Chapter 1

Introduction

Let us begin by briefly recalling some results from the articles [16] and [14]. Let G be a Lie group with finitely many connected components, and let $\mathfrak{g}$ denote its Lie algebra. Let $E \to B$ be a smooth principal G-bundle. Further let $EG \to BG$ be a classifying bundle for the group G. Denote $\mathfrak{g}^l = \mathfrak{g} \otimes \ldots \otimes \mathfrak{g}$ (l-times) and $I^l(G) = \{f : \mathfrak{g}^l \to \mathbb{R} | f \text{ is linear, symmetric, and } ad\text{–invariant}\}$. There is a universal Weil homomorphism $W : I^l(G) \to H^{2l}(BG, \mathbb{R})$ given by $P \to [P(\Omega_G)]$ where Ω_G is the curvature of any connection θ_G on $EG \to BG$.

Let θ be a connection on the G-bundle $E \to B$, and let Ω denote its curvature. Set $\phi_t = t\Omega + \frac{1}{2}(t^2 - t)[\theta, \theta]$. Chern and Simons define [16]

$$TP(\theta) = l \int_0^1 P(\theta \wedge \phi_t^{l-1}) dt \in \Omega^{2l-1}(E) \tag{1.1}$$

and show that

$$dTP(\theta) = P(\Omega^l) \tag{1.2}$$

They further show that

Proposition 1. *Let $\{\theta(s)\}$ be a 1-parameter family of smooth connections depending smoothly on the parameter 's'. Then*

$$\frac{d}{ds} TP(\theta(s))|_{s=0} = lP(\theta' \wedge \Omega^{l-1}) + exact \tag{1.3}$$

where $\theta' = \frac{d}{ds}\theta(s)|_{s=0}$

and

Proposition 2. *Let* $E \to B$ *be a* $G-$*bundle with connection* θ *and let* $P \in I_0^l(G)$. *If* $P(\Omega^l) = 0$, *then* $\exists\, U \in H^{2l-1}(B;\mathbb{R}/\mathbb{Z})$ *such that* $\widetilde{TP}(\theta) = \pi^*(U)$ *where* $\tilde{}$ *denotes mod* $\mathbb{Z}$ *reduction.*

Here $I_0^l = \{P \in I^l(G) | W(P) \in H^{2l}(BG,\mathbb{Z})_{\mathbb{R}}\}$ and $H^{2l}(BG,\mathbb{Z})_{\mathbb{R}}$ denotes the image of the change of coefficients map $H^{2l}(BG,\mathbb{Z}) \to H^{2l}(BG,\mathbb{R})$.

In [14], Cheeger and Simons define the abelian group

$$\hat{H}^k(B;\mathbb{R}/\mathbb{Z}) := \{f \in Hom(Z_{k-1}(B),\mathbb{R}/\mathbb{Z}) | f\circ\delta \in \Omega^k(B;\mathbb{R})\} \tag{1.4}$$

They show that there are exact sequences

$$0 \to H^{k-1}(B,\mathbb{R}/\mathbb{Z}) \xrightarrow{i_1} \hat{H}^k(B;\mathbb{R}/\mathbb{Z}) \xrightarrow{curv} \Omega^k_{cl}(B;\mathbb{R}) \to 0 \tag{1.5}$$

$$0 \to \frac{\Omega^{k-1}(B)}{\Omega_0^{k-1}(B)} \xrightarrow{\iota} \hat{H}^k(B;\mathbb{R}/\mathbb{Z}) \xrightarrow{ch} H^k(B,\mathbb{Z}) \to 0. \tag{1.6}$$

