Chapter 5

Dynamics of Prey-predator Model with Stage Structure in Prey Including Maturation and Gestation Delays [1]

5.1 Introduction

Ecology is the study of the distributions, abundance and relation of organisms and their interactions with the environment. The fundamental goal of ecological research is to study all the factors which affect the interaction of individual organism with each other or environment and to get the results which help to sustainable development of the ecosystem. The prey-predator relationship has been focused as an essential area of research in ecological systems [16]. The very beginning model on the prey-predator relationship is proposed and analyzed by Lotka and Volterra, which contains a pair of first order, nonlinear differential equations. After that, a tremendous amount of attempts have been made to extend the pioneering work [108, 43, 129, 9, 78, 207].

One crucial factor in the dynamics of prey-predator interaction is the intake rate of prey by predator population or functional response of predator. It helps to analyze a prey-predator dynamics with more accuracy. It is known that predation rate is dependent upon several aspects like age group, body size, nature of habitat, interference and help among individuals in particular species. If the prey is weak, small in size, immature or available in abundance then it consumed by a linear function, known as Holling type I functional response. There are many types of functional response: Holling type I-III, Beddingtion-DeAngelis, Crowley-Martin, ratio dependent, Hassel-Verley etc. Holling type I-III functional responses are prey dependent while all other are function of both prey and predator density. Crowley and Martin [38] considered that rate of predation will decrease at high density of predators due to interference among them.

[1] A considerable part of this chapter is published in *Nonlinear Dynamics*, **96**(4), 2653-2679, 2019.

This functional response is given by

$$f(x,y) = \frac{\alpha x}{(1+ax)(1+by)},$$

where α, a and b are positive parameters denoting capture rate, handling time and magnitude of interference among predators, respectively. Crowley-Martin functional response is involved in a significant number of studies [214, 50, 135]. The effect of intraspecific interference among predators has been investigated in prey-predator model with Holling type II functional response in [87], with Holling type III functional response in [73], with Beddington-DeAngelis type functional response in [129, 118, 224] and Hassell-Varley functional response is considered by Tian [206].

Reproduction rate and survival of most of the biological species is dependent upon age or the development stage of individuals. A life cycle of species can be divided into two groups, viz., mature and immature. The nature of a predator is completely different for a mature and immature prey. Predator has a good attraction on immature prey while mature prey has more escape ability then immature. Hence, the incorporation of the effect of past life of species is reasonable to obtain more realistic results. The analysis of stage structured models is done by large number of researchers, as a way to eliminate the shortcomings of classical Lotka-Volterra models [114, 227, 2, 3, 154, 71, 129, 166, 26]. Bosch and Gabriel [23] considered a predator-prey system where cyclic population fluctuations are due to the age structure in the predator species. It is shown that cannibalism is also a stabilizing mechanism when population oscillations are due to this age structure. Further, the effect of stage structure for prey in a predator-prey model under harvesting of adult prey and predator has been studied by Chakraborty *et al.* [26].

Hopf-bifurcation plays an important role to understand the complex dynamical behavior of the system. It gives such a threshold value of a parameter where the stability behavior of the system suddenly changes. Stability and Hopf-bifurcation for a prey-predator system including stage structure and refuge is analyzed by Wei and Fu [224]. Local Hopf-bifurcation in a time delayed system has been studied by Tripathi *et al.* [210], Bairagi and Jana [11] and Deng *et al.* [46]. They have shown the existence of periodic solution after a certain value of delay.

Time delay occurs in every biological moment, therefore models having delay are much more realistic in nature. Delay differential equation shows more complicated behavior than ordinary differential equation. The characteristic equation corresponding to the linearized system of delay differential equation has some exponential terms or quasi-polynomial [12]. Time delay can destabilize the system and emanates periodic solution [238, 46]. Predator species takes a time lag to introduce new born predators after consuming prey, which is known as gestation time delay. Besides that, in stage structured system maturation delay is the amount of time which is taken by a juvenile individual of a particular species to become adult and join the other class. Models including different type of discrete gestation delays are investigated by some researchers [140, 71, 66, 144, 11, 130, 210]. The effect of maturation delay on the dynamics of stage structured models is well studied in [98, 166, 10, 26, 48].

Song and Wei [195] have done bifurcation analysis for Chen's system with delayed feedback. They found that when the delay passes through certain values, chaotic oscillation is converted into a stable steady state or a stable periodic orbit. The dynamics of a stage structured population model with harvesting of mature prey is described by Zhang and Zhang [239]. Li and Li [116] have analyzed a prey-predator model with gestation delay and observed that system changes the stability behavior beyond a certain critical value of the delay. A delayed prey-predator model with Crowley-Martin type functional response including prey refuge has been considered by Maiti *et al.* [136]. Chakraborty *et al.* [26] analyzed the simultaneous effect of harvesting and delay on the dynamics of Leisle-Gower prey-predator system with stage structure in prey. They have shown that the delay not only can cause a stable equilibrium to become unstable, it can also cause a switching of stability sometimes. Some investigations have also been made with two or more delays involved in the model [235, 157, 120, 229, 64, 122, 46, 222]. In such a case, getting the eigenvalues from the characteristic equation is difficult due to presence of two or more exponential terms. Liao *et al.* [122] investigated a three-species predator-prey system with multiple delays and it is shown that Hopf-bifurcation at the positive equilibrium of the system can occur as the delay crosses its certain value. Further Hopf-bifurcation analysis for a ratio dependent predator-prey system with two delays and stage structure for predator is studied by Deng *et al.* [46].

To the best of author's knowledge, a prey-predator model with (i) stage structure in prey, (ii) Crowley-Martin type functional response and (iii) maturation and gestation delay have not been studied. Hence the main objective of this work is to analyze the dynamical complexity of a three dimensional prey-predator model incorporating stage structure in prey population with maturation and gestation delay. Motivated by this, we construct a delayed model in section 5.2, followed by some assumptions. The rest part of the manuscript is organized as follows: In section 5.3, the dynamics of non-delayed model is analyzed. We discuss positivity, boundedness and permanence of solution, existence of equilibrium points and their stability analysis. Section 5.4 is devoted to local stability and Hopf-bifurcation of delayed system. We investigate Hopf-bifurcation through local stability of positive equilibrium considering delay as a bifurcation parameter. In section 5.5, we analyze the direction and stability of Hopf-bifurcation using normal form method and center manifold theorem. Numerical simulation has been carried out to illustrate our theoretical findings in section 5.6.

5.2 Construction of Mathematical Model

We consider a three dimensional prey-predator with stage structure. Let x, y and z be the population size of immature prey, mature prey and predator species, respectively. The formulation of mathematical model is based on some following assumptions:

- We assume that the prey population is classified into two stage groups, immature and mature. The birth rate of immature prey is proportional to the density of mature prey population and mature prey grows logistically.
- Since the escape ability and battle efficiency of an immature prey is very less due to their small body size and unawareness. Therefore, predator has a good attraction on immature prey. It is assumed that the predators consume immature prey population by Holling type-I (linear mass-action principal) functional response.
- On the other hand predator has some searching time and handling time on mature prey and there is some magnitude of intraspecific interference among them. Due to this reason we assume that the interaction between mature and predator species is followed by Crowley-Martin type functional response.
- An immature prey becomes mature after time τ_1 (maturation time delay) and joins the mature prey group. It is assumed that $re^{-d\tau_1}x(t-\tau_1)$ be the number of immature preys who were born at the time $(t-\tau_1)$ and survive at time t, (Jana *et al.* [90], Chakraborty *et al.* [26] and Kar & Matsuda [98]).
- It is known that each organism requires a certain amount of time to reproduce their progeny. Because of this the increment in predator population due to predation does not appear immediately after consuming prey. It takes some time (say τ_2) which can be regarded as gestation period of predator or reaction time of predation. Therefore, it is assumed that the change rate of predator depends upon the number of individuals present at time $(t-\tau_2)$.

Keeping all above aspects in view, a suitable mathematical model can be governed as follows:

$$\begin{aligned}
\frac{dx}{dt} &= ry - dx - re^{-d\tau_1}x(t-\tau_1) - \rho xz,\\
\frac{dy}{dt} &= sy\left(1-\frac{y}{K}\right) + re^{-d\tau_1}x(t-\tau_1) - \frac{\alpha yz}{(1+ay)(1+bz)},\\
\frac{dz}{dt} &= -\delta_0 z - \delta_1 z^2 + c_1\rho x(t-\tau_2)z(t-\tau_2) + \frac{c_2\alpha y(t-\tau_2)z(t-\tau_2)}{(1+ay(t-\tau_2))(1+bz(t-\tau_2))},
\end{aligned} \tag{5.1}$$

subject to the non negative conditions $x(\xi) = \phi_1(\xi) \geq 0,\ y(\xi) = \phi_2(\xi) \geq 0,\ z(\xi) = \phi_3(\xi) \geq 0, \xi \in [-\tau, 0], \tau = \tau_1 + \tau_2$, where $\phi_j(\xi) \in C([-\tau, 0] \to R_+),\ (j = 1, 2, 3)$, where τ_1 and τ_2 denote maturation time delay of the prey and gestation time delay of the predator, respectively.

Model (5.1) takes the following form in the absence of both the delays ($\tau_1 = \tau_2 = 0$).

$$
\begin{aligned}
\frac{dx}{dt} &= ry - dx - rx - \rho xz,\\
\frac{dy}{dt} &= sy\left(1 - \frac{y}{K}\right) + rx - \frac{\alpha yz}{(1+ay)(1+bz)},\\
\frac{dz}{dt} &= -\delta_0 z - \delta_1 z^2 + c_1 \rho xz + \frac{c_2 \alpha yz}{(1+ay)(1+bz)},\\
x(0) &= x_0 \ \geq\ 0,\ y(0) = y_0 \ \geq\ 0,\ z(0) = z_0 \ \geq\ 0.
\end{aligned}
\tag{5.2}
$$

The biological meaning of all parameters and variables in above model is provided in Table 5.1.

Table 5.1: Variables and parameters used in model (5.1), and (5.2).

Variables/ Parameters	**Biological meaning**	**Units**
x	Density of immature prey	Number per unit area (tons)
y	Density of mature Prey	Number per unit area (tons)
z	Density of predator	Number per unit area (tons)
r	Growth rate of immature prey	Per day
d	Mortality rate of immature prey population	Per day
ρ	Attack rate of predator on immature prey	Per day
s	Intrinsic growth rate of mature prey	Per day
K	Carrying capacity of the prey population	Number per unit area (tons)
α	Attack rate of predator on mature prey	Per day
a	Handling time	Per day
b	Magnitude of interference among predators	Per day
c_1	Conversion efficiency of z on x	Constant & $0 < c_1 < 1$
c_2	Conversion efficiency of z on y	Constant & $0 < c_2 < 1$
δ_0	Natural death rate of predator	Per day
δ_1	Coefficient of intraspecific interference among predators	Per day

τ_1	Maturation delay for prey population	Days
τ_2	Gestation delay for predator population	Days

5.3 Dynamics of non-delayed model

In this section, we analyze the dynamics of non-delayed system (5.2). First we show that our system has positive and bounded solution, which refers that a restriction on populations to grow exponentially as a effect of limited space and resources in nature. Then we find out all feasible equilibrium points of the system and then their local and global stability is shown under certain conditions.

5.3.1 Positivity and boundedness of the solution

Theorem 5.3.1. *The model system (5.2) has a unique and non-negative solution with initial values* $\{(x(0),y(0),z(0)) \in R^3_+\}$*, where* $R^3_+ = \{(x_1,x_2,x_3) : x_i \geq 0, i = 1,2,3\}$*. Further, the set*

$$\Omega = \left\{(x,y,z) : 0 \leq x+y \leq \frac{f_{max}}{\delta},\ 0 \leq y + \frac{1}{c_2}z \leq \frac{M_0}{\delta^*}\right\}$$

is a positive invariant set for all the solutions initiating in the interior of the positive octant, where $f_{max} = (s+2r)^2\frac{K}{4s}$, $\delta = min\{r,d\}$, $M_0 = (2s+r)f_{max} + \frac{1}{c_2\delta_1}(\frac{c_1}{2}\rho f_{max})^2$, $\delta^* = min\{s,\delta_0\}$.

Proof. The model system (5.2) can be written in the matrix form

$$\dot{X} = G(X),$$

where $X = (x_1,x_2,x_3)^T = (x,y,z)^T \in R^3$, and G(X) is given by

$$G(X) = \begin{bmatrix} G_1(X) \\ G_2(X) \\ G_3(X) \end{bmatrix} = \begin{bmatrix} ry - dx - rx - \rho xz \\ sy\left(1-\frac{y}{K}\right) + rx - \frac{\alpha yz}{(1+ay)(1+bz)} \\ -\delta_0 z - \delta_1 z^2 + c_1\rho xz\frac{c_2\alpha yz}{(1+ay)(1+bz)} \end{bmatrix}.$$

Since $G : R^3_+ \to R^3$ is locally Lipschitz-continuous in Ω and $X(0) = X_0 \in R^3_+$, the fundamental theorem of ordinary differential equation guarantees the local existence and uniqueness of the solution. Since $[G_i(X)]_{x_i(t)=0,\ x\in R^3_+} \geq 0$, it follows [193, 223] that $X(t) \geq 0$ for all $t \geq 0$. In fact, from the last equation of model (5.2), it can easily be seen that $\dot{z}|_{z=0} \geq 0$ and hence $z(t) \geq 0$ for all $t \geq 0$. Secondly, $\dot{y}|_{y=0} = rx \geq 0$ and hence $y(t) \geq 0$ for all $t \geq 0$. If this is not true, then assume that there exist a $t_1 > 0$ with $t_1 = inf\{t : y(t) = 0,\ t > 0\}$, such that $\dot{y}(t_1)|_{y(t_1)=0} = rx(t_1) < 0$. But we also have $y(t_1) = 0,\ y(t) > 0$ with $t \in [0,t_1)$ and $x(t_1) < 0$. Since $x(0) \geq 0$, there is a $t_2 > 0$ with $t_2 = inf\{t : x(t) = 0,\ t \in [0,t_1)\}$. Hence by the definition of t_2, $\dot{x}(t_2) \leq 0$. But $\dot{x}(t_2) = ry(t_2) > 0$, which is a contradiction to our assumption. From the first equation of model (5.2), we have $\dot{x}|_{x=0} = ry \geq 0$, and therefore $x(t) \geq 0$ for all $t \geq 0$.

Now, suppose

$$W_1(t) = x(t) + y(t),$$

then we have

$$\frac{dW_1(t)}{dt} = \frac{dx}{dt} + \frac{dy}{dt} \leq ry - dx + sy\Big(1 - \frac{y}{K}\Big)$$
$$= sy\Big(1 - \frac{y}{K}\Big) + 2ry - ry - dx.$$

Let $f(y) = sy(1 - \frac{y}{K}) + 2ry$, then f has maximum value $f_{max} = (s+2r)^2 \frac{K}{4s}$ at $y = \frac{K}{2s}(s+2r)$. Hence, it follows that

$$\frac{dW_1(t)}{dt} \leq f_{max} - \delta W_1(t),$$

where $\delta = min\{r,d\}$,
which implies

$$\limsup_{t\to\infty} W_1(t) \leq \frac{f_{max}}{\delta}.$$

Again, suppose

$$W_2(t) = y(t) + \frac{1}{c_2} z(t),$$

then we have

$$\frac{dW_2(t)}{dt} = \frac{dy}{dt} + \frac{1}{c_2}\frac{dz}{dt} \leq 2sy - \delta^*\Big(y + \frac{z}{c_2}\Big) + \Big(rx - \frac{\delta_1}{c_2}z^2 + \frac{c_1}{c_2}\rho xz\Big),$$

where $\delta^* = min\{s,\delta_0\}$,
Hence, it follows that

$$\frac{dW_2(t)}{dt} + \delta^* W_2(t) \leq (2sy + rx) + \frac{1}{c_2\delta_1}\Big(\frac{c_1}{2}\rho x\Big)^2 - \frac{1}{c_2}\Big(\sqrt{\delta_1}z - \frac{c_1}{2\sqrt{\delta_1}}\rho x\Big)^2$$
$$\leq (2s+r)\frac{f_{max}}{\delta} + \frac{1}{c_2\delta_1}\Big(\frac{c_1}{2}\rho\frac{f_{max}}{\delta}\Big)^2 =: M_0,$$

which implies

$$\limsup_{t\to\infty} W_2(t) \leq \frac{M_0}{\delta^*}.$$

We also note that if $W_1(t) \geq \frac{f_{max}}{\delta}$ and $W_2(t) \geq \frac{M_0}{\delta^*}$, then $\frac{dW_1(t)}{dt} \leq 0, \frac{dW_2(t)}{dt} \leq 0$.

