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ELECTRICAL MEASUREMENTS

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Dr. F. K. Richtmyer was consulting editor of the series from its inception in 1929 until his death in 1939.

ELECTRICAL MEASUREMENTS

Precise Comparisons of Standards and Absolute Determinations of the Units

BY

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To

MY WIFE

PREFACE

This book presents those methods which are capable of giving precise results in absolute electrical measurements, and gives the details of methods which permit the accurate comparison of electrical standards. The increasing demand for precision in electrical measurements has emphasized the need for a text that places special emphasis on accuracy of measurement. This is especially the case in regard to those absolute electrical measurements which are used to establish the values of the electrical units. This book has been prepared to make available the technique that has been developed in this field.

The discussion of the methods is based on the assumption that an accuracy of one part in a million is desirable in every type of measurement. This accuracy was selected as the most precise that was demanded in any electrical measurement. However, since the preparation of the manuscript was begun, there has been a pronounced increase in the precision of measurements in several fields. Comparisons of resistance standards are now made to one part in ten million, and the same can be said concerning standard cells. Standards frequencies are now disseminated which have this accuracy. The need for this extreme accuracy is exceptional, but its use indicates the trend in electrical measurements and shows that an accuracy of one part in a million should be the goal in most fields. Moreover the use of the same standard of accuracy in all fields of electrical measurements greatly facilitates comparisons between them.

The original plan of the book contemplated including only those methods which are directly concerned with the establishing of the electrical units. However, the desirability of including the comparisons of standards soon became apparent, as there are no texts available which indicate the precautions that must be observed when comparisons are to be made with an accuracy of a part in a million, and such comparisons must frequently be made in the absolute methods. Hence the final plan includes, for each of the electrical quantities, at least one method for intercomparing two different values of this quantity. An attempt is made, not to include all the methods that can be used in precise comparisons, but to give, under each heading, one or two methods that will give accurate results. In a few cases, where suitable methods were unknown to me, new methods have been outlined that I believe are capable of giving accurate results. In such cases a statement is made that experimental proof of the attainable accuracy is lacking.

Many subjects are omitted that might properly be included in a treatise on precise electrical measurements. Perhaps the one that is most conspicuous because of its absence is that of galvanometers. In very few of the cases where a galvanometer is required is there any statement of the characteristics that the instrument should possess. This omission has been dictated partly by the requirements of space, and partly by the desire to lay emphasis on other features of the methods. The subject might well be treated in a separate text. Other omitted subjects are amplifiers, electrometers, sources of alternating current, standards of frequency, methods of measuring the power factor of capacitors.

For many years I was associated with the late Dr. E. B. Rosa who published many papers on the subject of absolute electrical measurements and who was an acknowledged authority in this field. Much of my training and enthusiasm for this work came from my association with him and with my former teacher, Professor H. S. Carhart.

In the direct preparation of this treatise I have had the assistance of a number of my associates. The entire manuscript was read by Miss C. Matilda Sparks and by Dr. R. W. Curtis. Parts of the proof were read by Dr. C. Snow, by Dr. A. V. Astin, and by Dr. R. W. Curtis. The criticisms received have resulted in decided improvements in the text.

HARVEY L. CURTIS.

WASHINGTON, D. C.,
December, 1936.

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CHAPTER XIX

ELECTRICAL MEASUREMENTS

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CHAPTER I

INTRODUCTION

Electrical measurements require an established system of electrical units. The recognized units—ohm, ampere, volt, etc.—are all defined in terms of the mechanical units and an arbitrarily assigned value to the permeability of a vacuum. A description of the experimental methods that may be used for the determination of the values of these units is the primary purpose of this book. Such determinations involve precise comparisons of mechanical and electrical standards, so that a description of methods of making these comparisons has been included.

The measurement of an electrical quantity in terms of other quantities is called an *absolute measurement.* Unfortunately, usage has not established a definite meaning for this term. Hence a statement is made of the interpretation herein applied.

1. Definition of Absolute Electrical Measurements.—Absolute electrical measurements comprise those measurements in which an electrical quantity is determined in terms of two or more other physical quantities. In some cases these physical quantities are entirely electrical, as when an electric current is measured in terms of resistance and electromotive force; in other cases the physical quantities are partly electrical and partly mechanical, illustrated by the measurement of capacitance in terms of resistance and time; in still other cases the physical quantities that must be measured are entirely mechanical, an example being the measurement of current in terms of the square root of force, *i.e.*, in terms of length, mass, and time.

The physical quantities used in absolute electrical measurements are not confined to electrical and mechanical quantities but may include magnetic and thermal quantities. For example, to measure a current by the deflection of a tangent galvanometer, the earth's magnetic field at that point must be known; or to measure a current by the heat produced in a known resistance, the temperature rise which this heat produces in a mass of known thermal capacity must be measured.

Absolute electrical measurements are essential for establishing the electrical units in terms of the mechanical units; they are often useful in determining the constancy of electrical standards and they must be used for the measurement of those quantities for which standards cannot be maintained.

Absolute electrical measurements as here defined include all measurements of electrical quantities except the intercomparison of two quantities of the same kind. Such comparative electrical measurements are simpler and more precise than absolute measurements, so they are universally employed when a suitable standard is available or when only the ratio of the two quantities is required. Comparative measurements are required in many of the absolute methods.

2. Establishing the Electrical Units.—The primary use of absolute electrical measurements is for establishing the electrical units in terms of the mechanical units. The electrical units are readily defined in terms of mechanical units, but the experimental realization of units which accurately conform to the definitions requires elaborate apparatus in the hands of skilled experimenters. The establishment of the electrical units is so difficult and costly that relatively few determinations have ever been made, and the results of these determinations are not identical, so that some uncertainty exists as to the absolute values of the electrical units. In order to estimate the accuracy with which these values have been or can be determined, the theory underlying each proposed method must be developed, the possible errors in the experimental procedure must be analyzed, and the accuracy with which the necessary electrical and mechanical measurements can be made must be studied.

The absolute value of the electrical units cannot possibly be determined with greater accuracy than can be attained in comparisons involving the fundamental mechanical units on which the electrical units are based, namely, the centimeter, gram, and second. In recent years, absolute electrical measurements have been so perfected that the error in the final result is sometimes introduced through errors in the mechanical measurements. Hence it is necessary to discuss the reliability of the fundamental mechanical standards and to consider the accuracy that can be attained by their use. Moreover, it is sometimes necessary to employ the derived mechanical units of force, energy, and power. The methods most often employed in the precise mechanical measurement of these quantities require that the acceleration of gravity shall be known at the place where the observations are made. Hence a discussion of the accuracy which has been attained in determining the value of gravity becomes essential.

Very few, if any, of the experimental methods for establishing the electrical units entirely fulfill the exact conditions laid down in the fundamental definitions so that the experimental values obtained by any method are generally influenced by certain approximations which are inherent in the method. To estimate the accuracy of a result, the extent of these approximations must be considered, and, if they are not negligible, corrections must be made on account of them. To ensure that no approximation has been overlooked, as wide a variation as possible should be made in the values of the different physical constants involved in the method.

3. Concrete Electrical Standards.—In order that the results of absolute electrical measurements may be preserved, concrete electrical standards are essential. Concrete standards are of two types, reproducible standards and working standards. A reproducible standard is a standard whose value, when once established, can be reproduced by means of an instrument made of specified materials and with specified dimensions. In most countries reproducible standards form the basis of the legal definitions of the electrical units. A working standard is a standard that will maintain a value for a period of time. A reproducible standard may also be a working standard, but is seldom convenient for that purpose. A perfect working standard is one that can be depended upon for an indefinitely long period to have a definite value whenever subjected to particular external conditions, such as temperature, pressure, and the like. No perfect working standards are known, but some make a reasonably close approach to perfection.

4. Scope.—While concrete standards and their measurement must be given careful consideration, the main purpose of this treatise is to outline the methods by which the electrical units can be accurately established from the mechanical units. These methods are not numerous. However, to obtain satisfactory results by their use requires attention to many details. Hence each method will be carefully described; the difficulties in the experimental procedure will be indicated; the assumptions that have been made in developing the theory will be outlined; and the accuracy that can be obtained by its use will be estimated.

CHAPTER II

DEFINITIONS AND PRINCIPLES

5. Systems of Units.—There are six electrical quantities, *viz.*, current, quantity, electromotive force or potential difference, resistance, capacitance, and inductance, each of which requires a unit. When electrical measurements were being developed, arbitrary units were selected for each of these quantities. However, in 1851, Weber showed that it is possible to develop systems of electrical units which are based on the mechanical units. Only two of the many possible systems will be considered, *viz.*, the electromagnetic system, on which the practical electrical units are based, and the electrostatic system.[1] These two systems have only one point of difference, *viz.*, the method of determining quantity of electricity or current from the mechanical units. However, as this is the starting point of each system, all the units are affected.

The method of defining the units of the electromagnetic system which was used when the system was established starts with magnetic-pole strength and then uses the magnetic pole in defining other units such as magnetic force and electric current. However, the poles of the most perfect magnet are so far from fulfilling the requirements of the fundamental definition that, in practical measurements, methods are now used which do not involve a magnetic pole. Hence, in the definitions which follow, this fiction of a magnetic pole is not used, but instead the electromagnetic force of attraction or repulsion of two conductors carrying currents is taken as the starting point. This has the advantage of using in the fundamental definitions the same principle that is most often employed in experimentally establish-

[1] Two other systems are occasionally used. The electrodynamic system as suggested by Ampère differs from the electromagnetic system in that unit current in the electrodynamic system is less than unit current in the electromagnetic system in the ratio of 1 to 2, and the other units are converted by some power of 2. The Heaviside-Lorenz system was devised to remove the factor of 4π from the more important electromagnetic equations.

ing the units. Since the force between conductors depends on the magnetic properties of the medium surrounding the conductors, the fundamental definition must specify this medium.

The units of the electrostatic system are defined in terms of the force between two charged bodies. The force depends on the dielectric constant of the medium that surrounds the charged bodies; in this case also the medium must be specified.

6. Definitions of Electrical Units.—In order to define the absolute unit of any of the electrical quantities, it is necessary to know the physical law that connects this quantity with the other physical quantities on which the unit is based. Hence in the following definitions the physical law will first be stated, then the units derived from these laws. As a means of showing the similarities, the two systems will be defined in adjacent columns insofar as there is any difference between them.

THE ELECTRICAL UNITS OF THE ELECTROMAGNETIC AND ELECTROSTATIC SYSTEMS

Electromagnetic

1. *Basic Unit—Electric Current.* A. FUNDAMENTAL LAW.[1] The force dF of attraction or repulsion between two conductors of infinitesimal lengths ds_1 and ds_2 (see Fig. 1), carrying currents I_1 and I_2, which make an angle ϵ with each

Electrostatic

1. *Basic Unit—Quantity of Electricity.* A. FUNDAMENTAL LAW.[2]—The force of attraction or repulsion between two point charges e_1 and e_2, separated by a distance r, is

$$F = \frac{e_1 e_2}{\kappa r^2}$$

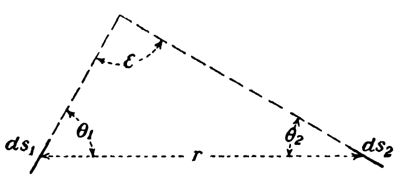

FIG. 1.—Diagram showing the geometrical quantities required to describe the force between two infinitesimal conductors when carrying current. The figure shows the elements lying in a plane, but the formula given in the text is applicable when the elements are not coplanar.

[1] First proposed by Ampère. See Maxwell, "Electricity and Magnetism," 3d ed., Art. 526.

[2] First established by Coulomb. See Maxwell, *op. cit.*, Arts. 38 and 215.

Electromagnetic (*Continued*)

other, and which make angles θ_1 and θ_2 with the line of length r that joins their centers, is

$$dF = \frac{\mu I_1 I_2}{r^2}(2 \cos \epsilon + 3 \cos \theta_1 \cos \theta_2) ds_1\, ds_2$$

The force is in the direction of the line joining the elements, and its magnitude depends on the permeability μ of the medium surrounding the elements.

B. A SPECIFIC APPLICATION OF THE FUNDAMENTAL LAW.—Let a circuit consisting of two long parallel wires of length b, separated a distance a, carry a constant current I. Then the force of repulsion, found by integrating the fundamental equation, is

$$F = \frac{2\mu I^2 b}{a}$$

This assumes that end effects are negligible.

C. DEFINITION OF CURRENT.—When equal currents flow through two conductors which have, in a vacuum, a force of attraction (or repulsion) F, the value I of each of the currents in c.g.s. electromag-

Electrostatic (*Continued*)

and is in the direction of the line joining the charges. The magnitude of the force depends on the dielectric constant κ of the medium surrounding the charges.

B. A SPECIFIC APPLICATION OF THE FUNDAMENTAL LAW.—The force of repulsion F between two parallel plates, each of area A and each having identical charges Q, is found, by integrating over the surfaces, to be

$$F = \frac{2\pi Q^2}{\kappa A}$$

This assumes the plates to be so large relative to their distance apart that edge effects can be neglected.

C. DEFINITION OF QUANTITY. When two equal point charges are 1 cm apart in a vacuum and repel each other with a force of 1 dyne, the value of each charge is, by definition, one c.g.s. electrostatic unit of

Electromagnetic (*Continued*)

netic units is defined as

$$I = \sqrt{\frac{F}{\int\int[(2 \cos \epsilon + 3 \cos \theta_1 \cos \theta_2)/r^2]ds_1\, ds_2}}$$

In the c.g.s. system, the permeability of a vacuum is taken as unity. As an example, the current in c.g.s. electromagnetic units flowing in a circuit consisting of two infinitely long parallel conductors is unity if the force of repulsion is 2 dynes per centimeter length when the distance between conductors is 1 cm and when the surrounding space is devoid of all matter.

2. *Quantity of Electricity.* A. FUNDAMENTAL LAW. Quantity of electricity equals the time integral of the current. If i is the instantaneous value of the current and Q is the total quantity that passes a cross section in time T, then

$$Q = \int_0^T idt$$

B. DEFINITION.—Unit quantity of electricity in the c.g.s. electromagnetic system passes a section of a conductor when a unit current is maintained for 1 second.

Electrostatic (*Continued*)

quantity of electricity. In this system, the dielectric constant of a vacuum is taken as unity.

2. *Electric Current.* A. FUNDAMENTAL LAW.—Current is the time rate at which quantity of electricity passes a given cross section of a conductor. If I is the current in a conductor, Q the quantity that passes a cross section, and t the time, then

$$I = \frac{dQ}{dt}$$

B. DEFINITION.—Unit current in the c.g.s. electrostatic system flows in a conductor when electricity passes any section at the rate of one c.g.s. unit of quantity per second.

C. COMPLEMENTARY RELATIONSHIPS.—Each of the preceding sets of definitions gives units for measuring electricity at rest and in motion. In the electromagnetic system the units are based on electricity in motion. A measure of quantity of electricity is obtained by measuring the current that flows for a definite time. If the current flows into a capacitor, the quantity at rest on the condenser is determined. In the electrostatic system the order is reversed. The units are based on electricity at rest while current is measured by the rate of change of quantity.

3. *Potential Difference and Electromotive Force.* A. FUNDAMENTAL LAW OF POTENTIAL DIFFERENCE.—The potential difference between two equipotential surfaces A and B is proportional to the work required to move a positive electric charge from A at lower potential to B at higher potential. If a negative charge is transferred from A to B or a positive charge from B to A, work is done by the electric system.

B. DEFINITION OF POTENTIAL DIFFERENCE.—In either of the c.g.s. systems, unit potential difference exists between two equipotential surfaces when 1 erg of work is required to transfer unit quantity of positive electricity from the surface of lower potential to the surface of higher potential. Since unit quantity of electricity in the electromagnetic system represents a different amount of electricity than unit quantity in the electrostatic system, unit potential difference in one system does not represent the same potential difference as unit potential difference in the other system.

C. FUNDAMENTAL LAW OF ELECTROMOTIVE FORCE.—The electromotive force in a circuit is the electric force that causes electricity to move around the circuit. The maintaining of electricity in motion through a circuit requires that energy shall be continually supplied to the circuit, since the energy of electricity in motion is continually transferred to other forms of energy. The electromotive force in a circuit multiplied by the electric current that is flowing is proportional to the rate at which energy is being introduced into the circuit. Since the energy of the electric current is transformed into other forms of energy at the same rate that energy is introduced into the circuit, the electromotive force can be measured by determining either the power output of a circuit or the power input.

D. DEFINITION OF ELECTROMOTIVE FORCE.—In either of the c.g.s. systems, unit electromotive force is present in a circuit when energy at the rate of 1 erg per second must be introduced to produce one c.g.s. unit of current in the circuit.

E. RELATION BETWEEN ELECTROMOTIVE FORCE AND POTENTIAL DIFFERENCE.—Potential difference is defined with respect to two equipotential surfaces; electromotive force is defined with respect to a circuit. A potential difference can be measured directly by an instrument connected between the two surfaces in question. An electromotive force can be determined only by measurements on a circuit in which a current is flowing, and these measurements merely give the resultant of all the electromotive forces in the circuit. However, this strict differentiation is not always observed. For example, the potential difference between the terminals of a standard cell is often called the electromotive force of the cell. Failure to distinguish between the two seldom leads to a misunderstanding, since the same unit is used for measuring the one as for the other.

4. *Resistance.* A. FUNDAMENTAL LAW.—The resistance of a circuit is the ratio of the electromotive force in the circuit to the current in it. The resistance of a portion of a circuit in which no electromotive force exists is equal to the potential difference between the two ends of this portion divided by the current in the circuit. If the portion under consideration is a metallic conductor, the resistance is independent of the current flowing and hence is a natural constant of that particular metallic conductor.

B. DEFINITION.—The resistance of a conductor is unity if there is unit potential difference between its terminals when there is unit current in the conductor. Both potential difference and current must be measured in the same system of units, and the value of the resistance applies to that system only.

5. *Inductance.* A. FUNDAMENTAL LAW.—The electromotive force that is induced in a circuit by a changing electric current is proportional to the product of the inductance and the time rate of change of the current. If the induced electromotive force and the changing current are in the same circuit, the inductance is called *self inductance.* If the induced electromotive force and changing current are in separate circuits, the inductance is called *mutual inductance.*

B. DEFINITION.—The inductance is unity if the induced electromotive force is unity when the current changes at the rate of one unit per second, all units being in the same system.

6. *Capacitance.* A. FUNDAMENTAL LAW.—The quantity of electricity on each of two insulated conductors (positive on one, negative on the other) is proportional to the product of the capacitance of the system and the difference of potential between the conductors.

B. DEFINITION.—The capacitance of a system consisting of two insulated conductors and the dielectric surrounding them is unity if unit quantity of positive electricity on one conductor, and unit quantity of negative electricity on the other conductor produce unit potential difference between the two conductors, all units being in the same system.

7. Names of Units.—No one of the twelve units (six in each system) already defined has a universally recognized name,

TABLE I.—NAMES OF UNITS IN THE DIFFERENT SYSTEMS OF ELECTRICAL UNITS[1]

Quantity	Name of practical unit	Number of practical units to equal 1 c.g.s. electromagnetic unit	Names proposed: c.g.s. electromagnetic	Names proposed: c.g.s. electrostatic
Current	Ampere	10	Abampere	Statampere
Quantity of electricity	Coulomb	10	Abcoulomb	Statcoulomb
Electromotive force	Volt	10^{-8}	Abvolt	Statvolt
Resistance	Ohm	10^{-9}	Abohm	Statohm
Inductance	Henry	10^{-9}	Abhenry / Centimeter	Stathenry
Capacitance	Farad	10^{9}	Abfarad	Statfarad / Centimeter

[1] The prefixes *ab* and *abstat* were proposed by Kennelly [see *Trans. A.I.E.E.*, **22**, 534 (1903)]. The latter has generally been shortened to *stat*. No one of the proposed names has been adopted by any international organization.

although names have at times been proposed and have received some recognition. The units in practical use are all decimal multiples or submultiples of the corresponding units of the c.g.s. electromagnetic system. The relationship between the practical and c.g.s. electromagnetic system, as well as the more important suggested names for the c.g.s. units of both the electrostatic and electromagnetic system, are given in Table I.

8. Relationships between the Units of the Two C.G.S. Systems. The definitions show that there is only one fundamental difference between the electromagnetic and electrostatic systems, *i.e.*, the method of defining current (or quantity). Hence if the ratio between the units of current in the two systems is experimentally determined, the ratios of all other units are immediately known. For example, consider a circuit in which the values of the current and electromotive force, and hence the resistance, are each unity in the electromagnetic system. Assume that the current in this circuit has been experimentally determined to be v electrostatic units. It follows, since the power is 1 erg per second, that the electromotive force expressed in the electrostatic system is $1/v$ and that the resistance is $1/v^2$. Similar reasoning can be extended to the other units. The ratios of the numerics when a definite amount of any of the electrical quantities is measured in the different systems of units are given in the following table. Experimental methods of determining v are given in Chapter XXI.

TABLE II.—FACTORS REQUIRED TO CONVERT DATA EXPRESSED IN ELECTROMAGNETIC OR PRACTICAL UNITS TO CORRESPONDING VALUES IN ELECTROSTATIC UNITS

Quantity	To convert from electromagnetic units to electrostatic units, multiply by	To convert from practical units to electrostatic units, multiply by
Current	v	$v/10$
Quantity of electricity	v	$v/10$
Electromotive force	$1/v$	$10^8/v$
Resistance	$1/v^2$	$10^9/v^2$
Inductance	$1/v^2$	$10^9/v^2$
Capacitance	v^2	$v^2/10^9$

The latest experimental value of v is 2.9979×10^{10}. Hence $v^2 = 8.9874 \times 10^{20}$; $1/v = 3.3357 \times 10^{-11}$; and $1/v^2 = 1.1127 \times 10^{-21}$.

9. Principles Involved in Experimentally Establishing the Units.—The fundamental laws on which the units are based show that the six units of the electromagnetic system can be divided into three classes: (1) those units, *viz.*, current and electromotive force, whose definitions introduce at least one mechanical unit other than time; (2) those units, *viz.*, inductance and quantity, which are defined in terms of other electrical units and the mechanical unit of time; (3) those units, *viz.*, resistance and capacitance, which are defined in terms of other electrical units only.

The most straightforward method of experimentally determining the units of the electromagnetic system, and hence of the practical system, would be first to measure a current, as it depends solely on the mechanical units of force and length, and then to measure electromotive force in terms of current and the mechanical unit of power. The other units follow from these either by electrical measurements alone or by electrical measurements combined with the measurement of time. However, this method, while satisfactory for measuring current, does not lend itself to the precise measurement of electromotive force since there is no method of completely converting electrical power to mechanical power. More accurate results can be obtained by indirect methods.

While the straightforward method of establishing the units bases the experimental procedure directly on the fundamental definitions, yet an equally logical and more practical method is to combine the principles on which the definitions are based in any manner that will improve the experimental procedure. However, any possible combination necessarily requires that two electrical units be determined by measurements which involve either a mass or a length, or both, and which may involve time; that at least one other unit involves time in addition to one or both of the preceding units; and that a maximum of three units can be obtained by electrical measurements from those previously established.

Two different experimental procedures for establishing all the units of the electromagnetic system have given results of high precision. In one the units of current and resistance are independently determined in terms of the mechanical units, the only electrical quantity involved being the permeability of space. Then the unit of capacitance is obtained from resistance and time. The unit of electromotive force is derived from resistance and current, and that of inductance from resistance and capacitance. The unit of quantity is obtained either from current and time or from capacitance and electromotive force.

In the second procedure, the units of time and inductance are determined from the mechanical units and the permeability of space; the unit of resistance is determined from inductance and time, while the remaining three units are obtained in the same manner as in the first procedure.

To establish the electrostatic units, there are two possible procedures. One is to determine them from the mechanical units, using the basic definitions and combinations of them; the other is to derive them from the electromagnetic units by multiplying each electromagnetic unit by the correct function of v (given in Table II). The second procedure, requiring a measurement in mechanical units of only one electrostatic unit, is almost universally employed.

10. Concrete Standards of the Units.—The maintenance of the electrical units is accomplished largely by means of concrete standards. From its nature, no standard of current is possible. Also no feasible method of constructing a standard of quantity has been evolved. Hence the entire system of electrical units is maintained by the concrete standards of resistance, electromotive force, capacitance, and inductance. There is but one possible intercomparison using electrical measurements alone among these four standards, namely, an inductance can be measured in terms of capacitance and resistance. Thus the number of electrical standards that are absolutely required to maintain an electrical system of units independent of mechanical units is three, one of which is electromotive force.

The unit of resistance can be maintained to a high degree of accuracy by means of concrete standards. Such a standard generally consists of a coil of alloy wire or of a definite length of metallic strip. The value in the electromagnetic system of a standard of resistance can be directly determined by absolute measurements.

The unit of electromotive force can also be accurately maintained by concrete standards. A standard cell is generally used as such a standard. Such cells can be quite accurately reproduced. However, the absolute value of a standard cell can be determined only by using Ohm's law in connection with absolute measurements of both current and resistance.

A concrete standard of capacitance consists of two sets of metallic plates which are separated by a dielectric, usually air or mica. An air capacitor is not so stable as a standard resistor, so the absolute value of such a standard must frequently be determined in terms of resistance and time. A mica capacitor can be used only in connection with alternating-current measurements, and in such measurements the capacitance depends, to some extent, on the frequency of the alternating current. The capacitance of a mica capacitor is determined either by comparison with an air capacitor or in terms of resistance and inductance.

A concrete standard of inductance consists usually of a coil (or coils) of copper wire wound inductively on a nonmagnetic form. If a coil is so constructed that its dimensions can be accurately measured, and its inductance computed from these dimensions, the coil is an absolute standard as well as a concrete standard. However, an absolute standard is necessarily large, and hence inconvenient to use. Laboratory standards are generally more compact, and their values are determined either by comparison with an absolute standard or by measurement in terms of resistance and capacitance. These laboratory standards of inductance are not so stable as standards of resistance and electromotive force.

11. Supplementary Electrical Units.—In addition to the six electrical units already defined, there are a considerable number of units which involve both electrical and mechanical units. These follow so directly from the established units that no special consideration needs to be given them. A partial list is given in Table III.

TABLE III.—SUPPLEMENTARY ELECTRICAL UNITS

Name of quantity	Unit	Symbol
Current density	Amperes/$\overline{\text{cm}^2}$	J
Surface density of charge	Coulomb/$\overline{\text{cm}^2}$	σ
Volume density of charge	Coulomb/$\overline{\text{cm}^3}$	ρ
Electric intensity or potential gradient	Volts/cm	F, R
Resistivity	Ohm-cm	ρ
Conductance	Mho	g
Conductivity	Mho/cm	γ
Dielectric constant	Farads/cm	κ or ϵ
Dielectric polarization	Coulombs/$\overline{\text{cm}^2}$	D

12. Definitions of the Magnetic Units.—No satisfactory concrete standards for the magnetic units have been developed, but these units can readily be established and maintained by means of the electrical units. When approached from this point of view, the two basic magnetic concepts are magnetic intensity and magnetic induction. Both of these quantities are vectors, and at any point where one has a value, the other has a value also. Hence in any region in which there is a vector field of magnetic intensity, a vector field of magnetic induction exists also. The direction of the vectors in any vector field may be represented graphically by means of lines so drawn that at any point the tangent to a line is in the direction of the vector at that point.

Some vector fields have the added property that, in a given region, the space may be divided into tubes, each of which has a series of vector lines as its boundary and each of which has the property that the areas of any two cross sections are inversely proportional to the values of the vector at those cross sections. Such tubes will either be closed tubes or will end in the boundary of the region. Such a vector field is called a *tubular field* or a *solenoidal field* (from the Greek *solen*, a tube). Examples of tubular vector fields are magnetic induction in any region whatsoever, and magnetic intensity when there is no magnetic material in the field.

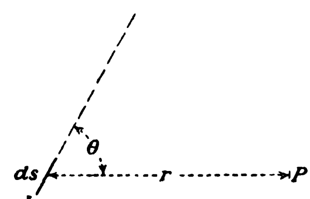

FIG. 2.—Diagram to show the geometric quantities required to describe the magnetic intensity at P which is caused by a current I in the conductor ds. The magnetic intensity at P is perpendicular to the plane that contains r and ds.

Since both magnetic intensity and magnetic induction occupy the same region around a conductor, a graphical representation of both is not always feasible, and the attempt has frequently led to misunderstandings. Thus, in the following definitions no reference is made to lines or tubes of magnetic intensity or magnetic induction. Also the positive and negative directions of the magnetic vectors are entirely conventional. As this convention is of no importance in absolute electrical measurements, all reference to it will be omitted in the definitions.

1. *Magnetic Intensity or Magnetic Force.* A. FUNDAMENTAL LAW.—The infinitesimal magnetic intensity dH_p at a point P, which is caused by a current I in an infinitesimal length ds of a conductor that is at a distance r from the point P and that makes an angle θ with the line joining P to ds, is proportional to $\frac{Ids}{r^2} \sin \theta$ and is in a direction perpendicular to the plane that is determined by ds and the line joining ds to the point P (see Fig. 2). The value and direction of the magnetic intensity at P which results from the total circuit is found by the integration of the above expression completely around the circuit. Expressed as a vector equation:

$$\mathbf{H}_p = KI\left(\int\right)\frac{[ds \times \mathbf{r}]}{r^3}$$

where K = proportionality factor which depends on the units employed.

B. EXAMPLES.—1. The magnetic intensity at a point P, which is at a distance r from the axis of an infinite straight wire carrying a current I, is proportional to $2I/r$ and is perpendicular to the plane passing through P and the axis of the wire.

2. The magnetic intensity at a point P (see Fig. 3), which is at a distance b from every point on the circumference of a circle of radius r when a current I flows in a conductor that is coincident with the circumference, is proportional to $2\pi I r^2/b^3$ and is in the direction of the perpendicular from P to the plane of the circle. If P is at the center of the circle, r is equal to b, and the magnetic intensity is $2\pi I/r$.

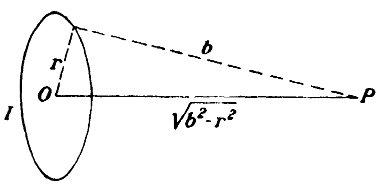

FIG. 3.—Diagram to show the geometrical quantities that are used in the equation for the magnetic intensity at a point P on the axis of a circle which has a current I in its circumference. The direction of the magnetic intensity is the same as the direction of the axis of the circle.

3. The magnetic intensity at any point inside an infinitely long solenoid having N turns per unit length and carrying a current I is proportional to $4\pi NI$ and is in the direction of the axis of the solenoid.

C. DEFINITION.—The value of the magnetic intensity in oersteds results from the substitution of numerical values in any integration of the fundamental law, provided the unit of length is the centimeter, the current is expressed in c.g.s. electromagnetic units, and the proportionality factor is taken as unity. The same result is obtained if the current is in amperes and the proportionality factor is ten.

2. *Magnetic Induction.* A. FUNDAMENTAL LAW.—The magnetic induction at a point P is a vector, the value of the component in any direction being proportional to the electromotive force that is induced in a conductor of length ds which is moving with unit velocity through the point in a direction perpendicular to itself and to the direction of the component. By obtaining the values of the components along three rectangular axes, the magnitude and direction of the magnetic induction at the point P can be obtained by the vector addition of these components. If one axis is in the direction of the magnetic induction, the components along the two axes at right angles are zero. In such a case, the electromotive force induced in ds when moving as indicated is the maximum that can be obtained by moving ds with unit velocity through P and is proportional to the magnitude of the magnetic induction (see Fig. 4).

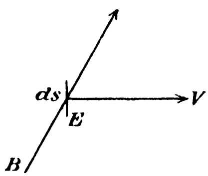

FIG. 4.—Diagram to show the direction of the magnetic induction relative both to the direction of a conductor **ds** and to its velocity **V**, when the value of the magnetic induction **B** is determined from the electromotive force E induced in the conductor.

The fundamental law is expressed by the vector equation

$$E = K\mathbf{B}[\mathbf{V} \times d\mathbf{s}]$$

where $\mathbf{B}$ = the magnetic induction.
$d\mathbf{s}$ = the length of an element of a circuit.
E = the induced electromotive force in $d\mathbf{s}$.
$\mathbf{V}$ = the velocity of $d\mathbf{s}$.
K = a proportionality constant.

B. DEFINITION.—The magnetic induction at a point is 1 gauss when the maximum electromotive force that can be induced in a conductor 1 cm long moving through the point with a velocity of 1 cm per second is 1 c.g.s. unit of electromotive force. The direction of the magnetic induction is perpendicular to the plane in which the conductor moves.

C. EFFECT OF MEDIUM.—In a vacuum, the magnetic induction is in the same direction as the magnetic intensity and has, in the c.g.s. electromagnetic system, the same numerical value. In any isotropic medium of infinite extent, the magnetic induction is in the same direction as the magnetic intensity, but the numerical values of the two are not the same. In a crystal, the magnetic induction may not even have the same direction as the magnetic intensity.

3. *Magnetic Permeability.* A. BASIC CONCEPT.—In most isotropic materials the quotient of the magnetic induction divided by the magnetic intensity is a constant, which is called the *permeability* of the material. For ferromagnetic materials, this quotient is not a constant but is still called *permeability.* When reference is made to the permeability of one of these materials, either the magnetic intensity or the magnetic induction should be stated.

Since in the electromagnetic system magnetic intensity and magnetic induction have the same dimensional equations, permeability is a dimensionless constant in this system. In other systems, such as the electrostatic, permeability is not dimensionless.

B. DEFINITION.—The permeability at a point in an isotropic medium is the quotient of the value of the magnetic induction at the point divided by the value of the magnetic intensity. In the c.g.s. electromagnetic system, the permeability of a vacuum is, by definition, unity. If the practical system of electrical units is extended to the magnetic units, the permeability of a vacuum is 10^{-7}.

4. *Magnetic Flux.* A. UNDERLYING PRINCIPLE.—Magnetic flux is the surface integral over some particular surface of the normal component of the magnetic induction. If the surface is not a closed surface, then it must end in a closed line which is the boundary of the surface. But one can conceive of an infinite number of other surfaces that would be bounded by this same closed line. Since magnetic induction is a tubular vector, the surface integral over each one of these surfaces is the same as that over every other one. Hence magnetic flux is determined by the closed line that bounds the surface.

B. DEFINITION.—The magnetic flux in maxwells within a closed line is equal to the surface integral of the normal component of the magnetic induction in gausses taken over any surface that is bounded by the line.

C. INDUCED ELECTROMOTIVE FORCE.—If the boundary line is the center of an electric conductor, then any change in the flux caused either by a change in the value of the magnetic induction or by a relative motion of the conductor and magnetic induction will induce, in the conductor, an electromotive force which depends on the rate of change of the flux. This is a direct consequence of the definition of magnetic induction and the fact that it is a tubular vector. It follows that in any electric circuit the induced electromotive force in c.g.s. units is equal to the time rate of change of the magnetic flux in maxwells.

5. *Magnet Pole.* A. THE POLES OF A SOLENOID.—The magnetic induction at any point outside a long solenoid of infinitesimal cross section in which there is an electric current is equal to the vector sum of two magnetic inductions, one of which is directly proportional to the flux through the solenoid and inversely proportional to the square of the distance from one end and is directed toward this end; the other magnetic induction bears a similar relation to the opposite end of the solenoid but is directed away from that end. The poles of a solenoid are the two points, one at each end, which appear to be the source of the external field of magnetic induction. In a solenoid having an infinitesimal cross section, the poles are at the ends, and their value is proportional to the flux through the solenoid. The poles of a solenoid having a finite cross section are on the axis a short distance inside the ends.

B. DEFINITION OF POLE STRENGTH.—The pole strength of a long solenoid of cross section dS, having N turns of wire per centimeter through which a current I in c.g.s. units is flowing, is equal to $NIdS$ provided the solenoid is in a vacuum. Since the flux through such a solenoid is $4\pi NIdS$, it follows that the total flux is 4π times the pole strength.

C. EFFECT OF MEDIUM.—If the solenoid is immersed in a medium of permeability μ, then the flux and the pole strength are μ times the values in a vacuum. If the magnetic material fills only the solenoid, then the magnetic flux through it can be experimentally determined by measuring the electromotive force induced in a loop of wire which fits the outside of the solenoid when the flux decreases to zero at a known rate. With some magnetic materials in the solenoid, the flux does not decrease to zero when the current becomes zero. These materials are then permanent magnets and the pole strength is defined as the magnetic flux divided by 4π.

D. HISTORICAL.—The unit magnetic pole was originally defined as that pole which would repel an equal pole placed at a distance of 1 cm from it with a force of 1 dyne. This is equivalent to the preceding definition and gives a better mental picture. However, it is necessary to use the one given to develop a logical system of units when the magnetic units are based on the electrical units.

13. The C.G.S. Magnetic Units Compared to the Practical Units. The c.g.s. magnetic units are of convenient size for laboratory use, and hence have been given the names indicated. The magnetic units corresponding to the practical electrical units are seldom used. As a result, in many formulas that contain both electric and magnetic quantities, factors are introduced for converting the values of the electric quantities in practical units to their values in c.g.s. electromagnetic units. As an example, the magnetic intensity H, in oersteds inside a long solenoid, is given by the equation

$$H = 4\pi \frac{NI}{10}$$

where N = number of turns per centimeter.
I = current in amperes.
10 = factor for converting the current to c.g.s. electromagnetic units.

14. Supplementary Magnetic Units.—In addition to the magnetic units already defined, there are a number of others which follow from combinations of these with each other or with mechanical units. A partial list is given in Table IV. Only those are included which apply generally. In addition there are several units which apply only to ferromagnetic materials.

