Studies in Graceful Labelings of Graphs and its Variations

THESIS

submitted in partial fulfilment
of the requirements for the degree of

DOCTOR OF PHILOSOPHY

by

FERNANDES JESSICA

under the supervision of

Dr. Tarkeshwar Singh

BITS Pilani
Pilani | Dubai | Goa | Hyderabad

BIRLA INSTITUTE OF TECHNOLOGY AND SCIENCE, PILANI
2016

BIRLA INSTITUTE OF TECHNOLOGY AND SCIENCE, PILANI

CERTIFICATE

This is to certify that the thesis entitled **Studies in Graceful Labelings of Graphs and its Variations** and submitted by **Fernandes Jessica** ID No. **2010PHXF019G** for award of Ph.D. of the Institute embodies original work done by her under my supervision.

Signature of the Supervisor
Dr. TARKESHWAR SINGH
Associate Professor

Date: 20/01/2016

ACKNOWLEDGEMENT

I would like to acknowledge the favour of the **Divine Providence** who has encouraged me in numerous ways, provided help through various people and whose faithfulness is marvelous.

I would like to thank my Supervisor and Guide **Dr. Tarkeshwar Singh** who encouraged and directed me towards doing research. He supported me throughout my thesis with his patience and knowledge whilst allowing me the room to work in my own way. It is his initiation and motivation that this work came into existence and I am greatly indebted to him. One simply could not wish for a better or friendlier supervisor.

I would like to express my deepest gratitude to my collaborator **Prof. S. Arumugam**, Director n-CARDMATH, Kalasalingam University, Tamil Nadu. I have been amazingly fortunate to have his excellent guidance and encouragement. I am thankful to him for providing me with an excellent research atmosphere which motivated me towards my research. He patiently corrected my writing and was instrumental in completing my thesis. I hope that one day I would become as good an advisor to my students as Prof. Arumugam has been to me.

I also take this opportunity to thank **Prof. Sasikumar Punnekkat**, Director, Birla Institute of Technology and Science Pilani (BITS Pilani) K K Birla Goa Campus, **Prof. Srinivasan Ashwin**, Deputy Director, BITS Pilani K K Birla Goa Campus, **Prof. V. S. Rao** acting Vice Chancellor, BITS Pilani, **Prof. B. N. Jain**, Ex-Vice Chancellor, BITS Pilani, **Dr. Kumar Mangalam Birla**, Chancellor, BITS Pilani, for giving me the opportunity to serve in this prestigious institute as a faculty while pursuing my research.

I wish to thank the Department Research Committee (DRC) members and the DRC Convenor **Dr. Anil Kumar**. I also express my sincere thanks to **Prof. S. K. Verma**, Dean, Academic Research (Ph.D. Programme), BITS Pilani, **Dr. Prasanta Kumar Das**, Associate Dean, Academic Research, BITS Pilani K K Birla Goa Campus for providing the necessary facilities.

My sincere thanks also goes to my Doctoral Advisory Committee (DAC) members **Dr. Bharat Deshpande**, head of the department of Computer Science and Information Systems and **Dr. Prasanna Kumar**, head of the department of Mathematics, BITS Pilani K K Birla Goa Campus for preliminary assessment of the thesis and helpful suggestions during my progress seminars. I also thank the faculty members and research scholars of the department of Mathematics, BITS Pilani K K Birla Goa Campus for their kind cooperation. My special thanks are to **Dr. Prasanna Kumar**, head of the department of Mathematics, BITS Pilani K K Birla Goa Campus for his support and help whenever sorted.

I heartily thank my friends for their prayers and constant encouragement to finish this work successfully. I am also thankful to my fellow colleagues whose challenges and productive criticism, especially at the progress seminars, have provided improvement in the presentation of the work. I extend my thanks to Mr. K. Alaguraj, Kalasalingam University, Tamil Nadu, for typesetting the thesis in an excellent manner.

I would also like to thank my parents late Joaquim Conçeicao Fernandes and Mercy Fernandes, two elder brothers Gipson Fernandes and Josy Fernandes, my in-laws, neices and nephews for their constant love, confidence, good wishes and for being with me against all odds in this journey of education.

Finally, I would like to thank my beloved husband Lincoln Pereira. He has always been there cheering me up and extending his whole hearted support throughout my research. I am forever indebted to my children Zayde, Gretchen and Harvey for their love and understanding and for giving me a happy and joyous environment at home and owe apology to them for depriving them of my love and time during this journey of chasing my dream.

This thesis is heartily dedicated to my father Joaquim Conçeicao Fernandes who took the lead to heaven before the completion of this work and my mother Mercy Fernandes who stood by me at all times.

FERNANDES JESSICA

ABSTRACT

By a graph $G = (V, E)$ we mean a finite undirected graph with neither loops nor multiple edges. The order $|V|$ and size $|E|$ of G are denoted by p and q respectively. For graph theoretic terminology, we refer to Chartrand and Lesniak [24].

In Chapter 1, we collect some basic definitions and theorems on graphs which are needed for the subsequent chapters.

A (p, q)-graph $G = (V, E)$ is said to be *k-hypergraceful* if there exists a decomposition of G into *edge induced subgraphs* $G_1, G_2, \ldots, G_k$ having sizes $m_1, m_2, \ldots, m_k$ respectively, and an injective labeling $f : V(G) \to \{0, 1, \ldots, q\}$, such that when each edge $uv \in E(G)$ is assigned the absolute difference $|f(u) - f(v)|$, the set of integers received by the edges of G_i is precisely $\{1, 2, \ldots, m_i\}$ for each $i \in \{1, 2, \ldots, k\}$. The decomposition $\{G_i\}$, if it exists, is then called a *hypergraceful decomposition* of G and f is called a *k-hypergraceful labeling* of G. Further, G is said to be *hypergraceful* if it possesses a hypergraceful decomposition. When $k = 1$, the above definition yields the well known notion of graceful graphs and $k = 2$ corresponds to the extension of the notion of graceful graphs to the realm of sigraphs as studied in ([10–12, 55]). Chapter 2 deals with k-hypergraceful complete graphs. We discuss the existance of k-hypergraceful labeling of the complete graph K_p where $k = (p-4)$ if

and only if $p \geq 8$. We also give a $(p-3)$-hypergraceful decomposition of K_p for $p \geq 4$. We prove that K_p is $(p-2)$-hypergraceful for $p \geq 3$ and finally show that K_p is $(p-1)$-hypergraceful for $p \geq 2$. In this chapter, we also provide all nonisomorphic 3-hypergraceful decompositions of the complete graph of order 5.

In Chapter 3 we define (k, d)-Skolem graceful graph as follows: A graph $G = (V, E)$ is said to be (k, d)-*Skolem graceful* if there exists a bijection $f : V(G) \rightarrow \{1, 2, \ldots, |V|\}$ such that the induced edge labeling g_f defined by $g_f(uv) = |f(u) - f(v)|$, $\forall uv \in E$, is a bijection from E to $\{k, k + d, \ldots, k + (q - 1)d\}$, where k and d are positive integers. Such a labeling f is called (k, d)-*Skolem graceful labeling* of G. We present several basic results on (k, d)-Skolem graceful graphs and prove that nK_2 is $(2, 1)$-Skolem graceful if and only if $n \equiv 0$ or $3 (\text{mod } 4)$. We prove that nK_2 is $(1, 2)$-Skolem graceful. Finally, we close the chapter with the observation that a $(1, 1)$-Skolem graceful labeling of G gives a Skolem graceful labeling. A Skolem graceful labeling of nK_2 gives the Skolem sequence, a $(2, 1)$-Skolem graceful labeling of nK_2 gives the $(2, n)$ Langford sequence and a $(k, 1)$-Skolem graceful labeling of nK_2 gives a perfect sequence.

In Chapter 4, we introduce the notion of (k, d)-hooked Skolem graceful graph as follows: A (p, q) graph $G = (V, E)$ is said to be (k, d)-*hooked Skolem graceful* if there exists a bijection $f : V(G) \rightarrow \{1, 2, \ldots, p - 1, p + 1\}$ such that the induced edge labeling $g_f : E \rightarrow \{k, k + d, k + 2d, \ldots, k + (q - 1)d\}$ defined by

$g_f(uv) = |f(u) - f(v)|$, $\forall uv \in E$ is also bijective, where k and d are positive integers. Such a labeling f is called *(k,d)-hooked Skolem graceful labeling* of G. We observe that when $k = d = 1$, this notion coincides with that of hooked Skolem graceful labeling of the graph G ([57]). It follows from the definition that if a graph G is (k,d)-hooked Skolem graceful, then $q \leq p-1$. In this chapter, we give a necessary condition for a graph G to be (k,d)-hooked Skolem graceful. We prove that nK_2 is $(2,1)$-hooked skolem graceful if and only if $n \equiv 1$ or $2 (\text{mod } 4)$.

The *gracefulness* $grac(G)$ of a graph G with $V(G) = \{v_1, v_2, \ldots, v_p\}$ and without isolated vertices is defined as the smallest positive integer k for which it is possible to label the vertices of G with distinct elements from the set $\{0, 1, \ldots, k\}$ in such a way that edges receive distinct labels ([24]). In Chapter 5, we define a new measure of gracefulness of graphs as follows: Let $G = (V, E)$ be a (p, q) graph. Let $f : V(G) \to \mathbb{N} \cup \{0\}$ be an injection such that the edge induced function g_f defined on E by $g_f(uv) = |f(u) - f(v)|$ is also injective. Let $c(f) = \max \{i : 1, 2, \ldots, i$ are edge labels under $f\}$. Let $m(G) = \max_f c(f)$, where the maximum is taken over all f. Then $m(G)$ is called the *m-gracefulness* of G. This new measure $m(G)$ determines how close G is to being graceful. Note that if G is a graceful graph, $m(G) = q$ and $grac(G) = q$. One may observe that $grac(G)$ measures gracefulness of the graph G from above q, whereas $m(G)$ measures gracefulness of G from below q. In this chapter we prove that there are infinitely many nongraceful graphs G with $m(G) = q-1$, we determine $m(G)$ for a few families of non-graceful graphs. We give necessary conditions for a (p, q)-eulerian graph and the complete graph K_p to have m-gracefulness $q-1$ and $q-2$. Using this, we prove that K_5 is the only complete graph to have m-gracefulness $q-1$. We also give an upper bound for the highest possible vertex label of K_p if $m(K_p) = q-2$. We have proved that $m(K_6) = 13 = q-2$, which is also shown in optimal Golomb ruler ([20, 21]).

In Chapter 6, we extend the notion of additively graceful graphs to sigraphs as follows: Let $S = (V, E)$ be a (p, m, n)-sigraph with $E = E^+ \cup E^-$, Assume $|E^+| = m$ and $|E^-| = n$ where $m+n = q$. Let $f : V \to \{0, 1, ..., m+\lceil\frac{(n+1)}{2}\rceil\}$ be an injective mapping and let the induced edge function be defined as $g_{f^-}(uv) = f(u)+f(v) \; \forall \; uv \in E^-$ and $g_{f^+}(uv) = |f(u)-f(v)| \; \forall \; uv \in E^+$. If $g_{f^-}(uv) = \{1, 2, ..., n\}$ and $g_{f^+}(uv) = \{1, 2, ..., m\}$, then f is called an *additively graceful labeling* of S. The sigraph which admits such a labeling is called an *additively graceful sigraph.* One can easily see that when $n = 0$, f is a graceful labeling of S, and when $m = 0$, f is an additively graceful labeling of S. We give some necessary or sufficient conditions for a sigraph to be additively graceful. We give a necessary and sufficient condition for K_4 to be additively graceful. We obtain some necessary conditions for eulerian sigraphs, complete bipartite sigraphs and complete sigraphs to be additively graceful.

In Chapter 7, we give an efficient method of embedding any connected graph G of order p as an induced subgraph of an eulerian graceful graph H whose order is $O(p^2)$. We also consider the following analogous problem for sigraphs: *Given a sigraph S and a graph theoretic property $\mathcal{P}$, is it possible to embed S in a graceful sigraph S_1 having the property $\mathcal{P}$?* We prove the existence of such an embedding where S_1 is eulerian, hamiltonian, planar or triangle-free. We prove that every signed tree can be embedded in a graceful signed tree.

Chapter 8 gives a conclusion of the study carried out and a brief summary of areas and problems for further research.

Contents

List of Tables

List of Figures

LIST OF SYMBOLS

$B(r, s)$	bistar
χ	chromatic number
ω	clique number
$N[v]$	closed neighborhood of v
$K_{r,s}$	complete bipartite graph
K_n	complete graph on n vertices
$\overline{G}$	complement of the graph G
nK_2	n copies of K_2
C_n	cycle on n vertices
$deg(v)$	degree of vertex v
γ	domination number
β_1	edge independence number
$E(G)$	edge set of graph G
F_k	friendship graph on k triangles
$m(G)$	m-gracefulness of graph G
$\lfloor x \rfloor$	greatest positive integer not greater than the real number x
β_0	independence number
$grac(G)$	index of gracefulness of graph G

$M_G(g_f)$	largest edge label induced by a labeling f
$M_G(f)$	largest vertex label under f
Δ	maximum degree
δ	minimum degree
$\eta(S)$	negation of sigraph S
$N(v)$	open neighborhood of v
p	order of a graph
P_n	path on n vertices
$M'_G(g_f)$	second largest edge label induced by a labeling f
$M'_G(f)$	second largest vertex label under f
$g_f(E^-)$	set of negative edge labels of a sigraph
$g_f(E^+)$	set of positive edge labels of a sigraph
Z_k	signed cycle on k vertices
q	size of a graph
$\lceil x \rceil$	smallest positive integer not smaller than the real number x
T_n	tree on n vertices
H_k	triangular snake on $2k+1$ vertices
$V(G)$	vertex set of graph G
W_n	wheel on n vertices

CHAPTER 1

INTRODUCTION

1.1 INTRODUCTION

This chapter is a collection of some basic definitions, literature review of the research topic which contains definitions and theorems on graphs which are useful for the subsequent chapters, some of the gaps in existing research and our objective for research. We also give an overview of the remaining chapters. For graph theoretic terminology, we refer to Chartrand and Lesniak [24].

In Section 1.2 we give a brief outline of the basic definitions in graph theory. Section 1.3 presents a review of several variations of graph labelings and known results, based on which we identify certain gaps in existing research and we give our objectives for present research. Finally an overview of the organization of the remaining chapters of the thesis is given in Section 1.4.

1.2 BASIC GRAPH THEORY

In this section we present the basic definitions and theorems in graph theory.

Definition 1.2.1. A graph G is a finite nonempty set of objects called *vertices* together with a set of unordered pairs of distinct vertices of G called *edges*. The vertex set and the edge set of G are denoted by $V(G)$ and $E(G)$ respectively. The edge $e = \{u, v\}$ is said to join the vertices u and v. We write $e = uv$ and say that u and v are *adjacent vertices* or u is a *neighbor* of v in G; u and e are incident, as are v and e. If e_1 and e_2 are distinct edges of G incident with a common vertex, then e_1 and e_2 are *adjacent edges*.

The set of all neighbors of v is the *open neighborhood* of v and is denoted by $N(v)$; the set $N[v] = N(v) \cup \{v\}$ is the *closed neighborhood* of v in G. If $S \subseteq V$, then $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$. We observe that $N[v] \neq N(v)$ but we can have $N[S] = N(S)$ such as in the case where S is the set of two vertices from K_3.

The number of vertices in G is called the *order* of G and the number of edges in G is called the *size* of G. A graph of order p and size q is called a (p, q)-*graph*. A graph is *trivial* if its vertex set is a singleton.

Definition 1.2.2. The *complement* $\overline{G}$ of a graph G is the graph with vertex set $V(G)$ such that two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G.

Definition 1.2.3. The *degree* of a vertex v in a graph G is defined to be the number of edges incident with v and is denoted by $\deg(v)$. A vertex of degree zero in G is an *isolated vertex* and a vertex of degree one is a *pendent vertex* or a *leaf*. An edge e in a graph G is called a *pendent edge* if it is incident with a pendent vertex. Any vertex which is adjacent to a pendent vertex is called a *support vertex*.

The minimum of $\{\deg(v) : v \in V(G)\}$ is denoted by $\delta(G)$ or simply δ and the maximum of $\{\deg(v) : v \in V(G)\}$ is denoted by $\Delta(G)$ or simply Δ.

A graph G is called *r-regular* if every vertex of G has degree r. A graph is said to be *regular* if it is r-regular for some nonnegative integer r. In particular, a 3-regular graph is called a *cubic graph*.

Definition 1.2.4. A graph H is called a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph H of a graph G is a *proper subgraph* of G if either $V(H) \neq V(G)$ or $E(H) \neq E(G)$. A *spanning subgraph* of G is a subgraph H of G with $V(H) = V(G)$.

For a set S of vertices of G, the *induced subgraph* is the maximal subgraph of G with vertex set S and is denoted by $\langle S \rangle$. Thus two vertices of S are adjacent in $\langle S \rangle$ if and only if they are adjacent in G. The induced subgraph $\langle S \rangle$ is also denoted by $G[S]$.

Similarly, for a subset E' of $E(G)$, the *edge induced subgraph* $\langle E' \rangle$ is the subgraph of G whose vertex set is the set of end vertices of edges in E' and whose edge set is E'.

Let v be a vertex of a graph G and $|V(G)| \geq 2$. Then the induced subgraph $\langle V(G) \backslash \{v\} \rangle$ is denoted by $G - v$ and it is the subgraph of G obtained by the removal of v and the edges incident with v. If $e \in E(G)$, the spanning subgraph with edge set $E(G) \backslash \{e\}$ is denoted by $G - e$ and it is the subgraph of G obtained by the removal of the edge e.

For any two disjoint subsets A, B in V, let $[A, B]$ denote the set of all edges with one end in A and the other end in B.

Definition 1.2.5. A graph G_1 is *isomorphic* to a graph G_2 if there exists a one-to-one mapping ϕ, called an *isomorphism*, from $V(G_1)$ onto $V(G_2)$ such that ϕ preserves adjacency; that is, $uv \in E(G_1)$ if and only if $\phi u \phi v \in E(G_2)$.

If G_1 is isomorphic to G_2, then we say G_1 and G_2 are *isomorphic* or *equal* and write $G_1 = G_2$.

It is easy to see that isomorphism is an equivalence relation on graphs; hence, this relation divides the collection of all graphs into equivalence classes, two graphs being *nonisomorphic* if they belong to different equivalence classes.

Definition 1.2.6. A graph G is said to be *embedded* in a graph G' if there exists an induced subgraph of G' which is isomorphic to G.

If G is embedded in a graph G', it is sometimes convenient to regard G itself as an induced subgraph of G'.

A subset $S \subseteq V$ is said to be *independent* if no two vertices in S are adjacent. The *independence number* $\beta_0(G)$ is the maximum cardinality of an independent set in G. The *clique number* $\omega(G)$ of a graph G is the maximum order among the complete subgraphs of G. Clearly $\omega(G) = \beta_0(\overline{G})$ for every graph G.

Definition 1.2.7. A subset $M \subseteq E$ is said to be an *independent set of edges* or a *matching* if no two edges in M are adjacent. The *edge independence number* $\beta_1(G)$ is the maximum cardinality of a matching in G. If $M \subseteq E(G)$ is a matching in a graph G with the property that every vertex of G is incident with an edge of M, then M is called a *perfect matching* in G.

Clearly, if G has a perfect matching M, then G has even order and $\langle M \rangle$ is a 1-regular spanning subgraph of G.

Definition 1.2.8. A *walk* W in a graph G is an alternating sequence $u_0, e_1, u_1, \ldots, u_{n-1}, e_n, u_n$ of vertices and edges of G, beginning and ending with vertices, such that $e_i = u_{i-1}u_i$, for $1 \leq i \leq n$. We denote the walk simply by the sequence of vertices, so that $W = (u_0, u_1, \ldots, u_{n-1}, u_n)$. This walk joins u_0 and u_n, and it is sometimes called a u_0-u_n walk. A u_0-u_n walk is *closed* or *open* depending on whether $u_0 = u_n$ or $u_0 \neq u_n$. A u_0-u_n *trail* is a u_0-u_n walk in which no edge is repeated. A nontrivial closed trail of a graph G is referred to as a *circuit* of G. A u_0-u_n walk is called a u_0-u_n path if all the vertices $u_0, u_1, \ldots, u_n$ are distinct. The vertices u_0 and u_n are called origin and terminus of the path respectively and $u_1, u_2, \ldots, u_{n-1}$ are called its internal vertices. A path on n vertices is denoted by P_n.

Definition 1.2.9. A *cycle* of length $n \geq 3$ in a graph G is a sequence $(u_0, u_1, \ldots, u_{n-1}, u_0)$ of vertices of G such that for $0 \leq i \leq n-2$, the vertices u_i and u_{i+1} are adjacent, u_{n-1} and u_0 are adjacent and $u_0, u_1, \ldots, u_{n-1}$ are distinct. A cycle on n vertices is denoted by C_n. A cycle C_n of length n is called *even* or *odd* according as n is even or odd. A cycle of length n is an n-cycle.

The graph F_k, obtained by identifying one vertex of each of the k copies of C_3, is called *friendship graph* or *Dutch k-windmill*.

A *triangular snake* is the graph obtained from a path $v_1, v_2, \ldots, v_n$ by joining v_i and v_{i+1} to a new vertex w_i for $i = 1, 2, \ldots, n-1$.

Definition 1.2.10. A graph G is said to be *connected* if every pair of vertices of G are joined by a path. A maximal connected subgraph of G is called a *component* of G.

A graph G having more than one component is called a *disconnected* graph. An edge e of a connected graph G is called a *cut-edge* if $G - e$ is disconnected. A vertex v of a connected graph G is called a *cut-vertex* if $G - v$ is disconnected.

Definition 1.2.11. A connected acyclic graph is called a *tree*. A graph having exactly one cycle is called a *unicyclic graph*. A disconnected graph in which each component is a tree is called a *forest*.

Definition 1.2.12. An *eulerian trail* of a graph G is an open trail of G containing all of the edges of G, while an *eulerian circuit* of G is a closed eulerian trail. A graph possessing an eulerian circuit is called an *eulerian graph*.

The following theorem gives a characterization of eulerian graphs.

Theorem 1.2.13. [24] *A nontrivial connected graph G is eulerian if and only if degree of every vertex of G is even.*

Definition 1.2.14. A cycle of a graph G containing every vertex of G is called a *hamiltonian cycle* of G; thus, a *hamiltonian graph* is one that possesses a hamiltonian cycle.

A graph is *planar* if it can be embedded in a plane and *nonplanar* otherwise.

Definition 1.2.15. A *bipartite graph* is a graph G whose vertex set $V(G)$ can be partitioned into two nonempty subsets X and Y such that each edge of G has one end in X and the other end in Y. The pair (X, Y) is called a *bipartition* of G. If further, every vertex in X

is adjacent to all the vertices of Y, then G is called a *complete bipartite graph*. The complete bipartite graph with bipartition (X, Y) such that $|X| = r$ and $|Y| = s$ is denoted by $K_{r,s}$. In particular, the graph $K_{1,n}$ is called a *star*.

Definition 1.2.16. A graph G is *complete* if every pair of distinct vertices of G are adjacent in G. A complete graph on p vertices is denoted by K_p.

Definition 1.2.17. Let $G_1, G_2, \ldots, G_k$ be a set of k disjoint graphs. Then the *join graph* $G_1 + G_2 + \cdots + G_k$ is obtained from $G_1, G_2, \ldots, G_k$ by joining every vertex of G_i with every vertex of G_j, whenever $i \neq j$.

For $n \geq 4$, the *wheel* on n vertices, denoted by W_n, is defined to be the graph $K_1 + C_{n-1}$.

Definition 1.2.18. A *proper coloring* of a graph G is an assignment of colors to the vertices of G in such a way that no two adjacent vertices receive the same color. The *chromatic number* $\chi(G)$ is the minimum number of colors required for a proper coloring of G.

Definition 1.2.19. A vertex v in a graph G is said to *dominate* itself and each of its neighbors, that is, v dominates the vertices in its closed neighborhood $N[v]$. A set S of vertices of G is a *dominating set* of G if every vertex of G is dominated by at least one vertex of S.

Equivalently, the set S of vertices of G is a dominating set if every vertex in $V(G) - S$ is adjacent to at least one vertex in S. The minimum cardinality among the dominating sets of G is called the *domination number* of G and is denoted by $\gamma(G)$. A dominating set of cardinality $\gamma(G)$ is then referred to as a *minimum dominating set*.

Definition 1.2.20. A *decomposition* of a graph G is a collection $\{H_i\}$ of nonempty subgraphs such that $H_i = \langle E_i \rangle$ for some (nonempty) subset E_i of $E(G)$, where E_i is a partition of $E(G)$. Thus no subgraph H_i in a decomposition of G contains isolated vertices.

Definition 1.2.21. A *signed graph* S (or simply *sigraph*) is a graph $G = (V, E)$ together with a function $s : E \to \{+, -\}$ called its signing function, which assigns a sign $+$ or $-$ to each edge. The graph G is called the underlying graph of the sigraph S. The set of all positive and negative edges of S are denoted by E^+ and E^- respectively so that $E^+ \cup E^- = E(S)$ is the edge set of S. Further, if $|E^+| = m$ and $|E^-| = n$ so that $m + n = q$, then S is called a (p, m, n)-sigraph. An all-positive sigraph S is one in which $E^+(S) = E(S)$ and an all-negative sigraph is one in which $E^-(S) = E(S)$. A sigraph is said to be homogeneous if it is either all-positive or all-negative, and heterogeneous otherwise.

By Z_k, we mean a (k, m, n)-signed cycle. For a sigraph S, $\eta(S)$ is a sigraph obtained from S by changing the sign of each edge of S to its opposite and is called the *negation of* S.

1.3 LITERATURE REVIEW OF RESEARCH TOPIC

In this section we give a review of several variations of graph labelings, definitions and known results.

1.3.1 GRAPH LABELING PROBLEMS

One of the most active fields of current research in the subject of Discrete Mathematics is the *theory of graphs*. Graph labeling is one of the fastest growing research areas within graph theory. New results are being discovered and published at a rapidly increasing rate. Further we have an enormous number of conjectures and open problems in graph labelings. For an excellent and up to date dynamic survey on graph labeling we refer to Gallian [29]. Unless mentioned otherwise, all graphs considered here will be finite and simple.

Very commonly encountered instances of a graph in the above sense are a *road network* (in certain restricted sense) where we ignore multiple road-connections between any two junctions, one-ways, loop-ways, etc. [28], the *electricity* or *water supply network* in a city; ([23, 25, 65]), the *railway* or *communication network* in a country [28], *chemical bond structure of a molecule* [15], computer network in an organization [25], even a *social network* that represents a group of persons and *interpersonal* relationship existing amongst them [2]. The structure or the *topology* of such networks may be represented by graphs for analytical purposes when a problem related to their structures is encountered in practice [60]. For such applied aspects of graph theory, one may consult specialized references such as Acharya ([3, 4, 6]), Balaban [15], Chartrand [23], Chen [25], Harary et al. [33], Jensen and Gutin [36], Roberts [49]; it would be rather unwieldy to provide such references exhaustively as they are quite numerous.

Often, one encounters a need to *label* the *elements* (i.e., vertices or edges or both) of a given graph $G = (V, E)$. For instance, in the road network of a city the *junctions* (represented by vertices) and the *roads* (represented by edges) are generally named (*labeled*) for one to locate them for various practical purposes. A *sociogram*, as another instance, is a graph whose edges are labeled as being *positive* or *negative* according to whether the two interacting persons forming a given edge have a qualitatively *positive* or *negative* type of interpersonal relationship; such a network has been called a *signed graph* or simply, a *sigraph* in the literature ([2, 6, 16, 17, 33, 63]). In fact, *sigraphs* were first discovered by Harary [33] as appropriate prototype models to represent structures of cognitive interpersonal relationships in a social group. Ever since, *sigraphs* have received much attention in *social psychology* because of their extensive use in modeling a variety of cognition-based social processes ([1, 2, 4, 17, 27, 33]). Further intensive study of the topic has been due to their subsequently discovered strong connections with many classical mathematical systems ([4,61–64]) used in solving a variety of problems of theoretical and practical interest.

There are basically two types of labelings of graphs, namely;

- **Quantitative Labelings** (Assignment of some numbers to the elements of graph): Let N be any set with a binary operation $*$ defined on it. If an injection $f : V(G) \to N$ induces $f^* : E(G) \to N$ where $f^*(uv) = f(u) * f(v)$, $\forall u, v \in V(G)$, then such a labeling f is called a vertex labeling of G.

  This labeling has inspired research in a wide variety of applications in radio-astronomy, development of missile guidance codes, spectral characterization of materials using X-ray crystallography etc., under certain constraints.

- **Qualitative Labelings** (Assignment of qualitative nature to the vertices or edges of graph): If $\sigma : V(G) \to \{+, -\}$, then we have *marked graph* as mathematical model and is denoted by $S = (G, \sigma)$. If $\rho : E(G) \to \{+, -\}$ then we have *signed graph* as mathematical model [51] and is denoted by $S = (G, \rho)$.

  These labelings have inspired research in unrelated areas of human enquiry such as conflict resolutions in social psychology, electrical circuit theory, energy crises etc., ([2,6,23,25,49,65]).

Thus utilization of labeled graph models require imposing of additional constraints which characterize the problem being investigated. The necessary constraints arise naturally in studying the wide variety of seemingly unrelated practical applications for which labeled graphs provide underlying mathematical models. Some embodiments of this theory are as follows:

- The design of certain important classes of good non periodic codes for pulse radar and missile guidance is equivalent to labeling the vertices of a complete graph such that all the edge labels are distinct. The vertex labels then determine the time positions at which pulses are transmitted. Corresponding radar pulse and missile-guidance code problems have been the subject of investigation for several years.

- Determination of crystal structures from X-ray diffraction data has long been a concern of crystallographers. The ambiguities inherent in this procedure are now beginning to be understood. In some cases, the same diffraction information may correspond to more than one structure. This problem is mathematically equivalent to determining all labelings of the appropriate graph which produce a pre-specified set of edge numbers.

- Methods of encoding the integers from 0 to $b^n - 1$ using n digit vectors from the b-symbol alphabet have been devised to minimize the seriousness of errors occurring in a single digit. These encodings have been extensively investigated. The corresponding graph problem involves labeling the vertices of the square lattice grid, b on a side in n dimensions with integers from 0 to $b^n - 1$, in a way that optimizes some statistical function (typically the mean or the variance) of the edge numbers.

Most graph labeling methods trace their origin to the one introduced by Rosa [50].

Definition 1.3.1. Let G be a graph of order p and size q. A graceful labeling of G is an injection $f : V \to \{0, 1, \ldots, q\}$ such that when each edge uv is assigned the label $g_f(uv) = |f(u) - f(v)|$, the resulting edge labels are all distinct. Such a function g_f is called the induced edge function and a graph which admits such a labeling is called a *graceful graph*.

Rosa [50] called such a labeling as a β-valuation and Golomb [31] subsequently called it graceful labeling.

The following result is due to Golomb.

Theorem 1.3.2. [31] *A necessary condition for a (p, q)-graph $G = (V, E)$ to be graceful is that, it be possible to partition its vertex set $V(G)$ into two subsets V_0 and V_e such that there are exactly $\lceil q/2 \rceil$ edges each of which joins a vertex of V_0 with one of V_e.*

The following is a necessary condition for graceful graphs.

Theorem 1.3.3. [31] *Suppose that integers, not necessarily distinct are assigned to the vertices of a graph G, and that each edge of G is given an edge number equal to the absolute difference of the vertex numbers at its end points. Then the sum of the edge numbers around any circuit of G is even.*

Graceful labeling is reported to have come from a problem in *mechanical engineering* which requires *notching a bar* so that distances between any two notches are all distinct. A problem modeled by Golomb [31] as one on *nonredundant distance measurement* [20] using what is known as a 'nonredundant ruler': It is a ruler with p marks placed on it end-to-end so that all the $\binom{p}{2}$ distances that can be measured by the calibration are distinct. If the maximum distances measured by such a ruler is least possible then the ruler is called a *Golomb ruler* after its discoverer ([20, 21]). Furthermore, if the distances measured by the ruler are all the first $\binom{p}{2}$ natural numbers then it is called *graceful*. It is well known that a *graceful Golomb ruler with more than four marks does not exist*. It is equivalent to the following statement.

Theorem 1.3.4. [31] *A complete graph K_p is graceful if and only if $p \leq 4$.*

Rosa [50] determined the graceful cycles.

Theorem 1.3.5. [50] *A cycle C_n of order n is graceful if and only if $n \equiv 0$ or $3 \pmod 4$.*

The following theorem determines the graceful friendship graphs.

Theorem 1.3.6. [18, 37] *A friendship graph F_k on k triangles is graceful if and only if $k \equiv 0$ or $1 \pmod 4$.*

A necessary condition for an eulerian graph to be graceful was discovered by Rosa [50].