This shows that all solutions of system (5.2) starting in Ω remains in Ω for all $t > 0$. □

Theorem 5.3.2. *Let the following inequalities hold true:*

$$s > \alpha c_2 \frac{M_0}{\delta^*}, \qquad c_1 \rho x_a > \delta_0.$$

Then model system (5.2) is uniformly persistence, where, x_a is defined in the proof.

Proof. Permanence or uniform persistence of a system implies that all species will be present in future and none of them will become extinct if they are initially present. System (5.2) is said to be uniformly persistence if there are positive constants M_1 and M_2 such that each positive solution $X(t) = (x(t), y(t), z(t))$ of the system with positive initial conditions satisfies

$$M_1 \le \liminf_{t\to\infty} X(t) \le \limsup_{t\to\infty} X(t) \le M_2.$$

Keeping the above in view, if we define

$$M_2 = max\left\{\frac{f_{max}}{\delta}, c_2 \frac{M_0}{\delta^*}\right\},$$

then from Theorem 5.3.1, it follows that

$$\limsup_{t\to\infty} X(t) \le M_2.$$

This also shows that for any sufficiently small $\varepsilon > 0$, there exists a $T > 0$ such that for all $t \ge T$, the following holds:

$$x(t) < \frac{f_{max}}{\delta} + \varepsilon, \;\; y(t) < \frac{f_{max}}{\delta} + \varepsilon, \;\; z(t) < c_2 \frac{M_0}{\delta^*} + \varepsilon.$$

Now from the second equation of model system (5.2), for all $t \ge T$, we can write

$$\frac{dy}{dt} \ge sy - \frac{sy^2}{K} - \alpha\left(c_2 \frac{M_0}{\delta^*} + \varepsilon\right)y$$
$$= \left(s - \alpha\left(c_2 \frac{M_0}{\delta^*} + \varepsilon\right)\right)y - \frac{s}{K}y^2.$$

Hence, it follows that

$$\liminf_{t\to\infty} y(t) \ge \frac{K}{s}\left(s - \alpha\left(c_2 \frac{M_0}{\delta^*} + \varepsilon\right)\right),$$

which is true for every $\varepsilon > 0$, thus

$$\liminf_{t\to\infty} y(t) \ge \frac{K}{s}\left(s - \alpha c_2 \frac{M_0}{\delta^*}\right) =: y_a,$$

where $s > \alpha c_2 \frac{M_0}{\delta^*}$.
Now from the first equation of model system (5.2), we obtain

$$\frac{dx}{dt} \geq ry_a - dx - rx - \rho x\Big(c_2\frac{M_0}{\delta^*} + \varepsilon\Big),$$

$$= ry_a - \Bigg(d + r + \rho\Big(c_2\frac{M_0}{\delta^*} + \varepsilon\Big)\Bigg)x,$$

which implies

$$\liminf_{t\to\infty} x(t) \geq \frac{ry_a}{d + r + \rho(c_2\frac{M_0}{\delta^*} + \varepsilon)},$$

which is true for every $\varepsilon > 0$, thus

$$\liminf_{t\to\infty} x(t) \geq \frac{ry_a}{d + r + \rho c_2\frac{M_0}{\delta^*}} =: x_a.$$

Third equation of model system (5.2) yields

$$\frac{dz}{dt} \geq -\delta_0 z - \delta_1 z^2 + c_1\rho x_a z.$$

Hence

$$\liminf_{t\to\infty} z(t) \geq \frac{1}{\delta_1}(-\delta_0 + c_1\rho x_a) =: z_a,$$

where $c_1\rho x_a > \delta_0$.

Taking $M_1 = min\{x_a, y_a, z_a\}$, the theorem follows. □

Remark 5.3.1. *The above analysis shows that the system is persistence if intrinsic growth rate of mature prey is greater than a threshold value and death rate of predator population is less than a threshold value.*

5.3.2 Equilibrium points

It can be investigated that model (5.2) has three equilibria, namely, $E_0(0,0,0)$, $E_*(x_*, y_*, 0)$ and $E^*(x^*, y^*, z^*)$. The equilibrium E_0 exists trivially. We will show the existence of planer equilibrium E_* and interior equilibrium E^*.
Existence of $E_*(x_*, y_*, 0)$: Here x_* and y_* are solution of the following system:

$$ry - (d+r)x = 0, \ \ sy\Big(1 - \frac{y}{K}\Big) + rx = 0,$$

which yields

$$y_* = \frac{K}{s}\left(s + \frac{r^2}{d+r}\right), \ \ x_* = \frac{ry_*}{d+r}.$$

Clearly, x_* and y_* both are positive. Thus, the equilibrium $E_*(x_*, y_*, 0)$ exists always.
Existence of $E^*(x^*, y^*, z^*)$: Here x^*, y^* and z^* are the positive solutions of the following algebraic equations:

$$\begin{aligned}
&ry - dx - rx - \rho xz = 0,\\
&sy\big(1 - \frac{y}{K}\big) + rx - \frac{\alpha yz}{(1+ay)(1+bz)} = 0,\\
&-\delta_0 - \delta_1 z + c_1\rho x + \frac{c_2\alpha y}{(1+ay)(1+bz)} = 0.
\end{aligned}$$

From the first equation of above system, we get $x = \frac{ry}{d+r+\rho z}$. Putting the value of x in last two equation of above system, we obtain

$$s\left(1 - \frac{y}{K}\right) + \frac{r^2}{d+r+\rho z} - \frac{\alpha z}{(1+ay)(1+bz)} = 0, \tag{5.3}$$

$$-\delta_0 - \delta_1 z + \frac{c_1\rho r y}{d+r+\rho z} + \frac{c_2\alpha y}{(1+ay)(1+bz)} = 0. \tag{5.4}$$

From equation (5.3) we note the following:

- When $y = 0$, then we obtain
$$m_1 z^2 + m_2 z + m_3 = 0, \tag{5.5}$$
where $m_1 = (bs-\alpha)\rho$, $m_2 = (bs-\alpha)d + (bs-\alpha)r + br^2 + s\rho$, $m_3 = ds + r^2 + sr$.
Equation (5.5) has a positive root z_1 under the following inequality:
$$(bs - \alpha) < 0. \tag{5.6}$$

- When $z = 0$, then $y = K(s + \frac{r^2}{d+r}) =: y_1 > 0$.

- We have
$$\frac{dz}{dy} = -\frac{\frac{s}{K} + \frac{\alpha a z}{(1+ay)^2(1+bz)}}{\frac{r^2\rho}{(d+r+\rho z)^2} + \frac{\alpha}{(1+ay)(1+bz)^2}} < 0.$$

The above analysis shows that equation (5.3) passes through the points $(y_1, 0)$ and $(0, z_1)$ and z is a decreasing function of y under condition (5.6).

Now we note the following from equation (5.4):

- When $y = 0$, then $z = -\frac{\delta_0}{\delta_1} =: z_2 < 0$.

- When $z = 0$, then we obtain
$$n_1 y^2 + n_2 y + n_3 = 0, \tag{5.7}$$

where $n_1 = c_1 a r \rho$, $n_2 = c_2 d\alpha - ad\delta_0 + c_2\alpha r - a\delta_0 r + c_1 r\rho$, $n_3 = -\delta_0(d+r)$. Equation (5.7) always has a positive root y_2 (say).

- We also have

$$\frac{dz}{dy} = \frac{\frac{c_1\rho r}{d+r+\rho z} - \frac{c_2\alpha}{(1+ay)^2(1+bz)}}{-\delta_1 + \frac{c_1 r\rho^2 y}{(d+r+\rho z)^2} - \frac{c_2\alpha b y}{(1+ay)(1+bz)^2}}.$$

It is evaluated that $\frac{dz}{dy} > 0$ if the following condition holds

$$max\left\{b, \frac{\rho}{d+r}\right\} < \frac{c_2\alpha b(d+r)}{c_1\rho r(1+a_0\frac{M_0}{\delta^*})^3}, \tag{5.8}$$

where $a_0 = max\{a,b\}$.

This shows that equation (5.4) passes through the point $(y_2, 0)$ and $(0, z_2)$, z is increasing function in y under condition (5.8).

From above analysis we conclude that isoclines (5.3) and (5.4) intersect a unique point (y^*, z^*) if

$$y_2 < y_1. \tag{5.9}$$

Now we are in a position to state the following theorem.

Theorem 5.3.3. *The positive equilibrium $E^*(x^*, y^*, z^*)$ exists uniquely under conditions (5.6),(5.8) and (5.9).*

5.3.3 Stability Analysis

The local stability of each existing equilibrium point can be investigated by computing the Jacobian matrix corresponding to the system and further by evaluating it at each equilibrium point. Then using the Routh-Hurwitz criterion, we get following results:

- The equilibrium point $E_0(0,0,0)$ is saddle point.

- The Jacobian matrix evaluated at $E_*(x_*, y_*, 0)$ is given by

$$J|_{E_*} = \begin{bmatrix} -d-r & r & -\rho x_* \\ r & -\frac{sy_*}{K} - \frac{r^2}{d+r} & -\frac{\alpha y_*}{(1+ay_*)} \\ 0 & 0 & -\delta_0 + c_1\rho x_* + \frac{c_2\alpha y_*}{1+ay_*} \end{bmatrix}$$

and it is noted that

  - $E_*(x_*, y_*, 0)$ is locally asymptotically stable if $\delta_0 > c_1\rho x_* + \frac{c_2\alpha y_*}{1+ay_*}$.

- $E_*(x_*,y_*,0)$ is saddle point having stable manifold along x and y-axis and unstable manifold along z-axis if $\delta_0 < c_1\rho x_* + \frac{c_2\alpha y_*}{1+ay_*}$.

- In order to study the local stability behavior of positive equilibrium, let $J|_{E^*}$ be the variational matrix evaluated at $E^*(x^*,y^*,z^*)$.

$$J|_{E^*} = \begin{bmatrix} \frac{-ry^*}{x} & r & -\rho x^* \\ r & -\frac{sy^*}{K} - \frac{rx^*}{y^*} + \frac{a\alpha y^* z^*}{(1+ay^*)^2(1+bz^*)} & -\frac{\alpha y^*}{(1+ay^*)(1+bz^*)^2} \\ c_1\rho z^* & \frac{c_2\alpha z^*}{(1+ay^*)^2(1+bz^*)} & -\delta_1 z^* - \frac{c_2 c\alpha y^* z^*}{(1+ay^*)(1+bz^*)^2} \end{bmatrix}.$$

The characteristic equation corresponding to the above variational matrix is

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0, \tag{5.10}$$

where A_1, A_2 and A_3 are given by
$A_1 = -(j_{11} + j_{22} + j_{33})$,
$A_2 = j_{11}j_{22} + j_{22}j_{33} + j_{33}j_{11} - j_{12}j_{21} - j_{23}j_{32} - j_{13}j_{31}$,
$A_3 = j_{12}j_{21}j_{33} + j_{11}j_{23}j_{32} + j_{13}j_{31}j_{22} - j_{11}j_{22}j_{33} - j_{12}j_{23}j_{31} - j_{13}j_{21}j_{32}$,

and j_{mn} for $m,n = 1,2,3$ represents an entry, in $J|_{E^*}$, in m^{th} row and n^{th} column. Using the Routh-Hurwitz criteria, all eigenvalues of $J|_{E^*}$ have negative real part if and only if $A_1 > 0$, $A_3 > 0$ and $A_1A_2 > A_3$.
Thus we can state the following theorem.

Theorem 5.3.4. *The system (5.2) is locally asymptotically stable around the interior equilibrium E^* if and only if $A_1 > 0$, $A_3 > 0$ and $A_1A_2 > A_3$.*

In the next theorem, we are able to find sufficient condition under which E^* is globally asymptotically stable.

Theorem 5.3.5. *The system (5.2) is globally asymptotically stable around the interior equilibrium E^* under following condition:*

$$r^2\left(1 + \frac{c_2 x^*(1+bz^*)}{c_1 y^*(1+ay^*)}\right)^2 < 4(d+r)\left(\frac{s}{K} - \frac{\alpha a z^*}{(1+ay^*)(1+bz^*)}\right). \tag{5.11}$$

Proof. Consider the following Lyapunov function about E^*

$$V(x,y,z) = \frac{1}{2}(x-x^*)^2 + l_1\left(y - y^* - y^*\ln\frac{y}{y^*}\right) + l_2\left(z - z^* - z^*\ln\frac{z}{z^*}\right),$$

where l_1 and l_2 are positive constants to be determined in the subsequent steps. Now, differentiating V with respect to time along the solutions of system (5.2)

$$\frac{dV}{dt} = (x-x^*)\frac{dx}{dt} + l_1\Big(\frac{y-y^*}{y}\Big)\frac{dy}{dt} + l_2\Big(\frac{z-z^*}{z}\Big)\frac{dz}{dt}.$$

After a little algebraic manipulation, we get

$$\begin{aligned}\frac{dV}{dt} &= -(d+r+\rho z)(x-x^*)^2 + l_1\left[-\frac{s}{K} - \frac{rx}{yy^*} + \frac{\alpha a z^*}{(1+ay)(1+ay^*)(1+bz^*)}\right](y-y^*)^2 \\ &\quad - l_2\left[\delta_1 + \frac{c_2\alpha b y^*}{(1+bz)(1+ay^*)(1+bz^*)}\right](z-z^*)^2 + \left[r + l_1\frac{r}{y^*}\right](x-x^*)(y-y^*) \\ &\quad + \left[-l_1\frac{\alpha}{(1+ay)(1+bz)(1+bz^*)} + l_2\frac{c_2\alpha}{(1+ay)(1+bz)(1+ay^*)}\right](y-y^*)(z-z^*) \\ &\quad + [-\rho x^* + l_2 c_1 \rho](x-x^*)(z-z^*).\end{aligned}$$

Choosing $l_1 = \frac{c_2 x^*(1+bz^*)}{c_1(1+ay^*)}$ and $l_2 = \frac{x^*}{c_1}$, we obtain

$$\begin{aligned}\frac{dV}{dt} &\le -(d+r)(x-x^*)^2 - \frac{c_2 x^*(1+bz^*)}{c_1(1+ay^*)}\left[\frac{s}{K} - \frac{\alpha a z^*}{(1+ay^*)(1+bz^*)}\right](y-y^*)^2 \\ &\quad + \left[r + \frac{c_2 r x^*(1+bz^*)}{c_1 y^*(1+ay^*)}\right](x-x^*)(y-y^*),\end{aligned}$$

which implies $\frac{dV}{dt}$ is negative definite under condition (5.11). Hence, the positive equilibrium E^* is globally asymptotically stable under condition (5.11). □

5.4 Analysis of delayed model

In this section we shall investigate the dynamics of the delayed system (5.1).

5.4.1 Local stability and Hopf-bifurcation

Rewritting model system (5.1) as

$$\frac{dU(t)}{dt} = F(U(t), U(t-\tau_1), U(t-\tau_2)),$$

where $U(t) = [x(t), y(t), z(t)]^T$, $U(t-\tau_i) = [x(t-\tau_i), y(t-\tau_i), z(t-\tau_i)]^T$, $i = 1, 2$.
Let $x(t) = x^* + x'(t)$, $y(t) = y^* + y'(t)$, $z(t) = z^* + z'(t)$. Then linearizing system (5.1) about the interior equilibrium solution $E^*(x^*, y^*, z^*)$, we have

$$\frac{dZ}{dt} = P'Z(t) + Q'Z(t-\tau_1) + R'Z(t-\tau_2),$$

where

$$P' = \left(\frac{\partial F}{\partial U(t)}\right)_{E^*}, \; Q' = \left(\frac{\partial F}{\partial U(t-\tau_1)}\right)_{E^*}, \; R' = \left(\frac{\partial F}{\partial U(t-\tau_2)}\right)_{E^*},$$

and $\qquad Z(t) = [x'(t), y'(t), z'(t)]^T.$

Thus, the variational matrix of the system (5.1) at E^* is given by

$$J = P' + Q'e^{-\lambda\tau_1} + R'e^{-\lambda\tau_2}.$$

After a little calculation, we obtain

$$J = \begin{bmatrix} a_1 - \eta e^{-\lambda\tau_1} & r & -\rho x^* \\ \eta e^{-\lambda\tau_1} & a_2 & a_3 \\ c_1\rho z^* e^{-\lambda\tau_2} & a_4 e^{-\lambda\tau_2} & a_5 + a_6 e^{-\lambda\tau_2} \end{bmatrix},$$

where

$a_1 = -d - \rho z^*$, $a_2 = s - \frac{2sy^*}{K} - \frac{\alpha z^*}{(1+ay^*)^2(1+bz^*)}$, $a_3 = \frac{\alpha y^*}{(1+ay^*)(1+bz^*)^2}$,
$a_4 = \frac{c_2\alpha z^*}{(1+ay^*)^2(1+bz^*)}$, $a_5 = -\delta_0 - 2\delta_1 z^*$, $a_6 = c_1\rho x^* + c_2 a_3$, $\eta = re^{-d\tau_1}$,

and its corresponding characteristic equation is

$$\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 + (\lambda^2 + b_4\lambda + b_5)\eta e^{-\lambda\tau_1} + (b_6\lambda^2 + b_7\lambda + b_8)e^{-\lambda\tau_2} + (b_6\lambda + b_9)\eta e^{-\lambda(\tau_1+\tau_2)} = 0, \tag{5.12}$$

where

$b_1 = -(a_1 + a_2 + a_5)$, $b_2 = a_1a_2 + a_2a_5 + a_1a_5$, $b_3 = -a_1a_2a_5$, $b_4 = -(r + a_2 + a_5)$,
$b_5 = a_2a_5 + ra_5$, $b_6 = -a_6$, $b_7 = a_2a_6 - a_3a_4 + a_1a_6 + c_1\rho^2 x^* z^*$,
$b_8 = -a_1a_2a_6 + a_1a_3a_4 - rc_1\rho a_3 z^* - c_1\rho^2 a_2 x^* z^*$, $b_9 = a_2a_6 - a_3a_4 + ra_6 + \rho a_4 x^*$.