TABLE IV.—SUPPLEMENTARY MAGNETIC UNITS

Name of quantity	Definition	Unit	Symbol
Magnetomotive force......	Line integral of magnetic intensity	Gilbert	F
Reluctivity...............	Reciprocal of permeability		
Reluctance...............	$\frac{\text{Magnetomotive force}}{\text{Magnetic flux}}$	......	R

CHAPTER III

HISTORY OF THE ELECTRICAL UNITS

15. Arbitrary Units.—The first electrical units were entirely arbitrary. During the latter part of the eighteenth century each physicist described the capacitance of his Leyden jar by the amount of liquid that the jar would hold. In the early part of the nineteenth century when current electricity was first studied, some experimenters adopted, as a standard of resistance, a given length of a particular size of iron wire. These arbitrary units had no direct connection with the units of any other branch of physics.

16. Development of the Absolute System.—The first step in the direction of connecting the electrical units with the mechanical units was taken by Gauss[1] in 1832. Gauss devised a method, which is still in extensive use, for measuring the horizontal intensity of the earth's magnetic field in terms of length, mass, and time. The next step was taken by Kohlrausch[2] in 1849, who measured a resistance in terms of mechanical units. The final step was taken by W. Weber,[3] who in 1851 showed the possibility of a connected system of electrical units based on the mechanical units. The experimental values that Weber obtained were never used to any extent, but the principles that were employed in obtaining these values form the basis of the present systems of electrical units.

17. The British Association Committee.—In 1861 the British Association for the Advancement of Science appointed a committee on Standards of Electrical Resistance. This committee[4]

[1] Paper in Latin read before the Royal Scientific Society of Göttingen. See "Collected Works," Vol. V, p. 79. German translation in *Pogg. Ann.*, **28**, 241 (1833).

[2] *Pogg. Ann.*, **76**, 412 (1849).

[3] *Pogg. Ann.*, **82**, 337 (1851).

[4] The names of many illustrious men appear on the roll of this committee. A few are Clerk Maxwell, Lord Kelvin, Joule, Matthiessen, and Latimer Clark. The complete reports of the committee, extending from 1862 to

laid the foundation for the practical electrical units. After a careful consideration of the many possible systems of units, the committee decided to base the electrical units on the centimeter-gram-second system of mechanical units, using the electromagnetic system proposed by Weber, and to sponsor research for obtaining experimental values. In 1864 the experimental work of Maxwell, Stewart, and Jenkin had reached a point where the

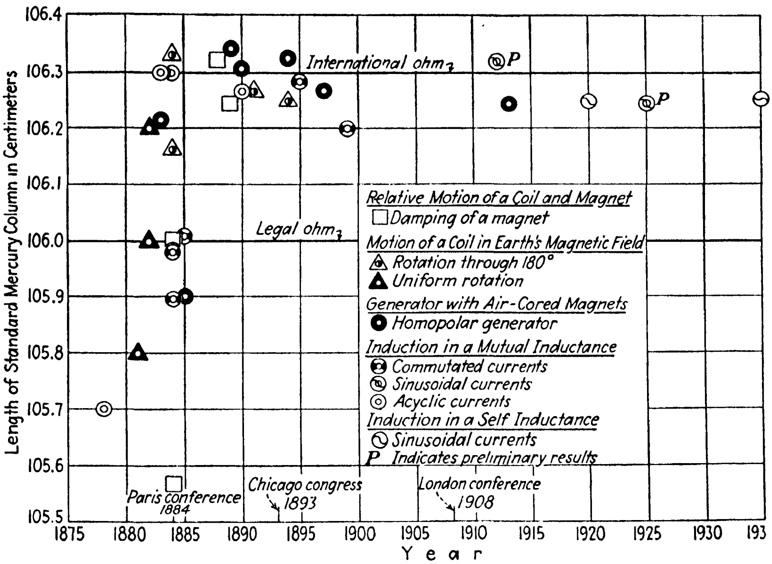

FIG. 5.—Absolute determinations of the ohm. The method employed in each determination is indicated by an appropriate symbol.

committee felt justified in issuing the British Association unit of resistance, which was intended to represent 10^9 absolute units. Some measurements were made to determine absolute values of the units of current, electromotive force, and capacitance, but standards were never issued by the committee.

18. Absolute Determinations of the Ohm.—The British Association unit of resistance was generally accepted for more than a decade. In 1878, Rowland announced that this unit differed from the absolute value by nearly 1.5 per cent. As this error was nearly ten times the expected amount, experimental work on

1912, have been collected in one volume, "British Association Reports on Electrical Standards," Cambridge University Press, 1913.

the absolute ohm was greatly stimulated. The continually increasing accuracy is well shown by the points on the diagram of Fig. 5. Here the results have all been reduced to a common basis, *viz*, the length of the standard mercury column, *i.e.*, a column with a cross section of 1 sq mm and measured at 0°C.

19. The Legal and International Ohms.—The accuracy which has been attained at different times in the absolute measurement of the ohm is well shown by the values that were adopted by different international bodies. In 1884 an international commission meeting in Paris decided that the experimental results then available did not justify fixing the length of the standard mercury column with an accuracy greater than 1 per cent, and it adopted 106 cm as the most probable length. This unit of resistance was called the *legal ohm*. Nine years later the Chicago Electrical Congress adopted 106.3 cm as the most probable length of the standard mercury column. This was called the *international ohm*. The London Electrical Conference in 1908 concluded that the length of the standard mercury column corresponding to an absolute ohm was not then known to a hundredth of a centimeter. To make the international ohm more definite, the conference decided arbitrarily to add two zeros to the length of the mercury column adopted at Chicago, making it 106.300 cm. The international ohm has been legalized in most countries of the civilized world.

20. Recent Absolute Values of the Ohm.—In 1914, Smith[1] at the National Physical Laboratory of England published the results of a series of experiments on the absolute value of the ohm. These experiments were carried out with great care under very favorable conditions, so that the result obtained was immediately accepted as being more accurate than any which had been obtained up to that time. Smith obtained as the length of the standard mercury column 106.245 ± 0.004 cm. About the same time, Grueneisen and Giebe, at the Physikalisch-Technische Reichsanstalt in Germany, made a careful determination of the ohm, although the published description[2] did not appear until 1921. By means of wire resistance standards which were taken to the National Physical Laboratory, a direct comparison was made with Smith's results. This comparison showed that the

[1] *Phil. Trans.*, **214A**, 212 (1914).

[2] *Ber. physik.-tech. Reich*, **5**, 5 (1921).

difference in the two absolute measurements was only four parts in a hundred thousand. A determination by Curtis, Moon, and Sparks[1] at the Bureau of Standards in 1936 gave a result that differs from the results of Smith and of Grueneisen and Giebe by thirty or forty parts in a million. An exact comparison with the earlier results is not possible because the recent results were not compared with the mercury ohm and changes of unknown amount have taken place in the wire standards of the National Laboratories during the twenty years that intervened between the experiments.

21. Absolute Measurements of the Ampere.—The absolute measurement of current for the purpose of establishing a consistent system of electrical units was first undertaken by W. Weber,[2] who used an electrodynamometer, the constants of which could be computed from its dimensions. Weber's results on the ohm and ampere served to fix the value of the Daniell cell, then commonly used as the standard of electromotive force, as 1.1 volts. The British Association committee, during the first decade of its existence, gave scant attention to the experimental determination of the ampere, probably because the tangent galvanometer was in general use, so that many laboratories were equipped regularly to measure current in absolute units. Between 1875 and 1890, one or two "galvanometer houses" were built for the sole purpose of housing tangent galvanometers. No magnetic material was used in the construction of these houses, since the undistorted magnetic field of the earth at the center of the galvanometer was considered essential. Relatively few results from these elaborate instruments have been preserved. They were constructed with the idea of regularly measuring current in absolute measure. With the advent of electrical machinery, particularly the electric street car, the earth's magnetic field was subject to such large disturbances that the tangent galvanometer rapidly fell into disuse.

The results obtained in recent absolute measurements of current have been expressed as the number of milligrams of silver deposited each second by unit current on the cathode of an electrolytic cell having silver nitrate as the electrolyte. Results of the most important determinations are shown in Fig. 6. The

[1] *J. Research N.B.S.*, **16**, 2 (1936).
[2] *Pogg. Ann.*, **82**, 337 (1851).

results obtained before 1880 had an accuracy of about 1 per cent. The accuracy gradually increased until in 1900 the results could be depended upon to nearly 0.1 per cent. A group of determinations about 1910 showed very satisfactory agreement. Two of these determinations were carried out with much greater care than any previous determination. The first of these experiments was performed by Ayrton, Mather, and Smith[1] at the National Physical Laboratory in England; the second was performed by Rosa, Dorsey, and Miller[2] at the Bureau of Standards in the

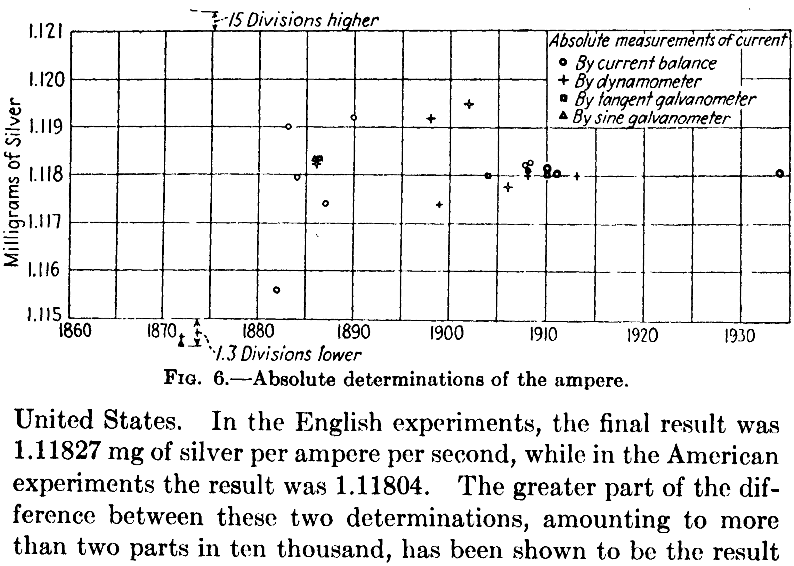

FIG. 6.—Absolute determinations of the ampere.

United States. In the English experiments, the final result was 1.11827 mg of silver per ampere per second, while in the American experiments the result was 1.11804. The greater part of the difference between these two determinations, amounting to more than two parts in ten thousand, has been shown to be the result of an error in measuring the silver deposit in the English experiments. The actual difference between the two absolute current determinations was probably about four parts in a hundred thousand, but lack of dependable concrete standards did not permit of an exact comparison of the two results.

The most recent result is by Curtis and Curtis,[3] who employed some of the apparatus used by Rosa, Dorsey, and Miller. Their result differs by only two parts in a million from that previously obtained in the same laboratory, but this agreement is admittedly fortuitous.

[1] *Phil. Trans.* **207A**, 463 (1908).
[2] *Bull. B. S.*, **8**, 269 (1912).
[3] *J. Research N.B.S.*, **12**, 665 (1934).

22. Standards of Electromotive Force.—Standards of electromotive force have generally consisted of some of the types of primary cells. For several decades preceding 1870, the Daniell cell was used as a standard cell. In 1872 Latimer Clark proposed a cell using mercury and zinc amalgam as the two electrodes, which has since been known as the *Clark cell.* This cell was a very great improvement over the Daniell cell. A still better cell was introduced by Weston in 1892. The *Weston cell* contains cadmium instead of zinc as used in the Clark cell. There are two types of this cell in common use—the Weston unsaturated cell and the Weston saturated cell. The Weston unsaturated cell has a small temperature coefficient, and can readily be made in portable form. The Weston saturated cell has an exceedingly stable electromotive force when kept at constant temperature; hence it is universally used as a primary standard of electromotive force.

To determine in absolute measure a cell's electromotive force, it is compared with the difference in potential between the terminals of a resistance through which a known current is flowing. This measurement gives the electromotive force in absolute units if both the current and resistance are known in absolute units. Hence an error either in the unit of resistance or in the unit of current will introduce an error in the unit of electromotive force.

23. Absolute Measurement of Inductance and Capacitance. The absolute measurement of inductance and capacitance has not been considered of sufficient importance to merit the attention of international electrical congresses and conferences. Also little attention has been given to comparing the results from different laboratories with regard to the absolute measurement of these two units. Since the working standards of both inductance and capacitance are not so stable as are the standards of resistance and electromotive force, the national standardizing laboratories maintain their units of inductance and capacitance by making absolute measurements at frequent intervals, say once or twice a year.

The first precise method for the absolute measurement of capacitance was described by Maxwell[1] in 1873. The accuracy of the method has been greatly improved, so that it is still one of

[1] "Electricity and Magnetism," Vol. II, Art. 776.

the most extensively used methods for measuring a capacitance in terms of resistance and time.

In 1891, Max Wien[1] discussed several alternating-current bridge methods, some of which were capable of indicating capacitance in absolute measure. They are now universally used to measure a capacitance in terms of resistance and inductance.

Two quite distinct methods of establishing the unit of inductance have been developed. One method is to compute the inductance of a circuit having a definite geometric form from the mechanical dimensions of the circuit and the permeability of the surrounding medium. The second method is to measure the inductance in terms of resistance and capacitance. Both methods were originally developed by Maxwell about 1870. Since that time both methods have been improved. The early formulas for computing the inductances of simple circuits have been extended so that now the inductance of such circuits can be computed with great exactness, and additional formulas have been developed for more complicated circuits. Also the measurement of inductance in terms of resistance and capacitance has become more accurate since alternating-current bridges were introduced.

24. Ratio of Electromagnetic to Electrostatic Unit of Current.—Another absolute measurement of great importance is the determination of the ratio of the electromagnetic to the electrostatic unit of current (or quantity). The first determination of this constant was made by Weber and Kohlrausch in 1856, who obtained a value of 3.107×10^{10} in the c.g.s. system. In 1863, Maxwell and Jenkin in their report to the British Association on electrical units discussed the different methods that could be used to measure this ratio, and soon afterward Maxwell made an experimental determination. His attempt to explain why the value of the ratio is the same as the velocity of light was the starting point in the development of the electromagnetic theory of light. Both of these constants have been determined a number of times since the days of Maxwell. The latest determinations show that the difference, if any, between the ratio of the units and the velocity of light is less than one part in ten thousand.

[1] *Wied. Ann.*, **44**, 681 (1891).

CHAPTER IV

THE STANDARDS FOR MAINTAINING THE ELECTRICAL UNITS

The results of any absolute electrical measurement are normally expressed in terms of the concrete electrical standards of the laboratory in which the measurement is made. The usefulness of a result will depend quite as much on the reliability of the standards used for maintaining the units as on the technique of the measurement. Hence a discussion of standards and their maintenance properly belongs to the subject of absolute measurements.

25. Reproducible Standards.—In the early days of absolute electrical measurements, when the accuracy attained was seldom more than 1 per cent, results could be expressed satisfactorily in terms of reproducible standards, which, in turn, could be used to fix the values of the concrete standards of any laboratory. Thus the ohm came to be expressed in terms of the length of a column of mercury having a cross section of 1 sq mm when its resistance was measured at 0°C; the ampere, in terms of the number of milligrams of silver deposited per second; and the volt in terms of the electromotive force of a cell of a definite chemical composition. For the accuracy required in those times, these standards could be reproduced in any laboratory having ordinary equipment. As the accuracy of absolute measurements increased, increasing difficulty was experienced in expressing results in terms of these reproducible standards. At the present time, the labor involved in setting up a reproducible standard to give the accuracy that can be obtained in an absolute measurement is nearly the same as that required to make the absolute measurement. Hence the present-day tendency[1] is to express the result of an absolute measurement in terms of the concrete standards of the laboratory in which it is made and to transport standards to other laboratories.

[1] Since this was written, the International Committee of Weights and Measures has adopted the absolute units, effective Jan. 1, 1940.

26. Legal Standards.—The reproducible standards still have some importance since, in most countries, the legal definitions of the electrical quantities are based on these reproducible standards. Any discussion of these legal definitions[1] is quite outside the province of this book. However, it is well to observe that in most countries legal definitions were adopted in the decade following the Chicago Electrical Congress of 1893. At that time the electrical units could be maintained more accurately by means of reproducible standards than by absolute measurements. As a natural result the reproducible standards are frequently referred to in legal literature. Until the laws of all countries can be modified, the reproducible standards cannot be entirely discarded.

27. Standards of National Laboratories.—Every laboratory which is equipped to make electrical measurements has its own standards. At the present time the values of these standards are generally obtained either directly or indirectly by comparison with the concrete standards of some one of the national standardizing laboratories, such as the National Bureau of Standards in the United States, the National Physical Laboratory in England, and the Physikalisch-Technische Reichsanstalt in Germany. As a result, the methods of establishing and maintaining the electrical units in these national laboratories by means of standards become of prime importance. Unfortunately, these methods have not, in all cases, been published. The following description is believed to be substantially correct, though satisfactory references cannot be given in all cases.

28. The Standards of Resistance.—The standards of resistance of all the national laboratories are coils of manganin wire. The first satisfactory coils of this material were made at the Reichsanstalt, where the value of four 1-ohm standards of this material was determined in 1892–1893 by reference to reproducible mercury standards, using the length of the standard mercury column which was adopted at the Chicago Electrical Congress, *viz.*, 106.3 cm. Since 1897 the mean value of these four 1-ohm coils

[1] In many countries these legal definitions are not consistent among themselves. For example, in the United States the ohm is defined in terms of a mercury column which is a reproducible unit, with the absolute ohm given as a practical equivalent, while the ampere is defined in terms of the absolute unit of current, with the reproducible unit given as a practical equivalent.

has been *assumed* to remain constant. These four coils constitute the primary concrete standard of the ohm in Germany. From time to time[1] these primary concrete standards have been compared with the mercury ohm, but the observed changes in the concrete standards have been too small to warrant changing their value.

The National Bureau of Standards in 1908 sent several resistance coils to the Reichsanstalt for certification. On their return, these coils, with the values assigned at the Reichsanstalt, were taken as the primary concrete standards of the bureau. Since that time, the unit of resistance has been maintained by means of manganin coils, the average value of which has been *assumed* to remain constant. At present, ten 1-ohm coils constitute the standard group. A comparison with the mercury ohm was made in 1911–1912, but values of the standards were not changed.

The National Physical Laboratory established its unit of resistance in 1903 by means of mercury resistance standards. Coils of manganin and of platinum-silver were used as concrete standards, but these were corrected yearly until 1910. In that year, the Washington unit[2] was definitely adopted, and has since been maintained by means of four 1-ohm manganin coils. The assumption is made that the average value of these coils has not changed. Comparisons with the mercury ohm were made in 1912 and in 1924, but at each time the indicated change in the mean values of the manganin coils was less than the experimental error of setting up the mercury ohm. Hence, since 1910, the concrete standard of resistance of the National Physical Laboratory has been the mean value of the four 1-ohm coils referred to above.

The above description of the establishing and maintaining of the unit of resistance in the different laboratories shows that the unit of resistance as maintained by the concrete standards must be expected to be different in the several national laboratories.

[1] Five comparisons with the mercury ohm have been made at the Reichsanstalt. The last, by Steinwehr and Schulze, was published in 1927.

[2] The Washington unit was the unit agreed upon by the technical committee of the International Electrical Commission meeting at Washington in 1910. It represented the mean of mercury ohm determinations at the Physikalisch-Technische Reichsanstalt and the National Physical Laboratory.

An interchange of standards shows that the maximum value of this difference was about twenty parts in a million in 1930.

29. The Standards of Electromotive Force.—The London Electrical Conference of 1908 adopted the Weston normal cell[1] as the concrete standard of electromotive force and authorized a special technical committee to fix its value. This committee met at the National Bureau of Standards in 1910. The value adopted was 1.0183 volts at 20°C. This became effective in most countries on Jan. 1, 1911. As the technical committee above referred to distributed cells of known value to all the important national laboratories, the unit of electromotive force was, at that time, substantially the same throughout the civilized world. However, the method of maintaining the unit has not been the same in all the laboratories, so that discrepancies now exist.

At the National Bureau of Standards the unit of electromotive force has been maintained by a group of about twenty Weston saturated cells, with an even larger number in reserve. All these cells have been maintained at constant temperature, usually 28°C. Occasionally a cell in the standard group will show a decided change in its electromotive force as compared to the mean of the group. It is then replaced by a cell from the reserve group. However, the present primary group of twenty cells contains fifteen cells that have been members of the group since 1906, more than a quarter of a century.

At the National Physical Laboratory, the primary group of Weston saturated cells consists of three sets of cells, each set having been constructed at different times. Under normal conditions, a new set is constructed every year; after proper aging, this set is introduced into the primary group, and at the same time the oldest set in the group is discarded. Recently the new cells have all been of the acid type, which has a slightly different electromotive force from the normal type, the value of which was fixed by the technical committee at Washington in 1910.

[1] The Weston normal cell is the name applied to the typical saturated cadmium cell with a neutral electrolyte. If several saturated cadmium cells are made from materials of the highest purity, the electromotive forces of the different cells generally differ by 10 microvolts, and sometimes differ by as much as 100 microvolts. The average electromotive force of a group of ten or twenty cells differs very little from the average electromotive force of another group. The typical cell is presumed to have the average electromotive force of an infinitely large group of cells.

At the Physikalisch-Technische Reichsanstalt a group of ten cells is used to maintain the unit of electromotive force. In 1910, the mean value of this group at 20°C was determined to be 1.018300 international volts. The mean of this group was assumed to remain constant until 1930, when silver voltameter determinations showed that the mean had decreased by 100 parts in a million. A new value was then taken for the mean of the group, which still is used to maintain the unit.

While in 1911 all countries had the same unit of electromotive force, that fortunate condition does not exist at the present time. By means of cells taken to the various national laboratories, the differences in the unit of electromotive force have been determined, at least approximately. In 1930, the date of the last extensive international comparison, a typical cell, transported to the different national laboratories, would have been assigned values as follows:

	Volts
France	1.018247
Japan	1.018278
England	1.018286
United States	1.018296
Mean	1.018300
Germany	1.018348
U.S.S.R.	1.018350

The maximum difference is nearly one part in ten thousand. This difference is several times as large as the difference in the concrete standards of resistance.

30. The Concrete Standards of Capacitance.—A laboratory does not generally maintain a single concrete standard of capacitance as is the case with resistance and electromotive force. Instead, a series of concrete standards are maintained, the value of each one being independently determined by absolute measurements. For the higher values of capacitance mica condensers are used as standards, while for the lower values air condensers are the standards, the dividing line for the two kinds of condensers being in the neighborhood of 0.001 microfarad. Neither kind of condenser is as stable as a manganin resistor or even a Weston normal cell. Hence the national laboratories make provision for frequent absolute measurement of the capacitance of each of their concrete standards. One international comparison of standards of capacitance[1] has been made, with agreement to about one part in ten thousand.

31. The Concrete Standards of Inductance.—The concrete standards of inductance are of two kinds, self inductance and mutual inductance. Methods of establishing and maintaining the values of these standards are very different in different laboratories. As examples, methods used at the National Physical Laboratory and at the National Bureau of Standards will be described.

At the National Physical Laboratory a concrete standard of mutual inductance is maintained, in terms of which all self inductors and mutual inductors are calibrated. This concrete standard is so constructed that its mutual inductance can be computed from its dimensions. This computed value is in absolute henrys. To obtain the inductance in international henrys, the computed value is divided by 1.00052, which is the factor that has been experimentally determined in that laboratory for converting from international henrys to absolute henrys. This concrete standard of mutual inductance is used at the National Physical Laboratory not only to standardize all inductors but also, in connection with standards of resistance, to establish the capacitance of any mica condenser that is under test.

At the National Bureau of Standards concrete standards of both self and mutual inductance are maintained, having values ranging from 10 microhenrys to 1 henry. These inductors are not considered as primary standards, and inductors are not certified in terms of them. An unknown inductor is measured in terms of resistance and capacitance. At the same time, a concrete standard of inductance of about the same value as the unknown is measured by the same apparatus. In this way the concrete standards are used as a check on the method of measurement.

[1] *B.S. J. Research*, **8**, 431 (1932), and *N.P.L. Collected Researches*, **24**, 59 (1932).

CHAPTER V

THE INTERNATIONAL ELECTRICAL UNITS

The international electrical units are a set of arbitrary units which, at the time of their adoption in 1893 and also at the time of their modification at the London Electrical Conference in 1908, corresponded to the best experimental values of the absolute units; they were supposed to be easily reproducible with greater precision than could be obtained by absolute measurements. In 1908, the two arbitrary units adopted were the ohm and the ampere. The volt was to be obtained by Ohm's law from the ohm and ampere. The coulomb, farad, and henry were to be obtained from the preceding units by absolute measurements.

32. The International Ohm Defined.—The international ohm was defined by the London Electrical Conference as "the resistance offered to an unvarying electric current by a column of mercury at the temperature of melting ice, 14.4521 grammes in mass, of a constant cross-sectional area, and of a length of 106.300 centimeters."[1] The experimental realization of this unit

[1] The procedure to be followed in determining the resistance of a column of mercury in international ohms is given in a set of specifications which may be summarized as follows:

The glass tubes for containing the mercury column must be straight. The bore must be nearly uniform and circular, and must have a cross-sectional area of approximately 1 sq mm. The mercury must have a resistance of approximately 1 ohm.

The calibration correction must not exceed five parts in ten thousand.

Measurements of length of tube, of mass of mercury in tube, and of electrical resistance shall be corrected to 0°C.

The mass of mercury required to fill a tube shall be taken as that confined in the tube when plane surfaces are placed in contact with the ends of the tube.

For the electrical measurements, the tube shall be fitted with spherical end vessels, approximately 4 cm in diameter, which carry the current and potential terminals. These terminals shall be thin platinum wires fused into the vessels, the current terminals being diametrically opposite the ends of the tube and each potential terminal 90° from its current terminal.

The resistance R_s of an end vessel shall be computed by the formula with an accuracy of one part in a thousand is comparatively easy; an accuracy of one part in ten thousand can be obtained by using many precautions, but no laboratory has yet succeeded in establishing the unit with an accuracy of one part in a hundred thousand.

33. Experimental Difficulties of the Mercury Ohm.—The experimental difficulties in connection with the realization of the international ohm[1] center about the glass tube that holds the thread of mercury. Since no tube with a uniform bore has ever been constructed, any tube which is to be used must be calibrated to determine the correction for lack of uniformity. This calibration is effected by means of a short thread of mercury which is introduced into the tube. The length of this thread, when it is near one end, is read; then the thread is moved a distance equal to its length and its length read again. This process of moving and reading is continued until the opposite end of the tube is reached. From these readings a calibration correction is computed. This calibration correction of a tube is the factor by which the resistance of a uniform tube, having the same length and average cross section as the actual tube, must be multiplied to give the resistance of the column of mercury in the actual tube. This correction factor is always slightly greater than unity, good tubes always having a value less than 1.000200, while the very best tubes on record have values near 1.000020.

The filling of the tube to determine the mass of mercury which it will hold is an entirely different operation from filling the tube to measure the electrical resistance. Hence each oper-

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$$R_s = \frac{0.8}{1063\pi r}$$

where r is the radius of the bore of the tube at the end which enters the vessel.

For a determination of the international ohm, at least five tubes shall be used, each of which shall be filled at least three times.

[1] Recent descriptions of the construction of mercury ohm standards are the following: von Steinwehr and Schulze, Die Quecksilbernormale der Physikalisch-Technischen Reichsanstalt für das Ohm, *Mitteilung aus d.P.T.R.*, 1927; Wolff, Shoemaker, and Briggs, Construction of Primary Mercurial Resistance Standards, *Bull. B. S.*, **12**, 375 (1915); F. E. Smith, On the Construction of Some Mercury Standards of Resistance, *Phil. Trans.*, **204A**, 57 (1904).

ation must be carried out several times, and the average value of each must be taken. In making the filling for determining the mass, the tube is evacuated, then is filled with mercury, and flat glass plates are fitted to the ends so that the mercury column completely fills the tube between these end plates. For making the resistance measurements, spherical bulbs, the resistances of which must be computed, are attached to the ends of the tube. As the resistance of the mercury in the bulb is very largely determined by the resistance of a small volume very close to the end of the tube, the computed resistance depends entirely on the radius of the tube at the end which enters the bulb. The radius is generally determined from the data taken in making the calibration correction. A tube somewhat longer than necessary is calibrated. The tube is then cut at a point where the cross section has been determined. The radius is computed on the assumption that the tube is circular.

The maximum error that each one of the above factors is likely to introduce into a determination can best be determined from a consideration of published data. This is summarized in Table V. By taking a number of observations on each factor

TABLE V.—MAXIMUM ERROR OF DIFFERENT FACTORS ENTERING INTO A DETERMINATION OF THE MERCURY OHM

Factor	Estimated Maximum Error, p.p.m.
Calibration correction	3
Mass of mercury	15
Resistance of mercury column	10
Length of tube	1
Resistance of terminal bulbs	10
Sum of all	39

and using a number of tubes, the probable error of a complete determination should be considerably less than the sum of the maximum errors, provided all systematic errors have been eliminated. That such is the case is shown by the data given in Table VI, which has been taken from the articles referred to at the beginning of this section. From these data, an accuracy of one part in a hundred thousand seems possible. However, international comparisons of wire standards indicate slightly larger discrepancies than would be expected from these data.

TABLE VI.—EXPERIMENTAL ERRORS IN MERCURY OHM DETERMINATIONS

Laboratory	Year of observation	Number of tubes	Maximum difference between tubes, p.p.m.	Average difference of each tube from mean of all tubes, p.p.m.
Physikalisch-Technische Reichsanstalt	1927	5 (new tubes)	20	6
Physikalisch-Technische Reichsanstalt	1927	10 (new and old)	70	19
National Bureau of Standards	1911	4	13	5
National Physical Laboratory	1904	10	36	14

34. The International Ampere Defined.—The international ampere was defined by the London Electrical Conference as "the unvarying electric current which, when passed through a solution of nitrate of silver in water, in accordance with specifications attached,[1] deposits silver at the rate of 0.00111800 of a gramme per second." As a result of this definition, only two accurate measurements are required, *viz.*, the time the current is flowing and the mass of silver deposited. However, there are numerous experimental difficulties in carrying out a determination of current by this method.

35. Experimental Difficulties of the Silver Voltameter.—The time during which the current is flowing must be measured with the same accuracy that is to be attained in the value of the current. The second signals from clocks and chronometers are often in error by 0.01 or 0.02 second, or a maximum error of 0.04 second in measuring any time interval. In order that

[1] The specifications may be summarized as follows: The electrolyte shall consist of from 15 to 20 parts by weight of silver nitrate in 100 parts of water, not less than 100 cc of fresh solution being used in a voltameter. The current density at the silver anode shall not exceed $\frac{1}{5}$ ampere/cm²; at the platinum cathode, $\frac{1}{50}$ ampere/cm². The silver deposited shall not exceed 30 per cent of that in the solution.

Provisions shall be made to prevent particles of silver which may become mechanically detached from the anode from reaching the cathode.

Before weighing, the cathode must be washed and dried.

0.04 second shall produce an error of only one in a hundred thousand, the time of a run must be 4000 seconds, which is more than an hour. During these 4000 seconds the current must be held constant. When the circuit is first closed, a few seconds elapse before the current can be brought to a definite value; thereafter no unusual trouble is experienced in maintaining the current to one part in a million. If there is an average error of 1 per cent in the current during the first 10 seconds, the error in the silver deposit of a run which is continued for 4000 seconds is more than two parts in a hundred thousand.

The mass of silver must also be determined with the same accuracy as is expected of the current. If a current of about 1 ampere has continued for 4000 seconds, the mass of the silver will be about 5 g. This silver must be weighed to one part in a hundred thousand while still contained in the platinum bowl in which it is deposited. A balance suitable for such a weighing must have a capacity of 100 g and a sensitivity of 0.05 mg. Since the absolute mass is required, double weighing must be used or a substitution method employed. Unless platinum weights are used for weighing the bowl and silver weights for weighing the silver, a correction must be applied for the buoyancy of the air. One of the difficulties connected with the weighing is caused by the large surface of the deposit. On all surfaces exposed to the air a layer of moisture is deposited, the thickness of which depends on the humidity of the air. Quantitative data on this effect are lacking.

Less than a year after the adjournment of the London conference, Rosa and Vinal[1] discovered that the specifications for the silver voltameter are indefinite since the weight of the deposit may be changed by as much as ten parts in a hundred thousand by the filter paper which was often used to prevent particles of the anode from falling onto the cathode. All laboratories have now discarded the use of filter paper in voltameters, but no single method has been universally adopted for protecting the cathode from particles that may drop off the anode.[2]

The published data of Rosa and Vinal show that, with suitable voltameters, the error of a single determination is seldom more

[1] *Bull. B. S.*, **9**, 151 (1912).

[2] The different forms of voltameters are described by Rosa and Vinal, *Bull. B. S.*, **9**, 172 (1912), and **10**, 479 (1914).

than 100 parts in a million. The mean of ten results will probably be in error by less than forty parts in a million.

36. The International Volt Defined.—The international volt was defined by the London Electrical Conference as "the electrical pressure which, when steadily applied to a conductor whose resistance is one International Ohm, will produce a current of one International Ampere." This definition requires that the volt be determined from the ohm and ampere. However, since no concrete standard of current is possible, standard cells are employed as the concrete standards of electromotive force. Recognizing this situation, the London Electrical Conference adopted the provision that "the Weston Normal Cell[1] may be conveniently employed as a standard of electric pressure for the measurement both of E.M.F. and current."

37. The International Coulomb, Farad, and Henry.—The definitions of the International coulomb, farad, and henry are those given by the Chicago Electrical Congress since the London conference did not consider these units. "The international coulomb is the quantity of electricity transferred by a current of 1 international ampere in one second." "The international farad is the capacity of a condenser charged to a potential of 1 international volt by 1 international coulomb of electricity." "The henry is the induction in a circuit when the electromotive force induced in this circuit is 1 international[2] volt while the inducing current varies at the rate of 1 ampere per second."

[1] Specifications were attached which may be summarized as follows: The positive electrode is mercury, the negative electrode a cadmium amalgam containing 12.5 per cent by weight of cadmium; the electrolyte is a saturated aqueous solution of cadmium sulphate; the depolarizer is a paste of mercurous sulphate. The most suitable containing vessel is made from glass tubing and has the form of an H. Platinum leads are sealed into the bottoms of the two limbs and must be completely covered with the electrode material. The depolarizer is placed above the mercury; then a layer of cadmium sulphate crystals is introduced into each limb. The cell is filled above the cross tube with a saturated solution of cadmium sulphate, then hermetically sealed. The recommended formula for determining the electromotive force at different temperatures between 0 and 40°C is

$$E_t = E_{20} - 0.0000406(t - 20) - 0.00000095(t - 20)^2 + 0.00000001(t - 20)^3$$

[2] Apparently the word *international* was inadvertently omitted before *henry* and *ampere*.

CHAPTER VI

UNITS AND MEASUREMENTS OF THE FUNDAMENTAL MECHANICAL QUANTITIES

The mechanical units of length, mass, and time are the starting points of all absolute electrical measurements. Hence the accuracy obtainable in absolute electrical measurements cannot possibly be any greater than the permanence and reproducibility of the mechanical units. Absolute electrical measurements have now become so accurate that it is necessary to consider the possibility of errors that result from the mechanical units and measurements.

38. The International Bureau.—The standards of length and mass are preserved by the International Bureau of Weights and Measures. This bureau, which is supported by contributions from the governments of most of the civilized countries, is located at Sèvres, near Paris. The treaty[1] establishing the bureau provided that each country could obtain prototype standards of certified value. The units of the national laboratories depend on these prototype standards.

39. The Unit of Length.—The unit of length universally used in absolute electrical measurements is the meter, which is the distance between two parallel lines on a platinum-iridium bar with an x-shaped cross section called the *international prototype meter* and preserved at the International Bureau. Each nation has a similar bar, called a *national prototype meter*, which is the standard of that nation. A photograph of the prototype meter of the United States is reproduced in Fig. 7. All the national meters have been measured at the International Bureau, so that the length of each[2] at 0°C is known closer than 0.1 micron or one

[1] The treaty was amended in 1928 so that now the bureau and its governing bodies have jurisdiction over electrical standards. Equipment and methods are being developed for exercising this function.

[2] The length of the United States prototype meter bar (No. 27) at any temperature t between 0 and 35°C is

$$0.9999986 + 8.620 \times 10^{-6}t - 1.77 \times 10^{-9}t^2 \text{ meter.}$$

part in ten million. The temperature coefficient of each of the prototype meters has the same value, and is known with sufficient accuracy so that the computed length of a meter bar at any temperature between 0 and 35°C is in error by less than 0.1 micron.

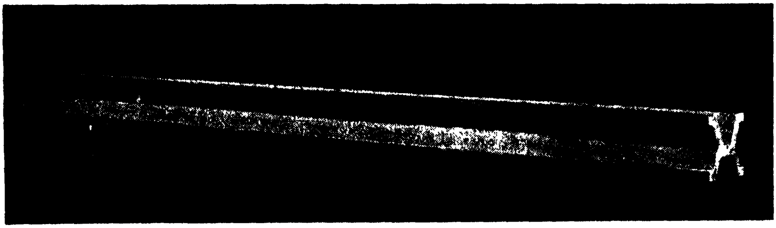

FIG. 7.—Photograph of the prototype meter of the United States.

