

Chapter 2

Modelling the Effect of Fear in a Prey-predator System with Prey Refuge and Gestation Delay¹

2.1 Introduction

The first theory on population dynamics is proposed by Malthus [137] in the end of eighteenth century. It is based on an idea that the rate of population expansion is proportional to the current population. The theory is named after ‘Malthusian Theory of Population’. But its unbounded solutions were unable to convey the accurate prediction in real scenario. On later the logistic growth model came into the frame that characterize natural restrictions to growth as a consequence of limited resources and space [215]. Thereafter Lotka [132] and Volterra [217] established a system of two nonlinear ordinary differential equations that represents population model of two interacting species and the equations have periodic solutions. It was a beginning of a book after which innumerable pages get added through tremendous attempts of several researchers [82, 199, 54, 149, 101]. Holling [85] defined the term ‘Functional Response’ that represents the rate of prey consumption by their predators. It makes the model more realistic in the ecological point of view. A lot of research has been done by choosing different functional response (Holling type-II in [94, 56], Holling type-III in [88, 101], Beddington-DeAngelis functional response in [131, 40, 208], Crowley-Martin functional response in [214, 191, 210]) to see their effect on the dynamics of system in last few decades.

Further, the Lotka-Volterra model is modified by considering various ecological factors, like Allee effect, stage structure, habitat complexity, feeding switching, gestation delay, additional food, etc. and examined the local and global behavior of the system and explored the effect of such ecological factors on the dynamics of the system. Over years our conception in prey-predator system was that the predators affect prey population by direct hunting because predation event are easy to observe in forest provinces. But recently, some studies show that

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predators do not affect prey population by only hunting, they also induce fear in prey individuals and due to that birth rate of prey reduces [220, 164, 155]. Due to the fear, induced by predators, prey individuals always scare to come into an open habitat, they do not get a free environment for their daily activities including mating. Hence predator's fear drops a negative effect on their reproduction rate. Thus, it is important to consider the cost of fear as the form of a decrease in reproduction. The general form of function representing fear effect can be written as $f(y, k)$, where y denotes the number of predator individuals and k is cost of induced fear by them. This function satisfies the following properties [220]:

- (a) $f(0, k) = f(y, 0) = 1$,
- (b) $\lim_{y \rightarrow \infty} f(y, k) = \lim_{k \rightarrow \infty} f(y, k) = 0$,
- (c) $\frac{\partial f}{\partial y} < 0$, $\frac{\partial f}{\partial k} < 0$.

Particularly, we can consider following functions:

$$f(y, k) = \frac{1}{1 + ky}, f(y, k) = e^{-ky}, f(y, k_1, k_2) = \frac{1}{1 + k_1y + k_2y^2}.$$

Wang *et al.* [220] proposed a prey-predator model with cost of fear into prey reproduction and discussed that high level of fear can stabilize the system by excluding the existence of periodic solutions. Further, the impact of fear in a tri-trophic food chain model with Holling type-II functional response is studied by Pandey *et al.* [164]. They concluded that system shows chaotic behavior for lower values of both the fear parameters and chaotic oscillations can be controlled by increasing these parameters.

Prey refuge is a good idea to reduce the possibilities of over-exploitation of prey biomass by predators. We divide the prey population into two categories: reserved and unreserved population. Reserved prey individuals are not accessible for predators whereas unreserved prey individuals are available for predation. Several research studies have been done with prey refuge [203, 190, 208, 68, 119, 56]. Most of the studies have shown that refuge have a stabilizing effect on the system. Tang *et al.* [203] considered a prey-predator model with a constant prey refuge and showed a global qualitative analysis to determine the global dynamics of the system. A Leslie-Gower prey-predator model with disease in prey including prey refuge is well studied by Sharma and Samanta [190]. Impact of additional food for predator on the dynamics of prey-predator model with prey refuge is investigated by Ghosh *et al.* [68]. They observed that predator extinction possibility in high prey refuge ecological systems may be removed by supplying additional food to predator population.

It is studied that the presence of a predator may alter the behavior and physiology of prey to such an extent that it can exert an effect on prey populations even more powerful than direct

predation ([124, 34, 123, 36]). So we realized that the prey-predator interaction should not be governed by the only predation, the cost of fear should also be considered. Moreover, prey refuge might be a good idea to reduce the over-exploitation of prey biomass by predators. Particularly, in this chapter, we are interested to pore over the dynamics of a prey-predator system with fear effect and prey refuge.

Time delay occurs in each biological phenomenon. The effect of delay in an ecological model can be shown by delay differential equations. Delayed models are much more realistic in nature. A delay differential equation demonstrates much more complex behavior than ordinary differential equation. In a prey-predator system, the effect of consumed prey into predator population does not appear instantaneously, there is some time lag that is gestation delay [210]. The dynamics of prey-predator systems with time delay and different kind of other parameters is studied by many researchers [107, 15, 144, 116, 89, 117, 133]. Ma and Wang [133] proposed and analyzed a delay induced predator-prey system with the effect of habitat complexity. Li *et al.* [117] examined the stability and Hopf-bifurcation of a delayed prey-predator model with Beddington-DeAngelis functional response. The roll of prey refuge and degree of mutual interference among predators is studied by Tripathi *et al.* [208]. Further, they have investigated the effect of gestation delay in the dynamics of the system. There are several studies carried out with multiple delays [120, 231, 111]. Li and Wei [120] have done stability and Hopf-bifurcation analysis of a prey-predator model with two maturation delays. Recently, Kundu and Maitra [111] have discussed about the dynamics of two prey and one predator system with cooperation among preys against predators incorporating three discrete delays. They have shown that all delays are capable to destabilize the system.

To the best of author's knowledge, an ecological model with (i) effect of fear in prey, (ii) prey refuge, (iii) Crowley-Martin functional response, (iv) gestation delay in predator have not been considered. Motivated by this, we formulate two mathematical models (non-delayed and delayed) in section 2.2. In section 2.3, we analyze the dynamics of non-delayed model via stability and Hopf-bifurcation. Analysis of delayed system is done in section 2.4. Properties of Hopf-bifurcation for the delayed system are given in section 2.5. We have also done some numerical simulations for both the non-delayed and delayed models to confirm our analytic findings. Section 2.6 is devoted to numerical simulations.

2.2 Evolution of Mathematical Model

We consider a ecosystem where prey and predator species survive. Prey population grows logistically in the absence of predation. Then it leads the following

$$\frac{dx}{dt} = rx - r_0x - r_1x^2,$$

where x represents the population of prey, r is the birth rate of prey, r_0 is the natural death rate of prey and r_1 denotes the death rate of prey due to competition among them. In the present chapter, we take

$$f(y, k) = \frac{1}{1 + ky}.$$

It may be noted that there are some arguments ([32]) that survival of prey may be affected due to fear effect of predator. But there is no such experimental evidences and hence we take the parameters r_0 and r_1 as constant in our model. Now we introduce the effect of fear and modify the expression (2.1) according to Wang *et al.* [220] and get

$$\frac{dx}{dt} = \frac{1}{1 + ky}rx - r_0x - r_1x^2,$$

where k is the cost of fear and y represents the predator population.

The Holling type-II functional response is only prey dependent functional response. This shows that the feeding rate of predators decreases with density of prey due to presence of handling and searching time. Crowley and Martin assumed that predator will decrease at high predator density due to interference among them for the common resources. It makes the functional response more realistic in the ecological point of view. The Crowley-Martin type functional response is given by

$$g(x, y) = \frac{\alpha x}{(1 + ax)(1 + by)},$$

where α is attack rate, a is handling time required per prey and b is magnitude of interference among predator individuals. Apart from this, it converts into classical (when $a = 0, b = 0$) and Holling type-II functional response (when $a > 0, b = 0$ or $a = 0, b > 0$). Now we divide the prey population into two categories: reserved population which is not accessible for predators and unreserved or open access population which are available for predation. This process is known as prey refuge. Let θx be the number of prey individuals which are reserved and $(1 - \theta)x$ the

number of prey individuals which are available for predation. Therefore, we modify Crowley-Martin type functional response incorporating the effect of prey refuge as:

$$g(x, y) = \frac{\alpha(1 - \theta)x}{(1 + a(1 - \theta)x)(1 + by)},$$

where θ ($0 \leq \theta \leq 1$) is refuge parameter.

Keeping above assumptions in mind, our mathematical model is governed as follows:

$$\begin{aligned} \frac{dx}{dt} &= \frac{1}{1 + ky}rx - r_0x - r_1x^2 - \frac{\alpha(1 - \theta)xy}{(1 + a(1 - \theta)x)(1 + by)}, \\ \frac{dy}{dt} &= \frac{c\alpha(1 - \theta)xy}{(1 + a(1 - \theta)x)(1 + by)} - \delta_0y - \delta_1y^2, \\ x(0) &= x_0 \geq 0, y(0) = y_0 \geq 0, \end{aligned} \quad (2.1)$$

where c ($0 < c < 1$) is the conversion coefficient representing the number of newly born predator for each hunted prey, δ_0 and δ_1 denote the natural death rate and crowding effect of predators, respectively.

As we have explained earlier that in real life, each organism needs an amount of time to reproduce their progeny. Due to this fact the increment in predators does not appear immediately after consuming prey. It is assumed that a predator individual takes τ time for gestation. Therefore, it seems reasonable to incorporate a gestation delay in the system. Here it is assumed that change rate of predators depends on the number of prey and predator present at some previous time [107]. Thus, the delay τ is considered in the numeric response only. The generalized model involving discrete time delay takes the following form

$$\begin{aligned} \frac{dx}{dt} &= \frac{1}{1 + ky}rx - r_0x - r_1x^2 - \frac{\alpha(1 - \theta)xy}{(1 + a(1 - \theta)x)(1 + by)}, \\ \frac{dy}{dt} &= \frac{c\alpha(1 - \theta)x(t - \tau)y(t - \tau)}{(1 + a(1 - \theta)x(t - \tau))(1 + by(t - \tau))} - \delta_0y - \delta_1y^2, \end{aligned} \quad (2.2)$$

subject to the non negative conditions $x(s) = \phi_1(s) \geq 0$, $y(s) = \phi_2(s) \geq 0$, $s \in [-\tau, 0]$, where $\phi_1(s) \in C([-\tau, 0] \rightarrow R_+)$, ($i = 1, 2$) and τ denotes gestation time delay of the predator.

Remark 2.2.1. From the first equation of model (2.1), we have $\frac{dx}{dt} \leq (r - r_0)x$. Assume $r < r_0$, which leads us to $\lim_{t \rightarrow \infty} x(t) = 0$ and consequently $\lim_{t \rightarrow \infty} y(t) = 0$ (as y is specialist predator and depends on only prey). This implies both the species will die out when $r < r_0$. This case is not substantial from biological point of view. Hence, throughout this chapter we consider the case when $r > r_0$.

2.3 Dynamics of Non-delayed Model (2.1)

In this section we shall show some preliminary properties of proposed non-delayed model (2.1) like positivity, boundedness and persistence of solutions. Then we shall investigate the local and global behavior of system near existing equilibrium points.

2.3.1 Positivity and boundedness of the solution

Positivity of solution shows that species exists and boundedness represents a restriction on species to grow them exponentially. The model (2.1) can be written as

$$\frac{dx}{dt} = x\phi_1(x, y), \quad \frac{dy}{dt} = y\phi_2(x, y),$$

where

$$\phi_1(x, y) = \frac{1}{1+ky}r - r_0 - r_1x - \frac{\alpha(1-\theta)y}{(1+a(1-\theta)x)(1+by)}, \quad \phi_2(x, y) = \frac{c\alpha(1-\theta)x}{(1+a(1-\theta)x)(1+by)} - \delta_0 - \delta_1y.$$

It follows that

$$\begin{aligned} x(t) &= x(0)e^{\int_0^t \phi_1(x(s), y(s)) ds} \geq 0, \\ y(t) &= y(0)e^{\int_0^t \phi_2(x(s), y(s)) ds} \geq 0. \end{aligned}$$

Thus, the solution $X(t) = (x(t), y(t))$ with positive initial condition $X(0) = X_0 = (x(0), y(0)) \in R_+^2$ remains positive throughout the region R_+^2 .

