Chapter 1 Introduction

1.1 Introduction

The formal introduction to calculus developed by Sir Issac Newton and Gottfried Wilhelm Leibniz dates back to the 17th century, even though traces of it can be found in India, Greek, China, and medieval Europe before 17th century. Since then, calculus becomes an integral part of many developments that are used in day to day life. Calculus has two subdivisions, integral and differential calculus, which are the mathematical study of integer order integration and differentiation of functions, respectively. In the same century, mathematicians like L'Hospital had thought about the half derivative of a function and wrote a letter to Leibniz about the same. Since then, many renowned mathematicians such as Riemann, Abel, and Liouville (Miller and Ross 1993; Podlubny 1999; Goyal and Mathur 2004; Goyal and Mathur 2006; Das *et al.* 2011) made notable contributions in the domain of fractional calculus.

Since the 20th century, theory and applications of fractional calculus are developing progressively. But only after the 1970s, the conferences particularly emphasized on fractional calculus held and one of the primary reasons is due to the works of B. Ross (Ross 1975; Miller and Ross 1993). The first conference on fractional calculus can be traced back to 1974, which was held at New Haven with proceedings edited in "First Conference on Fractional Calculus and its Applications" (Machado *et al.* 2011).

In the same year, the first book based entirely on fractional calculus "Theory and Applications of Differentiation and Integration to Arbitrary Order" was published by a chemist named Oldham along with a mathematician named Spanier applied in the area of heat and mass transfer and termed the present form of fractional operators, fractional integrals as semi-integrals and fractional derivatives as semi-derivatives which led to a new era for fractional calculus (Oldham and Spanier 1974). This collaborative work took nearly six years before the first publication on fractional calculus.

Later in the year 1987, a book entitled "Fractional Integrals and Derivatives" by Samko, Kilbas and Marichev were published in the Russian language which is considered to be the encyclopedia of fractional calculus and the book is translated and published in the English language in the year 1993 (Samko *et al.* 1993). Since then, a numerous publication in high reputed journals was published both in theory and wide range of applications to the fractional calculus and is further expected to grow exponentially in the coming years.

The term fractional calculus, which allows any integral or derivative of positive real order, is a misnomer and is only labeled due to its historical reasons. Fractional calculus deals with operators and equations of integrodifferential of convolution type, which manifests singular kernels of power-law type (Miller and Ross 1993). Further, this area of mathematical analysis is highly correlated with pseudo-differential operator theory, which consists of topics like integral transforms, special transcendental functions, control theory, and stochastic processes (Podlubny 1999). There is a vast scope for special functions in the area of fractional calculus, along with integral transforms. For example, special functions in the field of mathematical physics can be viewed as "generalized fractional integrals or derivatives" of three elementary functions (Kiryakova 2010; Qureshi *et al.* 2018). Further, transmutation operators can be interpreted as a generalized Laplace-type integral transform.

Fractional calculus was restricted to the development of the theoretical part for several decades, and the application aspect was untouched for a long time. But the scenario was changed and humongous research activities were carried over in the last few decades, and applications were extended to many diversified areas from physics to control systems, and finance to economics. Some of the major areas of application to fractional calculus include fractional system engineering of control theory. Fractional calculus is highly used in the field of control and signal processing, and there is a tremendous rise with respect to the number of publications in electrical engineering and applied mathematical journals since the celebrated works of Oustaloup that deals with fractional calculus applications in view of frequency responses (Oustaloup 2014; Li *et al.* 2020). Igor Podlubny makes another notable contribution in 1999 by emphasizing various other applications in automated control (Podlubny 1999). Oliver (1893) introduced fractional calculus applications in electromagnetic theory in 1920, and Machado et al. (2011) used fractional operators onto elasticity problems.

Other application of fractional calculus includes developments of optimal control of dynamical systems of fractional order along with calculus of variations, different varieties of engineering materials like tissues in humans and animals, polymers, gels along with scientific uses in engineering, fractional wave and diffusion study with applications in plasma physics, applications in biomedical and biotechnology, machinery tools such as brakes and modeling of thermal engineering systems, signal, and image processing (Ferdi 2012; Vosika *et al.* 2013; Moafi *et al.* 2016; Shah and Agashe 2016a; Yang *et al.* 2016; Prasad and Krushna 2017; Datsko *et al.* 2019; Li *et al.* 2020) which have shown grave improvement the application of fractional calculus in the last decade. One of the extensive areas of applications of fractional calculus is viscoelasticity, which has the advantage of memory property (Mahiuddin *et al.* 2020). Considering the number of research articles published in recent years, it can be concluded that fractional calculus allured the curiosity not only among engineers, mathematicians, physicists but also biologists, financial analysts and social scientists (Scalas 2000; Jeng and Huang 2015; Huang and Bae 2017; Harms and Stefanovits 2019).

Intending to contribute for a sustainable future, this study aims to explore the prospects of applying fractional calculus for sustainable development goals (SDG). Since there are multiple dimensions associated with SDGs, a short review is conducted at the primordium of this study to identify the crucial domains which need more emphasis on the current scenario.

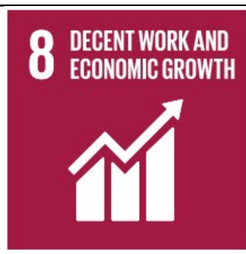
1.2 Sustainable Development Goals

SDGs become prior agenda for most of the countries, especially for emerging nations like India and China, after the world representatives met at United Nations in New York in September 2015 (Stafford-Smith *et al.* 2017). Seventeen goals and 169 targets were set out, which is to be achieved by or before the year 2030 with particular emphasis on the 17th goal, i.e., Partnerships for the Goals. Most of the goals and targets of SDG goals are interlinked. For example, Industry, Innovation and Infrastructure effects Climate Action, Decent Work, and Economics (Stafford-Smith *et al.* 2017).

Many renowned journals such as the "Lancet" criticized SDGs for the failures and overambitious projects. They described as "fairy tales, dressed in the bureaucratese of intergovernmental narcissism, adorned with the robes of multilateral paralysis, and poisoned by the acid of nation-state failure." Further, the same journal praised SDGs for the collective approval from UN nations on the proposed list of targets and goals. UN expressed its views on the success of SDG by saying that the goals can only be achieved by productive collaborations of government and non-government institutes within the country. SDGs can be viewed as an extension to previously proposed Millennium Development Goals (MDGs) with additional dimensions such as environmental, economic, and social parameters. These socio-economic parameters are further linked to time horizon where the short to medium time achievements of the wellbeing of humans shouldn't reflect long term achievements by destroying environmental and social aspects which the human and other living organisms depends on. For better understanding, a brief overview of SDGs is discussed in Table 1.1.

Table 1.1. Sustainable development goals and targets

 No Poverty	Eradicating Poverty in all forms, which includes cultural, political, socio-economic, and environmental dimensions of poverty in all nations, is one of the essential goals in SDG. Implementation of appropriate social protection systems is the most important of all the targets, which is to be achieved by the year 2030.
 Zero Hunger	The goal of zero hunger is multi-dimensional due to the inclusion of various parameters like nutrition and food security with consideration of the impact on the environment. This goal is one of the challenging tasks to achieve due to its coupling nature in biophysical processes with socio-economic processes. There is no evidence to tackle scale up food security successfully.
 Good Health and Well-being	Health lifestyle is being promoted with extra care offered to persons with disabilities around the globe and groups with various other disadvantages. Some of the privileges suggested for inclusive growth include subsided transport, free or subsidized education, employment, etc. One of the crucial targets includes eradication of epidemics such as HIV-AIDS, Malaria, Tuberculosis, and COVID-19.
 Quality Education	Free and equitable education to all, irrespective of any discrimination in primary and secondary school, is the main important target in this goal. Encourage students to pursue higher studies by providing scholarships to least developed, and developing countries is another target to be implemented by the year 2020.

 Gender Equality	Recognize and eliminate all forms of discrimination, especially to women and children, is on the prior agenda of this goal. Adopt various policies and frameworks to empower women to achieve sustainable growth of women are emphasized in one of the targets.
 Clean Water and Sanitation	Universal clean, sustainable, and affordable drinking water is to be achieved by the year 2030. Protecting nature and ecosystem is one of the targets which are to be completed by the year 2020. International support for building sanitation in developing countries by improving water efficiencies is also included in this goal.
 Affordable and Clean Energy	Increase reliability on sustainable energy, sharing research, and technology among the nation will improve the target of clean and affordable energy. One of the targets in this goal includes support to developing and least developed countries by providing infrastructural and technology for inclusive growth.
 Decent Work and Economic Growth	The target of seven percent in gross domestic growth rate in the least developed countries is to be achieved every year. Provide enhanced methodologies and frameworks for proper planning of financial assets to low- and middle-income families for the economic growth of countries. Promote universal decent work along with youth employment.
 Industry, Innovation and Infrastructure	This goal emphasizes on developing quality and reliable infrastructure to improve overall economic development and promote different types of enterprises and small-scale industries. Encourage research development by increasing the research workers per one million population along with the increment of research spending.

 Reduced Inequalities	One of the targets of these goals is to uplift the income of the bottom forty percent of the population at a higher pace than the nation's average by the year 2030. Active representation and involvement of developing countries in global financial and economic institutions is another motive of this goal. Introduce improved policies and framework related to social protection and wages to achieve equality
 Sustainable Cities and Communities	Improve amenities of slums and provide basic and affordable housing to all for the growth of sustainable cities. Improve the transport system by increasing public transportation. Introduce innovative frameworks that interlink urban, semi-urban, and rural areas, along with the protection of heritage sites.
 Responsible Consumption and Production	Ensure sustainable production and consumption by providing a 10-year framework to all nations with the lead of developed countries. Further, the effective utilization of natural resources is a must for sustainable consumption. Reduce, Recycle, and Reuse must be the priority to reduce waste by 2030. Develop sustainable tourism, which will create jobs by promoting local products.
 Climate Action	Improve adaptive capacity related to climate hazards and all forms of natural disasters in every nation. Special emphasis on national policies and planning should be given to climate change. Accumulative hundred billion dollars every year till 2020 to UN framework convention for climate change by developed countries to help developing countries for the green climate.

 Life Below Water	Reduce marine pollution of all forms, which are mainly caused due to land-based activities by the year 2025. Optimize ocean acidification. Regulate illegal fishing by the year 2020. A minimum of ten percent of the marine areas must be preserved as per national and international maritime laws. Small scale artisanal fishers must be provided with access to marine markets.
 Life on Land	Ensure sustainable utilization and conservation of terrestrial ecosystems giving more importance to forests, mountains, dry, and wetlands by international agreements by the year 2020. Action must be taken against desertification. Further, this SDG is highly correlated with other SDGs.
 Peace, Justice and Strong Institutions	This goal enhances the human well-being and inclusive and sustainable growth of society. This SDG signifies the effective tackle of drug utilization, various forms of abuse, law, and order, good governance, etc. The targets in this goal may profoundly influence economic growth and human development index. There is a great need to improve the existing policies and methodologies for sustainable growth and achieve peace and harmony in the society and nation.
 Partnerships for the Goals	Partnerships of goals are most crucial among all SDGs since it deals with the implementation and functioning of other SDGs. One of the essential targets of this goal is finance and technology and provides a new framework labeled “SMART” for the better assessment of MDGs.

Note: Imagery used in Table 1.1 are taken from https://unstats.un.org/sdgs/report/2019/

From Table 1.1, it can be inferred that 17 different goals, which were grouped under the umbrella of SDGs, have different dimensions, including technology, socio-economic and environmental aspects. Additionally, it is also noticed that there is a strong interdependency of each goal with the other. In this regard, a few of the goals which can trigger the other are identified, and due importance is given in this work. "Innovation Industries and Infrastructure," "Decent work and Economic growth," and "Justice and Strong Institutions" are the three different goals chosen for this doctoral work. At a macroscopic level, since various elements embodied the chosen goals, to address a problem to the root, a few of the specific aspects in each of the goals and are of current importance are selected and emphasized in this study. The best attempt has been made in the present research to systematically address the aforementioned research gaps through a mathematical decision framework to provide appropriate solutions in the best possible ways. Findings of the past research works conducted on these goals are presented in the following sections.

