Chapter 2

Transient Analysis of Variant Abandon and Vacation Policy

"Synchronous consensus applies to real-time systems, in which dedicated hardware means that messages will always be passed with specific timing guarantees."

Betsy Beyer

2.1 Introduction

In an active redundancy, a fault-tolerant machining system uses identical operating units functioning in parallel. Fault tolerance is a non-critical condition adapted by the machining system to continue working appropriately when some of its operating units fail randomly. The study of machine repair problems has remarkably increased and become significant with the development of production industries. Many diverse models have been well-thought-out by researchers, and comprehensive surveys on machine repairable problems have been done (*cf.* Sztrik and Bunday [165]; Haque and Armstrong [59]; Shekhar et al. [159]). The main apprehension in all past studies is about the reliability characteristics of the fault-tolerant machining system.

Passive redundancy is a backup arrangement of functional capabilities in the machining system that would be inoperable in a fault-free environment. The redundant machining system comprises the spare unit(s) that switch to the operable state automatically as the operating unit fails randomly. For the preventive maintenance of the fault-tolerant machining system, the backup arrangement of spare units has been preferred in general and useful to enhance or maintain high system reliability and availability. The spare unit in passive redundancy is categorized into three classifications relying on its failure attributes as follows: cold, warm, and hot. Cold spare means that the redundant unit does not fail while it is in an inoperable state, the warm spare means that the redundant unit may fail at the inoperable state with rate less than the failure rate of an operating unit, and the hot spare means that the redundant unit may also fail randomly in the inoperable state at the indistinguishable rate of failure of an operating unit. The scientific study of the redundant repairable machining systems was advanced by numerous researches (*cf.* Huang et al.[66]).

In graceful degradation, fault-tolerant machining system functions continuously as a whole even when some of its operating units break down. The graceful degradation is alluded as (m,M) degradable machining system (*cf.* Jain et al.[84]). For normal mode, every unit in the machining system comprises M operating units must work. In short mode, a machining system may also continue in function even it has lesser operating units in work than the required (M) but more than the least number of required operating units (m). The load sharing with the lesser number of operating units may lead to more likely of their failure.

The likelihood of a spare unit for being failed in switching from a standby state to the operating state on the failure of the operating units always exists due to some automation failure. The switching failure leads to diminishing the reliability of the fault-tolerant machining system and requires a special research investigation. Accordingly, the present study considers the switching failure possibility of the spare unit(s). Lewis [116] was first to introduce the concept of the switching failure from standby state to operating state of spare units in his text on the reliability of the spare provisioning system. The pioneer investigations with mathematical modeling of spare units' switching failure in machining system can be found in the contribution of the specialized researchers (*cf.* Jain et al. [85]; Jain et al. [82]; Jain and Rani [80]). Ke et al. [98] used the supplementary variable method with a recursive methodology to derive the stationary probabilities of the machine repair problem with general repair times of failed units and spare unit' switching failure. Kuo and Ke [111] compared the cost/benefit ratio and availability among different spare provisioning frameworks with switching failure and an unreliable server. Sadeghi and Roghanian [149] employed a Markov process, and Laplace transforms to determine the explicit expressions of mean time-to-failure and steady-state availability of warm spare repairable machining system with the imperfect switching mechanism.

The fault-tolerant machining system with repairman' multiple vacations has been studied extensively in earlier research. In the classical multiple vacation policy, the repairman is not available for repairing the failed units for some random period after completion of a required job. It reduces the idle time of the repairman, the expected cost of service, heat-up of the processor in computing and processing devices. But this may also prompt the loss, dissatisfaction or delay in repairing of the prospective failed units due to lack of just-in-time availability (*cf.* Zhang and Guo [205]).

Servi and Finn [152] introduced a new sort of semi-vacation policy for server called working vacation (WV) policy, in which the server continues to work at a slower rate rather than completely terminates during a vacation period in single-server Markovian queue of gateway router in fiber communication networks and derived the probability generating function (PGF) of the queue size and sojourn time in steady-state. In working vacations, the server may accomplish other jobs simultaneously for additional revenue and also quite reasonable for the repair of the failed units. A working vacation, a bit of an oxymoron in nature for a repairman, is the ideal balance between service and vacation. The major and iconic contributions in studying working vacation policy can be found in the following scientific literature (*cf.* Ke and Wu [100]; Yang and Wu [194]; Jain et al. [84]). The detailed overview of working vacation with variants of service and arrival was furnished by Chandrasekaran et al. [21]. Jain et al. [83] used Newton-quasi method and a direct search method for optimal analysis of the machining system operating under admission control and server working vacation policy.

In the working vacation policy, it is the usual strategy that server recommences service with the normal rate only when the system has a waiting failed unit at the end of a vacation. Without a doubt, such a proposition seems significantly restrictive in real-time. To overcome this imperative, Li and Tian [118] acclimated the vacation interruption schedule in an $M/M/1$ queue with working vacations. Under this sort of working vacation scheduling, if there are more than pre-specified threshold customers waiting at the moment of a service consummation in the vacation period, the server terminates his vacation and renders the service at a normal rate; otherwise, continues the vacation until the vacation epoch ends. Attributable to the strong application in the stochastic service systems, many useful and satisfactory theoretical researches are exhibited. The major work done on vacation interruption with different queueing variants are found in following research articles (*cf.* Li et al. [117]; Baba [13]; Zhang and Hou [201]; Liu et al. [125]). The realistic phenomenon vacation interruption schedule for the fault-tolerant machining system is also included in this chapter.

In general, the caretaker of failed units is impatience in a long waiting queue, although the repairman continuously provides repair to the failed units and more prompt in the absence of a repair facility. To diminish discouragement in the absence of in-house repairman due to vacation, in this investigation, the assumption of a different kind of source of abandonment for the benefit to waiting caretaker is also considered. The extraneous repair facility at additional expenses is made available following Poisson process when in-house repairman is on working vacation and provides the intended repair to failed units. In present assumption, caretakers abandon the machining system simultaneously for the seek of service by extraneous repair facility following some point process instead of the standard supposition of one by one independent abandonment. At the abandonment epochs, every waiting caretaker may abandon the waiting queue with probability p independent to the state of the others. The synchronized abandon follows binomial distribution at the abandonment epochs. To enrich the present literature on synchronized abandonment, some keynote works of pioneer researchers, mathematicians, etc, are available in following articles (*cf.* Yang et al. [196]). Shekhar et al. [153] analyzed the geometric reneging of failed units in the machining system with the provision of spare units and studied the various type of reliability characteristics.

Maintaining a high or required level of reliability and/or availability is a fundamental imperative of a fault-tolerant machining system. Markovian analyses are more likely in the dynamic system and may be well-suited in practice in designing such a system. It is essential to examine a redundant Markovian repairable machining system with switching failures, vacation interruption, and synchronized abandonment behavior of failed units. The derivation of the explicit expressions of the reliability function and the mean time-to-system failure $(MTTF)$ is also shown. The literature survey on reliability modeling and analysis is summarized in the following articles (*cf.* Lv et al. [126]; Ke et al. [67]; EI-Damcese and Shama [39]). Ahmad et al. [2] studied various modeling and analysis techniques that could aid in the study of the reliability and presented rich literature review.

Generally, finite population queueing models with vacation have usually accentuated steady-state or equilibrium performance. Inferring steady-state measures of system performances do not bode well for systems that never approach equilibrium. The steady-state is appropriate to study the performance of systems on a long time scale while the transient-state is valuable for studying the dynamical behavior of systems over a limited time horizon. The transient analysis helps to comprehend the behavior of a system when the parameters involved are perturbed. Stationary results are mainly used within the system design process. However, due to both variability and vulnerability of the failure of the unit, robust performance, in general, is difficult to accomplish simultaneously within the design process. Thus, some administrative decisions have to be taken with the aim of dynamically adapting the resource allocation for the system load. The administrative actions can often be based on transient analytical results for a queueing model precisely representing the system of interest. Moreover, transient analytical results are specifically useful for studying the finite-time characteristics of queueing systems. Some phenomenal articles on transient analysis are enriched by following researchers (*cf.* Jain and Preeti [78]; Ammar [7]). Recently, Shekhar et al. [156] employed a Runge-Kutta approach for the transient investigation and sensitivity analysis of the redundant repairable system with probabilistic spare switching failure and geometric reneging.

The content of this chapter varies from recently published works in (i) it considers the spare switching failures, vacation interruption of the server and synchronized abandon of the failed units; (ii) it presents advanced methodology to determine the transient-state probabilities in very less computational time; and (iii) it performs sensitivity analysis for the reliability characteristics with different system parameters.

The investigation of abandon of the failed units in the fault-tolerant machining system within the class of queueing systems with modified working vacations, including its interruption, is a new endeavor. The purpose of this investigation is to accomplish three objectives. The first is to present the mathematical model of state-of-the-art multi-unit system design, having various realistic issues related to a machining system. The second is to present the mathematical approach, a Laplace transform method for deriving the transient-state probability that the system has failed on or before time from which the reliability characteristics and queueing characteristics can be attained. The third is to perform a parametric examination of reliability and queueing attributes and sensitivity analysis of the system reliability and the mean time-to-failure along with changes in explicit estimations of the system design parameters.

The rest of this chapter is organized as follows. In section (2.2), the detailed description of the quasi-birth and death process of the model is given along with practical justification and governing equations. In the next section (2.3), the queue-size distribution and the state probabilities of the machining system are determined in a transient-state using the theory of Laplace Transforms, Eigenvalues, and Eigenvectors. The next section (2.4) describes various performance indices viz queueing characteristics and reliability characteristics to check the legitimacy of the model. Section (2.5) describes the procedure to audit the sensitivity using the theory of calculus. In section (2.6), we correlate speculation of the studied model with previous valid and published models as special cases. In section (2.7) and (2.8), procedure and simulation results are presented for sensitivity analysis for performance measures with respect to system parameters, respectively. The chapter concludes with section (2.9), where the conceivable speculations and future extension are discussed.

2.2 Model Description

2.2.1 Assumptions and Notations

In this investigation, the redundant repairable machining system are considered wherein M indistinguishable operating units are working simultaneously in parallel with the provision of S warm spare units and a reliable repairman. For the normal functioning of the machining system, all M units are required, alluded as normal mode, but will also be functioning safely in short mode, if there is at least $m\,(1 < m < M)$ operating units in the machining system. Hence, the number of failures of the unit allowed is $L = M + S - m + 1$. The fault-tolerant machining system is referred to as (m,M) machining system. System reliability is investigated according to the assumptions that the system is safe when at least m operating units are working. Following are some more assumptions and notations which are considered for modeling purpose:

Failure Process

- Each operating (or spare) unit fails autonomously to the failure condition of the other, and the time-to-failure of each operating (or spare) unit is exponentially distributed with the rate $\lambda\,(\nu, 0 < \nu < \lambda)$. A failed unit is sent for repairing instantaneously.

- On failure of an operating unit, the available spare unit switches to the working state promptly and has the same functioning and failure qualities as an operating unit. There may be a plausibility of switching failure from the standby state to working state with the likelihood q for all available spare units.

- On the exhaust of all spare units, the time-to-failure of each operating unit is exponentially distributed with an increased rate of λ_d. Hence, the fault-tolerant machining system alludes to a degraded system.

Vacation Policy

- When there is no failed unit to be repaired in the machining system, the repairman takes the working vacation of a random period. The time-for-on vacation also follows an exponential distribution with the meantime $1/\theta$. The repairman follows a multiple vacation policy in which the repairman takes another vacation of random duration if he finds no waiting failed units to be repaired in the machining system at the end of the vacation time.

- In short mode, due to the overload of failed units in the system, vacation time interrupts in short immediately, and the repairman continues to repair with a normal rate.

Repair Process

- Failed units are repaired in the order of breakdowns, with a time-to-repair which is exponentially distributed with parameter μ_b. The repaired unit is equivalent to the new unit and sent in the pool of operating units, or spare units instantaneously relies on a state of the system.

- During working vacation, the repairman keeps on repairing the failed units, and time-to-repair also follows an exponential distribution with rate parameter $\mu_v\,(\mu_v < \mu_b)$.

Abandon Policy

- During the working vacation epoch of the repairman, the system relies on an extraneous repair facility also to avoid the long waiting queue of failed units at some additional expenses. The time-to-arrival of the extraneous repair facility follows an exponential distribution with parameter ξ.

- On arrival of the extraneous repair facility, some or all failed units may abandon the system following a point process. Each failed unit may abandon the system independent of the other failed units with the probability of abandoning p.

All events operating/spare unit failure, switching failure, repair, abandon, repairman vacation, vacation interruption, etc. are independent of each other.

2.2.2 Practical Justification of the Model

In the improvement of the internet of things (IoT), cloud technology is a greatly evolving issue from the small endeavor to worldwide enterprises. The cloud model, infrastructure as a service (IaaS), is made of exceedingly versatile and mechanized computing resources and assets. Through virtualization innovation, IaaS is completely self-service for accessing and monitoring virtual machines (VM) like a computer, networking, storage, and other services, and it enables organizations to buy or get on rent resources on-demand and as-needed as opposed to purchasing the equipment outright physically. IaaS gives indistinguishable advancements and abilities as a customary service center without having owned infrastructure physically and without botheration to maintain or manage all of it. Some outstanding IaaS suppliers are DigitalOcean, Linode, Rackspace, Amazon Web Services (AWS), Cisco Metapod, Microsoft Azure, Google Compute Engine (GCE).