Theorem 1.3.7. [50] *If G is a graceful eulerian graph of size q, then $q \equiv 0$ or $3 (\mathrm{mod}\ 4)$.*

All the graceful Golomb rulers with less than five marks are known in the literature. In general, nonredundant rulers of least length (i.e., a Golomb ruler) with more than 26 marks are yet to be found and the problem of generating all of them having a given length is known to be computationally hard ([20, 21, 32, 44]). However, the actual story of representing the first p natural numbers as differences of pairs of terms of an integer sequence of shortest possible length (called *difference basis* of p) appears to have started much earlier in the works of Singer [54] and Brauer [22] from pure combinatorial number theoretic considerations. One of the still unsolved problems on graceful graphs is the now famous *Ringel-Kotzig Conjecture* ([**?**, 19, 38, 48]):

Conjecture 1.3.8. All trees are graceful.

Several classes of graceful and nongraceful graphs have been reported in the literature. For more details see Gallian [29].

Definition 1.3.9. [24] The *gracefulness* $grac(G)$ of a graph G with $V(G) = \{v_1, v_2, \ldots, v_p\}$ and without isolated vertices is defined as the smallest positive integer k for which it is possible to label the vertices of G with distinct elements from the set $\{0, 1, \ldots, k\}$ in such a way that edges receive distinct labels.

Such a vertex labeling always exists. One of which is to label v_i by 2^{i-1}. Hence for every graph G of order p and size q without isolated vertices, $q \leq grac(G) \leq 2^{p-1}$. If G is a graph of size q with $grac(G) = q$, then G is graceful. However Golomb [31] has proved the existance of such a labeling in which the maximum vertex label is $O(p^2)$.

Theorem 1.3.10. [31] *For the complete graph K_p, there exists an injection $f : V(K_p) \to \mathbb{N}$ such that the induced edge function g_f is also injective and with maximum vertex label $O(p^2)$.*

Remark 1.3.11. For any graph G of order p, the injection function $f : V(K_p) \to \mathbb{N}$ with maximum vertex label $O(p^2)$, gives an injection on $V(G)$ such that g_f is injective and the maximum vertex label is $O(p^2)$.

Acharya et al. [7] have proved the following.

Theorem 1.3.12. [7] *Any graph G can be embedded as an induced subgraph of a graceful graph.*

The above theorem shows the impossibility of obtaining a forbidden subgraph characterization for graceful graphs.

Corollary 1.3.13. [7] *The problem of deciding whether chromatic number $\chi \leq k$, where $k \geq 3$ is NP complete even for graceful graphs.*

Acharya et al. [7] have also considered the following problem:

Problem 1.3.14. Let $\mathcal{P}$ be a graph theoretic property. Given a graph G having the property $\mathcal{P}$, is it possible to embed G as an induced subgraph of a graceful graph H having the property $\mathcal{P}$?

They presented an affirmative answer to the above problem for some graph theoretic properties such as planarity, triangle-free graphs etc. We list their results below.

Theorem 1.3.15. [7] *Any triangle-free graph G can be embedded as an induced subgraph of a triangle-free graceful graph.*

Theorem 1.3.16. [7] *Any planar graph G can be embedded as an induced subgraph of a planar graceful graph.*

Theorem 1.3.17. [7] *Any connected graph can be embedded as an induced subgraph of a hamiltonian graceful graph.*

Theorem 1.3.18. [7] *Any tree T can be embedded in a graceful tree T_1.*

As stated in [7] these results are particularly important, especially in the context of an unpublished result of Erdös which states that *most graphs are nongraceful* ([29], page 4).

In most of the above embedding results, to get any missing edge label we add a pendent vertex with that label as vertex label and join it to a vertex with label 0. Thus the graceful graph into which G is embedded is mostly a noneulerian graph. Hence the following problem naturally arises.

Problem 1.3.19. Can every connected graph be embedded as an induced subgraph in an eulerian graceful graph?

Rao and Sahoo [47] obtained an affirmative answer for the above problem. However, in their proof the number of vertices in the eulerian graceful graph is $O(3^p)$.

The notion of graceful labeling has been extended to sigraphs by Acharya and Singh [10].

Definition 1.3.20. Let S be a (p, m, n)-sigraph. For any injection $f : V(S) \to \{0, 1, \ldots, q = m + n\}$, the induced edge labeling g_f is defined by $g_f(uv) = s(uv)|f(u) - f(v)|$, $\forall u, v \in V(S)$. The function f is said to be a *graceful labeling* of S if $g_f(E^+) = \{1, 2, \ldots, m\}$ and $g_f(E^-) = \{-1, -2, \ldots, -n\}$. A sigraph which admits a graceful labeling is called a *graceful sigraph*.

We observe that if $E^- = \phi$, the graceful labeling of S is simply the graceful labeling of the underlying graph G.

Acharya and Singh [9] have proved the following result.

Theorem 1.3.21. [9] *Every sigraph can be embedded in a graceful sigraph.*

For further results on graceful sigraphs, one may refer to Acharya and Singh ([8–14, 55, 56, 58]).

The study of graph decomposition has been one of the most important topics in graph theory and also play an important role in the study of Combinatorial designs. In this connection Rosa proposed the following conjecture.

Conjecture 1.3.22. K_{2n+1} can be cyclically decomposed into $2n+1$ subgraphs isomorphic to a given tree with n edges.

Intuitively, such a cyclic decomposition is accomplished by

- arbitrarily choosing tree T_n with n edges;
- identifying the edges of T_n with a suitable set of edges in K_{2n+1};
- rotating each vertex and edge of T_n, $2n$ times from its original position.

Definition 1.3.23. [46] A (p, q)-graph $G = (V, E)$ is said to be *k-hypergraceful* if there exists a decomposition of G into *edge induced subgraphs* G_1, G_2, $\ldots$, G_k having sizes m_1, m_2, $\ldots$, m_k respectively, and an injective labeling $f : V(G) \to \{0, 1, \ldots, q\}$, such that when each edge $uv \in E(G)$ is assigned the absolute difference $|f(u) - f(v)|$, the set of integers received by the edges of G_i is precisely $\{1,$

2, ..., $m_i\}$ for each $i \in \{1, 2, \ldots, k\}$. The decomposition $\{G_i\}$, if it exists, is then called a *hypergraceful decomposition* of G and f is called a *k-hypergraceful labeling* of G. Further, G is said to be *hypergraceful* if it possesses a hypergraceful decomposition.

If f is a hypergraceful labeling of a graph G, then f^* defined by $f^*(u) = q - f(u), \forall u \in V(G)$ is also a hypergraceful labeling of G and is called the *complement* of the hypergraceful labeling of f. We have $(f^*)^* = f$. Also, it is clear, in general that in the above definition $G_1, G_2, \ldots, G_k$ may be replaced by $G_{\sigma(1)}, G_{\sigma(2)}, \ldots, G_{\sigma(k)}$ for any permutation σ of the set $\{1, 2, \ldots, k\}$. As such, in a more general setting, the notion of hypergraceful decomposition of graphs was first introduced by Acharya [5].

It is immediate that the case $k = 1$ in the above definition yields the well known notion of graceful graphs. The case $k = 2$ corresponds to the extension of the notion of graceful graphs to the realm of sigraphs as studied in ([8–14, 46, 55, 56, 58]). For the case $k = 2$, k-hypergraceful labelings of complete graph has been investigated by Rao et al. [46].

Theorem 1.3.24. [46] *A necessary condition for a (p, q)-graph $G = (V, E)$ to be k-hypergraceful is that it be possible to partition its vertex set V into two subsets V_o and V_e such that for each integer $i = \{1, 2, \ldots, k\}$ there are exactly $\lfloor \frac{m_i+1}{2} \rfloor$ edges of G_i each of which joins a vertex of V_o with one of V_e, where $\lfloor x \rfloor$ denotes the greatest integer not greater than the real number x.*

Lemma 1.3.25. [46] *If for no integer j, $0 \leq j \leq k$, $p-2j$ is a perfect square, then K_p is not k-hypergraceful with respect to any decomposition of K_p.*

Remark 1.3.26. If for some integer j, there exists a k-hypergraceful decomposition of K_p, for which $p-2j$ is a perfect square, then j represents the number of G_i's with odd size.

By the *negation of a sigraph S*, we mean a sigraph $\eta(S)$ which is obtained from S by changing the sign of every edge to its opposite. If a sigraph S is graceful with a graceful labeling f, then the negation of the sigraph S is also graceful under the same f.

Lemma 1.3.27. [46] *If any integer p is such that none of p, $p-2$, $p-4$ is a perfect square, then no sigraph on K_p is graceful.*

Theorem 1.3.28. [46]

1. *No sigraph on K_p, $p \geq 6$ is graceful.*
2. *Every sigraph on K_p, $p \leq 3$ is graceful.*
3. *A sigraph on K_4 is graceful if and only if the number of negative edges in it is not three.*
4. *A sigraph S on K_5 is graceful if and only if S satisfies one of the following statements:*
   (a) The number n of negative edges in S is 1,

(b) $n = 3$ *and the three negative edges in* S *are not incident at the same vertex,*

(or) $\eta(S)$ *satisfies similar conditions with* n *replaced by* m*, the number of positive edges in* S.

A *Steiner triple system* $S(2, 3, n)$ is an n-element set S together with a set of 3-element subsets of S called *blocks*, with the property that each 2-element subset of S is contained in exactly one block.

While studying the structure of Steiner triple systems, Skolem [59] considered the following problem: *Is it possible to distribute the numbers* $1, 2, \ldots, 2p$ *into* p *pairs* (a_i, b_i) *such that* $b_i - a_i = i$ *for* $i = 1, 2, \ldots, p$*?*

In the sequel, a set of pairs of this kind is called 1, +1 system because the difference $b_i - a_i$ begins with 1 and increases by 1 when i increases by 1.

Theorem 1.3.29. [59] *A 1, +1 system exists if and only if* $p \equiv 0$ *or* $1 (\mathrm{mod}\ 4)$.

A 1, +1 system is also known as Skolem sequence, which is defined as follows:

Definition 1.3.30. Let $\langle C_i \rangle$ be a sequence of $2p$ terms, where $1 \leq C_i \leq p$. If each number i occurs exactly twice in the sequence and $|j_2 - j_1| = i$ if $i = C_{j_1} = C_{j_2}$ then $\langle C_i \rangle$ is called a *Skolem sequence*.

This concept was used by Lee and Shee [41] to introduce the notion of Skolem gracefulness of graphs.

Definition 1.3.31. A *Skolem graceful labeling* of a graph $G = (V, E)$ is a bijection $f : V \to \{1, 2, \ldots, p\}$ such that the induced labeling $g_f : E \to \{1, 2, \ldots, q\}$ defined by $g_f(uv) = |f(u) - f(v)|$, $\forall uv \in E$, is also a bijection. If such a labeling exists, then the graph G is called a *Skolem graceful graph*.

If a graph G with p vertices and q edges, is graceful then $q \geq p - 1$, while if it is Skolem graceful, then $q \leq p - 1$. Thus, as noted in [41], Skolem graceful labelings nearly complement graceful labelings, and a graph with $q = p - 1$ is graceful if and only if it is Skolem graceful.

Theorem 1.3.32. [41] *The graph nK_2 is Skolem graceful if and only if $n \equiv 0$ or $1 \pmod 4$.*

O'Keefe [43] extended the methods of Skolem sequences for $k \equiv 2$ or $3 \pmod 4$ by showing that the numbers $1, 2, \ldots, 2k - 1, 2k + 1$ can be distributed into k disjoint pairs (a_i, b_i) such that $b_i = a_i + i$ for $i = 1, 2, \ldots, k$. Motivated by this, Shalaby [42] defined the notion of hooked Skolem sequences.

Definition 1.3.33. A *hooked Skolem sequence* of order k is a sequence $(c_1, c_2, \ldots, c_{2k+1})$ of $2k + 1$ integers satisfying the following conditions:

1. For every $r \in \{1, 2, \ldots, k\}$ there exist exactly two elements c_i and c_j such that $c_i = c_j = r$.

2. If $c_i = c_j = r$ with $i < j$, then $j - i = r$.

3. $c_{2k} = 0$.

A hooked Skolem graceful graph [57] is defined as follows:

Definition 1.3.34. A (p, q) graph $G = (V, E)$ is said to be *hooked Skolem graceful* if there exists a bijection $f : V(G) \to \{1, 2, \ldots, p - 1, p+1\}$ such that the induced edge labeling $g_f : E \to \{1, 2, 3, \ldots, q\}$ defined by $g_f(uv) = |f(u) - f(v)|$, $\forall uv \in E$ is also bijective. Such a labeling f is called *hooked Skolem graceful labeling* of G.

A sequence $(a_1, a_2, \ldots, a_n)$ of integers $1, 2, \ldots, n$ is called a $(2, n)$ *Langford sequence* if for a_i appearing first at the i^{th} place, the next appearance of a_i is at $(a_i + i + 1)^{th}$ place (see [40]) .

Priday [45] and Davies [26] have proved that a $(2, n)$ Langford sequence exists if and only if $n \equiv 0$ or $3 (\bmod\ 4)$. Gillespie and Utz [30] generalized the concept of Langford sequence as follows:

Definition 1.3.35. Let k and n be positive integers with $k \geq 2$. The sequence $\alpha = (b_1, b_2, \ldots, b_{kn})$ is a (k, n) *Langford sequence (or a (k, n)-sequence)* provided that it consists of k appearances of i $(1 \leq i \leq n)$ and consecutive occurrences of i are separated by i elements of the sequence.

Priday [45] and Davies [26] introduced the concept of a perfect sequence as follows:

Definition 1.3.36. A sequence of m consecutive positive integers $\{d, d+1, \ldots, d+m-1\}$ is said to be *perfect* if the integers $\{1, 2, \ldots, 2m\}$

can be arranged into disjoint pairs $\{(a_i, b_i) : 1 \leq i \leq m\}$ so that $\{b_i - a_i : 1 \leq i \leq m\} = \{d, d+1, \ldots, d+m-1\}$.

Following Davies, Simpson [53] defined hooked sequence as follows:

Definition 1.3.37. A sequence of m consecutive positive integers $\{d, d+1, \ldots, d+m-1\}$ for which there is a partition of the set $\{1, 2, \ldots, 2m-1, 2m+1\}$ into m pairs (a_i, b_i) such that the m numbers $b_i - a_i$, $1 \leq i \leq m$ are all of the integers $d, d+1, \ldots, d+m-1$ is called a hooked sequence.

Hegde [34] introduced the notion of additively graceful graph as follows:

Definition 1.3.38. A (p, q) graph $G = (V, E)$ with $q \geq 1$ and $p \geq 2$ is said to be *additively graceful* if it admits an injective function $f : V \to \{0, 1, \ldots, \lceil \frac{(q+1)}{2} \rceil\}$ such that $\{f(u) + f(v) : uv \in E\} = \{1, 2, \ldots, q\}$.

Hegde [34] proved the following results.

Theorem 1.3.39. [34] *If G is an additively graceful (p, q)-graph then $q \geq 2p - 4$ and this bound is the best possible.*

Theorem 1.3.40. [34] *The complete graph K_p is additively graceful if and only if $2 \leq p \leq 4$.*

Theorem 1.3.41. [34] *An additively graceful graph is either K_2 or $K_{1,2}$ or has a triangle.*

Theorem 1.3.42. [34] *If an eulerian (p, q)-graph G is additively graceful then $q \equiv 0$ or $3 (\text{mod } 4)$.*

Theorem 1.3.43. [34] *A unicyclic graph G is additively graceful if and only if G is isomorphic to either C_3 or to the graph obtained by joining a unique vertex to any one vertex of C_3.*

1.3.2 GAPS IN EXISTING RESEARCH

Based on the literature survey on graph labeling problems given in Section 1.3.1, we identify certain gaps in the existing research.

Many graphs are known to be nongraceful. One can look into computing $grac(G)$ for certain classes of nongraceful graphs.

Decomposition of complete graphs into some specific class of graphs have many applications in coding theory, so one can characterize graphs having the hypergraceful decompositions and find hypergraceful labeling of certain classes of graphs like complete bipartite graph, complete graph, eulerian graph, hamiltonian graph, planar graph etc.

Given a sigraph S and a graph theoretic property $\mathcal{P}$, is it possible to embed S in a graceful sigraph S_1 having the property $\mathcal{P}$?

The sigraph S_1 could be eulerian, hamiltonian, planar or triangle-free etc. The problem of embedding any graph into an eulerian graceful graph is recently solved by Rao [47], but the complexity of the algorithm is $O(3^p)$, however one can look for a simpler labeling.

The concept of Skolem graceful graphs was introduced by Lee and Shee [41] and characterization of Skolem graceful graphs is still an open problem.

The concept of additively graceful labeling of graphs was introduced by Hegde [34], one can introduce a similar concept for sigraphs.

1.3.3 OBJECTIVES

Our objective is to find hypergraceful labeling of complete graphs. We study the various labelings of graphs like graceful labeling, Skolem graceful labeling, additively graceful labeling etc. We give a method for embedding a sigraph into a graceful sigraph having a specific property, like planarity, eulerian, triangle-free and hamiltonian, we also give an efficient method for embedding any graph into an eulerian graceful graph. We study Skolem graceful graphs in a more general setup as (k, d)-Skolem graceful graphs and (k, d)-hooked Skolem graceful graphs. We also introduce the concept of m-gracefulness of graphs motivated by gracefulness grac(G) of a graph G. Motivated by the concept of additively graceful graphs introduced by Hegde [34], we generalize this concept to the realm of sigraphs.

1.4 ORGANIZATION OF THE THESIS

In Chapter 1 we give some basic definitions, literature review of the research topic, some of the gaps in existing research and our objective for research.

In Chapter 2 we prove that the complete graph K_p is $(p-4)$-hypergraceful if and only if $p \geq 8$, $(p-3)$-hypergraceful for $p \geq 4$, $(p-2)$-hypergraceful for $p \geq 3$ and $(p-1)$-hypergraceful for $p \geq 2$. We also give all nonisomorphic 3-hypergraceful decompositions of K_5.

In Chapter 3 we define (k,d)-Skolem graceful graph as follows: A graph $G = (V, E)$ is said to be (k,d)-*Skolem graceful* if there exists a bijection $f : V(G) \to \{1, 2, \ldots, |V|\}$ such that the induced edge labeling g_f defined by $g_f(uv) = |f(u) - f(v)|$ $\forall uv \in E$, is a bijection from E to $\{k, k+d, \ldots, k+(q-1)d\}$, where k and d are positive integers. Such a labeling f is called (k,d)-*Skolem graceful labeling* of G. In this chapter we present several basic results on (k,d)-Skolem graceful graphs and prove that nK_2 is $(2,1)$-Skolem graceful if and only if $n \equiv 0$ or $3 \pmod 4$. We also prove that nK_2 is $(1,2)$-Skolem graceful. We finally close the chapter with the observation that $(1,1)$-Skolem graceful labeling of nK_2 produces a Skolem sequence, $(2,1)$-Skolem graceful labeling of nK_2 produces a $(2,n)$ Langford sequence and $(k,1)$-Skolem graceful labeling of nK_2

gives a perfect sequence.

In Chapter 4 we introduce the notion of (k,d)-hooked Skolem graceful graph as follows: A (p,q) graph $G = (V,E)$ is said to be *(k,d)-hooked Skolem graceful* if there exists a bijection $f : V(G) \rightarrow \{1, 2, \ldots, p-1, p+1\}$ such that the induced edge labeling $g_f : E \rightarrow \{k, k+d, k+2d, \ldots, k+(q-1)d\}$ defined by $g_f(uv) = |f(u) - f(v)|$, $\forall uv \in E$ is also a bijection, where k and d are positive integers. Such a labeling f is called *(k,d)-hooked Skolem graceful labeling* of G. We observe that when $k = d = 1$, this notion coincides with that of hooked Skolem graceful labeling of the graph G. We present some basic results and prove that nK_2 is $(2,1)$-hooked Skolem graceful if and only if $n \equiv 1$ or $2 (\text{mod } 4)$.

In Chapter 5 we define a new measure of gracefulness $m(G)$ of a graph G and determine the same for some families of nongraceful graphs. Let $G = (V,E)$ be a (p,q) graph. Let $f : V(G) \rightarrow \mathbb{N} \cup \{0\}$ be an injection such that the edge induced function g_f defined on E by $g_f(uv) = |f(u) - f(v)|$ is also injective. Let $c(f) = \max\{i : 1, 2, \ldots, i \text{ are edge labels under } f\}$. Let $m(G) = \max_f c(f)$, where the maximum is taken over all f. Then $m(G)$ is called the *m-gracefulness* of G. This new measure $m(G)$ determines how close G is to being graceful. We prove that there are infinitely many nongraceful graphs G with $m(G) = q-1$, we prove that for $n \equiv 1$ or $2 (\text{mod } 4)$, $m(C_n) = n-1$ and $grac(G) = n+1$. We also show that $m(F_k) = 3k-1 = q-1$ and $grac(F_k) = 3k+1 = q+1$ for $k \equiv 2$ or $3 (\text{mod } 4)$, where F_k is the friendship graph with k

triangles. We give necessary conditions for a (p, q)-eulerian graph and the complete graph K_p to have m-gracefulness $q-1$ and $q-2$. Using this, we prove that K_5 is the only complete graph to have m-gracefulness $q-1$. We thus prove that $m(K_6) = 13 = q-2$, which is also shown in optimal Golomb ruler. We also give an upper bound for the highest possible vertex label of K_p if $m(K_p) = q-2$.

The concept of additively graceful graphs is extended to the realm of sigraphs in Chapter 6. Let $S = (V, E)$ be a (p, m, n) sigraph with $E = E^+ \cup E^-$, assume $|E^+| = m$ and $|E^-| = n$ where $m + n = q$. Let $f : V \to \{0, 1, ..., m + \lceil \frac{(n+1)}{2} \rceil\}$ be an injective mapping and let the induced edge function be defined as $g_{f^-}(uv) = f(u) + f(v)\ \forall\ uv \in E^-$ and $g_{f^+}(uv) = |f(u) - f(v)|\ \forall\ uv \in E^+$. If $g_{f^-}(uv) = \{1, 2, ..., n\}$ and $g_{f^+}(uv) = \{1, 2, ..., m\}$, then f is called an *additively graceful labeling* of the sigraph S. The sigraph which admits such a labeling is called an *additively graceful sigraph.* One can easily see that when $n = 0$, f is a graceful labeling of S, and when $m = 0$, f is an additively graceful labeling of S. This chapter gives some necessary or sufficient conditions for additively graceful sigraphs and some results on eulerian sigraphs, complete bipartite sigraphs and complete sigraphs.

In Chapter 7 we consider the following problem: *Given a sigraph S and a graph theoretic property $\mathcal{P}$, is it possible to embed S in a graceful sigraph S_1 having the property $\mathcal{P}$?* We give the existence of such an embedding where S_1 is eulerian, hamiltonian, planar or triangle-free. This chapter also proves that every signed tree can be embedded in a graceful signed tree. We also give an efficient procedure of embedding a graph G of order p as an induced subgraph of an eulerian graceful graph H whose order is $O(p^2)$.

In Chapter 8, we give a summary of all the results obtained by us and indicate scope for further research.

CHAPTER 2

ON k-HYPERGRACEFUL COMPLETE GRAPHS

2.1 INTRODUCTION

The notion of hypergraceful decomposition of graphs was first introduced by Acharya [5]. A (p, q)-graph $G = (V, E)$ is said to be *k-hypergraceful* if there exists a decomposition of G into *edge induced subgraphs* G_1, G_2, $\ldots$, G_k having sizes m_1, m_2, $\ldots$, m_k respectively, and an injective labeling $f : V(G) \to \{0, 1, \ldots, q\}$, such that when each edge $uv \in E(G)$ is assigned the absolute difference $|f(u) - f(v)|$, the set of integers received by the edges of G_i is precisely $\{1,\ 2,\ \ldots,\ m_i\}$ for each $i \in \{1,\ 2,\ \ldots,\ k\}$. The decomposition $\{G_i\}$, if it exists, is then called a *hypergraceful decomposition* of G and f is called a *k-hypergraceful labeling* of G. Further, G is said to be *hypergraceful* if it possesses a hypergraceful decomposition. When $k = 1$, the above definition yields the well known notion of graceful graphs and $k = 2$ corresponds to the extension of the notion of graceful graphs to the realm of sigraphs as studied in ([10–12, 55]).

Note that, in the definition of k-hypergraceful labeling, if G is a complete graph on p vertices, then the injective labeling f is a function $f : V(G) \to \{0, 1, \ldots, q^*\}$, where $q^* = \max\{m_i : 1 \leq i \leq k\}$. Characterization of k-hypergraceful complete graphs for $k = 2$ and some partial results for $k \geq 3$ are obtained by Rao et al. [46]. They have proved that no sigraph on the complete graph K_p, $p \geq 6$, is graceful and also have given a characterization of graceful sigraphs on K_p, $p \leq 5$.

Following are examples of k-hypergraceful graphs.

Example 2.1.1. The graph in Figure 2.1 is k-hypergraceful, where $k = 1, 2, 3$ and 4.

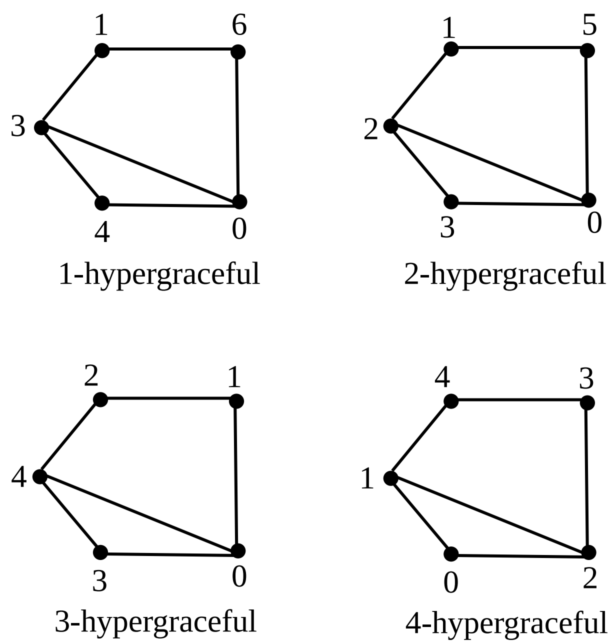


Figure 2.1: k-hypergraceful labeling of graph

Example 2.1.2. The friendship graph F_3 is known to be nongraceful, it is also known that no sigraph on F_3 is graceful. Therefore

F_3 is not 1-hypergraceful nor 2-hypergraceful. Figure 2.2 gives the 3-hypergraceful and 4-hypergraceful labelings of F_3.

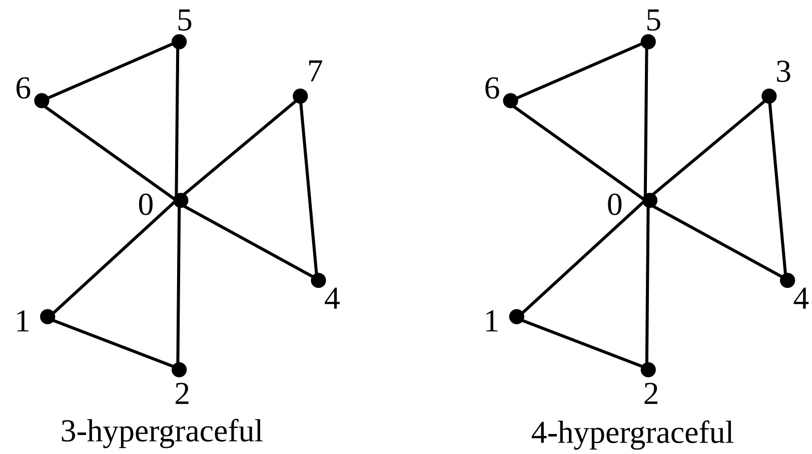


Figure 2.2: k-hypergraceful labeling of F_3 for $k = 3, 4$

In this chapter, we prove that the complete graph K_p is $(p-4)$-hypergraceful if and only if $p \geq 8$, $(p-3)$-hypergraceful for $p \geq 4$, $(p-2)$-hypergraceful for $p \geq 3$ and $(p-1)$-hypergraceful for $p \geq 2$. We also determine all possible nonisomorphic hypergraceful decompositions of the complete graph of order 5.

2.2 k-HYPERGRACEFUL LABELINGS OF COMPLETE GRAPHS

In this section we discuss the existance of k-hypergraceful labelings of the complete graph K_p where $k = (p-4)$ if and only if $p \geq 8$, $k = (p-3)$ for $p \geq 4$, $k = (p-2)$ for $p \geq 3$ and $k = (p-1)$ for $p \geq 2$. We present our results through a series of lemmas. We use the following notations in the proof of the lemmas.

Let $\pi = (a_1, a_2, \ldots, a_t)$ be a sequence of positive integers with $a_1 \leq a_2 \leq \cdots \leq a_t$. If a_i occurs r_i times in the sequence, then we write the sequence as $\pi = (a_1^{r_1}, a_2^{r_2}, \ldots, a_s^{r_s})$ where $1 \leq s \leq t$ and $\sum_{i=1}^{s} r_i = t$.

Lemma 2.2.1. *The complete graph K_p is $(p-4)$-hypergraceful if $p \geq 8$ and p is even.*

Proof. It is sufficient to provide a hypergraceful labeling for one possible decomposition of K_p, when p is even and $p \geq 8$. We label the vertices of K_p as $\{0, 3, 4, 6, 8, 9, \ldots, p+3\}$ and the edge labels of K_p are obtained as the absolute difference of its end vertex labels from $\{1, 2, 3, \ldots, p+3\}$. Let π_p denote the sequence of the corresponding edge labels. One can easily verify that $\pi_8 = (1^4, 2^4, 3^4, 4^3, 5^3, 6^3, 7^2, 8^2, 9^1, 10^1, 11^1)$ and $\pi_{10} = (1^6, 2^6, 3^6, 4^5, 5^4, 6^4, 7^3, 8^3, 9^3, 10^2, 11^1, 12^1, 13^1)$. The sequence π_8 determines the following 4-hypergraceful decomposition of K_8 given in Figure 2.3.

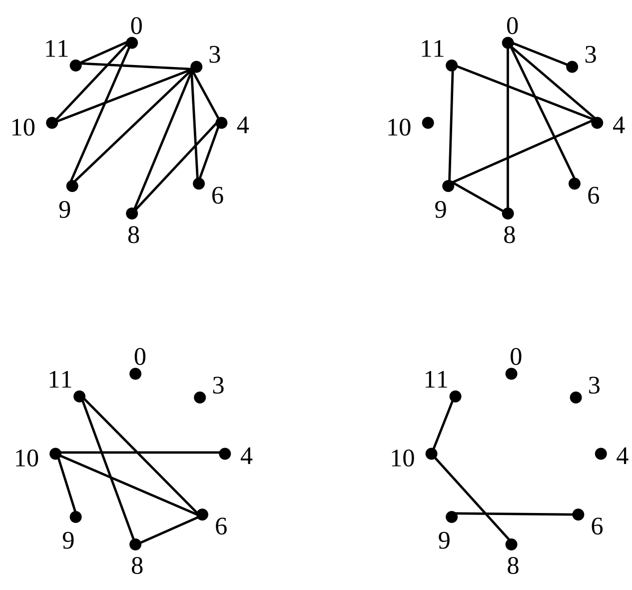


Figure 2.3: 4-hypergraceful decomposition of K_8

The sequence π_{10} determines the following 6-hypergraceful decomposition of K_{10} given in Figure 2.4.

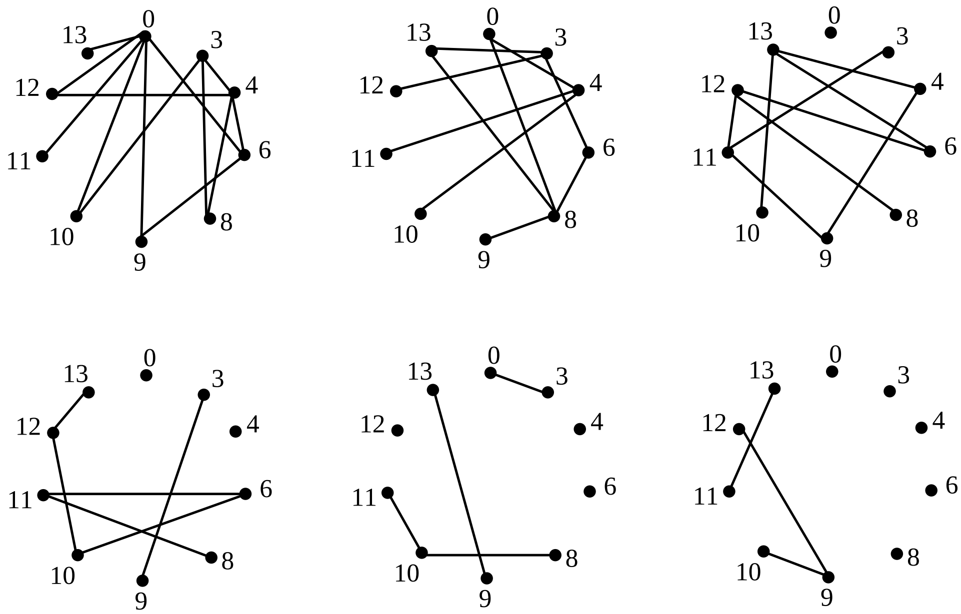


Figure 2.4: 6-hypergraceful decomposition of K_{10}

For $p \geq 12$, we claim that:

$$\pi_p = (1^{r_1}, 2^{r_2}, 3^{r_3}, \ldots, (p+3)^{r_{p+3}}), \tag{2.1}$$

$$\text{where } r_i = \begin{cases} p-4, & 1 \leq i \leq 3; \\ p-5, & i = 4; \\ p-6, & i = 5, 6; \\ p-8, & i = 7; \\ p-i, & 8 \leq i \leq p-4; \\ 4, & i = p-3; \\ 3, & i = p-2, p-1; \\ 2, & i = p; \\ 1, & i = p+1, p+2, p+3. \end{cases}$$

We prove this by induction on p.