1.2.1 Industry, Innovation and Infrastructure and its impact on climate change

One of the utmost importance to sustainable development goals listed in 17 points is industrialization and is of grave significance to developing countries like India. Proportional Integral Derivative (PID) controllers are one of the crucial components in automatic control process based industries due to its regulatory properties, which can synchronize temperature, pressure, and various industrial process variables. PID controllers are used in most automatic process control applications in industry today to regulate flow, temperature, pressure, level, and many other industrial process variables. Further, these controllers act as a workhorse in many modern control systems to regulate various tasks automatically that otherwise would have done manually. The emissions from these industries impact climate, which is another sustainability goal. Control systems like PID controllers are one of the factors affecting the greenhouse environment management problem (Hu *et al.* 2014; Wang *et al.* 2015).

There is a great need to eliminate greenhouse emissions, which can be reduced by distributed energy resources, which has a tremendous impact on production in the recent decade. One of the reasons for significant fluctuations in renewable resources is climate change. Fractional order control systems controllers improve the efficiency of power systems and their working conditions. Many researchers proposed a wide variety of Fractional order control systems such as Fractional order Proportional Integral Derivative (FOPID), Fuzzy fractional PID controllers, etc. (Wang *et al.* 2015; Moafi *et al.* 2016; Shah and Agashe 2016a; Puchalski *et al.* 2020). These controllers have numerous advantages as compared to conventional PID controllers like effective analysis and simulation of dynamic behavior in the case of microgrids and enhanced generation distribution along with the ability of plug and play. Energy storage helps for a better dynamical response of control systems with the inclusion of fractional order controllers (McMillan 2012; Moafi *et al.* 2016).

Excessive temperature emitted from industries can cause a severe threat to the environment, chemical reactors, quality of the product, and plant safety. Hence regulating temperature should be a priority to achieve sustainable goals. Considering all the above-mentioned factors, there is a great need to develop sustainable and environmentally friendly controllers by the year 2030, as proposed by the UN. PID controllers are used in many industries, which have a direct impact on many of the sustainable goals like environment, labor, and wages, infrastructure, etc. Hence many controllers that reduce rise time, overshoot, and settling time, are to be opted to minimize the impact on the environment (Wang *et al.* 2015; Mahdavian *et al.* 2016).

Hence robust PID controllers must be used to effectively manage tuning parameters for achieving both climate and industrialization related sustainable goals.

1.2.2 Studies on Decent work and economic growth

The finance sector is crucial, especially for second and third world nations, and can be achieved by easy and efficient systemic incentives over a long time. This will consequently serve the purpose of SDG with an emphasis on the products and service sector.

Financial education, irrespective of gender, is one of the crucial SDG targets (Man-Kit 2015). One of the most crucial sectors among SDG's goals is financial incentives. Investing in a country's growth through the equity market at initial stages or developing stages for a long time horizon is particularly beneficial to lower and middle-income countries for sustainable investment growth (Stafford-Smith *et al.* 2017). Proper construction of the methodological framework mentioned in one of the targets of "No poverty" SDG will not only eradicate poverty, but consistent investments will lead to higher purchasing power per capita and higher Human Development Index (HDI) which leads to achieve food security. Sustainable investments practices are exponentially increasing over the past decade. Management of several resources and risks are identified as one of the most important SDG's skills which are included in many of sustainable targets. Hence a strategic management of portfolio are proposed in the study (Rutkauskas *et al.* 2008; Stafford-Smith *et al.* 2017)

1.2.3 Peace, Justice and Strong Institutions

After building frameworks for poverty elimination and sustainable growth, United Nations had given a proposal included in the 16th goal which has 56 targets to reduce evils existing in the society such as civil society participation in talking corruption, bribery within the agencies protecting law enforcements, analyzing corruption and its impact on society, cybercrime, gender discrimination, money-laundering and terror financing by the year 2030. Further, reductions of crime rates and induce harmony and peace in the society is also included in the goals. A branch of General Assembly, Open Working Group (OWG), had agreed to the proposal given by the UN. To achieve peace and harmony, good governance, along with efficient deliverance of justice by jurisdictional institutes, are necessary (Rosa 2017). Hence these institutes must come up with new methodologies for sustainable and inclusive growth of society. Subsequently, the African Union has also agreed to increase the security and include peace as key pillars with the help of the Common African Position (CAP). Since then, several other institutes like Secretary-General and Development Agenda had reaffirmed the significance of this peace agenda. Even after the unanimous agreement of several nations and organizations, the agenda remains sensitive to some of the UN member nations due to the conflicts and insecurity among the nations. Some governments fear the inclusion of this proposal since some countries might violate sovereignty.

Even after the proposal of goals and targets, many experts think that there is a great need for advanced strategies and enhanced methodologies with improved language. Some experts think that a universal framework, along with a sensitive context to all countries, might be the solution to achieve peace. A framework that considers the previous effect in inputs while predicting is useful for faster achievements of sustainable goals.

The emphasized SDG's goals can be improved by the application of fractional calculus in the existing methodologies and frameworks. Fractional order control systems introduced by Podlubny (Podlubny 1999) had proved to yield better results by tuning parameters in comparison to conventional PID controllers. The current research recommends a revised version of models by adding a new dimension with the help of integrating fractional calculus to attain most of the sustainable goals by 2030. Further, fractional calculus inherits memory property, which considers the previous events in the input variables of an algorithm, which improves the precision of a model (Sun *et al.* 2011; Gómez-Aguilar *et al.* 2019). Considering the challenges and the targets for achieving SDGs, the following objectives are proposed with the application of fractional calculus.

1.3 The Objective of the Thesis:

The objective of the thesis is to perceive the applications and possible postulates of fractional calculus in three different areas of sustainable development goals which are related to innovative industries and its impact on climate change, financial planning depending upon perception and framework of Peace, Justice, and Strong Institutions.

1. To address the dimensional mismatch and explore the applications of generalized fractional calculus operators in the area of fractional order PID controllers.
2. To apply the nature-based optimization technique for optimizing the parameters of fractional order PID controller.
3. To analyze the possible postulates of fuzzy fractional calculus for sustainable growth in equity markets.
4. To analyze fractional differential equations methods in crime transmission.

The first and second objective is addressed in Chapter 2, and the third objective is discussed in Chapter 3 and Chapter 4 with the fourth objective, a new fractional differential equations method is proposed in Chapter 5. Some of the topics used in the present research work are presented below in order to discuss the above-mentioned problems in a systematic perspective way.

1.4 Scope of the Research

A novel dimensionally balanced fractional order proportional integral derivative controller (DBFOPID) is proposed in the thesis by considering the dimensional aspects of fractional order proportional integral derivative controller (FOPID). The proposed fractional order controller helps to achieve sustainable goals such as Industry, Innovation, and Infrastructure by replacing the PID controller and altering frequency variations in multiple area power systems. Further, the proposed DBFOPID controller helps to achieve the sustainable goal of climate action by minimizing greenhouse emissions, which is a consequence of optimizing overshoot, settling time, and rise time.

The perception of the investor in dynamic markets is neglected in most of the studies. To address this, three portfolios are introduced, which enables novice investors to manage their portfolios based on perception. Various parameters such as P-index and multi-criterial volatility (MCV) not only help to measure the performance of the sector/market but also aid the UN to achieve SDGs efficiently.

Further, a novel enhanced fractional crime transmission model is introduced, which can address the actual crime in society and the rate of transmission. This fractional crime transmission model, which is inherited by memory property, helps jurisdictional institutions capture the crime transmission. The methodology helps to achieve many targets of Peace, Justice, and Strong Institutions by the year 2030.

1.5 Major Contributions of Thesis

Major contributions of this thesis work are summarized below:

1.5.1 New Fractional PID Controller to Mitigate Frequency Variations in Power Systems

It is observed that the dimensional aspects which should be of grave importance in any study associated with FOPID have been overlooked. In this regard, the present study is emphasized to develop a new FOPID controller, which is dimensionally balanced. It is introduced as a dimensionally balanced fractional order proportional integral derivative controller (DBFOPID). The proposed controller has been analyzed on an isolated and two area power systems. Furthermore, the optimized parametric values of the variables affecting overshoot, settling time and rise time are estimated using PSO. To evaluate the optimum values of these parameters, integral of the time-weighted absolute error performance index has been used. The conventional PID controller controlled AGC, and the proposed DBFOPID controller systems are compared. Further, the sensitivity of the system to human controllable parameters to mitigate frequency variations like overshoot, settling time and rise time are studied.

1.5.2 A Framework for Efficient Portfolio Management for Trading under Uncertain Environment using Fuzzy TOPSIS

A framework for efficient portfolio management for trading under uncertain environment is developed in the thesis with the help of the MCDM technique Fuzzy TOPSIS. The proposed framework can be used to determine the dominance of various sectors among the chosen set of alternatives. The dominance of these sectors are analyzed by considering various aspects such as Return on equity, Book value per share, Price earnings ratio, and Price to book ratio. For illustration, the dominance of various sectors is evaluated using the proposed framework. The historical stats of various sectors, corresponding to the aforementioned criteria are collected from various sources and the dominance is studied using the proposed framework. The findings of this study facilitate novice users in understanding the relative dominance of each sector in context to today's scenario.

1.5.3 Hierarchy of Sectors in BSE SENSEX for Optimal Equity Investments using Fuzzy AHP

Authors have taken up a study to identify the best sector in BSE SENSEX for investments. Fuzzy Analytical Hierarchy Process (AHP) is used to evaluate the dominance of various sectors, including Automobile, Finance, Information technology, Oil, Pharmaceuticals, and Power. Four crucial derivatives criteria's Return on equity, Book value per share, Price-earnings ratio, Price to book ratio are considered to study the dominance of each sector. The results of this study help in prioritizing the sectors for future investments.

1.5.4 A Novel Methodology for Perception-based Portfolio Management using Fractional Lion Algorithm

Studies of portfolio optimization have gained momentum over the recent past after Markowitz's portfolio model. Subsequent advances in portfolio optimization techniques helped novice investors and decision-makers for building an effective portfolio in view of various financial ratios. Though the developed techniques help for the effective management of the portfolio, leverage to the perception of investor/user is a major shortcoming. In view of this, the current research focuses on developing a novel technique for perception based portfolio management. The philosophy of Multi-criteria decision making and Fractional calculus are blended in developing the proposed novel technique. It is expected that the proposed technique empowers investors to manage monetary funds based on their perception. The proposed technique is not just restricted to equity markets but can also be extended for management in various fields; such as network structures, supply chain management, research institutes feasibility; and large-scale rooftop photovoltaic. For a better understanding of the proposed technique, an illustrative example is demonstrated by considering the equity portfolio selection.

1.5.5 Underlying Dynamics of Crime Transmission with Memory

Various studies suggest different mathematical models of integer order differential equations to predict crime. But these models do not inherit non-local property, which depicts behavior changes due to contact with criminals for a long period. To overcome this, a fractional order mathematical model of crime transmission is proposed in this study. The proposed model considers the previous effects of the input while predicting the crime growth rate. Abstract compartmental parameters of fractional crime transmission equation, which illustrates varies stages of criminal activity, were employed to analyze crime contagion in the society. The present study demonstrates the progression of the flow of population by classifying into three systems based on involvement in crime and imprisonment by considering the criminal history of an individual. Subsequently, the equilibria of the three-dimensional fractional crime transmission model are evaluated using phase-plane analysis. The Lyapunov function is employed to determine threshold conditions to achieve a crime-free society.

In subsequent sections, a brief history of fractional calculus along with widely used fractional order operators is discussed. The important property of fractional calculus 'memory property' along with non-local property, is also discussed in upcoming sections.

1.6 Fractional Calculus

The study that deals with various properties of derivatives and integrals of arbitrary order in the real or complex field is fractional calculus. Even though fractional calculus is a generalization of integer-order derivatives, the generalizations may not be too trivial even for basic elementary functions such as exponential, sine and cosine functions. The integer-order n^{th}-derivative of these functions can be given by (1.1) – (1.3).

$$\frac{d^n e^{\mu x}}{dx^n} = \mu^n e^{\mu x} \tag{1.1}$$

$$\frac{d^n \sin(\mu x)}{dx^n} = \mu^n \sin\left(\frac{n\pi}{2} + \mu x\right) \tag{1.2}$$

$$\frac{d^n \cos(\mu x)}{dx^n} = \mu^n \cos\left(\frac{n\pi}{2} + \mu x\right) \tag{1.3}$$

These equations hold true for $n \in \mathbb{N}$, and is not true $\forall n \in \mathbb{R}$. Hence there is a great need to consider special conditions and formulate a formal definition. There are different Fractional order derivatives introduced since the introduction of Riemann- Liouville fractional operators, but only a few definitions are widely used since the introduction, such as Caputo and Grunwald–Letnikov fractional operators. But for defining these fractional operators, some fundamental functions are required for a better understanding of these fractional operators. One of the fundamental functions which are commonly utilized Gamma and Mittag-Leffler functions are defined in subsequent sections.