To fulfill the desired level of the quality of the service (QoS), a failed virtual machine (VM) is replaced immediately by an available warm spare VM, and the failed VM,s malfunctioning is repaired at in-house service center immediately. The switching of a warm spare VM may also fail due to an error in the kernel program or root filesystem. VM repair server may exhibit working vacation and provide a slower service rate to restore software or hardware efficiency or to avoid the problem, such as abnormal program execution or disk hot-swap. This vacation may be interrupted as and when required. To overcome the load of failed VM, the service center may opt services of outsourced repair servers on rent basis, which abandons some failed VM from service center for early restoration.

The studied model assumptions are suitable for such cloud service providers. The current investigation gives the platform for analyzing the reliability issues, predicting the just-in-time (JIT) service quality, and state-of-the-art infrastructure design.

2.2.3 Chapman Kolmogorov Differential-Difference Equations

Initially, at time $t = 0$, the system has all operating (or spare) units in the normal state, i.e., no failed units since the system has just started in operation. Firstly, the transient-state governing equations are formulated and then spectral method as well as an efficient MATLAB program is used to compute the transient-state probabilities.

The states of the machining system are structured in notation by the pair

$$\{(I(t), N(t)) \cup F(t) | I(t) = 0, 1; N(t) = 0, 1, 2, ..., L; t \geq 0\}$$

where

$$I(t) \equiv \begin{cases} 0; & \text{Repairman is in working vacation state at time } t \\ 1; & \text{Repairman is in a busy state at time } t \end{cases}$$

$$N(t) \equiv \text{Number of failed units in the system at time } t$$

and

$$F(t) \equiv \text{The system is in failed state at time } t$$

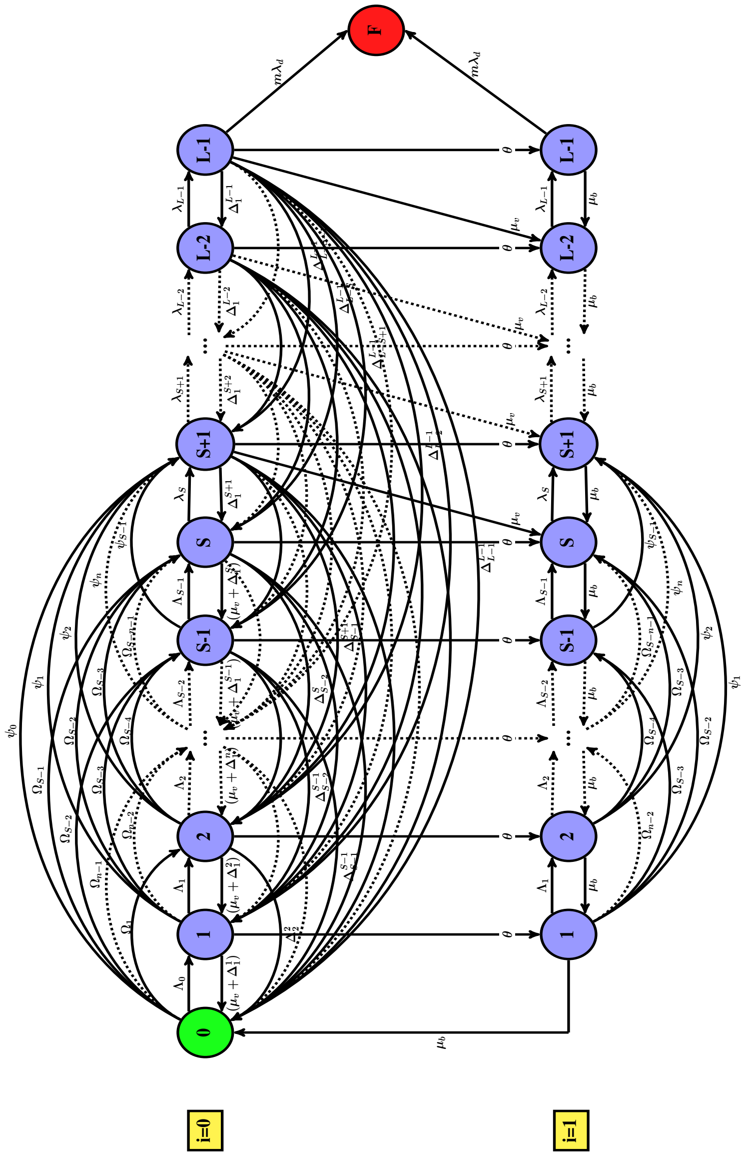

Figure 2.1: The state transition diagram

The transient-state probabilities at time $t\,(t \geq 0)$ are defined as follows:

$$P_{0,n}(t) = \text{Prob}\,\{I(t) = 0, N(t) = n\}\,; n = 0,1,2,...,L-1$$
$$P_{1,n}(t) = \text{Prob}\,\{I(t) = 1, N(t) = n\}\,; n = 1,2,...,L-1$$
$$P_F(t) = \text{Prob}\,\{\text{The system is in failed state at time } t\}$$

Referring state change in state-transition diagram drawn in Fig. (2.1), transient-state Chapman-Kolmogorov differential-difference equations of the studied MRP with working vacation interruption and abandon of failed units are as follows:

Case I: When repairman is on working vacation $(I(t) = 0)$

$$\frac{dP_{0,0}(t)}{dt} = -(M\lambda + S\nu)P_{0,0}(t) + \mu_\nu P_{0,1}(t) + \sum_{i=1}^{L-1} p^i \xi P_{0,i}(t) + \mu_b P_{1,1}(t) \tag{2.1}$$

$$\begin{aligned}\frac{dP_{0,1}(t)}{dt} =& -(M\lambda + (S-1)\nu + \mu_\nu + \xi + \theta)P_{0,1}(t) + (M\lambda(1-q) + S\nu)P_{0,0}(t) \\ &+ \mu_\nu P_{0,2}(t) + \sum_{i=1}^{L-1} \binom{i}{i-1} p^{i-1}(1-p)\xi P_{0,i}(t)\end{aligned} \tag{2.2}$$

$$\begin{aligned}\frac{dP_{0,n}(t)}{dt} =& -(M\lambda + (S-n)\nu + \mu_\nu + \xi + \theta)P_{0,n}(t) \\ &+ (M\lambda(1-q) + (S-n+1)\nu)P_{0,n-1}(t) + \sum_{i=0}^{n-2} M\lambda(1-q)q^{n-i-1}P_{0,i}(t) \\ &+ \mu_\nu P_{0,n+1}(t) + \sum_{i=n}^{L-1} \binom{i}{i-n} p^{i-n}(1-p)^n \xi P_{0,i}(t); \quad 2 \leq n \leq S-1\end{aligned} \tag{2.3}$$

$$\begin{aligned}\frac{dP_{0,S}(t)}{dt} =& -(M\lambda_d + \mu_\nu + \xi + \theta)P_{0,S}(t) + (M\lambda(1-q) + \nu)P_{0,S-1}(t) \\ &+ \sum_{i=0}^{S-2} M\lambda(1-q)q^{S-i-1}P_{0,i}(t) + \sum_{i=S}^{L-1} \binom{i}{i-S} p^{i-S}(1-p)^S \xi P_{0,i}(t)\end{aligned} \tag{2.4}$$

$$\frac{dP_{0,S+1}(t)}{dt} = -\left((M-1)\lambda_d+\mu_v+\xi+\theta\right)P_{0,S+1}(t)+M\lambda_d P_{0,S}(t)+\sum_{i=0}^{S-1}M\lambda q^{S-i}P_{0,i}(t) + \sum_{i=S+1}^{L-1}\binom{i}{i-S-1}p^{i-S-1}(1-p)^{S+1}\xi P_{0,i}(t) \tag{2.5}$$

$$\frac{dP_{0,n}(t)}{dt} = -\left((M+S-n)\lambda_d+\mu_v+\xi+\theta\right)P_{0,n}(t)+(M+S-n+1)\lambda_d P_{0,n-1}(t) + \sum_{i=n}^{L-1}\binom{i}{i-n}p^{i-n}\,(1-p)^n\xi\;P_{0,i}(t);\quad S+2\le n\le L-2 \tag{2.6}$$

$$\frac{dP_{0,L-1}(t)}{dt} = -\left(m\lambda_d+\mu_v+\xi+\theta\right)P_{0,L-1}(t)+(m+1)\lambda_d P_{1,L-2}(t) + (1-p)^{L-1}\xi P_{0,L-1}(t) \tag{2.7}$$

Case II: When the repairman is in the busy state $(I(t)=1)$

$$\frac{dP_{1,1}(t)}{dt} = -\left(M\lambda+(S-1)\nu+\mu_b\right)P_{1,1}(t)+\mu_b P_{1,2}(t)+\theta P_{0,1}(t) \tag{2.8}$$

$$\frac{dP_{1,n}(t)}{dt} = -\left(M\lambda+(S-n)\nu+\mu_b\right)P_{1,n}(t)+\left(M\lambda(1-q)+(S-n+1)\nu\right)P_{1,n-1}(t)) + \mu_b P_{1,n+1}(t)+\sum_{i=1}^{n-2}M\lambda(1-q)q^{n-i-1}P_{1,i}(t)+\theta P_{0,n}(t);\, 2\le n\le S-1 \tag{2.9}$$

$$\frac{dP_{1,S}(t)}{dt} = -(M\lambda_d+\mu_b)P_{1,S}(t)+\left(M\lambda(1-q)+\nu\right)P_{1,S-1}(t)+\mu_b P_{1,S+1}(t) + \sum_{i=1}^{S-2}M\lambda(1-q)q^{S-i-1}P_{1,i}(t)+\theta P_{0,S}(t)+\mu_v P_{0,S+1}(t) \tag{2.10}$$

$$\frac{dP_{1,S+1}(t)}{dt} = -\left((M-1)\lambda_d+\mu_b\right)P_{1,S+1}(t)+M\lambda_d P_{1,S}(t)+\mu_b P_{1,S+2}(t)\sum_{i=1}^{S-1}M\lambda q^{S-i}P_{1,i}(t) + \theta P_{0,S+1}(t)+\mu_v P_{0,S+2}(t) \tag{2.11}$$

$$\begin{aligned}\frac{dP_{1,n}(t)}{dt} = & -((M+S-n)\lambda_d+\mu_b)P_{1,n}(t)+(M+S-n+1)\lambda_d P_{1,n-1}(t) \\ & +\mu_b P_{1,n+1}(t)+\theta P_{0,n}(t)+\mu_v P_{0,n+1}; \quad S+2\leq n\leq L-2\end{aligned} \tag{2.12}$$

$$\frac{dP_{1,L-1}(t)}{dt} = -(m\lambda_d+\mu_b)P_{1,L-1}(t)+(m+1)\lambda_d P_{1,L-2}(t)+\theta P_{0,L-1}(t) \tag{2.13}$$

Case III: When the system is in down state (F)

$$\frac{dP_F(t)}{dt} = m\lambda_d P_{0,L-1}(t)+m\lambda_d P_{1,L-1}(t) \tag{2.14}$$

2.3 Transient Probabilities Analysis

In this section, the transient-state probabilities for the state of the machining system are computed from the system of differential difference Eq^{n}'s. (2.1)-(2.14) summarized in previous section following model description. For this purpose, the Laplace transforms of the state probabilities of number of failed units in the system and their derivatives are defined as:

$$\tilde{P}_{i,n}(s) = L(P_{i,n}(t)) = \int_0^{\infty} e^{-st}P_{i,n}(t)\,dt$$

and

$$L\left(\frac{dP_{i,n}(t)}{dt}\right) = s\tilde{P}_{i,n}(s)-P_{i,n}(0)$$

where s is Laplace variable. For the notational ease, the state probabilities are denoted in single subscript index as pursues:

$$\begin{aligned}&[P_{0,0}(t)\ldots P_{0,L-1}(t)]^T \equiv [\pi_1(t)\ldots\pi_L(t)]^T \\ &[P_{1,1}(t)\ldots P_{1,L-1}(t)]^T \equiv [\pi_{L+1}(t)\ldots\pi_{2L-1}(t)]^T \\ &P_F(t)\equiv\pi_{2L}(t)\end{aligned}$$

Hence, the corresponding Laplace transforms of state probabilities are $\tilde{\pi}_r(s) = L\{\pi_r(t)\},\ 1\leq r\leq 2L$. We also define column vectors of order $2L$ having state proprobabilities as elements as follows

$$\tilde{\mathbf{\Pi}}(s) = [\tilde{\pi}_1(s),\tilde{\pi}_2(s),\tilde{\pi}_3(s),\tilde{\pi}_4(s),\ldots,\tilde{\pi}_{2L-1}(s),\tilde{\pi}_{2L}(s)]^T$$

$$\mathbf{\Pi}(0) = [\pi_1(0),\pi_2(0),\pi_3(0),\pi_4(0),\ldots,\pi_{2L-1}(0),\pi_{2L}(0)]^T$$

On applying Laplace transforms of the state probabilities and their derivatives as defined above, the system of differential-difference Eq^{n}'s. (2.1)-(2.14) is changed to system of linear equations which can be expressed in the matrix form as follows

$$\mathbf{\Lambda}(s)\tilde{\mathbf{\Pi}}(s) = \mathbf{\Pi}(0) \tag{2.15}$$

where $\mathbf{\Lambda}(s)$ is the square matrix of order $2L$ having coefficients as elements. The Cramer's rule is employed for solving the matrix equation and obtain an explicit expression of Laplace transform of state probabilities $\tilde{\pi}_r(s)$ as

$$\tilde{\pi}_r(s) = \frac{|\mathbf{\Lambda}_r(s)|}{|\mathbf{\Lambda}(s)|}; 1 \leq r \leq 2L \tag{2.16}$$

where $\mathbf{\Lambda}_r(s)$ is also square matrix of order $2L$ that is generated from $\mathbf{\Lambda}(s)$ by substituting r^{th} column with the right hand side column vector $\mathbf{\Pi}(0)$. For the explicit expression for state probabilities, inverse Laplace transform is taken as

$$\pi_r(t) = L^{-1}(\tilde{\pi}_r(s))$$

For this purpose, the explicit expression of $\tilde{\pi}_r(s)$ from Eq^{n}.(2.16) is obtained as follows. Since $\mathbf{\Lambda}(s)$ is a coefficient matrix generated from balanced in-flows and out-flows rates which makes it singular in nature, it is primarily noted that $s = 0$ is one latent root of characteristics equation $|\mathbf{\Lambda}(s)| = 0$. which is obtained from the denominator $|\mathbf{\Lambda}(s)|$.

