is obtained for the unknown $\boldsymbol{\sigma}$ of order $O(ht^{-1})$.

3. Moreover, we aim to provide a comprehensive VEM analysis for both formulations. Due to the presence of the term $(b_0(\boldsymbol{x}; t, s)u(s), \nabla \cdot \boldsymbol{\chi})$ within the integral, the analysis of this formulation becomes complicated.

## 6.2 Mixed Virtual Element Formulation without Resolvent Kernel

**Lemma 6.2.1.** *Let* $(u, \boldsymbol{\sigma})$ *be the solution of* (6.1.1a)-(6.1.1b)*and* $u_0 \in L^2(\mathcal{D})$*; Then the following estimates hold for* $t \in (0, T]$ *and* $j \in \{1, 2\}$*:*

1. $t\|\boldsymbol{\sigma}(t)\|^2 + \int_0^t s\|u_s(s)\|^2 ds \leq C\|u_0\|^2,$
2. $t^2\|u_t(t)\|^2 + \int_0^t s^2\|\boldsymbol{\sigma}_s(s)\|^2 ds \leq C\|u_0\|^2,$
3. $\|\widehat{u}(t)\|_2 + t\|u(t)\|_2 \leq C\|u_0\|,$
4. $\|u_t(t)\|_j \leq Ct^{-(1+\frac{j}{2})}\|u_0\|,$
5. $\|\nabla \cdot \widehat{\boldsymbol{\sigma}}\| \leq C\|u_0\|.$

*Proof.* Differentiate both (6.1.1a) and (6.1.1b) with respect to the variable $t$ to obtain the following:

$$(p_{tt}, \phi) - (\nabla \cdot \boldsymbol{\sigma}_t, \phi) = 0 \quad \forall \phi \in \mathcal{Q}, \tag{6.2.1a}$$

$$(\mu\boldsymbol{\sigma}_t, \boldsymbol{\chi}) + (u_t, \nabla \cdot \boldsymbol{\chi}) - \frac{d}{dt}\left(\int_0^t (b_0(\boldsymbol{x}; t, s)u(s), \nabla \cdot \boldsymbol{\chi}) ds\right) - \frac{d}{dt}\left(\int_0^t (\nabla b_0(x; t, s)u(s), \boldsymbol{\chi}) ds\right) = 0 \quad \forall \boldsymbol{\chi} \in \mathcal{V} \tag{6.2.1b}$$

Substitute $\phi = t\left(u_t - \frac{d}{dt}\left(\int_0^t (b_0(\boldsymbol{x}; t, s)u(s)) ds\right)\right)$ in (6.1.1a) and $\boldsymbol{\chi} = t\boldsymbol{\sigma}$ in (6.2.1b). Combine the resulting expressions and apply Young's inequality to complete the proof of 1. Substitute $\phi = t^2\left(u_t - \frac{d}{dt}\left(\int_0^t b_0(\boldsymbol{x}; t, s)u(s) ds\right)\right)$ and $\boldsymbol{\chi} = t^2\boldsymbol{\sigma}_t$ into (6.2.1a) and (6.2.1b), respectively. Combine the resulting expressions and apply Young's inequality to complete the proof of 2. For the proofs of 3, 4 and 5 please refer to [14]. □

Now, we proceed to establish optimal error estimates. This necessitates addressing the memory term, for which we introduce a novel projection known as the mixed intermediate projection.

### 6.2.1 Mixed Intermediate Projection

Define $\mathscr{I}^Q u$ as in (4.2.1a) and (4.2.1b). To estimate $\|u_t - \mathscr{I}^Q u_t\|$ and $\|u - \mathscr{I}^Q u\|$, our primary emphasis will be on deriving the estimate for $\|\widehat{u} - \widehat{\mathscr{I}^Q u}\|$.

**Theorem 6.2.1.** *For $u(t) \in H_0^1(\mathcal{D}) \cap H^2(\mathcal{D})$, where $t > 0$, and with an initial condition $u_0 \in L^2(\mathcal{D})$, there exists a positive constant $C$ independent of the parameter $h$, under which the following estimate holds true:*

$$\|\widehat{u}(t) - \widehat{\mathscr{I}^Q u}(t)\| + h\|\widehat{\boldsymbol{\sigma}}(t) - \widehat{\mathscr{I}^V \boldsymbol{\sigma}}(t)\| \leq Ch^2\|u_0\|. \tag{6.2.2}$$

*Proof.* Consider

$$\begin{aligned}
\boldsymbol{\Theta} &:= \boldsymbol{\sigma} - \mathscr{I}^V \boldsymbol{\sigma} = \boldsymbol{\sigma} - \boldsymbol{\Pi}_h^F \boldsymbol{\sigma} + \boldsymbol{\Psi}_h \quad \text{where } \boldsymbol{\Psi}_h := \boldsymbol{\Pi}_h^F \boldsymbol{\sigma} - \mathscr{I}^V \boldsymbol{\sigma} \in V_h^k, \\
\eta &:= u - \mathscr{I}^Q u = u - \Pi_k^0 u + \nu_h \quad \text{where } \nu_h := \Pi_k^0 u - \mathscr{I}^Q u \in Q_h^k.
\end{aligned}$$

First we prove that $\nabla \cdot \boldsymbol{\Psi}_h = 0$,

$$\begin{aligned}
\|\nabla \cdot \boldsymbol{\Psi}_h\|^2 &= (\nabla \cdot (\boldsymbol{\Pi}_h^F \boldsymbol{\sigma} - \mathscr{I}^V \boldsymbol{\sigma}), \nabla \cdot \boldsymbol{\Psi}_h) \\
&= (\nabla \cdot (\boldsymbol{\Pi}_h^F \boldsymbol{\sigma} - \boldsymbol{\sigma}), \nabla \cdot \boldsymbol{\Psi}_h) + (\nabla \cdot (\boldsymbol{\sigma} - \mathscr{I}^V \boldsymbol{\sigma}), \nabla \cdot \boldsymbol{\Psi}_h) \\
&= (\Pi_k^0(\nabla \cdot \boldsymbol{\sigma}) - \nabla \cdot \boldsymbol{\sigma}, \nabla \cdot \boldsymbol{\Psi}_h) \quad \text{Using (1.5.8) and (4.2.1b)} \\
&= 0.
\end{aligned}$$

In a similar way, we can prove that $\nabla \cdot \widehat{\boldsymbol{\Psi}}_h = 0$. Now, we define the dual problem:

$$-\nabla \cdot (a\nabla\xi) = \Lambda \quad \text{in } \mathcal{D}; \quad \xi = 0 \quad \text{on } \partial\mathcal{D}, \tag{6.2.3}$$

which satisfy the following regularity condition $\|\xi\|_2 \leq \|\Lambda\|$. Consider $\boldsymbol{\Phi} = a\nabla\xi$, then the mixed variational formulation corresponding to (6.2.3) is; Find $(\boldsymbol{\Phi}, \xi) \in \mathcal{V} \times \mathcal{Q}$ such that:

$$
\begin{aligned}
(\mu\boldsymbol{\Phi}, \boldsymbol{\chi}) + (\xi, \nabla\cdot\boldsymbol{\chi}) = 0 \quad \forall \boldsymbol{\chi} \in \mathcal{V}, \\
-(\nabla\cdot\boldsymbol{\Phi}, \phi) = (\Lambda, \phi) \quad \forall \phi \in \mathcal{Q}.
\end{aligned} \tag{6.2.4}
$$

Our aim is to determine $\|\widehat{\nu}_h\|$. To achieve this, we integrate and rewrite (4.2.1a) and (4.2.1b) as follows:

$$
\begin{aligned}
(\mu\widehat{\boldsymbol{\Theta}}, \boldsymbol{\chi}_h) + (\nabla\cdot\boldsymbol{\chi}_h, \widehat{\nu}_h) = \widehat{\mathcal{F}}(\boldsymbol{\chi}_h) \quad \forall \boldsymbol{\chi}_h \in V_h^k, \\
(\nabla\cdot\widehat{\boldsymbol{\Theta}}, \phi_h) = 0 \quad \forall q_h \in Q_h^k,
\end{aligned} \tag{6.2.5}
$$

where

$$
\begin{aligned}
\widehat{\mathcal{F}}(\boldsymbol{\chi}_h) = {} & a_h(\widehat{\mathscr{I}^V\boldsymbol{\sigma}}, \boldsymbol{\chi}_h) - (\mu\widehat{\mathscr{I}^V\boldsymbol{\sigma}}, \boldsymbol{\chi}_h) - \int_0^t\int_0^s \big(b_0(\boldsymbol{x};s,z)(\mathscr{I}^Q u - u)(z), \nabla\cdot\boldsymbol{\chi}_h\big)\,dzds \\
& - \int_0^t\int_0^s \big((\nabla b_0(\boldsymbol{x};s,z)\mathscr{I}^Q u(z), \boldsymbol{\Pi}_k^0\boldsymbol{\chi}_h) - (\nabla b_0(\boldsymbol{x};s,z)u(z), \boldsymbol{\chi}_h)\big)\,dzds.
\end{aligned}
$$

Consider the dual problem (6.2.4) along with $\Lambda = \widehat{\nu}_h$. Substitute $\phi = \widehat{\nu}_h$ into (6.2.4), leads to the following:

$$
\begin{aligned}
\|\widehat{\nu}_h\|^2 &= \big(\widehat{\nu}_h, -\nabla\cdot(\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}} a\nabla\xi)\big) \\
&= (\mu\widehat{\boldsymbol{\Theta}}, \boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi)) - \widehat{\mathcal{F}}\left(\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi)\right) \\
&= (\mu\widehat{\boldsymbol{\Theta}}, \boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi) - a\nabla\xi) - \widehat{\mathcal{F}}\left(\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi)\right) + (\nabla\cdot\widehat{\boldsymbol{\Theta}}, \Pi_k^0\xi - \xi) \\
&\leq C\left(h\|\widehat{\boldsymbol{\Theta}}\|\|\xi\|_2 + h^2\|\nabla\cdot\widehat{\boldsymbol{\Theta}}\|\|\xi\|_2 + \widehat{\mathcal{F}}\left(\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi)\right)\right),
\end{aligned} \tag{6.2.6}
$$

where $\widehat{\mathcal{F}}\left(\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi)\right)$ can be solved as:

$$
\begin{aligned}
& \widehat{\mathcal{F}}(\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi)) \\
&= \Big(a_h(\widehat{\mathscr{I}^V\boldsymbol{\sigma}} - \boldsymbol{\Pi}_k^0\widehat{\boldsymbol{\sigma}}, \boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi)) - (\mu(\widehat{\mathscr{I}^V\boldsymbol{\sigma}} - \boldsymbol{\Pi}_k^0\widehat{\boldsymbol{\sigma}}), \boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi)) \\
&\quad + a_h(\boldsymbol{\Pi}_k^0\widehat{\boldsymbol{\sigma}}, \boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi)) - (\mu\boldsymbol{\Pi}_k^0\widehat{\boldsymbol{\sigma}}, \boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi))\Big) \\
&\quad - \left(\int_0^t\int_0^s \big(b_0(\boldsymbol{x};s,z)(\mathscr{I}^Q u - u)(z), \nabla\cdot\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi)\big)\,dzds\right) \\
&\quad - \left(\int_0^t\int_0^s \big((\nabla b_0(\boldsymbol{x};s,z)\mathscr{I}^Q u(z), \boldsymbol{\Pi}_k^0(\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi)) - (\nabla b_0(\boldsymbol{x};s,z)u(z), \boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi))\big)\,dzds\right)
\end{aligned}
$$

$$= I + II + III, \tag{6.2.7}$$

where

$$I = \Big(a_h(\widehat{\mathscr{I}^V\boldsymbol{\sigma}} - \boldsymbol{\Pi}_k^0\widehat{\boldsymbol{\sigma}}, \boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi)) - (\mu(\widehat{\mathscr{I}^V\boldsymbol{\sigma}} - \boldsymbol{\Pi}_k^0\widehat{\boldsymbol{\sigma}}), \boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi))$$
$$+ a_h(\boldsymbol{\Pi}_k^0\widehat{\boldsymbol{\sigma}}, \boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi)) - (\mu\boldsymbol{\Pi}_k^0\widehat{\boldsymbol{\sigma}}, \boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi))\Big),$$
$$II = \left(\int_0^t\int_0^s \big(b_0(\boldsymbol{x};s,z)(\mathscr{I}^Q u - u)(z), \nabla\cdot\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi)\big)\,dzds\right),$$
$$III = \left(\int_0^t\int_0^s \big((\nabla b_0(\boldsymbol{x};s,z)\mathscr{I}^Q u(z), \boldsymbol{\Pi}_k^0(\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi)) - (\nabla b_0(\boldsymbol{x};s,z)u(z), \boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi))\big)\,dzds\right).$$

The initial expression $I$ on the right-hand side of equation (6.2.7) can be addressed by using (4.1.2) in the following manner:

$$|I| \le C(\|\widehat{\mathscr{I}^V\boldsymbol{\sigma}} - \boldsymbol{\Pi}_k^0\widehat{\boldsymbol{\sigma}}\| + h|\widehat{\boldsymbol{\sigma}}|_1)\|\boldsymbol{\Pi}_k^0(\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi)) - \boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi)\|$$
$$\le C\left(h\|\widehat{\boldsymbol{\Psi}}_h\| + h^2\|u_0\|\right)\|\xi\|_2 \quad \text{[Using Lemma 6.2.1]}. \tag{6.2.8}$$

For the second term on the right-hand side of equation (6.2.7), we proceed as follows:

$$|II| = \int_0^t \Big(\Big(b_0(\boldsymbol{x};s,s)(\widehat{\mathscr{I}^Q u} - \widehat{u})(s), \nabla\cdot\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi)\Big)$$
$$- \int_0^s \Big(b_{0z}(\boldsymbol{x};s,z)(\widehat{\mathscr{I}^Q u} - \widehat{u})(z), \nabla\cdot\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi)\Big)\,dz\Big)\,ds$$
$$\le C\int_0^t\left(h^2\|\widehat{u}(s)\|_2 + \|\widehat{\nu}_h(s)\| + \int_0^s \big(h^2\|\widehat{u}(z)\|_2 + \|\widehat{\nu}_h(z)\|\big)\,dz\right)ds\|\xi\|_2 \quad \text{[Using (1.5.7)]}$$
$$\le C\left(h^2 t\|u_0\| + \int_0^t \|\widehat{\nu}_h(s)\|ds\right)\|\xi\|_2. \tag{6.2.9}$$

The third term on the right-hand side of equation (6.2.7) can be resolved through integration by parts as follows:

$$|III|$$
$$= \int_0^t \Big((\nabla b_0(\boldsymbol{x};s,s)\widehat{\mathscr{I}^Q u}(s), \boldsymbol{\Pi}_k^0(\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi))) - (\nabla b_0(\boldsymbol{x};s,s)\widehat{u}(s), \boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi))\Big)\,ds$$

$$
\begin{aligned}
&-\int_0^t\int_0^s \Big( (\nabla b_{0z}(\boldsymbol{x};s,z)\widehat{\mathscr{I}^Q u}(z), \boldsymbol{\Pi}_k^0(\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi))) - (\nabla b_{0z}(\boldsymbol{x};s,z)\widehat{u}(z), \boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi))\Big)\, dzds \\
&= \int_0^t (\nabla b_0(\boldsymbol{x};s,s)(\widehat{\mathscr{I}^Q u}-\widehat{u})(s), \boldsymbol{\Pi}_k^0(\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi)))ds \\
&\quad -\int_0^t\int_0^s \Big(\nabla b_{0z}(\boldsymbol{x};s,z)(\widehat{\mathscr{I}^Q u}-\widehat{u})(z), \boldsymbol{\Pi}_k^0(\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi))\Big)\, dzds \\
&\quad +\int_0^t (\nabla b_0(\boldsymbol{x};s,s)\widehat{u}(s) - \boldsymbol{\Pi}_k^0(\nabla b_0(\boldsymbol{x};s,s)\widehat{u}(s)), \boldsymbol{\Pi}_k^0(\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi)) - \boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi))ds \\
&\quad -\int_0^t\int_0^s \Big(\nabla b_{0z}(\boldsymbol{x};s,z)\widehat{u}(z) - \boldsymbol{\Pi}_k^0(\nabla b_{0z}(\boldsymbol{x};s,z)\widehat{u}(z)), \boldsymbol{\Pi}_k^0(\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi)) - \boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi)\Big)\, dzds \\
&\le C_{\nabla b_0, \nabla b_{0s}} \int_0^t \left(\|\widehat{\nu}_h(s)\| + h^2|\widehat{u}(s)|_2\right) ds\|\xi\|_2 \\
&\le C\left(h^2 t\|u_0\| + \int_0^t \|\widehat{\nu}_h(s)\|ds\right)\|\xi\|_2 \quad \text{[Using Lemma 6.2.1]}. \qquad (6.2.10)
\end{aligned}
$$

Using (6.2.8)-(6.2.10) in (6.2.7), we arrive at the following:

$$
\widehat{\mathcal{F}}(\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi)) \le C\left(h\|\widehat{\boldsymbol{\Psi}}_h\| + h^2\|u_0\| + \int_0^t \|\widehat{\nu}_h(s)\|ds\right)\|\xi\|_2. \qquad (6.2.11)
$$

To derive an estimate for $\|\nabla\cdot\widehat{\boldsymbol{\Theta}}\|$, we proceed as follows:

$$
\begin{aligned}
\|\nabla\cdot\widehat{\boldsymbol{\Theta}}\|^2 &= (\nabla\cdot\widehat{\boldsymbol{\Theta}}, \nabla\cdot(\widehat{\boldsymbol{\sigma}} - \boldsymbol{\Pi}_h^F\widehat{\boldsymbol{\sigma}}) + \nabla\cdot(\boldsymbol{\Pi}_h^F\widehat{\boldsymbol{\sigma}} - \widehat{\mathscr{I}^V\boldsymbol{\sigma}})) \\
&\le \|\nabla\cdot\widehat{\boldsymbol{\Theta}}\|\|\nabla\cdot(\widehat{\boldsymbol{\sigma}} - \boldsymbol{\Pi}_h^F\widehat{\boldsymbol{\sigma}})\| \quad \text{[Using (6.2.5)]} \\
\|\nabla\cdot\widehat{\boldsymbol{\Theta}}\| &\le C\|\nabla\cdot\widehat{\boldsymbol{\sigma}}\| \\
&\le C\|u_0\| \quad \text{[Using Lemma 6.2.1]}. \qquad (6.2.12)
\end{aligned}
$$

Now, substitute (6.2.11), (6.2.12) in (6.2.6) , and use Grönwall's lemma to arrive at the following:

$$
\|\widehat{\nu}_h\| \le C\left(h^2\|u_0\| + h\|\widehat{\boldsymbol{\Theta}}\|\right). \qquad (6.2.13)
$$

From the definition of mixed intermediate projection (4.2.1a), we observe:

$$
a_h(\boldsymbol{\Psi}_h, \boldsymbol{\chi}_h) = [a_h(\boldsymbol{\Pi}_h^F\boldsymbol{\sigma}, \boldsymbol{\chi}_h) - (\mu\boldsymbol{\sigma}, \boldsymbol{\chi}_h)] + (\mathscr{I}^Q u - u, \nabla\cdot\boldsymbol{\chi}_h)
$$

$$+\int_0^t \big(b_0(\boldsymbol{x};t,s)(u-\mathscr{I}^Q u)(s), \nabla\cdot\boldsymbol{\chi}_h\big)\,ds$$
$$-\int_0^t \big[(\nabla b_0(x;t,s)\mathscr{I}^Q u(s), \boldsymbol{\Pi}_k^0\boldsymbol{\chi}_h) - (\nabla b_0(x;t,s)u(s), \boldsymbol{\chi}_h)\big]\,ds. \tag{6.2.14}$$