1.6.1 Gamma Function

A factorial is restricted to take integer values as inputs. But the generalization can be extended to complex number 'z' and can be defined by (1.4) in the theory of fractional calculus.

$$\Gamma(z) = \int_0^\infty e^{-\tau}\,\tau^{z-1} d\tau \tag{1.4}$$

The region of convergence is on the right- half of the complex plane. One of the vital properties of this function is that it satisfies the following equation (1.5).

$$\Gamma(z) = (z-1)\Gamma(z-1) \tag{1.5}$$

Thus from (1.5) it is inferred that Gamma function $\Gamma(z)$ is a generalization of factorial function $n!$. The relation between these two is $\Gamma(n+1) = n!\ \forall n \in \mathbb{N}$.

1.6.2 Mittag-Leffler Function

This function is another crucial function, which plays a vital role in the theory of Fractional order differential equations. This function is the generalization of an exponential function and can be defined as following (1.6).

$$E_{a,b}(z) = \sum_{l=0}^{\infty} \frac{z^l}{\Gamma(al+b)} \tag{1.6}$$

where $a, b > 0$. This is two parametric representation of Mittag-Leffler Function defined by Wiman in 1905 and can be reduced to one parameter function originally defined by Mittag-Leffler in 1903 by replacing $b = 1$. Detailed work on this function and its vital role in fractional calculus can be seen in (Humbert and Agarwal 1953; Khan and Ahmed 2013).

Now we are ready to define differential and integral operators of fractional orders. Two of the widely adopted definitions of Fractional order operators are given by Riemann- Liouville and Caputo, which are briefed in the following sections. Riemann-Liouville derivative came from the generalization of the Cauchy formula for repeated integration and is defined in subsequent sections.

1.6.3 Riemann-Liouville Fractional Integral operator:

Let $Re(\alpha) > 0$ and f be piecewise continuous on interval $(0, \infty)$ *and* integrable on any finite subinterval of $[0, \infty)$. Then for $x > 0$ we define Riemann-Liouville fractional integral as (1.7)

$$ {}^{RL}_{a}J^{\alpha}_{x}f(x) = \frac{1}{\Gamma(\alpha)}\int_{a}^{x}(x-t)^{\alpha-1}f(t)dt \tag{1.7} $$

If we put $\alpha = 0$, in (1.7) then the above equation is Riemann's definition of fractional integration and if $\alpha = -\infty$, then it represents Liouville's definition of fractional integration.

1.6.4 Riemann-Liouville Fractional Differential operator:

The fractional order derivative of a function $f(x)$ of order $\alpha \in \mathbb{R}$ by using fractional order integral can be mathematically expressed as (1.8). Let m be an integer such that $m-1 \le \alpha < m$, then in order to get α^{th}-derivative of $f(x)$ we first integrate $f(x)$ fractionally up to order $m-\alpha$, then differentiate m times, i.e

$$ {}^{RL}_{a}D^{\alpha}_{t}f(x) = \begin{cases} \frac{d^m}{dt^m}\left\{\frac{1}{\Gamma(m-\alpha)}\int_{0}^{x}(x-t)^{m-\alpha-1}f(t)dt\right\}, when\ m-1 \le \alpha < m. \\ \frac{d^m f(x)}{dx^m} \qquad , when\ \alpha = m \end{cases} \tag{1.8} $$

Example 1.1: For a function $f(x) = x^n$, $t > -1$, then ${}^{RL}_{0}D^{\alpha}_{t}f(x) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}x^{n-\alpha}$, with $Re\ \alpha > 0, x > 0$ and Riemann-Liouville (RL) half derivative for different powers of x^n , $n = 1,2 \dots ,10$ is shown in Figure 1.1.

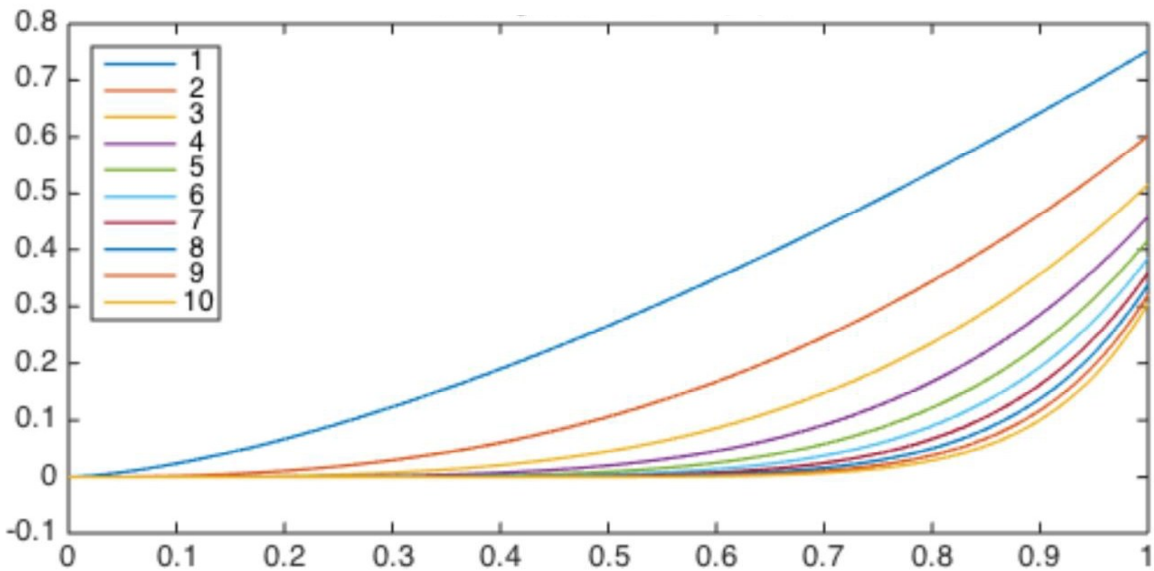


Figure 1.1 RL half derivative for different powers of $x^n, n = 1,2 \dots ,10$

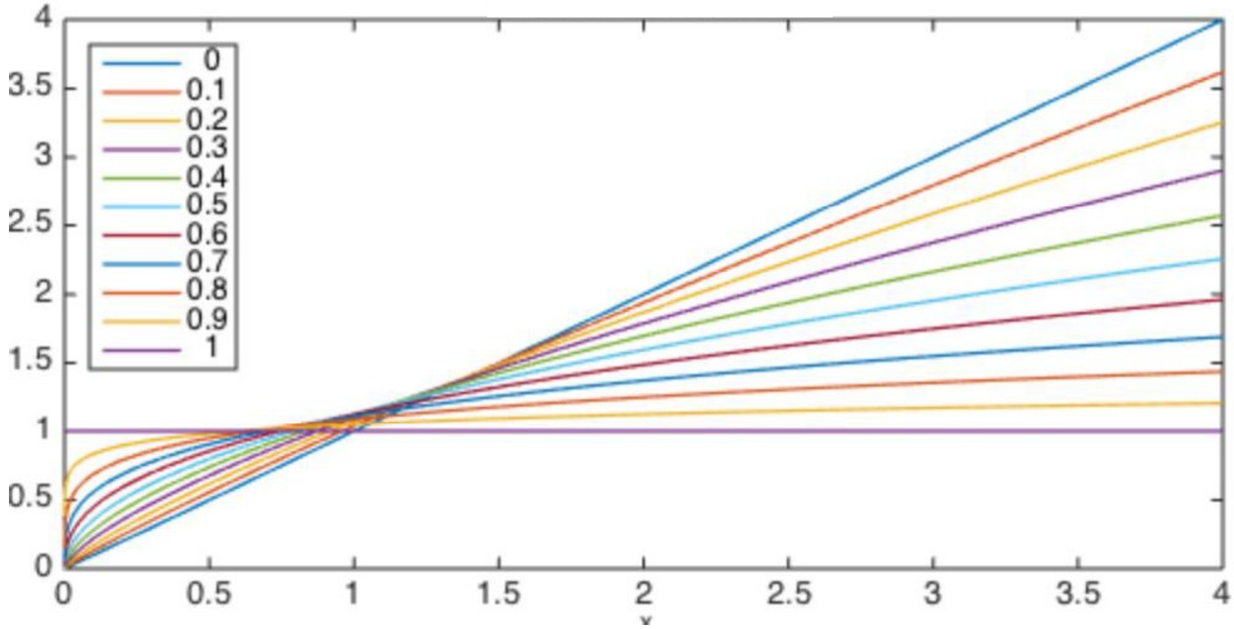


Figure 1.2. Different fractional orders of RL derivatives for $f(x) = x$

Different fractional orders of RL derivatives for the function $f(x) = x$ can be depicted from Figure 1.2. For $t = 0$, the half derivative of the function $f(x)$ is $\frac{2\sqrt{x}}{\sqrt{\pi}}$ and comparisons with the first-order derivative and integral are presented in Figure 1.3. Hence it can be concluded that the RL fractional derivative of a constant function is not necessarily be 0.

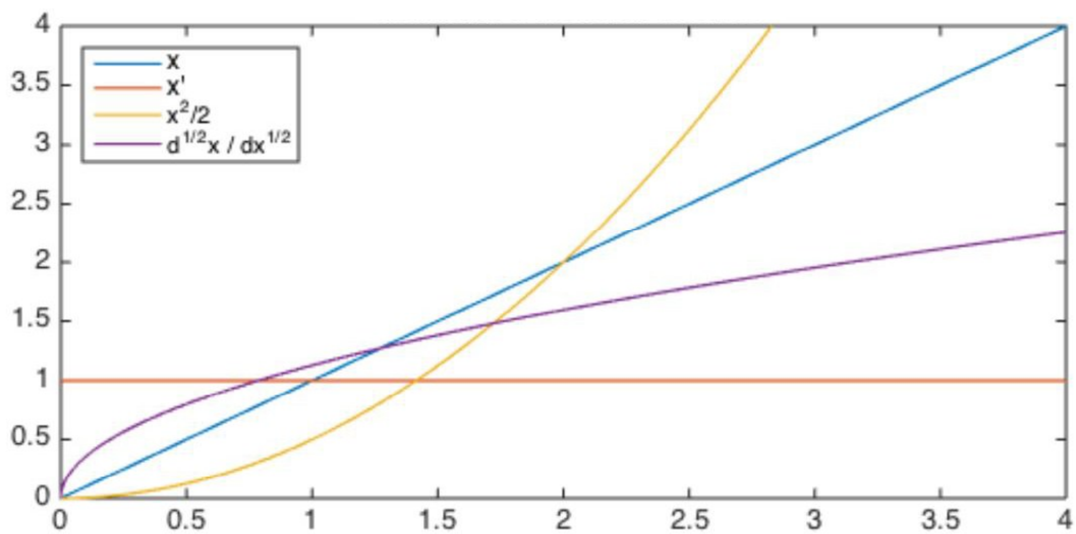


Figure 1.3. Comparison of Integral, Derivative and RL half-derivative of $f(x) = x$

Some of the important properties of RL fractional operators (Khan and Ahmed 2013) are briefed below. Figure 1.1-Figure 1.3 are implemented using open-source software package Chebfun® available online at https://www.chebfun.org/.

Theorem 1.1: If a function $f(x) \in L_1[0,\infty)$, i.e., $f(x)$ is a measurable function and satisfies $\int_0^\infty |f(y)|\, dy < \infty$, along with $\alpha, \beta \geq 0$, then the following is true for RL operators (Oldham and Spanier 1974; Podlubny 1999).

Semi-group property

$$ {}_{a}^{RL}J_t^{\alpha}\left\{ {}_{a}^{RL}J_t^{\beta} f(x) \right\} = {}_{a}^{RL}J_t^{\alpha+\beta} f(x) \tag{1.9} $$

Commutative property

$$ {}_{a}^{RL}J_t^{\alpha}\{ {}_{a}^{RL}J_t^{\beta} f(x)\} = {}_{a}^{RL}J_t^{\beta}\{ {}_{a}^{RL}J_t^{\alpha} f(x)\} \tag{1.10} $$

Theorem 1.2: If two functions $f(x)$ and $g(x)$ defined and RL differentiable on an interval $[a,b]$ and, $\sigma_1, \sigma_2 \in \mathbb{R}$, then the following properties (1.11) – (1.13) are true (Podlubny 1999).