Let $s = -l$ is one of the other unknown latent roots of $|\mathbf{\Lambda}(s)| = 0$. Therefore

$$\mathbf{\Lambda}(-l) = \boldsymbol{A} - l\boldsymbol{I} \tag{2.17}$$

where $\boldsymbol{A} = \mathbf{\Lambda}(0)$ and $\boldsymbol{I}$ is an identity matrix of order $2L$. Now expression in Eq^{n}. (2.15) can also be written as

$$\mathbf{\Lambda}(-l)\tilde{\mathbf{\Pi}}(s) = (\boldsymbol{A} - l\boldsymbol{I})\tilde{\mathbf{\Pi}}(s) = \mathbf{\Pi}(0) \tag{2.18}$$

Since $l_h(\neq 0); h = 1, 2, ..., 2L-1$ are $2L-1$ distinct latent roots of $|\boldsymbol{A} - l\boldsymbol{I}| = 0$ that may be real or complex in nature. Let $l_1, l_2, ..., l_{n_1}$ are n_1 real latent roots and $l_{n_1+1}, \bar{l}_{n_1+1}, ..., l_{n_1+n_2}, \bar{l}_{n_1+n_2}$ are $2n_2$ complex latent roots in conjugate pair such that $n_1 + 2n_2 = 2L - 1$. Therefore,

$$|\mathbf{\Lambda}(s)| = s\left(\prod_{h=1}^{n_1}(s + l_h)\right)\left(\prod_{h=1}^{n_2}\left(s^2 + \left(l_{n_1+h} + \bar{l}_{n_1+h}\right)s + l_{n_1+h}\bar{l}_{n_1+h}\right)\right) \tag{2.19}$$

Hence, Eq^n. (2.16) rewritten as

$$\tilde{\pi}_r(s) = \frac{|\mathbf{\Lambda}_r(s)|}{|\mathbf{\Lambda}(s)|}$$
$$= \frac{|\mathbf{\Lambda}_r(s)|}{s\left(\prod_{h=1}^{n_1}(s+l_h)\right)\left(\prod_{h=1}^{n_2}\left(s^2+\left(l_{n_1+h}+\bar{l}_{n_1+h}\right)s+l_{n_1+h}\bar{l}_{n_1+h}\right)\right)};1 \le r \le 2L \tag{2.20}$$

which can also be expressed in partial fraction form as

$$\tilde{\pi}_r(s) =$$
$$= \frac{a_{0,r}}{s} + \sum_{h=1}^{n_1}\frac{a_{h,r}}{(s+l_h)} + \sum_{h=1}^{n_2}\frac{b_{h,r}s+c_{h,r}}{\left(s^2+\left(l_{n_1+h}+\bar{l}_{n_1+h}\right)s+l_{n_1+h}\bar{l}_{n_1+h}\right)}; \quad 1 \le r \le 2L \tag{2.21}$$

where $a_{0,r}$, $a_{h,r}$, $b_{h,r}$ and $c_{h,r}$ are constants. The value of these constants can be obtained using algebra of matrix theory as follows

$$a_{0,r} = \frac{|\mathbf{\Lambda}_r(0)|}{\prod_{h=1}^{n_1}(l_h)\prod_{h=1}^{n_2}\left(l_{n_1+h}\bar{l}_{n_1+h}\right)} \tag{2.22}$$

$$a_{h,r} =$$
$$\frac{|\mathbf{\Lambda}_r(-l_h)|}{(-l_h)\prod_{\substack{g=1\\g\neq h}}^{n_1}(l_g-l_h)\prod_{g=1}^{n_2}\left(l_h^2+\left(l_{n_1+g}+\bar{l}_{n_1+g}\right)(-l_h)+l_{n_1+g}\bar{l}_{n_1+g}\right)} \quad h=1,2,...,n_1 \tag{2.23}$$

$$b_{h,r}(-l_{n_1+h})+c_{h,r} =$$
$$\frac{\left|\mathbf{\Lambda}_r(-l_{n_1+h})\right|}{(-l_{n_1+h})\prod_{g=1}^{n_1}(l_g-l_{n_1+h})\prod_{\substack{g=1\\g\neq h}}^{n_2}\left((-l_{n_1+h})^2+\left(l_{n_1+g}+\bar{l}_{n_1+g}\right)(-l_{n_1+h})+l_{n_1+g}\bar{l}_{n_1+g}\right)};$$
$$h=1,2,...,n_2 \tag{2.24}$$

On taking inverse Laplace transform of expression in Eq^n. (2.21), an explicit expression of the transient-state probabilities are obtained as follows:

$$\pi_r(t) = a_{0,r} + \sum_{h=1}^{n_1} a_{h,r} e^{-l_h t} + \sum_{h=1}^{n_2} \left[b_{h,r} e^{-u_h t} \cos v_h t + \frac{c_{h,r} - b_{h,r} u_h}{v_h} e^{-u_h t} \sin v_h t \right]; \; 1 \le r \le 2L \tag{2.25}$$

where u_h and v_h are real and imaginary parts of respective complex latent root l_{n_1+h} and $a_{0,r}$, $a_{h,r}$, $b_{h,r}$ and $c_{h,r}$ are already computed constants in Eq^n's. (2.23)-(2.24).

2.4 Performance Measures

In this section, some performance measures are delineated that are certainly difficult in computing directly due to assumption of point process abandonment which introduces state-heterogeneous transition rate matrices. These measures help researchers to know the behavior of the machining system and the extent of the impact of other variables on it.

Let Y be the arbitrary variable representing the time-to-failure of the machining system and $P_F(t)$ represents the probability that the system has failed at or before time t. The reliability is characterized as the probability that the system will perform its intended job without failure under stated specific condition for a stated period of time. Therefore, the reliability of the system is given by

$$R_Y(t) = 1 - P_F(t); t \ge 0 \tag{2.26}$$

Thus the mean time-to-failure $(MTTF)$ of the system is defined as

$$MTTF = \int_{t=0}^{\infty} R_Y(t)dt = \int_{t=0}^{\infty} (1 - P_F(t))dt$$

$$= \lim_{s \to 0} \left[\frac{1 - a_{0,2L}}{s} - \sum_{h=1}^{n_1} \frac{a_{h,2L}}{s + l_h} - \sum_{h=1}^{n_2} \frac{b_{h,2L} s + c_{h,2L}}{s^2 + (l_{n_1+h} + \bar{l}_{n_1+h})s + l_{n_1+h}\bar{l}_{n_1+h}} \right]$$

$$= -\sum_{h=1}^{n_1} \frac{a_{h,2L}}{l_h} - \sum_{h=1}^{n_2} \frac{c_{h,2L}}{l_{n_1+h}\bar{l}_{n_1+h}} \tag{2.27}$$

For the important elucidation of the studied MRP model from the application perspective, some more performance measures are defined as

- The expected number of failed units in the machining system at time t

$$E_N(t) = E(N(t)) = \sum_{i=0}^{1}\sum_{n=i}^{L-1} nP_{i,n}(t) + LP_F(t) \tag{2.28}$$

- The expected throughput of the machining system at time t

$$TP = \sum_{n=1}^{L-1} \mu_v P_{0,n}(t) + \sum_{n=1}^{L-1} \mu_b P_{1,n}(t) \tag{2.29}$$

- The expected number of spare units in the machining system at time t

$$E_S(t) = \sum_{i=0}^{1}\sum_{n=i}^{S} (S-n)P_{i,n}(t) \tag{2.30}$$

- The expected number of operating units in the machining system at time t

$$E_O(t) = M\sum_{i=0}^{1}\sum_{n=i}^{S} P_{i,n}(t) + \sum_{i=0}^{1}\sum_{n=S+1}^{L-1} (M+S-n)P_{i,n}(t) \tag{2.31}$$

- The expected carrying load of failed units in the machining system at time t

$$\begin{aligned} \lambda_{eff}(t) = &\sum_{i=0}^{1}\sum_{n=i}^{S} (M\lambda + (S-n)\nu)P_{i,n}(t) \\ &+ \sum_{i=0}^{1}\sum_{n=S+1}^{L-1} (M+S-n)\lambda_d P_{i,n}(t) \end{aligned} \tag{2.32}$$

- The expected waiting time for the failed unit in the machining system at time t

$$E_W(t) = \frac{E_N(t)}{\lambda_{eff}} \tag{2.33}$$

- The expected delay time in the service by the repairman at time t

$$E_D(t) = \frac{E_N(t)}{TP} \tag{2.34}$$

- The effective switching failure rate of spare unit at time t

$$SR(t) = \sum_{i=0}^{1}\sum_{n=i}^{S-1} M\lambda q P_{i,n}(t) \tag{2.35}$$

- The effective reneging rate of the failed units during vacation period at time t

$$RR(t) = \sum_{n=1}^{L-1} (1-p^n)\xi P_{0,n}(t) \tag{2.36}$$

- Failure frequency of the system at time t

$$FF(t) = m\lambda_d \sum_{i=0}^{1} P_{i,L}(t) \tag{2.37}$$

- Vacation interruption frequency of the system at time t

$$VF(t) = \sum_{n=S+1}^{L-1} \mu_v P_{0,n}(t) \tag{2.38}$$

- Machining system availability at time t

$$MA(t) = 1 - \frac{E_N(t)}{M+S} \tag{2.39}$$

2.5 Sensitivity Analysis

The affectability of reliability function $R_Y(t)$ and mean time-to-failure $MTTF$ with respect to diverse system parameters can be investigated on applying the principle of calculus as follows:

$$\frac{\partial \mathbf{\Lambda}(s)}{\partial \theta}\tilde{\mathbf{\Pi}}(s)+\mathbf{\Lambda}(s)\frac{\partial \tilde{\mathbf{\Pi}}(s)}{\partial \theta}=0 \tag{2.40}$$

$$\frac{\partial \tilde{\mathbf{\Pi}}(s)}{\partial \theta}=-(\mathbf{\Lambda}(s))^{-1}\frac{\partial \mathbf{\Lambda}(s)}{\partial \theta}\tilde{\mathbf{\Pi}}(s) \tag{2.41}$$

The derivative of reliability function $R_Y(t)$ is calculated as follows:

$$\begin{aligned}\Phi_\theta(t)=\frac{\partial R_Y(t)}{\partial \theta}=-\frac{\partial P_F(t)}{\partial \theta}=L^{-1}\left(-\frac{\partial \tilde{P}_F(s)}{\partial \theta}\right)\\ =L^{-1}\left(-\frac{\partial \tilde{\pi}_{2L}(s)}{\partial \theta}\right)\end{aligned} \tag{2.42}$$

where $\frac{\partial \tilde{P}_F(s)}{\partial \theta}$ or $\frac{\partial \tilde{\pi}_{2L}(s)}{\partial \theta}$ is derived using numerical scheme of first principle of derivative and the program in $MATLAB$ code from the Eq^n. (2.26). For the relative sensitivity analysis of reliability function, we evaluate

$$\begin{aligned}\Omega_\theta(t)&=\frac{\frac{\partial R_Y(t)}{R_Y(t)}}{\frac{\partial \theta}{\theta}}\\ &=\frac{\partial R_Y(t)}{\partial \theta}\cdot\frac{\theta}{R_Y(t)}\\ &=\Phi_\theta(t)\cdot\frac{\theta}{R_Y(t)}\end{aligned} \tag{2.43}$$

For the sensitivity analysis of $MTTF$ with respect to the system parameter θ, the derivative of $MTTF$ expression given in Eq^n. (2.27) is evaluated firstly as

$$\begin{aligned}\Delta_\theta=\frac{\partial MTTF}{\partial \theta}=\frac{\partial\left(\int_{t=0}^{\infty} R_Y(t)dt\right)}{\partial \theta}=\lim_{s\to 0}\left[\int_{t=0}^{\infty}\frac{R_Y(t)}{\partial \theta}e^{-st}dt\right]\\ =\lim_{s\to 0}\left[-\frac{\partial \bar{P}_F(s)}{\partial \theta}\right]=\lim_{s\to 0}\left[-\frac{\partial \bar{\pi}_{2L}(s)}{\partial \theta}\right]\end{aligned} \tag{2.44}$$

The relative sensitivity of $MTTF$ can be examined by evaluating the ratio

$$\Gamma_\theta=\frac{\partial MTTF/MTTF}{\partial \theta/\theta}=\frac{\partial MTTF}{\partial \theta}\cdot\frac{\theta}{MTTF}=\Delta_\theta\frac{\theta}{MTTF} \tag{2.45}$$

The affectability of both reliability and mean time-to-failure are investigated by exploring with diverse numerical simulation.