Remark 5.4.1. *When $\tau_1 = \tau_2 = 0$, then the characteristic equation (5.12) is same as the characteristic equation (5.10) of the non-delayed model system (5.2) studied earlier.*

Case (1): $\tau_1 = 0$, $\tau_2 > 0$. Then equation (5.12) becomes

$$\lambda^3 + d_1\lambda^2 + d_2\lambda + d_3 + (d_4\lambda^2 + d_5\lambda + d_6)e^{-\lambda\tau_2} = 0, \tag{5.13}$$

where

$d_1 = b_1 + r$, $d_2 = b_2 + rb_4$, $d_3 = b_3 + rb_5$ $d_4 = b_6$, $d_5 = b_7 + rb_6$, $d_6 = b_8 + rb_9$.

Let $i\omega$ $(\omega > 0)$ be a root of equation (5.13), then it follows that

$$\begin{aligned}(-d_4\omega^2+d_6)\cos(\omega\tau_2)+d_5\omega\sin(\omega\tau_2)&=d_1\omega^2-d_3,\\ d_5\omega\cos(\omega\tau_2)-(-d_4\omega^2+d_6)\sin(\omega\tau_2)&=\omega^3-d_2\omega.\end{aligned} \tag{5.14}$$

This leads to a six order equation in ω and has at most six real positive roots, which ensures that there are only finite intervals for the roots to cross the imaginary axis.

$$\omega^6+p_1\omega^4+p_2\omega^2+p_3=0, \tag{5.15}$$

where
$p_1=d_1^2-d_4^2-2d_2,\ p_2=d_2^2-d_5^2-2d_1d_3+2d_4d_6,\ p_3=d_3^2-d_6^2.$
If we put $\omega^2=v$, then equation (5.15) becomes

$$v^3+p_1v^2+p_2v+p_3=0.$$

Let $g(v)=v^3+p_1v^2+p_2v+p_3$.

Without loss of generality it is assumed that ω_1 is a positive root of equation (5.15). Substituting ω_1 into equation (5.14), we obtain

$$\tau_{2_i}=\frac{1}{\omega_1}\cos^{-1}\left[\frac{(d_1\omega_1^2-d_3)(-d_4\omega_1^2+d_6)+(\omega_1^3-d_2\omega_1)\omega_1d_5}{(-d_4\omega_1^2+d_6)^2+\omega_1^2{d_5}^2}\right]+\frac{2i\pi}{\omega_1},\ i=0,1,2\ldots. \tag{5.16}$$

$(H_1):\ g'(\omega_1^2)\neq 0.$
Let $\lambda(\tau_{2_i})=\pm i\omega_1$ be the root of equation (5.13), a little calculation yields

$$Re\left[\frac{d\lambda}{d\tau_2}\right]^{-1}_{\lambda=i\omega_1,\ \tau_2=\tau_{2_i}}=\frac{g'(\omega_1^2)}{(-d_4\omega_1^2+d_6)^2+\omega_1^2{d_5}^2}>0.$$

But sign of $\left[\frac{d}{d\tau_2}Re(\lambda)\right]_{\lambda=i\omega_1,\ \tau_2=\tau_{2_i}}$ is same as the sign of $\left[Re\left(\frac{d\lambda}{d\tau_2}\right)\right]_{\lambda=i\omega_1,\ \tau_2=\tau_{2_i}}$.
Hence, the transversality condition can be obtained under (H_1)

$$\left[\frac{d}{d\tau_2}(Re(\lambda))\right]_{\tau_2=\tau_{2_i}} > 0,$$

Thus, we can state the following theorem.

Theorem 5.4.1. *For system (5.1), with $\tau_1=0$ and assuming that (H_1) holds, there exists a positive number τ_{2_0} such that the equilibrium E^* is locally asymptotically stable when $\tau_2 < \tau_{2_0}$*

and unstable when $\tau_2 > \tau_{2_0}$. *Furthermore system (5.1) undergoes a Hopf-bifurcation at* E^* *when* $\tau_2 = \tau_{2_0}$.

Case (2): $\tau_1 > 0,\ \tau_2 = 0$. Then equation (5.12) becomes

$$\lambda^3 + e_1\lambda^2 + e_2\lambda + e_3 + (e_4\lambda^2 + e_5\lambda + e_6)e^{-\lambda\tau_1} = 0,$$

where
$e_1 = b_1 + b_6,\ e_2 = b_2 + b_7,\ e_3 = b_3 + b_8,\ e_4 = \eta,\ e_5 = \eta(b_4 + b_6),\ e_6 = \eta(b_5 + b_9)$.
Again above characteristic equation can be written as

$$P(\lambda,\tau_1) + Q(\lambda,\tau_1)e^{-\lambda\tau_1} = 0, \tag{5.17}$$

where
$P(\lambda,\tau_1) = \lambda^3 + e_1\lambda^2 + e_2\lambda + e_3,\ Q(\lambda,\tau_1) = e_4\lambda^2 + e_5\lambda + e_6$.

The necessary condition for a stability change of positive equilibrium E^* is that the characteristic equation (5.17) has purely imaginary roots. It is also noted that $P(\lambda,\tau_1)$ and $Q(\lambda,\tau_1)$ are delay-dependent terms as the positive equilibrium is function of delay. Therefore, it is a little difficult to handle the characteristic equation with delay-dependent coefficients. The main complication germinates during investigation of the existence of a purely imaginary root $\lambda = i\omega$ of (5.17).

Here, we follow the technique developed by Berreta and Kuang [14]. Let $\tau_{1_{max}}$ be the maximum value of τ_1 for which the positive equilibrium E^* exists. For $\tau_1 \in [0, \tau_{1_{max}}]$, we assume the following:

- $P(0,\tau_1) + Q(0,\tau_1) = e_3 + e_6 \neq 0$.
- $P(i\omega,\tau_1) + Q(i\omega,\tau_1) = e_3 + e_6 + \omega^2(e_1 + e_4) - i(\omega^3 - e_2\omega - e_5\omega) \neq 0$.
- $$\lim_{|\lambda|\to\infty}\left|\frac{Q(\lambda,\tau_1)}{P(\lambda,\tau_1)}\right| = \lim_{|\lambda|\to\infty}\frac{e_4\lambda^2 + e_5\lambda + e_6}{\lambda^3 + e_1\lambda^2 + e_2\lambda + e_3} = 0 < 1.$$
- $F(\omega,\tau_1) = |P(i\omega,\tau_1)|^2 - |Q(i\omega,\tau_1)|^2$ is a polynomial of degree 6. Therefore, it has finite number of positive roots.
- Each positive root of $F(\omega,\tau_1) = 0$ is continuous and differentiable in τ_1 whenever it exists (by implicit function theorem).

Let $\lambda(\tau_1) = \xi(\tau_1) + i\omega(\tau_1)$ be an eigenvalue of the system. Then stability will switch for $Re(\lambda) = 0$. Therefore, we substitute $\lambda = i\omega$ in (5.17) and separating the real and imaginary parts, we obtain

$$\begin{aligned} (e_4\omega^2 - e_6)\cos(\omega\tau_1) - e_5\omega\sin(\omega\tau_1) &= -e_1\omega^2 + e_3, \\ e_5\omega\cos(\omega\tau_1) + (e_4\omega^2 - e_6)\sin(\omega\tau_1) &= \omega^3 - e_2\omega, \end{aligned} \tag{5.18}$$

which gives

$$\sin(\omega\tau_1) = \frac{e_5\omega(e_1\omega^2 - e_3) + (e_4\omega^2 - e_6)(\omega^3 - e_2\omega)}{e_5^2\omega^2 + (e_6 - e_4\omega^2)^2}, \tag{5.19}$$

$$\cos(\omega\tau_1) = \frac{e_5\omega(\omega^3 - e_2\omega) - (e_4\omega^2 - e_6)(e_1\omega^2 - e_3)}{e_5^2\omega^2 + (e_6 - e_4\omega^2)^2}. \tag{5.20}$$

Squaring and adding both the equations of (5.18), we get

$$h(\omega, \tau_1) := \omega^6 + q_1\omega^4 + q_2\omega^2 + q_3 = 0, \tag{5.21}$$

where
$q_1 = e_1^2 - 2e_2 - e_4^2,\ q_2 = e_2^2 - e_5^2 - 2e_1e_3 + 2e_4e_6,\ q_3 = e_3^2 - e_6^2.$

Thus, ω can be evaluated from equation (5.21) in terms of τ_1. For each τ_1, equation (5.21) has at most six real positive roots, which indicates that the roots can cross the imaginary axis at most six times.

Now, we consider $I = \{\tau_1 : \tau_1 > 0 \text{ and } \omega(\tau_1) \text{ is positive root of } (5.21)\}$. So, if $\tau_1 \in I^c$, then there is no positive solution of (5.21) and thus no stability changes occur (Chakraborty *et al.*[26]).

Now, for any $\tau_1 \in I$, we define $\theta(\tau_1) \in (0, 2\pi)$ as the solution of (5.21). Then we can write

$$\sin(\theta(\tau_1)) = \frac{e_5\omega(e_1\omega^2 - e_3) + (e_4\omega^2 - e_6)(\omega^3 - e_2\omega)}{e_5^2\omega^2 + (e_6 - e_4\omega^2)^2} = \frac{\phi}{|Q(i\omega, \tau_1)|^2}, \tag{5.22}$$

$$\cos(\theta(\tau_1)) = \frac{e_5\omega(\omega^3 - e_2\omega) - (e_4\omega^2 - e_6)(e_1\omega^2 - e_3)}{e_5^2\omega^2 + (e_6 - e_4\omega^2)^2} = \frac{\psi}{|Q(i\omega, \tau_1)|^2}, \tag{5.23}$$

where ϕ and ψ are continuous and differentiable functions of τ_1 such that $\phi^2 + \psi^2 = |P(i\omega, \tau_1)|^4$ and $|Q(i\omega, \tau_1)|^2 = |P(i\omega, \tau_1)|^2$. Substituting $\omega = \omega(\tau_1)$ in (5.22) and (5.23), $\theta(\tau_1) \in (0, 2\pi)$

can be determined as follows:

$$\theta(\tau_1)=\begin{cases}\tan^{-1}\frac{\phi}{\psi}, & \text{if } \sin(\theta(\tau_1))>0,\ \cos(\theta(\tau_1))>0;\\ \frac{\pi}{2}, & \text{if } \sin(\theta(\tau_1))=1,\ \cos(\theta(\tau_1))=0;\\ \pi+\tan^{-1}\frac{-\phi}{\psi}, & \text{if } \cos(\theta(\tau_1))<0;\\ \frac{3\pi}{2}, & \text{if } \sin(\theta(\tau_1))=-1,\ \cos(\theta(\tau_1))=0;\\ 2\pi+\tan^{-1}\frac{\phi}{\psi}, & \text{if } \sin(\theta(\tau_1))<0.\end{cases}$$

Here, we notice that for $\tau_1 \in I$, $\theta(\tau_1)$ defined as above is continuous at τ_1. Furthermore if $\theta(\tau_1) \in (0,2\pi), \tau_1 \in I$, then $\theta(\tau_1)$ is also differentiable at τ_1. Now from (5.19), (5.20) and above, we have $\omega(\tau_1)\tau_1 = \theta(\tau_1)+2n\pi$, $n \in N_0$. We can define the maps $\tau_{1_n}(\tau_1) = \frac{1}{\omega(\tau_1)}(\theta(\tau_1)+2n\pi)$, $n \in N_0$, $\tau_1 \in I$, where $\omega(\tau_1)$ is a positive root of $h(\omega,\tau_1)=0$.

Let us introduce the function $S_n : I \to R$ given by

$$S_n = \tau_1 - \tau_{1_n}(\tau_1),\ \ \tau_1 \in I,\ n \in N_0,$$

which are continuous and differentiable at τ_1. It can be noted that the values of τ_1 $(\in I)$, $S_n(\tau_1) = 0$, $n \in N_0$, consequently stability switches under the transversality condition. Transversality condition is similar to Pathak *et al.* [166] and Chakraborty *et al.* [26]. Thus, we can now state the following theorems.

Theorem 5.4.2. *Let $\omega(\tau_1)$ be the positive root of equation (5.17) for $\tau_1 \in I$ and for some $\tau_{1_0} \in I$ and $n \in N_0$, $S_n(\tau_{1_0}) = 0$. Then a pair of simple conjugate pure imaginary root $\lambda_\pm(\tau_{1_0}) = \pm i\omega(\tau_{1_0})$ of (5.17) exists at $\tau_1 = \tau_{1_0}$, which crosses the imaginary axis from left to right if $\sigma(\tau_{1_0}) > 0$ and crosses the imaginary axis from right to left if $\sigma(\tau_{1_0}) < 0$, where*

$$\sigma(\tau_1) = sign\left\{\frac{d}{d\tau_1}Re(\lambda)\Big|_{\lambda=i\omega}\right\}.$$

Theorem 5.4.3. *Let $A_1 > 0$, $A_3 > 0$ and $A_1A_2 > A_3$. Further, let $\omega(\tau_1)$ be a positive root of (5.17) defined for $\tau_1 \in I$ and for some $\tau_{1_0} \in I$ and $n \in N_0$, $S_n(\tau_{1_0}) = 0$. Then the system (5.1) exhibits a Hopf-bifurcation near the positive equilibrium E^*, provided $\sigma(\tau_{1_0}) \neq 0$. As τ_1 increases from zero to positive direction the system would be stable to unstable, and then unstable back to stable. But this pattern would be repeated at most six times.*

Case (3): τ_1 is fixed in the interval $(0,\tau_{1_0})$ and $\tau_2 > 0$.
We consider equation (5.12) with τ_1 as fixed in its stable interval $(0,\tau_{1_0})$ and τ_2 as a variable parameter. Let $i\omega(\omega > 0)$ be a root of characteristic equation (5.12). Separating real and

imaginary parts, we have

$$-b_1\omega^2+b_3+(-\omega^2+b_5)\eta\cos(\omega\tau_1)+b_4\eta\omega\sin(\omega\tau_1)=\sin(\omega\tau_2)[b_9\eta\sin(\omega\tau_1) -b_6\eta\omega\cos(\omega\tau_1)-b_7\omega]+\cos(\omega\tau_2)[-b_9\eta\cos(\omega\tau_1)-b_6\eta\omega\sin(\omega\tau_1)-(b_6\omega^2+b_8)], \tag{5.24}$$

$$-\omega^3+b_2\omega-(\omega^2-b_5)\eta\sin(\omega\tau_1)+b_4\eta\omega\cos(\omega\tau_1)=\sin(\omega\tau_2)[b_9\eta\cos(\omega\tau_1) +b_6\eta\omega\sin(\omega\tau_1)+(-b_6\omega^2+b_8)]+\cos(\omega\tau_2)[b_9\eta\sin(\omega\tau_1)-b_6\eta\omega\cos(\omega\tau_1)-b_7\omega]. \tag{5.25}$$

Squaring and adding (5.24) and (5.25) to eliminate τ_2, we obtain

$$f_1^2+f_2^2+f_3^2+f_4^2-f_5^2-f_6^2-f_7^2-f_8^2+2(f_1f_2+f_3f_4-f_5f_7-f_6f_8)\cos(\omega\tau_1) +2(f_1f_3-f_2f_4-f_6f_7+f_5f_8)\sin(\omega\tau_1)=0, \tag{5.26}$$

where
$f_1=-b_1\omega^2+b_3,\ f_2=(-\omega^2+b_5)\eta,\ f_3=b_4\eta\omega,\ f_4=-\omega^3+b_2\omega,$
$f_5=b_9\eta,\ f_6=b_6\eta\omega,\ f_7=-b_6\omega^2+b_8,\ f_8=b_7\omega.$

