Chapter 4

The Commuting Graphs

The commuting graph has been attracted a great attention in last two decades and published a large number of articles (see Bates et al. [2003a]; Bundy [2006]; Giudici and Parker [2013]; Segev [2001]). The commuting graph $\Delta(G, \Omega)$ of a group G is a simple undirected graph whose vertex set is $\Omega \subseteq G$ and two vertex x, y are adjacent if and only if $xy = yx$. The commuting graphs were introduced by Brauer and Fowler [1955] with $\Omega = G \setminus \{e\}$. In recent years, the commuting graphs of various algebraic structures have become a topic of research for many mathematicians, see Ali et al. [2016]; Araújo et al. [2015]; Shitov [2016, 2018] and references therein. Araújo et al. [2011] calculated the diameter of commuting graphs of various ideals of full transformation semigroup. Also, for every natural number $n \geq 2$, a finite semigroup whose commuting graph has diameter n has been constructed in Araújo et al. [2011]. Iranmanesh and Jafarzadeh [2008] studied diameter, girth, clique number, independence number etc. of the commuting graph associated with symmetric group and alternating group. Tolue [2020] introduced the twin non-commuting graph by partitioning the vertices of non-commuting graph and studied the graph-theoretic properties of twin non-commuting graph of AC-group and dihedral group. The distant properties as well as detour distant properties of the commuting graph on the dihedral group D_{2n} were investigated by Ali et al. [2016]. Moreover, they obtained metric dimension of the commuting graph on D_{2n} and its resolving polynomial. Recently, Kakkar and Rawat [2018] studied the detour distance properties and obtained the resolving polynomial of the commuting graph of generalized dihedral group.

In this chapter, we discuss graph-theoretic and algebraic properties of commuting graph associated with groups and semigroups (in particular, Brandt semigroups). We begin with the investigation of the commuting graph $\Delta(G, \Omega)$, with $\Omega = G$, denoted by $\Delta(G)$. In Section 4.1, we calculate the minimum degree, edge connectivity of commuting graph of a finite group G. Also, we give a formulae of matching number of $\Delta(G)$ when $|G|$ is odd, otherwise we give bound of matching number of $\Delta(G)$. Further, we prove that if the automorphism group of commuting graphs of two groups are isomorphic and one of them is an AC-group then other is also an AC-group. Moreover, we investigate the other graph-theoretic properties of commuting graph for a finite group G, viz., boundary graph $\partial(\Delta(G))$, interior graph $Int(\Delta(G))$, eccentric graph of $Ecc(\Delta(G))$. In this connection, we prove that for finite group G, $\partial(\Delta(G)) = Ecc(\Delta(G))$. Section 4.2 comprises the study of various graph invariants of $\Delta(SD_{8n})$ viz. Hamiltonian, perfectness, independence number, clique number, vertex connectivity, edge connectivity, vertex covering number, edge covering number etc. Moreover, we study the Laplacian spectrum, metric dimension, resolving polynomial and the detour properties of the commuting graph of SD_{8n}. In Section 4.3, we study the commuting graph of Brandt semigroups B_n - an important class of inverse semigroups. The content of Sections 4.1 and 4.2 is accepted for publication in SCIE journal *"Bulletin of the Malaysian Mathematical Sciences Society"*, To appear, DOI: https://doi.org/10.1007/s40840-021-01111-0.

4.1 The Commuting Graph of Groups

In this section, we investigate the commuting graph of an arbitrary group G. First, we show that the edge connectivity and the minimum degree of $\Delta(G)$ are equal. For $a \in G$, let $cl(a)$ be the conjugacy class of G containing a. The centre of G is denoted by $Z(G)$ and the centralizer of a is denoted by

$$C_G(a) = \{b \in G : ab = ba\}.$$

The following remark follows from the definition of $\Delta(G)$.

Remark 4.1.1. In the commuting graph of a group G, we have $\mathrm{N}[x] = C_G(x)$ for each $x \in G$.

Theorem 4.1.2. *Let G be a finite group and $t = \max\{|cl(a)| : a \in G\}$. Then*

$$\kappa'(\Delta(G)) = \delta(\Delta(G)) = \frac{|G|}{t} - 1.$$

Proof. In view of Remark 4.1.1, $\delta(\Delta(G)) = r - 1$, where $r = \min\{|C_G(a)| : a \in G\}$. For a graph Γ, since $\kappa'(\Gamma) \leq \delta(\Gamma)$ we obtain $\kappa'(\Delta(G)) \leq r-1$. By Menger's theorem (cf. [Bondy et al., 1976, Theorem 3.2]), to prove another inequality, it is sufficient to show that there exist at least $r-1$ internally edge disjoint paths between arbitrary pair of vertices. Let x and y be the distinct pair of vertices in $\Delta(G)$. Suppose $|C_G(x) \cap C_G(y)| = q$. For $z \in C_G(x) \cap C_G(y)$, we have $x \sim z$ and $y \sim z$. Then $\Delta(G)$ contains at least q internally edge disjoint paths between x and y. Further there exist $x_1, x_2, \ldots, x_{r-q-1} \in C_G(x) \setminus C_G(y)$ and $y_1, y_2, \ldots, y_{r-q-1} \in C_G(y) \setminus C_G(x)$. Consequently, we get $x \sim x_i \sim e \sim y_i \sim y$ internally edge disjoint paths between x and y which are $r-q-1$ in total. Thus, we have at least $r-1$ internally disjoint paths between x and y. Since for $x \in G$, we have $|cl(x)| = \frac{|G|}{|C_G(x)|}$. Hence, $\kappa'(\Delta(G)) = \delta(\Delta(G)) = \frac{|G|}{t} - 1$, where $t = \max\{|cl(a)| : a \in G\}$. □

In the following theorem, we obtain the matching number of $\Delta(G)$.

Theorem 4.1.3. *Let G be a finite group and let t be the number of involutions in $G \setminus Z(G)$. Then*

(i) $$\alpha'(\Delta(G)) = \begin{cases} \dfrac{|G|-1}{2} & \text{if } |G| \text{ is odd;} \\ \\ \dfrac{|G|}{2} & \text{if } |G| \text{ is even and } t \leq |Z(G)|; \end{cases}$$

(ii) *for $t > |Z(G)|$ and G is of even order, we have*

$$\frac{|G| + |Z(G)| - t}{2} \leq \alpha'(\Delta(G)) \leq \frac{|G|}{2}.$$

Proof. (i) Let G be a finite group of odd order. Observe that for $x \in G \backslash \{e\}$, we have $x \neq x^{-1}$ as $o(x) > 2$ and $x \sim x^{-1}$. Thus, $M = \{(x, x^{-1}) : x \neq e \in G\}$ is a matching of order $\dfrac{|G|-1}{2}$ in $\Delta(G)$. On the other hand, the order of a largest matching in a graph of order n is $\left\lfloor \dfrac{n}{2} \right\rfloor$. Hence, we get $\alpha'(\Delta(G)) = \dfrac{|G|-1}{2}$.

Now we assume that G is of even order and $t \leq |Z(G)|$. Note that $x \in Z(G)$ if and only if $x^{-1} \in Z(G)$. Consider the set $A = \{a \in G \setminus Z(G) : o(a) = 2\}$ whose cardinality is t. Further, we denote the edges with ends a_i and z_i by ϵ_i, where $a_i \in A$ and $z_i \in Z(G)$. Let $M = \{\epsilon_i : 1 \leq i \leq t\} \bigcup \{(x, x^{-1}) : x \neq e \in G \setminus Z(G)\}$ is a matching such that $G \setminus G_M \subseteq Z(G)$, where $G_M = \{x \in G : (\exists\, x' \in G),\ (x, x') \in M\}$. Clearly, $|G \setminus G_M|$ is even as both $|G|$ and $|G_M|$ are even. Consequently, $\mathcal{M} = M \cup \{(x, x') : x \neq x' \text{ and } x, x' \in G_M\}$ is a matching of size $\dfrac{|G|}{2}$. Since $\alpha'(\Delta(G)) \leq \dfrac{|G|}{2}$, we have the result.

(ii) Suppose $|G|$ is even and $t > |Z(G)|$. By the proof of part (i), we have a matching $\mathcal{M}$ of size at least $\dfrac{|G| + |Z(G)| - t}{2}$. Thus, we get the desired inequality. □

In view of Lemma 1.2.10(ii), we have the following corollary.

Corollary 4.1.4. *For a finite group G and let t be the number of involutions in $G \setminus Z(G)$, we have*

(i) $\beta'(\Delta(G)) = \begin{cases} \dfrac{|G|+1}{2} & \textit{if } |G| \textit{ is odd;} \\ \\ \dfrac{|G|}{2} & \textit{if } |G| \textit{ is even and } t \leq |Z(G)|; \end{cases}$

(ii) *for* $t > |Z(G)|$ *and* G *is of even order, we have*

$$\frac{|G|}{2} \leq \beta'(\Delta(G)) \leq \frac{|G| - |Z(G)| + t}{2}.$$

For $x \in G \setminus Z(G)$, $C_G(x)$ is called *maximal centralizer* if there is no $y \in G \setminus Z(G)$ such that $C_G(x)$ is a proper subgroup of $C_G(y)$. In the following proposition, we compute the vertex connectivity of $\Delta(G)$.

Proposition 4.1.5. *Let* G *be a finite non-abelian group such that, for some* $x \in G$, $C_G(x)$ *is a maximal centralizer and an abelian subgroup of* G. *Then* $\kappa(\Delta(G)) = |Z(G)|$.

Proof. Suppose $C_G(x)$ is an abelian subgroup of G for some $x \in G$. Clearly, $C_G(x) \neq G$ as G is a non-abelian group. For $y \in C_G(x)$, we have $zy = yz$ for all $z \in C_G(x)$ as $C_G(x)$ is an abelian subgroup of G gives $C_G(x) \subseteq C_G(y)$. Since $C_G(x)$ is a maximal centralizer so $C_G(x) = C_G(y)$ for all $y \in C_G(x) \setminus Z(G)$. As G is a non-abelian group, there exists $z \in G$ such that $xz \neq zx$. It follows that there is no path between x and z in the subgraph induced by the vertices of $G \setminus Z(G)$. For instance if there is a path $x = x_1 \sim x_2 \sim \cdots \sim x_r = y$ for some $r > 1$ then it follows that $x_i \in C_G(x)$ for all i, where $1 \leq i \leq r$. Thus, $xz = zx$; a contradiction. Thus, the subgraph induced by the vertices of $G \setminus Z(G)$ is disconnected so $\kappa(\Delta(G)) \leq |Z(G)|$.

If there exists a vertex cut-set $\mathcal{O}$ which do not contain $Z(G)$, then there exists $a \in Z(G)$ such that $a \notin \mathcal{O}$. For distinct $x, y \in G \setminus \mathcal{O}$, we have $x \sim a \sim y$. Consequently, the subgroup induced by the vertices of $G \setminus \mathcal{O}$ is connected implies the set $\mathcal{O}$ is not a vertex cut-set; a contradiction. Thus, any vertex cut-set always contains $Z(G)$. Consequently, $\kappa(\Delta(G)) \geq |Z(G)|$ and hence $\kappa(\Delta(G)) = |Z(G)|$. $\square$

A group G is called an AC-group if the centralizer of every non-central element is abelian.

Theorem 4.1.6. *Let G be a finite group such that $\Delta(G) \cong \Delta(H)$ for some AC-group H. Then G is an AC-group.*

Proof. Suppose $\Delta(G) \cong \Delta(H)$, where H is an AC-group. By [Dutta and Kanti Nath, 2017, Lemma 2.1], the subgraph induced by the vertices of $H \setminus Z(H)$ is $\bigcup_{i=1}^{r} K_{|X_i|-|Z(H)|}$, where $X_1, X_2, \ldots, X_r$ are the distinct centralizers of non-central elements of H. Note that for each $x \in Z(H)$, we get $x \sim y$ for all $y \in H$. Therefore, we have $\Delta(G) \cong \Delta(H) = K_{|Z(H)|} \vee \left(\bigcup_{i=1}^{r} K_{|X_i|-|Z(H)|} \right)$. If $x \in G \setminus Z(G)$, then clearly $\mathrm{N}[x] = X_i$ for some i. The subgraph induced by the vertices of X_i is complete follows that $C_G(x)$ is an abelian subgroup of G. Thus, G is an AC-group. □

Proposition 4.1.7. *Let K be a clique in $\Delta(G)$. Then $\omega(\Delta(G)) = |K|$ if and only if K is a commutative subgroup of maximum size in G.*

Proof. Let K be any clique of maximum size such that $x, y \in K$. Then xy commutes to every element of K. Consequently, we get $xy \in K$ as $|K|$ is maximum. Note that the identity element e of G is in K. If $x \in K$, then $x^{-1} \in \langle x \rangle$ so x^{-1} commutes with every element of K. Since K is a clique of maximum size we obtain $x^{-1} \in K$. Therefore K forms a subgroup of G. Clearly, K is a commutative subgroup of maximum size. Converse part is straightforward. □

The proof of the following lemma follows from the definition of complete vertex.

Lemma 4.1.8. *An element x is a complete vertex in $\Delta(G)$ if and only if $C_G(x)$ is a commutative subgroup of G.*

Proposition 4.1.9. *For any group G, we have*

$$Ecc(\Delta(G)) = \begin{cases} \Delta(G) \setminus \{e\} & \text{if } |Z(G)| = 1; \\ \Delta(G) & \text{Otherwise.} \end{cases}$$

Proof. Let $x \in G \setminus \{e\}$. Then $d(x, e) = ecc(e)$ so that x is an eccentric vertex for e. Thus, each non identity element of G is an eccentric vertex of $\Delta(G)$. If $|Z(G)| > 1$, then there exists $x \in Z(G) \setminus \{e\}$. Note that e is an eccentric vertex for x. Thus the result holds. For $Z(G) = \{e\}$, one can observe that e is not an eccentric vertex of $\Delta(G)$. For instance, if e is an eccentric vertex for some $y \in G$, then $d(x, e) = 1 = ecc(y)$. As $y \in G \setminus Z(G)$, there exists $z \in G$ such that $z \nsim y$ gives $ecc(y) > 1$; a contradiction. □

Corollary 4.1.10. *Let G be a group with $|Z(G)| > 1$. Then $\Delta(G)$ is an eccentric graph.*

In the next lemma, for each $x \in G$, we obtain the condition on $y \in G$ such that y is a boundaryboundary vertex of x.

Lemma 4.1.11. *An element y is a boundary vertex of x in $\Delta(G)$ if and only if one of the following hold:*

(i) $y \notin C_G(x)$

(ii) $C_G(y) \subseteq C_G(x)$.

Proof. If $x \in Z(G)$, then by the definition of boundary vertex the result holds. Now, let $x \in G \setminus Z(G)$. Suppose y is a boundary vertex of x. On contrary, we assume that $y \in C_G(x)$ and $C_G(y) \nsubseteq C_G(x)$. Then there exists $z \in C_G(y)$ such that $z \notin C_G(x)$. Consequently, we get $d(x, z) > 1$ and $d(x, y) = 1$; a contradiction. On the other hand, we assume that y satisfy either (i) or (ii). Suppose $y \notin C_G(x)$. Since e is adjacent to all the vertices of $\Delta(G)$ so diameter of $\Delta(G)$ is at most two. Therefore, $d(x, y) = 2$ as $x \nsim y$ implies $d(x, y) \geq d(x, z)$ for all $z \in G$. Consequently, y is a boundary vertex of x. If $C_G(y) \subseteq C_G(x)$, then clearly y is a boundary vertex of x. □

Proposition 4.1.12. *For the graph $\Delta(G)$, we have $\partial(\Delta(G)) = Ecc(\Delta(G))$.*

Proof. For $x \neq e$, we have $C_G(x) \subseteq C_G(e)$, so by Lemma 4.1.11, x is a boundary vertex of e. Therefore, x is a boundary vertex of $\Delta(G)$. If $|Z(G)| > 1$, there exists $x \neq e \in Z(G)$ so e is a boundary vertex of $\Delta(G)$. For $Z(G) = \{e\}$, note that e is not a boundary vertex of $\Delta(G)$. For instance, if e is a boundary vertex of x for some $x \in G \setminus \{e\}$, then $\mathrm{d}(x, y) \leq \mathrm{d}(x, e) = 1$ for all $y \in \mathrm{N}(e) = G \setminus \{e\}$. Consequently, we get $x \in Z(G)$ which is not possible. Thus, by Proposition 4.1.9, the result holds. □

Lemma 4.1.13 ([Ali et al., 2016, Lemma 1.2]). *For any group G, $Cen(\Delta(G))$ is the subgraph induced by the vertices of $Z(G)$.*

Further, we characterize the group G such that $Int(\Delta(G)) = Cen(\Delta(G))$.

Theorem 4.1.14. *Let G be a non-abelian group with $|G| > 2$. Then $Int(\Delta(G)) = Cen(\Delta(G))$ if and only if G is an AC-group.*

Proof. In view of Lemma 4.1.13, we show that $Int(\Delta(G))$ is the subgraph induced by the vertices of $Z(G)$ if and only G is an AC-group. First we assume that G is an AC-group. We claim that v is an interior point if and only if $v \in Z(G)$. Let $v \in G \setminus Z(G)$. Since $C_G(v)$ is a commutative subgroup of G as G is an AC-group so by Theorems 1.2.11, 1.2.12 and Lemma 4.1.8, v is not an interior point of $\Delta(G)$. On the other hand, we assume that $v \in Z(G)$. Then clearly v is not a complete vertex as G is a non-abelian group. In view of Theorems 1.2.11 and 1.2.12, v is an interior point. Thus, $Int(\Delta(G))$ is the subgraph induced by the vertices of $Z(G)$.

Suppose $Int(\Delta(G))$ is the subgraph induced by the vertices of $Z(G)$. Let $x \in G \setminus Z(G)$. Then x is not an interior point in $\Delta(G)$ implies x is a complete vertex (cf. Theorems 1.2.11, 1.2.12 and Lemma 4.1.8). Consequently, $C_G(x)$ is an abelian subgroup of G. Thus, G is an AC-group. □

Since the dihedral group D_{2n} is an AC-group, we have the following corollary.

Corollary 4.1.15 ([Ali et al., 2016, Theorem 2.11]). *For $n > 1$, the interior and the center of the commuting graph of the dihedral group D_{2n} are equal.*

4.2 Commuting Graph of the Semidihedral Group SD_{8n}

In this section, we obtain various graph invariants of $\Delta(SD_{8n})$ viz. vertex connectivity, independence number, edge connectivity, matching number, clique number etc. As a consequence, we obtain the vertex covering number and the edge covering number of $\Delta(SD_{8n})$. Further, we also study the Laplacian spectrum, resolving polynomial and the detour properties of $\Delta(SD_{8n})$ in various subsections. For $n \geq 2$, the *semidihedral group* SD_{8n} is a group of order $8n$ with presentation

$$SD_{8n} = \langle a, b : a^{4n} = b^2 = e, ba = a^{2n-1}b \rangle.$$

First note that

$$ba^i = \begin{cases} a^{4n-i}b & \text{if } i \text{ is even,} \\ a^{2n-i}b & \text{if } i \text{ is odd.} \end{cases} \tag{4.1}$$

Thus, every element of $SD_{8n} \setminus \langle a \rangle$ is of the form a^ib for some $0 \leq i \leq 4n-1$. We denote the subgroups $H_i = \langle a^{2i}b \rangle = \{e, a^{2i}b\}$ and $T_j = \langle a^{2j+1}b \rangle = \{e, a^{2n}, a^{2j+1}b, a^{2n+2j+1}b\}$. Then we have

$$SD_{8n} = \langle a \rangle \cup \left(\bigcup_{i=0}^{2n-1} H_i \right) \cup \left(\bigcup_{j=0}^{n-1} T_j \right).$$

Further, note that $Z(SD_{8n}) = \begin{cases} \{e, a^{2n}\} & \text{when } n \text{ is even,} \\ \{e, a^n, a^{2n}, a^{3n}\} & \text{otherwise.} \end{cases}$

Lemma 4.2.1. *In* $\Delta(SD_{8n})$,

(i) *for even* n, *we have* $\mathrm{N}[x] = SD_{8n}$ *if and only if* $x \in \{e, a^{2n}\}$.

(ii) *for odd* n, *we have* $\mathrm{N}[x] = SD_{8n}$ *if and only if* $x \in \{e, a^n, a^{2n}, a^{3n}\}$.

Proof. The result follows from Remark 4.1.1. □

By Theorem 1.2.1, we have the following corollary.

Corollary 4.2.2. *The commuting graph of SD_{8n} is not Eulerian.*

The following remark will be useful in the sequel.

Remark 4.2.3. For even n and $1 \leq i \leq 4n$, we have a^ib commutes with a^jb if and only if $j = 2n + i$.

Proof. Let $1 \leq i \neq j \leq 4n$. In view of (4.1), we have

- $a^iba^{2n+i}b = \begin{cases} a^{2n} & \text{when } i \text{ is even,} \\ e & \text{when } i \text{ is odd,} \end{cases}$ and $a^{2n+i}ba^ib = \begin{cases} a^{2n} & \text{when } i \text{ is even,} \\ e & \text{when } i \text{ is odd;} \end{cases}$
- $a^iba^{n+i}b = \begin{cases} a^{3n} & \text{when } i \text{ is even,} \\ a^n & \text{when } i \text{ is odd,} \end{cases}$ and $a^{n+i}ba^ib = \begin{cases} a^{n} & \text{when } i \text{ is even,} \\ a^{3n} & \text{when } i \text{ is odd;} \end{cases}$
- $a^iba^{3n+ib} = \begin{cases} a^{n} & \text{when } i \text{ is even,} \\ a^{3n} & \text{when } i \text{ is odd,} \end{cases}$ and $a^{3n+i}ba^ib = \begin{cases} a^{3n} & \text{when } i \text{ is even,} \\ a^{n} & \text{when } i \text{ is odd;} \end{cases}$
- $a^iba^jb = \begin{cases} a^{4n+i-j} & \text{when } j \text{ is even,} \\ a^{2n+i-j} & \text{when } j \text{ is odd,} \end{cases}$ and $a^jba^ib = \begin{cases} a^{4n+j-i} & \text{when } i \text{ is even,} \\ a^{2n+j-i} & \text{when } i \text{ is odd.} \end{cases}$

Further notice that $a^iba^jb = a^jba^ib$ if and only if $|i - j| = 2n$. Thus the result holds. □

Similar to the Remark 4.2.3, we have the following remarks.

