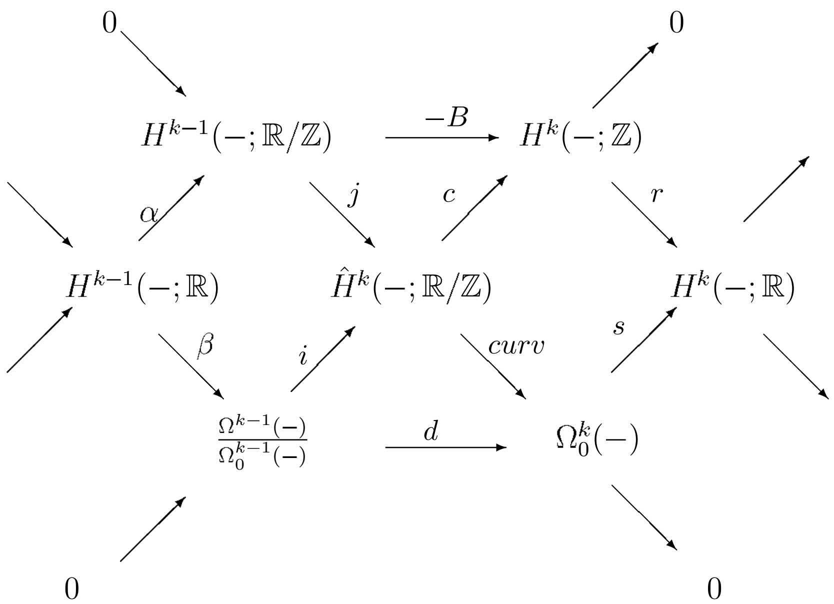
Chapter 3

Uniqueness of differential characters and differential K-theory

As discussed in the introduction, there are several models (e.g. [27, 7, 8, 32, 31, 14])of differential characters and differential K-theory (e.g. [33, 9, 46]). In order to compare different constructions, it helps to have an axiomatic characterization. For the case of differential characters, Simons and Sullivan show [45] that the differential character functor is uniquely determined up to a natural equivalence by the following diagram



For a precise statement, see Proposition 36. In [46], Simons and Sullivan construct a model of differential K-theory and show that there is a corresponding hexagon diagram. They ask whether this hexagon diagram uniquely determines K-theory. In this chapter[1] we show that for a fixed compact manifold M, the hexagon diagram indeed uniquely determines the differential K-theory groups.

Our argument is based on the recent work *Proc. Math. Sci. 129, 70(219)* [41] of Rakesh Pawar. In that article, Rakesh Pawar find necessary and sufficient conditions for the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & & & 0 & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & P & \longrightarrow & E & \longrightarrow & R & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & H & & & & F & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & S & \longrightarrow & G & \longrightarrow & Q & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & 0 & & & & 0 & &
\end{array}$$

Diagram 1

with short exact rows and columns to extend to

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P & \xrightarrow{\nu} & E & \longrightarrow & R & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle j} & & \downarrow & & \\
0 & \dashrightarrow & H & \overset{i}{\dashrightarrow} & X & \overset{m}{\dashrightarrow} & F & \dashrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle n} & & \downarrow & & \\
0 & \longrightarrow & S & \longrightarrow & G & \longrightarrow & Q & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Diagram 2

with short exact rows and columns. He further gives necessary and sufficient conditions for such an extension to be unique up to isomorphism.

[1]This chapter is based on the author's article 'Uniqueness of differential characters and differential K-theory via homological algebra' arXiv:2005.02056 a revised version of which has been accepted for publication in *Journal of Homotopy and Related Structures*.

In this chapter we refine and extend Rakesh Pawar's results. We give necessary and sufficient conditions for the extension to be unique up to an isomorphism compatible with all the other maps in the diagram. We also state necessary and sufficient conditions for this isomorphism itself to be unique.

This chapter is organised as follows. In section 3.1 we summarise the results of the article [41] that we need for our purposes. Thereafter in section 3.2 we extend these results and apply them to discuss the uniqueness results for differential characters, differential K-theory, and generalised differential cohomology theories.

A strong and general result showing the uniqueness of differential refinements of generalized cohomology theories (and hence differential K-theory, in particular) has been proved in [10] by a different approach. For a proof of uniqueness of differential character functor defined on smooth spaces, see [1].

3.1 Statement of Rakesh Pawar's results

In this section, we summarise the results of [41] that we need for present purposes. Let us begin by recalling some standard preliminary definitions and results from homological algebra (see, for example, [48, 43]). If $\mathcal{A}$ is an abelian category with enough projectives, then for any two objects P, Q in $\mathcal{A}$, one can consider the groups $Ext^n(Q, P)$ as the derived functor of the Hom functor.

Alternatively, one can consider the group of Yoneda extensions of P by Q as follows. Consider the set of long exact sequences $\zeta : 0 \to P \to X_n \to \cdots \to X_1 \to Q \to 0$. If $\zeta' : 0 \to P \to X'_n \to \cdots \to X'_1 \to Q \to 0$ is another such extension, a map $f : \zeta \to \zeta'$ is a collection of maps $f_i : X_i \to X'_i$ such that the diagram

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & P & \longrightarrow & X_n & \longrightarrow & \dots & \longrightarrow & X_1 & \longrightarrow & Q & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id_P} & & \downarrow{\scriptstyle f_n} & & & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle id_Q} & & \\ 0 & \longrightarrow & P & \longrightarrow & X'_n & \longrightarrow & \dots & \longrightarrow & X'_1 & \longrightarrow & Q & \longrightarrow & 0 \end{array}$$

commutes. Define an equivalence relation $\zeta \sim \eta \iff \exists$ a finite zigzag chain $\zeta \to \alpha_1 \leftarrow \alpha_2 \to \alpha_3 \leftarrow \cdots \to \eta$. Quotient of the set of extensions considered above by this equivalence relation gives us the set of Yoneda extensions $Ext^n_{Yoneda}(Q,P)$. On this set, define addition as $\zeta + \zeta' = [0 \to P \to Y_n \to X'_{n-1} \oplus X_{n-1} \to \cdots \to X'_2 \oplus X_2 \to Y_1 \to Q \to 0]$. Here Y_1 is the pullback $X_1 \times_Q X'_1$, and Y_n is the quotient by a skew diagonal copy of P, of the pushout of $P \to X_n$ and $P \to X'_n$. The set $Ext^n_{Yoneda}(Q,P)$ becomes an abelian group under this operation. If the category $\mathcal{A}$ has enough projectives, then $Ext^n_{Yoneda}(Q,P)$ is isomorphic to $Ext^n(Q,P)$ (see, for example, section 3.4 of [48]). Throughout this article we shall assume that the category $\mathcal{A}$ has enough projectives. Let us now discuss Rakesh Pawar's criterion for the existence of an extension of diagram 1 to diagram 2.

