Studies in Graceful Labelings of Graphs and its Variations

A SYNOPSIS

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1 INTRODUCTION

By a graph $G = (V, E)$ we mean a finite undirected graph, with neither loops nor multiple edges, of order p and size q. For graph theoretic terminology, we refer to Chartrand and Lesniak [15].

Graph labeling is one of the fastest growing research areas within graph theory. New results are being discovered and published at a rapidly increasing rate. Further we have an enormous number of conjectures and open problems in graph labelings. For an excellent and up to date dynamic survey on graph labeling we refer to Gallian [19].

Most of the graph labeling methods trace their origin to the concept of β-valuation introduced by Rosa [33]. The same concept was introduced by Golomb [20], who called it a graceful labeling. Various other types of graph labelings such as harmonious labeling, magic labeling, antimagic labeling, prime labeling, additively graceful labeling, hypergraceful labeling, Skolem graceful labeling and cordial labeling have been investigated by several authors.

The concept of graph labeling has a wide range of applications to other branches of science such as conflict resolutions in social psychology, electrical circuit theory, energy crises, X-ray crystallography, coding theory, astronomy, communication networks design, cryptography and circuit design ([2, 6, 14, 16, 32, 40]). For such applied aspects of graph theory, one may consult specialized references such as Acharya ([3, 4, 6]), Balaban [11], Chartrand [14], Chen [16], Harary et al. [21], Jensen and Gutin [24] and Roberts [32]. A *sociogram*, as another instance, is a graph whose edges are labeled as being *positive* or *negative* according to whether the two interacting persons forming a given edge have a qualitatively *positive* or *negative* type of inter-personal relationship; such a network has been called a *signed graph* or simply, a *sigraph* in the literature ([2, 6, 12, 13, 21, 39]). In fact, *sigraphs* were first discovered by Harary [21] as appropriate prototype models to represent structures of cognitive inter-personal relationships in a social group. Ever since, *sigraphs* have received much attention in *social psychology* because of their extensive use in modeling a variety of cognition-based social processes ([1, 2, 4, 13, 18, 21]).

Informally, by a graph labeling we mean an assignment of numbers to graph elements such as vertices or edges or both subject to some conditions. The labeling is called vertex labeling or an edge labeling or a total labeling according as the domain of the labeling is V or E or $V \cup E$ respectively. Further conditions to be satisfied by any such labeling are normally expressed on the basis of some values (weights) of an evaluating function. One situation is the labeling of vertices, using distinct elements from the set $\{0, 1, \ldots, q\}$, in such a way that for every edge in G, the induced edge labels, defined by the absolute difference of its vertex labels, are distinct. Such a labeling is called a graceful labeling. Another situation is to assign labels to the vertices of G from the set $\{0, 1, \ldots, \lceil \frac{(q+1)}{2} \rceil\}$, such that the edge induced labels, defined by the sum of its vertex labels, are all distinct and from the set $\{1, 2, \ldots, q\}$. A graph which admits such a labeling is called an additively graceful graph.

The notion of graceful labeling can be extended to sigraphs. A sigraph S on p vertices, m positive and n negative edges denoted by $S(p, m, n)$ is said to be graceful if there exists an injective function $f : V \to \{0, 1, \ldots, q\}$ such that the induced edge labeling g_f defined by $g_f(uv) = s(uv)|f(u) - f(v)|$ gives $g_f(E^+) = \{1, 2, \ldots, |E^+|\}$ and $g_f(E^-) = \{-1, -2, \ldots, -|E^-|\}$, where $s : E \to \{+, -\}$ is the function which assigns a sign $+$ or $-$ to each edge, E^+ and E^- denote the set of all positive and negative edges of S respectively. The notion of graceful labeling of sigraphs is given by Acharya and Singh [8]. In this study we concentrate mainly on graceful labelings and its variations.

Hypergraceful decomposition of graphs was first introduced by Acharya [5]. A (p, q)-graph $G = (V, E)$ is said to be *k-hypergraceful* if there exists a decomposition of G into *edge induced subgraphs* G_1, G_2, $\ldots$, G_k having sizes m_1, m_2, $\ldots$, m_k respectively, and an injective labeling $f : V(G) \to \{0, 1, \ldots, q\}$, such that when each edge $uv \in E(G)$ is assigned the absolute difference $|f(u) - f(v)|$, the set of integers received by the edges of G_i is precisely $\{1,\ 2,\ \ldots,\ m_i\}$ for each $i \in \{1,\ 2,\ \ldots,\ k\}$. The decomposition $\{G_i\}$, if it exists, is then called a *hypergraceful decomposition* of G and f is called a k-

hypergraceful labeling of G. Further, G is said to be *hypergraceful* if it possesses a hypergraceful decomposition. When $k = 1$, the above definition yields the well known notion of graceful graphs and $k = 2$ corresponds to the extension of the notion of graceful graphs to the realm of sigraphs as studied in ([8–10, 36]). Characterization of k-hypergraceful complete graphs for $k = 2$ and some partial results for $k \geq 3$ are obtained by Rao et al. [30].

