Introduction

The study of graphical depiction of groups and semigroups is an interesting topic for researchers which leads various results and questions. By using graphical depiction of groups and semigroups, mathematician found more regular graphs with a large number of vertices for a given diameter and for a given number of edges per nodes than were previously known. This allows us the construction of cosmic networks, while meeting designed criteria of a fixed number of nearest neighbours and a fixed maximum communication time between arbitrary nodes. Other than mathematics, the study of graphs has various applications in other branches also. For instance, the Laplacian spectrum of a graph has many applications in chemistry, whereas the spectral radius are used in computer networks to protect personal data in some database. The metric dimension of a graph is widely applicable in pharmaceutical chemistry (see Cameron and Van Lint [1991]; Chartranda et al. [2000]), robot navigation (see Khuller et al. [1994]) etc. The concept of detour distance has applications in channel assignment in FM. In particular, detour index of a graph is widely used in chemical theory (see Zhou and Cai [2010]). Some more applications which motivate us to study graphs associated with algebraic structures: It is used to design and analysis of topologies of interconnection networks that is helpful to connect processors in a super computer by using the Cayley graphs. The study of graphs constructed over algebraic structures is more interesting and is a large research area. Among these classes of graphs, Cayley graph is very well-studied due to its symmetric behaviour and it was introduced by Cayley in 1878. Many other classes of graphs such as intersection graph (Zelinka [1973]), commuting graph (Ambrozie et al. [2013]), Cayley graph (Lu et al. [2014]), zero divisor graph (Chakrabarty, Ghosh, Mukherjee and Sen [2009]), prime graph (Tong-Viet [2014]), power graph (Kelarev and Quinn [2002]), enhanced power graph (Aalipour et al. [2017]) and cyclic graph (Ma et al. [2013]) etc. associated with algebraic structures (viz. lattices, Hilbert spaces, fields, rings, groups and semigroups etc.) have been extensively studied by various researchers. Motivated with the applications of some of these graphs and their connections with respective algebraic objects, we are keen to develop further results. This thesis contributes the further study of cyclic graph, enhanced power graph and commuting graph associated with groups and semigroups.

The commutativity plays an important role in studying solvability of groups and hence in Galois theory of solvability by radicals (see Grove [1980]). The notion of *commuting graph* first appeared in the seminal paper of Brauer and Fowler [1955] and they proved that there exists a subgroup H of a finite group G such that $|H| > \sqrt[3]{|G|}$. For a group G and $\Omega \subseteq G$, the commuting graph $\Delta(G, \Omega)$ is the simple undirected graph whose vertex set is Ω and two distinct vertices x, y are adjacent if $xy = yx$. For $\Omega = G$, we write $\Delta(G, \Omega)$ by $\Delta(G)$. Further, the non commuting graph was considered by Neumann [1976] and solved a problem posed by Erdös: Let G be a group such that its non commuting graph, denoted by $\overline{\Delta(G)}$- the complement of $\Delta(G)$, contains no infinite complete subgraph; is there then a finite bound on the cardinality of complete subgraphs of $\overline{\Delta(G)}$. Segev [1999, 2001]; Segev and Seitz [2002] used combinatorial parameters of certain commuting graphs to establish long standing conjectures in the theory of division algebras. A variant of commuting graphs on groups has played an important role in classification of finite simple groups (see Aschbacher [2000]). Commuting graphs associated with groups is also used to establish some NSSD (non-singular with a singular deck) molecular graph (see Hayat et al. [2019]).