Linearity

$$ {}_{0}^{RL}D_t^{\alpha}\big(\sigma_1 f(x) + \sigma_2 g(x)\big) = \sigma_1\, {}_{0}^{RL}D_t^{\alpha}\big(f(x)\big) + \sigma_2\, {}_{0}^{RL}D_t^{\alpha}\big(g(x)\big) \tag{1.11} $$

The rule for power zero

$$ {}_{0}^{RL}D_t^{0}\big(f(x)\big) = f(x) \tag{1.12} $$

Product rule

$$ {}_{0}^{RL}D_t^{\alpha}[f(x)g(x)] = \sum_{l=0}^{\infty} \binom{\alpha}{l} [{}_{0}^{RL}D_t^{l} f(x)][{}_{0}^{RL}D_t^{\alpha-l}(g(x)] \tag{1.13} $$

It should be noted that semi-group property doesn't always satisfy for RL derivatives i.e.

$$ {}_{a}^{RL}D_t^{\alpha}\{ {}_{a}^{RL}D_t^{\beta} f(x)\} = {}_{a}^{RL}D_t^{\alpha+\beta} f(x) \tag{1.14} $$

is not necessarily true. For example, let $f(x) = 1$, a constant identity function with $\alpha = 1$ and $\beta = 0.5$ doesn't satisfy (1.14).

***Theorem 1.3*:** If $D^n f(x) \in C([0,\infty))$, where $C([0,\infty))$ denotes set of continuous functions on the interval $[0,\infty)$ with α strictly positive number, then (1.15) holds

$$ J_x^{\alpha+n}\big({}_{0}^{RL}D_t^{\alpha} f(x)\big) = J_x^{\alpha} f(x) - F_n(x,\alpha) \tag{1.15} $$

and for positive x, (1.16) holds

$$ {}_{0}^{RL}D_t^{\alpha}\big(J_x^{\alpha} f(x)\big) = J_x^{\alpha}\big({}_{0}^{RL}D_t^{\alpha} f(x)\big) + F_n(x,\alpha - n) \tag{1.16} $$

where,

$$F_n(x,\alpha) = \sum_{l=0}^{n-1} \frac{x^{\alpha+l} f^{(l)}(0)}{\Gamma(\alpha+l+1)}$$

1.6.5 Laplace transformation for RL fractional derivatives

The Laplace transformation $\mathcal{L}$ for RL fractional derivatives is shown in (1.17)

$$\mathcal{L}\{{}_0^{RL}D_t^{\alpha} f(x); s\} = s^{\alpha} F(s) - \sum_{l=0}^{k-1} s^l \, [D^{\alpha-l-1} f(x)]_{x=0} \tag{1.17}$$

It can be observed from (1.17) that the Riemann–Liouville operators require initial conditions containing the limit values of Riemann–Liouville fractional derivative at the origin of time $x = 0$, whose physical interpretation is not lucid. Due to this initial condition, RL derivatives are practically insignificant in the application on initial value problems and boundary value problems. Hence the area of fractional calculus has been untouched for many centuries due to this shortcoming. Here $F(s)$ is the Laplace transform of a function $f(x)$.

If $F(s)$ is the Laplace transform of a function $f(x)$ and let $G(s)$ be the Laplace transform of a function $g(x)$, then the following (1.18) holds

$$\mathcal{L}\left\{\int_0^x f(x-t)g(t)dt\right\} = F(s)G(s) \tag{1.18}$$

To restore original function from Laplace transform (1.17) for any fractional operator, an inverse Laplace transformation (1.19) is utilized

$$f(x) = \mathcal{L}^{-1}\{F(s); x\} = \int_{c-i\infty}^{c+i\infty} e^{sx} F(s) ds \, ; c = Re(s) > d \tag{1.19}$$

where d belongs to the right half-plane of the absolute convergence of Laplace integral.

1.6.6 Caputo Differential operator:

Due to the lack of physical interpretation with initial value problems, RL fractional operators are practically insignificant. In 1967, a practically applicable solution was proposed by Caputo in his research work (Caputo 1967). In the case of Caputo fractional derivatives, the initial conditions take similar configuration as that of integer differential equations, which are useful for various practical applications. This can be mathematically expressed as (1.20)

$${}_0^C D_t^{\alpha} f(x) = J^{n-\alpha} D^n f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau \,, \quad n-1 < \alpha < n, \tag{1.20}$$

The relation between RL fractional differential operators to that of Caputo's fractional differential operator is given in (1.21).

$$ {}_{0}^{C}D_{t}^{\alpha}f(x) = {}^{RL}_{0}D_{t}^{\alpha}f(x) - \sum_{l=0}^{n-1} \frac{x^{l-\alpha}}{\Gamma(l-\alpha+1)}, \quad n-1 < \alpha < n, \tag{1.21} $$

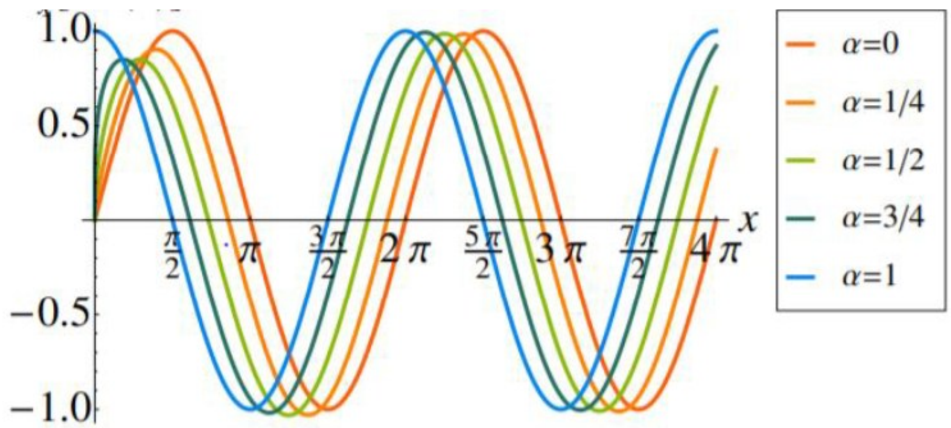


Figure 1.4. Caputo order fractional derivative of $f(x) = sin(x)$ with orders ranging from 0 to 1 (Shchedrin et al. 2018)

The Caputo fractional operators on elementary functions such as sine, cosine, arcsine and arctan can be depicted from the following Figure 1.4 – Figure 1.7 (Shchedrin *et al.* 2018).

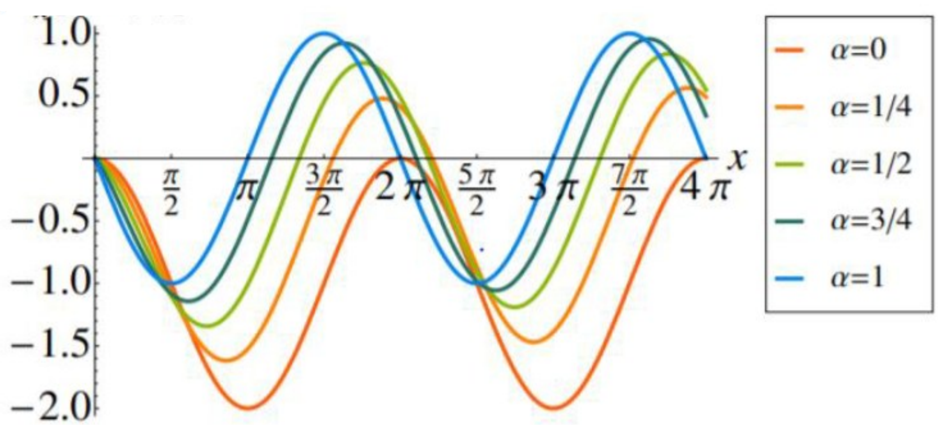


Figure 1.5. Caputo order fractional derivative of $f(x) = cos(x)$ with orders (Shchedrin et al. 2018)

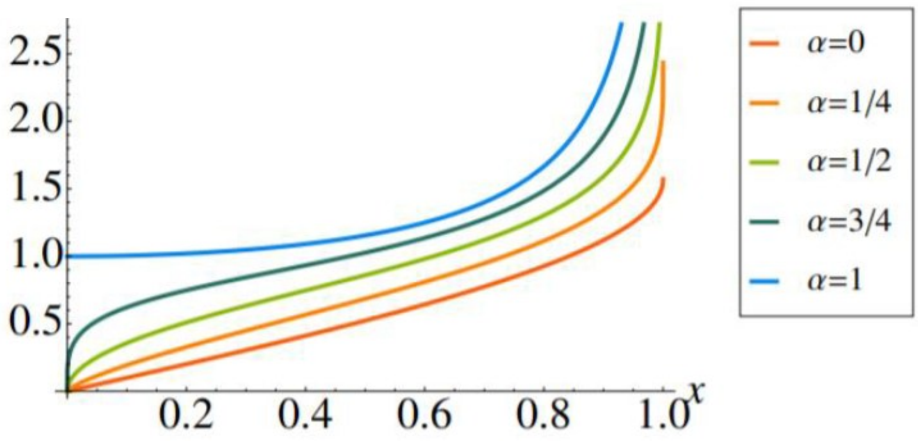


Figure 1.6. Caputo order fractional derivative of $f(x) = arcsin(x)$ with orders ranging from 0 to 1 (Shchedrin et al. 2018)

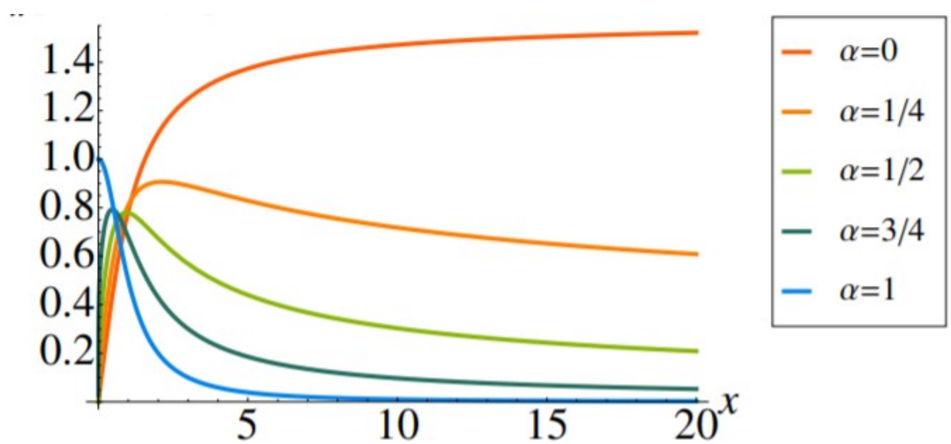


Figure 1.7. Caputo order fractional derivative of $f(x) = arctan(x)$ with orders ranging from 0 to 1 (Shchedrin et al. 2018)

Gaussian functions are commonly used functions in the field of Statistics, Machine Learning, Economics and Finance. With the increasing use of fractional calculus in these fields, the Caputo fractional derivatives on Gaussian functions for different orders are presented in Figure 1.8 and Figure 1.9 (Shchedrin *et al.* 2018).