Here $\Omega_0^{k-1}(B)$ denotes the set of closed $(k-1)-$forms on B with integral periods. For a description of the maps in these sequences, see [14]. Given a $G-$bundle $E \to B$, there is a map $B \xrightarrow{f} BG$ such that $f^*(EG \to BG)$ is isomorphic to $E \to B$. The map f is unique up to homotopy. Hence there are well defined induced maps in cohomology making the diagram

$$\begin{array}{ccc} H^*(BG,\mathbb{Z}) & \longrightarrow & H^*(B,\mathbb{Z}) \\ \downarrow & & \downarrow \\ H^*(BG,\mathbb{R}) & \longrightarrow & H^*(B,\mathbb{R}) \end{array}$$

commutative. There is also the Chern–Weil map $W : I^*(G) \to H^*(BG,\mathbb{R})$. The image of $I_0^*(G)$ under this map lies in the image of the map $H^*(BG,\mathbb{Z}) \to H^*(BG,\mathbb{R})$. If the bundle $E \to B$ has a connection θ, there is also a map $I^*(G) \to \Omega^*_{cl}(M,\mathbb{R})$. These maps fit together in the diagram

$$\begin{array}{ccccc} I^k(G) & \xrightarrow{W} & H^{2k}(BG,\mathbb{R}) & \xleftarrow[r]{} & H^{2k}(BG,\mathbb{Z}) \\ \downarrow{\scriptstyle W} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ \Omega^{2k}_{cl}(B;\mathbb{R}) & \xrightarrow{deRh} & H^{2k}(B;\mathbb{R}) & \xleftarrow[r]{} & H^{2k}(B;\mathbb{Z}) \end{array}$$

Let $K^{2k}(G)$ denote the pull-back of the upper horizontal line i.e.

$$K^{2k}(G) = \{(P,u) \in I^k(G) \times H^{2k}(BG,\mathbb{Z}) | W(P) = r(u)\} \tag{1.7}$$

and similarly let $R^{2k}(B;\mathbb{Z})$ be the pullback of the lower horizontal line i.e.

$$R^{2k}(B;\mathbb{Z}) = \{(\omega,\alpha) \in \Omega^{2k}_{cl}(B) \times H^{2k}(B;\mathbb{Z}) | deRh(\omega) = r(\alpha)\} \tag{1.8}$$

Then the pair (W, f^*) induces a map $K^{2k}(G) \to R^{2k}(B;\mathbb{Z})$.
Cheeger–Simons show that this map factors through $\hat{H}^{2k}(B;\mathbb{R}/\mathbb{Z}) \xrightarrow{(curv,ch)} R^{2k}(B;\mathbb{Z})$ i.e. $\exists K^{2k}(G) \to \hat{H}^{2k}(B;\mathbb{R}/\mathbb{Z})$ such that

$$\begin{array}{ccc} & & \hat{H}^{2k}(B;\mathbb{R}/\mathbb{Z}) \\ & \nearrow & \downarrow{\scriptstyle (curv,ch)} \\ K^{2k}(G) & \longrightarrow & R^{2k}(B;\mathbb{Z}) \end{array}$$

commutes. The discussion can be paraphrased as follows. Choose a compatible pair (P,u). (We say that P and u are compatible if $(P,u) \in K^{2k}(G)$.) Then given a bundle $\alpha : E \to B$, and a connection θ on it, the Cheeger–Simons construction yields a differential character $\hat{\alpha}$ with $ch(\hat{\alpha}) = u(\alpha)$ and $curv(\hat{\alpha}) = P(\Omega)$. Thus, this differential character α contains information about the *integral* cohomology of the bundle as well as the Chern–Weil *form* (not just the real de-Rham cohomology class). We denote this differential character by $cs_{P,u}(E \to B, \theta)$. When the connection is flat, the curvature of this differential character vanishes and thus, by the virtue of the exact sequence 1.5, the differential character is the image of a unique element of $H^{2k-1}(B;\mathbb{R}/\mathbb{Z})$ under i_1. In this way, the Cheeger–Simons construction yields an element of the $\mathbb{R}/\mathbb{Z}$ cohomology group for a flat connection.