Equation (5.26) is a transcendental equation in complicated form. It is not easy to predict about the nature of roots. Without going detailed analysis with (5.26) it is assumed that there exist at least one positive root ω_0. Equation (5.24) and (5.25) can be re-written as

$$A_1\sin(\omega_0\tau_2)-B_1\cos(\omega_0\tau_2)=C_1, \tag{5.27}$$

$$B_1\sin(\omega_0\tau_2)+A_1\cos(\omega_0\tau_2)=C_2, \tag{5.28}$$

where
$A_1=b_9\eta\sin(\omega_0\tau_1)-b_6\eta\omega_0\cos(\omega_0\tau_1)-b_7\omega_0,$
$B_1=b_9\eta\cos(\omega_0\tau_1)+b_6\eta\omega_0\sin(\omega_0\tau_1)+(-b_6\omega_0^2+b_8),$
$C_1=-b_1\omega_0^2+b_3+(-\omega_0^2+b_5)\eta\cos(\omega_0\tau_1)+b_4\eta\omega_0\sin(\omega_0\tau_1),$
$C_2-\omega_0^3+b_2\omega_0-(-\omega_0^2+b_5)\eta\sin(\omega_0\tau_1)+b_4\eta\omega_0\cos(\omega_0\tau_1).$

Equation (5.27) and (5.28) leads to

$$\tau'_{2_j}=\frac{1}{\omega_0}\cos^{-1}\left[\frac{A_1C_2-B_1C_1}{A_1^2+B_1^2}\right]+\frac{2j\pi}{\omega_0},\ j=0,1,2,\dots. \tag{5.29}$$

Now we will verify the transversality condition of Hopf-bifurcation. Differentiating equation (5.24) and (5.25) with respect to τ_2 and substitute $\tau_2 = \tau'_{2_0}$, we obtain

$$\begin{aligned} P\left[\frac{d(Re\,\lambda)}{d\tau_2}\right]_{\tau_2=\tau'_{2_0}} + Q\left[\frac{d\omega_0}{d\tau_2}\right]_{\tau_2=\tau'_{2_0}} &= R, \\ -Q\left[\frac{d(Re\,\lambda)}{d\tau_2}\right]_{\tau_2=\tau'_{2_0}} + P\left[\frac{d\omega_0}{d\tau_2}\right]_{\tau_2=\tau'_{2_0}} &= S, \end{aligned} \tag{5.30}$$

where

$$\begin{aligned} P = &-3\omega_0^2 + b_2 + (b_4 + \omega_0\tau_1 - b_5\tau_1)\eta\cos(\omega_0\tau_1) + (2 - b_4\tau_1)\eta\omega_0\sin(\omega_0\tau_1) + (b_7 + b_6\omega_0^2\tau'_{2_0} \\ &- b_8\tau'_{2_0})\cos(\omega_0\tau'_{2_0}) + (2b_6 - b_7\tau'_{2_0})\omega_0\sin(\omega_0\tau'_{2_0}) + b_6\eta\cos(\omega_0(\tau_1 + \tau'_{2_0})) - b_9\eta(\tau_1 + \tau'_{2_0}) \\ &\cos(\omega_0(\tau_1 + \tau'_{2_0})) - b_6\eta\omega_0(\tau_1 + \tau'_{2_0})\sin(\omega_0(\tau_1 + \tau'_{2_0})), \\ Q = &-2b_1\omega_0 - (2 - b_4\tau_1)\eta\omega_0\cos(\omega_0\tau_1) + (b_4 + \omega_0\tau_1 - b_5\tau_1)\eta\sin(\omega_0\tau_1) + (2b_6 - b_7\tau'_{2_0}) \\ &\omega_0\cos(\omega_0\tau'_{2_0}) + (b_7 + b_6\omega_0^2\tau'_{2_0} - b_8\tau'_{2_0})\sin(\omega_0\tau'_{2_0}) - b_9\eta(\tau_1 + \tau'_{2_0})\sin(\omega_0(\tau_1 + \tau'_{2_0})) \\ &+ b_6\eta\omega_0(\tau_1 + \tau'_{2_0})\cos(\omega_0(\tau_1 + \tau'_{2_0})) + b_6\eta\sin(\omega_0(\tau_1 + \tau'_{2_0})), \\ R = &(b_8 - b_6\omega_0^2)\omega_0\sin(\omega_0\tau'_{2_0}) - b_7\omega_0^2\cos(\omega_0\tau'_{2_0}) + b_9\eta\omega_0\sin(\omega_0(\tau_1 + \tau'_{2_0})) \\ &- b_6\eta\omega_0^2\cos(\omega_0(\tau_1 + \tau'_{2_0})), \\ S = &(b_8 - b_6\omega_0^2)\omega_0\cos(\omega_0\tau'_{2_0}) + b_7\omega_0^2\sin(\omega_0\tau'_{2_0}) - b_9\eta\omega_0\cos(\omega_0(\tau_1 + \tau'_{2_0})) \\ &+ b_6\eta\omega_0^2\sin(\omega_0(\tau_1 + \tau'_{2_0})). \end{aligned}$$

Solving equation (5.30) for $\left[\frac{d(Re\,\lambda)}{d\tau_2}\right]_{\tau_2=\tau'_{2_0}}$, it is obtained

$$\left[\frac{d(Re\,\lambda)}{d\tau_2}\right]_{\tau_2=\tau'_{2_0},\ \lambda=\iota\omega_0} = \frac{PR - QS}{P^2 + Q^2}.$$

$(H_2): PR - QS \neq 0.$

Theorem 5.4.4. *For system (5.1), with $\tau_1 \in (0, \tau_{1_0})$ and assuming that (H_2) holds, there exists a positive number τ'_{2_0} such that E^* is locally asymptotically stable when $\tau_2 < \tau'_{2_0}$ and unstable when $\tau_2 > \tau'_{2_0}$. Furthermore, system (5.1) undergoes a Hopf-bifurcation at E^* where $\tau_2 = \tau'_{2_0}$.*

Case (4): τ_2 fixed in the interval $(0, \tau_{2_0})$ and $\tau_1 > 0$.
The characteristic equation of the system (5.1) at $E^*(x^*, y^*, z^*)$ is

$$P_1(\lambda, \tau_1) + Q_1(\lambda, \tau_1) = 0, \tag{5.31}$$

where
$P_1(\lambda,\tau_1) = \lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 + (b_6\lambda^2 + b_7\lambda + b_8)e^{-\lambda\tau_2}$,
$Q_1(\lambda,\tau_1) = [\lambda^2 + b_4\lambda + b_5 + (b_6\lambda + b_9)e^{-\lambda\tau_2}]\eta$.

Here, we notice that $P_1(\lambda,\tau_1)$ and $Q_1(\lambda,\tau_1)$ are delay dependent as the interior equilibrium components x^* and y^* having delay term. Similar to Case (2), let τ_{1_m} be the maximum value of τ_1 for which the interior equilibrium E^* exists. For $\tau_1 \in [0, \tau_{1_m}]$, we assume

- $P_1(0,\tau_1) + Q_1(0,\tau_1) = b_3 + b_8 + (b_5 + b_9)\eta \neq 0.$
- $P_1(i\omega,\tau_1) + Q_1(i\omega,\tau_1) = -(b_1+\eta)\omega^2 + b_5\eta + b_3 + (-b_6\omega^2 + b_9\eta + b_8)\cos(\omega\tau_2) + (b_7 + b_6\eta)\omega\sin(\omega\tau_2) - i(-\omega^3 + b_4\eta\omega + b_2\omega - (b_6\omega^2 + b_6\eta\omega + b_8)\sin(\omega\tau_2) + (b_7\omega + b_9\eta)\cos(\omega\tau_2)) \neq 0.$
- $$\lim_{|\lambda|\to\infty}\left|\frac{Q_1(\lambda,\tau_1)}{P_1(\lambda,\tau_1)}\right| = 0 < 1.$$
- $$F'(\omega,\tau_1) = |P_1(i\omega,\tau_1)|^2 - |Q_1(i\omega,\tau_1)|^2 = pol(\omega) + T,$$

  where $pol(\omega)$ is a polynomial of degree 6 in ω and T is a trigonometric function. Therefore, it has finite number of positive roots.
- Each positive root of $F'(\omega,\tau_1) = 0$ is continuous and differentiable in τ_1 whenever it exists (by implicit function theorem).

Let $\lambda(\tau_1) = \xi(\tau_1) + i\omega(\tau_1)$ be an eigenvalue of the system. It change the stability when $Re(\lambda) = 0$. Therefore, we substitute $\lambda = i\omega$ in (5.31) and separating the real and imaginary parts, we obtain

$$A_2\sin(\omega\tau_1) - B_2\cos(\omega\tau_1) = D_1, \tag{5.32}$$

$$B_2\sin(\omega\tau_1) + A_2\cos(\omega\tau_1) = D_2, \tag{5.33}$$

where
$A_2 = b_9\eta\sin(\omega\tau_2) - b_6\eta\omega\cos(\omega\tau_2) - b_4\eta\omega$,
$B_2 = b_9\eta\cos(\omega\tau_2) + b_6\eta\omega\sin(\omega\tau_2) - (-\omega^2 + b_5)\eta$,
$D_1 = -b_1\omega^2 + b_3 + (-b_6\omega^2 + b_8)\cos(\omega\tau_2) + b_7\omega\sin(\omega\tau_2)$,
$D_2 = -\omega^3 + b_2\omega - (-b_6\omega^2 + b_8)\sin(\omega\tau_2) + b_7\omega\cos(\omega\tau_2)$,
which gives

$$sin(\omega\tau_1) = \frac{A_2D_1 + B_2D_2}{A_2^2 + B_2^2}, \quad cos(\omega\tau_1) = \frac{A_2D_2 - B_2D_1}{A_2^2 + B_2^2}.$$

Squaring and adding equations (5.32) and (5.33), we get

$$A_2^2 + B_2^2 - D_1^2 - D_2^2 = 0. \tag{5.34}$$

Thus, ω can be evaluated from equation (5.34) in terms of τ_1. For each τ_1, equation (5.34) has at most six real positive roots, which indicates that the roots can cross the imaginary axis at most six times.

Further analysis can be carried out as in Case (2).

5.5 Direction and Stability of Hopf-bifurcation

In previous section, we have obtained the condition under which the system undergoes Hopf-bifurcation for the different combinations of the delay τ_1 and τ_2. In this section we will determine the direction, stability and period of the bifurcated periodic solutions at $\tau_1 = \tau'_{1_0}$ using center manifold theorem and normal form theory following the concept of Hassard *et al.* [80].

Without loss of generality, we assume that $\tau_2^* < \tau'_{1_0}$, where $\tau_2^* \in (0, \tau_{2_0})$. Let

$$x_1(t) = x(t) - x^*,\ y_1(t) = y(t) - y^*,\ z_1(t) = z(t) - z^*,$$

and still denote $x_1(t), y_1(t), z_1(t)$ by $x(t), y(t), z(t)$. Let $\tau_1 = \tau'_{1_0} + \mu,\ \mu \in R$ so that Hopf-bifurcation occurs at $\mu = 0$. We normalize the delay with scaling $t \mapsto (\frac{t}{\tau_1})$, then system (5.1) can be re-written as

$$\dot{U}(t) = \tau_1\left(P'U(t) + R'U\left(t - \frac{\tau_2^*}{\tau_1}\right) + Q'U(t-1) + f'(x,y,z)\right), \tag{5.35}$$

where $U(t) = (x(t), y(t), z(t))^T$

$$P' = \begin{bmatrix} a_1 & r & -\rho x^* \\ 0 & a_2 & a_3 \\ 0 & 0 & -a_5 \end{bmatrix},\ Q' = \begin{bmatrix} -\eta & 0 & 0 \\ \eta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ R' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ c_1\rho z^* & a_4 & a_6 \end{bmatrix},\ f'(x,y,z) = \begin{bmatrix} f'_1 \\ f'_2 \\ f'_3 \end{bmatrix}.$$

The nonlinear term f'_1, f'_2 and f'_3 are given by

$$f'_1 = -\rho x(t)y(t)\ ,$$

$$f_2' = -\frac{2s}{K}y^2(t) + \frac{2\alpha a z^*}{(1+ay^*)^3(1+bz^*)}y^2(t) + \frac{2\alpha b y^*}{(1+ay^*)(1+bz^*)^3}z^2(t) - \frac{\alpha y(t)z(t)}{(1+ay^*)^2(1+bz^*)^2}$$
$$- \frac{6\alpha a^2 z^*}{(1+ay^*)^4(1+bz^*)}y^3(t) - \frac{6\alpha b^2 y^*}{(1+ay^*)(1+bz^*)^4}z^3(t) + \frac{2\alpha a}{(1+ay^*)^3(1+bz^*)^2}y^2(t)z(t)$$
$$+ \frac{2\alpha b}{(1+ay^*)^2(1+bz^*)^3}y(t)z^2(t) + \dots,$$

$$f_3' = -2\delta_1 z^2(t) - \frac{2c_2\alpha a z^*}{(1+ay^*)^3(1+bz^*)}y^2\Big(t-\frac{\tau_2^*}{\tau_1}\Big) - \frac{2c_2\alpha b y^*}{(1+ay^*)(1+bz^*)^3}z^2\Big(t-\frac{\tau_2^*}{\tau_1}\Big)$$
$$+ \frac{c_2\alpha}{(1+ay^*)^2(1+bz^*)^2}y\Big(t-\frac{\tau_2^*}{\tau_1}\Big)z\Big(t-\frac{\tau_2^*}{\tau_1}\Big) + \frac{6c_2\alpha b^2 y^*}{(1+ay^*)(1+bz^*)^4}z^3\Big(t-\frac{\tau_2^*}{\tau_1}\Big)$$
$$+ \frac{6c_2\alpha a^2 z^*}{(1+ay^*)^4(1+bz^*)}y^3\Big(t-\frac{\tau_2^*}{\tau_1}\Big) - \frac{2\alpha a}{(1+ay^*)^3(1+bz^*)^2}y^2\Big(t-\frac{\tau_2^*}{\tau_1}\Big)z\Big(t-\frac{\tau_2^*}{\tau_1}\Big)$$
$$- \frac{2\alpha b}{(1+ay^*)^2(1+bz^*)^3}y\Big(t-\frac{\tau_2^*}{\tau_1}\Big)z^2\Big(t-\frac{\tau_2^*}{\tau_1}\Big) + \dots.$$

The linearization of equation (5.35) around the origin is given by

$$\dot{U}(t) = \tau_1\big(P'U(t) + R'U\big(t-\frac{\tau_2^*}{\tau_1}\big) + Q'U(t-1)\big).$$

For $\phi = (\phi_1, \phi_2, \phi_3)^T \in C([-1,0], R^3)$, define

$$L_\mu(\phi) = (\tau_1+\mu)\big(P'\phi(0) + R'\phi\big(-\frac{\tau_2^*}{\tau_1}\big) + Q'\phi(-1)\big).$$

By the Riesz representation theorem, there exists a 3×3 matrix $\zeta(\theta,\mu)$, $(-1 \le \theta \le 0)$ whose element are of bounded variation function such that

$$L_\mu(\phi) = \int_{-1}^{0} d\zeta(\theta,\mu)\phi(\theta) \;\; for\; \phi \in C([-1,0], R^3). \tag{5.36}$$

In fact, we can obtain

$$\zeta(\theta,\mu) = \begin{cases} (\tau'_{1_0}+\mu)(P'+R'+Q'), & \text{if } \theta = 0 \\ (\tau'_{1_0}+\mu)(R'+Q'), & \text{if } \theta \in \big[-\frac{\tau_2^*}{\tau_1}, 0\big) \\ (\tau'_{1_0}+\mu)Q', & \text{if } \theta \in \big(-1, -\frac{\tau_2^*}{\tau_1}\big) \\ 0, & \text{if } \theta = -1. \end{cases}$$