From the vertex labels of K_{12}, we have $\pi_{12} = (1^8, 2^8, 3^8, 4^7, 5^6, 6^6, 7^4, 8^4, 9^4, 10^3, 11^3, 12^2, 13^1, 14^1, 15^1)$, which is the same as Equation 2.1 when $p = 12$. Thus the result is true for $p = 12$.

We now assume that the result is true for $p-2$. That is, $\pi_{p-2} = (1^{p-6}, 2^{p-6}, 3^{p-6}, 4^{p-7}, 5^{p-8}, 6^{p-8}, 7^{p-10}, 8^{p-10}, 9^{p-11}, 10^{p-12}, \ldots, (p-6)^4, (p-5)^4, (p-4)^3, (p-3)^3, (p-2)^2, (p-1)^1, p^1, (p+1)^1)$.

Now consider K_p, since $K_p = K_{p-2} + K_2$, and the labels of the two new vertices are $(p+2)$ and $(p+3)$, the sequence of edge labels of the additional $2p-3$ edges is given by $(1^2, 2^2, 3^2, \ldots, (p-6)^2, (p-5)^1, (p-4)^1, (p-3)^1, (p-2)^1, (p-1)^2, p^1, (p+2)^1, (p+3)^1)$. Thus $\pi_p = (1^{p-4}, 2^{p-4}, 3^{p-4}, 4^{p-5}, 5^{p-6}, 6^{p-6}, 7^{p-8}, 8^{p-8}, 9^{p-9}, \ldots, (p-6)^6, (p-5)^5, (p-4)^4, (p-3)^4, (p-2)^3, (p-1)^3, p^2, (p+1)^1, (p+2)^1, (p+3)^1)$. This proves Equation 2.1.

If r_i denotes the number of repetitions of the label i in π_p, then we have $p-4 = r_1 \geq r_2 \geq r_3 \geq \cdots \geq r_{p+3} = 1$ and $\sum_{i=1}^{p+3} r_i = \binom{p}{2}$. Further, the maximum repetition of an edge label is $p-4$. Therefore π_p determines a $(p-4)$-hypergraceful decomposition of K_p, $p \geq 12$. Hence K_p is $(p-4)$-hypergraceful for $p \geq 8$ and even. $\square$

Lemma 2.2.2. *The complete graph K_p is $(p-4)$-hypergraceful if $p = 4t+1$ for every positive integer $t \geq 2$.*

Proof. We provide a hypergraceful labeling for one possible decomposition of K_p, where $p = 4t+1$ for every positive integer $t \geq 2$.

We label the vertices of K_p as $\{0,\ 2,\ 4,\ 5,\ 8,\ 9,\ \ldots,\ p+3\}$ and the edge labels of K_p are obtained as the absolute difference of its end vertex labels from $\{1,\ 2,\ 3,\ \ldots,\ p+3\}$. Let π_p denote the sequence of the corresponding edge labels. One can easily verify that $\pi_9 = (1^5, 2^5, 3^4, 4^4, 5^3, 6^3, 7^3, 8^3, 9^2, 10^2, 11^1, 12^1)$. The sequence π_9 determines a 5-hypergraceful decomposition of K_9 as given in Figure 2.5.

For $p \geq 13$, we claim that:

$$\pi_p = (1^{r_1}, 2^{r_2}, 3^{r_3}, \ldots, (p+3)^{r_{p+3}}), \tag{2.2}$$

$$\text{where } r_i = \begin{cases} p-4, & i = 1, 2; \\ p-5, & i = 3, 4; \\ p-6, & i = 5; \\ p-7, & i = 6; \\ p-8, & i = 7; \\ p-i, & 8 \leq i \leq p-4; \\ 4, & i = p-3, p-2; \\ 3, & i = p-1; \\ 2, & i = p, p+1; \\ 1, & i = p+2, p+3. \end{cases}$$

We prove this by induction on p.

For $p = 13$, from the vertex labels of K_{13}, we have $\pi_{13} = (1^9, 2^9, 3^8, 4^8, 5^7, 6^6, 7^5, 8^5, 9^4, 10^4, 11^4, 12^3, 13^2, 14^2, 15^1, 16^1)$, which is

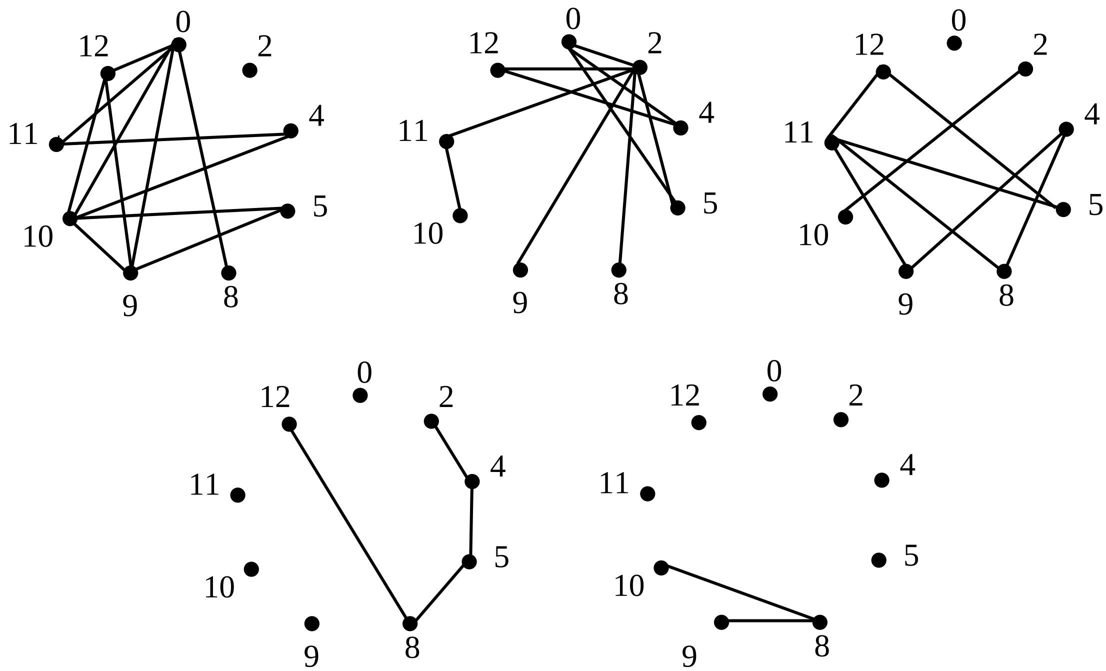


Figure 2.5: 5-hypergraceful decomposition of K_9

the same as Equation 2.2 when $p = 13$. Thus the result is true for $p = 13$.

We now assume that the result is true for $p-4$. That is, $\pi_{p-4} = (1^{p-8}, 2^{p-8}, 3^{p-9}, 4^{p-9}, 5^{p-10}, 6^{p-11}, 7^{p-12}, 8^{p-12}, 9^{p-13}, \ldots, (p-8)^4, (p-7)^4, (p-6)^4, (p-5)^3, (p-4)^2, (p-3)^2, (p-2)^1, (p-1)^1)$.

Consider K_p, since $K_p = K_{p-4} + K_4$, and the labels of the four new vertices are p, $p+1$, $p+2$ and $p+3$, the sequence of edge labels of the additional $4p-10$ edges is given by $(1^4, 2^4, 3^4, \ldots, (p-8)^4, (p-7)^3, (p-6)^2, (p-5)^2, (p-4)^2, (p-3)^2, (p-2)^3, (p-1)^2, p^2, (p+1)^2, (p+2)^1, (p+3)^1)$. Thus $\pi_p = (1^{p-4}, 2^{p-4}, 3^{p-5}, 4^{p-5}, 5^{p-6}, 6^{p-7}, 7^{p-8}, 8^{p-8}, 9^{p-9}, \ldots, (p-4)^4, (p-3)^4, (p-2)^4, (p-1)^3, p^2, (p+1)^2, (p+2)^1, (p+3)^1)$. This proves Equation 2.2.

If r_i denotes the number of repetitions of the label i in π_p, then we have $p-4 = r_1 \geq r_2 \geq r_3 \geq \cdots \geq r_{p+3} = 1$ and $\sum_{i=1}^{p+3} r_i = \binom{p}{2}$. Further, the maximum repetition of an edge label is $p-4$. Hence π_p determines a $(p-4)$-hypergraceful decomposition of K_p, where $p = 4t+1$ for every positive integer $t \geq 2$. □

Lemma 2.2.3. *The complete graph K_p is $(p-4)$-hypergraceful, if $p = 4t+3$ for every positive integer $t \geq 3$.*

Proof. It is sufficient to provide a hypergraceful labeling for one possible decomposition of K_p, where $p = 4t+3$ for every positive integer $t \geq 3$. We label the vertices of K_p as $\{0, 3, 4, 6, 8, 9, \ldots, p+3\}$ and the edge labels of K_p are obtained as the absolute difference of its end vertex labels from $\{1, 2, 3, \ldots, p+3\}$.
We claim that the sequence of the corresponding edge labels

$$\pi_p = (1^{r_1}, 2^{r_2}, 3^{r_3}, \ldots, (p+3)^{r_{p+3}}), \qquad (2.3)$$

$$\text{where } r_i = \begin{cases} p-4, & 1 \leq i \leq 3; \\ p-5, & i = 4; \\ p-6, & i = 5, 6; \\ p-8, & i = 7; \\ p-i, & 8 \leq i \leq p-4; \\ 4, & i = p-3; \\ 3, & i = p-2, p-1; \\ 2, & i = p; \\ 1, & i = p+1, p+2, p+3. \end{cases}$$

We prove this by induction on p.

From the vertex labels we have $\pi_{15} = (1^{11}, 2^{11}, 3^{11}, 4^{10}, 5^9, 6^9, 7^7, 8^7, 9^6, 10^5, 11^4, 12^4, 13^3, 14^3, 15^2, 16^1, 17^1, 18^1)$, which is the same as Equation 2.3 when $p = 15$. Thus the result is true for $p = 15$.

We now assume that the result is true for $p-4$. That is, $\pi_{p-4} = (1^{p-8}, 2^{p-8}, 3^{p-8}, 4^{p-9}, 5^{p-10}, 6^{p-10}, 7^{p-12}, 8^{p-12}, 9^{p-13}, \ldots, (p-8)^4, (p-7)^4, (p-6)^3, (p-5)^3, (p-4)^2, (p-3)^1, (p-2)^1, (p-1)^1)$.

Since $K_p = K_{p-4} + K_4$, and the labels of the four new vertices are p, $p+1$, $p+2$ and $p+3$, the sequence of edge labels of the additional $4p-10$ edges is given by $(1^4, 2^4, 3^4, \ldots, (p-8)^4, (p-7)^3, (p-6)^3, (p-5)^2, (p-4)^2, (p-3)^3, (p-2)^2, (p-1)^2, p^2, (p+1)^1, (p+2)^1, (p+3)^1)$. Thus $\pi_p = (1^{p-4}, 2^{p-4}, 3^{p-4}, 4^{p-5}, 5^{p-6}, 6^{p-6}, 7^{p-8}, 8^{p-8}, 9^{p-9}, \ldots, (p-4)^4, (p-3)^4, (p-2)^3, (p-1)^3, p^2, (p+1)^1, (p+2)^1, (p+3)^1)$. This proves Equation 2.3.

If r_i denotes the number of repetitions of the label i in π_p, then we have $p-4 = r_1 \geq r_2 \geq r_3 \geq \cdots \geq r_{p+3} = 1$ and $\sum_{i=1}^{p+3} r_i = \binom{p}{2}$. Further, the maximum repetition of an edge label is $p-4$. Hence π_p determines a $(p-4)$-hypergraceful decomposition of K_p, where $p = 4t+3$ for every positive integer $t \geq 3$. □

Lemma 2.2.4. *The complete graph K_{11} is 7-hypergraceful.*

Proof. In this case, we shall provide a hypergraceful labeling of the complete graph K_{11} with respect to the 7-hypergraceful decomposition G_1, G_2, G_3, G_4, G_5, G_6 and G_7. The hypergraceful labeling of K_{11} is $\{0, 4, 6, 7, 9, 10, 11, 12, 13, 14, 15\}$ and the edge labels obtained as the absolute differences of its end vertices are as follows:

$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$,

$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$,

$\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$,

$\{1, 2, 3, 4, 5, 6, 7\}$,

$\{1, 2, 3, 4, 5, 6\}$,

$\{1, 2, 3, 4\}$ and

$\{1, 2, 3\}$.

Hence, K_{11} is 7-hypergraceful. □

Lemma 2.2.5. *The complete graph K_7 is not 3-hypergraceful.*

Proof. By Lemma 1.3.25 and Remark 1.3.26, it is easy to check that the only possible 3-hypergraceful decompositions G_1, G_2, G_3 of K_7 with sizes (m_1, m_2, m_3) are (1, 1, 19), (1, 3, 17), (1, 5, 15), (1, 7, 13), (1, 9, 11), (3, 3, 15), (3, 5, 13), (3, 7, 11), (3, 9, 9), (5, 5, 11), (5, 7, 9) and (7, 7, 7). We prove that none of these twelve decompositions have a 3-hypergraceful labeling of K_7.

Case 1: (1, 1, 19).

In this case, we have to assign the labels to the vertices of

K_7 from the set $\{0, 1, \ldots, 19\}$ such that the sequence of edge labels is $(1^3, 2^1, 3^1, \ldots, 19^1)$. A possible set of labels of the vertices of K_7 could be one from the two sets given in Table 2.1.

Vertex labels	Edge labels	Repetitions
{0, 1, 2, 6, 7, 16, 19}	6	2
{0, 1, 2, 6, 12, 16, 19}	10	2

Table 2.1: Vertex labeling of K_7 for the decomposition (1, 1, 19)

In both the cases shown in Table 2.1, the number in the third column violates the definition of 3-hypergracefulness of a graph.

Case 2: (1, 3, 17).

We have to assign the labels to the vertices of K_7 from the set $\{0, 1, \ldots, 17\}$ such that the sequence of edge labels is $(1^3, 2^2, 3^2, 4^1, 5^1, \ldots, 17^1)$. A possible set of labels of the vertices of K_7 could be one from the five sets given in Table 2.2.

Vertex labels	Edge labels	Repetitions
{0, 1, 2, 8, 11, 14, 17}	9	2
{0, 1, 2, 6, 10, 14, 17}	4	3
{0, 1, 2, 6, 7, 14, 17}	6	2
{0, 1, 2, 3, 8, 13, 17}	5	2
{0, 1, 2, 3, 9, 13, 17}	8	2

Table 2.2: Vertex labeling of K_7 for the decomposition (1, 3, 17)

In each of the cases shown in Table 2.2, the number in the third column violates the definition of 3-hypergracefulness of a graph.

Case 3: (1, 5, 15).

In this case, we have to assign the labels to the vertices of K_7 from the set $\{0, 1, \ldots, 15\}$ such that the sequence of edge labels is $(1^3, 2^2, 3^2, 4^2, 5^2, 6^1, 7^1, \ldots, 15^1)$. A possible set of labels of the vertices of K_7 could be one from the four sets given in Table 2.3.

Vertex labels	Edge labels	Repetitions
{0, 1, 2, 4, 9, 12, 15}	11	2
{0, 1, 6, 8, 11, 13, 15}	7	3
{0, 1, 4, 5, 6, 13, 15}	9	2
{0, 1, 4, 5, 7, 13, 15}	8	2

Table 2.3: Vertex labeling of K_7 for the decomposition (1, 5, 15)

In each of the cases shown in Table 2.3, the number in the third column violates the definition of 3-hypergracefulness.

Case 4: (1, 7, 13).

Here, we have to assign the labels to the vertices of K_7 from the set {0, 1, ..., 13} such that the sequence of edge labels is $(1^3, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2, 8^1, 9^1, \ldots, 13^1)$. A possible set of labels of the vertices of K_7 could be one from the seven sets given in Table 2.4.

Vertex labels	Edge labels	Repetitions
{0, 1, 2, 3, 6, 9, 13}	3	3
{0, 1, 2, 6, 7, 10, 13}	6	3
{0, 1, 6, 7, 9, 11, 13}	6	3
{0, 1, 4, 6, 8, 11, 13}	7	3
{0, 1, 4, 5, 8, 11, 13}	3	3
{0, 1, 4, 5, 6, 11, 13}	5	3
{0, 1, 4, 5, 7, 11, 13}	6	3

Table 2.4: Vertex labeling of K_7 for the decomposition (1, 7, 13)

In each of the cases shown in Table 2.4, the number in the third column violates the definition of 3-hypergracefulness of a graph.

Case 5: (1, 9, 11).

In this case, the available vertex labels for K_7 is from the set $\{0, 1, \ldots, 11\}$, so that the sequence of edge labels is $(1^3, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2, 8^2, 9^2, 10^1, 11^1)$. A possible set of labels of the vertices of K_7 could be one from the four sets given in Table 2.5.

Vertex labels	Edge labels	Repetitions
{0, 1, 2, 6, 8, 9, 11}	2	3
{0, 1, 2, 5, 8, 9, 11}	3	3
{0, 1, 2, 3, 7, 9, 11}	2	3
{0, 1, 2, 3, 4, 9, 11}	2	3

Table 2.5: Vertex labeling of K_7 for the decomposition (1, 9, 11)

None of the cases shown in Table 2.5, satisfy the definition of 3-hypergracefulness of K_7.

Case 6: (3, 3, 15).

In this case, we have to assign the labels to the vertices of K_7 from the set $\{0, 1, \ldots, 15\}$ such that the sequence of edge labels is $(1^3, 2^3, 3^3, 4^1, 5^1, \ldots, 15^1)$. A possible set of labels of the vertices of K_7 could be one from the two sets given in Table 2.6.

Vertex labels	Edge labels	Repetitions
{0, 1, 2, 5, 9, 12, 15}	10	2
{0, 1, 2, 3, 7, 11, 15}	8	2

Table 2.6: Vertex labeling of K_7 for the decomposition (3, 3, 15)

In both the cases shown in Table 2.6, the number in the third column violates the definition of 3-hypergracefulness of a graph.

Case 7: (3, 5, 13).

In this case, the vertices of K_7 can be labeled from the set

$\{0, 1, \ldots, 13\}$, so that the sequence of edge labels is $(1^3, 2^3, 3^3, 4^2, 5^2, 6^1, 7^1, \ldots, 13^1)$. A possible set of labels of the vertices of K_7 could be one from the four sets given in Table 2.7.

Vertex labels	Edge labels	Repetitions
{0, 1, 2, 3, 4, 9, 13}	9	2
{0, 1, 2, 3, 5, 9, 13}	8	2
{0, 1, 4, 6, 9, 11, 13}	9	2
{0, 1, 2, 5, 6, 10, 13}	8	2

Table 2.7: Vertex labeling of K_7 for the decomposition (3, 5, 13)

In each of the four cases shown in Table 2.7, the number in the third column violates the definition of 3-hypergracefulness of a graph.

Case 8: (3, 7, 11).

In this case, we have to assign the labels to the vertices of K_7 from the set $\{0, 1, \ldots, 11\}$ such that the sequence of edge labels is $(1^3, 2^3, 3^3, 4^2, 5^2, 6^2, 7^2, 8^1, 9^1 \ldots, 11^1)$. A possible set of labels of the vertices of K_7 could be one from the four sets given in Table 2.8.

Vertex labels	Edge labels	Repetitions
{0, 1, 2, 5, 7, 8, 11}	6	3
{0, 1, 2, 4, 6, 8, 11}	2	4
{0, 1, 2, 6, 7, 8, 11}	6	3
{0, 1, 4, 6, 7, 9, 11}	5	3

Table 2.8: Vertex labeling of K_7 for the decomposition (3, 7, 11)

In each of the cases shown in Table 2.8, the number in the third column violates the definition of 3-hypergracefulness of a graph.

Case 9: (3, 9, 9).

In order to get the sequence of edge labels as $(1^3, 2^3, 3^3, 4^2,$

$5^2, 6^2, 7^2, 8^2, 9^2)$, we have to assign the labels to the vertices of K_7 from the set $\{0, 1, 2, \ldots, 9\}$, which is not possible as we cannot get two edges with label 9.

Case 10: (5, 5, 11).

In this case, in order to get the sequence of edge labels as $(1^3, 2^3, 3^3, 4^3, 5^3, 6^1, 7^1, \ldots, 11^1)$, we have to assign the labels from the set $\{0, 1, \ldots, 11\}$ to the vertices of K_7. A possible set of labels of the vertices of K_7 could be one from the two sets given in Table 2.9.

Vertex labels	Edge labels	Repetitions
$\{0, 1, 2, 3, 5, 7, 11\}$	2	4
$\{0, 1, 4, 6, 7, 9, 11\}$	6	2

Table 2.9: Vertex labeling of K_7 for the decomposition (5, 5, 11)

In both the cases shown in Table 2.9, there is repetition of edge labels and hence K_7 is not 3-hypergraceful in this case also.

Case 11: (5, 7, 9).

To get the sequence of edge labels as $(1^3, 2^3, 3^3, 4^3, 5^3, 6^2, 7^2, 8^1, 9^1)$, we have to assign the labels from the set $\{0, 1, \ldots, 9\}$ to the vertices of K_7. A possible set of labels of the vertices of K_7 could be one from the four sets given in Table 2.10.

Vertex labels	Edge labels	Repetitions
$\{0, 1, 2, 3, 4, 7, 9\}$	1	4
$\{0, 1, 2, 3, 5, 7, 9\}$	4	4
$\{0, 1, 2, 4, 6, 7, 9\}$	2	4
$\{0, 1, 2, 5, 6, 7, 9\}$	1	4

Table 2.10: Vertex labeling of K_7 for the decomposition (5, 7, 9)

In each of the cases shown in Table 2.10, the number in the third column violates the definition of 3-hypergracefulness of a graph.

Case 12: (7, 7, 7).

In this case, to get the sequence of edge labels $(1^3, 2^3, 3^3, 4^3, 5^3, 6^3, 7^3)$, we have to assign the labels to the vertices of K_7 from the set $\{0,\ 1,\ 2,\ \ldots,\ 7\}$. One can easily see that no labeling from this set can give three edges with label 7.

Thus we see that none of the above decompositions have a 3-hypergraceful labeling of K_7. Hence K_7 is not 3-hypergraceful. $\square$

Theorem 2.2.6. *The complete graph K_p is $(p-4)$-hypergraceful if and only if $p \geq 8$.*

Proof. Let $p \geq 8$, then by Lemmas 2.2.1−2.2.4 the result follows. Conversely, suppose K_p is $(p-4)$-hypergraceful and let $p < 8$. Therefore $p = 5$, 6 or 7. By Theorem 1.3.4, K_5 is nongraceful; by Theorem 1.3.28, K_6 is not 2-hypergraceful and by Lemma 2.2.5, K_7 is not 3-hypergraceful. This gives a contradiction to our assumption that K_p is $(p-4)$-hypergraceful. Therefore $p \geq 8$. This completes the proof. $\square$

Lemma 2.2.7. *The complete graph K_p is $(p-3)$-hypergraceful, for $p \geq 7$.*

Proof. We provide a hypergraceful labeling for one possible decomposition of K_p, where $p \geq 7$. We label the vertices of K_p as $\{0,\ 2,\ 5,$

6, 7, 8, 9, ..., $p+2\}$ and the edge labels of K_p are obtained as the absolute difference of its end vertex labels from $\{1, 2, 3, \ldots, p+2\}$. Let π_p denote the sequence of the corresponding edge labels. We claim that:

$$\pi_p = (1^{r_1}, 2^{r_2}, 3^{r_3}, \ldots, (p+2)^{r_{p+2}}), \tag{2.4}$$

$$\text{where } r_i = \begin{cases} p-3, & i = 1, 2; \\ p-4, & i = 3; \\ p-5, & i = 4; \\ p-i, & 5 \leq i \leq p-2; \\ 2, & i = p-1, p; \\ 1, & i = p+1, p+2. \end{cases}$$

We prove this by induction on p.

From the vertex labeling we have $\pi_7 = (1^4, 2^4, 3^3, 4^2, 5^2, 6^2, 7^2, 8^1, 9^1)$, which is the same as Equation 2.4 when $p = 7$. Thus the result is true for $p = 7$.

We now assume that the result is true for $p-1$. That is, $\pi_{p-1} = (1^{p-4}, 2^{p-4}, 3^{p-5}, 4^{p-6}, 5^{p-6}, 6^{p-7}, 7^{p-8}, \ldots, (p-3)^2, (p-2)^2, (p-1)^2, p^1, (p+1)^1)$.

Now consider K_p, since $K_p = K_{p-1} + K_1$, and the label of the new vertex is $(p+2)$, the sequence of edge labels of the additional $p-1$ edges is given by $(1^1, 2^1, 3^1, \ldots, (p-4)^1, (p-3)^1, p^1, (p+2)^1)$. Thus $\pi_p = (1^{p-3}, 2^{p-3}, 3^{p-4}, 4^{p-5}, 5^{p-5}, 6^{p-6}, \ldots, (p-3)^3, (p-2)^2, (p-1)^2, p^2, (p+1)^1, (p+2)^1)$. This proves Equation 2.4.

If r_i denotes the number of repetitions of the label i in π_p, then we have $p-3 = r_1 \geq r_2 \geq r_3 \geq \cdots \geq r_{p+2} = 1$ and $\sum_{i=1}^{p+2} r_i = \binom{p}{2}$. Further, the maximum repetition of an edge label is $p-3$. Hence π_p determines a $(p-3)$-hypergraceful decomposition of K_p, where $p \geq 7$. $\square$

Lemma 2.2.8. *The complete graph K_6 is 3-hypergraceful.*

Proof. We provide one 3-hypergraceful labeling of K_6, given by {0, 1, 3, 4, 5, 7}. One can easily verify that the corresponding sequence of induced edge labels is $(1^3, 2^3, 3^3, 4^3, 5^1, 6^1, 7^1)$. Therefore the decomposition G_1, G_2 and G_3 of K_6 with sizes (m_1, m_2, m_3) is (4, 4, 7) and hence the proof. $\square$

Theorem 2.2.9. *The complete graph K_p is $(p-3)$-hypergraceful for $p \geq 4$.*

Proof. By Theorems 1.3.4, 1.3.28 and Lemmas 2.2.7 and 2.2.8, the result follows. $\square$

Theorem 2.2.10. *The complete graph K_p is $(p-2)$-hypergraceful, for $p \geq 3$.*

Proof. It is sufficient to provide a hypergraceful labeling for one possible decomposition of K_p, where $p \geq 3$. We label the vertices of K_p as $\{0, 2, 3, 4, 5, \ldots, p\}$ and the edge labels of K_p are obtained as the absolute difference of its end vertex labels from $\{1, 2, 3, \ldots, p\}$. Let π_p denote the sequence of the corresponding edge labels. We claim that:

$$\pi_p = (1^{r_1}, 2^{r_2}, 3^{r_3}, \ldots, (p+2)^{r_p}), \tag{2.5}$$

$$\text{where } r_i = \begin{cases} p-2, & i = 1; \\ p-i, & 2 \leq i \leq p-1; \\ 1, & i = p. \end{cases}$$

We prove this by induction on p.

From the vertex labels we have $\pi_3 = (1^1, 2^1, 3^1)$, which is the same as Equation 2.5 when $p = 3$. Thus the result is true for $p = 3$.

We now assume that the result is true for $p-1$. That is, $\pi_{p-1} = (1^{p-3}, 2^{p-3}, 3^{p-4}, \ldots, (p-2)^1, (p-1)^1)$.

Since $K_p = K_{p-1} + K_1$, and the label of the new vertex is p, the sequence of edge labels of the additional $p-1$ edges is given by $(1^1, 2^1, 3^1, \ldots, (p-3)^1, (p-2)^1, p^1)$. Thus $\pi_p = (1^{p-2}, 2^{p-2}, 3^{p-3}, 4^{p-4}, \ldots, (p-3)^3, (p-2)^2, (p-1)^1, p^1)$. This proves Equation 2.5.

If r_i denotes the number of repetitions of the label i in π_p, then we have $p-2 = r_1 \geq r_2 \geq r_3 \geq \cdots \geq r_p = 1$ and $\sum_{i=1}^{p} r_i = \binom{p}{2}$. Further, the maximum repetition of an edge label is $p-2$. Hence π_p determines a $(p-2)$-hypergraceful decomposition of K_p, where $p \geq 3$. $\square$

Theorem 2.2.11. *The complete graph K_p, $p \geq 2$ is $(p-1)$-hypergraceful.*

Proof. It suffices to provide a $(p-1)$-hypergraceful labeling of K_p, $p \geq 2$. We label the vertices of K_p from the set $\{0, 1, 2, \ldots, p-1\}$. It can be easily verified that the sequence of edge labels $\pi_p = (1^{p-1}, 2^{p-2}, 3^{p-3}, \ldots, (p-2)^2, (p-1)^1)$. Hence K_p, $p \geq 2$ is $(p-1)$-hypergraceful. $\square$

2.3 3-HYPERGRACEFUL DECOMPOSITION OF K_5

In this section, we shall provide all nonisomorphic 3-hypergraceful decompositions of the complete graph of order 5. By Lemma 1.3.25 and Remark 1.3.26, the possible 3-hypergraceful decompositions G_1, G_2, G_3 of K_5 with sizes (m_1, m_2, m_3) are (1, 1, 8), (1, 2, 7), (1, 3, 6), (1, 4, 5), (2, 2, 6), (2, 3, 5), (2, 4, 4) and (3, 3, 4). Out of these the possible four decompositions of K_5 having a 3-hypergraceful labeling are given in Table 2.11.

Case	Decomposition	3-Hypergraceful labeling
1	(1, 2, 7)	{0, 4, 5, 6, 7}
2	(1, 3, 6)	{0, 3, 4, 5, 6}
3	(1, 4, 5)	{0, 1, 3, 4, 5}
4	(2, 3, 5)	{0, 2, 3, 4, 5}

Table 2.11: 3-hypergraceful labeling of K_5

In each of these cases, it is easy to check that the given labeling of K_5 is a 3-hypergraceful labeling.

Case 1: (1, 2, 7).

Since $m_1 = 1$, $m_2 = 2$ and $m_3 = 7$, the corresponding edge sequence is given by $(1^3, 2^2, 3^1, 4^1, 5^1, 6^1, 7^1)$. Hence, there are three ways to get edge label 1; two ways to get edge label 2; one way to get edge labels 3, 4, 5, 6 and 7, so there are six ways to get the edge labels of G_3. Having chosen the graph G_3, now there are two ways to get edge label 1 and one way to get edge label 2, so there are two ways to get edge labels 1 and 2 of G_2. Having chosen the graphs G_3 and G_2, there is only one way to get the graph G_1. Therefore there are in all twelve possible 3-hypergraceful decompositions of K_5. Out of these only three are nonisomorphic. The only possible nonisomorphic 3-hypergraceful decompositions (1, 2, 7) of K_5 with hypergraceful labeling is shown in Figure 2.6.

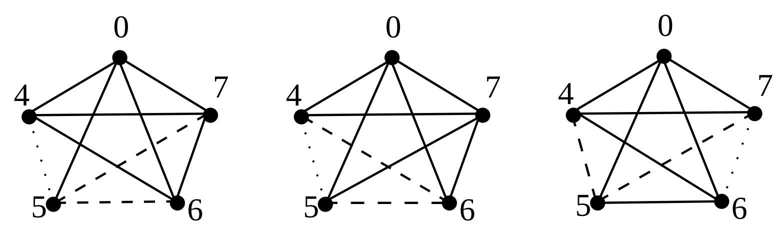


Figure 2.6: Nonisomorphic 3-hypergraceful labelings of K_5 for (1, 2, 7)

Figure 2.7 gives the possible nonisomorphic decompositions G_1, G_2, G_3 of K_5 with sizes (1, 2, 7).