2.6 Special Cases

Following aforesaid methodology for computing transient state probabilities, reliability, mean time-to-failure and queueing characteristics analysis have also been done in earlier publication and results are resembling as a special case. Some special cases are highlighted as follows:

a If $q = 0$, $\theta = 0$, $\mu_v = \mu_b$, the system reduced to machine repair problem with spare and resembles with the model of Hsieh and Wang [62] for $R = 1$.

b If $\theta = 0$, $\mu_v = \mu_b$, the studied model resembles with model considered by Shekhar et al. [153] without geometric reneging.

c For $q = 0$, $mu_v = 0$, the studied model reduces to special case of machining system which is examined by Jain et al. [79].

d For $q = 0$, $\theta = 0$ in studied MRP without vacation, the model resembles with the results of Kumar et al. [108] findings for $F = 0$.

2.7 Illustrative Example

To comprehend the proposed solution methodology to evaluate the transient-state probabilities and associated performance indices, numerical illustrative example is provided by considering the accompanying default parameter(s) $M = 4$, $S = 3$, $m = 2$, $\lambda = 0.8$, $\lambda_d = 1.2$, $\nu = 0.5$, $\xi = 0.4$, $\mu_v = 6$, $\mu_b = 8$, $\theta = 0.7$, $p = 0.4$, $q = 0.5$ at the epoch $t = 10$. Hence, the value of the system threshold $L = M + S - m + 1 = 6$. For the Eq^n. (2.15), the coefficients square matrix of order $2L$, $\mathbf{\Lambda}(s)$, is structured as

$$\mathbf{\Lambda}(s) = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{A}_3 & \mathbf{A}_4 \\ \mathbf{B}_1 & \mathbf{B}_2 & \mathbf{B}_3 & \mathbf{B}_4 \\ \mathbf{C}_1 & \mathbf{C}_2 & \mathbf{C}_3 & \mathbf{C}_4 \\ \mathbf{D}_1 & \mathbf{D}_2 & \mathbf{D}_3 & \mathbf{D}_4 \end{bmatrix}$$

where $\mathbf{A}_4$, $\mathbf{B}_3$, $\mathbf{B}_4$, $\mathbf{C}_1$, $\mathbf{C}_4$, and $\mathbf{D}_1$ are null sub-matrices of an appropriate order and other sub-matrices are as follows

$$\mathbf{A}_1 = \begin{bmatrix} -4.7+s \end{bmatrix}; \mathbf{A}_2 = \begin{bmatrix} 7.6 & 0.64 & 0.256 & 0.1024 & 0.04096 \end{bmatrix}; \mathbf{A}_3 = \begin{bmatrix} 8 & 0 & 0 & 0 & 0 \end{bmatrix};$$

$$\mathbf{B}_1 = \begin{bmatrix} 3.1 \\ 0.8 \\ 0.4 \\ 0.4 \\ 0 \end{bmatrix}; \mathbf{B}_2 = \begin{bmatrix} -12.5+s & 7.92 & 1.152 & 0.6144 & 0.3072 \\ 2.6 & -12.96+s & 7.728 & 1.3824 & 0.9216 \\ 0.8 & 2.1 & -14.636+s & 1.3824 & 1.3824 \\ 0.8 & 1.6 & 4.8 & -13.7816+s & 1.0368 \\ 0 & 0 & 0 & 3.6 & -12.789+s \end{bmatrix};$$

$$\mathbf{C}_2 = \begin{bmatrix} 0.7 & 0 & 0 & 0 & 0 \\ 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0.7 & 6 & 0 \\ 0 & 0 & 0 & 0.7 & 6 \\ 0 & 0 & 0 & 0 & 0.7 \end{bmatrix};$$

$$\mathbf{C}_3 = \begin{bmatrix} -12.2+s & 8 & 0 & 0 & 0 \\ 2.6 & -11.7+s & 8 & 0 & 0 \\ 0.8 & 2.1 & -12.8+s & 8 & 0 \\ 0.8 & 1.6 & 4.8 & -11.6+s & 8 \\ 0 & 0 & 0 & 3.6 & -10.4+s \end{bmatrix};$$

$$\mathbf{D}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 2.4 \end{bmatrix}; \mathbf{D}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 2.4 \end{bmatrix}; \mathbf{D}_4 = \begin{bmatrix} s \end{bmatrix}$$

For applying discussed methodology, following latent roots are obtained that are listed in Table (2.1). It is observed that some latent roots are real and as well complex in nature that exists in conjugate pair. Since coefficient matrix $\mathbf{\Lambda}(s)$ is obtained by balancing the inflow and outflow of the rates among different states of the system, one latent root is always zero. It is noted that $n_1 = 6$ and $n_2 = 3$. For computing transient-state probabilities, constants $a_{h,r}$, $b_{h,r}$, $c_{h,r}$; $r = 1, 2, ..., 2L$ are obtained from Eq^n (2.23)-(2.24) and summarized in Table (2.2). On taking inverse Laplace transform, the transient-state probabilities $\pi_r; r = 1, 2, ..., 2L$ are computed from the Eq^n (2.25) and tabulated in Table (2.3). From the obtained transient-state probabilities, the numeric value of the following performance measures are also computed that are discussed in section (2.4). The numerical values of the performance measures are illustrated in Table (2.4). The summary of the simulation and sensitivity analysis of these performance measures for a different value of different parameters are presented in the next section.

2.8 Numerical Result

In this section, the code is developed in MATLAB computer program to provide numerical insight of the impacts of the different administering parameters on reliability and queueing attributes of the considered machine repair problem (MRP) with working vacation and synchronized abandon. The parameters for machining system are fixed as follows $M = 10$, $S = 6$, $m = 3$, $\lambda = 0.5$, $\lambda_d = 1$, $\nu = 0.2$, $\xi = 0.08$, $\mu_v = 8$, $\mu_b = 10$, $\theta = 0.8$, $p = 0.2$, $q = 0.1$, $t = 50$. The numerical results are depicted in Figs. (2.2)-(2.13) and tabulated in Tables (2.5)-(2.7). Figs. (2.2)-(2.3) depict the variation of system reliability ($R_Y(t)$) with respect to time t for various different system parameters. The obvious result of decrements in the value of reliability of the machining system with respect to increased time prompts. It clearly observed that how the failure of any type, failure of an operating unit (λ, λ_d)/spare unit (ν), switching failure probability (q), number of operating units (M) is more critical for the system reliability. It is observed that providing the better corrective maintenance facility like more number of spare units (S), higher repair rate (μ_v, μ_b), less number of minimum number of operating units (m) increase the reliability of the system but incur some additional cost. The random vacation time (θ) is also a prominent factor for maintaining the better system reliability. It is also observed that the arrival of extraneous service facility (ξ) and synchronized abandonment rate (p) are not critical parameters for maintaining the system reliability. It is advice to the system manager to maintain the preventive time measures frequently to avoid the failure of the unit to reduce the expensive corrective maintenance.

Fig. (2.4) depicts the variation of the mean-time-to-failure ($MTTF$) of the machining system graphically. It is clearly notable that $MTTF$ reduces with increase in the value M, m, λ, λ_d, ν and q. These results are obvious and validate the present modeling and methodology. Mean-time-to-failure can also be increased to some extent by using some preventive maintenance policy. It is observed from the Fig. (2.4) that $MTTF$ increases with S, μ_v, μ_b and θ.

Fig. (2.5) and Table (2.5) summarize very important finding in term of the sensitivity of the governing parameters for system reliability and system mean-time-to-failure respectively. The positive and negative magnitude show increment and decrements in the characteristics measures with increasing the value of system parameters. Fig. (2.5) depicts the behavior of the system reliability with respect to system parameters for a wide range of duration. In comparison the absolute magnitude in infant mortality phase of the system, $t = 1$, it is observed that the reliability is more sensitive in order $q > \lambda > \lambda_d > \nu > \theta > \mu_v > \xi > \mu_b > p$. It is due to a failure in automation, manufacturing defects in the unit, designing failure etc. As time enhanced from $t = 100$ to $t = 500$, system get stabilize and reliability of the system sensitive in order $\lambda_d > \lambda > q > \nu > \mu_b > \mu_v > \theta > \xi > p$ with substantial magnitude. For higher time, $t > 1000$, reliability of the machining system is sensitive in same order $\lambda_d > \lambda > q > \nu > \mu_b > \mu_v > \theta > \xi > p$ and magnitude is very less. This prompts that machining system gets stabilize completely, working as steady-state and system affect due to random failure of units only. Fig. (2.5) also depicts relative sensitivity of the parameters for the system reliability. Table (2.5) portrays the sensitivity and relative sensitivity of the mean-time-to-failure of the system. For $M = 10, S = 6, m = 3$, $MTTF$ is sensitive in the order of $\lambda_d > \lambda > q > \nu > \mu_b > \mu_v > \theta > \xi > p$. But for $M = 8$ order of sensitivity is change to $q > \lambda > \lambda_d > \nu > \mu_b > \mu_v > \theta > \xi > p$. And for $S = 8$ and $m = 5$ this order of sensitivity of $MTTF$ changed to $\lambda > \lambda_d > q > \nu > \mu_b > \mu_v > \theta > \xi > p$ and $\lambda > q > \lambda_d > \nu > \mu_b > \mu_v > \theta > \xi > p$ respectively. These observations suggest how system design and specification is also important for machining system.

The changes in another important reliability characteristics, failure frequency $FF(t)$, are depicted in the Fig. (2.6) and (2.7) prompt that $FF(t)$ is increasing with time due to unit wear and tear. Failure of the system is more prominent when the values of λ, λ_d, ν, q, M and m are high since they give high value for failure frequency. Failure frequency could be diminished to some limit by providing better corrective and preventive maintenance facility.

Some more established queueing characteristics behavior with governing system parameters are charted in Tables (2.6)-(2.7) by taking a different set of input parameter along with following default value of governing parameters $M = 10$, $S = 6$, $m = 3$, $\lambda = 0.5$, $\lambda_d = 1.0$, $\nu = 0.2$, $\xi = 0.09$, $\mu_v = 8$, $\mu_b = 10$, $\theta = 0.8$, $p = 0.2$, $q = 0.1$, $t = 50$. We summarize the changed value for expected number of spare units ($E_S(t)$), expected number of operating units ($E_O(t)$), effective failure rate of unit (λ_{eff}), expected waiting time ($E_W(t)$), expected delay time ($E_D(t)$), effective switching failure rate ($SR(t)$), effective reneging rate ($RR(t)$) and vacation interruption frequency ($VF(t)$). The obvious trends for expectation and effective rates are observed which validate the studied model. Vacation interruption frequency is high for higher failure rates, vacation time, system specification.

In a nutshell, the findings from the present investigation are as follows:

- Reliability characteristics are more dipped with failure rates of units. It is advisable for the system designer to follow preventive measures prominently.

- Protuberant reliability characteristics are achieved by enhancing corrective measures. It is noticeable that investing some budget for better corrective measures is good for maintaining the required level of reliability of the machining system.

- The various types of queueing characteristics give a quick insight into designing the specification of the machining system.
- Vacation interruption is a good policy for the system manager to maintain high indices.
- Synchronized reneging with extraneous service facility to stabilize the machining system, but it may require additional investment.
- Automation of switching of the spare unit should properly be checked.

2.9 Conclusion

In this chapter, the synchronized abandonment phenomena of the failed units in the fault-tolerant machining system is studied with repairman,s modified multiple working vacations and their interruption. More specifically, binomially distributed synchronized abandonment of the failed units on the availability of an extraneous repair facility is considered, wherein the extraneous repair facility follows the Poisson process when in-house repairman is on vacation. The modified multiple working vacation policy of the repairman with threshold-based controlled vacation interruption is also incorporated. The computationally efficient spectral method is employed to derive the stationary distributions for the number of failed units in the machining system in continuous time for the Markovian models of the fault-tolerant machining system numerically. The illustrative example is added to validate the methodology to determine the transient-state probabilities via Laplace transform. It would be motivating to consider extensions of this procedure for the study of another fault-tolerant machining systems and queueing models with this type of binomial transition. Sensitivity analysis is also presented to determine the substantial parameter(s) for the fault-tolerant machining system. It is demonstrated how reliability can be improved by providing sufficient spares as standbys or increasing the repair rates, setting the threshold for vacation interruption and vacation time. It is recommended through the present investigation that system designer should opt for some preventive maintenance policy to diminish the likelihood of any failure like the breakdown of operating units/spare units, switching failure, etc. The present studied fault-tolerant machining system has potential application in modeling the manufacturing systems, industrial systems, and others.

Moreover, we consider the case where the service and inter-arrival times are exponentially distributed. The instances where those times are not exponential are essential in practice. The threshold-based policy can be set for the vacation interruption. The fellow researchers can extend this work to a single vacation policy wherein repairman takes just one vacation and remains in the system after vacation even if there is no waiting failed units in the system. One can extend this investigation for unreliable repairman also. These cases are left for future study.