Iranmanesh and Jafarzadeh [2008] studied diameter, girth, clique number, independence number etc. of the commuting graph associated with symmetric group and alternating group. Araújo et al. [2011] calculated the diameter of commuting graphs of various ideals of full transformation semigroup. Also, in the same paper, for every natural number $n \geq 2$, a finite semigroup has been constructed such that the diameter of its commuting graph is n. The distant properties as well as detour distant properties of the commuting graph on the dihedral group D_{2n} were investigated by Ali et al. [2016]. Moreover, they obtained the metric dimension of the commuting graph on D_{2n} and its resolving polynomial. Later, the commuting graphs of various algebraic structures have become a topic of research for many mathematicians (see Araújo et al. [2015]; Ali et al. [2016]; Dolžan [2016]; Shitov [2018] etc. and references therein). Kakkar and Rawat [2018] studied the detour distance properties and obtained the resolving polynomial of the commuting graph of generalized dihedral group. Let T be a transversal of the center $Z(G)$ of a finite non-abelian group G. For $\Omega = T \setminus Z(G)$, Pezzott and Nakaoka [2019] proved that the commuting graph $\Delta(G, \Omega)$ is connected strongly regular if and only if G is isoclinic to an extraspecial 2-group of order at least 32. Further, they classified the groups G such that $\Delta(G, \Omega)$ is disconnected strongly regular. Ali and Li [2019] studied the connectivity and the spectral radius of the commuting graph of dihedral and dicyclic groups. Recently, Tolue [2020] introduced the twin non-commuting graph by partitioning the vertices of non-commuting graph and studied the graph-theoretic properties of twin non-commuting graph of AC-group and dihedral group.

The *directed power graph* $\overrightarrow{\mathcal{P}(S)}$ of a semigroup S, introduced by Kelarev and Quinn [2000, 2002], is a graph whose vertex set is S and there is an arc from a vertex u to another vertex v if $v = u^n$ for some positive integer $n \in \mathbb{N}$. Following this, Chakrabarty, Ghosh and Sen [2009] defined (undirected) *power graph* $\mathcal{P}(S)$ of a semigroup S is the simple undirected graph whose vertex set is S and two distinct vertices x, y are adjacent if one of them is power of other. We refer the reader for more results on power graph to survey paper Abawajy et al. [2013]. Aalipour et al. [2017] characterize the finite group G such that the graphs $\mathcal{P}(G)$ and $\Delta(G)$ are not equal and then they introduced a new graph between power graph and commuting graph called *enhanced power graph*. The *enhanced power graph* $\mathcal{P}_e(G)$ of a group G is an undirected and simple graph whose vertices are elements of G and two vertices are adjacent if they belong to the same cyclic subgroup. The clique number of enhanced power graph of an arbitrary group G was obtained by Aalipour et al. [2017] in terms of orders of elements of G. Bera and Bhuniya [2017] characterized the abelian groups and the non abelian p-groups having dominatable enhanced power graphs. Dupont et al. [2017b] determined the rainbow connection number of enhanced power graph of a finite group G. Further Dupont et al. [2017a] studied the enhanced quotient graph of a finite group G. Ma and She [2020] investigated the metric dimension of an enhanced power graph of finite groups. Further, the enhanced power graphs have been studied by various researchers. Zahirović et al. [2020] proved that two finite abelian groups are isomorphic if their enhanced power graphs are isomorphic. Also, they supplied a characterization of finite nilpotent groups whose enhanced power graphs are perfect. Recently, Bera et al. [2021] gave an upper bound for the vertex connectivity of enhanced power graph of any finite abelian group G. Moreover, they classified the finite abelian group G such that their proper enhanced power graph is connected.