Then equation (5.36) is satisfied.
For $\phi \in C^1([-1,0],R^3)$, define the operator $A(\mu)$ as

$$A(\mu)\phi(\theta)=\begin{cases}\frac{d\phi(\theta)}{d\theta}, & \text{if } \theta\in[-1,0)\\ \int_{-1}^{0}[d\zeta(\xi,\mu)]\phi(\xi), & \text{if } \theta=0,\end{cases}$$

and

$$R(\mu)\phi(\theta)=\begin{cases}0, & \text{if } \theta\in[-1,0)\\ h'(\mu,\phi), & \text{if } \theta=0,\end{cases}$$

where

$$h'(\mu,\phi)=(\tau'_{1_0}+\mu)\begin{bmatrix}h'_1\\ h'_2\\ h'_3\end{bmatrix},\ \phi=(\phi_1,\phi_2,\phi_3)^T\in C([-1,0],R^3),$$

$$h'_1=-\rho x(0)y(0)\ ,$$

$$\begin{aligned}h'_2=&-\frac{2s}{K}y^2(0)+\frac{2\alpha a z^*}{(1+ay^*)^3(1+bz^*)}y^2(0)+\frac{2\alpha b y^*}{(1+ay^*)(1+bz^*)^3}z^2(0)-\frac{\alpha y(0)z(0)}{(1+ay^*)^2(1+bz^*)^2}\\&-\frac{6\alpha a^2 z^*}{(1+ay^*)^4(1+bz^*)}y^3(0)-\frac{6\alpha b^2 y^*}{(1+ay^*)(1+bz^*)^4}z^3(0)+\frac{2\alpha a}{(1+ay^*)^3(1+bz^*)^2}y^2(0)z(0)\\&+\frac{2\alpha b}{(1+ay^*)^2(1+bz^*)^3}y(0)z^2(0)+\dots,\end{aligned}$$

$$\begin{aligned}h'_3=&-2\delta_1 z^2(0)-\frac{2c_2\alpha a z^*}{(1+ay^*)^3(1+bz^*)}y^2\Big(-\frac{\tau_2^*}{\tau_1}\Big)-\frac{2c_2\alpha b y^*}{(1+ay^*)(1+bz^*)^3}z^2\Big(-\frac{\tau_2^*}{\tau_1}\Big)\\&+\frac{c_2\alpha}{(1+ay^*)^2(1+bz^*)^2}y\Big(-\frac{\tau_2^*}{\tau_1}\Big)z\Big(-\frac{\tau_2^*}{\tau_1}\Big)+\frac{6c_2\alpha b^2 y^*}{(1+ay^*)(1+bz^*)^4}z^3\Big(-\frac{\tau_2^*}{\tau_1}\Big)\\&+\frac{6c_2\alpha a^2 z^*}{(1+ay^*)^4(1+bz^*)}y^3\Big(-\frac{\tau_2^*}{\tau_1}\Big)-\frac{2\alpha a}{(1+ay^*)^3(1+bz^*)^2}y^2\Big(-\frac{\tau_2^*}{\tau_1}\Big)z\Big(-\frac{\tau_2^*}{\tau_1}\Big)\\&-\frac{2\alpha b}{(1+ay^*)^2(1+bz^*)^3}y\Big(-\frac{\tau_2^*}{\tau_1}\Big)z^2\Big(-\frac{\tau_2^*}{\tau_1}\Big)+\dots.\end{aligned}$$

Then system (5.1) is equivalent to the following operator equation

$$\dot{U}_t=A(\mu)U_t+R(\mu)U_t\ ,$$

where $U_t=U(t+\theta)$ for $\theta\in[-1,0]$.
For $\psi\in C^1([0,1],(R^3)^*)$, define

$$A^*\psi(s_1)=\begin{cases}-\frac{d\psi(s_1)}{ds_1}, & \text{if } s_1\in(0,1])\\ \int_{-1}^{0}\psi(-\xi)d\zeta(\xi,0), & \text{if } s_1=0,\end{cases}$$

and a bilinear form

$$\langle \psi(s_1), \phi(\theta) \rangle = \overline{\psi}(0)\phi(0) - \int_{-1}^{0} \int_{\xi=0}^{\theta} \overline{\psi}(\xi - \theta) d\zeta(\theta) \phi(\xi) d\xi \ ,$$

where $\zeta(\theta) = \zeta(\theta, 0)$, $A = A(0)$ and A^* are adjoint operators. From the discussion in previous section, we know that $\pm i\omega_0 \tau'_{1_0}$ are the eigenvalues of $A(0)$ and therefore they are also eigenvalues of A^*. It is not difficult to verify that the vectors $q(\theta) = (1, \alpha_1, \beta_1)^T e^{i\omega_0 \tau'_{1_0} \theta}$ $(\theta \in [-1, 0])$ and $q^*(s_1) = \frac{1}{D}(1, \alpha_1^*, \beta_1^*) e^{i\omega_0 \tau'_{1_0} s_1}$ $(s_1 \in [0, 1])$ are the eigenvectors of $A(0)$ and A^* corresponding to the eigenvalue $i\omega_0 \tau'_{1_0}$ and $-i\omega_0 \tau'_{1_0}$ respectively, where

$$\langle q^*(s_1), q(\theta) \rangle = 1, \ \langle q^*(s_1), \overline{q}(\theta) \rangle = 1,$$

$$\alpha_1 = \frac{\eta e^{-i\omega_0 \tau'_{1_0}} + a_3 \beta_1}{i\omega_0 - a_2}, \ \beta_1 = \frac{(i\omega_0 - a_2) c_1 \rho z^* e^{-i\omega_0 \frac{\tau_2^*}{\tau'_{1_0}}} + a_4 \eta e^{-i\omega_0 \left(\tau'_{1_0} + \frac{\tau_2^*}{\tau'_{1_0}}\right)}}{(i\omega_0 - a_2)\left(i\omega_0 + a_5 - a_6 e^{-i\omega_0 \frac{\tau_2^*}{\tau'_{1_0}}}\right) - a_3 a_4 e^{-i\omega_0 \frac{\tau_2^*}{\tau'_{1_0}}}},$$

$$\alpha_1^* = \frac{r + \beta_1^* a_4 e^{-i\omega_0 \frac{\tau_2^*}{\tau'_{1_0}}}}{-i\omega_0 - a_2}, \ \beta_1^* = \frac{\rho x^* (i\omega_0 + a_2) + a_3 r}{(i\omega_0 + a_2)\left(i\omega_0 - a_5 + a_6 e^{-i\omega_0 \frac{\tau_2^*}{\tau'_{1_0}}}\right) - a_3 a_4 e^{-i\omega_0 \frac{\tau_2^*}{\tau'_{1_0}}}},$$

$$\overline{D} = \left[1 + \alpha_1 \overline{\alpha_1^*} + \beta_1 \overline{\beta_1^*} + \tau'_{1_0}\left((-\eta + \overline{\alpha_1^*}\eta) e^{-i\omega_0 \tau'_{1_0}} + \overline{\beta_1^*} \frac{\tau_2^*}{\tau'_{1_0}} (c_1 \rho z^* + \alpha_1 a_4 + \beta_1 a_6) e^{-i\omega_0 \frac{\tau_2^*}{\tau'_{1_0}}}\right)\right].$$

Following the algorithms explained in Hassard *et al.* [80] and using a computation process similar to that in Song and Wei [195], which is used to obtain the properties of Hopf-bifurcation, we obtain

$$g_{20} = -\frac{2\tau'_{1_0}}{\overline{D}} \left[\rho \beta_1 + \overline{\alpha_1^*} \frac{s}{K} \alpha_1^2 + \alpha \overline{\alpha_1^*} \alpha_1 \beta_1 + \overline{\beta_1^*} \delta_1 \beta_1^2 - \overline{\beta_1^*} c_1 \rho \beta_1 e^{-2i\omega_0 \tau_2^*} - \overline{\beta_1^*} c_2 \alpha \alpha_1 \beta_1 e^{-2i\omega_0 \tau_2^*}\right],$$

$$g_{11} = -\frac{\tau'_{1_0}}{\overline{D}} \left[\rho \overline{\beta_1} + \rho \beta_1 + \alpha \alpha_1 \overline{\alpha_1^*} \overline{\beta_1} + \alpha \overline{\alpha_1^*} \beta_1 \overline{\alpha_1} - \overline{\beta_1^*} c_1 \rho (\overline{\beta_1} + \beta_1) - \overline{\beta_1^*} c_2 \alpha (\alpha_1 \overline{\beta_1} + \overline{\alpha_1} \beta_1)\right],$$

$$g_{02} = -\frac{2\tau'_{1_0}}{\overline{D}} \left[\rho \overline{\beta_1} + \overline{\alpha_1^*} \frac{s}{K} \overline{\alpha_1}^2 + \alpha \overline{\alpha_1^*} \overline{\alpha_1} \overline{\beta_1} + \overline{\beta_1^*} \delta_1 \overline{\beta_1}^2 - \overline{\beta_1^*} c_1 \rho \overline{\beta_1} e^{2i\omega_0 \tau_2^*} - \overline{\beta_1^*} c_2 \alpha \overline{\alpha_1} \overline{\beta_1} e^{2i\omega_0 \tau_2^*}\right],$$

$$g_{21} = -\frac{2\tau'_{1_0}}{\overline{D}}\Big[\rho W_{11}^{(3)}(0) + \frac{1}{2}\rho W_{20}^{(3)}(0) + \frac{1}{2}\rho W_{20}^{(1)}(0)\overline{\beta_1} + \rho\beta_1 W_{11}^{(1)}(0) + \overline{\alpha_1^*}\frac{s}{K}\Big\{2\alpha_1 W_{11}^{(0)}(0)$$
$$+ \alpha_1 W_{20}^{(2)}(0)\Big\} + \alpha\overline{\alpha_1^*}\Big\{\alpha_1 W_{11}^{(3)}(0) + \frac{1}{2}\overline{\alpha_1}W_{20}^{(3)}(0) + \frac{1}{2}\overline{\beta_1}W_{20}^{(2)}(0) + \beta_1 W_{11}^{(2)}(0) - \alpha_1^2 a\overline{\beta_1}$$
$$-2a\alpha_1\overline{\alpha_1}\beta_1 - 2b\alpha_1\beta_1\overline{\beta_1} - b\overline{\alpha_1}\beta_1^2\Big\} + \overline{\beta_1^*}\delta_1\Big\{2\beta_1 W_{11}^{(3)}(0) + \overline{\beta_1}W_{20}^{(3)}(0)\Big\} - \overline{\beta_1^*}c_1\rho\Big\{W_{11}^{(3)}\Big(-\frac{\tau_2^*}{\tau'_{1_0}}\Big)$$
$$e^{-i\omega_0\tau_2^*} + \frac{1}{2}W_{20}^{(3)}\Big(-\frac{\tau_2^*}{\tau'_{1_0}}\Big)e^{i\omega_0\tau_2^*} + \frac{1}{2}\overline{\beta_1}W_{20}^{(1)}\Big(-\frac{\tau_2^*}{\tau'_{1_0}}\Big)e^{i\omega_0\tau_2^*} + \beta_1 W_{11}^{(1)}\Big(-\frac{\tau_2^*}{\tau'_{1_0}}\Big)e^{-i\omega_0\tau_2^*}\Big\}$$
$$-\overline{\beta_1^*}c_2\alpha\Big\{\alpha_1 W_{11}^{(3)}\Big(-\frac{\tau_2^*}{\tau'_{1_0}}\Big)e^{-i\omega_0\tau_2^*} + \frac{1}{2}\overline{\alpha_1}W_{20}^{(3)}\Big(-\frac{\tau_2^*}{\tau'_{1_0}}\Big)e^{i\omega_0\tau_2^*} + \frac{1}{2}\overline{\beta_1}W_{20}^{(2)}\Big(-\frac{\tau_2^*}{\tau'_{1_0}}\Big)e^{i\omega_0\tau_2^*}$$
$$+\beta_1 W_{11}^{(2)}\Big(-\frac{\tau_2^*}{\tau'_{1_0}}\Big)e^{-i\omega_0\tau_2^*} - \big(a\alpha_1^2\overline{\beta_1} + 2a\alpha_1\overline{\alpha_1}\beta_1 + 2b\beta_1\overline{\beta_1}\alpha_1 + b\beta_1^2\overline{\alpha_1}\big)e^{-i\omega_0\tau_2^*}\Big\}\Big],$$

where
$W_{20}(\theta) = \frac{ig_{20}}{\omega_0\tau'_{1_0}}q(0)e^{i\omega_0\tau'_{1_0}\theta} + \frac{i\overline{g}_{02}}{3\omega_0\tau'_{1_0}}\overline{q}(0)e^{-i\omega_0\tau'_{1_0}\theta} + E_1 e^{2i\omega_0\tau'_{1_0}\theta}$,
$W_{11}(\theta) = -\frac{ig_{11}}{\omega_0\tau'_{1_0}}q(0)e^{i\omega_0\tau'_{1_0}\theta} + \frac{i\overline{g}_{11}}{\omega_0\tau'_{1_0}}\overline{q}(0)e^{-i\omega_0\tau'_{1_0}\theta} + E_2$,
$E_1 = (E_1^{(1)}, E_1^{(2)}, E_1^{(3)})^T \in R^3$ and $E_2 = (E_2^{(1)}, E_2^{(2)}, E_2^{(3)})^T \in R^3$ are constant vectors, computed as:

$$E_1 = 2\begin{bmatrix} 2i\omega_0 - a_1 + \eta e^{-2i\omega_0\tau'_{1_0}} & -r & \rho x^* \\ -\eta e^{-2i\omega_0\tau'_{1_0}} & 2i\omega_0 - a_2 & -a_3 \\ -c_1\rho z^* e^{-2i\omega_0\tau_2^*} & -a_4 e^{-2i\omega_0\tau_2^*} & 2i\omega_0 + a_5 - a_6 e^{-2i\omega_0\tau_2^*}\end{bmatrix}^{-1} \begin{bmatrix} \rho\beta_1 \\ \frac{s}{K}\alpha_1^2 + \alpha\alpha_1\beta_1 \\ \delta_1\beta_1^2 - c_1\rho\beta_1 e^{-2i\omega_0\tau_2^*} - c_2\alpha\alpha_1\beta_1 e^{-2i\omega_0\tau_2^*}\end{bmatrix},$$

$$E_2 = 2\begin{bmatrix} -a_1 + \eta & -r & \rho x^* \\ -\eta & -a_2 & -a_3 \\ -c_1\rho z^* & -a_4 & -a_6 + a_5\end{bmatrix}^{-1}\begin{bmatrix} \frac{1}{2}\rho(\beta_1 + \overline{\beta_1}) \\ \frac{s}{K}\alpha_1\overline{\alpha_1} + \frac{1}{2}\alpha(\overline{\alpha_1}\beta_1 + \alpha_1\overline{\beta_1}) \\ \delta_1\beta_1\overline{\beta_1} - \frac{1}{2}c_1\rho(\beta_1 + \overline{\beta_1}) - \frac{1}{2}c_2\alpha(\overline{\alpha_1}\beta_1 + \alpha_1\overline{\beta_1})\end{bmatrix}.$$

Consequently, g_{ij} can be expressed by the parameters and delays τ'_{1_0} and τ_2^*. Thus, these standard results can be computed as:

$$c_1(0) = \frac{i}{2\omega_0\tau'_{1_0}}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2}, \quad \mu_2 = -\frac{Re(c_1(0))}{Re(\lambda'(\tau'_{1_0}))},$$

$$\beta_2 = 2Re(c_1(0)), \quad T_2 = -\frac{Im(c_1(0)) + \mu_2 Im(\lambda'(\tau'_{1_0}))}{\omega_0\tau'_{1_0}}.$$

These expressions give a description of the bifurcating periodic solution in the center manifold of system (5.1) at critical values $\tau_1 = \tau_{1_0}$ which can be stated in the form of following theorem:

Theorem 5.5.1.
- *If* $\mu_2 > 0(< 0)$, *then the Hopf-bifurcation is supercritical (subcritical).*
- *If* $\beta_2 > 0(< 0)$, *then the bifurcated periodic solutions are unstable (stable).*
- *If* $T_2 > 0(< 0)$, *then the period increases (decreases) .*

5.6 Numerical Simulation

In this section, we will present some numerical simulations to verify our theoretical findings, obtained in previous section using MATLAB 2018b.

5.6.1 Non-delayed system

For the model (5.2), parameters are chosen as follows:

$$\begin{aligned} &r = 3,\ K = 100,\ s = 4,\ d = 0.7,\ \rho = 0.57,\ \alpha = 1.3,\\ &a = 0.03,\ b = 0.04,\ c_1 = 0.67,\ c_2 = 0.84,\ \delta_0 = 1.8,\ \delta_1 = 0.025, \end{aligned} \tag{5.37}$$

with initial conditions $x(0) = y(0) = z(0) = 1$.
For the above set of parameters, there exists a positive equilibrium point $E^*(0.846, 1.8755, 5.1766)$. It is also noted that the condition in Theorem 5.3.4 is satisfied for the set of parameters chosen in (5.37). So, the system (5.2) is asymptotically stable around the positive equilibrium, which is shown in Fig. 5.1. This figure shows that density of populations fluctuate for some finite time and then settle down to their respective equilibrium levels. Now, if we increase the value of parameter a keeping value of other parameters same as in (5.37), then the time of fluctuations also increases and the system (5.2) undergoes a Hopf-bifurcation at a critical value of parameter $a = a^* = 0.0616$. The system (5.2) becomes unstable for further increment in the value of parameter a. In Fig. 5.2 we have drown time series evolution and phase portrait of species at $a = 0.07$. This figure depicts that the system (5.2) is unstable for $a = 0.07 > a^* = 0.0616$ and periodic solution exists.