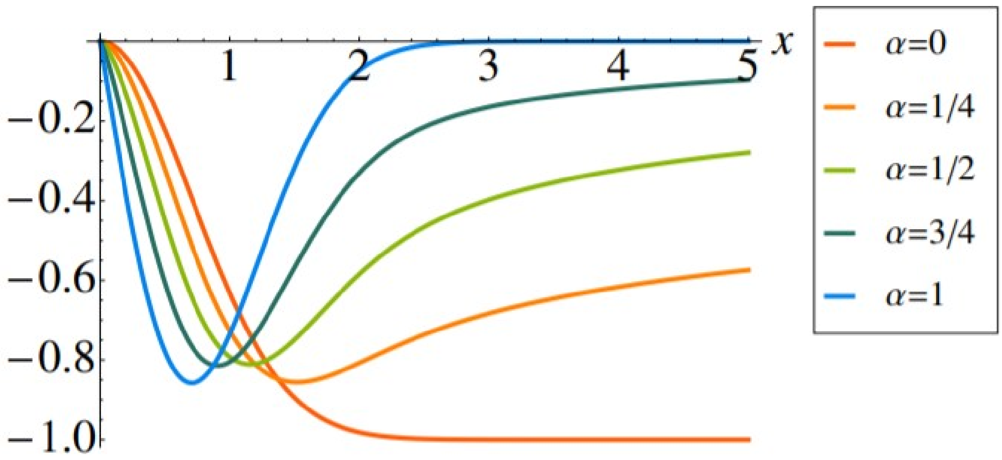


Figure 1.8. Caputo order fractional derivative of $f(x) = e^{-x^2}$ with orders ranging from 0 to 1 (Shchedrin et al. 2018)

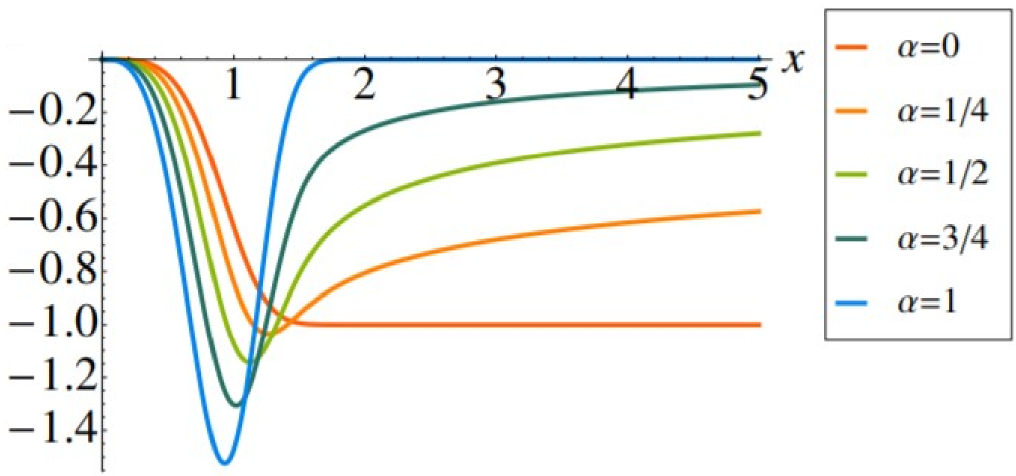


Figure 1.9. Caputo order fractional derivative of $f(x) = e^{-x^4}$ with orders ranging from 0 to 1 (Shchedrin et al. 2018)

The wide range of practical applications of Caputo operators extends to the fields of atomic spectroscopy, quantum optics and electrodynamics. One of the commonly utilized functions in these fields includes the Lorentzian function (Shchedrin *et al.* 2018). This function can be defined by (1.22)

$$f_L(x,\gamma) = \frac{1}{\pi}\frac{\frac{\gamma}{2}}{x^2 + \frac{\gamma^2}{4}} \tag{1.22}$$

The graphical depiction of Lorentzian function is displayed in Figure 1.10

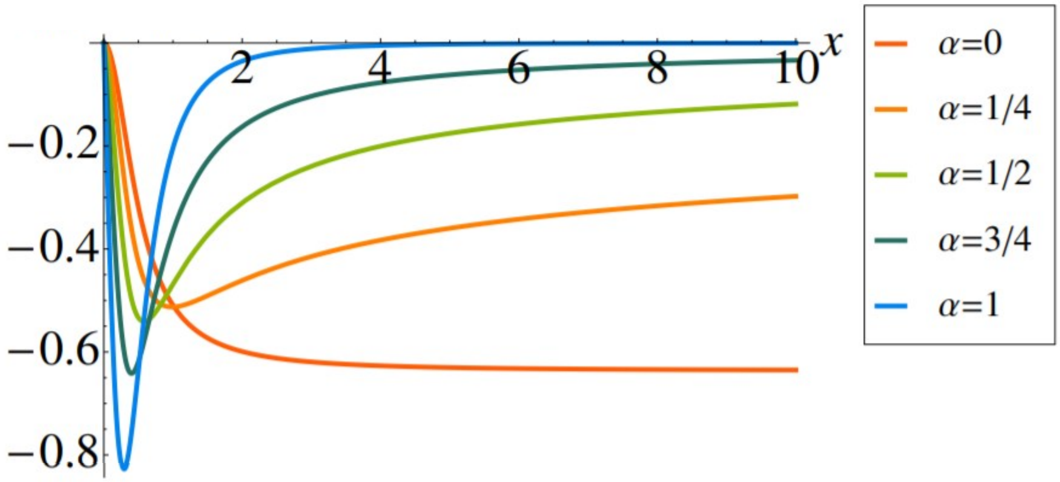


Figure 1.10. Caputo order fractional derivative of Lorentzian function $f_L(x, \gamma)$ with orders ranging from 0 to 1 (Shchedrin et al. 2018)

Although similar solutions are presented in Banach spaces by El-Sayed (El-Sayed 1999).

1.6.7 Properties of Caputo Differential Operator:

The Caputo derivative assumes the function to be in the class of C^n, i.e., n times smooth, whereas Riemann-Liouville Fractional Differential operator assumes the function considered to be integrable. But the Caputo fractional derivative of a constant is zero and in the case of Riemann-Liouville Fractional Derivate, the constant function is not always zero, which is evident from Example 1.1.

Linearity property

Let the function $f(x)$ be a n^{th}-order differentiable function then Caputo fractional operator obeys linearity property (1.23)

$$ {}_0^C D_t^{\alpha}\big(\sigma_1 f(x) + \sigma_2 g(x)\big) = \sigma_1 {}_0^C D_t^{\alpha}\big(f(x)\big) + \sigma_2 {}_0^C D_t^{\alpha}\big(g(x)\big) \qquad (1.23) $$

Semi-group property

Unlike RL fractional operator, the semi-group property is satisfied by Caputo fractional operator as shown in (1.24) if ${}_0^C D_t^{\alpha} f(x) \in C([0, \infty))$

$$ {}_0^C D_t^{\alpha}\left({}_0^C D_t^{\alpha} f(x)\right) = {}_0^C D_t^{\alpha+\beta} f(x) \qquad (1.24) $$

Leibniz rule

The Leibniz rule for Caputo operators are given below (1.25)

$$ {}_0^C D_t^{\alpha}\big(f(x)g(x)\big) $$

$$= \frac{(x-t)^{-\alpha}}{\Gamma(1-\alpha)} g(t)\big(f(x)-f(t)\big) + \left({}_0^C D_t^{\alpha} g(x)\right) f(x) + \sum_{l=0}^{\infty} \binom{\alpha}{l} \left({}_0^C D_t^{\alpha-l} g(x)\right) {}_0^C D_t^{\alpha} f(x) \qquad (1.25)$$

1.6.8 Laplace transformation for Caputo fractional derivatives

The Laplace transformation for Caputo fractional derivatives is given by (1.26)

$$\mathcal{L}\{{}_0^C D_t^{\alpha} f(x); s\} = s^{\alpha} F(s) - \sum_{i=0}^{k-1} s^{\alpha-i-1} f^{(i)}(0) \qquad (1.26)$$

The fractional operators are just not limited to these two operators, but there are wide operators utilized, but the most common excluding these two derivatives is Grunwald–Letnikov fractional derivative defined in the next subsection. It can be observed from (1.25) that Caputo fractional operator can be practically applied in initial value problems and boundary value problems due to its integer order initial conditions.

1.6.9 Grunwald–Letnikov fractional derivative

Like Caputo fractional derivative, Grunwald–Letnikov fractional derivative also assumes n-times differentiable for existence. For $\alpha \in (0,1)$, Grunwald–Letnikov fractional derivative is defined as (1.27)

$$D_G^{\alpha} f(x) = \lim_{h \to 0} \frac{1}{h} \sum_{l=0}^{\left[\frac{x}{h}\right]} \omega_l^{\alpha} f(x - hl) \qquad (1.27)$$

where, $\left[\frac{x}{h}\right]$ is the greatest integer function of $\frac{x}{h}$ and

$$\omega_l^{\alpha} = \frac{\Gamma(-\alpha + l)}{\Gamma(-\alpha)\Gamma(l+1)}$$

Further, Grunwald–Letnikov fractional operators provide easy and effective approximation in the area of numerical analysis. The values obtained from each of fractional derivatives may not be unique at non-integer points and for initial value problems, and the examples of fractional integrals and derivatives for basic elementary functions are presented in the next section. Further, Grunwald–Letnikov fractional operators are used in nature-based optimization problems such as Particle Swarm Optimization, Artificial Bee Colony, and Genetic Algorithms for smooth position update of particles.

1.6.10 Fractional Operators of Elementary Functions

This section presents Riemann-Liouville, Caputo and Grunwald–Letnikov integrals and derivatives of elementary functions such as Exponential, Sine and Cosine functions with pictorial depiction of the same.

1.6.10.1 Exponential function

The α^{th}- fractional derivative for Riemann-Liouville, Caputo and Grunwald–Letnikov operators for an exponential function $e^{\lambda(x-a)}$, for $x > a$ can be obtained by (1.28)

$$\begin{aligned} J_a^{\alpha} e^{\lambda(x-a)} &= (x-a)^{\alpha} E_{1,1+\alpha} \lambda(x-a) \\ {}^{RL}D_a^{\alpha} e^{\lambda(x-a)} &= (x-a)^{-\alpha} E_{1,1-\alpha} \lambda(x-a) \\ {}^{C}D_a^{\alpha} e^{\lambda(x-a)} &= \lambda^{\lceil\alpha\rceil}(x-a)^{\lceil\alpha\rceil-\alpha} E_{1,\lceil\alpha\rceil-\alpha+1} \lambda(x-a) \\ {}^{GL}D_a^{\alpha} e^{\lambda(x-a)} &= \lambda^{\alpha} e^{\lambda(x-a)} \end{aligned} \quad (1.28)$$

where $\lceil\alpha\rceil = [\alpha] + 1$ is the ceiling functions of α.

For detailed results see (Garrappa *et al.* 2019).

The exponential function graph of RL fractional operators is shown in Figure 1.11.

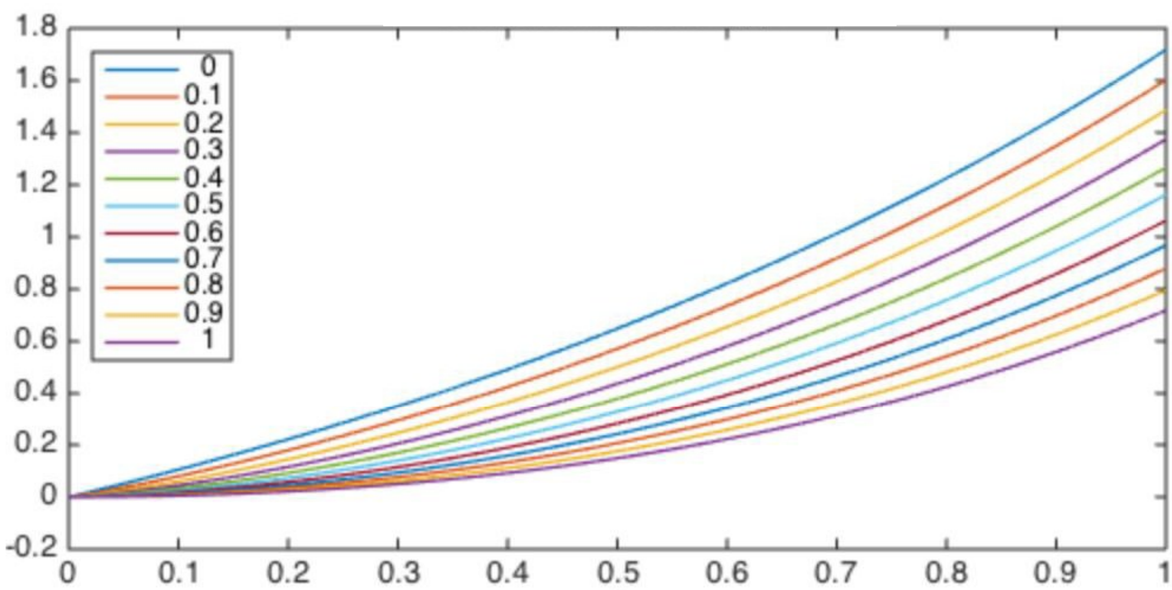


Figure 1.11. RL fractional derivatives for the function $e^x - 1$ with order variying from 0 to 1

1.6.10.2 Sine function

Similarly, α^{th}- fractional derivative of trigonometric function sine can be obtained from the following equation (1.29). Further, the graph of different Riemann-Liouville fractional order derivatives for the function of $\sin(x)$ is shown in Figure 1.12.