Case 2: (1, 3, 6)

Here $m_1 = 1$, $m_2 = 3$ and $m_3 = 6$ and the corresponding sequence of edge labels is $(1^3, 2^2, 3^2, 4^1, 5^1, 6^1)$. Therefore, there are three ways to get edge label 1; two ways to get edge label 2; two

G_1

G_2

G_3

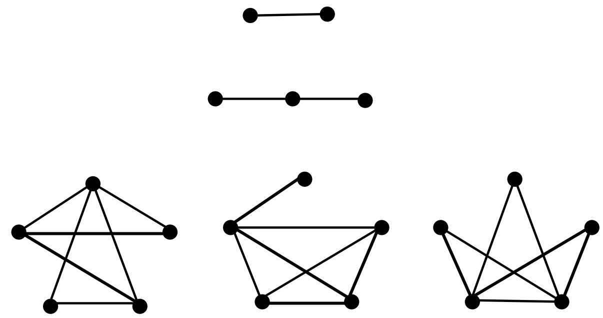


Figure 2.7: Nonisomorphic 3-decompositions of K_5 with sizes (1, 2, 7)

ways to get edge label 3 and one way to get edge labels 4, 5 and 6, so there are twelve ways to get edge labels of G_3. Having chosen the graph G_3, there are now two ways to get edge label 1; one way to get edge labels 2 and 3 so there are two ways to get edge labels 1, 2 and 3 for G_2. Having chosen the graphs G_3 and G_2, there is only one way to get the graph G_1. Therefore, in this case, there are twenty-four possible 3-hypergraceful decompositions of K_5. Out of these only nine are nonisomorphic. The only possible nonisomorphic 3-hypergraceful decompositions (1, 3, 6) of K_5 with hypergraceful labeling is shown in Figure 2.8.

Figure 2.9 gives the possible nonisomorphic decompositions G_1, G_2, G_3 of K_5 with sizes (1, 3, 6).

Case 3: (1, 4, 5)

In this case, the edge sequence is given by $(1^3, 2^2, 3^2, 4^2, 5^1)$. Hence, there are forty-eight ways to get the edge labels of G_1, G_2 and G_3. Out of these there are only fifteen nonisomorphic 3-hypergraceful decompositions of K_5.

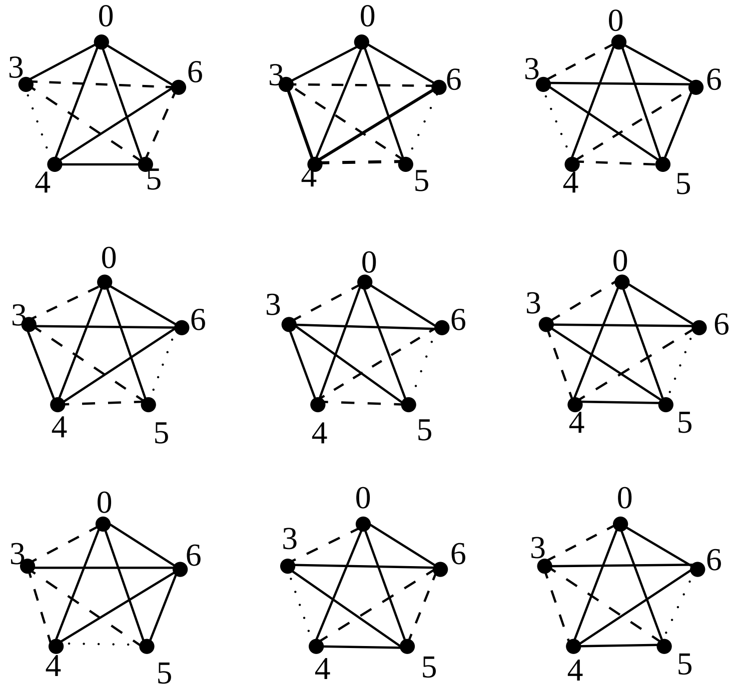


Figure 2.8: Nonisomorphic 3-hypergraceful labeling of K_5 for (1, 3, 6)

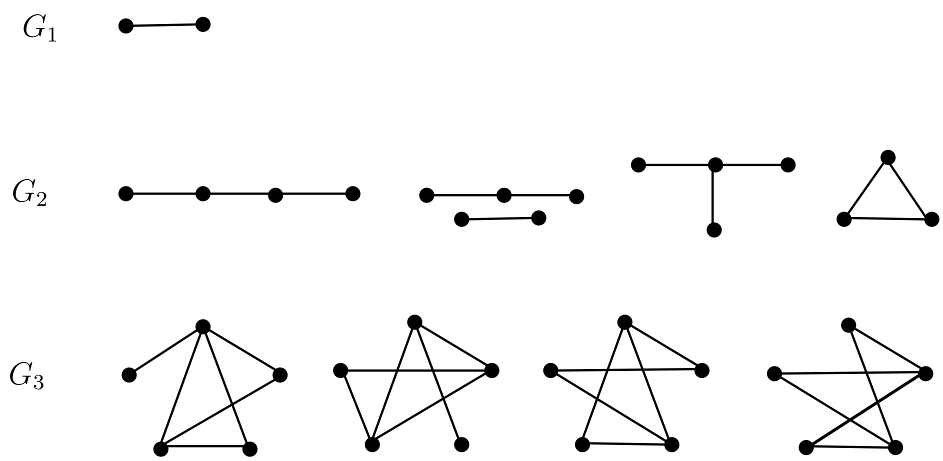


Figure 2.9: Nonisomorphic 3-decompositions of K_5 with sizes (1, 3, 6)

The possible nonisomorphic decompositions G_1, G_2, G_3 of K_5 with sizes (1, 4, 5) is shown in Figure 2.10.

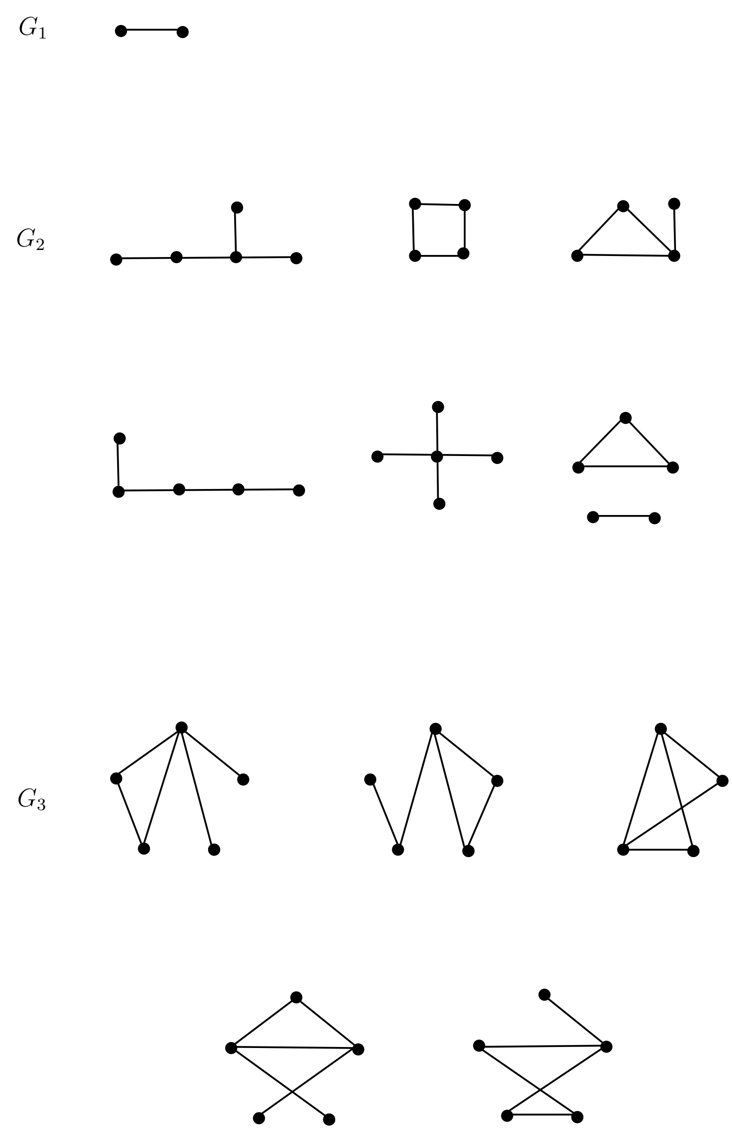


Figure 2.10: Nonisomorphic 3-decompositions of K_5 with sizes (1, 4, 5)

Case 4: (2, 3, 5)

In this case, the corresponding edge sequence is $(1^3, 2^3, 3^2, 4^1, 5^1)$. Therefore, there are seventy-two ways to get the edge labels of G_1, G_2 and G_3. Out of these there are only twenty-seven nonisomorphic 3-hypergraceful decompositions of K_5.

The possible nonisomorphic decompositions G_1, G_2, G_3 of K_5 with sizes (2, 3, 5) is shown in Figure 2.11.

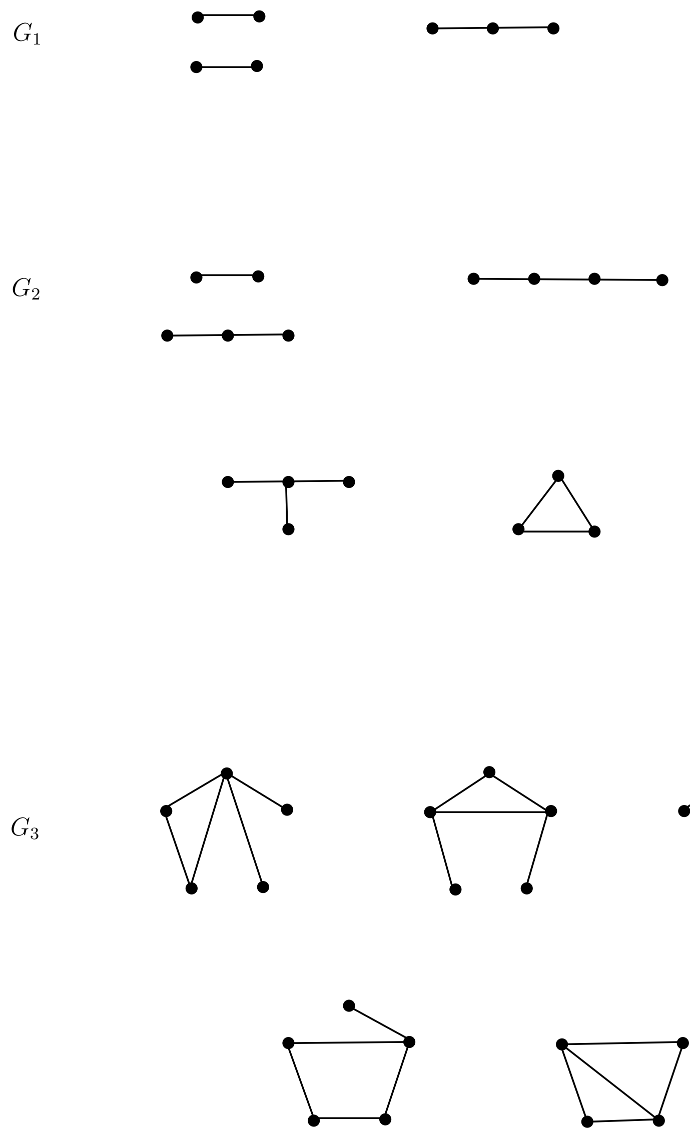


Figure 2.11: Nonisomorphic 3-decompositions of K_5 with sizes (2, 3, 5)

Remark 2.3.1. In general finding nonisomorphic k-hypergraceful decompositions of K_p, $p \geq 5$ seems to be a difficult problem.

CHAPTER 3

ON (k, d)-SKOLEM GRACEFUL GRAPHS*

3.1 INTRODUCTION

While studying the structure of Steiner triple systems, Skolem [59] considered the following problem: *Is it possible to distribute the numbers* $1, 2, \ldots, 2n$ *into* n *pairs* (a_i, b_i) *such that we have* $b_i - a_i = i$ *for* $i = 1, 2, \ldots, n$*?*

In the sequel, a set of pairs of this kind is called a 1, +1 system because the difference $b_i - a_i$ begins with 1 and increases by 1 when i increases by 1.

Example 3.1.1. For $n = 5$ the set of numbers is {1, 2, ..., 10}. Now {(1,2), (4,6), (7,10), (5,9), (3,8)} is a 1, +1 system.

One can see easily that such a system does not always exist.

---

*The content of this chapter has been published in *Electronic Notes in Discrete Mathematics* **48** (2015) 81–88.

Example 3.1.2. When $n = 3$, the set of numbers is {1, 2, 3, 4, 5, 6}, then possible number of pairs are as follows, showing that 1, +1 system does not exist for $n = 3$.

(1,2), (3,4), (5,6) and the corresponding differences are 1, 1, 1.

(2,3), (4,5), (1,6) and the corresponding differences are 1, 1, 5.

(1,3), (4,6), (2,5) and the corresponding differences are 2, 2, 3.

(2,4), (3,6), (1,5) and the corresponding differences are 2, 3, 4.

(3,5), (1,4), (2,6) and the corresponding differences are 2, 3, 4.

Hence the following question naturally arises: *For which n such a 1, +1 system of pairs exist?*

Skolem [59] proved that a 1, +1 system exists if and only if $n \equiv 0$ or $1 (\text{mod } 4)$. A 1, +1 system is also known as Skolem sequence, which is defined as follows: Let $< C_i >$ be a sequence of $2n$ terms, where $1 \leq C_i \leq n$. If each number i occurs exactly twice in the sequence and $|j_2 - j_1| = i$ if $i = C_{j_1} = C_{j_2}$ then $< C_i >$ is called a *Skolem sequence.*

This concept was used by Lee and Shee [41] to introduce the notion of Skolem gracefulness of graphs. *A Skolem graceful labeling* of a graph $G = (V, E)$ is a bijection $f : V \to \{1, 2, \ldots, p\}$ such that the induced labeling $g_f : E \to \{1, 2, \ldots, q\}$ defined by $g_f(uv) = |f(u) - f(v)| \; \forall uv \in E$ is also a bijection. If such a labeling exists, then the graph G is called a *Skolem graceful graph.*

If a graph G with p vertices and q edges is graceful, then $q \geq p-1$, while if it is Skolem graceful, then $q \leq p-1$. Thus, as noted in [41], Skolem graceful labelings nearly complement graceful labelings, and a graph with $q = p-1$ is graceful if and only if it is Skolem graceful.

A sequence $(a_1, a_2, \ldots, a_n)$ of integers $1, 2, \ldots, n$ is called a *(2,n) Langford sequence* if for a_i appearing first at the i^{th} place, the next appearance of a_i is at $(a_i + i + 1)^{th}$ place [40].

Example 3.1.3. The $(2,3)$ Langford sequence is $(3,1,2,1,3,2)$ and the $(2,4)$ Langford sequence is $(4,1,3,1,2,4,3,2)$.

Priday [45] and Davies [26] have proved that a $(2,n)$ Langford sequence exists if and only if $n \equiv 0$ or $3(\text{mod } 4)$. Gillespie and Utz [30] generalized the concept of Langford sequence as follows: Let k and n be positive integers with $k \geq 2$. The sequence $\alpha = (b_1, b_2, \ldots, b_{kn})$ is a (k,n) *Langford sequence* (or an (k,n)-*sequence)* provided that it consists of k appearances of i $(1 \leq i \leq n)$ and consecutive occurrences of i are separated by i elements of the sequence.

Priday [45] and Davies [26] introduced the concept of a perfect sequence. A sequence of m consecutive positive integers $\{d, d+1, \ldots, d+m-1\}$ is said to be *perfect* if the integers $\{1, 2, \ldots, 2m\}$ can be arranged into disjoint pairs $\{(a_i, b_i) : 1 \leq i \leq m\}$ so that $\{b_i - a_i : 1 \leq i \leq m\} = \{d, d+1, \ldots, d+m-1\}$.

Motivated by this, we define (k,d)-Skolem graceful graph as follows:

Definition 3.1.4. A graph $G = (V, E)$ is said to be (k, d)-*Skolem graceful* if there exists a bijection $f : V(G) \to \{1, 2, \ldots, p\}$ such that the induced edge labeling g_f defined by $g_f(uv) = |f(u) - f(v)|\ \forall uv \in E$, is a bijection from E to $\{k, k+d, \ldots, k+(q-1)d\}$, where k and d are positive integers. Such a labeling f is called (k, d)-*Skolem graceful labeling* of G.

In this chapter, we observe that a $(1, 1)$-Skolem graceful labeling of G is a Skolem graceful labeling. A $(1, 1)$-Skolem graceful labeling of nK_2 gives a Skolem sequence, a $(2, 1)$-Skolem graceful labeling of nK_2 gives a $(2, n)$ Langford sequence and a $(k, 1)$-Skolem graceful labeling of nK_2 gives a perfect sequence. We present several basic results on (k, d)-Skolem graceful graphs and prove that nK_2 is $(2, 1)$-Skolem graceful if and only if $n \equiv 0$ or $3 (\text{mod } 4)$. We also prove that nK_2 is (1,2)-Skolem graceful.

3.2 (k, d)-SKOLEM GRACEFUL GRAPHS

It follows from the definition that if G is (k, d)-Skolem graceful, then $q \leq p - 1$. For any two disjoint subsets A and B of V, we denote by $m(A, B)$ the number of edges of G with one end in A and the other end in B. Following is a necessary condition for a graph G to be (k, d)-Skolem graceful.

Theorem 3.2.1. *Let k and d be two positive integers which are not simultaneously even. If G is (k, d)-Skolem graceful, then $V(G)$ can* *be partitioned into two subsets V_o and V_e satisfying the following conditions.*

1. $m(V_o, V_e) = \lfloor \frac{q+1}{2} \rfloor$ *if k and d are both odd.*

2. $m(V_o, V_e) = \lfloor \frac{q}{2} \rfloor$ *if k is even and d is odd.*

3. $m(V_o, V_e) = q$ *if k is odd and d is even.*

Proof. Let f be a (k, d)-Skolem graceful labeling of G. Let $V_o = \{u \in V(G) : f(u) \text{ is odd}\}$ and $V_e = V(G) - V_o$. Then $\forall uv \in E$, $g_f(uv)$ is odd if and only if $u \in V_0$ and $v \in V_e$ or vice versa. Hence the result follows. □

Definition 3.2.2. A graph G is said to be arbitrarily Skolem graceful if G is (k, d)-Skolem graceful for all possible values of k and d.

In the following theorem we investigate the existence of (k, d)-Skolem graceful labeling for nK_2.

Theorem 3.2.3. *If the graph nK_2 is (k, d)-Skolem graceful, then one of the following holds.*

1. *If* $n \equiv 0 \pmod 4$, *then k and d can be even or odd.*

2. *If* $n \equiv 1 \pmod 4$, *then k is odd.*

3. *If* $n \equiv 2 \pmod 4$, *then d is even.*

4. *If* $n \equiv 3 \pmod 4$, *then $k + d$ is odd.*

Proof. Let f be a (k,d)-Skolem graceful labeling of nK_2. Let $e_i = u_iv_i$ be the components of nK_2 and let $f(u_i) = a_i, f(v_i) = b_i$ and $b_i > a_i, 1 \le i \le n$. Since the set of vertex labels is $\{1, 2, \ldots, 2n\}$ and the set of edge labels is $\{k, k+d, \ldots, k+(n-1)d\}$, we have

$$\sum_{i=1}^{n}(b_i - a_i) = k + (k+d) + \cdots + k + (n-1)d. \tag{3.1}$$

$$\sum_{i=1}^{n} b_i + \sum_{i=1}^{n} a_i = 1 + 2 + \cdots + 2n = n(2n+1). \tag{3.2}$$

On adding (3.1) and (3.2) we have

$$\sum_{i=1}^{n} b_i = \tfrac{1}{4}[2n(2n+1) + 2nk + n(n-1)d].$$

Thus 4 divides $[4n^2+2n+2nk+n(n-1)d]$ and the result follows. $\square$

Theorem 3.2.4. *The graph nK_2 is $(1,2)$-Skolem graceful.*

Proof. Let $e_i = a_ib_i$ be the edges of nK_2 with $b_i > a_i, 1 \le i \le n$. We define the vertex labeling f as follows: For $i = 1, 2, \ldots, n$,

$$f(a_i) = i$$

and

$$f(b_i) = 2n + 1 - i.$$

The edge induced function g_f is defined by $g_f(e_i) = b_i - a_i = 2n + 1 - 2i$ for $1 \le i \le n$. It can be easily verified that $g_f(E)$ has the required properties to qualify f to be a $(1,2)$-Skolem graceful labeling of nK_2. This completes the proof. $\square$

Example 3.2.5. In Figure 3.1 we give a $(1,2)$-Skolem graceful labeling of $6K_2$ and $7K_2$.

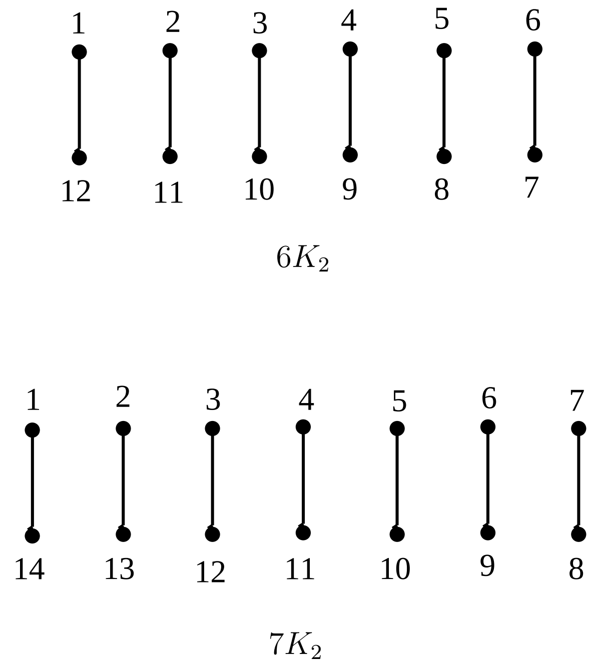


Figure 3.1: $(1,2)$-Skolem graceful labeling of $6K_2$ and $7K_2$

Theorem 3.2.6. *The graph nK_2 is $(2,1)$-Skolem graceful if and only if $n \equiv 0$ or $3 \pmod 4$.*

Proof. If nK_2 is $(2,1)$-Skolem graceful, then it follows from Theorem 3.2.3 that $n \equiv 0$ or $3 \pmod 4$. Conversely, let $n \equiv 0$ or $3 \pmod 4$. Let $e_i = a_ib_i$ be the edges of nK_2 with $b_i > a_i, 1 \leq i \leq n$.

Case 1. $n \equiv 0 \pmod 4$.

We define the vertex labeling f as follows:

$$f(a_i) = \begin{cases} i, & \text{for } 1 \leq i \leq \frac{n}{2}; \\ \frac{n}{2} - 2 + i, & \text{for } \frac{n}{2} + 1 \leq i \leq \frac{3n}{4}; \\ \frac{n}{2} - 1 + i, & \text{for } \frac{3n}{4} + 1 \leq i \leq n; \end{cases}$$

$$f(b_i) = \begin{cases} n - 1 - i, & \text{for } 1 \leq i \leq \frac{n}{4} - 1; \\ \frac{5n}{4}, & \text{for } i = \frac{n}{4}; \\ n - i, & \text{for } \frac{n}{4} + 1 \leq i \leq \frac{n}{2} - 1; \\ \frac{3n}{2} - 1, & \text{for } i = \frac{n}{2}; \\ 2n, & \text{for } i = \frac{n}{2} + 1; \\ \frac{5n}{2} - i, & \text{for } \frac{n}{2} + 2 \leq i \leq n - 1; \\ 2n - 1, & \text{for } i = n. \end{cases}$$

Case 2. $n \equiv 3(\text{mod } 4)$.

In this case, we define the vertex labeling f as follows:

$$f(a_i) = \begin{cases} i, & \text{for } 1 \leq i \leq \frac{n+1}{2}; \\ \frac{n+1}{2} - 2 + i, & \text{for } \frac{n+3}{2} \leq i \leq \frac{3(n+1)}{4}; \\ \frac{n+1}{2} - 1 + i, & \text{for } \frac{3(n+1)}{4} + 1 \leq i \leq n; \end{cases}$$

$$f(b_i) = \begin{cases} n-i, & \text{for } 1 \leq i \leq \frac{n-3}{4}; \\ \frac{5n+1}{4}, & \text{for } i = \frac{n+1}{4}; \\ n+1-i, & \text{for } \frac{n+1}{4}+1 \leq i \leq \frac{n-1}{2}; \\ \frac{3(n+1)}{2}, & \text{for } i = \frac{n+1}{2}; \\ \frac{3n+1}{2}, & \text{for } i = \frac{n+3}{2}; \\ \frac{5(n+1)}{2} - i, & \text{for } \frac{n+3}{2}+1 \leq i \leq n. \end{cases}$$

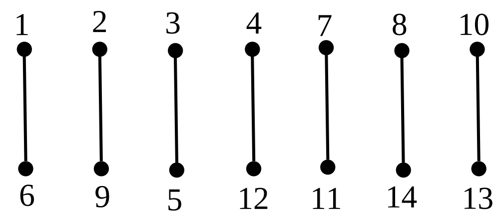
In each case, it can be easily verified that the induced edge function g_f defined by $g_f(e_i) = b_i - a_i$ has the required properties to qualify f to be a $(2,1)$-Skolem graceful labeling of nK_2 and the cases exhaust all the possibilities. This completes the proof. □

Based on the (k,d)-Skolem graceful labeling of nK_2, we define a (k,d)-Skolem sequence as follows:

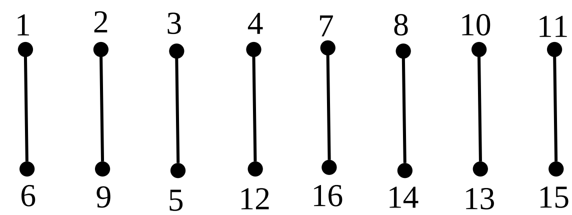
Definition 3.2.7. Consider the (k,d)-Skolem graceful labeling of nK_2 and construct a sequence $\langle s_i \rangle$, $1 \leq s_i \leq 1+(n-1)d$, of $2n$ terms as follows: If $|j_2 - j_1| = l$, where j_1 and j_2 are the vertex labels of a component of nK_2, then let $s_{j_1} = s_{j_2} = l-(k-1)$. Such a sequence is called a (k,d)-*Skolem sequence.*

From the above construction, we observe that a $(1,1)$-Skolem sequence corresponds to a Skolem sequence, a $(2,1)$-Skolem sequence corresponds to a $(2,n)$ Langford sequence and a $(k,1)$-Skolem sequence is a perfect sequence.

Example 3.2.8. Figure 3.2 gives $(2,1)$-Skolem graceful labeling of $7K_2$ and $8K_2$.



$7K_2$



$8K_2$

Figure 3.2: $(2,1)$-Skolem graceful labeling of $7K_2$ and $8K_2$

Thus the (2,7) Langford sequence is (4, 6, 1, 7, 1, 4, 3, 5, 6, 2, 3, 7, 2, 5) and the (2,8) Langford sequence is (4, 6, 1, 7, 1, 4, 8, 5, 6, 2, 3, 7, 2, 5, 3, 8).

CHAPTER 4

ON (k, d)-HOOKED SKOLEM GRACEFUL GRAPHS

4.1 INTRODUCTION

Skolem [59] proved that a Skolem sequence does not exist if $n \equiv 2$ or $3(\mathrm{mod}\ 4)$. O'Keefe [43] extended the concept of Skolem sequence for $n \equiv 2$ or $3(\mathrm{mod}\ 4)$ by showing that the numbers $1, 2, \ldots, 2n-1, 2n+1$ can be distributed into n disjoint pairs (a_i, b_i) such that $b_i = a_i + i$ for $i = 1, 2, \ldots, n$. Motivated by this, Shalaby [42] defined the notion of hooked Skolem sequences. A hooked Skolem sequence (HS) of order n is a sequence $(c_1, c_2, \ldots, c_{2n+1})$ of $2n+1$ integers satisfying the following conditions:

1. For every $r \in \{1, 2, \ldots, n\}$ there exist exactly two elements c_i and c_j such that $c_i = c_j = r$.
2. If $c_i = c_j = r$ with $i < j$, then $j - i = r$.
3. $c_{2n} = 0$.

In [53], a *hooked sequence* is defined as a sequence of m consecutive positive integers $\{d, d+1, \ldots, d+m-1\}$ for which there is a partition of the set $\{1, 2, \ldots, 2m-1, 2m+1\}$ into m pairs (a_i, b_i) such that the m numbers $b_i - a_i$, $1 \leq i \leq m$ are all of the integers $d, d+1, \ldots, d+m-1$. Where a_i and b_i are interpreted as the two positions in the sequence where $b_i - a_i$ appears. For example $48574365387 * 6$ and $64758463573 * 8$ are hooked sequences where $d = 3$ and $m = 6$.

In [57], a hooked Skolem graceful graph is defined as follows: A (p, q) graph $G = (V, E)$ is said to be *hooked Skolem graceful* if there exists a bijection $f : V(G) \to \{1, 2, \ldots, p-1, p+1\}$ such that the induced edge labeling $g_f : E \to \{1, 2, 3, \ldots, q\}$ defined by $g_f(uv) = |f(u) - f(v)|$, $\forall uv \in E$ is also bijective. Such a labeling f is called *hooked Skolem graceful labeling* of G.

In this chapter, we introduce the notion of (k, d)-hooked Skolem graceful graph as follows:

Definition 4.1.1. A (p, q) graph $G = (V, E)$ is said to be (k, d)-*hooked Skolem graceful* if there exists a bijection $f : V(G) \to \{1, 2, \ldots, p-1, p+1\}$ such that the induced edge labeling $g_f : E \to \{k, k+d, k+2d, \ldots, k+(q-1)d\}$ defined by $g_f(uv) = |f(u) - f(v)|$, $\forall uv \in E$ is also bijective, where k and d are positive integers. Such a labeling f is called (k, d)-*hooked Skolem graceful labeling* of G.

We observe that when $k = d = 1$, this notion coincides with that of hooked Skolem graceful labeling of the graph G [57].

In the next section we present some basic results and prove that nK_2 is $(2,1)$-hooked Skolem graceful if and only if $n \equiv 1$ or $2 (\mod 4)$.

4.2 (k, d)-HOOKED SKOLEM GRACEFUL GRAPHS

It follows from the definition that if G is (k, d)-hooked Skolem graceful, then $q \leq p - 1$. For any two disjoint subsets A and B of V, we denote by $m(A, B)$ the number of edges of G with one end in A and the other end in B. Following is a necessary condition for a graph G to be (k, d)-hooked Skolem graceful.

Theorem 4.2.1. *Let k and d be two positive integers which are not simultaneously even. If G is (k, d)-hooked Skolem graceful, then $V(G)$ can be partitioned into two subsets V_o and V_e satisfying the following conditions.*

1. *$m(V_o, V_e) = \lfloor \frac{q+1}{2} \rfloor$ if k and d are both odd.*
2. *$m(V_o, V_e) = \lfloor \frac{q}{2} \rfloor$ if k is even and d is odd.*
3. *$m(V_o, V_e) = q$ if k is odd and d is even.*

Proof. Let f be a (k, d)-hooked Skolem graceful labeling of G. Let $V_o = \{u \in V(G) : f(u) \text{ is odd}\}$ and $V_e = V(G) - V_o$. Then $\forall uv \in E$,

$g_f(uv)$ is odd if and only if $u \in V_o$ and $v \in V_e$ or vice versa. Hence the result follows. $\square$

In the following theorem, we investigate the existence of (k, d)-hooked Skolem graceful labeling for nK_2.

Theorem 4.2.2. *If nK_2 is (k, d)-hooked Skolem graceful, then one of the following holds.*

1. *$n \equiv 1 (\text{mod } 4)$, then k is even.*
2. *$n \equiv 2 (\text{mod } 4)$, then d is odd.*
3. *$n \equiv 3 (\text{mod } 4)$, then both k and d are even or they are odd.*

Proof. Let f be a (k, d)-hooked Skolem graceful labeling of nK_2. Let $e_i = u_i v_i$ be the components of nK_2 and let $f(u_i) = a_i$, $f(v_i) = b_i$ and $b_i > a_i$, $1 \leq i \leq n$. Since the set of vertex labels is $\{1, 2, \ldots, 2n-1, 2n+1\}$ and the set of edge labels is $\{k, k+d, \ldots, k+(n-1)d\}$, we have

$$\sum_{i=1}^{n} (b_i - a_i) = k + (k+d) + \cdots + k + (n-1)d \tag{4.1}$$

$$\begin{aligned} \sum_{i=1}^{n} b_i + \sum_{i=1}^{n} a_i &= 1 + 2 + \cdots + (2n-1) + (2n+1) \\ &= n(2n-1) + (2n+1) \end{aligned} \tag{4.2}$$

On adding (4.1) and (4.2) we have,

$$\sum_{i=1}^{n} b_i = \frac{1}{4}\left\{2nk + 4n^2 + 2n + 2 + n(n-1)d\right\}$$

Thus 4 divides $\left\{2nk + 4n^2 + 2n + 2 + n(n-1)d\right\}$ and the result follows. $\square$

The following theorem gives the necessary and sufficient condition for nK_2 to be $(2,1)$-hooked Skolem graceful.