Remark 4.2.4. For even n and $1 \leq i \leq 4n$, we have a^ib commutes with a^j if and only if $j \in \{2n, 4n\}$.

Remark 4.2.5. For odd n and $1 \leq i \leq 4n$, we have a^ib commutes with a^j if and only if $j \in \{n, 2n, 3n, 4n\}$.

Remark 4.2.6. For odd n and $1 \leq i \leq 4n$, we have a^ib commutes with a^jb if and only if $j \in \{n + i, 2n + i, 3n + i\}$.

In view of the Remarks 4.2.3 - 4.2.6, we obtain the neighbourhood of each vertex of $\Delta(SD_{8n})$.

Lemma 4.2.7. *In $\Delta(SD_{8n})$, for even n, we have*

(i) $\mathrm{N}[x] = \{e, a^{2n}, a^i b, a^{2n+i}b\}$ *if and only if* $x \in \{a^i b, a^{2n+i}b\}$*, where* $1 \leq i \leq 4n$.

(ii) $\mathrm{N}[x] = \langle a \rangle$ *if and only if* $x \in \langle a \rangle \setminus \{e, a^{2n}\}$.

Lemma 4.2.8. *In $\Delta(SD_{8n})$, for odd n, we have*

(i) $\mathrm{N}[x] = \{e, a^n, a^{2n}, a^{3n}, a^i b, a^{n+i}b, a^{2n+i}b, a^{3n+i}b\}$ *if and only if* $x \in \{a^i b, a^{n+i}b, a^{2n+i}b, a^{3n+i}b\}$*, where* $1 \leq i \leq 4n$.

(ii) $\mathrm{N}[x] = \langle a \rangle$ *if and only if* $x \in \langle a \rangle \setminus \{e, a^n, a^{2n}, a^{3n}\}$.

In view of Section 4.1 and Lemmas 4.2.7, 4.2.8, we have the following proposition.

Proposition 4.2.9. *The commuting graph of SD_{8n} satisfies the following properties:*

(i) *sdim*$(\Delta(SD_{8n}))$ *is* $8n - 2$

(ii) $\omega(\Delta(SD_{8n})) = 4n$

(iii) $Cen(\Delta(SD_{8n})) = Int(\Delta(SD_{8n}))$

(iv) $\Delta(SD_{8n})$ *is an eccentric graph*

(v) $\Delta(SD_{8n})$ *is a closed graph*

(vi) $\kappa'(\Delta(SD_{8n})) = \begin{cases} 3 & \text{if } n \text{ is even ;} \\ 7 & \text{if } n \text{ is odd.} \end{cases}$

Proof. (i) One can observe that the graph $\widehat{\Delta}(SD_{8n})$ is a star graph. Thus by Theorem 1.2.15, the result holds.

(ii) For $1 \leq i \leq 4n$, note that the element $a^i b$ is commute with at most eight elements of $\Delta(SD_{8n})$. Since the commutative subgroup generated by a is of size

$4n$. It follows that any commutative subgroup of SD_{8n} of maximum size does not contain the elements of the form $a^i b$. Thus, $\langle a \rangle$ is a commutative subgroup of SD_{8n} of maximum size $4n$ and hence the result holds (cf. Proposition 4.1.7).

(iii) For any $x \in SD_{8n} \setminus Z(SD_{8n})$, we have $\mathrm{N}[x] = \langle x \rangle$. By Remark 4.1.1, $C_G(x)$ is a commutative subgroup of SD_{8n}. Thus by Theorem 4.1.14, $Int(\Delta(SD_{8n})) = Cen(\Delta(SD_{8n}))$.

(iv) Since $|Z(\Delta(SD_{8n}))| > 1$ (see Lemma 4.2.1) so that by Proposition 4.1.9, the result holds.

(v) Note that for non-adjacent vertices x and y, we have $|\mathrm{N}(x)|+|\mathrm{N}(y)| < |V(\Delta(SD_{8n}))| = 8n$ (cf. Lemmas 4.2.7 and 4.2.8). Consequently, $\deg(x) + \deg(y) < |V(\Delta(SD_{8n}))|$ for all non-adjacent vertices x and y. Thus, by [Ali et al., 2016, Lemma 2.15], the result holds.

(vi) For even n, by Lemma 4.2.7, note that $\delta(\Delta(SD_{8n})) = \min\{|C_G(x)| : x \in SD_{8n}\} - 1 = 3$ and for odd n, by Lemma 4.2.8, note that $\delta(\Delta(SD_{8n})) = \min\{|C_G(x)| : x \in SD_{8n}\} - 1 = 7$. Thus, by Theorem 4.1.2, we have the result. □

As a consequence of Lemmas 4.2.7 and 4.2.8, we have the following proposition.

Proposition 4.2.10. *For $n \geq 1$, we have*

$$\Delta(SD_{8n}) \cong \begin{cases} K_2 \vee (K_{4n-2} \cup 2nK_2) & \text{if } n \text{ is even ;} \\ K_4 \vee (K_{4n-4} \cup nK_4) & \text{if } n \text{ is odd.} \end{cases}$$

Now, we obtain the automorphism group of the commuting graph of SD_{8n}. First, we recall the notion of wreath product.

Semidirect product: Let N be a normal subgroup of the group G and let H be a subgroup of G. Then G is said to be the *internal semidirect product* of N and H if $G = HN$ and $H \cap N = \varnothing$. We denote it by $G = H \ltimes N$ or $G = N \rtimes H$. For each $h \in H$, conjugation in N by an element h, yields an automorphism h^α of N and $\alpha : h \mapsto h^\alpha$ is a homomorphism from H to $\mathrm{Aut}(N)$.

Conversely, suppose H and N are two groups, together with a homomorphism $\alpha : H \to \mathrm{Aut}(N)$. The *external semidirect product* of H and N is the set $\{(h, n) : h \in H, n \in N\}$ with the binary operation

$$(h_1, n_1)(h_2, n_2) = (h_1 h_2, n_1^{h_2^\alpha} n_2),$$

where $n_1^{h_2^\alpha} = h_2^\alpha(n_1)$, forms a group. We shall denote it by $H \ltimes_\alpha N$.

Wreath product: Let F and H be groups and H acts on the sets $\{1, 2, \dots, n\}$. Then there is an action of H on F^n (F^n is the direct product of n copies of F) by

$$(f_1, f_2, \dots, f_n)^h = (f_{1h}, f_{2h}, \dots, f_{nh}).$$

Define a map $\phi : H \to \mathrm{Aut}(F^n)$ by $\phi(h) = \phi_h$, where

$$\phi_h(f_1, f_2, \dots, f_n) = (f_{1h}, f_{2h}, \cdots, f_{nh}).$$

The semidirect product $H \ltimes_\phi F^n$ is known as the *wreath product* of F by H, and is written as $F \wr H$.

Theorem 4.2.11 ([Ashrafi et al., 2017, Theorem 2.2])**.** *Suppose* $\Gamma = n_1\Gamma_1 \cup n_2\Gamma_2 \cup \cdots \cup n_t\Gamma_t$ *with* $\Gamma_i \neq \Gamma_j$ *for* $i \neq j$. *Then*

$$\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\Gamma_1) \wr S_{n_1} \times \mathrm{Aut}(\Gamma_2) \wr S_{n_2} \times \cdots \times \mathrm{Aut}(\Gamma_t) \wr S_{n_t}.$$

Remark 4.2.12. If A is the set of all vertices adjacent to every vertex in a graph Γ and $\Gamma - A$ is the subgraph of Γ induced by the vertices of $V(\Gamma) - A$, then $\mathrm{Aut}(\Gamma)$ is isomorphic to $S_{|A|} \times \mathrm{Aut}(\Gamma - A)$.

By Proposition 4.2.10 and Theorem 4.2.11, we have the following theorem.

Theorem 4.2.13. *For* $n \in \mathbb{N}$, *we have*

$$\mathrm{Aut}(\Delta(SD_{8n})) = \begin{cases} S_2 \times ((S_{4n-2} \wr S_1) \times (S_2 \wr S_{2n})) & \text{if } n \text{ is even };\\ S_4 \times ((S_{4n-4} \wr S_1) \times (S_4 \wr S_n)) & \text{if } n \text{ is odd.} \end{cases}$$

Next, we obtain the vertex connectivity, independence number and the matching number of $\Delta(SD_{8n})$.

Theorem 4.2.14. *In the graph* $\Delta(SD_{8n})$*,*

(i) *the vertex connectivity of* $\Delta(SD_{8n})$ *is given below:*

$$\kappa(\Delta(SD_{8n})) = \begin{cases} 2 & \text{if } n \text{ is even }; \\ 4 & \text{if } n \text{ is odd.} \end{cases}$$

(ii) *the independence number of* $\Delta(SD_{8n})$ *is given below:*

$$\alpha(\Delta(SD_{8n})) = \begin{cases} 1 & \text{if } n = 1; \\ 2n+1 & \text{if } n \text{ is even}; \\ n+1 & \text{otherwise.} \end{cases}$$

(iii) *the matching number of* $\Delta(SD_{8n})$ *is* $4n$*.*

Proof. (i) In view of Remark 4.1.1 and Lemmas 4.2.7, 4.2.8, $C_G(ab)$ is an abelian subgroup of SD_{8n} and maximal centralizer. By Proposition 4.3.22, $\kappa(\Delta(SD_{8n})) = |Z(SD_{8n})|$. Thus, we have the result.

(ii) Suppose n is even. Consider the set $I = \{a^ib : 1 \leq i \leq 2n\} \cup \{a\}$. In view of Lemmas 4.2.1 and 4.2.7, I is an independent in $\Delta(SD_{8n})$ of size $2n+1$. If there exists another independent set I' such that $|I'| > 2n+1$, then there exist $x, y \in I'$ such that either $x, y \in \langle a \rangle$ or $x, y \in \{a^ib, a^{2n+i}b\}$ for some i, where $1 \leq i \leq 2n$ as $SD_{8n} = \langle a \rangle \cup (\bigcup_{i=1}^{2n} \{a^ib, a^{2n+i}b\})$. In both cases, we have $x \sim y$ (see Lemma 4.2.7); a contradiction of the fact that I' is an independent in $\Delta(SD_{8n})$. Thus the result holds.

On the other hand, we assume that n is odd. By using the Lemma 4.2.8 and similar to even n case, we get an independent set $I = \{a^ib : 1 \leq i \leq n\} \cup \{a\}$ of the maximum size $n+1$.

(iii) In view of Lemmas 4.2.7 and 4.2.8, $a^ib \sim a^{2n+i}b$ for all i, where $1 \leq i \leq 2n$. Consider the set $\mathcal{M} = \{(a^ib, a^{2n+i}b) \in E(\Delta(SD_{8n})) : 1 \leq i \leq 2n\} \cup \{(a^i, a^{2n+i}) \in E(\Delta(SD_{8n})) : 1 \leq i \leq 2n\}$ which forms a matching of size $4n$. Consequently, we get $\alpha'(\Delta(SD_{8n})) \geq 4n$. It is well known that $\alpha'(\Delta(SD_{8n})) \leq \frac{|V(\Delta(SD_{8n}))|}{2} = 4n$. Thus, $\alpha'(\Delta(SD_{8n})) = 4n$. □

In view of Lemma 1.2.10, we have the following corollary.

Corollary 4.2.15. *For $n \geq 1$,*

(i) *the vertex covering number of $\Delta(SD_{8n})$ is given below:*

$$\beta(\Delta(SD_{8n})) = \begin{cases} 7 & \text{if } n = 1; \\ 6n-1 & \text{if } n \text{ is even }; \\ 7n-1 & \text{otherwise.} \end{cases}$$

(ii) *the edge covering number of $\Delta(SD_{8n})$ is $4n$.*

Now, we investigate perfectness and Hamiltonian property of $\Delta(SD_{8n})$.

Theorem 4.2.16. *The commuting graph of SD_{8n} is perfect.*

Proof. In view of Theorem 1.2.4, it is enough to show that $\Delta(SD_{8n})$ does not contain a hole or antihole of odd length at least five. Note that neither hole nor antihole can contain any element of $Z(\Delta(SD_{8n}))$ (cf. Remark 1.2.5). First suppose that $\Delta(SD_{8n})$ contains a hole C given by $x_1 \sim x_2 \sim \cdots \sim x_{2l+1} \sim x_1$, where $l \geq 2$. Note that any hole can contain at most two elements of $\langle a \rangle$, otherwise C contains a triangle which is not possible. In view of Lemmas 4.2.7 and 4.2.8, $\mathrm{N}[x] = \langle a \rangle$ if and only if $x \in \langle a \rangle \setminus Z(\Delta(SD_{8n}))$. It follows that $x_i \notin \langle a \rangle$ for all i, where $1 \leq i \leq 2l+1$. Consequently, we get $a^ib \in C$ for some i. If n is even, then we must have $a^ib \sim a^{2n+i}b$ in $\Delta(SD_{8n})$ as $\mathrm{N}[a^ib] = \mathrm{N}[a^{2n+i}b] = \{e, a^{2n}, a^ib, a^{2n+i}b\}$ (cf. Lemma 4.2.7). As a result a^ib is adjacent with only one element in C; a contradiction. In case of odd n, there exist $x, y \in C \cap \mathrm{N}(a^ib)$. Note that

$\mathrm{N}[a^ib] = \mathrm{N}[a^{n+i}b] = \mathrm{N}[a^{2n+i}b] = \mathrm{N}[a^{3n+i}b] = \{e, a^n, a^{2n}, a^{3n}, a^ib, a^{n+i}b, a^{2n+i}b, a^{3n+i}b\}$ (cf. Lemma 4.2.8) implies $x, y \in \{a^{n+i}b, a^{2n+i}b, a^{3n+i}b\}$. Therefore, we have x, y and a^ib forms a triangle in C; a contradiction. Thus, $\Delta(SD_{8n})$ does not contain any hole of odd length at least five.

Now assume that C' is an antihole of length at least 5 in $\Delta(SD_{8n})$, that is, we have a hole $y_1 \sim y_2 \sim \cdots \sim y_{2l+1} \sim y_1$, where $l \geq 2$, in $\overline{\Delta(SD_{8n})}$. Clearly, $y_i \notin Z(\Delta(SD_{8n}))$ for all i, where $1 \leq i \leq 2l+1$. Suppose $y_i \in \langle a \rangle$ for some i. Then clearly $y_{i-1}, y_{i+1} \in SD_{8n} \setminus \langle a \rangle$, otherwise $y_i \sim y_{i-1}$ and $y_i \sim y_{i+1}$ in $\Delta(SD_{8n})$; a contradiction. Further note that for $1 \leq j \leq 2l+1$ and $j \notin \{i-1, i, i+1\}$, we have $y_j \in \langle a \rangle$. For instance, if $y_j \in SD_{8n} \setminus (\langle a \rangle \cup \{y_{i-1}, y_i, y_{i+1}\})$ for some j, then $y_j \sim y_i$ in $\overline{\Delta(SD_{8n})}$ as $y_i \notin Z(\Delta(SD_{8n}))$ (see Lemmas 4.2.7 and 4.2.8); a contradiction. Therefore, there exists $y_j \in \langle a \rangle$ gives $y_j \sim y_{i-1}$ and $y_j \sim y_{i+1}$ in $\overline{\Delta(SD_{8n})}$. As a result $\{y_j, y_{i-1}, y_i, y_{i+1}\}$ forms a cycle of length four in $\overline{\Delta(SD_{8n})}$; a contradiction. Thus, $y_i \notin \langle a \rangle$ for all i.

If n is even, then $a^ib \sim a^jb$ for all $j \neq 2n+i$ in $\overline{\Delta(SD_{8n})}$ (see Lemma 4.2.7) implies C' is not an antihole; again a contradiction. Now we assume that n is odd. Let $y_1 = a^ib$ for some i. Then we have $y_3, y_4 \in \mathrm{N}(a^ib) = \{e, a^n, a^{2n}, a^{3n}, a^{n+i}b, a^{2n+i}b, a^{3n+i}b\}$ gives $y_3 \nsim y_4$ in C' (see Lemma 4.2.8); a contradiction. Thus, $\Delta(SD_{8n})$ does not contain any antihole of odd length at least five. □

Theorem 4.2.17. *The commuting graph of SD_{8n} is Hamiltonian if and only if $n \in \{1, 3\}$.*

Proof. First suppose that n is even. By Lemma 4.2.7, $\{e, a^{2n}\}$ is a vertex cut set in $\Delta(SD_{8n})$ so that by deletion of these vertices, the connected components of the subgraph induced by the vertices $V(\Delta(SD_{8n})) \setminus \{e, a^{2n}\}$ are $\{a^ib, a^{2n+i}b\}$ and $\langle a \rangle \setminus \{e, a^{2n}\}$ where $1 \leq i \leq 2n$. Therefore, it is impossible to construct Hamiltonian cycle in $\Delta(SD_{8n})$. Thus, $\Delta(SD_{8n})$ is not Hamiltonian graph when n is even. Now we assume that n is odd. By Lemma 4.2.8, $\{e, a^n, a^{2n}, a^{3n}\}$ is a vertex cut set in $\Delta(SD_{8n})$ so that by deletion of these vertices, the connected components of the subgraph induced by the vertices $V(\Delta(SD_{8n})) \setminus \{e, a^n, a^{2n}, a^{3n}\}$ are $\{a^ib, a^{n+i}b, a^{2n+i}b, a^{3n+i}b\}$ and $\langle a\rangle \setminus \{e, a^n, a^{2n}, a^{3n}\}$ where $1 \leq i \leq n$ which are $n+1$ in total. It follows that for the construction of Hamiltonian cycle in $\Delta(SD_{8n})$, we required at least $n+1$ element from the vertex cut set. Thus, for $n > 3$, $\Delta(SD_{8n})$ is not Hamiltonian graph. For $n = 1, 3$, in view of Lemma 4.2.8, we have $\deg(x) \geq \frac{|V(\Delta(SD_{8n}))|}{2}$ for all $x \in SD_{8n}$. Thus, by [West, 1996, Theorem 7.2.8], $\Delta(SD_{8n})$ is Hamiltonian. □

4.2.1 Laplacian Spectrum

In this subsection, we investigate the Laplacian spectrum of $\Delta(SD_{8n})$. Consequently, we provide the number of spanning trees of $\Delta(SD_{8n})$. In the following theorem, we obtain the characteristic polynomial of $L(\Delta(SD_{8n}))$.

Theorem 4.2.18. *For even n, the characteristic polynomial of the Laplacian matrix of $\Delta(SD_{8n})$ is given by*

$$\Phi(L(\Delta(SD_{8n})), x) = x(x-8n)^2(x-4)^{2n}(x-2)^{2n}(x-4n)^{4n-3}.$$

Proof. The Laplacian matrix $L(\Delta(SD_{8n}))$ is the $8n \times 8n$ matrix given below, where the rows and columns are indexed in order by the vertices $e = a^{4n}, a^{2n}, a, a^2, \ldots, a^{2n-1}$,

$a^{2n+1}, a^{2n+2}, \ldots, a^{4n-1}$ and then $ab, a^2b, \ldots, a^{4n}b$.

$$L(\Delta(SD_{8n})) = \begin{pmatrix} 8n-1 & -1 & -1 & -1 & \cdots\cdots & -1 & -1 & \cdots\cdots & -1 \\ -1 & 8n-1 & -1 & -1 & \cdots\cdots & -1 & -1 & \cdots\cdots & -1 \\ -1 & -1 & & & & & & & \\ -1 & -1 & & A & & & \mathcal{O} & & \\ \vdots & \vdots & & & & & & & \\ \vdots & \vdots & & & & & & & \\ -1 & -1 & & & & & & & \\ -1 & -1 & & & & & & & \\ \vdots & \vdots & & \mathcal{O}' & & & B & & \\ \vdots & \vdots & & & & & & & \\ -1 & -1 & & & & & & & \end{pmatrix}$$

where $A = 4nI_{4n-2} - J_{4n-2}$, $B = \begin{vmatrix} 3I_{2n} & -I_{2n} \\ -I_{2n} & 3I_{2n} \end{vmatrix}$, $\mathcal{O}$ is the zero matrix of size $(4n-2) \times (4n)$ and $\mathcal{O}'$ is the transpose matrix of $\mathcal{O}$. Then the characteristic polynomial of $L(\Delta(SD_{8n}))$ is

$$\begin{vmatrix} x-(8n-1) & 1 & 1 & 1 & \cdots\cdots & 1 & 1 & \cdots\cdots & 1 \\ 1 & x-(8n-1) & 1 & 1 & \cdots\cdots & 1 & 1 & \cdots\cdots & 1 \\ 1 & 1 & & & & & & & \\ 1 & 1 & & (xI_{4n-2}-A) & & & \mathcal{O} & & \\ \vdots & \vdots & & & & & & & \\ \vdots & \vdots & & & & & & & \\ 1 & 1 & & & & & & & \\ 1 & 1 & & & & & & & \\ \vdots & \vdots & & \mathcal{O}' & & & (xI_{4n}-B) & & \\ \vdots & \vdots & & & & & & & \\ 1 & 1 & & & & & & & \end{vmatrix}.$$

Apply row operation $R_1 \to (x-1)R_1 - R_2 - \cdots - R_{8n}$ and then expand by using first row, we get $\Phi(L(\Delta(SD_{8n})), x) =$

$$\frac{x(x-8n)}{(x-1)}\begin{vmatrix} x-(8n-1) & 1 & 1 & \cdots\cdots & 1 & 1 & \cdots\cdots & 1 \\ 1 & & & & & & & \\ \vdots & & (xI_{4n-2}-A) & & & & \mathcal{O} & \\ \vdots & & & & & & & \\ 1 & & & & & & & \\ 1 & & & & & & & \\ \vdots & & \mathcal{O}' & & & & (xI_{4n}-B) & \\ \vdots & & & & & & & \\ 1 & & & & & & & \end{vmatrix}.$$

Again, apply row operation $R_1 \to (x-2)R_1 - R_2 - R_3 - \cdots - R_{8n-1}$ and then expand by using first row, we get

$$\Phi(L(\Delta(SD_{8n})), x) = \frac{x(x-8n)^2}{(x-2)}\begin{vmatrix} xI_{4n-2}-A & \mathcal{O} \\ \mathcal{O}' & xI_{4n}-B \end{vmatrix}.$$

By using Schur's decomposition theorem (see Cvetkovic et al. [2009]), we have

$$\Phi(L(\Delta(SD_{8n})), x) = \frac{x(x-8n)^2}{(x-2)}|xI_{4n-2}-A| \cdot |xI_{4n}-B|.$$

Clearly, $|xI_{4n}-B| = \begin{vmatrix} (x-3)I_{2n} & I_{2n} \\ I_{2n} & (x-3)I_{2n} \end{vmatrix}$. Again by using Schur's decomposition theorem, we obtain

$$|xI_{4n}-B| = |(x-3)I_{2n}||(x-3)I_{2n} - \frac{1}{(x-3)}I_{2n}| = (x-4)^{2n}(x-2)^{2n}.$$

Now we obtain $|xI_{4n-2}-A| = |xI_{4n-2} - (4nI_{4n-2} - J_{4n-2})|$. It is easy to compute the characteristic polynomial of the matrix J_{4n-2} is $x^{4n-3}(x-4n+2)$. It is well known that if $f(x) = 0$ is any polynomial and λ is an eigenvalue of the matrix P, then $f(\lambda)$ is an eigenvalue of the matrix $f(P)$. Consequently, the eigenvalues of the matrix A are $4n$ and 2. Note that if x is an eigenvector of J_n corresponding to the eigenvalue 0, then x is also an eigenvector of the matrix A corresponding to eigenvalue $4n$. since dimension of the null space of J_{4n-2} is $4n-3$ so that the multiplicity of the eigenvalue $4n$ in the characteristic polynomial of the matrix A is $4n-3$. Thus, $|xI_{4n-2} - A| = (x-4n)^{4n-3}(x-2)$ and hence the result holds. □

Corollary 4.2.19. *For even n, the Laplacian spectrum of $\Delta(SD_{8n})$ is given by*

$$\begin{pmatrix} 0 & 2 & 4 & 4n & 8n \\ 1 & 2n & 2n & 4n-3 & 2 \end{pmatrix}.$$

By [Brauer and Fowler, 1991, Corollary 4.2], we have the following corollary.