Lemma 2.3.1. *The set $\Omega = \{(x, y) : 0 \leq x \leq \frac{r}{r_1}, 0 \leq x + \frac{1}{c}y \leq \frac{r^2}{r_1\delta}\}$ is a region of attraction for all solutions initiating in the positive quadrant, where $\delta = \min\{r_0, \delta_0\}$*

Proof. The first equation of model (2.1) implies

$$\frac{dx}{dt} \leq rx - r_1x^2,$$

which yields

$$\limsup_{t \rightarrow \infty} x(t) \leq \frac{r}{r_1}.$$

Again let $L(t) = x(t) + \frac{1}{c}y(t)$,

then we have

$$\frac{dL}{dt} = \frac{dx}{dt} + \frac{1}{c} \frac{dy}{dt} = \frac{rx}{1+ky} - r_0x - r_1x^2 - \frac{\delta_0}{c}y - \frac{\delta_1}{c}y^2 \leq \frac{r^2}{r_1} - \delta L,$$

where $\delta = \min\{r_0, \delta_0\}$.

Hence it follows that

$$\limsup_{t \rightarrow \infty} L(t) \leq \frac{r^2}{r_1 \delta}.$$

We also note that if $x > \frac{r}{r_1}$ and $L > \frac{r^2}{r_1 \delta}$, then $\frac{dx}{dt} < 0$ and $\frac{dL}{dt} < 0$. This shows that solutions of system (2.1) are bounded and Ω is a positive invariant set. \square

Persistence of a system characterizes that all species will survive in future time and none of them will extinct if they are initially present. Now we shall establish some conditions for uniform persistence of model (2.1).

Theorem 2.3.2. *Model system (2.1) is uniformly persistence if following inequalities are satisfied:*

$$r > \left(1 + k \frac{cr^2}{r_1 \delta}\right) \left(r_0 + \alpha(1 - \theta) \frac{cr^2}{r_1 \delta}\right), \quad \frac{c\alpha(1 - \theta)x_i}{(1 + a(1 - \theta)x_i)(1 + b \frac{cr^2}{r_1 \delta})} > \delta_0,$$

where x_i is defined in the proof of this theorem.

Proof. System (2.1) is said to be uniformly persistence if there are positive constants M and N such that each positive solution $X(t) = (x(t), y(t))$ of the system with positive initial conditions satisfies

$$M \leq \liminf_{t \rightarrow \infty} X(t) \leq \limsup_{t \rightarrow \infty} X(t) \leq N.$$

According to the above, if we define

$$N = \max\left\{\frac{r}{r_1}, \frac{cr^2}{r_1 \delta}\right\},$$

then from Lemma 2.3.1, we have

$$\limsup_{t \rightarrow \infty} X(t) \leq N.$$

This also shows that for any sufficiently small $\varepsilon > 0$, there exists a $T > 0$ such that for all $t \geq T$, the following holds:

$$x(t) < \frac{r}{r_1} + \varepsilon, \quad y(t) < \frac{cr^2}{r_1 \delta} + \varepsilon.$$

Now from the first equation of model (2.1), for all $t \geq T$, we can write

$$\begin{aligned} \frac{dx}{dt} &\geq \frac{rx}{1 + k(\frac{cr^2}{r_1 \delta} + \varepsilon)} - r_0x - r_1x^2 - \alpha(1 - \theta)x\left(\frac{cr^2}{r_1 \delta} + \varepsilon\right) \\ &= \left(\frac{r}{1 + k(\frac{cr^2}{r_1 \delta} + \varepsilon)} - r_0 - \alpha(1 - \theta)\left(\frac{cr^2}{r_1 \delta} + \varepsilon\right)\right)x - r_1x^2. \end{aligned}$$

It follows that

$$\liminf_{t \rightarrow \infty} x(t) \geq \frac{1}{r_1} \left(\frac{r}{1 + k \left(\frac{cr^2}{r_1 \delta} + \varepsilon \right)} - r_0 - \alpha(1 - \theta) \left(\frac{cr^2}{r_1 \delta} + \varepsilon \right) \right).$$

For sufficiently small $\varepsilon > 0$, we have

$$\liminf_{t \rightarrow \infty} x(t) \geq \frac{1}{r_1} \left(\frac{r}{1 + k \frac{cr^2}{r_1 \delta}} - r_0 - \alpha(1 - \theta) \frac{cr^2}{r_1 \delta} \right) =: x_i.$$

Therefore, the prey population of system (2.1) is persistent under the condition given below:

$$r > \left(1 + k \frac{cr^2}{r_1 \delta} \right) \left(r_0 + \alpha(1 - \theta) \frac{cr^2}{r_1 \delta} \right).$$

Now from the second equation of model (2.1), we can write

$$\begin{aligned} \frac{dy}{dt} &\geq \frac{c\alpha(1 - \theta)(x_i + \varepsilon)y}{(1 + a(1 - \theta)(x_i + \varepsilon))(1 + b \left(\frac{cr^2}{r_1 \delta} + \varepsilon \right))} - \delta_0 y - \delta_1 y^2 \\ &= \left(\frac{c\alpha(1 - \theta)(x_i + \varepsilon)}{(1 + a(1 - \theta)(x_i + \varepsilon))(1 + b \left(\frac{cr^2}{r_1 \delta} + \varepsilon \right))} - \delta_0 \right) y - \delta_1 y^2, \end{aligned}$$

which implies

$$\liminf_{t \rightarrow \infty} y(t) \geq \frac{1}{\delta_1} \left(\frac{c\alpha(1 - \theta)(x_i + \varepsilon)}{(1 + a(1 - \theta)(x_i + \varepsilon))(1 + b \left(\frac{cr^2}{r_1 \delta} + \varepsilon \right))} - \delta_0 \right).$$

For sufficiently small $\varepsilon > 0$, we have

$$\liminf_{t \rightarrow \infty} y(t) \geq \frac{1}{\delta_1} \left(\frac{c\alpha(1 - \theta)x_i}{(1 + a(1 - \theta)x_i)(1 + b \frac{cr^2}{r_1 \delta})} - \delta_0 \right).$$

For persistence of the predator population of system (2.1), we must have

$$\frac{c\alpha(1 - \theta)x_i}{(1 + a(1 - \theta)x_i)(1 + b \frac{cr^2}{r_1 \delta})} > \delta_0.$$

Taking $M = \min \left\{ r - \left(1 + k \frac{cr^2}{r_1 \delta} \right) \left(r_0 + \alpha(1 - \theta) \frac{cr^2}{r_1 \delta} \right), \frac{c\alpha(1 - \theta)x_i}{(1 + a(1 - \theta)x_i)(1 + b \frac{cr^2}{r_1 \delta})} - \delta_0 \right\}$, the theorem follows. \square

Remark 2.3.1. From the above theorem, we can conclude that the fear parameter k governs the persistence of the system.

2.3.2 Existence of equilibrium points

It is investigated that system (2.1) has three equilibria $E_0(0,0)$, $E_1(x_*,0)$ and $E^*(x^*,y^*)$. The trivial equilibrium $E_0(0,0)$ always exists.

- **Existence of $E_1(x_*,0)$:** It is easy to see that x_* is given by

$$x_* = \frac{1}{r_1}(r - r_0).$$

- **Existence of interior equilibrium $E^*(x^*,y^*)$:** It may be seen that x^* and y^* are the positive solutions of the following system of algebraic equations:

$$\begin{aligned} \frac{r}{1+ky} - r_0 - r_1x - \frac{\alpha(1-\theta)y}{(1+a(1-\theta)x)(1+by)} &= 0, \\ \frac{c\alpha(1-\theta)x}{(1+a(1-\theta)x)(1+by)} - \delta_0 - \delta_1y &= 0. \end{aligned} \quad (2.3)$$

From the first equation of system (2.3), the following points are noted:

- (i) When $y = 0$, then we get $r - r_0 - r_1x = 0$, which leads to

$$x = \frac{1}{r_1}(r - r_0) = x_*.$$

- (ii) When $x = 0$, then it leads to the following quadratic equation

$$m_1y^2 + m_2y + m_3 = 0, \quad (2.4)$$

where $m_1 = k(r_0b + \alpha(1-\theta))$, $m_2 = r_0k + b(r_0 - r) + \alpha(1-\theta)$, $m_3 = -(r - r_0)$. Eq. (2.4) has always a positive root y_1 .

- (iii)

$$\frac{dy}{dx} = -\frac{r_1 - \frac{\alpha a(1-\theta)^2 y}{(1+a(1-\theta)x)^2(1+by)}}{\frac{rk}{(1+ky)^2} + \frac{\alpha(1-\theta)}{(1+a(1-\theta)x)(1+by)^2}}.$$

It can be seen that $\frac{dy}{dx} < 0$ under the condition

$$r_1^2 \delta > \alpha a(1-\theta)^2 cr^2. \quad (2.5)$$

The above analysis shows that the first equation of system (2.3) passes through the points $(x_*, 0)$, $(0, y_1)$ and decreases under condition (2.5).

From the second equation of system (2.3), we note the following:

(i) When $y = 0$, then $x = \frac{\delta_0}{(1-\theta)(c\alpha - \delta_0 a)} =: x_1$.

(ii) When $x = 0$, then $y = -\frac{\delta_0}{\delta_1} =: y_2 < 0$.

(iii)

$$\frac{dy}{dx} = \frac{\frac{c\alpha(1-\theta)}{(1+a(1-\theta)x)^2(1+by)}}{\frac{c\alpha(1-\theta)bx}{(1+a(1-\theta)x)(1+by)^2} + \delta_1} > 0.$$

This shows that the second equation of system (2.3) passes through the points $(x_1, 0)$ $(0, y_2)$ and increases as $\frac{dy}{dx} > 0$.

From above analysis it can be concluded that system (2.3) has unique positive solution (x^*, y^*) if, in addition to condition (2.5), the following inequality holds true:

$$x_* > x_1, \quad (2.6)$$

Hence, we can state the following theorem.