While studying the structure of Steiner triple systems, Skolem [38] considered the following problem: *Is it possible to distribute the numbers* $1, 2, \ldots, 2p$ *into* p *pairs* (a_i, b_i) *such that we have* $b_i - a_i = i$ *for* $i = 1, 2, \ldots, p$*?* In the sequel, a set of pairs of this kind is called 1, +1 system because the difference $b_i - a_i$ begins with 1 and increases by 1 when i increases by 1. Skolem [38] proved that a 1, +1 system exists if and only if $p \equiv 0$ or $1 (\mathrm{mod}\ 4)$. A 1, +1 system is also known as Skolem sequence, which is defined as follows: Let $< C_i >$ be a sequence of $2n$ terms, where $1 \leq C_i \leq n$. If each number i occurs exactly twice in the sequence and $|j_2 - j_1| = i$ if $i = C_{j_1} = C_{j_2}$ then $< C_i >$ is called a Skolem sequence. This concept was used by Lee and Shee [26] to introduce the notion of Skolem gracefulness of graphs. A *Skolem graceful labeling* of a graph $G = (V, E)$ is a bijection $f : V \to \{1, 2, \ldots, p\}$ such that the induced labeling $g_f : E \to \{1, 2, \ldots, q\}$ defined by $g_f(uv) = |f(u) - f(v)|\ \forall uv \in E$, is also a bijection. If such a labeling exists, then the graph G is called a *Skolem graceful graph*. If a graph G with p vertices and q edges is graceful, then $q \geq p - 1$, while if it is Skolem graceful, then $q \leq p - 1$. Thus, as noted in [26], Skolem graceful labelings nearly complement graceful labelings, and a graph with $q = p - 1$ is graceful if and only if it is Skolem graceful. A sequence $(a_1, a_2, \ldots, a_n)$ of integers $1, 2, \ldots, n$ is called a $(2, n)$ *Langford sequence* if for a_i appearing first at the i^{th} place, the next appearance of a_i is at $(a_i + i + 1)^{th}$ place [25]. For example the $(2, 3)$ Langford sequence is $(3, 1, 2, 1, 3, 2)$ and the $(2, 4)$ Langford sequence is $(4, 1, 3, 1, 2, 4, 3, 2)$. Priday [29] and Davies [17] have proved that a $(2, n)$ Langford sequence exists if and only if $n \equiv 0$ or $3(\bmod 4)$. Priday [29] and Davies [17] also introduced the concept of a perfect sequence. A sequence of m consecutive positive integers $\{d, d+1, \ldots, d+m-1\}$ is said to be *perfect* if the integers $\{1, 2, \ldots, 2m\}$ can be arranged into disjoint pairs $\{(a_i, b_i) : 1 \leq i \leq m\}$ so that $\{b_i - a_i : 1 \leq i \leq m\} = \{d, d+1, \ldots, d+m-1\}$.

Skolem [38] proved that a Skolem sequence does not exist if $n \equiv 2$ or $3(\text{mod } 4)$. O'Keefe [28] extended the concept of Skolem sequence for $n \equiv 2$ or $3(\text{mod } 4)$ by showing that the numbers $1, 2, \ldots, 2n-1, 2n+1$ can be distributed into n disjoint pairs (a_i, b_i) such that $b_i = a_i + i$ for $i = 1, 2, \ldots, n$. Motivated by this, Shalaby [27] defined the notion of hooked Skolem sequences.

A hooked Skolem graceful graph [37] is defined as follows: A (p, q) graph $G = (V, E)$ is said to be *hooked Skolem graceful* if there exists a bijection $f : V(G) \to \{1, 2, \ldots, p-1, p+1\}$ such that the induced edge labeling $g_f : E \to \{1, 2, 3, \ldots, q\}$ defined by $g_f(uv) = |f(u) - f(v)|, \forall uv \in E$ is also bijective. Such a labeling f is called *hooked Skolem graceful labeling* of G.

Bloom and Golomb considered two interesting and significant problems.