Corollary 4.2.20. *For even n, the number of spanning trees of $\Delta(SD_{8n})$ is $2^{14n-3}n^{4n-2}$.*

Theorem 4.2.21. *For odd n, the characteristic polynomial of the Laplacian matrix of $\Delta(SD_{8n})$ is given by*

$$\Phi(L(\Delta(SD_{8n})), x) = x(x-8n)^4(x-4)^n(x-8)^{3n}(x-4n)^{4n-5}.$$

Proof. The Laplacian matrix $L(\Delta(SD_{8n}))$ is the $8n \times 8n$ matrix given below, where the rows and columns are indexed by the vertices $e = a^{4n}, a^{3n}, a^{2n}, a^n, a, a^2, \ldots, a^{n-1}$, $a^{n+1}, a^{n+2}, \ldots, a^{2n-1}, a^{2n+1}, a^{2n+2}, \ldots, a^{3n-1}, a^{3n+1}, a^{3n+2}, \ldots, a^{4n-1}$ and then ab, a^2b, $\ldots, a^{4n}b$.

$$L(\Delta(SD_{8n})) = \begin{pmatrix} 8n-1 & -1 & -1 & -1 & \cdots\cdots & -1 & -1 & \cdots\cdots & -1 \\ -1 & 8n-1 & -1 & -1 & \cdots\cdots & -1 & -1 & \cdots\cdots & -1 \\ -1 & -1 & 8n-1 & -1 & \cdots\cdots & -1 & -1 & \cdots\cdots & -1 \\ -1 & -1 & -1 & 8n-1 & \cdots\cdots & -1 & -1 & \cdots\cdots & -1 \\ -1 & -1 & -1 & -1 & & & & & \\ -1 & -1 & -1 & -1 & & & & & \\ -1 & -1 & -1 & -1 & A & & & \mathcal{O} & \\ \vdots & \vdots & \vdots & \vdots & & & & & \\ -1 & -1 & -1 & -1 & & & & & \\ \vdots & \vdots & \vdots & \vdots & \mathcal{O}' & & & B & \\ -1 & -1 & -1 & -1 & & & & & \end{pmatrix}$$

where $A = 4nI_{(4n-4)} - J_{(4n-4)}$, $B = \begin{pmatrix} 7I_n & -I_n & -I_n & -I_n \\ -I_n & 7I_n & -I_n & -I_n \\ -I_n & -I_n & 7I_n & -I_n \\ -I_n & -I_n & -I_n & 7I_n \end{pmatrix}$, $\mathcal{O}$ and $\mathcal{O}'$ are defined in Theorem 4.2.18. Then the characteristic polynomial of $L(\Delta(SD_{8n}))$ is

$$\begin{vmatrix} x-(8n-1) & 1 & 1 & 1 & \cdots & 1 \; 1 & \cdots 1 \\ 1 & x-(8n-1) & 1 & 1 & \cdots & 1 \; 1 & \cdots 1 \\ 1 & 1 & x-(8n-1) & 1 & \cdots & 1 \; 1 & \cdots 1 \\ 1 & 1 & 1 & x-(8n-1) & \cdots & 1 \; 1 & \cdots 1 \\ 1 & 1 & 1 & 1 & & & \\ 1 & 1 & 1 & 1 & xI-A & & \mathcal{O} \\ \vdots & \vdots & \vdots & \vdots & & & \\ \vdots & \vdots & \vdots & \vdots & & & \\ 1 & 1 & 1 & 1 & & & \\ 1 & 1 & 1 & 1 & & & \\ \vdots & \vdots & \vdots & \vdots & \mathcal{O}' & & xI-B \\ \vdots & \vdots & \vdots & \vdots & & & \\ 1 & 1 & 1 & 1 & & & \end{vmatrix}$$

Apply the following row operations consecutively

- $R_1 \to (x-1)R_1 - R_2 - \cdots - R_{8n}$
- $R_2 \to (x-2)R_2 - R_3 - \cdots - R_{8n}$
- $R_3 \to (x-3)R_3 - R_4 - \cdots - R_{8n}$
- $R_4 \to (x-4)R_4 - R_5 - \cdots - R_{8n}$

and then expand, we get

$$\Phi(L(\Delta(SD_{8n})),x) = \frac{x(x-8n)^4}{(x-4)} \begin{vmatrix} xI-A & \mathcal{O} \\ \mathcal{O}' & xI-B \end{vmatrix} = \frac{x(x-8n)^4}{(x-4)} |xI-A||xI-B|.$$

By the similar argument used in the proof of Theorem 4.2.18, we obtain $|xI-A| = (x-4n)^{4n-5}(x-4)$. To get

$$|xI-B| = \begin{vmatrix} (x-7)I_n & I_n & I_n & I_n \\ I_n & (x-7)I_n & I_n & I_n \\ I_n & I_n & (x-7)I_n & I_n \\ I_n & I_n & I_n & (x-7)I_n \end{vmatrix},$$

apply the following row operations consecutively $R_i \to (x-5)R_i - R_{i+1} - \cdots - R_{4n}$ where $1 \leq i \leq n$ and then on solving, we get

$$|xI-B| = \frac{(x-4)^n(x-8)^n}{(x-5)^n} \begin{vmatrix} (x-7)I_n & I_n & I_n \\ I_n & (x-7)I_n & I_n \\ I_n & I_n & (x-7)I_n \end{vmatrix}.$$

Again apply the following row operations consecutively

- For $1 \leq i \leq n$, $R_i \to (x-6)R_i - R_{i+1} - \cdots - R_{3n}$
- For $n+1 \leq i \leq 2n$, $R_i \to (x-7)R_i - R_{i+1} - \cdots - R_{3n}$

and then expand, we obtain

$$|xI-B| = \frac{(x-4)^n(x-8)^{3n}}{(x-7)^n} |(x-7)I_n| = (x-4)^n(x-8)^{3n}.$$

Thus, the result holds. □

Corollary 4.2.22. *For odd n, the Laplacian spectrum of $\Delta(SD_{8n})$ is given by*

$$\begin{pmatrix} 0 & 4 & 8 & 4n & 8n \\ 1 & n & 3n & 4n-5 & 4 \end{pmatrix}.$$

By [Brauer and Fowler, 1991, Corollary 4.2], we have the following corollary.

Corollary 4.2.23. *For odd n, the number of spanning trees of $\Delta(SD_{8n})$ is $2^{19n-1}n^{4n-2}$.*

4.2.2 Resolving Polynomial

In this subsection, we obtain the resolving polynomial of $\Delta(SD_{8n})$.

Proposition 4.2.24. *The metric dimension of* $\Delta(SD_{8n})$ *is given below:*

$$\dim(\Delta(SD_{8n})) = \begin{cases} 6n-2 & \text{when } n \text{ is even };\\ 7n-2 & \text{otherwise.} \end{cases}$$

Proof. First we assume that n is even. In view of Lemma 4.2.7, we get twin-sets $\langle a\rangle \setminus \{e, a^{2n}\}, \{e, a^{2n}\}$ and $\{a^ib, a^{2n+i}b\}$ where $1 \leq i \leq 2n$. By Remark 1.2.13, any resolving set in $\Delta(SD_{8n})$ contains at least $6n-2$ vertices. Now we provide a resolving set of size $6n-2$. By Lemma 4.2.7, one can verify that the set $R_{\text{even}} = \{a^ib : 1 \leq i \leq 2n\} \cup \{a^i : i \neq 1, 2n\}$ is a resolving set of size $6n-2$. Consequently, $\dim(\Delta(SD_{8n})) = 6n-2$. We may now suppose that n is odd. By Lemma 4.2.8, note that $\langle a\rangle \setminus \{e, a^n, a^{2n}, a^{3n}\}, \{e, a^n a^{2n}, a^{3n}\}$ and $\{a^ib, a^{n+i}b, a^{2n+i}b, a^{n+3i}b\}$, where $1 \leq i \leq n$, are twin sets in $\Delta(SD_{8n})$. In view of Remark 1.2.13, any resolving set in $\Delta(SD_{8n})$ contains at least $7n-2$ vertices. Further, it is routine to verify that the set $R_{\text{odd}} = \{a^ib, a^{n+i}b, a^{2n+i}b : 1 \leq i \leq n\} \cup \{a^i : i \neq 1, 2n\}$ is a resolving set of size $7n-2$. Thus, $\dim(\Delta(SD_{8n})) = 7n-2$. □

Theorem 4.2.25. *For even n, the resolving polynomial of* $\Delta(SD_{8n})$ *is given below:*

$$\beta(\Delta(SD_{8n}), x) = x^{8n} + 8nx^{8n-1} + 2^{2n+2}(2n-1)x^{6n-2} + \sum_{i=6n-1}^{8n-2} r_i x^i,$$

where $r_i = 2^{8n-i}\left\{\binom{2n+1}{8n-i} + (2n-1)\binom{2n+1}{8n-i-1}\right\}$ *for* $6n-1 \leq i \leq 8n-2$.

Proof. In view of Proposition 4.2.24, we have $\dim(\Delta(SD_{8n})) = 6n-2$. It is sufficient to find the resolving sequence $(r_{6n-2}, r_{6n-1}, \ldots, r_{8n-2}, r_{8n-1}, r_{8n})$. By the proof of Proposition 4.2.24, any resolving set R satisfies the following:

- $|R \cap (\langle a\rangle \setminus \{e, a^{2n}\})| \geq 4n-3$;
- $|R \cap \{e, a^{2n}\}| \geq 1$;

- $|R \cap \{a^ib, a^{2n+i}b\}| \geq 1$ where $1 \leq i \leq 2n$.

For $|R| = i \geq 6n-2$, there exist $v_1, v_2, \ldots, v_{8n-i} \in SD_{8n} \setminus R$. Therefore we have one of the following:

(i) $v_j \in \langle a \rangle \setminus \{e, a^{2n}\}$ for some j and
$v_1, v_2, \ldots, v_{j-1}, v_{j+1}, v_{j+2}, \ldots, v_{8n-i} \in \left(\bigcup_{i=1}^{2n} \{a^ib, a^{2n+i}b\} \right) \cup \{e, a^{2n}\}$.

(ii) $v_1, v_2, \ldots, v_{8n-i} \in \left(\bigcup_{i=1}^{2n} \{a^ib, a^{2n+i}b\} \right) \cup \{e, a^{2n}\}$.

For $i = 6n-2$, (ii) does not hold so $v_j \in \langle a \rangle \setminus \{e, a^{2n}\}$ and $v_1, v_2 \ldots, v_{j-1}, v_{j+1}, v_{j+2}, \ldots, v_{8n-i} \in \left(\bigcup_{i=1}^{2n} \{a^ib, a^{2n+i}b\} \right) \cup \{e, a^{2n}\}$. Therefore, we obtain $r_{6n-2} = 2^{2n+1}(4n-2)$. Now for fixed i, $6n-1 \leq i \leq 8n-2$, we get $r_i = 2^{8n-i}\left\{\binom{2n+1}{8n-i} + (2n-1)\binom{2n+1}{8n-i-1}\right\}$. By Proposition 1.2.14, $r_{8n-1} = 8n$ and $r_{8n} = 1$. □

Theorem 4.2.26. *For odd n, the resolving polynomial of $\Delta(SD_{8n})$ is given below:*

$$\beta(\Delta(SD_{8n}), x) = x^{8n} + 8nx^{8n-1} + 2^{2n+4}(n-1)x^{7n-2} + \sum_{i=7n-1}^{8n-2} r_i x^i,$$

where $r_i = 2^{16n-2i}\left\{\binom{n+1}{8n-i} + (n-1)\binom{n+1}{8n-i-1}\right\}$ for $7n-1 \leq i \leq 8n-2$.

Proof. In view of Proposition 4.2.24, we have $\dim(\Delta(SD_{8n})) = 7n-2$. It is sufficient to find the resolving sequence $(r_{7n-2}, r_{7n-1}, \ldots, r_{8n-2}, r_{8n-1}, r_{8n})$. By the proof of Proposition 4.2.24, any resolving set R satisfies the following:

- $|R \cap (\langle a \rangle \setminus \{e, a^n, a^{2n}, a^{3n}\})| \geq 4n-5$;

- $|R \cap \{e, a^n, a^{2n}, a^{3n}\}| \geq 3$;

- $|R \cap \{a^ib, a^{n+i}b, a^{2n+i}b, a^{3n+i}b\}| \geq 3$ where $1 \leq i \leq n$.

For $|R| = i \geq 7n-2$, there exist $v_1, v_2, \ldots, v_{8n-i} \in SD_{8n} \setminus R$. Therefore we have one of the following

(i) $v_j \in \langle a \rangle \setminus \{e, a^n, a^{2n}, a^{3n}\}$ for some j and

$$v_1, v_2, \ldots, v_{j-1}, v_{j+1}, v_{j+2}, \ldots, v_{8n-i} \in \left(\bigcup_{i=1}^{n} \{a^i b, a^{n+i} b, a^{2n+i} b, a^{3n+i} b\} \right) \cup \{e, a^n, a^{2n}, a^{3n}\}$$

(ii) $v_1, v_2, \ldots, v_{8n-i} \in \left(\bigcup_{i=1}^{n} \{a^i b, a^{n+i} b, a^{2n+i} b, a^{3n+i} b\} \right) \cup \{e, a^n, a^{2n}, a^{3n}\}$.

For $i = 7n - 2$, (ii) does not hold so $v_j \in \langle a \rangle \setminus \{e, a^{2n}\}$ and

$$v_1, v_2, \ldots, v_{j-1}, v_{j+1}, v_{j+2}, \ldots, v_{8n-i} \in \left(\bigcup_{i=1}^{n} \{a^i b, a^{n+i} b, a^{2n+i} b, a^{3n+i} b\} \right) \cup \{e, a^n, a^{2n}, a^{3n}\}.$$

Therefore, we have $r_{7n-2} = 4^{n+1}(4n - 4)$. Now for fixed i, $7n - 1 \leq i \leq 8n - 2$, we get

$$r_i = 2^{16n-2i} \left\{ \binom{n+1}{8n-i} + (n-1) \binom{n+1}{8n-i-1} \right\}.$$

By Proposition 1.2.14, $r_{8n-1} = 8n$ and $r_{8n} = 1$. □

4.2.3 Detour Distance Properties

In this subsection, we study the detour distance properties of $\Delta(SD_{8n})$ viz. detour radius, detour eccentricity, detour degree, detour degree sequence and detour distance degree sequence of each vertex.

Theorem 4.2.27. *In $\Delta(SD_{8n})$, we have for each $x \in Z(SD_{8n})$,*

$$ecc_D(x) = \begin{cases} 4n + 1 & \text{when } n \text{ is even;} \\ 4n + 11 & \text{when } n \text{ is odd.} \end{cases}$$

and for each $x \in SD_{8n} \setminus Z(SD_{8n})$,

$$ecc_D(x) = \begin{cases} 4n + 3 & \text{when } n \text{ is even;} \\ 4n + 15 & \text{when } n \text{ is odd.} \end{cases}$$

Proof. We split our proof in two cases depend on n.

Case 1: n is even. First note that $x \sim y$ for $x \in Z(SD_{8n})$ and $y \in SD_{8n} \setminus \{x\}$; $x' \sim y'$ for all distinct $x', y' \in \langle a \rangle \setminus \{e, a^{2n}\}$; for each $1 \leq i \leq 4n$, $a^i b \sim a^{2n+i} b$, $a^i b \nsim a^j b$ for all $j \neq 2n + i$, $a^i b \nsim a^j$ for all $a^j \in \langle a \rangle \setminus \{e, a^{2n}\}$. Thus we have (i) for each $x \in Z(SD_{8n})$, there is a x - y detour of length $4n-1$ for all $y \in Z(SD_{8n})\setminus\{x\}$; a x - y detour of length $4n+1$ for all $y \in SD_{8n} \setminus Z(SD_{8n})$ as $Z(SD_{8n}) = \{e, a^{2n}\}$; (ii) for each $1 \le i \le 4n$, there is a a^ib - $a^{2n+i}b$ detour of length $4n+1$; for distinct $1 \le i, j \le 4n$ and $j \neq 2n+i$, a a^ib - a^jb detour of length $4n+3$; for each $1 \le i \le 4n$ and for each $a^j \in \langle a\rangle \setminus Z(SD_{8n})$, a a^ib - a^j detour of length $4n+3$; and (iii) for distinct $1 \le i, j < 4n$ and $i, j \neq 2n$, there is a a^i - a^j detour of length $4n+1$.

Case 2: n is odd. First note that $x \sim y$ for $x \in Z(SD_{8n})$ and $y \in SD_{8n} \setminus \{x\}$; $x' \sim y'$ for all distinct $x', y' \in \langle a\rangle \setminus \{e, a^n, a^{2n}, a^{3n}\}$; for each $1 \le i \le n$ and for each $j \in \{n+i, 2n+i, 3n+i\}$, $a^ib \sim a^jb$; for each $1 \le i \le 4n$, $a^ib \nsim a^jb$ for all $j \notin \{n+i, 2n+i, 3n+i\}$, $a^ib \nsim a^j$ for all $a^j \in \langle a\rangle\setminus\{e, a^n, a^{2n}, a^{3n}\}$. Thus we have (i) for each $x \in Z(SD_{8n})$, there is a x - y detour of length $4n+7$ for all $y \in Z(SD_{8n})\setminus\{x\}$; a x - y detour of length $4n+11$ for all $y \in SD_{8n}\setminus Z(SD_{8n})$; (ii) for each $1 \le i \le 4n$, there is a a^ib - a^jb detour of length $4n+11$ for all $j \in \{n+i, 2n+i, 3n+i\}$; for distinct $1 \le i \le 4n$ and $j \notin \{n+i, 2n+i, 3n+i\}$, a a^ib - a^jb detour of length $4n+15$; for each $1 \le i \le 4n$ and for each $a^j \in \langle a\rangle \setminus Z(SD_{8n})$, a a^ib - a^j detour of length $4n+15$; and (iii) for distinct $1 \le i, j < 4n$ and $i, j \notin \{n, 2n, 3n, 4n\}$, there is a a^i - a^j detour of length $4n+11$. □

By the definition of $rad_D(\Delta(SD_{8n}))$ and $diam_D(\Delta(SD_{8n}))$, we have the following corollary.

Corollary 4.2.28. *In $\Delta(SD_{8n})$, we have*

(i) $$rad_D(\Delta(SD_{8n})) = \begin{cases} 4n+1 & \text{if } n \text{ is even;} \\ 4n+11 & \text{if } n \text{ is odd.} \end{cases}$$

(ii) $$diam_D(\Delta(SD_{8n})) = \begin{cases} 4n+3 & \text{if } n \text{ is even;} \\ 4n+15 & \text{if } n \text{ is odd.} \end{cases}$$

The *detour degree* $d_D(v)$ of v is the number $|D(v)|$, where $D(v) = \{u \in V(\Gamma) : d_D(u, v) = ecc_D(v)\}$. The *average detour degree* $D_{av}(\Gamma)$ of a graph Γ is the quotient of the sum of the detour degrees of all the vertices of Γ and the order of Γ. The detour degrees of the vertices of a graph written in non-increasing order is said to be the *detour degree sequence* of graph Γ, denoted by $D(\Gamma)$. For a vertex $x \in V(\Gamma)$, we denote $D_i(x)$ be the number of vertices at a detour distance i from the vertex x, then the sequence $D_0(x), D_1(x), D_2(x), \ldots, D_{ecc_D(x)}(x)$ is called *detour distance degree sequence of a vertex* x, denoted by $dds_D(x)$. In the remaining part of this paper, (a^r, b^s, c^t) denote a occur r times, b occur s times and c occur t times in the sequence. Now we have the following remark.

Remark 4.2.29 ([Ali et al., 2016, Remark 2.6])**.** In a graph Γ, we have

(i) $D_0(v) = 1$ and $D_{ecc_D}(v) = d_D(v)$.

(ii) The length of sequence $dds_D(v)$ is one more than the detour eccentricity of v.

(iii) $\sum_{i=0}^{ecc_D(v)} D_i(v) = |\Gamma|$.