Theorem 2.3.3. *The interior equilibrium $E^*(x^*, y^*)$ exists uniquely if (2.5) and (2.6) hold.*

2.3.3 Stability analysis

The local behavior of system around any existing equilibrium point is closely related to the behavior of corresponding variational (Jacobian) system. Therefore, we compute the Jacobian matrix. Further this matrix is evaluated at each equilibrium point. Then using the Routh-Hurwitz criterion, it is concluded that

- The equilibrium point $E_0(0, 0)$ is saddle point having stable manifold along the y -axis and unstable manifold along the x -axis.
- The Jacobian matrix, computed at $E_1(x_*, 0)$ is

$$J|_{E_1(x_*, 0)} = \begin{bmatrix} -(r - r_0) & -\left(\frac{2k(r-r_0)}{r_1} + \frac{\alpha(1-\theta)x_*}{1+a(1-\theta)x_*}\right) \\ 0 & \frac{c\alpha(1-\theta)x_*}{1+a(1-\theta)x_*} - \delta_0 \end{bmatrix}.$$

It is noted that the equilibrium point $E_1(x_*, 0)$ is

– locally asymptotically stable if

$$\frac{c\alpha(1-\theta)x_*}{1+a(1-\theta)x_*} < \delta_0.$$

– saddle point having stable manifold along the x -axis and unstable manifold along the y -axis if

$$\frac{c\alpha(1-\theta)x_*}{1+a(1-\theta)x_*} > \delta_0.$$

• The Jaxobian matrix, computed at interior equilibrium $E^*(x^*, y^*)$ is given by

$$J|_{E^*(x^*, y^*)} = \begin{bmatrix} -\left[r_1 x^* - \frac{\alpha a(1-\theta)^2 x^* y^*}{(1+a(1-\theta)x^*)^2 (1+by^*)} \right] & -\left[\frac{krx^*}{(1+ky^*)^2} + \frac{\alpha(1-\theta)x^*}{(1+a(1-\theta)x^*)(1+by^*)^2} \right] \\ \frac{c\alpha(1-\theta)y^*}{(1+a(1-\theta)x^*)^2 (1+by^*)} & -\left[\delta_1 y^* + \frac{c\alpha(1-\theta)bx^*y^*}{(1+a(1-\theta)x^*)(1+by^*)^2} \right] \end{bmatrix}.$$

$$\lambda^2 + A_1\lambda + A_2 = 0, \quad (2.7)$$

where

$$A_1 = \left[r_1 x^* - \frac{\alpha a(1-\theta)^2 x^* y^*}{(1+a(1-\theta)x^*)^2 (1+by^*)} \right] + \left[\delta_1 y^* + \frac{c\alpha(1-\theta)bx^*y^*}{(1+a(1-\theta)x^*)(1+by^*)^2} \right],$$

$$A_2 = \left[r_1 x^* - \frac{\alpha a(1-\theta)^2 x^* y^*}{(1+a(1-\theta)x^*)^2 (1+by^*)} \right] \left[\delta_1 y^* + \frac{c\alpha(1-\theta)bx^*y^*}{(1+a(1-\theta)x^*)(1+by^*)^2} \right] + \frac{c\alpha(1-\theta)y^*}{(1+a(1-\theta)x^*)^2 (1+by^*)} \left[\frac{krx^*}{(1+ky^*)^2} + \frac{\alpha(1-\theta)x^*}{(1+a(1-\theta)x^*)(1+by^*)^2} \right].$$

Now using the Routh-Hurwitz criterion, both the eigenvalues of $J|_{E^*(x^*, y^*)}$ have negative real part iff

$$A_1 > 0, \quad A_2 > 0. \quad (2.8)$$

Thus, we can state the following theorem.

Theorem 2.3.4. *The interior equilibrium point $E^*(x^*, y^*)$ is locally asymptotically stable in the xy -plane iff inequalities in (2.8) hold.*

It can easily be noted that (2.8) holds if

$$r_1 > \frac{\alpha a(1-\theta)^2 y^*}{(1+a(1-\theta)x^*)^2 (1+by^*)}. \quad (2.9)$$

Remark 2.3.2. *The interior equilibrium point $E^*(x^*, y^*)$ is locally asymptotically stable in the xy -plane if (2.9) holds.*

In the characteristic Eq. (2.7), if we assume that $A_2 < 0$, then both the eigenvalues are real and have opposite sign. Thus, the following theorem can be stated.

Theorem 2.3.5. *If $A_2 < 0$, then $E^*(x^*, y^*)$ is a saddle point.*

Again if we assume that $A_1 < 0$ and $A_2 > 0$, then eigenvalues are either real and positive or complex conjugate with positive real part. Thus, the following theorem follows.

Theorem 2.3.6. *If $A_1 < 0$ and $A_2 > 0$, then $E^*(x^*, y^*)$ is unstable.*

Now, we shall show the existence of limit cycle in the system via Hopf-bifurcation. In the characteristic Eq. (2.7), we assume that $A_1 = 0$, $A_2 > 0$. This gives us a critical value of fear parameter $k = k^*$, where

$$k^* = \frac{r - G}{Gy^*},$$

where

$$G = r_0 + 2rx^* + \frac{\alpha(1 - \theta)y^*}{(1 + a(1 - \theta)x^*)^2(1 + by^*)} + \delta_1 y^* + \frac{c\alpha(1 - \theta)bx^*y^*}{(1 + a(1 - \theta)x^*)(1 + by^*)^2}.$$

Theorem 2.3.7. *If $k = k^*$, which is defined above and $A_2 > 0$, then system (2.1) goes through a Hopf-bifurcation near the interior equilibrium point $E^*(x^*, y^*)$.*

Proof. We have $\text{tr}(J|_{E^*}) = 0$ and $\det(J|_{E^*}) = A_2 > 0$ at $k = k^*$, which implies the eigenvalues are purely imaginary. We also have

$$\left[\frac{d}{dk} \left(\text{tr}(J|_{E^*}) \right) \right]_{k=k^*} = -\frac{ry^*}{(1 + ky^*)^2} < 0.$$

Hence, it is noted that

- When $A_1 > 0$ and $A_2 > 0$, then $k > k^*$ and E^* is locally asymptotically stable.
- When $A_1 < 0$ and $A_2 > 0$, then $k < k^*$ and E^* is unstable.

Hence, by the Andronov-Hopf bifurcation theorem, system (2.1) has a Hopf-bifurcation near the interior equilibrium point $E^*(x^*, y^*)$. \square

In the following theorem we give a sufficient condition for global asymptotic stability of the unique positive equilibrium $E^*(x^*, y^*)$ of the system (2.1). It is assumed that conditions in Theorem 2.3.3 are satisfied.

Theorem 2.3.8. Assume that following inequality holds

$$\left(\frac{kr}{1+ky^*}\right)^2 < 4\left(r_1 - \frac{\alpha a(1-\theta)^2 y^*}{(1+a(1-\theta)x^*)(1+by^*)}\right)\left(\delta_1 + \frac{c\alpha(1-\theta)bx^*}{(1+a(1-\theta)x^*)(1+by^*)(1+b\frac{cr^2}{r_1\delta})}\right), \quad (2.10)$$

then the positive equilibrium $E^*(x^*, y^*)$ is globally asymptotically stable.

Proof. Choose a positive definite function about E^* as

$$V(x, y) = x - x^* - x^* \ln \frac{x}{x^*} + \beta \left(y - y^* - y^* \ln \frac{y}{y^*} \right),$$

where $\beta > 0$ to be specified later. Now, differentiating V with respect to t along the solutions of system (2.1), we have

$$\frac{dV}{dt} = \left(\frac{x-x^*}{x} \right) \frac{dx}{dt} + \beta \left(\frac{y-y^*}{y} \right) \frac{dy}{dt}.$$

On a little algebraic manipulation, we obtain

$$\begin{aligned} \frac{dV}{dt} = & \left(-r_1 + \frac{\alpha a(1-\theta)^2 y^*}{(1+a(1-\theta)x)(1+a(1-\theta)x^*)(1+by^*)} \right) (x-x^*)^2 \\ & - \beta \left(\delta_1 + \frac{c\alpha(1-\theta)bx^*}{(1+a(1-\theta)x^*)(1+by^*)(1+by)} \right) (y-y^*)^2 + \left(-\frac{kr}{(1+ky)(1+ky^*)} \right. \\ & \left. - \frac{\alpha(1-\theta)}{(1+by^*)(1+a(1-\theta)x)(1+by)} + \frac{\beta c\alpha(1-\theta)}{(1+a(1-\theta)x^*)(1+a(1-\theta)x)(1+by)} \right) (x-x^*)(y-y^*). \end{aligned}$$

Now choosing $\beta = \frac{(1+a(1-\theta)x^*)}{c(1+by^*)}$, we get

$$\begin{aligned} \frac{dV}{dt} = & -\left(r_1 - \frac{\alpha a(1-\theta)^2 y^*}{(1+a(1-\theta)x)(1+a(1-\theta)x^*)(1+by^*)} \right) (x-x^*)^2 - \frac{(1+a(1-\theta)x^*)}{c(1+by^*)} \left(\delta_1 \right. \\ & \left. + \frac{c\alpha(1-\theta)bx^*}{(1+a(1-\theta)x^*)(1+by^*)(1+by)} \right) (y-y^*)^2 - \frac{kr}{(1+ky)(1+ky^*)} (x-x^*)(y-y^*), \end{aligned}$$

which implies $\frac{dV}{dt}$ is negative definite under condition (2.10). Hence, E^* is globally asymptotically stable under condition (2.10). \square

2.3.4 Periodic orbit

In this section, we shall obtain certain conditions for the existence and non-existence of periodic orbit of system (2.1).

Theorem 2.3.9. Assume that

$$a(1 - \theta)(r - r_0) < r_1, \quad (2.11)$$

holds. Then the system (2.1) has no periodic solution in the interior of the xy -plane.

Proof. Let

$$f_1(x, y) = \frac{1}{1 + ky}rx - r_0x - r_1x^2 - \frac{\alpha(1 - \theta)xy}{(1 + a(1 - \theta)x)(1 + by)},$$

$$f_2(x, y) = \frac{c\alpha(1 - \theta)xy}{(1 + a(1 - \theta)x)(1 + by)} - \delta_0y - \delta_1y^2.$$

In the interior of the positive quadrant of the xy -plane, consider a continuously differentiable function as

$$\phi(x, y) = \frac{(1 + a(1 - \theta)x)(1 + by)}{xy}.$$

Now, we have

$$\begin{aligned} \nabla &= \frac{\partial}{\partial x}(f_1\phi) + \frac{\partial}{\partial y}(f_2\phi) \\ &= \frac{ra(1 - \theta)(1 + by)}{y(1 + ky)} - \frac{(1 + by)}{y} [r_1 + r_0a(1 - \theta) + 2r_1a(1 - \theta)x] - \frac{1 + a(1 - \theta)x}{x} [\delta_1 + b\delta_0 + 2b\delta_1y] \\ &= \frac{(1 + by)}{y} \left[\frac{ra(1 - \theta)}{(1 + ky)} - r_1 - r_0a(1 - \theta) - 2a(1 - \theta)r_1x \right] - \frac{1 + a(1 - \theta)x}{x} [\delta_1 + b\delta_0 + 2b\delta_1y]. \end{aligned}$$

It is noted that $\nabla < 0$ if $a(1 - \theta)(r - r_0) < r_1$. By Dulac-Bendixson criterion, we conclude that system (2.1) has no periodic orbit under assumption (2.11). \square

Theorem 2.3.10. If any one of the following holds:

- $A_2 < 0$,
- $A_1 < 0$ and $A_2 > 0$.

Then system (2.1) has at least one periodic orbit.

Proof. In Lemma 2.3.1, it is shown that solutions of system (2.1) are bounded and Ω is a positively invariant set. Above inequalities implies that E^* is unstable. Thus, the positive limit set does not contain any equilibrium point. Hence, by Poincare-Bendixson Theorem, system (2.1) admits at least a periodic solution. \square

2.4 Dynamics of Delayed Model (2.2)

Under our analysis similar to Section 2.3, it can easily be seen that all the solutions of delayed system (2.2) initiated from positive initial pair are positive and uniformly bounded.

2.4.1 Local stability and Hopf-bifurcation

Model system (2.2) can be rewritten as

$$\frac{dU(t)}{dt} = F(U(t), U(t - \tau)),$$

where $U(t) = [x(t), y(t)]^T$, $U(t - \tau) = [x(t - \tau), y(t - \tau)]^T$.

Let $x(t) = x^* + x'(t)$, $y(t) = y^* + y'(t)$. Then linearizing system (2.2) about the interior equilibrium solution $E^*(x^*, y^*)$, we have

$$\frac{dZ}{dt} = PZ(t) + QZ(t - \tau),$$

where

$$P = \left(\frac{\partial F}{\partial U(t)} \right)_{E^*}, \quad Q = \left(\frac{\partial F}{\partial U(t - \tau)} \right)_{E^*},$$

and $Z(t) = [x'(t), y'(t)]^T$.