Using the sequences $0 \to P \to E \to R \to 0$ and $0 \to P \to H \to S \to 0$, we construct the short exact sequence $0 \to P \oplus P \to E \oplus H \to R \oplus S \to 0$. Now pushout this short exact sequence along the map $\nabla : P \oplus P \to P$ given by $(p_1, p_2) \mapsto p_1 + p_2$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P \oplus P & \longrightarrow & E \oplus H & \longrightarrow & R \oplus S & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \nabla} & & \downarrow & & \| & & \\ 0 & \longrightarrow & P & \longrightarrow & W & \longrightarrow & R \oplus S & \longrightarrow & 0 \end{array}$$

to obtain the short exact sequence $0 \to P \to W \to R \oplus S \to 0$. Denote the class of this sequence in $Ext^1(R \oplus S, P)$ by $[W]$. Next, pull back the exact sequence $0 \to R \to F \to Q \to 0$ by the map $G \to Q$ to get:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \longrightarrow & Y & \longrightarrow & G & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id_R} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & R & \longrightarrow & F & \longrightarrow & Q & \longrightarrow & 0 \end{array}$$

Applying the Snake lemma, Rakesh Pawar obtains

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & R & \xrightarrow{id_R} & R & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & S & \longrightarrow & Y & \longrightarrow & F & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id_S} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & S & \longrightarrow & G & \longrightarrow & Q & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

The injective maps $R \to Y$, and $S \to Y$ induce a map $R \oplus S \to Y$ making the sequence $0 \to R \oplus S \to Y \to Q \to 0$ exact. Applying the functor $Hom(-, P)$, one obtains the long exact sequence

$$Hom(Y,P) \xrightarrow{\zeta} Hom(R \oplus S, P) \xrightarrow{\alpha} Ext^1(Q,P) \xrightarrow{\beta} Ext^1(Y,P) \xrightarrow{\gamma} Ext^1(R \oplus S, P) \xrightarrow{\delta_Y} Ext^2(Q,P). \tag{3.1}$$

(Here ζ denotes the restriction map.) Thereafter, Rakesh Pawar shows (lemma 3.4 of [41]) that diagram 2 exists if and only if the following diagram exists:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P & \longrightarrow & W & \longrightarrow & R \oplus S & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P & \dashrightarrow & X & \dashrightarrow & Y & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & Q & = & Q & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

In turn, this diagram exists if and only if the pullback of $[0 \to P \to X \to Y \to 0] \in Ext^1(Y,P)$ under the map $R \oplus S \to Y$ equals $[0 \to P \to W \to R \oplus S \to 0]$. Therefore the question of extending diagram 1 to diagram 2 is equivalent to finding an element $[X] := [0 \to P \to X \to Y \to 0] \in Ext^1(Y,P)$ which maps to $[0 \to P \to W \to R \oplus S \to 0]$ under the map γ i.e. $\gamma([X]) = [W]$. By using the long exact sequence 3.1, $[X] \in Ext^1(Y,P)$ exists if and only if $\delta_Y([W]) = 0$. In lemma 3.6 of [41], it is shown that $\delta_Y([W])$ = Baer sum of $[E] \cup [F]$ and $[H] \cup [G]$. Thus,

Proposition 26. *(Theorem 1.1, [41]) Let $\mathcal{A}$ be an abelian category. Let Diagram 1 have exact rows and columns of objects of $\mathcal{A}$. The diagram 1 extends to diagram 2 with exact rows and columns if and only if the Baer sum of $[E] \cup [F]$ and $[H] \cup [G]$ is zero in $Ext^2(Q,P)$.*

In [41], this proposition is stated for small categories, however as noted in Remark 3.3 of [41], the result holds good for general abelian categories.

The exactness of the sequence 3.1 yields a necessary and sufficient condition for the uniqueness of the extension. Towards this end, define:

Definition 27. *Let $\mathcal{S} \subset Ext^1(Y,P)$ be the set $\{[X'] \in Ext^1(Y,P) | (X', i', j', m', n')$ is an extension of diagram 1 with all rows and columns exact$\}$.*

Proposition 28. *Assume that Diagram 1 admits an extension (X, i, j, m, n). Then $\mathcal{S} = \Big\{[X] + \beta(\lambda) | \lambda \in Ext^1(Q,P)\Big\} = [X] + Im(\beta)$.*

Proof. By the discussion above Proposition 26, the set $\mathcal{S}$ is given by the solutions of the equation $\gamma([X']) = [W]$. $[X'] \in Ext^1(Y,P)$ is a solution to this equation, if and only if $\gamma([X']) = [W] = \gamma([X])$. Equivalently $[X'] - [X] \in Ker(\gamma)$. From the exactness of 3.1, we have $Ker(\gamma) = Im(\beta)$. Hence $[X'] \in \mathcal{S}$ if and only if $[X'] \in [X] + Im(\beta)$. □

Proposition 29. *Suppose diagram 1 admits an extension (X, i, j, m, n) to diagram 2. Then the following are equivalent:*

1. *For any other extension X', $[X'] = [X] \in Ext^1(Y,P)$*

2. *The map α is surjective*

Proof. In the view of the Proposition 28,
(1) is true
$\Leftrightarrow Im(\beta) = 0$
$\Leftrightarrow Ker(\beta) = Ext^1(Q,P)$
$\Leftrightarrow Im(\alpha) = Ext^1(Q,P)$ (by the exactness of sequence 3.1). □

Remark 30. *We wish to state that the Proposition 28 is a reformulation of corollary 3.7 of [41]. As noted in [41], the map α can be described in the following way. If $\phi \in Hom(R \oplus S, P)$, then push-forward the exact sequence $0 \to R \oplus S \to Y \to Q \to 0$ by the map $\phi : R \oplus S \to P$ to obtain $\alpha(\phi) := \phi_*([0 \to R \oplus S \to Y \to Q \to 0])$. There is an alternate description of the map α as follows. Write $\phi = f \oplus g$ where $f : R \to P$ and $g : S \to P$. We have two exact sequences $0 \to R \to F \to$*

$Q \to 0$ and $0 \to S \to G \to Q \to 0$. Upon applying the functor $Hom(-, P)$, we obtain two connecting homomorphisms $\delta_F : Hom(R, P) \to Ext^1(Q, P)$ and $\delta_G : Hom(S, P) \to Ext^1(Q, P)$. One can define $\alpha(f \oplus g) :=$ Baer sum of $\delta_F(f)$ and $\delta_G(g)$. In the discussion above corollary 3.7 of the article [41], it is shown that these two descriptions of the map α coincide. Hence $Im(\delta_F + \delta_G) = Im(\alpha)$. Since the sequence 3.1 is exact, the map β induces an injective map $0 \to \frac{Ext^1(Q,P)}{Im(\alpha)=Im(\delta_F+\delta_G)} \xrightarrow{\beta} Ext^1(Y, P)$. For simplicity and parity with the notation in [41], the induced map on the quotient is also denoted by β. Thus, $\mathcal{S} = \left\{[X] + \beta(\lambda) | \lambda \in \frac{Ext^1(Q,P)}{Im(\delta_F+\delta_G)}\right\}$ as stated in corollary 3.7 of [41]. It follows that $[X]$ is uniquely determined in $Ext^1(Y, P)$ if and only if $Im(\alpha) = Im(\delta_F + \delta_G) = Ext^1(Q, P)$, since β is an injective map on the quotient $\frac{Ext^1(Q,P)}{Im(\alpha)=Im(\delta_F+\delta_G)}$.