To estimate $\|\widehat{\boldsymbol{\Psi}}_h\|$, we proceed by integrating (6.2.14) as:

$$\begin{aligned}a_h(\widehat{\boldsymbol{\Psi}}_h, \boldsymbol{\chi}_h) &= [a_h(\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}\widehat{\boldsymbol{\sigma}}, \boldsymbol{\chi}_h) - (\mu\widehat{\boldsymbol{\sigma}}, \boldsymbol{\chi}_h)] + (\widehat{\mathscr{I}^Q u} - \widehat{u}, \nabla\cdot\boldsymbol{\chi}_h)\\ &+ \left(\int_0^t\int_0^s \big(b_0(\boldsymbol{x};s,z)(u-\mathscr{I}^Q u)(z), \nabla\cdot\boldsymbol{\chi}_h\big)\,dzds\right)\\ &- \left(\int_0^t\int_0^s \big[(\nabla b_0(x;s,z)\mathscr{I}^Q u(z), \boldsymbol{\Pi}_k^0\boldsymbol{\chi}_h) - (\nabla b_0(x;s,z)u(z), \boldsymbol{\chi}_h)\big]\,dzds\right).\end{aligned} \tag{6.2.15}$$

To deal with the first expression on the right-hand side of (6.2.15), we use Lemma 6.2.1 to arrive at the following:

$$\begin{aligned}&a_h(\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}\widehat{\boldsymbol{\sigma}}, \boldsymbol{\chi}_h) - (\mu\widehat{\boldsymbol{\sigma}}, \boldsymbol{\chi}_h)\\ &= a_h(\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}\widehat{\boldsymbol{\sigma}} - \boldsymbol{\Pi}_k^0\widehat{\boldsymbol{\sigma}}, \boldsymbol{\chi}_h) - (\mu(\widehat{\boldsymbol{\sigma}} - \boldsymbol{\Pi}_k^0\widehat{\boldsymbol{\sigma}}), \boldsymbol{\chi}_h) + a_h(\boldsymbol{\Pi}_k^0\widehat{\boldsymbol{\sigma}}, \boldsymbol{\chi}_h) - (\mu\boldsymbol{\Pi}_k^0\widehat{\boldsymbol{\sigma}}, \boldsymbol{\chi}_h)\\ &\le C(\|\widehat{\boldsymbol{\sigma}} - \boldsymbol{\Pi}_k^0\widehat{\boldsymbol{\sigma}}\| + \|\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}\widehat{\boldsymbol{\sigma}} - \boldsymbol{\Pi}_k^0\widehat{\boldsymbol{\sigma}}\| + h|\widehat{\boldsymbol{\sigma}}|_1)\|\boldsymbol{\chi}_h\| \quad \text{[Using (3.1.3) and Lemma 4.1.1]}\\ &\le Ch\|u_0\|\|\boldsymbol{\chi}_h\| \quad \text{[Using (1.5.7) and Lemma 6.2.1]}.\end{aligned} \tag{6.2.16}$$

The last term on the right-hand side of equation (6.2.15) can be resolved in a similar manner as in equation (6.2.10), resulting in:

$$\begin{aligned}&\int_0^t\int_0^s \big((\nabla b_0(x;s,z)\mathscr{I}^Q u(z), \boldsymbol{\Pi}_k^0\boldsymbol{\chi}_h) - (\nabla b_0(x;s,z)u(z), \boldsymbol{\chi}_h)\big)\,dzds\\ &\qquad \le C\left(h^2 t\|u_0\| + \int_0^t \|\widehat{\nu}_h(s)\|ds\right)\|\boldsymbol{\chi}_h\|.\end{aligned} \tag{6.2.17}$$

Substitute $\boldsymbol{\chi}_h = \widehat{\boldsymbol{\Psi}}_h$ into (6.2.15), then utilize (3.1.3), (6.2.16), (6.2.17), and $\nabla\cdot\widehat{\boldsymbol{\Psi}}_h = 0$ to obtain:

$$\alpha_1\|\widehat{\boldsymbol{\Psi}}_h\|^2 \le C\left(h\|u_0\| + h^2 t\|u_0\| + \int_0^t \|\widehat{\nu}_h(s)\|ds\right)\|\widehat{\boldsymbol{\Psi}}_h\|.$$

Use of the triangle inequality along with (1.5.9) in the above equation gives us the following:

$$\|\widehat{\mathbf{\Theta}}\| \leq C\left(h\|u_0\| + \int_0^t \|\widehat{\nu}_h(s)\|ds\right). \tag{6.2.18}$$

Put (6.2.13) in (6.2.18) followed by the use of Grönwall's lemma gives us the following:

$$\|\widehat{\mathbf{\Theta}}\| \leq Ch\|u_0\|. \tag{6.2.19}$$

Use of (6.2.19) in (6.2.13) yields:

$$\|\widehat{\nu}_h\| \leq Ch^2\|u_0\|. \tag{6.2.20}$$

□

**Theorem 6.2.2.** *For $u(t) \in H_0^1(\mathcal{D}) \cap H^2(\mathcal{D})$, where $t > 0$, and with an initial condition $u_0 \in L^2(\mathcal{D})$, there exists a positive constant $C$ independent of the parameter $h$, under which the following estimates hold:*

$$\|u(t) - \mathscr{I}^Q u(t)\| + h\|\boldsymbol{\sigma}(t) - \mathscr{I}^V \boldsymbol{\sigma}(t)\| \leq Ch^2 t^{-1}\|u_0\|. \tag{6.2.21}$$

*Proof.* For the proof of $\|\mathbf{\Psi}_h\|$, we proceed by using (6.2.14). For the first and last expression on the right-hand side of (6.2.14), we follow a similar approach as in (6.2.16) and (6.2.10) and arrive at the following:

$$a_h(\mathbf{\Pi}_h^F \boldsymbol{\sigma}, \boldsymbol{\chi}_h) - (\mu\boldsymbol{\sigma}, \boldsymbol{\chi}_h) \leq Cht^{-1}\|u_0\|\|\boldsymbol{\chi}_h\|, \tag{6.2.22}$$

$$\begin{aligned}\int_0^t [(\nabla b_0(\boldsymbol{x}; t, s)\mathscr{I}^Q u(s), \mathbf{\Pi}_k^0 \boldsymbol{\chi}_h) - (\nabla b_0(\boldsymbol{x}; t, s)u(s), \boldsymbol{\chi}_h)]ds \\ \leq C\left(h^2\|u_0\| + \|\widehat{\nu}_h\| + \int_0^t \|\widehat{\nu}_h(s)\|ds\right)\|\boldsymbol{\chi}_h\|.\end{aligned} \tag{6.2.23}$$

Substitute $\boldsymbol{\chi}_h = \mathbf{\Psi}_h$ into (6.2.14), then utilize (3.1.3), (6.2.22), (6.2.23), the boundedness of $b_0(\boldsymbol{x}; t, s)$, and the fact that $\nabla \cdot \mathbf{\Psi}_h = 0$, to obtain the following:

$$\|\mathbf{\Psi}_h\| \leq C\left(ht^{-1}\|u_0\| + h^2\|u_0\| + \|\widehat{\nu}_h\| + \int_0^t \|\widehat{\nu}_h(s)\|ds\right). \tag{6.2.24}$$

Use of the triangle inequality, (6.2.20) and (1.5.9) in the above equation to arrive at the following:

$$\|\boldsymbol{\Theta}\| \leq Cht^{-1}\|u_0\|. \tag{6.2.25}$$

For the estimate of $\|\nu_h\|$, we rewrite (4.2.1a) and (4.2.1b) as:

$$(\mu\boldsymbol{\Theta}, \boldsymbol{\chi}_h) + (\nabla\cdot\boldsymbol{\chi}_h, \nu_h) = \mathcal{F}(\boldsymbol{\chi}_h) \quad \forall\boldsymbol{\chi}_h \in V_h^k,$$
$$(\nabla\cdot\boldsymbol{\Theta}, \phi_h) = 0 \quad \forall q_h \in Q_h^k,$$

where

$$\mathcal{F}(\boldsymbol{\chi}_h) = a_h(\mathscr{I}^V\boldsymbol{\sigma}, \boldsymbol{\chi}_h) - (\mu\mathscr{I}^V\boldsymbol{\sigma}, \boldsymbol{\chi}_h) - \int_0^t (b_0(\boldsymbol{x};t,s)(\mathscr{I}^Q u - u)(s), \nabla\cdot\boldsymbol{\chi}_h)ds$$
$$- \int_0^t ((\nabla b_0(\boldsymbol{x};t,s)\mathscr{I}^Q u(s), \boldsymbol{\Pi}_k^0\boldsymbol{\chi}_h) - (\nabla b_0(\boldsymbol{x};t,s)u(s), \boldsymbol{\chi}_h))ds.$$

Now, consider the dual problem (6.2.3) with $\Lambda = \nu_h$ and then put $\phi = \nu_h$ in (6.2.4), to arrive at the following:

$$\begin{aligned}\|\nu_h\|^2 &= (\mu\boldsymbol{\Theta}, \boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi)) - \mathcal{F}(\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi))\\ &\leq C\left(h\|\boldsymbol{\Theta}\|\|\xi\|_2 + h^2\|\nabla\cdot\boldsymbol{\Theta}\|\|\xi\|_2 + \mathcal{F}(\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi))\right).\end{aligned} \tag{6.2.26}$$

The term $\mathcal{F}(\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi))$ can be solved as:

$$\begin{aligned}&\mathcal{F}(\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi))\\ &= \big(a_h(\mathscr{I}^V\boldsymbol{\sigma} - \boldsymbol{\Pi}_k^0\boldsymbol{\sigma}, \boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi)) - (\mu(\mathscr{I}^V\boldsymbol{\sigma} - \boldsymbol{\Pi}_k^0\boldsymbol{\sigma}), \boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi))\\ &\quad + a_h(\boldsymbol{\Pi}_k^0\boldsymbol{\sigma}, \boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi)) - (\mu\boldsymbol{\Pi}_k^0\boldsymbol{\sigma}, \boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi))\big)\\ &\quad - \left(\int_0^t (b_0(\boldsymbol{x};t,s)(\mathscr{I}^Q u - u)(s), \nabla\cdot\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi))ds\right)\\ &\quad - \left(\int_0^t \big((\nabla b_0(\boldsymbol{x};t,s)\mathscr{I}^Q u(s), \boldsymbol{\Pi}_k^0(\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi)) - (\nabla b_0(\boldsymbol{x};t,s)u(s), \boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi))\big)\,ds\right)\\ &= IV + V + VI,\end{aligned} \tag{6.2.27}$$

where

$$\begin{aligned}
IV &= \big(a_h(\mathscr{I}^V\boldsymbol{\sigma} - \boldsymbol{\Pi}_k^0\boldsymbol{\sigma}, \boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi)) - (\mu(\mathscr{I}^V\boldsymbol{\sigma} - \boldsymbol{\Pi}_k^0\boldsymbol{\sigma}), \boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi)) \\
&\quad + a_h(\boldsymbol{\Pi}_k^0\boldsymbol{\sigma}, \boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi)) - (\mu\boldsymbol{\Pi}_k^0\boldsymbol{\sigma}, \boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi))\big)\,, \\
V &= \left(\int_0^t (b_0(\boldsymbol{x};t,s)(\mathscr{I}^Q u - u)(s), \nabla\cdot\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi))ds\right), \\
VI &= \left(\int_0^t \big((\nabla b_0(\boldsymbol{x};t,s)\mathscr{I}^Q u(s), \boldsymbol{\Pi}_k^0(\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi)) - (\nabla b_0(\boldsymbol{x};t,s)u(s), \boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi))\big)\,ds\right).
\end{aligned}$$

The initial expression on the right-hand side of equation (6.2.27) can be addressed by using (4.1.2) in the following manner:

$$\begin{aligned}
|IV| &\le C\left(\|\mathscr{I}^V\boldsymbol{\sigma} - \boldsymbol{\Pi}_k^0\boldsymbol{\sigma}\| + h|\boldsymbol{\sigma}|_1\right)\|\boldsymbol{\Pi}_k^0(\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi)) - \boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi)\| \\
&\le Ch^2t^{-1}\|u_0\|\|\xi\|_2 \quad [\text{Using (6.2.24) and Lemma 6.2.1}]. \qquad (6.2.28)
\end{aligned}$$

The second and the third term on the right-hand side of the (6.2.27) can be solved by using integration by parts and proceed in a similar manner as (6.2.9) and (6.2.10):

$$|V| \le Ch^2\|u_0\|\|\xi\|_2, \qquad (6.2.29)$$

$$|VI| \le Ch^2\|u_0\|\|\xi\|_2. \qquad (6.2.30)$$

For the estimate of $\nabla\cdot\boldsymbol{\Theta}$, we proceed in a similar way as (6.2.12) and use Lemma 6.2.1 to arrive at the following:

$$\|\nabla\cdot\boldsymbol{\Theta}\| \le Ct^{-1}\|u_0\|. \qquad (6.2.31)$$

Now, using (6.2.27)-(6.2.31) in (6.2.26) to achieve:

$$\|\nu_h\| \le Ch^2t^{-1}\|u_0\|.$$

Use of the triangle inequality and (1.5.7) completes the proof:

$$\|\eta\| \le Ch^2t^{-1}\|u_0\|.$$

□

**Theorem 6.2.3.** *For $u(t) \in H_0^1(\mathcal{D}) \cap H^2(\mathcal{D})$, $t > 0$, with an initial condition $u_0 \in L^2(\mathcal{D})$, then there exists a positive constant $C$ that is not dependent on the parameter $h$, and under these conditions, the following estimates are valid:*

$$\|u_t(t) - \mathscr{I}^Q u_t(t)\| + h\|\boldsymbol{\sigma}_t(t) - \mathscr{I}^V \boldsymbol{\sigma}_t(t)\| \leq Ch^2 t^{-2}\|u_0\|. \tag{6.2.32}$$

*Proof.* For the proof of (6.2.32), we differentiate (6.2.14) to achieve:

$$\begin{aligned} a_h(\boldsymbol{\Psi}_{ht}, \boldsymbol{\chi}_h) = &\big(a_h(\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}} \boldsymbol{\sigma}_t, \boldsymbol{\chi}_h) - (\mu\boldsymbol{\sigma}_t, \boldsymbol{\chi}_h)\big) + (b_0(\boldsymbol{x}; t, t)(u - \mathscr{I}^Q u)(t), \nabla \cdot \boldsymbol{\chi}_h) \\ &+ (\mathscr{I}^Q u_t - u_t, \nabla \cdot \boldsymbol{\chi}_h) - \int_0^t ((b_{0t}(\boldsymbol{x}; t, s)(u - \mathscr{I}^Q u)(s), \nabla \cdot \boldsymbol{\chi}_h) ds \\ &- \big((\nabla b_0(x; t, t)\mathscr{I}^Q u(t), \boldsymbol{\Pi}_k^0 \boldsymbol{\chi}_h) - (\nabla b_0(x; t, t)u(t), \boldsymbol{\chi}_h)\big) \\ &+ \int_0^t \big((\nabla b_{0t}(x; t, s)\mathscr{I}^Q u(s), \boldsymbol{\Pi}_k^0 \boldsymbol{\chi}_h) - (\nabla b_{0t}(x; t, s)u(s), \boldsymbol{\chi}_h)\big)\, ds. \end{aligned}$$

Now, we follow the similar arguments as in Theorem 6.2.2 to achieve:

$$\|\boldsymbol{\Theta}\| \leq Cht^{-2}\|u_0\|.$$

For the estimate of $\nu_{h,t}$, differentiate (4.2.1a)-(4.2.1b) to obtain:

$$\begin{aligned} (\mu\boldsymbol{\Theta}_t, \boldsymbol{\chi}_h) + (\nabla \cdot \boldsymbol{\chi}_h, \nu_{ht}) &= \mathcal{F}_t(\boldsymbol{\chi}_h) \quad \forall \boldsymbol{\chi}_h \in V_h^k, \\ (\nabla \cdot \boldsymbol{\Theta}_t, \phi_h) &= 0 \quad \forall q_h \in Q_h^k, \end{aligned}$$

where

$$\begin{aligned} \mathcal{F}_t(\boldsymbol{\chi}_h) = &a_h(\mathscr{I}^V \boldsymbol{\sigma}_t, \boldsymbol{\chi}_h) - (\mu\mathscr{I}^V \boldsymbol{\sigma}_t, \boldsymbol{\chi}_h) - (b_0(\boldsymbol{x}; t, t)(\mathscr{I}^Q u - u)(t), \nabla \cdot \boldsymbol{\chi}_h) \\ &+ \int_0^t (b_{0t}(\boldsymbol{x}; t, s)(\mathscr{I}^Q u - u)(s), \nabla \cdot \boldsymbol{\chi}_h) ds \\ &- \big((\nabla b_0(\boldsymbol{x}; t, t)\mathscr{I}^Q u(t), \boldsymbol{\Pi}_k^0 \boldsymbol{\chi}_h) - (\nabla b_0(\boldsymbol{x}; t, t)u(t), \boldsymbol{\chi}_h)\big) \end{aligned}$$

$$+ \int_0^t \big( (\nabla b_{0t}(\boldsymbol{x}; t, s) \mathscr{I}^Q u(s), \boldsymbol{\Pi}_k^0 \boldsymbol{\chi}_h) - (\nabla b_{0t}(\boldsymbol{x}; t, s) u(s), \boldsymbol{\chi}_h) \big) \, ds.$$

Consider the dual problem (6.2.3) with $\Lambda = \nu_{h,t}$ and proceed as in Theorem 6.2.2 to complete the proof of (6.2.32). □

### 6.2.2 A *prior* Error Estimates

Writing $u - u_h = \big(u - \mathscr{I}^Q u\big) + \big(\mathscr{I}^Q u - u_h\big) =: \eta + \eta_h$ and $\boldsymbol{\sigma} - \boldsymbol{\sigma}_h = \big(\boldsymbol{\sigma} - \mathscr{I}^V \boldsymbol{\sigma}\big) + \big(\mathscr{I}^V \boldsymbol{\sigma} - \boldsymbol{\sigma}_h\big) =: \boldsymbol{\Theta} + \boldsymbol{\Theta}_h$. Since, we already have the estimates for $\|\eta\|$ and $\|\boldsymbol{\Theta}\|$, our goal is to determine $\|\eta_h\|$ and $\|\boldsymbol{\Theta}_h\|$. To proceed, we use (6.1.1a)-(6.1.1b)and (6.1.3) to obtain:

$$(\eta_{h,t}, \phi_h) - (\nabla \cdot \boldsymbol{\Theta}_h, \phi_h) = -(\eta_t, \phi_h), \tag{6.2.33a}$$

$$\begin{aligned} a_h(\boldsymbol{\Theta}_h, \boldsymbol{\chi}_h) + (\eta_h, \nabla \cdot \boldsymbol{\chi}_h) &= \int_0^t (b_0(\boldsymbol{x}; t, s) \eta_h(s), \nabla \cdot \boldsymbol{\chi}_h) ds \\ &\quad + \int_0^t (\nabla b_0(\boldsymbol{x}; t, s) \eta_h(s), \boldsymbol{\Pi}_k^0 \boldsymbol{\chi}_h) ds. \end{aligned} \tag{6.2.33b}$$