1. Find largest graceful subgraph of the complete graph.

   This led to the limitation of the Design of a Communication Network.

2. Increase the maximum vertex label so that the induced edge labels are distinct.

   This resulted in finitely many counter examples to a "theorem" of S. Picard which was relied upon (erroneously) for some 35 years in the field of X-ray diffraction crystallography (cf.: [23]).

The second problem led to the concept of gracefulness of graph which is defined as follows: The *gracefulness* $grac(G)$ of a graph G with $V(G) = \{v_1, v_2, \ldots, v_p\}$ and without isolated vertices is defined as the smallest positive integer k for which it is possible to label the vertices of G with distinct elements from the set $\{0, 1, \ldots, k\}$ in such a way that edges receive distinct labels.

Hegde [22] introduced the notion of additively graceful graph as follows: A (p, q) graph $G = (V, E)$ with $q \geq 1$ and $p \geq 2$ is said to be *additively graceful* if it admits a labeling $f : V \to \{0, 1, \ldots, \lceil \frac{(q+1)}{2} \rceil\}$ such that the edge induced labels defined by $f^+(uv) = f(u) + f(v)$ are all distinct, and $f^+ : E \to \{1, 2, \ldots, q\}$ is a bijection. He has characterized some additively graceful graphs.

Acharya et al. [7] have considered the following problem:

Given a graph G, is it possible to embed G as an induced subgraph of a graceful graph H having a graph theoretic property $\mathcal{P}$? In [7], they have answered the problem for triangle-free graphs, planar graphs, hamiltonian graphs and trees. Rao and Sahoo [31] have proved that any graph can be embedded in a graceful eulerian graph. However, in their proof the number of vertices in the eulerian graceful graph is $O(3^p)$.

2 ORGANISATION OF THE THESIS

The thesis consists of 8 chapters.

In Chapter 1, we give some basic definitions and theorems on graphs which are needed for the subsequent chapters.

Synch-set codes (designed by Simmons [34]) are used to synchronize the relative annular positions of a photo-detector on one side of a rotating disk with a stationary target light source on the other side. A $S(p, \lambda)$-synch set is defined as a set of p distinct nonnegative integers for which no more than λ pairs have the same common difference and for which the maximum element is as small as possible. A synch-set designates positions for the p holes so that distance from the first to the last hole is minimized. Hence a synch-set represents a labeling of K_p with distinct positive integers such that the largest vertex label is minimized and an edge label is repeated at most λ times. For $\lambda = 1$, this is rephrasing of the Golomb ruler. For $\lambda = 2$, we get graceful labeling of complete signed graph ([30, 34]). In Chapter 2, we prove that the complete graph K_p is $(p-4)$-hypergraceful if and only if $p \geq 8$, $(p-3)$-hypergraceful for $p \geq 4$, $(p-2)$-hypergraceful for $p \geq 3$ and $(p-1)$-hypergraceful for $p \geq 2$. We also give all nonisomorphic 3-hypergraceful decompositions of K_5.

In Chapter 3, we define (k, d)-Skolem graceful graph as follows: A graph $G = (V, E)$ is said to be (k, d)-*Skolem graceful* if there exists a bijection $f : V(G) \to \{1, 2, \dots, p\}$ such that the induced edge labeling g_f defined by $g_f(uv) = |f(u) - f(v)|$, $\forall uv \in E$, is a bijection from E to $\{k, k+d, \dots, k+(q-1)d\}$, where k and d are positive integers. Such a labeling f is called (k, d)-*Skolem graceful labeling* of G. We present several basic results on (k, d)-Skolem graceful graphs and prove that nK_2 is $(2, 1)$-Skolem graceful if and only if $n \equiv 0$ or $3 \pmod 4$. We give necessary condition for a graph G to be (k, d)-Skolem graceful. We also prove that nK_2 is $(1, 2)$-Skolem graceful. Finally we close the chapter with the observation that a $(1, 1)$-Skolem graceful labeling of G is a Skolem graceful labeling. A Skolem graceful labeling of nK_2 gives the Skolem sequence, a $(2, 1)$-Skolem graceful labeling of nK_2 gives the $(2, n)$ Langford sequence and a $(k, 1)$-Skolem graceful labeling of nK_2 gives a perfect sequence.