Thus, in case $coker(\alpha) = 0$, if X_1 is another abelian group together with maps i_1, j_1, m_1, n_1, then $[X] = [X_1] \in Ext^1(Y, P)$. Equivalently there is an abelian group isomorphism $\phi : X \to X_1$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id_P} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle id_Y} & & \\ 0 & \longrightarrow & P & \longrightarrow & X_1 & \longrightarrow & Y & \longrightarrow & 0 \end{array}$$

commutes. Since Y is the pullback

$$\begin{array}{ccc} Y = F \times_Q G & \longrightarrow & F \\ \downarrow & & \downarrow \\ G & \longrightarrow & Q \end{array},$$

it follows that the morphism ϕ is compatible with (m, m_1) and (n, n_1) i.e. $m_1 \circ \phi = m$ and $n_1 \circ \phi = n$, and that $\phi \circ i|_P = i_1|_P$, $\phi \circ j|_P = j_1|_P$. (Here we are considering P as a subgroup of H via μ, and of E via ν.) Alternatively we could say that $\phi \circ i \circ \mu = i_1 \circ \mu$, and $\phi \circ j \circ \nu = j_1 \circ \nu$. However, we need a stronger compatibility result for our purposes: $\phi \circ i = i_1$, and $\phi \circ j = j_1$. We obtain this in the next section.

3.2 Existence and uniqueness results for differential cohomology theories

Let $(X_1, i_1, j_1, m_1, n_1)$ and $(X_2, i_2, j_2, m_2, n_2)$ be two extensions of Diagram 1. Let us say that an isomorphism $\phi : X_1 \to X_2$ is a *compatible isomorphism* between these two extensions if $\phi \circ i_1 = i_2, \phi \circ j_1 = j_2, m_2 \circ \phi = m_1$, and $n_2 \circ \phi = n_1$. The following proposition gives a necessary and sufficient criterion for an extension (assuming it exists) to be determined up to a compatible isomorphism.

Proposition 31. *Suppose* $(X_1, i_1, j_1, m_1, n_1)$ *is an extension of Diagram 1. Let* $E_1 = j_1(E) \subset X_1$*, and* $H_1 = i_1(H) \subset X_1$*. The following are equivalent:*

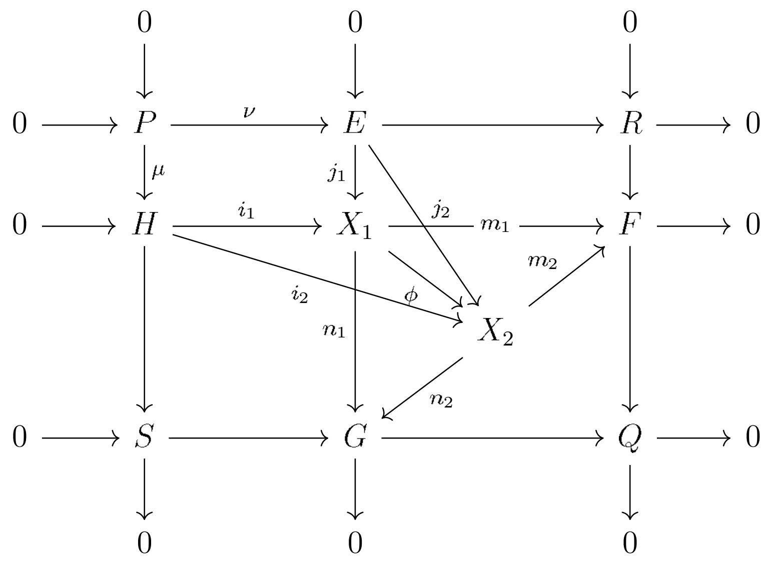
1. *For any other extension* $(X_2, i_2, j_2, m_2, n_2)$ *of the diagram, there exists an isomorphism* $\phi : X_1 \to X_2$ *compatible with the two extensions.*

2. *The map* $Hom(R \oplus S, P) \stackrel{\alpha}{\to} Ext^1(Q, P)$ *is surjective, and every homomorphism* $\lambda : E_1 + H_1 \to P$ *which vanishes on* $P_1 \subset E_1 + H_1$ *admits an extension* $\Lambda : X_1 \to P$*.*

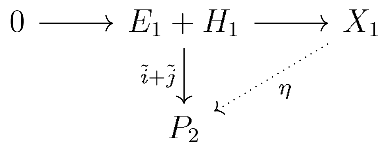
3. $Ext^1(Q, P) = 0$

Proof. Let us first show that (2) $\implies$ (1).

By Proposition 29, there is an isomorphism $\phi : X_1 \to X_2$ such that $m_2 \circ \phi = m_1$, and $n_2 \circ \phi = n_1$. The strategy is to find a morphism $\eta : X_1 \to X_2$ such that the morphism $\phi' \equiv \phi + \eta$ is a compatible isomorphism i.e. $\phi' \circ i_1 = i_2, \phi' \circ j_1 = j_2, m_2 \circ \phi' = m_1$, and $n_2 \circ \phi' = n_1$.