Integrate (6.2.33a) and (6.2.33b) from 0 to $t$ to obtain:

$$(\eta_h, \phi_h) - (\nabla \cdot \widehat{\boldsymbol{\Theta}}_h, \phi_h) = -(\eta, \phi_h), \tag{6.2.34a}$$

$$\begin{aligned} & a_h(\widehat{\boldsymbol{\Theta}}_h, \boldsymbol{\chi}_h) + (\widehat{\eta}_h, \nabla \cdot \boldsymbol{\chi}_h) - \Big( (b_0(\boldsymbol{x}; t, t) \widehat{\widehat{\eta}}_h, \nabla \cdot \boldsymbol{\chi}_h) + (\nabla b_0(\boldsymbol{x}; t, t) \widehat{\widehat{\eta}}_h, \boldsymbol{\Pi}_k^0 \boldsymbol{\chi}_h) \Big) \\ & + 2 \int_0^t \Big( (b_{0s}(\boldsymbol{x}; s, s) \widehat{\widehat{\eta}}_h(s), \nabla \cdot \boldsymbol{\chi}_h) + (\nabla b_{0s}(\boldsymbol{x}; s, s) \widehat{\widehat{\eta}}_h(s), \boldsymbol{\Pi}_k^0 \boldsymbol{\chi}_h) \Big) \, ds \\ & - \int_0^t \int_0^s \Big( (b_{0zz}(\boldsymbol{x}; s, z) \widehat{\widehat{\eta}}_h(z), \nabla \cdot \boldsymbol{\chi}_h) + (\nabla b_{0zz}(\boldsymbol{x}; s, z) \widehat{\widehat{\eta}}_h(z), \boldsymbol{\Pi}_k^0 \boldsymbol{\chi}_h) \Big) \, dz ds = 0. \end{aligned} \tag{6.2.34b}$$

Again integrating (6.2.34a) and (6.2.34b) to obtain:

$$(\widehat{\eta}_h, \phi_h) - (\nabla \cdot \widehat{\widehat{\boldsymbol{\Theta}}}_h, \phi_h) = -(\widehat{\eta}, \phi_h), \tag{6.2.35a}$$

$$a_h(\widehat{\widehat{\boldsymbol{\Theta}}}_h, \boldsymbol{\chi}_h) + (\widehat{\widehat{\eta}}_h, \nabla \cdot \boldsymbol{\chi}_h) - \int_0^t \Big( (b_0(\boldsymbol{x}; s, s) \widehat{\widehat{\eta}}_h(s), \nabla \cdot \boldsymbol{\chi}_h) + (\nabla b_0(\boldsymbol{x}; s, s) \widehat{\widehat{\eta}}_h(s), \boldsymbol{\Pi}_k^0 \boldsymbol{\chi}_h) \Big) \, ds$$

$$+ 2\int_0^t \int_0^s \Big( (b_{0z}(\boldsymbol{x};z,z)\widehat{\widehat{\eta}}_h(z), \nabla\cdot\boldsymbol{\chi}_h) + (\nabla b_{0z}(\boldsymbol{x};z,z)\widehat{\widehat{\eta}}_h(z), \boldsymbol{\Pi}_k^0\boldsymbol{\chi}_h)\Big)\,dzds$$
$$- \int_0^t\int_0^s\int_0^z \Big( (b_{0z'z'}(\boldsymbol{x};z,z')\widehat{\widehat{\eta}}_h(z'), \nabla\cdot\boldsymbol{\chi}_h) + (\nabla b_{0z'z'}(\boldsymbol{x};z,z')\widehat{\widehat{\eta}}_h(z'), \boldsymbol{\Pi}_k^0\boldsymbol{\chi}_h)\Big)\,dz'dzds = 0. \tag{6.2.35b}$$

### 6.2.3 Error Estimates for $u - u_h$

First, we prove some lemmas that will be used subsequently to prove $\|u-u_h\|$:

**Lemma 6.2.2.** *Let $\widehat{\eta}_h$ and $\widehat{\widehat{\eta}}_h$ satisfy (6.2.34b) and (6.2.35b), then there exists a positive constant $C$, such that the following estimates hold true:*

$$\|\widehat{\widehat{\eta}}_h(t)\|^2 + \int_0^t \|\widehat{\widehat{\boldsymbol{\Theta}}}_h(s)\|^2 ds \le Ch^4 t\|u_0\|^2, \tag{6.2.36}$$
$$\int_0^t \|\widehat{\eta}_h(s)\|^2 ds + \|\widehat{\widehat{\boldsymbol{\Theta}}}_h(t)\|^2 \le Ch^4 t\|u_0\|^2. \tag{6.2.37}$$

*Proof.* Put $\boldsymbol{\chi}_h = \widehat{\widehat{\boldsymbol{\Theta}}}_h$ and $\phi_h = \widehat{\widehat{\eta}}_h - \int_0^t \Pi_k^0(b_0(\boldsymbol{x};s,s)\widehat{\widehat{\eta}}_h(s))ds$ $+2\int_0^t\int_0^s \Pi_k^0(b_{0z}(\boldsymbol{x};z,z)\widehat{\widehat{\eta}}_h(z))dzds - \int_0^t\int_0^s\int_0^z \Pi_k^0(b_{0z'z'}(\boldsymbol{x};z,z')\widehat{\widehat{\eta}}_h(z'))dz'dzds$ in (6.2.35b) and (6.2.35a) and to arrive at the following:

$$\begin{aligned}
&(\widehat{\eta}_h, \widehat{\widehat{\eta}}_h) + a_h(\widehat{\widehat{\boldsymbol{\Theta}}}_h, \widehat{\widehat{\boldsymbol{\Theta}}}_h)\\
&= -\Big(\widehat{\eta}, \widehat{\widehat{\eta}}_h - \int_0^t \Pi_k^0(b_0(\boldsymbol{x};s,s)\widehat{\widehat{\eta}}_h(s))ds + 2\int_0^t\int_0^s \Pi_k^0(b_{0z}(\boldsymbol{x};z,z)\widehat{\widehat{\eta}}_h(z))dzds\Big)\\
&\quad + \Big(\widehat{\eta}, \int_0^t\int_0^s\int_0^z \Pi_k^0(b_{0z'z'}(\boldsymbol{x};z,z')\widehat{\widehat{\eta}}_h(z'))dz'dzds\Big)\\
&\quad + \Big(\widehat{\eta}_h, \int_0^t \Pi_k^0(b_0(\boldsymbol{x};s,s)\widehat{\widehat{\eta}}_h(s))ds - 2\int_0^t\int_0^s \Pi_k^0(b_{0z}(\boldsymbol{x};z,z)\widehat{\widehat{\eta}}_h(z))dzds\Big)\\
&\quad + \Big(\widehat{\eta}_h, \int_0^t\int_0^s\int_0^z \Pi_k^0(b_{0,z',z'}(\boldsymbol{x};z,z')\widehat{\widehat{\eta}}_h(z'))dz'dzds\Big)\\
&\quad + \int_0^t\int_0^s\int_0^z (\nabla b_{0z'z'}(\boldsymbol{x};z,z')\widehat{\widehat{\eta}}_h(z'), \boldsymbol{\Pi}_k^0\widehat{\widehat{\boldsymbol{\Theta}}}_h)dz'dzds\\
&\quad + \int_0^t (\nabla b_0(\boldsymbol{x};s,s)\widehat{\widehat{\eta}}_h(s), \boldsymbol{\Pi}_k^0\widehat{\widehat{\boldsymbol{\Theta}}}_h)ds - 2\int_0^t\int_0^s (\nabla b_{0z}(\boldsymbol{x};z,z)\widehat{\widehat{\eta}}_h(z), \boldsymbol{\Pi}_k^0\widehat{\widehat{\boldsymbol{\Theta}}}_h)dzds.
\end{aligned}$$

Employ (3.1.3), apply Young's inequality, consider the boundedness of $t$, and subsequently utilize the kickback argument to rewrite the above equation as:

$$\frac{d}{dt}\|\widehat{\widehat{\eta}}_h\|^2 + \|\widehat{\widehat{\mathbf{\Theta}}}_h\|^2 \leq C\left(\|\widehat{\eta}\|^2 + \|\widehat{\eta}_h\|^2 + \|\widehat{\widehat{\eta}}_h\|^2 + \int_0^t \|\widehat{\widehat{\eta}}_h(s)\|^2 ds\right).$$

By integrating the above equation from 0 to $t$ and then applying Grönwall's lemma, we obtain the following:

$$\|\widehat{\widehat{\eta}}_h\|^2 + \int_0^t \|\widehat{\widehat{\mathbf{\Theta}}}_h(s)\|^2 ds \leq C\int_0^t \left(\|\widehat{\eta}(s)\|^2 + \|\widehat{\eta}_h(s)\|^2\right) ds.$$

Use of (6.2.2) gives us the following:

$$\|\widehat{\widehat{\eta}}_h\|^2 + \int_0^t \|\widehat{\widehat{\mathbf{\Theta}}}_h(s)\|^2 ds \leq C\left(h^4 t\|u_0\|^2 + \int_0^t \|\widehat{\eta}_h(s)\|^2\right) ds. \tag{6.2.38}$$

Put $\phi_h = \widehat{\eta}_h - \Pi_k^0(b_0(\boldsymbol{x};t,t)\widehat{\widehat{\eta}}_h) + 2\int_0^t \Pi_k^0(b_{0s}(\boldsymbol{x};s,s)\widehat{\widehat{\eta}}_h(s))ds$ $-\int_0^t\int_0^s \Pi_k^0(b_{0zz}(\boldsymbol{x};s,z)\widehat{\widehat{\eta}}_h(z))dzds$ and $\boldsymbol{\chi}_h = \widehat{\widehat{\mathbf{\Theta}}}_h$ in (6.2.35a) and (6.2.34b) to arrive at the following:

$$\begin{aligned}
&\|\widehat{\eta}_h\|^2 + a_h(\widehat{\mathbf{\Theta}}_h, \widehat{\widehat{\mathbf{\Theta}}}_h) \\
&= (\widehat{\eta}_h, \Pi_k^0(b_0(\boldsymbol{x};t,t)\widehat{\widehat{\eta}}_h) - 2\int_0^t \Pi_k^0(b_{0s}(\boldsymbol{x};s,s)\widehat{\widehat{\eta}}_h(s))ds + \int_0^t\int_0^s \Pi_k^0(b_{0zz}(\boldsymbol{x};s,z)\widehat{\widehat{\eta}}_h(z))dzds) \\
&\quad - (\widehat{\eta}, \widehat{\eta}_h - \Pi_k^0(b_0(\boldsymbol{x};t,t)\widehat{\widehat{\eta}}_h) + 2\int_0^t \Pi_k^0(b_{0s}(\boldsymbol{x};s,s)\widehat{\widehat{\eta}}_h(s))ds \\
&\quad - \int_0^t\int_0^s \Pi_k^0(b_{0zz}(\boldsymbol{x};s,z)\widehat{\widehat{\eta}}_h(z))dzds) + (\nabla b_0(\boldsymbol{x};t,t)\widehat{\widehat{\eta}}_h, \mathbf{\Pi}_k^0\widehat{\widehat{\mathbf{\Theta}}}_h) \\
&\quad - 2\int_0^t (\nabla b_{0s}(\boldsymbol{x};s,s)\widehat{\widehat{\eta}}_h(s), \mathbf{\Pi}_k^0\widehat{\widehat{\mathbf{\Theta}}}_h)ds + \int_0^t\int_0^s (\nabla b_{0zz}(\boldsymbol{x};s,z)\widehat{\widehat{\eta}}_h(z), \mathbf{\Pi}_k^0\widehat{\widehat{\mathbf{\Theta}}}_h)dzds.
\end{aligned}$$

Employ (3.1.3), apply Young's inequality, consider the boundedness of $t$, and subsequently utilize the kickback argument to rewrite the above equation as:

$$\|\widehat{\eta}_h\|^2 + \frac{\alpha_1}{2}\frac{d}{dt}\|\widehat{\widehat{\mathbf{\Theta}}}_h\|^2 \leq C\left(\|\widehat{\eta}\|^2 + \|\widehat{\widehat{\eta}}_h\|^2 + \|\widehat{\widehat{\mathbf{\Theta}}}_h\|^2 + \int_0^t \|\widehat{\widehat{\eta}}_h(s)\|^2 ds\right).$$

By integrating the above equation from 0 to $t$ and then applying Grönwall's lemma, we obtain the following:

$$\int_0^t \|\widehat{\eta}_h(s)\|^2 ds + \|\widehat{\widehat{\boldsymbol{\Theta}}}_h(s)\|^2 \leq C \int_0^t \left( \|\widehat{\eta}(s)\|^2 + \|\widehat{\widehat{\eta}}_h(s)\|^2 \right) ds.$$

Use of (6.2.2) and (6.2.38) followed by Grönwall's lemma completes the proof of (6.2.37). Use of (6.2.37) in (6.2.38) completes the proof of (6.2.36). □

**Lemma 6.2.3.** *Let $\eta_h$ and $\widehat{\boldsymbol{\Theta}}_h$ satisfy (6.2.33b) and (6.2.34a), then there exists a positive constant $C$, such that the following estimates hold true:*

$$t^2\|\widehat{\boldsymbol{\Theta}}_h(t)\|^2 + t\|\widehat{\eta}_h(t)\|^2 + \int_0^t s^2\|\eta_h(s)\|^2 ds \leq Ch^4 t\|u_0\|^2.$$

*Proof.* Put $\phi_h = t^2 \left( \eta_h - \int_0^t \Pi_k^0(b_0(\boldsymbol{x};t,s)\eta_h(s))ds \right)$ and $\boldsymbol{\chi}_h = t^2\widehat{\boldsymbol{\Theta}}_h$ in (6.2.34a) and (6.2.33b) to arrive at the following:

$$\begin{aligned}
&t^2\|\eta_h\|^2 + \frac{\alpha_1}{2}\frac{d}{dt}t^2\|\widehat{\boldsymbol{\Theta}}_h\|^2 \\
&\leq -t^2 \left( \eta, \eta_h - \int_0^t \Pi_k^0(b_0(\boldsymbol{x};t,s)\eta_h(s))ds \right) + t^2 \left( \eta_h, \int_0^t \Pi_k^0(b_0(\boldsymbol{x};t,s)\eta_h(s))ds \right) \\
&\quad + ta_h(\widehat{\boldsymbol{\Theta}}_h, \widehat{\boldsymbol{\Theta}}_h) + t^2 \int_0^t (\nabla b_0(\boldsymbol{x};t,s)\eta_h(s), \boldsymbol{\Pi}_k^0\widehat{\boldsymbol{\Theta}}_h)ds.
\end{aligned}$$

Using (3.1.3), Young's inequality and integration by parts gives us the following:

$$\begin{aligned}
&t^2\|\eta_h\|^2 + \frac{\alpha_1}{2}\frac{d}{dt}t^2\|\widehat{\boldsymbol{\Theta}}_h\|^2 \\
&\leq ta_h(\widehat{\boldsymbol{\Theta}}_h, \widehat{\boldsymbol{\Theta}}_h) - t^2 \left( \eta, \eta_h - \Pi_k^0(b_0(\boldsymbol{x};t,t)\widehat{\eta}_h(t)) + \int_0^t \Pi_k^0(b_{0s}(\boldsymbol{x};t,s)\widehat{\eta}_h(s))ds \right) \\
&\quad + t^2 \left( \eta_h, \Pi_k^0(b_0(\boldsymbol{x};t,t)\widehat{\eta}_h(t)) - \int_0^t \Pi_k^0(b_{0s}(\boldsymbol{x};t,s)\widehat{\eta}_h(s))ds \right) \\
&\quad + t^2 \left( \nabla b_0(\boldsymbol{x};t,t)\widehat{\eta}_h(t) - \int_0^t \nabla b_{0s}(\boldsymbol{x};t,s)\widehat{\eta}_h(s)ds, \boldsymbol{\Pi}_k^0\widehat{\boldsymbol{\Theta}}_h \right).
\end{aligned} \tag{6.2.39}$$

Now, put $\phi_h = t\left(\widehat{\eta}_h - \Pi_k^0(b_0(\boldsymbol{x};t,t)\widehat{\widehat{\eta}}_h) + 2\int_0^t \Pi_k^0(b_{0s}(\boldsymbol{x};s,s)\widehat{\widehat{\eta}}_h(s))ds\right.$ $\left.-\int_0^t\int_0^s \Pi_k^0(b_{0zz}(\boldsymbol{x};s,z)\widehat{\widehat{\eta}}_h(z))dzds\right)$ and $\boldsymbol{\chi}_h = t\widehat{\boldsymbol{\Theta}}_h$ in (6.2.34a) and (6.2.34b) and use the fact $\frac{d}{dt}t(\widehat{\eta}_h,\widehat{\eta}_h) = (\widehat{\eta}_h,\widehat{\eta}_h) + 2t(\eta_h,\widehat{\eta}_h)$ to arrive at the following:

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}t\|\widehat{\eta}_h\|^2 - \frac{1}{2}\|\widehat{\eta}_h\|^2 + ta_h(\widehat{\boldsymbol{\Theta}}_h,\widehat{\boldsymbol{\Theta}}_h)\\
&\leq t(\nabla b_0(\boldsymbol{x};t,t)\widehat{\widehat{\eta}}_h, \boldsymbol{\Pi}_k^0\widehat{\boldsymbol{\Theta}}_h) - 2t\int_0^t(\nabla b_{0s}(\boldsymbol{x};s,s)\widehat{\widehat{\eta}}_h(s), \boldsymbol{\Pi}_k^0\widehat{\boldsymbol{\Theta}}_h)ds\\
&\quad + t\int_0^t\int_0^s(\nabla b_{0zz}(\boldsymbol{x};s,z)\widehat{\widehat{\eta}}_h(z), \boldsymbol{\Pi}_k^0\widehat{\boldsymbol{\Theta}}_h)dzds + t\left(\eta_h, \int_0^t\int_0^s \Pi_k^0(b_{0zz}(\boldsymbol{x};s,z)\widehat{\widehat{\eta}}_h(z))dzds\right)\\
&\quad + t\left(\eta_h, \Pi_k^0(b_0(\boldsymbol{x};t,t)\widehat{\widehat{\eta}}_h) - 2\int_0^t \Pi_k^0(b_{0s}(\boldsymbol{x};s,s)\widehat{\widehat{\eta}}_h(s))ds\right)\\
&\quad - t\left(\eta, \widehat{\eta}_h - \Pi_k^0(b_0(\boldsymbol{x};t,t)\widehat{\widehat{\eta}}_h) + 2\int_0^t \Pi_k^0(b_{0s}(\boldsymbol{x};s,s)\widehat{\widehat{\eta}}_h(s))ds\right)\\
&\quad + t\left(\eta, \int_0^t\int_0^s \Pi_k^0(b_{0zz}(\boldsymbol{x};s,z)\widehat{\widehat{\eta}}_h(z))dzds\right). \qquad (6.2.40)
\end{aligned}
$$

Add (6.2.39) and (6.2.40) to arrive at:

$$
\begin{aligned}
&\frac{\alpha_1}{2}\frac{d}{dt}t^2\|\widehat{\boldsymbol{\Theta}}_h\|^2 + \frac{1}{2}\frac{d}{dt}t\|\widehat{\eta}_h\|^2 + t^2\|\eta_h\|^2\\
&\leq \frac{1}{2}\|\widehat{\eta}_h\|^2 + t\alpha_2\|\widehat{\boldsymbol{\Theta}}_h\|^2 - t^2\left(\eta, \eta_h - \Pi_k^0(b_0(\boldsymbol{x};t,t)\widehat{\eta}_h(t)) + \int_0^t \Pi_k^0(b_{0s}(\boldsymbol{x};t,s)\widehat{\eta}_h(s))ds\right)\\
&\quad + t^2\left(\eta_h, \Pi_k^0(b_0(\boldsymbol{x};t,t)\widehat{\eta}_h(t)) - \int_0^t \Pi_k^0(b_{0s}(\boldsymbol{x};t,s)\widehat{\eta}_h(s))ds\right)\\
&\quad + t^2\left(\nabla b_0(\boldsymbol{x};t,t)\widehat{\eta}_h(t) - \int_0^t \nabla b_{0s}(\boldsymbol{x};t,s)\widehat{\eta}_h(s)ds, \boldsymbol{\Pi}_k^0\widehat{\boldsymbol{\Theta}}_h\right)\\
&\quad + t(\nabla b_0(\boldsymbol{x};t,t)\widehat{\widehat{\eta}}_h, \boldsymbol{\Pi}_k^0\widehat{\boldsymbol{\Theta}}_h) - 2t\int_0^t(\nabla b_{0s}(\boldsymbol{x};s,s)\widehat{\widehat{\eta}}_h(s), \boldsymbol{\Pi}_k^0\widehat{\boldsymbol{\Theta}}_h)ds\\
&\quad + t\int_0^t\int_0^s(\nabla b_{0zz}(\boldsymbol{x};s,z)\widehat{\widehat{\eta}}_h(z), \boldsymbol{\Pi}_k^0\widehat{\boldsymbol{\Theta}}_h)dzds\\
&\quad + t\left(\eta_h, \int_0^t\int_0^s \Pi_k^0(b_{0zz}(\boldsymbol{x};s,z)\widehat{\widehat{\eta}}_h(z))dzds\right)\\
&\quad + t\left(\eta_h, \Pi_k^0(b_0(\boldsymbol{x};t,t)\widehat{\widehat{\eta}}_h) - 2\int_0^t \Pi_k^0(b_{0s}(\boldsymbol{x};s,s)\widehat{\widehat{\eta}}_h(s))ds\right)
\end{aligned}
$$

$$-t\left(\eta, \widehat{\eta}_h - \Pi_k^0(b_0(\boldsymbol{x};t,t)\widehat{\widehat{\eta}}_h) + 2\int_0^t \Pi_k^0(b_{0s}(\boldsymbol{x};s,s)\widehat{\widehat{\eta}}_h(s))ds\right)$$
$$+t\left(\eta, \int_0^t\int_0^s \Pi_k^0(b_{0zz}(\boldsymbol{x};s,z)\widehat{\widehat{\eta}}_h(z))dzds\right).$$