Theorem 4.2.3. *The graph nK_2 is $(2,1)$-hooked Skolem graceful if and only if $n \equiv 1$ or $2 (\text{mod } 4)$.*

Proof. If nK_2 is $(2,1)$-hooked Skolem graceful, then it follows from Theorem 4.2.2 that $n \equiv 1$ or $2(\text{mod } 4)$.

Conversely, let $n \equiv 1$ or $2(\text{mod } 4)$. Let $e_i = a_i b_i$ be the edges of nK_2 with $b_i > a_i$, $1 \leq i \leq n$.

Case 1: $n \equiv 2(\text{mod } 4)$.

Let $n = 4r - 2$, where r is a positive integer. For $r = 1,\ 2$ and 3, the $(2,1)$-hooked Skolem graceful labeling of $2K_2$, $6K_2$ and $10K_2$ are given in Figure 4.1.

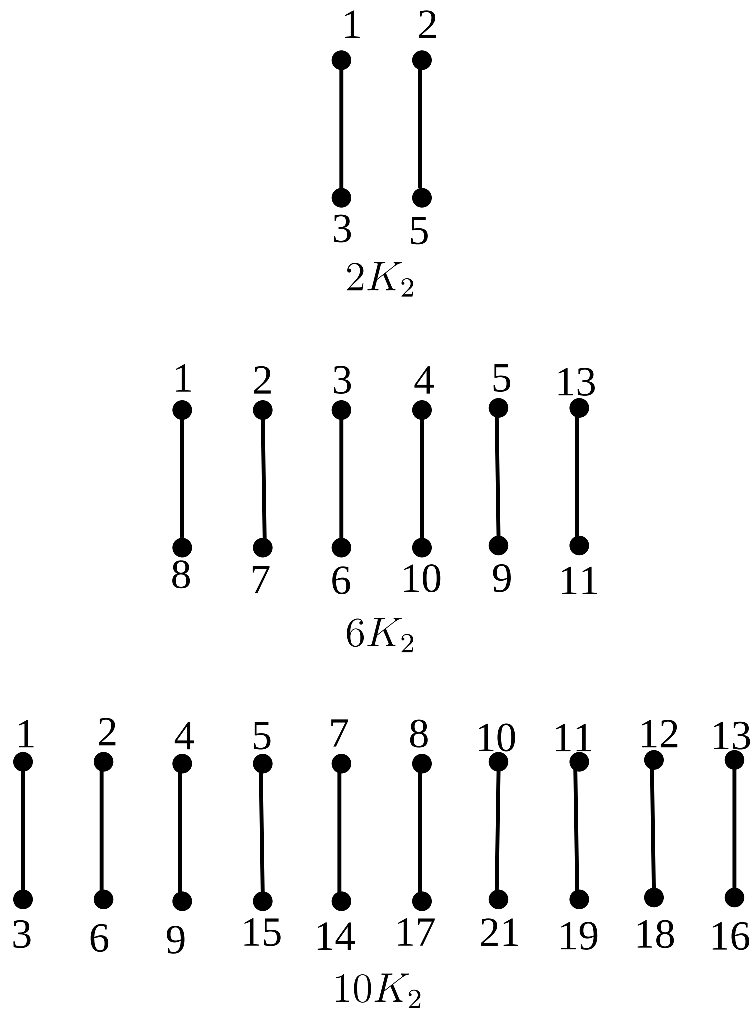


Figure 4.1: $(2, 1)$-hooked Skolem graceful labeling of $2K_2$, $6K_2$ and $10K_2$

For $r \geq 4$, we define the vertex labeling f as follows:

$$f(a_i) = \begin{cases} i, & \text{for } i = 1, 2; \\ i + 1, & \text{for } 3 \leq i \leq 2r - 2; \\ \frac{n+4}{2}, & \text{for } i = 2r - 1; \\ \frac{3n+2}{4}, & \text{for } i = 2r; \\ \frac{n-4}{2} + i, & \text{for } 2r + 1 \leq i \leq n. \end{cases}$$

$$f(b_i) = \begin{cases} 3, & \text{for } i = 1; \\ \frac{n+2}{2}, & \text{for } i = 2; \\ n + 2 - i, & \text{for } 3 \leq i \leq r; \\ n + 1 - i, & \text{for } r + 1 \leq i \leq 2r - 3; \\ \frac{3n}{2}, & \text{for } i = 2r - 2; \\ \frac{3n-2}{2}, & \text{for } i = 2r - 1; \\ \frac{7n-2}{4}, & \text{for } i = 2r; \\ 2n + 1, & \text{for } i = 2r + 1; \\ \frac{5n+4}{2} - i, & \text{for } 2r + 2 \leq i \leq 3r; \\ \frac{5n+2}{2} - i, & \text{for } 3r + 1 \leq i \leq n. \end{cases}$$

Case 2: $n \equiv 1 (\text{mod } 4)$.

Let $n = 4r - 3$, where r is a positive integer. For $r = 1$ and 2 the $(2,1)$-hooked Skolem graceful labelings of K_2 and $5K_2$ are given in Figure 4.2.

For $r \geq 3$, we define the vertex labeling f as follows:

$$f(a_i) = \begin{cases} i, & \text{for } 1 \leq i \leq 2r - 1; \\ \frac{n-3}{2} + i, & \text{for } 2r \leq i \leq 3r - 2; \\ \frac{n-1}{2} + i, & \text{for } 3r - 1 \leq i \leq n; \end{cases}$$

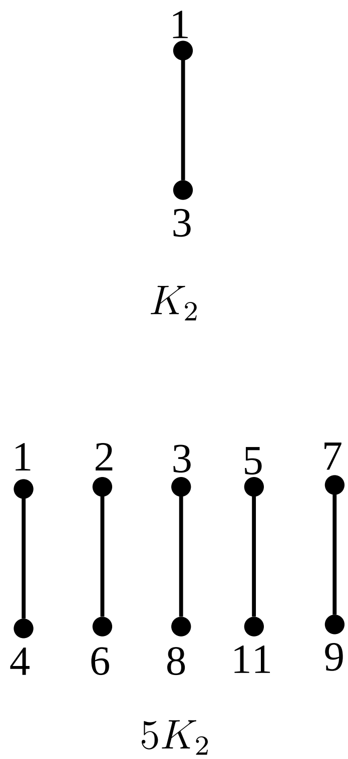


Figure 4.2: $(2,1)$-hooked Skolem graceful labeling of K_2 and $5K_2$

$$f(b_i) = \begin{cases} n-i, & \text{for } 1 \le i \le r-1; \\ n-1+i, & \text{for } i = r, n; \\ n+1-i, & \text{for } r+1 \le i \le 2r-2; \\ \frac{3n+1}{2}, & \text{for } i = 2r-1; \\ 2n+1, & \text{for } i = 2r; \\ \frac{5n+1}{2} - i, & \text{for } 2r+1 \le i \le n-1. \end{cases}$$

In each case, it can be easily verified that the induced edge function g_f defined by $g_f(e_i) = b_i - a_i$ has the required properties to qualify f to be a $(2,1)$-hooked Skolem graceful labeling of nK_2 and the cases exhaust all the possibilities. This completes the proof. $\square$

CHAPTER 5

A NEW MEASURE FOR GRACEFULNESS OF GRAPHS*†

5.1 INTRODUCTION

Bloom and Golomb considered two interesting and significant problems. One is to find largest graceful subgraph of the complete graph, which led to the limitation of the Design of a Communication Network and the other is to increase the maximum vertex label so that the induced edge labels are distinct, which resulted in finitely many counter examples to a "theorem" of S. Picard which was relied upon (erroneously) for some 35 years in the field of X-ray diffraction crystallography [35].

The second problem led to the concept of gracefulness of a graph. In [24], the gracefulness $grac(G)$ of a graph G with $V(G) = \{v_1, v_2, \ldots, v_p\}$ without isolates is defined as the smallest positive integer k for which it is possible to label the vertices of G with

*The content of this chapter has been published in *Electronic Notes in Discrete Mathematics* 48 (2015) 275–280.

†The content of this chapter to appear in *Lecture Notes in Computer Science*.

distinct elements from the set $\{0, 1, \ldots, k\}$ in such a way that when an edge is labeled with the absolute difference of the labels of its end vertices, then distinct edges receive distinct labels. Obviously $grac(G) \geq q$ and $grac(G) = q$ if and only if G is graceful. Thus $grac(G)$ gives a measure of gracefulness of G.

Motivated by this, in this chapter, we define a new measure of gracefulness of graphs and determine the same for some families of nongraceful graphs. We prove that there are infinitely many nongraceful graphs G with $m(G) = q - 1$. We give necessary conditions for a (p, q)-eulerian graph and the complete graph K_p to have m-gracefulness $q - 1$ and $q - 2$. Using this, we prove that K_5 is the only complete graph to have m-gracefulness $q - 1$. We also give an upper bound for the highest possible vertex label of K_p if $m(K_p) = q - 2$ and hence prove that $m(K_6) = 13 = q - 2$, which is also shown in optimal Golomb ruler [21].

Definition 5.1.1. Let G be a (p, q) graph. Let $f : V(G) \to \mathbb{N} \cup \{0\}$ be an injection such that the edge induced function g_f defined on E by $g_f(uv) = |f(u) - f(v)|$ is also injective. Let $c(f) = \max \{i : 1, 2, \ldots, i$ are edge labels under $f\}$. Let $m(G) = \max_f c(f)$, where the maximum is taken over all f. Then $m(G)$ is called the *m-gracefulness of G*, the labeling f is called the m-graceful labeling of G and the graph G is said to be m-graceful.

Example 5.1.2. Consider the triangular snake H_k on $2k+1$ vertices where k is a positive integer. It is known that H_k is nongraceful for $k \equiv 2$ or $3 (\mathrm{mod}\ 4)$. The labelings of H_k in Figure 5.1 gives $m(H_k) = 3k - 1 = q - 1$ and $grac(H_k) = 3k + 1 = q + 1$ for k =2, 3, 6.

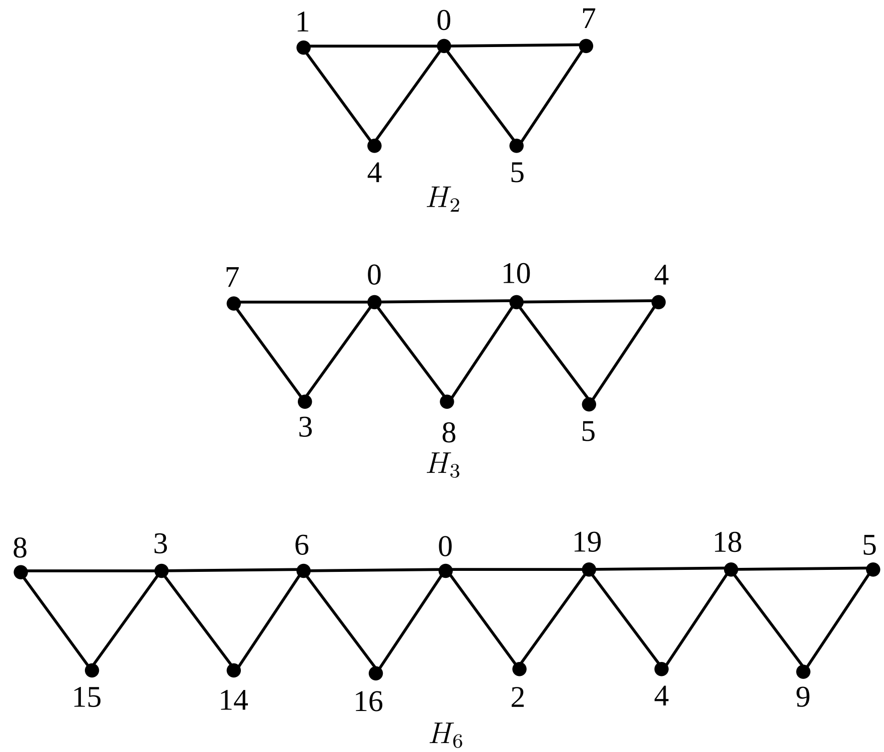


Figure 5.1: m-graceful labelings of triangular snakes H_k for $k = 2, 3, 6$

This new measure $m(G)$ determines how close G is to being graceful. We denote by $M_G(f)$ and $M_G(g_f)$, the largest vertex label and the largest induced edge label respectively, received by G under f. Note that the function $h : V \to \mathbb{N}$ defined by $h(v) = M_G(f) - f(v)\ \forall v \in V(G)$ is also an injective vertex labeling of the graph G, with the same set of induced edge labels $g_f(E)$. We therefore assume without loss of generality that $0 \in f(V)$. Also note that, $M_G(f) \geq grac(G)$ and if G is a graceful graph, then $m(G) = q$, $M_G(f) = q$ and $M_G(g_f) = q$. One may observe that $grac(G)$ measures gracefulness of the graph G from above q, whereas $m(G)$ measures gracefulness of G from below q.

5.2 m-GRACEFULNESS OF A GRAPH

In this section, we focus our study on $m(G)$ for nongraceful graphs.

All connected graphs of order at most four are known to be graceful. There are exactly three connected nongraceful graphs of order five and for each of them $m(G) = q-1$ and it is known that $grac(G) = q+1$. These three graphs with appropriate labelings for $grac(G)$ and $m(G)$ are given in Figure 5.2. If the label of a vertex is (a, b), then a is the label corresponding to $m(G)$ and b is the label corresponding to $grac(G)$.

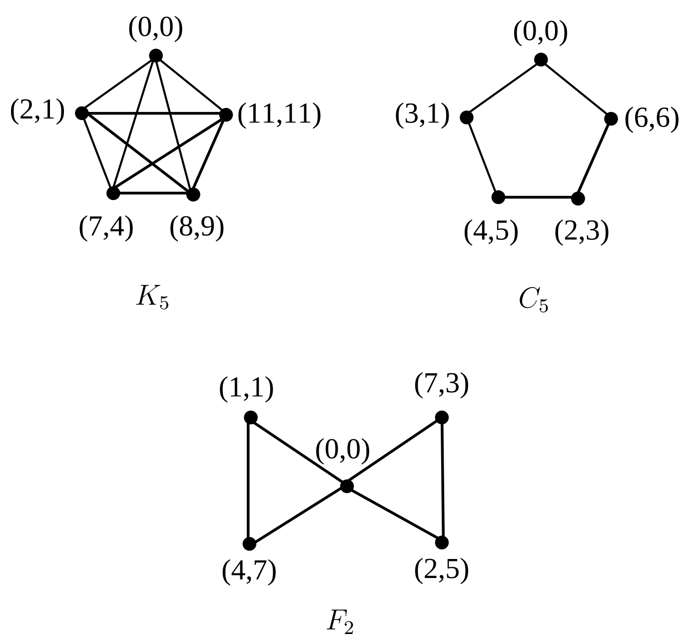


Figure 5.2: The three connected nongraceful graphs of order 5

The following theorem shows that there are infinitely many nongraceful graphs G with $m(G) = q - 1$.

Theorem 5.2.1. *There exist infinitely many nongraceful (p, q)-graphs having m-gracefulness $q - 1$ and grac $q + 1$.*

Proof. Consider the cycle C_5 having vertex set $\{v_1, v_2, v_3, v_4, v_5\}$ with two chords v_1v_3 and v_3v_5 as shown in Fig. 5.3.

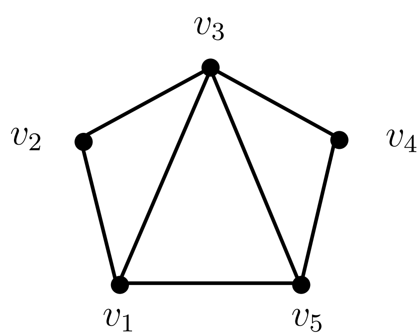


Figure 5.3: C_5 with 2 chords at a common vertex

For $k = 1, 2, \ldots$, construct graphs G_k by inserting $(2k-1)$ vertices $v_6, v_7, \ldots, v_{2k+4}$ and joining each of them to v_1 and v_5. Then G_k is an eulerian graph with order $2k+4$ and size $4k+5$ as shown in Fig. 5.4 and by Theorem 1.3.7, it is nongraceful. Hence $m(G_k) < q$. Now consider the labeling $f : V(G_k) \to \mathbb{N}$ defined by

$$f(v_i) = \begin{cases} i - 1 & \text{if } i = 1, 2 \\ 2k + 4 & \text{if } i = 3 \\ k + 2 & \text{if } i = 4 \\ 4k + 6 & \text{if } i = 5 \\ i - 4 & \text{if } 6 \le i \le k + 5 \\ i - 3 & \text{if } k + 6 \le i \le 2k + 4 \end{cases}$$

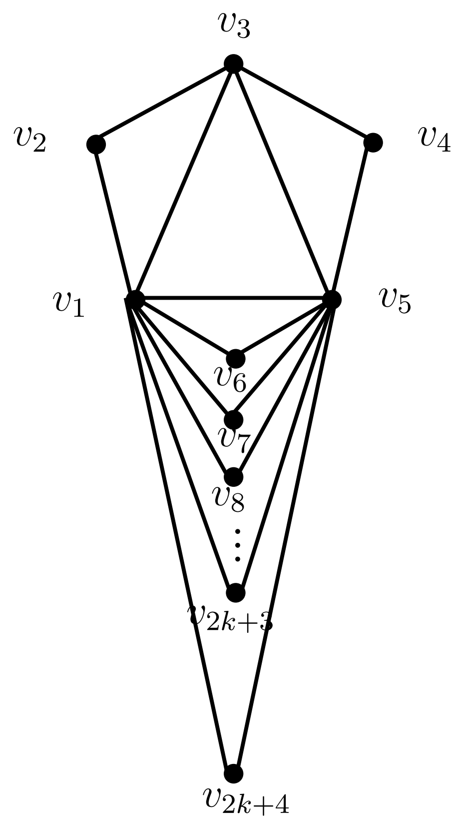


Figure 5.4: Nongraceful eulerian graph

It can be easily verified that the set of induced edge labels is $g_f(E) = \{1, 2, \ldots, 4k+3, 4k+4, 4k+6\}$. Hence $m(G_k) = q - 1$ and $grac(G_k) = q + 1$. $\square$

In the following theorem, we give a necessary condition for an eulerian (p, q)-graph to have m-gracefulness $q - 1$.

Theorem 5.2.2. *Let G be a (p, q)-eulerian graph with $m(G) = q - 1$. Then $q \equiv 2k$ or $(2k-1) (\mathrm{mod}\ 4)$, where $k = M_G(g_f) - q$.*

Proof. Let T be the sum of the edge labels of G. Then by Theorem 1.3.3, since G is eulerian and can be decomposed into cycles, T is an even number. Since $m(G) = q - 1$, $T = \frac{q(q-1)}{2} + (q+k)$ and this is even only when $q \equiv 2k(\text{mod } 4)$ or $q \equiv (2k-1)(\text{mod } 4)$. $\square$

We denote by $M'_G(g_f)$, the second largest edge label received by G under f. Note that if G is a graceful graph, then $M'_G(g_f) = q - 1$.

The following theorem gives a necessary condition for an eulerian graph to have m-gracefulness $q - 2$.

Theorem 5.2.3. *Let G be a (p, q)-eulerian graph with $m(G) = q-2$ under a labeling f. Then $q \equiv (2s+1)$ or $(2s+2)(\text{mod } 4)$, where $k = M_G(g_f) - q$ and $s = q + k - M'_G(g_f)$.*

Proof. Let T be the sum of the edge labels of G. Since G can be decomposed into cycles, it follows from Theorem 1.3.3 that T is an even number. Further $m(G) = q - 2$ implies $T = \frac{(q-1)(q-2)}{2} + (q + k - s) + (q + k)$ and this is even only when $q \equiv (2s+1)(\text{mod } 4)$ or $q \equiv (2s+2)(\text{mod } 4)$ where $1 \leq s \leq k$. $\square$

It is known that if $n \equiv 1$ or $2(\text{mod } 4)$, then the cycle C_n is nongraceful and in the following theorem we determine $m(C_n)$.

Theorem 5.2.4. *Let $n \equiv 1$ or $2(\text{mod } 4)$. Then $m(C_n) = n - 1$.*

Proof. Let $n = 4x + 2$ or $4x + 1$ according as $n \equiv 2(\text{mod } 4)$ or $n \equiv 1(\text{mod } 4)$. Let $C_n = (a_1, b_1, a_2, b_2, \ldots, a_{2x+1}, b_{2x+1})$ if $n \equiv 2(\text{mod } 4)$ and let $C_n = (a_1, b_1, a_2, b_2, \ldots, a_{2x}, b_{2x}, a_{2x+1})$ if $n \equiv 1(\text{mod } 4)$.

Let $f : V(C_n) \to \{0, 1, \ldots, n+1\}$ be defined as follows:

$$f(a_i) = \begin{cases} 0, & \text{for } \ i = 1; \\ i, & \text{for } 2 \leq i \leq 2x+1. \end{cases}$$

$$\text{and } f(b_i) = \begin{cases} n+2-i, & \text{for } 1 \leq i \leq x; \\ n+1-i, & \text{for } i \geq x+1. \end{cases}$$

It can be easily verified that f is injective, the induced edge function g_f is also injective, the highest vertex label used is $n+1$ and the set of induced edge labels is $\{1, 2, \ldots, n-2, n-1, n+1\}$. Hence $m(C_n) = n-1$ and the highest vertex label used to achieve this is $n+1$. □

Corollary 5.2.5. *$grac(C_n) = n+1$ for $n \equiv 1$ or $2 \pmod 4$.*

Example 5.2.6. Consider C_{21} and label its vertices as follows:

$$f(a_1) = 0,$$

$$f(a_i) = i \ \text{ for } i = 2, 3, \ldots, 11$$

$$\text{and } f(b_i) = \begin{cases} 23-i, & \text{for } i = 1, 2, \ldots, 5; \\ 22-i, & \text{for } i = 6, 7, \ldots, 10. \end{cases}$$

It is easy to check that the set of induced edge labels is $\{1,\ 2,\ 3,\ \ldots,\ 19,\ 20,\ 22\}$. Hence $m(C_{21}) = 20$ and $grac(C_{21}) = 22$.

Figure 5.5 gives the graph representation of m-graceful labeling of C_{21}.

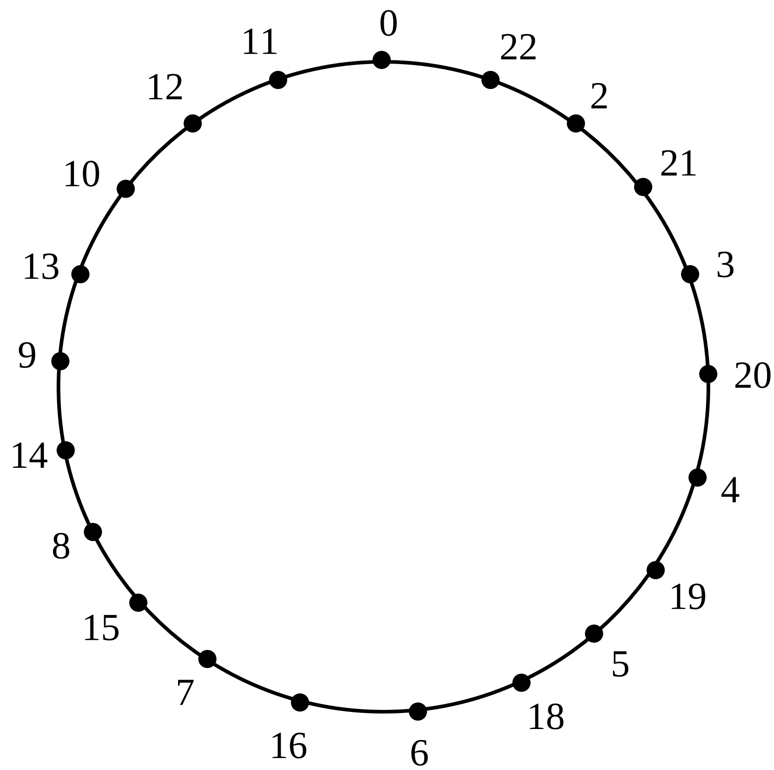


Figure 5.5: m-graceful labeling of C_{21}

Example 5.2.7. Consider C_{22} and label its vertices as follows:

$$f(a_1) = 0,$$

$$f(a_i) = i \quad \text{for } i = 2, 3, \ldots, 11$$

$$\text{and } f(b_i) = \begin{cases} 24 - i, & \text{for } i = 1, 2, \ldots, 5; \\ 23 - i, & \text{for } i = 6, 7, \ldots, 11. \end{cases}$$

It is easy to check that the set of induced edge labels is {1, 2, 3, ..., 20, 21, 23}. Hence $m(C_{22}) = 21$ and $grac(C_{22}) = 23$.

Figure 5.6 gives the graph representation of m-graceful labeling of C_{22}.

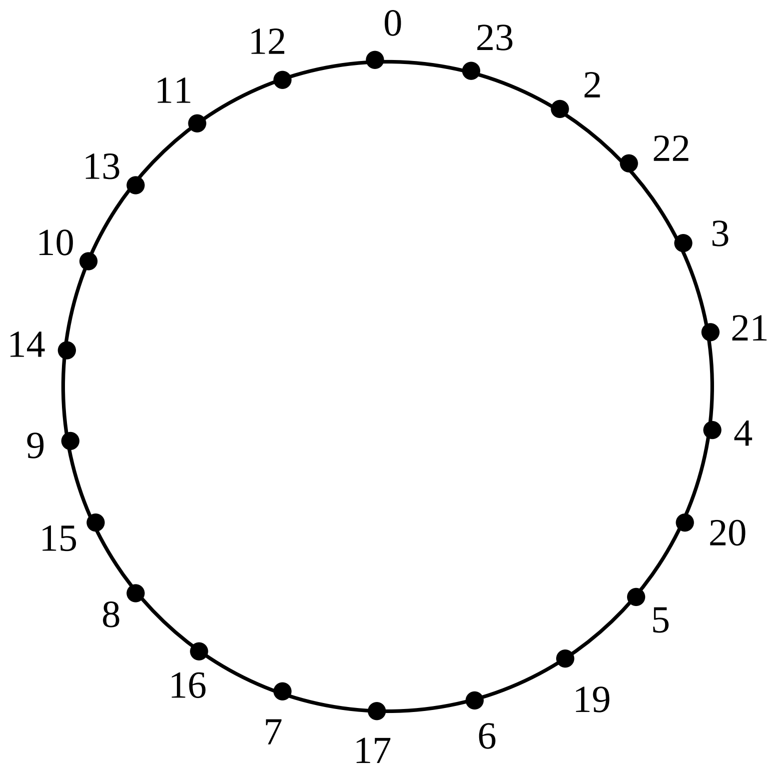


Figure 5.6: m-graceful labeling of C_{22}

Let F_k denote the friendship graph consisting of k triangles (a, u_i, v_i, a), $1 \leq i \leq k$. By Theorem 1.3.6, F_k is nongraceful if $k \equiv 2$ or $3 \pmod 4$ and in the following theorem we determine $m(F_k)$ for this k.

Theorem 5.2.8. *For the friendship graph F_k, $m(F_k) = 3k-1$ where $k \equiv 2$ or $3 \pmod 4$.*

Proof. Let $f(a) = 0$ where a is the central vertex of F_k. We have the following two cases:

Case 1: $k \equiv 2 \pmod 4$.

For F_2, F_6, F_{10} and F_{14} the labeling is given in the following tables.

u_i	1	2
v_i	4	7

Table 5.12: m-graceful labeling of F_2

u_i	1	2	3	4	5	6
v_i	15	19	11	13	12	16

Table 5.13: m-graceful labeling of F_6

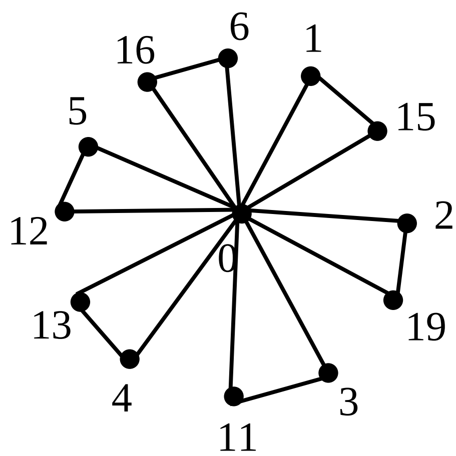


Figure 5.7: m-graceful labeling of F_6

u_i	1	2	3	4	5	6	7	8	9	10
v_i	13	31	14	25	22	26	23	27	24	28

Table 5.14: m-graceful labeling of F_{10}

u_i	1	2	3	4	5	6	7	8	9	10	11	12	13	14
v_i	19	43	35	26	36	23	37	24	38	25	39	33	40	34

Table 5.15: m-graceful labeling of F_{14}

For $k \geq 18$, we define f as follows:

$$f(u_i) = i, \quad \text{for} \quad i = 1, 2, \ldots, k;$$
$$f(v_1) = \frac{5k+6}{4};$$
$$f(v_2) = 3k+1;$$
$$f(v_k) = \frac{5k-2}{2};$$

$$f(v_{k-2}) = \frac{5k-4}{2};$$

$$f(v_{k-1-2i}) = 3k-2-i, \quad \text{for} \quad i = 0, 1, \ldots, \frac{k-4}{2};$$

$$f(v_{2i}) = \begin{cases} \frac{3k}{2}+i, & \text{for} \quad i = 2, 3, \ldots, \frac{k-10}{4}; \\ 2k-2, & \text{for} \quad i = \frac{k-6}{4}; \\ \frac{3k-2}{2}+i, & \text{for} \quad i = \frac{k-2}{4}, \frac{k+2}{4}, \ldots, \frac{k}{2}-2. \end{cases}$$

It can be easily verified that f is injective, the induced edge labeling g_f is also injective, the highest vertex label used is $3k+1$ and $m(F_k) = 3k-1$.

Case 2: $k \equiv 3 (\text{mod } 4)$.

For F_3, F_7 and F_{11}, the labeling is given in the following tables.

u_i	1	2	3
v_i	6	10	7

Table 5.16: m-graceful labeling of F_3

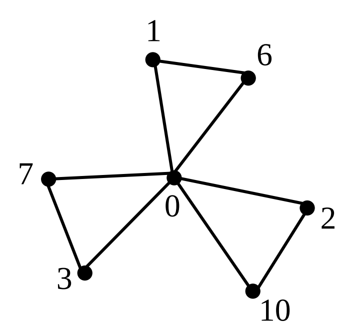


Figure 5.8: m-graceful labeling of F_3

u_i	1	2	3	4	5	6	7
v_i	9	22	17	15	18	16	19

Table 5.17: m-graceful labeling of F_7

u_i	1	2	3	4	5	6	7	8	9	10	11
v_i	14	34	27	19	28	18	29	25	30	26	31

Table 5.18: m-graceful labeling of F_{11}

For $k \geq 15$, we label the vertices as follows:

$$\begin{aligned}
f(u_i) &= i, \text{ for } i = 1, 2, \ldots, k; \\
f(v_1) &= \frac{5k+1}{4}; \\
f(v_2) &= 3k+1; \\
f(v_{k-1}) &= \frac{5k-3}{2}; \\
f(v_{k-3}) &= \frac{5k-5}{2}; \\
f(v_{k-2i}) &= 3k-2-i, \text{ for } i = 0, 1, \ldots, \frac{k-3}{2}; \\
f(v_{2i}) &= \begin{cases} \frac{3k-1}{2} + i, & \text{for } i = 2, 3, \ldots, \frac{k-7}{4}; \\ 2k-3, & \text{for } i = \frac{k-3}{4}; \\ \frac{3k-3}{2} + i, & \text{for } i = \frac{k+1}{4}, \frac{k+5}{4}, \ldots, \frac{k-5}{2}. \end{cases}
\end{aligned}$$

It can be easily verified that f is injective, the induced edge function g_f is also injective, the highest vertex label used is $3k+1$ and $m(F_k) = 3k-1$. □

Corollary 5.2.9. *$grac(F_k) = 3k+1 = q+1$ for $k \equiv 2$ or $3 (\bmod\ 4)$.*

5.3 m-GRACEFULNESS OF COMPLETE GRAPH K_p

In this section, we focus our study on the complete graph K_p. By Theorem 1.3.4, K_p for $p > 5$ is a nongraceful graph. Notice that for p even, K_p is noneulerian and as p increases, the task of finding the m-gracefulness of K_p is a difficult problem. We now proceed to investigate complete graphs K_p for which $m(K_p) = q - 1$.