For convenience, let us denote $P_1 \equiv i_1 \circ \mu(P) = j_1 \circ \nu(P) \subset X_1$, and $P_2 \equiv i_2 \circ \mu(P) = j_2 \circ \nu(P) \subset X_2$. Similarly let $E_1 = j_1(E), E_2 = j_2(E), H_1 = i_1(H)$, and $H_2 = i_2(H)$. Now consider $\tilde{j} = j_2 - \phi \circ j_1 : E \to X_2$. Note that $m_2 \circ \tilde{j} = m_2 \circ j_2 - m_2 \circ \phi \circ j_1 = m_2 \circ j_2 - m_1 \circ j_1 = 0$, and $n_2 \circ \tilde{j} = n_2 \circ j_2 - n_2 \circ \phi \circ j_1 = m_2 \circ j_2 - n_1 \circ j_1 = 0$. Thus $\tilde{j}(E) \subset P_2$. Similarly $\tilde{i}(H) \subset P_2$. Also note that since $\phi \circ i_1|_P = i_2|_P$ and $\phi \circ j_1|_P = j_2|_P$ (by the discussion after Proposition 29), we conclude that $\tilde{j}|_P = 0 = \tilde{i}|_P$. Hence, $\tilde{i} + \tilde{j} : E_1 + H_1 \to X_2$ is a well defined abelian group homomorphism taking values in P_2. Here we have identified E with E_1, and H with H_1, for notational simplicity we use the same notation for the maps $\tilde{i}$ and $\tilde{j}$. We thus have a commutative diagram:



By hypothesis, there exists an extension $\eta : X_1 \to X_2$. Let $\phi' \equiv \phi + \eta$. Then $\phi' \circ i_1 = \phi \circ i_1 + \eta \circ i_1 = \phi \circ i_1 + (i_2 - \phi \circ i_1) = i_2$, and $\phi' \circ j_1 = \phi \circ j_1 + \eta \circ j_1 = \phi \circ j_1 + (j_2 - \phi \circ j_1) = j_2$. Further since η takes values in P_2, $m_2 \circ \eta = 0$. Thus $m_2 \circ \phi' = m_2 \circ \phi + m_2 \circ \eta = m_2 \circ \phi = m_1$, and similarly $n_2 \circ \phi' = n_1$. ϕ' is an isomorphism by the three lemma. Hence ϕ' is a required isomorphism.

Proof that (1) $\implies$ (2): By Proposition 29, it follows that α is surjective. Thus it remains to show that every homomorphism $\lambda : E_1 + H_1 \to P$ which vanishes on $P_1 \subset E_1 + H_1$ admits an extension $\Lambda : X_1 \to P$. The proof is by contradiction.

Thus assume that there is a $\lambda : E_1 + H_1 \to P$ which vanishes on $P_1 \subset E_1 + H_1$ and which does not admit any extension to X_1. Put $X_2 = X_1, m_2 = m_1, n_2 = n_1$, and $i_2 = i_1 + \lambda|H; j_2 = j_1 + \lambda|E$. Then it follows that $(X_2, i_2, j_2, m_2, n_2)$ is an extension of diagram 1. By assumption, there is a compatible isomorphism $\phi : X_1 \to X_2$. Then $\Lambda := \phi - id_{X_1} : X_1 \to P$ is an extension of λ. (As previously, we identify E, H with $E_1 = j_1(E), H_1 = i_1(H)$ respectively for convenience.)

Now let us show that (2) $\iff$ (3). First note that $\frac{E_1+H_1}{P_1} \cong R \oplus S$. Thus maps λ from $E_1 + H_1$ to P which vanish on P_1 are in one-to-one correspondance with elements of $Hom(R \oplus S, P)$. And the extensions $\Lambda : X_1 \to P$ which vanish on P_1 are in one-to-one correspondance with elements of $Hom(\frac{X_1}{P_1}, P) = Hom(Y, P)$. (For the equality we have used that $0 \to P \to X \to Y \to 0$ is exact.). But the statement that every $\lambda \in Hom(R \oplus S, P)$ extends to an element of $Hom(Y, P)$ says precisely that the map ζ in 3.1 is surjective. By the exact sequence 3.1, we see that both ζ and α are surjective $\iff$ $Ext(Q, P) = 0$. $\square$

Corollary 32. *If P is an injective object, there exists an extension of the diagram 1. Further for any two such extensions, there exists a compatible isomorphism between them.*

Proposition 33. *Suppose $(X_1, i_1, j_1, m_1, n_1)$ is an extension of diagram 1. Let $(X_2, i_2, j_2, m_2, n_2)$ be any other such extension of diagram 1, and*

$$\mathcal{U} = \{\phi' : X_1 \to X_2 | \phi' \text{ is a compatible isomorphism}\}.$$

Suppose that the conditions (2) in Proposition 31 are satisfied, so that $\mathcal{U}$ contains at least one element ϕ. Then there is a free and transitive action of the group $Hom(Q, P)$ on $\mathcal{U}$. Thus elements of $\mathcal{U}$ are in one-to-one correspondance with elements of $Hom(Q, P)$. In particular, $\mathcal{U}$ equals the singleton set $\{\phi\}$ $\iff$ $\phi : X_1 \to X_2$ is the unique compatible isomorphism $\iff$ $Hom(Q, P) = 0$.

Proof. The proof is by diagram chasing. As in Proposition 31, we shall identify H (resp. E) with its image $H_1 \subset X_1$ (resp. $E_1 \subset X_1$) under the injection $i_1 : H \to X_1$

(resp. $j_1 : E \to X_1$), and let $P_1 \equiv i_1 \circ \mu(P) = j_1 \circ \nu(P) = E_1 \cap H_1 \subset X_1$. Similarily E_2, and H_2 denote the corresponding images in X_2 under j_2 and i_2.

The composition $X_1 \xrightarrow{m_1} F \to Q$ is surjective and is equal to the composition $X_1 \xrightarrow{n_1} G \to Q$. We claim that the kernel of this surjection $\psi : X_1 \to Q$ is precisely $E_1 + H_1$, hence $\frac{X_1}{E_1+H_1} \cong Q$. First note that if $e_1 = j_1(e) \in E_1$, then $\psi(e_1) = 0$ since ψ is the composite $X_1 \xrightarrow{n_1} G \to Q$, and $n_1 \circ j_1 = 0$. Similarily, $\psi(H_1) = 0$. Thus $E_1 + H_1 \subset ker(\psi)$. To see containment in the other direction, suppose that $\psi(x) = 0$ for some $x \in X_1$. Then the image of $m_1(x) \in F$ under the map $F \to Q$ is zero. Hence there is an element $r \in R$ which maps to $m_1(x)$ under the map $R \to F$. Now since $E \to R$ is surjective, we can pick a preimage $e \in E$ of r. Using the commutativity of the top-right square, we have $m_1(x - j_1(e)) = m_1(x) - m_1(j_1(e)) = 0$. Thus, by the exactness of the middle horizontal sequence, $\exists h \in H$, such that $i_1(h) = x - j_1(e)$ or $x = i_1(h) + j_1(e)$. Hence $ker(\psi) \subset i_1(H) + j_1(E) = H_1 + E_1$.