Proposition 4.2.30. *In $\Delta(SD_{8n})$, we have for each $x \in Z(SD_{8n})$*

$$d_D(x) = \begin{cases} 8n-2 & \text{when } n \text{ is even;} \\ 8n-4 & \text{when } n \text{ is odd;} \end{cases}$$

for each $1 \leq i \leq 4n$,

$$d_D(a^ib) = \begin{cases} 8n-4 & \text{when } n \text{ is even;} \\ 8n-8 & \text{when } n \text{ is odd;} \end{cases}$$

and for each $x \in \langle a \rangle \setminus Z(SD_{8n})$, $d_D(x) = 4n$.

Proof. Let $x \in Z(SD_{8n})$. In view of Theorem 4.2.27, $ecc_D(x) = 4n+1$ when n is even. Otherwise $ecc_D(x) = 4n+11$. In each case, by the proof of Theorem 4.2.27, one can observe that $D(x) = SD_{8n} \setminus Z(SD_{8n})$. Similar to $x \in Z(SD_{8n})$, for $x \in \langle a \rangle \setminus Z(SD_{8n})$ we obtain $D(x) = SD_{8n} \setminus \langle a \rangle$ (cf. Theorem 4.2.27). Now let

$x = a^i b$ for some i, where $1 \leq i \leq 4n$. Similar to $x \in Z(SD_{8n})$, when n is even, we get

$$D(a^i b) = \left(\{a^j b : 1 \leq j \leq 4n\} \setminus \{a^i b, a^{2n+i} b\}\right) \cup (\langle a \rangle \setminus Z(SD_{8n})).$$

and for odd n,

$$D(a^i b) = \left(\{a^j b : 1 \leq j \leq 4n\} \setminus \{a^{n+i} b, a^{2n+i} b, a^{3n+i} b, a^{4n+i} b\}\right) \cup (\langle a \rangle \setminus Z(SD_{8n})).$$

□

Corollary 4.2.31. *In $\Delta(SD_{8n})$, we have*

(i)

$$D(\Delta(SD_{8n})) = \begin{cases} ((4n)^{4n-2}, (8n-4)^{4n}, (8n-2)^2) & \text{if } n \text{ is even;} \\ ((4n)^{4n-4}, (8n-8)^{4n}, (8n-4)^4) & \text{if } n \text{ is odd.} \end{cases}$$

(ii)

$$D_{av}(\Delta(SD_{8n})) = \begin{cases} \frac{12n^2-2n-1}{2n} & \text{if } n \text{ is even;} \\ \\ \frac{2(3n^2-n-1)}{n} & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 4.2.32. *In $\Delta(SD_{8n})$, we have*

$$dds_D(\Delta(SD_{8n})) = \begin{cases} (1, 0^{4n-2}, 1, 0, 8n-2)^2, (1, 0^{4n}, 4n-1, 0, 4n)^{4n-2}, \\ (1, 0^{4n}, 3, 0, 8n-4)^{4n} \text{ if } n \text{ is even ;} \\ (1, 0^{4n+6}, 3, 0^3, 8n-4)^4, (1, 0^{4n+10}, 4n-1, 0^3, 4n)^{4n-4}, \\ (1, 0^{4n+10}, 7, 0^3, 8n-8)^{4n} \text{ if } n \text{ is odd.} \end{cases}$$

Proof. **Case 1:** n is even. For $x \in Z(SD_{8n})$, by the proof of Theorem 4.2.27 (**Case 1**), we have $ecc_D(x) = 4n + 1$ so $dds_D(x) = (1, \underbrace{0, 0, \ldots, 0}_{4n-2}, 1, 0, 8n - 2)$. For $x \in SD_{8n} \setminus Z(SD_{8n})$, again by the proof of Theorem 4.2.27 (**Case 1**), we have $ecc_D(x) = 4n + 3$. Thus

$$dds_D(x) = \begin{cases} (1, \underbrace{0, 0, \ldots, 0}_{4n}, 4n-1, 0, 4n) & \text{if } x \in \langle a \rangle \setminus Z(SD_{8n}) \\ (1, \underbrace{0, 0, \ldots, 0}_{4n}, 3, 0, 8n-4) & \text{if } x \in SD_{8n} \setminus \langle a \rangle. \end{cases}$$

Case 2: n is odd. For $x \in Z(SD_{8n})$, by the proof of Theorem 4.2.27 (**Case 1**), we have $ecc_D(x) = 4n + 11$ so $dds_D(x) = (1, \underbrace{0, 0, \ldots, 0}_{4n+6}, 3, 0, 0, 0, 8n - 4)$. For $x \in SD_{8n} \setminus Z(SD_{8n})$, again by the proof of Theorem 4.2.27 (**Case 2**), we have $ecc_D(x) = 4n + 15$. Consequently,

$$dds_D(x) = \begin{cases} (1, \underbrace{0, 0, \ldots, 0}_{4n+10}, 4n - 1, 0, 0, 0, 4n) & \text{if } x \in \langle a \rangle \setminus Z(SD_{8n}) \\ (1, \underbrace{0, 0, \ldots, 0}_{4n+10}, 7, 0, 0, 0, 8n - 8) & \text{if } x \in SD_{8n} \setminus \langle a \rangle. \end{cases}$$

□

4.3 Commuting Graph of Brandt Semigroup B_n

Since all completely 0-simple inverse semigroups are exhausted by Brandt semigroups, their consideration seems interesting and useful in various aspects. Brandt semigroups have been studied extensively by various authors (see Jackson and Volkov [2009]; Sadr [2009, 2012] and the references therein). The Brandt semigroup B_n is isomorphic to the Rees matrix semigroup $M^0(\{1, \ldots, n\}, 1, \{1, \ldots, n\}, I_n)$, where I_n is the $n \times n$ identity matrix (see Howie [1995]). Brandt semigroup B_n plays an important role in inverse semigroup theory and arises in number of different ways (see Ćirić and Bogdanović [2000]; Kátai-Urbán and Szabó [2006] and the references therein). Endomorphism seminear-rings on B_n have been classified by Gilbert and Samman [2010]. Further, various aspects of affine near-semirings generated by affine maps on B_n have been studied in Kumar [2014]. The combinatorial study of B_n have been related with theory of matroids and simplicial complexes in Margolis et al. [2018]. Various ranks of B_n have been obtained by Howie and Ribeiro [1999, 2000], where some of the ranks of B_n were obtained by using graph theoretic properties of some graph associated on B_n. Hao et al. [2011] obtained a necessary and sufficient condition for the components of Cayley graphs of Brandt semigroups to be strongly regular. Further, Khosravi and Khosravi [2012] characterized Cayley graphs of finite

Brandt semigroups and provided a condition to check whether a finite digraph is a Cayley graph of a finite Brandt semigroup.

Since the central elements of the semigroup S (or group) are always adjacent with all the elements of S in $\Delta(S)$, various authors (see Ali et al. [2016]; Araújo et al. [2015]) have studied commuting graphs by removing all central elements from its vertex set. In view of this, we study the commuting graph of $\Delta(B_n, \Omega)$ such that $\Omega = B_n \setminus Z(B_n)$. Note that $Z(B_n) = \{0\}$. Hereafter, we shall denote $\Delta(B_n, \Omega)$ by $\Delta(B_n)$ and any vertex $x \in V(\Delta(B_n))$ is denoted by (i, j), where $i, j \in [n]$.

In this section, we examine various graph invariants of $\Delta(B_n)$, viz. minimum degree, vertex covering number, edge covering number, independence number, matching number, vertex connectivity. Further, we calculate the chromatic number and strong metric dimension of commuting graph of Brandt semigroup B_n. In order to study algebraic aspects of $\Delta(B_n)$, we have investigated the automorphism group and endomorphism monoid of $\Delta(B_n)$ in Subsection 4.3.1. Finally, we ascertained a class of inverse semigroup whose commuting graph is Hamiltonian. This provides a partial answer to the question posed in Araújo et al. [2011].

We begin with the results concerning the neighbours (degree) of all the vertices of $\Delta(B_n)$.

Lemma 4.3.1. *In the graph $\Delta(B_n)$, we have the following:*

(i) $\mathrm{N}[(i,i)] = \{(j,k) \ : \ j,k \in [n], \ j,k \neq i\} \cup \{(i,i)\}$

(ii) $\mathrm{N}[(i,j)] = \{(i,l) \ : \ l \in [n], l \neq i,j\} \cup \{(l,j) \ : \ l \in [n], l \neq i,j\} \cup \{(k,l) \ : \ k,l \in [n], \ k \neq i,j \text{ and } l \neq i,j\} \cup \{(i,j)\}$, *where* $i \neq j$.

Proof. The result is straightforward for $n = 2$. Now, let $n \geq 3$. Note that $(k,l) \sim (i,i)$ in $\Delta(B_n)$ if and only if $k \neq i$ and $l \neq i$. Now, let (i,j) be a vertex of $\Delta(B_n)$, where $i \neq j$. Then vertices $(k,l) \sim (i,j)$ if and only if (k,l) satisfies one of the following:

(a) where $k \neq i, j$ and $l \neq i, j$

(b) where $k = i$ and $l \neq i, j$

(c) where $k \neq i, j$ and $l = j$. □

In view of the above proof, we have the following useful remark.

Remark 4.3.2. Two distinct vertices (i, j) and (k, i) are not adjacent in $\Delta(B_n)$.

Corollary 4.3.3. *In the commuting graph $\Delta(B_n)$, the degree of idempotent vertices is $(n-1)^2$ and the degree of non-idempotent vertices is $n(n-2)$.*

Corollary 4.3.4. *The minimum degree of $\Delta(B_n)$ is $n(n-2)$.*

Theorem 4.3.5. *For $n \geq 3$, the commuting graph $\Delta(B_n)$ satisfies the following properties:*

(i) *$\Delta(B_n)$ is not Eulerian*

(ii) *The girth of $\Delta(B_n)$ is 3*

(iii) *$\Delta(B_n)$ is Hamiltonian*

(iv) *$\Delta(B_n)$ is connected and* $\mathrm{diam}(\Delta(B_n)) = 2$.

Proof. (i) If n is even, then the $\deg(i, i) = (n-1)^2$ is odd. If n is odd, the vertex (i, j), where $i \neq j$, has degree $n(n-2)$, which is odd. Thus by Theorem 1.2.1, $\Delta(B_n)$ is not Eulerian.

(ii) In order to find the girth of $\Delta(B_n)$, note that $(1, 1) \sim (2, 2) \sim (3, 3) \sim (1, 1)$ is a smallest cycle with length 3. Thus, we have the result.

(iii) For $n = 3$, note that $(1, 2) \sim (1, 3) \sim (2, 3) \sim (2, 1) \sim (3, 1) \sim (3, 2) \sim (1, 1) \sim (2, 2) \sim (3, 3) \sim (1, 2)$ is a Hamiltonian cycle in $\Delta(B_n)$. Since the minimum degree of $\Delta(B_n)$ is $n(n-2)$ (cf. Lemma 4.3.3), for $n \geq 4$, we have $n(n-2) - \frac{n^2}{2} = \frac{n}{2}(n-4) \geq 0$. Hence, by [West, 1996, Theorem 7.2.8], $\Delta(B_n)$ is Hamiltonian.

(iv) From the Figure 1, note that the diam($\Delta(B_3)$) is 2. Now for $n \geq 4$, let $(a,b),(c,d) \in B_n$. If all of a,b,c,d are distinct, then $(a,b) \sim (c,d)$. If at most three from a,b,c,d are distinct, then there exists $i \neq a,b,c,d$ such that $(a,b) \sim (i,i) \sim (c,d)$ is a path. Thus, any two vertices of $\Delta(B_n)$ are connected by a path with maximum distance two.

□

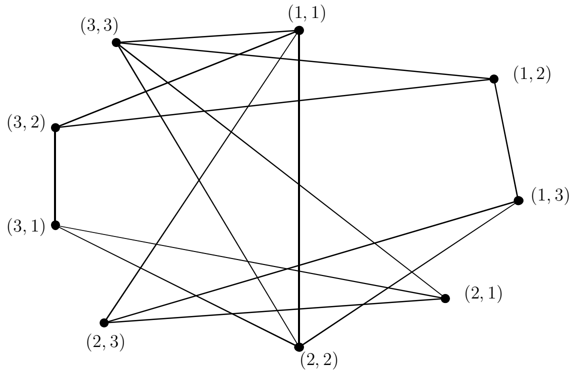

FIGURE 4.1: The commuting graph of B_3

Since $\Delta(B_n)$ is Hamiltonian so we have the following corollary.

Corollary 4.3.6. *For $n \geq 3$, the matching number of $\Delta(B_n)$ is $\left\lfloor \frac{n^2}{2} \right\rfloor$.*

In view of Lemma 1.2.10(ii), we have the following corollary.

Corollary 4.3.7. *For $n \geq 3$, the edge covering number of $\Delta(B_n)$ is $n^2 - \left\lceil \frac{n^2}{2} \right\rceil$.*

In the following theorem, we investigate the independence number, dominance number, planarity and perfectness of $\Delta(B_n)$.

Theorem 4.3.8. *For $n \geq 2$, we have*

(i) *the independence number of $\Delta(B_n)$ is 3.*

(ii) *the dominance number of $\Delta(B_n)$ is 3.*

(iii) *$\Delta(B_n)$ is planar if and only if $n = 2$.*

(iv) $\Delta(B_n)$ *is perfect if and only if* $n = 2$.

Proof. (i) For $n \geq 2$, first note that the set $I' = \{(1,1),(1,2),(2,1)\}$ is an independent set of $\Delta(B_n)$. In fact, I is of maximum size for $n = 2$. Thus, the result hold for $n = 2$. Now, to prove the result we show that any independent set in $\Delta(B_n)$, where $n \geq 3$, is of size at most 3. Let I be an independent subset of $\Delta(B_n)$. If I does not contain any non-idempotent vertex of $\Delta(B_n)$, then clearly $I = \{(i,i)\}$, for some $i \in [n]$. Thus, $|I| \leq 3$. We may now suppose (i,j), where $i \neq j$, belongs to I. Note that each of the set $A = \{(i,i),(j,j)\}$, $B = \{(x,i) : x \neq i\}$ and $C = \{(j,y) : y \neq j\}$ of vertices forms a complete subgraph of $\Delta(B_n)$. For $(k,l) \notin A \cup B \cup C$, we have $k \neq j$ and $l \neq i$. In this case, $(k,l) \sim (i,j)$. Thus, the independent set I must contained in $A \cup B \cup C$. Being an independent set I can contain at most one element from each of these sets. Consequently, $|I| \leq 3$.

(ii) By part (i), we have $\alpha(\Delta(B_n)) = 3$. Further by Lemma 1.2.3, the dominance number of $\Delta(B_n)$ is at most 3. Now we prove the result by showing that any dominating subset of $\Delta(B_n)$ contains at least three elements. Let D be a dominating subset of $\Delta(B_n)$. In view of Corollary 4.3.3, we do not have a vertex of $\Delta(B_n)$ whose degree is $n^2 - 1$ so that $|D| \neq 1$. Suppose $D = \{(a,b),(c,d)\}$. If a, b, c, d all are distinct, then it can be verified that the vertex (b,c) is not adjacent to any element of D; a contradiction for D to be a dominating set. If D contains an idempotent, say $(a,b) = (i,i)$, then clearly both c, d can not be equal to i. Without loss of generality, let $c \neq i$. Then it is easy to observe that (i,c) is not adjacent to all the elements of D; a contradiction. Thus, in this case ($|D| = 2$), the dominating set D can not contain any idempotent vertex and it can not be of the form $\{(a,b),(c,d)\}$, where all a, b, c, d are distinct. Now we have the following remaining cases:

Case 1: $c \in \{a,b\}$. Then the vertex (d,a) is not adjacent to any element of D; a contradiction.

Case 2: $d \in \{a,b\}$. Then the vertex (b,c) is not adjacent to any element of D; again a contradiction.

Thus, a dominating set of two elements in $\Delta(B_n)$ is not possible. Consequently, $|D| \geq 3$.

(iii) For $n = 2$, it is easy to observe that $\Delta(B_n)$ is planar. For the converse part, let $n \geq 3$. It is sufficient to show that some induced subgraph of $\Delta(B_n)$ is not planar. From the subgraph of $\Delta(B_n)$ in Figure 4.2, if we apply edge contraction on the vertices $(1, 2)$ and $(2, 1)$, then we get a complete bipartite graph $K_{3,3}$. Hence, by Theorem 1.2.7, $\Delta(B_n)$ is not planar.

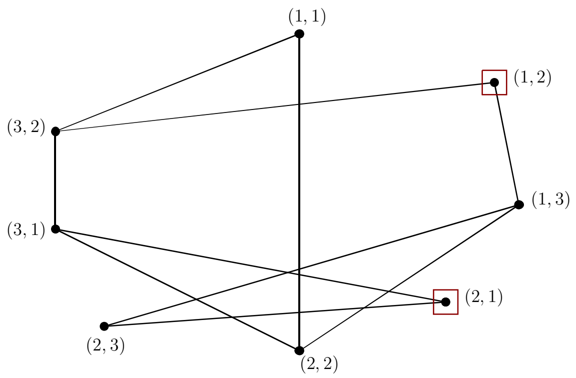

FIGURE 4.2: The subgraph of $\Delta(B_n)$

(iv) For $n = 2$, it is easy to verify that $\Delta(B_n)$ is perfect. On the other hand, let $n \geq 3$. In order to prove that $\Delta(B_n)$ is not perfect, we prove that the chromatic number and clique number of an induced subgraph of $\Delta(B_n)$ are not equal. In fact, we show that for the subgraph induced by $U = V(\Delta(B_3)) \setminus \{(3, 3)\}$, $\omega(\Delta(U)) = 2$ whereas $\chi(\Delta(U)) = 3$. By part (i), since the independence number of $\Delta(B_n)$ is 3, we must have at least three subsets in any chromatic partition of $\Delta(U)$. Thus, $\chi(\Delta(U)) \geq 3$. Further, note that the sets $A_1 = \{(1, 1), (1, 2), (2, 1)\}$, $A_2 = \{(2, 2), (2, 3), (3, 2)\}$ and $A_3 = \{(1, 3), (3, 1)\}$ forms a chromatic partition of the vertex set of $\Delta(U)$. Hence, $\chi(\Delta(U)) = 3$. □

In view of Lemma 1.2.10, we have the following consequences of Theorem 4.3.8(i).

Corollary 4.3.9. *For $n \geq 2$, the vertex covering number of $\Delta(B_n)$ is $n^2 - 3$.*

Notation: We denote $\mathcal{K}$ as the set of all cliques of $\Delta(B_n)$ having no idempotent element and $\mathcal{E}$ as the set of non-zero idempotents of B_n.

In order to obtain the clique number of $\Delta(B_n)$, the following lemma is useful.

Lemma 4.3.10. *For $K \in \mathcal{K}$, we have*

$$|K| \leq \begin{cases} \frac{n^2}{4} & \text{if } n \text{ is even;} \\ \\ \frac{n^2-1}{4} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Suppose K is of maximum size. Consider $A = \{a \in [n] : (a, y) \in K$ for some $y \in [n]\}$ and $B = \{b \in [n] : (x, b) \in K$ for some $x \in [n]\}$. If $t \in A \cap B$, then there exist $p, q \in [n]$ such that $(t, p), (q, t) \in K$. Since K is a clique, we get $(t, p) \sim (q, t)$ so that $(t, p)(q, t) = (q, t)(t, p)$. Consequently, $p = q = t$ gives $(t, t) \in K$; a contradiction. Thus, A and B are disjoint subsets of $[n]$ and so $A \times B$ does not contain an idempotent. If $(a, b), (c, d) \in A \times B$, then $a \neq d$ and $b \neq c$. As a consequence, $(a, b) \sim (c, d)$. Thus, $A \times B$ is a clique such that $K \subseteq A \times B$. Since K is a clique of maximum size which does not contain an idempotent, we get $K = A \times B$. If $|A| = k$, then $|B| = n - k$ because $A \times B$ is a clique of maximum size. Further, $|K| = |A||B| = k(n - k)$. If n is even, note that $|K| = \frac{n^2}{4}$ which attains at $k = \frac{n}{2}$. Otherwise, $|K| = \frac{n^2-1}{4}$ which attains at either $k = \frac{n-1}{2}$ or $k = \frac{n+1}{2}$. □

In view of the proof of Lemma 4.3.10, we have the following corollary.

Corollary 4.3.11. *For $n \geq 4$, there exists $K \in \mathcal{K}$ such that*

$$|K| = \begin{cases} \frac{n^2}{4} & \text{if } n \text{ is even;} \\ \\ \frac{n^2-1}{4} & \text{if } n \text{ is odd.} \end{cases}$$

Lemma 4.3.12. *For $n \in \{2, 3, 4\}$, the set $\mathcal{E}$ forms a clique of maximum size. Moreover, in this case $\omega(\Delta(B_n)) = n$.*

Proof. By Figure 4.1, note that $\{(1,1),(2,2)\}$ and $\{(1,1),(2,2),(3,3)\}$ forms a clique of maximum size for $n = 2$ and 3, respectively. Now, for $n = 4$, clearly $K = \{(1,1),(2,2),(3,3),(4,4)\}$ is a clique in $\Delta(B_4)$. Suppose K' is a clique of maximum size. If K' does not contain an idempotent, then by Lemma 4.3.10, $|K'| = 4$. Thus, K is also a clique of maximum size. On the other hand, we may now assume that K contains an idempotent. Without loss of generality, let $(4,4) \in K'$. Then $K' \setminus \{(4,4)\}$ is a clique of maximum size in $\Delta(B_3)$. Since $\{(1,1),(2,2),(3,3)\}$ is the only clique in $\Delta(B_3)$ of maximum size. Thus, $K' \setminus \{(4,4)\} = \{(1,1),(2,2),(3,3)\}$. Consequently, $K' = \{(1,1),(2,2),(3,3),(4,4)\} = K$. Hence, we have the result. □

From the proof of Lemma 4.3.10 and Lemma 4.3.12, we have the following remark.

Remark 4.3.13. For $n = 4$, let K be any clique in $\Delta(B_n)$ of size 4. Then K is either $\mathcal{E}$ or $K = A \times B$, where A and B are disjoint subset of $\{1,2,3,4\}$ of size two.