The number of wave lengths of red cadmium light in a meter has been determined with such accuracy that these wave lengths can be taken as a standard of length.[1] In fact, over long periods

[1] Five determinations have been made of the number of red cadmium wave lengths in a meter. The published results for air at 0°C and 760 mm pressure are:

Michelson and Benoit (1893) at International Bureau:

1 meter = 1 553 163.5 wave lengths

Benoit, Fabry, and Perot (1907) at International Bureau:

1 meter = 1 553 164.13 wave lengths

Watanabe and Imaizumi (1928) at Japanese Central Bureau of Weights and Measures:

1 meter = 1 553 164.37 wave lengths

Koesters and Lampe (1934) at the Physikalische-Technische Reichsanstalt:

1 meter = 1 553 164.70 wave lengths

Sears and Barrel (1934) at the National Physical Laboratory:

1 meter = 1 553 163.75 wave lengths

Michelson and Benoit did not correct their value for the humidity of the air—in fact, did not record the humidity. Fabry and Perot, by using a probable value of the humidity, correct Michelson and Benoit's value to

1 meter = 1 553 164.0

The value of Benoit, Fabry, and Perot is generally accepted, with the others giving valuable confirmation. The maximum variation is only seven parts in ten million.

of time, the wave lengths may prove to be more dependable than the meter bar.

40. Classification of Unknown Lengths.—The method to be employed in measuring an unknown length depends upon the distance to be measured and the nature and position of the defining points, lines, or surfaces. This discussion will cover only lengths between 1 mm and 1 m. If the unknown length is the distance between two points in a plane, or between two parallel lines in a plane, the measurement will generally be made by comparison with a bar which is subdivided to millimeters. If the unknown length is the distance between two parallel planes, or is a distance that can be defined by means of parallel planes (such as the diameter of a sphere or of a cylinder), the measurement may be made either by determining the number of wave lengths of red cadmium light in the unknown length or by comparing it with a suitable end standard.

41. Calibration of a Subdivided Meter.—The millimeters of a subdivided scale are never exactly uniform, and the total length of 1000 mm is never exactly 1 m. In scales of the highest quality, the individual millimeters may vary from a true millimeter by 1 or 2 microns, while the 1000 mm may differ from a meter by ±10 microns. In order to obtain the highest accuracy in the use of a subdivided scale, a complete calibration must be made which will give the true value of each subdivision of the scale. As the width of the scale lines is usually 3 or 4 microns, even in the best scales, an accuracy of calibration of 0.1 micron is all that can be expected in any single interval.

42. Length Measurements Using a Line Standard.—In order to compare the unknown distance between two points or between two parallel lines with a calibrated meter bar having millimeter divisions, some apparatus must be provided which will determine the small amount by which the unknown distance is greater, or less, than the distance between two of the millimeter divisions of the meter bar. A comparator[1] for the purpose usually consists of two micrometer microscopes mounted in a heavy metal frame or on a stiff rod. The object which contains the defining lines or points of the unknown distance is placed so that each line or point is at the focus of one of the microscopes. The position of

[1] A vertical comparator, or cathetometer, has been described by Moon, *J. Research N.B.S.*, **14**, 363 (1935).

each defining line is read by the micrometer of its microscope. The meter bar is then substituted for the unknown, and the positions of two millimeter lines are read by the micrometers. From the known distance between these two millimeter lines and from the readings and calibrations of the micrometers of the microscopes, the unknown length is determined. In precise measurements of length, the temperatures of the standard, of the unknown, and of the measuring apparatus must not vary appreciably during the measurements; the length of the standard must be known at the temperature of measurement; and the length of the unknown is determined for the one temperature only.

43. Measurements Using an End Standard.—In measuring the distance between two parallel planes, some mechanical instrument such as a micrometer is usually employed. Measurement directly in light waves requires special apparatus which will not be considered in this treatise. If the unknown distance is more than 1 cm, an end standard is employed which has a length approximately that of the unknown. The first step of the measurement consists in determining the length of the end standard; the second step is the comparison of the unknown with the end standard.

At the present time, the length of an end standard is usually determined either directly or indirectly in terms of light waves. Standardizing laboratories have sets of gage blocks, the length of each block having been determined in terms of light waves by interferometer methods. These blocks are generally made of hardened steel, each block having two highly polished plane surfaces which are parallel. The distance between the parallel faces of blocks having a length less than 1 decimeter can be measured in an interferometer with an accuracy of about 0.02 micron. Gage blocks can be used singly as end standards, or can be assembled end to end to measure lengths longer than any that can be measured in an interferometer. When two gage blocks are assembled by pressing them together with a sliding motion (wringing them together), the length of the combination is greater than the sum of the lengths by less than 0.01 micron.

While standards up to a ½ m in length can be assembled from a large set of gage blocks, such an assembly is not convenient to use in measuring an unknown length. Hence single-piece end standards of approximately the same length as the unknown are generally employed. Such standards may have their ends plane and parallel, but very often the ends are portions of the surface of a sphere having its center at the mid-point of the end standard.[1] The length of the end standard is determined by comparison with an assembly of gage blocks by means of an indicating device of which a micrometer is one type. If the temperature is controlled with sufficient accuracy, the length of an end standard can be determined to 0.1 micron. A micrometer[2] is also used to compare the unknown length with the end standard, and this comparison can be made to 0.1 micron provided the unknown distance is sufficiently definite to permit a measurement of this precision.

44. The Unit of Mass.—The primary unit of mass is a cylinder of platinum-iridium which is preserved at the International Bureau and which is called the *international prototype kilogram.* Each nation has a national prototype kilogram.[3] The uncertainty in the mass of these national prototype kilograms is less than 0.02 mg or two parts in a hundred million.

45. Calibration of a Set of Weights.—A calibration is required for any set of weights which is to be used in precision weighing. Such a calibration requires at least one standard, the value of which is accurately known, and one or more balances of high sensitivity. The accuracy obtainable is less for small weights than for the larger ones. For example, two kilograms can be compared to one or two parts in a hundred million, but the value of a gram weight can be determined only to one part in a million.

46. Measurements of Mass.—Precision measurements of mass are made by comparing the unknown mass with weights from a calibrated set by means of an equal-arm balance. To determine the value of an unknown mass to one part in a million requires several precautions.

a. Inequalities in the lengths of the two arms of the beam of the balance on which the comparison is being made must be eliminated either by double weighing or by a substitution method.

[1] A comprehensive discussion of the use of end standards is found in Rolt, "Gauges and Fine Measurements," The Macmillan Company.

[2] A micrometer suitable for precision measurements is described by Moon, *J. Optical Soc. Am. and R.S.I.*, **11,** 453 (1925).

[3] The prototype kilogram of the United States is No. 20, having a mass of 1 kg − 0.04 mg.

b. The buoyancy of the surrounding air must be considered. If the density of the unknown mass is exactly the same as the density of the weights with which it is compared, the buoyancy introduces no error. When the two densities are not equal, a correction can be computed provided data are available concerning the densities of the air, of the standard weights, and of the unknown mass. If the densities of the weights and of the unknown are very different, or if extreme accuracy is required, the densities must be determined with high precision and an exact formula for the buoyancy correction must be employed. However, by specifying that the accuracy shall not exceed one part in a million and by considering only materials of relatively high density, the correction can be rather easily determined.

The symbols used in the formula for the buoyancy correction are:

M_x = mass of the unknown object.
M_s = masses of the standard weights required to counterbalance M_x when the weighing is made on an equal-arm balance in air.
ρ_x = density of the unknown mass.
ρ_s = density of the standard weights.
ρ_a = density of the air = 0.0013 g/cc approximately.
g = the value of the acceleration of gravity.

The formula for force on the pans of an equal-arm balance is

$$M_x g\left(1 - \frac{\rho_a}{\rho_x}\right) = M_s g\left(1 - \frac{\rho_a}{\rho_s}\right)$$

so that

$$M_x = M_s\left(1 - \frac{\rho_a}{\rho_s}\right)\left(1 + \frac{\rho_a}{\rho_x} + \frac{\rho_a^2}{\rho_x^2} + \cdots\right)$$

If the density of the unknown is greater than 1.3 g/cc, so that ρ_a^2/ρ_x^2 is less than 10^{-6}, the approximate formula

$$M_x = M_s\left[1 + \rho_a\left(\frac{1}{\rho_x} - \frac{1}{\rho_s}\right)\right]$$

gives a value which is in error by less than one part in a million.

In order that M_x can be determined to one part in a million, the three densities ρ_a, ρ_s, and ρ_x must be known with sufficient accuracy so that the correction term $\rho_a\left(\frac{1}{\rho_x} - \frac{1}{\rho_s}\right)$ is correct to the sixth decimal place. Since ρ_a is a factor in this correction term, the precision with which ρ_a must be known is determined by the difference between the reciprocals of ρ_x and ρ_s. As an example, take $\rho_x = 2.2$ g/cc (the density of silica glass) and $\rho_s = 21.4$ g/cc (the density of platinum). Then

$$\frac{1}{\rho_s} - \frac{1}{\rho_x} = 0.408 \text{ cc/g}$$

Under standard conditions $\rho_a = 0.001293$ g/cc so that the correction term with these extremes of density is about 500 millionths. Hence, to obtain an accuracy of one part in a million in the mass of a piece of quartz when weighed with platinum weights, the density of air must be known to 0.2 per cent. On the other hand, if gold is weighed with platinum weights, ρ_a need be known only to 10 per cent to give the same accuracy.

To give an accuracy of one part in a million in M_x, both ρ_x and ρ_s must be known with sufficient accuracy so that their reciprocals are correct to within five in the fourth decimal place. It follows that the smaller the density, the more accurately it must be known. For example, with silica glass the density must be known to 0.1 per cent (*i.e.*, $d = 2.200 \pm 0.002$) whereas with platinum the density need be known only to 1 per cent (*i.e.*, $d = 21.4 \pm 0.2$). It should be noted that the accuracy with which the density of the unknown must be known is entirely independent of the density of the weights and vice versa. If a piece of silica glass is weighed against weights of the same material, the correction term is negligible only if the densities of the two are the same within 0.1 per cent, a condition that seldom occurs. Hence the density of the unknown and of the weights would have to be determined experimentally, whereas, if platinum weights had been used, the density of platinum is sufficiently uniform so that 21.4 g/cc can be used without making a determination. The only advantage in having the unknown and the weights of nearly the same density is that the density of the air need not be known with high accuracy.

The method of experimentally determining the density of a solid consists in weighing the body in air and then in a liquid of known density, usually water. The details are given in many textbooks on mechanical measurements. The density of the air is usually computed by the formula

$$\rho_a = 0.001293(1 - 0.003665t)\frac{(B_0 - b)}{760}$$

where t is the temperature in degrees centigrade, B_0 is the barometric pressure in millimeters of mercury under standard conditions, and b is a correction for humidity. If ρ_a is required to 0.2 per cent, t must be known to 0.5°C, while both B_0 and b must be known to 1.5 mm. To determine B_0 from the reading B of a mercurial barometer, the only necessary correction for laboratories near sea level is for the expansion of mercury. Hence

$$B_0 = B(1 - 0.0002t)$$

where t is the temperature of the mercury column. The value of b in millimeters when the relative humidity h is less than 90 per cent and the air temperature is between 19 and 31°C is given with sufficient approximation by the expression

$$b = (1.4t - 11)\frac{h}{300}$$

The value of h expressed in per cent ($h = 40$ for 40 per cent humidity) can be read with sufficient accuracy by means of a hair hygrometer provided the humidity has not exceeded 80 per cent since the last calibration of the instrument.

c. The humidity in the weighing room must be so low that no weighable amount of moisture is deposited either on the unknown or on the weights. Relatively few data are available concerning the amount of moisture deposited on metal surfaces from air at different humidities. These data indicate that the deposit on metals is of the same order of magnitude as that on quartz and glass. On clean surfaces of these materials the moisture deposit in air of 90 per cent humidity weighs from 10^{-5} to 10^{-7} g/cm². Judged from measurements of surface resistivity, at 20 per cent humidity the moisture deposit weighs about 10^{-12} g/cm². Assuming that these figures are applicable to metals, the standard kilogram, which has a surface of about 70 cm², might in very humid air have a deposit of nearly 1 mg of moisture, while in very dry air only one-millionth of a milligram would be deposited.

If the unknown is in a form which has a large surface relative to its volume, such as a bowl or tube, then the weight of the deposited moisture at humidities as low as 70 per cent may be as much as one-millionth of the weight of the bowl. The only certain method of elimination of the effect of moisture is to make the weighing in an atmosphere of low humidity. The weights and the unknown must be kept at this humidity for several hours to reach equilibrium.

d. Any electrostatic charge on either the unknown or the weights must be completely dissipated before the weighing is started. The charge seldom causes any error if the unknown and weights are of metal, but, if either is of an insulating material such as quartz, relatively large charges may be retained for several hours. The dissipation may be hastened by placing in the balance case a small amount of radioactive material.

47. The Unit of Time.—The unit of time is the mean solar second, which is 1/86 400 of the mean solar day or 1/86 164.100 of the sidereal day. The constancy of the second depends upon the uniformity of the rotation of the earth, which is exceedingly constant, though the change in its moment of inertia produced by the shifting of ice masses, by the depositing of silt, and by other changes in the earth's crust theoretically cause a slight effect on the time required for a revolution, though this has never been measured. The second is maintained by means of clocks and chronometers which are controlled either by the vibrations of an elastic body or by the swings of a gravity pendulum; generally one which has such a length that it makes a single swing in 1 second. The pendulum of a clock must be given an occasional impulse—on every swing in most clocks—to keep it in vibration. Even slight variations in the amount or phase of the impulse produce an effect on the period of the pendulum. Earth tremors which are nearly always present sometimes produce a measurable effect on individual seconds. Hence there is some question whether the individual beats of any clock agree closer than one part in a million. These errors tend to average out if a number of beats are considered. Over an entire day, the best clocks are not in error by more than 0.01 second or nearly one part in ten million.

48. Measurement of Time.—The precise measurement of a time interval is generally accomplished by comparing it either directly or indirectly with the beats of the seconds pendulum of a standard clock. The comparison is sometimes carried out by optical methods, but more often by means of electrical impulses which are produced each second in a circuit by some device attached to the clock. Most standard clocks have mechanical contacts which are closed (or opened) on each swing. The impulses produced in an electrical circuit by one of these mechanical contacts are less regular than the beats of the pendulum itself. In good clocks having mechanical contacts, the maximum variation of any electrical impulse from the mean of a large number is generally as large as 1 millisecond, and often is as large as 10 milliseconds. Hence with the best mechanical contacts an interval as long as 15 minutes must be measured to obtain an accuracy of one part in a million.

More accurate electrical impulses can be obtained from the pendulum of a clock by means of a photoelectric cell. A mirror is placed on the pendulum so that the light from the filament of an incandescent lamp is reflected, at the middle of the pendulum swing, onto a slit placed in front of the photoelectric cell. An optical system is arranged so that a small image of the filament is formed at the slit. With a slit of about the same width as the image, very accurate electrical impulses are produced, which can be amplified by a vacuum-tube amplifier without appreciably decreasing the accuracy. While the impulses produced by photoelectric cells are appreciably more regular than those from mechanical contacts, quantitative estimates of the regularity are not available.

For obtaining a standard of time shorter than a second, some form of vibrating system is usually employed. A tuning fork is satisfactory from 10^{-1} to 10^{-4} second. Quartz plates are used from 10^{-5} to 10^{-6} second. Oscillating electric circuits can be used to 10^{-8} second. For most uses, the vibrations must be maintained by energy supplied from some outside source. The method of applying the energy usually effects the period to some extent.

The period of a vibrating body must be determined by comparison with a standard clock. When the constancy of the vibrating body and of the clock are such that the comparison can be continued over a relatively long time interval, a very high accuracy is readily obtained. High accuracy can also be obtained in a few seconds provided very precise methods[1] of comparing coincidences are available.

In using a vibrating body for the accurate measurement of a time interval, several factors must be considered.

1. The vibrating body should not have two free periods which differ by a small amount, as the beats which are produced cause a change in amplitude so that the frequency may be slightly affected. As an example, the two prongs of a tuning fork should be adjusted until they have so nearly the same periods that they force each to vibrate with the same period rather than producing beats.

2. The temperature coefficient must be taken into consideration or the temperature controlled. The increase in frequency of a steel tuning fork with increase in temperature is about 100 parts in a million per degree centigrade; that of an elinvar tuning fork is about 10 parts in a million. The temperature coefficient of a quartz plate may be either positive or negative depending on the form and orientation with respect to the crystallographic axes. If properly designed, the temperature coefficient will be only a few parts in a million.

3. A method of mounting must be employed in which very little energy is transferred from the vibrating body to the mounting. With a well-balanced fork this condition is fulfilled by rigidly clamping the stem to a very heavy base. A quartz plate is supported at one or more points in the plane midway between its faces.

4. The method of driving a vibrating body must be adapted to it. Mechanical contacts on the vibrating body are not sufficiently constant to maintain an accurate frequency. Acoustic, magnetic, or electrostatic phenomena connected with the vibrating body are usually utilized to produce a feeble alternating current of electricity which is amplified and, if necessary, changed in phase, so that it will drive the vibrating body either by an electromagnet or by a piezoelectric condenser. The amplifying and driving circuits, together with their relationship to the vibrating body, should be so designed that slight changes in the outside energy supplied to the amplifier produces a minimum effect on the frequency.

[1] Moon, *B.S. J. Research*, **4**, 213 (1930).

5. The amplitude of vibration must remain constant. The frequency of vibrating bodies is slightly affected by the amplitude of the vibration. The frequency at different amplitudes has been measured by Moon for two freely vibrating forks. For a steel fork the increase in frequency was 80 parts in a million as the amplitude of the end of a prong decreased from 2.4 to 0.6 mm; for a similar elinvar fork the increase under the same conditions was 110 parts in a million. No data are available for quartz plates.

6. The air pressure must be held constant for the highest precision.

The meager data on the effect of air pressure on the frequency of a tuning fork[1] indicate that for a particular fork a change in air pressure of 1 cm of mercury produces a change in frequency of about four parts in a million. With quartz plates,[2] an increase in air pressure of 10 cm of mercury is required to increase the frequency by one part in a million.

[1] Tuma, *Wien. Ber.*, **98**, 1028 (1889).
[2] Hall, Heaton, and Lapham, *J. Research N.B.S.*, **14**, 85 (1935).

CHAPTER VII

UNITS AND MEASUREMENTS OF DERIVED MECHANICAL QUANTITIES

The fundamental definitions of the electrical quantities are stated in terms of force and energy, the units of which are derived from the fundamental mechanical units. In developing a system of absolute electrical measurements, either force, energy, or power must be measured in mechanical units. Hence the necessity of considering the accuracy with which such measurements can be made.

49. Measurement of Force.—Measurements of force are most accurately made by comparing the unknown force with the force exerted on a mass by the attraction of the earth. The mass can be accurately determined by comparison with standards as already discussed. The accuracy with which the acceleration of gravity is known at any place limits the accuracy that can be obtained in measuring a force at that place.

50. The Value of Gravity.—The value of the acceleration of gravity has been measured in terms of length and time at relatively few places. At the present time, gravity determinations throughout the entire world are referred to the value at a particular pier in the laboratory of the Geodetic Institute at Potsdam, Germany, where an elaborate absolute determination was made between 1894 and 1906 by means of several reversible pendulums of the Kater type. The result obtained was 981.274 ± 0.003 cm/sec^2. The computed probable error, as given above, was obtained from a number of observations on each of five different pendulums. No estimate was made of the effect of systematic errors, such as an uncertainty in the length of the meter bar to which all length measurements were referred. Taking all factors into account, the probability of the Potsdam value being in error by ten parts in a million is very small, but there is a reasonable probability that it may be in error by five parts in a million.

Before the Potsdam determination, values of gravity were mostly based on an absolute determination made in Vienna in 1884. By determining the period of an invariable pendulum at both Potsdam and Vienna, the value at Potsdam, based on the absolute determination in Vienna, was found to be sixteen parts in a million larger than the value as determined absolutely at Potsdam. The Vienna value is admittedly less accurate than the Potsdam value.

The value of gravity at any particular place (g_x) is compared with the value at Potsdam (g_p) by determining the period of a particular pendulum at Potsdam (T_p) and at the place in question (T_x). Then

$$g_x = \frac{g_p T_p^2}{T_x^2}$$

This equation assumes that the pendulum has remained constant in length and that all other conditions, such as temperature, air pressure, amplitude of vibration, rigidity of the pier, etc., have been the same at the two places.

There are many records where the difference in gravity between two places has been determined two or more times. Assuming a constant value at one place, the two values at the second place usually differ by three or four parts in a million, and differences as large as ten parts in a million are not uncommon. As an example, the values obtained by different observers on the Coast and Geodetic Pier in Washington by transporting pendulums from Potsdam are given in the following table:

TABLE VII.—GRAVITY DETERMINATIONS AT WASHINGTON

Observer	Year	Value, cm/sec²
Putnam	1900	980.113
Meinesz	1926	980.122
Miller	1929	980.119
Brown	1933	980.118

The possible error in the value of gravity at any place is the sum of the error in the absolute value and of the error in the comparison between that place and Potsdam. As a result there is an uncertainty in the value of gravity of from ten to twenty parts in a million at most places where gravity comparisons have been made.

An approximate value of gravity at any place can be computed by the formula

$$g = 978.03 + 4.92 \sin^2 \phi - 3.09 \times 10^{-4} h$$

where g = value of gravity in centimeters per second per second.
ϕ = latitude of place.
h = height of place above sea level in meters.

The computed value will seldom be in error by as much as 0.1 cm/sec² or 100 parts in a million.

51. Measurement of Energy.—Since mechanical standards of energy are not physically attainable, all mechanical measurements of energy are made in terms of more fundamental units. Two kinds of mechanical energy can be measured with precision, *viz.*, the energy of position relative to the earth and the kinetic energy of motion with respect to the earth.

The energy W of mass m, located at a small distance S above the surface of the earth, is given by the equation

$$W = mgS$$

The accuracy with which all these quantities can be measured has already been discussed. The kinetic energy of a body of mass m, having a linear velocity V, is

$$\text{K.E.} = \tfrac{1}{2} m V^2$$

The kinetic energy of a body rotating about an axis is

$$\text{K.E.} = \tfrac{1}{2} I \omega^2$$

where I is the moment of inertia about the axis of rotation and ω is the angular velocity. At the present time these energy equations are so seldom employed in precise measurements that a discussion of the accuracy of measurement is not essential.

Other forms of mechanical energy, of which the most important is the deformation of an elastic body, can be measured by comparison with one of the above types.

One of the important forms of energy is heat energy. However, measurement of heat energy directly in terms of mechanical energy involves very difficult laboratory technique. Hence the measurement of heat energy is generally accomplished by comparison with electrical energy. This is only one example of several that might be cited where energy is measured in terms of electrical units. No method has recently been used in absolute electrical measurements which required the measurement of mechanical energy.

52. Measurement of Power.—The measurement of mechanical power is even more difficult than the measurement of mechanical energy. The most practical mechanical method of measuring the power of a machine is to determine the mass that the machine can raise at a given velocity against the force of gravity. Most measurements of power are made in terms of the electrical units.

CHAPTER VIII

CONSTRUCTION OF RESISTANCE STANDARDS

In order to maintain the unit of resistance, suitable resistance standards are essential. Moreover, for convenience in use, standards must be available which have a wide range of values. In absolute measurements, standard resistors are used from 0.0001 ohm to 100 000 ohms. Naturally the construction varies to some extent with the value of the resistor. However, with one exception, the properties that are required are the same for all denominations.

53. Essential Characteristics of Resistance Standards.—The important characteristics of a resistance standard are: (1) permanence; (2) definiteness; (3) small temperature coefficient of resistance; (4) small load coefficient; (5) small thermoelectromotive force at terminals when carrying maximum current.

The permanence of a resistor requires that, at a given temperature, the resistance shall have a definite value which remains the same for a long period of time. No resistor completely satisfies this condition. Some standard resistors show a continuous change in resistance caused by physical changes in the resistance material, or chemical action between the resistance material and other materials that may come in contact with it. Other resistors show seasonal changes in resistance, which are generally caused by humidity effects in the form on which the resistance material is wound. The different methods that have been used to make standard resistors as permanent as possible will be discussed under the different types.

The definiteness of a resistor requires that the resistance have the same value under all ordinary conditions of use. This requirement is easily met for all resistors having a value of 10 ohms or more, but is difficult to meet with resistors of 0.01 ohm or less, the difficulty increasing with decreasing resistance. A lack of definiteness is most apparent in resistors which have both current and potential terminals. A resistor of this type has a definite value only when the current distribution in the cross sections that include the potential terminals is unaffected by the method used in connecting to the current terminals, and when each potential terminal is so constructed that it assumes the potential of a particular point of the resistor.

The temperature coefficient of resistance of a resistor depends largely on the characteristics of the resistance material, but is slightly influenced by the method of construction. There are several resistance alloys which have a zero temperature coefficient at a particular temperature, but no material is known for which the temperature coefficient is zero over any appreciable range of temperature. The alloys that have been most extensively used in standards are:

Alloy	Temperature Coefficient, p.p.m. per 1°C
German silver	360
Platinum-silver (⅓ Pt–⅔ Ag)	320
Manganin, Constantin	Zero at some temperature between 20 and 80°C

A gold-chromium alloy has recently been investigated[1] which can be so annealed that its temperature coefficient is zero between 20 and 30°C. The last three alloys are, from the viewpoint of temperature coefficient, entirely satisfactory for the most precise standards. By a careful choice of material, resistors can be constructed of either manganin or constantin so that they do not change by more than ten parts per million per degree centigrade in the ordinary range of working temperatures. The gold-chromium alloy has, when properly annealed, a zero temperature coefficient over a wider range of temperature than any of the other alloys.

The load coefficient of a resistor is the change of resistance per watt of power converted into heat in the resistor when the temperature of the surrounding medium is maintained constant. The load coefficient of a resistor results from the fact that the heat generated by the current keeps the temperature of the resistance material somewhat higher than the surrounding medium in which the temperature is measured. The load coefficient depends on the temperature coefficient of the resistance material, on the thermal resistance to the flow of heat from the resistance material to the

[1] Thomas, *B.S. J. Research*, **13**, 681 (1934).

outside fluid medium, and on the thermal capacity and velocity of the outside medium. The load coefficient for standard resistors of normal size, when immersed in oil, is generally less than ten parts per million per watt, and may be less than one part per million per watt. Hence the load coefficient does not need to be considered in making measurements with an accuracy of one part in a million on manganin or constantin resistors provided the expenditure of power in the resistor can be kept as low as a few hundredths of a watt. Where the expenditure of power is necessarily high, special coils are sometimes constructed.

The thermoelectromotive force at the terminals of a resistor when carrying maximum current depends on the materials of which it is constructed and on the method of construction. In constructing a two-terminal resistor, a resistance alloy is soldered or brazed to copper terminals. The two junctions form a thermocouple which may introduce a spurious electromotive force into the measuring circuit. To keep this electromotive force small, the resistance material must have a small thermoelectric power relative to copper and the construction must be such that the junctions are normally maintained at the same temperature. The thermoelectric power of manganin against copper is only 1 or 2 microvolts per degree centigrade, whereas that of constantin is about 20 microvolts per degree. To insure that the temperature difference between the junctions will always be small, the junctions should be placed near each other, where the rate of cooling will be as nearly as possible the same. With resistors having both current and potential terminals, the potential terminal at each end should be attached at a distance of several millimeters from the place of attaching the current terminals, since the Peltier effect causes more heat to be liberated at one junction of the resistance material with the copper of the current terminal than at the other junction. This heating keeps one junction at a different temperature from the other, but this temperature difference may not exist between points a few millimeters from the junctions.

The thermoelectromotive force in a well-constructed resistor, even of such an unfavorable material as constantin, is seldom more than a few microvolts. If the coil resistance is sufficiently high so that as much as 5 volts is normally impressed on it in making measurements, the presence of the few microvolts of thermoelectromotive force will not introduce an error of one part in a million in the measurement of the resistance. Generally the thermoelectromotive forces do not need to be considered for resistors of 1000 ohms and above.

54. Types of Resistance Standards.—Three types of resistance standards have been extensively used, *viz.*, the B.A. type, the Reichsanstalt type, and the B.S. oil-filled type. The B.S. double-walled type is the latest addition to the types that are in use in the national standardizing laboratories. Photographs of all four types are shown in Fig. 8. No type entirely fulfills all the requirements laid down in the preceding section.

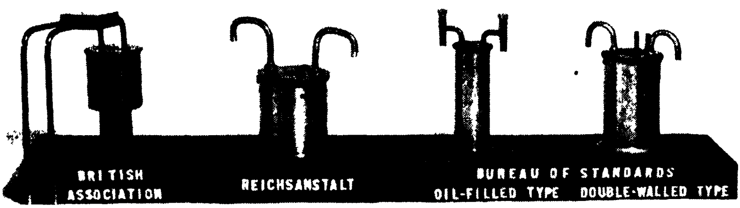

FIG. 8.—Types of standard resistors.

All the types have been made in coils of different denominations, but the greatest care has been exercised in the design and construction of 1-ohm resistors, since these are used at all national laboratories for maintaining the unit of resistance. Hence the following discussion of types will consider mainly 1-ohm coils.

The B.A. type was devised by a committee of the British Association.[1] The method of construction of a 1-ohm standard is shown in Fig. 9. The resistance material was platinum-silver, which was imbedded in paraffin. These standards were intended to be used in a water bath. The B.A. resistance standards were in general use throughout the world for nearly a quarter of a century, but now are of historic interest only.

The Reichsanstalt type of resistance standard was developed at the Physikalisch-Technische Reichsanstalt[2] in Berlin about

[1] See description in Report of Committee on Electrical Standards to British Association for the Advancement of Science, 1865.

[2] First description of Reichsanstalt resistors is by Feussner, *Z. Instrumentenk.*, **10**, 6 (1890). A more complete description is by St. Lindeck, B.A. Report of Standard Committee for 1892, Appendix IV.

1890. The construction of a 1-ohm standard is shown in Fig. 10. These standards have played a very important part in the development of precise electrical measurements and are still extensively

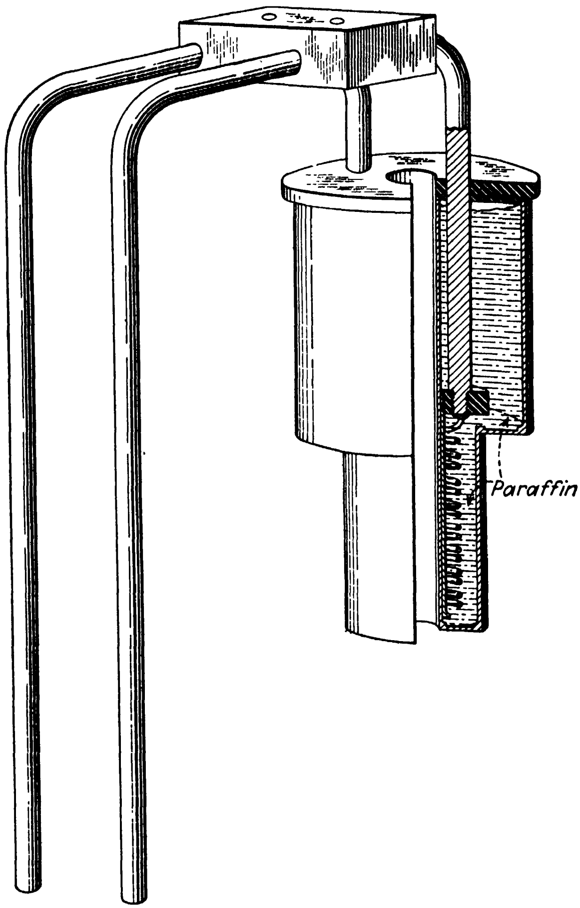

FIG. 9.—The B.A. type of standard resistor.

employed. Hence their construction and characteristics will be carefully considered.

The Reichsanstalt type of resistance standard uses manganin as a resistance material. For a standard having a resistance of 1 ohm, a manganin wire, insulated by silk, is wound bifilarly on a silk-covered brass cylinder; the silk is impregnated with shellac;

and the whole is baked at 140°C for 24 hours. After the baking, copper wires are silver-soldered to the manganin, and these copper wires are soft-soldered to the heavy copper terminals. The brass cylinder and copper terminals are mounted on a hard rubber top, to which is attached a perforated metal can for protecting the winding, yet permitting the cooling fluid, usually air or oil, to circulate over the shellacked surface. The copper terminals are amalgamated at the end so that they will make a low resistance

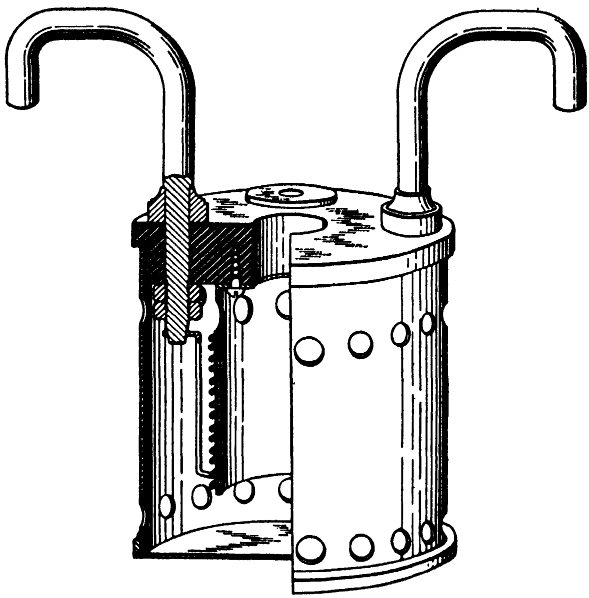

FIG. 10.—The Reichsanstalt (P.T.R.) type of standard resistor.

contact in mercury cups and thus provide a suitable current terminal. Potential terminals are also attached to all resistors having a value of 1 ohm or less.

Resistors of the P.T.R. type satisfy all the requirements laid down in the previous section except permanence. The permanence of resistors of this type increases as the resistors become older but each resistor is a law into itself. As an example, of four resistors which have been under observation since 1893 at the Reichsanstalt, the values relative to the mean of the four had changed in 1909 (sixteen years) by +6, +15, −58, and +37 parts in a million, while during the next sixteen years (ending in 1925) the change had been +5, −9, −19, +23 parts in a million for the same coils, the results being given in the same order. During the first sixteen years the average change from the mean was 29 parts in a million, compared with 14 parts for the second sixteen years. In addition to the changes over a long period of time, resistors of this type show, in many climates, changes from season to season. The cause of this seasonal change in resistance is the absorption of water from the air by the shellac, causing the shellac to swell, which in turn elongates the manganin wire, thus increasing its resistance. The change from winter to summer in a humid climate may be as much as 250 parts in a million. The above statements show that standard resistors of the Reichsanstalt type may not be as permanent as is required in precision electrical measurements.

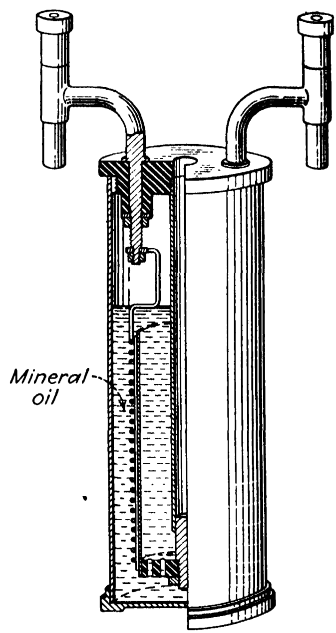

FIG. 11.—The B.S. oil-filled type of standard resistor.

The B.S. type of standard resistor, developed by Rosa[1] in 1908, is shown diagrammatically in Fig. 11. The materials and general construction features of the B.S. type are the same as in the Reichsanstalt type, the essential difference being that the coils of the B.S. type are sealed in an air-tight metal can which is nearly filled with a mineral oil. As the B.S. type of standard resistor is sealed, there is no effect of atmospheric humidity and hence no change in resistance with the seasons. However, the resistance does change over a period of years. As an example, seven resistors which were, in the sixteen years following 1909, in the group of ten coils used to maintain the unit of resistance at the National Bureau of Standards, showed changes of resistance relative to the maintained unit of +63, −21, −4, +21, −13, +9, +55 parts in a million, which is an average change of 27 parts in a million. These changes are not materially different from those observed during the first sixteen years of the life of the Reichsanstalt resistors.

[1] *Bull. B. S.*, **5**, 413 (1908).

Recently an attempt has been made to improve the B.S. type.[1] A bare manganin wire is wound on a metal form of the same size as the form on which it is to be mounted. It is then annealed at 500°C in a vacuum or an inert gas. After slowly cooling, the wire is slipped off the annealing form and mounted on another form which has an insulating coating, care being exer-

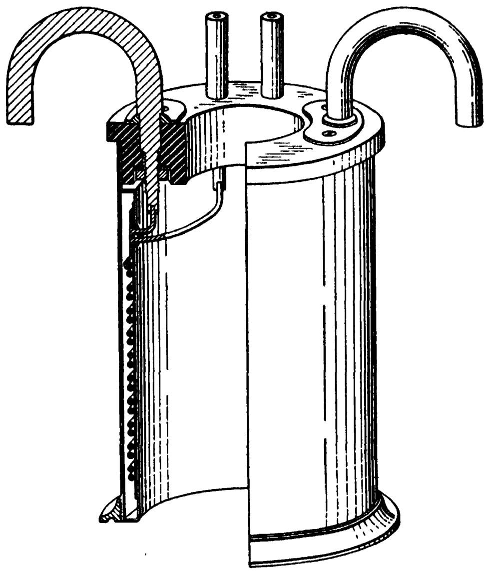

FIG. 12.—The B.S. double-walled type of standard resistor.

cised to see that no mechanical strains are introduced into the annealed manganin. The form is the inner cylinder of a double-walled container, as shown in Fig. 12. This is hermetically sealed and attached to a top which supports the current and potential terminals. Such resistors show great promise, but have not been under observation for a sufficiently long time to determine their permanence.