Employ (3.1.3), apply Young's inequality, consider the boundedness of $t$, and subsequently utilize the kickback argument to rewrite the above equation as:

$$\frac{d}{dt}t^2\|\widehat{\boldsymbol{\Theta}}_h\|^2 + \frac{d}{dt}t\|\widehat{\eta}_h\|^2 + t^2\|\eta_h\|^2$$
$$\leq C\left(t^2\|\eta\|^2 + \|\widehat{\eta}_h\|^2 + \|\widehat{\widehat{\eta}}_h\|^2 + t^2\|\widehat{\boldsymbol{\Theta}}_h\|^2 + \int_0^t \|\widehat{\eta}_h(s)\|^2 ds + \int_0^t \|\widehat{\widehat{\eta}}_h(s)\|^2 ds\right.$$
$$\left.+\int_0^t\int_0^s \|\widehat{\widehat{\eta}}_h(z)\|^2 dzds\right).$$

By integrating the above equation from 0 to $t$ and then applying Grönwall's lemma, we obtain the following:

$$t^2\|\widehat{\boldsymbol{\Theta}}_h\|^2 + t\|\widehat{\eta}_h\|^2 + \int_0^t s^2\|\eta_h(s)\|^2 ds$$
$$\leq C\int_0^t \left(\|\widehat{\eta}_h(s)\|^2 + s^2\|\eta(s)\|^2 + \|\widehat{\eta}_h(s)\|^2 + \|\widehat{\widehat{\eta}}_h(s\|^2\right) ds.$$

Using Lemma 6.2.2 and (6.2.21), we obtain:

$$t^2\|\widehat{\boldsymbol{\Theta}}_h\|^2 + t\|\widehat{\eta}_h\|^2 + \int_0^t s^2\|\eta_h(s)\|^2 ds \leq Ch^4 t\|u_0\|^2.$$

□

**Theorem 6.2.4.** *Let* $(u, \boldsymbol{\sigma})$ *and* $(u_h, \boldsymbol{\sigma}_h)$ *satisfy* (6.1.1a)-(6.1.1b) *and* (6.1.3)*, respectively. Then, there exists a positive constant* $C$ *independent of the* $h$ *such that for* $t \in (0, T]$*, the following estimate holds true:*

$$\|u(t) - u_h(t)\| \leq Ch^2 t^{-1}\|u_0\|.$$

*Proof.* By substituting $\phi_h = t^3\left(\eta_h - \int_0^t \Pi_k^0\left(b_0(\boldsymbol{x};t,s)\eta_h(s)\right)ds\right)$ into (6.2.33a) and

$\boldsymbol{\chi}_h = t^3\boldsymbol{\Theta}_h$ into (6.2.33b), and then adding them together, gives us:

$$
\begin{aligned}
& t^3(\eta_{h,t}, \eta_h) + t^3 a_h(\boldsymbol{\Theta}_h, \boldsymbol{\Theta}_h) \\
& = -t^3\left(\eta_t, \eta_h - \int_0^t \Pi_k^0\left(b_0(\boldsymbol{x};t,s)\eta_h(s)\right)ds\right) + t^3\int_0^t (\nabla b_0(\boldsymbol{x};t,s)\eta_h(s), \boldsymbol{\Pi}_k^0\boldsymbol{\Theta}_h)ds \\
& \quad + t^3\left(\eta_{h,t}, \int_0^t \Pi_k^0\left(b_0(\boldsymbol{x};t,s)\eta_h(s)\right)ds\right).
\end{aligned}
$$

Using (3.1.3), we rewrite the above equation as:

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{dt}t^3\|\eta_h\|^2 + \alpha_1 t^3\|\boldsymbol{\Theta}_h\|^2 \\
& \leq \frac{3}{2}t^2\|\eta_h\|^2 - t^3\left(\eta_t, \eta_h - \int_0^t \Pi_k^0\left(b_0(\boldsymbol{x};t,s)\eta_h(s)\right)ds\right) \\
& \quad + t^3\int_0^t (\nabla b_0(\boldsymbol{x};t,s)\eta_h(s), \boldsymbol{\Pi}_k^0\boldsymbol{\Theta}_h)ds + t^3\left(\eta_{h,t}, \int_0^t \Pi_k^0\left(b_0(\boldsymbol{x};t,s)\eta_h(s)\right)ds\right).
\end{aligned}
\tag{6.2.41}
$$

As we know

$$
\begin{aligned}
& \frac{d}{dt}\left(t^3\left(\eta_h, \int_0^t \Pi_k^0\left(b_0(\boldsymbol{x};t,s)\eta_h(s)\right)ds\right)\right) \\
& = t^3\left(\eta_{h,t}\int_0^t \Pi_k^0(b_0(\boldsymbol{x};t,s)\eta_h(s)ds)\right) \\
& \quad + 3t^2\left(\eta_h, \Pi_k^0(b_0(\boldsymbol{x};t,t)\widehat{\eta}_h(t)) - \int_0^t \Pi_k^0(b_{0s}(\boldsymbol{x};t,s)\widehat{\eta}_h(s))ds\right) \\
& \quad + t^3\left(\eta_h, \Pi_k^0(b_0(\boldsymbol{x};t,t)\eta_h(t)) + \Pi_k^0(b_{0t}(\boldsymbol{x};t,t)\widehat{\eta}_h(t)) - \int_0^t \Pi_k^0(b_{0,ts}(\boldsymbol{x};t,s)\widehat{\eta}_h(s))ds\right).
\end{aligned}
\tag{6.2.42}
$$

Using (6.2.42) in (6.2.41), we arrive at the following:

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{dt}(t^3\|\eta_h\|^2) + \alpha_1 t^3\|\boldsymbol{\Theta}_h\|^2 \\
& \leq \frac{3}{2}t^2\|\eta_h\|^2 - t^3\left(\eta_t, \eta_h - \Pi_k^0(b_0(\boldsymbol{x};t,t)\widehat{\eta}_h)(t) + \int_0^t \Pi_k^0(b_{0s}(\boldsymbol{x};t,s)\widehat{\eta}_h(s))ds\right)
\end{aligned}
$$

$$+ t^3 \left( \nabla b_0(\boldsymbol{x};t,t)\widehat{\eta}_h(t), \boldsymbol{\Pi}_k^0 \boldsymbol{\Theta}_h) - \int_0^t (\nabla b_{0s}(\boldsymbol{x};t,s)\widehat{\eta}_h(s), \boldsymbol{\Pi}_k^0 \boldsymbol{\Theta}_h) ds \right)$$
$$+ \frac{d}{dt}\left( t^3 \left( \eta_h, \Pi_k^0(b_0(\boldsymbol{x};t,t)\widehat{\eta}_h(t)) - \int_0^t \Pi_k^0(b_{0s}(\boldsymbol{x};t,s)\widehat{\eta}_h(s)) ds \right)\right)$$
$$- 3t^2 \left( \eta_h, \Pi_k^0(b_0(\boldsymbol{x};t,t)\widehat{\eta}_h(t)) - \int_0^t \Pi_k^0(b_{0s}(\boldsymbol{x};t,s)\widehat{\eta}_h(s)) ds \right)$$
$$- t^3 \left( \eta_h, \Pi_k^0(b_0(\boldsymbol{x};t,t)\eta_h(t)) + \Pi_k^0(b_{0t}(\boldsymbol{x};t,t)\widehat{\eta}_h(t)) - \int_0^t \Pi_k^0(b_{0,ts}(\boldsymbol{x};t,s)\widehat{\eta}_h(s)) ds \right).$$

Using Young's inequality followed by the kickback argument and integration from 0 to $t$, we arrive at the following:

$$t^3\|\eta_h\|^2 + \int_0^t s^3\|\boldsymbol{\Theta}_h(s)\|^2 ds$$
$$\leq C\left( \int_0^t (s^2\|\eta_h(s)\|^2 + s^4\|\eta_s(s)\|^2 + \|\widehat{\eta}_h(s)\|^2 + s^2\int_0^s \|\widehat{\eta}_h(z)\|^2 dz) ds + t^3\int_0^t \|\widehat{\eta}_h(s)\|^2 ds \right).$$

Using (6.2.32), Lemma 6.2.2 and Lemma 6.2.3, we arrive at the following:

$$t^3\|\eta_h\|^2 + \int_0^t s^3\|\boldsymbol{\Theta}_h(s)\|^2 ds \leq Ch^4 t\|u_0\|^2.$$

Now, using the triangle inequality, (6.2.21) in the above equation completes the proof. □

### 6.2.4 Estimates for $\sigma(t) - \sigma_h(t)$

In this subsection, we present the analysis to obtain the $L^2$- error estimate for the approximation of $\boldsymbol{\sigma}$.

$$\boldsymbol{\sigma} - \boldsymbol{\sigma}_h = \left(\boldsymbol{\sigma} - \boldsymbol{\Pi}_h^F\boldsymbol{\sigma}\right) + \left(\boldsymbol{\Pi}_h^F\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\right) =: \boldsymbol{\varpi} + \boldsymbol{\varpi}_h,$$
$$u - u_h = \left(u - \Pi_k^0 u\right) + \left(\Pi_k^0 u - u_h\right) =: \zeta + \zeta_h.$$

Using (6.1.1a), (6.1.1b)and (6.1.3) along with the properties of $\Pi_k^0$ and $\boldsymbol{\Pi}_h^F$ in the following manner:

$$(\zeta_{h,t}, \phi_h) - (\nabla\cdot\boldsymbol{\varpi}_h, \phi_h) = 0 \quad \forall \phi_h \in Q_h^k, \tag{6.2.43a}$$

$$(\mu\boldsymbol{\sigma}, \boldsymbol{\chi}_h) - a_h(\boldsymbol{\Pi}_h^F\boldsymbol{\sigma}, \boldsymbol{\chi}_h) + (\zeta_h, \nabla\cdot\boldsymbol{\chi}_h) - \int_0^t (b_0(\boldsymbol{x};t,s)(u-u_h)(s), \nabla\cdot\boldsymbol{\chi}_h)ds$$
$$+ a_h(\boldsymbol{\varpi}_h, \boldsymbol{\chi}_h) - \int_0^t \big((\nabla b_0(\boldsymbol{x};t,s)u(s), \boldsymbol{\chi}_h) - (\nabla b_0(\boldsymbol{x};t,s)u_h(s), \boldsymbol{\Pi}_k^0\boldsymbol{\chi}_h)\big)\, ds$$
$$= 0 \quad \forall \boldsymbol{\chi}_h \in V_h^k. \tag{6.2.43b}$$

Integrating (6.2.43a) and (6.2.43b) from 0 to $t$ to arrive at the following:

$$(\zeta_h, \phi_h) - (\nabla\cdot\widehat{\boldsymbol{\varpi}}_h, \phi_h) = 0 \quad \forall \phi_h \in Q_h^k, \tag{6.2.44a}$$

$$(\mu\widehat{\boldsymbol{\sigma}}, \boldsymbol{\chi}_h) - a_h(\boldsymbol{\Pi}_h^F\widehat{\boldsymbol{\sigma}}, \boldsymbol{\chi}_h) + (\widehat{\zeta}_h, \nabla\cdot\boldsymbol{\chi}_h) - \int_0^t (b_0(\boldsymbol{x};s,s)(\widehat{u}-\widehat{u}_h)(s), \nabla\cdot\boldsymbol{\chi}_h)ds$$
$$+ \int_0^t\int_0^s (b_{0z}(\boldsymbol{x};s,z)(\widehat{u}-\widehat{u}_h)(z), \nabla\cdot\boldsymbol{\chi}_h)dzds$$
$$- \int_0^t \big((\nabla b_0(\boldsymbol{x};s,s)\widehat{u}(s), \boldsymbol{\chi}_h) - (\nabla b_0(\boldsymbol{x};s,s)\widehat{u}_h(s), \boldsymbol{\Pi}_k^0\boldsymbol{\chi}_h)\big)\, d + a_h(\widehat{\varpi}_h, \boldsymbol{\chi}_h)s$$
$$+ \int_0^t\int_0^s \big((\nabla b_{0z}(\boldsymbol{x};s,z)\widehat{u}(z), \boldsymbol{\chi}_h) - (\nabla b_{0z}(\boldsymbol{x};s,z)\widehat{u}_h(z), \boldsymbol{\Pi}_k^0\boldsymbol{\chi}_h)\big)\, dzds = 0 \quad \forall \boldsymbol{\chi}_h \in V_h^k. \tag{6.2.44b}$$

Differentiate (6.2.43b) with respect to $t$ to arrive at the following:

$$(\mu\boldsymbol{\sigma}_t, \boldsymbol{\chi}_h) - a_h(\boldsymbol{\Pi}_h^F\boldsymbol{\sigma}_t, \boldsymbol{\chi}_h) + (\zeta_{h,t}, \nabla\cdot\boldsymbol{\chi}_h) - (b_0(\boldsymbol{x};t,t)(u-u_h)(t), \nabla\cdot\boldsymbol{\chi}_h)$$
$$- \int_0^t (b_{0t}(\boldsymbol{x};t,s)(u-u_h)(s), \nabla\cdot\boldsymbol{\chi}_h)ds$$
$$+ a_h(\boldsymbol{\varpi}_{h,t}, \boldsymbol{\chi}_h) - \big((\nabla b_0(\boldsymbol{x};t,t)u(t), \boldsymbol{\chi}_h) - (\nabla b_0(\boldsymbol{x};t,t)u_h(t), \boldsymbol{\Pi}_k^0\boldsymbol{\chi}_h)\big)$$
$$- \int_0^t \big((\nabla b_{0t}(\boldsymbol{x};t,s)u(s), \boldsymbol{\chi}_h) - (\nabla b_{0t}(\boldsymbol{x};t,s)u_h(s), \boldsymbol{\Pi}_k^0\boldsymbol{\chi}_h)\big)\, ds = 0 \quad \forall \boldsymbol{\chi}_h \in V_h^k. \tag{6.2.45}$$

First, we establish several lemmas that will be utilized later in the proof of $\|\boldsymbol{\sigma}(t) - \boldsymbol{\sigma}_h(t)\|$:

**Lemma 6.2.4.** *Let $\widehat{\boldsymbol{\varpi}}_h$ and $\widehat{\zeta}_h$ satisfies* (6.2.44a) *and* (6.2.44b)*, then there exists a positive constant $C$, such that the following estimates hold true:*

$$\|\widehat{\zeta}_h(t)\|^2 + \int_0^t \|\widehat{\boldsymbol{\varpi}}_h(s)\|^2 ds \le Ch^2 t\|u_0\|^2.$$

*Proof.* Put $\phi_h = \widehat{\zeta}_h - \int_0^t \Pi_k^0(b_0(\boldsymbol{x};s,s)(\widehat{u} - \widehat{u}_h)(s))ds + \int_0^t \int_0^s \Pi_k^0(b_{0z}(\boldsymbol{x};s,z)(\widehat{u} - \widehat{u}_h)(z))dzds$ and $\boldsymbol{\chi}_h = \widehat{\boldsymbol{\varpi}}_h$ into (6.2.44a) and (6.2.44b), and then add to arrive at the following:

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\widehat{\zeta}_h\|^2 + \alpha_1\|\widehat{\boldsymbol{\varpi}}_h\|^2 \\
&\leq \left(a_h(\boldsymbol{\Pi}_h^F\widehat{\boldsymbol{\sigma}}, \widehat{\boldsymbol{\varpi}}_h) - (\mu\widehat{\boldsymbol{\sigma}}, \widehat{\boldsymbol{\varpi}}_h)\right) \\
&\quad + \left(\zeta_h, \int_0^t \Pi_k^0(b_0(\boldsymbol{x};s,s)(\widehat{u} - \widehat{u}_h)(s))ds - \int_0^t \int_0^s \Pi_k^0(b_{0z}(\boldsymbol{x};s,z)(\widehat{u} - \widehat{u}_h)(z))dzds\right) \\
&\quad + \int_0^t \left((\nabla b_0(\boldsymbol{x};s,s)\widehat{u}(s), \widehat{\boldsymbol{\varpi}}_h) - (\nabla b_0(\boldsymbol{x};s,s)\widehat{u}_h(s), \boldsymbol{\Pi}_k^0(\widehat{\boldsymbol{\varpi}}_h))\right) ds \\
&\quad - \int_0^t \int_0^s \left((\nabla b_{0z}(\boldsymbol{x};s,z)\widehat{u}(z), \widehat{\boldsymbol{\varpi}}_h) - (\nabla b_{0z}(\boldsymbol{x};s,z)\widehat{u}_h(z), \boldsymbol{\Pi}_k^0(\widehat{\boldsymbol{\varpi}}_h))\right) dzds.
\end{aligned}
$$

By using (3.1.3), (1.5.9), (1.5.7), Lemma 4.1.1, Young's inequality, and then applying the kickback argument, we arrive at the following:

$$
\begin{aligned}
&\frac{d}{dt}\|\widehat{\zeta}_h\|^2 + \|\widehat{\boldsymbol{\varpi}}_h\|^2 \\
&\leq C\Bigg(\left(\|\boldsymbol{\Pi}_k^0\widehat{\boldsymbol{\sigma}} - \widehat{\boldsymbol{\sigma}}\|^2 + \|\boldsymbol{\Pi}_h^F\widehat{\boldsymbol{\sigma}} - \boldsymbol{\Pi}_k^0\widehat{\boldsymbol{\sigma}}\|^2 + h|\widehat{\boldsymbol{\sigma}}|_1^2 + \int_0^t \left(h^4\|\widehat{u}(s)\|_2^2 + \|\widehat{\zeta}_h(s)\|^2\right) ds\right) \\
&\quad + \left(\zeta_h, \int_0^t \Pi_k^0(b_0(\boldsymbol{x};s,s)(\widehat{u} - \widehat{u}_h)(s))ds - \int_0^t \int_0^s \Pi_k^0(b_{0z}(\boldsymbol{x};s,z)(\widehat{u} - \widehat{u}_h)(z))dzds\right)\Bigg).
\end{aligned}
$$

