Chapter 5 Underlying Dynamics of Crime Transmission with Memory

5.1 Introduction

Crime can occur in various forms like homicide and assault, which is prevalent in most of the third world countries, especially in India, due to the rapidly growing population and high growth rate in towns and cities. Crime can be defined as a deleterious act towards the society against the wish of the residing government territory and which, upon conviction, is punishable by fine, imprisonment, and death. Most governments across the globe have proposed various methods and predictive methodologies to tackle crime primarily related to the crime committed towards discrimination against various gender, caste, creed, and religion. Most of the nations publicize data governed by bureaus such as the National crime records bureau in India. These records help judiciary and police departments to curb crime up to a certain extent. Even after these different methodologies and predictive analysis, the crime is on a higher side. Hence numerous researchers had proposed various mathematical models to help tackle crime.

There is a high correlation between crime and the economy of a state, which could impact the GDP, stock markets, which in turn leads to a crisis in economy and unemployment. (Mittal *et al.* 2019) had proposed indicators related to economy and crime in India.

A detailed understanding of current statistics and its correlation with spatial and temporal factors would benefit in the appropriate utilization of resources. Various studies suggest different mathematical models of integer order differential equations predict crime. But these models do not inherit non-local property, which depicts behavior changes due to contact with criminals for a long period. To overcome this, a Fractional order mathematical model of crime transmission is proposed in this study. The proposed model considers the previous effects of the input while predicting the crime growth rate. A mathematical model of crime transmission inherited with memory property is proposed in this study to analyze crime congestion. Abstract compartmental parameters of fractional crime transmission equation, which illustrates various stages of criminal activity, were employed to analyze crime contagion in the society. The present study demonstrates the progression of the flow of population by classifying into three systems based on involvement in crime and imprisonment by considering the criminal history of an individual. Subsequently, the equilibria of the three-dimensional fractional crime transmission model are evaluated using phase-plane analysis. The Lyapunov function is employed to determine stability and threshold conditions to achieve a crime-free society.

The rise in crime hurts the employment and economy of any nation (Mittal *et al.* 2019). A criminally active individual's contact may impact the behavior of others adversely. Therefore, the future state is highly correlated to the criminal history of an individual in the transmission phase. Several mathematical models were introduced to eradicate crime and enhance the proper functioning of jurisdictional institutions. All previous studies are formulated based on integer-order compartmental crime models as their preliminary investigation. Hence there is a great need to introduce a fractional order crime transmission mathematical model to overcome the memory-less property in conventional crime spread models to reflect the impact of memory in criminal transmissions. The integer-order models of crime spread generally are either of ordinary differential equations or partial differential equations. The communication of illegal activities from one criminal to another individual can be reflected through a fractional order crime transmission model, which is proposed in this study and can be further utilized in dynamical processes. A mathematical crime transmission model that is integrated with memory inheritance helps to combat the actual crimes in society and formulate policies in the current justice system. This fractional model distinguishes from other crime spread models by the indulgence of non-local property. Various stages of criminal activities are considered to model crime behavior. Further, it is worth noting that fractional order derivatives have physical engineering processes superiority to that of integer-orders (Sardar *et al.* 2015).

The current investigation aims to provide a novel fractional order dynamical system of the spread of crime, incarceration, and recidivism by using abstract parameters for transitions. Further, differential operators of fractional order are hired to include memory property and fractional derivatives are utilized as memory index (Atangana and Gómez-Aguilar 2018). The procedure proposed in Dokoumetzidis et al. (2010) is adopted in the present study to fractionalize the crime differential system and to analyze a few dynamical aspects of the fractionalized system.

The considered fractional order model is three-dimensional, and the equilibrium points are determined by using the method of phase plane (McMillon *et al.* 2014) in the current study. Further, equilibrium points are classified into low and high crime equilibriums. The tipping point, which classifies low and high equilibrium points in the crime transmission fractional-model, is identified to be threshold value. The obtained equilibria points help to analyze the impact of criminal activity in the community by altering crucial parameters such as people's incarceration and the release of criminals. One of the critical conclusions of the current study is to show that a minute growth in the imprisonment rate tends to lower the spread of crime. Adoption of the proposed model by jurisdiction departments and allies may help to tackle crime transmission effectively. The fractional crime model is built on system models by balancing the compartmental models of retiring flux and incoming flux.

5.2 Literature Review

The application of system models on crime initiated with the path-breaking work by the Department of Law Enforcement of President's commission in the United States (Douglas 1967). This work led to the effective management of operating costs of various jurisdictions departments and made amendments in policies accordingly. Subsequently, advanced mathematical and statistical models were developed by operations researchers to control serious offenders discharged at different phases into the society (Blumstein and Larson 1969). These advanced models mainly focused on recidivism processes. Many of Blumstein's works concentrated on cost-effective mathematical models that help to eradicate crime through the strategic deployment of police forces. Further, the introduced model helped the policymakers to optimize the implementations of several police personals which reduce the adverse effects of crime (Blumstein 2020). In the subsequent decades, statistical and mathematical models based on careers of criminals integrated with punishments were introduced and analyzed (González-Parra *et al.* 2018; Monteiro 2020; Shen and Li 2020). A person's involvement in crime is investigated in the works of Blumstein along with his collaborators consequently (Blumstein *et al.* 1988; Barnett *et al.* 1989; Blumstein 2018). Different views on the correlation between the impact of imprisonment period and crime were raised over a while (Durlauf and Nagin 2011). Consequently, various amendments were followed in the policies in departments of prison and police using the above-mentioned studies. Long periods of incarceration drew criticism based on ethical grounds (Turner *et al.* 2018). Many studies and organizations also criticized the ineffective implementation of policies and crime involvement of policymakers and political involvement (Gehring *et al.* 2016).

One of the effective ways to eradicate crime is to sanction strategic plans through particular dissuasions (Nagin 2013; Chalfin and McCrary 2017). Other studies of Blumstein (Blumstein

2002; Blumstein 2017) conclude that the rise in violence and criminal possession among youths are due to drugs and various other harmful pharmaceutical consumption for the drastic spread of crime in various parts of the world. Freedom and colleagues (Freeman *et al.* 1996) studied the congestion of crime for limited surroundings along with monetary involvement. Wang et al. 2005 analyzed crime equilibrium by evaluating the asymptotic stability of crime.

The game-theoretical approach was also utilized to model crime by considering different paradigms (Short *et al.* 2010; Segovia and Smith-Miles 2018). The game of snowdrift was used to impose levy fines on offenders (Jiang *et al.* 2013). The simulations obtained from Monte Carlo and its variants helped in the identification of cost detecting crimes (Perc *et al.* 2013). A theoretical model based on the evolutionary game approach helped to evaluate the outcomes of long term exposure to imprisonments and recidivists (Berenji *et al.* 2014).