The load coefficient of a resistor of either B.S. type is higher than that of a resistor of the Reichsanstalt type provided all the coils are made from resistance materials which have the same temperature coefficient. This results from the fact that the heat produced by the measuring current is not transferred to the cooling fluid so readily in the B.S. types as in the Reichsanstalt type.

[1] Thomas, *B.S. J. Research*, **5**, 295 (1930).

CHAPTER IX

CONSTRUCTION OF STANDARD CELLS

Standard cells serve to maintain the unit of electromotive force. While any number of cells can be used in series provided they are sufficiently insulated, yet the range of values of electromotive force that can be obtained by standard cells is quite limited.

55. Essential Characteristics of Standard Cells.—The important characteristics of a standard cell are: (1) permanence; (2) low temperature coefficient; (3) quick response to changes of temperature; (4) low internal resistance; (5) negligible polarization for small currents through the cell; (6) reproducibility.

A cell is permanent when, at a definite temperature, it has a definite electromotive force at any time within a period of years. No type of cell is entirely satisfactory as regards permanence. The electromotive force of a cell is affected by slight amounts of some impurities which may be present in the chemicals of which the cell is constructed and possibly by the physical condition of the chemicals, such as size of the crystal grains. Apparently the effect of impurities and crystal size may change with time, causing a change in electromotive force. Also the glass container may dissolve slightly, producing an impurity in the electrolyte. In unsaturated cells, a change in concentration of some of the solutions may affect the electromotive force.

Low temperature coefficient is essential for cells which are to be used in precision electrical measurements. As temperature control and measurement cannot be made closer than 0.01°C except with great difficulty, the temperature coefficient of a standard cell should be less than 100 parts in a million if measurements are to be made to one part in a million.

The electromotive force of a cell which has been subjected to a change in temperature slowly changes for several hours or even days after the temperature has become stationary. This phenomenon is known as *hysteresis*. For some unknown reason, some cells show much greater hysteresis than others. A quick response to temperature changes (small hysteresis) is very desirable.

If the internal resistance of a cell is too high, difficulty is experienced in obtaining sufficient sensitivity in the measuring apparatus. However, a resistance of a few hundred ohms does not generally affect the accuracy of measurement, and may be of some advantage in preventing an excessive current if an accidental short circuit occurs. Most cells are made of such dimensions that the resistance is more than 100 ohms.

The polarization of a cell for a small current is determined by the characteristics of the chemicals of which it is constructed. In the construction of all standard cells a depolarizer is used, but often this is so slightly soluble that chemical action is slow. For such cells the current that should be allowed to flow through the cell may be very small. As the common methods of measurement require that a small current shall flow through the cell, it is important that this small current should not affect the electromotive force of the cell.

56. Types of Standard Cells.—The chemical constituents and important characteristics of the various types of standard cells are given in Table VIII. All the different types have been made in many forms. Perhaps the most common form is the H-shape cell suggested by Lord Rayleigh for the Clark cell and illustrated below for the Weston cell. This form is suitable for all types of cells except the Daniell cell.

The cell which was first suggested by the B.A. Committee was the Daniell cell, then in common use. This cell is not permanent and new cells must be set up at least once a fortnight. The accuracy of reproduction is about 1 per cent.

The Clark cell has a rather short life. The leads to the electrodes are platinum wires sealed through glass. The zinc slowly alloys with the platinum, causing it to swell and crack the glass. Hence Clark cells never last more than a few years, too short a time to make an extended study of the constancy of their electromotive force. This tendency to crack, combined with the relatively large temperature coefficient of electromotive force, has caused the Clark cell to be discarded in practically all laboratories. The Carhart-Clark cell is a modification of the Clark cell which has a much lower temperature coefficient but an equally short life.

TABLE VIII.—COMPOSITION AND CHARACTERISTICS OF TYPES OF STANDARD CELLS

Name of cell	Year proposed	Negative electrode	Electrolyte	Depolarizer	Positive electrode	Electromotive force, absolute volts at 20°C	Temperature coefficient	
							Microvolts per °C	Parts per million per °C
Daniell	1836	Zinc	Zinc sulphate	Copper sulphate	Copper	1.08	+ 34	+ 31
Clark	1872	Zinc amalgam	Zinc sulphate saturated solution	Mercurous sulphate	Mercury	1.428	−1290	−904
Carhart-Clark	1891	Zinc amalgam	Zinc sulphate saturated at 0°C	Mercurous sulphate	Mercury	1.436	− 530	−370
Weston saturated	1892	Cadmium amalgam	Cadmium sulphate saturated solution	Mercurous sulphate	Mercury	1.0187	− 37	− 37
Weston unsaturated	1903	Cadmium amalgam	Cadmium sulphate saturated at 4°C	Mercurous sulphate	Mercury	1.0190	− 10	− 10
Weston acid	1908	Cadmium amalgam	Cadmium sulphate saturated solution in $n/10$ sulphuric acid	Mercurous sulphate	Mercury	1.0186	− 37	− 37

The various types of Weston cell are, for the most part, similar to the types of Clark cell with cadmium substituted for zinc. A diagram showing the H-form is given in Fig. 13. Weston cells are universally employed at the present time. The Weston saturated cell, often called the *saturated cadmium cell,* is used in the national laboratories as the primary standard of electromotive force. It is also extensively used in commercial and college laboratories. On account of the extensive use of this cell, its properties will be considered in detail, giving special attention to the essential properties discussed in the preceding section.

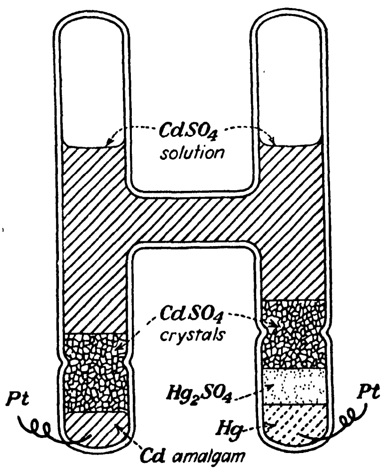

FIG. 13.—The H-form of Weston saturated standard cell.

Weston saturated cells are not absolutely permanent as regards voltage. At McGill University,[1] the average voltage of a group of seven Weston saturated cells decreased 80 microvolts in twenty years, 75 per cent of this decrease occurring in the first five years and 50 per cent in the first eighteen months. After the cells in the above group were ten years old, the change was nearly linear, being about 1 microvolt per year. The temperature coefficient (37 microvolts per 1°C) of the Weston saturated cell is sufficiently low for precise measurements, and the hysteresis with changes in temperature is not troublesome for most cells, though an occasional one, for some unknown reason, exhibits considerable hysteresis. The internal resistance of Weston saturated cells of the usual form is a few hundred ohms. Also the electromotive force is not affected by currents of a microampere or less through the cell, which is sufficient to permit its use as a standard. Another desirable characteristic in a type of cell is that new cells, constructed of highly purified materials, shall all have the same electromotive force. The maximum dif-

[1] Shaw, Reilley, and Clark, The Ageing of Standard Cells, *Phil. Trans.*, **A229**, 125 (1930).

ference in electromotive force between the cells of a group of ten new Weston saturated cells will generally be as much as 10 microvolts and may be as large as 100 microvolts.

The Weston acid cell has all the desirable properties of the Weston saturated cell, and in addition appears to be more permanent. A group of seven acid cells at McGill University showed a change of about 1 microvolt per year for the first five years. If these acid cells improve with age in the same manner as do the saturated cells, then old cells of this type will be excellent standards.

Weston unsaturated cells are not so permanent as other types of Weston cells, decreasing by 30 to 40 microvolts per year. On the other hand, unsaturated cells have a lower temperature coefficient than any other type of standard cell. The temperature coefficient varies somewhat from cell to cell, but is never more than -10 microvolts per degree centigrade, and is usually about half this value. Unsaturated cells are used in laboratories where the highest precision is not required and where the low temperature coefficient is particularly useful. Unsaturated cells should be frequently compared against a more permanent standard. Outside of standardizing laboratories, they are almost universally employed, partly on account of their portability.

CHAPTER X

CONSTRUCTION OF STANDARDS OF CAPACITANCE AND INDUCTANCE

Standards of capacitance and inductance, while not of such fundamental importance as standards of resistance and electromotive force, are required in many kinds of electrical measurements. Standards of capacitance can be so constructed that the capacitance in electrostatic units can be computed from the mechanical dimensions; and standards of inductance can be so constructed that the inductance in electromagnetic units can be likewise computed. Such standards will be discussed later, but the present description will be confined to those standards which are used to maintain the units of capacitance and inductance.

57. Essential Requirements of Standard Capacitors.—A capacitor to be suitable as a standard of capacitance must possess the following properties: (1) permanence; (2) definiteness; (3) high insulation resistance; (4) capacitance independent of applied voltage; (5) small temperature coefficient of capacitance; (6) small pressure coefficient of capacitance. In addition, it is desirable that there should be no energy loss in the capacitor when it forms part of an alternating-current circuit and that the capacitance should be independent of the method of measurement.

A capacitor is permanent when its capacitance as measured by a particular method is, for a given temperature and atmospheric pressure, constant over a period of years. In order to be permanent, the relative positions of the two sets of plates of the capacitor must remain fixed, and the dielectric between the plates must not change its properties. This last may be accomplished by using a solid dielectric such as mica, which is permanent and which does not change its position relative to the plates, or by using a fluid dielectric of definite composition, such as air, which may circulate but which always fills the space between the plates with a dielectric having the same properties.

A two-terminal capacitor has a definite capacitance only when one of its sets of plates completely surrounds the other set so that the presence of outside bodies does not affect the electrostatic field between the two sets of plates. Such a capacitance cannot be measured because there are no leads for connecting to the measuring apparatus. When leads are connected and the outside set of plates is grounded, that portion of the electrostatic field resulting from the charge on the lead that connects to the inner set of plates may be affected by the presence of outside bodies. Since any change in the electrostatic field affects the capacitance, there is an indefiniteness in the capacitance of every capacitor which has only two terminals. By proper design, the indefiniteness of a capacitor may generally be made less than 1 micromicrofarad. Hence this indefiniteness is important only in capacitors of small value.

High insulation resistance is particularly important in capacitors that are to be used in direct-current measurements. The insulation resistance of a capacitor is equal to the applied voltage divided by the leakage current between terminals. This current consists of two parts: (1) the part through the dielectric, and (2) the part between the leads. When the leads end in terminals mounted on a piece of solid insulating material, the current over the surface and through the body of this material may be appreciable. For large capacitors the second part is usually negligible relative to that through the dielectric, while for small condensers with fixed terminals the second part may be the more important of the two.

The energy loss in a capacitor which is connected in an alternating-current circuit may be divided into three parts: (1) the energy loss caused by the current in the insulation of the capacitor, (2) the energy loss caused by the current in the plates and the leads to the plates, and (3) the energy loss in the dielectric caused by the alternating electrostatic field. The energy loss in joules per second produced by the current through the insulation is E^2/R, where E is the voltage at the terminals and R is the insulation resistance. For good condensers, the insulation resistance is so high that this loss is negligible. The energy loss in the plates cannot readily be computed since the resistance of the plates and distribution of charge on them need to be considered. However, the energy loss per second in the leads is equal to I^2r, where I is the current in the leads and r is their resistance. For a perfect capacitance, $I = 2\pi fCE$ (f = frequency, C = capacitance, and

E = potential difference), so that the energy loss *per second* for a given voltage applied to the capacitor increases approximately as the square of the frequency; hence the loss per cycle increases directly as the frequency. For most standard capacitors the plate and lead loss is negligible for frequencies below 1000 cycles per second, and in many does not become appreciable until the frequency is above 1 000 000 cycles per second. The loss in the dielectric varies greatly with the kind of dielectric. If the dielectric is a vacuum, the loss is zero. With air and the other permanent gases, the loss is so exceedingly small as to be negligible even in the most precise measurements. With most liquids and solids, the loss is appreciable and for some is very large. Among solids, fused silica, mica, and paraffin have small loss; among liquids, highly refined oils have small loss.

The capacitance of a capacitor should be independent of the method of measurement that is employed to obtain the capacitance. When capacitors are used at extremely high frequencies, the apparent capacitance may be affected by the inability of the charge to distribute itself uniformly over the plates. However, the principle cause of change of capacitance with method of measurement is electrostatic absorption in the dielectric. Only vacuum and air capacitors are entirely free from absorption, so that such capacitors are always employed when the capacitance is required to be independent of the method of measurement. However, capacitors which have absorption are often used as standards of capacitance. The absorption is perfectly definite, so that the apparent capacitance is definite when the essential features of the method of measurement are specified; in alternating-current measurements, the essential features are the frequency and wave form of the applied electromotive force.

The capacitance of almost every capacitor is independent of the applied voltage unless the voltage approaches that required for breakdown. Occasionally a capacitor will show a decided change of capacitance with voltage, but this has usually been observed in types that are not now used as standards. When observed, a capacitor is considered defective.

The temperature coefficient of capacitance of a capacitor depends on the expansion coefficient of its different parts and on the change of dielectric constant with temperature of its dielectric. These effects are so involved that they are seldom used to determine the temperature coefficient of a capacitor. An experimental determination is more accurate.

The pressure coefficient is readily computed for a condenser having air as a dielectric by the formula

$$C_p = C_{760}\left[1 + 0.000585\left(\frac{p - 760}{760}\right)\right]$$

where C_p is the capacitance at a pressure p in millimeters of mercury and C_{760} is the capacitance at a pressure of 760 mm. In case the air pressure is reduced to half normal, the decrease in capacitance is less than three parts in ten thousand. For condensers having dielectrics other than a gas, the pressure coefficient must be determined experimentally.

58. Types of Capacitors.—There are only two types of capacitors that are now used to any extent as standards, *viz.*, air capacitors and mica capacitors. Air capacitors are bulky, and hence find greatest use where a small capacitance is required. Mica capacitors are seldom used in values smaller than 0.001 microfarad or larger than 1 microfarad. The properties of air capacitors are quite different from those of mica capacitors.

A fixed-value air capacitor consists of two sets of plates, one set being held in a definite position relative to the other set by means of one or more pieces of solid insulating material. This solid material should be so placed that no part is in a strong electrostatic field. The permanence of an air condenser depends on the permanence of the plates and of the solid insulating material. If fused silica or porcelain is used for the insulating material, changes therein will probably be negligible with time. However, all metal plates are likely to warp owing to slow annealing. Air capacitors with brass plates are seldom constant to one part in ten thousand over a period of a year and a change of a part in a thousand is not unusual. In a variable air capacitor one set of plates can move relative to the other by turning in bearings or sliding in guides, with a scale to indicate the relative position. In such capacitors, the permanence is affected by the wear on the bearings or guides.

The definiteness of an air capacitor depends on the completeness with which one set of plates surrounds (shields) the other set and on the method employed in attaching a lead to the inner set of plates. In capacitors of recent manufacture the lack of completeness in shielding seldom introduces any indefiniteness in the capacitance, but the lead to the inner plates may introduce an uncertainty of one or more micromicrofarads into the measured capacitance. This is obviated by a three-terminal capacitance, examples of which will be discussed in a later chapter.

The insulation resistance of an air capacitor depends almost entirely on the supports that maintain the relative position of the two sets of plates. If the supports are properly made from a suitable insulating material, the insulation resistance will be as high as 10^{13} ohms. Likewise the energy loss in alternating-current circuits is principally in the supports. By proper design the loss can be made so small that, even when the capacitance is only 100 micromicrofarads, the power factor (which is equal to the energy loss during a cycle divided by 2π times the product of the capacitance and the square of the applied voltage) will be about 10^{-5}. Again the capacitance of an air capacitor is independent of the method of measurement, unless the dielectric absorption in the supports appreciably affects the measured capacitance. With the best capacitors, the capacitance when measured with 100 charges and discharges per second is never more than 0.1 micromicrofarad greater than the capacitance measured with 1000 charges and discharges per second. The capacitance of a properly designed air condenser is independent of voltage provided there is no corona.

A mica capacitor consists of an assembly of sheets of metal foil which are separated by thin sheets of mica. This assembly is impregnated with some wax, such as paraffin, to remove the air, and is mounted between heavy metal plates to increase its permanency. Each sheet of metal foil has an "ear" which projects beyond the edge of the mica. The ears of alternate sheets project to opposite edges of the mica sheet which separates them. These ears serve to connect the sheets to the terminals of the capacitor. The dielectric is mainly mica, with very thin layers of paraffin on the surfaces. Mica capacitors have greater permanence than air capacitors, but the capacitance of most of them changes by a small amount in the course of a year. Unless a mica capacitor has been under observation for several years so that its permanence is assured, the capacitance cannot be depended on to remain constant within one part in a thousand for a period of a year.

An unshielded mica capacitor may have a somewhat indefinite capacitance and this indefiniteness may be augmented by mounting it near other unshielded capacitors or by connecting it to large terminal blocks. For a properly shielded capacitor, the indefiniteness does not exceed 1 or 2 micromicrofarads, but, for unshielded capacitors with large terminal blocks, the indefiniteness may be 10 or even 100 times this amount.

The insulation resistance of a properly mounted mica capacitor is almost entirely determined by the current that flows through the dielectric. This dielectric current is determined by the resistivity of the mica and the size and thickness of the mica between the plates. The capacitance is determined by the dielectric constant of the mica and the size and thickness of the mica plates. It follows that the product of the resistance and capacitance is proportional to the product of the resistivity and dielectric constant, and hence should be a constant for all mica condensers. This is approximately the case, the value varying from 5000 to 50 000 megohms in a microfarad (or ohms in a farad).

The energy loss in a mica condenser, when used in an alternating-current circuit in which the frequency is less than 1000 cycles per second, is almost entirely in the dielectric, but at frequencies above 1000 the loss resulting from the plate resistance sometimes becomes an appreciable part of the total. The dielectric loss per cycle for a given applied voltage is nearly independent of the frequency, decreasing by a factor of two or three as the frequency increases from 60 to 1000 cycles per second. It follows that the dielectric loss per second increases with the frequency, but a little less slowly than in direct proportion to it. For a good mica condenser, the power factor at 60 cycles per second is less than 9×10^{-4} (phase difference about 3′), and for some exceptional condensers is as low as 2×10^{-4}. At 1000 cycles a good condenser has a power factor less than 5×10^{-4}, while exceptional condensers may have a power factor as low as 1×10^{-4}.

The effect of any given method of measurement on the apparent capacitance of a mica capacitor can be expressed by stating the difference between the capacitance as obtained by the method of measurement and the geometric capacitance. The geometric capacitance[1] can be determined either by measuring with direct

[1] For a more complete discussion of geometric capacitance, see Curtis, *Bull. B. S.*, **6**, 478 (1910).

current, using several short times of charge or of discharge and extrapolating to zero time, or with alternating current, using several frequencies and extrapolating to infinite frequency (zero period). The apparent capacitance by any method of measurement is always larger than the geometric capacitance. For a good mica capacitor, the capacitance measured at 60 cycles per second may be five parts in ten thousand larger than the geometric capacitance, while for some exceptional capacitors the increase may be only two or three parts in ten thousand.

An occasional mica condenser will have a capacitance that changes with the applied voltage. Such condensers are not suitable for standards.

The temperature coefficient of capacitance of a mica capacitor depends on the method used in measuring the capacitance. The temperature coefficient of the geometric capacitance is usually negative, and may be as large as 50 parts in a million per degree centigrade. The temperature coefficient of the capacitance measured with alternating current of audio frequencies is algebraically larger by a small amount than that for the geometric capacitance. For power frequencies, the temperature coefficient has a still larger algebraic value than for audio frequencies, but even it may be negative. At 60 cycles per second, the temperature coefficient is generally less than twenty parts in a million per degree centigrade and is more often negative than positive.

The pressure coefficient of capacitance of mica capacitors is quite different for different instruments, varying from -2 to $+35$ (average about 15) parts in a million for an increase in barometric pressure of 1 cm.[1] Hence the normal changes in barometric pressure from day to day at one place may cause a change in capacitance of one part in ten thousand. Also a capacitor must be expected to decrease in capacitance by one part in a thousand if taken from sea level to an elevation of 700 m (about 2000 feet).

From the data already presented, the conclusion can be drawn that, in general, mica capacitors cannot be depended upon as standards when an accuracy greater than one part in a thousand is required. However, individual capacitors frequently show a constancy over a limited time of one part in ten thousand. Such capacitors can be obtained only by selection after observation for a considerable period of time.

[1] Curtis, *Bull. B. S.*, **8**, 459 (1910).

59. Essential Requirements of a Standard Inductor.—An inductor, to be suitable as a standard of inductance, must possess the following characteristics: (1) permanence; (2) definiteness; (3) small temperature coefficient of inductance. In addition, the resistance should be low and the change of inductance with frequency small. The type of construction required to meet the above characteristics differs somewhat for a standard of self inductance from that for a standard of mutual inductance.

The permanence of an inductor depends upon the stability of the form on which the wire is wound, of the insulation which separates the wires, and of the wire itself. The form is usually placed under considerable stress when the wire is wound on the inductor. Some materials flow slowly under this stress, causing a change in the inductance. On the other hand, if the wire slowly elongates under the tension introduced in the winding (as is likely if soft-drawn copper wire has been used), the stress in the form is released and a change occurs in the inductance. Also any change in the insulation which will permit a relative change in position between the turns of wire causes a change in inductance.

The definiteness of an inductor indicates the accuracy that can be expected when its inductance is measured in two different laboratories, or in the same laboratory on two different occasions, when the only difference in the inductance is the way in which the leads are connected. In some cases, leads are permanently attached to the inductor, but most inductors have binding posts to which the leads are attached. The latter are not so definite as the former, since with the latter the leads cannot be placed in a second measurement in exactly the same position relative to the inductor as they were in the first measurement. The definiteness of an inductor is mainly determined by the position of the binding posts which are provided for attaching the leads. On a self inductor the two binding posts should be close together, and so placed that there will normally be no appreciable mutual inductance between the leads and the coil. On a mutual inductor the two pairs of binding posts should be so located that there will normally be no appreciable mutual inductance between the two pairs of leads, or between either lead and the opposite coil.

The temperature coefficient of inductance of an inductor results from the change in the dimensions of the winding on the inductor with a change in its temperature. The wire is always wound under tension, thus compressing the form and the insulation underneath the wire, and a change in temperature will generally affect the tension, since the coefficients of linear expansion of the form and insulation are seldom the same as that of the wire. Hence the temperature coefficient of inductance depends not only on coefficients of linear expansion but also on the compressibility of the form and insulation. This analysis shows the difficulty of predicting the temperature coefficient of inductance.

In most applications it is desirable to have the resistance of each of the windings of the inductor as low as feasible, but, to decrease the resistance, the size of the inductor must be increased. Experience has shown that the most useful inductance standard is one having a reasonable resistance and a moderate size. For a self-inductance standard, the ratio of the inductance to the resistance, called the *time constant* (L/R), of the inductor is generally as large as $1/1000$ second, or 1 millisecond, and seldom more than 10 milliseconds. The time constant of each coil of a mutual inductance should have about the same value.

The change of inductance with frequency may be caused by capacitance between the turns of the winding, by dielectric loss in the insulating material between turns, and by skin effect in the wire with which the inductor is wound. All three of these causes are always present, but generally the capacitance is the most important. If the other effects are negligible, the self inductance L_f of a coil at any frequency f is given approximately by the equation

$$L_f = L_0(1 + 4\pi^2 f^2 K L_0)$$

where L_0 is the inductance at zero frequency and K is called the distributed capacitance of the coil. The value of K and L_0 are obtained by measuring L_f at two frequencies, substituting the values in the equation and solving the resulting equations. A similar equation can be written for a mutual inductance. The effect of dielectric loss on an inductance is generally small, but not always negligible. As the loss changes with frequency in a manner which is different for different dielectrics, it is not possible to express the effect by means of an equation. However, the statement can be made that there may be a term which depends on the first power of the frequency. The skin effect produces a much smaller change in inductance than in resistance. Skin effect produces a decrease in inductance with increasing frequency and this decrease is approximately proportional to the square of the frequency.

60. Types of Inductors.—Standard inductors are of two types: fixed standards and variable standards. Each type may be designed as a self inductance or as a mutual inductance. Many different forms of each type are possible, but only the more common will be described.

61. Fixed Standards of Self Inductance.—Two forms of fixed standards of self inductance are in common use: toroidal coils and

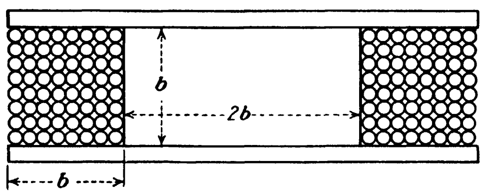

FIG. 14.—Dimensions of an inductor to give maximum self inductance for a given length and size of wire.

circular coils with rectangular section. Toroidal coils have the advantage of a relatively small external field, but they are difficult to wind. A discussion of a circular coil of rectangular section will indicate the important points to be considered in designing a fixed standard of self inductance.

A given length of wire when wound into a circular coil of rectangular section (see Fig. 14) will have a maximum inductance if the section is a square and the inside diameter is twice the side of the square section.[1] Under this condition, the inductance L in henrys is given with sufficient accuracy for design purposes by the formula

$$L = 24.5bn^2 \times 10^{-9} = \frac{24.5b^5}{D^4} \times 10^{-9} \text{ henrys}$$

where n is the number of turns of wire, b is the side of the square in centimeters, and D is the diameter of the wire over the insulation in centimeters. A given inductance can be made from wire of

[1] For details see a paper by Brooks, *B.S. J. Research*, **7**, 293 (1931).

any diameter (D) by winding the coil in a channel of the proper width (b); all the other dimensions have the proportions given above. The resistance R, in ohms, of such a coil is given by the equation

$$R = \frac{12\rho b^3}{q^2 d^4} \text{ ohms}$$

where ρ is the resistivity of the material of the wire (1.724×10^{-6} ohm-cm for copper) and $q = D/d$ is the ratio of the diameter over the insulation (D) to the diameter of the wire (d).

By algebraic operations with the last two equations and by inserting the numerical value of ρ for copper, the following equations can be obtained.

$$b = 29.1q\sqrt{\frac{L}{R}}$$

$$n = 1185\sqrt[4]{\frac{LR}{q^2}}$$

$$d = 0.844\sqrt[8]{\frac{L^3q^2}{R^5}}$$

As an example, design an inductor to have an inductance of 1 henry and a resistance of 100 ohms, using wire in which the ratio q is 1.16. Then

$$b = 2.91q = 3.38 \text{ cm}$$
$$n = 3747\sqrt{q} = 4033 \text{ turns}$$
$$d = 0.475\sqrt[4]{q} = 0.493 \text{ mm}$$

62. Fixed Standards of Mutual Inductance.—In designing a mutual inductance standard, several factors may need to be considered in addition to the mutual inductance between the two windings. The most important of these are: the resistance of the primary and of the secondary, the self inductance of the primary and of the secondary, and the capacitance between the primary and secondary. With so many variables, the number of types of mutual inductance standards is very large. Only a few will be discussed here.

A common type of mutual inductor is made by winding two wires side by side in a circular coil of rectangular section. In such an inductor, the self inductance of the primary is the same as that of the secondary and only slightly greater than the mutual inductance. It follows that the self inductance of the two windings in parallel is a very little less than the self inductance of either coil alone. While this type of coil has the advantage of a large mutual inductance for a small amount of wire, so that the resistances of the primary and secondary can be kept small, it has the disadvantage that the capacitance between the two windings is large.

An easy method of designing a coil of this type is to design a self inductor, by the method of the preceding section, to have a self inductance which is four times the mutual inductance desired.

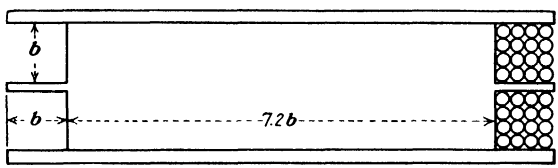

FIG. 15.—Dimensions of two coils of the size (placed as close together as possible) when the mutual inductance is a maximum for a given length and size of wire.

Then, instead of winding a single wire, wind two wires in parallel, keeping the two wires well insulated and bringing their ends out to separate terminals. The mutual inductance between the two windings will be a little less than one-fourth of the computed self inductance, and the resistance of each winding one-half the computed resistance of the self inductor.

Another type of mutual inductor consists of two coaxial coils wound on the same form (see Fig. 15). Such an inductor has the advantage that the capacitance between the windings is small, but the resistance and self inductance of both the primary and the secondary for a given mutual inductance are greater than in the type just described.

The design of such an inductor must start by arbitrarily selecting some value for the distance between the two coils. Unless there is some reason for making the capacitance between the primary and secondary unusually small, the best design is that which places the coils as close together as possible, so that the distance between the centers of the coils is equal to a side of the square cross section of each coil. With this condition and with the condition for maximum mutual inductance,[1] the mutual inductance M is given by the equation

$$M = 77bn^2 \times 10^{-9} = \frac{77b^5}{D^4} \times 10^{-9} \text{ henrys}$$

where b = length in centimeters of a side of the square cross section.
n = number of turns on each coil.
D = diameter in centimeters of wire over the insulation.

The self inductance L of each coil is given by the equation

$$L = 120bn^2 \times 10^{-9} = \frac{120b^5}{D^4} \times 10^{-9} \text{ henrys}$$

63. Variable Standards of Inductance.—A variable inductor consists of two coils, one of which is stationary and the other movable with a single degree of freedom so that one number denotes the relative position of the two coils. The most usual method of accomplishing the required motion is to have one coil rotate about some chosen axis, which cannot be the axis of the coil. When the two coils are connected in series, a variable self inductor results; if the terminals of each coil are brought out to

[1] The maximum mutual inductance for a given length of wire l (the same for each coil) can be approximately determined as follows:

Let a = mean radius of coils.
b = side of square which is the cross section of each coil and also the mean distance between the coils.
n = number of turns in each coil.
D = diameter of wire over the insulation.
$n = b^2/D^2 = l/2\pi a$

$$M = 4\pi a n^2\left(\ln \frac{8a}{b} - 2\right) \text{ approx.}$$

$$= \frac{l^2}{2\pi a}\left(\ln \frac{128\pi a^3}{lD^2} - 4\right)$$

Since l and D are constants,

$$\frac{dM}{da} = -\frac{l^2}{2\pi a^2}\left(\ln \frac{128\pi a^3}{lD^2} - 7\right)$$

For a maximum value of M, $\frac{dM}{da} = 0.$

$$\therefore \frac{64a^2}{b^2} = \epsilon^7 = 1096$$

$$a = 4.1b$$

$$2a - b = 7.2b = \text{diameter of form}$$

separate binding posts, a variable mutual inductor is obtained. There are two types in common use, the Ayrton and Perry[1] and the Brooks.[2] The coils of the Ayrton and Perry type cover equatorial zones on two concentric spherical shells, the smaller of which can rotate about an axis which lies in the mid-plane of both zones and which passes through their common center. On account of the difficulty of constructing such surfaces, the spherical shells are often replaced by short cylindrical shells, the rotation of one cylinder taking place around a line perpendicular to its axis. In the Brooks type there are four fixed coils and two movable coils, each coil having the shape of a link in a chain. Each movable coil has twice as many turns as each fixed coil. The coils are mounted in pairs in three sheets of insulating material, which can be so assembled that the sheet carrying the movable coils rotates between the other two. In each sheet the two coils are mounted symmetrically with respect to the axis of rotation, the long axes of the two links being perpendicular to a line through the axis of rotation. The coils in each sheet are so connected in series that the current flows in opposite directions around the two coils, and in the assembly the two fixed sheets are so placed that the coils in which the current has the same direction will be placed one above the other.

The cylindrical form of the Ayrton and Perry type has the advantages of ease of construction and of compactness. The Brooks type is less affected by external magnetic fields, and, when it is properly designed, the inductance change is nearly proportional to the angular motion throughout a large part of the rotation. The resistance of the Brooks type for a given maximum inductance can more readily be made small than is the case with the Ayrton and Perry type. For both types the ratio of the maximum inductance to the minimum inductance is about 10 to 1.

[1] *Electrician*, **85**, 546 (1895).
[2] Brooks and Weaver, *Bull. B. S.*, **13**, 569 (1917).

CHAPTER XI

METHODS OF COMPARING STANDARDS

While absolute electrical measurements are concerned with measuring one quantity in terms of the units of other quantities, in the setups for making such measurements the necessity often arises for comparing two electrical standards of the same kind. Hence methods will be described which will permit the intercomparison with high accuracy of standards of each of the electrical quantities. Only one or two methods will be given for each quantity, these being selected because they are capable of giving precise results. In all cases the methods will give the details for comparing one standard with another of the same value. Enough methods will be given to cover the range of values of standards usually required in absolute electrical measurements. In some cases consideration will be given to comparing two standards, the values of which have a ratio of 10 to 1.

64. Comparison of Resistances.—The comparison of resistance standards can be made with high accuracy only when proper precautions are taken. For standards of low value, *i.e.*, 1 ohm and below, special precautions must be taken to eliminate the resistance of the leads and contacts. For standards of high value, *i.e.*, 1 megohm and above, the leakage resistance between the terminals, which is usually variable, may appreciably affect the measured resistance. Perhaps the simplest to measure are standards of intermediate value.

In all comparisons of high accuracy, the temperature of the standard must be carefully measured, as even standards of high quality may have a temperature coefficient of twenty parts in a million per degree centigrade in the useful temperature range. In order that measurements to one part in a million will have meaning, the temperature of the standard coil must be measured to one-twentieth of a degree. The most satisfactory arrangement is to keep the standard coils in an oil bath, the temperature of which is controlled by a thermostat. The oil in the bath must be well stirred to insure that the standard is at the same temperature as the oil. If there is an appreciable amount of heat generated in the standard by the measuring current, the temperature of the resistance material will be higher than the temperature of the oil by an amount that depends on the rate at which heat is generated in the coil. The rate at which electrical energy is converted into heat is called the load (p) on the coil. The resistance $R_{t,p}$ of a standard in a bath maintained at a temperature t, when the electrical load on it is p, can be computed by the formula

$$R_{t,p} = R_{25,0}\{1 + \alpha[(t + \eta p) - 25] + \beta[(t + \eta p) - 25]^2\} \quad (1)$$

where $R_{25,0}$ is the resistance at 25°C with zero load, and α, β, and η are experimental constants. The constant η gives the degrees centigrade per watt that the temperature of the coil is above the temperature of the bath. For sealed standards of the B.S. type in which the generated heat is first absorbed by the oil within the case, the value of η may be taken as 0.8°C per watt if the measurement is made in less than a minute after the current is applied.

Instead of determining η as a constant of the coil, it is customary to maintain the coil in a bath which is kept at a particular temperature and to determine the effect of load at this temperature. Then ${}_tR_p = {}_tR_0(1 + \lambda p)$, where λ, the load coefficient at temperature t, is a function of α, β, and η. There are so many factors which may influence the load coefficient of a coil that it should be experimentally determined under the same conditions as those under which the coil is being used.

65. Comparison of Resistance Standards of Intermediate Value.—Resistance standards of intermediate value can be compared with an accuracy of one part in a million by a Wheatstone bridge, which is shown diagrammatically in Fig. 16. Current from the battery E divides at the junction of the bridge into two parts, i_1 and i_2. The resistance C is adjusted until no current flows through the galvanometer. Then the potential drop over the resistance A equals that over resistance C, and the drop over B equals that over D. Hence

$$Ai_1 = Ci_2$$
$$Bi_1 = Di_2$$

so that $A/B = C/D$ or $A = BC/D$. The accuracy with which the resistance A is determined depends on the precision with which C can be adjusted and on the accuracy with which B, C, and D are known. The precision of adjustment of C depends primarily on the sensitivity of the galvanometer. Galvanometers with critical damping are now available which permit an adjustment of C to one part in a million, even with relatively small currents in the bridge arms. The accuracy with which B, C, and D are known depends on the kind of resistors used. If they are

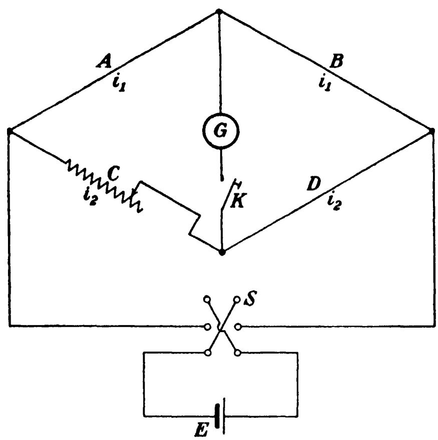

FIG. 16.—Diagram of a Wheatstone bridge.

commercial resistance boxes or bridges, the nominal value of each will probably be correct within 0.1 per cent, so that the error in A will be less than 0.3 per cent; but if they are standard resistors having values known within one part in a million, the error in the value of A which is caused by errors in B, C, and D will not be so much as three parts in a million.

In balancing the bridge, the reversing switch S is first closed and the galvanometer circuit is closed momentarily by the tapping key K. The resistance C is adjusted between taps until the bridge is nearly balanced. Then K is kept closed and C is adjusted until the reversal of S produces no permanent deflection of the galvanometer. There are generally some residual inductances in the resistances, so that the galvanometer may receive a slight "kick" on reversal of the current, even after a balance has been obtained. This "kick" must be neglected. The purpose of balancing for reversal of the current is to eliminate the effect of any thermal electromotive forces that may be present in the arms of the bridge or in the galvanometer circuit. The elimination is complete if the thermal electromotive forces remain constant during the two balances. However, it is desirable to keep them small. For this reason the key placed in the galvanometer circuit should be free from thermal electromotive forces. Another useful precaution is to wait several seconds after making an adjustment of C to insure that the thermal electromotive forces at the contacts have become stable.