Integrating form 0 to $t$ to arrive at the following:

$$
\begin{aligned}
&\|\widehat{\zeta}_h\|^2 + \int_0^t \|\widehat{\boldsymbol{\varpi}}_h(s)\|^2 ds \\
&\leq C\Bigg(h^2 t\|u_0\|^2 + \int_0^s \|\widehat{\zeta}_h(s)\|^2 ds + \int_0^t \bigg(\zeta_h(s), \int_0^s \Pi_k^0(b_0(\boldsymbol{x};z,z)(\widehat{u} - \widehat{u}_h)(z))dz \\
&\quad - \int_0^s \int_0^z \Pi_k^0(b_{0,z'}(\boldsymbol{x};z,z')(\widehat{u} - \widehat{u}_h)(z'))dz'dz\bigg) ds\Bigg) \\
&\leq C\Bigg(h^2 t\|u_0\|^2 + \bigg(\widehat{\zeta}_h(t), \int_0^t \Pi_k^0(b_0(\boldsymbol{x};s,s)(\widehat{u} - \widehat{u}_h)(s))ds \\
&\quad - \int_0^t \int_0^s \Pi_k^0(b_{0z}(\boldsymbol{x};s,z)(\widehat{u} - \widehat{u}_h)(z))dzds\bigg)
\end{aligned}
$$

$$-\int_0^t \left( \widehat{\zeta}_h(s), \Pi_k^0(b_0(\boldsymbol{x};s,s)(\widehat{u}-\widehat{u}_h)(s)) - \int_0^s \Pi_k^0(b_{0z}(\boldsymbol{x};s,z)(\widehat{u}-\widehat{u}_h)(z))dz \right) ds \Bigg).$$

By applying Young's inequality, followed by the kickback argument, and then utilizing Grönwall's lemma, we arrive at the following:

$$\|\widehat{\zeta}_h\|^2 + \int_0^t \|\widehat{\boldsymbol{\varpi}}_h(s)\|^2 ds \le Ch^2 t\|u_0\|^2.$$

□

**Lemma 6.2.5.** *Let $\widehat{\boldsymbol{\varpi}}_h$ and $\zeta_h$ satisfies* (6.2.44a) *and* (6.2.43b)*, then there exists a positive constant $C$, such that the following estimates hold true:*

$$t\|\widehat{\boldsymbol{\varpi}}_h(t)\|^2 + \int_0^t s\|\zeta_h(s)\|^2 ds \le Ch^2 t\|u_0\|^2.$$

*Proof.* Put $\phi_h = t\left(\zeta_h - \int_0^t \Pi_k^0\left(b_0(\boldsymbol{x};t,s)(u-u_h)(s)\right) ds\right)$ and $\boldsymbol{\chi}_h = t\widehat{\boldsymbol{\varpi}}_h$ in (6.2.44a) and (6.2.43b) and add to arrive at the following:

$$\begin{aligned}
&t\|\zeta_h\|^2 + \frac{1}{2}\frac{d}{dt}(t\|\widehat{\boldsymbol{\varpi}}_h\|^2) \\
&\le \frac{1}{2}\|\widehat{\boldsymbol{\varpi}}_h\|^2 + t\left(\zeta_h, \Pi_k^0(b_0(\boldsymbol{x};t,t)(\widehat{u}-\widehat{u}_h)(t)) - \int_0^t \Pi_k^0(b_{0s}(\boldsymbol{x};t,s)(\widehat{u}-\widehat{u}_h)(s))ds\right) \\
&\quad + t\left(a_h(\boldsymbol{\Pi}_h^F\boldsymbol{\sigma}, \widehat{\boldsymbol{\varpi}}_h) - (\mu\boldsymbol{\sigma}, \widehat{\boldsymbol{\varpi}}_h)\right) - t\left(\nabla b_0(\boldsymbol{x};t,t)\widehat{u}(t) - \int_0^t (\nabla b_{0s}(\boldsymbol{x};t,s)\widehat{u}(s)ds, \widehat{\boldsymbol{\varpi}}_h)\, ds\right) \\
&\quad + t\left(\nabla b_0(\boldsymbol{x};t,t)\widehat{u}_h(t) - \int_0^t \left(\nabla b_{0s}(\boldsymbol{x};t,s)\widehat{u}_h(s)ds, \boldsymbol{\Pi}_k^0(\widehat{\boldsymbol{\varpi}}_h)\right) ds\right).
\end{aligned}$$

Using Young's inequality, Lemma 4.1.1, Lemma 6.2.1 followed by the kickback argument, we arrive at the following:

$$\begin{aligned}
&t\|\zeta_h\|^2 + \frac{d}{dt}(t\|\widehat{\boldsymbol{\varpi}}_h\|^2) \\
&\le C\Bigg(\|\widehat{\boldsymbol{\varpi}}_h\|^2 + t\|\widehat{\zeta}\|^2 + t\|\widehat{\zeta}_h\|^2 + t^2h^2\|\widehat{u}\|_2^2 + t\int_0^t \left(\|\widehat{\zeta}(s)\|^2 + \|\widehat{\zeta}_h(s)\|^2 + h^4\|\widehat{u}(s)\|_2^2\right) ds \\
&\quad + t^2\left(\|\boldsymbol{\Pi}_k^0\boldsymbol{\sigma} - \boldsymbol{\sigma}\|^2 + \|\boldsymbol{\Pi}_h^F\boldsymbol{\sigma} - \boldsymbol{\Pi}_k^0\boldsymbol{\sigma}\|^2 + h^2|\boldsymbol{\sigma}|_1^2\right)\Bigg).
\end{aligned}$$

By integrating from 0 to $t$ and applying Lemma 6.2.4 along with the boundedness of $t$, and subsequently employing Grönwall's lemma, we obtain the following:

$$t\|\widehat{\boldsymbol{\varpi}}_h\|^2 + \int_0^t s\|\zeta_h(s)\|^2 ds \leq Ch^2 t\|u_0\|^2.$$

□

**Lemma 6.2.6.** *Let $\boldsymbol{\varpi}_h$ and $\zeta_h$ satisfies (6.2.43a) and (6.2.43b), then there exists a positive constant $C$, such that the following estimates hold true:*

$$t^2\|\zeta_h\|^2 + \int_0^t s^2\|\boldsymbol{\varpi}_h(s)\|^2 ds \leq Ch^2 t\|u_0\|^2.$$

*Proof.* Substitute $\phi_h = t^2\left(\zeta_h - \int_0^t \Pi_k^0(b_0(\boldsymbol{x};t,s)(u-u_h)(s))ds\right)$ and $\boldsymbol{\chi}_h = t^2\boldsymbol{\varpi}_h$ into (6.2.43a) and (6.2.43b) and then add. By applying Young's inequality and then utilizing the kickback argument, we arrive at the following:

$$\begin{aligned}
&\frac{d}{dt}(t^2\|\zeta_h\|^2) + t^2\|\boldsymbol{\varpi}_h\|^2 \\
&\leq C\Big(t\|\zeta_h\|^2 + t^2\left(\|\boldsymbol{\Pi}_k^0\boldsymbol{\sigma} - \boldsymbol{\sigma}\|^2 + \|\boldsymbol{\Pi}_h^F\boldsymbol{\sigma} - \boldsymbol{\Pi}_k^0\boldsymbol{\sigma}\|^2 + h^2|\boldsymbol{\sigma}|_1^2 + \|\widehat{\zeta}\|^2 + \|\widehat{\zeta}_h\|^2 + h^2\|\widehat{u}\|_2\right) \\
&\quad + t^2\int_0^t\left(\|\widehat{\zeta}(s)\|^2 + \|\widehat{\zeta}_h(s)\|^2 + h^2\|\widehat{u}(s)\|_2^2\right)ds + t^2\left(\zeta_{h,t}, \int_0^t \Pi_k^0(b_0(\boldsymbol{x};t,s)(u-u_h)(s))ds\right)\Big).
\end{aligned} \tag{6.2.46}$$

As we know

$$\begin{aligned}
&\frac{d}{dt}\left(t^2\left(\zeta_h, \int_0^t \Pi_k^0(b_0(\boldsymbol{x};t,s)(u-u_h)(s))ds\right)\right) \\
&= t^2\left(\zeta_{h,t}, \int_0^t \Pi_k^0(b_0(\boldsymbol{x};t,s)(u-u_h)(s))ds\right) \\
&\quad + 2t\left(\zeta_h, \Pi_k^0(b_0(\boldsymbol{x};t,t)(\widehat{u}-\widehat{u}_h)(t)) - \int_0^t \Pi_k^0(b_{0s}(\boldsymbol{x};t,s)(\widehat{u}-\widehat{u}_h)(s))ds\right) \\
&\quad + t^2\left(\zeta_h, \Pi_k^0(b_0(\boldsymbol{x};t,t)(u-u_h)(t)) + \Pi_k^0(b_{0t}(\boldsymbol{x};t,t)(\widehat{u}-\widehat{u}_h)(t))\right) \\
&\quad - t^2\left(\zeta_h, \int_0^t \Pi_k^0(b_{0,ts}(\boldsymbol{x};t,s)(\widehat{u}-\widehat{u}_h)(s))ds\right).
\end{aligned} \tag{6.2.47}$$

Using (6.2.47) in (6.2.46) along with the Lemma 6.2.4, Lemma 6.2.5, Young's inequality, and the kickback argument, we arrive at the following:

$$
\begin{aligned}
&\frac{d}{dt}(t^2\|\zeta_h\|^2) + t^2\|\boldsymbol{\varpi}_h\|^2 \\
&\leq C\left(t\|\zeta_h\|^2 + h^2\|u_0\|^2\right. \\
&\quad \left.+\frac{d}{dt}\left(t^2\left(\zeta_h, \Pi_k^0(b_0(\boldsymbol{x};t,t)(\widehat{u}-\widehat{u}_h)(t)) - \int_0^t \Pi_k^0(b_{0s}(\boldsymbol{x};t,s)(\widehat{u}-\widehat{u}_h)(s))ds\right)\right)\right).
\end{aligned}
$$

Integrating the above equation from 0 to $t$. Then, apply Lemma 6.2.4 and Lemma 6.2.5, followed by the utilization of Grönwall's lemma, yields the following:

$$
t^2\|\zeta_h\|^2 + \int_0^t s^2\|\boldsymbol{\varpi}_h(s)\|^2 ds \leq Ch^2 t\|u_0\|^2.
$$

□

**Theorem 6.2.5.** *Let $(u, \boldsymbol{\sigma})$ and $(u_h, \boldsymbol{\sigma}_h)$ satisfy (6.1.1a)-(6.1.1b)and (6.1.3), respectively. Then, there exists a positive constant $C$ independent of the $h$ such that for $t \in (0, T]$, the following estimate holds true:*

$$
\|\boldsymbol{\sigma}(t) - \boldsymbol{\sigma}_h(t)\| \leq Cht^{-1}\|u_0\|.
$$

*Proof.* Put $\phi_h = t^3\zeta_{h,t} - \Pi_k^0\left(b_0(\boldsymbol{x};t,t)(u-u_h)(t)\right) - \int_0^t \Pi_k^0\left(b_{0t}(\boldsymbol{x};t,s)(u-u_h)(s)\right) ds$ and $\boldsymbol{\chi}_h = t^3\boldsymbol{\varpi}_h$ into (6.2.43a) and (6.2.45), and then add to arrive at the following:

$$
\begin{aligned}
&t^3\|\zeta_{h,t}\|^2 + \frac{1}{2}\frac{d}{dt}(t^3\|\boldsymbol{\varpi}_h\|^2) \\
&\leq t^3\left(\zeta_{h,t}, \Pi_k^0(b_0(\boldsymbol{x};t,t)(u-u_h)(t)) + \int_0^t \Pi_k^0(b_{0t}(\boldsymbol{x};t,s)(u-u_h)(s))ds\right) + \frac{3}{2}(t^2\|\boldsymbol{\varpi}_h\|^2) \\
&\quad + t^3\left(a_h(\boldsymbol{\Pi}_h^F\boldsymbol{\sigma}_t, \boldsymbol{\varpi}_h) - (\mu\boldsymbol{\sigma}_t, \boldsymbol{\varpi}_h) + (\nabla b_0(\boldsymbol{x};t,t)u(t), \boldsymbol{\varpi}_h) - (\nabla b_0(\boldsymbol{x};t,t)u_h(t), \boldsymbol{\Pi}_k^0(\boldsymbol{\varpi}_h))\right) \\
&\quad + t^3\int_0^t \left((\nabla b_{0t}(\boldsymbol{x};t,s)u(s), \boldsymbol{\varpi}_h) - (\nabla b_{0t}(\boldsymbol{x};t,s)u_h(s), \boldsymbol{\Pi}_k^0(\boldsymbol{\varpi}_h)\right) ds.
\end{aligned}
$$

By employing integration by parts, Lemma 4.1.1, Lemma 6.2.1, Young's inequality followed by the kickback argument, we arrive at the following:

$$
\begin{aligned}
t^3\|\zeta_{h,t}\|^2 + \frac{1}{2}\frac{d}{dt}(t^3\|\boldsymbol{\varpi}_h\|^2) \leq C \Big( & t^3 \left(\|\zeta\|^2 + \|\zeta_h\|^2 + \|\widehat{\zeta}\|^2 + \|\widehat{\zeta}_h\|^2\right) \\
& + t^2\|\boldsymbol{\varpi}_h\|^2 + t^3 \int_0^t \left(\|\widehat{\zeta}(s)\|^2 + \|\widehat{\zeta}_h(s)\|^2\right) ds \\
& + t^4(\|\boldsymbol{\Pi}_k^0\boldsymbol{\sigma}_t - \boldsymbol{\sigma}_t\|^2 + \|\boldsymbol{\Pi}_h^F\boldsymbol{\sigma}_t - \boldsymbol{\Pi}_k^0\boldsymbol{\sigma}_t\|^2 + h^2|\boldsymbol{\sigma}_t|_1^2) \\
& + t^4(\|\zeta\|^2 + \|\zeta_h\|^2 + h^4t^{-2}\|u_0\|^2) \\
& + t^4 \int_0^t \left(\|\widehat{\zeta}(s)\|^2 + \|\widehat{\zeta}_h(s)\|^2 + h^4\|\widehat{u}(s)\|_2^2\right) ds \\
& + t^4 \left(\|\widehat{\zeta}\|^2 + \|\widehat{\zeta}_h\|^2 + h^4\|\widehat{u}\|_2^2\right)\Big).
\end{aligned}
$$

Integrating from 0 to $t$, and then using Lemma 6.2.4, Lemma 6.2.5, and Lemma 6.2.6, we arrive at the following:

$$
\int_0^t s^3\|(\Pi_k^0 p_t - u_{h,t})(s)\|^2 ds + t^3\|\boldsymbol{\Pi}_h^F\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|^2 \leq Ch^2t\|u_0\|^2. \tag{6.2.48}
$$

Now, applying the triangle inequality along with (1.5.9), Lemma 6.2.1, and (6.2.48), we arrive at the following:

$$
\|\boldsymbol{\sigma}(t) - \boldsymbol{\sigma}_h(t)\| \leq Cht^{-1}\|u_0\|.
$$

This completes the proof of the theorem. □

**Remark 6.2.1.** *Utilizing the triangle inequality along with Lemma 6.2.6 and (1.5.7), we achieve an estimate of $\|u - u_h\| \leq Cht^{-\frac{1}{2}}$ which is not optimal. That's why introducing the mixed intermediate projection is important, which helps to enhance the optimal convergence order of $\|u - u_h\|$ to $h^2t^{-1}$.*

*Due to the non-smooth initial data imposing regularity constraints on the solution $u$, the exact solution $u$ can only be in $H^2(\mathcal{D})$, see [62]. And to achieve an order $O(h^2t^{-3/2})$ for $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|$, we require $u \in H^3(\mathcal{D})$ which is not possible. Instead, sticking with the mixed intermediate projection allowed us to attain an approximation of $O(ht^{-3/2})$. But for optimal convergence, we use the Fortin operator to achieve $O(ht^{-1})$.*

## 6.3 Mixed Virtual Element Formulation Using Resolvent Kernel

**Lemma 6.3.1.** *Let* $(u, \boldsymbol{\sigma})$ *be the solution of* (6.1.2) *and* $u_0 \in L^2(\mathcal{D})$*; Then the following estimates hold for* $t \in (0, T]$ *and* $j \in \{1, 2\}$*:*

1. $t\|u(t)\|_1^2 + \int_0^t s\|u_s(s)\|^2 ds \leq C\|u_0\|^2,$
2. $t^2\|u_t(t)\|^2 + \int_0^t s^2\|u_s(s)\|_1^2 ds \leq C\|u_0\|^2,$
3. $\|\widehat{u}(t)\|_2 + t\|u(t)\|_2 \leq C\|u_0\|,$
4. $\|u_t(t)\|_j \leq Ct^{-(1+\frac{j}{2})}\|u_0\|,$
5. $\|\nabla \cdot \widehat{\boldsymbol{\sigma}}\| \leq C\|u_0\|.$

*Proof.* Proof of this follows from [14, 63, 67]. □

### 6.3.1 Mixed Ritz Volterra Projection

Below, we present the estimates for the mixed R.V. projection when the initial data is not smooth.

**Theorem 6.3.1.** *For* $u(t) \in H_0^1(\mathcal{D}) \cap H^2(\mathcal{D})$*, where* $t > 0$*, and with an initial condition* $u_0 \in L^2(\mathcal{D})$*, there exists a positive constant* $C$ *independent of the parameter* $h$*, under which the following estimates hold:*

$$\|u - \tilde{u}\| + h\|\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}\| \leq Ch^2 t^{-1}\|u_0\|. \tag{6.3.1}$$

*Proof.* Consider

$$\boldsymbol{\theta} := \boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma} - \boldsymbol{\Pi}_h^F \boldsymbol{\sigma} + \boldsymbol{\psi}_h \quad \text{where } \boldsymbol{\psi}_h := \boldsymbol{\Pi}_h^F \boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}} \in V_h^k,$$
$$\rho := u - \tilde{u} = u - \Pi_k^0 u + \tau_h \quad \text{where } \tau_h := \Pi_k^0 u - \tilde{u} \in Q_h^k.$$

By the definition of the mixed R.V. projection (3.2.3), we have:

$$\begin{aligned}
& a_h(\boldsymbol{\psi_h}, \boldsymbol{\chi}_h) + \int_0^t \mathcal{K}_h(t,s;\boldsymbol{\psi_h}(s),\boldsymbol{\chi}_h)ds + (\nabla\cdot\boldsymbol{\chi}_h, \tau_h) \\
& = (\nabla\cdot\boldsymbol{\chi}_h, \Pi_k^0 u - u) + \left(\int_0^t \mathcal{K}_h(t,s;\boldsymbol{\Pi_h^F}\boldsymbol{\sigma}(s),\boldsymbol{\chi}_h)ds - \int_0^t (\mathcal{K}(t,s)\boldsymbol{\sigma}(s),\boldsymbol{\chi}_h)ds\right) \\
& \quad + \left(a_h(\boldsymbol{\Pi_h^F}\boldsymbol{\sigma},\boldsymbol{\chi}_h) - (\mu\boldsymbol{\sigma},\boldsymbol{\chi}_h)\right), \qquad (6.3.2)\\
& (\nabla\cdot\boldsymbol{\psi_h}, \phi_h) = 0. \qquad (6.3.3)
\end{aligned}$$

Put $\boldsymbol{\chi}_h = \boldsymbol{\psi}_h$ in (6.3.2) and $\phi_h = \tau_h$ in (6.3.3), then subtract and use (3.1.3) to arrive at the following:

$$\begin{aligned}
& \alpha_1\|\boldsymbol{\psi}_h\|^2 \\
& \leq \left(a_h(\boldsymbol{\Pi_h^F}\boldsymbol{\sigma},\boldsymbol{\psi_h}) - (\mu\boldsymbol{\sigma},\boldsymbol{\psi_h})\right) + \left(\int_0^t \mathcal{K}_h(t,s;\boldsymbol{\Pi_h^F}\boldsymbol{\sigma}(s),\boldsymbol{\psi_h})ds - \int_0^t (\mathcal{K}(t,s)\boldsymbol{\sigma}(s),\boldsymbol{\psi_h})ds\right) \\
& \quad + (\nabla\cdot\boldsymbol{\psi_h}, \Pi_k^0 u - u) - \int_0^t \mathcal{K}_h(t,s;\boldsymbol{\psi_h}(s),\boldsymbol{\psi_h})ds. \qquad (6.3.4)
\end{aligned}$$