$$J_a^{\alpha}\sin\big(\lambda(x-a)\big)=\frac{(x-a)^{\alpha}}{2i}\Big(E_{1,1+\alpha}(i\lambda(x-a))-E_{1,1+\alpha}\big(-i\lambda(x-a)\big)\Big),$$

$$^{RL}D_a^{\alpha}\big(sin\lambda(x-a)\big)=\frac{(x-a)^{\alpha}}{2i}\Big(E_{1,1-\alpha}(i\lambda(x-a))-E_{1,1-\alpha}\big(-i\lambda(x-a)\big)\Big), \tag{1.29}$$

$$^{C}D_a^{\alpha}\big(sin\lambda(x-a)\big)=$$

$$i^{\lceil\alpha\rceil}\lambda^{\lceil\alpha\rceil}\frac{(x-a)^{\lceil\alpha\rceil-\alpha}}{2i}\Big(E_{1,\lceil\alpha\rceil-\alpha+1}(i\lambda(x-a))-(-1)^{\lceil\alpha\rceil}E_{1,\lceil\alpha\rceil-\alpha+1}\big(-i\lambda(x-a)\big)\Big),$$

$$^{GL}D_a^{\alpha}sin\lambda(x-a)=\lambda^{\alpha}\sin\left(\lambda(x-a)+\alpha\frac{\pi}{2}\right)$$

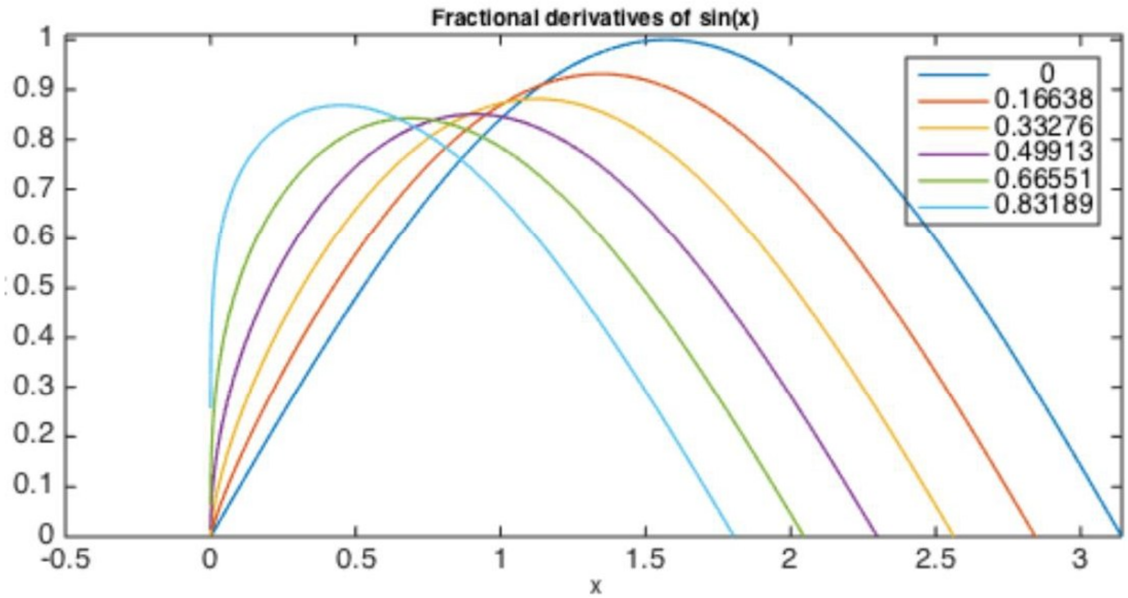


Figure 1.12. RL fractional order derivatives for the function $sin(x)$

1.6.10.3 Cosine function

Similarly, α^{th}- fractional derivative of trigonometric function cosine can be obtained from the following equation (1.30).

$$J_a^{\alpha}\cos\big(\lambda(x-a)\big)=\frac{(x-a)^{\alpha}}{2}\Big(E_{1,1+\alpha}(i\lambda(x-a))+E_{1,1+\alpha}\big(-i\lambda(x-a)\big)\Big),$$

$$^{RL}D_a^{\alpha}\big(cos\lambda(x-a)\big)=\frac{(x-a)^{\alpha}}{2}\Big(E_{1,1-\alpha}(i\lambda(x-a))+E_{1,1-\alpha}\big(-i\lambda(x-a)\big)\Big), \tag{1.30}$$

$$^{C}D_a^{\alpha}\big(cos\lambda(x-a)\big)=$$

$$i^{\lceil\alpha\rceil}\lambda^{\lceil\alpha\rceil}\frac{(x-a)^{\lceil\alpha\rceil-\alpha}}{2}\Big(E_{1,\lceil\alpha\rceil-\alpha+1}(i\lambda(x-a))+(-1)^{\lceil\alpha\rceil}E_{1,\lceil\alpha\rceil-\alpha+1}\big(-i\lambda(x-a)\big)\Big)$$

Further, the graph of different Riemann-Liouville fractional order derivatives for the function $\cos(x)$ is shown in Figure 1.13 (Garrappa *et al.* 2019).

The graphs of Riemann-Liouville derivative, Caputo and Grunwald–Letnikov fractional derivatives of the sine function is presented in Figure 1.14 for the 0.7^{th} and 1.7^{th} order derivative with $\lambda = 1.5$.

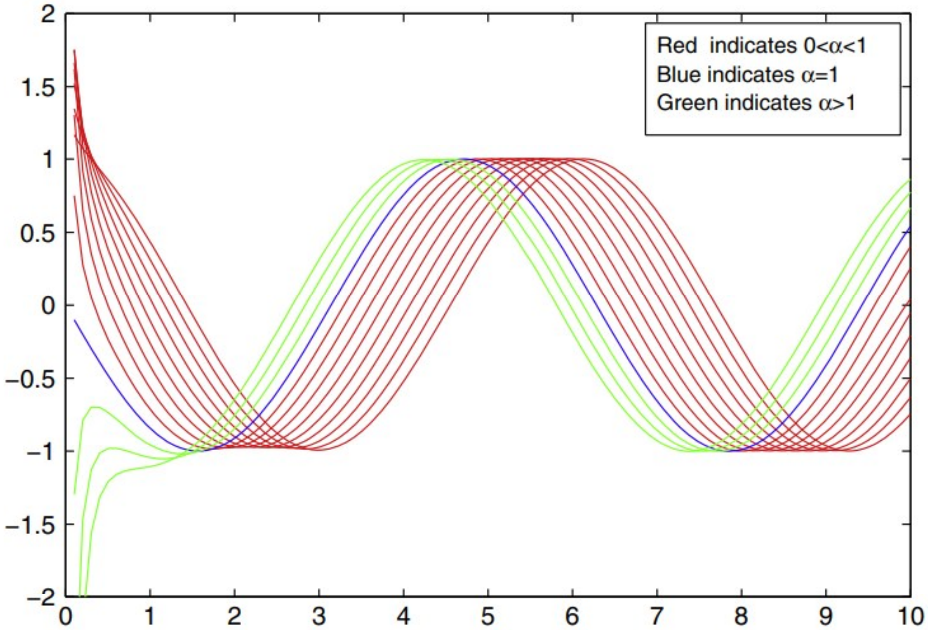


Figure 1.13. Different ranges of Caputo fractional order derivatives for the function $cos(x)$

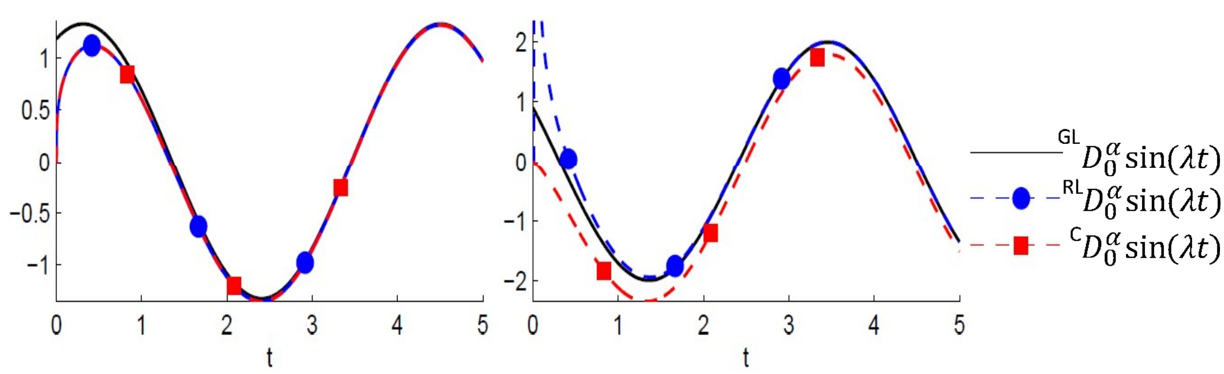


Figure 1.14. Comparing Grunwald–Letnikov, Riemann-Liouville, and Caputo fractional derivative of a sine function of order 0.7 and 1.7

1.6.11 Memory property of fractional calculus

Any integer-order derivative considers the only instantaneous rate of change, which is a local property. For example, for the first-order derivative of a differentiable function $'G'$ with respect to a variable $'t'$ represented by $'\dot{G}'$ is given in the following equation (1.31).

$$\dot{G}(t) = \lim_{\Delta t \to 0} \frac{G(t) - G(t-\Delta t)}{\Delta t}, t > 0 \tag{1.31}$$

It can be observed that (1.31) considers only two variable values, whereas in Caputo fractional operator in (1.20) $'t'$ actually considers all the values since origin or $'0'$. This property of fractional operator which is non-local in nature synonymously called memory effect and can be utilized in any of the mathematical models which impact from previous history (Sun *et al.* 2011; Damor *et al.* 2013; Tarasova and Tarasov 2016). The memory property can be delineated by memory function which is based on integro-differential kernel. Ludwig Boltzmann introduced a model inherited by memory property in the year 1874 in the field of viscoelastic media. Since then, many of mathematical models of transmissions use fractional operators since the future events may be largely altered by historical events (Sardar *et al.* 2015; Ullah *et al.* 2018; Veeresha *et al.* 2019; Öztürk and Özköse 2020).

With the increasing number of applications of fractional calculus, singular kernel fractional derivatives have become important to science and engineering. Other than RL derivatives and Caputo derivatives, other operators such as Canavati, Cossar, Liouville, Marchaud, and Riesz are examples of singular kernel fractional derivatives. Recently, singular kernel conformable and tempered fractional derivatives were introduced (Caputo and Fabrizio 2015; Saad 2019). The existing single kernel fractional derivative equations are not inherited with an exact solution. To address the downside of single kernel fractional derivatives, a non-singular kernel fractional derivative was introduced by replacing the singular kernel with non-singular kernels. For example, widely used RL fractional derivatives and Caputo derivatives have singular kernels. Caputo fractional derivative in (1.20) has a kernel of $(x - \tau)^{n-\alpha-1}$ with singularity at $x = \tau$. This kernel is replaced with an exponential function $\exp(\frac{\alpha-n+1}{\alpha-n}x)$ to remove singularity at $x = \tau$.

A new non-singular kernel fractional derivative, which presumes two non-identical representations with respect to temporal and space was introduced to describe the viscoelastic materials, and electromagnetic systems (Atangana and Baleanu 2016). The initial representation considered is enacted on time variable and transforms fractional derivatives into integer powers using Laplace transformation. Whereas, the second representation deals with the spatial aspect of the variable and mostly uses the Fourier transformation onto a non-local fractional derivative.

1.7 Fuzzy Set Theory

Vague real-life concepts like high contrast, tall person, medium saturation are perceived reasoning and logic of a person and are not well-defined. Fuzzy logic gives a formal expression that enables a person's reasoning. It is a decision making tool that represents both logic and imprecision exists in real day to day life. The concepts of Fuzzy logic are used in many fields such as business analytics, control systems, industrial, image processing and intelligent systems.

The introduction of the fuzzy set theory dates back to 1965, where Zadeh gave a mathematical representation for mapping abstract concepts to computable entities defined as fuzzy sets (Zadeh 1965). The theory of fuzzy sets gives a solution to define imprecise and vague data or information to simulate the perception and logical reasoning of an individual. Fuzzy sets are just not restricted to imprecise or vague data but can also be used to enhance conventional modeling techniques for better performance.

A crisp set is a well defined collection of objects $y \in Y$ and for a given set $A \subseteq Y$, only one condition among $y \in A$, $y \notin A$ satisfies. This can be otherwise given by an indicator or characteristic function $\mu_A$which is defined as (1.32).

$$\mu_A(y) = \begin{cases} 1, when\ y \in A \\ 0, when\ y \notin A \end{cases} \tag{1.32}$$

A fuzzy set is defined on the closed interval $[0,1]$ i.e. $\mu_{\tilde{A}}: Y \to [0,1]$ and is formally defined in subsequent definitions. Here Y can be viewed as a universe of discourse which is defined as the set of all possible range for fuzzy system input.