Table 2.1: Latent roots of the coefficient matrix

l_1	0.0000	l_5	18.1810	l_9	12.5786 - 1.1792i
l_2	0.0974	l_6	17.7420	l_{10}	7.2602 + 0.1045i
l_3	2.8689	l_7	15.7285 - 1.2556i	l_{11}	7.2602 - 0.1045i
l_4	20.0425	l_8	12.5786 + 1.1792i	l_{12}	15.7285 + 1.2556

Table 2.2: Constants $a_{h,r}$,$b_{h,r}$,$c_{h,r}$

r	1	2	3	4	5	6	7	8	9	10	11	12
$a_{1,r}$	0	0	0	0	0	0	1	0	0	0	0	0
$a_{2,r}$	0.3710	0.1493	0.0797	0.0351	0.0421	0.0119	-1.0437	0.0636	0.0831	0.0929	0.0847	0.0304
$a_{3,r}$	0.2083	0.1607	0.1021	0.0452	0.0562	0.0204	0.0469	-0.1154	-0.1487	-0.1355	-0.1638	-0.0764
$a_{4,r}$	0.0005	-0.0003	0.0001	0.0000	0.0000	0.0000	0.0000	-0.0007	0.0007	-0.0005	0.0003	-0.0001
$a_{5,r}$	-0.0752	-0.0537	0.0704	-0.0237	0.0200	-0.0133	0.0002	0.1727	-0.1244	0.0385	-0.0234	0.0120
$a_{6,r}$	0.1293	-0.0032	-0.0505	0.0209	-0.0214	0.0156	-0.0006	-0.2042	0.1417	-0.0363	0.0199	-0.0112
$b_{1,r}$	0.0315	-0.0395	0.0064	0.0050	0.0109	-0.0177	0.0012	-0.0005	-0.0087	0.0092	-0.0078	0.0100
$b_{2,r}$	0.0844	-0.0674	-0.0330	0.0009	0.0055	0.0532	-0.0026	-0.0022	-0.0117	0.0056	0.0055	-0.0382
$b_{3,r}$	0.2501	-0.1458	-0.1751	-0.0834	-0.1133	-0.0701	-0.0013	0.0867	0.0679	0.0262	0.0846	0.0736
$c_{1,r}$	0.5455	-0.5927	0.0777	0.0689	0.1899	-0.2909	0.0182	-0.1071	-0.0955	0.1591	-0.1432	0.1701
$c_{2,r}$	1.1787	-0.8326	-0.4470	0.0013	0.0491	0.6778	-0.0286	-0.1690	-0.1411	0.1206	0.0982	-0.5072
$c_{3,r}$	1.8936	-1.0937	-1.3183	-0.6283	-0.8536	-0.5292	-0.0112	0.6425	0.5039	0.1959	0.6396	0.5588

Table 2.3: Transient-state probabilities

$P_{0,0}(t)$	$P_{0,1}(t)$	$P_{0,2}(t)$	$P_{0,3}(t)$	$P_{0,4}(t)$	$P_{0,5}(t)$	$P_{1,1}(t)$	$P_{1,2}(t)$	$P_{1,3}(t)$	$P_{1,4}(t)$	$P_{1,5}(t)$	$P_F(t)$
0.1402	0.0564	0.0301	0.0132	0.0159	0.0045	0.6057	0.0240	0.0314	0.0351	0.0320	0.0115

Table 2.4: Performance measures

$E_N(t)$	TP	$E_S(t)$	$E_O(t)$	$\lambda_{eff}(t0$	$E_W(t)$	$E_D(t)$
4.2544	1.7925	0.6427	1.4971	1.5893	2.6768	2.3735
$SR(t)$	$RR(t)$	$FF(t)$	$MA(t)$	$VF(t)$	$R_Y(t)$	$MTTF$
0.4513	0.3659	2.9076	0.3922	0.1225	0.3943	10.7047

Table 2.5: Sensitivity and relative sensitivity of the MTTF of the system

	M,S,m	Θ								
		λ	λ_d	ν	ξ	μ_v	μ_b	θ	p	q
Δ_Θ	(10,6,3)	-1289.45	-1564.23	-316.36	31.09	45.67	163.99	33.45	10.72	-1129.71
	(8,6,3)	-2367.82	-1821.61	-734.37	70.49	105.48	187.24	74.96	23.23	-2435.25
	(9,6,3)	-1801.96	-1744.08	-494.89	48.57	71.89	182.21	52.04	16.37	-1701.72
	(10,5,3)	-746.44	-1125.45	-159.21	17.55	28.11	111.69	17.17	6.11	-656.34
	(10,8,3)	-3323.90	-2915.06	-1025.57	85.92	107.17	339.97	104.18	28.94	-2884.46
	(10,6,4)	-489.92	-459.61	-120.83	11.75	17.37	48.94	12.68	4.08	-429.98
	(10,6,5)	-249.78	-172.12	-62.11	5.98	8.92	18.75	6.49	2.10	-219.85
Ξ_Θ	(10,6,3)	-2.7124	-6.5809	-0.2662	0.0105	1.5372	6.8993	0.1126	0.0090	-0.4753
	(8,6,3)	-3.1953	-4.9164	-0.3964	0.0152	2.2775	5.0535	0.1619	0.0125	-0.6573
	(9,6,3)	-2.9901	-5.7881	-0.3285	0.0129	1.9087	6.0470	0.1382	0.0109	-0.5648
	(10,5,3)	-2.1503	-6.4841	-0.1835	0.0081	1.2956	6.4350	0.0791	0.0070	-0.3781
	(10,8,3)	-3.8215	-6.7029	-0.4716	0.0158	1.9713	7.8172	0.1916	0.0133	-0.6633
	(10,6,4)	-2.7417	-5.1442	-0.2705	0.0105	1.5550	5.4773	0.1135	0.0091	-0.4813
	(10,6,5)	-2.7812	-3.8330	-0.2766	0.0107	1.5883	4.1763	0.1156	0.0093	-0.4896

Table 2.6: System performance measures for different parameters

Parameter	$E_S(t)$	$E_O(t)$	$\lambda_e ff$	$E_W(t)$	$E_D(t)$	$SR(t)$	$RR(t)$	$VF(t)$
$t = 10$	3.5563	9.3608	5.8378	0.5171	0.4731	0.4080	0.0276	0.0604
$t = 50$	3.0001	7.8975	4.9254	0.9615	0.8796	0.3442	0.0232	0.0509
$t = 100$	2.4258	6.3858	3.9826	1.6344	1.4952	0.2783	0.0188	0.0412
$M = 8$	3.9106	6.9538	4.3545	0.6636	0.6091	0.3322	0.0268	0.0321
$M = 9$	3.4980	7.5298	4.6640	0.7881	0.7212	0.3473	0.0257	0.0426
$M = 10$	3.0001	7.8975	4.9254	0.9615	0.8796	0.3442	0.0232	0.0509
$S = 5$	2.1587	7.2206	4.5189	1.1360	1.0362	0.2970	0.0197	0.0724
$S = 6$	3.0001	7.8975	4.9254	0.9615	0.8796	0.3442	0.0232	0.0509
$S = 8$	4.7942	8.8042	5.5874	0.7508	0.6903	0.4097	0.0281	0.0247
$m = 3$	3.0001	7.8975	4.9254	0.9615	0.8796	0.3442	0.0232	0.0509
$m = 4$	2.1570	5.6111	3.4818	2.0001	1.8487	0.2471	0.0168	0.0368
$m = 5$	1.2538	3.2011	1.9706	4.4926	4.2154	0.1432	0.0098	0.0214
$p = 0.05$	2.9888	7.8832	4.9179	0.9677	0.8838	0.3433	0.0250	0.0513
$p = 0.20$	3.0001	7.8975	4.9254	0.9615	0.8796	0.3442	0.0232	0.0509
$p = 0.90$	3.0346	7.9327	4.9445	0.9450	0.8710	0.3467	0.0052	0.0500
$q = 0.05$	3.3381	8.3278	5.1377	0.7870	0.7489	0.1874	0.0255	0.0429
$q = 0.10$	3.0001	7.8975	4.9254	0.9615	0.8796	0.3442	0.0232	0.0509
$q = 0.90$	0.2188	2.7176	2.2126	5.2984	3.6883	0.3471	0.0008	0.0407

Table 2.7: System performance measures for different parameters

Parameter	$E_S(t)$	$E_O(t)$	λ_eff	$E_W(t)$	$E_D(t)$	$SR(t)$	$RR(t)$	$VF(t)$
$\lambda = 0.25$	5.0147	9.7481	3.5128	0.3399	0.3151	0.2404	0.0249	0.0096
$\lambda = 0.50$	3.0001	7.8975	4.9254	0.9615	0.8796	0.3442	0.0232	0.0509
$\lambda = 0.75$	1.3056	5.5397	4.7620	1.7559	1.6341	0.2756	0.0112	0.0579
$\nu = 0.1$	3.2605	8.1507	4.7369	0.9008	0.8198	0.3624	0.0238	0.0471
$\nu = 0.2$	3.0001	7.8975	4.9254	0.9615	0.8796	0.3442	0.0232	0.0509
$\nu = 0.4$	2.5302	7.4031	5.1669	1.0862	1.0008	0.3091	0.0215	0.0568
$\lambda_d = 0.6$	3.9488	9.8511	5.7568	0.3803	0.3488	0.4540	0.0310	0.0579
$\lambda_d = 0.8$	3.6799	9.3958	5.6181	0.5053	0.4610	0.4227	0.0286	0.0588
$\lambda_d = 1.0$	3.0001	7.8975	4.9254	0.9615	0.8796	0.3442	0.0232	0.0509
$\xi = 0.04$	2.9924	7.8876	4.9202	0.9657	0.8825	0.3436	0.0116	0.0512
$\xi = 0.08$	3.0001	7.8975	4.9254	0.9615	0.8796	0.3442	0.0232	0.0509
$\xi = 0.12$	3.0077	7.9073	4.9305	0.9573	0.8767	0.3449	0.0349	0.0507
$\mu_v = 5$	1.9625	6.4225	4.1246	1.6914	1.5505	0.2525	0.0187	0.0679
$\mu_v = 6$	2.3235	7.0169	4.4523	1.3774	1.2609	0.2872	0.0207	0.0643
$\mu_v = 8$	3.0001	7.8975	4.9254	0.9615	0.8796	0.3442	0.0232	0.0509
$\mu_b = 10$	3.0001	7.8975	4.9254	0.9615	0.8796	0.3442	0.0232	0.0509
$\mu_b = 12$	3.8058	9.2625	5.6464	0.4986	0.4536	0.4258	0.0307	0.0672
$\mu_b = 16$	4.2689	9.7854	5.8867	0.3257	0.2962	0.4660	0.0361	0.0789
$\theta = 0.5$	2.9281	7.8084	4.8785	1.0005	0.9155	0.3382	0.0252	0.0627
$\theta = 0.8$	3.0001	7.8975	4.9254	0.9615	0.8796	0.3442	0.0232	0.0509
$\theta = 1.2$	3.0759	7.9893	4.9736	0.9218	0.8430	0.3505	0.0211	0.0393