Theorem 4.3.14. *For $n > 4$, the clique number of $\Delta(B_n)$ is given below:*

$$\omega(\Delta(B_n)) = \begin{cases} \frac{n^2}{4} & \text{if } n \text{ is even;} \\ \\ \frac{n^2-1}{4} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. In view of Lemma 4.3.10, it is sufficient to prove that any clique of maximum size in $\Delta(B_n)$ contains only non-idempotent vertices. Suppose K is a clique of maximum size such that K contains m idempotents viz. $(i_1,i_1),(i_2,i_2),\ldots,(i_m,i_m)$. Without loss of generality, we assume that $\{i_1,i_2,\ldots,i_m\} = \{n-m+1, n-m+2,\ldots,n\}$. For $1 \le r \le m$, K contains (i_r,i_r) and no element of the form (x,i_r) or (i_r,x) $(x \in [n]), x \neq i_r$ is in K. Thus $K \setminus \{(i_1,i_1),\ldots,(i_m,i_m)\}$ is a clique in $\Delta(B_{n-m})$ which does not contain any idempotent. Clearly, $|K \setminus \{(i_1,i_1),\ldots,(i_m,i_m)\}| =$

$\omega(\Delta(B_{n-m}))$. Then by Corollary 4.3.11

$$|K \setminus \{(i_1, i_1), \ldots, (i_m, i_m)\}| = \begin{cases} \frac{(n-m)^2}{4} & \text{if } n-m \text{ is even;} \\ \\ \frac{(n-m)^2-1}{4} & \text{if } n-m \text{ is odd.} \end{cases}$$

Thus,

$$|K| = \begin{cases} \frac{(n-m)^2}{4} + m & \text{if } n-m \text{ is even;} \\ \frac{(n-m)^2-1}{4} + m & \text{if } n-m \text{ is odd.} \end{cases}$$

Since $n > 4$ and for $m > 0$, one can observe that

$$|K| < \begin{cases} \frac{n^2}{4} & \text{if } n \text{ is even;} \\ \frac{n^2-1}{4} & \text{if } n \text{ is odd;} \end{cases}$$

a contradiction of the fact that K is a clique of maximum size (see proof of Lemma 4.3.10). Thus K has no idempotent. □

By the proof of Lemma 4.3.10 and Theorem 4.3.14, we have the following remarks.

Remark 4.3.15. For $n > 4$, let K be a clique of maximum size in $\Delta(B_n)$. Then all elements of K are non-idempotent.

Remark 4.3.16. For $n > 4$ and $(i, j) \notin \mathcal{E}$, there exists a clique K of maximum size such that $(i, j) \in K$.

In view of Lemma 4.3.1, note that for each vertex $\widehat{v}$ of $\widehat{\Delta}(B_n)$ we have $\widehat{v} = \{v\}$. Thus, $\omega(\widehat{\Delta}(B_n)) = \omega(\Delta(B_n))$. Hence by Theorems 1.2.15 and 4.3.14, we have the following result.

Theorem 4.3.17. *For $n \geq 2$, we have*

$$\operatorname{sdim}(\Delta(B_n)) = \begin{cases} \frac{3n^2}{4} & \textit{if } n \textit{ is even;} \\ \\ \frac{3n^2+1}{4} & \textit{if } n \textit{ is odd.} \end{cases}$$

Now we obtain the chromatic number of $\Delta(B_n)$. For $n \in \mathbb{N}$, we write $n = 3a + r$ where $0 \leq r \leq 2$ and $a \in \mathbb{N}$. Consider

$$\mathcal{A}_{m,x} = \{(m + x, m), (m, n - 2m + 2 - x), (n - 2m + 2 - x, m + x)\}$$

and

$$\mathcal{B}_{\ell,y} = \{(\ell, \ell - y), (\ell - y, 2n - 2\ell + 2 + y), (2n - 2\ell + 2 + y, \ell)\}.$$

In order to obtain $\chi(\Delta(B_n))$, first we prove the following claims which will be useful to obtain the chromatic partition of $V(\Delta(B_n))$.

Claim 4.3.18. *Let $n = 3a$. Then*

(i) *for $m \in \{1, 2, \ldots, a\}$ and $x \in \{0, 1, 2, \ldots, n - 3m + 1\}$, $\mathcal{A}_{m,x}$ are the disjoint independent subsets of $\Delta(B_n)$.*

(ii) *for $\ell \in \{2a + 1, 2a + 2, \ldots, n\}$ and $y \in \{0, 1, 2, \ldots, 3\ell - 2n - 3\}$, $\mathcal{B}_{\ell,y}$ are the disjoint independent subsets of $\Delta(B_n)$.*

Proof. (i) For $m \in \{1, 2, \ldots, a\}$ and $x \in \{0, 1, 2, \ldots, n - 3m + 1\}$, note that $m, m + x, n - 2m + 2 - x \in [n]$. Thus, $\mathcal{A}_{m,x} \subseteq B_n$. By Remark 4.3.2, any pair of vertices in $\mathcal{A}_{m,x}$ are not adjacent and so each $\mathcal{A}_{m,x}$ is an independent subset of $\Delta(B_n)$. Now we prove that any two distinct subsets $\mathcal{A}_{m_1,x_1}$ and $\mathcal{A}_{m_2,x_2}$ are disjoint. If possible, let $(m_1 + x_1, m_1) \in \mathcal{A}_{m_2,x_2}$. Clearly, $(m_1 + x_1, m_1) \neq (m_2 + x_2, m_2)$. Then either $(m_1 + x_1, m_1) = (m_2, n - 2m_2 + 2 - x_2)$ or $(m_1 + x_1, m_1) = (n - 2m_2 + 2 - x_2, m_2 + x_2)$. If $(m_1 + x_1, m_1) = (m_2, n - 2m_2 + 2 - x_2)$, we get $m_1 + x_1 = m_2$ and $m_1 = n - 2m_2 + 2 - x_2$. As a consequence, $x_2 = (n - 3m_2 + 1) + x_1 + 1 \geq n - 3m_2 + 1$; a contradiction of $x_2 \leq n - 3m_2 + 1$. Similarly, for $(m_1 + x_1, m_1) = (n - 2m_2 + 2 - x_2, m_2 + x_2)$, we get $x_1 = n - 3m_1 + 2 + x_2 > n - 3m_1 + 1$; a contradiction. Thus, $(m_1 +$

$x_1, m_1) \notin \mathcal{A}_{m_2,x_2}$. Analogously, one can check that $(m_2 + x_2, m_2) \notin \mathcal{A}_{m_1,x_1}$. Now, if $(m_1, n-2m_1+2-x_1) \in \mathcal{A}_{m_2,x_2}$, then either $(m_1, n-2m_1+2-x_1) = (m_2, n-2m_2+2-x_2)$ or $(m_1, n-2m_1+2-x_1) = (n-2m_2+2-x_2, m_2+x_2)$. For $(m_1, n-2m_1+2-x_1) = (m_2, n-2m_2+2-x_2)$, we obtain $m_1 = m_2$ and $x_1 = x_2$; a contradiction so $(m_1, n-2m_1+2-x_1) = (n-2m_2+2-x_2, m_2+x_2)$. Thus, we have $x_2 = n-3m_2+2+x_1$ implies $x_2 > n-3m_2+1$; a contradiction. Therefore, $(m_1, n-2m_1+2-x_1) \notin \mathcal{A}_{m_2,x_2}$. By replacing m_1, x_1 with m_2, x_2 respectively, we get $(m_2, n-2m_2+2-x_2) \notin \mathcal{A}_{m_1,x_1}$. If $(n-2m_1+2-x_1, m_1+x_1) \in \mathcal{A}_{m_2,x_2}$, then $(n-2m_1+2-x_1, m_1+x_1) = (n-2m_2+2-x_2, m_2+x_2)$. As a consequence, $m_1 + x_1 = m_2 + x_2$ and $2m_1 + x_1 = 2m_2 + x_2$ gives $m_1 = m_2$ and $x_1 = x_2$; a contradiction. Thus, $\mathcal{A}_{m_1,x_1} \cap \mathcal{A}_{m_2,x_2} = \varnothing$.

(ii) For $\ell \in \{2a+1, 2a+2, \dots, n\}$ and $y \in \{0, 1, 2, \dots, 3\ell - 2n - 3\}$, note that $\ell, \ell - y, 2n - 2\ell + 2 + y \in [n]$. Thus $\mathcal{B}_{\ell,y} \subseteq B_n$. By Remark 4.3.2, any pair of vertices in $\mathcal{B}_{\ell,y}$ are not adjacent and so each $\mathcal{B}_{\ell,y}$ is an independent subset of $\Delta(B_n)$. Now we prove that any two distinct subsets $\mathcal{B}_{\ell_1,y_1}$ and $\mathcal{B}_{\ell_2,y_2}$ are disjoint. If possible, let $(\ell_1, \ell_1 - y_1) \in \mathcal{B}_{\ell_2,y_2}$. Clearly $(\ell_1, \ell_1 - y_1) \neq (\ell_2, \ell_2 - y_2)$. Then, we get either $(\ell_1, \ell_1 - y_1) = (\ell_2 - y_2, 2n - 2\ell_2 + 2 + y_2)$ or $(\ell_1, \ell_1 - y_1) = (2n - 2\ell_2 + 2 + y_2, \ell_2)$. If $(\ell_1, \ell_1 - y_1) = (\ell_2 - y_2, 2n - 2\ell_2 + 2 + y_2)$, then $\ell_1 = \ell_2 - y_2$ and $\ell_1 - y_1 = 2n - 2\ell_2 + 2 + y_2$ so $\ell_1 - y_1 = 2n - 2(\ell_1 + y_2) + 2 + y_2$. Therefore, we get $y_1 = (3\ell_1 - 2n - 3) + y_2 + 1$ which is not possible as $y_1 \leq 3\ell_1 - 2n - 3$. As a result, $(\ell_1, \ell_1 - y_1) = (2n - 2\ell_2 + 2 + y_2, \ell_2)$ gives $\ell_1 = 2n-2\ell_2+2+y_2$ and $\ell_1 - y_1 = \ell_2$. Consequently, $\ell_2+y_1 = 2n-2\ell_2+2+y_2$ implies $y_2 = 3\ell_2 - 2n - 3 + y_1 + 1$; a contradiction of $y_2 \leq 3\ell_2 - 2n - 3$. Thus, $(\ell_1, \ell_1 - y_1) \notin \mathcal{B}_{\ell_2,y_2}$. Analogously, one can show that $(\ell_2, \ell_2 - y_2) \notin \mathcal{B}_{\ell_1,y_1}$. If $(\ell_1 - y_1, 2n - 2\ell_1 + 2 + y_1) \in \mathcal{B}_{\ell_2,y_2}$, then either $(\ell_1 - y_1, 2n - 2\ell_1 + 2 + y_1) = (\ell_2 - y_2, 2n - 2\ell_2 + 2 + y_2)$ or $(\ell_1 - y_1, 2n - 2\ell_1 + 2 + y_1) = (2n - 2\ell_2 + 2 + y_2, \ell_2)$. Suppose $(\ell_1 - y_1, 2n - 2\ell_1 + 2 + y_1) = (\ell_2 - y_2, 2n - 2\ell_2 + 2 + y_2)$, we obtain $\ell_1 - y_1 = \ell_2 - y_2$ and $2n - 2\ell_1 + 2 + y_1 = 2n - 2\ell_2 + 2 + y_2$. It follows that $2(\ell_2 - \ell_1) = \ell_2 - \ell_1$ which is possible only if $\ell_1 = \ell_2$ and $y_1 = y_2$; a contradiction. Therefore, we get $(\ell_1 - y_1, 2n - 2\ell_1 + 2 + y_1) = (2n - 2\ell_2 + 2 + y_2, \ell_2)$ and this implies $\ell_1 - y_1 = 2n - 2(2n - 2\ell_1 + 2 + y_1) + 2 + y_2$. As a result, $y_1 = 3\ell_1 - 2n - 3 + y_2 + 1$ which is not possible as $y_1 \leq 3\ell_1 - 2n - 3$. Thus, $(\ell_1 - y_1, 2n - 2\ell_1 + 2 + y_1) \notin \mathcal{B}_{\ell_2, y_2}$. In a similar lines one can show that $(\ell_2 - y_2, 2n - 2\ell_2 + 2 + y_2) \notin \mathcal{B}_{\ell_1, y_1}$. Thus, if $\mathcal{B}_{\ell_1, y_1} \cap \mathcal{B}_{\ell_2, y_2} \neq \varnothing$, then we must have $(2n - 2\ell_1 + 2 + y_1, \ell_1) = (2n - 2\ell_2 + 2 + y_2, \ell_2)$. It follows that $\ell_1 = \ell_2$ and $y_1 = y_2$; again a contradiction. Hence, the result hold.

□

The proof of the following claims is in the similar lines to the proof of the **Claim 4.3.18**, hence omitted.

Claim 4.3.19. *Let $n = 3a + 1$. Then*

(i) *for $m \in \{1, 2, \dots, a\}$ and $x \in \{0, 1, 2, \dots, n - 3m + 1\}$, $\mathcal{A}_{m,x}$ are the disjoint independent subsets of $\Delta(B_n)$.*

(ii) *for $\ell \in \{2a + 2, 2a + 3, \dots, n\}$ and $y \in \{0, 1, 2, \dots, 3\ell - 2n - 3\}$, $\mathcal{B}_{\ell,y}$ are the disjoint independent subsets of $\Delta(B_n)$.*

Claim 4.3.20. *Let $n = 3a + 2$. Then*

(i) *for $m \in \{1, 2, \dots, a+1\}$ and $x \in \{0, 1, 2, \dots, n - 3m + 1\}$, $\mathcal{A}_{m,x}$ are the disjoint independent subsets of $\Delta(B_n)$.*

(ii) *for $\ell \in \{2a + 3, 2a + 4, \dots, n\}$ and $y \in \{0, 1, 2, \dots, 3\ell - 2n - 3\}$, $\mathcal{B}_{\ell,y}$ are the disjoint independent subsets of $\Delta(B_n)$.*

In view of above claims, a visual representation of $\mathcal{A}_{m,x}$ and $\mathcal{B}_{\ell,y}$ can be observed in the matrix given in Figure 4.3. Independent sets $\mathcal{A}_{1,x}, \mathcal{A}_{2,x}, \cdots$ covers the vertices through dashed triangles, whereas the independent sets $\mathcal{B}_{n,y}, \mathcal{B}_{n-1,y}, \cdots$ covers the vertices of $V(\Delta(B_n))$ on doted triangles as shown in Figure 4.3.

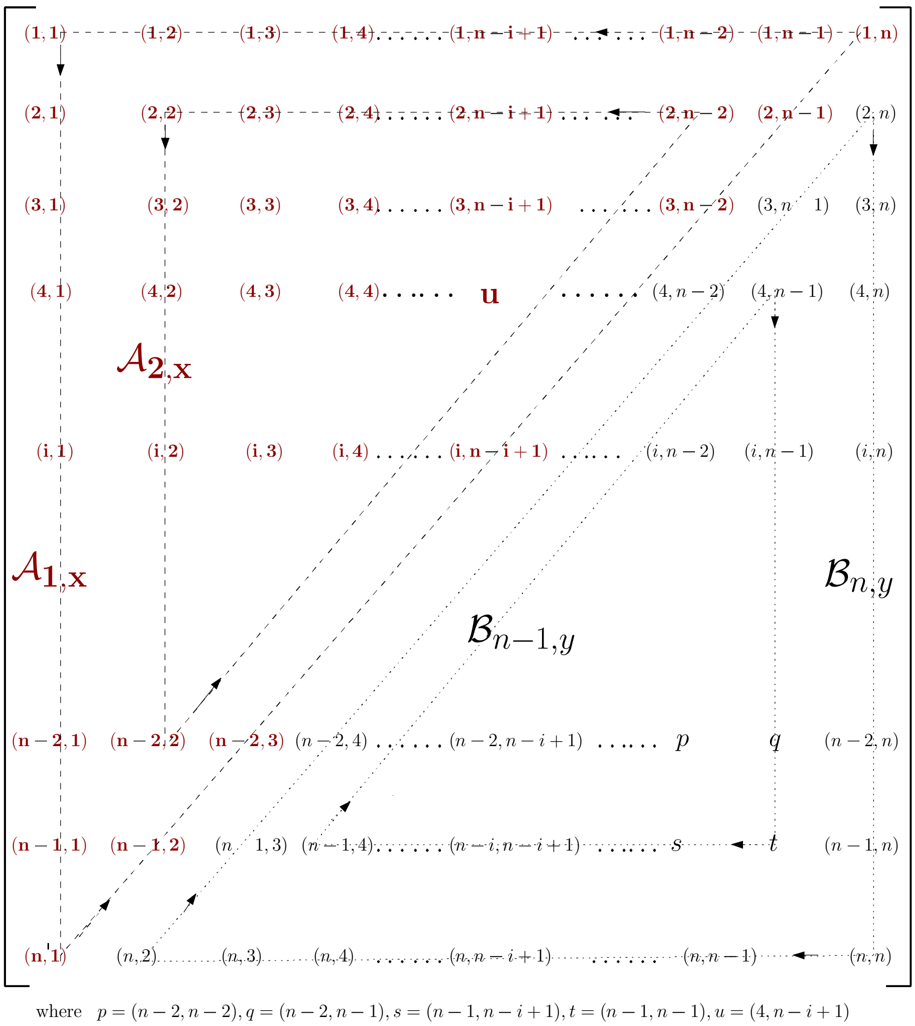

FIGURE 4.3: Visual Representation of $\mathcal{A}_{m,x}$ and $\mathcal{B}_{\ell,y}$

Theorem 4.3.21. *For $n \geq 2$, we have $\chi(\Delta(B_n)) = \left\lceil \frac{n^2}{3} \right\rceil$.*

Proof. For $n = 2$, it is straightforward that $\chi(\Delta(B_n)) = 2$. Since $\chi(\mathcal{G}) \geq \frac{|V(\mathcal{G})|}{\alpha(\mathcal{G})}$ (cf. [West, 1996, Proposition 5.1.7]) so that by Lemma 4.3.8, we have $\chi(\Delta(B_n)) \geq \frac{n^2}{3}$. In order to obtain the result, we shall provide a partition of $V(\Delta(B_n))$ into $\left\lceil \frac{n^2}{3} \right\rceil$ independent subsets. Now, we have the following cases:

Case 1: $n = 3a$. First we prove that the sets $\mathcal{A}_{m,x}$ and $\mathcal{B}_{\ell,y}$, where m, x, ℓ, y are given in **Claim 4.3.18** are disjoint with each other. Note that for $(i, j) \in \mathcal{A}_{m,x}$ and $(k, t) \in \mathcal{B}_{\ell,y}$, we have $i + j \leq n + 1$ and $k + t \geq n + 2$. Thus, $\mathcal{A}_{m,x} \cap \mathcal{B}_{\ell,y} = \varnothing$. Now we shall show that $\mathcal{A}_{m,x}$ and $\mathcal{B}_{\ell,y}$ forms a partition of $V(\Delta(B_n))$. It is sufficient to show that $|(\cup\mathcal{A}_{m,x}) \cup (\cup\mathcal{B}_{\ell,y})| = n^2$. If $m = a, a-1, \ldots, 1$ then $x \in \{0,1\}$, $x \in \{0,1,2,3,4\}, \ldots, x \in \{0,1,2,\ldots,n-2\}$, respectively. Thus, the total number of sets of the form $\mathcal{A}_{m,x}$ is $2+5+\cdots+n-1 = \frac{n(n+1)}{6}$. Similarly, the total number of the sets of the form $\mathcal{B}_{\ell,y}$ is $1+4+\cdots+n-2 = \frac{n(n-1)}{6}$. Consequently, we have $(\cup\mathcal{A}_{m,x}) \cup (\cup\mathcal{B}_{\ell,y}) = V(\Delta(B_n))$. Thus, we have a partition of $V(\Delta(B_n))$ into $\frac{n^2}{3}$ independent sets. Therefore, $\chi(\Delta(B_n)) \leq \frac{n^2}{3}$. Hence, $\chi(\Delta(B_n)) = \frac{n^2}{3}$.

Case 2: $n = 3a+1$. By the similar arguments used in **Case 1**, the sets $\mathcal{A}_{m,x}$ and $\mathcal{B}_{\ell,y}$, where m, x, ℓ, y are given in **Claim 4.3.19** are disjoint with each other. Now, we shall show that the number of elements in the union of sets given in **Claim 4.3.19** is n^2-1. If $m = a, a-1, \ldots, 1$ then $x \in \{0,1,2\}$, $x \in \{0,1,2,3,4,5\}, \ldots, x \in \{0,1,2,\ldots,n-2\}$, respectively. Thus, the total number of the sets of the form $\mathcal{A}_{m,x}$ is $3+6+\cdots+n-1 = \frac{(n+2)(n-1)}{6}$. Similarly, the total number of sets of the form $\mathcal{B}_{\ell,y}$ is $2+5+\cdots+n-2 = \frac{n(n-1)}{6}$. Consequently, we get $|(\cup\mathcal{A}_{m,x}) \cup (\cup\mathcal{B}_{\ell,y})| = n^2-1$. Note that the set $C = \{(a+1, a+1)\}$ is disjoint with $\mathcal{A}_{m,x}$ and $\mathcal{B}_{\ell,y}$. Thus, the sets $\mathcal{A}_{m,x}$, $\mathcal{B}_{\ell,y}$ and C forms a partition of $V(\Delta(B_n))$. Therefore, $\chi(\Delta(B_n)) \leq \frac{n^2-1}{3}+1 = \left\lceil \frac{n^2}{3} \right\rceil$. Hence, $\chi(\Delta(B_n)) = \left\lceil \frac{n^2}{3} \right\rceil$.