Lemma 5.3.1. *If $m(K_p) = q-1$ under a labeling f with $M_G(g_f) = q+k,\ k \geq 1$, then none of the vertices of K_p can be assigned a label t, where $0 < t < k+1$ or $q-1 < t < q+k$.*

Proof. Since $m(K_p) = q - 1$, the set of induced edge labels is given by

$$g_f(E) = \{1, 2, \ldots, q-1, q+k\}. \tag{5.1}$$

Let u and v be vertices of K_p for which $f(u) = 0$ and $f(v) = q + k$. If there exists a vertex w with $f(w) = t$, where $0 < t < k+1$ or $q-1 < t < q+k$, then either $q-1 < g_f(vw) < q+k$ or $q-1 < g_f(uw) < q+k$, a contradiction to the set of induced edge labels given in (5.1). □

Observation 5.3.2. *Let f be a m-graceful labeling of K_p. If 0 and $2t$ are vertex labels, then t and $4t$ cannot be vertex labels, since otherwise the edge label t or $2t$ is repeated. Hence it follows that if $m(K_p) = q-1$ under a labeling f, then $M_G(f) \neq 2(q-1)$.*

Lemma 5.3.3. *If $m(K_p) = q-1$ under a labeling f with $M_G(g_f) = q+k,\ k \geq 1$, then no two vertices of K_p can be labeled $k+t$ and $q-t$, where $1 \leq t \leq \left\lfloor \frac{q-k-1}{2} \right\rfloor$.*

Proof. Since $M_G(g_f) = q + k$, there exist two vertices u and $v \in V(K_p)$ with $f(u) = 0$ and $f(v) = q + k$. If there exist x and $y \in V(K_p)$ such that $f(x) = k+t$ and $f(y) = q-t$ for $1 \leq t \leq \left\lfloor \frac{q-k-1}{2} \right\rfloor$, then $g_f(uy) = g_f(vx) = q - t$, which is a contradiction. $\square$

The following theorem gives an upper bound for the highest vertex label $M_G(f)$ that can be used for the vertices of K_p if $m(K_p) = q - 1$.

Theorem 5.3.4. *If $m(K_p) = q-1$ under a labeling f, then $M_G(f) \leq 2(q-p) + 3$.*

Proof. Let $m(K_p) = q - 1$ with $M_G(f) = q + k,\ k \geq 1$. By Lemma 5.3.1, $f(V) \subseteq A = \{0, k+1, k+2, \ldots, q-2, q-1, q+k\}$ and by Lemma 5.3.3, the set

$$B = \begin{cases} A - \{0, q+k\} & \text{if } q+k \text{ is odd} \\ A - \{0, \frac{q+k}{2}, q+k\} & \text{if } q+k \text{ is even} \end{cases}$$

can be partitioned into $\left\lfloor \frac{q-k-1}{2} \right\rfloor$ disjoint pairs of labels $\{k+t, q-t\}$, $1 \leq t \leq \left\lfloor \frac{q-k-1}{2} \right\rfloor$ such that only one of the labels from each pair can be used for the remaining $(p-2)$ vertices of K_p. Therefore $\left\lfloor \frac{q-k-1}{2} \right\rfloor \geq p-2$. It follows that $k \leq q - 2p + 3$ and hence $M_G(f) = q + k \leq 2(q-p) + 3$. $\square$

Observation 5.3.5. *It follows from the above theorem that if $m(K_p) = q-1$ under the labeling f, then $q+1 \leq grac(K_p) \leq M_G(f) \leq 2(q-p)+3$.*

Theorem 5.3.6. *The m-gracefulness of the complete graph K_p is $q-1$ if and only if $p = 5$.*

Proof. Let $p = 5$, if we label the vertices of K_5 from the set $\{0, 3, 4, 9, 11\}$, then the set of induced edge labels obtained is $\{1, 2, \ldots, 8, 9, 11\}$. Hence $m(K_5) = 9 = q - 1$.

Conversely, let $m(K_p) = q - 1$ under the labeling f and let $M_G(f) = q + k$. By Theorem 5.3.4, $1 \leq k \leq q - 2p + 3$. Suppose $p \neq 5$. Let $\{v_1, v_2, \ldots, v_p\} = V(K_p)$, with $f(v_1) = 0$ and $f(v_2) = q + k$. Since $m(K_p) = q - 1$, there exists a vertex say, $v_3 \in V(K_p)$ such that, either $f(v_3) = k+1$ or $f(v_3) = q-1$. Without loss of generality, let $f(v_3) = q - 1$. Hence $\{0, q+k, q-1\} \subset f(V)$. Consider Fig. 5.9 for the graphical representation of all the possible vertex labelings of K_p.

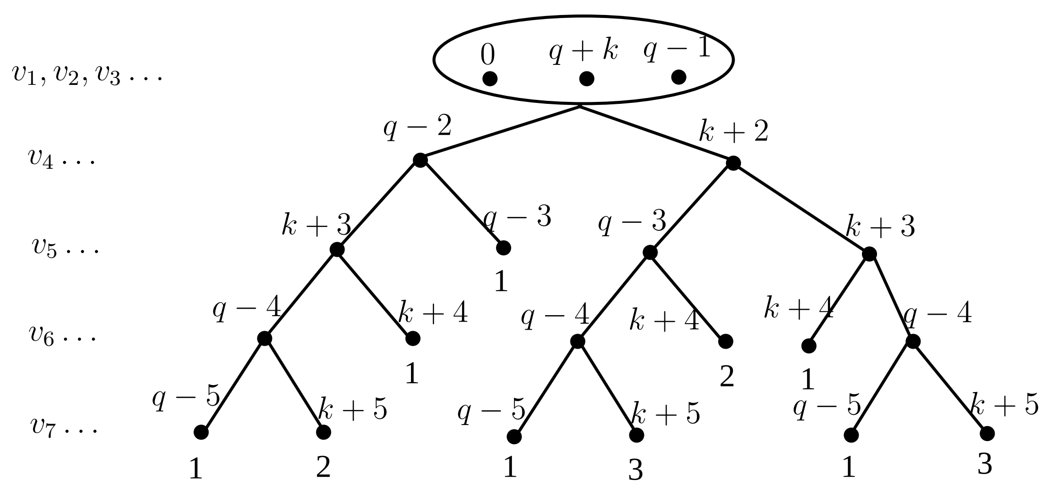


Figure 5.9: Graphical representation of possible vertex labels of K_p if $m(K_p) = q - 1$

In the figure, the number above the vertex v_i, $1 \leq i \leq 7$ is its label under f. If by assignment of this label to v_i, any edge label is repeated, then that edge label is indicated under v_i. At each level, having assigned a label to the vertex v_i, $3 \leq i \leq 6$, note that $q-(i-1)$ is not an induced edge label. As a consequence, by Lemma 5.3.3, either $f(v_{i+1}) = q-(i-1)$ or $f(v_{i+1}) = k+(i-1)$ for $3 \leq i \leq 6$. Also note that, the vertex v_7 cannot be assigned any label without resulting in repetition of edge labels. Hence $p \leq 6$. By our assumption, $p \neq 5$ and since K_p is graceful if and only if $p \leq 4$, p must be 6. Hence $q = 15$ and since $grac(K_6) = 17$, by Theorem 5.3.4, $2 \leq k \leq 6$.

Figure 5.9 gives $f(v_6) = q-4$. Therefore the set of possible vertex labels of K_6 are as follows:

$$f(V) = \{0, q+k, q-1, q-2, k+3, q-4\} \tag{5.2}$$

$$f(V) = \{0, q+k, q-1, k+2, q-3, q-4\} \tag{5.3}$$

and

$$f(V) = \{0, q+k, q-1, k+2, k+3, q-4, \} \tag{5.4}$$

Tables 5.19, 5.20 and 5.21 give the vertex labelings of K_6 for $2 \leq k \leq 6$ corresponding to (5.2), (5.3) and (5.4) respectively.

k	Vertex labels	Edge labels	No. of Repetitions
2	{0, 5, 11, 13, 14, 17}	3,6	2
3	{0, 6, 11, 13, 14, 18}	5,7	2
4	{0, 7, 11, 13, 14, 19}	6,7	2
5	{0, 8, 11, 13, 14, 20}	3,6	2
6	{0, 9, 11, 13, 14, 21}	2	2

Table 5.19: Vertex labeling of K_6 with $f(v_4) = q-2$ and $f(v_5) = k+3$

k	Vertex labels	Edge labels	No. of Repetitions
2	{0, 4, 11, 12, 14, 17}	3	2
3	{0, 5, 11, 12, 14, 18}	6, 7	2
4	{0, 6, 11, 12, 14, 19}	5, 6, 8	2
5	{0, 7, 11, 12, 14, 20}	7	2
6	{0, 8, 11, 12, 14, 21}	3, 4	2

Table 5.20: Vertex labeling of K_6 with $f(v_4) = k + 2$ and $f(v_5) = q - 3$

k	Vertex labels	Edge labels	No. of Repetitions
2	{0, 4, 5, 11, 14, 17}	3, 6	2
3	{0, 5, 6, 11, 14, 18}	5, 6	2
4	{0, 6, 7, 11, 14, 19}	5, 7, 8	2
5	{0, 7, 8, 11, 14, 20}	3, 6, 7	2
6	{0, 8, 9, 11, 14, 21}	3	2

Table 5.21: Vertex labeling of K_6 with $f(v_4) = k + 2$ and $f(v_5) = k + 3$

The last column of each of the tables, gives a contradiction to the fact that $m(K_p) = q - 1$. Hence $p \neq 6$, so that $p = 5$. From Fig. 5.9, $f(V) = \{0, q+k, q-1, k+2, k+3\}$ for $k = 1$ is a m-graceful labeling of K_5. □

We now give some necessary conditions for the m-gracefulness of K_p to be $q - 2$, using which we find an upper bound for the highest vertex label of K_p.

Lemma 5.3.7. *If $m(K_p) = q - 2$ under a labeling f, $M_G(g_f) = q + k$, $k \geq 1$ and $M'_G(g_f) = q + k - s$, $1 \leq s \leq k$, then none of the vertices of K_p can be assigned a label t where $0 < t < s$, $s < t < k + 2$, $q - 2 < t < q + k - s$ or $q + k - s < t < q + k$.*

Proof. Since f is a m-graceful labeling of K_p, the set of induced edge labels is given by,

$$g_f(E) = \{1, 2, 3, \ldots, q-2, q+k-s, q+k\}. \tag{5.5}$$

Therefore, there exist vertices u and v of K_p for which $f(u) = 0$ and $f(v) = q+k$. Suppose there exists $x \in V(K_p)$ with $f(x) = t$, where $0 < t < s$, $s < t < k+2$, $q-2 < t < q+k-s$ or $q+k-s < t < q+k$. If $0 < t < s$ or $s < t < k+2$, then $q+k-s < g_f(vx) < q+k$ or $q-2 < g_f(vx) < q+k-s$ respectively, if $q-2 < t < q+k-s$ or $q+k-s < t < q+k$, then $q-2 < g_f(ux) < q+k-s$ or $q+k-s < g_f(ux) < q+k$ respectively. Either of the cases give a contradiction to the set of induced edge labels given in (5.5). $\square$

Lemma 5.3.8. *If $m(K_p) = q-2$ under a labeling f with $M_G(g_f) = q+k$, $k \geq 1$ and $M'_G(g_f) = q+k-s$, $1 \leq s \leq k$, then no two vertices of K_p can be labeled $k+t$ and $q-t$, where $2 \leq t \leq \left\lfloor \frac{q-k-3}{2} \right\rfloor$.*

Proof. Since $m(K_p) = q-2$, the set of induced edge labels is $g_f(E) = \{1, 2, 3, \ldots, q-2, q+k-s, q+k\}$ and by Lemma 5.3.7, $f(V) \subseteq \{0, s, k+2, k+3, \ldots, q-2, q+k-s, q+k\}$. Since $q+k \in g_f(E)$, there exists two vertices u and v of K_p with $f(u) = 0$ and $f(v) = q+k$. Now, if there exist two vertices, w and x with $f(w) = q-t$ and $f(x) = k+t$ for $2 \leq t \leq \left\lfloor \frac{q-k-3}{2} \right\rfloor$, then $g_f(uw) = q-t$ and $g_f(vx) = q-t$, which is a contradiction to the fact that f is an m-graceful labeling. Therefore only one of the vertex labels from each pair $\{k+t, q-t\}$ for $2 \leq t \leq \left\lfloor \frac{q-k-3}{2} \right\rfloor$ can be assigned to the vertices of K_p. $\square$

Theorem 5.3.9. *If $m(K_p) = q-2$ under a labeling f, then $M_G(f) \leq 2(q-p)+1$.*

Proof. Let $M_G(g_f) = q+k,\ k \geq 1$ and $M'_G(g_f) = q+k-s$, $1 \leq s \leq k$. Since f is a m-graceful labeling of K_p, the set of induced edge labels is $g_f(E) = \{1, 2, \ldots, q-2, q+k-s, q+k\}$. Let u and $v \in V(K_p)$ such that $f(u) = 0$ and $f(v) = q+k = M_G(f)$. Let $w \in V(K_p)$ with $f(w) = q+k-s$. By Lemma 5.3.7, $f(V) \subseteq A = \{0, s, k+2, k+3, \ldots, q-3, q-2, q+k-s, q+k\}$ and by Lemma 5.3.8, the set

$$B = \begin{cases} A - \{0, s, q+k-s, q+k\} & \text{if } q+k \text{ is odd} \\ A - \{0, s, \frac{q+k}{2}, q+k-s, q+k\} & \text{if } q+k \text{ is even} \end{cases}$$

can be partitioned into $\left\lfloor \frac{q-k-3}{2} \right\rfloor$ disjoint pairs of labels $\{k+t, q-t\}$ for $2 \leq t \leq \left\lfloor \frac{q-k-3}{2} \right\rfloor$ such that only one of the labels from each of these pairs can be used for the remaining $(p-3)$ vertices of K_p. Therefore $\left\lfloor \frac{q-k-3}{2} \right\rfloor - 1 \geq p-3$. It follows that $k \leq q-2p+1$ and hence $M_G(f) = q+k \leq 2(q-p)+1$. $\square$

Observation 5.3.10. *If $m(K_p) = q-2$ under a labeling f, then $q+1 \leq grac(G) \leq M_G(f) \leq 2(q-p)+1$.*

Theorem 5.3.11. *For the complete graph K_6 we have $m(K_6) = 13 = q-2$.*

Proof. It is known that K_6 is nongraceful and by Theorem 5.3.6, $m(K_6) \neq q-1$. Hence $m(K_6) \neq 15$ or 14. If we label the vertices of

K_6 either from the set $\{0, 1, 4, 10, 12, 17\}$ or $\{0, 4, 6, 9, 16, 17\}$, then the set of induced edge labels is $\{1, 2, 3, \ldots, 12, 13, 16, 17\}$. Hence $m(K_6) = 13 = q - 2$ and the highest vertex label used is 17. $\square$

Corollary 5.3.12. $grac(K_6) = 17 = q + 2$.

Problem 5.3.13. Is K_6 the only complete graph with m-gracefulness $q - 2$?

Problem 5.3.14. Determine the exact value of $m(K_p)$ for $p \geq 7$.

Observation 5.3.15. *1. From Theorem 5.2.1, we observe that, there are infinitely many graphs with the property that* $grac(G) - q = q - m(G)$.

2. From Theorem 5.2.4 we observe that, for $n \equiv 1$ *or* $2 (\mathrm{mod}\ 4)$, $grac(C_n) - q = q - m(C_n)$.

3. From Theorem 5.2.8 we observe that, for $k \equiv 2$ *or* $3 (\mathrm{mod}\ 4)$, $grac(F_k) - q = q - m(F_k)$.

4. Also from Theorem 5.3.6 and Theorem 5.3.11, we observe that, $grac(K_p) - q = q - m(K_p)$ *for* $p = 5, 6$.

Therefore the following problem naturally arises.

Problem 5.3.16. Is it true that $grac(G) - q = q - m(G)$?

CHAPTER 6

ADDITIVELY GRACEFUL SIGNED GRAPHS

6.1 INTRODUCTION

Hegde [34] introduced the notion of additively graceful graphs. He characterized some additively graceful graphs, gave a lower bound on number of edges of an additively graceful graph and some necessary or sufficient conditions for a graph to be additively graceful. In this chapter, we extend this notion to the realm of sigraphs as follows:

Definition 6.1.1. Let $S = (V, E)$ be a (p, m, n)-sigraph with $E = E^+ \cup E^-$, Assume $|E^+| = m$ and $|E^-| = n$ where $m + n = q$. Let $f : V \to \{0, 1, ..., m + \lceil\frac{(n+1)}{2}\rceil\}$ be an injective mapping and let the induced edge function be defined as $g_{f^-}(uv) = f(u) + f(v)\ \forall\ uv \in E^-$ and $g_{f^+}(uv) = |f(u) - f(v)|\ \forall\ uv \in E^+$. If $g_{f^-}(uv) = \{1, 2, ..., n\}$ and $g_{f^+}(uv) = \{1, 2, ..., m\}$, then f is called an *additively graceful labeling* of S. The sigraph which admits such a labeling is called an *additively graceful sigraph*.

Example 6.1.2. Figure 6.1 gives examples of additively graceful sigraphs.

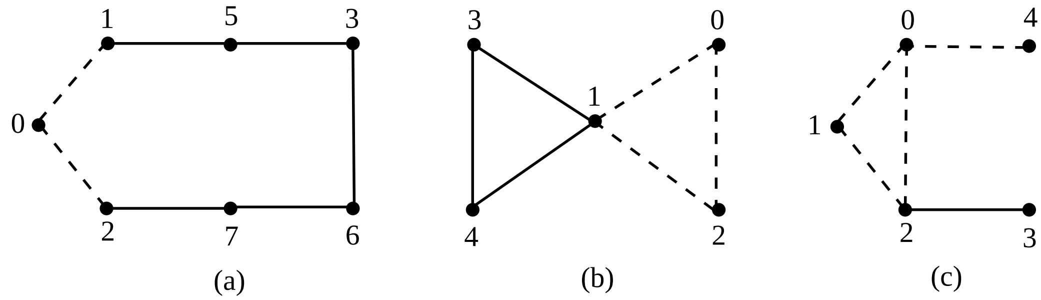


Figure 6.1: Additively graceful sigraphs

One can easily see that when $n = 0$, f is a graceful labeling of S, and when $m = 0$, f is an additively graceful labeling of S. In this chapter, we give some necessary or sufficient conditions for a sigraph to be additively graceful. We also obtain some necessary conditions for eulerian sigraphs, complete bipartite sigraphs and complete sigraphs to be additively graceful.

In a sigraph S, any maximal subgraph C in which all edges are positive is called a positive section of C. Similarly we define a negative section.

6.2 BASIC RESULTS

In this section, we present some basic results on additively graceful sigraphs.

Theorem 6.2.1. *If S is an additively graceful sigraph then there exists a partition of $V(G)$ into V_o and V_e such that $m^+(V_o, V_e) =$*

$\lfloor \frac{m+1}{2} \rfloor$ *and* $m^-(V_o, V_e) = \lfloor \frac{n+1}{2} \rfloor$ *where* $m^+(V_o, V_e)$ *and* $m^-(V_o, V_e)$ *are the number of positive and negative edges of* S *respectively each of which joins a vertex of* V_o *with one of* V_e*.*

Proof. Let S be an additively graceful sigraph, with an additively graceful labeling f. Let $V_o = \{u \in V(S) : f(u) \text{ is odd}\}$ and $V_e = V(S) - V_o$. Therefore every edge receiving an odd label must join a vertex of V_o with one of V_e. Since the number of edges of S with odd positive labels is $\lfloor \frac{m+1}{2} \rfloor$ and the number of edges of S with odd negative labels is $\lfloor \frac{n+1}{2} \rfloor$, it follows that $m^+(V_o, V_e) = \lfloor \frac{m+1}{2} \rfloor$ and $m^-(V_o, V_e) = \lfloor \frac{n+1}{2} \rfloor$. □

Theorem 6.2.2. *If a* (p, m, n)*-sigraph* S *is additively graceful then* $2m + n \geq 2p - 4$ *and this bound is the best possible.*

Proof. Let f be an additively graceful labeling of S. Since the highest vertex label is $m + \lceil \frac{n+1}{2} \rceil$, we have

$$p - 1 \leq m + \left\lceil \frac{n+1}{2} \right\rceil.$$

Now we have the following two cases:

Case 1: n is odd.

Then

$$p - 1 \leq m + \frac{n+1}{2}.$$

Hence

$$2p - 2 \leq 2m + n + 1.$$

Therefore

$$2m + n \geq 2p - 3.$$

Case 2: n is even.

Then

$$p - 1 \leq m + \frac{n}{2} + 1.$$

Hence

$$p - 2 \leq \frac{2m + n}{2}.$$

Therefore

$$2m + n \geq 2p - 4.$$

The result follows from Case 1 and Case 2 and equality holds for the sigraph in Figure 6.1(c) . □

We have the following result from [34] as a corollary.

Corollary 6.2.3. *If S is an all negative sigraph which admits an additively graceful labeling then $n \geq 2p - 4$.*

Lemma 6.2.4. *If a sigraph S is additively graceful, then the sum of all edge labels of any circuit C in S is even.*

Proof. Let S be an additively graceful sigraph with an additively graceful labeling f. Let $P_1, P_2, \ldots, P_k$ and $Q_1, Q_2, \ldots, Q_k$ denote the positive and the negative sections of C respectively. For $1 \leq i \leq k$, let $P_i = (a_{i,1}, a_{i,2}, \ldots, a_{i,r_i})$, $Q_i = (b_{i,1}, b_{i,2}, \ldots, b_{i,s_i})$. Where $a_{i,j}$, $j = 1, 2, \ldots, r_i$ denote the vertices of the positive section P_i; $b_{i,t}$, $t = 1, 2, \ldots, s_i$ denote the vertices of the negative section Q_i; $a_{i,r_i} = b_{i,1}$; $b_{i,s_i} = a_{(i+1),1}$ and $b_{k,s_k} = a_{1,1}$. Now, for $1 \leq i \leq k$, $\sum\limits_{j=1}^{r_i-1} |f(a_{i,j}) - f(a_{i,(j+1)})| \equiv [f(a_{i,1}) - f(a_{i,r_i})](\text{ mod } 2)$ and $\sum\limits_{j=1}^{s_i-1} [f(b_{i,j}) + f(b_{i,(j+1)})] \equiv [f(b_{i,1}) + f(b_{i,s_i})](\text{mod } 2)$. Hence if T denotes the sum of the edge labels of the edges of C, then $T \equiv \{\sum\limits_{i=1}^{k} [f(a_{i,1}) - f(a_{i,r_i})] + \sum\limits_{i=1}^{k} [f(b_{i,1}) + f(b_{i,s_i})]\}(\text{ mod } 2) = 2(f(a_{1,1}) + f(a_{2,1}) + \cdots + f(a_{k-1,1}) + f(a_{k,1}))(\text{ mod } 2) \equiv 0(\text{mod } 2)$. Therefore T is even. $\square$

We have the following result from [31] as a corollary.

Corollary 6.2.5. *Let $G = (V, E)$ be a graph and $f : V \to \mathbb{N}$ be any function. Let $g_f(uv) = |f(u) - f(v)|$ for any edge uv of G. Then the sum of the edge labels of all the edges on any circuit of G is even.*

Theorem 6.2.6. *The sigraph S obtained from the all-negative cycle $C_3 = (v_1, v_2, v_3, v_1)$ by adding k positive pendent edges $v_3w_1, v_3w_2, \ldots, v_3w_k$ is additively graceful.*

Proof. Define $f : V \to \{0, 1, \ldots, k+2\}$ by $f(v_i) = i - 1$ for $i = 1, 2, 3$ and $f(w_i) = i + 2$ for $1 \leq i \leq k$. This gives an additively graceful labeling of S. $\square$

Theorem 6.2.7. *Let S be an additively graceful sigraph and let H be the subgraph induced by the set of all negative edges of S. If H is connected then either H is a star or contains a triangle.*

Proof. Let f be an additively graceful labeling of S. Let uv_1 be a negative edge of S with label 1, so that $f(u) = 0$ and $f(v_1) = 1$. If all the negative edges of S are incident with u, then H is a star. If not, let t be the least positive integer such that the negative edge xy with label t is not incident with u. Hence all negative edges e_i with labels i, $1 \leq i \leq t-1$ are incident with u. Let $e_i = uv_i$ and $f(v_i) = i$, $1 \leq i \leq t-1$. Since xy is a negative edge with label t, $1 \leq f(x),\ f(y) \leq t-1$. Hence $x = v_i$ and $y = v_j$ for $1 \leq i, j \leq t-1$ and $\{u, x, y\}$ is a triangle in H. □

6.3 ADDITIVELY GRACEFUL LABELINGS OF COMPLETE SIGRAPHS, COMPLETE BIPARTITE SIGRAPHS AND EULERIAN SIGRAPHS

In this section, we present some results on additively graceful labeling of sigraphs on complete graphs, complete bipartite graphs and eulerian graphs.

Theorem 6.3.1. *Let S be a (p, m, n)-eulerian sigraph. A necessary condition for S to be additively graceful is that $m^2 + n^2 + m + n \equiv 0 (\text{mod } 4)$.*

Proof. Let f be an additively graceful labeling of a (p, m, n)-eulerian sigraph. Then the m positive edges are labeled as $1, 2, \ldots, m$ and n negative edges are labeled as $1, 2, \ldots, n$. Since sum of the edge labels along the eulerian circuit is even, by Lemma 6.2.4, we have

$\frac{m(m+1)}{2} + \frac{n(n+1)}{2} \equiv 0(\text{mod } 2)$. Hence $m^2 + n^2 + m + n \equiv 0(\text{mod } 4)$. □

Corollary 6.3.2. *If a (k, m, n)-signed cycle Z_k, $k = m + n \geq 3$ is additively graceful, then $m^2 + n^2 + m + n \equiv 0(\text{mod } 4)$.*

Corollary 6.3.3. *If the signed cycle $Z_k, k = m+n \geq 3$ is such that $k \equiv 1(\text{mod } 4)$, then Z_k is not an additively graceful sigraph.*

Corollary 6.3.4. *If S is a graceful eulerian all-positive (p, m)-sigraph, then $m \equiv 0$ or $3(\text{mod } 4)$.*

Corollary 6.3.5. *If S is an additively graceful eulerian all-negative (p, n)-sigraph, then $n \equiv 0$ or $3(\text{mod } 4)$.*

Theorem 6.3.6. *Let $p \geq 5$ be a positive integer such that none of p, $p-2$, $p-4$ is a perfect square. Then no sigraph on K_p is additively graceful.*

Proof. Suppose there exists a sigraph S on K_p which is additively graceful. By Theorem 6.2.1, there exists a partition of the vertex set $V(S)$ into two subsets V_o and V_e such that $|V_o| = a$, $|V_e| = b$, $a + b = p$ and

$$ab = \left\lfloor \frac{m+1}{2} \right\rfloor + \left\lfloor \frac{n+1}{2} \right\rfloor.$$

For different parities of m and n we have the following three cases:

Case 1: m and n are odd.

In this case, we have

$$ab = \frac{m+1}{2} + \frac{n+1}{2} = \frac{m+n+2}{2}.$$

Hence

$$2ab = m + n + 2 = \frac{p(p-1)}{2} + 2.$$

Now

$$a - b = \sqrt{(a+b)^2 - 4ab} = \sqrt{p-4}.$$

Since $a - b$ is an integer it follows that $p - 4$ is a perfect square, which is a contradiction.

Case 2: One of m and n is odd and the other is even.

Without loss of generality, we assume that m is even and n is odd. In this case, we have

$$ab = \frac{m}{2} + \frac{n+1}{2} = \frac{m+n+1}{2}.$$

Hence

$$2ab = m + n + 1 = \frac{p(p-1)}{2} + 1.$$

Now

$$a - b = \sqrt{(a+b)^2 - 4ab} = \sqrt{p-2}.$$

Since $a - b$ is an integer it follows that $p - 2$ is a perfect square, which is a contradiction.

Case 3: m and n are even.

In this case, we have

$$ab = \frac{m}{2} + \frac{n}{2} = \frac{m+n}{2}.$$

Hence

$$2ab = m + n = \frac{p(p-1)}{2}.$$

Now

$$a - b = \sqrt{(a+b)^2 - 4ab} = \sqrt{p}$$

Since $a - b$ is an integer it follows that p is a perfect square, which is a contradiction. □

Lemma 6.3.7. *All sigraphs on K_p, $p \leq 3$ are additively graceful.*

Proof. The additively graceful labeling of all sigraphs on K_p, $p \leq 3$ are shown in Figure 6.2

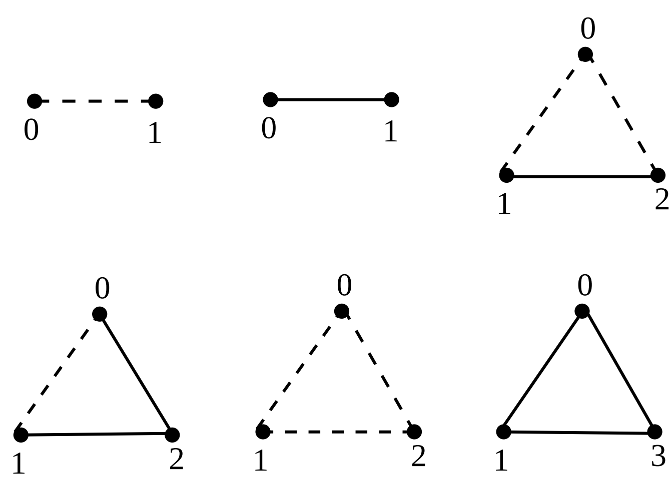


Figure 6.2: Additively graceful sigraphs on K_2 and K_3

□

Lemma 6.3.8. *A (p, m, n)-sigraph S on the complete graph K_4 is additively graceful if and only if either $n = 0$ or the subgraph H induced by the set of all negative edges is isomorphic to K_4, $K_4 - e$, K_3, P_3 or P_2.*

Proof. If H is isomorphic to K_4, $K_4 - e$, K_3, P_3 or P_2, then the labeling of S is given in Figure 6.3.

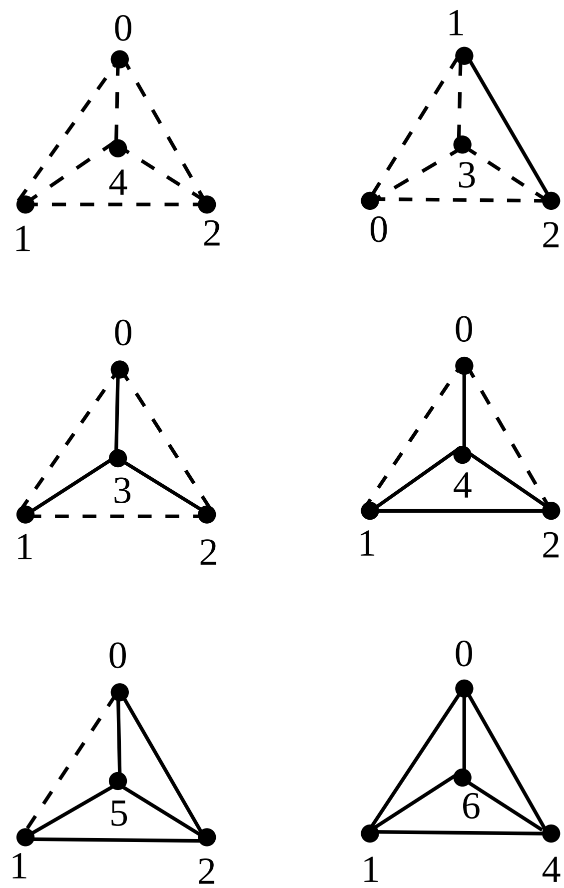


Figure 6.3: Additively graceful sigraphs on K_4

Conversely, suppose S is additively graceful. If $n = 0$, then there is nothing to prove. Hence we assume that $n > 0$. We claim that $n \neq 4$. Suppose $n = 4$. It follows from Theorem 6.2.7 that H is isomorphic to the graph given in Figure 6.4(a). There are exactly

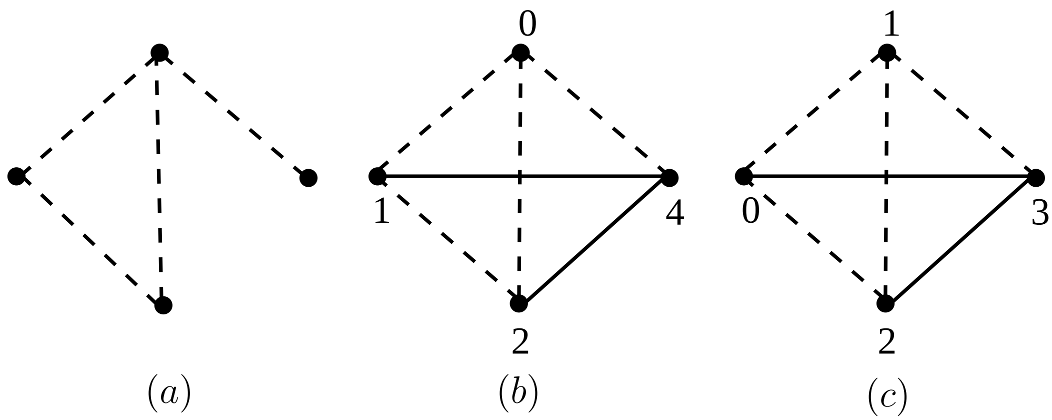


Figure 6.4: The subgraph H of K_4 induced by 4 negative edges and the corresponding labeling

two possible labelings which give the set of induced edge labels $\{1, 2, 3, 4\}$ for the negative edges in H and this labeling is given in Figure 6.4(b) and (c) . The set of induced positive edge labels is not equal to $\{1, 2\}$. Thus S is not additively graceful. Hence $n \neq 4$. If $n = 6$, then $H = K_4$ and if $n = 5$, then $H = K_4 - e$. If $n = 3$, then it follows from Theorem 6.2.7 that $H = K_{1,3}$ or K_3. If $H = K_{1,3}$, then the centre of the star gets the label 0 and the 3 pendent vertices receive the labels 1, 2, 3. But in this case the set of labels of the positive edges is not equal to $\{1, 2, 3\}$. Hence, if $n = 3$, then $H = K_3$. If $n = 2$, then it follows from Theorem 6.2.7 that $H = P_3$. If $n = 1$, then $H = P_2$ □

Lemma 6.3.9. *If a sigraph on K_5 is additively graceful then the number of its negative edges is odd.*

Proof. Let S be an additively graceful sigraph on K_5. Then by Theorem 6.3.1, we have $m^2 + n^2 + m + n \equiv 0 (\text{mod } 4)$. Since $q = m + n = 10$ for K_5, substituting $m = (10 - n)$ we get $2(n^2 - 10n +$

$55) \equiv 0 (\text{mod } 4)$ which implies that $n^2 - 10n + 55$ is even. Hence n is odd. $\square$

Example 6.3.10. Figure 6.5 gives the additively graceful labelings for the sigraphs on K_5 for n =1, 3, 5, 7, 9.