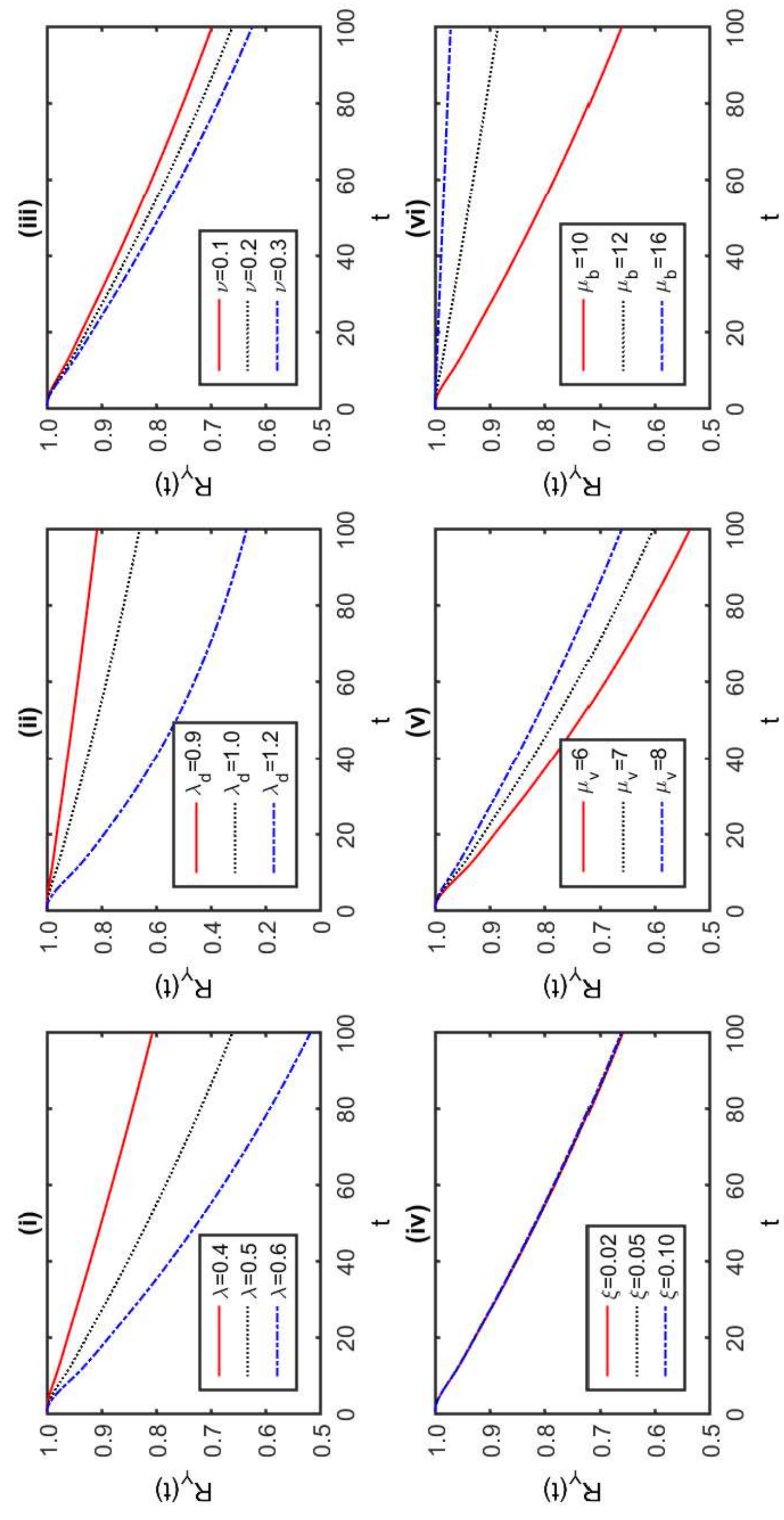

Figure 2.2: Reliability of the system $(R_Y(t))$ wrt t for different parameters

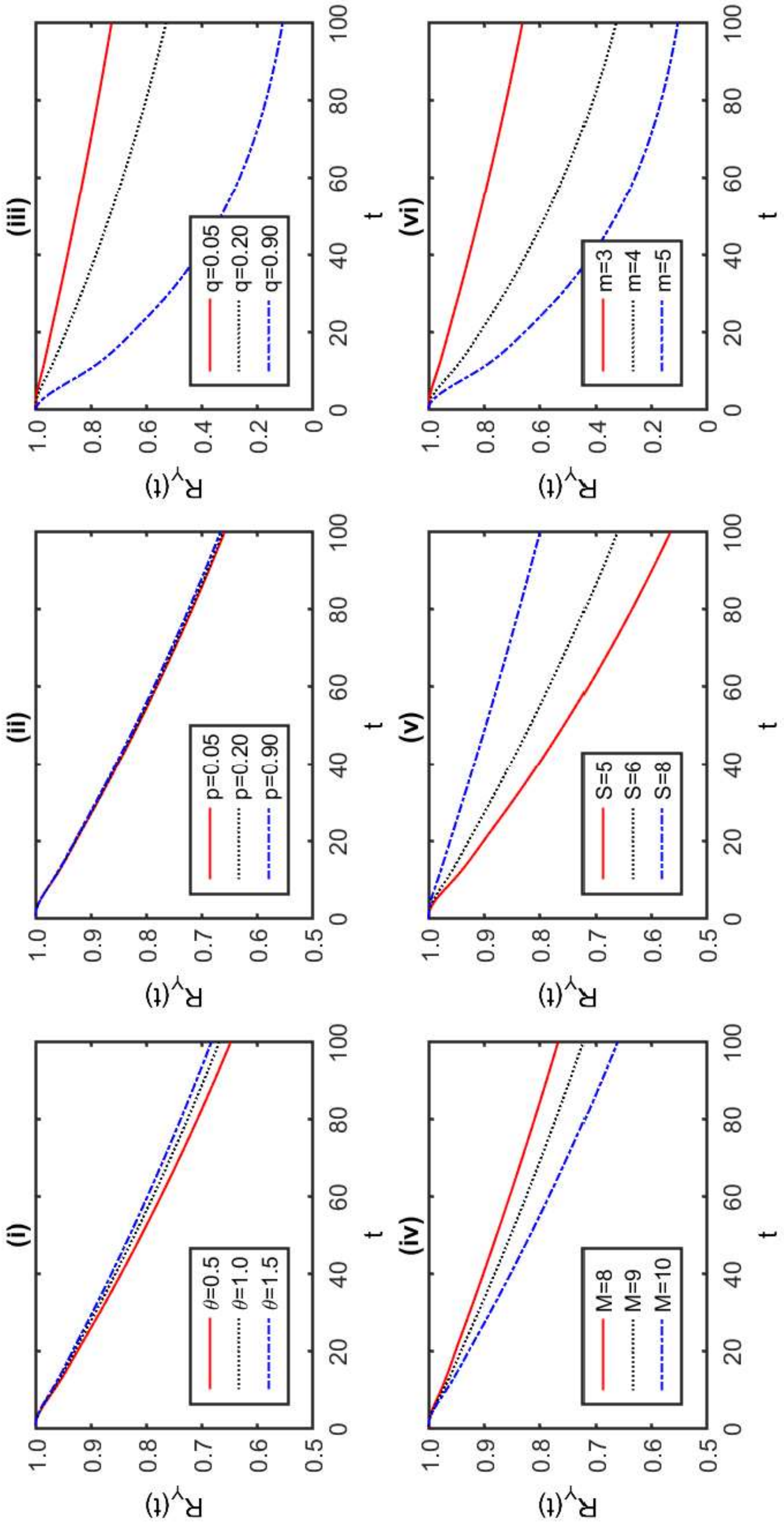

Figure 2.3: Reliability of the system $(R_Y(t))$ wrt t for different parameters

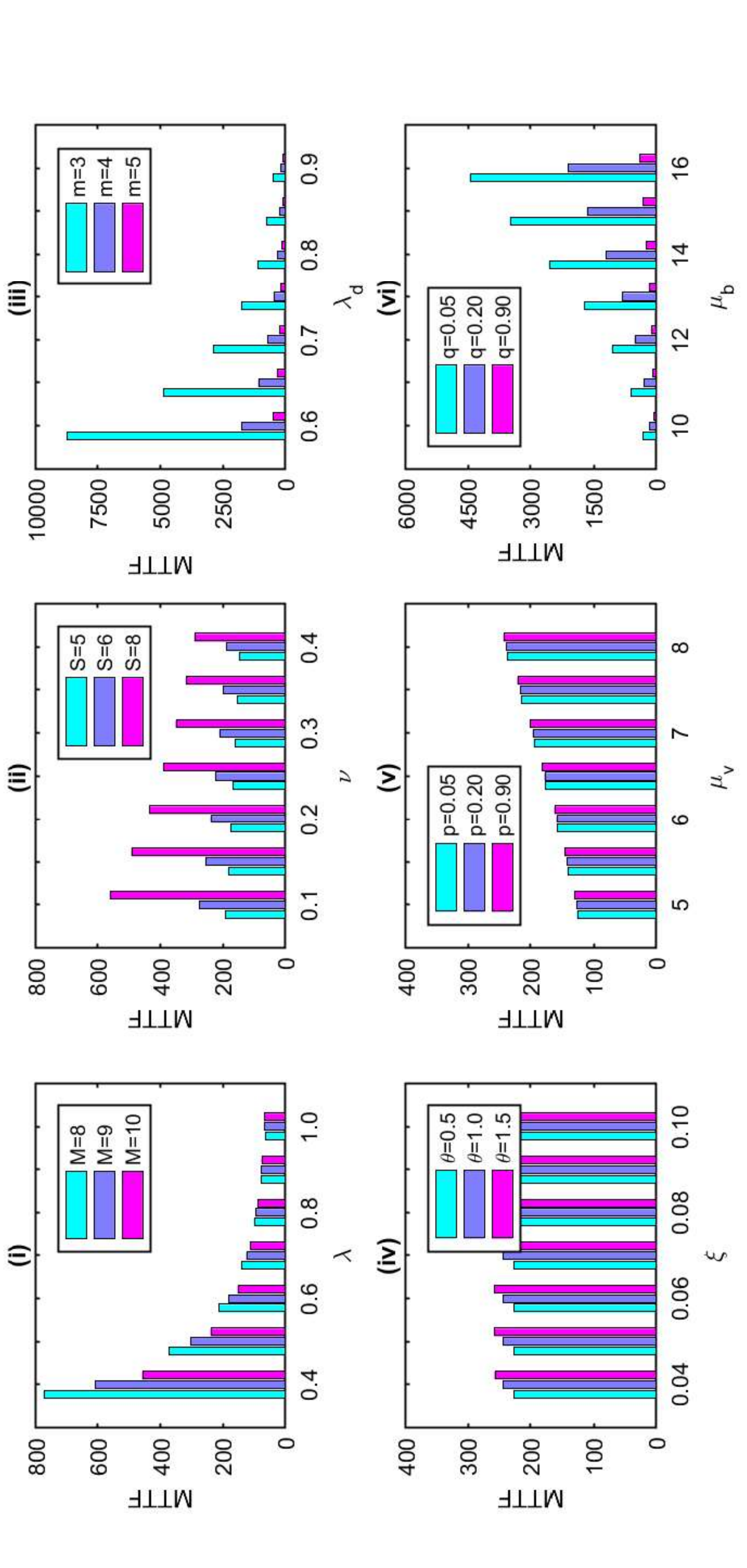

Figure 2.4: Mean time-to-failure $(MTTF)$ of the system for different parameters

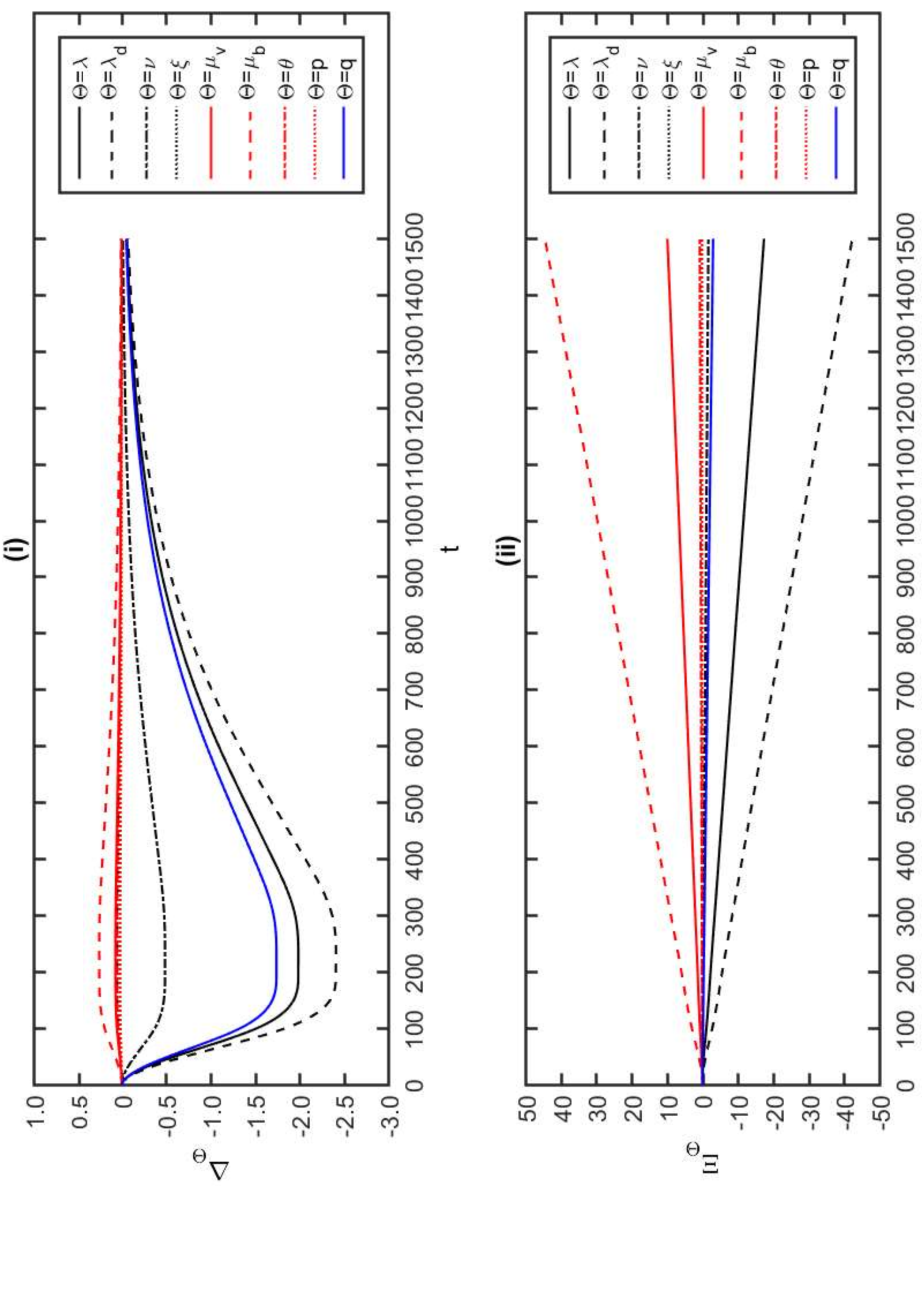

Figure 2.5: Sensitivity and relative sensitivity of the reliability of the system