The game-theoretic approaches discussed in the literature review were based on the population approach. The present study acknowledges the population-based approach of the spread of disease based on fractional order models. After the full success of fractional order dynamical models based on diseases, these models are widely adopted in the spread of influenza globally which helped to cure different diseases such as measles. The threshold spread of malaria which was modeled by Sir Richard Ross showed that sickness could be stopped without eliminating each mosquito in the contaminated area. The study assumes homogenous thresholds for evaluating the crime congestion for the fractional dynamical system.

5.3 Model Description of Crime Transmission

The present study demonstrates the progression of the flow of population by classifying into three systems based on involvement in crime and imprisonment by considering the criminal history of an individual. Subsequently, the framework of the fractional model is explained. Figure 5.1 depicts the fundamental components of the conventional three-stage crime transmission model through a flowchart of three criminal activity stages of an individual.

1. A: Criminally inactive individuals and non-offenders.
2. O: Persons indulged in crime but aren't imprisoned.
3. P: Prisoners

Each compartmental parameters of the fractional crime equation are given below:

ν: The conversion rate of an individual from criminally inactive state 'A' to criminally active state 'O.' This can be interpreted as the initial involvement rate in crime. This rate is obtained due to the contact and influence of criminally active individuals. The conversion rate is analogously referred to as "contact rate" in the field of mathematical biology. This conversion rate is directly proportional to the individual's contact to criminally active people, which are obtained from various theories of criminology based on the design that patterns social communion's effects on active involvement in the crime. Some of the theories, such as theories of social learning and labeling acts, help to understand these patterns. The method of social learning interprets social communications to be a significant factor in acquiring knowledge related to crime transmission. Fabricating identities through interactions are emphasized in labeling theory. Criminally active teenagers or youth population influenced by criminally active surroundings are discussed in subcultural theory.

η: This rate can be interpreted as the recovery rate where individuals convalesce to criminally inactive state 'A' from criminally active state 'O.' The imprisonment of an individual is not involved in this stage. The positive rate of η can be achieved through rehabilitation or the impact of general deterrence. Socially responsible organization or individual involvements help to reform criminals who are not incarcerated lead a healthy life.

λ: This rate is entirely dependent on jurisdiction and police systems. This can be referred to as the incarcerated rate of criminals from state 'O' to state 'P' where individuals committed crimes are imprisoned.

ζ: The release rate of prisoners from the imprisoned state 'P' to criminally active state 'O' but not incarcerated. This can be interpreted as prisoners either been escaped from prison or released by the jurisdiction's order and continue to commit the crime.

μ: The rate where prisoners released from imprisonment and lead a criminal free life which can be defined as the rate where individuals return to the criminally inactive state 'A' from incarcerated state 'P'. Blumstein's (Blumstein *et al.* 1988; Blumstein 2002; Blumstein 2017) works established this state of being redeemed.

The proposed fractional model can be depicted in the two-dimensional system, which is discussed in subsequent sections. The system dynamics of the crime transmission model assumes a closed population (constant population) of humans. One dimensional compartment is reduced using the relation of conservation for three-dimensional crime models by substituting '$K - O - P$' in place of 'A'. A flow-diagram of the three-stage crime transmission model is presented in Figure 5.1 which depicts the fundamental components of the conventional three-stage crime transmission model through a flowchart of three criminal activity stages of an individual.

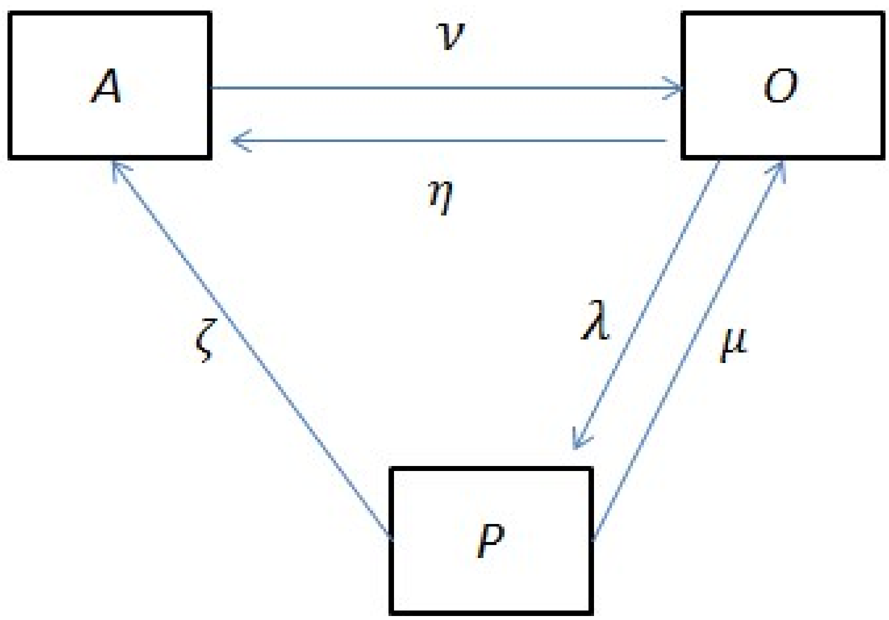

Figure 5.1. Flow-diagram of three-stage crime transmission model

The three-stage crime transmission model is shown in (5.1)

$$\dot{A} = \eta O - \nu \frac{AO}{K-P} + \zeta P \tag{5.1a}$$

$$\dot{O} = -\eta O + \nu \frac{AO}{K-P} - \lambda O + \mu P \tag{5.1b}$$

$$\dot{P} = \lambda O - (\zeta + \mu) P \tag{5.1c}$$

$$A + O + P = K \tag{5.1d}$$

(5.1) which represents a three dimensional crime transmission model that can be reduced to two dimensional model by replacing the parameters of non-offenders i.e. $'A'$ with other parameters in (5.1d) for visual depiction in 2D graph. Considering three cases, by varying slopes obtained by the isoclines of the parameters persons indulged in crime but aren't imprisoned $'O'$ and prisoners 'P' (both who committed a crime) possess one steady-state for crime free equilibrium. Graphs of different cases of three-stage crime transmission model where the isoclines of $\dot{O} = 0$ and $\dot{P} = 0$

are presented in Figure 5.2 - Figure 5.4.