Thus, the variational matrix of the system (2.2) at E^* is given by

$$J = P + Qe^{-\lambda\tau} = \begin{bmatrix} a_1 & a_2 \\ ca_3e^{-\lambda\tau} & a_4 + ca_5e^{-\lambda\tau} \end{bmatrix},$$

where

$$a_1 = \frac{r}{1+ky^*} - r_0 - 2r_1x^* - \frac{\alpha(1-\theta)y^*}{(1+a(1-\theta)x^*)^2(1+by^*)}, \quad a_2 = -\frac{rky^*}{(1+ky^*)^2} - \frac{\alpha(1-\theta)x^*}{(1+a(1-\theta)x^*)(1+by^*)^2},$$

$$a_3 = \frac{\alpha(1-\theta)y^*}{(1+a(1-\theta)x^*)^2(1+by^*)}, \quad a_4 = -\delta_0 - 2\delta_1y^*, \quad a_5 = \frac{\alpha(1-\theta)x^*}{(1+a(1-\theta)x^*)(1+by^*)^2}.$$

Characteristic equation of above Jacobian matrix is given by

$$\lambda^2 + b_1\lambda + b_2 + (b_3\lambda + b_4)e^{-\lambda\tau} = 0, \quad (2.12)$$

where

$$b_1 = -(a_1 + a_4), \quad b_2 = a_1a_4, \quad b_3 = -ca_5, \quad b_4 = c(a_1a_5 - a_2a_3).$$

Case (1): $\tau = 0$. Then characteristic equation becomes

$$\lambda^2 + (b_1 + b_3)\lambda + (b_2 + b_4) = 0. \quad (2.13)$$

Remark 2.4.1. The characteristic equation (2.13) is same as the characteristic Eq. (2.7) of the non-delayed model system (2.1), studied earlier.

All the roots of characteristic Eq. (2.13) have negative real part if and only if

$$(H_1) : b_1 + b_3 > 0, b_2 + b_4 > 0.$$

Case (2): $\tau > 0$. Let $i\omega$ ($\omega > 0$) be a root of Eq. (2.12), then it follows

$$-\omega^2 + b_1 i\omega + (b_3 i\omega + b_4)(\cos(\omega\tau) - i\sin(\omega\tau)) + b_2 = 0.$$

On equating real and imaginary parts, we obtain

$$\begin{aligned} b_3\omega \sin(\omega\tau) + b_4 \cos(\omega\tau) &= \omega^2 - b_2, \\ b_4 \sin(\omega\tau) - b_3\omega \cos(\omega\tau) &= b_1\omega, \end{aligned} \quad (2.14)$$

which leads to

$$z^2 + pz + q = 0, \quad (2.15)$$

where $p = b_1^2 - b_3^2 - 2b_2$, $q = b_2^2 - b_4^2$ and $z = \omega^2$.

Let $h(z) = z^2 + pz + q$.

$$(H_2) : p > 0, q > 0.$$

Remark 2.4.2. • If (H_2) holds, then Eq. (2.15) has no positive roots. Hence, all the roots of (2.12) have negative real part and hence $E^*(x^*, y^*)$ is asymptotically stable for all $\tau > 0$ under conditions (H_1) and (H_2) .

• If (H_1) fails and (H_2) holds true, then E^* is unstable for all $\tau > 0$.

$$(H_3) : q < 0.$$

If (H_1) and (H_3) hold, then Eq. (2.15) has a unique positive root ω_0^2 . Substitution of ω_0 into Eq. (2.12) gives us

$$\begin{aligned} b_3\omega_0 \sin(\omega_0\tau) + b_4 \cos(\omega_0\tau) &= \omega_0^2 - b_2, \\ b_4 \sin(\omega_0\tau) - b_3\omega_0 \cos(\omega_0\tau) &= b_1\omega_0, \end{aligned}$$

which yields

$$\tau_i = \frac{1}{\omega_0} \cos^{-1} \left[\frac{b_4(\omega_0^2 - b_2) - b_1 b_3 \omega_0^2}{b_3^2 \omega_0^2 + b_4^2} \right] + \frac{2i\pi}{\omega_0}, \quad i = 0, 1, 2, \dots \quad (2.16)$$

$$(H_4) : p < 0, q > 0, p^2 > 4q.$$

If (H_1) and (H_4) hold, then Eq. (2.15) has two positive roots ω_1^2 and ω_2^2 , substituting $\omega_{1,2}^2$ into

Eq. (2.14), we have

$$\tau_j^{1,2} = \frac{1}{\omega_{1,2}} \cos^{-1} \left[\frac{b_4(\omega_{1,2}^2 - b_2) - b_1 b_3 \omega_{1,2}^2}{b_3^2 \omega_{1,2}^2 + b_4^2} \right] + \frac{2j\pi}{\omega_{1,2}}, \quad j = 0, 1, 2, \dots$$

Let $\lambda(\tau)$ be a root of Eq. (2.12) satisfying $Re\{\lambda(\tau_i)\} = 0$. Then differentiating Eq. (2.12) with respect to τ , we obtain.

$$\left(\frac{d\lambda}{d\tau} \right)^{-1} = \frac{(2\lambda + b_1)e^{\lambda\tau}}{(b_3\lambda + b_4)\lambda} + \frac{b_3}{(b_3\lambda + b_4)\lambda} - \frac{\tau}{\lambda},$$

$$\left[\frac{d\lambda}{d\tau} \right]_{\lambda=i\omega_0}^{-1} = \frac{(2i\omega_0 + b_1)e^{i\omega_0\tau}}{(b_3i\omega_0 + b_4)i\omega_0} + \frac{b_3}{(b_3i\omega_0 + b_4)i\omega_0} - \frac{\tau}{i\omega_0},$$

$$Re \left[\frac{d\lambda}{d\tau} \right]_{\lambda=i\omega_0}^{-1} = \frac{-b_3\omega_0^2(b_1 \cos(\omega_0\tau) - 2\omega_0 \sin(\omega_0\tau)) + C\omega_0(2\omega_0 \cos(\omega_0\tau) + b_1 \sin(\omega_0\tau)) - b_3^2\omega_0^2}{b_3^2\omega_0^4 + b_4^2\omega_0^2},$$

where $\sin(\omega_0\tau)$ computed as

$$\sin(\omega_0\tau) = \frac{b_1 b_4 \omega_0 + b_3 \omega_0 (\omega_0^2 - b_2)}{b_3^2 \omega_0^2 + b_4^2}.$$

After a little calculation, we obtain

$$Re \left[\frac{d\lambda}{d\tau} \right]_{\lambda=i\omega_0}^{-1} = \frac{h'(\omega_0^2)}{b_3^2 \omega_0^2 + b_4^2}.$$

But $sign \left[\frac{d}{d\tau} Re(\lambda) \right]_{\lambda=i\omega_0} = sign \left[Re \left(\frac{d\lambda}{d\tau} \right) \right]_{\lambda=i\omega_0}$

$(H_5) : h'(\omega_0^2) \neq 0$.

Hence $\left[\frac{d}{d\tau} Re(\lambda) \right]_{\lambda=i\omega_0} \neq 0$ under condition (H_5) .

Now we are in position to state the following theorem.

Theorem 2.4.1. *For system (2.2), assume that (H_1) , (H_3) and (H_5) hold. Then there exists a positive number τ_0 such that the equilibrium E^* is locally asymptotically stable when $\tau < \tau_0$ and unstable when $\tau > \tau_0$. Furthermore, system undergoes a Hopf-bifurcation at E^* when $\tau = \tau_0$.*

2.4.2 Global stability

Theorem 2.4.2. *The interior equilibrium E^* of the delayed system (2.2) is globally asymptotically stable under condition (2.10).*

Proof. We consider a Lyapunov function about E^* as

$$V_1(x, y) = x - x^* - x^* \ln \frac{x}{x^*} + \frac{(1 + a(1 - \theta)x^*)}{c(1 + by^*)} \left(y - y^* - y^* \ln \frac{y}{y^*} \right).$$

Differentiating V_1 with respect to t along the solutions of system (2.2)

$$\begin{aligned} \frac{dV_1}{dt} &= \left(\frac{x - x^*}{x} \right) \frac{dx}{dt} + \frac{(1 + a(1 - \theta)x^*)}{c(1 + by^*)} \left(\frac{y - y^*}{y} \right) \frac{dy}{dt} \\ &= (x - x^*) \left[\frac{r}{1 + ky} - r_0 - r_1 x - \frac{\alpha(1 - \theta)y}{(1 + a(1 - \theta)x)(1 + by)} \right] + (y - y^*) \left[\frac{c\alpha(1 - \theta)x}{(1 + a(1 - \theta)x)(1 + by)} \right. \\ &\quad \left. - \delta_0 - \delta_1 y \right] + \frac{(1 + a(1 - \theta)x^*)}{c(1 + by^*)} \left(\frac{y - y^*}{y} \right) \frac{c\alpha(1 - \theta)x(t - \tau)y(t - \tau)}{(1 + a(1 - \theta)x(t - \tau))(1 + by(t - \tau))} \\ &\quad - \frac{(1 + a(1 - \theta)x^*)}{c(1 + by^*)} (y - y^*) \frac{c\alpha(1 - \theta)x}{(1 + a(1 - \theta)x)(1 + by)}. \end{aligned}$$

On simplification, one can obtain

$$\begin{aligned} \frac{dV_1}{dt} &= - \left(r_1 - \frac{\alpha a(1 - \theta)^2 y^*}{(1 + a(1 - \theta)x)(1 + a(1 - \theta)x^*)(1 + by^*)} \right) (x - x^*)^2 - \frac{(1 + a(1 - \theta)x^*)}{c(1 + by^*)} \left(\delta_1 \right. \\ &\quad \left. + \frac{c\alpha(1 - \theta)bx^*}{(1 + a(1 - \theta)x^*)(1 + by^*)(1 + by)} \right) (y - y^*)^2 - \frac{kr}{(1 + ky)(1 + ky^*)} (x - x^*)(y - y^*) \\ &\quad + \frac{(1 + a(1 - \theta)x^*)}{c(1 + by^*)} \left(\frac{y - y^*}{y} \right) \frac{c\alpha(1 - \theta)x(t - \tau)y(t - \tau)}{(1 + a(1 - \theta)x(t - \tau))(1 + by(t - \tau))} \\ &\quad - \frac{(1 + a(1 - \theta)x^*)}{c(1 + by^*)} (y - y^*) \frac{c\alpha(1 - \theta)x}{(1 + a(1 - \theta)x)(1 + by)}. \end{aligned}$$

Now let

$$\begin{aligned} V_2 &= V_1 + c\alpha(1 - \theta) \int_{t-\tau}^t \left[\frac{x(s)y(s)}{(1 + a(1 - \theta)x(s))(1 + by(s))} - \frac{x^*y^*}{(1 + a(1 - \theta)x^*)(1 + by^*)} \right. \\ &\quad \left. - \frac{x^*y^*}{(1 + a(1 - \theta)x^*)(1 + by^*)} \ln \left(\frac{x(s)y(s)(1 + a(1 - \theta)x^*)(1 + by^*)}{x^*y^*(1 + a(1 - \theta)x(s))(1 + by(s))} \right) \right] ds. \end{aligned}$$

Then we have

$$\begin{aligned} \frac{dV_2}{dt} &= \frac{dV_1}{dt} + \frac{c\alpha(1 - \theta)xy}{(1 + a(1 - \theta)x)(1 + by)} \left[1 - \frac{x(t - \tau)y(t - \tau)}{xy} \frac{(1 + a(1 - \theta)x)(1 + by)}{(1 + a(1 - \theta)x(t - \tau))(1 + by(t - \tau))} \right. \\ &\quad \left. - \frac{x^*y^*(1 + a(1 - \theta)x)(1 + by)}{xy(1 + a(1 - \theta)x^*)(1 + by^*)} \ln \left(\frac{xy(1 + a(1 - \theta)x(t - \tau))(1 + by(t - \tau))}{x(t - \tau)y(t - \tau)(1 + a(1 - \theta)x)(1 + by)} \right) \right]. \end{aligned}$$