Case 3: $n = 3a+2$. By the similar concept used in **Case 1**, the sets $\mathcal{A}_{m,x}$ and $\mathcal{B}_{\ell,y}$, where m, x, ℓ, y are given in **Claim 4.3.20** are disjoint with each other. Now, we shall show that the number of elements in the union of above defined sets is n^2-1. If $m = a+1, a, \ldots, 1$ then $x \in \{0\}$, $x \in \{0,1,2,3\}, \ldots, x \in \{0,1,2,\ldots,n-2\}$, respectively. Thus, the total number of sets of the form $\mathcal{A}_{m,x}$ is $1+4+\cdots+n-1 = \frac{n(n+1)}{6}$. Similarly, the total number of the sets of the form $\mathcal{B}_{\ell,y}$ is $3+6+\cdots+n-2 = \frac{(n+1)(n-2)}{6}$. Consequently, we get $|(\cup\mathcal{A}_{m,x}) \cup (\cup\mathcal{B}_{\ell,y})| = n^2-1$. Note that the set $C = \{(2a+2, 2a+2)\}$ is disjoint with $\mathcal{A}_{m,x}$ and $\mathcal{B}_{\ell,y}$. Thus, the sets $\mathcal{A}_{m,x}$, $\mathcal{B}_{\ell,y}$ and C forms a partition of $V(\Delta(B_n))$. Therefore, $\chi(\Delta(B_n)) \leq \frac{n^2-1}{3}+1 = \left\lceil \frac{n^2}{3} \right\rceil$ so that $\chi(\Delta(B_n)) = \left\lceil \frac{n^2}{3} \right\rceil$. □

Theorem 4.3.22. *For $n \geq 3$, the vertex connectivity of $\Delta(B_n)$ is $n(n-2)$.*

Proof. By Theorem 1.2.9 and Corollary 4.3.3, we have $\kappa(\Delta(B_n)) \leq n(n-2)$. By Menger's theorem (cf. [Bondy et al., 1976, Theorem 3.2]), to prove another inequality, it is sufficient to show that there exist at least $n(n-2)$ internally disjoint paths between arbitrary pair of vertices. Let (a, b) and (c, d) be arbitrary pair of vertices in $V(\Delta(B_n))$. Now consider

$$A = \{(b, x) : x \in [n]\} \cup \{(x, a) : x \in [n]\}$$

and

$$B = \{(d, x) : x \in [n]\} \cup \{(x, c) : x \in [n]\}.$$

Note that $|A| = |B| = 2n-1$ and each element of A and B is not adjacent with (a, b) and (c, d), respectively (see Remark 4.3.2). If $T = A \cup B \cup \{(a, b), (c, d)\}$, then note that every element of $T' = V(\Delta(B_n)) \setminus T$, commutes with (a, b) and (c, d). Thus, for each element (x, y) of T', we have a path $(a, b) \sim (x, y) \sim (c, d)$. Consequently, there are at least $|T'|$ many internally disjoint paths between (a, b) and (c, d). We show that there exist $n(n-2)$ internally disjoint paths between (a, b) and (c, d) in the following cases.

Case 1: Both (a, b) and (c, d) are distinct idempotents. Clearly $a = b, c = d$ and $a \neq c$. Then, we have $A \cap B = \{(a, c), (c, a)\}$ so that $|T'| = n^2 - 4n + 4$. As a consequence, we get $n^2 - 4n + 4$ internally disjoint paths between (a, b) and (c, d). Furthermore, for $x \in [n] \setminus \{a, c\}$, we have $(a, a) \sim (c, x) \sim (a, x) \sim (c, c)$ and $(a, a) \sim (x, c) \sim (x, a) \sim (c, c)$ internally disjoint paths between (a, b) and (c, d) which are $2n - 4$ in total. Thus, there are at least $n^2 - 2n$ internally disjoint paths between (a, b) and (c, d).

Case 2: Either (a, b) or (c, d) is idempotents. Without loss of generality, let $c = d$. Further, we have the following subcases.

Subcase 2.1: $c \neq a, b$. Then $A \cap B = \{(b, c), (c, a)\}$ so that $|T'| = n^2 - 4n + 3$. Consequently, we get $n^2 - 4n + 3$ internally disjoint paths between (a, b) and (c, d).

In addition to that, for $x \in [n] \setminus \{a, b, c\}$, we have

$$(a, b) \sim (c, x) \sim (b, x) \sim (c, c),$$

$$(a, b) \sim (x, c) \sim (x, a) \sim (c, c)$$

internally disjoint paths between (a, b) and (c, d) which are $2n-6$ in total. Further, we have three more paths between (a, b) and (c, d) as follows:

$$(a, b) \sim (c, b) \sim (a, a) \sim (c, c),$$

$$(a, b) \sim (a, c) \sim (b, b) \sim (c, c),$$

$$(a, b) \sim (c, c).$$

Thus, there are at least $n^2 - 2n$ internally disjoint paths between (a, b) and (c, d).

Subcase 2.2: $c = a$ or $c = b$. First suppose $c = a$. Then, we have $A \cap B = \{(x, a) : x \in [n]\}$ so that $|T'| = n^2 - 3n + 2$. Therefore, $\Delta(B_n)$ contains $n^2 - 3n + 2$ internally disjoint paths between (a, b) and (c, d). Additionally, for $x \in [n] \setminus \{a, b\}$, we have $n-1$ internally disjoint paths $(a, b) \sim (a, x) \sim (b, x) \sim (a, a)$ between (a, b) and (c, d). Thus, there are at least $n^2 - 2n$ internally disjoint paths between (a, b) and (c, d). Similarly, for $c = b$, at least $n^2 - 2n$ internally disjoint paths between (a, b) and (c, d) can be obtained.

Case 3: Both (a, b) and (c, d) are non-idempotent element. Clearly, $a \neq b$ and $c \neq d$. Further, we have the following subcases.

Subcase 3.1: a, b, c, d all are distinct. Then, we have $A \cap B = \{(b, c), (d, a)\}$ so that $|T'| = n^2 - 4n + 2$. Thus, there are $n^2 - 4n + 2$ internally disjoint paths between (a, b) and (c, d). In addition to that, for $x \in [n] \setminus \{a, b, c, d\}$, we have $(a, b) \sim (x, c) \sim (x, a) \sim (c, d)$ and $(a, b) \sim (d, x) \sim (b, x) \sim (c, d)$ internally disjoint paths between (a, b) and (c, d) which are $2n - 8$ in total. Moreover, we have six additional paths between (a, b) and (c, d) as follows:

$$(a, b) \sim (a, c) \sim (b, b) \sim (c, d),$$

$$(a,b) \sim (c,c) \sim (b,d) \sim (c,d),$$

$$(a,b) \sim (d,c) \sim (a,a) \sim (c,d),$$

$$(a,b) \sim (d,d) \sim (b,a) \sim (c,d),$$

$$(a,b) \sim (d,b) \sim (c,a) \sim (c,d),$$

$$(a,b) \sim (c,d).$$

Thus, there are at least $n^2 - 2n$ internally disjoint paths between (a,b) and (c,d).

Subcase 3.2: $c \in \{a,b\}$. If $c = a$, then $A \cap B = \{(x,a) \ : \ x \in [n]\}$ so that $|T'| = n^2 - 3n$. Therefore, $\Delta(B_n)$ contains $n^2 - 3n$ internally disjoint paths between (a,b) and (c,d). Additionally, for $x \in [n] \setminus \{a,b,d\}$, we have $(a,b) \sim (d,x) \sim (b,x) \sim (a,d)$ internally disjoint paths between (a,b) and (c,d) which are $n-3$ in total. Besides these paths, we have three paths between (a,b) and (c,d) as follows:

$$(a,b) \sim (d,b) \sim (a,a) \sim (b,d) \sim (a,d),$$

$$(a,b) \sim (d,d) \sim (b,b) \sim (a,d),$$

$$(a,b) \sim (a,d).$$

Thus, there are at least $n^2 - 2n$ internally disjoint paths between (a,b) and (c,d). On the other hand $c = b$. Now we have the two possibilities (i) $d = a$ (ii) a,b,d are distinct. If $d = a$, then $A \cap B = \{(b,b),(a,a)\}$ so that $|T'| = n^2 - 4n + 4$. Consequently, we get $n^2 - 4n + 4$ internally disjoint paths between (a,b) and (c,d). In addition to that, for $x \in [n] \setminus \{a,b\}$, we have $(a,b) \sim (x,b) \sim (x,a) \sim (b,a)$ and $(a,b) \sim (a,x) \sim (b,x) \sim (b,a)$ internally disjoint paths between (a,b) and (c,d) which are $2n-4$ in total. Thus, we get at least $n^2 - 2n$ internally disjoint paths between (a,b) and (c,d). For distinct a,b and d, we get $A \cap B = \{(d,a),(b,b)\}$ so that $|T'| = n^2 - 4n + 4$. Consequently, we get $n^2 - 4n + 4$ internally disjoint paths between (a, b) and (c, d). Additionally, for $x \in [n] \setminus \{a, b, d\}$, we have $2n - 6$ internally disjoint paths

$$(a, b) \sim (x, b) \sim (x, a) \sim (b, d),$$

$$(a, b) \sim (d, x) \sim (b, x) \sim (b, d)$$

between (a, b) and (c, d). Besides these paths, we have two more paths $(a, b) \sim (d, b) \sim (a, a) \sim (b, d)$ and $(a, b) \sim (d, d) \sim (b, a) \sim (b, d)$. Thus, there are at least $n^2 - 2n$ internally disjoint paths between (a, b) and (c, d).

Subcase 3.3: $d \in \{a, b\}$. If $d = a$, then $A \cap B = \{(b, c), (a, a)\}$ so that $|T'| = n^2 - 4n + 4$. Consequently, we get $n^2 - 4n + 4$ internally disjoint paths between (a, b) and (c, d). In addition to that, for $x \in [n] \setminus \{a, b, c\}$, we have $(a, b) \sim (a, x) \sim (b, x) \sim (c, a)$ and $(a, b) \sim (x, c) \sim (x, a) \sim (c, a)$ internally disjoint paths between (a, b) and (c, d) which are $2n-6$ in total. Moreover, we have two paths $(a, b) \sim (a, c) \sim (b, b) \sim (c, a)$ and $(a, b) \sim (c, c) \sim (b, a) \sim (c, a)$ between (a, b) and (c, d). Thus, there are at least $n^2 - 2n$ internally disjoint paths between (a, b) and (c, d). On the other hand, let $d = b$. Then $A \cap B = \{(b, x) : x \in [n]\}$ so that $|T'| = n^2 - 3n$. As a consequence, we get $n^2 - 3n$ internally disjoint paths between (a, b) and (c, d). Furthermore, for $x \in [n] \setminus \{a, b, c\}$, we have $n-3$ internally disjoint paths $(a, b) \sim (x, c) \sim (x, a) \sim (c, b)$ between (a, b) and (c, d). Besides these paths, we have three more paths between (a, b) and (c, d) as follows:

$$(a, b) \sim (c, c) \sim (a, a) \sim (c, b),$$

$$(a, b) \sim (a, c) \sim (b, b) \sim (c, a) \sim (c, b),$$

$$(a, b) \sim (c, b).$$

Thus, there are at least $n^2 - 2n$ internally disjoint paths between (a, b) and (c, d). □

In view of Lemma 4.3.1 and since $\kappa(\mathcal{G}) \leq \kappa'(\mathcal{G}) \leq \delta(\mathcal{G})$, we have the following corollary.

Corollary 4.3.23. *For $n \geq 3$, the edge connectivity of $\Delta(B_n)$ is $n(n-2)$.*

4.3.1 Algebraic Properties of $\Delta(B_n)$

In order to study algebraic aspects of the graph $\Delta(B_n)$, in this subsection we obtain automorphism group (see Theorem 4.3.30) and endomorphism monoid (see Theorem 4.3.41) of $\Delta(B_n)$. For $n = 1$, the group $\mathrm{Aut}(\Delta(B_n))$ is trivial. For the remaining subsection, we assume $n \geq 2$.

Lemma 4.3.24. *Let $x \in V(\Delta(B_n))$ and $f \in \mathrm{Aut}(\Delta(B_n))$. Then x is an idempotent if and only if xf is an idempotent.*

Proof. Since f is an automorphism, we have $\deg(x) = \deg(xf)$. By Corollary 4.3.3, the result holds. □

Lemma 4.3.25. *For $f \in \mathrm{Aut}(\Delta(B_n))$ and $i, j, k, k' \in [n]$ such that $(i,i)f = (k,k)$ and $(j,j)f = (k',k')$, we have either $(i,j)f = (k,k')$ or $(i,j)f = (k',k)$.*

Proof. For $i \neq j$, suppose that $(i,j)f = (x,y)$. Clearly, $(i,j) \nsim (i,i)$ so that $(x,y) = (i,j)f \nsim (i,i)f = (k,k)$. Since $(x,y) \nsim (k,k)$, we get either $x = k$ or $y = k$. Similarly, for $(i,j) \nsim (j,j)$, we have either $x = k'$ or $y = k'$. Thus, by Lemma 4.3.24, we have $(x,y) = (k,k')$ or $(x,y) = (k',k)$. □

Lemma 4.3.26. *For $\sigma \in S_n$, let $\phi_\sigma : V(\Delta(B_n)) \to V(\Delta(B_n))$ defined by $(i,j)\phi_\sigma = (i\sigma, j\sigma)$. Then $\phi_\sigma \in \mathrm{Aut}(\Delta(B_n))$.*

Proof. It is easy to verify that ϕ_σ is a permutation on $V(\Delta(B_n))$. Now we show that ϕ_σ preserves adjacency. Let $(i,j), (x,y) \in V(\Delta(B_n))$ such that $(i,j) \sim (x,y)$. Now,

$$\begin{aligned}
(i,j) \sim (x,y) &\Longleftrightarrow x \neq j \text{ and } y \neq i \\
&\Longleftrightarrow \text{ for } \sigma \in S_n, \text{ we have } x\sigma \neq j\sigma \text{ and } y\sigma \neq i\sigma \\
&\Longleftrightarrow (i\sigma, j\sigma) \sim (x\sigma, y\sigma) \\
&\Longleftrightarrow (i,j)\phi_\sigma \sim (x,y)\phi_\sigma.
\end{aligned}$$

Hence, $\phi_\sigma \in \mathrm{Aut}(\Delta(B_n))$. □

Lemma 4.3.27. *Let* $\alpha : V(\Delta(B_n)) \to V(\Delta(B_n))$ *be a mapping defined by* $(i,j)\alpha = (j,i)$. *Then* $\alpha \in \mathrm{Aut}(\Delta(B_n))$.

Proof. It is straightforward to verify that α is a one-one and onto map on $V(\Delta(B_n))$. Note that

$$\begin{aligned} (i,j) \sim (x,y) &\iff x \neq j \text{ and } y \neq i \\ &\iff (j,i) \sim (y,x) \\ &\iff (i,j)\alpha \sim (x,y)\alpha. \end{aligned}$$

Hence, $\alpha \in \mathrm{Aut}(\Delta(B_n))$. □

Remark 4.3.28. For ϕ_σ and α, defined in Lemma 4.3.26 and 4.3.27, we have $\phi_\sigma \circ \alpha = \alpha \circ \phi_\sigma$.

Proposition 4.3.29. *For each* $f \in \mathrm{Aut}(\Delta(B_n))$, *we have either* $f = \phi_\sigma$ *or* $f = \phi_\sigma \circ \alpha$ *for some* $\sigma \in S_n$.

Proof. Since $f \in \mathrm{Aut}(\Delta(B_n))$, by Lemma 4.3.24, note that there exists a permutation $\sigma : [n] \to [n]$ such that $i\sigma = j \iff (i,i)f = (j,j)$, determined by f. Thus, we have $(i,i)f = (i\sigma, i\sigma)$ for all $i \in [n]$. Let $j \neq i$. Then by Lemma 4.3.25, we get either $(i,j)f = (i\sigma, j\sigma)$ or $(i,j)f = (j\sigma, i\sigma)$.

Case 1: $(i,j)f = (i\sigma, j\sigma)$. We show that for any $(k,l) \neq (i,j)$, where $k \neq l$, we have $(k,l)f = (k\sigma, l\sigma)$ so that $f = \phi_\sigma$. We have the following subcases:

Subcase 1.1: $k = i$. Clearly, $l \neq j$. Then $(i,j) \sim (k,l)$ so that $(i\sigma, j\sigma) = (i,j)f \sim (k,l)f$. We must have $(k,l)f = (k\sigma, l\sigma)$.

Subcase 1.2: $l = j$. Clearly, $k \neq i$. Then $(i,j) \sim (k,l)$ so that $(i\sigma, j\sigma) = (i,j)f \sim (k,l)f$. We must have $(k,l)f = (k\sigma, l\sigma)$.

Subcase 1.3: $l = i$. Note that $(i,j) \nsim (k,l)$ so that $(i\sigma, j\sigma) = (i,j)f \nsim (k,l)f$. We must have $(k,l)f = (k\sigma, l\sigma)$.

Subcase 1.4: $k = j$. Note that $(i,j) \nsim (k,l)$ so that $(i\sigma, j\sigma) = (i,j)f \nsim (k,l)f$. We must have $(k,l)f = (k\sigma, l\sigma)$.

Subcase 1.5: $k, l \in [n] \setminus \{i, j\}$. By *Subcase 1.1*, we get $(i, l)f = (i\sigma, l\sigma)$. Thus, by *Subcase 1.2* we get $(k, l)f = (k\sigma, l\sigma)$.

Case 2: $(i, j)f = (j\sigma, i\sigma)$. Let, if possible, there exists $(k, l) \neq (i, j)$, where $k \neq l$, such that $(k, l)f = (k\sigma, l\sigma)$. Then by **Case 1**, we get $(i, j)f = (i\sigma, j\sigma)$. Consequently, $i = j$; a contradiction. Thus, for any $(k, l) \neq (i, j)$, we have $(k, l)f = (l\sigma, k\sigma)$ so that $f = \phi_\sigma \circ \alpha$. □

Theorem 4.3.30. *For $n \geq 2$, we have* $\mathrm{Aut}(\Delta(B_n)) \cong S_n \times \mathbb{Z}_2$. *Moreover,* $|\mathrm{Aut}(\Delta(B_n))| = 2(n!)$.

Proof. In view of Lemmas 4.3.26, 4.3.27 and 4.3.29, note that the underlying set of the automorphism group of $\Delta(B_n)$ is

$$\mathrm{Aut}(\Delta(B_n)) = \{\phi_\sigma \; : \; \sigma \in S_n\} \cup \{\phi_\sigma \circ \alpha \; : \; \sigma \in S_n\},$$

where S_n is a symmetric group of degree n. Note that the groups $\mathrm{Aut}(\Delta(B_n))$ and $S_n \times \mathbb{Z}_2$ are isomorphic under the assignment $\phi_\sigma \mapsto (\sigma, \bar{0})$ and $\phi_\sigma \circ \alpha \mapsto (\sigma, \bar{1})$. Since, all the elements in $\mathrm{Aut}(\Delta(B_n))$ are distinct, we have $|\mathrm{Aut}(\Delta(B_n))| = 2(n!)$. □

A mapping f from a graph $\mathcal{G}$ to $\mathcal{G}'$ is said to be a *homomorphism* if $x \sim y$, then $xf \sim yf$ for all $x, y \in V(\mathcal{G})$. If $\mathcal{G}' = \mathcal{G}$, then we say f is an *endomorphism*. Note that the set $\mathrm{End}(\mathcal{G})$ of all endomorphisms on $\mathcal{G}$ forms a monoid with respect to the composition of mappings. First we obtain the endomorphism monoid of $\Delta(B_n)$ for $n \in \{2, 3\}$. The following remark is useful in the sequel.

Remark 4.3.31. Let $f \in \mathrm{End}(\mathcal{G})$ and K be a clique of maximum size in $\mathcal{G}$. Then Kf is again a clique of maximum size.

Lemma 4.3.32. $\mathrm{End}(\Delta(B_2)) = \{f : V(\Delta(B_2)) \to V(\Delta(B_2)) \; : \; \mathcal{E}f = \mathcal{E}\}$, where $\mathcal{E} = \{(1, 1), (2, 2)\}$.

Proof. For $x, y \in V(\Delta(B_2))$, note that $x \sim y$ if and only if x, y belongs to $\mathcal{E}$. Hence, we have the result. □

For $\sigma \in S_3$, we define the mappings f^σ and g^σ on $V(\Delta(B_3))$ by

- $(i,i) \xmapsto{f^\sigma} (i\sigma, i\sigma), (1,2) \xmapsto{f^\sigma} (1\sigma, 1\sigma), (1,3) \xmapsto{f^\sigma} (3\sigma, 3\sigma), (2,3) \xmapsto{f^\sigma} (2\sigma, 2\sigma), (2,1) \xmapsto{f^\sigma} (1\sigma, 1\sigma), (3,1) \xmapsto{f^\sigma} (3\sigma, 3\sigma)$, $(3,2) \xmapsto{f^\sigma} (2\sigma, 2\sigma)$, and
- $(i,i) \xmapsto{g^\sigma} (i\sigma, i\sigma), (1,2) \xmapsto{g^\sigma} (2\sigma, 2\sigma), (3,2) \xmapsto{g^\sigma} (3\sigma, 3\sigma), (3,1) \xmapsto{g^\sigma} (1\sigma, 1\sigma), (2,1) \xmapsto{g^\sigma} (2\sigma, 2\sigma), (2,3) \xmapsto{g^\sigma} (3\sigma, 3\sigma)$, $(1,3) \xmapsto{g^\sigma} (1\sigma, 1\sigma)$, respectively.

It is routine to verify that $f^\sigma, g^\sigma \in \mathrm{End}(\Delta(B_3))$.

Lemma 4.3.33. $\mathrm{End}(\Delta(B_3)) = \mathrm{Aut}(\Delta(B_3)) \cup \{f^\sigma \ : \ \sigma \in S_3\} \cup \{g^\sigma \ : \ \sigma \in S_3\}$, where f^σ and g^σ are the endomorphisms on $V(\Delta(B_3))$ as defined above.