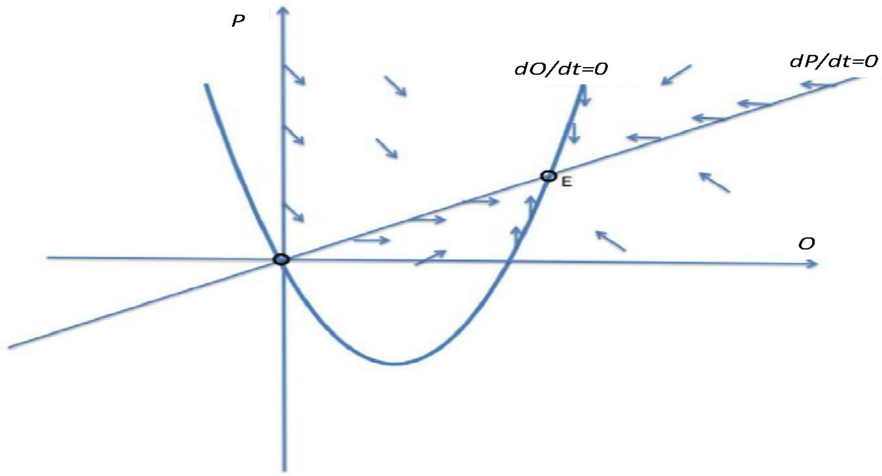

Figure 5.2. Case where the zeroes of $\dot{O} = 0$ isocline are non-negative

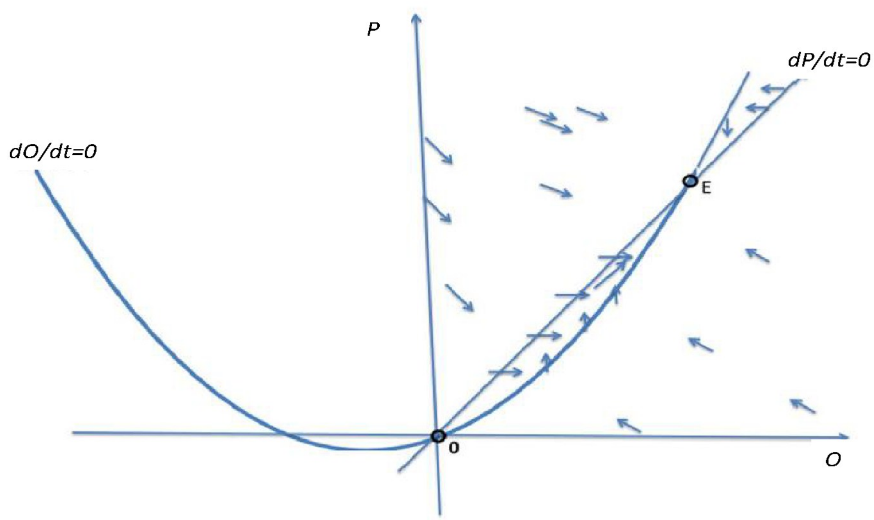

Figure 5.3. Case where the zeroes of $\dot{O} = 0$ isocline are negative and slope less than the slope of $\dot{P} = 0$ isocline

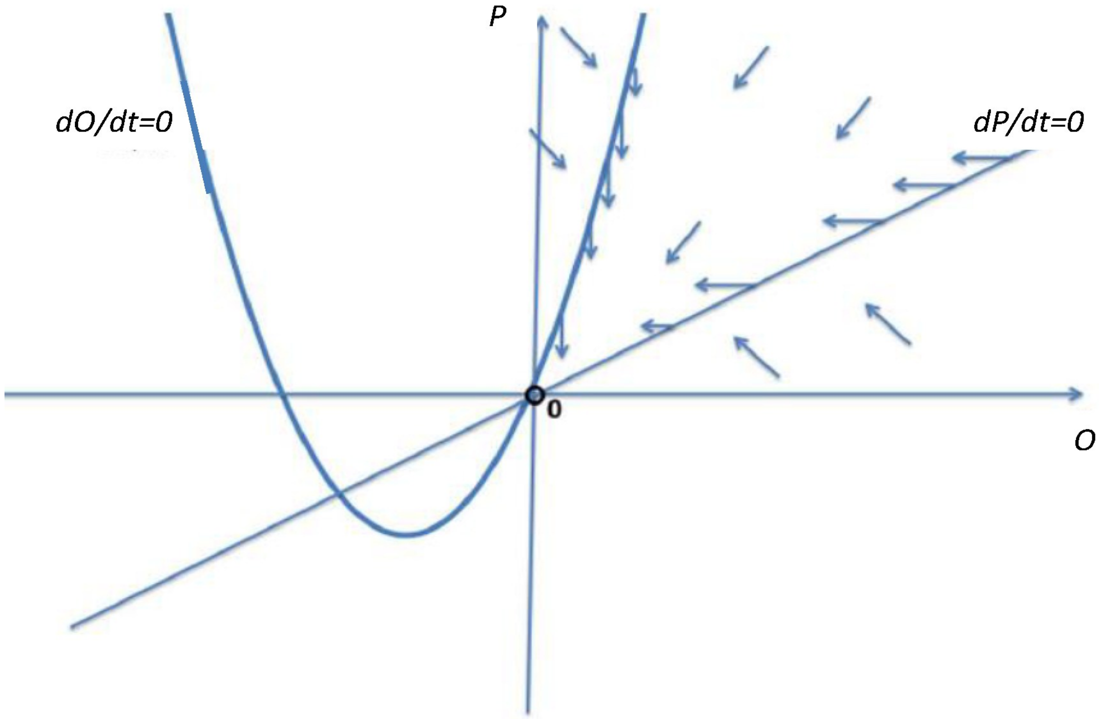

Figure 5.4. Case where the zeroes of $\dot{O} = 0$ isocline are negative and slope greater than slope of $\dot{P} = 0$ isocline

Further, a five-stage crime transmission model is presented in Figure 5.5, which includes two parameters in addition to the three-stage transmission model which are persons who commit crime even after been previously arrested O_2 and prisoners who are released R. These five parameter crime transmission models consider a high-level underlying perspective of crime transmission.

ν_1: The conversion rate of an individual from criminally inactive state 'A' to criminally active state 'O_1'. This can be an initial involvement rate in crime. This rate is obtained due to the contact and influence of criminally active individuals. This rate may also help with theories such as theories of social learning and labeling to understand patterns of criminal behavior. Fabricating identities through interactions are emphasized in labeling theory.