Noting that $\frac{c\alpha(1-\theta)xy}{(1+a(1-\theta)x)(1+by)} = \frac{c\alpha(1-\theta)x^*y^*}{(1+a(1-\theta)x^*)(1+by^*)}$ at E^* , we have

$$\begin{aligned} \frac{dV_2}{dt} = & \frac{dV_1}{dt} + \frac{c\alpha(1-\theta)x^*y^*}{(1+a(1-\theta)x^*)(1+by^*)} \left[1 - \frac{x(t-\tau)y(t-\tau)}{xy} \frac{(1+a(1-\theta)x)(1+by)}{(1+a(1-\theta)x(t-\tau))(1+by(t-\tau))} \right. \\ & \left. + \ln \left(\frac{x(t-\tau)y(t-\tau)(1+a(1-\theta)x)(1+by)}{xy(1+a(1-\theta)x(t-\tau))(1+by(t-\tau))} \right) \right]. \end{aligned}$$

Using $n = 1$ in the result obtained by Manna and Chakrabarty [139], it follows that

$$\begin{aligned} \left[1 - \frac{x(t-\tau)y(t-\tau)}{xy} \frac{(1+a(1-\theta)x)(1+by)}{(1+a(1-\theta)x(t-\tau))(1+by(t-\tau))} \right. \\ \left. + \ln \left(\frac{x(t-\tau)y(t-\tau)(1+a(1-\theta)x)(1+by)}{xy(1+a(1-\theta)x(t-\tau))(1+by(t-\tau))} \right) \right] \leq 0. \end{aligned}$$

Thus $\frac{dV_2}{dt} \leq 0$ with equality iff $x = x^*$, $y = y^*$. The invariant subset S within the set $S = \{(x, y) | x = x^*, y = y^*\}$. Thus $S = \{E^*\}$. Using LaSalle invariance principle, E^* is globally asymptotically stable under condition (2.10). \square

Remark 2.4.3. E^* is globally asymptotically stable for delayed system (2.2) if it is globally asymptotically stable for non-delayed system (2.1).

2.5 Direction and Stability of Hopf-bifurcation

In the previous section, we have obtained that when gestation delay τ crosses the threshold value τ_0 , the model (2.2) becomes unstable and system undergoes a hopf-bifurcation. In this section, we will discuss the properties of the bifurcated periodic solutions using the center manifold theorem and normal form theory [80].

Let us assume that $\tau = \tau_0 + \mu$, $\mu \in \mathbb{R}$, so that the Hopf-bifurcation occurs at $\mu = 0$. Considering the transformation

$$x_1(t) = x(t) - x^*, \quad y_1(t) = y(t) - y^*,$$

and still denote $x_1(t), y_1(t)$ by $x(t), y(t)$, respectively, our delayed model (2.2) is transformed into the following functional differential equation in $C = C([-1, 0], \mathbb{R}^2)$

$$\dot{U}(t) = \tau \left(PU(t) + QU(t-1) + f(x, y) \right), \quad (2.17)$$

where $U(t) = (x(t), y(t))^T$,

$$P = \begin{bmatrix} a_1 & a_2 \\ 0 & a_4 \end{bmatrix}, Q = \begin{bmatrix} 0 & 0 \\ ca_3 & ca_5 \end{bmatrix}, f(x, y) = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}.$$

The nonlinear term f_1, f_2 are given by

$$\begin{aligned} f_1 = & -2r_1x^2(t) + \frac{2\alpha a(1-\theta)^2y^*}{(1+a(1-\theta)x^*)^3(1+by^*)}x^2(t) + \frac{2k^2x^*}{(1+ky^*)}y^2(t) \\ & + \frac{2\alpha b(1-\theta)x^*}{(1+a(1-\theta)x^*)(1+by^*)^3}y^2(t) - \frac{rk}{(1+ky^*)^2}x(t)y(t) - \frac{\alpha(1-\theta)x(t)y(t)}{(1+a(1-\theta)x^*)^2(1+by^*)^2} \\ & - \frac{6\alpha a^2(1-\theta)^3y^*}{(1+a(1-\theta)x^*)^4(1+by^*)}x^3(t) - \frac{6k^3x^*}{(1+ky^*)^4}y^3(t) - \frac{6\alpha b^2(1-\theta)x^*}{(1+a(1-\theta)x^*)(1+by^*)^4}y^3(t) \\ & + \frac{2\alpha a(1-\theta)^2}{(1+a(1-\theta)x^*)^3(1+by^*)^2}x^2(t)y(t) + \frac{2rk^2}{(1+ky^*)^3}x(t)y^2(t) \\ & + \frac{2\alpha b(1-\theta)}{(1+a(1-\theta)x^*)^2(1+by^*)^3}x(t)y^2(t) + \dots, \end{aligned}$$

$$\begin{aligned} f_2 = & -2\delta_1y^2(t) - \frac{2c\alpha a(1-\theta)^2y^*}{(1+a(1-\theta)x^*)^3(1+by^*)}x^2(t-\tau) - \frac{2c\alpha b(1-\theta)x^*}{(1+a(1-\theta)x^*)(1+by^*)^3}y^2(t-\tau) \\ & + \frac{c\alpha(1-\theta)}{(1+a(1-\theta)x^*)^2(1+by^*)^2}x(t-\tau)y(t-\tau) + \frac{6c\alpha a^2(1-\theta)^3y^*}{(1+a(1-\theta)x^*)^4(1+by^*)}x^3(t-\tau) \\ & + \frac{6c\alpha b^2(1-\theta)x^*}{(1+a(1-\theta)x^*)(1+by^*)^4}y^3(t-\tau) - \frac{2c\alpha a(1-\theta)^2}{(1+a(1-\theta)x^*)^3(1+by^*)^2}x^2(t-\tau)y(t-\tau) \\ & - \frac{2c\alpha b(1-\theta)}{(1+a(1-\theta)x^*)^2(1+by^*)^3}x(t-\tau)y^2(t-\tau) + \dots. \end{aligned}$$

The linearization of equation (2.17) around the origin is given by

$$\dot{U}(t) = \tau(PU(t) + QU(t-1)).$$

For $\phi = (\phi_1, \phi_2)^T \in C([-1, 0], R^2)$, we define

$$L_\mu(\phi) = (\tau + \mu)(P\phi(0) + Q\phi(-1)).$$

By the Riesz representation theorem, there exists a 2×2 matrix $\zeta(\theta, \mu)$, $(-1 \leq \theta \leq 0)$ whose elements are of bounded variation function such that

$$L_\mu(\phi) = \int_{-1}^0 d\zeta(\theta, \mu)\phi(\theta) \text{ for } \phi \in C([-1, 0], R^2).$$

In fact, we can choose

$$\zeta(\theta, \mu) = (\tau_0 + \mu)[P\delta(\theta) - Q\delta(\theta + 1)],$$

where δ is the Dirac-delta function.

For $\phi \in C^1([-1, 0], \mathbb{R}^2)$, we define

$$A(\mu)\phi(\theta) = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & \text{if } \theta \in [-1, 0) \\ \int_{-1}^0 [d\zeta(\xi, \mu)]\phi(\xi), & \text{if } \theta = 0, \end{cases}$$

and

$$R(\mu)\phi(\theta) = \begin{cases} 0, & \text{if } \theta \in [-1, 0) \\ h(\mu, \phi), & \text{if } \theta = 0, \end{cases}$$

where

$$h(\mu, \phi) = (\tau_0 + \mu) \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}, \quad \phi = (\phi_1, \phi_2)^T \in C([-1, 0], \mathbb{R}^2),$$

$$\begin{aligned} h_1 = & -2r_1x^2(0) + \frac{2\alpha a(1-\theta)^2y^*}{(1+a(1-\theta)x^*)^3(1+by^*)}x^2(0) + \frac{2k^2x^*}{(1+ky^*)}y^2(0) \\ & + \frac{2\alpha b(1-\theta)x^*}{(1+a(1-\theta)x^*)(1+by^*)^3}y^2(0) - \frac{rk}{(1+ky^*)^2}x(0)y(0) - \frac{\alpha(1-\theta)x(0)y(0)}{(1+a(1-\theta)x^*)^2(1+by^*)^2} \\ & - \frac{6\alpha a^2(1-\theta)^3y^*}{(1+a(1-\theta)x^*)^4(1+by^*)}x^3(0) - \frac{6k^3x^*}{(1+ky^*)^4}y^3(0) - \frac{6\alpha b^2(1-\theta)x^*}{(1+a(1-\theta)x^*)(1+by^*)^4}y^3(0) \\ & + \frac{2\alpha a(1-\theta)^2}{(1+a(1-\theta)x^*)^3(1+by^*)^2}x^2(0)y(0) + \frac{2rk^2}{(1+ky^*)^3}x(0)y^2(0) \\ & + \frac{2\alpha b(1-\theta)}{(1+a(1-\theta)x^*)^2(1+by^*)^3}x(0)y^2(0) + \dots, \end{aligned}$$

$$\begin{aligned} h_2 = & -2\delta_1y^2(0) - \frac{2c\alpha a(1-\theta)^2y^*}{(1+a(1-\theta)x^*)^3(1+by^*)}x^2(-\tau) - \frac{2c\alpha b(1-\theta)x^*}{(1+a(1-\theta)x^*)(1+by^*)^3}y^2(-\tau) \\ & + \frac{c\alpha(1-\theta)}{(1+a(1-\theta)x^*)^2(1+by^*)^2}x(-\tau)y(-\tau) + \frac{6c\alpha a^2(1-\theta)^3y^*}{(1+a(1-\theta)x^*)^4(1+by^*)}x^3(-\tau) \\ & + \frac{6c\alpha b^2(1-\theta)x^*}{(1+a(1-\theta)x^*)(1+by^*)^4}y^3(-\tau) - \frac{2c\alpha a(1-\theta)^2}{(1+a(1-\theta)x^*)^3(1+by^*)^2}x^2(-\tau)y(-\tau) \\ & - \frac{2c\alpha b(1-\theta)}{(1+a(1-\theta)x^*)^2(1+by^*)^3}x(-\tau)y^2(-\tau) + \dots \end{aligned}$$

Then system (2.2) is equivalent to the following operator equation

$$\dot{U}_t = A(\mu)U_t + R(\mu)U_t,$$

where $U_t = U(t + \theta)$ for $\theta \in [-1, 0]$.