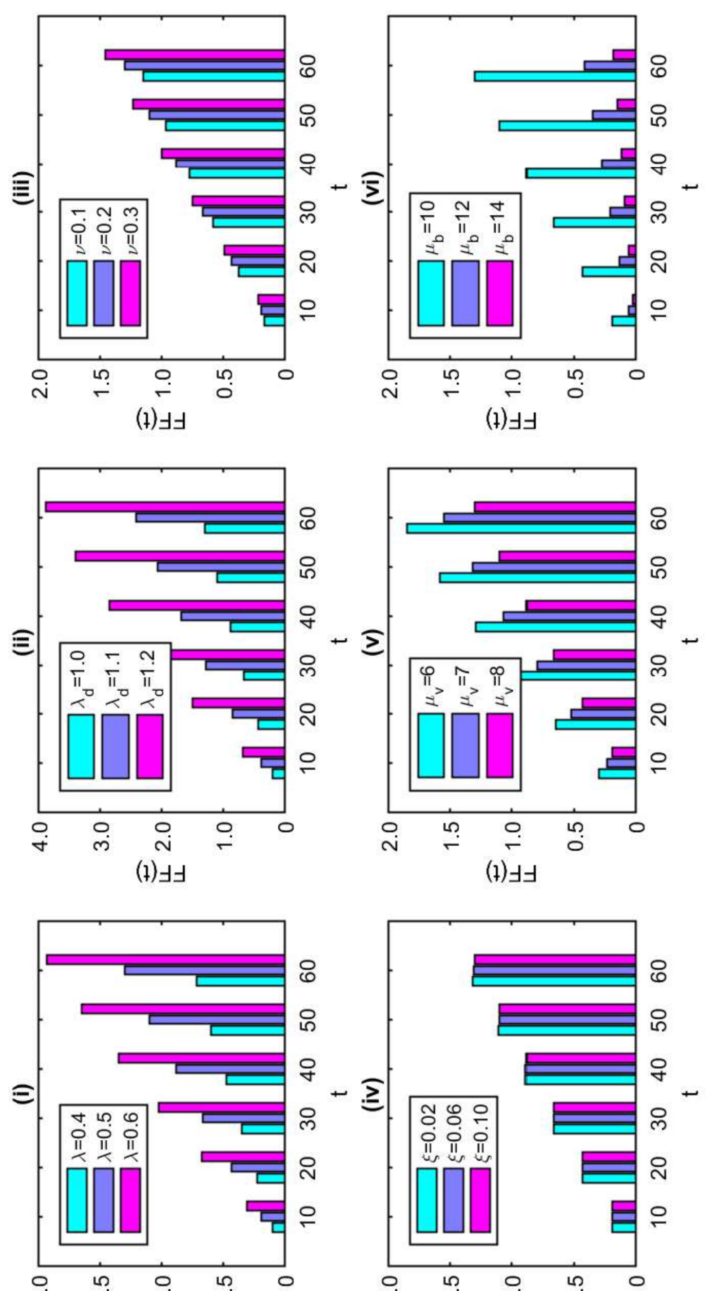

Figure 2.6: Failure frequency of the system $(FF(t))$ wrt t for different parameters

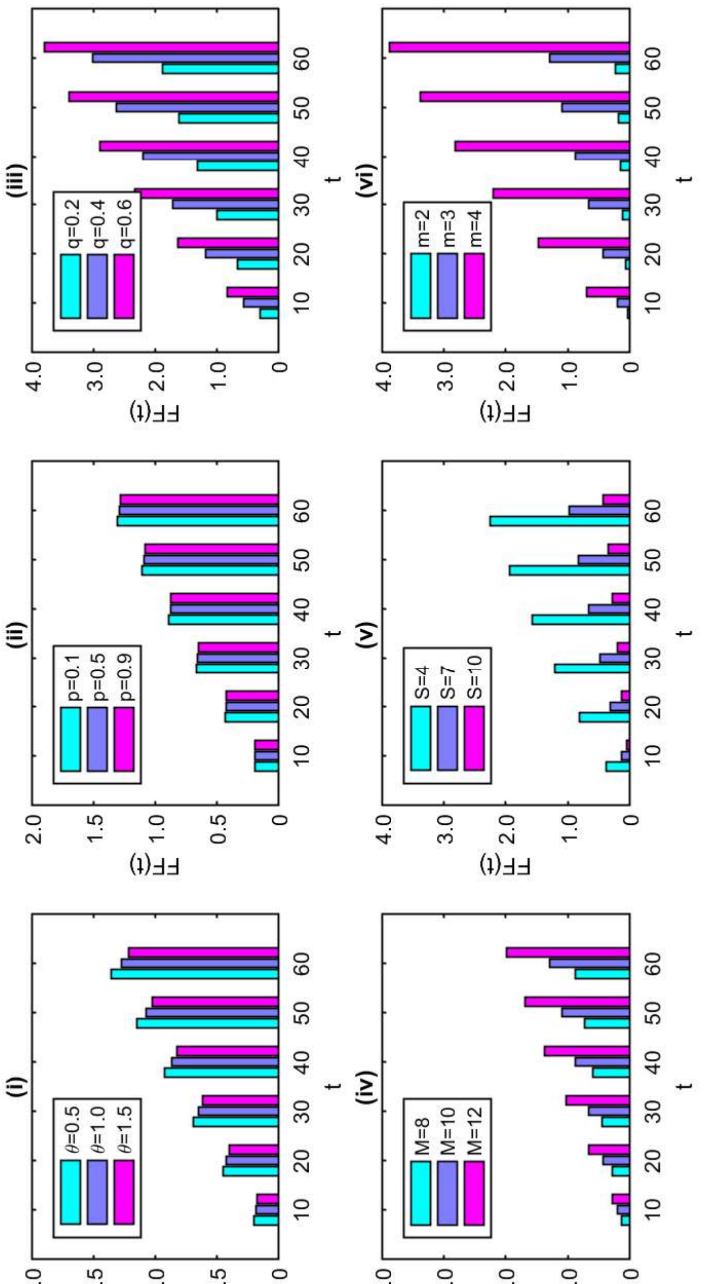

Figure 2.7: Failure frequency of the system $(FF(t))$ wrt t for different parameters

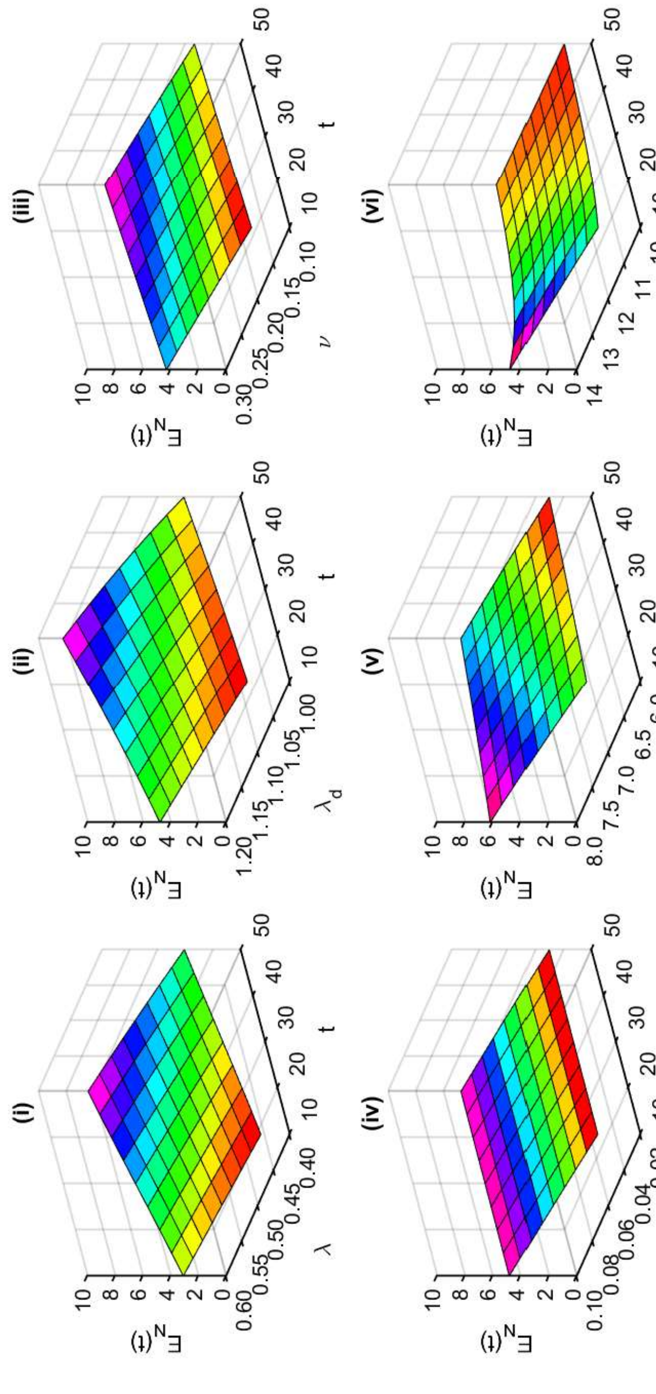

Figure 2.8: Expected number of the failed units in the system $(E_N(t)$ wrt t for different parameters

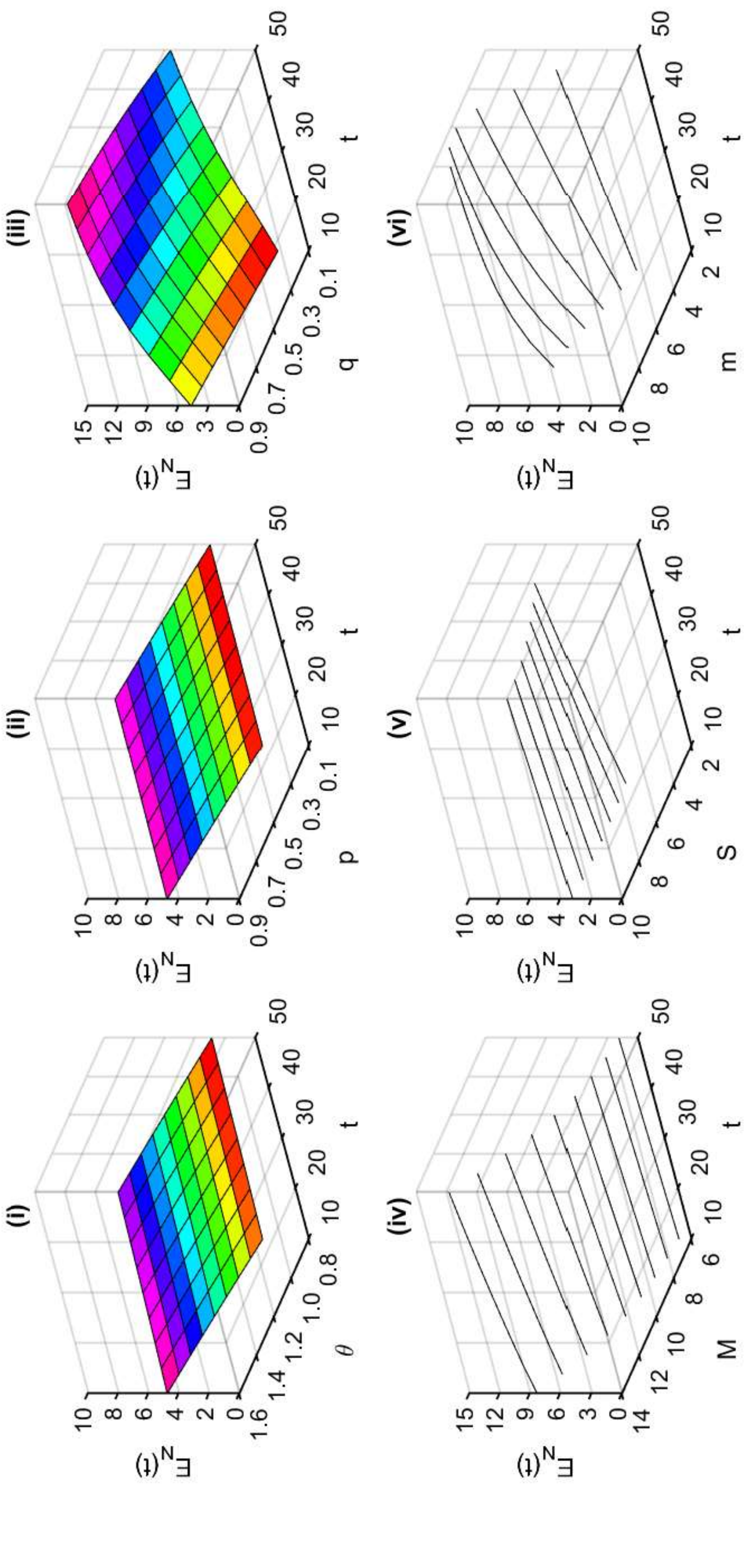

Figure 2.9: Expected number of the failed units in the system $(E_N(t))$ wrt t for different parameters