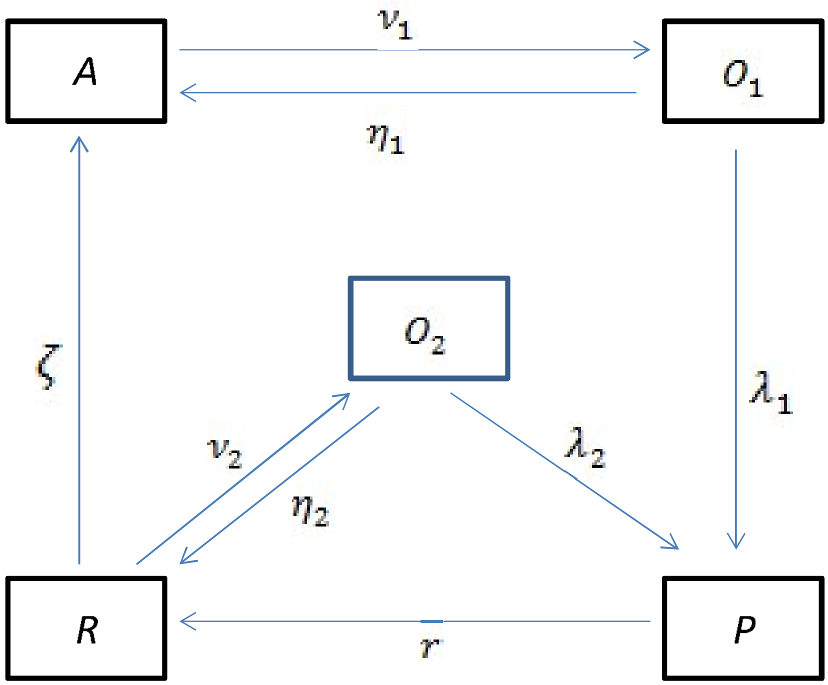

Figure 5.5. Flow-diagram of five stage crime transmission model

ν_2: The conversion rate of an individual from criminally inactive state 'A' to criminally active state 'O_2'. This repeated involvement rate in crime and serious offense to society.

η_1: This rate can be interpreted as the recovery rate where individuals convalesce to criminally inactive state 'A' from criminally active state 'O_1'. This rate is contrary to ν_1 and the imprisonment of an individual is not involved in this stage. The positive rate of η can be achieved through rehabilitation or the impact of general deterrence. Socially responsible organization or individual involvements help to reform criminals who are not incarcerated lead a healthy life.

η_2: This rate can be interpreted as the recovery rate where individuals convalesce to criminally inactive state 'A' from criminally active state 'O_2'. This rate is contrary to ν_2 and the imprisonment of an individual is involved in this stage.

λ_1: Alike λ in three parameters crime transmission model, this rate is entirely dependent on jurisdiction and police systems. This can be referred to as the incarcerated rate of criminals from state 'O_1' to state 'P' where individuals committed crimes are imprisoned.

λ_2: This is the incarcerated rate of criminals from state 'O_2' to state 'P' where individuals committed crimes are imprisoned for multiple times.

r: This is defined as the releasing rate of prisoners incarcerated in the state 'P' to the released state

'R'. This rate is also entirely dependent on jurisdiction and police systems.

With the aforementioned parameters for the five-stage crime transmission model, the mathematical model for the same is shown in (5.2) which also depicts repeated offenders to the society and much more complex to solve compared three-stage crime transmission model given in (5.1)

$$\dot{A} = \eta_1 O_1 - \nu_1 \frac{A(O_1+O_2)}{K-P} - \nu_1 A + \zeta R \tag{5.2a}$$

$$\dot{O} = -\eta_1 O_1 + \nu_1 \frac{A(O_1+O_2)}{K-P} + \nu_1 A - \lambda_1 O_1 \tag{5.2b}$$

$$\dot{P} = \lambda_1 O_1 + \lambda_2 O_2 - rP \tag{5.2c}$$

$$\dot{R} = rP + \eta_2 O_2 - \nu_2 \frac{R(O_1+O_2)}{K-P} - \nu_2 R - \zeta R \tag{5.2d}$$

$$\dot{O}_2 = \nu_2 \frac{R(O_1+O_2)}{K-P} + \nu_2 R - \lambda_2 O_2 - \eta_2 O_2 \tag{5.2e}$$

$$A + O_1 + O_2 + P = K \tag{5.2f}$$

Unlike the three-stage crime transmission model, five-stage crime transmission models do not possess a crime-free equilibrium due to the conversion of law-abiding citizens to serious offenders and with the extra parameter consideration in the five-stage crime transmission model which is recommitting of crime or offense after the release of criminals from prisons. This five-stage crime transmission model is much practical in comparison to three-stage crime transmission models. (5.2) relaxes the condition that the prisoners cannot commit crime after prison release. Further, this stage models consider the fact that some law-abiding citizens can commit crime even without the contact of criminals with the addition of $-\nu_1 A$. Subsequently, the five-stage crime transmission model differentiate repeated offenders O_2 with first time crime offenders O_1. The inclusion of new compartmental rates λ_1, λ_2 and ζ helps us to investigate the social criminal patterns across the world.

5.4 Impact of Length of Term in Prison on Crime

There is a fraction of population who are not imprisoned but are offenders $\frac{O_1+O_2+R}{K-P}$ and the remaining population who are not imprisoned and law-abiding citizens $\frac{A}{K-P}$ are not dependent on the prison release rate 'r'. Hence the length of the term of prison is $= \frac{1}{r}$. The values of equilibrium for (5.2) shows that all the parameters A, O_1, O_2, P, R depends r and hence implies that by increasing duration of prison length, the equilibrium grows and values of other parameters slash.

5.5 Fractional Order Crime Transmission Model

Unlike the crime spread models discussed in the literature review, the fractional crime transmission model helps to evaluate the systematic consequences of crime contamination, desistance and the rates of imprisonment. It should be noted that the current study is restricted to only three states defined earlier and doesn't differentiate other criminal actions such as a detainment, conviction and life imprisonment. Further, the frequency of the illegal activity of the offender is not comprehended. In future studies, more advanced models by considering more systems are intended to develop.

Caputo fractional order operators are hired in the present study due to its practical applicability (Sene 2020). Further, the dimensions of the entire population-based parameters A, O, P and K introduced fractional crime transmission model are in line with the convectional integer-order crime transmission model (McMillon *et al.* 2014; González-Parra *et al.* 2018). Consequently, the gaining flux and the leaving flux from one compartment to another (which can be interpreted as some fractional order rate) must not violate the balance of the population. Also, care is taken to fractionalize the conventional crime model to satisfy the power-law kind of system dynamics. Hence keeping all the necessary precautions, we propose a novel fractional order crime transmission model shown in (5.1)

For a given system of fractional order non-linear equation ${}_0^{\zeta}D_t^{\alpha}x(t) = f(t, x(t))$, where α is the fractional order in fractional crime transmission model, which lies in $(0,1]$, the equilibrium points of this system is obtained by evaluating $f\big(t, x(t)\big) = 0$.

5.6 Analyses of the Proposed Fractional Crime Transmission Model

The model shown in Figure 5.6 is a fractional order 3-D model partitioned into three sections: non-offenders A indulged in crime/offenders O and the imprisoned population P.

(5.3a) depicts the change in the population of non-offenders. The term $\eta^{\alpha}\, O$ represents the flux from O into A. This flux is observed when some criminals stop indulging in crime, which can either because of any fear of law or because of deterrence.