The bound for the first term on the right-hand side of (6.3.4) follows from (6.2.22) whereas the second term can be solved by using integration by parts and then proceed similarly to (6.2.16):

$$\begin{aligned}
\int_0^t \mathcal{K}_h(t,s;\boldsymbol{\Pi_h^F}\boldsymbol{\sigma}(s),\boldsymbol{\psi_h})ds - \int_0^t (\mathcal{K}(t,s)\boldsymbol{\sigma}(s),\boldsymbol{\psi_h})ds
& = \left(\mathcal{K}_h(t,t;\widehat{\boldsymbol{\Pi_h^F}\boldsymbol{\sigma}}(t),\boldsymbol{\psi_h}) - (\mathcal{K}(t,t)\widehat{\boldsymbol{\sigma}}(t),\boldsymbol{\psi_h})\right) \\
& \quad - \int_0^t \left(\mathcal{K}_{hs}(t,s;\widehat{\boldsymbol{\Pi_h^F}\boldsymbol{\sigma}}(s),\boldsymbol{\psi_h}) - (\mathcal{K}_s(t,s)\widehat{\boldsymbol{\sigma}}(s),\boldsymbol{\psi_h})\right)ds \\
& \leq Ch\left(|\widehat{\boldsymbol{\sigma}}|_1 + \int_0^t |\widehat{\boldsymbol{\sigma}}(s)|_1 ds\right)\|\boldsymbol{\psi_h}\| \\
& \leq Ch\|u_0\|\|\boldsymbol{\psi_h}\| \quad \text{[Using Lemma 6.3.1]}. \qquad (6.3.5)
\end{aligned}$$

The fourth term on the right-hand side of (6.3.4) can be dealt with by using integration by parts as:

$$\int_0^t \mathcal{K}_h(t,s;\boldsymbol{\psi_h}(s),\boldsymbol{\psi_h})ds = \mathcal{K}_h(t,t;\widehat{\boldsymbol{\psi}}_{\boldsymbol{h}}(t),\boldsymbol{\psi_h}) - \int_0^t \mathcal{K}_{hs}(t,s;\widehat{\boldsymbol{\psi}}_{\boldsymbol{h}}(s),\boldsymbol{\psi_h})ds. \quad (6.3.6)$$

Using the fact $\nabla\cdot\boldsymbol{\psi}_h = 0$ along with (6.2.22), (6.3.5) and (6.3.6), we rewrite (6.3.4) as:

$$\|\boldsymbol{\psi}_h\| \leq C\left(ht^{-1}\|u_0\| + \|\widehat{\boldsymbol{\psi}}_{\boldsymbol{h}}\| + \int_0^t \|\widehat{\boldsymbol{\psi}}_{\boldsymbol{h}}(s)\|ds\right). \quad (6.3.7)$$

Substitute (6.3.19) in (6.3.7) followed by the use of the triangle inequality gives us the following:

$$\|\boldsymbol{\vartheta}\| \leq Cht^{-1}\|u_0\|. \quad (6.3.8)$$

Now, to achieve $\|u-\tilde{u}\|$, we proceed by using the definition of mixed R.V. projection (3.2.3) as:

$$(\mu\boldsymbol{\vartheta},\boldsymbol{\chi}_h) + (\nabla\cdot\boldsymbol{\chi}_h,\tau_h) = \mathcal{F}(\boldsymbol{\chi}_h) \quad \forall\boldsymbol{\chi}_h\in V_h^k,$$
$$(\nabla\cdot\boldsymbol{\vartheta},\phi_h) = 0 \quad \forall q_h\in Q_h^k,$$

where

$$\mathcal{F}(\boldsymbol{\chi}_h) = a_h(\tilde{\boldsymbol{\sigma}},\boldsymbol{\chi}_h) - (\mu\tilde{\boldsymbol{\sigma}},\boldsymbol{\chi}_h) + \int_0^t \mathcal{K}_h(t,s;\tilde{\boldsymbol{\sigma}}(s),\boldsymbol{\chi}_h)ds - \int_0^t (\mathcal{K}(t,s)\boldsymbol{\sigma}(s),\boldsymbol{\chi}_h)ds. \quad (6.3.9)$$

Consider the dual problem (6.2.3) with $\Lambda=\tau_h$ and put $\phi=\tau_h$ in (6.2.4) to achieve:

$$\begin{aligned}\|\tau_h\|^2 &= (\tau_h, -\nabla\cdot(\boldsymbol{\Pi}_h^F a\nabla\xi))\\ &= (\mu\boldsymbol{\vartheta},\boldsymbol{\Pi_h^F}(a\nabla\xi)-a\nabla\xi) + (\nabla\cdot\boldsymbol{\vartheta},\Pi_k^0\xi-\xi) - \mathcal{F}(\boldsymbol{\Pi_h^F}(a\nabla\xi)). \quad (6.3.10)\end{aligned}$$

Now, by using (1.5.7) and (1.5.9) , we arrive at:

$$(\mu\boldsymbol{\vartheta},\boldsymbol{\Pi_h^F}(a\nabla\xi)-a\nabla\xi) \leq Ch\|\boldsymbol{\vartheta}\|\|\xi\|_2, \quad (6.3.11)$$
$$(\nabla\cdot\boldsymbol{\vartheta},\Pi_k^0\xi-\xi) \leq Ch^2\|\nabla\cdot\boldsymbol{\vartheta}\|\|\xi\|_2, \quad (6.3.12)$$

where $\nabla \cdot \boldsymbol{\vartheta}$ can be estimated in a similar way as (6.2.31). Now, (6.3.9), can be rewritten as:

$$
\begin{aligned}
&\mathcal{F}(\mathbf{\Pi}_h^F(a\nabla\xi)) \\
&= \big(a_h(\tilde{\boldsymbol{\sigma}} - \mathbf{\Pi}_k^0\boldsymbol{\sigma}, \mathbf{\Pi}_h^F(a\nabla\xi)) - (\mu(\tilde{\boldsymbol{\sigma}} - \mathbf{\Pi}_k^0\boldsymbol{\sigma}), \mathbf{\Pi}_h^F(a\nabla\xi))\big) \\
&\quad + \big(a_h(\mathbf{\Pi}_k^0\boldsymbol{\sigma}, \mathbf{\Pi}_h^F(a\nabla\xi)) - (\mu\mathbf{\Pi}_k^0\boldsymbol{\sigma}, \mathbf{\Pi}_h^F(a\nabla\xi))\big) \\
&\quad + \Big(\mathcal{K}_h(t,t;(\widehat{\tilde{\boldsymbol{\sigma}}} - \mathbf{\Pi}_k^0\widehat{\boldsymbol{\sigma}})(t), \mathbf{\Pi}_h^F(a\nabla\xi))ds - (\mathcal{K}(t,t)(\widehat{\tilde{\boldsymbol{\sigma}}} - \mathbf{\Pi}_k^0\widehat{\boldsymbol{\sigma}})(t), \mathbf{\Pi}_h^F(a\nabla\xi))\Big) \\
&\quad - \left(\int_0^t \mathcal{K}_{hs}(t,s;(\widehat{\tilde{\boldsymbol{\sigma}}} - \mathbf{\Pi}_k^0\widehat{\boldsymbol{\sigma}})(s), \mathbf{\Pi}_h^F(a\nabla\xi))ds - \int_0^t (\mathcal{K}_s(t,s)(\widehat{\tilde{\boldsymbol{\sigma}}} - \mathbf{\Pi}_k^0\widehat{\boldsymbol{\sigma}})(s), \mathbf{\Pi}_h^F(a\nabla\xi))ds\right) \\
&\quad + \big(\mathcal{K}_h(t,t;\mathbf{\Pi}_k^0\widehat{\boldsymbol{\sigma}}(t), \mathbf{\Pi}_h^F(a\nabla\xi))ds - (\mathcal{K}(t,t)\mathbf{\Pi}_k^0\widehat{\boldsymbol{\sigma}}(t), \mathbf{\Pi}_h^F(a\nabla\xi))ds\big) \\
&\quad - \left(\int_0^t \mathcal{K}_{hs}(t,s;\mathbf{\Pi}_k^0\widehat{\boldsymbol{\sigma}}(s), \mathbf{\Pi}_h^F(a\nabla\xi))ds - \int_0^t (\mathcal{K}_s(t,s)\mathbf{\Pi}_k^0\widehat{\boldsymbol{\sigma}}(s), \mathbf{\Pi}_h^F(a\nabla\xi))ds\right) \\
&\quad - \int_0^t (\mathcal{K}(t,s)\boldsymbol{\vartheta}(s), \mathbf{\Pi}_h^F(a\nabla\xi) - a\nabla\xi)ds - (\mathcal{K}(t,t)\widehat{\boldsymbol{\vartheta}}(t), a\nabla\xi) \\
&\quad + \int_0^t (\mathcal{K}_s(t,s)\widehat{\boldsymbol{\vartheta}}(s), a\nabla\xi)ds.
\end{aligned}
$$

By using the dual norm approach for the last term of the above equation and for all the remaining terms, we use integration by parts and follow similar steps as in (6.2.27) to arrive at the following:

$$
\mathcal{F}(\mathbf{\Pi}_h^F(a\nabla\xi)) \leq C\left(h^2t^{-1}\|u_0\| + \int_0^t \|\widehat{\boldsymbol{\vartheta}}(s)\|_{-1}\right).
$$

Now, using (6.3.26) and (6.3.27), we arrive at the following:

$$
\mathcal{F}(\mathbf{\Pi}_h^F(a\nabla\xi)) \leq Ch^2t^{-1}\|u_0\|. \tag{6.3.13}
$$

Using (6.3.11)- (6.3.13) in (6.3.10) we arrive at the following:

$$
\|\tau_h\| \leq Ch^2t^{-1}\|u_0\|.
$$

Use of the above equation along with the triangle inequality and (6.3.8) completes the proof of the (6.3.1). $\square$

**Theorem 6.3.2.** *For $u(t) \in H_0^1(\mathcal{D}) \cap H^2(\mathcal{D})$, where $t > 0$, and with an initial condition $u_0 \in L^2(\mathcal{D})$, there exists a positive constant $C$ independent of the parameter $h$, under which the following estimates hold:*

$$\|\widehat{u}(t) - \widehat{\tilde{u}}(t)\| + h\|\widehat{\boldsymbol{\sigma}}(t) - \widehat{\tilde{\boldsymbol{\sigma}}}(t)\| \leq Ch^2 t\|u_0\|, \tag{6.3.14}$$

$$\|u_t(t) - \tilde{u}_t(t)\| + h\|\boldsymbol{\sigma}_t(t) - \tilde{\boldsymbol{\sigma}}_t(t)\| \leq Ch^2 t^{-2}\|u_0\|. \tag{6.3.15}$$

*Proof.* To estimate $\|\widehat{\boldsymbol{\psi}}_h\|^2$, we integrate (6.3.2) and (6.3.3), and arrive at the following:

$$\begin{aligned}
&a_h(\widehat{\boldsymbol{\psi}}_h, \boldsymbol{\chi}_h) + \int_0^t \mathcal{K}_h(s,s;\widehat{\boldsymbol{\psi}}_h(s), \boldsymbol{\chi}_h)ds - \int_0^t\int_0^s \mathcal{K}_{hz}(s,z;\widehat{\boldsymbol{\psi}}_h(z), \boldsymbol{\chi}_h)dzds + (\nabla\cdot\boldsymbol{\chi}_h, \widehat{\tau}_h)\\
&= (\nabla\cdot\boldsymbol{\chi}_h, \Pi_k^0\widehat{u} - \widehat{u}) + \left(\int_0^t \mathcal{K}_h(s,s;\widehat{\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}\boldsymbol{\sigma}}(s), \boldsymbol{\chi}_h)ds - \int_0^t (\mathcal{K}(s,\widehat{s)\boldsymbol{\sigma}(}s), \boldsymbol{\chi}_h)ds\right)\\
&\quad - \left(\int_0^t\int_0^s \mathcal{K}_{hz}(s,z;\widehat{\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}\boldsymbol{\sigma}}(z), \boldsymbol{\chi}_h)dzds - \int_0^t\int_0^s (\mathcal{K}_z(s,\widehat{z)\boldsymbol{\sigma}(}z), \boldsymbol{\chi}_h)dzds\right)\\
&\quad + \left(a_h(\widehat{\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}\boldsymbol{\sigma}}, \boldsymbol{\chi}_h) - (\mu\widehat{\boldsymbol{\sigma}}, \boldsymbol{\chi}_h)\right),
\end{aligned} \tag{6.3.16}$$

$$(\nabla\cdot\widehat{\boldsymbol{\psi}}_h, \phi_h) = 0. \tag{6.3.17}$$

Put $\boldsymbol{\chi}_h = \widehat{\boldsymbol{\psi}}_h$ and $\phi_h = \widehat{\tau}_h$ in (6.3.16) and (6.3.17) and subtract to arrive at the following:

$$\begin{aligned}
\alpha_1\|\widehat{\boldsymbol{\psi}}_h\|^2 &= \left(a_h(\widehat{\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}\boldsymbol{\sigma}, \boldsymbol{\psi}_h}) - (\mu\widehat{\boldsymbol{\sigma}}, \widehat{\boldsymbol{\psi}}_h)\right) + (\nabla\cdot\widehat{\boldsymbol{\psi}}_h, \Pi_k^0\widehat{u} - \widehat{u})\\
&\quad + \left(\int_0^t \mathcal{K}_h(s,s;\widehat{\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}\boldsymbol{\sigma}(s), \boldsymbol{\psi}_h})ds - \int_0^t (\mathcal{K}(s,s)\widehat{\boldsymbol{\sigma}}(s), \widehat{\boldsymbol{\psi}}_h)ds\right)\\
&\quad - \left(\int_0^t\int_0^s \mathcal{K}_{hz}(s,z;\widehat{\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}\boldsymbol{\sigma}(z), \boldsymbol{\psi}_h})dzds - \int_0^t\int_0^s (\mathcal{K}_z(s,z)\widehat{\boldsymbol{\sigma}}(z), \widehat{\boldsymbol{\psi}}_h)dzds\right)\\
&\quad - \int_0^t \mathcal{K}_h(s,s;\widehat{\boldsymbol{\psi}}_h(s), \widehat{\boldsymbol{\psi}}_h)ds + \int_0^t\int_0^s \mathcal{K}_{hz}(s,z;\widehat{\boldsymbol{\psi}}_h(z), \widehat{\boldsymbol{\psi}}_h)dzds.
\end{aligned} \tag{6.3.18}$$

Using the similar arguments as in (6.2.22), (6.3.5), (6.3.6), boundedness of $t \leq T$ and the fact that $\nabla\cdot\widehat{\boldsymbol{\psi}}_h$, (6.3.18) can be written as:

$$\|\widehat{\boldsymbol{\psi}}_h\| \leq C\left(h\|u_0\| + \int_0^t \|\widehat{\boldsymbol{\psi}}_h\|ds\right).$$

The use of Grönwall's lemma gives us the following:

$$\|\widehat{\boldsymbol{\psi}}_h\| \leq Ch\|u_0\|. \tag{6.3.19}$$

Use of the triangle inequality, (6.3.19) and (1.5.9) gives us the following:

$$\|\widehat{\boldsymbol{\vartheta}}\| \leq Ch\|u_0\|.$$

Now, we proceed by using the definition of mixed R.V. projection (3.2.3) as:

$$(\mu\widehat{\boldsymbol{\vartheta}}, \boldsymbol{\chi}_h) + (\nabla\cdot\boldsymbol{\chi}_h, \widehat{\tau}_h) = \widehat{\mathcal{F}}(\boldsymbol{\chi}_h) \quad \forall\boldsymbol{\chi}_h \in V_h^k,$$
$$(\nabla\cdot\widehat{\boldsymbol{\vartheta}}, \phi_h) = 0 \quad \forall q_h \in Q_h^k,$$

where

$$\widehat{\mathcal{F}}(\boldsymbol{\chi}_h) = a_h(\widehat{\tilde{\boldsymbol{\sigma}}}, \boldsymbol{\chi}_h) - (\mu\widehat{\tilde{\boldsymbol{\sigma}}}, \boldsymbol{\chi}_h) + \int_0^t \mathcal{K}_h(s,s;\widehat{\tilde{\boldsymbol{\sigma}}}(s), \boldsymbol{\chi}_h)ds - \int_0^t (\mathcal{K}(s,s)\widehat{\boldsymbol{\sigma}}(s), \boldsymbol{\chi}_h)ds$$
$$- \int_0^t\int_0^s \mathcal{K}_{hz}(s,z;\widehat{\tilde{\boldsymbol{\sigma}}}(z), \boldsymbol{\chi}_h)dzds - \int_0^t\int_0^s (\mathcal{K}_z(s,z)\widehat{\boldsymbol{\sigma}}(z), \boldsymbol{\chi}_h)dzds. \tag{6.3.20}$$

Consider (6.2.3) with $\Lambda = \hat{\tau}_h$, then put $\phi = \widehat{\tau}_h$ in (6.2.4) to arrive at the following:

$$\begin{aligned}\|\widehat{\tau}_h\|^2 &= (\widehat{\tau}_h, -\nabla\cdot(\boldsymbol{\Pi}_h^F a\nabla\xi)) \\ &= (\mu\widehat{\boldsymbol{\vartheta}}, \boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi) - a\nabla\xi) + (\nabla\cdot\widehat{\boldsymbol{\vartheta}}, \Pi_k^0\xi - \xi) - \widehat{\mathcal{F}}(\boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi)).\end{aligned} \tag{6.3.21}$$

Now, by using (1.5.7) and (1.5.9) , we arrive at:

$$(\mu\widehat{\boldsymbol{\vartheta}}, \boldsymbol{\Pi}_{\boldsymbol{h}}^{\boldsymbol{F}}(a\nabla\xi) - a\nabla\xi) \leq Ch\|\widehat{\boldsymbol{\vartheta}}\|\|\xi\|_2, \tag{6.3.22}$$
$$(\nabla\cdot\widehat{\boldsymbol{\vartheta}}, \Pi_k^0\xi - \xi) \leq Ch^2\|\nabla\cdot\widehat{\boldsymbol{\vartheta}}\|\|\xi\|_2. \tag{6.3.23}$$

where, $\|\nabla\cdot\widehat{\boldsymbol{\vartheta}}\|$ can be estimated in a similar way to (6.2.12) as below:

$$\|\nabla\cdot\widehat{\boldsymbol{\vartheta}}\| \leq C\|u_0\|. \tag{6.3.24}$$

Now, (6.3.20), can be rewritten as:

$$\begin{aligned}
&\widehat{\mathcal{F}}(\mathbf{\Pi}_h^F(a\nabla\xi)) \\
&= \Big(a_h(\widehat{\widetilde{\boldsymbol{\sigma}}} - \mathbf{\Pi}_k^0\widehat{\boldsymbol{\sigma}}, \mathbf{\Pi}_h^F(a\nabla\xi)) - (\mu(\widehat{\widetilde{\boldsymbol{\sigma}}} - \mathbf{\Pi}_k^0\widehat{\boldsymbol{\sigma}}), \mathbf{\Pi}_h^F(a\nabla\xi))\Big) \\
&\quad + \big(a_h(\mathbf{\Pi}_k^0\widehat{\boldsymbol{\sigma}}, \mathbf{\Pi}_h^F(a\nabla\xi)) - (\mu\mathbf{\Pi}_k^0\widehat{\boldsymbol{\sigma}}, \mathbf{\Pi}_h^F(a\nabla\xi))\big) \\
&\quad + \left(\int_0^t \mathcal{K}_h(s,s;(\widehat{\widetilde{\boldsymbol{\sigma}}} - \mathbf{\Pi}_k^0\widehat{\boldsymbol{\sigma}})(s), \mathbf{\Pi}_h^F(a\nabla\xi))ds - \int_0^t (\mathcal{K}(s,s)(\widehat{\widetilde{\boldsymbol{\sigma}}} - \mathbf{\Pi}_k^0\widehat{\boldsymbol{\sigma}})(s), \mathbf{\Pi}_h^F(a\nabla\xi))ds\right) \\
&\quad - \int_0^t\int_0^s \mathcal{K}_{hz}(s,z;(\widehat{\widetilde{\boldsymbol{\sigma}}} - \mathbf{\Pi}_k^0\widehat{\boldsymbol{\sigma}})(z), \mathbf{\Pi}_h^F(a\nabla\xi))dzds \\
&\quad + \int_0^t\int_0^s (\mathcal{K}_z(s,z)(\widehat{\widetilde{\boldsymbol{\sigma}}} - \mathbf{\Pi}_k^0\widehat{\boldsymbol{\sigma}})(z), \mathbf{\Pi}_h^F(a\nabla\xi))dzds \\
&\quad + \left(\int_0^t \mathcal{K}_h(s,s;\mathbf{\Pi}_k^0\widehat{\boldsymbol{\sigma}}(s), \mathbf{\Pi}_h^F(a\nabla\xi))ds - \int_0^t (\mathcal{K}(s,s)\mathbf{\Pi}_k^0\widehat{\boldsymbol{\sigma}}(s), \mathbf{\Pi}_h^F(a\nabla\xi))ds\right) \\
&\quad - \int_0^t\int_0^s \mathcal{K}_{hz}(s,z;\mathbf{\Pi}_k^0\widehat{\boldsymbol{\sigma}}(z), \mathbf{\Pi}_h^F(a\nabla\xi))dzds \\
&\quad + \int_0^t\int_0^s (\mathcal{K}_z(s,z)\mathbf{\Pi}_k^0\widehat{\boldsymbol{\sigma}}(z), \mathbf{\Pi}_h^F(a\nabla\xi))dzds - \int_0^t (\mathcal{K}(s,s)(\widehat{\boldsymbol{\vartheta}})(s), \mathbf{\Pi}_h^F(a\nabla\xi) - a\nabla\xi)ds \\
&\quad + \int_0^t\int_0^s (\mathcal{K}_z(s,z)(\widehat{\boldsymbol{\vartheta}})(z), \mathbf{\Pi}_h^F(a\nabla\xi) - a\nabla\xi)dzds \\
&\quad - \int_0^t (\mathcal{K}(s,s)(\widehat{\boldsymbol{\vartheta}})(s), a\nabla\xi)ds - \int_0^t\int_0^s (\mathcal{K}_z(s,z)(\widehat{\boldsymbol{\vartheta}})(z), a\nabla\xi)dzds. \qquad (6.3.25)
\end{aligned}$$