For $\psi \in C^1([0, 1], (R^2)^*)$, we define

$$A^* \psi(s) = \begin{cases} -\frac{d\psi(s)}{ds}, & \text{if } s \in (0, 1] \\ \int_{-1}^0 \psi(-\xi) d\zeta^T(\xi, 0), & \text{if } s = 0, \end{cases}$$

and a bilinear form

$$\langle \psi(s), \phi(\theta) \rangle = \bar{\psi}(0)\phi(0) - \int_{-1}^0 \int_{\xi=0}^{\theta} \bar{\psi}(\xi - \theta) d\zeta(\theta) \phi(\xi) d\xi,$$

where $\zeta(\theta) = \zeta(\theta, 0)$, $A = A(0)$ and A^* are adjoint operators. From the discussion in the previous section, we know that $\pm i\omega_0\tau_0$ are the eigenvalues of $A(0)$ and therefore they are also eigenvalues of A^* . It is not difficult to verify that the vectors $q(\theta) = (1, \alpha_1)^T e^{i\omega_0\tau_0\theta}$ ($\theta \in [-1, 0]$) and $q^*(s) = D(1, \alpha_1^*) e^{i\omega_0\tau_0 s}$ ($s \in [0, 1]$) are the eigenvectors of $A(0)$ and A^* corresponding to the eigenvalue $i\omega_0\tau_0$ and $-i\omega_0\tau_0$ respectively, where

$$\langle q^*(s), q(\theta) \rangle = 1, \quad \langle q^*(s), \bar{q}(\theta) \rangle = 1,$$

$$\alpha_1 = \frac{ca_3 e^{-i\omega_0\tau_0}}{i\omega_0 - a_4 - ca_5 e^{-i\omega_0\tau_0}}, \quad \alpha_1^* = -\frac{a_2}{i\omega_0 + a_4 + a_5 e^{-i\omega_0\tau_0}},$$

$$\bar{D} = \frac{1}{[1 + \alpha_1 \bar{\alpha}_1^* + \tau_0 (ca_3 \bar{\alpha}_1^* + ca_5 \alpha_1 \bar{\alpha}_1^*) e^{-i\omega_0\tau_0}]}.$$

Following the algorithms explained in Hassard *et al.* [80] and using a computation process similar to that in Song and Wei [195], which is used to obtain the properties of Hopf-bifurcation, we obtain

$$g_{20} = 2\tau_0 \bar{D} \left[-r\alpha_1 k + \frac{1}{2} W_{20}^{(1)}(0) - r_1 - \alpha(1-\theta)\alpha_1 + c\alpha(1-\theta)\alpha_1^* \alpha_1 e^{-2i\omega_0\tau_0} - \delta_1 \alpha_1^2 \alpha_1^* \right],$$

$$g_{11} = \tau_0 \bar{D} \left[-k(\alpha_1 + \bar{\alpha}_1) - 2r_1 - \alpha(1-\theta)(\alpha_1 + \bar{\alpha}_1) + c\alpha(1-\theta)\bar{\alpha}_1^*(\alpha_1 + \bar{\alpha}_1) - 2\delta_1 \alpha_1^* \alpha_1 \bar{\alpha}_1 \right],$$

$$g_{02} = 2\tau_0 \bar{D} \left[-r\bar{\alpha}_1 k + \frac{1}{2} W_{02}^{(1)}(0) - r_1 - \alpha(1-\theta)\bar{\alpha}_1 + c\alpha(1-\theta)\alpha_1^* \bar{\alpha}_1 e^{2i\omega_0\tau_0} - \delta_1 \bar{\alpha}_1^2 \alpha_1^* \right],$$

$$\begin{aligned}
g_{21} = & 2\tau_0 \bar{D} \left[-rW_{11}^{(2)}(0) - \frac{1}{2}rkW_{20}^{(2)}(0) - \frac{1}{2}rk\bar{\alpha}_1W_{20}^{(1)}(0) - rk\alpha_1W_{11}^{(1)}(0) - 2r_1W_{11}^{(1)}(0) \right. \\
& - r_1W_{20}^{(1)}(0) - \alpha(1-\theta)(W_{11}^{(2)}(0) + \frac{1}{2}W_{20}^{(2)}(0) + \frac{1}{2}\bar{\alpha}_1W_{20}^{(1)}(0) + \alpha_1W_{11}^{(1)}(0)) \\
& + \alpha(1-\theta)(\alpha_1 + \bar{\alpha}_1)(a(1-\theta) + b\alpha_1) + \alpha(1-\theta)\alpha_1(a(1-\theta) + b\bar{\alpha}_1) + \bar{\alpha}_1^*c\alpha(1-\theta) \\
& \left(W_{11}^{(2)}(-1)e^{-i\omega_0\tau_0} + \frac{1}{2}W_{20}^{(2)}(-1)e^{i\omega_0\tau_0} + \frac{1}{2}\bar{\alpha}_1W_{20}^{(1)}(-1)e^{i\omega_0\tau_0} + \alpha_1W_{11}^{(1)}(-1)e^{-i\omega_0\tau_0} \right) \\
& - \bar{\alpha}_1^*c\alpha(1-\theta)e^{-i\omega_0\tau_0}(\alpha_1 + \bar{\alpha}_1)(a(1-\theta) + \alpha_1b) - \bar{\alpha}_1^*c\alpha(1-\theta)e^{-i\omega_0\tau_0}\alpha_1(a(1-\theta) + \bar{\alpha}_1b) \\
& \left. - \delta_1\bar{\alpha}_1^*(2\alpha_1W_{11}^{(2)}(0) + \bar{\alpha}_1W_{20}^{(2)}(0)) \right],
\end{aligned}$$

where

$$W_{20}(\theta) = \frac{ig_{20}}{\omega_0\tau_0}q(0)e^{i\omega_0\tau_0\theta} + \frac{i\bar{g}_{02}}{3\omega_0\tau_0}\bar{q}(0)e^{-i\omega_0\tau_0\theta} + E_1e^{2i\omega_0\tau_0\theta},$$

$$W_{11}(\theta) = -\frac{ig_{11}}{\omega_0\tau_0}q(0)e^{i\omega_0\tau_0\theta} + \frac{i\bar{g}_{11}}{\omega_0\tau_0}\bar{q}(0)e^{-i\omega_0\tau_0\theta} + E_2,$$

$E_1 = (E_1^{(1)}, E_1^{(2)})^T \in \mathbb{R}^2$ and $E_2 = (E_2^{(1)}, E_2^{(2)})^T \in \mathbb{R}^2$ are constant vectors, computed as:

$$E_1 = 2 \begin{bmatrix} 2i\omega_0 - a_1 & -a_2 \\ -ca_3\eta e^{-2i\omega_0\tau_0} & 2i\omega_0 - a_4 - ca_5 e^{-2i\omega_0\tau_0} \end{bmatrix}^{-1} \begin{bmatrix} r - r_1 - \alpha(1-\theta)\alpha_1 \\ c\alpha(1-\theta)\alpha_1 e^{-2i\omega_0\tau_0} \end{bmatrix},$$

$$E_2 = 2 \begin{bmatrix} -a_1 & -a_2 \\ -ca_3 & -a_4 - ca_5 \end{bmatrix}^{-1} \begin{bmatrix} r - r_1 - \frac{1}{2}\alpha(1-\theta)(\alpha_1 + \bar{\alpha}_1) - \delta_1\alpha_1^2 \\ \frac{1}{2}c\alpha(1-\theta)(\alpha_1 + \bar{\alpha}_1) - \delta_1\alpha_1\bar{\alpha}_1 \end{bmatrix}.$$

Consequently, g_{ij} can be expressed by the parameters and delays τ_0 . Thus, these standard results can be computed as:

$$c_1(0) = \frac{i}{2\omega_0\tau_0} \left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3} \right) + \frac{g_{21}}{2}, \quad \mu_2 = -\frac{Re(c_1(0))}{Re(\lambda'(\tau_0))},$$

$$\beta_2 = 2Re(c_1(0)), \quad T_2 = -\frac{Im(c_1(0)) + \mu_2 Im(\lambda'(\tau_0))}{\omega_0\tau_0}.$$

These expressions give a description of the bifurcating periodic solution in the center manifold of system (2.2) at critical values $\tau = \tau_0$ which can be stated in the form of following theorem:

Theorem 2.5.1. • If $\mu_2 > 0 (< 0)$, then the Hopf-bifurcation is supercritical (subcritical).

• If $\beta_2 > 0 (< 0)$, then the bifurcated periodic solutions are unstable (stable).

• If $T_2 > 0 (< 0)$, then the period increases (decreases).

2.6 Numerical Simulation

In order to show the theoretical results derived in previous sections computationally, we carry out numerical simulation over the following set of values of parameters corresponding to system (2.1) and (2.2).

Table 2.1: List of parameters for model (2.1) and (2.2)

Parameters	Biological meaning	Numerical value	Unit	Source
r	Birth rate of prey population	3	day ⁻¹	[138]
r_0	Death rate of prey population	0.03	day ⁻¹	-
r_1	Coefficient of intraspecific interference among prey individuals	0.1	meter ² day ⁻¹ individual ⁻¹	-
k	Cost of fear	30	predator individual ⁻¹	[220]
θ	Prey refuge parameter	0.3	Constant	[100]
a	Handling time	0.2	day	[135]
b	Magnitude of interference among predators	0.01	day ⁻¹	-
α	Attack rate	2	day ⁻¹	[167, 46],
c	Conversion efficiency of y on x	0.4	Constant	[220, 155]
δ_0	Death rate of predator population	0.5	day ⁻¹	[184]
δ_1	Coefficient of intraspecific interference among predator individuals	0.025	day ⁻¹	-

For the above set of parameters, system (2.1) has three equilibrium points, namely trivial equilibrium point $E_0(0,0)$, axial equilibrium point $E_1(29.7,0)$ and positive equilibrium point $E^*(1.0357,0.2174)$. The eigenvalues of the Jacobian matrices evaluated at E_0 and E_1 are $(2.97, -0.5)$ and $(-2.97, 4.1064)$, respectively. Therefore E_0 and E_1 both are saddle points. Again the Jacobian matrix at the positive equilibrium E^* has complex conjugate eigenvalues

with negative real part as the inequalities given by (2.8) hold. Hence, the interior equilibrium E^* is locally asymptotically stable (Theorem 2.3.4). It is illustrated by Fig. 2.1.

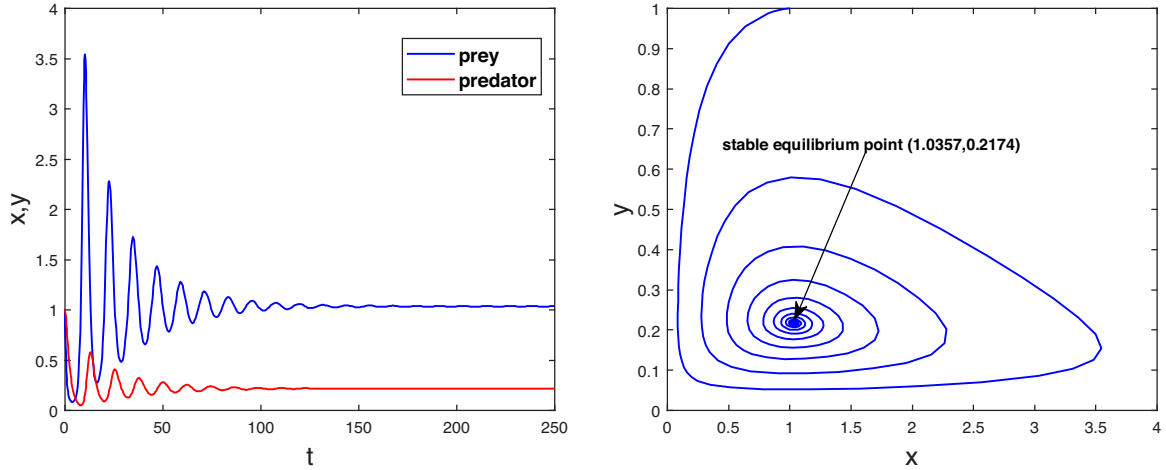


Fig. 2.1: Solution curves (left) and phase diagram (right) for the model system (2.1) for the set of values of parameters chosen in Table 2.1.

Throughout in this chapter, the cost of fear k and prey refuse θ are crucial parameters. Therefore it seems reasonable to study the effect of variation of these two parameters on the dynamics of system (2.1). In Theorem 2.3.7 we have shown analytically that system (2.1) undergoes Hopf-bifurcation at $k = k^*$. $k^* = 1.8998$ is computed by formula, derived in section 2.3. To determine the direction of Hopf-bifurcation we computed the value of σ^* by the formula given in Wang *et al.* [220]. The value of σ^* is $-0.0759 < 0$. By Perko [169] (Theorem 1 on page 34), Hopf-bifurcation is supercritical as $\sigma^* < 0$. The instable behavior of solutions and existence of stable limit cycle at $k = 1 < k^* = 1.8998$ is depicted in Fig. 2.2. In Fig. 2.3 we draw the bifurcation diagram for both the species with respect to fear parameter k . It can easily be seen that oscillations move off from the system as we increase the value of k . For $k = 30 > k^*$, time series of species and phase portrait are depicted in Fig. 2.1 which show the stable behavior of the system around E^* .