Proof. Let $\psi \in \mathrm{End}(\Delta(B_3))$. By Figure 4.1, note that $\{(1,1),(2,2),(3,3)\}$ is the only clique of maximum size in $\Delta(B_3)$. Since the image of a clique of maximum size under an endomorphism is again a clique of maximum size, we get $(i,i)\psi$ is an idempotent element for all $i \in \{1,2,3\}$. Also note that restriction of ψ to $\mathcal{E} = \{(1,1),(2,2),(3,3)\}$ is a bijective map from $\mathcal{E}$ to $\mathcal{E}$. If $(i,i)\psi = (j,j)$ for some $j \in \{1,2,3\}$, then define $\sigma : \{1,2,3\} \to \{1,2,3\}$ by $i\sigma = j$. Consequently, $\sigma \in S_3$. Suppose $(i,j)\psi$ is an idempotent element for some distinct $i,j \in \{1,2,3\}$. Without loss of generality, let $i = 1$ and $j = 2$. Since $(1,2) \sim (3,3)$ we have $(1,2)\psi \sim (3,3)\psi = (3\sigma, 3\sigma)$. Consequently, $(1,2)\psi \in \{(1\sigma,1\sigma),(2\sigma,2\sigma)\}$. If $(1,2)\psi = (1\sigma, 1\sigma)$, then $\psi = f^\sigma$. Otherwise, $\psi = g^\sigma$. On the other hand, if $(i,j)\psi$ is a non-idempotent for all $i \neq j$. Let $(i,j)\psi = (x,y)$, where $x \neq y$. For $k \neq i,j$, we have $(x,y) = (i,j)\psi \sim (k,k)\psi$. Thus, $(i,j)\psi$ is either $(i\sigma, j\sigma)$ or $(j\sigma, i\sigma)$. By the similar argument used in Proposition 4.3.29, we have $\psi \in \mathrm{Aut}(\Delta(B_3))$. $\square$

Now, we obtain $\mathrm{End}(\Delta(B_n))$ for $n \geq 4$. We begin with few definitions and necessary results. If $\mathcal{G}'$ is a subgraph of $\mathcal{G}$, then a homomorphism $f : \mathcal{G} \to \mathcal{G}'$ such that $xf = x$ for all $x \in \mathcal{G}'$ is called a *retraction* of $\mathcal{G}$ onto $\mathcal{G}'$ and $\mathcal{G}'$ is said to be a *retract* of $\mathcal{G}$. A subgraph $\mathcal{G}'$ of $\mathcal{G}$ is said to be a *core* of $\mathcal{G}$ if and only if it admits no proper retracts (cf. Hell and Nešetřil [1992]). Let $X \subset A$, $Y \subseteq B$ and f be any mapping from the set A to B such that $Xf \subseteq Y$. We write the *restriction map* of

f from X to Y as $f_{X\times Y}$ i.e $f_{X\times Y} : X \to Y$ such that $xf_{X\times Y} = xf$.

Proposition 4.3.34 ([J. Cameron, 2006, Proposition 2.4]). *A graph $\mathcal{G}$ is a core if and only if* $\mathrm{End}(\mathcal{G}) = \mathrm{Aut}(\mathcal{G})$.

Lemma 4.3.35. *Let f be a retraction of $\Delta(B_4)$. Then a non-idempotent element maps to a non-idempotent element of B_4 under f.*

Proof. Let, if possible there exists a non-idempotent element (i, j) of B_4 such that $(i, j)f$ is an idempotent element. In order to get a contradiction, first we show that $(a, b)f \in \mathcal{E} = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$ for all $a \neq b \in \{1, 2, 3, 4\}$. Without loss of generality, we may assume that $i = 1$ and $j = 2$. In view of Remark 4.3.13 , any clique K in $\Delta(B_4)$ of maximum size is either $K = \mathcal{E}$ or $K = A \times B$, where A and B are disjoint subsets of $\{1, 2, 3, 4\}$ of size two. Therefore, $\Delta(B_4)$ has two cliques of maximum size which contains $(1, 2)$ viz. $K_1 = \{1, 3\} \times \{2, 4\}$ and $K_2 = \{1, 4\} \times \{2, 3\}$. Note that for disjoint subsets A and B of $\{1, 2, 3, 4\}$, the clique $A \times B$ does not contain an idempotent element. Since $(1, 2)f$ is an idempotent element and by Remark 4.3.31, we have $K_1f = K_2f = \mathcal{E}$. By using the other elements of $(K_1f \cup K_2f) \setminus \{(1, 2)f\}$, in a similar manner, one can observe that the image of remaining non-idempotent elements belongs to $\mathcal{E}$. Thus, $(a, b)f \in \mathcal{E}$ for all $a \neq b \in [n]$. Now, we show that for any two distinct $x, y \in \{1, 2, 3, 4\}$, $(x, y)f$ is either (x, x) or (y, y). Since image of non-idempotent element is an idempotent so that $(x, y)f = (p, p)$ for some $p \in \{1, 2, 3, 4\}$. Note that $p \in \{x, y\}$. Otherwise, $(p, p) \sim (x, y)$ implies $(p, p) = (p, p)f \sim (x, y)f = (p, p)$; which is not possible. Now suppose $(1, 2)f = (1, 1)$. Since $(1, 2) \sim (1, k)$ for $k \neq 1, 2$, we get $(1, 1) = (1, 2)f \sim (1, k)f$. Consequently, $(1, k)f = (k, k)$. Similarly, we get $(2, k)f = (2, 2)$. Therefore, $(2, 3)f = (2, 4)f = (2, 2)$. We get a contradiction as $(2, 4) \sim (2, 3)$. Similarly, we get a contradiction when $(1, 2)f = (2, 2)$. Hence, the result hold. □

Lemma 4.3.36. *For $n \geq 5$, let $f \in \mathrm{End}(\Delta(B_n))$. Then a non-idempotent element maps to a non-idempotent element of B_n under f.*

Proof. Let (i, j) be a non-idempotent element of B_n. By Remark 4.3.16, there exists a clique K of maximum size which contains (i, j). In view of Remarks 4.3.15 and 4.3.31, all the elements of Kf are non-idempotent. Thus, $(i, j)f$ is a non-idempotent element. □

Proposition 4.3.37. *For $n \geq 4$, let $\mathcal{G}'$ be a retract of $\Delta(B_n)$ such that $(i, i) \in \mathcal{G}'$ for all $i \in [n]$. Then $\mathcal{G}' = \Delta(B_n)$.*

Proof. Since $\mathcal{G}'$ is a retract of $\Delta(B_n)$, there exists a homomorphism $f : \Delta(B_n) \to \mathcal{G}'$ such that $xf = x$ for all $x \in V(\mathcal{G}')$. Let (i, j) be a non-idempotent element of B_n. Then $(i, j)f$ is a non-idempotent element of B_n (cf. Lemmas 4.3.35 and 4.3.36). Let $(i, j)f = (x, y)$, where $x \neq y$. For $k \in [n] \setminus \{i, j\}$, we have $(i, j) \sim (k, k)$. Since $(k, k) \in \mathcal{G}'$, we get $(x, y) \in N[(k, k)]$. By Lemma 4.3.1(i), $x, y \neq k$. Consequently, $(x, y) \in \{(i, j), (j, i)\}$. Thus, either $(i, j)f = (i, j)$ or (j, i). Now to prove $\mathcal{G}' = \Delta(B_n)$, we show that f is an identity map. Since $(i, i) \in \mathcal{G}'$, it is sufficient to prove that for any $i, j \in [n]$ such that $i \neq j$, we have $(i, j)f = (i, j)$. Let if possible, $(i, j)f = (j, i)$ for some $i \neq j$. Then $(j, i)f = (j, i)$. For $p \in [n] \setminus \{i, j\}$, note that $(j, p)f = (j, p)$ because if $(j, p)f = (p, j)$, then $(j, p) \sim (j, i)$ implies $(j, p)f = (p, j) \nsim (j, i) = (j, i)f$; a contradiction. Further, note that $(i, p)f \notin \{(i, p), (p, i)\}$ which is not possible. For instance, if $(i, p)f = (i, p)$ then $(i, p) \sim (i, j)$ gives $(i, p)f \sim (i, j)f$. Consequently, we get $(i, p) \sim (j, i)$; a contradiction. On the other hand, if $(i, p)f = (p, i)f$ then $(i, p) \sim (j, p)$ gives $(i, p)f = (p, i) \nsim (j, p) = (j, p)f$; a contradiction. Hence, f is an identity map so that $\mathcal{G}' = \Delta(B_n)$. □

To obtain the $\mathrm{End}(\Delta(B_n))$, following Lemmas will be useful.

Lemma 4.3.38. *For $n \geq 4$, let f be a retraction of $\Delta(B_n)$ onto $\mathcal{G}'$. Then there exists a clique K of maximum size in $\mathcal{G}'$ such that $K = A \times B$ where A and B forms a partition of $[n]$. Moreover,*

(i) *if n is even then $|A| = |B| = \frac{n}{2}$, or*

(ii) *if n is odd then either* $|A| = \frac{n-1}{2}, |B| = \frac{n+1}{2}$ *or* $|A| = \frac{n+1}{2}, |B| = \frac{n-1}{2}$.

Proof. Let f be a retraction on $\Delta(B_n)$. For $n \geq 4$, in view of Corollary 4.3.11, Lemma 4.3.12 and Theorem 4.3.14, $\Delta(B_n)$ contains a clique K' of maximum size such that all the elements of K' are non-idempotent. By Remark 4.3.31 and Lemmas 4.3.35, 4.3.36, $K'f$ is a clique of maximum size and all of its elements are non-idempotents. Now consider $K'f = K$, by the proof of Lemma 4.3.10, we get $K = A \times B$ where A and B forms a partition of $[n]$ together with (i) or (ii). □

In the following lemma, we provide the possible images of non-idempotent elements of B_n under a retraction.

Lemma 4.3.39. *Let f be a retraction of $\Delta(B_n)$ onto $\mathcal{G}'$, where $n \geq 4$. Then for $p \neq q \in [n]$, we have*

$$(p, q)f \in \{(t, p) : t \in A\} \cup \{(q, t) : t \in B\} \cup \{(p, q)\},$$

for some partition $\{A, B\}$ of $[n]$. Moreover,

(i) *if $p \in A$, then $(p, q)f \neq (t, p)$ for any $t \in A$.*

(ii) *if $q \in B$, then $(p, q)f \neq (q, t)$ for any $t \in B$.*

Proof. In view of Lemma 4.3.38, there exists a clique $K = A \times B$ of maximum size in $\mathcal{G}'$ for some partition $\{A, B\}$ of $[n]$. Suppose $(p, q)f = (x, y)$. Then, by Lemmas 4.3.35 and 4.3.36, we have $x \neq y$. If $(p, q)f = (p, q)$ then there is nothing to prove. Now let $(p, q)f = (x, y)$ where $(x, y) \neq (p, q)$. If $x, y \notin \{p, q\}$, then $(p, q) \sim (x, y)$ gives $(p, q)f = (x, y)f = (x, y)$; a contradiction. Then either $x \in \{p, q\}$ or $y \in \{p, q\}$. If $x = p$, then clearly $y \notin \{p, q\}$. Consequently, $(p, q) \sim (x, y)$ provides again a contradiction. Therefore, $x \neq p$. Similarly, one can show that $y \neq q$. It follows that $(p, q)f = (x, y)$ where either $x = q$ or $y = p$. Now observe that if $y = p$, then $x \in A$. If possible, let $x \in B$. Then for $\alpha \in A \setminus \{q\}$, $(\alpha, x)f = (\alpha, x)$ as $(\alpha, x) \in A \times B \subseteq \mathcal{G}'$. Since $x \neq p$ as $x \neq y$, we get $(p, q) \sim (\alpha, x)$ so that

$(p,q)f = (x,p) \sim (\alpha,x) = (\alpha,x)f$; a contradiction of Remark 4.3.2. In a similar manner it is not difficult to observe if $x = q$, then $y \in B$.

To prove addition part of the lemma, suppose $p \in A$ and $(p,q)f = (t,p)$ for some $t \in A$. For $r \in B$ such that $r \neq q$, we have $(p,q) \sim (p,r)$ and $(p,r)f = (p,r)$ as $(p,r) \in K \subseteq \mathcal{G}'$. Consequently, we get $(p,q)f = (t,p) \sim (p,r) = (p,r)f$; a contradiction of Remark 4.3.2. Thus, $(p,q)f \neq (t,p)$. Using similar argument, observe that for $q \in B$, $(p,q)f \neq (q,t)$ for any $t \in B$. Thus, the result hold. □

Theorem 4.3.40. *For $n = 4$, we have* $\mathrm{End}(\Delta(B_n)) = \mathrm{Aut}(\Delta(B_n))$.

Proof. In view of Proposition 4.3.34, we show that $\Delta(B_n)$ is a core. For that it is sufficient to show $\Delta(B_n)$ admits no proper retract (cf. Hell and Nešetřil [1992]). On contrary, suppose $\Delta(B_n)$ admits a proper retract $\mathcal{G}'$. Then there exists a homomorphism $f : \Delta(B_n) \to \mathcal{G}'$ such that $xf = x$ for all $x \in \mathcal{G}'$. Since the set $\mathcal{E} = \{(1,1),(2,2),(3,3),(4,4)\}$ forms a clique of maximum size as $\omega(\Delta(B_4)) = 4$ (cf. Lemma 4.3.12) so that $\mathcal{E}f$ is a clique of size 4 (see Remark 4.3.31). By Remark 4.3.13, we have either $\mathcal{E}f = \mathcal{E}$ or $\mathcal{E}f = A \times B$ where $A, B \subseteq \{1,2,3,4\}$ with $|A| = |B| = 2$. If $\mathcal{E}f = \mathcal{E}$, then by Proposition 4.3.37, $\mathcal{G}' = \Delta(B_n)$; a contradiction. Thus, $\mathcal{E}f = A \times B$. Let $(1,1)f = (i,j)$ where $i \neq j$. Then $(i,j)f = (i,j)$ as $(i,j) \in \mathcal{G}'$. Note that either $i = 1$ or $j = 1$. If both $i, j \neq 1$, then $(i,j) \sim (1,1)$. Consequently, $(1,1)f \sim (i,j)f$ which is not possible as $(i,j)f = (1,1)f = (i,j)$. Without Loss of generality, we assume that $i = 1$ and $j = 2$. Similarly, $(2,2)f \in \{(2,k),(k,2)\}$ for some $k \neq 1,2$. Since $(2,2)f \sim (1,2) = (1,1)f$ as $(1,1) \sim (2,2)$. If $(2,2)f = (2,k)$, then $(2,k) \sim (1,2)$; a contradiction of Remark 4.3.2 so $(2,2)f = (k,2)$ for some $k \neq 1,2$. Without loss of generality, we suppose $k = 3$. In the same way, we get $(3,3)f = (3,4)$ and $(4,4)f = (1,4)$. Therefore, we have $A = \{1,3\}$ and $B = \{2,4\}$. In view of Lemma 4.3.39, $(2,4)f \in \{(1,2),(3,2),(2,4)\}$. Since $(1,1) \sim (2,4)$ so that $(1,1)f = (1,2) \sim (2,4)f$ gives $(2,4)f = (3,2)$. Similarly, we get $(2,3)f = (3,4)$. Again by Lemma 4.3.39, we have $(1,3)f \in \{(3,2),(3,4),(1,3)\}$. For $(1,3) \sim (2,3)$ and $(1,3) \sim (2,4)$ we obtained $(1,3)f \sim (3,4)$ and $(1,3)f \sim (3,2)$. Consequently, we get a contradiction of Remark 4.3.2. □

Theorem 4.3.41. *For $n \geq 5$, we have* $\mathrm{End}(\Delta(B_n)) = \mathrm{Aut}(\Delta(B_n))$.

Proof. In order to prove the result, we show that $\Delta(B_n)$ is a core (see Proposition 4.3.34). For that it is sufficient to show $\Delta(B_n)$ admits no proper retract (cf. Hell and Nešetřil [1992]). On contrary, suppose $\Delta(B_n)$ admits a proper retract $\mathcal{G}'$. Then there exists an onto homomorphism $f : \Delta(B_n) \to \mathcal{G}'$ such that $xf = x$ for all $x \in \mathcal{G}'$. In view of Lemma 4.3.38, there exists a clique $K = A \times B$ where A and B forms a partition of $[n]$. Without loss of generality, we may assume that $A = \{1, 2, \ldots, t\}$ and $B = \{t+1, t+2, \ldots, n\}$ where $t \in \{\frac{n}{2}, \frac{n-1}{2}, \frac{n+1}{2}\}$. Consider the set

$$X = \{i \in A \setminus \{1\} : (1, i)f = (1, i)\} \cup \{1 : (2, 1)f = (2, 1)\}.$$

The following claims will be useful in the sequel.

Claim 4.3.42. (i) *For $i \in X$ and $r \neq i \in A$, we have $(r, i)f = (r, i)$.*

(ii) *For $i \in A \setminus X$ and $r \neq i \in A$, we have $(r, i)f = (i, s)$ for some $s \in B$.*

Proof of Claim (i) Let $i \neq 1 \in X$. Then $(1, i)f = (1, i)$. If $r \in A \setminus \{1, i\}$, then we have either $(r, i)f = (r, i)$ or $(r, i)f = (i, s)$ where $s \in B$ (cf. Lemma 4.3.39). Now, we assume that $(r, i)f = (i, s)$ for some $s \in B$. Since $(r, i)f \sim (1, i)f$ as $(r, i) \sim (1, i)$ so that $(i, s) \sim (1, i)$; a contradiction of Remark 4.3.2. Thus, $(r, i)f = (r, i)$ for all $r \neq i \in A$. Similarly one can observe that if $i = 1 \in X$ and $r \neq i \in A$, we have $(r, 1)f = (r, 1)$.

(ii) First suppose $i \neq 1 \in A \setminus X$. In view of Lemma 4.3.39, we have either $(1, i)f = (1, i)$ or $(1, i)f = (i, s)$ for some $s \in B$. Note that $(1, i)f \neq (1, i)$ as $i \in A \setminus X$ so $(1, i)f = (i, s)$ for some $s \in B$. If $r \in A \setminus \{1, i\}$, then we have either $(r, i)f = (r, i)$ or $(r, i)f = (i, s')$ where $s' \in B$ (cf. Lemma 4.3.39). Suppose $(r, i)f = (r, i)$. Since $(r, i)f \sim (1, i)f$ as $(r, i) \sim (1, i)$ so that $(r, i) \sim (i, s)$; a contradiction of Remark 4.3.2. Thus, $(r, i)f = (i, s')$ for some $s' \in B$. Similarly, one can observe that if $i = 1 \in A \setminus X$ and $r \neq i \in A$, we have $(r, 1)f = (1, s)$ for some $s \in B$.

In view of X we have the following cases.

Case 1: Suppose $|X| > |A \setminus X|$. Then $|X| \geq 2$ as $n \geq 5$. In order to get a contradiction of the fact that $\mathcal{G}'$ is a proper retract of $\Delta(B_n)$, we prove that f is an identity map in this case. First we show that each non-idempotent element of $\Delta(B_n)$ maps to itself under f through the following claim.

Note: If $n > 5$, then $|A| \geq 3$. For $n = 5$, we have either $|A| = 2$, $|B| = 3$ or $|A| = 3$, $|B| = 2$. If $|A| = 2$ and $|B| = 3$, then $X = A = \{1, 2\}$. This case we will discuss separately in the following claim (vi). Therefore, in part (ii) to (v), we assume that $|A| \geq 3$.

Claim 4.3.43. (i) *For $p \in A, q \in B$, we have $(p, q)f = (p, q)$.*

(ii) *If $p \neq q$ such that $(p, q)f = (a, p)$ for some $a \in A$, then $a \in A \setminus X$.*

(iii) *For $p \in B, q \in A$, we have $(p, q)f = (p, q)$.*

(vi) *For $p, q \in B$, we have $(p, q)f = (p, q)$.*

(v) *For $p, q \in A$, we have $(p, q)f = (p, q)$.*

(vi) *For $n = 5$, $|A| = 2$, $|B| = 3$ and $p \neq q$, we have $(p, q)f = (p, q)$.*

Proof of Claim: (i) Since $K = A \times B$ is contained in $\mathcal{G}'$ so that $(p, q)f = (p, q)$ for all $p \in A, q \in B$.

(ii) On contrary, we assume that $a \in X$. Clearly, $a \neq p$ (cf. Lemmas 4.3.35 and 4.3.36). If $p \in A$, then by Claim 4.3.42(i), we get $(p, a)f = (p, a)$. Note that $q \neq a$, otherwise $(p, q)f = (p, q) = (q, p)$ implies $p = q$; a contradiction. Consequently, $(p, q) \sim (p, a)$ gives $(p, a)f = (p, a) \sim (a, p) = (p, q)f$; a contradiction of Remark 4.3.2. Thus, $p \in B$. For $r \in A \setminus \{a, q\}$, by Claim 4.3.42(i), we have $(r, a)f = (r, a)$. Since $(p, q) \sim (r, a)$ as $a \neq p$ and $r \neq q$ so that $(p, q)f = (a, p) \sim (r, a) = (r, a)f$ which is not possible. Thus, $a \notin X$.

(iii) Let $p \in B$ and $q \in A$. First suppose that $q \in X$. Then by Lemma 4.3.39, $(p, q)f \in \{(s, p) : s \in A\} \cup \{(q, s) : s \in B\} \cup \{(p, q)\}$. For $r \neq q \in A$, we have $(r, q)f = (r, q)$ (cf. Claim 4.3.42(i)). Note that $(p, q)f \neq (q, s)$ for any $s \in B$. For instance, if $(p, q)f = (q, s)$ for some $s \in B$, then $(p, q)f = (q, s) \sim (r, q) = (r, q)f$ as $(p, q) \sim (r, q)$, where $r \neq q \in A$; a contradiction of Remark 4.3.2. It follows that $(p, q)f \in \{(s, p) : s \in A\} \cup \{(p, q)\}$. Suppose $(p, q)f = (s, p)$ for some $s \in A$. Note that $s \in A \setminus X$ (see part (ii)). Now we claim that for any $j \neq q \in X$, we have $(p, j)f = (s', p)$ for some $s' \in A \setminus X$. In view of Lemma 4.3.39, $(p, j)f \in \{(s', p) : s \in A\} \cup \{(j, s') : s' \in B\} \cup \{(p, j)\}$. Note that $(p, j)f \neq (p, j)$ because $(p, q) \sim (p, j)$ but $(p, q)f = (s, p) \nsim (p, j)$ (cf. Remark 4.3.2). In a similar manner, of $(p, q)f \neq (q, s)$ for any $s \in B$, one can show that $(p, j)f \neq (j, s')$ for any $s' \in B$. It follows $(p, j)f = (s', p)$ for some $s' \in A$. By part (ii), we get $(p, j)f = (s', p)$ for some $s' \in A \setminus X$. Since the subgraph induced by the vertices of the form (p, j) where $j \in X$ forms a clique. Consequently, for any $i \neq j \in X$, we get $(p, i)f = (s, p)$ and $(p, j)f = (s', p)$ are distinct for some $s, s' \in A \setminus X$. Therefore, we have $|X| \leq |A \setminus X|$; a contradiction. Thus, $(p, q)f = (p, q)$ for all $p \in B$ and $q \in X$.