For the last two terms in the right-hand side of (6.3.25), we use the dual norm approach, whereas all the remaining terms can be solved in a similar way to (6.2.27) as:

$$\begin{aligned}
\widehat{\mathcal{F}}(\mathbf{\Pi}_h^F(a\nabla\xi)) &\leq Ch^2\left(|\widehat{\boldsymbol{\sigma}}(t)|_1 + \int_0^t |\widehat{\boldsymbol{\sigma}}(s)|_1 ds\right)\|\xi\|_2 + \int_0^t \|\widehat{\boldsymbol{\vartheta}}\|_{-1}ds\|a\nabla\xi\|_1 \\
&\quad + \int_0^t\int_0^s \|\widehat{\boldsymbol{\vartheta}}(z)\|_{-1}dzds\|a\nabla\xi\|_1 \\
&\leq Ch^2\|u_0\|\|\xi\|_2 + \int_0^t \|\widehat{\boldsymbol{\vartheta}}\|_{-1}ds\|a\nabla\xi\|_1.
\end{aligned}$$

Now, we need to find the $\|\widehat{\boldsymbol{\vartheta}}\|_{-1}$, and for that, we proceed as by considering $\boldsymbol{\varkappa} \in (H^1(\mathcal{D}))^2$:

$$(\mu\widehat{\boldsymbol{\vartheta}}, \boldsymbol{\varkappa})$$

$$
\begin{aligned}
&= a_h(\widehat{\widetilde{\boldsymbol{\sigma}}}, \boldsymbol{\Pi}_{\boldsymbol{k}}^{\mathbf{0}}\boldsymbol{\varkappa}) - (\mu\widehat{\widetilde{\boldsymbol{\sigma}}}, \boldsymbol{\Pi}_{\boldsymbol{k}}^{\mathbf{0}}\boldsymbol{\varkappa}) + \int_0^t \mathcal{K}_h(s,s;\widehat{\widetilde{\boldsymbol{\sigma}}}(s), \boldsymbol{\Pi}_{\boldsymbol{k}}^{\mathbf{0}}\boldsymbol{\varkappa})ds \\
&\quad - \int_0^t (\mathcal{K}(s,s)\widehat{\widetilde{\boldsymbol{\sigma}}}(s), \boldsymbol{\Pi}_{\boldsymbol{k}}^{\mathbf{0}}\boldsymbol{\varkappa})ds - (\nabla\cdot\boldsymbol{\Pi}_{\boldsymbol{k}}^{\mathbf{0}}\boldsymbol{\varkappa}, \widehat{\tau}_h) - \int_0^t\int_0^s \mathcal{K}_{hz}(s,z;\widehat{\widetilde{\boldsymbol{\sigma}}}(z), \boldsymbol{\Pi}_{\boldsymbol{k}}^{\mathbf{0}}\boldsymbol{\varkappa})dzds \\
&\quad - \int_0^t\int_0^s (\mathcal{K}_z(s,z)\widehat{\widetilde{\boldsymbol{\sigma}}}(z), \boldsymbol{\Pi}_{\boldsymbol{k}}^{\mathbf{0}}\boldsymbol{\varkappa})dzds + (\mu\widehat{\boldsymbol{\vartheta}}, \boldsymbol{\varkappa} - \boldsymbol{\Pi}_{\boldsymbol{k}}^{\mathbf{0}}\boldsymbol{\varkappa}) - \int_0^t (\mathcal{K}\widehat{\boldsymbol{\vartheta}}(s), \boldsymbol{\Pi}_{\boldsymbol{k}}^{\mathbf{0}}\boldsymbol{\varkappa})ds \\
&\le C\left(h^2\|u_0\| + \|\widehat{\tau}_h\| + \int_0^t \|\boldsymbol{\vartheta}(s)\|_{-1}ds\right)\|\boldsymbol{\varkappa}\|_1.
\end{aligned}
$$

Using Grönwall's lemma, we arrive at:

$$
\|\widehat{\boldsymbol{\vartheta}}\|_{-1} \le C(h^2\|u_0\| + \|\widehat{\tau}_h\|). \tag{6.3.26}
$$

Using (6.3.22)-(6.3.26) in (6.3.21), we have:

$$
\|\widehat{\tau}_h\| \le C\left(h^2\|u_0\| + \int_0^t \left(h^2\|u_0\| + \|\widehat{\tau}_h(s)\|ds\right)\right).
$$

An application of Grönwall's lemma yields:

$$
\|\widehat{\tau}_h\| \le Ch^2\|u_0\|. \tag{6.3.27}
$$

Now, using the triangle inequality, (1.5.7) and Lemma 6.3.1, we arrive at the following:

$$
\|\widehat{\rho}\| \le Ch^2\|u_0\|.
$$

The proof of (6.3.15) follows a similar argument to that of (6.3.14). □

### 6.3.2 A *prior* Error-Estimates

Use (6.1.2) and (6.1.4) to have the error equation as:

$$
(\rho_{h,t}, \phi_h) - (\nabla\cdot\boldsymbol{\vartheta}_h, \phi_h) = -(\rho_t, \phi_h), \tag{6.3.28}
$$

$$
a_h(\boldsymbol{\vartheta}_h, \boldsymbol{\chi}_h) + \int_0^t \mathcal{K}_h(t,s;\boldsymbol{\vartheta}_h(s), \boldsymbol{\chi}_h)ds + (\nabla\cdot\boldsymbol{\chi}_h, \rho_h) = 0. \tag{6.3.29}
$$

Integrate (6.3.28) and (6.3.29) and arrive at:

$$(\rho_h, \phi_h) - (\nabla \cdot \widehat{\boldsymbol{\vartheta}}_h, \phi_h) = -(\rho, \phi_h), \quad (6.3.30)$$

$$a_h(\widehat{\boldsymbol{\vartheta}}_h, \boldsymbol{\chi}_h) + \mathcal{K}_h(t,t; \widehat{\widehat{\boldsymbol{\vartheta}}}_h(t), \boldsymbol{\chi}_h) - 2\int_0^t \mathcal{K}_{hs}(s,s; \widehat{\widehat{\boldsymbol{\vartheta}}}_h(s), \boldsymbol{\chi}_h) ds$$
$$+ \int_0^t \int_0^s \mathcal{K}_{hzz}(s,z; \widehat{\widehat{\boldsymbol{\vartheta}}}_h(z), \boldsymbol{\chi}_h) dz ds + (\nabla \cdot \boldsymbol{\chi}_h, \widehat{\rho}_h) = 0. \quad (6.3.31)$$

Again integrating (6.3.30) and (6.3.31) to get:

$$(\widehat{\rho}_h, \phi_h) - (\nabla \cdot \widehat{\widehat{\boldsymbol{\vartheta}}}_h, \phi_h) = -(\widehat{\rho}, \phi_h), \quad (6.3.32)$$

$$a_h(\widehat{\widehat{\boldsymbol{\vartheta}}}_h, \boldsymbol{\chi}_h) + \int_0^t \mathcal{K}_h(s,s; \widehat{\widehat{\boldsymbol{\vartheta}}}_h(s), \boldsymbol{\chi}_h) ds - 2\int_0^t \int_0^s \mathcal{K}_{hz}(z,z; \widehat{\widehat{\boldsymbol{\vartheta}}}_h(z), \boldsymbol{\chi}_h) dz ds$$
$$+ \int_0^t \int_0^s \int_0^z \mathcal{K}_{hz'z'}(z,z'; \widehat{\widehat{\boldsymbol{\vartheta}}}_h(z'), \boldsymbol{\chi}_h) dz' dz ds + (\nabla \cdot \boldsymbol{\chi}_h, \widehat{\widehat{\rho}}_h) = 0. \quad (6.3.33)$$

Now, we present some lemmas, which will be used in the proof of $\|u - u_h\|$.

**Lemma 6.3.2.** *Let $\widehat{\widehat{\rho}}_h$ and $\widehat{\widehat{\boldsymbol{\vartheta}}}_h$ satisfies (6.3.32) and (6.3.33), then there exists a positive constant $C$, such that the following estimates hold true:*

$$\|\widehat{\widehat{\rho}}_h\|^2 + \int_0^t \|\widehat{\widehat{\boldsymbol{\vartheta}}}_h(s)\|^2 ds \le Ch^4 t \|u_0\|^2, \quad (6.3.34)$$

$$\|\widehat{\widehat{\boldsymbol{\vartheta}}}_h(s)\|^2 + \int_0^t \|\widehat{\rho}_h(s)\|^2 ds \le Ch^4 t \|u_0\|^2. \quad (6.3.35)$$

*Proof.* Put $\phi_h = \widehat{\widehat{\rho}}_h$ and $\boldsymbol{\chi}_h = \widehat{\widehat{\boldsymbol{\vartheta}}}_h$ in (6.3.32) and (6.3.33) respectively, and add to achieve:

$$(\widehat{\rho}_h, \widehat{\widehat{\rho}}_h) + \alpha_1 \|\widehat{\widehat{\boldsymbol{\vartheta}}}_h\|^2$$
$$\le -(\widehat{\rho}, \widehat{\widehat{\rho}}_h) - \int_0^t \mathcal{K}_h(s,s; \widehat{\widehat{\boldsymbol{\vartheta}}}_h(s), \widehat{\widehat{\boldsymbol{\vartheta}}}_h) ds + 2\int_0^t \int_0^s \mathcal{K}_{hz}(z,z; \widehat{\widehat{\boldsymbol{\vartheta}}}_h(z), \widehat{\widehat{\boldsymbol{\vartheta}}}_h) dz ds$$
$$- \int_0^t \int_0^s \int_0^z \mathcal{K}_{hz'z'}(z,z'; \widehat{\widehat{\boldsymbol{\vartheta}}}_h(z'), \widehat{\widehat{\boldsymbol{\vartheta}}}_h) dz' dz ds.$$

Using Young's inequality, followed by the kickback argument, we arrive at the following:

$$\frac{d}{dt}\|\widehat{\rho}_h\|^2 + \|\widehat{\widehat{\boldsymbol{\vartheta}}}_h\|^2 \leq C\left(\|\widehat{\rho}\|^2 + \|\widehat{\rho}_h\|^2 + \int_0^t \|\widehat{\widehat{\boldsymbol{\vartheta}}}_h(s)\|^2 ds + \int_0^t \int_0^s \|\widehat{\widehat{\boldsymbol{\vartheta}}}_h(z)\|^2 dz ds \right.$$
$$\left. + \int_0^t \int_0^s \int_0^z \|\widehat{\widehat{\boldsymbol{\vartheta}}}_h(z')\|^2 dz' dz ds \right).$$

Integrate the above equation from 0 to $t$, followed by the use of Grönwall's lemma and (6.3.14) completes the proof of (6.3.34).

Now, for the proof of (6.3.35), put $\phi_h = \widehat{\rho}_h$ in (6.3.32) and $\boldsymbol{\chi}_h = \widehat{\widehat{\boldsymbol{\vartheta}}}_h$ in (6.3.31) and add to obtain:

$$\|\widehat{\rho}_h\|^2 + \frac{\alpha_1}{2}\frac{d}{dt}\|\widehat{\widehat{\boldsymbol{\vartheta}}}_h\|^2 \leq -(\widehat{\rho}, \widehat{\rho}_h) + \mathcal{K}_h(t, t; \widehat{\widehat{\boldsymbol{\vartheta}}}_h(t), \widehat{\widehat{\boldsymbol{\vartheta}}}_h) - 2\int_0^t \mathcal{K}_{hs}(s, s; \widehat{\widehat{\boldsymbol{\vartheta}}}_h(s), \widehat{\widehat{\boldsymbol{\vartheta}}}_h) ds$$
$$+ \int_0^t \int_0^s \mathcal{K}_{hzz}(s, z; \widehat{\widehat{\boldsymbol{\vartheta}}}_h(z), \widehat{\widehat{\boldsymbol{\vartheta}}}_h) dz ds.$$

Using Young's inequality and the kickback argument, we arrive at the following:

$$\|\widehat{\rho}_h\|^2 + \frac{d}{dt}\|\widehat{\widehat{\boldsymbol{\vartheta}}}_h\|^2 \leq C\left(\|\widehat{\rho}\|^2 + \|\widehat{\widehat{\boldsymbol{\vartheta}}}_h\|^2 + \int_0^t \|\widehat{\widehat{\boldsymbol{\vartheta}}}_h(s)\|^2 ds\right).$$

Integrating the above equation form 0 to $t$ gives us the following:

$$\int_0^t \|\widehat{\rho}_h(s)\|^2 ds + \|\widehat{\widehat{\boldsymbol{\vartheta}}}_h(s)\|^2 ds \leq C\left(\int_0^t \left(\|\widehat{\rho}(s)\|^2 ds + \|\widehat{\widehat{\boldsymbol{\vartheta}}}_h(s)\|^2\right) ds\right).$$

Use of Grönwall's lemma and (6.3.14) completes the proof of (6.3.35). □

**Lemma 6.3.3.** *Let $\rho_h$, $\widehat{\rho}_h$ and $\widehat{\boldsymbol{\vartheta}}_h$ satisfies (6.3.30), (6.3.32) and (6.3.31), then there exists a positive constant $C$, such that the following estimates hold true:*

$$t\|\widehat{\rho}_h\|^2 + \int_0^t s\|\widehat{\boldsymbol{\vartheta}}_h(s)\|^2 ds \leq Ch^4 t\|u_0\|^2, \tag{6.3.36}$$
$$t^2\|\widehat{\boldsymbol{\vartheta}}_h\|^2 + \int_0^t s^2\|\rho_h(s)\|^2 ds \leq Ch^4 t\|u_0\|^2. \tag{6.3.37}$$

*Proof.* Put $\phi_h = t\widehat{\rho}_h$ and $\boldsymbol{\chi}_h = t\widehat{\boldsymbol{\vartheta}}_h$ in (6.3.30) and (6.3.31) and add to arrive at the following:

$$\frac{1}{2}\frac{d}{dt}t\|\widehat{\rho}_h\|^2 + \alpha_1 t\|\widehat{\boldsymbol{\vartheta}}_h\|^2$$
$$= -t(\rho, \widehat{\rho}_h) + \frac{1}{2}\|\widehat{\rho}_h\|^2 - t\mathcal{K}_h(t,t;\widehat{\widehat{\boldsymbol{\vartheta}}}_h(t), \widehat{\boldsymbol{\vartheta}}_h) + 2t\int_0^t \mathcal{K}_{hs}(s,s;\widehat{\widehat{\boldsymbol{\vartheta}}}_h(s), \widehat{\boldsymbol{\vartheta}}_h)ds$$
$$- t\int_0^t\int_0^s \mathcal{K}_{hzz}(s,z;\widehat{\widehat{\boldsymbol{\vartheta}}}_h(z), \widehat{\boldsymbol{\vartheta}}_h)dzds.$$

Using the Young's inequality, kickback argument and then integrating from 0 to $t$ to arrive at the following:

$$t\|\widehat{\rho}_h\|^2 + \int_0^t s\|\widehat{\boldsymbol{\vartheta}}_h(s)\|^2 ds$$
$$\leq C\int_0^t \left(\|\widehat{\rho}_h\|^2 + s^2\|\rho(s)\|^2 ds + \|\widehat{\widehat{\boldsymbol{\vartheta}}}_h(s)\|^2 ds + s\int_0^s \|\widehat{\widehat{\boldsymbol{\vartheta}}}_h(z)\|^2 dz\right) ds.$$

Using (6.3.34) and (6.3.35), we arrive at the following:

$$t\|\widehat{\rho}_h\|^2 + \int_0^t s\|\widehat{\boldsymbol{\vartheta}}_h(s)\|^2 ds \leq Ch^4 t\|u_0\|^2.$$

Now, put $\phi_h = t^2\rho_h$ and $\boldsymbol{\chi}_h = t^2\widehat{\boldsymbol{\vartheta}}_h$ in (6.3.30) and (6.3.29) to arrive at:

$$t^2\|\rho_h\|^2 + t^2 a_h(\boldsymbol{\vartheta}_h, \widehat{\boldsymbol{\vartheta}}_h) + t^2\int_0^t \mathcal{K}_h(t,s;\boldsymbol{\vartheta}_h(s), \widehat{\boldsymbol{\vartheta}}_h)ds = -t^2(\rho, \rho_h).$$

Note that $\frac{d}{dt}(t^2 a_h(\widehat{\boldsymbol{\vartheta}}_h, \widehat{\boldsymbol{\vartheta}}_h)) = 2ta_h(\widehat{\boldsymbol{\vartheta}}_h, \widehat{\boldsymbol{\vartheta}}_h) + t^2 2a_h(\boldsymbol{\vartheta}_h, \widehat{\boldsymbol{\vartheta}}_h)$, hence:

$$t^2\|\rho_h\|^2 + \frac{\alpha_1}{2}\frac{d}{dt}(t^2\|\widehat{\boldsymbol{\vartheta}}_h\|^2)$$
$$\leq C\Big(t\|\widehat{\boldsymbol{\vartheta}}_h\|^2 - t^2(\rho, \rho_h) + t^2\mathcal{K}_h(t,t;\widehat{\boldsymbol{\vartheta}}_h(t), \widehat{\boldsymbol{\vartheta}}_h) - t^2\mathcal{K}_{h,t}(t,t;\widehat{\boldsymbol{\vartheta}}_h(t), \widehat{\boldsymbol{\vartheta}}_h)$$
$$+ t^2\int_0^t \mathcal{K}_{hss}(t,s;\widehat{\widehat{\boldsymbol{\vartheta}}}_h(s), \widehat{\boldsymbol{\vartheta}}_h)ds\Big).$$