The next term -$\nu^{\alpha}\frac{AO}{K-P}$ represents the change in the population of non-offenders due to their interaction with offenders. The term $\zeta^{\alpha}P$ displays the difference when prisoners after release join back in society. Here it has been assumed that non-offenders indulge in crimes only after interacting with criminals. This assumption is necessary for the existence of a crime-free equilibrium.

$${}_{0}^{C}D_{t}^{\alpha}(A) = \eta^{\alpha} O - \nu^{\alpha}\frac{AO}{K-P} + \zeta^{\alpha}P \qquad (5.3a)$$

$${}_{0}^{C}D_{t}^{\alpha}(O) = -\eta^{\alpha} O + \nu^{\alpha}\frac{AO}{K-P} - \lambda^{\alpha}O + \mu^{\alpha}P \qquad (5.3b)$$

$${}_{0}^{C}D_{t}^{\alpha}(P) = \lambda^{\alpha} O - (\zeta^{\alpha} + \mu^{\alpha})P \qquad (5.3c)$$

$$A + O + P = K \qquad (5.3d)$$

where α is the fractional order in fractional crime transmission model, which lies in $(0,1]$.

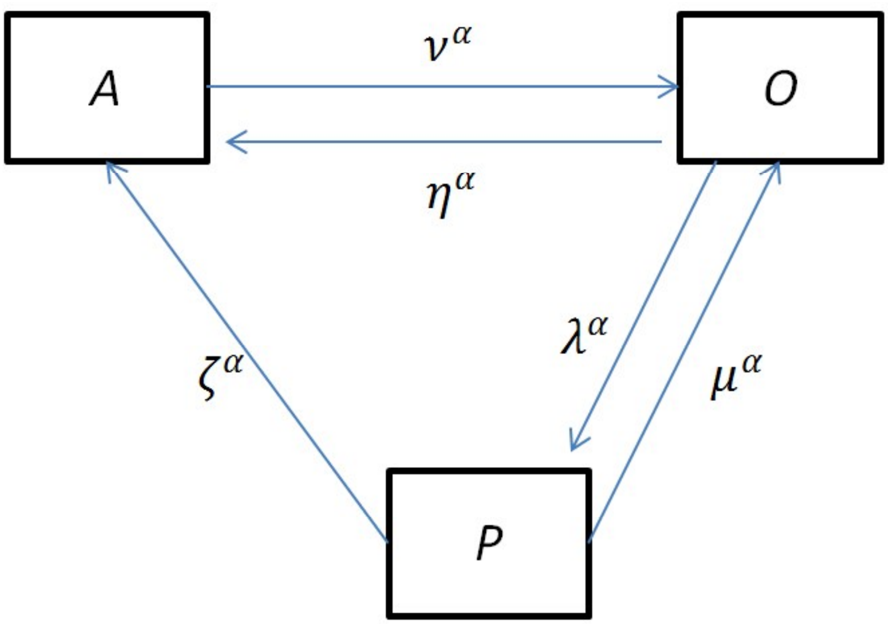

Figure 5.6. Flow diagram of a fractional crime transmission model

(5.3b) shows the change in the population of offenders O. The first term is due to flux out from O due to reformation, and the second term is for flux is due to the interaction of offenders with non-offenders, the third term arrives due to imprisonment of offenders and last term represents recidivism after release from prison.

(5.3c) describes the dynamics of the imprisoned population P.

(5.3d) though there exist many factors that influence the growth of the population, the population is assumed to be constant over time in the case of a negligible growth rate.

It can be seen that the above equations in the model are dimensionally consistent with the crime model given by the system (5.3). The model is well-posed since it satisfies the Lipschitz condition. There are three equations in the model, which can be reduced to the system of two equations. We replace A in (5.3a) by $K-O-P$ and then analyze the equilibrium of this obtained system of two variables O and P by the method of phase plane analysis given by (5.4).

$${}_0^CD_t^\alpha(O) = -\eta^\alpha O + \nu^\alpha \frac{(K-O-P)O}{(K-P)} + -\lambda^\alpha O + \mu^\alpha P \tag{5.4a}$$

$${}_0^CD_t^\alpha(P) = \lambda^\alpha O - (\zeta^\alpha + \mu^\alpha)P \tag{5.4b}$$

5.7 Phase Plane Analysis

This section illustrates the geometric analysis of the isoclines with different order $\alpha \in (0,1]$ obtained from the above 2-D system, for solving the equations and evaluating the point of crime-free equilibria. The crime-free equilibrium points are evaluated by solving ${}_0^CD_t^\alpha(O) = 0$ and ${}_0^CD_t^\alpha(P) = 0$ isoclines. From (5.4), it can be viewed that for any order α, ${}_0^CD_t^\alpha(P) = 0$ is a line passing through origin and ${}_0^CD_t^\alpha(O) = 0$ is a concave-up parabola with zeros at $O=0$ and $O = K\frac{\nu^\alpha-(\lambda^\alpha+\eta^\alpha)}{\nu^\alpha}$ in the OP plane.

We now consider three cases for the analysis, according to the relative positions of the isoclines.

Case 1: $\nu^\alpha - (\lambda^\alpha+\eta^\alpha) > 0$

In this case, the isoclines corresponding to a particular α intersect each other twice and both the intercepts of the isoclines ${}_0^CD_t^\alpha(O) = 0$ lie on the negative O-axis. As the order α increases, the slope of parabolic isoclines also increases. Each pair of isoclines is intersecting at origin and some other point in quadrant 1 of OP-plane as in Figure 5.7, and hence we get three non-trivial equilibrium points corresponding to three values of $\alpha = 0.25, 0.5, 0.75$. The same behavior is observed for different values of α. It is visible that the origin 0 is an unstable steady-state, and all the arrows move towards the other point. Hence that is the point of endemic equilibria E_α.

Case 2: $\nu^\alpha - (\lambda^\alpha+\eta^\alpha) < 0$ and $\frac{(\lambda^\alpha+\eta^\alpha)-\nu^\alpha}{\mu^\alpha} < \frac{\lambda^\alpha}{\zeta^\alpha+\mu^\alpha}$

In this case, non-trivial zero of ${}_0^CD_t^\alpha(O) = 0$ is negative, and its slope is less than the slope of ${}_0^CD_t^\alpha(P) = 0$ at the origin. Each pair of isoclines for any particular α is again intersecting at origin and some other point in quadrant 1 of OP-plane, as in Figure 5.8. But in this case, as the order α increases, the slope of parabolic isoclines decreases. The origin O is again an unstable steady-state, and the other intersection point is a globally stable point of endemic equilibria E_α.

Case 3: $\nu^\alpha - (\lambda^\alpha+\eta^\alpha) < 0$ and $\frac{(\lambda^\alpha+\eta^\alpha)-\nu^\alpha}{\mu^\alpha} > \frac{\lambda^\alpha}{\zeta^\alpha+\mu^\alpha}$

In this case, non-trivial zero of ${}_0^CD_t^\alpha(O) = 0$ is negative, and its slope is greater than the slope of

${}_{0}^{C}D_{t}^{\alpha}(P)=0$ at the origin. Each pair of isoclines for any particular α is intersecting at origin and some other point in quadrant 3 of OP-plane, as in Figure 5.9. It follows that there exists only one point for the analyses of the equilibrium, and it can be inferred from the figure that the origin is globally stable steady-state.