Abdollahi and Hassanabadi [2007] introduced the notion of noncyclic graph of a group and studied its graph-theoretic parameters. The noncyclic graph of a group G is the simple undirected graph whose vertex set is $G \setminus T_G$, where $T_G = \{x \in G :$ the subgroup $\langle x, y\rangle$ is cyclic for all $y \in G\}$ and two distinct vertices x, y are adjacent if the subgroup $\langle x, y\rangle$ is not cyclic. They proved that if G is a nilpotent group and H is a group with their noncyclic graphs are isomorphic and $|T_G| = |T_H| = 1$, then H is a nilpotent group. Abdollahi and Hassanabadi [2009] characterize groups G such that clique number of its noncyclic graph is at most 4. Moreover, they prove that G is solvable if the clique number of noncyclic graph is at most 31. Later, Ma et al. [2019a] classified all groups whose noncyclic graph is free from $K_{1,n}$, where $3 \leq n \leq 6$. Recently, Ma and Su [2020], proved that for a non negative integer k, there are at most finitely many finite noncyclic groups whose noncyclic graphs have (non)orientable genus k. The complement of noncyclic graph of G is called the cyclic graph of G. Further, the notion of cyclic graph was slightly modified by Ma et al. [2013] as follows. The *cyclic graph* $\Gamma(G)$ is the simple undirected graph whose vertex set is G and two distinct vertices x and y are adjacent if the subgroup $\langle x, y\rangle$ is cyclic. They investigated the graph-theoretic properties of $\Gamma(G)$. Afkhami et al. [2014] extended the notion of cyclic graph on semigroups and determined the structure of cyclic graph $\Gamma(S)$ of a finite semigroup S.

The benefaction of the thesis has been arranged into five chapters after presenting the necessary preliminaries in Chapter 1.

Chapter 2: The Cyclic Graph of Semigroups

Chapter 3: The Enhanced Power Graphs

Chapter 4: The Commuting Graphs

Chapter 5: Equality of Graphs

Chapter 6: Conclusion and Future Research Work

Chapter 2 : The cyclic graph of a group has been studied by various researchers. However, there are few research papers on the study of cyclic graphs associated with semigroups. Since cyclic graphs and enhanced powers graphs of finite group are equal, in this chapter, we study the cyclic graph $\Gamma(S)$ of a semigroup S. First, we provide the structure of $\Gamma(S)$ for an arbitrary semigroup S. Then the basic properties of $\Gamma(S)$, namely: completeness, bipartite, tree and regularity etc. are discussed. Further, we investigate the chromatic number of $\Gamma(S)$ and it is proved that the chromatic number of $\Gamma(S)$, where S is a semigroup of unbounded exponent, is at most countable (see Theorem 2.3.7). We provide an upper bound of the chromatic number of $\Gamma(S)$, where S is a semigroup of bounded exponent. For a semigroup S with exponent n, we provide an expression of $\omega(\Gamma(S))$ (cf. Theorem 2.3.5). We give a formulae of $\alpha(\Gamma(S))$, where S is a finite monogenic semigroup (cf. Theorem 2.4.1). Finally, we provide a bound of independence number of $\Gamma(S)$, where S is of bounded exponent (see Theorem 2.4.2). The results of Section 2.2, concerning the chromatic number of $\Gamma(S)$, are published in SCIE journal *"Graphs and Combinatorics"*, Springer.

Chapter 3 : The notion of enhanced power graph of a group has been introduced in 2017 by Aalipour et al. The power graph of a group (or semigroup) is a spanning subgraph of enhanced power graph of group (or semigroup). Further, the power graph has an interesting applications and this attracts various researchers to study enhanced power graph. In this chapter, we study enhanced power graph associated with groups and semigroups. First we investigate the graph-theoretic properties viz. minimum degree, independence number and matching number and then determine them when G is a finite abelian p-group, the dihedral group D_{2n}, the semidihedral group SD_{8n}, the dicyclic group Q_{4n}, U_{6n} or V_{8n}. As a consequence, we obtain the vertex covering number and matching number of $\mathcal{P}_e(G)$, where G is aforesaid group. If G is any of these groups, we prove that $\mathcal{P}_e(G)$ is perfect and then obtain its strong metric dimension. Additionally, we give an expression for the independence number of $\mathcal{P}_e(G)$ for any finite abelian group G. These results along with certain known equalities yield the edge connectivity, vertex covering number and edge covering number of enhanced power graphs of the respective groups as well.