Remark 3. *Note that even though the above discussion is carried out for the case of principal $G-$bundles, it can be written down for vector bundles too using the correspondence between principal $U(n)-$bundles and rank n complex vector bundles.*

In [34], Jaya Iyer considers the following question:

Question 1. *Given a flat connection on a bundle, the Cheeger–Simons construction associates an element of the $\mathbb{R}/\mathbb{Z}$ cohomology of the base manifold. Can this construction be generalised to associate such invariants to families of flat connections?*

In [34] Jaya Iyer associates an element of $H^{2p-r-1}(B, \mathbb{R}/\mathbb{Z})$ to an element of r-th simplicial homology of the simplicial set of relatively flat connections on the bundle $E \times B$. More precisely, she defines the simplicial set $\mathcal{D}(E)$ whose $r-$simplices are $(r+1)$-tuples $(D^0, \ldots, D^r)$ of relatively flat connections[1]. The collection $\{D^0, \ldots, D^r\}$ of connections is said to be relatively flat if $\Sigma_j t_j D^i$ is flat for any choice of t_i's such that $\Sigma t_j = 1$. She then constructs maps $\rho_{p,r} : H_r(\mathcal{D}(E)) \to H^{2p-r-1}(B, \mathbb{R}/\mathbb{Z})$ for $p > r+1, r \geq 1$. (Recall that given a simplicial set $\mathcal{S}$, the Dold-Kan construction yields a chain complex $S_\bullet$ with $S_n = \mathbb{Z}[\mathcal{S}_n]$ and the boundary operator given by the alternating face map. The homology groups of the simplicial set $\mathcal{S}$ are defined to be the homology groups of this chain complex $H_r(\mathcal{S}) := H_r(S_\bullet)$)
In the chapter 2 of this thesis, we extend Jaya Iyer's work (henceforth [34] is referred to as 'Jaya Iyer's paper/work') to include the $p < r$ case, using the construction of fiber integration of differential characters developed by Bär and Becker in [1]. Given an element u compatible with P as above, we construct maps $\tilde{\psi}_{P,u,r} : H_r(\mathcal{D}(E)) \to H^{2p-r-1}(B, \mathbb{R}/\mathbb{Z})$ for $p \neq r, r+1$. If Σ is an r-cycle in $\mathcal{D}(E)$, we consider the bundle $E \times \Sigma \to B \times \Sigma$ (we use the same symbol Σ to denote the cycle and its geometric realization), and endow it with a certain smooth connection. We apply the Cheeger–Simons theory to this data to obtain a differential character $h_{B\times\Sigma} \in \hat{H}^{2p}(B \times \Sigma)$, and apply fiber integration along the fibers of the bundle $B \times \Sigma \to B$ to get a differential character in $\hat{H}^{2p-r}(B)$. This map vanishes on the boundaries, thereby

[1] To avoid duplicity, we shall choose any ordering on the set of all connections and require that $D^0 \leq D^1 \leq \ldots \leq D^r$

yielding a map on homology. We find that these maps do not depend on the choice of $u \in H^{2p}(BGL(n,\mathbb{R}),\mathbb{Z})$ but only on the choice of the polynomial P, and that the characteristic class of the differential character $\tilde{\psi}_{P,u,r}([\Sigma])$ is zero. Thus we deduce that $\tilde{\psi}_{P,u,r}([\Sigma])$ is in the image of the inclusion $\frac{\Omega^{2p-r-1}}{\Omega_0^{2p-r-1}} \xrightarrow{\iota} \hat{H}^{2p-r}(B)$. Using a result of Bär and Becker [1], we calculate a representative differential form. We find that for the $p > r+1$ case, this form equals the one constructed in [34] thereby yielding an alternate derivation of the results of [34]. We find that the curvature of these differential characters is zero, and hence the characters are in the image of the map $H^{2p-r-1}(B,\mathbb{R}/\mathbb{Z}) \to \hat{H}^{2p-r}(B)$. When $p = r$, the curvature may not vanish, and we get a map $\tilde{\psi}_{r,r} : H_r(\mathcal{D}(E)) \to \hat{H}^r(B)$. Our method extends to the $p < r$ case, and we show that the fiber integrals vanish in this case.