Using Young's inequality, kickback argument and integrating from 0 to $t$ to arrive at the following:

$$t^3\|\widehat{\boldsymbol{\vartheta}}_h\|^2 + \int_0^t s^2\|\rho_h(s)\|^2 ds \leq C\int_0^t \left(s\|\widehat{\boldsymbol{\vartheta}}_h(s)\|^2 ds + s^2\|\rho(s)\|^2 + s^2\int_0^s \|\widehat{\widehat{\boldsymbol{\vartheta}}}(z)\|^2 dz\right) ds.$$

Use of Grönwall's lemma, (6.3.1), (6.3.36) and (6.3.34) completes the proof of (6.3.37). □

**Theorem 6.3.3.** *Let* $(u, \boldsymbol{\sigma})$ *and* $(u_h, \boldsymbol{\sigma}_h)$ *satisfy* (6.1.2) *and* (6.1.4)*, respectively. Then, there exists a positive constant* $C$ *independent of the* $h$ *such that for* $t \in (0, T]$*, the following estimates are valid:*

$$\|u(t) - u_h(t)\| \leq Ch^2 t^{-1} \|u_0\|.$$

*Proof.* Consider

$$u - u_h = \rho + \rho_h \text{ where } \rho_h = \tilde{u} - u_h,$$
$$\boldsymbol{\sigma} - \boldsymbol{\sigma}_h = \boldsymbol{\vartheta} + \boldsymbol{\vartheta}_h \text{ where } \boldsymbol{\vartheta}_h = \tilde{\boldsymbol{\sigma}} - \boldsymbol{\sigma}_h.$$

Since, we already have the estimates of $\|\rho\|$ and $\|\boldsymbol{\vartheta}\|$, we need to find $\|\rho_h\|$ and $\|\boldsymbol{\vartheta}_h\|$. Put $\phi_h = t^3 \rho_h$ and $\boldsymbol{\chi}_h = t^3 \boldsymbol{\vartheta}_h$ in (6.3.28) and (6.3.29) respectively, and then add to arrive at the following:

$$\begin{aligned}
&t^3(\rho_{h,t}, \rho_h) + t^3 a_h(\boldsymbol{\vartheta}_h, \boldsymbol{\vartheta}_h) + t^3 \int_0^t \mathcal{K}_h(t, s; \boldsymbol{\vartheta}_h(s), \boldsymbol{\vartheta}_h) ds = -t^3(\rho_t, \rho_h), \\
&\frac{1}{2}\frac{d}{dt}(t^3 \|\rho_h\|^2) + \alpha_1 t^3 \|\boldsymbol{\vartheta}_h\|^2 \\
&\leq \frac{3}{2} t^2 \|\rho_h\|^2 - t^3(\rho_t, \rho_h) + t^3 \mathcal{K}_h(t, t; \widehat{\boldsymbol{\vartheta}}_h(t), \boldsymbol{\vartheta}_h) - t^3 \mathcal{K}_{h,t}(t, t; \widehat{\widehat{\boldsymbol{\vartheta}}}_h(s), \boldsymbol{\vartheta}_h) \\
&\quad + t^3 \int_0^t \mathcal{K}_{hss}(t, s; \widehat{\widehat{\boldsymbol{\vartheta}}}_h(s), \boldsymbol{\vartheta}_h) ds.
\end{aligned}$$

Using Young's inequality and then kickback arguments, we obtain:

$$\frac{d}{dt}(t^3 \|\rho_h\|^2) + t^3 \|\boldsymbol{\vartheta}_h\|^2 = C \left( t^2 \|\rho_h\|^2 + t^4 \|\rho_t\|^2 + t^3 \|\widehat{\boldsymbol{\vartheta}}_h\|^2 + t^3 \|\widehat{\widehat{\boldsymbol{\vartheta}}}_h\|^2 + t^3 \int_0^t \|\widehat{\widehat{\boldsymbol{\vartheta}}}_h(s)\|^2 ds \right).$$

Now, integrate the above equation from 0 to $t$ to arrive at:

$$t^3 \|\rho_h\|^2 + \int_0^t s^3 \|\boldsymbol{\vartheta}_h(s)\|^2 ds$$

$$\leq C\int_0^t \left(s^2\|\rho_h(s)\|^2 + s^4\|\rho_s(s)\|^2 + s^3\|\widehat{\boldsymbol{\vartheta}}_h(s)\|^2 + s^3\|\widehat{\widehat{\boldsymbol{\vartheta}}}_h(s)\|^2 + s^3\int_0^s \|\widehat{\widehat{\boldsymbol{\vartheta}}}_h(z)\|^2 dz\right) ds. \tag{6.3.38}$$

Use of (6.3.15), (6.3.34) and (6.3.37), yields:

$$\|\rho_h\| \leq Ch^2 t^{-1}\|u_0\|. \tag{6.3.39}$$

Now, we use the triangle inequality, (6.3.1) and (6.3.39) to arrive at the following:

$$\|u - u_h\| \leq Ch^2 t^{-1}\|u_0\|.$$

□

**Theorem 6.3.4.** *Let* $(u, \boldsymbol{\sigma})$ *and* $(u_h, \boldsymbol{\sigma}_h)$ *satisfy* (6.1.2) *and* (6.1.4)*, respectively. Then, there exists a positive constant* $C$ *independent of the* $h$ *such that for* $t \in (0, T]$*,*

$$\|\boldsymbol{\sigma}(t) - \boldsymbol{\sigma}_h(t)\| \leq Cht^{-1}\|u_0\|.$$

To prove this result, use similar arguments as in [14].

## 6.4 Conclusions

Given the advantages of both VEM and mixed methods, we used a mixed VEM approach to tackle a linear PIDE with non-smooth initial data. This chapter explored two formulations of the mixed VEM approach for the PIDE. We introduced two new projections, mixed intermediate projection and mixed Ritz Volterra projection, which includes a memory term and derived estimates for the same. By the repeated use of the integral operator, the Fortin operator, and the properties of $L^2$ projection, we established the error estimates for the two unknowns in both formulations.

# Chapter 7

# Conclusions

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This chapter highlights the critical assessment of the outcomes, emphasizing the contributions made by this research and the methodologies employed to achieve them. Further, we discuss the possible extensions and the scope for further investigations in this direction.

## 7.1 Crtical Review of the Results

In this thesis, we studied VEM and mixed VEM for PIDE. In contrast to the conventional FEM, the suggested approach permits the presence of hanging nodes during mesh generation, eliminates the need for explicit construction of nodal basis functions, effectively manages complicated domains, and allows higher-order polynomials. Consequently, these features enhance accuracy and adaptability when dealing with convex and non-convex polygonal meshes. Since this is the case where we don't require the basis functions explicitly, we need suitable projectors from the local VEM spaces to some polynomial spaces to construct the element-by-element necessary local matrices.

In Chapter 2, we developed and analyzed the confirming VEM for linear PIDE with smooth initial data. For the discrete formulation, we defined the local bilinear forms with the help of $\Pi_k^0$ projection and discussed their consistency and stability. The analysis for the semi-discrete and fully-discrete cases required the introduction of a new R.V. projection

(2.2.1) to deal with the integral term. For the $H^1$ estimates (2.2.2) of R.V. projection, we first split $u - R^h u$ into two parts, $u - u^I$ and $u_I - R^h u$, where the function $u_I$ is an interpolant of $u$, in $W_h$ and satisfies

$$\text{dof}_n(u_I) = \text{dof}_n(u) \quad n = 1, ..., N_{\text{dof}},$$

where $\text{dof}_n(\cdot)$ indicates the operator which relates the $n$th degree of freedom to each smooth enough function, and $N_{\text{dof}}$ stands for the number of degrees of freedom. Whereas to find out the $L^2$ estimate (2.2.2) of R.V. projection, we used the dual problem: Let $\phi \in H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})$, with $\mathcal{D}$ to be convex and bounded, be the solution of

$$-\nabla \cdot (a\nabla\phi) + a_0\phi = \rho \quad \text{in } \mathcal{D} \quad \phi = 0 \quad \text{on } \partial\mathcal{D},$$

where $\rho = R^h u - u$. We also estimated the time derivative of R.V. projection, and then by using the (2.0.2) and (2.1.1) along with the R.V. estimates, we proved the optimal order of convergence.

Then, we also studied the fully-discrete case(2.3.1), where the time derivative is approximated by the backward Euler method and the integral term is dealt with the left rectangular rule. Several numerical experiments (Section 2.4) are presented to confirm the computational efficiency of the proposed scheme and validate the theoretical findings. Moreover, in order to show the real application of VEMs, numerical experiments are conducted with local mesh refinements (Fig.2.3), which are necessary to reduce the overall computational cost but may not be possible in the context of conforming FEMs.

Chapter 3 developed and analyzed the mixed VEM for PIDE by using a resolvent kernel. An aspect of our focus in (1.3.1) involves the determination of the flux or velocity in conjunction with the pressure. The conventional Galerkin method tends to suffer from reduced accuracy because it calculates this quantity through post-processing from the approximated solution. In contrast, mixed methods offer a direct estimation of this physical parameter, resulting in solutions that are locally conservative. Another benefit of employing a mixed approach in this context is the capability to introduce an additional physically significant variable, which can be computed directly without introducing new sources of error. We introduced $\boldsymbol{\sigma}(\boldsymbol{x}, t)$, defined by (1.3.4) and and rewrite (1.3.1) as:

$$u_t(\boldsymbol{x}, t) - \nabla \cdot \boldsymbol{\sigma}(\boldsymbol{x}, t) = f(\boldsymbol{x}, t).$$

Considering the computational advantages of the formulation described in [14], we analyzed this formulation for the semi-discrete (3.1.2) and fully-discrete case (3.3.1). To deal with the integral term, mixed R.V. projection (3.2.3) is introduced. The time derivative of mixed R.V. projection (3.2.27), (3.2.28) is also estimated along with the estimates of mixed R.V. projection (3.2.5), (3.2.6). Semi-discrete error estimates are established by using (1.3.8) and (3.1.2). Furthermore a step-by-step superconvergence of $\Pi_k^0 u - u_h$ is proved (Section 3.2.2).

A fully-discrete case has also been developed (3.3.1) and analyzed (Theorem 3.3.1) for this formulation. We also conduct numerical experiments (Section 3.4) to validate the effectiveness of the introduced mixed virtual element scheme for the PIDE.

In Chapter 4, we analyzed the mixed VEM for PIDE. This chapter focused on the variational formulation that eliminates the need for a resolvent kernel. We adopt the formulation outlined by [15]. It's important to note that due to the additional term beneath the integral, this formulation may take a longer computational time when compared to the approach presented in [14]. Nevertheless, this formulation offers a broader scope of applicability, even in cases where finding a resolvent kernel proves challenging. By using this formulation, we can attain the necessary convergence, as illustrated through numerical experiments. To deal with the integral term, we define a mixed Intermediate projection (4.2.1b) and find out the estimates (Theorem 3.2.1) for the same. Then, semi-discrete error estimates are derived by using (1.3.10a) and (4.1.1). Furthermore, the fully-discrete case (4.3.1) has also been examined, and the error estimates have been derived. We presented a systematic analysis that outlines the step-by-step process for achieving super convergence (4.2.20) of the discrete solution, with an order of $O(h^{k+2})$. Several computational experiments are discussed to validate the proposed scheme's computational efficiency and support the theoretical conclusions.

In Chapter 5, we analyzed the VEM for the linear PIDE with the non-smooth initial data. Chapter 2 focuses on establishing optimal error estimates for smooth initial data, but for less regular initial data, a different analysis is required. For this, we first define an Intermediate projection (5.2.1) with an integral term accordingly. By the repetitive use of the integration by parts and some regularity results, we first derive the estimates for $\hat{u} - \widehat{I^h u}$ (5.2.3), and then by using this estimate, we derive the estimates (5.2.2) of the Intermediate projection in terms of the initial data is in $L^2(\mathcal{D})$. Finally, we proved some lemmas, which helped us to establish the optimal error estimates.

Chapter 6 explored two formulations of the mixed VEM approach for the PIDE. We introduced two new projections, mixed Intermediate projection and mixed Ritz Volterra projection, which includes a memory term and derived estimates for the same. By the repeated use of the integral operator, the Fortin operator, and the properties of $L^2$ projection, we established the error estimates for the two unknowns in both formulations.

## 7.2 Possible Extensions and Future Problems

- *VEM for hyperbolic integro-differential equations:* In Chapter 2, we developed and analyzed the confirming VEM for linear PIDE. The results of this can be easily extended to the hyperbolic integro-differential equations: Find $u(\boldsymbol{x}, t)$ such that

$$
\begin{aligned}
u_{tt}(\boldsymbol{x}, t) - \nabla \cdot (a(\boldsymbol{x}) \nabla u(\boldsymbol{x}, t) - \textstyle\int_0^t b(\boldsymbol{x}; t, s) \nabla \quad & u(\boldsymbol{x}, s) ds) \\
&= f(\boldsymbol{x}, t) \quad (\boldsymbol{x}, t) \in \mathcal{D} \times (0, T], \\
u(\boldsymbol{x}, t) &= 0 \quad (\boldsymbol{x}, t) \in \partial\mathcal{D} \times [0, T], \\
u(\boldsymbol{x}, 0) &= u_0(\boldsymbol{x}) \quad \boldsymbol{x} \in \mathcal{D},
\end{aligned}
\tag{7.2.1}
$$

- *Mixed VEM for hyperbolic integro-differential equations:* In Chapter 3 and Chapter 4, we developed and analyzed the mixed VEM for linear PIDE. The results of this can be easily extended to the hyperbolic integro-differential equations (7.2.1). By introducing $\boldsymbol{\sigma}(\boldsymbol{x}, t)$, defined by (1.3.4) and rewrite (7.2.1) as:

$$u_{tt}(\boldsymbol{x}, t) - \nabla \cdot \boldsymbol{\sigma}(\boldsymbol{x}, t) = f(\boldsymbol{x}, t).$$

- *Fully-discrete case for the non-smooth data:* In Chapter 5 and Chapter 6, we study the VEM and mixed VEM for the linear PIDE with non-smooth initial data. Our future work includes the extension of this for the fully-discrete case.

- *Mixed VEM for strongly damped wave equation:* We plan to introduce and analyze mixed VEM for strongly damped wave equation defined on a domain $\mathcal{D} \subset \mathbb{R}^2$ such that:

$$\begin{aligned} u_{tt} - \alpha \nabla \cdot (a(\boldsymbol{x}) \nabla u_t) - \nabla \cdot (a(\boldsymbol{x}) \nabla u) = f(\boldsymbol{x}, t) \quad (\boldsymbol{x}, t) \in \mathcal{D} \times (0, \infty, ) \\ u(\boldsymbol{x}, t) = 0 \quad (\boldsymbol{x}, t) \in \partial \mathcal{D} \times [0, \infty), \\ u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}) \quad \boldsymbol{x} \in \mathcal{D}, \\ u_t(\boldsymbol{x}, 0) = u_1(\boldsymbol{x}) \quad \boldsymbol{x} \in \mathcal{D}. \end{aligned}$$

- *Mixed VEM for weakly damped wave equation:* We plan to introduce and analyze mixed VEM for strongly damped wave equation defined on a domain $\mathcal{D} \subset \mathbb{R}^2$ such that:

$$\begin{aligned} u_{tt} - \alpha u_t - \nabla \cdot (a(\boldsymbol{x}) \nabla u) = f(\boldsymbol{x}, t) \quad (\boldsymbol{x}, t) \in \mathcal{D} \times (0, \infty, ) \\ u(\boldsymbol{x}, t) = 0 \quad (\boldsymbol{x}, t) \in \partial \mathcal{D} \times [0, \infty), \\ u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}) \quad \boldsymbol{x} \in \mathcal{D}, \\ u_t(\boldsymbol{x}, 0) = u_1(\boldsymbol{x}) \quad \boldsymbol{x} \in \mathcal{D}. \end{aligned}$$

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# List of Research Publications

## Journal Publications

1. S Yadav, **M Suthar**, and S Kumar "A Conforming Virtual Element Method for Parabolic Integro-Differential Equations", ***Computational Methods in Applied Mathematics*** https://doi.org/10.1515/cmam-2023-0061 (2023).

2. **M Suthar**, and S Yadav, "Mixed Virtual Element Method for Linear Parabolic Integro-Differential Equations", ***International Journal of Numerical Analysis and Modeling*** https://doi.org/10.4208/ijnam2024-1020 (2024).

3. **M Suthar**, S Yadav, and S Kumar, "Mixed Virtual Element Method for Integro-Differential Equations of Parabolic Type." ***Journal of Applied Mathematics and Computing*** https://doi.org/10.1007/s12190-024-02066-8 (2024).

4. **M Suthar**, and S Yadav, "Two Mixed Virtual Element Formulations for Parabolic Integro-Differential Equations with Nonsmooth Initial Data" (**Under Revision**).

5. **M Suthar**, and S Yadav, "A Conforming Virtual Element Method for Parabolic Integro-Differential Equations with Non-smooth Initial Data" (**Under preparation**).

# Conferences/Workshops

## Conferences

1. Attended **International Conference and 22nd Annual Convention of Vijnana Parishad of India on Advances in Operations Research Statistics and Mathematics (AOSM)** organized by BITS Pilani, Pilani Campus (Dec 2019). **(Attended)**.

2. **International Conference on Frontiers in Industrial and Applied Mathematics (FIAM)** organized by Department of Mathematics SLIET, Longowal (Dec 2021). **(Presented)**.

3. **International Conference on Computational Partial Differential Equations and Applications (ICCPDEA)** organized by BML Munjal University, Gurgaon (Sept 2022). **(Attended)**.

4. **International Conference on Differential Equations and Control Problems 2023 (ICDECP23)**, organized by IIT Mandi (June 2023). **(Presented)**.

# Workshops

1. **International Workshop on Stochastic Simulation and Its Applications (WSSA-2019)** organized by BITS Pilani, Pilani Campus (Dec 24-27, 2019).

2. Mini-Symposium on **LaTeX and MATLAB** organized by BITS Pilani, Pilani Campus (Feb 02, 2020).

3. Workshop on **Adaptive finite element method** organized by IIT Bombay, Scheme for Promotion of Academic and Research Collaboration (SPARC) (July 2-7, 2021).

4. **International Workshop on Recent Advancements in Data Envelopment Analysis and Applications (IWRADEAA-2021)** organized by BITS Pilani, Pilani Campus (July 10-11, 2021).

5. Workshop on **Numerical Methods for Differential Equations and Applications** organized by BITS Pilani, Pilani Campus (March 27-28, 2023).

6. Mini-Symposium on **Fixed point theorems and their applications to differential equations** organized by BITS Pilani, Pilani Campus (Dec 05, 2023).

7. **A Three Days International Workshop on Virtual Element Analysis: Scientific Computation and Applications** organized by VIT Vellore (Jan 10-12, 2024).

# Biography of the Candidate

Ms. Meghana completed her B.Sc. from Kurukshetra University, Kurukshetra, and subsequently her M.Sc. in Mathematics from Guru Jambheshwar University of Science & Technology, Hisar. She has qualified the CSIR-UGC Junior Research Fellowship (JRF) test and the National Eligibility Test (NET) in Mathematical Sciences in December 2018. In 2019, she joined as a full-time Ph.D. scholar in the Department of Mathematics at Birla Institute of Technology and Science (BITS) Pilani, India under the supervision of Prof. Sangita Yadav. Her research interests include numerical methods for PDEs, Finite Element Method, Virtual Element Method. She has served as a teaching assistant for the various courses of mathematics at BITS Pilani, Pilani Campus. Contact her at `p20190039@pilani.bits-pilani.ac.in`.

# Biography of the Supervisor

Prof. Sangita Yadav is an Associate Professor in the Department of Mathematics at Birla Institute of Technology and Science Pilani, Pilani campus. She earned her Ph.D. degree from the Indian Institute of Technology, Bombay. Her research interests include numerical analysis, scientific computing, finite element methods, discontinuous Galerkin methods, multigrid methods, and adaptive methods. She has several research publications in reputed international journals. She is a member of the Indian Mathematical Society (IMS) and the Indian Society of Industrial and Applied Mathematics (ISIAM). She has also attended several international conferences and symposiums.