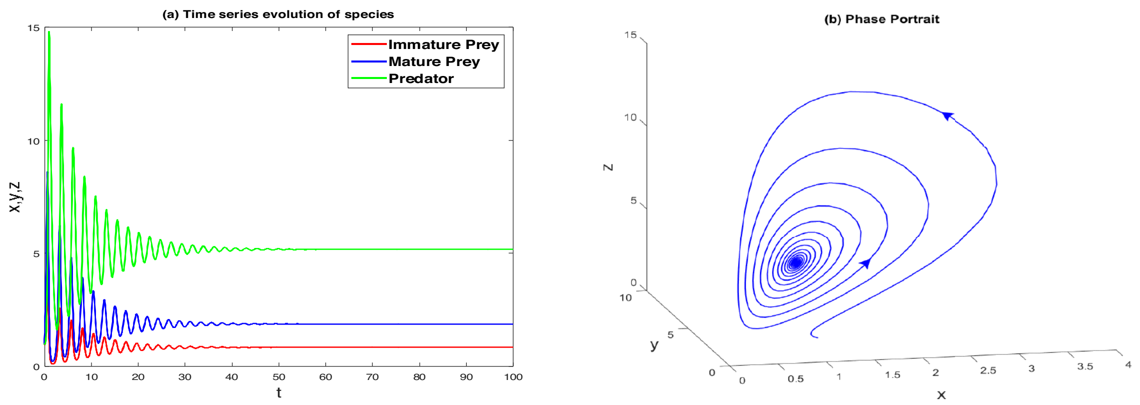

Fig. 5.1: Local asymptotic stablility of system for the set of parameters chosen in (5.37) (a) Time series evolution of x, y and z, (b) phase portrait.

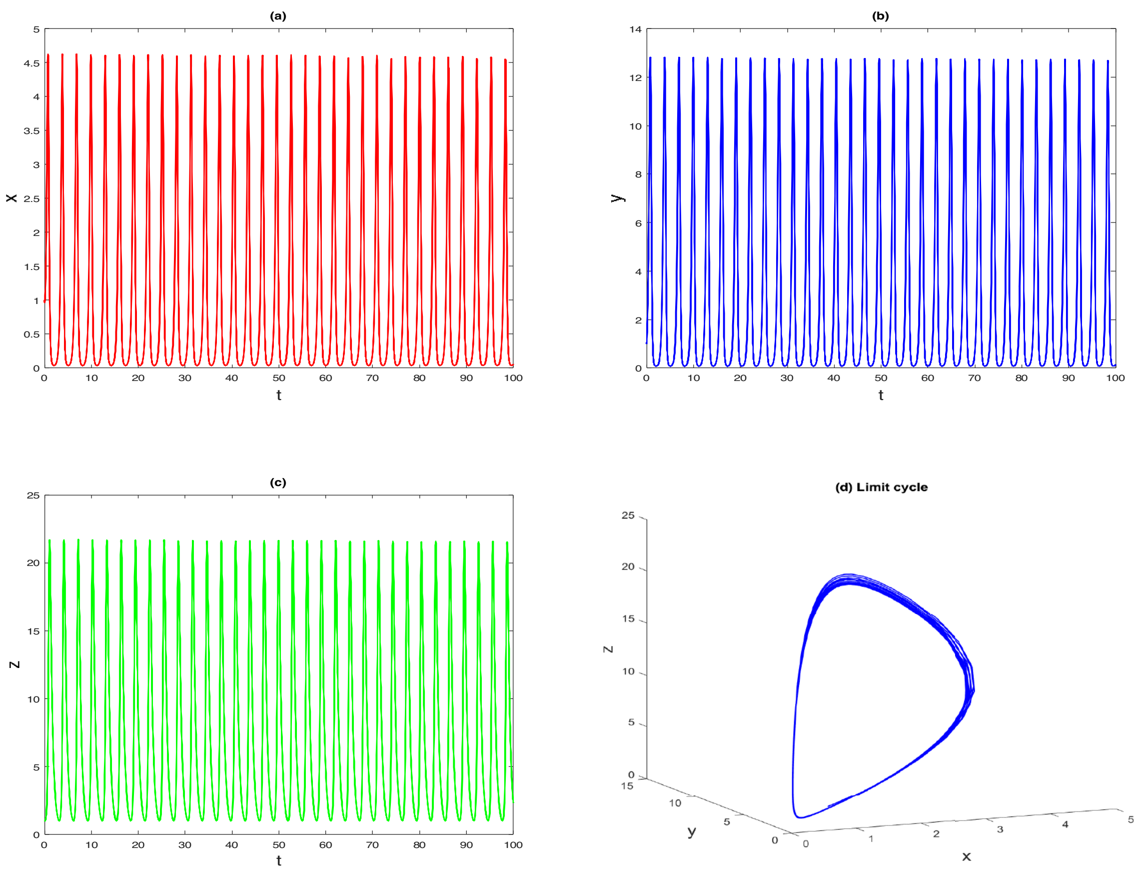

Fig. 5.2: E^* is unstable when $a = 0.07 > a^*$ (a-c) Time series evolution of species, (d) existence of periodic solution.

Similarly the system (5.2) shows Hopf-bifurcation on decreasing the value of parameter $b = b^* = 0.0029185$. Therefore the positive equilibrium point E^* is locally asymptotically stable for

$b > 0.0029185$ (shown in Fig. 5.1) and unstable for $b < 0.0029185$. Fig. 5.3 depicts that the positive equilibrium point is unstable at $b = 0.002 < b^*$.

Fig. 5.4 is bifurcation diagrams of the system (5.2) with respect to parameter a, which make us more clear that the stable interior equilibrium point bifurcates into a stable limit cycle and unstable interior equilibrium point on increasing the parameter a. The region of stability and instability for the set of parameters (5.37) with respect to parameters a and b has been drawn in Fig. 5.5. For each value of parameter b there is a critical value of parameter a, where Hopf-bifurcation occurs in the system.

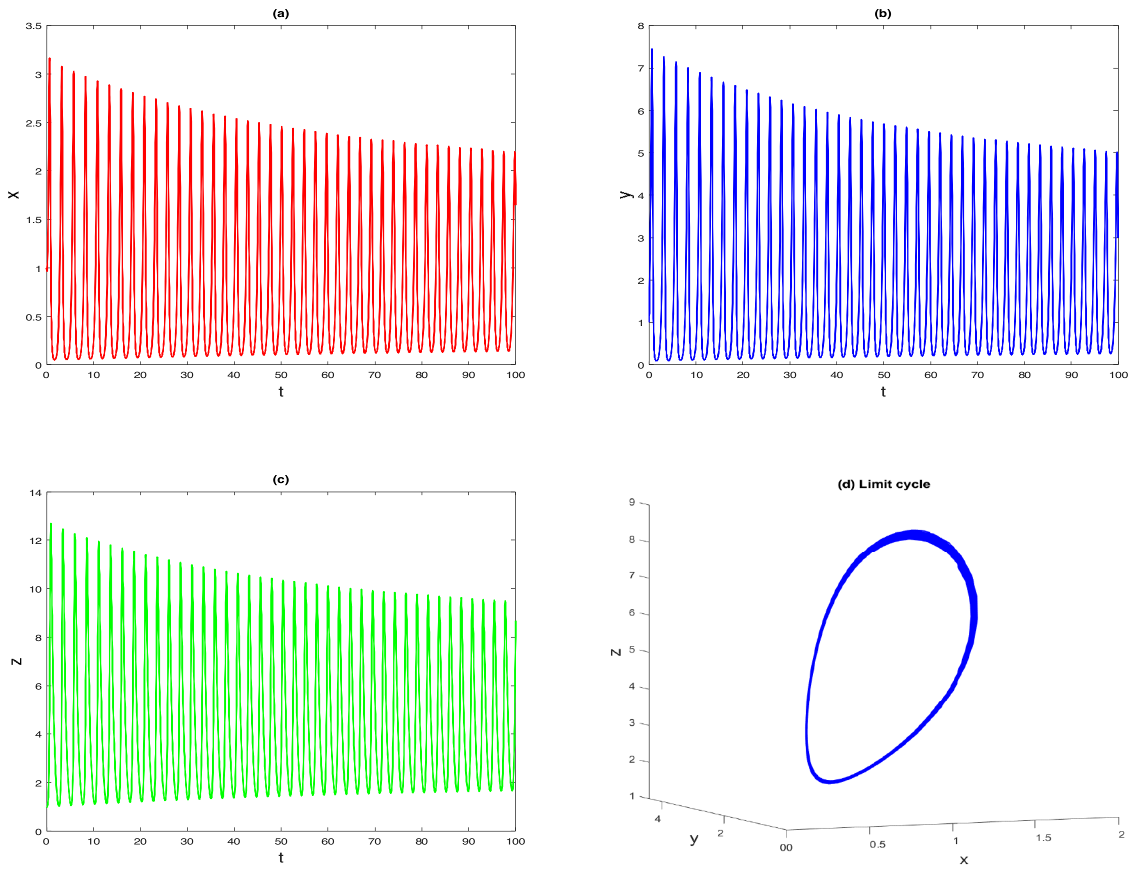

Fig. 5.3: E^* is unstable when $b = 0.002 < b^*$ (a-c) Time series evolution of species, (d) existence of periodic solution.

We have analyzed the sensitivity of parameter α and s with respect to each other. It is observed that the parameter s may have multiple bifurcation point (switching of stability more than one time) in different range of α, which is shown in Table 5.2. From the table it can be noted that if $\alpha \in (0, 0.886)$, then the system (5.2) is stable for all $s > 0$ i.e. there is no bifurcation point where stability of the system switches. If $\alpha \in (0.886, 1.835)$, system change the stability once with respect to the parameter s. The bifurcation diagram at $\alpha = 1.5$, considering s as the bifurcation parameter is depicted in Fig. 5.6 that demonstrates the above phenomenon. The stability of the system (5.2) switches twice with respect to s when $\alpha \in (1.835, 4)$. For $\alpha = 2.6$, bifurcation diagram is presented in Fig. 5.7. From the figure it can be observed that for $\alpha = 2.6 \in (1.835, 4)$ stability of the system switches twice. Initially at $s = 0$, the system is stable around its positive equilibrium point. Now, if we increase the value of parameter s then it is observed that the system losses its stability at a certain value of s and becomes unstable. Again if we increase the value of s then it gets the stability back and remains stable for further increment in parameter s. (see Fig. 5.7).

Table 5.2: Multiplicity of stability switching of system (5.2) with respect to parameter s in different range of α, when $c_2 = 0.35$ and rest of parameters have same value as in (5.37).

Range of α	multiplicity of stability switching	Nature of positive equilibrium point at $s = 0$
(0, 0.886)	0	stable
(0.886, 1.835)	1	unstable
(1.835, 4)	2	stable

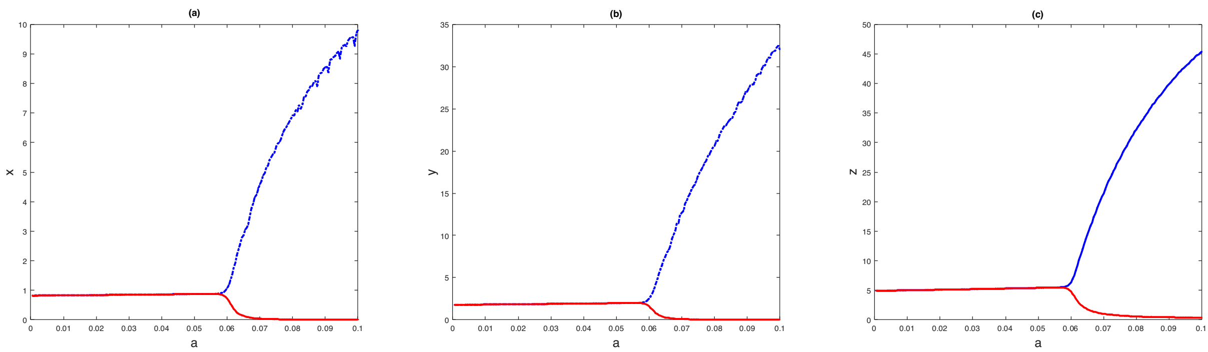

Fig. 5.4: Bifurcation diagram of x, y and z with respect to parameter a. Other parametric values are same as in (5.37).

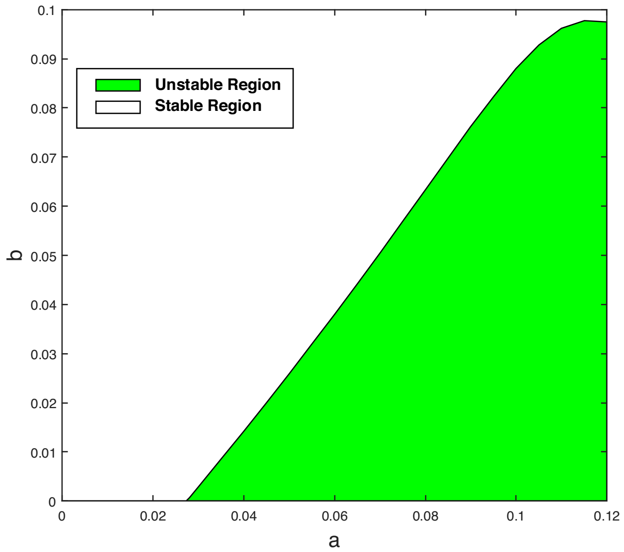

Fig. 5.5: Stability and instability regions of model (5.2) for the set of values of parameters in (5.37) with respect to parameters a and b.

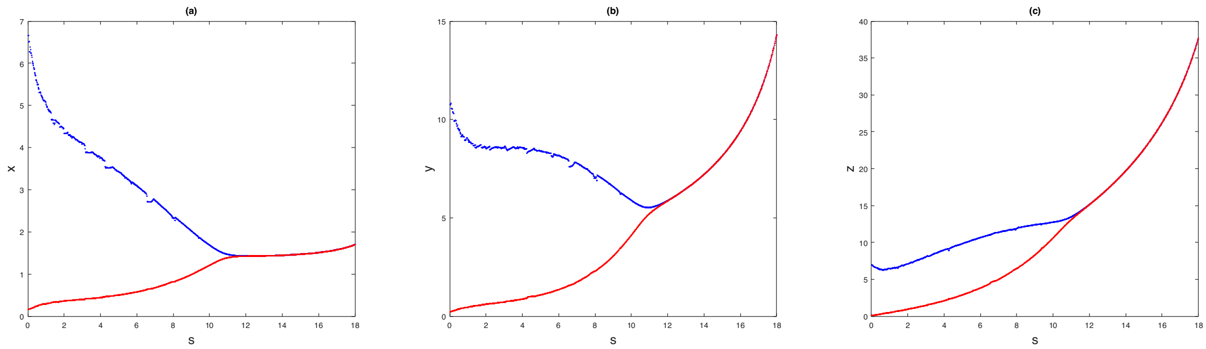

Fig. 5.6: Stability of the system (5.2) switches only once with respect to parameter s when $\alpha = 1.5$.