Next, we have studied the enhanced power graph $\mathcal{P}_e(S)$ of a semigroup S. First, we give the structure of $\mathcal{P}_e(S)$ (see Theorem 3.2.1) and then we characterize the semigroup S such that $\mathcal{P}_e(S)$ is complete, bipartite, tree, regular, null graph, acyclic graph, etc. Moreover, we discuss the planarity of $\mathcal{P}_e(S)$ (cf. Theorem 3.2.12) and then we obtain the minimum degree and independence number of $\mathcal{P}_e(S)$. Finally, in order to compare the results of Section 2.2 of Chapter 2 with $\mathcal{P}_e(S)$, we have constructed a semigroup S such that the chromatic number of $\mathcal{P}_e(S)$ is uncountable.

The results on enhanced power graphs associated with groups are published in SCIE journal *"Communications in Algebra"*, Taylor & Francis and in SCOPUS indexed journal *"Discrete Mathematics, Algorithms and Applications"*, World Scientific.

Chapter 4 : In this chapter, we discuss graph-theoretic and algebraic properties of the commuting graph associated with groups and semigroups (in particular, Brandt semigroups). First we consider the commuting graph $\Delta(G)$ of an arbitrary group G. We determine the edge connectivity and the minimum degree of $\Delta(G)$ and prove that both are equal (see Theorem 4.1.2). Then other graph invariants, namely: matching number, clique number, boundary vertex, of $\Delta(G)$ are studied. Also, we give necessary and sufficient condition on the group G such that the interior and center of $\Delta(G)$ are equal (see Theorem 4.1.14). Further, we investigate the commuting graph of the semidihedral group SD_{8n}. In this connection, we discuss various graph invariants of $\Delta(SD_{8n})$ including, vertex connectivity, independence number, matching number and detour properties. We also obtain the Laplacian spectrum, metric dimension and resolving polynomial of $\Delta(SD_{8n})$.

The commuting graph associated with semigroups, viz. symmetric inverse semigroups, full transformation semigroups, have been studied. Further to study the commuting graph associated with semigroups, we consider the commuting graph on an important class of inverse semigroups, namely: Brandt semigroup B_n. In this connection, we determine the minimum degree, girth, diameter, dominance number of $\Delta(B_n)$ and then investigate the properties including Eulerian, Hamiltonian, planarity, perfectness. The clique number, strong metric dimension, chromatic number of $\Delta(B_n)$ are also obtained. Thereafter, we calculate the vertex connectivity of

$\Delta(B_n)$ (see Theorem 4.3.22). Further, we obtain the automorphism group and endomorphism monoid of $\Delta(B_n)$, respectively. We show that $\mathrm{Aut}(\Delta(B_n)) \cong S_n \times \mathbb{Z}_2$ for all $n \geq 2$ (cf. Theorem 4.3.30) and $\mathrm{End}(\Delta(B_n)) = \mathrm{Aut}(\Delta(B_n))$ for all $n \geq 4$ (cf. Theorem 4.3.41). Finally, we ascertained a class of inverse semigroups whose commuting graph is Hamiltonian (see Theorem 4.3.46). This provides a partial answer to the question posed in Araújo et al. [2011].

The results obtained for $\Delta(G)$, where G is an arbitrary group or the semidihedral group, are published in SCIE journal *" Bulletin of the Malaysian Mathematical Sciences Society"*, Springer. Some partial results of $\Delta(B_n)$ are published in the book *"Semigroups, Categories and Partial Algebras"* Springer Proceedings in Mathematics and Statistics.

Chapter 5 : Aalipour et al. [2017] characterize finite groups such that an arbitrary pair of graphs $\mathcal{P}(G), \mathcal{P}_e(G)$ and $\Delta(G)$ are equal, in order to extend their results from groups to semigroups, in this chapter, we classify finite semigroups such that the pair of graphs, viz. $\mathcal{P}(S), \Gamma(S), \mathcal{P}_e(S)$ and $\Delta(S)$ are equal.

Chapter 6 : The thesis is summarized in this chapter and concluded with some future research work.