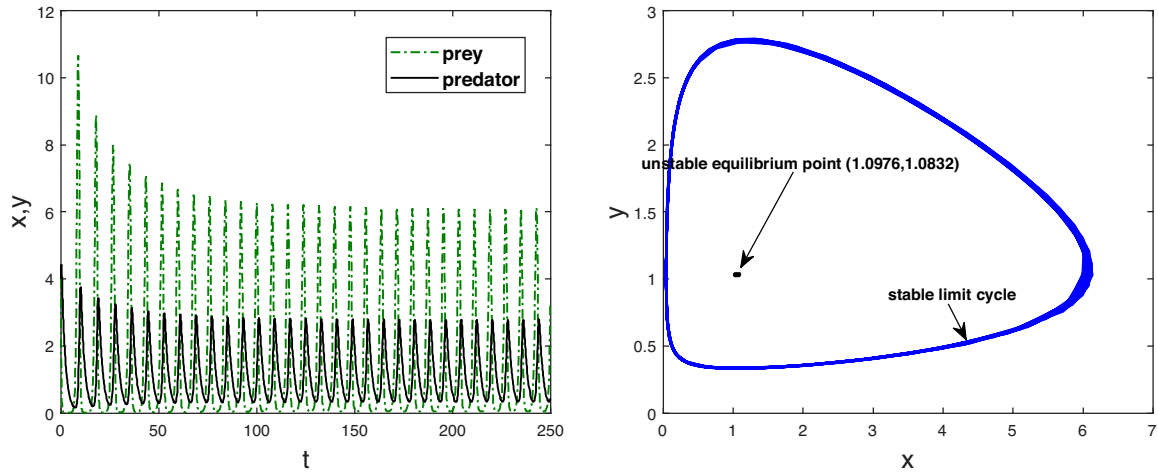


Fig. 2.2: Instable behavior of solutions and existence of stable limit cycle for $k = 1 (< k^*)$. Other parametric values are same as in Table 2.1.

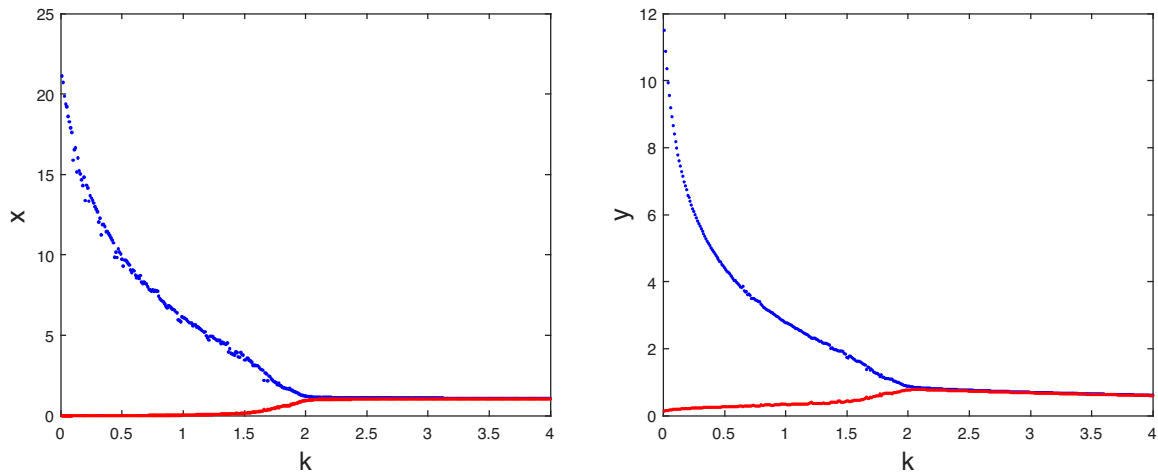


Fig. 2.3: Bifurcation diagram of both the populations with respect to cost of fear k .

The effect of variation of refuge parameter θ on the steady state level of prey and predator species is shown in Fig. 2.4. It is observed that x^* always increases with θ . On the other hand y^* increases initially with θ but beyond a threshold value of θ it starts decreasing and vanishes at $\theta = 1$ which represents extinction of predators as a consequence of inaccessibility of food resources. For the set of values of parameters chosen in Table 1, the threshold value of θ is computed as $\theta^* = 0.5781$. This shows that prey refuge makes the positive effect on both the species if it is under the critical level. But beyond this level, it is harmful to predators.

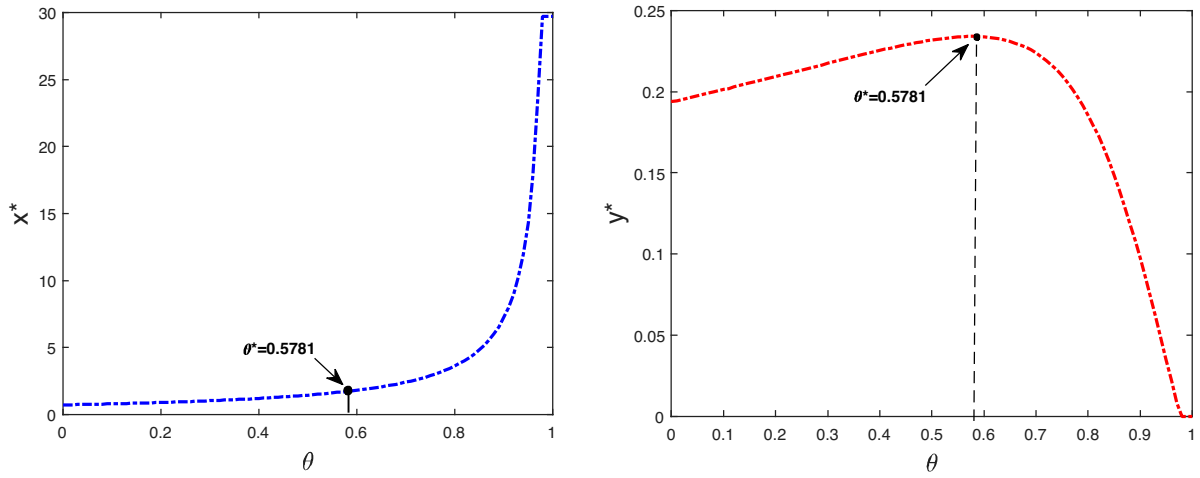


Fig. 2.4: Effect of refuge parameter θ on the steady state level of populations.

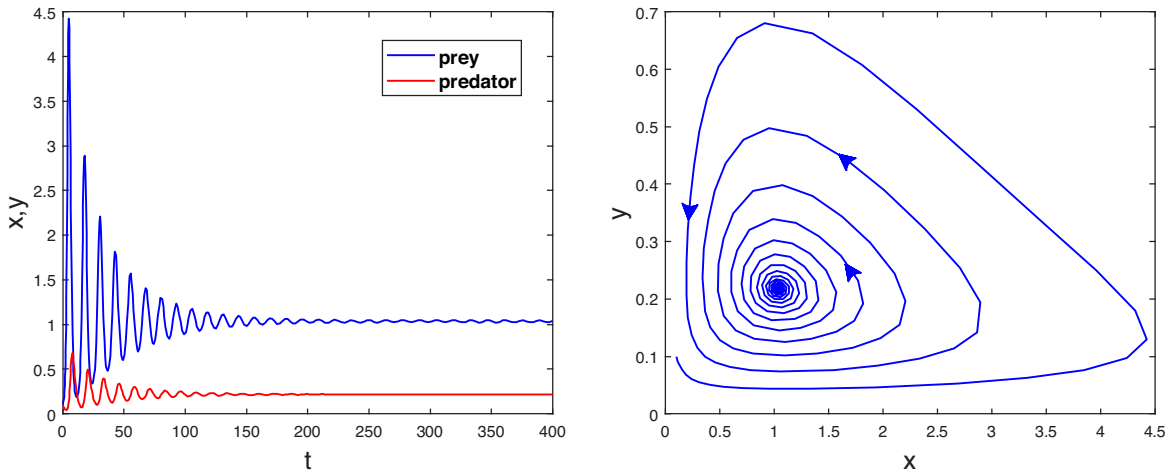


Fig. 2.5: Time evolution of x & y and phase portrait when $\tau = 0.1 (< \tau^*)$. Figure shows stable behavior of system (2.2) around E^* .

In order to validate the analytical findings derived for delayed system (2.2), we consider the same set of values of parameters given in Table 1. The introduction of gestation delay does not affect the equilibrium point. Therefore $E^*(1.0357, 0.2174)$ is an interior equilibrium point of model system (2.2). For the chosen values of parameters, hypotheses (H_1) and (H_3) hold. Apart from these, the transversality condition is also satisfied. Assuming $i = 0$ in Eq. (2.16), we obtain

$$\omega_0 = 0.4751, \quad \tau_0 = 0.3609.$$

According to Theorem 2.4.1, system (2.2) undergoes a Hopf-bifurcation at $\tau = \tau_0$, E^* is locally asymptotically stable for $\tau < \tau_0$ and unstable for $\tau > \tau_0$. In Fig. 2.5 we draw time series of

solutions (left) and phase portrait (right) of the system at $\tau = 0.1 < \tau_0$, which shows stable behavior of the system around E^* . System remains stable unless the value of τ does not exceed τ_0 . The positive equilibrium point E^* losses its stability at the critical value of $\tau = \tau_0 = 0.3609$ and remains unstable for further increment in τ . Fig. 2.6 depicts the time series of solutions and existence of stable limit cycle showing instable behavior of system at $\tau = 0.4$.

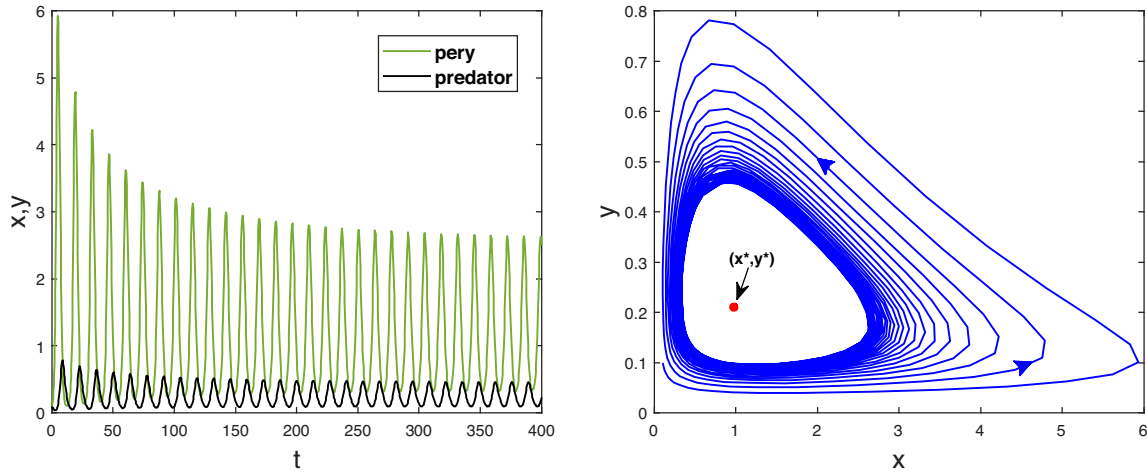


Fig. 2.6: Oscillations of solutions x & y (left) and existence of stable limit cycle (right) around E^* when $\tau = 0.4 (> \tau^*)$. The other parameters are same as given in Table 2.1.