5.6.2 Delayed system

In order to validate the analytical predictions derived for delayed system, the values of parameters are chosen as follows:

$$\begin{aligned} &r = 3,\ K = 100,\ s = 4,\ d = 0.7,\ \rho = 0.3,\ \alpha = 1.3, \\ &a = 0.03,\ b = 0.04,\ c_1 = 0.67,\ c_2 = 0.1,\ \delta_0 = 1.8,\ \delta_1 = 0.25, \end{aligned} \tag{5.38}$$

Case-I: When $\tau_1 = 0$ and $\tau_2 > 0$, then the positive equilibrium point E^* is given by $E^*(11.8818, 24.0078, 7.8722)$. Taking $i = 0$ in equation (5.16), our computer simulation gives

$$\omega_1 = 2.2763, \quad \tau_{2_0} = 0.1229,$$

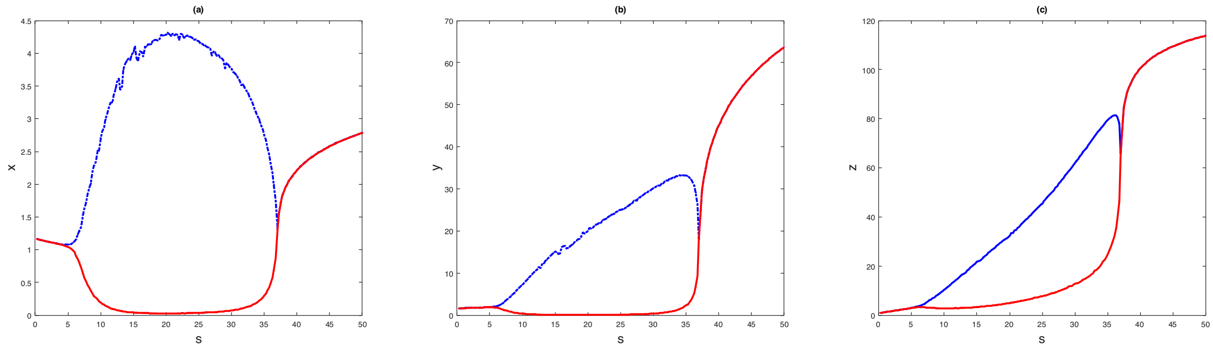

Fig. 5.7: Stability of the system (5.2) switches twice with respect to parameter s when $\alpha = 2.6$ and rest of parameters are same as in (5.37).

and the transversality condition is also satisfied. Hence from the Theorem 5.4.1, it follows that the system (5.1) undergoes a Hopf-bifurcation at $\tau_2 = \tau_{2_0}$. Therefore the system is locally asymptotically stable around the positive equilibrium E^* when $\tau_2 < \tau_{2_0}$ and unstable when $\tau_2 > \tau_{2_0}$. The time series evolution and phase portrait of system (5.1) is shown in Fig. 5.8 for $\tau_2 = 0.05 (< \tau_{2_0})$, which confirms the system is stable when $\tau_2 < \tau_{2_0}$. In Fig. 5.9, the time series solution and phase portrait trajectory have been drawn for $\tau_2 = 0.15 (> \tau_{2_0})$. The figure shows instability of the system when $\tau_2 > \tau_{2_0}$.

Table 5.3: Effect of gestation delay τ_2 on critical value of parameters a and b. Other values are as in (5.37).

τ_2	a^*	b^*
0	0.0616	0.0029185
0.01	0.0498	0.0182
0.02	0.03912	0.0302
0.03	0.0286	0.04107
0.04	0.01877	0.0509
0.05	0.0089	0.05968
0.0596	0	0.0676

In the Table 5.3, we have studied that how the gestation delay τ_2 change the critical value of parameters a and b where system undergoes Hopf-bifurcation. Here it is remarkable that as

τ_2 increases, value of a^* decreases while value of b^* increases.

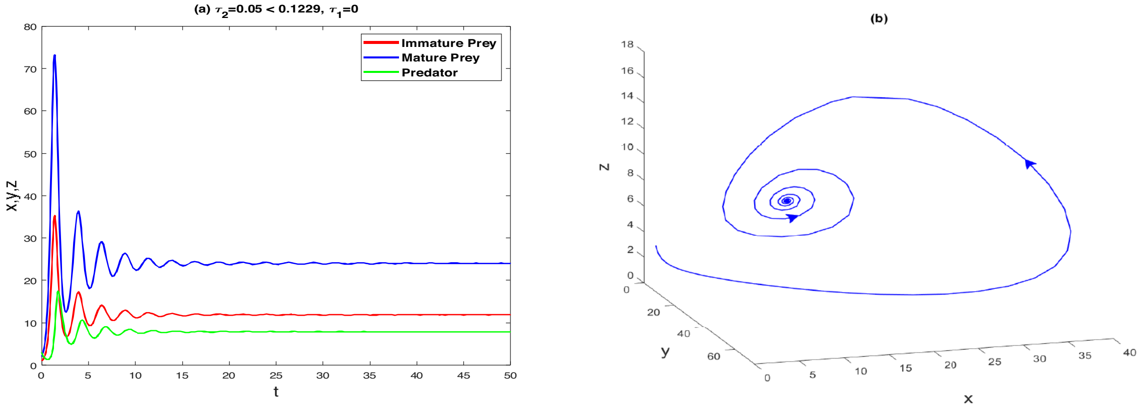

Fig. 5.8: E^* is stable when $\tau_2 = 0.05 < \tau_{2_0}$ (a) Time series evolution of x, y and z, (b) phase portrait.

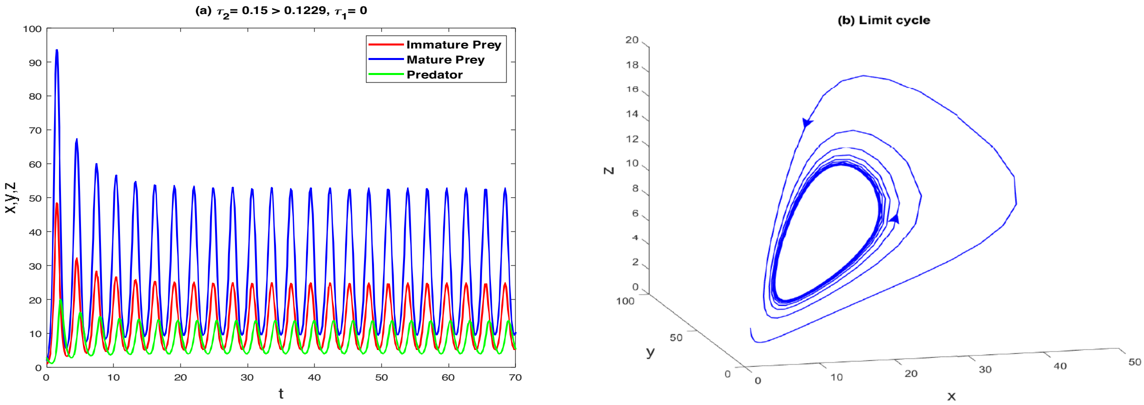

Fig. 5.9: E^* is unstable when $\tau_2 = 0.15 > \tau_{2_0}$ (a) Time series evolution of x, y and z, (b) existence of stable periodic solution.

Case-II: When $\tau_1 > 0$ and $\tau_2 = 0$, then interior equilibrium is delay dependent and transversality condition is satisfied, so the system will show the Hopf-bifurcation with respect to τ_1. Through some computation we get $\tau_{1_0} = 0.8216$.

According to Theorem 5.4.3, we can see that the system (5.1) is locally asymptotically stable around E^* when $\tau_1 = 0.5 < \tau_{1_0}$, which is shown in Fig. 5.10. The interior equilibrium E^* loses its stability and system undergoes a Hopf-bifurcation as the value of τ_1 passes through τ_{1_0}. The instability behavior of E^* for $\tau_1 = 1.5(> \tau_{1_0})$ is shown in Fig. 5.11. It is noted that that if we further increase the value of τ_1, then at $\tau_1 = 1.9083$, the system again change the stability behavior. Fig. 5.12 shows the system is stable at $\tau_1 = 2.7 > 1.9083$. Further the behavior of the system switches at $\tau_1 = 3.8377,\ 5.0067$ and 6.4309. The switching of stability at above values of parameter τ_1 can be seen in the bifurcation diagram, presented in Fig. 5.13. From the figure, it is clear that the stability is switching less than six times, as followed by theory presented.

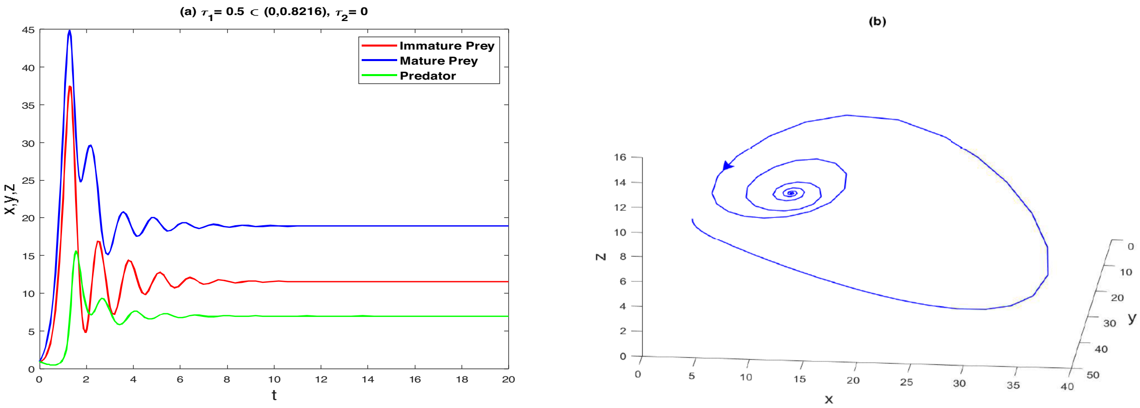

Fig. 5.10: E^* is stable when $\tau_1 = 0.5 \in (0, 0.8216)$ and $\tau_2 = 0$ (a) Time series evolution of x, y and z, (b) phase portrait.

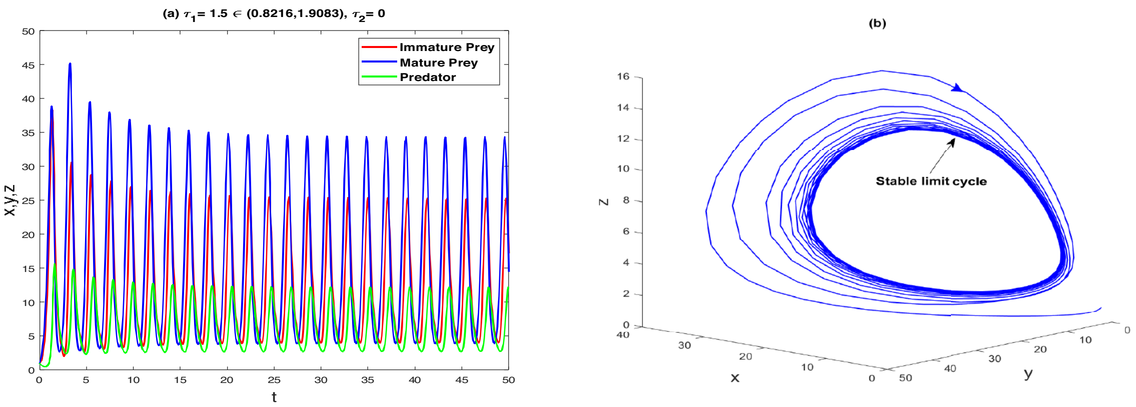

Fig. 5.11: E^* is unstable when $\tau_1 = 1.5 \in (0.8216, 1.9083)$ and $\tau_2 = 0$ (a) Time series evolution of x, y and z, (b) phase portrait.

Case-III: When $\tau_1 \in (0, \tau_{1_0})$ and $\tau_2 > 0$. Let $\tau_1^* = 0.4 \in (0, 0.8216)$ and choosing τ_2 as a parameter. In such a case, the positive equilibrium point $E^*(x^*, y^*, z^*)$ is given by

$$x^* = 11.6118, \; y^* = 19.7958, \; z^* = 7.1569.$$

Transversality condition is also satisfied so Hopf-bifurcation occurs in the system with respect to τ_2. Taking $j = 0$ in equation (5.29), a little computation yields

$$\omega_0 = 1.8931, \; \tau'_{2_0} = 0.2651.$$

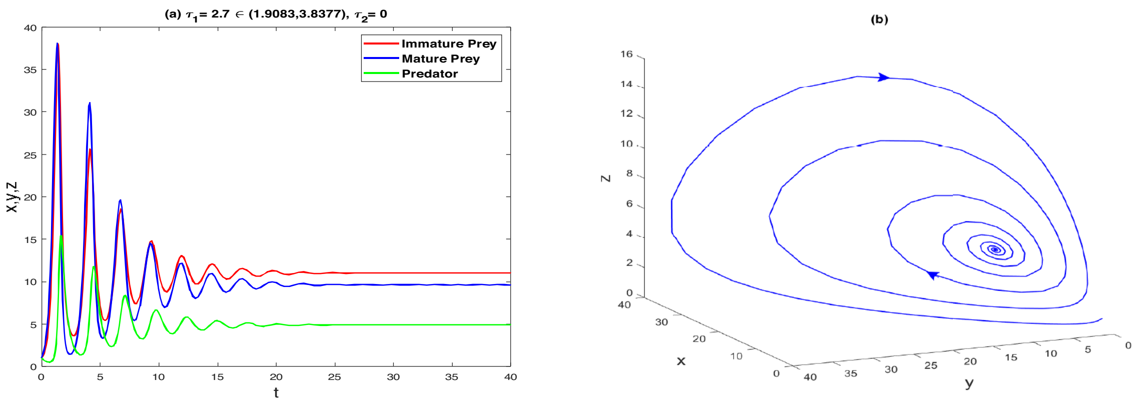

Fig. 5.12: E^* is stable when $\tau_1 = 2.7 \in (1.9083, 3.8377)$ and $\tau_2 = 0$ (a) Time series evolution of $x,\ y$ and z, (b) phase portrait.

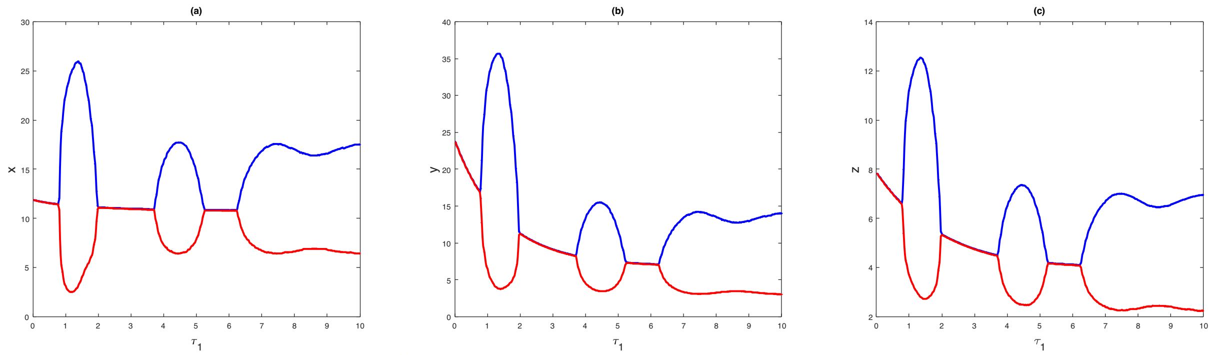

Fig. 5.13: Bifurcation diagram of $x,\ y$ and z with respect to τ_1 when $\tau_2 = 0$. Stability of the system (5.1) switches five times.

For $\tau_2 \in (0, \tau'_{2_0})$, the system is locally asymptotically stable but as τ_2 crosses the critical value τ'_{2_0} the system undergoes a Hopf-bifurcation and system becomes unstable. The stability and instability behavior of the system is depicted in Fig. 5.14 and 5.15 for $\tau_2 = 0.22(< \tau'_{2_0})$ and $0.3(> \tau'_{2_0})$, respectively. Fig. 5.16 is the bifurcation diagram with respect to bifurcation parameter τ_2. It can be seen from this figure that the system (5.1) changes its stability nature around the positive equilibrium point E^* at $\tau_2 = \tau'_{2_0} = 0.2651$.

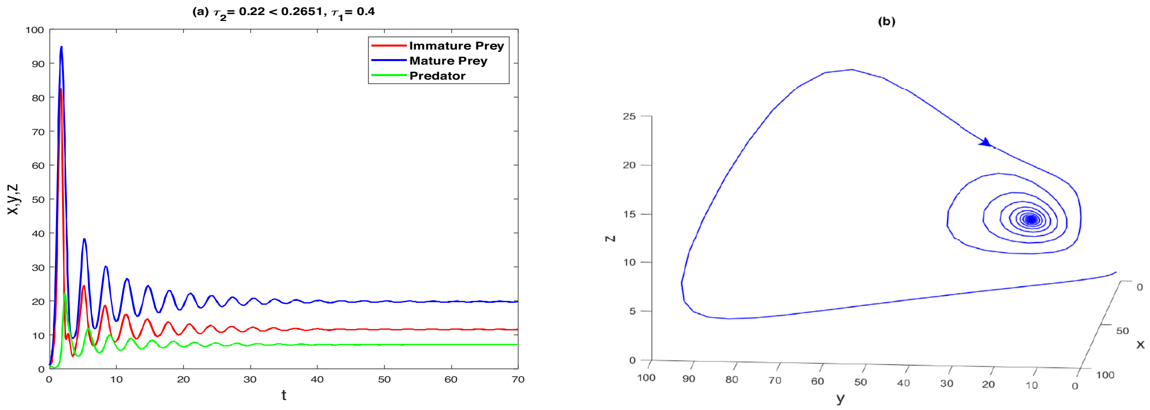

Fig. 5.14: E^* is stable when $\tau_2 = 0.22 < \tau_2^*$ and $\tau_1 = 0.4$ is fixed (a) Time series evolution of $x,\ y$ and z, (b) phase portrait.