If A and B are standard resistors of the same nominal value, the value of one in terms of the other can be obtained with high accuracy, even though D and C are not accurately known. To accomplish this, interchange A and B in the bridge and readjust by varying C, taking care not to change the large-valued coils of C, but inserting or removing only small-valued coils. Let the new value of the resistance be $C \pm \Delta C$. Then

$$\frac{A}{B} = \frac{D}{C \pm \Delta C} \tag{2}$$

Multiplying this equation by the one obtained before reversing A and B,

$$\frac{A^2}{B^2} = \frac{CD}{CD \pm D\Delta C} = \frac{1}{1 \pm \frac{\Delta C}{C}} \tag{3}$$

or

$$B = A\sqrt{1 \pm \frac{\Delta C}{C}} = A\left(1 \pm \frac{\Delta C}{2C}\right) \text{ approx.} \tag{4}$$

The approximation is correct to one part in a million if $\Delta C/C < 0.001$ and if both ΔC and C are known to one part in a thousand. From the preceding equation,

$$\frac{B - A}{A} = \pm\frac{\Delta C}{2C} \tag{5}$$

In the above description, the assumption is made that all contact resistances are negligible. In general this is not the case, so that a discussion of methods of eliminating their effect is necessary. All standard resistors are constructed with copper terminals (see Fig. 8, page 60), which, when the resistor is measured, are set into mercury cups in a copper block, both terminals and cups being amalgamated. Since the resistivity of mercury is more than fifty times that of copper, and since that of copper amalgam is considerably more than that of mercury, the resistance of the mercury and amalgam between the two copper pieces may be greater than that of the copper terminals. The thickness of the copper amalgam slowly increases with time, so that occasionally both cups and terminals should be cleaned and reamalgamated. With freshly amalgamated surfaces, the cup resistance should be only 1 or 2 microhms, but with cups of several years standing it may become as much as 20 microhms. It follows that a 10-ohm coil is definite to one part in a million only when the contacts are kept in good condition.

The contacts in the *C* and *D* arms frequently introduce errors. The *D* arm does not need to be known and the only requirement is that the contacts shall remain constant while the two measurements are being made. Since the *C* arm, however, must be adjusted, variations in the contact resistances in this arm may become important. If a plug resistance box is used, there may be an uncertainty of 100 microhms whenever a plug is removed or inserted. Likewise, there is a variability of about 100 microhms in the contact resistance of each dial of a dial resistance box. If *D* is 1000 ohms, then a six-dial resistance box, having ten steps per dial, with the largest dial having 100-ohm steps, would make a convenient resistor for the *C* arm. The variability of the contact resistance in the dials of such a resistor should not be so large as 0.001 ohm, the smallest resistance that can be read.

A better arrangement for the *C* arm is to use two or more fixed resistances in *C*, the smaller ones being shunted by resistance boxes. The effect of contact resistances in the resistance box used as a shunting resistor on the resistance in the *C* arm decreases as the square of the ratio of the shunted resistance to the sum of the shunting and shunted resistances. Hence, by keeping the shunted resistance small relative to the shunting resistance, contact resistances become negligible.

A still better arrangement for the *C* arm is to have a fixed resistance in series with a special resistance box in which the resistance steps of each dial are effected by a change in a shunt resistance. The electrical connections and the values of the various resistances of a box of this type are shown diagrammatically in Fig. 17. As the illustrated box has steps of a ten-thousandth of an ohm, it can be used to measure to one part in a million in an arm of 100 ohms. For comparing two 10-ohm or two 100-ohm coils, greater sensitivity is secured by having the C and D arms 100 ohms each, rather than the 1000 ohms which is necessary if all contacts are in series, as suggested in the first arrangement on page 88. The contact resistances in the illustrated box will not cause an uncertainty of more than 10 microhms in the box resistance.

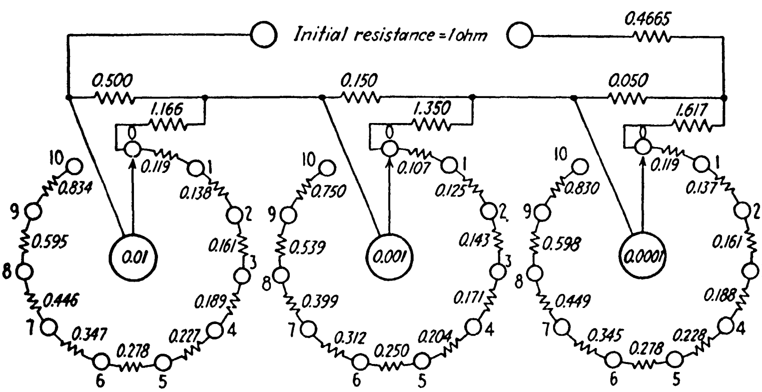

FIG. 17.—Diagram of a resistance box in which the resistance range is from 1.0000 to 1.1110 ohms in steps of 0.0001 ohm. The contact resistance in the dials should not cause an uncertainty of more than 0.00001 ohm in the value of any resistance.

The Wheatstone bridge can be used to determine the value of standards of one decimal value in terms of decimal standards of one order higher or one order lower. As an example, consider the comparison of a 100-ohm standard with a 10-ohm standard. Place the 100-ohm coil, R_{100}, in the A arm and the 10-ohm coil, R_{10}, in the B arm, being careful to keep lead and contact resistances so low that they will not affect the resistance of either arm by one part in a million. The D arm is made about 10 ohms and the C arm is adjusted by means of the illustrated shunt box until the bridge balances. Then $R_{100}/R_{10} = C/D$. To determine C/D with an accuracy of one part in a million, provide ten additional 10-ohm standards, which are numbered from one to ten and designated as R_{10}^{I}, R_{10}^{II}, R_{10}^{III}, . . . R_{10}^{X}. Substitute each coil in turn for R_{10}, and balance by adjusting C by amounts $\Delta_1 C$, $\Delta_2 C$, etc., so that the values of the ten numbered coils are $R_{10}^{\text{I}} = R_{10}\left(1 - \frac{\Delta_1 C}{C}\right)$; $R_{10}^{\text{II}} = R_{10}\left(1 - \frac{\Delta_2 C}{C}\right)$; etc. Then place the R_{10} coil in the B arm and the other ten in series in the A arm. When the bridge is balanced by adjusting the C arm, designate the value of the C arm by $C + \Delta_0 C$. Combining the twelve equations and neglecting second-order terms.

$$\frac{C}{D} = \frac{R_{100}}{R_{10}} = 10 - \frac{10\Delta_0 C + \Delta_1 C + \Delta_2 C + \cdot\cdot\cdot + \Delta_{10} C}{C} \quad (6)$$

If all the ΔC's are small, so that calibration errors are negligible, and if all the comparisons have been made with an accuracy of one part in a million, the ratio of R_{100} to R_{10} and hence of C to D will be known with the same accuracy.

Once the ratio of C to D has been obtained with the required accuracy when the ratio is 10 to 1, the bridge can be used to determine the ratio of a 1000-ohm standard to a 100-ohm standard. As the sensitivity of a bridge decreases when there is a large difference in the resistances of the various arms, the comparison of a 10 000-ohm standard with a 1000-ohm standard cannot be made with high accuracy when the values of C and D are 100 and 10 ohms, respectively. To overcome this difficulty, the 1000- and 100-ohm standards, the ratio of which has been accurately determined, can be used to determine the ratio of a new C and D, which have nominal values of 10 000 and 1000 ohms, respectively. While the method outlined requires only one standard in each decade above the 10-ohm standards where eleven are required, yet at least two standards should be available in each decade to permit of suitable check measurements.

66. Comparison of Resistance Standards of Low Value.—For comparing resistance standards having a value of 1 ohm or less where the resistance of the leads and contacts must be considered, a Thomson double bridge[1] is often employed. If properly adjusted, an accuracy of one part in a million in the comparison of two nearly equal resistances in the range from $1/1000$ to 1 ohm can be attained. The bridge is shown diagrammatically in Fig. 18. The two resistances to be compared are A and B, each of which has current terminals (designated by I with a subscript)

[1] See references in articles by Wenner, *Bull. B. S.*, **8**, 580 (1912) and **11**, 65 (1911).

and potential terminals (designated by P with a subscript). The resistances to be considered are those portions of the resistance material lying between the points where the potential terminals are joined to it. The resistances β and D usually have the same nominal value of 100 ohms or more, while the resistances α and C are adjustable. To facilitate bridge settings, α and C are often mechanically connected together, so that a change in one produces the same nominal change in the other. The small resistance between the terminal of D and the junction of the potential

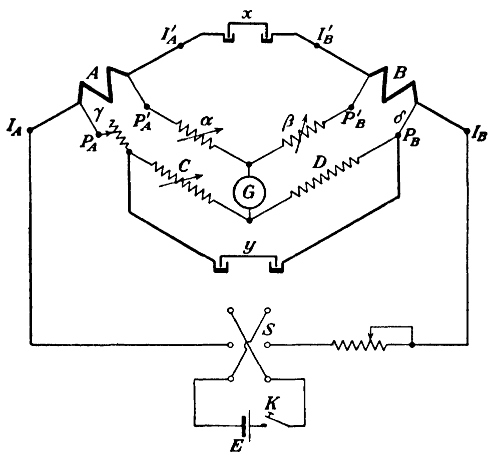

FIG. 18.—Diagram of a Thomson double bridge.

terminal with the resistance material of B is designated by δ, while the similar resistance between C and A is designated by γ. In γ is a low-valued adjustable resistance, the value of which does not need to be known. The resistance of the branch which includes the link x is designated as X and that of the branch including the link y as Y. The adjustments are made by a series of approximations. In the different bridge arrangements, the battery resistance must be adjusted to give a suitable current through the bridge. The different bridge arrangements, which are accomplished by the links at x and y, are as follows:

1. With y out and x in, balance by adjusting C and α, keeping them approximately equal.
2. With both y and x in, balance by adjusting γ only.
3. With both y and x out, balance by adjusting β only. Repeat in this order until the bridge is balanced for all three

arrangements. In every case the final balance is made when using the reversing switch S, thus avoiding the effect of thermal electromotive forces.

The equations which show the relationships of the resistances under the three conditions of balance are:

With both x and y in:

$$\frac{(C + \gamma)Y + \gamma(C + D)}{(D + \delta)Y + \delta(C + D)} = \frac{(A + \alpha)X + A(\alpha + \beta)}{(B + \beta)X + B(\alpha + \beta)} \tag{7}$$

With x in and y out, $Y = \infty$.

$$\frac{C + \gamma}{D + \delta} = \frac{(A + \alpha)X + A(\alpha + \beta)}{(B + \beta)X + B(\alpha + \beta)} \tag{8}$$

With both x and y out, $X = Y = \infty$.

$$\frac{C + \gamma}{D + \delta} = \frac{A + \alpha}{B + \beta} \tag{9}$$

Solving these three equations

$$\frac{A}{B} = \frac{C}{D} \tag{10}$$

If A and B have the same nominal value, then they should be reversed in the bridge and a new set of balances obtained, care being taken that D is not altered and that C is varied only in the smaller dials, as was described when interchanging coils in a Wheatstone bridge. By the new balance, $A/B = D/(C + \Delta C)$, so that, as with the Wheatstone bridge,

$$\frac{A - B}{B} = \frac{\Delta C}{2C} \tag{11}$$

If A and B have the nominal ratio of 1 to 10, the smaller may be replaced by a 10-ohm standard and the larger by a 100-ohm standard, the ratio between these two standards having been determined by means of the method described under the Wheatstone bridge. In these large-valued standards, a single terminal serves for both current terminal and potential terminal. The bridge is adjusted exactly as when A and B were in the bridge, leaving D fixed and adjusting C in the lower dials by an amount $\Delta'C$. If the ratio of the 100-ohm standard to the 10-ohm standard is $10 + \xi$, then

$$\frac{A}{B} = 10 + \xi - \frac{\Delta'C}{D} \tag{12}$$

If the ratio of the large-valued standards is known to one part in a million and if $\Delta' C$ is less than 0.1 per cent of C, while $\Delta' C$ and D are both accurate to 0.1 per cent, then the maximum error in the ratio of A to B is two parts in a million.

While the Thomson bridge as described can theoretically be used for resistances of any value, yet practically it cannot be used for extremely low resistances when the value of the connecting resistance, X, is as large as that of the standards being compared. An accuracy of one part in a million can be attained with resistances of $\frac{1}{1000}$ ohm and larger. There is no object in using the Thomson bridge for standards having a value of 10 ohms and larger since a Wheatstone bridge is simpler and just as accurate. However, the Thomson bridge is required for the comparison of a 10-ohm standard with a 1-ohm standard.

67. Measurement of the Load Coefficient of a Resistor.—The load coefficient of a standard resistor having a value of 10 ohms or more can be measured by means of a Wheatstone bridge, provided two auxiliary standards, as nearly identical as possible with the first, are available. Three separate measurements are required to determine the load coefficient. In these measurements the load on the auxiliary standards is kept constant, while that on the standard is varied.

First the three standards are intercompared by means of a Wheatstone bridge with a definite load, p, in watts on each standard. For convenience in computing the load on each coil, let the electromotive force of the battery of the Wheatstone bridge be represented by $2E$. Also let the battery be so connected that one part of the current from it flows through the A and B arms and the other part through the C and D arms. Then if the resistance in the battery arm of the bridge is negligible, and if the standard (or one of the auxiliaries) is in the A arm and an equal resistance is in the B arm, the power expended in the standard (load on the standard) and on each auxiliary coil is equal to E^2/R watts. Represent the resistance, under this load, of the standard under investigation by R and that of the measured resistance of the two auxiliary resistors by $R + \delta_1$ and $R + \delta_2$.

Next place the standard in the A arm of a Wheatstone bridge, place the two auxiliary resistances *in series* in the B arm, and increase the battery voltage to $3E$. The standard and the two auxiliary resistances now have the same load, p, as when intercompared. To introduce the load coefficient into the equation for this balance of the bridge, let the resistance of the standard at zero load be R_0, so that, with load p, $R = R_0(1 + \lambda p)$, where λ is the load coefficient. When the bridge is balanced,

$$\frac{R}{2R + \delta_1 + \delta_2} = \frac{R_0(1 + \lambda p)}{2R + \delta_1 + \delta_2} = \frac{C}{D} \tag{13}$$

Now put R in the B arm and the two auxiliary standards *in parallel* in the A arm, keeping the battery voltage at $3E$. The load on the auxiliary standards and on the C and D arms is the same as in the previous measurement but on the standard it is now $4p$. The ratio of the A arm to the B arm is nearly the same as in the first arrangement, so that the bridge is balanced by a small adjustment ΔC of C. Then

$$\frac{\frac{1}{4}(2R + \delta_1 + \delta_2)}{R_0(1 + 4\lambda p)} = \frac{C + \Delta C}{D} \tag{14}$$

Dividing this equation by the previous one, inserting for R the value $R_0(1 + \lambda p)$, and solving while neglecting second order terms,

$$3\lambda p = \frac{\delta_1 + \delta_2}{R} - \frac{\Delta C}{C} \tag{15}$$

A determination of the load coefficient of a standard of 100 ohms can be made by using a Wheatstone bridge with unequal arms in conjunction with a 10-ohm coil of known load coefficient. Insert the 100-ohm standard in the A arm of a Wheatstone bridge, with the 10-ohm standard in the B arm. In the C arm place ten 100-ohm coils in series and in the D arm place a similar 100-ohm coil. Balance the bridge by shunting one of the 100-ohm coils with a high resistance. The load on each of the coils in the C and D arms is only one-hundredth of that in the 100-ohm standard in the A arm, so that the change in resistance of C and D arms with load can be neglected. Hence, using two loads, p_1 and p_2, on the 100-ohm standard, and representing the load coefficients of the 100-ohm standard by λ' and those of the 10-ohm standard by λ,

$$\frac{100(1 + \lambda' p_1)}{10\left(1 + \frac{\lambda p_1}{10}\right)} = \frac{C}{D} \quad \text{and} \quad \frac{100(1 + \lambda' p_2)}{10\left(1 + \frac{\lambda p_2}{10}\right)} = \frac{C + \Delta C}{D} \tag{16}$$

Solving,

$$\lambda' = \frac{\lambda}{10} + \frac{\Delta C}{C(p_2 - p_1)} \tag{17}$$

Similar methods can be applied to standards of still larger value.

No entirely satisfactory method of determining the load coefficient of a standard having a resistance of 1 ohm or less has been developed. If the load coefficient of a 1-ohm standard is desired, and if a $\frac{1}{10}$-ohm standard is available, having approximately the same size and type of construction and approximately the same temperature coefficient as the 1-ohm standard, then the load coefficients of the two standards are approximately the same. These two standards can be compared in a Thomson bridge at two different loads, the bridge arrangement being such that the loads on the α, β, C, and D arms is so small as not to affect their resistances. The resulting bridge equations are the same as in the Wheatstone bridge of the preceding paragraph. Hence, since $\lambda' = \lambda$,

$$\lambda = \frac{10\Delta C}{9C(p_2 - p_1)} \tag{18}$$

Another method of determining the load coefficient of a low-valued standard is to find a second standard of the same denomination, which has a different temperature coefficient from the first standard but which is identical in construction with the first standard. The load coefficients may be assumed proportional to the temperature coefficients at the temperature of measurement so that $\alpha_1/\alpha_2 = \lambda_1/\lambda_2$. Comparing the two standards in a Thomson bridge at two loads, p_1 and p_2, the result is the same as for the Wheatstone bridge so that

$$\lambda_1 - \lambda_2 = \frac{\Delta C}{C(p_2 - p_1)} \tag{19}$$

But from the above relationship,

$$\lambda_1 - \lambda_2 = \lambda_1 \times \frac{\alpha_1 - \alpha_2}{\alpha_1} \tag{20}$$

Hence

$$\lambda_1 = \frac{\alpha_1}{\alpha_1 - \alpha_2} \times \frac{\Delta C}{C(p_2 - p_1)} \tag{21}$$

The accuracy of this method is greatest when the value of α_2 is very near zero.

68. Comparison of the Electromotive Forces of Standard Cells. The comparison of two nearly equal electromotive forces is usually made by balancing the difference between the two electromotive forces against the drop in potential of a known current over a known resistance. Several types of potentiometers for making comparisons of electromotive forces are available, but in general these instruments are more elaborate than is necessary for the comparison of standard cells, and they do not always give the required accuracy. Two comparatively simple methods will be described which will permit the comparison of two similar standard cells with an accuracy of one part in a million.

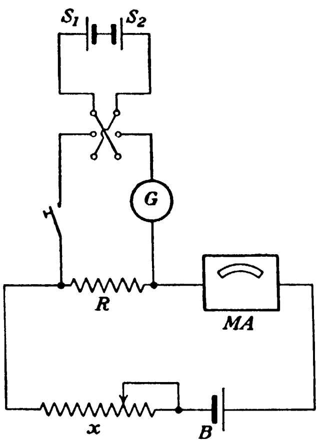

FIG. 19a.—Diagram of the milliameter method for measuring the difference in the electromotive forces of two similar standard cells.

A diagram for comparing two cells S_1 and S_2 when their electromotive forces differ by less than 100 microvolts is shown in Fig. 19a. The current from the battery B is regulated by the resistance x and measured by the milliammeter MA. The standard cells S_1 and S_2 with their electromotive forces opposed are connected to the potential terminals of the standard resistor R having a resistance of 0.001 ohm. The current is adjusted by the resistance x until there is no deflection of the galvanometer G. Then if the reading of the milliammeter is I,

$$E_1 - E_2 = IR \tag{22}$$

where E_1 and E_2 are the electromotive forces of the standard cells S_1 and S_2. If the value of $E_1 - E_2$ is less than 100 microvolts, this method will give results accurate to 1 microvolt, since R is easily maintained with much higher accuracy than required, and the readings of a good milliammeter will not be in error by 1 per cent. It is very rapid in operation, but only the differences are obtained. A numerical addition or subtraction is required in reducing the data. The principal sources of error are the thermal electromotive forces in the standard cell circuit.

The principle underlying the second method differs from that of the preceding in that the current is maintained constant and the resistance varied to obtain a balance. However, resistances of low value which can be varied by definite amounts are not available. To avoid this difficulty and to obtain potential drops as low as 1 microvolt, two circuits are arranged from one battery

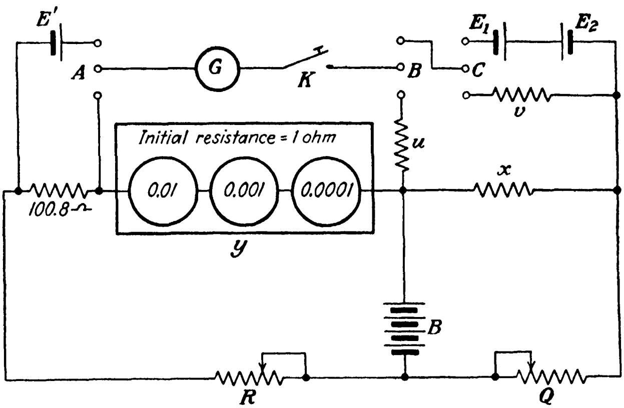

FIG. 19b.—Diagram of a potentiometer method of measuring the difference in the electromotive forces of two similar standard cells.

and, where the currents separate, a resistance is inserted in each circuit so that the difference between the two potential drops is used in balancing against the difference in electromotive forces of the two cells. The current in one circuit is maintained at a definite value by making the potential drop over a known resistance equal to the electromotive force of a standard cell.

The setup for this method, which is shown in Fig. 19b, is suggested for standard cell measurements as being somewhat simpler than standard potentiometers which embody the same principles but which are designed for more general uses. The diagram indicates a number of separate pieces of apparatus, but many of these can be collected into a single instrument if desired. Especial attention must be given to the galvanometer, which should have a current sensitivity as great as 10^{-9} ampere per milliradian and a critical damping resistance of about 1000 ohms.

A storage battery B having an electromotive force of 10 or more volts produces current in two circuits, one in the lower left of the diagram and the other in the lower right. The current in the left circuit is adjusted by the resistance R and is maintained at 0.01 ampere by balancing the potential drop in a resistance of 101.8 ohms against the electromotive force of a Weston cell E'. The current in the right circuit is adjusted by the resistance Q and is maintained at such a value that the potential drop over the resistance x is just equal to that over the resistance box y when all dials are set at zero. This resistance box y is of the shunt dial type shown in Fig. 17 and has an initial resistance of 1 ohm with dials which increase this resistance in steps of 0.01, 0.001, and 0.0001 ohm. The difference in the electromotive forces of E_1 and E_2 is balanced against the difference in the potential drops over x and y, the necessary adjustment being made by varying y. The switches A, B, and C are arranged to permit a single galvanometer to be used in making all the necessary adjustments. The resistances u and v are provided so that the proper damping resistance can be introduced into the galvanometer circuit.

Three independent adjustments are required in making a comparison.

1. The first adjustment is for the purpose of making the current in the left circuit 0.01 ampere. Hence the switch A is up, B down, and C open, so that the galvanometer G is in series with the standard cell E', which has an electromotive force of 1.018 + volts (a Weston saturated cell). This electromotive force is balanced against the potential drop in the resistance by adjusting R until the galvanometer shows no deflection on closing the key K. The current through the left arm will then be $\frac{1}{100}$ ampere. The current must be adjusted to this value with an accuracy of one part in a thousand. Since B is 10 volts or more, $\frac{1}{10}$ ohm steps in R will suffice to make the adjustment with the required accuracy.

2. The second adjustment is for the purpose of making the potential drop across x equal and in opposite direction to that across y when the dials are on zero. Hence the switch A is down, B up, and C down, so that the galvanometer is connected across the resistances x and y. With the dials of y on zero, the resistance of Q is adjusted until there is no deflection of the galvanometer on closing the galvanometer key. Since x is

approximately 1 ohm, the same as the zero resistance of y, Q must be adjustable with the same accuracy as R.

3. The third adjustment is to balance the difference in the electromotive forces of E_1 and E_2 against the difference in the potential drops over x and y. Hence A is down, B up, and C up, so that the standard cells, connected to oppose each other, are connected across the resistances x and y. It is necessary that the cell with the larger electromotive force be in the position indicated by E_1. The shunt-dial resistor y is adjusted until there is no deflection of the galvanometer on closing the key. Since the current in the left-hand arm is not changed from 0.01 ampere by as much as one part in a thousand by adjusting the dials of y, $E_1 - E_2 = 0.01D$, where D is the reading of the dials of the shunt resistor. Since the maximum value of D is 0.111 ohm, $E_1 - E_2$ cannot be measured if it is greater than 1110 microvolts.

The two sources of error that may enter in a measurement by this method are thermal electromotive forces in the standard cell circuit and lack of steadiness of the current in the two arms. The thermal electromotive forces are from two sources: first, those which result from some temperature gradient in the apparatus and which are relatively stable; second, those which are caused by the frictional heating of the switches and dials as they are operated. The first, so long as they remain constant, are completely eliminated by the procedure just outlined. The second source is unimportant since an electromotive force in any one of the dials of the shunt-dial resistor produces a current through the circuit within the resistor of which it is a part. The electromotive force, as measured between the terminals of the resistor, is determined by the fall of potential of this current over the shunted resistance. If the value of the shunted resistance is r and that of the shunt resistance is s, and if an electromotive force e is present in the shunt resistance, then the electromotive force at the terminals of the resistor is $er/(r + s)$. Now the thermal electromotive force from a pair of well-designed contacts is seldom more than 1 microvolt. Using this electromotive force and the resistance of the shunt-dial resistor given in Fig. 17, the maximum value of the computed electromotive force at the terminals of the resistor is about 0.5 microvolt.

While the experimental procedure eliminates all constant electromotive forces in the galvanometer circuit, yet good design requires the materials and construction to be such that the thermal electromotive forces are a minimum. All connecting wires and binding posts, as well as all conducting parts of the galvanometer, should be of pure copper. The key and switches should be of copper, except the contacts of the key, which, if small, may be of silver, gold, or platinum. The resistance materials in x and y should be of manganin or other alloy having a small thermoelectric power with respect to copper. The two ends of each resistance coil should be hard-soldered to copper wires, and the two junctions should be kept close together so that their temperatures will be the same.

The second source of error is a lack of steadiness of the current in either of the two arms. This steadiness depends on the constancy not only of the electromotive force of B but also of the resistances. However, as variations in current of one part in ten thousand will not affect the final result by a microvolt, the required steadiness is readily attained. With such large variations in current permissible, any storage battery which is in good condition will have a sufficiently steady electromotive force. Resistance boxes using manganin as the resistance material are sufficiently stable for use in R and Q. The change in current caused by operating the dials of the shunt-dial box is negligible.

With the apparatus as described and reasonable precautions in operation, the difference in electromotive forces of two standard cells can be determined to 1 microvolt. Such accuracy in intercomparison is useful only when the cells are continuously maintained at a definite temperature. As the temperature coefficient of a saturated cadmium cell is about forty parts in a million per degree, the temperature should be controlled to 0.01°C.

69. Comparison of Currents.—The comparison of two currents is usually effected by determining the fall in potential over known resistances which are inserted in the circuits. The fall in potential over each resistance may be measured by means of a potentiometer, in which case $I_1/I_2 = E_1R_2/E_2R_1$. If this comparison is to be made to one part in a million, the resistances must be known and the potential drop measured to that accuracy. As measurements with a potentiometer to one part in a million are extremely difficult, a direct comparison of the potential drop over known resistances is sometimes desirable.

A setup for the comparison of the currents of two circuits which are so well insulated that they can be connected together at one point is shown in Fig. 20. In the upper circuit there is a standard resistance, R_2, with two potential terminals, while the lower resistance has three potential terminals so arranged that R_1 is at least 1000 times greater than r. Across r is connected a high-resistance slide wire having 1000 divisions, the connecting resistances from the resistance material in the standard to the effective end of the slide wire being equal to a and b

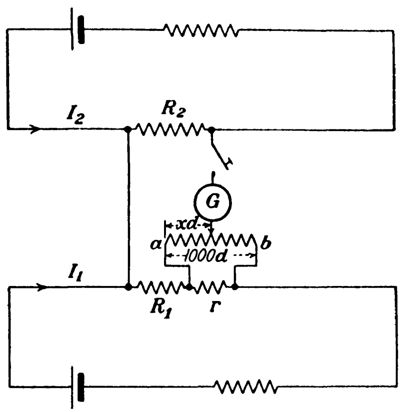

FIG. 20.—Diagram of a setup for the comparison of two currents.

divisions of the slide wire, respectively. One potential terminal of R_2 is connected to a corresponding potential terminal of R_1. If the circuits are well insulated, there will be no current in the connecting conductor. The second potential terminal of R_2 is connected through a suitable galvanometer to the contact on the slide wire. When the slide-wire contact is adjusted until no current flows through the galvanometer,

$$\frac{I_2}{I_1} = \frac{R_1}{R_2} + \frac{r(x + a)}{R\left(1000 + a + b + \frac{r}{z}\right)} \qquad (23)$$

where x is the reading of the slide wire and z is the resistance of one division of the slide wire. If the resistances can be intercompared with sufficient accuracy and if there are no thermoelectromotive forces in the galvanometer circuit, the ratio of the currents can be determined to one part in a million.

As an example, consider that I_2, being about 2 amperes' is a little more than twice the value of I_1. Choose a suitable 1-ohm standard for R_2 and a 2-ohm standard for R_1. There should be four potential terminals for R_1, so that it is equivalent to two 1-ohm coils in series with a $\frac{1}{1000}$-ohm section for r. Both R_1 and R_2 should be mounted in an oil bath, the temperature of which is controlled by a suitable thermostat; otherwise the temperature coefficient of both must be known, and the temperature of the oil must be frequently measured so that the resistance can be reduced to a common temperature. Also the load coefficient must be known or shown to be negligible. The resistance of each of the 1-ohm sections of R_1 can be compared with R_2 with an accuracy of one part in a million by means of a Thomson bridge. If r is $\frac{1}{1000}$ ohm, its value is required only with an accuracy of one part in a thousand, which can readily be determined by a Thomson bridge, or by numerous other methods.

The galvanometer must have a high voltage sensitivity and a low external critical damping resistance. If the damping resistance is greater than 3 ohms, the damping should be made critical by means of a copper resistance in series with the galvanometer.

The effect of thermoelectromotive forces in the galvanometer circuit is not so easily eliminated. All parts of the circuit should either be copper or an alloy having a low thermoelectric power against copper. The most suitable alloy for the resistance standards and slide wire is manganin; all other parts, including binding posts, keys, etc., should be copper. Even then, if the contact rests on the slide wire while adjustments are being made, a temperature gradient may be established, which disappears in less than 20 seconds after the motion ceases. If possible, the currents in both circuits should be reversed, so that a mean of the two readings will eliminate the effect of any permanent thermoelectromotive force. With care, thermoelectromotive forces may be reduced to 1 microvolt or less.

70. Comparison of Capacitances.—The comparison of capacitances is generally accomplished by the use of an alternating electromotive force. The method to be employed in any case will depend to some extent on the frequency of the alternating current and the values of the capacitances to be compared. This discussion will be limited to frequencies below 3000 cycles per second and to capacitances greater than 0.001 microfarad. Also the capacitor will be supposed to have, at all frequencies, a power factor of less than 6×10^{-4} (phase difference less than 2′), which will be the case with a mica condenser of good quality.

In Fig. 21 is a diagram of an alternating-current bridge suitable for comparing an unknown capacitance C_X having a power factor $\cos \theta_X$ with a standard capacitance C_S of approximately the same capacitance as C_X and a power factor $\cos \theta_S$. The standard C_S is inserted in the upper right-hand arm; C_3 is chosen to have the same nominal value as C_S; R_1 is set at some convenient value;[1] C_1, a calibrated air capacitor having a maximum capacitance of about 1000 micromicrofarads, is set near the middle of its range; and the bridge is balanced by adjusting R_2 and C_2. The switch S is now thrown to connect the detecting instrument to the grounded arm, and Q_1 and K_1 (or Q_2 and K_2), having values of the same order of magnitude as R_1 and C_1, are adjusted for zero current through the detecting instrument. The galvanometer is again connected to the bridge, which is again balanced. This process is repeated until the detecting instrument shows no current for either position of the switch S. The unknown capacitance C_X is now inserted in place of C_S, and the bridge and grounding arms are balanced, but this time by adjusting R_1 and C_1 so that, as pointed out in the footnote, the computations will be simplified.

The solution of the bridge equations is simplified by the use of the notation of imaginary numbers, in which the impedance of a condenser is represented by the expression

$$\frac{(\cos \theta - j)}{\omega C}$$

[1] The most desirable resistance value for R_1, and hence for R_2, will depend on the frequency. For low frequencies the resistance should be relatively high, between 1000 and 10 000 ohms, so that the necessary phase-angle adjustment can be made with the variable air condenser C_1. For higher frequencies, the resistance of R_1 should be lower, between 100 and 1000 ohms, so that an excessively precise setting of C_1 will not be required. In any case the computation of results will be simplified if the numeric which represents the resistance of R_1 bears a decimal relationship to the numeric which represents the capacitance of C_S; *e.g.*, if $C_S = 0.1002456$ microfarad, let $R_1 = 100.2456$ ohms. Then, if the procedure is followed as indicated, the value of C_X can be deduced from the corresponding reading of R_1 without any numerical calculation.

where $\cos\theta$ = power factor.

$j = \sqrt{-1}$.

ω = angular velocity of the alternating electromotive force (2π times the frequency).

C = capacitance.

Let the effective values of the alternating currents in the three left-hand branches be represented by I_1, I_2, and I_3, respectively, and those in the three right-hand branches by I_4, I_5, and I_6

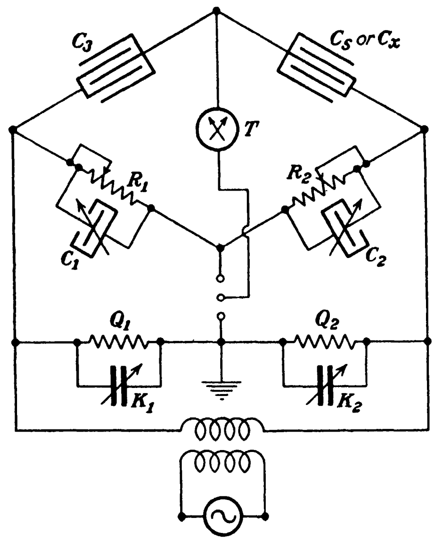

FIG. 21.—Diagram of a setup for the comparison of two capacitances.

Then
$$I_1\left(\frac{-j}{\omega C_1}\right) = I_2R_1 = I_3\frac{(\cos\theta_3 - j)}{\omega C_3} \qquad (24)$$

and
$$I_4\left(\frac{-j}{\omega C_2}\right) = I_5R_2 = I_6\frac{(\cos\theta_S - j)}{\omega C_S} \qquad (25)$$

Also
$$I_1 + I_2 = I_4 + I_5 \qquad \text{and} \qquad I_3 = I_6$$

By determining the values of I_1, I_2, I_4, and I_5 in terms of I_3 and I_6 from Eqs. (24) and (25) and substituting in the equations for the currents,

$$\left(\frac{1}{R_1} + j\omega C_1\right)\left(\frac{\cos\theta_3 - j}{C_3}\right) = \left(\frac{1}{R_2} + j\omega C_2\right)\left(\frac{\cos\theta_S - j}{C_S}\right) \qquad (26)$$

Separating into the real and imaginary parts, and solving for C_S and $\cos\theta_S$,

$$C_S = \frac{C_3R_1}{R_2} \times \frac{1 - \omega C_2R_2 \cos \theta_S}{1 - \omega C_1R_1 \cos \theta_3} \tag{27}$$

$$\cos \theta_S = \cos \theta_3 + \omega(R_1C_1 - R_2C_2) \tag{28}$$

Small terms are neglected in Eq. (28).

When the bridge is balanced with the capacitor C_X inserted in place of C_S, the value of R_1 becomes R_1', that of C_1 becomes C_1', and the equations for capacitance C_X and power factor $\cos \theta_X$ are:

$$C_X = \frac{C_3R_1'}{R_2} \times \frac{1 - \omega C_2R_2 \cos \theta_X}{1 - \omega C_1'R_1' \cos \theta_3} \tag{29}$$

$$\cos \theta_X = \cos \theta_3 + \omega(R_1'C_1' - R_2C_2) \tag{30}$$

Combining Eqs. (27) and (29) and neglecting second order terms

$$C_X = C_S\frac{R_1'}{R_1} \times \frac{1 - \omega(C_1R_1 \cos \theta_3 + C_2R_2 \cos \theta_X)}{1 - \omega(C_1'R_1' \cos \theta_3 + C_2R_2 \cos \theta_S)} \tag{31}$$

Also, by combining Eqs. (28) and (30),

$$\cos \theta_X = \cos \theta_S + \omega(R_1'C_1' - R_1C_1) \tag{32}$$

For use in computation, the following approximate formula for C_X is more convenient and sufficiently accurate, *viz.*,

$$C_X = \frac{C_SR_1'}{R_1}[1 + \omega(C_1'R_1' - C_1R_1)(\cos \theta_S - \omega C_1R_1)] \tag{33}$$

This formula assumes that the impedance of the capacitor is represented as a capacitance in series with a resistance.

The detecting instrument for the capacitance bridge represented in Fig. 21 should be a vibration galvanometer for frequencies below 500 cycles, but for the higher frequencies a telephone receiver may be used. In either case an amplifier may be employed provided sufficient care is exercised in its design. The alternating electromotive force impressed on the bridge should be approximately a sine wave, but a few per cent of harmonics will not appreciably affect the measured value of the capacitance or power factor.

The substitution method as outlined does not require that the resistances be calibrated with high accuracy. However, the adjustment of R_1 must be accomplished by changing the dials of low value, without changing the main portion of the resistance.

Also the capacitance of the leads to C_X and C_S should be kept constant, so that no correction on this account is necessary.

A capacitor does not have a definite capacitance unless it is surrounded by a metallic shield which is connected to one terminal. In the diagram all the shields are shown connected to the terminals nearest to the galvanometer connection. The resistances do not need to be shielded unless great accuracy is required in the power-factor measurement.