It is observed that model system (2.2) has chaotic solution for higher values of gestation delay ($\tau > 12.6$). The time series and chaotic attractor for $\tau = 18$ are presented in Fig. 2.7. To explore the possibility of occurrence of chaos, bifurcation diagram is plotted with respect to delay parameter τ (see Fig. 2.8). The chaotic nature of the model system is again confirmed by Maximum Lyapunov exponent in Fig. 2.9. It is the average exponential rates of divergence or convergence of nearby orbits in phase space [176]. If it is positive then two nearby orbits move too far apart with time, this implies chaotic behavior of solution. If Maximum Lyapunov exponent is negative then two nearby orbits converges to each other, this implies a stable solution or a periodic attractor. In Fig. 2.9 it can be noted that Maximum Lyapunov exponent becomes positive at $\tau = 12.6$.

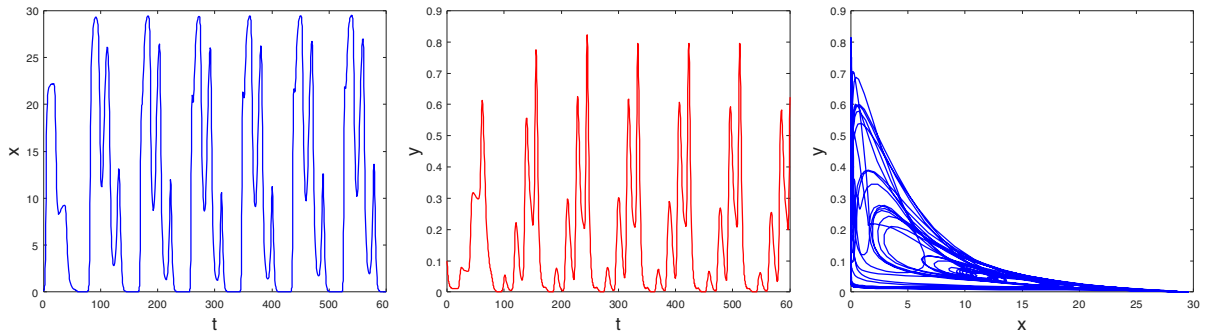


Fig. 2.7: Time evolution of x & y and chaotic attractor when $\tau = 18$.

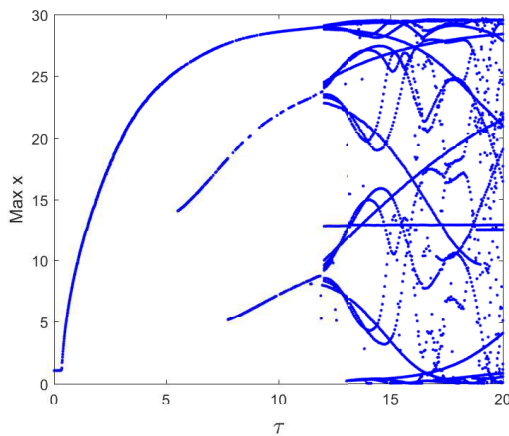


Fig. 2.8: Bifurcation diagram of prey population with respect to gestation delay τ .

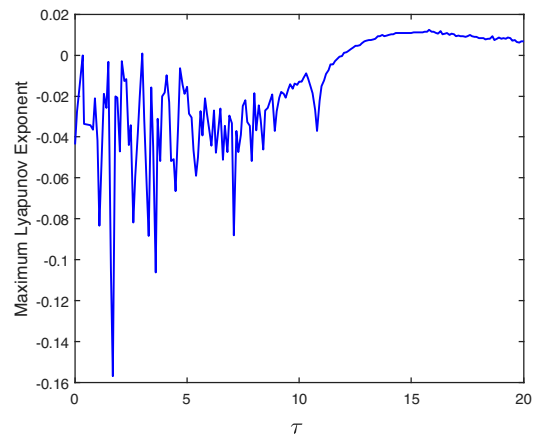


Fig. 2.9: Maximum Lyapunov exponent with respect to τ corresponding to Fig. 2.8.

Sensitivity to initial conditions is an elemental property of chaos. In a chaotic system small perturbation of the current trajectory may lead to significantly different future behavior. In Fig. 2.10 the initial conditions for y are kept the same but those for x are different 0.1 and 0.11. As can be seen, even the slightest difference in initial values causes significant changes in time evolution of species.

In Fig. 2.11, stability and instability regions are depicted in $k\tau$ -plane. In green region of the plane, the system (2.2) is locally asymptotically stable (stable focus) around its positive equilibrium and in white region the positive equilibrium is unstable and system has periodic solutions. Fig. 2.12 shows the different type of stability behavior of system (2.2) for higher cost of fear ($k > 10$) in $k\tau$ -plane. In the lower region, the system is locally asymptotically stable around its interior equilibrium. In green region the interior equilibrium is unstable and system has periodic solutions. The separation curve of both the regions has the points where the system undergoes Hopf-bifurcation. In white region system (2.2) has period doubling oscillations and in orange region system shows higher periodic and chaotic oscillations.

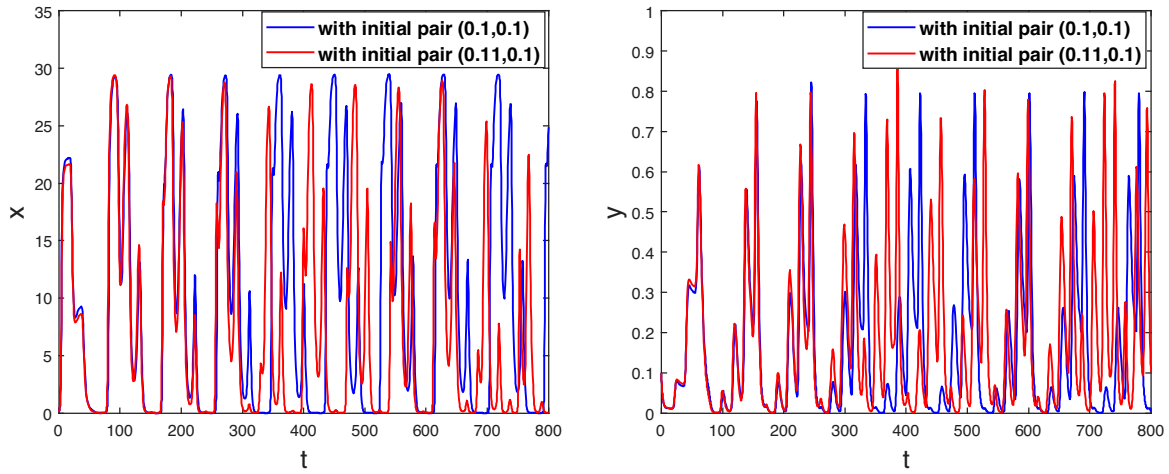


Fig. 2.10: Sensitivity of solutions of system (2.2) with respect to initial condition when $\tau = 18$.

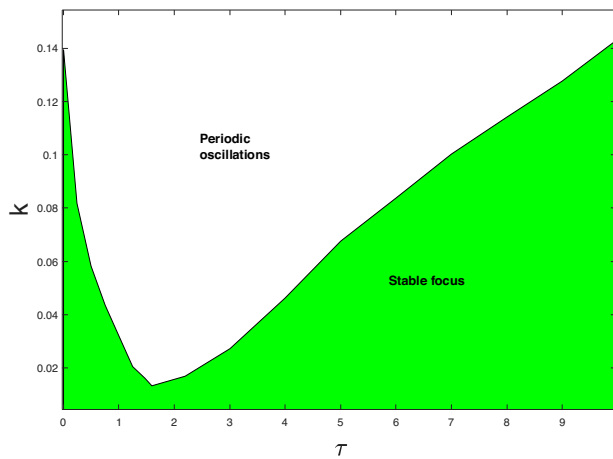


Fig. 2.11: Stability and instability regions of equilibrium E^* with respect to k and τ .

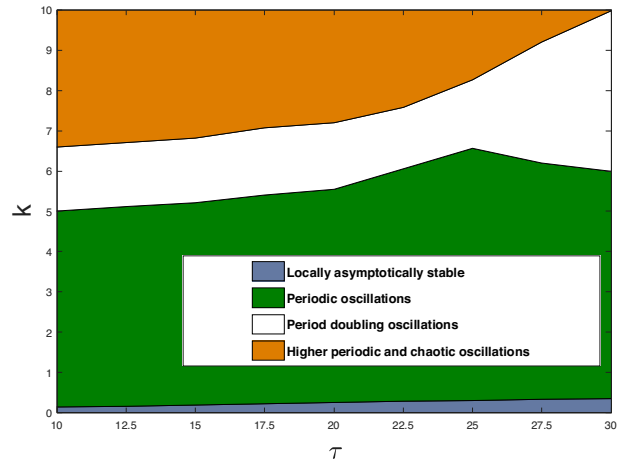


Fig. 2.12: Chaotic and non-chaotic regions of model (2.2) in $k\tau$ -plane.

2.7 Conclusion

In this chapter, a prey-predator model is proposed with the effect of fear in prey induced by predators and prey refuge. It is assumed that prey population grows logistically and predators consume prey population under Crowley-Martin type functional response. Further the gestation delay is introduced in the system to get much rich dynamics.

In the non-delayed system, first we have proved that our proposed model is biologically well behaved by showing the positivity of solutions, initiated from a positive initial pair and boundedness of solutions in a compact region Ω in R_+ . Then we get some sufficient conditions

for persistence of the system. It is obtained that system (2.1) is persistent if growth rate of prey population is greater than a threshold value and death rate of predator population is less than a threshold value. These threshold values depend upon refuge factor as well as on the fear parameter. In order to investigate the dynamics of system it is noted that system (2.1) has three kind of equilibria. $E_0(0,0)$ which exists trivially, $E_1(x_*,0)$ exists if $r > r_0$ holds and interior equilibrium point $E^*(x^*,y^*)$ exists uniquely under conditions (2.5) and (2.6). The local stability behavior is carried out of the system around each equilibria. It is observed that if $E_1(x_*,0)$ exists (does not exist), then $E_0(0,0)$ is a saddle point (locally asymptotically stable). The unique interior equilibrium point E^* is locally asymptotically under a sufficient condition (2.9). Throughout in this chapter, the fear parameter is a crucial parameter. So we have done the Hopf-bifurcation analysis of the system with respect to fear parameter and derived a formula for critical value of the parameter k where Hopf-bifurcation occurs and system switches its stability behavior. The analysis of direction of Hopf-bifurcations shows that periodic solutions exist on the side of $k < k^*$. This implies that oscillations can be controlled from the system by increasing the cost of fear k . We also performed the global stability of positive equilibrium by choosing a suitable Lyapunov function. A brief discussion on the existence of limit cycles has been done using Bendixson-Dulac criterion and Poincare-Bendixson Theorem. The impact of refuge parameter on the dynamics of system is also investigated and we found that conservation of prey individuals under 57.81% promotes both the species. But still if we keep conserving them, then it produces a negative effect on predators.

Delay shows much more realistic dynamics as it is involved in most of the biological phenomena. In our system, the reproduction of predators after consuming prey is not instantaneous. Therefore, to obtain some rich outcomes we have added the gestation delay into our system. For the delayed system, we examined the local stability via Hopf-bifurcation and derived a critical value τ_0 of τ where Hopf-bifurcation occurs. System is stable if $\tau < \tau_0$ and as τ crosses τ_0 system destabilizes and remains unstable for further increment in τ . This analysis is demonstrated by Figs. 2.5 and 2.6. The global stability behavior of delayed system has been discussed and it is observed that if the non-delayed system is globally asymptotically stable then the delayed system is also globally asymptotically stable around the positive equilibrium. We also observed the chaotic behavior of the system for the large values of the delayed parameter. Moreover chaotic behavior of delayed system is confirmed by evaluating maximum Lyapunov exponent. In Figs. 2.11 and 2.12, we have drawn several regions in $k\tau$ -plane that show different stability nature of system (2.2).