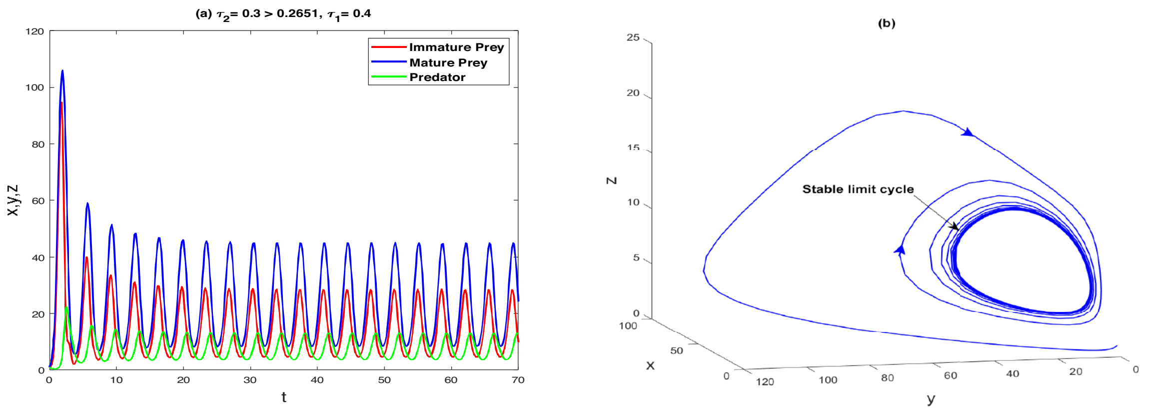

Fig. 5.15: E^* is unstable when $\tau_2 = 0.3 > \tau_2^*$ and $\tau_1 = 0.4$ is fixed (a) Time series evolution of $x,\ y$ and z, (b) existence of periodic solution.

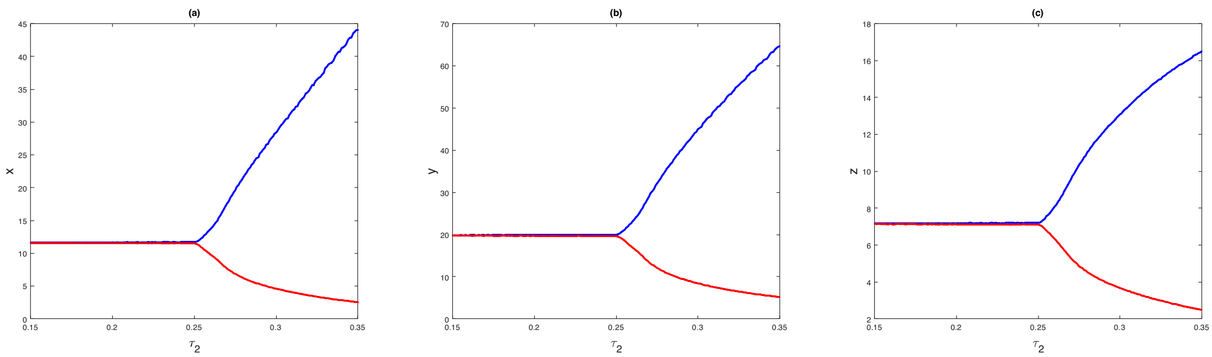

Fig. 5.16: Bifurcation diagram of $x,\ y$ and z with respect to τ_2 when $\tau_1 = 0.4$ is fixed.

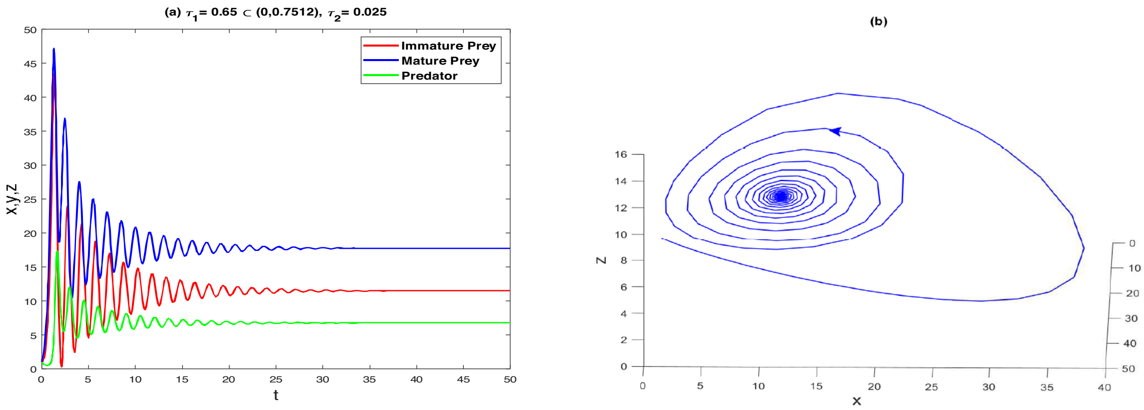

Fig. 5.17: E^* is stable when $\tau_1 = 0.65 \in (0, 0.7512)$ and $\tau_2 = 0.025$ (a) Time series evolution of x, y and z, (b) phase portrait.

Case-IV: When $\tau_2 \in (0, \tau_{2_0})$ and $\tau_1 > 0$. Let $\tau_2^* = 0.025 \in (0, 0.1229)$ and assuming τ_1 as a parameter, we obtain $\tau'_{1_0} = 0.7512$, $\tau_1^{(2)} = 2.64$ and $\tau_1^{(3)} = 3.4307$ as critical values of τ_1 where stability of the system (5.1) switches. The system is stable for $\tau_1 \in (0, \tau'_{1_0}) \bigcup (\tau_1^{(2)}, \tau_1^{(3)})$ and unstable for $\tau_1 \in (\tau'_{1_0}, \tau_1^{(2)}) \bigcup (\tau_1^{(3)}, \infty)$. Stability behavior of system (5.1) is presented in Figs. 5.17,5.18,5.19 and 5.20 for different values of τ_1 picked up from each of above intervals. Bifurcation diagram (Fig. 5.21) makes sure that stability of the system is switching three times. The figure shows that at $\tau_1 = 0$ the system is stable around the positive equilibrium and it becomes unstable at $\tau_1 = 0.7512$ and we obtain periodic solution. The system obtains stability on increasing the value of τ_1 beyond 2.64. It remains stable till $\tau_1 = 3.4307$ and thereafter becomes unstable for further increment in τ_1. By the algorithm derived in section 5.5, we can obtain

$$c_1(0) = -0.01179 + 0.001618i, \ \mu_2 = 0.1649, \ \beta_2 = -0.02358, \ T_2 = 0.01859.$$

Since $\mu_2 > 0$, the Hopf-bifurcation is supercritical. $\beta_2 < 0$ implies that the bifurcated periodic solution is stable. $T_2 > 0$ shows that the period of bifurcated periodic solution increases.

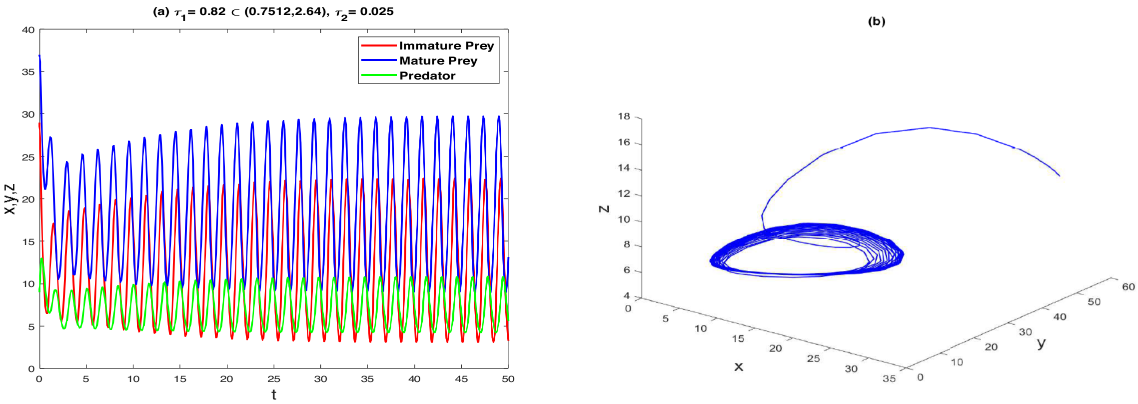

Fig. 5.18: E^* is unstable when $\tau_1 = 0.82 \in (0.7512, 2.64)$ and $\tau_2 = 0.025$ (a) Time series evolution of $x,\ y$ and z, (b) Existence of stble limit cycle.

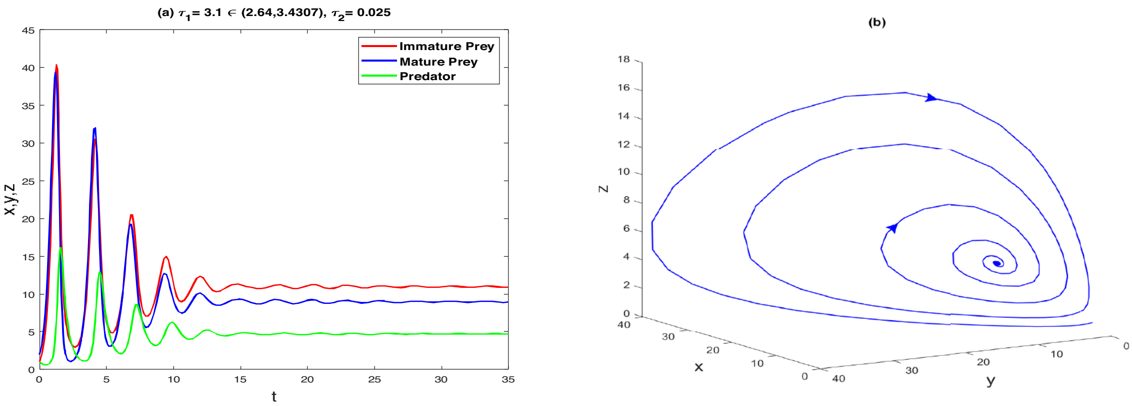

Fig. 5.19: E^* is stable when $\tau_1 = 3.1 \in (2.64, 3.4307)$ and $\tau_2 = 0.025$ (a) Time series evolution of $x,\ y$ and z, (b) phase portrait.

5.7 Conclusion

In this chapter, a three dimensional mathematical model is proposed and analyzed to understand the dynamical complexity of prey-predator interaction including stage structure in prey population. The mature prey grows logistically and consumed by predator with Crowley-Martin functional response. The growth rate of immature prey population is proportional to the number of existing mature prey. It is consumed by the predator with linear mass action principle. It is assumed that the immature prey joins mature prey class after a certain time lag (maturation delay) τ_1 with mortality rate d. As the reproduction of predator after consuming prey is not instantaneous process. There is some time required, known as gestation delay. So, for more realistic situations, we have considered the gestation delay for predator population in our model. First we discussed the positivity, boundedness and persistence of the solution for the non-delayed model (5.2). The system is confined within a compact set Ω in the non-negative octant, which represents a natural restrictions to growth as a consequence of limited resources. The model is persistence if intrinsic growth rate of prey population is greater than a threshold value whilst death rate of predator population is less than a threshold value. The local and global behavior of the system around its interior equilibrium point is also investigated. Further, we have studied the Hopf-bifurcation with respect to different parameters. The multiplicity of stability switching for the different range of parameter α with respect to parameter s is obtained in the Table 5.2 and illustrated by Figs. 5.6 and 5.7.

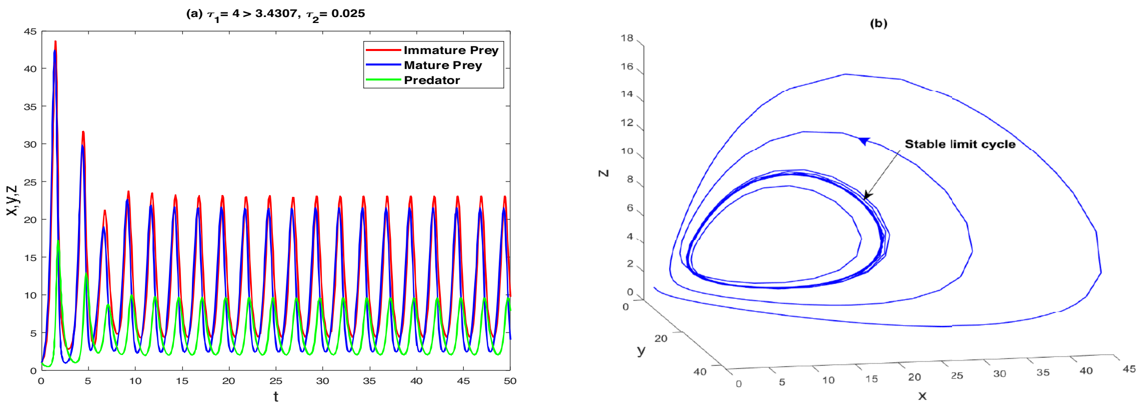

Fig. 5.20: E^* is stable when $\tau_1 = 4 > 3.4307$ and $\tau_2 = 0.025$ (a) Time series evolution of $x,\ y$ and z, (b) phase portrait (existence of stable limit cycle).

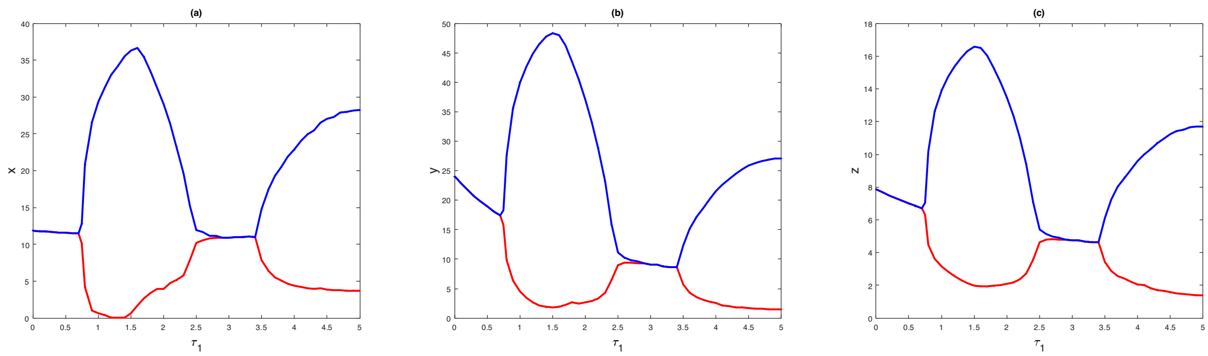

Fig. 5.21: Bifurcation diagram of $x,\ y$ and z with respect to parameter τ_1 when $\tau_2 = 0.025$ is fixed.

In order to analyze the dynamical behavior of delayed system (5.1), we discussed Hopf-bifurcation via local stability taking delay as a bifurcation parameter. We have shown the existence of Hopf-bifurcation for possible combinations of both the delays. Case (1): $\tau_1 = 0,\ \tau_2 > 0$, Case (2): $\tau_1 > 0,\ \tau_2 = 0$, Case (3): τ_1 is fixed in its stable interval and τ_2 as bifurcation parameter, Case (4): τ_2 is fixed in its stable interval and τ_1 as bifurcation parameter.

In each of above cases we note that when time delay increases, oscillations in species increases and beyond a critical value of delay system becomes unstable and periodic solution occurs in the system. This proves that the time delay can cause a stable system to become unstable. Here one notable result is that for Case (2) and Case (4) (when τ_1 as a bifurcation parameter) system becomes unstable as delay crosses a threshold value of τ_1. But if we further increase the value of τ_1 then system changes the stability again and becomes stable. In other words stability of the system switches more than once. According to our theoretical analysis, the stability can switch at most six times, which is followed by the system. Hence the time delay can cause a stable equilibrium to become unstable and even a switching of stability. The direction and stability of Hopf-bifurcation also have been investigated using normal form method and center manifold theorem. For the set of values of parameters chosen in (5.38), our numerical experiments show that the Hopf-bifurcation is supercritical, the bifurcated periodic solution is stable and its period increases.

Our numerical simulation and graphical illustration is based on some biologically feasible data to validate the analytical findings. Switching of stability with respect to delay τ_1 can be seen in the bifurcation diagrams 5.13 and 5.21. We hope that this chapter will help to understand the dynamics of prey-predator system with stage structure in prey population including maturation delay and gestation delay.