In conclusion, when the second O-intercept of ${}_{0}^{C}D_{t}^{\alpha}(O)=0$ then the isocline is negative, i.e. $\nu^{\alpha}-(\lambda^{\alpha}+\eta^{\alpha})<0$, and the slope of ${}_{0}^{C}D_{t}^{\alpha}(O)=0$. Further, it can be observed that the isocline is greater than the slope of ${}_{0}^{C}D_{t}^{\alpha}(P)=0$ at the origin, as in Figure 5.9,

$$\frac{(\lambda^{\alpha}+\eta^{\alpha})-\nu^{\alpha}}{\mu^{\alpha}}>\frac{\lambda^{\alpha}}{\zeta^{\alpha}+\mu^{\alpha}}\ ,\text{ or }\ \frac{\nu^{\alpha}}{\eta^{\alpha}+\lambda^{\alpha}\left(\frac{\zeta^{\alpha}}{\zeta^{\alpha}+\mu^{\alpha}}\right)}<1 \tag{5.5}$$

If the system of proposed equations satisfies condition (5.5) then it converges to the equilibrium point, which depicts a crime-free state. Or else, the system leads to an equilibrium with high-crime.

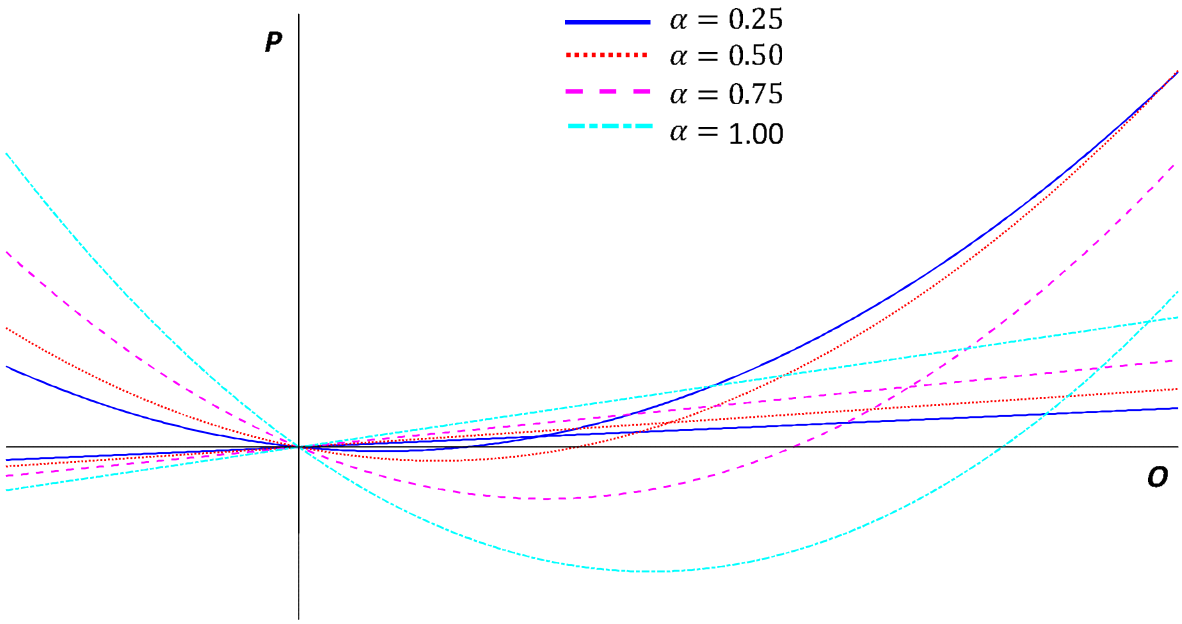

Figure 5.7. Fractional crime model of Case 1

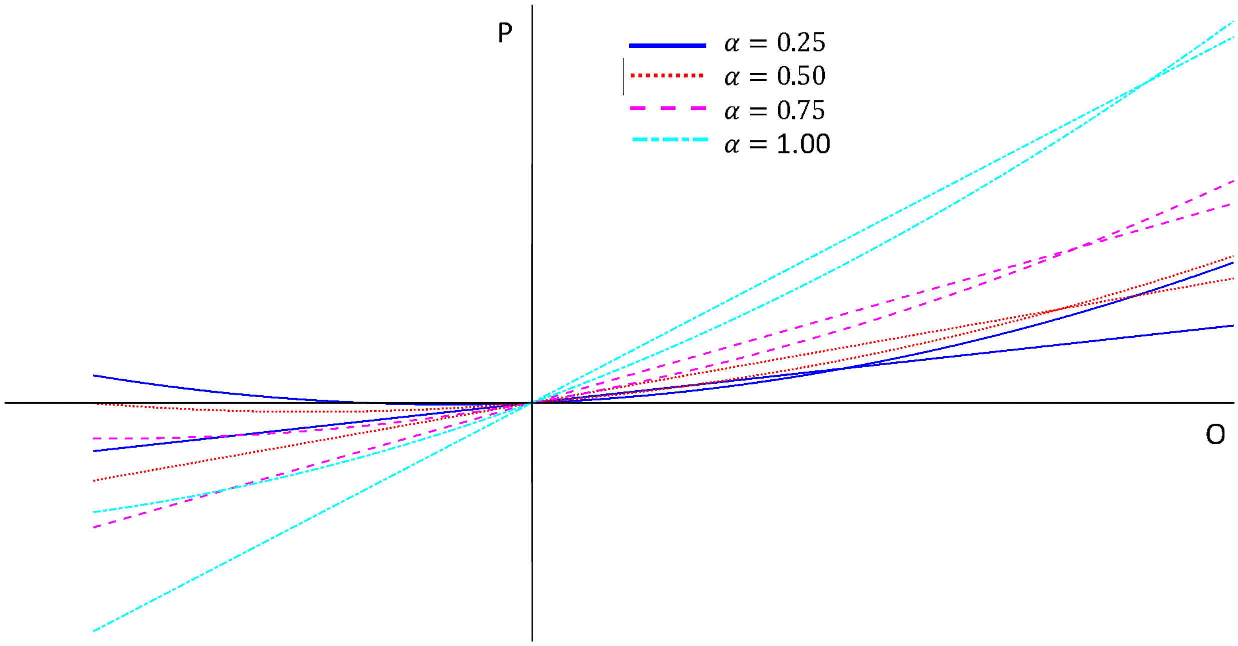

Figure 5.8. Fractional crime model of Case 2

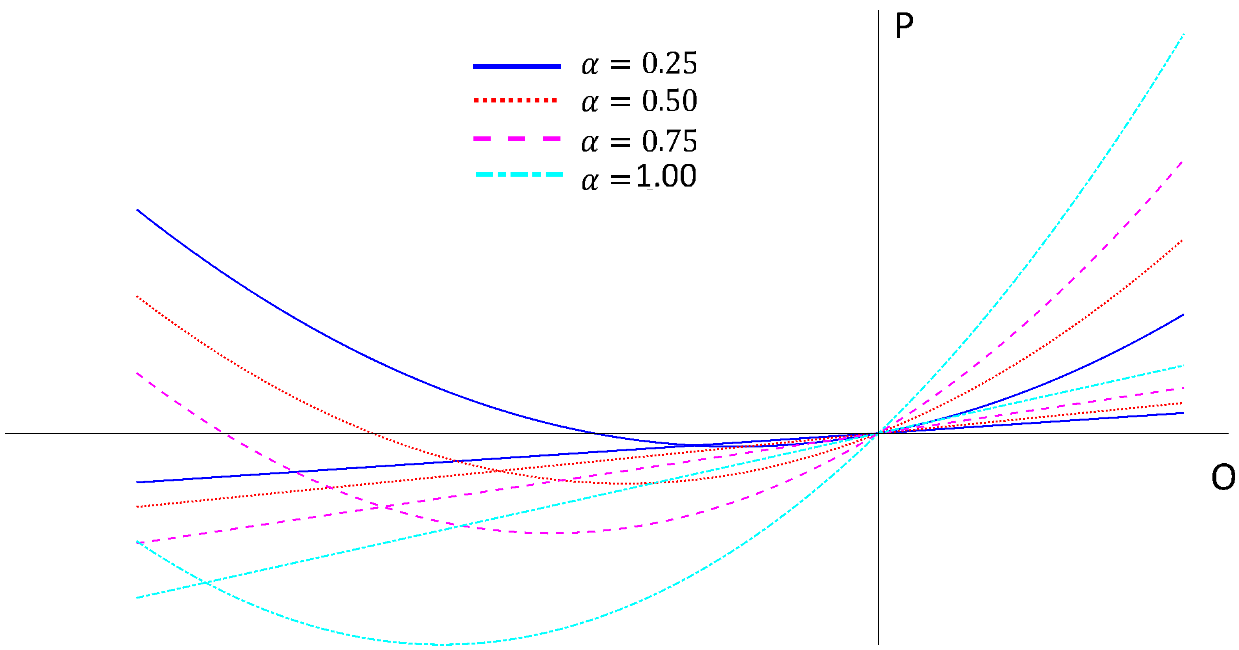

Figure 5.9. Fractional crime model of Case 3

Hence from Figure 5.7, Figure 5.8 and Figure 5.9, it can be inferred that the system attains stability for smaller values of α, which can be observed from intersections of pair of isoclines. For example, the intersection of pair of isoclines for $\alpha = 0.25$, attains stability faster than $\alpha = 1$ in all the cases, as evident from the above three Figure 5.7, Figure 5.8 and Figure 5.9.

5.7.1 Threshold Condition for the system

The inequality (5.3) acts as a borderline for the case in which the solution leads to the equilibrium point depicting crime-free state from the case in which they tend to the equilibrium with high-crime. The numerator of the fraction on the left side of the inequality represents the rate of indulging in a crime, and the denominator is the rate of cessation of a crime as the denominator is the sum of η^α (rate of withdrawal from a felony) and $\lambda^\alpha(\frac{\zeta^\alpha}{\zeta^\alpha+\mu^\alpha})$, which is the product of the term λ^α, the rate of imprisonment which contributes to the reduction of criminal activities, and the term $\frac{\zeta^\alpha}{\zeta^\alpha+\mu^\alpha}$ acts as a non-recidivism factor. So for crime-free equilibrium, the rate of involvement in a crime should be less than the rate of coming out of the class of criminals. So numerator in the inequality should be less than the denominator for crime-free equilibrium. Here we introduce the term basic reproduction number denoted by R_0 which interprets the number of persons an offender can influence and convince to indulge in criminal activities. Hence for smaller values of α, the threshold value is smaller. Hence it can be concluded that the fractional order system attains crime-free equilibrium faster in comparison to the integer-order crime transmission model. Therefore, the proposed fractional crime transmission model helps to achieve crime free society more swiftly than the existing models and will be useful to Jurisdiction systems upon adoption.