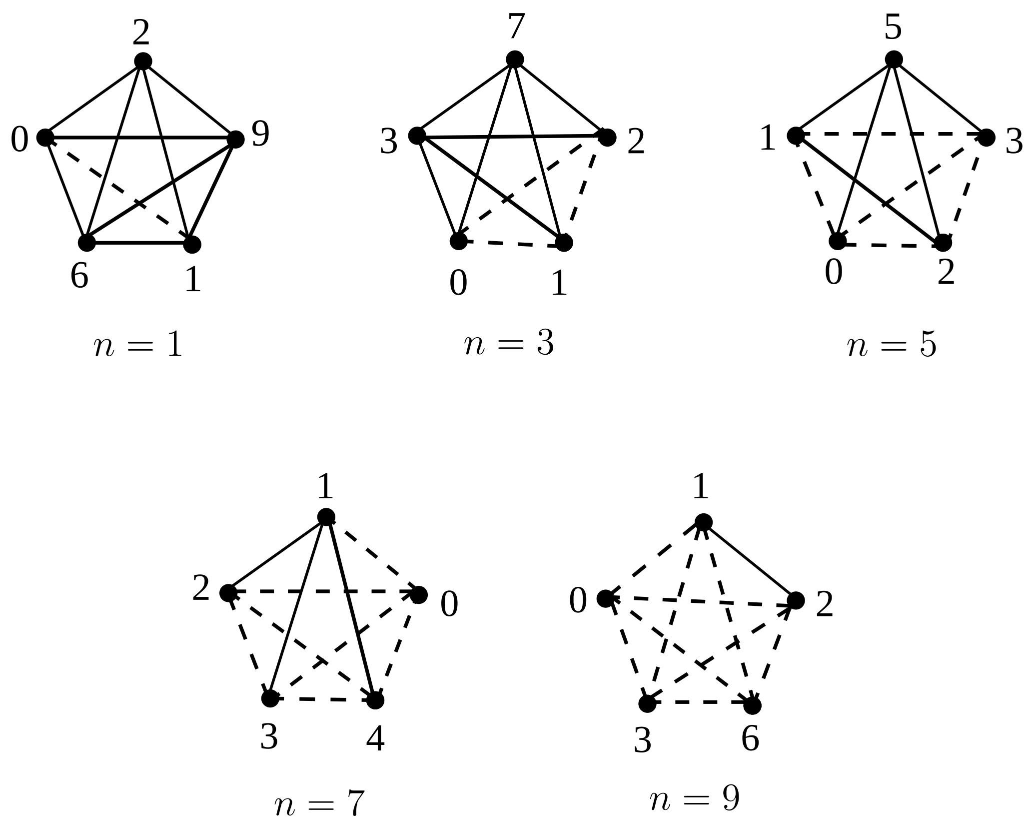


Figure 6.5: Additively graceful sigraphs on K_5 for $n = 1, 3, 5, 7, 9$

Theorem 6.3.11. *Let G be the complete bipartite graph $K_{2,t}$ with bipartition $V_1 = \{v_1, v_2\}$ and $V_2 = \{w_1, w_2, \ldots, w_t\}$. Let S be the sigraph obtained from G by assigning positive sign to all edges incident with v_1 and negative sign to all edges incident with v_2. Then S is additively graceful.*

Proof. Define $f : V \to \{0, 1, \ldots, t + \lceil \frac{(t+1)}{2} \rceil\}$ by $f(v_1) = t + 1$, $f(w_i) = i$ for $i = 1, 2, \ldots, t$ and $f(v_2) = 0$. This gives an additively graceful labeling of S. $\square$

Example 6.3.12. Consider the graph $K_{2,6}$ with bipartition $V_1 = \{v_1, v_2\}$ and $V_2 = \{w_1, w_2, \ldots, w_6\}$. Let S be the sigraph obtained from $K_{2,6}$ by assigning positive sign to all edges incident with v_1 and negative sign to all edges incident with v_2. Figure 6.6 gives the additively graceful labeling of S.

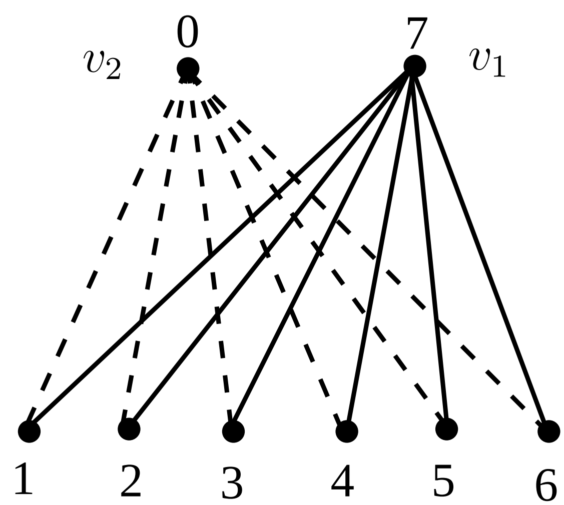


Figure 6.6: Additively graceful labeling of the sigraph on $K_{2,6}$

Theorem 6.3.13. *Let G be the complete bipartite graph $K_{2,2s+1}$ with bipartition $V_1 = \{v_1, v_2\}$ and $V_2 = \{w_1, w_2, \ldots, w_{2s+1}\}$. Let S be the sigraph obtained from G by assigning positive sign to all but one edge. Then S is additively graceful.*

Proof. Let the negative edge be v_1w_1.

Define $f : V \to \{0, 1, \ldots, 4s+2\}$ by

$$f(v_1) = 0,\ f(v_2) = 2 \text{ and}$$
$$f(w_i) = \begin{cases} 1, & \text{for } i = 1; \\ 4s+9-4i, & \text{for } 2 \le i \le s+1; \\ 8(s+1)-4i, & \text{for } s+2 \le i \le 2s+1. \end{cases}$$

It can be easily verified that f is an additively graceful labeling of S.

□

Example 6.3.14. Figure 6.7 gives additively graceful labeling of the sigraph on $K_{2,7}$ with one negative edge.

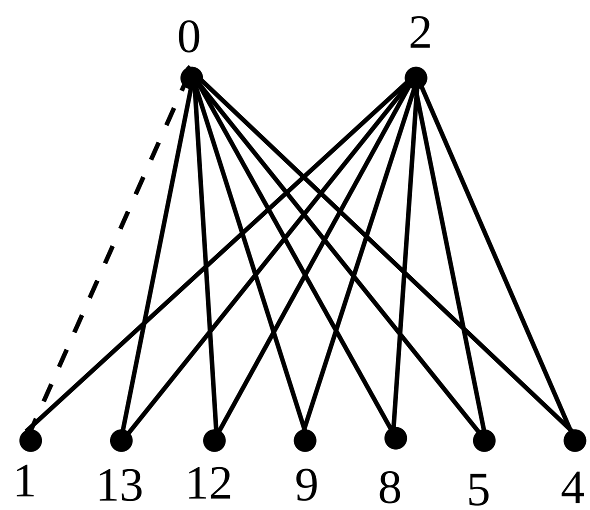


Figure 6.7: Additively graceful labeling of the sigraph on $K_{2,7}$

Theorem 6.3.15. *Let G be a star $K_{1,t}$ with bipartition $V_1 = \{u\}$ and $V_2 = \{w_1, w_2, \ldots, w_t\}$. Let S be the sigraph obtained from G by assigning positive sign to all but one edge. Then S is additively graceful.*

Proof. Let the negative edge be uw_1. Define $f : V \rightarrow \{0, 1, \ldots, t\}$ by $f(u) = 1,\ f(w_1) = 0$ and $f(w_i) = i,$ for $2 \leq i \leq t$. It can be easily verified that f is an additively graceful labeling of S. $\square$

Example 6.3.16. Figure 6.8 gives additively graceful labeling of the sigraph on $K_{1,6}$ with one negative edge.

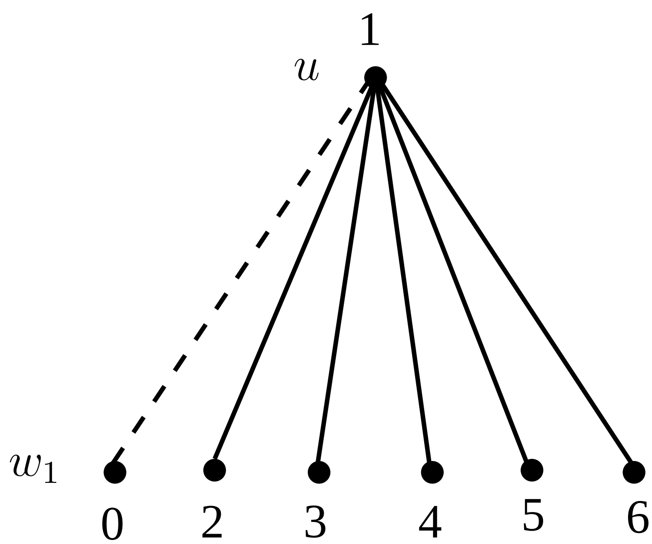


Figure 6.8: Additively graceful labeling of the sigraph on $K_{1,6}$

CHAPTER 7

EMBEDDING OF GRAPHS INTO GRACEFUL GRAPHS

7.1 INTRODUCTION

Acharya et al. [7] have considered the following problem: *Given a graph G, is it possible to embed G as an induced subgraph of a graceful graph H having a graph theoretic property $\mathcal{P}$?* In [7], they have answered the problem in an affirmative way for triangle-free graphs, planar graphs, hamiltonian graphs and trees. In most of the embedding results, to get any missing edge label we add a pendent vertex with that label as vertex label and join it to a vertex with label 0. Thus the graceful graph into which G is embedded is mostly a noneulerian graph. Hence the following problem naturally arises. *Can every connected graph be embedded as an induced subgraph in an eulerian graceful graph?*

Rao and Sahoo [47] obtained an affirmative answer for the above problem. However, in their proof the number of vertices in the eulerian graceful graph is $O(3^p)$. In this chapter, we obtain a more efficient embedding of a graph G of order p as an induced subgraph of an eulerian graceful graph H whose order is $O(p^2)$. In [31], Golomb has given an upper bound for the largest vertex label for the complete graph K_p to be $O(p^2)$ such that the vertex labels and the induced edge labels are distinct. Hence, one can easily see that for any graph G of order p, the injection $f : V(K_p) \to \mathbb{N}$ with largest vertex label $O(p^2)$, gives an injection on $V(G)$ such that g_f is injective and the largest vertex label is $O(p^2)$.

We also consider the following analogous problem for sigraphs: *Given a sigraph S and a graph theoretic property $\mathcal{P}$, is it possible to embed S in a graceful sigraph S_1 having the property $\mathcal{P}$?* We prove the existence of such an embedding where S_1 is eulerian, hamiltonian, planar or triangle-free. We prove that every signed tree (in short *sitree*) can be embedded in a graceful sitree.

7.2 EMBEDDING OF GRAPHS

In this section, we give an efficient algorithm to embed a connected graph as an induced subgraph of an eulerian graceful graph.

Theorem 7.2.1. *Every connected graph G can be embedded as an induced subgraph of an eulerian graceful graph H.*

Proof. Since any connected graph G can be embedded as an induced subgraph of an eulerian graph, we may assume without loss of generality that G is eulerian. Let $f : V(G) \to \mathbb{N} - \{1\}$ be an injection such that the edge induced function g_f is injective. Let $M_G(f) = max_{v \in V(G)} f(v)$, let $M_G(g_f)$ be the largest edge label induced by f and let $c \in V$ be such that $f(c) = M_G(f)$. Without loss of generality we assume that $M_G(g_f)$ and $M_G(f)$ are of same parity, since otherwise we can replace $f(v)$ by $f(v)+1$ for every $v \in V(G)$. Insert two vertices a and b in G with $f(a) = 0$ and $f(b) = 1$. Let $r = 1$.

Let i be the first missing edge label. If $i+1$ is not a missing edge label, we insert a new vertex i_r with $f(i_r) = f(c) + i$ and join i_r to c. We replace r by $r + 1$. If $i + 1$ is also a missing edge label, we insert a vertex with label $i + 1$ if $i + 1$ is also a missing vertex label and join it to a and b. If $i + 1$ is a vertex label of G, then join the vertex with label $i + 1$ to both a and b. We repeat this process until all the edge labels $1, 2, \ldots, f(c) - 2$ are created.

Now $r - 1$ is the number of pendent edges attached to c. We denote the resulting graph by G_1 and let the labeling of G_1 be f_1.

If $r = 1$, then join c to a and b. If a and b both have odd degree, then to make it even, we insert a vertex with label $M_{G_1}(f_1) + 2$ and join it to a and b. The resulting graph is the required eulerian graph with graceful labeling f_1 having G as an induced subgraph. Now suppose $r > 1$. If r is even then join c to b. If r is odd, then join c to both a and b. Let G_2 be the resulting graph with order p_2 and size q_2 and labeling f_2. Let $k = 1$. If q_2

and $f_2(i_k)$ have same parity join i_k to b and otherwise join i_k to a. If $j = \lfloor \frac{f_2(i_k)-q_2-1}{2} \rfloor > 0$, then insert j vertices with labels $q_2 + 2t$, $1 \leq t \leq j$ and join each of these vertices to a and b. Let $k = k + 1$. Repeat this process until $k = r$. Finally if the vertex i_{r-1} was joined to b, then insert a vertex labeled $f_2(i_{r-1}) + 1$ and join it to a and b. If the degrees of a and b are odd then insert a vertex with label $M_{G_2}(f_2) + 2$ and join it to a and b. This gives the required eulerian graceful graph H having G as an induced subgraph. $\square$

We give an illustration of embedding a connected non-graceful graph G as an induced subgraph of an eulerian graceful graph H.

Illustration 7.2.2. Figure 7.1 gives a connected eulerian graph G with an injective labeling f such that $f(c) = 12$ and $M_G(f)$ and $M_G(g_f)$ have the same parity.

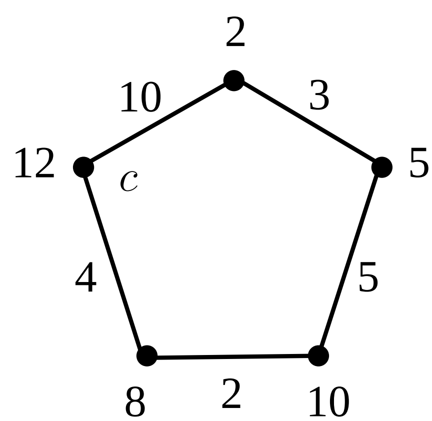


Figure 7.1: Connected eulerian nongraceful graph G

The missing vertex labels and edge labels of G under f are respectively, $\overline{f}(V) = \{0, 1, 3, 4, 6, 7, 9, 11\}$ and $\overline{g}_f(E) = \{1, 6, 7, 8, 9, 11, 12\}$.

Insert two vertices a and b in G with $f(a) = 0$ and $f(b) = 1$. Let $r = 1$. For the first missing edge label $i = 1$, $i + 1$ is not a missing edge label. Therefore, we insert a new vertex i_1 with $f(i_1) = f(c) + i = 13$ and join it to vertex c. Replace r by $r + 1$, that is, $r = 2$. Now, $i = 6$ and $i + 1 = 7$ is a missing edge label and also a missing vertex label, so we insert a vertex with label 7 and join it to vertices a and b. Similarly, we insert a vertex with label 9 and join it to a and b. At this stage, we have the edge labels 1, 2, 3, $\ldots$, $f(c) - 2 = 10$ in G and there are $r - 1$ pendent edges attached to c. Since $r = 2$ is even, we join vertex c to vertex b by an edge. Let the resulting graph be G_2 with labeling f_2, order p_2 and size q_2 as shown in Figure 7.2.

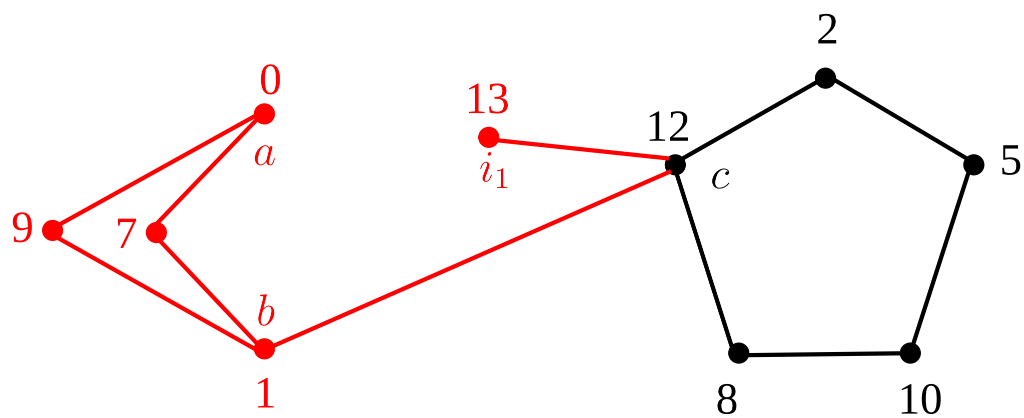


Figure 7.2: Graph G_2 with labeling f_2

Let $k = 1$, since $q_2 = 11$ and $f_2(i_1) = 13$ have the same parity, we join i_1 to b. Notice that, $j = \lfloor \frac{f_2(i_1) - q_2 - 1}{2} \rfloor = 0$ and by incrementing k by 1, we have, $k = r$. Since the vertex $i_{r-1} = i_1$ was joined to b, we insert a vertex labeled $f_2(i_1) + 1 = 14$ and join it to vertices a and b.

Note that, we have obtained a graceful graph but it is not eulerian as the degree of a and b is odd. Hence we insert a vertex with label $M_{G_2}(f_2) + 2 = 16$ and join it to a and b. The graph thus obtained, say H is eulerian and graceful as shown in Figure 7.3, having G as an induced subgraph.

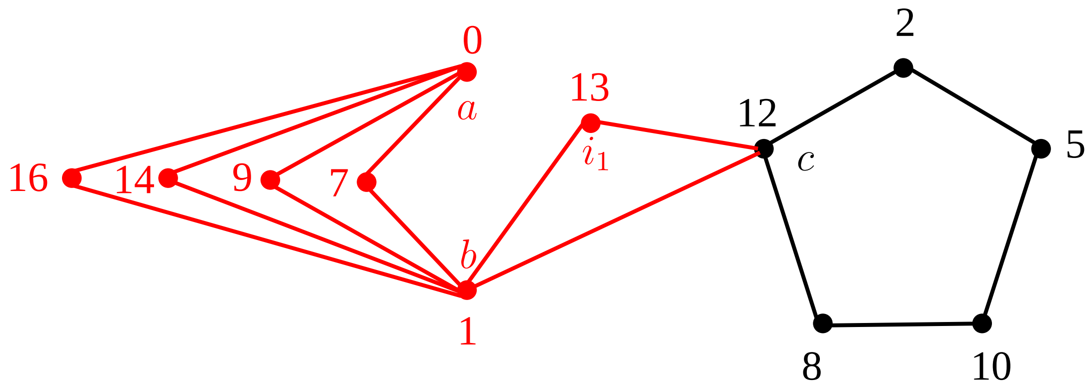


Figure 7.3: Eulerian graceful graph H with G as an induced subgraph

Theorem 7.2.3. *Let G be a connected graph of order p. Let H be an eulerian graceful graph constructed in Theorem 7.2.1 having G as an induced subgraph. Then the number of vertices in H is $O(p^2)$.*

Proof. It follows from Remark 1.3.11 that we can choose an injection $f : V \to \mathbb{N}$ such that g_f is injective and $M_G(f)$ is $O(p^2)$. Hence the number of missing edge labels is $M_G(f) - q$. In the worst case let $M_G(f) - 2$ be a missing edge label. By the construction given in Theorem 7.2.1, we insert a vertex with label $2M_G(f) - 2$ and join it to the vertex c with label $M_G(f)$ and also to a or b. Now the number of missing edge labels is $2M_G(f) - q - 4$. To obtain the required eulerian graceful graph H we insert a maximum of $\frac{2M_G(f)-q-4}{2} + 5$ vertices. Hence the number of vertices in H is $O(p^2)$. $\square$

7.3 EMBEDDING OF SIGRAPHS

In this section, we prove that any sitree can be embedded in a graceful sitree. We also prove that any connected sigraph can be embedded as an induced subsigraph of an eulerian graceful sigraph as well as hamiltonian sigraph.

Acharya and Singh [10], have shown that not every sitree is graceful. The following theorem gives a method of embedding any sitree into a graceful sitree.

Theorem 7.3.1. *Every sitree can be embedded in a graceful sitree.*

Proof. Let S be a sitree with an injection $f : V \to \mathbb{N} \cup \{0\}$ such that the edge induced function g_f is injective and let $M_S(f) = max\{f(v) : v \in V\}$. Let $v_0 \in V$ and let $f(v_0) = 0$. Let $m_1 = M^-(S) + M^+(S)$, where $M^-(S)$ is the absolute value of the smallest negative edge label and $M^+(S)$ is the largest positive edge label. Let

$$l = \begin{cases} 0, & \text{if } M_S(f) \leq m_1; \\ M_S(f) - m_1, & \text{if } M_S(f) > m_1. \end{cases}$$

Let $\overline{f}(V)$ denote the set of missing vertex labels of S, therefore $\overline{f}(V) = \{0, 1, 2, \ldots, M_S(f)\} - f(V)$. Let $E_1 = \{1, 2, \ldots, M^+(S) + l\} - g_{f^+}(S)$ and $E_2 = \{1, 2, \ldots, M^-(S)\} - g_{f^-}(S)$, where $g_{f^+}(S) = \{|f(u) - f(v)| : uv \in E^+\}$ and $g_{f^-}(S) = \{|f(u) - f(v)| : uv \in E^-\}$.

For each $r \in (E_1 \cup E_2) \cap \overline{f}(V)$, insert a vertex v_r with

$f(v_r) = r$ and join it to the vertex v_0 using a positive edge if $r \in E_1$ or using a negative edge if $r \in E_2$.

Now, let j denote the smallest missing vertex label of S and let i denote the largest number in the set $E_1 \cup E_2$. If $j < i$, let $k = i + j$ else let $k = j - i$. Insert a vertex v_j with label j. If k is not a vertex label of S, then insert another vertex v_k with label k, if there exists a vertex in S with label k, then call it v_k. Join the vertex v_j to the vertex v_k using a positive edge if $i \in E_1$ or using a negative edge if $i \in E_2$. Repeat this process by taking j to be the smallest missing vertex label if v_k had not been inserted in the previous step or by taking $j = k$ otherwise, until there is no missing vertex label. This gives the required graceful sitree. $\square$

Remark 7.3.2. In the above embedding if S is a sigraph which is embedded in a graceful sigraph, we observe that if S is triangle-free then S_1 is also triangle-free and if S is planar then S_1 is also planar. Hence it follows that any triangle-free sigraph can be embedded in a graceful triangle-free sigraph and any planar sigraph can be embedded in a graceful planar sigraph.

We illustrate the procedure of embedding a sitree in a graceful sitree as given in Theorem 7.3.1.

Illustration 7.3.3. Consider the sitree S with an injective labeling f as shown in Figure 7.4. From the figure, $f(v_0) = 0$, $M_S(f) = 18$, $M^+(S) = 6$ and $M^-(S) = 7$, therefore, $m_1 = 13$ and hence $l =$

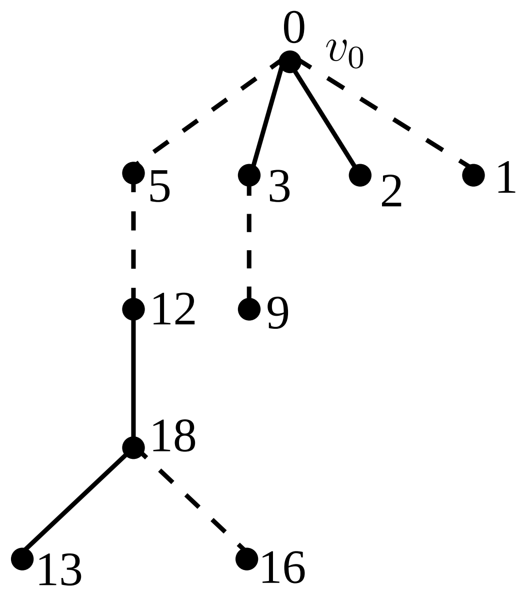


Figure 7.4: Sitree S with an injective labeling f

5. The set $E_1 = \{1, 4, 7, 8, 9, 10, 11\}$, $E_2 = \{3, 4\}$ and the set of missing vertex labels is $\overline{f}(V) = \{4, 6, 7, 8, 10, 11, 14, 15, 17\}$. Since $(E_1 \cup E_2) \cap \overline{f}(V) = \{4, 7, 8, 10, 11\} = R$ (say), we insert $|R| = 5$ vertices v_4, v_7, v_8, v_{10} and v_{11} with labels $4, 7, 8, 10, 11$. We join each of the vertices v_7, v_8, v_{10} and v_{11} to the vertex v_0 by a positive edge and the vertex v_4 to v_0 by a negative edge as shown in Figure 7.5.

Now, we have $E_1 = \{1, 4, 9\}$, $E_2 = \{3\}$ and $\overline{f}(V) = \{6, 14, 15, 17\}$. Therefore, $j = 6$ and $i = 9$, hence $k = i + j = 15$. Insert a vertex v_6 with label 6. Since 15 is not a vertex label of S, we insert another vertex v_{15} with label 15 and join it to v_6 by a positive edge, as $9 \in E_1$. Now let $j = k = 15$, i is 4, hence $k = 11$. Join v_{15} to v_{11} by a positive edge, as $4 \in E_1$. Similarly, we insert vertices v_{14} and v_{17} and join v_{14} to v_{11} by a negative edge and v_{17} to v_{16} by a positive edge. The resulting sigraph T is a graceful sitree as shown in Figure 7.6, with S as an induced subsitree.

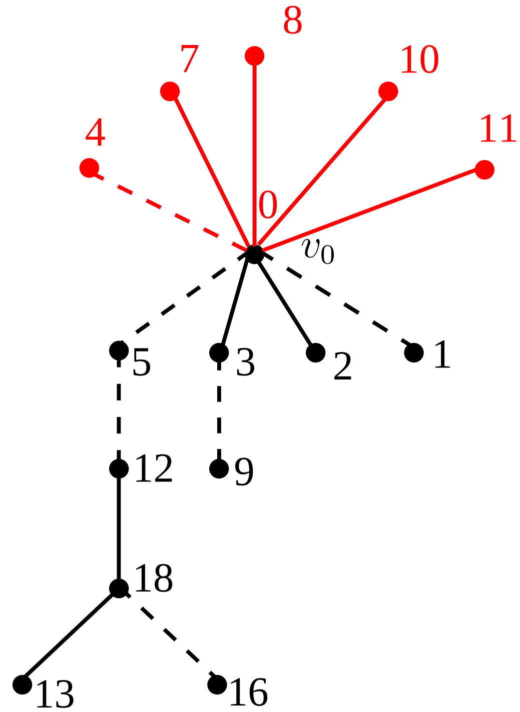


Figure 7.5: The sigraph S_1

Theorem 7.3.4. *Every connected sigraph S can be embedded in a graceful eulerian sigraph H.*

Proof. Since any connected sigraph S can be embedded as an induced subsigraph of an eulerian sigraph, we may assume without loss of generality that S is eulerian. Let $f : V \to \mathbb{N} - \{1\}$ be an injective function such that the edge induced function g_f is also injective. Let $M_S(f) = max\{f(v) : v \in V(S)\}$ and let $M^-(S)$ be the absolute value of the smallest negative edge label of S. Let $c \in V(S)$ be such that $f(c) = M_S(f)$. Without loss of generality we assume that $M^-(S)$ and $M_S(f)$ are of same parity, since otherwise we can replace $f(v)$ by $f(v) + 1$ for every $v \in V(S)$. Insert two vertices a and b in S with $f(a) = 0$ and $f(b) = 1$.

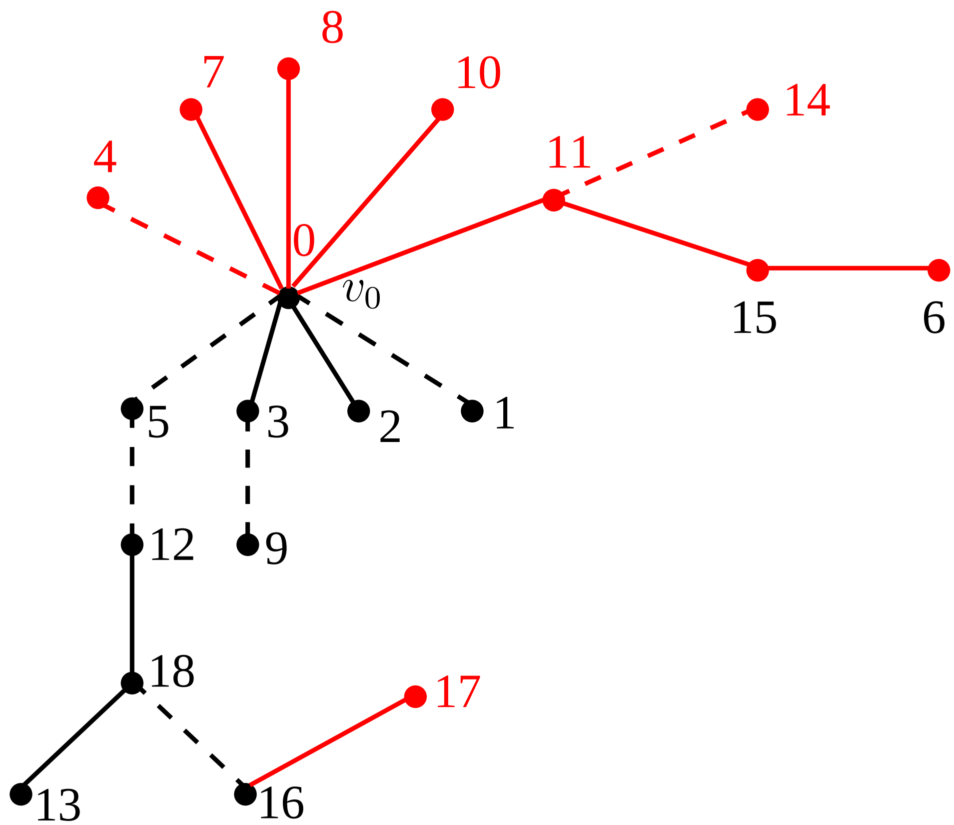


Figure 7.6: Graceful sitree T with S as an induced subsitree

Let $r = 1$. Let $E_1 = \{1, 2, \ldots, f(c) - 2\} - g_{f^-}(S)$ where $g_{f^-}(S) = \{|f(u) - f(v)| : uv \in E^-\}$ and let i be the smallest value in E_1. If $i + 1$ is not present in E_1, we insert a new vertex i_r with $f(i_r) = f(c) + i$ and join i_r to c by a negative edge. We replace r by $r + 1$. If $i + 1$ is also in E_1 and $i + 1$ is also a missing vertex label, then we insert a vertex with label $i + 1$ and join it to a and b by negative edges. If $i + 1$ is a vertex label, then join the vertex with label $i + 1$ to both a and b by negative edges. We repeat this process until E_1 is empty.