Two capacitances having a value of 1 microfarad can be compared at frequencies of 60 and 1000 cycles per second with an accuracy of one part in a million. However, capacitors are influenced by so many external conditions that comparisons with the highest accuracy are seldom of any importance. As an example, a change in atmospheric pressure of 1 cm of mercury changes the capacitance of many capacitors by twenty parts in a million.

71. Comparison of Self Inductances.—The comparison of two self inductances is not readily accomplished by a direct comparison in an alternating-current bridge, as the balancing of such a bridge is difficult unless a variable self inductance is employed. A satisfactory method for making a comparison of self inductances is to measure first the one and then the other in terms of resistance and capacitance in an alternating-current bridge. A suitable bridge will be described later when discussing the absolute measurement of inductance.

72. Comparison of Mutual Inductances.—The precise comparison of mutual inductances is of value only when the comparison can be made under the same conditions as those under which the unknown instrument will be used. The most common method is to use alternating current of a given frequency and to connect a terminal of the secondary to a terminal of the primary. In this case, the potential difference between the terminals of the secondary, which serves to define the mutual inductance, includes not only that resulting from electromagnetic induction but also that produced by the potential drop of the current which flows in the secondary as a result of the capacitance between the coils. The effect of this capacitance current will be discussed in more detail in a later chapter. For the comparison of mutual inductances, the assumption is made that, for a given connection between the primary and secondary, the instantaneous potential difference, e_s, at the terminals of the secondary is given by the equation[1]

$$e_s = i_p[j\omega(M + \omega^2\tau) + \omega^2\sigma] \tag{34}$$

where i_p = the instantaneous value of the sinusoidal alternating current in the primary.

ω = the angular velocity of the alternating current = 2π times frequency.

M = the effective mutual inductance.

τ = a constant which determines the increase of mutual inductance with frequency.

σ = a constant which, when multiplied by the square of the angular velocity, indicates the component of the potential difference that is in phase with the current.

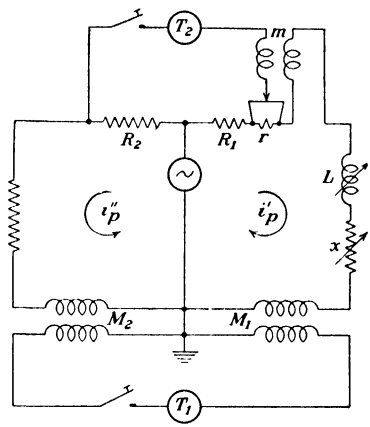

FIG. 22.—Diagram of a setup for the comparison of two mutual inductances.

The diagram of connections for a method of comparing two mutual inductances is shown in Fig. 22. The secondaries of the two mutual inductances are connected in series with a vibration galvanometer T_1 (or telephone) while the primaries are in parallel and energized by a common alternating electromotive force. The current in one of the primaries is varied in magnitude and phase by means of the variable resistance x and variable self inductance L, until there is no current in the galvanometer T_1. Then the two currents are compared by balancing the potential drop produced by I_1 in R_1 against the potential drop produced by I_2 in R_2, the resistance balance being obtained by a contact on a slide wire shunting a portion of R_1, and the phase being adjusted by means of the small mutual inductance, m. The method of comparing the currents is the same as described in Sec. 69, except

[1] The effect of distributed capacitance in a mutual inductance has been investigated by Butterworth, *N.P.L. Coll. Researches*, **18**, 41, or *Proc. Phy. Soc. London*, **33**, 312 (1921).

that in this case alternating currents are compared so that the variable mutual inductance m must be introduced to correct for any difference in phase between the currents.

The following nomenclature, in addition to that given on the preceding page is employed in developing the equations for comparing the constants of the two coils:

M_1 and M_2 = mutual inductances of the first and second coils, respectively.

σ_1 and σ_2 = in-phase constants of the first and second coils, respectively.

τ_1 and τ_2 = frequency coefficients of M_1 and M_2, respectively.

i_p' and i_p'' = currents in the primaries of coils 1 and 2, respectively.

e_s' and e_s'' = potential differences at the terminals of the secondaries of coils 1 and 2, respectively.

Then, since $e_s' = e_s''$ when the current in T_1 is zero,

$$i_p'[\omega^2\sigma_1 + j\omega(M_1 + \omega^2\tau_1)] = i_p''[\omega^2\sigma_2 + j\omega(M_2 + \omega^2\tau_2)] \quad (35)$$

When there is no current in T_2,

$$i_p'[R_1 + j\omega(l_1 + m)] = i_p''[R_2 + j\omega l_2] \quad (36)$$

where l_1 and l_2 are the residual inductances of R_1 and R_2, respectively. Dividing (35) by (36), simplifying and separating into real and imaginary parts,

Real: $R_1\sigma_2 - (l_1 + m)(M_2 + \omega^2\tau_2) = R_2\sigma_1 - l_2(M_1 + \omega^2\tau_1)$ (37)

Imaginary:

$$R_1(M_2 + \omega^2\tau_2) + \omega^2\sigma_2(l_1 + m) = R_2(M_1 + \omega^2\tau_1) + \omega^2\sigma_1 l_2 \quad (38)$$

From Eq. (37),

$$\frac{\sigma_2}{\sigma_1} = \frac{R_2}{R_1} + \frac{(l_1 + m)(M_2 + \omega^2\tau_2) - l_2(M_1 + \omega^2\tau_1)}{\sigma_1 R_1} \quad (39)$$

From Eq. (38),

$$\frac{M_2}{M_1} = \frac{R_2}{R_1} + \frac{\omega^2[R_2\tau_1 - R_1\tau_2 + \sigma_1 l_2 - \sigma_2(l_1 + m)]}{R_1 M_1} \quad (40)$$

The terms in ω^2 are negligible if the frequency is sufficiently low. With a low frequency the ratio of the mutual inductances is equal to the ratio of the resistances. However, the ratio of the sigmas can be determined only if the residual inductances in the resistors are also known. For determining τ_2, a second set of observations at a higher frequency is required.

The method has the following advantages:

1. Each balance (x and L or r and m) can be made accurately and rapidly since in each case the adjustments for magnitude and phase are independent of each other.

2. The two balances are nearly independent of the frequency, so that harmonics in the electromotive force of the generator have but slight effect on the result.

3. The ratio of the mutual inductances is equal to the ratio of the resistances of two resistors which can be constructed of a resistance alloy and hence can be accurately intercompared.

Some of the disadvantages are:

1. The detectors are not at earth potential.

2. There are four inductances which must be arranged so that there is no electromagnetic induction between any two of them and so that the capacitance and inductance of their leads does not influence the comparison.

The method has not had sufficient use to determine the accuracy that can be obtained with it. Probably with a low frequency and great care in locating the leads, an accuracy of one part in a million can be attained.

CHAPTER XII

MEASUREMENTS INVOLVING TWO OR MORE ELECTRICAL QUANTITIES

Some electrical measurements involve the measurement of one electrical quantity in terms of two or more other electrical quantities. In some cases such a measurement is required, since a standard of the quantity to be measured cannot be maintained; in other cases a measurement of this type is used to establish a standard of an electrical quantity. The most important cases under this type are: a current in terms of resistance and electromotive force; a quantity of electricity in terms of capacitance and electromotive force; and an inductance in terms of capacitance and resistance. A single method for each case will be described, giving particular attention to the precautions that must be observed to obtain high accuracy.

73. Measurement of Current in Terms of Resistance and Electromotive Force.—The precise measurement of an unvarying current is usually made by comparing the difference in potential between two terminals of a resistor in which there is the unknown current with the electromotive force of one or more standard cells. To increase the range of currents that can be measured, a potentiometer is often used to measure the potential difference across the resistance, but this decreases the accuracy, since the calibration of a potentiometer is subject to some uncertainties. The method to be described is capable of measuring a current to one part in a million provided the current is sufficiently steady and the values of the standard cell and of the standard resistor are known with that accuracy.

The most suitable source of steady current is a lead storage battery. The battery must have a very large current capacity relative to the current to be drawn from it. The rated capacity of the battery in ampere-hours should be at least 50 times the number of amperes to be measured. The battery should be well insulated from ground, and should not be subject to rapid temperature variations. The setup for measuring a current I

in terms of a resistance and electromotive force is shown in Fig. 23. The method has many things in common with the method for comparing two currents (see page 101). It is suitable only for determining those values of the current for which standard resistors and standard cells can be employed. A resistance standard is divided into two parts, R and r, such that R is very nearly equal to an ohm or to some multiple or submultiple of an ohm, while r is about one-thousandth of R. A slide wire having 1000 divisions and a resistance of at least $10R$ is connected in parallel with r. The connecting resistances between the ends of the slide wire and the junctions of the potential terminals of r with the resistance material are equal to a and b slide-wire divisions, respectively. Then when the slide-wire setting x is such that there is no deflection of the galvanometer, the current can be computed by the following formula:

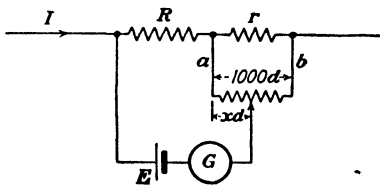

FIG. 23.—Diagram of a setup to measure a current in terms of resistance and electromotive force.

$$I = \frac{E}{R}\left(1 - \frac{r}{R} \times \frac{x + a}{1000 + a + b}\right) \tag{1}$$

This formula is not exact, but the neglected terms do not introduce an error of one part in a million.

To determine I to one part in a million, both E and R must be known with that accuracy in terms of the units of the system in which the measurements are being made. However, the other resistances need only be known to one part in a thousand.

A source of error is the thermoelectromotive forces that may be present in the circuit. The methods for minimizing the effect of such electromotive forces were discussed in describing the method for comparing two currents.

74. Measurement of Quantity of Electricity in Terms of Capacitance and Electromotive Force.—A definite quantity of electricity is obtained by charging a standard air capacitor with a standard cell. The air capacitor must have a high insulation resistance and must be practically free from absorption. If the capacitance of the capacitor is not more than a few microfarads, a single charge drawn from the standard cell will not affect its electromotive force. In case the capacitor is to be frequently charged, or if it is to be charged to a high voltage, the setup shown in Fig. 24 can be used. A battery B having a constant electromotive force causes a current I to flow through the standard resistors R_1 and R_2. The current is adjusted until there is no current through the galvanometer which is in the standard cell circuit. Then $E = IR_1$, where E is the electromotive force of the standard cell. But the potential to which the condenser is charged is IR_2. Hence the quantity Q on the condenser is given by the equation

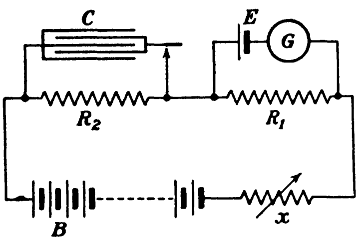

FIG. 24.—Diagram of a setup for determining a quantity of electricity in terms of capacitance and electromotive force.

$$Q = IR_2C = \frac{ECR_2}{R_1} \qquad (2)$$

Since R_2 and R_1 are standard resistors, their ratio can be determined with high accuracy. The value of E can also be accurately determined. Hence the accuracy with which Q can be measured will largely depend on the accuracy with which C can be determined. This will be discussed in a later chapter.

A possible source of error is leakage in the capacitor. As soon as the capacitor is disconnected from the battery, the current flowing through the insulation decreases the charge on the capacitor. If the capacitor has a capacitance of C microfarads and an insulation resistance of $\bar{R}$ megohms, the time t in seconds in which one-millionth of the charge will be neutralized is given by the equation,

$$t = \bar{R}C \times 10^{-6} \qquad (3)$$

Even with very good capacitors, the time t is measured in milliseconds.

Another possible cause of error is thermoelectromotive forces which may be present either in the standard cell circuit or in the capacitor circuit. The thermoelectromotive forces in the capacitor circuit will be negligible if the potential applied to the capacitor is 10 volts or more, while the effect of those in the standard cell circuit can be minimized by the methods already discussed in describing the method for comparing two currents.

75. Measurement of Self Inductance in Terms of Resistance and Capacitance.—A self inductance can be compared with resistance and capacitance by any type of changing current, the most satisfactory type being a sinusoidal alternating current having a frequency less than 3000 cycles per second. Several bridge methods for making this comparison are possible, one of the simplest having been proposed by Maxwell,[1] but adapted by Wien[2] for use with alternating current.

The diagram for a Maxwell-Wien alternating-current bridge is shown in Fig. 25. The inductance L is in one arm of a bridge, the resistance of this arm being Q. The other arms have resistances P, R, and S and inductances l_p, l_r, and l_s, respectively. Also the inductance of the part of the Q arm not included in the inductor is l_q. The R arm is shunted by a capacitance C, a part of which is continuously variable. A grounded third arm permits the galvanometer to be maintained at earth potential.

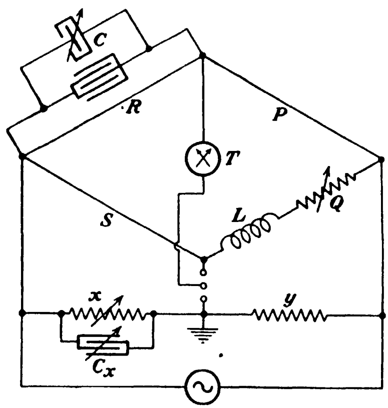

FIG. 25.—Diagram of a Maxwell-Wien alternating-current bridge.

In choosing the values of the resistances and capacitance for measuring a given inductance, the approximate equations for a balance, *viz.*, $L = CPS$ and $PS = RQ$, are sufficiently accurate to serve as guides. Moreover, the best sensitivity will be obtained if $P = S$ and if C is as large as possible. Also there is some gain in sensitivity by making R large and Q small, but such an arrangement of resistances has the disadvantage of requiring that the small resistance Q, which will be mostly of copper having a large temperature coefficient, must be adjusted and maintained with extreme precision. As some correction terms can be more easily eliminated by making $P = R$, the most common arrangement of resistances is to have all arms of equal resistance, *i.e.*, $P = Q = R = S$. Hence, in selecting the elements for the bridge arms, make the capacitance C as large as convenient and have the resistance of each arm as nearly as possible equal to $\sqrt{L/C}$.

[1] "Electricity and Magnetism," 3d ed., Art. 778.

[2] *Ann. Phys.*, **44**, 711 (1891).

The detecting instrument T for a Maxwell-Wien bridge is either a vibration galvanometer or a telephone receiver. The vibration galvanometer is invariably used for frequencies below 500 cycles per second, and may be used up to 1000 cycles per second, or even somewhat higher, though the sensitivity decreases as the frequency increases. The telephone is satisfactory as a detecting instrument between 500 and 3000 cycles per second.

The source of alternating current must give a nearly pure sine-wave electromotive force and must be capable of supplying several watts of power at a voltage suitable for application to the bridge. The commercial electrical power supply is a suitable source for one frequency provided it is maintained at constant frequency, as for operating synchronous clocks. A vacuum-tube oscillator is a convenient source of power for the higher frequencies, but the current should be passed through a wave filter to ensure a sufficiently pure sine-wave current.

The bridge is balanced by adjusting Q and C. When a preliminary balance has been obtained, the detecting instrument is connected to the third arm, and the resistance and capacitance of this arm are adjusted for balance. The detecting instrument is again connected to the bridge and a second balance is made. The detecting instrument is in this way alternated between the bridge and third arm, until the adjustments are such that there is no current through the detecting instrument with either connection. The equations which represent the relations between the quantities when the bridge is balanced include not only the resistances P, Q, R, and S, the capacitance C, and the inductance L, but also the residual inductances of the resistances, l_p, l_r, l_s, and l_q (l_q is the inductance of that part of the Q arm outside the inductance L) and the power factor, $\cos\theta$, of the condenser C. The equations are

$$\begin{aligned}(1 + \cos^2\theta)\{QR - PS + \omega^2[l_pl_s - l_r(L + l_q)]\}\\ = \omega CPSR\cos\theta - \omega^2C(PSl_r + RSl_p + RPl_s)\\ - \omega^3C(Rl_pl_s + Sl_pl_r + Pl_sl_r)\cos\theta + \omega^4Cl_rl_pl_s \quad (4)\end{aligned}$$

and

$$\begin{aligned}(1 + \cos^2\theta)[R(L + l_q) + Ql_r - Sl_p - Pl_s]\\ = C[PRS + \omega(SRl_p + PRl_s + PSl_r)\cos\theta\\ - \omega^2(Rl_pl_s + Sl_pl_r + Pl_sl_r) - \omega^3l_pl_rl_s\cos\theta] \quad (5)\end{aligned}$$

Only the second of these equations is used in computing the inductance. In this equation several of the terms are negligibly small. Air capacitors and good mica capacitors have a power factor less than 10^{-3}, so that $\cos^2 \theta$ is negligible relative to unity. The time constant (ratio of L/R) of each of the resistors, if the resistances are 1000 ohms or less, will usually be less than 10^{-6} second. If ω is less than 1000 (frequency less than 150 cycles per second), the reactance of each of the resistors is less than one-thousandth of its resistance, so that, if $\omega l_p = \alpha P$, $\omega l_r = \beta R$, and $\omega l_s = \gamma S$, α, β, and γ are each less than 10^{-3}. Substituting these values for ωl_p, etc., in the equation for inductance, PRS becomes a factor of each term in the square brackets on the right-hand side of the equation. When this factor is taken outside the brackets, the sum of the terms remaining inside differs from unity by less than three parts in a million under the conditions assumed above. Hence the inductance, even at a frequency as high as 150 cycles per second ($\omega = 1000$), is given within a few parts in a million by the equation

$$L = CPS + \frac{1}{R}(Sl_p - Ql_r + Pl_s - Rl_q) \tag{6}$$

If P and R are interchanged, and if a new balance is obtained by adjusting Q and C, changing them by ΔQ and ΔC, respectively, but without changing l_q, then

$$L = CRS + \Delta CRS + \frac{1}{P}(Sl_r - Ql_p + Rl_s - Pl_q) \tag{7}$$

Adding the two equations, dividing by two, and in the correction terms putting $P = R$ and $S = Q$

$$L = S\left(\frac{P + R}{2}\right)\left(C + \frac{\Delta C}{2}\right) + l_s - l_q \tag{8}$$

In discussing this method, the assumption has been made that any capacitance in parallel with any of the arms of the bridge may be treated as a part of the residual inductance in the arm. This is a legitimate assumption for all the arms except the arm containing the inductance. Even in this arm, the assumption is satisfactory for the resistance in series with the inductance, but must be modified for the inductance itself. If there is in parallel with the inductance a capacitance K (caused by the leads to the coil, binding posts on the bridge, etc.), then the inductance as obtained by the previous method is

$$L = S\left(\frac{P+R}{2}\right)\left(C + \frac{\Delta C}{2}\right)(1 - \omega^2 KL) + l_s - l_q \qquad (9)$$

With a suitable choice of leads, K can be kept less than 10^{-11} farads. With $\omega = 1000$, inductances as large as 0.1 henry can be measured to one part in a million without requiring a correction on account of K. For large inductances at relatively high frequencies, the correction for K must be made with great care.

The method just described, of correcting for the residual inductances in the resistances requires that the resistors used in the Q and S arms be calibrated for inductance as well as resistance. To avoid this, a substitution inductor may be constructed having the same resistance as the inductor, but of such a form that its inductance l can be computed from its mechanical dimensions. After the bridge has been balanced in the usual way with the inductor in the Q arm of the bridge, the inductor is removed, the substitution inductor is inserted, and the bridge is again balanced by reducing the capacitance, the new value being designated as c. To insure that a balance can readily be obtained, an auxiliary inductor with a small inductance, l_a, is inserted in the Q arm before measurements are started and is left there during both adjustments. This inductor must be so placed that there is no mutual inductance either between it and the inductor being measured or between it and the substitution inductor. The equations for the two bridge balances are

$$L + l_a = CPS(1 - \omega^2 KL) + \frac{1}{R}(Sl_p + Pl_s - Ql_r - Rl_q) \qquad (10)$$

$$l + l_a = cPS + \frac{1}{R}(Sl_p + Pl_s - Ql_r - Rl_q) \qquad (11)$$

Subtracting these equations, and adding to the second member of the resulting equation the negligibly small quantity $\omega^2 KLcPS$,

$$L - l = PS(C - c)(1 - \omega^2 KL) \qquad (12)$$

The substitution inductor usually takes the form of a return circuit of parallel wires of a high-resistance material. If the wires are of small size, placed about 1 cm apart, the inductance in microhenrys is approximately one-hundredth the total length in centimeters of the wire in the circuit, the value being in error by less than 50 per cent in the most unfavorable case. A more exact solution can be obtained from the formula

$$l = 0.004S\left(\frac{1}{4} + \ln \frac{2d}{D}\right) \tag{13}$$

where l = inductance, in microhenrys.
S = length of the circuit, in centimeters, (one-half the total length of wire).
d = distance between the wires, in centimeters.
D = diameter of the wire, in centimeters.
ln signifies the natural logarithm.

A still more exact formula is given in the next chapter.

76. Measurement of a Mutual Inductance in Terms of Resistance and Capacitance.—A satisfactory method of measuring a mutual inductance in terms of resistance and capacitance is a modification of the method for measuring self inductance which has just been described. A diagram of the setup is shown in Fig. 26. One of the coils of the mutual inductor, usually the one with the largest self inductance, is connected in one arm of a bridge. The second coil of the inductor is connected to the terminals of a double-pole, double-throw switch[1] so that the bridge current can be sent either through this coil or through a by-pass. The secondary is generally so connected that the electromotive force induced in the primary by the current in the secondary is in the opposite direction to the electromotive force resulting from the self inductance of the primary. With the switch thrown to by-pass the secondary, the bridge is balanced by varying Q and C. This is the self-inductance bridge already described, so that no further discussion of its limitations is required. For this balance, the value of the capacitance will be

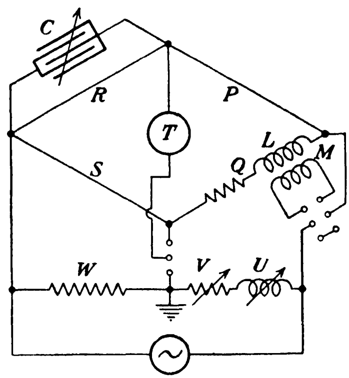

FIG. 26.—Adaption of the Maxwell-Wien bridge to the measurement of a mutual inductance.

[1] A single-pole double-throw switch may be used when the two coils of the inductor are permanently connected together.

designated as C_L and the resistance of the Q arm as Q_L. The switch is then thrown to include the secondary in the supply circuit, and a new balance obtained by varying C and Q. For this balance, the values will be designated as C_M and Q_M. In both cases the third arm (the Wagner ground) must also be balanced.

In the first position of the switch, the equation for the inductance is, when the capacitor has a negligible phase difference,

$$L = C_L PS + \frac{1}{R}(Sl_p - Q_L l_r + Pl_s - Rl_q) + \text{terms in } \omega^2 \quad (14)$$

where l_p, l_r, l_s, and l_q = inductances in the resistances P, R, S, and Q.

In the second position of the switch, with the mutual inductance so connected that the electromotive force produced by it in the Q arm is opposite to that produced by the self inductance,

$$L = M\left(1 + \frac{S}{R}\right) + C_M PS + \frac{1}{R}(Sl_p - Q_M l_r + Pl_s - Rl_q') + \text{terms in } \omega^2 \quad (15)$$

where l_q' is the residual inductance of Q_M.

Subtracting these equations and solving for M,

$$M = (C_L - C_M)\frac{PRS}{R + S} + \frac{(Q_M - Q_L)l_r}{R + S} + \frac{R(l_q' - l_q)}{R + S} + \text{terms in } \omega^2 \quad (16)$$

This method is satisfactory only if the frequency is so low that Q_L differs but little from Q_M, and terms in ω^2 are negligible. This can be observed experimentally by making measurements at decreasing frequencies until there are two frequencies which give the same value for M. For most inductances there is no change below 100 ~.

The resistances P, R, and S should all be of a type which will have little change of resistance with frequency, so that they can be calibrated by direct current against precision standards. The capacitances C_L and C_M will require calibration. It is preferable to use air condensers, which can be measured directly in terms of resistance and time.

It should be noticed that L must be considerably larger than M—twice as large if $R = S$. If the self inductance of the coil of the inductor in the Q arm is not sufficient, any convenient self inductance may be added to the arm. In such a case, care must be exercised to insure that there is no mutual inductance between this inserted inductance and the coil in the supply circuit. This can be tested by altering the position of the inserted coil and observing whether there is a change in bridge balance when there is current through the secondary.

The leads can be so arranged that there is no correction to the mutual inductance on their account. This is readily accomplished by placing the double-pole switch at some distance from the mutual inductance and connecting the two by a pair of twisted conductors which have a negligible mutual inductance with other parts of the circuit. Any mutual inductance between parts that remain in the same positions during both measurements produces no effect on the result.

The third arm is indicated as consisting of a fixed resistance W in one arm and a variable resistance V in series with a variable self inductance U in the other arm. This representation was chosen because variable self inductances are available in most laboratories. However, as this inductance will be changed in going from one bridge arrangement to the other, there is a possibility of producing an error in the final result by its use. Unless this variable self inductance can be placed at a considerable distance from the mutual, all inductances should be omitted from the third arm, and a variable capacitance should be placed in parallel with the resistance W.

The capacitances between the various parts of the inductor have not been considered in the preceding derivation. When two terminals are connected together, the capacitances produce approximately the same effect on the inductance as a single capacitance connected between the terminals that are not connected together A bridge with such a capacitance, K, is shown in Fig. 27. In this bridge, N and L indicate the self inductance of the primary and secondary, respectively; W and V the corresponding resistances; U the resistance in series with the secondary; and the remaining letters have the same significance as in Fig. 26. In deriving the equation for this bridge, the values of C and U

are indicated as C_M and U_M when the current flows through the primary as shown in the figure and as C_L and U_L when the current

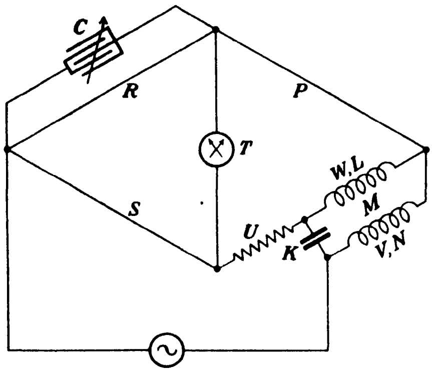

FIG. 27.—A mutual-inductance bridge with capacitance between the primary and secondary.

does not flow through the primary. Then, approximately

$$M = \frac{PRS}{R + S}(C_L - C_M) - K\left(WV + \frac{RW^2}{R + S}\right) + \frac{W^2K}{R + S}\{S(LN - M^2) + C_MRS[VL + WN - M(W + V) + P(L + N - 2M)]\} \quad (17)$$

By measuring at two frequencies, the value of K can be determined. This value can then be used to determine the mutual inductance M.

This method has not had sufficient use to determine the accuracy that can be obtained with it. An important feature is that the mutual inductance is evaluated in terms of resistances and capacitances that can be accurately measured.

CHAPTER XIII

ABSOLUTE MEASUREMENT OF CAPACITANCE

The absolute measurement of capacitance involves the measurement of time in addition to the measurement of electrical quantities. Various methods involving a resonance circuit in which $\omega^2 LC$ equals unity (ω = angular velocity of alternating current, L = inductance in series with capacitance C) have been proposed. There is difficulty in obtaining a precise result with such methods, because the effect of distributed capacitance in the inductor is not readily determined. A more precise method is the one proposed by Maxwell, in which a capacitance is charged and discharged in one arm of a bridge circuit. This is a precise method, but is applicable only to a capacitor that has no absorption, *i.e.*, to an air or vacuum capacitor.

77. Absolute Capacitance by Maxwell's Method.—The absolute measurement of capacitance by Maxwell's method[1] employs a network similar to a Wheatstone bridge, but with one arm replaced by an apparatus for charging and discharging a capacitance a definite number of times per second. The diagram of connections is given in Fig. 28. The resistances of three arms are designated as P, Q, and S. The capacitance C with the commutator a,h,b for charging and discharging it a definite number of times per second constitutes the fourth arm. The balancing of the bridge consists in adjusting either one of the resistances, usually S, or the number of times per second that the capacitance is charged and discharged until the integral current through the galvanometer is zero.

To understand the operation of the bridge, consider the flow of currents during the different positions of the contactor a,h,b. Starting when the vibrator h has left b and is moving towards a, there is a continuous current in all the bridge arms including

[1] For a more critical study of the method, see paper by Curtis and Moon, Absolute Measurement of Capacitance by Maxwell's Method, *Sci. Pap. B.S.*, **22**, 487 (1927), or one by Yoneda and Yamaguchi on the same subject, published in *Researches Electro-Tech. Lab. Tôkyô, Japan*, No. 355 (1933).

the galvanometer arm. The instant that *h* touches *a*, there is a rush of current to charge the capacitance, the current through the galvanometer having a direction opposite to that of the continuous current. The charging current becomes negligibly small before *h* leaves *a*, and the capacitance current does not again flow through the galvanometer until the vibrator, having touched *b* for the purpose of discharging the capacitance, has again touched *a*. If the period of the galvanometer is long relative to the period of the vibrator, the actual movement of the galvanometer coil is small during a cycle of the vibrator.

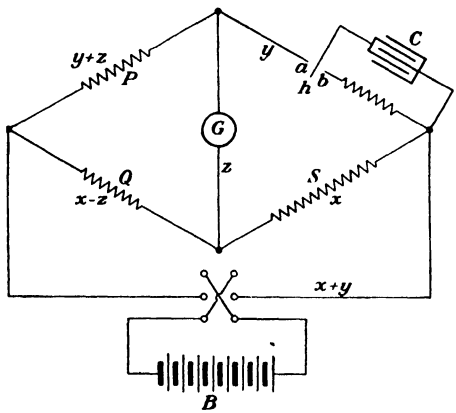

FIG. 28.—Maxwell's method for the absolute measurement of capacitance.

The galvanometer coil makes a small oscillation during each period of the vibrator. The charging current gives the coil an impulsive torque in one direction, while the continuous current maintains a uniform torque in the opposite direction. In the steady state which results after the battery key has been closed for several seconds, the uniform torque of the continuous current acting with or against the torque of the suspension acts to bring the coil at the end of a period to the position occupied by it when it received the impulse at the beginning of the period. Then a new impulse starts a second period. The impulse gives the coil an angular velocity. The uniform torque retards the motion of the coil, so that at the middle of the period the velocity of the coil is zero, then slowly increases until just before the next impulse, when the velocity has the same value as at the beginning, although in the opposite direction. When the impulse is applied, there is a sudden reversal of velocity. If a lamp and scale are used to observe the movements of the galvanometer coil, the oscillation of the coil will make a band of light on the scale which will be bright at the edge, where the velocity of the coil changes more slowly than at the other edge. Theory shows that the bridge is balanced when the galvanometer zero is in the band of light at a distance of one-third of its width from the bright edge.

A galvanometer, to be suitable for a detecting instrument in a Maxwell bridge, must have a period which is long relative to the period of the commutator for charging and discharging the capacitance and must correctly integrate the current. Other desirable properties are that both the resistance of the galvanometer and its external critical damping resistance shall be small and the current sensitivity high. The period of the galvanometer should be about 1000 times that of the vibrator. With this ratio of periods, the width of band on the observing scale produced by the oscillations of the galvanometer coil will be so narrow that the exact position of the galvanometer zero in the band is not important.

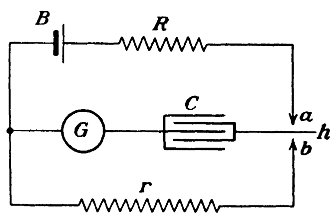

FIG. 29.—Setup for determining that a galvanometer integrates correctly.

A galvanometer correctly integrates the current only when its coil is symmetrical with respect to its magnet. A suitable setup for determining the correct position is shown diagrammatically in Fig. 29. The vibrator used in the bridge setup is so connected that a capacitor C is charged by the battery B through the galvanometer G and the resistance R and is discharged through the galvanometer and the resistance r. If the insulation is good, the integral of the current through the galvanometer is zero. By making R about 100 ohms and r 10 000 ohms or more, the maximum value of the discharge current will be much less than that of the charging current, so that the conditions of the Maxwell bridge are approximated. In such a setup, there will often be a deflection of the galvanometer, but by turning the torsion head to which the galvanometer suspension is attached, the deflection can be made to disappear. Then the galvanometer integrates correctly so that the coil is in a symmetrical position with respect to the magnet. However, with many galvanometers the setting must be made so accurately that slight changes in temperature may require a resetting. Galvanometers using magnets of low permeability and high retentivity are most suitable.

The formula for computing the capacitance can be derived by applying Kirchhoff's laws to the circuits in the bridge, together with the requirement that the integral of the galvanometer current is zero. The currents in the arms are indicated by x, y, z, and their sums and differences, as shown in Fig. 28, and the quantity on the capacitor at any instant t is q. The potential drops around three closed circuits give the following equations:

$$Q(x - z) - Gz - P(y + z) = 0 \tag{1}$$

$$Gz + Sx = \frac{q}{C} \tag{2}$$

$$Q(x - z) + Sx + B(x + y) = E \tag{3}$$

Solving these equations for y, noting that $y = dq/dt$ and letting $S' = S + Q + B$ and $P' = P + Q + G$, there results the differential equation

$$\frac{dq}{dt} + \frac{P'S' - Q^2}{C[P(SQ + S'G) + B(P'S + QG)]}q = \frac{E(QG + P'S)}{P(SQ + S'G') + B(P'S + QG)} \tag{4}$$

Representing the coefficient of q by α and that of E by β, this equation becomes

$$\frac{dq}{dt} + \alpha q = E\beta \tag{5}$$

Solving,

$$q = \frac{E\beta}{\alpha} + K\epsilon^{-\alpha t} \tag{6}$$

where K = constant of integration.
ϵ = base of the natural logarithms.

Let $q = 0$ when $t = 0$, then

$$K = -\frac{E\beta}{\alpha}$$

so that

$$q = \frac{E\beta}{\alpha}(1 - \epsilon^{-\alpha t}) \tag{7}$$

and

$$y = \frac{dq}{dt} = E\beta\epsilon^{-\alpha t} \tag{8}$$

Eliminating x between Eqs. (1) and (2) and substituting in the resulting equation the values of q and y from Eqs. (7) and (8), the resulting equation is

$$(QG + P'S)z = E\beta\left[\frac{Q}{C\alpha} - \left(\frac{Q}{C\alpha} + PS\right)e^{-\alpha t}\right] \tag{9}$$

Multiplying both sides of the equation by dt and integrating from 0 to T, the result is

$$\begin{aligned}(QG + P'S)\int_0^T zdt &= E\beta\left[\frac{Qt}{C\alpha} + \left(\frac{Q}{C\alpha} + PS\right)\frac{1}{\alpha}e^{-\alpha t}\right]_0^T \\ &= E\beta\left[\frac{QT}{C\alpha} - \left(\frac{Q}{C\alpha^2} + \frac{PS}{\alpha}\right)(1 - e^{-\alpha T})\right]\end{aligned} \tag{10}$$

In order that $\int_0^T zdt$ shall be zero, the term in the square brackets must be zero since neither E nor β can be zero. If T is so large that $e^{-\alpha T}$ is negligibly small relative to unity,

$$\frac{QT\alpha}{C} - \left(\frac{Q}{C} + PS\alpha\right) = 0 \tag{11}$$

Substituting the value of α and solving for C,

$$C = \frac{QT(P'S' - Q^2)}{(PS' + BQ)(P'S + QG)} \tag{12}$$

$$= \frac{QT}{PS}\left[\frac{1 - \frac{Q^2}{P'S'}}{\left(1 + \frac{BQ}{PS'}\right)\left(1 + \frac{GQ}{P'S}\right)}\right] = \frac{QT}{PS}F \tag{13}$$

The term within the square brackets, which is indicated as F, is nearly equal to unity provided Q, G, and B are small. In this case an approximate value is obtained by dividing the numerator by the denominator and taking only first-order terms of the quotient, so that

$$F = 1 - \frac{Q^2}{P'S'} - \frac{BQ}{PS'} - \frac{GQ}{P'S} \tag{14}$$

A further approximation is obtained by putting $S' = S$, so that, on substituting the value of P',

$$F = 1 - \frac{Q}{S}\left(\frac{Q + G}{P + Q + G} + \frac{B}{P}\right) \tag{15}$$

The value of F should not differ from unity by more than $1/1000$, since it depends on the resistances G and B, neither of which can be measured with precision. To accomplish this, both P and Q must be small. However, the sensitivity is decreased if P and Q are made too small, so that generally the values of both P and Q lie between one-hundredth and one-thousandth of S. This may be written, in the customary case where P equals Q, as

$$100Q < S < 1000Q$$

In deriving the formula, the electromotive forces produced by the self inductance of the galvanometer and by the motion of the galvanometer coil in a magnetic field have been neglected since the integral of each over a complete cycle is zero.

In deriving the equation for the capacitance, the assumption was made in evaluating Eq. (10) that $e^{-\alpha T}$ is negligibly small relative to unity. This is equivalent to assuming that the condenser is completely charged during each cycle. A table of exponentials will show that, for $e^{-\alpha T}$ to be less than 10^{-6},

$$\alpha T \geqq 14$$

However the time during which the condenser must be completely charged is less than half the time of a complete cycle. Substituting the value of α obtained from Eqs. (4) and (5) and the value of T from Eq. (13), and assuming that $F = 1$, $P = Q$, and B and G are negligibly small, then

$$\alpha T = \frac{2S + Q}{Q}$$

or the condenser will be completely charged in the time $T/2$ if $S \geqq 13Q$. For other reasons, S is generally much larger than required, as has already been shown.