Thereafter in section 2.3, we examine the relationship of our construction to other constructions in the literature. Biswas and Lopez [4] consider the set of smooth maps $Maps(S,\mathcal{F})$ where S is a smooth null-cobordant manifold of dimension r, and $\mathcal{F}$ is the space of flat connections on the principal G-bundle $E \to B$. Using the formalism of Atiyah bundle and the bundle of connections, they construct certain forms $\beta_k^p \in \Omega^k(\mathcal{A},\Omega^{2r-k}(B))$ where $\mathcal{A}$ is the space of all smooth connections on the bundle $E \to B$ considered as an infinite dimensional Fréchet manifold. Using these forms they define certain maps $\Lambda_{r+1}^p : Maps(S,\mathcal{F}) \to H^{2p-r-1}(B,\mathbb{R})$. They show that the maps can be described as $[f : S \to \mathcal{F}] \mapsto [\fint_T P(\Omega)]$ where $\bar{f} : T \to \mathcal{A}$ is an extension of f to a manifold T whose boundary is S, and Ω is the curvature of a certain connection on $E \times T \to B \times T$. They prove that these maps are well defined i.e. the result is independent of the choice of the extension $\bar{f}$ or the manifold T, that Λ_{r+1}^p are closed forms which determine elements of the cohomology group $H^{2p-r-1}(B,\mathbb{R})$, and that $[\Lambda_{r+1}^p(f_0)] = [\Lambda_{r+1}^p(f_1)] \in H^{2p-r-1}(B,\mathbb{R})$ whenever f_0, f_1 are homologous. In section 2.3, we argue that from the viewpoint of fiber integration these results proved in [4] can be obtained (modulo $\mathbb{Z}$) directly as consequences of some properties of fiber integration 11 proved in the article [1].

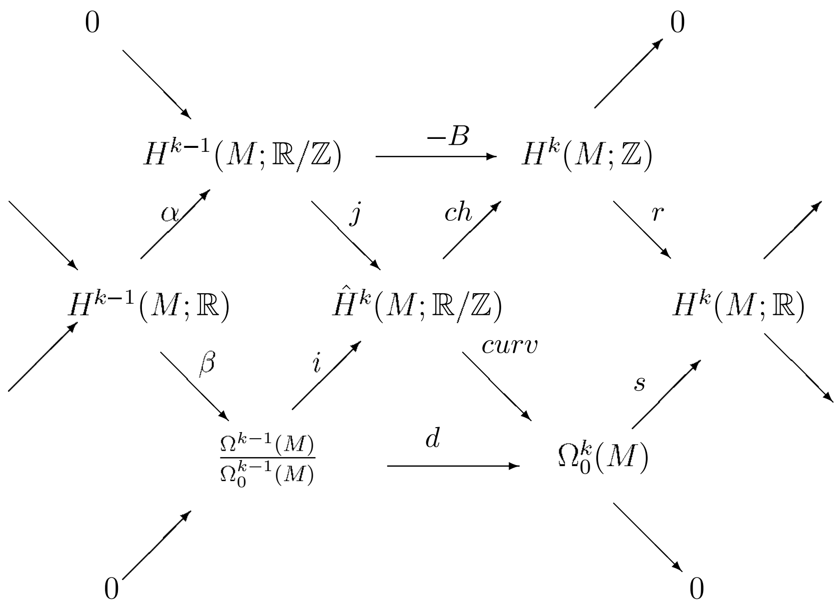
Thereafter, we discuss the relation of our results with those of [13]. In [13], the authors consider the principal G-bundle $E \times \mathcal{A} \to B \times \mathcal{A}$ and endow it with the canonical connection $\mathbb{A}$. If $\mathcal{G}$ is a subgroup of $Gau(E)$ which acts freely on $\mathcal{A}$, then $\mathcal{A} \to \mathcal{A}/\mathcal{G}$ is a principal bundle. They prove that a choice of connection $\mathcal{U}$ on $\mathcal{A} \to \mathcal{A}/\mathcal{G}$ yields a connection $\underline{\mathcal{U}}$ on $(E \times \mathcal{A})/\mathcal{G} \to B \times \mathcal{A}/\mathcal{G}$, and apply Cheeger–Simons theory to this bundle to get maps $\chi_r : H_r(\mathcal{F}/\mathcal{G}, \mathbb{Z}) \times H_{2p-r-1}(B, \mathbb{Z}) \to \mathbb{R}/\mathbb{Z}$ which are independent of the choice of the connection $\mathcal{U}$. They also prove that their approach using differential characters yields the same results as in [4] on cycles in $\mathcal{F}/\mathcal{G}$ that come from cycles in $\mathcal{F}$.