In Chapter 4, we introduce the notion of (k, d)-hooked Skolem graceful graph as follows: A (p, q) graph $G = (V, E)$ is said to be (k, d)-*hooked Skolem graceful* if there exists a bijection

$f : V(G) \to \{1, 2, \ldots, p-1, p+1\}$ such that the induced edge labeling $g_f : E \to \{k, k+d, k+2d, \ldots, k+(q-1)d\}$ defined by $g_f(uv) = |f(u) - f(v)|$, $\forall uv \in E$ is also bijective, where k and d are positive integers. Such a labeling f is called *(k,d)-hooked Skolem graceful labeling* of G. We observe that when $k = d = 1$, this notion coincides with that of hooked Skolem graceful labeling of the graph G [37]. It follows from the definition that if a graph G is (k,d)-hooked Skolem graceful, then $q \leq p-1$. In this chapter, we give a necessary condition for a graph G to be (k,d)-hooked Skolem graceful. We also prove that nK_2 is $(2,1)$-hooked skolem graceful if and only if $n \equiv 1$ or $2 (\bmod\ 4)$.

Motivated by gracefulness of graph, in Chapter 5, we define a new measure of gracefulness of graphs called m-gracefulness of graph as follows: Let $G = (V, E)$ be a (p,q) graph. Let $f : V(G) \to \mathbb{N} \cup \{0\}$ be an injection such that the edge induced function g_f defined on E by $g_f(uv) = |f(u) - f(v)|$ is also injective. Let $c(f) = \max\{i : 1, 2, \ldots, i \text{ are edge labels under } f\}$. Let $m(G) = \max_f c(f)$, where the maximum is taken over all f. Then $m(G)$ is called the *m-gracefulness* of G. This new measure $m(G)$ determines how close G is to being graceful. Note that if G is a graceful graph, $m(G) = q$ and $grac(G) = q$. One may observe that $grac(G)$ measures gracefulness of the graph G from above q, whereas $m(G)$ measures gracefulness of G from below q. In this chapter we prove that there are infinitely many nongraceful graphs G with $m(G) = q-1$, we determine $m(G)$ for a few families of non-graceful graphs, in particular we find $m(C_n)$ for $n \equiv 1$ or $2 (\bmod\ 4)$ and $m(F_k)$ for $k \equiv 2$ or $3 (\bmod\ 4)$, where F_k

is the friendship graph with k triangles. We give necessary conditions for a (p, q)-eulerian graph and the complete graph K_p to have m-gracefulness $q-1$ and $q-2$. Using this, we prove that K_5 is the only complete graph to have m-gracefulness $q-1$. We also give an upper bound for the highest possible vertex label of K_p if $m(K_p) = q-2$. We prove that $m(K_6) = 13 = q-2$, which is also shown in optimal Golomb ruler.

In Chapter 6, we extend the notion of additively graceful graphs introduced by Hegde, to the realm of sigraphs as follows: Let $S = (V, E)$ be a (p, m, n)-sigraph with $E = E^+ \cup E^-$, Assume $|E^+| = m$ and $|E^-| = n$ where $m+n = q$. Let $f : V \rightarrow \{0, 1, ..., m + \lceil \frac{(n+1)}{2} \rceil\}$ be an injective mapping and let the induced edge function be defined as $g_{f^-}(uv) = f(u) + f(v)\ \forall\ uv \in E^-$ and $g_{f^+}(uv) = |f(u) - f(v)|\ \forall\ uv \in E^+$. If $g_{f^-}(uv) = \{1, 2, ..., n\}$ and $g_{f^+}(uv) = \{1, 2, ..., m\}$, then f is called an *additively graceful labeling* of S. The sigraph which admits such a labeling is called an *additively graceful sigraph*. One can easily see that when $n = 0$, f is a graceful labeling of S, and when $m = 0$, f is an additively graceful labeling of S. We give some necessary or sufficient conditions for a sigraph to be additively graceful. We also give a necessary and sufficient condition for K_4 to be additively graceful. We also obtain some necessary conditions for eulerian sigraphs, complete bipartite sigraphs and complete sigraphs to be additively graceful.

In Chapter 7, we obtain a more efficient embedding of a graph G of order p as an induced subgraph of an eulerian graceful graph H whose order is $O(p^2)$. We also consider the following problem for sigraphs: *Given a sigraph S and a graph theoretic property $\mathcal{P}$, is it possible to embed S in a graceful sigraph S_1 having the property $\mathcal{P}$?* We prove the existence of such an embedding where S_1 is eulerian, hamiltonian, planar or triangle-free. We also prove that every signed tree can be embedded in a graceful signed tree.

Chapter 8 gives a conclusion of the study carried out and a brief summary of areas and problems for further research.

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LIST OF PUBLICATIONS

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