The capacitor must also be completely discharged. A resistance of a few ohms placed in the discharge circuit will prevent an oscillatory discharge and thus may decrease the time required for a complete discharge.

The contactor *a,h,b* for charging and discharging the condenser must operate very uniformly at a rate that can be accurately determined. Several types of rotating commutators have been used, but vibration contactors are superior. A suitable type is shown in Fig. 30. A flat spring with a weight on the end is mounted to vibrate between platinum contacts. The assembly is mounted on one prong of a tuning fork, the other prong being suitably balanced. The tuning fork may be self-driven, in which case its frequency must be accurately measured whenever the fork is used, or it may be driven by an alternating current having its frequency controlled by a piezoelectric oscillator so that its frequency will probably be accurate to one part in ten million. The latter method can be used only when the natural frequency of the fork does not differ from the frequency of the alternating current by more than a few parts in ten thousand. Since the leads to the contactor will affect the frequency of the fork because of their weight and stiffness, they should be small and flexible.

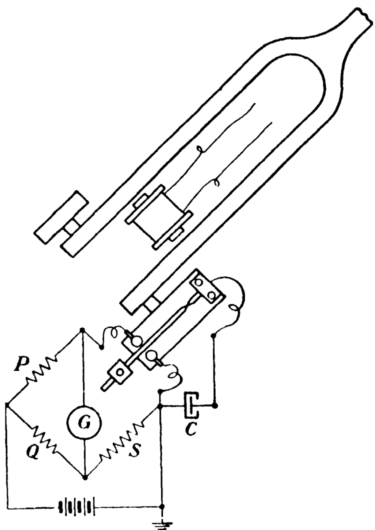

Fig. 30.—A vibration contactor of the inertia type for charging and discharging the capacitor in a Maxwell bridge for the absolute measurement of capacitance.

The formula for the capacitance may be written $C = QF/nPS$ where n, the number of charges and discharges per second, equals $1/T$. In this form, the three resistances Q, P, and S must all be accurately known. To avoid the necessity of calibrating each of these resistances, the capacitor and contactor are replaced by a standard resistance of such value that only a slight change in the S arm is necessary to balance the network as a Wheatstone bridge. Let the change of resistance in S be δS. Then

$$P(S + \delta S) = QR$$

Substituting in the capacitance equation

$$C = \frac{F\left(1 + \frac{\delta S}{S}\right)}{nR} \tag{16}$$

If $P = Q$ approximately, then $R = S$, and, when δS is less than one-thousandth of S, Eq. (16) becomes

$$C = \frac{F}{n(R - \delta S)} \tag{16a}$$

with an accuracy of one part in a million. This reduces the number of resistances which must be known accurately to one, *viz.*, R.

The accuracy that can be attained by this method is limited by the accuracy with which the galvanometer will integrate the current and by the accuracy with which the correction factor F can be determined. Under favorable conditions the uncertainty in a single measurement of a 0.25-microfarad air capacitor will not exceed one or two parts in a million.

CHAPTER XIV

COMPUTATION OF THE INDUCTANCE OF A CIRCUIT FROM ITS DIMENSIONS

The inductance of a circuit of known geometric shape can be accurately computed from its dimensions only if the shape of the circuit is one for which a suitable formula has been derived[1] and if the permeabilities of all bodies near the circuit are known. However, very few of the shapes for which formulas are available are of such a type that their mechanical construction can be carried out with the necessary precision or their dimensions measured with the required accuracy. Only those shapes will be considered which are suitable for precise standards. In all cases all bodies near the circuit will be considered as having the same permeability, which is generally that of a vacuum (unity in the c.g.s.m. system).

78. Principles Used in Deriving Inductance Formulas.—An inductance formula is an equation which connects an induced electromotive force in a circuit with the rate of change of the current that induces it. Such a formula involves the dimensions and relative positions of the induced and inducing circuits. The first step in the derivation of an inductance formula is the determination of the equation connecting the current in a circuit with the magnetic intensity at any point in its magnetic field. But the magnetic induction is equal to the magnetic intensity multiplied by the permeability of the medium. Hence the derivation of an inductance formula for a given circuit consists in determining the equation that expresses the magnetic intensity at any point in the surrounding region when there is unit current in the circuit, in multiplying the magnetic intensity by the permeability of the medium to obtain the magnetic induction, and in integrating the magnetic induction to determine the magnetic flux through the circuit in which the electromotive

[1] The most complete compilation of formulas is that by Rosa and Grover, *Bull. B. S.*, **8**, 1 (1912).

force is induced. Since the induced electromotive force is equal both to the rate of change of magnetic flux and to the product of the inductance multiplied by the rate of change of the current, the magnetic flux resulting from unit current is numerically equal to the inductance. But the magnetic flux through a circuit produced by unit current in the same or another circuit depends only on the dimensions of the circuit or circuits and, when two circuits are involved, their relative position. Hence the derivation of an inductance formula is primarily a problem in geometry.

79. Inductance of a Portion of a Circuit.—In order to compute the inductance of a portion of a circuit, an expression for the magnetic field of a portion of a circuit must be known. Several different laws have been suggested for determining the magnetic field produced by a portion of a circuit, all leading to the same result for the complete circuit. Since an experimental test can be made only on the complete circuit, there is no possible way of deciding between the suggested laws. The computed inductance for a part of a circuit depends on the law that is chosen for expressing the magnetic field, but, so long as the same law is used throughout, the inductance of a complete circuit is independent of the law that may be chosen to express the magnetic field.

Practically all inductance formulas for portions of a circuit have been developed on the assumption that the Law of Biot and Savart holds for portions of a circuit. This law states that the magnetic intensity $d\mathbf{H}$ produced at a point P, by a current I in an element $d\mathbf{s}$ of a circuit, may be represented by the vector equation

$$d\mathbf{H} = -\frac{I[\mathbf{R} \times d\mathbf{s}]}{r^3} \tag{1}$$

where r = magnitude of the vector $\mathbf{R}$ representing the distance from $d\mathbf{s}$ to P.

$[\mathbf{R} \times d\mathbf{s}]$ = vector product of $\mathbf{R}$ and $d\mathbf{s}$.

This law has been universally used in developing inductance formulas.

While the principles involved in developing a self-inductance formula are the same as those used for a mutual inductance, yet the mathematical procedure is generally somewhat different in the two cases. Several methods of attacking any particular problem are available. One or two of the best known will be outlined, using vector notation to simplify the mathematical expressions.

80. Neumann's Formula for Mutual Inductance.—In developing Neumann's formula for the mutual inductance between two circuits, each circuit will be treated as having infinitesimal cross section, so that at any point the circuit is completely defined by an element of length, $d\mathbf{s}$, which coincides with the axis of the circuit. If a current I flows in one circuit, any element of which can be represented by $d\mathbf{s}_1$, the magnetic intensity $\mathbf{H}_1$ at any point P can, by integration of the law of Biot and Savart, be determined from the equation

$$\mathbf{H}_1 = -I\int\frac{[\mathbf{R} \times d\mathbf{s}_1]}{r^3} \tag{2}$$

where $\mathbf{R}$ = vector of magnitude r joining $d\mathbf{s}_1$ to P, and the integration is to extend from one end to the other of the portion of the circuit under consideration. The vector potential $\mathbf{A}_1$ of the magnetic induction $\mathbf{B}_1$ at the point P is defined by the equation

$$\text{curl } \mathbf{A}_1 = \mathbf{B}_1 = \mu\mathbf{H}_1 \tag{3}$$

Combining Eqs. (2) and (3), and applying Stokes's theorem

$$\mathbf{A}_1 = \mu I\int\frac{d\mathbf{s}_1}{r} \tag{4}$$

In the media surrounding the circuit, the magnetic induction $\mathbf{B}$ has the value $\mu\mathbf{H}$ at every point. The magnetic flux through a circuit is, by definition, the surface integral of the magnetic induction over any surface which is bounded by the circuit. Hence the flux, ϕ_2, through a second circuit of which an element of a surface bounded by the circuit is represented by the vector $d\mathbf{S}_2$, is given by the equation

$$\phi_2 = \iint(\mathbf{B}_1 \cdot d\mathbf{S}_2) \tag{5}$$

But since $\mathbf{B}_1 = \text{curl } \mathbf{A}_1$,

$$\phi_2 = \iint(\text{curl } \mathbf{A}_1 \cdot d\mathbf{S}_2) \tag{6}$$

By Stokes's theorem this surface integral reduces to a line integral, so that

$$\phi_2 = \int(\mathbf{A}_1 \cdot d\mathbf{s}_2) \tag{7}$$

where $d\mathbf{s}_2$ is an element of the second circuit. Substituting the value of **A** from Eq. (4),

$$\phi_2 = \mu I_1 \int\int \frac{(d\mathbf{s}_1 \cdot d\mathbf{s}_2)}{r} \tag{8}$$

Since the mutual inductance is the flux through a second circuit caused by unit current in the first circuit,

$$M = \mu \int\int \frac{(d\mathbf{s}_1 \cdot d\mathbf{s}_2)}{r} = \mu \int\int \frac{ds_1\, ds_2 \cos \epsilon}{r} \tag{9}$$

where ϵ is the angle between $d\mathbf{s}_1$ and $d\mathbf{s}_2$. This is Neumann's integral, which has been extensively employed in developing mutual inductance formulas.

As derived above, Neumann's integral requires that the integration be carried completely around the second circuit, since there is no meaning to the flux through a portion of a circuit. However, the final formula shows that mutual inductance is a reciprocal relationship between two circuits. Hence Neumann's integral may be extended to indicate the mutual inductance between portions of circuits, so that the limits of the integrations are the ends of the two portions under consideration.

81. Methods for Deriving Self-inductance Formulas.—Neumann's integral is not directly applicable to the derivation of formulas of self inductance, since the fundamental assumption that the circuit can be considered as concentrated in a line cannot be applied to a single circuit. In developing a formula for the self inductance of a circuit, a relationship between current and magnetic intensity that is applicable within the conductor as well as outside the conductor is required. A convenient relationship is

$$\text{curl } \mathbf{H} = 4\pi \mathbf{J} \tag{10}$$

where **H** is the magnetic intensity at a point at which the current density is **J**. Also the Helmholz equation for the instantaneous value of an electromotive force e in a circuit is

$$e = Ri + L\frac{di}{dt} \tag{11}$$

where i = instantaneous value of the current.
R = resistance of the circuit.
L = inductance of the circuit.

Multiplying each term of this equation by idt, and integrating between the limits 0 and T, when the value of i has the limits 0 and I, the following equation results:

$$\int_0^T ei\,dt = R\int_0^T i^2\,dt + \frac{LI^2}{2} \tag{12}$$

The left-hand integral represents the total energy that has been supplied to the circuit through the electromotive force e. The first term of the right-hand member gives the energy that has been converted into heat in time T, while the last term represents the energy that has been stored in the magnetic field. But the energy of the magnetic field is

$$\frac{1}{8\pi}\int\int\int(\mathbf{B}\cdot\mathbf{H})dv \tag{13}$$

where the integral is to be taken throughout the magnetic field. Hence

$$LI^2 = \frac{1}{4\pi}\int\int_\infty\int(\mathbf{B}\cdot\mathbf{H})dv \tag{14}$$

To simplify the treatment, a third vector field **A**, known as the vector potential of **B**, will be introduced. This vector field will be defined by the relations

$$\text{curl } \mathbf{A} = \mathbf{B} \tag{15}$$

$$\text{div } \mathbf{A} = 0 \tag{16}$$

Since

$$\mathbf{H} \text{ curl } \mathbf{A} = \mathbf{A} \text{ curl } \mathbf{H} + \text{div } [\mathbf{A}\times\mathbf{H}] \tag{17}$$

the equation for the self inductance becomes

$$LI^2 = \frac{1}{4\pi}\int\int_\infty\int\{(\mathbf{A} \text{ curl } \mathbf{H}) + \text{div } [\mathbf{A}\times\mathbf{H}]\}dv \tag{18}$$

The second term of the integral is zero. This can be demonstrated by changing the volume integral, by Gauss's theorem, to a surface integral which can be shown to be zero.

$$\iiint_{\infty} \text{div}[\mathbf{A} \times \mathbf{H}]dv = \iint_{\infty} [\mathbf{A} \times \mathbf{H}]dS \tag{19}$$

The surface integral must be considered as taken over the surface of an infinitely large volume such as an infinitely large sphere. In spherical coordinates $dS = R^2 \sin\theta\, d\theta\, d\varphi$. Since the electric circuit is of finite dimensions, the magnetic intensity at a distance R, which is large relative to the dimensions of the circuit, is given by the equation $\mathbf{H} = K_1/R^2$. Also $\mathbf{A} = K_2/R$, where K_1 and K_2 are finite constants depending on the circuit. Substituting these values in the surface integral of Eq. (19) and taking the maximum value of the vector product,

$$\iiint_{\infty} \text{div}\ [\mathbf{A} \times \mathbf{H}]dv \gtreqless \iint_{\infty} \frac{K_1K_2 \sin\theta\, d\theta\, d\varphi}{R}$$

Since the numerator of the surface integral is finite, the integral becomes zero when the denominator R becomes infinite.

Substituting Eq. (10) in (18)

$$LI^2 = \iiint_{\infty} (\mathbf{A} \cdot \mathbf{J})dv \tag{20}$$

Now since $\mathbf{J}$ is zero at all points outside the conductor, the integral need be taken only through the volume of the conductor in which the current is flowing. But before Eq. (20) can be integrated, $\mathbf{A}$ must be determined as a function of $\mathbf{J}$ and the coordinates.

The value of $\mathbf{A}$ can be determined from the equations

$$\text{curl}\ \mathbf{A} = \mathbf{B} = \mu\mathbf{H} \qquad \text{curl}\ \mathbf{H} = 4\pi\mathbf{J}$$
$$\text{div}\ \mathbf{A} = 0 \qquad \text{div}\ \mathbf{H} = 0 \qquad \text{div}\ \mathbf{J} = 0$$

Then $$\text{curl}^2\ \mathbf{A} = \mu\ \text{curl}\ \mathbf{H} = 4\pi\mu\mathbf{J} \tag{21}$$

But $$\text{curl}^2\ \mathbf{A} = -\nabla^2\mathbf{A} + \nabla\ \text{div}\ \mathbf{A} \tag{22}$$

Hence $$-\nabla^2\mathbf{A} = 4\pi\mu\mathbf{J}$$

This last equation is a vector equation which, when expressed in rectangular coordinates, takes the form

$$i\nabla^2A_x + j\nabla^2A_y + k\nabla^2A_z = -4\pi\mu(iJ_x + jJ_y + kJ_z) \tag{23}$$

The solution of this vector equation requires the solution of the three scalar equations:

$$\begin{aligned} \nabla^2 A_x &= -4\pi\mu J_x \\ \nabla^2 A_y &= -4\pi\mu J_y \\ \nabla^2 A_z &= -4\pi\mu J_z \end{aligned} \tag{24}$$

each of which is of the form of Poisson's equation as used to express the potential in terms of volume density. The solution of these equations (given in texts on Newtonian potential) is of the form

$$A_x = \mu \int\int\int \frac{J_{x_1}\, dx_1\, dy_1\, dz_1}{\sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2}} \tag{25}$$

with similar equations for A_y and A_z, where A_x, A_y, and A_z are the components of A at a point having the coordinates x, y, and z and J_{x_1} is the x component of the current density at any other point x_1, y_1, z_1, while the integration extends over the volume of the conductor. The values A_x, A_y, and A_z can be substituted in Eq. (20), which, when expressed in rectangular coordinates has the form

$$LI^2 = \int\int\int (A_x J_x + A_y J_y + A_z J_z) dx\, dy\, dz \tag{26}$$

If the current density J is uniform throughout the conductor, the equation which results from substituting Eq. (25) in Eq. (26) can be simplified. For this case, the value of J_x at any point x,y,z, is $J_x = Jf(x,y,z)$, where $f(x,y,z)$ is a function of the coordinates only, but depends on the particular circuit under consideration. Similarly $J_y = J\phi(x_1,y_1,z_1)$ and $J_z = J\psi(x_1,y_1,z_1)$ Hence

$$A_x = J\mu \int\int\int \frac{f(x_1,y_1,z_1)dx_1\, dy_1\, dz_1}{\sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2}} = J\mu B_x \tag{27}$$

$$A_y = J\mu \int\int\int \frac{\phi(x_1,y_1,z_1)dx_1\, dy_1\, dz_1}{\sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2}} = J\mu B_y \tag{28}$$

$$A_z = J\mu \int\int\int \frac{\psi(x_1,y_1,z_1)dx_1\, dy_1\, dz_1}{\sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2}} = J\mu B_z \tag{29}$$

These integrals can be evaluated either completely or with any desired approximation for any circuit in which the functions f, ϕ, and ψ can be stated, and for which suitable limits can be set. The evaluated integrals will be represented by B_x, B_y, and B_z, as indicated in the equations.

The values for the components of A and J may be substituted in Eq. (26) so that

$$LI^2 = \mu\iiint[JB_x \cdot Jf(x,y,z) + JB_y \cdot J\phi(x,y,z) + JB_z \cdot J\psi(x,y,z)]dx\,dy\,dz \quad (30)$$
$$= \mu J^2\iiint[B_x f(x,y,z) + B_y\phi(x,y,z) + B_z\psi(x,y,z)]dx\,dy\,dz \quad (31)$$

Since the current density has been assumed uniform throughout the circuit, the cross section S of the conductor must be a constant. Hence $I = JS$. Substituting in the above,

$$L = \frac{\mu}{S^2}\int\int\int[B_x f(x,y,z) + B_y\phi(x,y,z) + B_z\psi(x,y,z)]dx\,dy\,dz \quad (32)$$

Cylindrical coordinates are more convenient than rectangular coordinates for deriving self-inductance formulas of circuits in which the conductor is a round wire. In transforming to cylindrical coordinates the equations for circuits in which the current density is uniform throughout the conductor, the components of the current density at any point having the cylindrical coordinates r,θ,z are J_r, J_θ, and J_z and those of the vector potential at the same point are A_r, A_θ, and A_z. Since J is a constant, $J_r = Jf_r(r,\theta,z)$, $J_\theta = J\phi_\theta(r,\theta,z)$, and $J_z = J\psi(r,\theta,z)$.

Making the necessary substitutions and transformations,

$$A_r = J\mu\int\int\int\frac{F(r_1,\theta_1,z_1)r_1\,dr_1\,d\theta_1\,dz_1}{\sqrt{r^2 - 2rr_1\cos(\theta - \theta_1) + r_1^2 + (z - z_1)^2}} = J\mu B_r \quad (33)$$

where $F(r_1,\theta_1,z_1) = f_r(r_1,\theta_1,z_1)\cos(\theta - \theta_1) + \phi_\theta(r_1,\theta_1,z_1)\sin(\theta - \theta_1)$

$$A_\theta = J\mu\int\int\int\frac{\Phi(r_1,\theta_1,z_1)r_1\,dr_1\,d\theta_1\,dz_1}{\sqrt{r^2 - 2rr_1\cos(\theta - \theta_1) + r_1^2 + (z - z_1)^2}} = J\mu B_\theta \quad (34)$$

where $\Phi(r_1,\theta_1,z_1) = \phi_\theta(r_1,\theta_1,z_1)\cos(\theta - \theta_1) - f_r(r_1,\theta_1,z_1)\sin(\theta - \theta_1)$

$$A_z = J\mu\int\int\int\frac{\psi(r_1,\theta_1,z_1)r_1\,dr_1\,d\theta_1\,dz_1}{\sqrt{r^2 - 2rr_1\cos(\theta - \theta_1) + r_1^2 + (z - z_1)^2}} = J\mu B_z \quad (35)$$

In these equations the values of the components of the vector potential at a point r,θ,z, are obtained by integration through the volume of the conductor of certain functions which are used in representing the value of J at any point r_1,θ_1,z_1 and of the distance between r,θ,z and r_1,θ_1,z_1.

Substituting the cylindrical components of **A** and **J** in Eq. (20) and inserting the value of I when J is a constant, the equation for self inductance is

$$L = \frac{\mu}{S^2}\int\int\int [B_r f_r(r,\theta,z) + B_\theta \phi_\theta(r,\theta,z) + B_z \psi(r,\theta,z)] r\, dr\, d\theta\, dz \quad (36)$$

In the above developments of self-inductance integrals, no mention has been made of the Biot-Savart Law, but it was tacitly assumed in the integrations for obtaining the components of A. Hence the results by the above integrals, when applied to a portion of a circuit, give results which correspond to those obtained by Neumann's integral.

Another method of deriving a formula for the self inductance of a conductor of uniform cross section is to assume that it is divided into an infinite number of filaments. By Neumann's integral, a formula for the mutual inductance between any two filaments can be derived. The induced electromotive force in any arbitrarily chosen filament is the sum of the electromotive forces resulting from the changing currents in each of the filaments. Hence if an equation can be written to express the electromotive force induced in the chosen filament by a second arbitrarily chosen filament, the total induced electromotive force can be obtained by integrating over the entire cross section of the conductor. To make this integration, the current in a filament must be expressed in terms of the current density and the area of the filament, so that the total electromotive force in the filament depends on the rate of change of current density. Now the induced electromotive force is not the same in all filaments, even when the current distribution is uniform, so that the average value for all filaments must be obtained by again integrating over the cross section and dividing by its area. Then this average induced electromotive force, divided by the rate of change of the total current in the circuit, is the self inductance.[1]

82. Derivation of Inductance Formulas to Illustrate Methods. The application of the foregoing methods to complicated circuits is outside the province of this book, since the integrations required are often difficult. However, the principles can be

[1] For an application of this method, see an article by Curtis, An Integration Method of Deriving the Alternating Current Resistance and Inductance of Conductors, *Sci. Pap. B.S.*, **16**, 93 (1920).

illustrated by choosing circuits in which the integrals can be readily evaluated. In the following inductance formulas, the permeability of the region surrounding the inductor is taken as unity, so that permeability does not appear explicitly in the formulas.

a. Mutual Inductance between Equal Parallel Wires.—The formula for the mutual inductance between two equal parallel wires, as shown in Fig. 31, can be derived by the application of Neumann's integral, *viz.*,

$$M = \int\int \frac{ds_1\, ds_2 \cos \epsilon}{r}$$

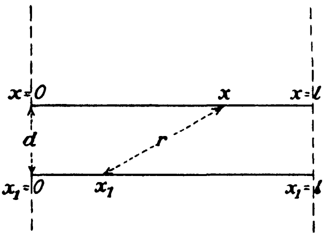

FIG. 31.—Diagram to illustrate the nomenclature used in deriving the formula for the mutual inductance between parallel wires.

Let $ds_1 = dx_1$, $ds_2 = dx$, $\cos \epsilon = 1$. Then $r = \sqrt{d^2 + (x - x_1)^2}$, and the limits of integration for both circuits are 0 and l. Then

$$M = \int_0^l dx_1 \int_0^l \frac{dx}{\sqrt{d^2 + (x - x_1)^2}} \quad (37)$$

This can be integrated by putting $x - x_1 = z$ and changing the limits so that

$$M = \int_0^l dx_1 \int_{-x_1}^{l-x_1} \frac{dz}{\sqrt{d^2 + z^2}} = \int_0^l dx_1 \Big[\log (z + \sqrt{d^2 + z^2}) \Big]_{-x_1}^{l-x_1} \quad (38)$$

Inserting the limits and separating into two integrals,

$$M = \int_0^l dx_1 \ln [l - x_1 + \sqrt{d^2 + (l - x_1)^2}] - \int_0^l dx_1 \ln [-x_1 + \sqrt{d^2 + x_1^2}] \quad (39)$$

In the first integral, substitute $u = l - x_1 + \sqrt{d^2 + (l - x_1)^2}$, and in the second integral substitute $v = -x_1 + \sqrt{d^2 + x_1^2}$. Then

$$M = \frac{1}{2}\int_d^{l+\sqrt{l^2+d^2}} \left(1 + \frac{d^2}{u^2}\right) \ln u\, du + \frac{1}{2}\int_d^{-l+\sqrt{l^2+d^2}} \left(1 + \frac{d^2}{v^2}\right) \ln v\, dv \quad (40)$$

Integrating,

$$M = \frac{1}{2}\left[\left(u - \frac{d^2}{u}\right)\ln u - u - \frac{d^2}{u}\right]_d^{l+\sqrt{l^2+d^2}} + \frac{1}{2}\left[\left(v - \frac{d^2}{v}\right)\ln v - v - \frac{d^2}{v}\right]_d^{-l+\sqrt{l^2+d^2}} \quad (41)$$

Substituting limits and simplifying,

$$M = 2\left(l \ln \frac{l + \sqrt{l^2 + d^2}}{d} - \sqrt{l^2 + d^2} + d\right) \quad (42)$$

This is an exact solution, no approximations having been introduced. However, the assumption was made that the wires can be represented by geometric lines.

b. Mutual Inductance between Coaxial Circles.—Neumann's integral was used by Maxwell[1] to determine the mutual inductance between two coaxial circles. The circles are shown diagrammatically in Fig. 32. The larger circle of radius A has its center

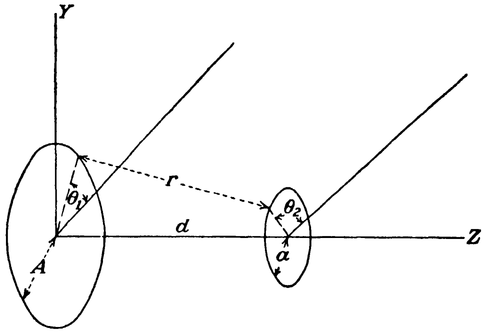

FIG. 32.—Diagram to illustrate the nomenclature used in developing the formula for the mutual inductance between two coaxial circles.

at the origin of coordinates and lies in a plane perpendicular to the z axis. The smaller circle of radius a has its center on, and its plane perpendicular to, the z axis at a distance d from the origin. The distance between any two points P_1 and P_2 selected at random on the two circles is r. Then

$$\begin{aligned} ds_1 &= A d\theta_1 \\ ds_2 &= a d\theta_2 \\ \epsilon &= \theta_2 - \theta_1 \\ r &= \sqrt{d^2 + A^2 + a^2 - 2Aa\cos(\theta_1 - \theta_2)} \end{aligned}$$

[1] "Electricity and Magnetism," 3d ed., Art. 701.

Hence by Neumann's formula,

$$M = \int_0^{2\pi}\int_0^{2\pi} \frac{Aa \cos(\theta_2 - \theta_1)\, d\theta_1\, d\theta_2}{\sqrt{d^2 + A^2 + a^2 - 2Aa\cos(\theta_1 - \theta_2)}} \qquad (43)$$

The integration is simplified by introducing a constant k where

$$k^2 = \frac{4Aa}{d^2 + (A + a)^2} \qquad (44)$$

The first integration involves only trigonometric functions, but the second integration requires the introduction of elliptic integrals. Carrying through this work,

$$M = 4\pi\sqrt{Aa}\left[\left(\frac{2}{k} - k\right)K - \frac{2}{k}E\right] \qquad (45)$$

where K and E are the complete elliptic integrals[1] of the first and second kinds, respectively, to modulus K. This is an exact formula.

c. Self Inductance of a Circular Ring.—The development of the formula for the self inductance of a circular ring having circular cross section is most easily accomplished by using cylindrical coordinates in which the center of coordinates is the center of the ring and the z axis coincides with the axis of the ring. In Fig. 33 is shown a ring and its cylindrical coordinates: r, θ, and z. The mean radius of the ring is a and the radius of the cross section is ρ.

Assuming the current density J to be uniform,

$$J_r = 0 \qquad J_\theta = J \qquad \text{and} \qquad J_z = 0$$

so that

$$f_r(r,\theta,z) = 0 \qquad \phi_\theta(r,\theta,z) = 1 \qquad \psi(r,\theta,z) = 0$$

Hence, by Eq. (33), using the limits of integration which can be seen from Fig. 33,

$$B_r = \int_{a-\rho}^{a+\rho} r_1\, dr_1 \int_{-\rho}^{\rho} dz_1 \int_0^{2\pi} \frac{\sin(\theta - \theta_1)\, d\theta_1}{\sqrt{r^2 + r_1^2 + (z - z)^2 - 2rr_1\cos(\theta - \theta_1)}} \qquad (46)$$

[1] Values of K and E for various values of k are given in Peirce, "Integral Tables," p. 117. More complete tables have been prepared by Nagaoka and Sakauri. A method of obtaining numerical values is given in Appendix I.

The substitution of the limits in the first integration reduces this to zero. Also, since $\psi(r_1,\theta_1,z_1) = 0$, $B_z = 0$, from Eq. (35). Substituting in Eq. (34),

$$B_\theta = \int_{a-\rho}^{a+\rho} r_1\,d\rho_1 \int_{-\rho}^{\rho} dz_1 \int_0^{2\pi} \frac{\cos(\theta - \theta_1)\,d\theta_1}{\sqrt{r^2 + r_1^2 + (z - z_1)^2 - 2rr_1\cos(\theta - \theta_1)}} \tag{47}$$

Substituting the values of B_θ and of $\phi_\theta(r,\theta,z)$ in Eq. (36) and inserting the appropriate limits,

$$L = \frac{1}{\pi^2\rho^4}\int_{a-\rho}^{a+\rho} rdr \int_{-\rho}^{\rho} dz \int_0^{2\pi} d\theta \int_{a-\rho}^{a+\rho} r_1dr_1 \int_{-\rho}^{\rho} dz_1 \int_0^{2\pi} Fd\theta_1 \tag{48}$$

where $$F = \frac{\cos(\theta - \theta_1)}{\sqrt{r^2 + r_1^2 + (z - z_1)^2 - 2rr_1\cos(\theta - \theta_1)}}$$

The evaluation of the above integrals is difficult since the first

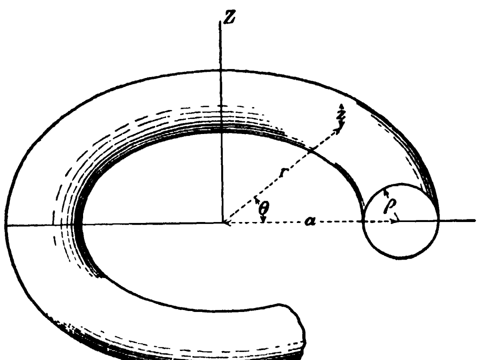

FIG. 33.—Diagram to illustrate the nomenclature used in developing the formula for the self inductance of a circular ring.

integral is an elliptic integral. However, by assuming that the radius of the wire, ρ, is small relative to the radius of the circle, a, so that terms smaller than ρ/a can be neglected, the value of L can be approximately determined as

$$L = 4\pi a\left(\ln\frac{8a}{\rho} - 1.75\right) \tag{49}$$

83. Important Formulas for Computing Inductance.—Only those formulas for computing inductance will be given which apply to forms that can be precisely constructed and accurately measured. However, a number of auxiliary formulas will be given which are of importance in computing the inductance of leads, of substitution inductances, etc. In all the formulas, if the dimensions are in centimeters, the computed inductance is in millimicrohenrys.

a. Mutual Inductance of Concentric, Coaxial Solenoids.—There are several formulas for the mutual inductance of concentric, coaxial solenoids, but all have been derived on the assumption that each solenoid is a current sheet. A current sheet is an idealized winding having properties such that the current is uniformly distributed over the surface of the cylinder and at every point flows in a direction perpendicular to an element of the cylinder. A current sheet can be most nearly approximated by winding a thin tape on a cylinder, leaving very little space between adjacent turns. However, there are many practical difficulties in making a winding of tape. The only practical solenoid consists of a helix of round wires wound on an insulating cylinder. Hence the importance of determining the difference between the mutual inductance of two current sheets and two helices made from round wire. Before we discuss the difference between the two types of windings, the current sheet formulas will be considered.

Two formulas for the mutual inductance between two coaxial and concentric current sheets have been developed[1] which are applicable to solenoids of any length or diameter. The one by Cohen uses complete and incomplete elliptic integrals; the other by Nagaoka uses q functions. Cohen's formula is in the form of the difference between two quantities which may be nearly equal, so that the computation of each must be made with great exactness. Hence it is not a practical formula for computation. Nagaoka's formula is very laborious to use, though giving accurate results.

In addition to the exact formulas, there are a number of series formulas, each best adapted to a particular region of coil dimensions. The region of coil dimensions of greatest importance in absolute electrical measurements may be described as an outer coil having a length greater than the diameter and an inner

[1] A collection of formulas of the mutual inductance of solenoids, together with a discussion of the types of solenoids to which they may be applied, is found in a paper by Rosa and Grover, *Bull. B. S.*, **8**, 53 (1912).

coil with dimensions not more than half those of the outer coil. The mutual inductance of the system just described can be computed by the formula of Roiti, but a formula recently given by Snow[1] is somewhat more convenient. This formula is given below, together with an example of computation by means of it.

The symbols employed for the dimensions can be understood by referring to Fig. 34:

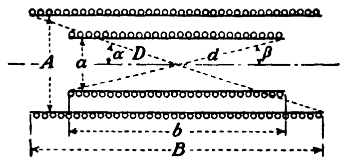

FIG. 34.—Diagram to illustrate the nomenclature used in the formula for the mutual inductance of concentric, coaxial solenoids.

A and a = diameters of the outer and inner solenoids, respectively.
B and b = lengths of the outer and inner solenoids, respectively.
D and d = diagonals of the outer and inner solenoids, respectively.
α and β = angles between the axis and the diagonals of the outer and inner solenoids, respectively.
N and n = total numbers of turns on the outer and inner solenoids, respectively.

$$D^2 = A^2 + B^2 \qquad d^2 = a^2 + b^2$$
$$\sin \alpha = \frac{A}{D} \qquad \sin \beta = \frac{a}{d}$$

The mutual inductance M in millimicrohenrys is given by the formula

$$M = \frac{\pi^2 N n a^2}{D}\left\{1 - \frac{\sin^2 \alpha}{2}\sum_{k=1}^{k=\infty}[C_k(\alpha)][C_{k+1}(\beta)]\left(\frac{d^2}{D^2}\right)^k\right\} \qquad (50)$$

where $C_k(\alpha)$ and $C_k(\beta)$ are a group of finite series of $\sin \alpha$ and $\sin \beta$, respectively.

The general expression for the kth series in the α group of series is

[1] Snow, Mutual Inductance and Torque between Two Concentric Solenoids, *B.S. J. Research*, **1**, 685 (1928). There is an error in the formula as originally published.

$$C_k(\alpha) = 1 + \sum_{r=1}^{r=k-1} \left[(-1)^r \frac{(k-1)(k-2)\cdots(k-r)}{\underline{|r+1}} \right] \left[\frac{(2k+3)(2k+5)\cdots(2k+2r+1)}{2^r \underline{|r}} \sin^{2r} \alpha \right] \quad (51)$$

The β group differs from the α group only by replacing α with β.

For convenience in calculation, the first six series of the α group are given below with numerical coefficients expressed as common fractions.

$$\left.\begin{aligned}
C_1(\alpha) &= 1 \\
C_2(\alpha) &= 1 - \frac{7}{4}\sin^2\alpha \\
C_3(\alpha) &= 1 - \frac{9}{2}\sin^2\alpha + \frac{33}{8}\sin^4\alpha \\
C_4(\alpha) &= 1 - \frac{33}{4}\sin^2\alpha + \frac{143}{8}\sin^4\alpha - \frac{715}{64}\sin^6\alpha \\
C_5(\alpha) &= 1 - 13\sin^2\alpha + \frac{195}{4}\sin^4\alpha - \frac{1105}{16}\sin^6\alpha \\
&\qquad + \frac{4199}{128}\sin^8\alpha \\
C_6(\alpha) &= 1 - \frac{75}{4}\sin^2\alpha + \frac{425}{4}\sin^4\alpha - \frac{8075}{32}\sin^6\alpha \\
&\qquad + \frac{33\,915}{128}\sin^8\alpha - \frac{52\,003}{512}\sin^{10}\alpha
\end{aligned}\right\} \quad (52)$$

The β group is identical with the above except that α is everywhere replaced by β.

The method of using the above formula for computing the mutual inductance of concentric and coaxial current sheets can be illustrated by applying it to a numerical example. Assume the following constants of the two solenoids:

	Large Solenoid	*Small Solenoid*
Length	$B = 60$ cm	$b = 30$ cm
Diameter	$A = 20$ cm	$a = 10$ cm
Number of turns	$N = 600$	$n = 300$

Compute the following constants of each solenoid:

Diagonal	$D = 20\sqrt{10}$ cm	$d = 10\sqrt{10}$ cm
	$\sin^2\alpha = 1/10$	$\sin^2\beta = 1/10$

The values for the series $C_k(\alpha)$ and $C_k(\beta)$ are

$$\begin{aligned} C_1(\alpha) &= C_1(\beta) = 1 \\ C_2(\alpha) &= C_2(\beta) = 0.825 \\ C_3(\alpha) &= C_3(\beta) = 0.59125 \\ C_4(\alpha) &= C_4(\beta) = 0.34258 \\ C_5(\alpha) &= C_5(\beta) = 0.1217 \\ C_6(\alpha) &= C_6(\beta) = -0.0394 \end{aligned}$$

Also

$$\frac{d^2}{D^2} = \frac{1}{4}.$$

Substituting in the formula for mutual inductance,

$$M = \frac{\pi^2 \cdot 600 \cdot 300 \cdot 100}{20\sqrt{10}} \left\{ 1 - \frac{1}{20} \left[\frac{1}{4}(0.825) + \tfrac{1}{16}(0.825)(0.59125) + \tfrac{1}{64}(0.59125)(0.34258) + \tfrac{1}{256}(0.3426)(0.1217) - \tfrac{1}{1024}(0.1217)(0.0394) \right] \right\}$$

Converting each term into a decimal fraction

$$\begin{aligned} M &= \pi^2 \cdot 9 \times 10^4 \cdot \sqrt{10} \{ 1 - 0.010312 - 0.001524 - 0.000158 - 0.000008 + 0.0000002 \} \\ &= 2\ 775\ 225 \text{ millimicrohenrys} \\ &= 2.775225 \text{ millihenrys} \end{aligned}$$

This same case was independently computed by the formula of Roiti, giving $M = 2.775225$ millihenrys. The complete agreement is accidental, as the series was not carried in either computation to more than the sixth decimal place, giving one part in a million in the result.