1.7.1 Fuzzy Set

A Fuzzy set $\tilde{A}$ is an ordered pair with objects or elements in the first tuple and the membership function (elements having grades of membership) in the second defined below in (1.33). Here $\tilde{A}$ acts as a fuzzy linguistic label to the element y.

$$\tilde{A} = \{(y, \mu_{\tilde{A}}(y)) | y \in Y\} \tag{1.33}$$

The membership function for crisp and fuzzy sets can be observed in Figure 1.15.

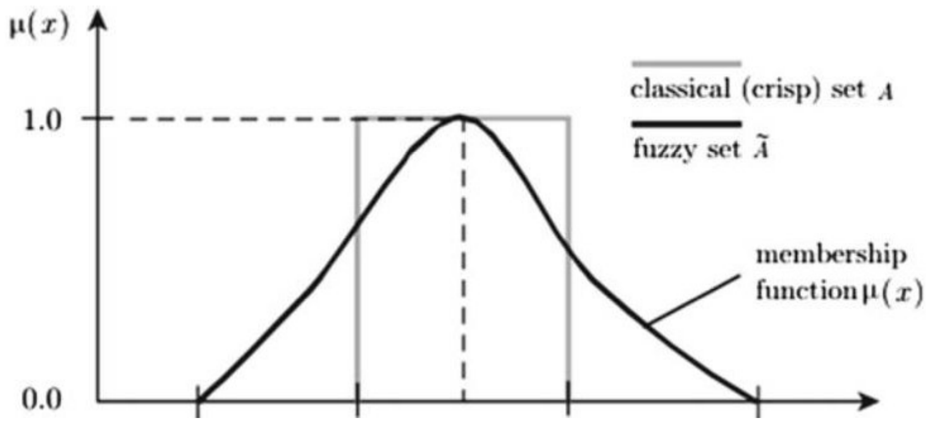


Figure 1.15. Membership function for crisp and fuzzy sets (Lee et al. 2010)

1.7.2 Support of a fuzzy set

Support for a fuzzy set $\tilde{A}$ denoted by $S(\widetilde{A})$ can be defined as the set of all $y \in Y$ which satisfies $\mu_{\tilde{A}}(y) > 0$.

1.7.3 α level cut

α-cut or α level cut for a fuzzy set $\tilde{A}$ denoted by $A_\alpha(y)$ is defined as following (1.34)

$$A_\alpha = \{y \in Y | \mu_{\tilde{A}}(y) \geq \alpha\} \tag{1.34}$$

1.7.4 Crossover point

Crossover point of U is an element that possesses a membership value of 0.5.

1.7.5 Height of a fuzzy set

The supremum of the fuzzy membership function is defined to be the height of a fuzzy set and if the Height of the membership function is 1, then the fuzzy set is called **normalized fuzzy set**.

1.7.6 Core of a fuzzy set

The set of all elements whose values of the fuzzy membership function is 1 is defined to be the core of a fuzzy set and is crisp in nature is shown by (1.35).

$$Core\left(\tilde{A}\right) = \{y \in Y | \mu_{\tilde{A}}(y) = 1\} \tag{1.35}$$

Figure 1.16 presents a pictorial representation of core, α level cut, crossover points and support of a fuzzy set.

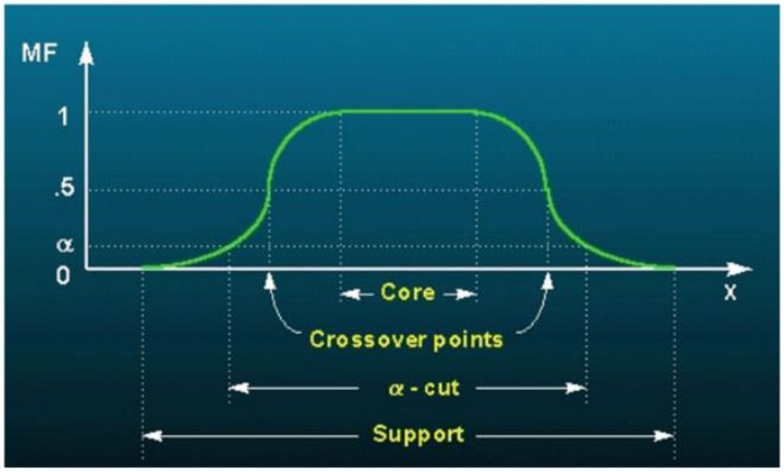


Figure 1.16. Intuitive diagram for basics of fuzzy sets (Lee et al. 2010)

1.7.7 Convex fuzzy set

A fuzzy set $\tilde{A}$ is said to be convex if it satisfies the condition (1.36)

$$\mu_{\tilde{A}}(\sigma f(y_1) + \sigma(1 - f(y_2)) \geq \min(\mu_{\tilde{A}} f(y_1), \mu_{\tilde{A}} f(y_2)); y_1, y_2 \in Y \ for\ all\ \sigma \in [0,1] \quad (1.36)$$

1.7.8 Examples

1. A fuzzy set with exponential fuzzy membership function is a convex fuzzy set for all real numbers since it satisfies (1.35). (Jovanović and Jovanović 2011).
2. A fuzzy set with sine or cosine functions as its fuzzy membership function is not a convex fuzzy set for all real numbers but is they are convex on the interval $[\pi, 2\pi]$ and $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ respectively.

1.7.9 Fuzzy Logic

Fuzzy logic is the generalization of Boolean logic or logic, which deals with concepts like partial truth or partial false unlike logic. It can be interpreted as the logic of appropriate reasoning rather than exact. Fuzzy logic formally defines the general usage of common sense and human reasoning, and the exact or complete reasoning is considered as a particular case of appropriate reasoning. With fuzzy logic, we can transform an input variable to obtain an output using the process of fuzzy inference, which involves steps like fuzzification and defuzzification techniques with the help of logical operations and membership function. In this thesis, fuzzifications and defuzzification techniques are used in Chapter 3 and Chapter 4 in the MCDM techniques of Fuzzy AHP and Fuzzy TOPSIS to transform the linguistic terms to fuzzy variables and vice-versa.

1.8 Types of Fuzzy Membership Functions

The memberships of fuzzy sets depend upon the user and the nature of the application. But most used and user-friendly membership functions are triangular, Gaussian, bell-shaped and trapezoidal functions (Sadollah 2018), which are easy to use.

1.8.1 Piecewise linear membership functions

These functions are one of the most commonly used and simplest fuzzy membership functions and commonly are in the type of either triangular or trapezoidal form (Sadollah 2018). Given three parameters e, f and g, the trapezoidal fuzzy membership function can be given by (1.37) and is shown by Figure 1.17.

$$\mu_{\tilde{A}}(y) = \begin{cases} 0, if\ y \leq a_1, \\ \frac{y-a_1}{a-a_1}, if\ a_1 \leq y \leq a, \\ 1,\ if\ a \leq y \leq b, \\ \frac{b_1-y}{b_1-b}, if\ b \leq y \leq b_1, \\ 0, if\ y \geq b_1, \end{cases} \tag{1.37}$$

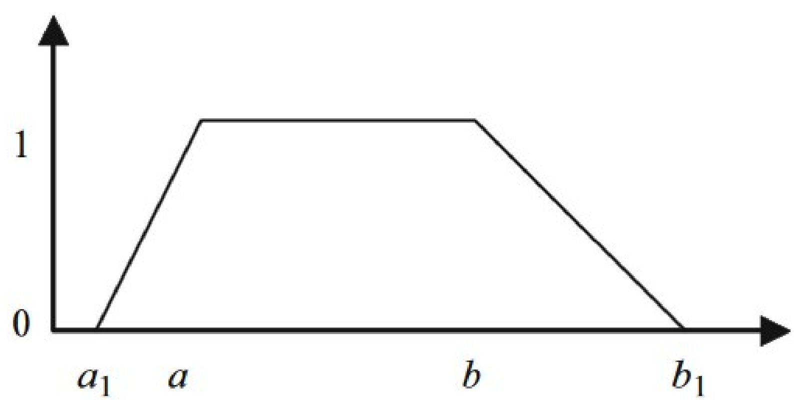


Figure 1.17. Trapezoidal fuzzy membership function

Another widely used piecewise linear membership function is triangular membership function, which can be given by (1.38) and its pictorial representation is shown in Figure 1.18.

$$\mu_{\tilde{A}}(y) = \begin{cases} 0, if\ y \leq a_1 \\ \frac{y-a_1}{a-a_1}, if\ a_1 \leq y \leq a \\ \\ \frac{b_1-y}{b_1-a}, if\ a \leq y \leq b_1 \\ 0, if\ y \geq b_1 \end{cases} \tag{1.38}$$

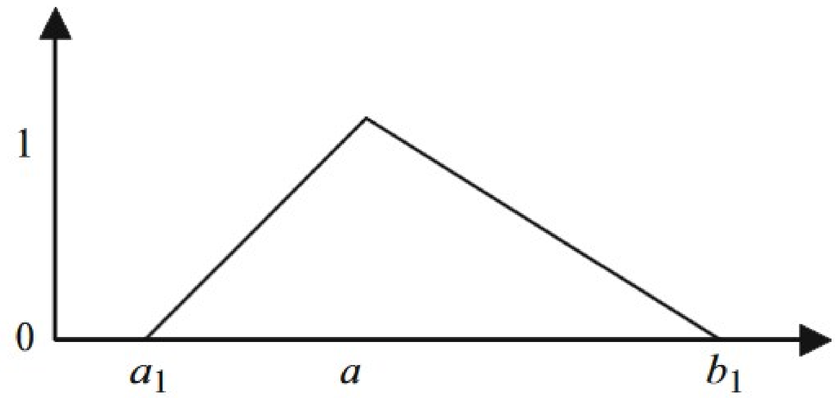


Figure 1.18. Triangular membership function

1.8.2 Gaussian membership function

Gaussian membership function can be given by equation (1.39), which is shown in Figure 1.19.

$$\tilde{A}(y) = ce^{\frac{(y-a)^2}{b}} \tag{1.39}$$

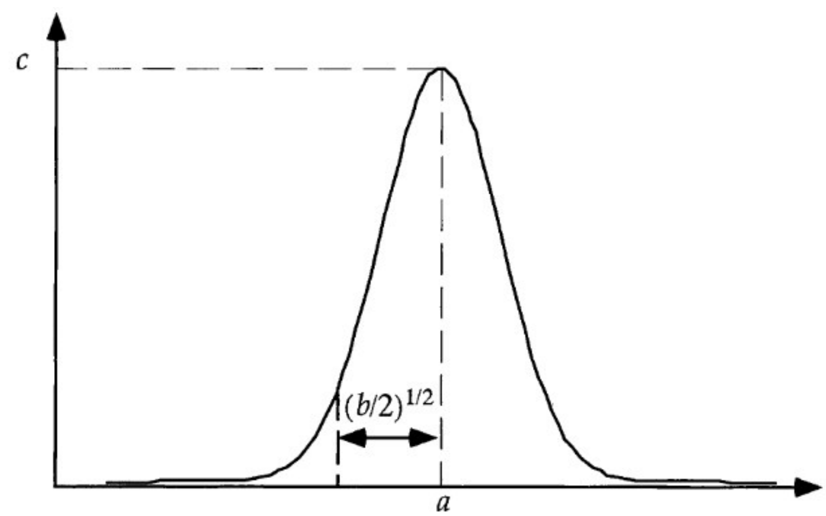


Figure 1.19. Gaussian membership function

1.8.3 Bell-shaped membership function

Bell-shaped or generalized bell-shaped membership function is a three parametric function that can be represented by Figure 1.20 and with equation (1.40).

$$\tilde{A}(y,a,b,c) = \frac{1}{1 + \left|\frac{y-c}{a}\right|^{2b}} \tag{1.40}$$

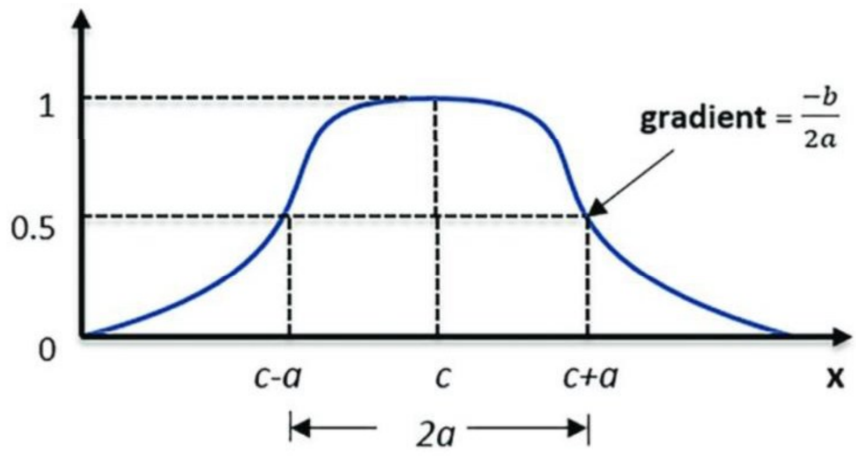


Figure 1.20. Bell-shaped membership function

1.9 MCDM techniques

Decisions made in real life for solving real-world problems are multifactorial and unorganized, which shouldn't be explained by considering one criterion which can obtain undesirable results. For achieving an optimal solution for realistic and better decision, MCDM techniques which are a branch of operations research constitutes several criteria are used to enhance decision making skills by utilizing decision support tools to tackle complex problems.