We now describe an action $\tau : Hom(Q, P) \times \mathcal{U} \to \mathcal{U}$ of $Hom(Q, P)$ on $\mathcal{U}$. For any $\eta \in Hom(Q, P)$, let $\tilde{\eta}$ denote the composition $X_1 \to \frac{X_1}{E_1+H_1} \cong Q \xrightarrow{\eta} P \xrightarrow{i_2 \circ \mu = j_2 \circ \nu} X_2$. We define $\tau(\eta, \phi) \equiv \phi + \tilde{\eta}$. Since $\tilde{\eta}$ takes values in $P_2 = ker(m_2) \cap ker(n_2)$, it follows that $m_2 \circ \tilde{\eta} = 0 = n_2 \circ \tilde{\eta}$. Hence $m_2 \circ (\phi + \tilde{\eta}) = m_2 \circ \phi = m_1$, and $n_2 \circ (\phi + \tilde{\eta}) = n_2 \circ \phi = n_1$. From the fact that $\tilde{\eta}$ vanishes on E_1 and H_1, it follows that $(\phi + \tilde{\eta}) \circ i_1 = \phi \circ i_1 = i_2$ and $(\phi + \tilde{\eta}) \circ j_1 = j_2$. An application of the five-lemma shows that $\phi + \tilde{\eta}$ is an isomorphism. The associativity of group action follows by noting that $\widetilde{\eta_1 + \eta_2} = \widetilde{\eta_1} + \widetilde{\eta_2}$.

If $\eta \neq 0$, then $\tilde{\eta} \neq 0$ since $X_1 \to \frac{X_1}{E_1+H_1}$ is surjective, and $P \xrightarrow{i_2 \circ \mu = j_2 \circ \nu} X_2$ is injective. Thus for $\eta \neq 0$, $\tau(\eta, \phi) \neq \phi$, thereby showing that the action is free.

We now show that the action is transitive. Suppose $\phi' : X_1 \to X_2$ is another isomorphism which is compatible i.e. $\phi' \circ i_1 = i_2 = \phi \circ i_1, \phi' \circ j_1 = j_2 = \phi \circ j_1, m_2 \circ \phi' = m_1 = m_2 \circ \phi$, and $n_2 \circ \phi' = n_1 = n_2 \circ \phi$. Since $(\phi' - \phi) \circ i_1 = 0 = (\phi' - \phi) \circ j_1$, it follows that $\phi' - \phi$ vanishes on $E_1 = j_1(E)$ and $H_1 = i_1(H)$, and induces a map $\chi : \frac{X_1}{E_1+H_1} \cong Q \to X_2$. Since $m_2 \circ (\phi' - \phi) = m_1 - m_1 = 0$, we conclude that $(\phi' - \phi)(X_1) \subset H_2$. Similarily, since $n_2 \circ (\phi' - \phi) = n_1 - n_1 = 0$, we have

$(\phi'-\phi)(X_1) \subset E_2$. Thus $(\phi'-\phi)(X_1) \subset E_2 \cap H_2 = P_2$. Hence, the map χ takes values in P_2, and thus is an element of $Hom(Q, P_2)$. Identifying P with P_2 via $i_2 \circ \mu = j_2 \circ \nu$, $\chi \in Hom(Q, P)$. By construction, $\phi' - \phi = \tilde{\chi}$. Hence $\phi' = \phi + \tilde{\chi} = \tau(\chi, \phi)$. □

Corollary 34. *With the same setup as in Proposition 33,* $Hom(G, H) = 0 \implies \phi$ *is unique.*

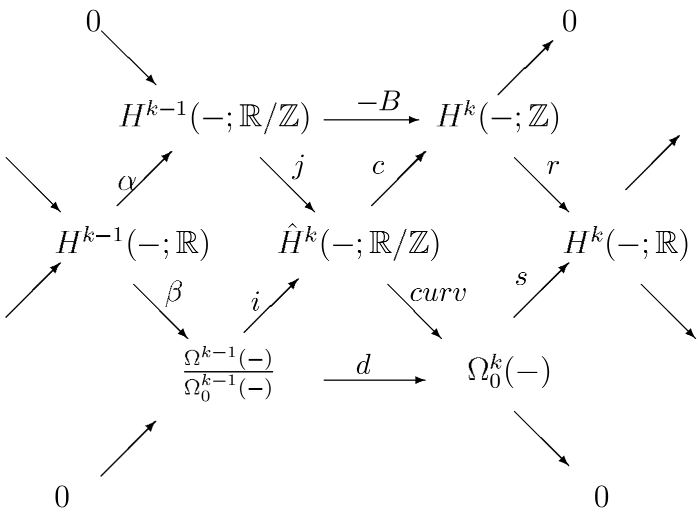
Proof. In view of Proposition 33, it suffices to show that $Hom(G, H) = 0 \implies Hom(Q, P) = 0$. To see this, suppose $\eta : Q \to P$ is a non-zero element of $Hom(Q, P)$. Then since $G \to Q$ is surjective, and $P \to H$ is injective, it follows that the composite $G \to Q \xrightarrow{\eta} P \to H$ is non-zero. But $Hom(G, H) = 0$ by assumption, therefore we have reached a contradiction. □

Let **Man** be the category of smooth manifolds and smooth maps between them, and let **Ab** be the category of abelian groups and group homomorphisms.

3.2.1 The case of differential characters

As noted in the introduction, the Deligne cohomology groups defined as hypercohomology of a certain double complex [7] are isomorphic to the differential character groups $\hat{H}^k(M; \mathbb{R}/\mathbb{Z})$ defined by Cheeger–Simons. Similarly, the de Rham-Federer currents [32, 31] too provide a model of differential cohomology. In order to compare various models, it is important to axiomatically characterise ordinary differential cohomology. In [45] Simons and Sullivan define:

Definition 35. *A functor* $\hat{H}^k$ *from* $\boldsymbol{Man}^{op}$ *to* $\boldsymbol{Ab}$*, together with natural transformations* $i, j, c, curv$ *is called a differential character functor if the following diagram in* $Fun(\boldsymbol{Man}^{op}, \boldsymbol{Ab})$



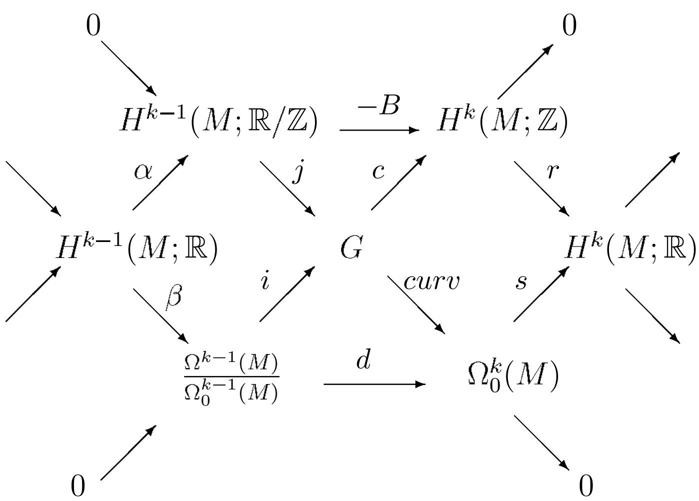
commutes and has exact diagonals.