Our results are a special case of their results for the case $\mathcal{G} = \{e\}$. (Though we state and prove our results for smooth vector bundles, the same discussion applies *mutatis mutandis* to smooth principal $G-$bundles.) However, since we deal with finite dimensional stratifolds, we do not need to assume that the Cheeger–Simons construction of a differential character given a smooth connection on a bundle holds good in case the base of the bundle is an infinite dimensional Fréchet manifold (see section 2.3 for more discussion on this point). Also our approach makes the relation between various constructions [4, 13, 34] more explicit. We find that all the three approaches yield the same invariant in $\mathbb{R}/\mathbb{Z}$ cohomology (up to a possible sign factor depending upon the orientation conventions).

The idea that fiber integration of differential characters can be employed to obtain invariants of families of connections was propounded in [24] (also see [37]). However as noted in [34], the fiber integration developed in [24] can not be used for the map $X \times \Delta^r \to X$ as the construction in [24] requires that the fibers be compact manifolds without boundary while Δ^r is not boundary-less and as not a manifold. In our case, as we shall see, the fibers are compact stratifolds without boundary. This enables us to make use of the fiber integration construction of [1]. In fact our results are a direct application of the fiber integration construction (lemma 41), and Proposition 54 of [1].

The study of invariants of flat connections is the subject of many articles, see for example [30, 37, 18].

In the next chapter (ch. 3) we discuss an axiomatic approach to differential cohomology theories. In addition to the Cheeger–Simons construction, there are several other constructions of differential refinements of ordinary singular cohomology in the literature, see for example [27, 7, 32, 31, 8]. It is natural to ask whether these constructions are equivalent. In [45], Simons and Sullivan give an axiomatic characterization of ordinary differential cohomology. They prove that the diamond/hexagon diagram with short exact diagonals [45]:

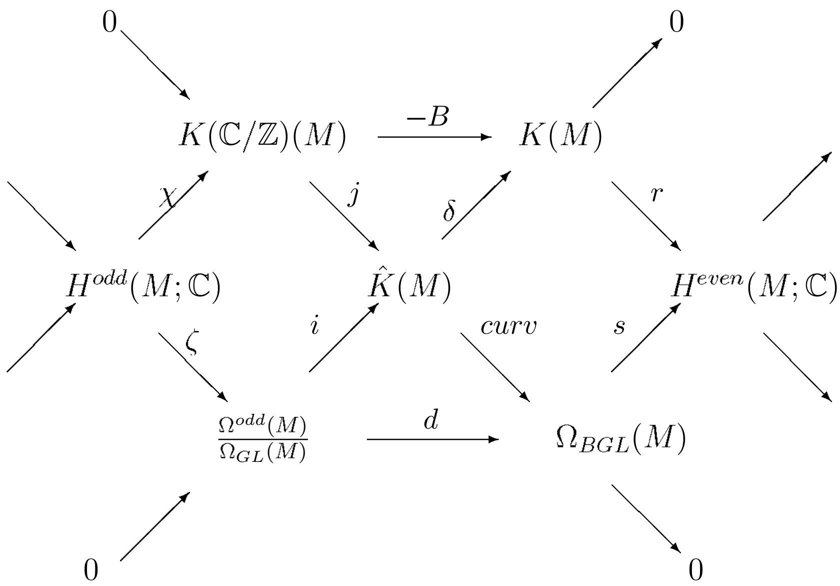


Differential cohomology hexagon diagram

uniquely characterizes the ordinary differential cohomology functor up to a natural isomorphism (for a precise statement, see Proposition 36). The long exact sequence of the upper arrows is the long exact sequence in cohomology corresponding to the short exact sequence $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0$. The maps s and β are induced by the de-Rham morphism.

This axiomatization of ordinary differential cohomology is useful as it establishes that different constructions of differential refinements of ordinary singular cohomology are equivalent. As an example, since the Deligne cohomology functor [7] - defined as the hypercohomology of a certain double complex- fits in the hexagon diagram [27], it follows that the Cheeger–Simons differential character functor is naturally isomorphic to the Deligne cohomology functor via a natural isomorphism compatible with the diagonal morphisms in the respective hexagon diagrams.