5.7.2 Endemic Equilibrium in system

For the system to be at equilibrium, ${}_0^C D_t^\alpha(O) = 0$ and ${}_0^C D_t^\alpha(P) = 0$, as from (5.4b) then we have $P = \frac{\lambda^\alpha}{\zeta^\alpha+\mu^\alpha} O$ and from (5.4a), we have

$$\frac{\nu^\alpha(K-O-P)}{(K-P)} - (\lambda^\alpha+\eta^\alpha) + \frac{\lambda^\alpha\,\mu^\alpha}{\zeta^\alpha+\mu^\alpha} = 0$$

$$\frac{\nu^\alpha(K-O-P)}{(K-P)} - (\eta^\alpha) - \lambda^\alpha\,\frac{\zeta^\alpha}{\zeta^\alpha+\mu^\alpha} = 0$$

Thus,

$$\frac{A}{K-P} = \frac{(K-O-P)}{(K-P)} = \frac{\eta^\alpha+\lambda^\alpha(\frac{\mu^\alpha}{\zeta^\alpha+\mu^\alpha})}{\nu^\alpha} \equiv \frac{1}{R_0} \tag{5.6}$$

The equilibrium prevalence $\frac{1}{R_0}$ depicted from (5.4) of non-offenders is just the ratio of the rate of withdrawal from criminal activities and the rate of indulging in illegal activities.

5.7.3 Crime-free stability of the Fractional order system using a Lyapunov function.

For the stability analysis of any fractional order non-linear system, applying the conventional Lyapunov method is ineffective (Aguila-Camacho *et al.* 2014; Xu *et al.* 2018). Hence an extension method for the Lyapunov function holds effective to evaluative stability for time-variant dynamical systems and most nonlinear fractional mathematical models. The following theorem can define this extension model.

Theorem 5.1 (Gallegos and Duarte-Mermoud 2016)**:** Given a fractional order dynamical system, the system is said to be stable if $|arg(eig(C))| > \alpha\frac{\pi}{2}$, where $eig(C)$ represents the eigen values of the matrix for fractional order crime transmission model.

Theorem 5.2 (Gallegos and Duarte-Mermoud 2016)**:** Let a fractional order non-autonomous dynamical system has an equilibrium point x_0 contained in domain $\varphi \subset \mathbb{R}^n$**.** Then for any function $U: [0, \infty) \times \varphi \rightarrow \mathbb{R}$ continuously differentiable along with the conditions $U_1(x) \leq U\big(t, x(t)\big) \leq U_2(x)$ and ${}_0^C D_t^\alpha(U\big(t, x(t)\big)) \leq -U_3(x)$, where $U_1(x)$, $U_2(x)$ *and* $U_3(x)$ are continuous positive definite functions on the domain φ for every $\alpha \in (0,1)$, the fractional non-autonomous dynamical system is uniformly asymptotically stable.