They prove (Theorem 1.1, [45]) that

Proposition 36. *If $\hat{H}_1(-)$ and $\hat{H}_2(-)$ are two differential character functors from* ***Man***^{op} *to* ***Ab*** *together with the natural transformations $(i_1, j_1, c_1, curv_1)$ and $(i_2, j_2, c_2, curv_2)$ (respectively), then there exists a unique natural equivalence $\psi : \hat{H}_1 \to \hat{H}_2$ which is compatible with the given maps i.e. $\psi \circ i_1 = i_2, \psi \circ j_1 = j_2, c_2 \circ \psi = c_1$, and $curv_2 \circ \psi = curv_1$.*

Here, we observe that the following proposition is a direct consequence of corollary 32.

Proposition 37. *Let M be a smooth manifold. Then there exists a group G together with maps $i, j, c, curv$ such that the following diagram commutes and has short exact diagonals:*



Furthermore, if G' is any other abelian group together with maps $i', j', c', curv'$ which make the diagram commute and have short exact diagonals, then there exists an isomorphism $\phi : G \to G'$ such that $\phi \circ i = i'$, $\phi \circ j = j'$, $c' \circ \phi = c$, and $curv' \circ \phi = curv$.

Proof. The diagram above can be redrawn (Diagram 36, p.25, [1])as:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \frac{H^{k-1}(M;\mathbb{R})}{H^{k-1}(M;\mathbb{Z})} & \longrightarrow & \frac{\Omega^{k-1}(M)}{\Omega_0^{k-1}(M)} & \longrightarrow & d\Omega^{k-1}(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle i} & & \downarrow & & \\
0 & \longrightarrow & H^{k-1}(M;\mathbb{R}/\mathbb{Z}) & \xrightarrow{j} & G & \xrightarrow{curv} & \Omega_0^k(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle c} & & \downarrow & & \\
0 & \longrightarrow & Ext(H_{k-1}(M;\mathbb{Z}),\mathbb{Z}) & \longrightarrow & H^k(M;\mathbb{Z}) & \longrightarrow & Hom(H_k(M;\mathbb{Z}),\mathbb{Z}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

The proposition follows from corollary 32 by noting that $P = \frac{H^{k-1}(M;\mathbb{R})}{H^{k-1}(M;\mathbb{Z})}$ is divisible, and hence injective. □

Note that the Proposition 37 is weaker than the Simons–Sullivan result 36. The full Simons-Sullivan theorem is a statement about functors. We therefore consider the category of functors $Fun(\mathbf{Man}^{op}, \mathbf{Ab})$. This is an abelian category, having enough projectives and enough injectives. Therefore by Proposition 31, we have the following

Proposition 38. *The following are equivalent:*

1. *The Simons–Sullivan hexagon diagram uniquely determines the functor $\hat{H}^n(-;\mathbb{R}/\mathbb{Z})$ up to a compatible natural equivalence.*

2. $Ext\Big(Hom(H_n(-,\mathbb{Z}),\mathbb{Z}), \frac{H^{n-1}(-;\mathbb{R})}{H^{n-1}(-;\mathbb{Z})_{\mathbb{R}}}\Big) = 0$

In this proposition, the Ext is taken in the $Fun(\mathbf{Man}^{op}, \mathbf{Ab})$ category. For simplicity of notation, let us denote $Hom_{Fun(\mathbf{Man}^{op},\mathbf{Ab})}(-,-)$ by $Nat(-,-)$.

Corollary 39. *Consider the case of differential characters. In this case, we have* $P = \frac{H^{k-1}(-;\mathbb{R})}{H^{k-1}(-;\mathbb{Z})_{\mathbb{R}}}$*, and* $Q = Hom(H_k(-;\mathbb{Z}), \mathbb{Z})$*. Assume that* $Ext(Q, P) = 0$*, so that (by Proposition 31) for any two differential character functors* $\hat{H}^k_1(-)$ *and* $\hat{H}^k_2(-)$*, there is a compatible natural isomorphism*

$$\psi : \hat{H}^k_1(-) \longrightarrow \hat{H}^k_2(-).$$

Then

1. ψ *is unique* $\iff Nat\Big(Hom(H_k(-;\mathbb{Z}), \mathbb{Z}), \frac{H^{k-1}(-;\mathbb{R})}{H^{k-1}(-;\mathbb{Z})}\Big) = 0.$
2. $Nat\Big(H^k(-;\mathbb{Z}), H^{k-1}(-;\mathbb{R}/\mathbb{Z})\Big) = 0 \implies \psi$ *is unique.*

Proof. The statement (1) follows by an application of Proposition 33. Corollary 34 implies the statement (2) as $G = H^k(-;\mathbb{Z})$, and $H = H^{k-1}(-;\mathbb{R}/\mathbb{Z})$. □

Remark 40. *If we regard* $H^k(-;\mathbb{Z})$ *and* $H^{k-1}(-;\mathbb{R}/\mathbb{Z})$ *as functors from the opposite category of CW-complexes to the category of abelian groups, a standard argument shows that*

$Nat\left(H^k(-;\mathbb{Z}), H^{k-1}(-;\mathbb{R}/\mathbb{Z})\right) = 0$*. A non-zero natural transformation in the category* $Fun(\mathbf{CW^{op}}, \boldsymbol{Ab})$*, yields a nontrivial cohomology operation from* $H^k(-;\mathbb{Z})$ *to* $H^{k-1}(-;\mathbb{R}/\mathbb{Z})$*. But since these functors are represented by* $[-, K(\mathbb{Z}, k)]$ *and* $[-, K(\mathbb{R}/\mathbb{Z}, k-1)]$*, the cohomology operations are in one-to-one correspondance with elements of* $[K(\mathbb{Z}, k), K(\mathbb{R}/\mathbb{Z}, k-1)] \simeq H^{k-1}(K(\mathbb{Z}, k); \mathbb{R}/\mathbb{Z}) = 0$*. Hence* $Nat(H^k(-;\mathbb{Z}), H^{k-1}(-;\mathbb{R}/\mathbb{Z})) = 0$ *However, this argument doesn't work in the category* $Fun(\boldsymbol{Man}^{op}, \boldsymbol{Ab})$ *since the spaces* $K(\mathbb{Z}, k)$ *and* $K(\mathbb{R}/\mathbb{Z}, k-1)$ *are not finite dimensional manifolds in general.*

As we have seen in the proof of Proposition 37 the corresponding question in the category **Ab** is trivial since $\frac{H^{n-1}(M;\mathbb{R})}{H^{n-1}(M;\mathbb{Z})_{\mathbb{R}}}$ is divisible and hence injective. However, it is difficult to see whether or not the functor $\frac{H^{n-1}(-;\mathbb{R})}{H^{n-1}(-;\mathbb{Z})_{\mathbb{R}}}$ is an injective object in $Fun(\mathbf{Man}^{op}, \mathbf{Ab})$.