Now we assume $q \in A \setminus X$. In view of Lemma 4.3.39, $(p, q)f \in \{(\alpha, p) : \alpha \in A\} \cup \{(q, \beta) : \beta \in B\} \cup \{(p, q)\}$. Suppose $(p, q)f = (\alpha, p)$ for some $\alpha \in A$. In fact $\alpha \in A \setminus X$ (see part (ii)). Choose $i \in X$ as $|X| > |A \setminus X|$, from above we get $(p, i)f = (p, i)$ as $p \in B$. Since $(p, q) \sim (p, i)$ so that $(p, q)f = (\alpha, p) \sim (p, i) = (p, i)f$ which is not possible. Therefore, we have $(p, q)f = (q, \beta)$ for some $\beta \in B$ if $(p, q)f \neq (p, q)$. Again for $i \in X$ and from the above we get $(\beta, i)f = (\beta, i)$. Since $(p, q) \sim (\beta, i)$ as $p, \beta \in B$ and $q, i \in A$ gives $(p, q)f = (q, \beta) \sim (\beta, i) = (\beta, i)f$; a contradiction of Remark 4.3.2. Thus, $(p, q)f = (p, q)$ $\forall p \in B$ and $q \in A \setminus X$ and hence the result hold.

(iv) Let $p \neq q \in B$. In view of Lemma 4.3.39, $(p, q)f \in \{(s, p) : s \in A\} \cup \{(p, q)\}$. Suppose $(p, q)f = (s, p)$ for some $s \in A$. Since $(p, s) \sim (p, q)$ so that $(p, s)f = (p, s) \sim (s, p) = (p, q)f$; a contradiction of Remark 4.3.2. Thus, $(p, q)f = (p, q)$ for all $p, q \in B$.

(v) By Claim 4.3.42(i), we have $(p, q)f = (p, q)$ when $q \in X$ so it is sufficient to prove the result for $q \in A \setminus X$. In view of Lemma 4.3.39, $(p, q)f \in \{(q, s) : s \in B\} \cup \{p, q)\}$. Suppose $(p, q)f = (q, s)$ for some $s \in B$. Then by (iv) part, we have $(s, x)f = (s, x)$ where $x \neq s \in B$. For $p, q \in A$ and $s, x \in B$, we get $(p, q) \sim (s, x)$ gives $(p, q)f = (q, s) \sim (s, x) = (s, x)f$; a contradiction of Remark 4.3.2. Thus, $(p, q)f = (p, q)$ for all $p \neq q \in A$.

(vi) Suppose $n = 5$, $|A| = 2$, $|B| = 3$ and $p \neq q$. Then $X = A$ so $(p, q)f = (p, q)$ for all $p, q \in A$ (see Claim 4.3.42(i)). If $p, q \in B$, then by Lemma 4.3.39, $(p, q)f \in \{(s, p) : s \in A\} \cup \{p, q)\}$. Suppose $(p, q)f = (s, p)$ for some $s \in A$. Then there exists $s' \in A$ as $|A| = 2$. Consequently, $(s', s)f = (s', s)$ and $(p, q) \sim (s', s)$ gives $(p, q)f = (s, p) \sim (s', s) = (s', s)f$ which is not possible. Thus, $(p, q)f = (p, q)$ for all $p, q \in B$. Now we suppose that $p \in B$ and $q \in A$. In view of Lemma 4.3.39, we have $(p, q)f \in \{(r, p) : r \in A\} \cup \{(q, r') : r' \in B\} \cup \{(p, q)\}$. Suppose $(p, q)f = (r, p)$ for some $r \in A = X$. For $\beta \in B \setminus \{p\}$, we get $(p, q) \sim (p, \beta)$ and $(p, \beta)f = (p, \beta)$ provides $(s, p) \sim (p, \beta)$ which is not possible. Therefore, $(p, q)f \in \{(q, r') : r' \in B\} \cup \{(p, q)\}$. Let $(p, q)f = (q, r')$ for some $r' \in B$. Since $|B| = 3$ so that there exists $z \in B \setminus \{p, r'\}$. As a consequence, we have $(r', z) \sim (p, q)$ and $(r', z)f = (r', z)$ implies $(r', z)f = (r', z) \sim (q, r') = (p, q)f$; a contradiction. Thus, $(p, q)f = (p, q)$ for all $p \neq q \in [n]$.

Thus, by Claim 4.3.43, we have $(p, q)f = (p, q)$ for all $p \neq q$. Now we show that $(p, p)f = (p, p)$ for all $p \in [n]$. On contrary assume that $(p, p)f = (x, y)$ for some $(x, y) \neq (p, p) \in B_n$. Then $(x, y)f = (x, y)$ as f is a retraction on $\Delta(B_n)$. Note that $x \neq y$. Otherwise, $(p, p) \sim (x, y)$ but $(p, p)f = (x, y)f = (x, y)$; a contradiction. Also, observe that $p \in \{x, y\}$. Otherwise, being an adjacent elements (x, y) and (p, p) have same images; again a contradiction. Without loss of generality assume that $x = p$. For $z \in [n] \setminus \{y, p\}$, we get $(p, p) \sim (y, z)$ so that $(p, p)f = (p, y) \sim (y, z) =$

$(y, z)f$; a contradiction of Remark 4.3.2. Thus, f is an identity map. Consequently, $\mathcal{G}' = \Delta(B_n)$; a contradiction. Thus, **Case 1** is not possible.

Case 2: Suppose $|X| \leq |A \setminus X|$. Then $X \neq A$. Now, we have the following subcases depend on n. In each subcase, we prove that $A = X$ which is a contradiction.

Subcase 1: n is even. The following claim will be useful in the sequel.

Claim 4.3.44. (i) *Let $i \in A \setminus X$. Then there exists a unique $s_i \in B$ such that the restriction map $f_{A_i \times B_{s_i}}$ of f is a bijection from $A_i = \{(r, i) : r \neq i \in A\}$ onto $B_{s_i} = \{(i, s) : s \neq s_i \in B\}$.*

(ii) *In view of part (i), for $Y = \{s_i \in B : i \in A \setminus X\}$, we have $Y = B$. Moreover, for $i \neq j \in A \setminus X$, we have $s_i \neq s_j$.*

(iii) *If $x \neq y \in B$, then $(x, y)f = (x, y)$.*

(iv) *If $i \neq j \in A$, then $(i, j)f = (i, j)$.*

Proof of Claim: (i) Let $i \in A \setminus X$. Then for $r \neq i \in A$, we have $(r, i)f = (i, s)$ for some $s \in B$ (see Claim 4.3.42(ii)). Consequently, $A_i f \subset \{(i, s) : s \in B\}$. Since f is one-one on A_i because A_i forms a clique, we get $|A_i f| = |A_i| = |A| - 1 = |B| - 1$ as n is even. Thus, there exists $s_i \in B$ such that $A_i f = B_{s_i}$, where $B_{s_i} = \{(i, s) : s \in B \setminus \{s_i\}\}$. Hence, $f_{A_i \times B_{s_i}}$ is a one-one map from A_i onto B_{s_i}.

(ii) Clearly $Y \subseteq B$. We show that $Y \subset B$ is not possible. On contrary, if $Y \subset B$ so there exists $s \in B \setminus Y$. Let $x \neq s \in B$. By Lemma 4.3.39, $(s, x)f \in \{(\alpha, s) : \alpha \in A\} \cup \{(s, x)\}$. We provide a contradiction for both the possibilities of $(s, x)f$. Suppose $(s, x)f = (\alpha, s)$ for some $\alpha \in A$. By Claim 4.3.43(ii), in fact we have $(s, x)f = (\alpha, s)$ for some $\alpha \in A \setminus X$. Then by part (i) there exists $s_\alpha \in B$ such that the map $f_{A_\alpha \times B_{s_\alpha}}$ is a bijection. As $s_\alpha \in Y$, $s \neq s_\alpha$ so that $(\alpha, s) \in B_{s_\alpha}$. Consequently, there exists $r_\alpha \neq \alpha \in A$ such that $(r_\alpha, \alpha)f = (\alpha, s)$. Now since $r_\alpha, \alpha \in A$ and $s, x \in B$ we get $(r_\alpha, \alpha) \sim (s, x)$ as A and B forms a partition of $[n]$ so that $(r_\alpha, \alpha)f \sim (s, x)f$. But $(r_\alpha, \alpha)f = (s, x)f = (\alpha, s)$ which is not possible.

It follows that $(s,x)f = (s,x)$. For $i \in A \setminus X$, there exists $s_i \in Y$ such that the map $f_{A_i \times B_{s_i}}$ is a bijection. Since $s \neq s_i$ as $s \notin Y$ gives $(i,s) \in B_{s_i}$. As a result, there exists $r \neq i \in A$ such that $(r,i)f = (i,s)$. For $r, i \in A$ and $s, x \in B$, we get $(s,x) \sim (r,i)$; again a contradiction as $(s,x)f = (s,x) \sim (i,s) = (r_i,i)f$. Hence, $Y = B$.

(iii) Let $x, y \in B$. Then by Lemma 4.3.39, $(x,y)f \in \{(\alpha,x) : \alpha \in A\} \cup \{(x,y)\}$. Suppose $(x,y)f = (\alpha,x)$ for some $\alpha \in A$. In fact $\alpha \in A \setminus X$ (see Claim 4.3.43(ii)). For $x \in B = Y$, there exists $i_x \in A \setminus X$ such that $f_{A_{i_x} \times B_x}$ is a bijection. If $\alpha \neq i_x \in A \setminus X$, then by part (i) there exists $s_\alpha \in B \setminus \{x\}$ such that the restriction map $f_{A_\alpha \times B_{s_\alpha}}$ is a bijective map and $(\alpha,x) \in B_{s_\alpha}$. Consequently, we get $(r,\alpha)f = (\alpha,x)$ for some $r \neq \alpha \in A$. But $(x,y) \sim (r,\alpha)$ as $x, y \in B$ and $r, \alpha \in A$ gives $(x,y)f \neq (r,\alpha)f$. However, we have $(x,y)f = (r,\alpha)f$; a contradiction. It follows that $\alpha = i_x$. In view of Lemma 4.3.39, for $y' \in B \setminus \{x,y\}$, note that $(x,y')f \in \{(\alpha',x) : \alpha' \in A\} \cup \{(x,y')\}$. Now observe that $(x,y')f \neq (x,y')$. If $(x,y')f = (x,y')$, then $(x,y) \sim (x,y')$ provides $(\alpha,x) \sim (x,y')$; a contradiction of Remark 4.3.2. Thus, $(x,y')f = (\alpha',x)$ for some $\alpha' \in A \setminus X$. Further note that $\alpha' \neq \alpha$. Otherwise, $(x,y) \sim (x,y')$ gives $(x,y)f \sim (x,y')$ but $(x,y)f = (x,y')f = (\alpha,x)$ which is not possible. Consequently, $\alpha' \neq i_x$. By the similar argument used for $\alpha \neq i_x$, we get $(r',\alpha')f = (\alpha',x)$ for some $r' \neq \alpha' \in A$. Since $(r',\alpha') \sim (x,y')$ we get $(r',\alpha')f \sim (x,y')f$ but $(r',\alpha')f = (x,y')f = (\alpha',x)$ is not possible. Hence, $(x,y)f = (x,y)$ for all $x \neq y \in B$.

(iv) Suppose $i \neq j \in A$. Then by Lemma 4.3.39, $(i,j)f \in \{(j,\beta) : \beta \in B\} \cup \{(i,j)\}$. If $(i,j)f = (j,\beta)$ for some $\beta \in B$ then for $x \in B \setminus \{\beta\}$ note that $(i,j) \sim (\beta,x)$ but $(i,j)f = (j,\beta) \nsim (\beta,x) = (\beta,x)f$ (cf. part (iii)). Thus, $(i,j)f = (i,j)$.

By Claim 4.3.44(iv), we get $A = X$. Therefore, **Case 2** is not possible when n is even.

Subcase 2: n is odd. By Lemma 4.3.38, we have either $|A| = \frac{n+1}{2}$, $|B| = \frac{n-1}{2}$ or $|A| = \frac{n-1}{2}$, $|B| = \frac{n+1}{2}$ (see proof of Lemma 4.3.10). First we prove the following claim.

Claim 4.3.45. (i) *If $x \neq y \in B$, then $(x, y)f = (x, y)$.*

(ii) *If $x \in B$ and $i \in A$, then $(x, i)f = (x, i)$.*

Proof of Claim: (i) First, we suppose that $|A| = \frac{n+1}{2}$ and $|B| = \frac{n-1}{2}$. Let $x \neq y \in B$. Then by Lemma 4.3.39, we get either $(x, y)f = (i, x)$ for some $i \in A$ or $(x, y)f = (x, y)$. Let if possible, $(x, y)f = (i, x)$ for some $i \in A$. In fact $i \in A \setminus X$ (cf. Claim 4.3.43(ii)). Also, for $r \neq i \in A$ and $i \in A \setminus X$, by Claim 4.3.42(ii), we get $(r, i)f = (i, s)$ for some $s \in B$. As a result, $A_i f \subseteq B_i$ where $A_i = \{(r, i) : r \neq i \in A\}$ and $B_i = \{(i, s) : s \in B\}$. Since A_i forms a clique, we have f is one-one on A_i. Moreover, $|A_i f| = |A_i| = |A| - 1 = |B| = |B_i|$. Therefore, we get a bijection $f_{A_i \times B_i}$ from A_i onto B_i. Then there exists $r \neq i \in A$ such that $(r, i)f = (i, x)$ for some $x \in B$. Note that $(x, y) \sim (r, i)$ but $(x, y)f = (r, i)f = (i, x)$ which is not possible. Thus, $(x, y)f = (x, y)$ for all $x \neq y \in B$.

On the other hand, we may assume that $|A| = \frac{n-1}{2}$ and $|B| = \frac{n+1}{2}$. Then $|B| \geq 3$. First, we claim that there exist $x, y \neq B$ such that $(x, y)f = (x, y)$. On contrary, we assume that $(x, y)f \neq (x, y)$ for all $x \neq y$ in B. Let $x \neq y \in B$. By Lemma 4.3.39 and Claim 4.3.43(ii), we have $(x, y)f = (\alpha, x)$ for some $\alpha \in A \setminus X$. Similarly, for any $y' \in B \setminus \{x, y\}$, we have $(x, y')f = (\alpha', x)$ for some $\alpha' \in A \setminus X$. It follows that $B_x f \subseteq A_x$ where $B_x = \{(x, z) : z \neq x \in B\}$ and $A_x = \{(i, x) : i \in A \setminus X\}$. Since the set B_x forms a clique so that f is one-one on B_x provide $|B_x f| = |B_x| = |B| - 1 = |A| = |A_x| = |A \setminus X|$. Consequently, we get $f_{B_x \times A_x}$ is a bijection and $X = \varnothing$. For $r \neq \alpha \in A$, we have $(r, \alpha)f = (\alpha, \beta)$ for some $\beta \in B$ (cf. Claim 4.3.42(i)). If $\beta = x$, then $(x, y)f = (r, \alpha)f = (\alpha, x)$ but $(x, y) \sim (r, \alpha)$ which is not possible. For $\beta \neq x$, by using the similar argument used for x, there exist the subsets B_β and A_β such that the restriction map $f_{B_\beta \times A_\beta}$ is a bijective map. As a consequence $(\alpha, \beta) \in A_\beta$ so that there exists $(\beta, s) \in B_\beta$ such that $(\beta, s)f = (\alpha, \beta)$. As $r, \alpha \in A$ and $\beta, s \in B$, $(r, \alpha) \sim (\beta, s)$ gives $(r, \alpha)f \sim (\beta, s)f$ but $(r, \alpha)f = (\beta, s)f = (\alpha, \beta)$ which is not possible. Thus, there exist $p \neq q \in B$ such that $(p,q)f = (p,q)$.

For any $w \in B \setminus \{p,q\}$, we have either $(p,w)f = (p,w)$ or $(p,w)f = (i,p)$ for some $i \in A$. Since $(p,q) \sim (p,w)$ so that $(p,q)f = (p,q) \sim (p,w)f$ implies $(p,w)f \neq (i,p)$ for any $i \in A$. Therefore, $(p,w)f = (p,w)$. Consider the subsets $A' = A \cup \{p\}$ and $B' = B \setminus \{p\}$ of $[n]$. Note that A' and B' are the disjoint subsets of $[n]$ with $|A'| = \frac{n+1}{2}$ and $|B'| = \frac{n-1}{2}$ so $A' \times B'$ forms a clique of maximum size in $\mathcal{G}'$. If $|X| > |A' \setminus X|$, then in Claim 4.3.43(iv), replace A and B with A' and B' respectively, we get $(a,b)f = (a,b)$ for all $a,b \in B'$. For $|X| \leq |A' \setminus X|$, by using the similar concept used above we have $(a,b)f = (a,b)$ for all $a,b \in B'$. Since $(p,w)f = (p,w)$ for all $w \neq x \in B$ so that $(a,b)f = (a,b)$ for all $a,b \in B$ and $b \neq x$. If possible, let $(a,p)f \neq (a,p)$, then by Lemma 4.3.39, $(a,p)f = (l,a)$ for some $l \in A$. Choose $\beta \in B \setminus \{a,p\}$ so $(a,\beta) \sim (a,p)$ and $(a,\beta)f = (a,\beta)$ as $a,\beta \in B'$ we obtained $(a,\beta)f = (a,\beta) \sim (l,a) = (a,p)$; a contradiction of remark 4.3.2. Hence, $(a,b)f = (a,b)$ for all $a,b \in B$.

(ii) Let $x \in B$ and $i \in A$. Then by Lemma 4.3.39, we have $(x,i)f \in \{(\alpha,x) : \alpha \in A\} \cup \{(i,\beta) : \beta \in B\} \cup \{(x,i)\}$. Note $(x,i)f \neq (\alpha,x)$ for any $\alpha \in A$. For instance if $(x,i)f = (\alpha,x)$ for some $\alpha \in A$, then $(x,y) \sim (x,i)$ where $y \neq x \in B$ gives $(x,y)f \sim (x,i)f$. By part (i), we get $(x,y)f = (x,y)$ so $(x,y) \sim (\alpha,x)$; a contradiction of Remark 4.3.2. On the other hand now we get a contradiction for $(x,i)f = (i,\beta)$ for some $\beta \in B$. If $\beta = x$ then for $\gamma \neq x \in B$, we have $(x,\gamma)f = (x,\gamma)$ (by part (i)). Since $(x,i) \sim (x,\gamma)$ but $(x,i)f = (i,x) \nsim (x,\gamma) = (x,\gamma)f$ which is not possible so $\beta \neq x$. For $n \geq 5$, we have $|B| \geq 2$. If $|B| = 2$, then $|A| = 3$. There exists $j,k \in A \setminus \{i\}$. Consequently, $(j,i)f = (i,y)$ and $(k,i)f = (i,z)$ for some $y,z \in B$. Because if $(j,i)f = (j,i)$ (cf. Lemma 4.3.39) then $(x,i) \sim (j,i)$ gives $(x,i)f = (i,\beta) \sim (j,i) = (j,i)f$; a contradiction of Remark 4.3.2. Similarly, $(k,i)f = (k,i)$ is not possible. Note that $\{(x,i),(j,i),(k,i)\}$ forms a clique of size 3 so that $\{(x,i)f,(j,i)f,(k,i)f\} =$
$\{(i,y),(i,z),(i,s)\}$. Consequently, β, y, z are the elements of B. Thus, $|B| \geq 3$; a

contradiction of $|B| = 2$. It follows that $|B| \geq 3$. For $z \in B \setminus \{x, s\}$ we have $(x, i) \sim (\beta, z)$. By part (i), $(\beta, z)f = (\beta, z)$. Consequently, $(x, i)f = (i, \beta) \sim (\beta, z) = (\beta, z)f$ which is not possible. Hence, $(x, i)f = (x, i)$.

Now if $x \in A$, then $i \in A \setminus X$. For $x \in B$, by Claim 4.3.45(ii), we have $(x, i)f = (x, i)$. Since $(1, i) \sim (x, i)$ so that $(1, i)f = (i, s) \sim (x, i) = (x, i)f$; a contradiction of Remark 4.3.2. Thus, $X \subset A$ is not possible. Consequently, $X = A$; a contradiction of **Case 2**. In view of **Case 1** and **Case 2** such X is not possible. Thus, $\Delta(B_n)$ admits no proper retract. Hence, $\Delta(B_n)$ is a core. □

Future work: The work in this chapter can be carried out for other class of semigroups viz. the semigroup of all partial maps on a finite set and its various subsemigroups. In view of Theorem 1.1.12; to investigate the commuting graph of finite 0-simple inverse semigroup, it is sufficient to investigate $\Delta(B_n(G))$. In this connection, the results obtained in this paper might be useful. For example, using the result of $\Delta(B_n)$, in particular Theorem 4.3.5(iii), we prove the following theorem which gives a partial answer to the problem posed in [Araújo et al., 2011, Section 6].

Theorem 4.3.46. *For $n \geq 3$, $\Delta(B_n(G))$ is Hamiltonian.*

Proof. Let $G = \{a_1, a_2, \ldots, a_m\}$. We show that there exists a Hamiltonian cycle in $\Delta(B_n(G))$. First note that if $(i, j) \sim (k, l)$ in $\Delta(B_n)$, then $(i, a, j) \sim (k, b, l)$ in $\Delta(B_n(G))$ for all $a, b \in G$. Let $G_{a_1} = \{(i, a_1, j) \ : \ i, j \in [n]\}$. Since $\Delta(B_n)$ is Hamiltonian (see Theorem 4.3.5(iii)), we assume that there exists a Hamiltonian cycle C . Corresponding to the cycle C, choose a Hamiltonian path P whose first vertex is (i, j) and the end vertex is (k, l). For the path P, there exists a Hamiltonian path in the subgraph induced by G_{a_1} whose first vertex is (i, a_1, j) and the end vertex is (k, a_1, l). Since $(i, j) \sim (k, l)$ in $\Delta(B_n)$, we have $(k, a_1, l) \sim (i, a_2, j)$. By the similar way, we get a Hamiltonian path in the subgraph induced by G_{a_2} whose first vertex is (i, a_2, j) and the end vertex is (k, a_2, l). On Continuing this process, we get a Hamiltonian path in $\Delta(B_n(G))$ with first vertex is (i, a_1, j) and the end vertex is (k, a_m, l). For $(i, j) \sim (k, l)$, we get $(i, a_1, j) \sim (k, a_m, l)$. Thus, $\Delta(B_n(G))$ is Hamiltonian. $\square$