In the previous section, the fractional order 2-D model is analyzed to find a threshold value using the phase plane method. Further, the Lyapunov function, along with Theorem 5.1 is used to determine the uniformly asymptotical stability of our proposed fractional crime transmission model.

Let $U = O + X.P$, be Lyapunov function for our proposed fractional crime transmission model where X is a constant such that the solutions for that 2-D fractional system move to lower level sets, and in this way, solutions are expected to move towards the origin.

We seek for $X > 0$ such that the fractional order derivative of U along with solutions by (5.7)

$$ {}_0^C D_t^\alpha(U) = {}_0^C D_t^\alpha(U(O(t), P(t)) = {}_0^C D_t^\alpha(O + X.P) \tag{5.7}$$

As the fractional order derivative is linear, therefore

$${}_0^C D_t^\alpha(O+XP) = {}_0^C D_t^\alpha O(t) + X\, {}_0^C D_t^\alpha P(t)$$

$$= -\eta^\alpha O + \nu^\alpha \frac{(K-O-P)O}{(K-P)} + -\lambda^\alpha \mathrm{O} + \mu^\alpha P + X\lambda^\alpha O - \mathrm{X}(\zeta^\alpha + \mu^\alpha)P$$

$$= O(\underbrace{\nu^\alpha \frac{(K-O-P)}{(K-P)} - \eta^\alpha - \lambda^\alpha + X\lambda^\alpha}_{first}) + \mathrm{P}(\underbrace{\mu^\alpha - \mathrm{X}(\zeta^\alpha + \mu^\alpha)}_{second})$$

Now for ${}_0^C D_t^\alpha(U)$ to always be negative, the above *first* and *second* expressions must be negative. As $\nu^\alpha \frac{(K-O-P)}{(K-P)} - \eta^\alpha - \lambda^\alpha + X\lambda^\alpha < \nu^\alpha - \eta^\alpha - \lambda^\alpha + X\lambda^\alpha$ which is negative if and only if $X < \frac{\eta^\alpha + \lambda^\alpha - \nu^\alpha}{\lambda^\alpha}$ and $\mu^\alpha - \mathrm{X}(\zeta^\alpha + \mu^\alpha) < 0$ if and only if $\frac{\mu^\alpha}{\zeta^\alpha + \mu^\alpha} < X$.

Hence, $\frac{\mu^\alpha}{\zeta^\alpha + \mu^\alpha} < X < \frac{\eta^\alpha + \lambda^\alpha - \nu^\alpha}{\lambda^\alpha}$ and thus, $\frac{\mu^\alpha}{\zeta^\alpha + \mu^\alpha} < \frac{\eta^\alpha + \lambda^\alpha - \nu^\alpha}{\lambda^\alpha}$, which is the same threshold condition obtained from the phase plane method of analysis. Since our Lyapunov function is bounded for considered constant population and the Caputo fractional derivative of U is negative, we can conclude by Theorem 5.1 that our fractional crime transmission model is not only time productive but also uniformly asymptotic stable.

5.8 Summary of Model Analysis

The proposed fractional order crime transmission model has a unique high-crime equilibrium point. The main contribution of our proposed model is the threshold. It is observed in phase plane analysis, that the system had both low and high-crime equilibrium. The assumption of inducement of a person in crime due to mutual interaction between offenders and non-offenders cannot be relaxed. Also, the considered fractional order system is solved analytically using the Lyapunov function and the threshold R_0 acts as a differentiating curve between low and high-crime equilibrium. The threshold obtained from the proposed dynamic system is sophisticated to that of the threshold for the conventional model.

The inequality $\frac{\nu^\alpha}{\eta^\alpha + \lambda^\alpha\left(\frac{\zeta^\alpha}{\zeta^\alpha + \mu^\alpha}\right)} \leq 1$ is a sufficient and necessary condition for the global convergence of the proposed fractional order model to the crime-free equilibrium. Further, it should be noted that R_0 is the reproduction number as it depicts the number of non-offenders, an offender can influence and convince for engagement in crime. For $R_0 > 1$, the crime-free equilibrium is unstable and is convergent to unique endemic equilibrium. The fraction of offenders and non-offenders can be given by $\frac{A}{K-P} = \frac{1}{R_0}$ and $\frac{O}{K-P} = 1 - \frac{1}{R_0}$ respectively.

From the previous statement, it can be depicted that the reproduction number is directly proportional to the fraction of the population of offenders. By analyzing the proposed model, it is possible to reduce the crime prevailing in society by evaluating the necessary level of imprisonment.

$$\frac{\nu^\alpha - \eta^\alpha}{\left(\frac{\zeta^\alpha}{\zeta^\alpha + \mu^\alpha}\right)} \leq \lambda^\alpha$$

If $\nu^\alpha < \eta^\alpha$, or $\nu < \eta$, i.e., rate of desistance is more than the rate of indulgement in criminal activities, then there is no need for imprisonment. So if rehab programs for offenders are implemented successfully and are more influential to that of rate of involvement in crime, then crime continuously decreases using fractional order crime transmission model. This is due to the smaller threshold of the proposed fractional order model in comparison with the conventional crime transmission model.

5.9 Conclusions

A mathematical model of crime transmission inherited with memory property is proposed in this study to analyze crime congestion. It is observed that both high and low crime-equilibria exist in our proposed model by varying degrees in fractional orders operators and found a tipping curve that helps to differentiate low-crime equilibrium from high-crime equilibria with the help of a threshold point. With the indulgence of non-local property, people's involvement in illegal and criminal activity can be derived from the threshold value. The value of the threshold indicates that the numbers of malefactors are highly sensitive to desistance alterations compared to imprisonment, hence suggesting a change in jurisdiction policy changes. The crime-free equilibrium can be achieved much faster with a fractional crime transmission model dynamical system compared to the conventional crime transmission model. Further, it is found that among the three considered cases, only Case 2 and Case 3 converge to crime-free equilibrium. The current study shows that a minute growth in the imprisonment rate tends to lower the spread of crime. Further, our fractional crime transmission model is not only time productive but also uniformly asymptotic stable.

5.10 Future Work and Scope

An optimized value of the fractional order of the system can be calculated by using heuristic optimization algorithms such as Particle swarm optimization, Genetic algorithms, and Quantum optimization for faster convergence to low crime-equilibria for crime-free society. Further, the restriction of parameters associated with heterogeneity such as caste, gender, creed can be relaxed and can increase the number of stages in justice systems. Different higher dimensional fractional order models can be constructed past crime contagion for effective results.