Choosing the best optimal solution from a given set of alternatives and ranking them in accordance to decision maker are the properties of MCDM (discrete) techniques which are addressed by using multi-attribute decision making techniques. Further, continuous MCDM techniques deal with an infinite set of alternatives that are addressed by using multi-objective decision making (MODM) techniques. Structuring, planning and solving problems are the functions of MCDM techniques. There is no optimal solution for the decision making problems and vary depending upon the decision maker.

The field of MCDM has been a topic of grave importance to many of the researchers since the 1970s. There are several MCDM-related organizations, including the International Society on Multi-criteria Decision Making. It is evident from the literature that several MCDM techniques were proposed to solve diverse problems (Jha and Puppala 2017; Yanık and Eren 2017). Some of the MCDM techniques include Analytic Hierarchy Process, Case-based Reasoning, Data

Envelopment Analysis, DEMATEL, ELECTRE, PROMETHEE, TOPSIS, and VIKOR (Wang and Chan 2013; Jha and Puppala 2017; Yanık and Eren 2017).

1.10 Fuzzy MCDM techniques

The day to day problems witnessed are not certain and are vague in nature. The criteria considered to solve these problems possess interactive and highly dependent features that are difficult to calculate by conventional MCDM techniques (Ertuğrul and Karakaşoğlu 2009; Wang and Chan 2013). Hence fuzzy MCDM techniques are useful to estimate the human subjective evaluation process. In the last few years, numerous attempts to handle this uncertainty, imprecision and subjectiveness have been carried out basically by fuzzy set theory, and the applications of fuzzy set theory to multi-criteria evaluation methods under the framework of utility theory have proven to be a practical approach

On the contrary to MCDM techniques, fuzzy MCDM techniques are employed to gauge the best alternatives in accordance with predetermined criteria. This technique is not restricted to an individual decision maker but can be extended to a group of decision makers. The techniques have various intermediate steps, such as alternatives versus criteria evaluation, criteria versus criteria evaluation, etc. The criteria weights are assessed by linguistic values, which are represented by fuzzy numbers. The linguistic variable can be defined as sentences or words in artificial or natural languages.

In the majority of fuzzy MCDM techniques, the values obtained at the final steps are in fuzzy form and proper defuzzification methods are applied to convert into crisp sets, which is discussed in subsequent sections. Further, in fuzzy MCDM problems, the final evaluation values of alternatives are still fuzzy numbers, and these fuzzy numbers need a proper ranking approach to defuzzify them into crisp values for decision making.

Among all the fuzzy MCDM techniques, fuzzy AHP is widely used in the field of finance, engineering and portfolio management since it effectively tackles decision-making problems involving numerous criteria and alternatives. In recent times, the fuzzy set theory is induced with type - 2 fuzzy sets and type- n fuzzy sets, which are inherited with various advantages.

1.11 Fuzzification

Fuzzification can be defined as the process of converting the crisp judgments into fuzzy sets within the input space, and the component which helps in functioning fuzzification is the fuzzifier. At the initial stage of the process, the fuzzification function is instigated to every input variable to measure the uncertainty involved. The fuzzification function is utilized to elucidate and evaluate input variables, which are a crisp real number transformed to many practical approximations of fuzzy number with respect to the crisp real number. Some of the commonly utilized fuzzification methods include rank-ordering, inference, neural network, intuition, and angular fuzzy sets.

In most of the scenarios, the variables are not in fuzzified form. Hence to convert the input variables which are crisp in nature, a singleton method of fuzzification is solicited for generating fuzzified form, and the input variables are indulged in the process of inference for measurement purposes. Singleton fuzzification transforms crisp value 'x_0' which is input variables to a fuzzy set whose support can be reduced to 'x_0'. The cost of the computation involved in the process is meager due to the simplified output calculation. Even though singleton fuzzification has its advantages, it alone may not be sufficient in heavy noise or corrupted input data. Non-singleton fuzzification is necessary to account for data uncertainty (Zimmermann 2001; Ross 2010). The fuzzification function 'f_f' is given by (1.41)

$$f_f: [-h, h] \to X \tag{1.41}$$

Here X represents power set of fuzzy sets $f_f(x_0)$ is a fuzzy approximation of the measurement 'x_0'.

1.12 Defuzzification

After exercising the fuzzy operations, defuzzification is done in the final stage to get a crisp output value, since most of the practical output values are crisp in nature. For example, in many engineering applications of fuzzy systems such as fuzzy controls, defuzzification is an essential part of getting an output value which is crisp to satisfy the rules of inference. Defuzzifier helps the output value transform to a crisp point from a fuzzy set value. There are many methods to determine the defuzzified score by the level of optimism. Some of the widely used defuzzification techniques include the center of sums, method of total integral value, weighted average method, maximum membership principle, and centroid method. The method of total integral value is considered for a wide range of benefits (Liou and Wang 1992). The estimation of a fuzzy number by using the method of total integral value can be mathematically given by (1.42). This defuzzification method is utilized in chapter 3.

$$I_T^A = 0.5[\lambda b_3 + (1-\lambda)b_1 + b_2], \qquad \lambda \epsilon [0,1] \tag{1.42}$$

where, λ is the optimism index, which indicates the decision maker's level of risk. The value λ is directly proportional to optimism degree. Generally, $\lambda = 0, 0.5, 1$ corresponds to pessimist, moderate and optimist, respectively.

Many researchers had proposed different strategies to explain the concept of defuzzification through nature based optimization techniques such as Particle Swarm Optimization, Artificial Bee Colony, and Genetic Algorithms for fuzzy clustering of data (Roychowdhury and Pedrycz 2001). Eventually, these clustering techniques have evolved and are integrated with advanced fractional calculus operators for much smoother updation of position. Various operators such as Reimann-Liouville, Caputo and Grunwald–Letnikov fractional operators are utilized depending upon the nature of optimization problems (Chander *et al.* 2018). This thesis further explores the fields of fractional modeling of criminal transmission and the application of fractional clustering algorithms in financial portfolio management. One of the widely used optimization techniques used in the areas of SDGs such as "Innovation Industries and Infrastructure," and "Decent work and economic growth" is Particle Swarm Optimization discussed in the next section.

1.13 Nature based Optimization Techniques

Optimization techniques are generally used to detect the best element from a given set of alternatives. These techniques are utilized in a wide range of disciples, such as computer science, engineering, economics, finance, and operations research. Through systematic selection from the given input values, the optimization technique enables the optimization problem to find the optimal point of a function.

The mathematical optimization dates back to Lagrange and Fermat, where they formulated a calculus-based optimization technique. While Gauss and Newton introduced optimization problems based on iterative methods. In the year 1947, Dantzig proposed a Simplex algorithm, which is a popular algorithm for linear programming. In the same year, John von Neumann proposed duality theory, which gives a bound to the optimal value.

However, real-life optimization problems involve high NP complexity. To address such complexities, many techniques such as the Bruss algorithm, Chaff algorithm, Cuckoo search, Firefly algorithm, Genetic algorithm, Gradient descent, and Particle swarm optimization (PSO) are developed. Among them, Cuckoo search, Firefly algorithm, and PSO grabbed the attention due to their serious problem tackling efficiency. Over the past decade, nature-based optimization techniques are adopted due to their physical applicability.

Almost all algorithms can be classified into four considerable clusters, namely bio-inspired algorithms, chemistry or physics-based algorithms, swarm intelligence, and the rest of all other techniques. Swarm intelligence algorithms evaluate the collective and emerging behavior of nature-based agents. The agent may not necessarily be intelligent but follow a self-organization pattern and collectively form an intelligent agent. These agents can be animal societies such as ants, bees, swarms, and wasps. Even birds, flocks, or fishes can also behave like multi-agents. These swarm-based algorithms are the most commonly used techniques among optimization problems, especially in the fields of control theory, finance and economics.

PSO (Zamani *et al.* 2009) is a swarm intelligence based algorithm that is evolved on population and designed by researching birds flocking and fish schooling. It is easily one of the most utilized and powerful optimization techniques in the recent decade due to its simple implementation and robust problem-solving technique such as maxima and minima, higher dimensionality, nonlinearity, and non-differentiability. With the increase in computational efficiency, PSO converges with an increase in instability.

PSO is solved by assuming the input data points as swarm particles moving in D dimensional space with the objective is to optimize a fitness problem. A particular position is assigned to all swarm particle using position vectors $P_i = (p_{i1}, p_{i2}, \dots, p_{iD})$ along the velocity vector $V_i = (v_{i1}, v_{i2}, \dots, v_{iD})$ with a maximum velocity vector $V_{max} = (v_{max1}, v_{max2}, \dots, v_{maxD})$. Each swarm remembers its self best position by the vector $B_i = (b_{i1}, b_{i2}, \dots, b_{iD})$. Here i denotes the particle index. The neighboring particles best position of all the neighbors stored can be given by the vector $B_n = (b_{n1}, b_{n2}, \dots, b_{nD})$. The updated velocity and position for each swarm can be mathematically shown by (1.43)

$$v_{id}^{(t+1)} = w v_{id}^{(t)} + c_1 r_1 \left(b_{id} - p_{id}^{(t)} \right) + c_2 r_2 (b_{nd} - p_{nd}^{(t)}) \quad (1.43)$$

$$p_{id}^{(t+1)} = p_{id}^{(t)} + v_{id}^{(t+1)}; \; d = 1,2, \dots, D$$

where the variable w represents inertia weights and is evaluated in accordance with (1.44)

$$w = w_{max} - \frac{w_{max} - w_{min}}{iter_{max}} \; iter \quad (1.44)$$

In general, the values of w_{max} and w_{min} are defaulted at 0.9 and 0.4 respectively during run.

The variable representation is given below

$iter_{max}, iter$ represents the maximum number of iterations and a present number of iterations, respectively.

c_1, c_2 - Acceleration constants of all particles which affected the convergence speed and defaulted to 2.0.

r_1, r_2 - Two random variables ranging from 0 to 1.

In most of the cases during PSO run, V_{max} is generally set to a maximum dynamic range of the variables in each dimension, $v_{dmax} = p_{dmax}$.

1.14 Organization of thesis

- A general introduction, along with background and need of SDG's is presented in Chapter-1. The objectives of the thesis and the mathematical techniques used in the thesis are also discussed in this chapter.
- A new fractional order controller proposed in Chapter 2. The proposed controller has been tested on an isolated and two area power systems to mitigate frequency variations such as affecting overshoot, settling time and rise time. Further, the sensitivity of the system to human controllable parameters to mitigate frequency variations like overshoot, settling time and rise time is analyzed and the findings are presented in this chapter.
- A portfolio management strategy is introduced in Chapter 3 and Chapter 4, which helps novice investors for managing their portfolio by the perception of an investor, is presented.
- Chapter 3 explicitly analyzes six sectors in BSE SENSEX using two fuzzy MCDM's, Fuzzy AHP and Fuzzy TOPSIS, to find the cumulative dominance of each sector. Various parameters that help to measure the performance of the sector/market are presented in this chapter.
- Chapter 4 constructs three portfolios based upon the perception of an investor using an advanced clustering algorithm known as Fractional Lion Algorithm (FLA) by using two introduced indicators named as P-index and MCV. The proposed methodology determines the maximum and minimum share that can be invested in a market/sector is proposed by considering investor's perception.
- A new fractional order mathematical model that captures the crime transmission is proposed in Chapter 5. Subsequently, the equilibria of the three-dimensional fractional crime transmission model are evaluated using phase-plane analysis. The Lyapunov

function is employed to determine stability and threshold conditions to achieve a crime-free society.