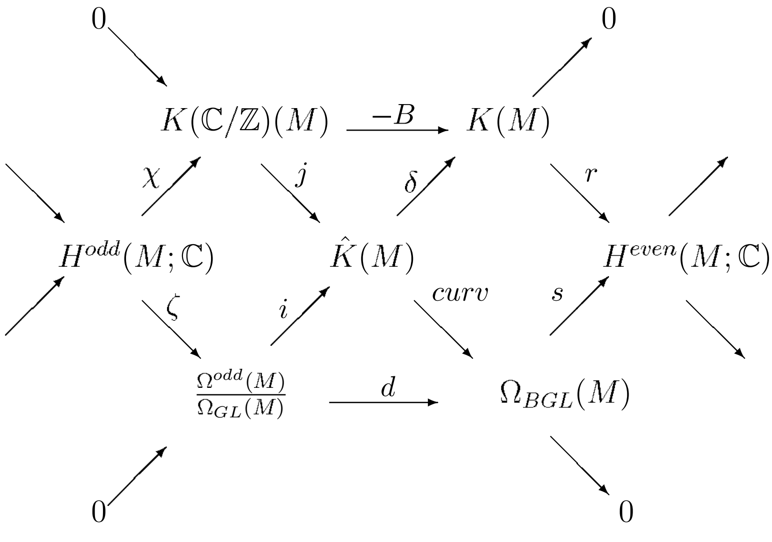
The existence of the differential character functor $\hat{H}^k : \mathbf{Man}^{op} \to \mathbf{Ab}$ is already known by the work [14] of Cheeger and Simons. Combining this with Proposition 26 applied in the category $Fun(\mathbf{Man}^{op}, \mathbf{Ab})$, we obtain

Proposition 41. *The Baer sum of* $[0 \to \frac{H^{k-1}(-,\mathbb{R})}{H^{k-1}(-,\mathbb{Z})} \to \frac{\Omega^{k-1}(-)}{\Omega_0^{k-1}(-)} \to \Omega_0^k(-) \to Hom(H_k(-;\mathbb{Z}),\mathbb{Z}) \to 0]$ *and* $[0 \to \frac{H^{k-1}(-,\mathbb{R})}{H^{k-1}(-,\mathbb{Z})} \to H^{k-1}(-,\mathbb{R}/\mathbb{Z}) \to H^k(-;\mathbb{Z}) \to Hom(H_k(-;\mathbb{Z}),\mathbb{Z}) \to 0]$ *vanishes in* $Ext^2(Hom(H_k(-;\mathbb{Z}),\mathbb{Z}), \frac{H^{k-1}(-,\mathbb{R})}{H^{k-1}(-,\mathbb{Z})})$.

3.2.2 The case of differential K-theory

Complex topological K-theory too admits a differential refinement called differential K-theory.For a survey of various models of differential K-theory, see [11]. In [46], Simons and Sullivan develop a model of differential K-theory for compact manifolds as the Grothendieck completion of the semigroup of 'structured vector bundles' and show that this group fits into a hexagon diagram:

Proposition 42. *The differential K-groups fit into the hexagon diagram*



Differential K-theory hexagon diagram

Here χ is reduction mod $\mathbb{Z}$, and ζ is induced by the de Rham map.

The sequence of arrows $H^{odd}(M;\mathbb{C}) \to K(\mathbb{C}/\mathbb{Z})(M) \to K(M) \to H^{even}(M;\mathbb{C})$ is the sequence associated to the short exact sequence $0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}/\mathbb{Z} \to 0$

where $K^{even/odd}(M;\mathbb{C})$ has been identified with $H^{even/odd}(M;\mathbb{C})$ via $ch \otimes \mathbb{C}$. For a description of rest of the terms and maps in this diagram, see [46]. Throughout this subsection, i.e. in the context of differential $K-$theory, M is assumed to be a compact manifold. Simons and Sullivan ask whether the diagram determines the groups $\hat{K}(M)$ up to isomorphism compatible with the other maps in the diagram. The following proposition provides a partial answer to their question.

Proposition 43. *If $\hat{K}'(M)$ is any other abelian group together with maps $i', j', \delta', curv'$ which makes the above hexagon diagram commute, and have short exact diagonals, then there exists an isomorphism $\phi : \hat{K}(M) \to \hat{K}'(M)$ such that $\phi \circ i = i', \phi \circ j = j', \delta' \circ \phi = \delta$, and $curv' \circ \phi = curv$.*

Proof. The diagram can be redrawn as

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \frac{H^{odd}(M;\mathbb{C})}{ker(\chi)=ker(\zeta)} & \longrightarrow & \frac{\Omega^{odd}(M)}{\Omega_{GL}(M)} & \longrightarrow & ker(s) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow i & & \downarrow & & \\ 0 & \longrightarrow & K(\mathbb{C}/\mathbb{Z}) & \xrightarrow{j} & \hat{K}(M) & \xrightarrow{curv} & \Omega_{BGL}(M) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \delta & & \downarrow & & \\ 0 & \longrightarrow & im(B) & \longrightarrow & K(M) & \longrightarrow & im(r) = im(s) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

The proposition follows from corollary 32 by noting that $P = \frac{H^{odd}(M;\mathbb{C})}{ker(\chi)=ker(\zeta)}$ is divisible and hence an injective object in **Ab**. □

This is weaker than the claim of uniqueness of the functor $\hat{K}$. Consider the functor category $Fun(\mathbf{Man}^{op}_{cpt}, \mathbf{Ab})$. From 31, we note the following

Proposition 44. *The following are equivalent:*

1. *If $\hat{K}'(-)$ is another functor from $\mathbf{Man}^{op}_{cpt}$ to **Ab** together with natural transformations $i', j', \delta', curv'$ which fit in the hexagon diagram with exact diagonals, then there is a natural equivalence $\phi : \hat{K}(-) \to \hat{K}'(-)$ such that $\phi \circ i = i', \phi \circ j = j', \delta' \circ \phi = \delta$, and $curv' \circ \phi = curv$.*

2. $Ext(Q, P) = 0$ *where* $P = \frac{H^{odd}(-;\mathbb{C})}{ker(\chi)=ker(\zeta)}$ *and* $Q = im(r)$ *are objects of* $Fun(\boldsymbol{Man}^{op}_{cpt}, \boldsymbol{Ab})$.