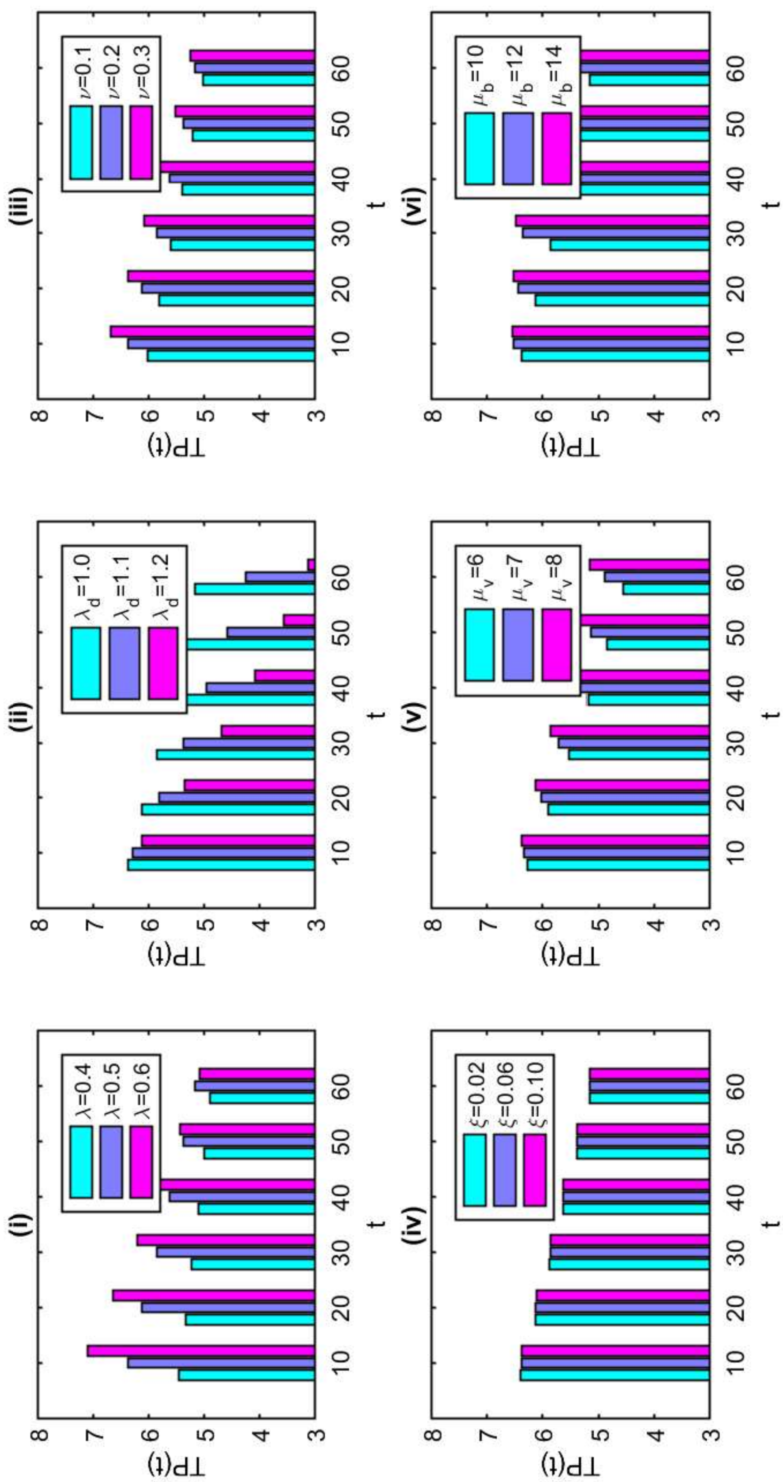

Figure 2.10: Expected throughput of the system $(TP(t))$ wrt t for different parameters

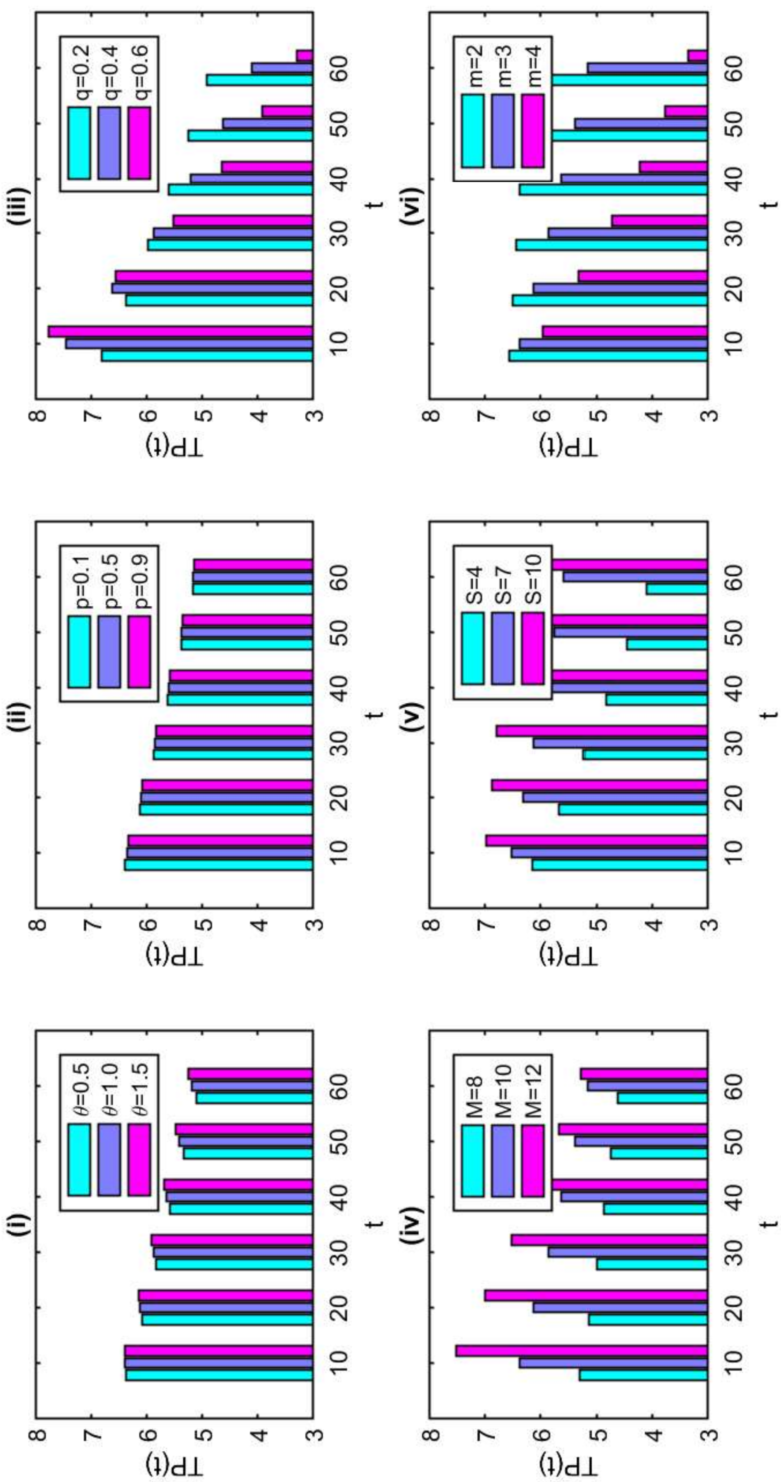

Figure 2.11: Expected throughput of the system $(TP(t))$ wrt t for different parameters

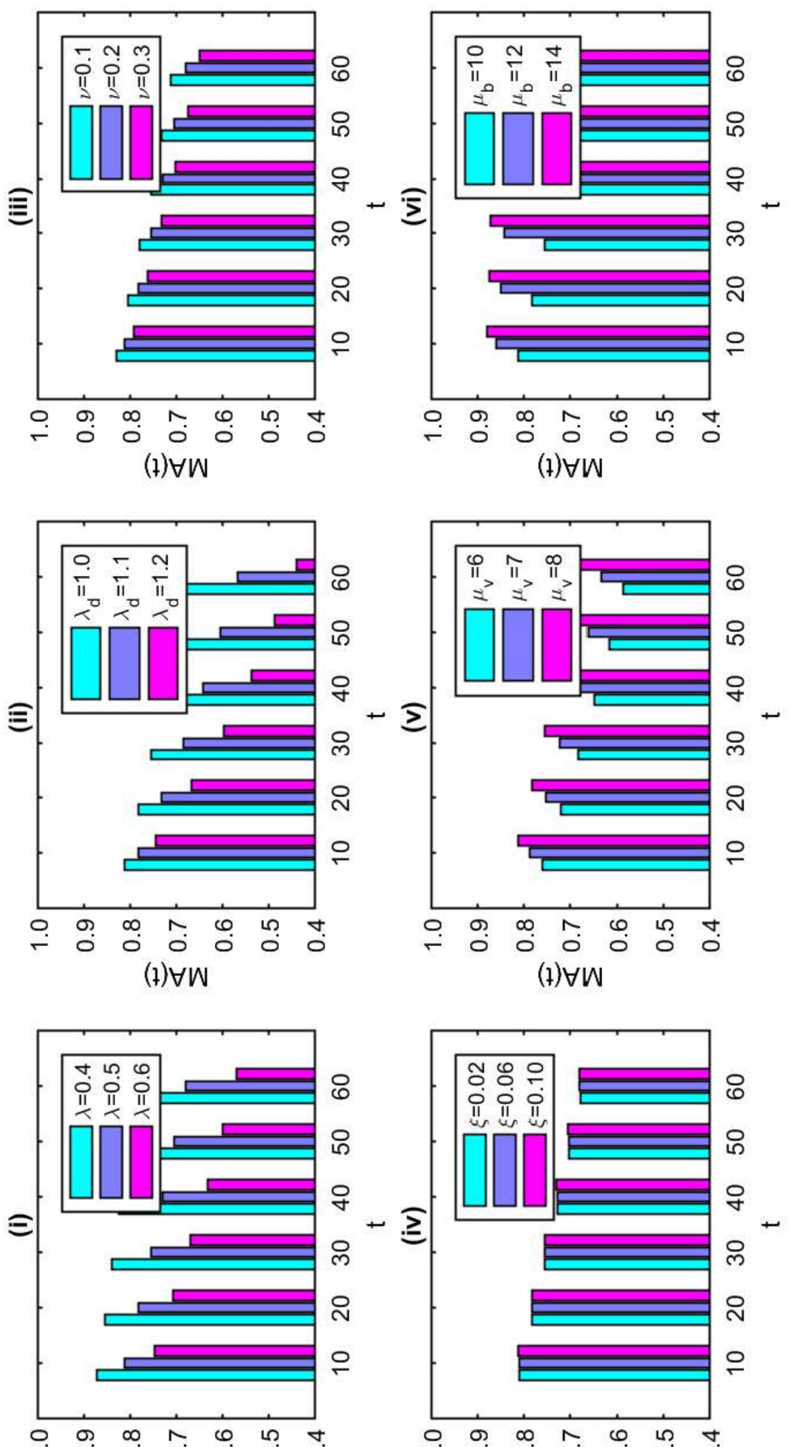

Figure 2.12: Machining system availability $(MA(t))$ wrt t for different parameters

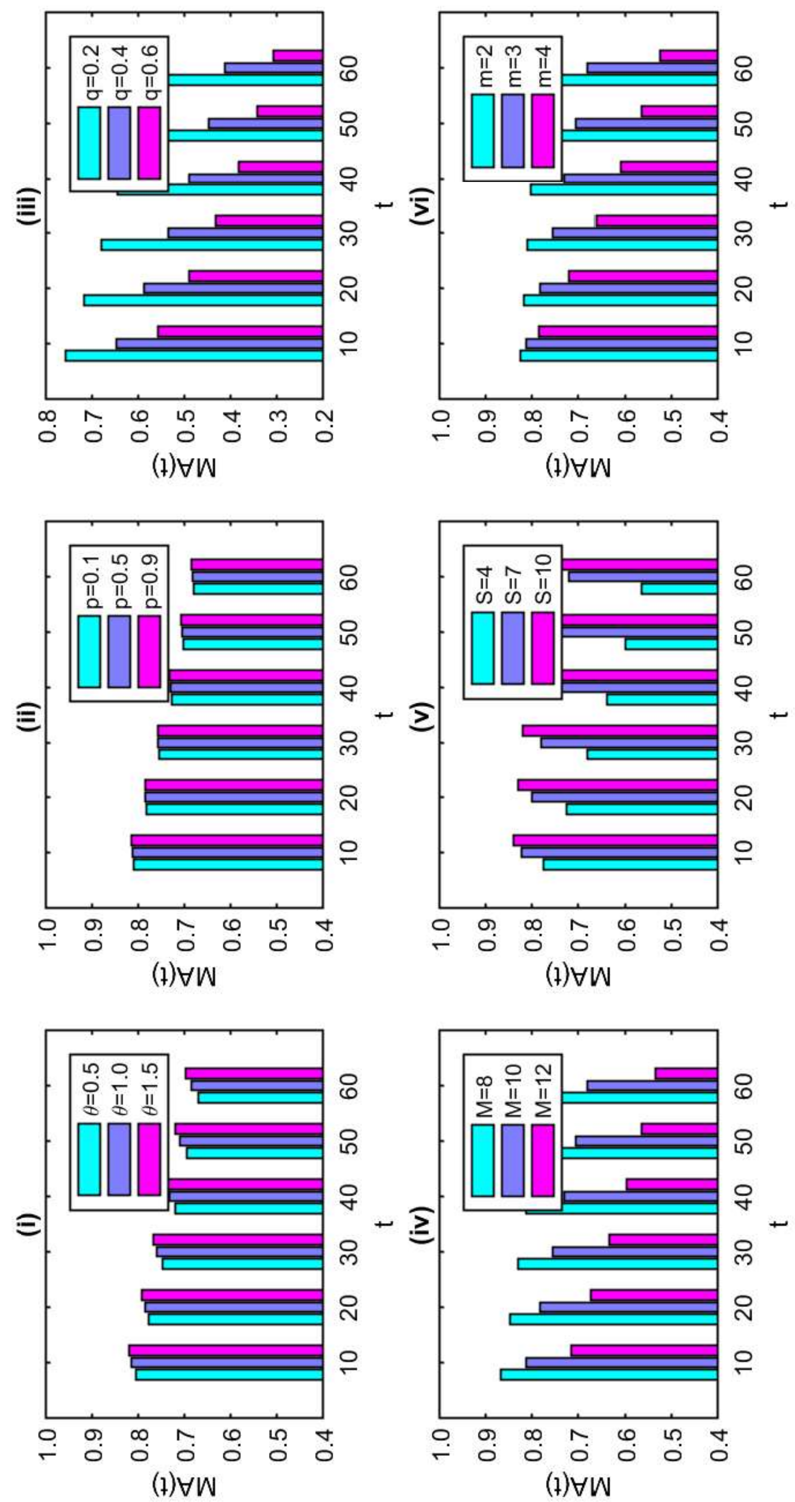

Figure 2.13: Machining system availiability $(MA(t))$ wrt t for different parameters