Now $r - 1$ is the number of pendent edges attached to c. We denote the resulting sigraph by S_1 and let the labeling of S_1 be

f_1. If $r = 1$, then join c to a and b by negative edges. If a and b both have odd degree, then to make it even, we insert a vertex with label $M_{S_1}(f_1)+2$ and join it to a and b using negative edges. The resulting graph is an eulerian sigraph having edge labels $-1, -2, \cdots - M_{S_1}(f_1)$ and having S as an induced subsigraph. Now suppose $r > 1$. If r is even, then join c to b by a negative edge. If r is odd, then join c to both a and b by negative edges. Let S_2 be the resulting sigraph with labeling f_2, let $k = 1$ and $M^-(S_2)$ be the absolute value of the smallest negative edge label of S_2. If $M^-(S_2)$ and $f_2(i_k)$ have same parity, join i_k to b by a negative edge and otherwise join i_k to a by a negative edge. If $j = \lfloor \frac{f_2(i_k) - M^-(S_2) - 1}{2} \rfloor > 0$, then insert j vertices with labels $M^-(S_2) + 2t$, $1 \le t \le j$ and join each of these vertices to a and b by negative edges. Replace k by $k+1$. Repeat this process until $k = r$. Finally if the vertex i_{r-1} was joined to b, then insert a vertex labeled $f_2(i_{r-1}) + 1$ and join it to a and b by negative edges. If the degrees of a and b are odd then insert a vertex with label $M_{S_2}(f_2) + 2$ and join it to a and b by negative edges.

Let the resulting sigraph be G^* and let the labeling of G^* be h^*. If $M_{G^*}(h^*)$ and $M^+(G^*)$ are of same parity, then we replace $h^*(v)$ by $h^*(v) + 2$ for every $v \in V(G^*)$. Otherwise we replace $h^*(v)$ by $h^*(v) + 3$ for every $v \in V(G^*)$. Insert two vertices u and v in G^* with $h^*(u) = 0$ and $h^*(v) = 1$. Let $w \in V(G^*)$ be such that $h^*(w) = M_{G^*}(h^*)$. Let $r = 1$. Let $E_2 = \{1, 2, \ldots, h^*(w) - 2\} - g_{f^+}(S)$ where $g_{f^+}(S) = \{|f(u) - f(v)| : uv \in E^+\}$ and let i be the smallest value in E_2. If $i + 1$ is not present in E_2, we insert a new vertex j_r with $h^*(j_r) = h^*(w) + i$ and join j_r to w by a positive edge. We replace r by $r+1$. If $i+1$ is also in E_2 and $i+1$ is also a missing vertex label, then we insert a vertex with label $i+1$ and join it to u and v by positive edges. If $i+1$ is a vertex label, then join the vertex with label $i+1$ to both u and v by positive edges. We repeat this process until E_2 is empty.

Now $r-1$ is the number of pendent edges attached to w. We denote the resulting sigraph by G_1 and let the labeling of G_1 be h_1. If $r=1$, then join w to u and v. If u and v both have odd degree in G_1, then to make it even , we insert a vertex with label $M_{G_1}(h_1)+2$ and join it to u and v using positive edges.The resulting graph is the required eulerian graceful sigraph with graceful labeling h_1 having S as an induced subsigraph. Now suppose $r>1$. If r is even then join w to v using a positive edge. If r is odd, then join w to both u and v using positive edges. Let G_2 be the resulting graph with labeling h_2, let $k=1$ and let the largest positive edge label of G_2 be $M^+(G_2)$. If $M^+(G_2)$ and $h_2(j_k)$ have same parity join j_k to v, and otherwise join j_k to u using a positive edge. If $d=\lfloor\frac{h_2(j_k)-M^+(G_2)-1}{2}\rfloor>0$, then insert d vertices with labels $M^+(G_2)+2t$, $1\leq t\leq d$ and join each of these vertices to u and v using positive edges. Replace k by $k+1$. Repeat this process until $k=r$. Finally if the vertex j_{r-1} was joined to v, then insert a vertex labeled $h_2(j_{r-1})+1$ and join it to u and v using positive edges. If the degrees of u and v are odd then insert a vertex with label $M_{G_2}(h_2)+2$ and join it to u and v using positive edges. This gives the required eulerian graceful sigraph H having S as an induced subsigraph. $\square$

We give an illustration of Theorem 7.3.4.

Illustration 7.3.5. Consider the sigraph S as shown in Figure 7.7. We shall embed S in a graceful eulerian sigraph H.

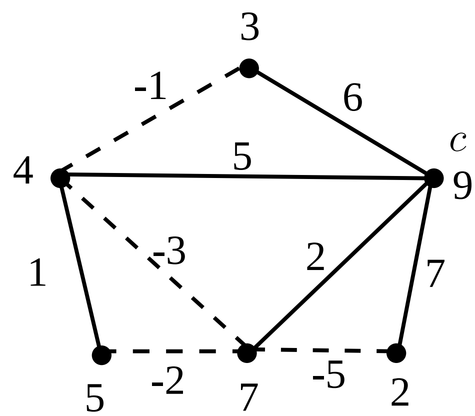


Figure 7.7: A connected sigraph S with an injective labeling f

From Figure 7.7, $M_S(f) = 9$ and $M^-(S) = 5$. Insert two vertices a and b in S with $f(a) = 0$ and $f(b) = 1$. Let $r = 1$ and $E_1 = \{4, 6, 7\}$. Since $i = 4$ and 5 is not present in E_1, insert a vertex i_1 with $f(i_1) = 13$, join it to c by a negative edge and let $r = 2$. Now, $i = 6$ and $7 \in E_1$, therefore we join the vertex with label 7 to both a and b using negative edges. At this stage E_1 is empty and there is one pendent edge attached to c. Since r is even, we join c to b by a negative edge. The resulting sigraph S_2 with labeling f_2 is shown in Figure 7.8.

Let $k = 1$. From Figure 7.8, $M^-(S_2) = 8$ and $f_2(i_1) = 13$, therefore we join i_1 to a by a negative edge. Since $j = \lfloor \frac{f_2(i_1) - M^-(S_2) - 1}{2} \rfloor = 2$, we insert two vertices with labels 10 and 12 and join each of them to a and b by negative edges and replace k by $k+1$. At this stage the degree of a and b is even. The resulting

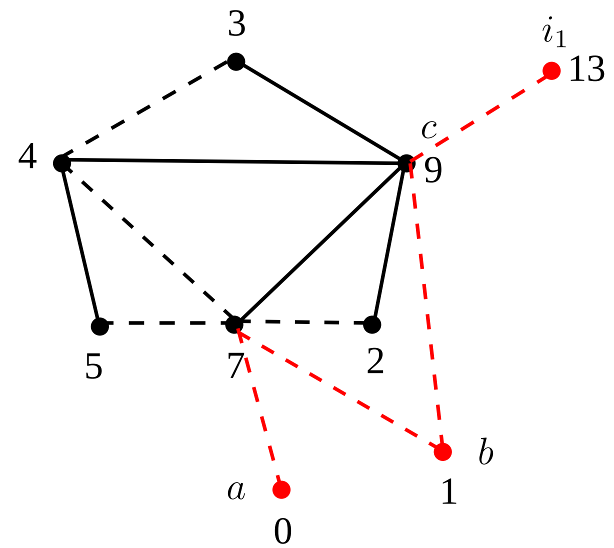


Figure 7.8: The sigraph S_2 with labeling f_2

graph G^* with labeling h^* is shown in Figure 7.9. In the figure, $h^*(v)$ has been replaced by $h^*(v) + 2$, since $M_{G^*}(h^*)$ and $M^+(G^*)$ have the same parity.

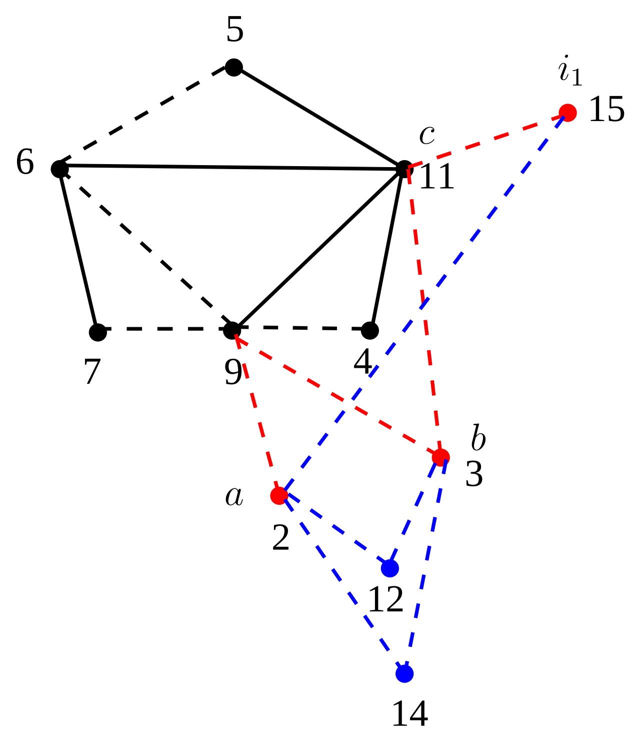


Figure 7.9: The sigraph G^* with labeling h^*

We insert two vertices u and v with $h^*(u) = 0$ and $h^*(v) = 1$. Since $M_{G^*}(h^*) = 15$, we shall call the vertex i_1 as w. Let $r = 1$ and $E_2 = \{3, 4, 8, 9, 10, 11, 12, 13\}$. For $i = 3$, since $4 \in E_2$ is a vertex label of G^*, we join the vertex with label 4 to u and v by positive edges. Similarly, we join the vertices with labels 9 and 11 to vertices u and v by positive edges. For $i = 12$, since 13 is not a vertex label of G^*, we insert a new vertex with label 13 and join it to vertices u and v using positive edges. At this stage, $r = 1$, so we join w to u and v and since the degree of u and v is odd, we insert another vertex with label 17 and join it to both u and v using positive edges. The resulting sigraph H as shown in Figure 7.10 is the required eulerian graceful sigraph having S as an induced subsigraph.

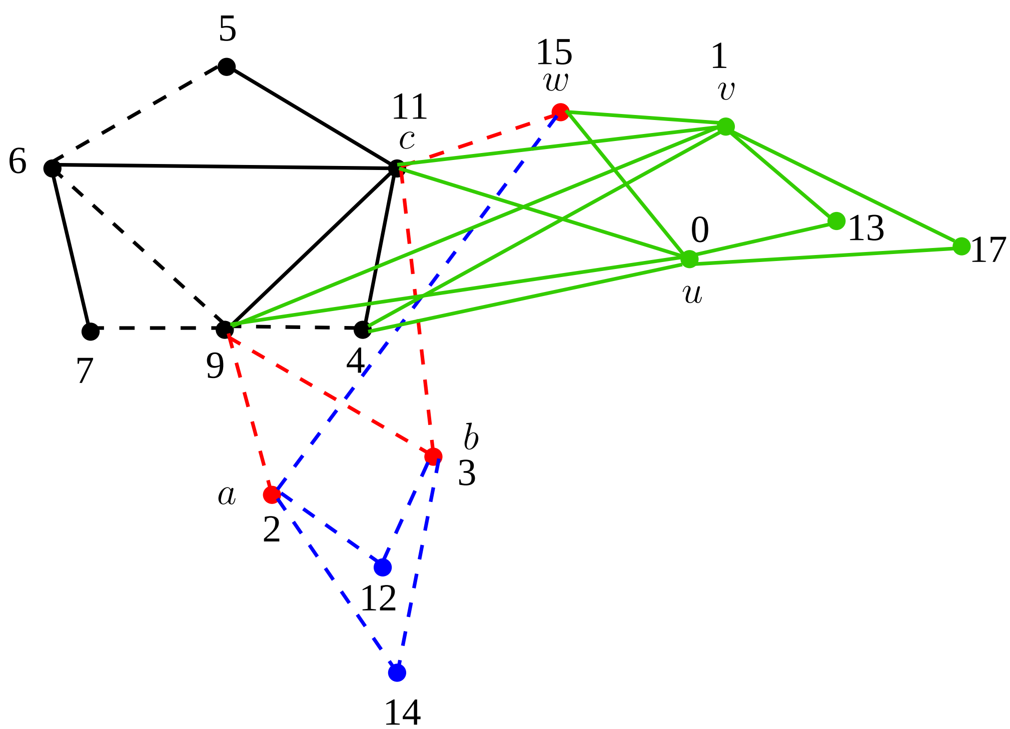


Figure 7.10: Eulerian graceful sigraph H having S as an induced subsigraph

Theorem 7.3.6. *Every sigraph can be embedded in a graceful hamiltonian sigraph.*

Proof. Let S be a sigraph. By Theorem 1.3.21, S can be embedded in a graceful sigraph H. Let p and q be the order and size of H and let m and n be the number of its positive and negative edges respectively. Let $H^* = (H \cup \overline{K}_r) + \overline{K}_s$, where $r = M_H(f) - (p-1)$ and $s = p+r$ and all the edges joining $\overline{K}_s$ and $(H \cup \overline{K}_r)$ are positive. Label the vertices of $\overline{K}_r$ from the set $\{0, 1, 2, \ldots, M_H(f)\} - f(V)$ and label the vertices of $\overline{K}_s$ with s new labels $(m+s)+is$ where $0 \leq i \leq (s-1)$. It is easy to verify that the resulting sigraph is hamiltonian as well as graceful. $\square$

CHAPTER 8

CONCLUSION AND FUTURE SCOPE OF WORK

Synch-set codes (designed by Simmons [52]) are used to synchronize the relative annular positions of a photo-detector on one side of a rotating disk with a stationary target light source on the other side. A $S(p, \lambda)$-synch set is defined as a set of p distinct nonnegative integers for which no more than λ pairs have the same common difference and for which the maximum element is as small as possible. A synch-set designates positions for the p holes so that distance from the first to the last hole is minimized. Hence a synch-set represents a labeling of K_p with distinct positive integers such that the largest vertex label is minimized and an edge label is repeated at most λ times. For $\lambda = 1$, this is rephrasing of the Golomb ruler. For $\lambda = 2$, we get graceful labeling of complete signed graph ([46, 52]).

In Chapter 2, we have proved that the complete graph K_p is $(p-4)$-hypergraceful if and only if $p \geq 8$, $(p-3)$-hypergraceful for $p \geq 4$, $(p-2)$-hypergraceful for $p \geq 3$ and $(p-1)$-hypergraceful for $p \geq 2$. The k-hypergraceful labeling of other classes of graphs like eulerian graphs is an open problem. We have given all nonisomorphic 3-hypergraceful decompositions of K_5. The problem of determining all nonisomorphic k-hypergraceful decompositions of K_p seems to be a difficult problem.

In Chapter 3, we have defined (k, d)-Skolem graceful graphs and given some necessary or sufficient condition for a graph G to be (k, d)-Skolem graceful. We have proved that nK_2 is $(2, 1)$-Skolem graceful if and only if $n \equiv 0$ or $3 (\text{mod } 4)$. We have proved that nK_2 is $(1, 2)$-Skolem graceful. Further we observe that $(1, 1)$-Skolem graceful labeling of nK_2 produces a Skolem sequence, $(2, 1)$-Skolem graceful labeling of nK_2 produces a $(2, n)$ Langford sequence and $(k, 1)$-Skolem graceful labeling of nK_2 gives a perfect sequence. A graph G is said to be arbitrarily Skolem graceful if G is (k, d)-Skolem graceful for all possible values of k and d. Determining the value of n for which nK_2 is arbitrarily Skolem graceful is an open problem. Hence another natural research area is to determine more families of arbitrarily Skolem graceful graphs.

In Chapter 4, we have introduced the notion of (k, d)-hooked Skolem graceful graphs and observe that $k = d = 1$ coincides with the notion of hooked Skolem graceful labeling of a graph G. We have given some necessary or sufficient conditions for a graph G to be (k, d)-hooked Skolem graceful. We have proved that nK_2 is $(2, 1)$-hooked Skolem graceful if and only if $n \equiv 1$ or $2 (\text{mod } 4)$. Determining the value of n for which nK_2 is (k, d)-hooked Skolem graceful for given values of k and d is an open problem.

Chapter 5, introduces a new measure of gracefulness $m(G)$ of a graph G. This new measure $m(G)$ determines how close G is to being graceful. It can be easily noted that $m(G) = q$ if G is graceful. We have proved that there are infinitely many nongraceful graphs G with $m(G) = q-1$, and for $n \equiv 1$ or $2 \pmod 4$, $m(C_n) = n-1 = q-1$ and this is achieved with $n+1$ as the highest vertex label, therefore we conclude that for $n \equiv 1$ or $2(\text{mod } 4)$, $grac(C_n) = n+1 = q+1$. We have also shown that $m(F_k) = 3k-1 = q-1$ and $grac(F_k) = 3k+1 = q+1$ for $k \equiv 2$ or $3(\text{mod } 4)$, where F_k is the friendship graph with k triangles. We have given necessary conditions for a (p, q)-eulerian graph and the complete graph K_p to have m-gracefulness $q-1$ and $q-2$. Using this, we have proved that K_5 is the only complete graph to have m-gracefulness $q-1$. We have also given an upper bound for the highest possible vertex label of K_p if $m(K_p) = q-2$. We have proved that $m(K_6) = 13 = q-2$, which is also shown in optimal Golomb ruler [21]. The problem of determining $m(G)$ for several other classes of nongraceful graphs remains open. We also have the following question which arises naturally: Is it true that $grac(G) - q = q - m(G)$?

Hegde [34] introduced the concept of additively graceful graphs and gave some necessary or sufficient conditions for the same. This concept is extended to the realm of sigraphs in Chapter 6 as additively graceful sigraphs. We have obtained some necessary or sufficient conditions for additively graceful sigraphs and some results on eulerian sigraphs, complete bipartite sigraphs and complete sigraphs have also been obtained. One can investigate other classes of additively graceful sigraphs or characterize additively graceful sigraphs.

In Chapter 7, we have obtained an efficient embedding of a graph G of order p as an induced subgraph of an eulerian graceful graph H whose order is $O(p^2)$. We have also considered the following problem for sigraphs: *Given a sigraph S and a graph theoretic property $\mathcal{P}$, is it possible to embed S in a graceful sigraph S_1 having the property $\mathcal{P}$?* We have proved the existence of such an embedding where S_1 is eulerian, hamiltonian, planar or triangle-free. We have proved that every sitree can be embedded in a graceful sitree. Investigation of other graph theoretic properties of the graph H which is an optimal graceful embedding of any given graph G is an interesting problem.

References

[1] Abelson R.P., Aronson E., McGuire W.J., Newcomb T.M., Rosenberg M.J. and Tannenbaum R.H. (Eds.) *Theories of cognitive consistency: A sourcebook*, Rand McNally, Chicago, 1968.

[2] Abelson R.P. and Rosenberg M.J., Symbolic psycho-logic: A model of attitudinal cognition, *Behav. Sci.*, **3** (1958), 1–13.

[3] Acharya B.D., *Applications of sigraphs in behavioural sciences*, 1980, MRI Tech. Rep. No. DST/HCS/409/79, sponsored by Deparment of Science and Technology, Govt. of India.

[4] Acharya B.D., A spectral criterion for cycle balance in networks, *J. Graph Theory*, **4**(1) (1980), 1–11.

[5] Acharya B.D., (k, d)-*graceful packings of a graph*, In: Technical Proc. of Group Discussion on Graph Labeling Problems (Eds.: Acharya B.D. and Hegde S.M.), National Institute of Technology, Karnataka, Surathkal, India, during August 16-25, 1999.

[6] Acharya B.D. and Acharya M., New algebraic models of social systems, *Indian J. Pure Appl. Math.*, **17**(2) (1986), 150–168.

[7] Acharya B.D., Rao S.B. and Arumugam S., *Embeddings and NP-complete problems for graceful graphs*, Labelings of Discrete Structures and Applications (Eds.: Acharya B.D., Arumugam S. and Rosa A.), 2008, 57–62, Narosa Publishing House.

[8] Acharya M. and Singh T., Construction of graceful signed graphs, *Defence Sci. J.*, **56**(5) (2006), 801–808.

[9] Acharya M. and Singh T., Embedding of signed graphs in graceful signed graphs, *ARS Combin.*, **111** (2013), 421–426.

[10] Acharya M. and Singh T. , Graceful signed graphs, *Czechoslovak Math. J.*, **54**(129) (2004), no.2, 291–302.

[11] Acharya M. and Singh T., Graceful signed graphs: II. The case of signed cycle with connected negative section, *Czechoslovak Math. J.*, **55**(130) (2005), no.1, 25–40.

[12] Acharya M. and Singh T., Graceful signed graphs. III. The case of signed cycles in which the negative sections form a maximum matching, *Graph Theory Notes N.Y.*, **45** (2003), 11–15.

[13] Acharya M. and Singh T., Graceful sigraphs: V. The case of union of signed cycles of length three with one vertex in common, *Inter. J. Management and Systems*, **20**(3) (2004), 245–254.

[14] Acharya M. and Singh T., Skolem graceful signed graphs, *Util. Math.*, **82** (2010), 97–109.

[15] Balaban A.T., *Chemical applications of graph theory*, Academic Press, New York, 1976.

[16] Behzad M. and Chartrand G.T., Line-coloring of signed graphs, *Elem. Math., Band*, **24**(3) (1969), 49–52.

[17] Beineke L.W. and Harary F., Consistency in marked digraphs, *J. Math. Psych.*, **18**(3) (1978), 260–269.

[18] Bermond J.C., Graceful graphs, radio antennae and French windmills, Proceedings One Day Combinatorics Conference, Open University, May 1978 (Pitman, London, 1979).

[19] Bloom G.S., *A chronology of the Ringel-Kotzig conjecture and the continuing quest to call all trees graceful*, (Ed.: Harary F.), Topics in Graph Theory, Ann. N.Y. Acad. Sci., **328** (1979), 32–51.

[20] Bloom G.S. and Golomb S.W., Applications of numbered undirected graphs, *Proc. IEEE*, **65**(4) (1977), 562–570.

[21] Bloom G.S. and Golomb S.W., Numbered complete graphs, unusual rulers and assorted applications, *Lecture Notes in Mathematics*, Springer-Verlag, New York, **642** (1978), 53–65.

[22] Brauer A., A problem of additive number theory and its application in electrical engineering, *J. Elisha Mitchell Sci. Soc.*, **61** (1945), 55–66.

[23] Chartrand G., *Graphs as mathematical models*, Prindle, Weber and Schmidt, Inc., Boston, Massachusetts, 1977.

[24] Chartrand G. and Lesniak L., *Graphs and digraphs*, Chapman and Hall/CRC, 2005.

[25] Chen W.K., *Applied graph theory: graphs and electrical networks*, North-Holland, Amsterdam, 1975.

[26] Davies R.O., On Langford's problem II, *Math. Gaz.*, **43** (1959), 253–255.

[27] Festinger L., *A Theory of cognitive dissonance*, Row Peterson, Evanston, Illinios, 1957.

[28] Frank H. and Frisch I.T., *Communication, transmission and transportation networks*, Addison-Wesley, Reading Massachusetts 1971.

[29] Gallian J.A., A dynamic survey of graph labeling, *Electron. J. Combin.*, Dynamic survey, (2005), DS #6: 1–148.

[30] Gillespie F.S. and Utz W.R., A generalized Langford Problem, *Fibonacci Quart.* **4** (1966), 184–186.

[31] Golomb S.W., How to number a graph, In: *Graph theory and computing* (Ed.: Read R.C.), Academic Press 1972, 23–37.

[32] Hale W.K., Frequency assignment: Theory and application, *Proc. IEEE*, **68**(12) (1980), 1497–1514.

[33] Harary F., Norman R.Z. and Cartwright D.W., *Structural models: An introduction to the theory of directed graphs*, Wiley, New York, 1965.

[34] Hegde S.M., Additively graceful graphs, *Nat. Acad. Sci. Letters*, **12**(11) (1989), 387–390.

[35] Hegde S.M., Labeled graphs and Digraphs: Theory and Applications, 12-01-2012 Research Promotion Workshop on Introduction to graph and geometric algorithms (IGGA).

[36] Jensen J.B. and Gutin G., *Digraphs: Theory, algorithms and applications*, Springer-Verlag, London, 2002.

[37] Koh K.M., Rogers D.G., Lee P.Y. and Toh C.W., On graceful graphs V:unions of graphs with one vertex in common, *Nanta Math.*, **12** (1979), 133–136.

[38] Kotzig A., On certain vertex-valuations of finite graphs, *Util. Math.*, **4** (1973), 261–290.

[39] Kotzig A., β-valuations of quadratic graphs with isomorphic components, *Util. Math.*, **7** (1975), 263–279.

[40] Langford C.D., Problem, *Math. Gaz.*, **42** (1958), pp.228.

[41] Lee S.M. and Shee S.C., On Skolem graceful graphs, *Discrete Math.*, **93** (1991), 195–200.

[42] Nabil Shalaby, *Skolem sequences*, The CRC Hand Book of Combinatorial Designs, (Eds.: Charles J.C. and Jeffery H.D.) 1996, 457–461.

[43] O'Keefe E.S. , Verification of a conjecture of Th. Skolem, *Math. Scand.*, **9** (1961), 80–82.

[44] Postpischil E. and Gilbert P., There are no new homometric Golomb ruler pairs with 12 marks or less, *Experiment. Math.*, **3**(2) (1994), 147–152.

[45] Priday C.J., On Langford's problem I, *Math. Gaz.*, **43** (1959), 250–253.

[46] Rao S.B., Acharya B.D., Singh T. and Acharya M., Hypergraceful complete graphs, *Australas. J. Combin.*, **48** (2010), 5–24.

[47] Rao S.B. and Sahoo U.K., Embeddings in eulerian graceful graphs, *Australas. J. Combin.*, **62** (2015), 128–139.

[48] Ringel G., *Problem 25*, In: Theory of graphs and its applications, Proceedings of symposium, Smolenice, 1963, Prague Publishing House of Czech. Acad. of Science, 1964, pp.162.

[49] Roberts F.S., *Structural analysis of energy system*, In: Energy: Mathematics and Models, SIAM, 1976, 84–101.

[50] Rosa A., *On certain valuations of the vertices of a graph*; In. Theory of Graphs, Internat. Symp., Rome, 1966, (Ed.: Rosentiehl P.), Gordon and Breach, New York and Dunod, Paris, 1967, 349–355.

[51] Sampathkumar E., Point signed and line signed graphs, *Nat. Acad. Sci. Letters*, **7**(3) (1984), 91–93.

[52] Simmons G.J., Synch-sets: A variant of difference sets, In: Proc. 5th S.E. Conf. Combin., Graph Theory and Computing, Winnipeg: *Util. Math.*, (1974), 625–645.

[53] Simpson J.E., Langford sequences: perfect and hooked, *Discrete Math.*, **44** (1983), 97–104.

[54] Singer J., A theorem in finite projective geometry and some applications to number theory, *Trans. Amer. Math. Soc.*, **43** (1938), 377–385.

[55] Singh T., *Advances in the theory of signed graphs*, Ph.D. Thesis, University of Delhi, Delhi, 2003.

[56] Singh T., Graceful sigraphs on C_3^k, *AKCE J. Graphs. Combin.*, **6**(1) (2009), 201–208.

[57] Singh T., *Hypergraceful graphs*, 2008, DST Project completion Rep. No.:SR/FTP/MS-01/2003, Department of Science and Technology, Govt. of India.

[58] Singh T. *Skolem and hooked Skolem graceful sigraphs*, Labelings of Discrete Structures and Applications (Eds.: Acharya B.D., Arumugam S. and Rosa A.), 2008, 155–164, Narosa Publishing House.

[59] Skolem T., On certain distributions of integers in pairs with given differences, *Math. Scand.*, **5** (1957), 57–68.

[60] White A.T. and Beineke L.W., *Topological graph theory*, In: Selected Topics in Graph Theory, (Eds.: Beineke L.W. and Wilson R.J.), Acad. Press, New York, 1978, 15–49.

[61] Zaslavsky T., A mathematical bibliography of signed and gain graphs and allied areas, *Electron. J. Combin.*, Dynamic survey, (2005), DS #6: 1–148.

[62] Zaslavsky T., Perpendicular dissection of space, *Discrete Comput. Geom.*, **27** (2002), 303–351.

[63] Zaslavsky T., Signed graphs, *Discrete Appl. Math.*, **4**(1) (1982), 47–74.

[64] Zaslavsky T., The geometry of root systems and signed graphs, *Amer. Math. Monthly*, **88**(2) (1981), 88–105.

[65] Zemanian A.H., Infinite electrical networks, *Proc. IEEE*, **64**(1) (1976), 6–17.

LIST OF PUBLICATIONS

LIST OF PUBLICATIONS COUNTED IN THESIS

1. Jessica Pereira, Singh T., Arumugam S., A new measure for gracefulness of graphs, *Electronic Notes in Discrete Mathematics* **48** (2015) 275–280.

2. Jessica Pereira, Singh T., Arumugam S., On (k, d)-Skolem graceful graphs, *Electronic Notes in Discrete Mathematics* **48** (2015) 81–88.

3. Jessica Pereira, Singh T., Arumugam S., m-Gracefulness of graphs, to appear in *Lecture Notes in Computer Science.*

LIST OF PAPERS PRESENTED IN CONFERENCES AND WORKSHOPS

1. Rao S. B., Singh T., Jessica Pereira, k-hypergraceful complete graphs, *XXI international conference on interdisciplinary mathematics, statistics and computational techniques*, organized by department of statistics, Punjab University, Chandigarh, India, Dec 15-17, 2012.

2. Singh T., Jessica Pereira, Additively graceful sigraphs, *XXI international conference on interdisciplinary mathematics, statistics and computational techniques*, organized by department of statistics, Punjab University, Chandigarh, India, Dec 15-17, 2012.

3. Jessica Pereira, Singh T., Arumugam S., A new measure for gracefulness of graphs, *Eighth international workshop on graph labelings-2014*, organized by n-CARDMATH, Kalasalingam University, Tamil Nadu, India, Dec 03-06, 2014.

4. Jessica Pereira, Singh T., Arumugam S., On (k, d)-Skolem graceful graphs, *Eighth international workshop on graph labelings-2014*, organized by n-CARDMATH, Kalasalingam University, Tamil Nadu, India, Dec 03-06, 2014.

5. Jessica Pereira, Singh T., Arumugam S., On (k, d)-hooked Skolem graceful graphs, *National conference on discrete mathematics and its applications*, organized by department of mathematics, Manonmaniam Sundaranar University, Tirunelveli, Tamil Nadu, India, July 15-17, 2015.

6. Jessica Pereira, Singh T., Arumugam S., m-Gracefulness of graphs, *International Conference on algorithms and discrete applied mathematics-2016*, organized by Department of Futures Studies, University of Kerala, Thiruvanthapuram, India, February 18-20, 2016.

BRIEF BIOGRAPHY OF CANDIDATE

FERNANDES JESSICA is currently serving as a Lecturer in the department of Mathematics, BITS Pilani K K Birla Goa Campus, Goa. She received her Bachelor's degree in Mathematics in 1999 from Goa University, Goa. She did her Masters degree in Mathematics in 2001 from Goa University, Goa for which she was awarded "Goa University Prize" for securing the highest marks. She received her Master of Philosophy in Mathematics in 2008 from Bharathiar University Coimbatore. She started her research career at Birla Institute of Technology and Science Pilani K K Birla Goa Campus in 2010. She has attended several international and national conferences and has presented papers. She is actively involved in research for the past 5 years. Two of her research papers have been published and one is accepted for publication. She is having more than eight years of teaching experience in BITS Pilani K K Birla Goa Campus, Goa.

BRIEF BIOGRAPHY OF SUPERVISOR

Dr. TARKESHWAR SINGH is an associate professor in the Mathematics Department, Birla Institute of Technology and Science Pilani K K Birla Goa Campus. He received his Ph.D. degree from University of Delhi, India. He has completed a project titled "Hypergraceful Graphs" funded by the Department of Science and Technology, Govt. of India. He is presently working on a project titled "Studies on Neighborhood Magic Graphs" which is also funded by the Department of Science and Technology, Govt. of India. He has published 20 research papers in national and international journals and 7 research articles in conference proceedings. He has attended 25 academic meetings and delivered invited talks. He has attended 15 national and international conferences and presented papers. He has guided 3 M.Phil. Students, 3 Masters Thesis and 30 Undergraduate Projects. He is presently guiding 3 Ph.D. students.