Like ordinary singular cohomology, other generalized cohomology theories too admit differential refinements. Given a generalized cohomology theory represented by a spectrum, Hopkins and Singer [33] give a prescription for constructing its differential refinement. In particular, one can construct a differential version of K-theory using their prescription. There are other models of differential K-theory (e.g. [9]) as well, for a survey see [11]. In [46], Simons and Sullivan construct another model of differential K-theory for compact manifolds in terms of structured vector bundles. They prove that their model of differential K-theory fits in the following hexagon diagram:



Differential K-theory hexagon diagram

They ask whether, like the case of differential characters,

Question 2. *Does the above hexagon diagram determine the differential K-theory functor (from the category of compact manifolds to the category of abelian groups) up to a natural isomorphism compatible with the respective diagonal morphisms?*

In [41], Rakesh Pawar determines necessary and sufficient conditions for the diagram

$$\begin{array}{ccccccccc} & & 0 & & & & 0 & & \\ & & \downarrow & & & & \downarrow & & \\ 0 & \longrightarrow & P & \longrightarrow & E & \longrightarrow & R & \longrightarrow & 0 \\ & & \downarrow & & & & \downarrow & & \\ & & H & & & & F & & \\ & & \downarrow & & & & \downarrow & & \\ 0 & \longrightarrow & S & \longrightarrow & G & \longrightarrow & Q & \longrightarrow & 0 \\ & & \downarrow & & & & \downarrow & & \\ & & 0 & & & & 0 & & \end{array}$$

Diagram 1

with short exact columns and rows to extend to

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & P & \xrightarrow{\nu} & E & \longrightarrow & R & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle j} & & \downarrow & & \\ 0 & \dashrightarrow & H & \overset{i}{\dashrightarrow} & X & \overset{m}{\dashrightarrow} & F & \dashrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle n} & & \downarrow & & \\ 0 & \longrightarrow & S & \longrightarrow & G & \longrightarrow & Q & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

Diagram 2

with short exact columns and rows. He further gives conditions for uniqueness of such extensions.

Here we extend (see Propositions 31, 33) Rakesh Pawar's results, and show that they imply (see Proposition 37) the existence and uniqueness of differential character groups $\hat{H}^k(M;\mathbb{R}/\mathbb{Z})$ for any fixed manifold M. Uniqueness of the functor $\hat{H}^k(-;\mathbb{R}/\mathbb{Z})$ is a stronger statement, for which we do not have a complete proof. However, we state a condition 38 which implies the full Simons-Sullivan result.

Similarly we note in Proposition 43 that for any compact manifold M, the differential K-theory groups are uniquely determined up to an isomorphism compatible with the respective diagonal maps, thereby partially answering the question 2 of Simons and Sullivan. We give necessary and sufficient conditions 44 for an affirmative answer to the Simons-Sullivan question in full generality. The method is quite general and applies to equivariant/twisted/generalised diffrerential cohomology theories as well.

This thesis is organised as follows. In chapter 2, we discuss the construction of the cohomological invariants associated to families of flat connections. In chapter 3, we turn to giving an alternate proof of uniqueness of differential characters and differential K-theory groups. Almost the entirety of the text of this thesis is taken verbatim from the author's two articles

- Invariants of families of flat connections using fiber integration of differential characters, *Lett. Math. Phys.* 110, 639–657 (2020). https://doi.org/10.1007/s11005-019-01234-3

- Uniqueness of differential characters and differential K-theory via homological algebra arXiv:2005.02056 (accepted for publication in *Journal of Homotopy and Related Structures*)

on which this thesis is based.

The study of differential cohomology theories is relevant in several ways to Quantum Field Theory and String Theory. The Chern–Simons form was one of the first examples of topological action functionals in Physics [49, 17]. Since then, differential refinements of ordinary and generalised cohomology theories have appeared in Physics in different contexts. The reader interested in exploring these connections could refer to [3, 26, 25, 29, 44, 23, 22] and the references therein.