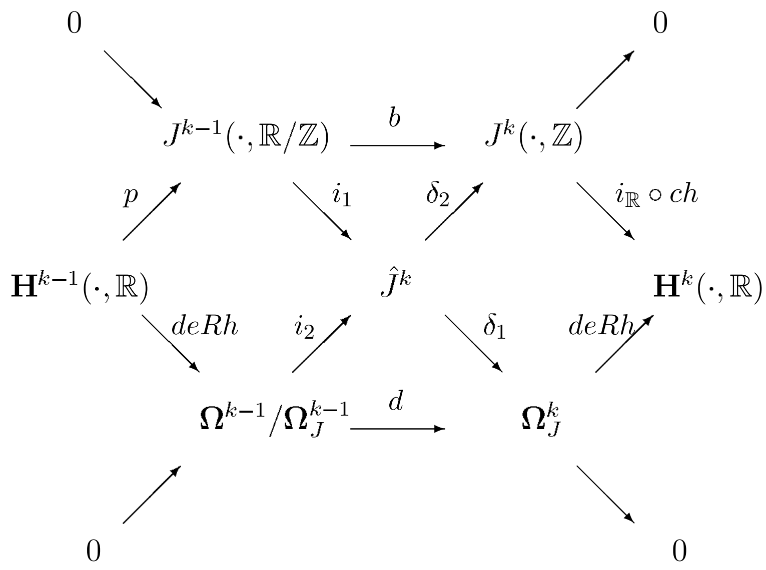
Since the existence of differential K-theory is well known, Proposition 26 shows that

Proposition 45. *The Baer sum of* $[0 \to P \to \frac{\Omega^{odd}(-)}{\Omega_{GL}(-)} \to \Omega_{BGL}(-) \to Q \to 0]$ *and* $[0 \to P \to K(-;\mathbb{C}/\mathbb{Z}) \to K(-;\mathbb{Z}) \to Q \to 0]$ *vanishes in* $Ext^2(Q, P)$ *where* $P = \frac{H^{odd}(-;\mathbb{C})}{ker(\chi)=ker(\zeta)}$ *and* $Q = im(r) = im(s)$.

Remark 46. *Though we have considered the case of even K-theory, we wish to remark that the same proof technique shows that for a fixed compact manifold* M, *the odd degree differential K-theory groups too are determined uniquely up to a compatible isomorphism. Note that this is weaker than uniqueness as a functor. A very general result about axiomatization of differential refinement of generalized cohomology theories (as functors) is proved in [10]. In particular, there it is shown that the even degree differential K-theory functor is uniquely determined by the axioms, and that the odd degree K-theory has multiple different inequivalent differential refinements (as functors).*

We further note that the technique above shows the uniqueness of differential generalized cohomology groups for any fixed manifold. Let $J = \sum \oplus J^k$ be any generalized graded cohomology theory. We use the Simons–Sullivan definition [47] of a differential refinement of J:

Definition 47. *A differential refinement of* J *is a functor* $\hat{J} : \boldsymbol{Man}^{op} \to \boldsymbol{Ab}$ *together with natural transformations* i_1, i_2, δ_1 *and* δ_2 *which make the diagram below commute and have short exact diagonals:*



Here

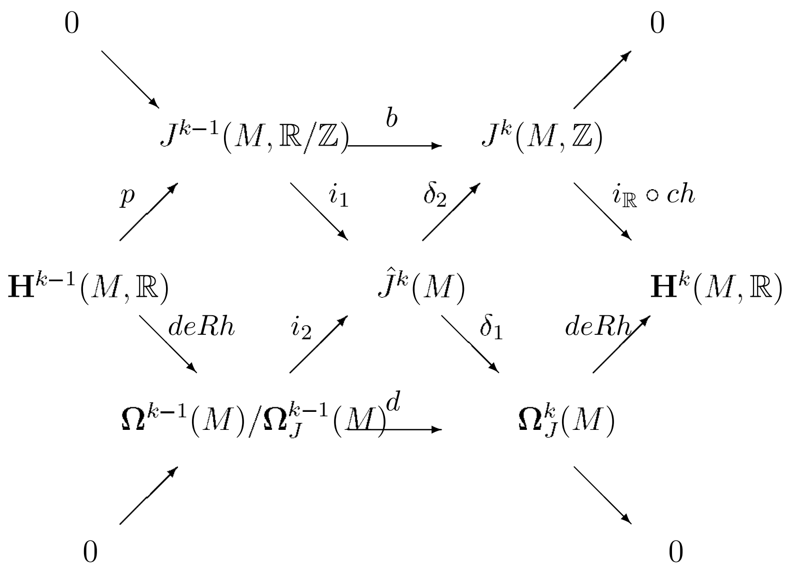
$$\mathbf{H}^k(\cdot\,,\mathbb{R}) = \sum_{j=0} \oplus H^j(\cdot\,, J^{k-j}(\text{point},\mathbb{R}))$$

$$\mathbf{\Omega}^k = \sum_{j=0} \oplus \Omega^j(\cdot\,, J^{k-j}(\text{point},\mathbb{R})), \quad \textit{and}$$

$$\mathbf{\Omega}_J^k = (\text{de Rham})^{-1}(\text{Im}(ch \circ i_{\mathbb{R}})).$$

Similar to the case of differential characters and differential K-theory discussed above, we have

Proposition 48. *For a fixed manifold M, there exist abelian groups $\hat{J}^k(M)$ for all $k > 0 \in \mathbb{Z}$ such that the following diagram is commutative and has short exact diagonals*



Moreover, this hexagon diagram uniquely determines the groups up to a compatible isomorphism.

In [33], Hopkins and Singer give a prescription for constructing a differential refinement of an arbitrary generalised cohomology theory. The existence of such a generalised differential cohomology theory implies (by using Proposition 26):

Proposition 49. *Let P be the functor $\frac{\mathbf{H}^{k-1}(-;\mathbb{R})}{ker(p)=ker(deRh)}$ and $Q = im(deRh) = im(i_{\mathbb{R}} \circ ch)$. Then the Baer sum of $[0 \to P \to J^{k-1}(-;\mathbb{R}/\mathbb{Z}) \to J^k(-;\mathbb{Z}) \to Q \to 0]$ and $[0 \to P \to \mathbf{\Omega}^{k-1}(-)/\mathbf{\Omega}_J^{k-1}(-) \to \mathbf{\Omega}_J^k(-) \to Q \to 0]$ vanishes in $Ext^2(Q,P)$.*

Remark 50. *Differential refinements of equivariant cohomology and twisted K-theory functors too fit into hexagon diagrams ([36, 42][40, 12]). We remark that, like in Propositions 37, and 43 for a fixed manifold M, the abelian group of equivariant differential characters (resp. twisted differential K-theory group) is determined uniquely up to a compatible isomorphism by the respective hexagon diagram.*