As mentioned previously, the formula just given is applicable only to coils which are long relative to their diameter. In the above example the length of each coil is three times the diameter.

The formula would not be applicable to coils appreciably shorter than this, but there is little probability that shorter coils would be used in absolute measurements.

In order to apply the current sheet formula for the mutual inductance of concentric and coaxial solenoids to actual windings consisting of turns of wire, the principles involved in developing the formula must be given consideration. For this purpose consider that the mutual inductance is defined as the electromotive force induced in the inner coil divided by the rate of change of current in the outer coil, and determine the effect on the induced electromotive force of having the solenoids made of wire of finite cross section and measurable spacing, instead of an idealized current sheet.

First consider the magnetic field produced by a current in the wires of the outer solenoid. Just inside the solenoid the lines of magnetic intensity are not straight, but have a wavy appearance. It has been shown by Snow[1] that the amplitude of these waves decreases rapidly as the distance from the wires increases, so that, if the pitch of the winding is 1 mm, at a distance of 1 cm from the wires the amplitude is less than one-millionth of the value of the magnetic intensity. Hence, if the diameter of the inner solenoid is less than the diameter of the outer by more than 2 cm, a negligible error is introduced because of waves in the lines of magnetic intensity.

A second consideration is the method to be adopted in determining the length of the outer solenoid. Since the axial component of the magnetic field of a current in a wire solenoid has no appreciable waves in the region under consideration, this component is the same as would be produced by the same current in a current sheet formed by flattening the wires into ribbons. Hence the length to be taken for the wire solenoid is the length of the current sheet produced by flattening the wires, which again is equal to the number of turns times the pitch. This is a length greater than the physical length of the winding of the solenoid.

A third consideration is the equivalent diameter of the outer solenoid when wound with wire. In the formula for computing the inductance, the diameter of the outer solenoid does not enter directly, but is required for computing the length of the diagonal. Hence the longer the solenoid, the less the accuracy required in

[1] *B.S. J. Research*, **1**, 697 (1928).

determining the equivalent diameter. The diameter of a wire solenoid from the center of the wire on one side to the center on the opposite side can be measured with high accuracy. This does not signify, however, that this is the diameter of the equivalent current sheet, for, although the magnetic field outside a wire carrying a current which is uniformly distributed over its cross section is the same as though the current were concentrated on the axis, the magnetic field outside a ribbon, which takes the place of a wire in a current sheet, cannot be considered as emanating from a filament at its center. As a first approximation, the diameter to the centers of the wires may be assumed to be equivalent to the current sheet diameter. This method of measuring the diameter of the outer solenoid may introduce an appreciable error in the computed mutual inductance between the two wire-wound solenoids.

A fourth consideration is the helical nature of the windings of the wire solenoids. Any element of a helical winding may be resolved into a circular part and an axial part. The circular part is the equivalent of the current sheet, while the axial component produces a magnetic field at right angles to the axial. If the inner solenoid were a true current sheet with the conductors exactly perpendicular to the axis, the magnetic field of the axial component of the outer solenoid would not produce an induced electromotive force in the inner solenoid. However, with the inner solenoid in the form of a helix, there is an axial component of the inner solenoid in which an electromotive force is induced. The magnitude of the induced electromotive force in the axial component of the inner solenoid can be approximated by assuming that it is equal to that which would be induced in a wire, lying in the common axis, by the normal current of the outer cylinder when in a wire lying in the surface of the outer solenoid and parallel to its axis. In the example just solved, the inductance caused by the axial components of the windings would be approximately 150 millimicrohenrys, or about 50 parts in a million. If both windings correspond to a right-hand screw or both to a left-hand screw, the inductance of the axial components adds to that of the current sheet. If one corresponds to a right-hand screw and the other to a left-hand screw, the inductance of the axial components subtracts from that of the current sheet. The formula for computing the mutual inductance of parallel wires is given in a later section.

A fifth consideration is the diameter of a current sheet which will be equivalent to the inner solenoid wound with wire. In a wire solenoid, the induced electromotive force is not the same at all points of a cross section. Hence it is necessary to determine the average value of the induced electromotive force, and then find the diameter of the current sheet in which this electromotive force would be induced. Since the magnetic field of the outer solenoid is nearly uniform over the region occupied by the inner solenoid, an approximate correction for the finite cross section of the wire of the inner solenoid can be obtained by determining the average electromotive force that is induced in a single turn of wire placed with its center at the center of the inner solenoid and its plane perpendicular to the axis of the solenoid when the current in the outer solenoid is changing at a known rate. The wire of this turn can be considered as made up of circular filaments, the electromotive force in each being equal to the area of the circle which it bounds multiplied by the rate of change of the magnetic induction (a constant for all the filaments). The average electromotive force is found by writing the general expression for the electromotive force induced in any filament, integrating over the cross section of the wire and dividing by the cross-sectional area. Carrying out this procedure for a turn of wire in which the axis of the turn has a diameter a and the wire a diameter w, those filaments which have a diameter equal to $a\sqrt{1 + \frac{w^2}{4a^2}}$ have an average electromotive force induced in them. This correction can then be applied to the mean diameter of the wire solenoid to obtain the diameter of the equivalent current sheet.

The formula for the mutual inductance of two concentric and coaxial solenoids wound with wire can be obtained by applying the corrections, as outlined, to the current sheet formula. The measurable dimensions of the solenoids, with symbols, are as follows:

Dimension	Symbols	
	Outer solenoid	Inner solenoid
Mean diameter, center of wire to center of wire	A	a
Diameter of wire	W	w
Pitch of winding	P	p
Number of turns	N	n

From the measured dimensions, the following constants can be determined:

Constant	Symbol and formula	
	Outer solenoid	Inner solenoid
Equivalent length.......	$B = NP$	$b = np$
Diagonal...............	$D = \sqrt{A^2 + B^2}$	$d = \sqrt{a^2 + b^2}$
Angular opening........	$\sin \alpha = A/D$	$\sin \beta = a/d$
Equivalent diameter....	....	$a_e = a\sqrt{1 + \frac{w^2}{4a^2}}$

The formula for the mutual inductance M of the concentric and coaxial wire solenoids is

$$M = \frac{\pi^2 N n a_e^2}{D}\left\{1 - \frac{\sin^2 \alpha}{2}\sum_{k=1}^{k=\infty}[C_k(\alpha)][C_k(\beta)]\left(\frac{d^2}{D^2}\right)^k\right\}$$
$$\pm \{(B + b) \ln [B + b + \sqrt{A^2 + (B + b)^2}]$$
$$- (B - b) \ln [B - b + \sqrt{A^2 + (B - b)^2}] - 2b \ln A$$
$$- \sqrt{A^2 + (B + b)^2} + \sqrt{A^2 + (B - b)^2}\} \quad (53)$$

where $C_k(\alpha)$ and $C_k(\beta)$ are the functions given under the current sheet formula and ln is the natural logarithm. The $\pm$ sign indicates that the result depends on the nature of the pitch of the two solenoids, the plus sign to be used if both are right-handed or both left-handed, the minus sign if one is right-handed and the other left-handed.

Since the above formula neglects certain corrections and estimates others, it is not to be expected that results of the highest accuracy can be obtained by its use. However, there is no reason to believe that the error in the computed inductance will be large; it is almost certainly less than 100 parts in a million, and possibly less than 10 parts.

b. The Mutual Inductance between a Helix and a Coaxial Circle Lying in an End Plane of the Helix.—A formula for computing the mutual inductance between a helix and the circumference of a coaxial circle which lies in a plane passing through *one of the end points* of the helix was derived by Jones.[1] The derivation

[1] *J.* Viriamu Jones, *Proc. Roy. Soc.* (*London*), **63**, 198 (1898).

assumes that the wires used in constructing the helix and the circle have an infinitesimal cross section. The number of turns in the helix does not need to be an integer.

The measured constants required in the computation are:

D = diameter of circle.
H = diameter of helix.
p = pitch of helix.
z = axial length of helix — pitch times the number of turns.

From the measured constants, the following constants are computed:

$$k^2 = \frac{4HD}{(H+D)^2 + 4z^2}$$

$$k' = \sqrt{1-k^2}$$

$$\beta = \sin^{-1}\left[\frac{H-D}{k'(H+D)}\right]$$

The formula is

$$M = \frac{\pi}{2p}\left[2z\sqrt{(H+D)^2+4z^2}\{K-E\} - (H^2-D^2)\left\{\frac{\pi}{2} + (K-E)[F(k',\beta)] - K[E(k',\beta)]\right\}\right] \quad (54)$$

where K = *complete* elliptic integral of the *first* kind to modulus k.
E = *complete* elliptic integral of the *second* kind to modulus k.
$F(k',\beta)$ = the *incomplete* elliptic integral of the *first* kind to modulus k' and amplitude β.
$E(k',\beta)$ = the *incomplete* elliptic integral of the *second* kind to modulus k' and amplitude β.

As elliptic integrals enter into several important inductance formulas, a discussion of their meaning and the methods of evaluating them will be given in Appendix I.

As the formula given is an exact mathematical derivation, any error therein must result from the assumptions on which the derivation rests. The important assumptions that cannot be fulfilled in an actual inductor are (1) that the wire used in constructing the helix has infinitesimal cross section; (2) that the form of the helix is geometrically perfect; (3) that the wire used in the circle has infinitesimal cross section; (4) that the circle has a true geometric form. An assumption that can be verified by electrical measurements is that the circle is coaxial with the helix. The effect on the computed inductance of the assumptions that cannot be fulfilled can be approximately determined by analysis.

The finite cross section of the wire of the helix will introduce no correction in the computed inductance provided the current in the wire is symmetrically distributed about the center. This follows from the general proposition that the lines of magnetic intensity around a wire carrying a symmetrically distributed current are circles. With alternating current of high frequency, the current distribution is not symmetrical about the center of each wire but depends on the frequency of the alternating current, on the diameter of the wire and its resistivity, and on the pitch of the winding. No formula is known for computing the mutual inductance which includes the above factors. Hence the inductance formula is applicable only at low frequency.

The significant imperfections in the form of the helix are ellipticity of cross section, conicality of supporting form, and irregularities in the pitch. The ellipticity will introduce little error if the mean diameter is measured. The effect of conicality, however, is not eliminated by taking the average diameter over the length, since the turns at the end next to the circle have more influence on the mutual inductance than those farther away. To obtain the mutual inductance in the case of a conical helix, consider that the helix is made up of a series of circles, the distance between them corresponding to the pitch of the helix, and the diameter of each corresponding to that of the helix at that point. Now the mutual inductance between the circle and the helix is so nearly equal to the sum of the mutual inductances between this circle and each of the circles of the series that correction terms can be computed on the assumption that they are exactly equal. The change in mutual inductance between two coaxial circles with their centers at a distance x, when the diameter of the smaller circle varies, is given by the equation

$$\frac{\partial M}{\partial a} = \frac{2\pi a}{\sqrt{(A + a)^2 + 4x^2}}\left[K + \frac{A^2 - a^2 - 4x^2}{(A + a)^2 + 4x^2}E\right] \qquad (55)$$

where K and E are complete elliptic integrals of the first and second kind, respectively, to modulus

$$k = 2\sqrt{Aa}/\sqrt{(a + A)^2 + 4x^2}.$$

To avoid computing k for each circle, a few computations can be made for different values of x, and these can be plotted in a curve, from which the value for each of the other circles can be determined. The correction for a circle is the product of its value of $\frac{\partial M}{\partial a}$ multiplied by its variation from the mean diameter. The total correction is the algebraic sum of the corrections for the individual circles.

The correction for the irregularities in the pitch is made in a manner similar to that just given for the variations in diameter. The helix is replaced by a series of circles. The relationship between the change of mutual inductance with the displacement of a circle $\frac{\partial M}{\partial x}$ and the distance x from the plane of the secondary circle is determined for several points, and a curve is plotted from which the effect of a displacement at any point can be estimated. Then, from the measured displacement δx of each helical turn, the correction to the mutual inductance is computed as $\left(\frac{\partial M}{\partial x}\right)_x \delta x$. The total correction is the algebraic sum of the separate turns. The formula for computing the change of mutual inductance with displacement is

$$\frac{\partial M}{\partial x} = \frac{8\pi x}{\sqrt{(A + a)^2 + 4x^2}}\left[\frac{A^2 + a^2 + 4x^2}{(A - a)^2 + 4x^2}E - K\right] \qquad (56)$$

where the symbols have the same meaning as in the preceding formula.[1]

The correction for the finite cross section of the secondary circle depends not only on the diameter of the wire but also on the diameter of the circle. A satisfactory discussion is difficult and will be reserved until the applications of the formula are considered. The same statement applies to the corrections for the imperfections in the geometric form of the secondary circle.

The above formula can be used to compute the mutual inductance between any solenoid and a circle lying in a plane perpendicular to the axis of the solenoid and passing through the center of the wire at an extreme end of the solenoid. However,

[1] For an application of these correction formulas, see paper by Dye and Hartshorn, *N.P.L., Coll. Researches*, **21**, 1 (1929).

the computation is quite laborious. A more convenient formula[1] is available when the length of the solenoid is larger than its diameter. This formula is

$$M = \frac{2\pi^2 a^2 n}{d}\left(1 + \frac{3a^2A^2}{8d^4} + \frac{5}{64}\frac{a^4A^4}{d^8}\chi_2 + \frac{35}{512}\frac{a^6A^6}{d^{12}}\chi_4 + \frac{63}{1024}\frac{a^8A^8}{d^{16}}\chi_6 + \cdots\right) \quad (57)$$

where a = smaller radius (solenoid or circle).
A = larger radius (circle or solenoid).
l = axial length of solenoid.
n = number of turns in length l.
$d = \sqrt{A^2 + l^2}$.
$\chi_2 = 3 - \frac{4l^2}{A^2}$.
$\chi_4 = \frac{5}{2} - \frac{10l^2}{A^2} + \frac{4l^4}{A^4}$.
$\chi_6 = \frac{35}{16} - \frac{35l^2}{2A^2} + \frac{21l^4}{A^4} - \frac{4l^6}{A^6}$.

c. The Self-inductance of a Helix and Its Leads.—The self inductance of a helix wound with round wire can be computed by a formula developed by Snow.[2] The constants needed for the computation are:

D = diameter of helix from center of wire on one side to center of wire on opposite side, in centimeters.
w = diameter of wire, in centimeters.
p = pitch of winding, in centimeters.
N = number of turns.
$l = pN$ = length of helix from beginning of first turn to end of last turn, in centimeters.

The computed constant is

$$k^2 = \frac{D^2}{D^2 + l^2}$$

The self inductance L of the helix in millimicrohenrys is given by the formula

[1] Rosa, *Bull. B. S.*, **3**, 209 (1907).
[2] *B. S. J. Research*, **9**, 419 (1932).

$$
\begin{aligned}
L = {} & \frac{4\pi N^2}{3}\sqrt{D^2 + l^2}\left[K - E + \frac{D^2}{l^2}(E - k)\right] \\
& + \pi D\left[2N\left(\ln\frac{p}{w} - 0.89473\right) + \frac{1}{3}\ln\frac{\pi D}{p}\right] \\
& - \frac{2\pi}{3}\sqrt{D^2 + l^2}\left(K - E - \frac{k^2 K}{2}\right) \\
& - \frac{2}{\pi}\sqrt{D^2 + l^2}\left(2 \pm \frac{\pi^2 w^2}{4p^2}\right)(E - k) \\
& + lP - \frac{\pi l}{2}\left(1 - \frac{l}{D}\tan^{-1}\frac{D}{l}\right) \qquad (58)
\end{aligned}
$$

where K = complete elliptic integral of the first kind to modulus k.

E = complete elliptic integral of the second kind to modulus k.

$$
\left.
\begin{aligned}
P &= \frac{D^2}{4l^2} + 2\ln\frac{4l}{D}, \qquad \text{when} \quad l \geqq D \\
&= \frac{3l}{D} - \frac{l}{D}\ln\frac{l}{D}, \qquad \text{when} \quad l \leqq D
\end{aligned}
\right\} \qquad (59)
$$

The first term is the same as the formula of Lorenz for a cylindrical current sheet, and it is the most important term of the equation. The second term is the principal correction term and generally needs to be determined to about 1 per cent of the accuracy required for the first term. The third term is important only for short coils, being equal to the mutual inductance between the end turns. The last three terms are usually negligible. In the term having a plus and minus sign ($\pm$), the negative sign is to be taken if the current density is assumed uniform over the cross section of the wire, the positive sign if the current density is assumed to vary inversely as the length of the helical filaments into which the wire is considered to be divided by surfaces having the lines of current flow as generators (the so-called natural distribution). Since there is no experimental method of determining which current distribution is correct, this term serves only to show the accuracy that can be expected.

The preceding formula has been developed on the assumption that the winding is a perfect helix. Small irregularities in the winding affect the computed inductance, requiring a small correction term. Before giving the formula for computing the effect of irregularities in this winding, the method of measurement to obtain the irregularities will be discussed.

Consider that the helix has n turns and that the radial leads are attached at the beginning of the first turn and at the end of the nth turn. Also consider that measurements of the axial position are to be made at q points on each turn, the positions on the different turns being on the same generators of the cylindrical form. From these q measurements, the average position of a turn is determined. The pitch p of the helix is obtained by dividing the average distance between the first turn and nth turn by $(n - 1)$. The deviation $\delta(r)$ of the rth turn from the position it is supposed to occupy is determined from the average distance A_r between the first and rth turns by the equation

$$A_r - p(r - 1) = \delta(r) \tag{60}$$

For values of r greater than $n/2$, the deviation may be determined from the distance ${}_nA_r$ between the nth turn and the rth turn. Then

$${}_nA_r - p(n - r) = -\delta(r) \tag{61}$$

When the deviation for the second half of the solenoid is determined by Eq. (61), a small error in the pitch will not introduce an appreciable error in the correction for irregularities in winding.

It should be noted that, when a circle is farther from the first turn than it should be, the deviation of that turn is positive, while, if it is nearer, the deviation is negative. The computation of the correction term is simplified by combining the deviations of turns which occupy symmetrical positions with respect to the two ends. Thus

$$\left.\begin{aligned} f(2) &= \delta(2) - \delta(n - 1) \\ f(3) &= \delta(3) - \delta(n - 2) \\ f(r) &= \delta(r) - \delta(n - r + 1) \\ f\left(\frac{n}{2}\right) &= \delta\left(\frac{n - 1}{2}\right) - \delta\left(\frac{n + 1}{2}\right) \\ f(n - 1) &= -f(2) = \delta(n - 1) - \delta(2) \end{aligned}\right\} \tag{62}$$

The second half of the series repeats the first half but in reverse order and with the sign changed.

The formula[1] for computing the correction term δL is derived by assuming (1) that each turn of the helix can be replaced by a circular hoop located at the average axial position of the turn, and (2) that the radial component of the magnetic field of the helix is identical with that of a current sheet having the same number of turns. The correction term is given by the equation

$$\delta L = \frac{4\pi D}{p}\sum_{r=2}^{n-1}\psi(r)\cdot f(r) \tag{63}$$

where $f(r)$ is the experimental function described above and $\psi(r)$ is a theoretical function given by the equation

$$\psi(r) = \frac{2}{k}(K - E) - kK \tag{64}$$

where $k^2 = \dfrac{D^2}{D^2 + p^2(r-1)^2}$ and K and E are complete elliptic integrals of the first and second kind, respectively, to modulus k. Because of the antisymmetry in the values of $f(r)$, the equation for δL may be written

$$\delta L = \frac{4\pi D}{p}\sum_{r=2}^{\frac{n}{2}} f(r)[\psi(r) - \psi(n - r + 1)] \tag{65}$$

If the values of $f(r)$ are in centimeters, the value of δL will be in millimicrohenrys.

The computation of the function $\psi(r)$ by elliptic integrals is somewhat tedious. While this function has the same form as the elliptic integral portion of the formula for the mutual inductance between two circles, for which numerous tables are available,[2] yet the computation is laborious, on account of the difficulty of determining k from D, p, and r. A simpler and sufficiently accurate method of obtaining $\psi(r)$ is to compute it from some of the series formulas which have been developed for computing the mutual inductance between circles.

The following set of formulas can be used for any value of r from 2 to infinity. To use these formulas, first compute the

[1] Snow, The Effect of Small Variations in Pitch upon the Inductance of a Solenoid, *B.S. J. Research*, 6, 777 (1931).

[2] Nagaoka and Sakurai, *Sci. Pap. Inst. Phys. Chem. Research*, Table 2 (1927).

value of an auxiliary constant λ from the equation

$$\lambda = \frac{p(r-1)}{D}$$

Then the formulas and limits are

$$\left.\begin{array}{lll} & \textit{Formulas} & \textit{Limits} \\ (1) & \psi(r) = \left(1 + \frac{3}{4}\lambda^2\right)\left(\ln\frac{4}{\lambda}\right) - \left(2 + \frac{1}{4}\lambda^2\right) & 0 < \lambda < 0.4 \\ (2) & \psi(r) = \frac{\pi}{2}[(\sqrt{1+\lambda^2} - \lambda)^3 & \\ & \qquad + \frac{3}{8}(\sqrt{1+\lambda^2} - \lambda)^7] & 0.4 < \lambda < 2 \\ (3) & \psi(r) = \frac{\pi}{16\lambda^3}\left(1 - \frac{3}{4\lambda^2}\right) & 2 < \lambda < \infty \end{array}\right\} \quad (66)$$

The accuracy required in the computation of $\psi(r)$ is not great, but will depend on the precision of winding and on the pitch. As an example, consider a coil having a pitch of 1 mm (1000 microns) and a maximum value of $\delta(r)$ of 50 microns. Also assume that the position of each turn can be measured to 1 micron, so that an error of 2 per cent in $\delta(r)$ can be expected. Hence if $\psi(r)$ is computed with an accuracy of 1 per cent, no appreciable error in δL will result from the uncertainty in $\psi(r)$. The above formulas will give this accuracy with the possible exception of a few points near the boundary limits of the formulas, and the errors that these points can introduce will be quite negligible in the summation. Instead of computing for every value of r, a few may be computed and the values of the remainder determined by plotting a curve.

The inductance of the leads of a helix must be computed and added to the computed inductance of the helix to obtain the total inductance up to the point of connection to the outside circuit. A simple method of locating the leads is shown in Fig. 35, in which the two leads are symmetrical with respect to the two ends of the helix. Each lead consists of three straight parts, designated as PQ, QR, and RS, with the same symbols primed for the symmetrical lead. The self inductance of each of these parts must be computed, and also the mutual inductance between each part and every other part, including the helix. The construction is such that the mutual inductance between each of the following pairs is zero: PQ and helix; $P'Q'$ and helix; RS and helix; $R'S'$ and helix; PQ and QR; $P'Q'$ and $Q'R'$; QR and RS; $Q'R'$ and $R'S'$; PQ and $R'Q'$; $P'Q'$ and RQ; QR and $R'S'$; and $Q'R'$ and RS. In addition, the mutual inductance between PQ and $P'Q'$ is generally negligible, while that between PQ and RS is nearly equal and opposite to that between PQ and $R'S'$; likewise that between $P'Q'$ and $R'S'$ opposes that between $P'Q'$ and RS. Hence, of the twenty-one possible mutual inductances, only

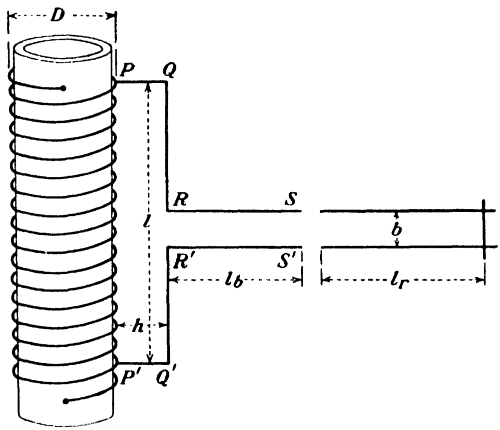

FIG. 35.—The position of the leads to a helix. At the right is the substitution inductor.

four need be computed, *viz.*, $M_{SS'}$ between RS and $R'S'$, $M_{QQ'}$ between QR and $Q'R'$, M_{Rh} between QR and helix, and $M_{R'h}$ between $Q'R'$ and helix. As the latter two are identical, only three computations are necessary.

The total inductance of a helix including leads may be written

$$L = L_h + 2l_{PQ} + 2l_{QR} + 2l_{RS} - 2M_{SS'} - 4M_{Rh} + 2M_{QQ'} \quad (67)$$

where L_h = self inductance of the helix.

l_{PQ}, l_{RQ}, and l_{RS} = self inductances of straight wires.

$M_{SS'}$, M_{Rh}, and $M_{QQ'}$ = mutual inductances as already indicated.

A formula has been given for L_h, and below are given formulas for computing the self and mutual inductance of the straight parts of the circuit. The part that needs special consideration is M_{Rh}, which is the mutual inductance between a helix and the straight wire parallel to its axis and of nearly half its length.

The mutual inductance between the helix and the two leads QR and $Q'R'$ can be approximated by computing the mutual inductance between a wire at the axis of the helix and a wire extending from Q to Q'. A more exact formula is

$$2M_{Kh} = lP - lQ - \frac{4}{\pi}\left(\frac{D+h}{k}\right)(E - k) \tag{68}$$

where
$$P = \frac{D^2}{4l^2} + 2 \ln \frac{4l}{D}, \text{ if } l \geqq D.$$
$$= \frac{3l}{D} - \frac{l}{D} \ln \frac{l}{D}, \text{ if } l < D.$$
$$Q = \frac{D^2}{l^2} + 2 \ln \frac{2l}{D}, \text{ if } l \geqq 2D.$$
$$= \frac{3l}{2D} - \frac{l}{2D} \ln \frac{l}{2D}, \text{ if } l < 2D.$$
$$k^2 = \frac{D^2}{D^2 + l^2}.$$

E = complete elliptical integral of second kind to modulus k.

d. Self Inductance of a Straight Wire.—The self inductance L, in millimicrohenrys, of a straight cylindrical wire of length l and diameter d is

$$L = 2l \ln \left(\frac{2l + \sqrt{4l^2 + d^2}}{d}\right) + d + \frac{l}{2} - \sqrt{4l^2 + d^2} \tag{69}$$

If the length is large relative to the diameter, then approximately

$$L = 2l \ln \frac{4l}{d} + d - \frac{3l}{2} \tag{70}$$

In this equation, the largest neglected term is $d^2/4l$.

e. Mutual Inductance between Parallel Wires.—The mutual inductance M between any two parallel wires of lengths l and m, r distance apart and with distance q between one pair of corresponding ends (see Fig. 36) is given by the formula

$$\begin{aligned} M = {} & (q + m) \ln [q + m + \sqrt{r^2 + (q + m)^2}] - \sqrt{r^2 + (q + m)^2} \\ & + (q - l) \ln [q - l + \sqrt{r^2 + (q - l)^2}] - \sqrt{r^2 + (q - l)^2} \\ & - (m + q - l) \ln [m + q - l + \sqrt{r^2 + (m + q - l)^2}] \\ & \qquad + \sqrt{r^2 + (m + q - l)^2} \\ & - q \ln (q + \sqrt{r^2 + q^2}) + \sqrt{r^2 + q^2} \end{aligned} \tag{71}$$

This formula is derived on the assumption that the diameters of the wires are so small relative to their distance apart that the induced electromotive force in a filament at the center of a wire is equal to the average electromotive force over the cross section of the wire.

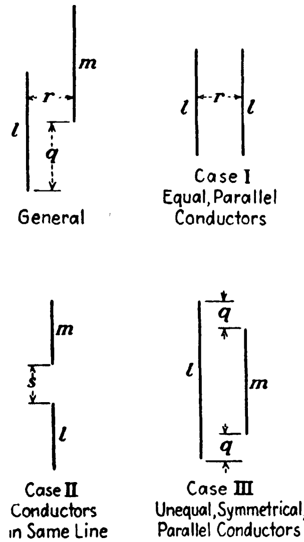

FIG. 36.—Nomenclature for the mutual inductance between parallel wires.

In using the value obtained by this formula, care must be exercised to ensure that the correct sign of the mutual inductance is taken. The formula is derived on the assumption that, for an increasing current in one conductor, the direction of the induced electromotive force in the other conductor is the same as the direction of the current in the first conductor. The above general formula simplifies for special cases.

Case I. Equal, parallel conductors:

$$q = 0, \qquad l = m$$

$$\therefore M = 2l \ln\left(\frac{l + \sqrt{r^2 + l^2}}{r}\right) - 2\sqrt{r^2 + l^2} + 2r \qquad (72)$$

Case II. Conductors in the same line:

$$r = 0, \qquad q = l + s$$

where s is the distance between the two adjacent ends.

$$M = (s + l + m) \ln (s + l + m) + s \ln s - (s + m) \ln (s + m) - (s + l) \ln (s + l) \qquad (73)$$

Case IIA. Equal adjacent conductors in the same line:

$$r = 0, \qquad s = 0, \qquad l = m$$

$$M = 2l \ln 2 = 1.386l \qquad (74)$$

Case III. Unequal symmetrical parallel conductors:

$$l > m \qquad q = \frac{l - m}{2}$$

$$M = (l + m) \ln [l + m + \sqrt{4r^2 + (l + m)^2}] - (l - m) \ln [l - m + \sqrt{4r^2 + (l - m)^2}] - 2m \ln 2r + \sqrt{4r^2 + (l - m)^2} - \sqrt{4r^2 + (l + m)^2} \qquad (75)$$

CHAPTER XV

THE DESIGN AND CONSTRUCTION OF ABSOLUTE STANDARDS OF INDUCTANCE

In order to produce an inductance which can be accurately computed from measured dimensions, attention must be given both to the design of the standard and to its construction. The design of a standard involves not only its form and dimensions, but also the materials of which it is to be made. The construction of a standard requires the use of methods which can be so applied to the materials selected by the designer that the resulting product will closely conform to the desired form and dimensions.

The designs will be considered for only three forms of inductors, *viz.:* (1) a mutual inductor consisting of two concentric, coaxial solenoids, (2) Campbell's mutual inductor, and (3) a self inductor consisting of a single-layer solenoid. All these forms require the construction of one or more large solenoids consisting of a cylindrical tube of some insulating material on which a helix of wire is wound.

84. Materials Suitable for the Supporting Cylinder of a Solenoid.—The materials which are suitable for the insulating cylinder of a solenoid are not numerous. The essential properties are: permanence over long periods of time, small susceptibility, high surface and volume resistivity, low coefficient of thermal expansion, and suitable mechanical qualities to permit machining or grinding. There are other desirable qualities, such as, low dielectric constant, small energy absorption in alternating-current fields, and small compressibility.

Materials which approximately fulfill the essential requirements are statuary marble, electrical porcelain, low-expansion glass, and silica glass (fused quartz). Statuary marble can be obtained in any desired size, and it can be readily machined. However, its insulating properties are only reasonably good, its thermal expansion is relatively high, and some samples apparently change their dimensions slightly with time when subjected to pressure.

Electrical porcelain is difficult to work, and some specimens contain enough iron to affect the susceptibility. Low-expansion glass is difficult to work. Silica glass is also difficult to work and in large sizes is available only in a porous form. The properties of these four materials are listed in Table IX.

TABLE IX.—PROPERTIES OF MATERIALS SUITABLE FOR THE SUPPORTING CYLINDER OF A SOLENOID

Properties	Statuary marble	Electrical porcelain	Low-expansion glass[1]	Silica glass (fused quartz)
Susceptibility	-0.4×10^{-6}	Variable	-1.0×10^{-6}	-0.4×10^{-6}
Surface resistivity, 70 per cent humidity, ohms	10^8	10^{10}	10^{10}	10^{14}
Volume resistivity, ohm-cm	10^{11}	10^{14}	10^{16}	10^{18}
Linear expansion, p.p.m. for 1°C	5	4	3	0.5
Dielectric constant	8.3	6 to 7	6 to 8	3.5
Power factor	0.05	No data	0.01	0.0002
Method of working	Machine tools	Grind	Grind	Grind
Permanence	Accurate data lacking. Probable order, starting with the most permanent, is (1) silica glass, (2) porcelain, (3) low-expansion glass, (4) marble			

[1] The data are for chemical pyrex.

Of these four materials, silica glass is the most suitable, but until recently sufficiently large pieces have not been available. Marble has been most often used, but is probably the least desirable of the four.

85. Construction of a Cylindrical Form.—In order to construct a cylindrical form, marble may be turned, but glass and porcelain must be ground.

The method of turning marble in a lathe is well known. Methods for grinding a glass or porcelain cylinder are not so well known. Probably sufficient accuracy can be obtained in the highest grade of commercial grinding machines, such as those used to grind the rollers for paper mills. A procedure for grinding a cylinder has been developed,[1] based on the methods used in optical shops for grinding lenses. The apparatus required is not expensive and in the hands of a skilled workman gives excellent results.

[1] Curtis, Moon, and Sparks, *J. Research N.B.S.*, **16**, 1 (1936).

In the optical-shop method, the cylinder, roughly ground inside and out in a lathe or grinding machine, is mounted so that it can rotate slowly about a horizontal axis. The axis of the cylinder does not need to coincide exactly with the axis of rotation, but the mounting of the cylinder must be of such a type that there is negligible distortion. A simple mounting consists of inflated rubber tubing wound on two flanged pulleys attached to the supporting axis. The pulleys are about ¾ inch less in

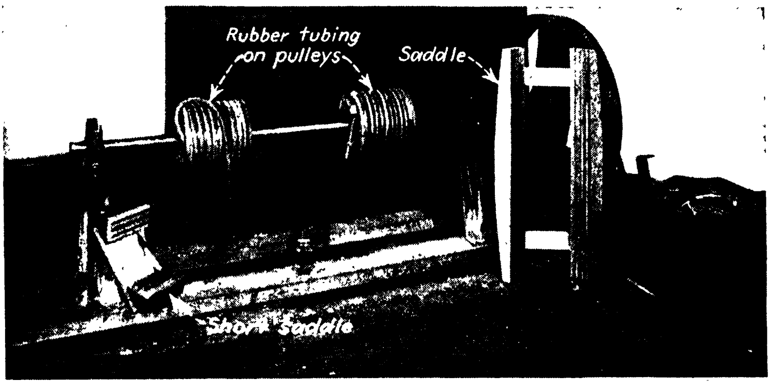

FIG. 36a.—The pulleys used to mount a cylinder and the laps (saddles) used in grinding it.

diameter than the inside of the cylinder to be ground, with ¼-inch flange on each edge and a face several inches wide. Rubber tubing of ½-inch diameter is wound on each pulley and the ends are suitably fastened. The tubing is then evacuated, so that it collapses, allowing the form to be easily slipped over the pulleys. The tubing is then inflated, causing the form to be held firmly to the supporting axis. For minimum distortion, the distance between the centers of the pulleys should be about five-eighths the length of the form to be ground. A photograph of a mounting which has been used is shown in Fig. 36a.

The grinding is accomplished by covering the rotating cylinder with a suspension of carborundum in oil or water and lightly pressing a hand-controlled saddle against the cylinder. The grinding parts of the saddle are six brass or aluminum bars, each of which is about two-thirds the length of the cylinder. The bars are mounted in a wooden or light metal frame in two sets of three each so that they conform to the outer surface of the cylinder, the centers of the two sets being separated by about 90°. With the rotating cylinder covered by the carborundum suspension, the saddle is slowly and uniformly moved from end to end of the cylinder, but with a stroke that carries about half the saddle beyond the end of the cylinder. After a definite number of strokes, the saddle is turned end for end and grinding is continued for the same number of strokes. At frequent intervals the cylinder must be cleaned and the diameter measured. If the

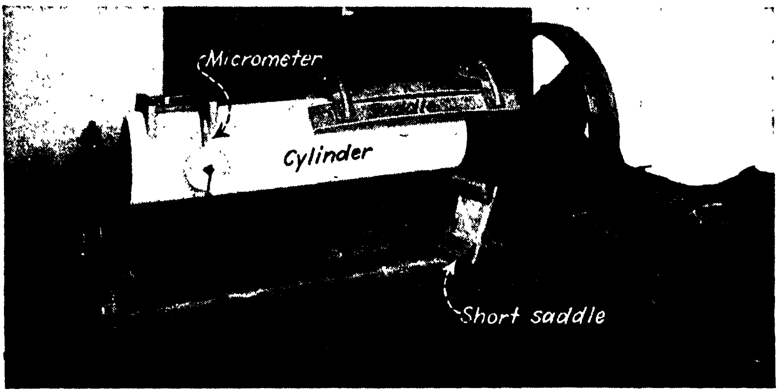

FIG. 37.—Setup for grinding a cylinder by the optical-shop method.

measurements show that the center is grinding more than the ends, at the next grinding the length of stroke of the saddle is increased; if the center is grinding less than the ends, the stroke is decreased. If the measurements show that the grinding is not sufficiently rapid in one or more restricted regions, a short saddle can be employed to reduce these regions. A setup for grinding a cylinder is shown in Fig. 37.

The measurements of diameter must be made when the variations of temperature throughout the cylinder are so small that they do not appreciably affect its shape. With a low-expansion material like silica glass, the temperature inequalities produced by grinding are not sufficient to produce measurable distortion, but with other kinds of glass and with most porcelains the distortions may be such that several hours must elapse between grinding and measuring.

The grinding should start with a fairly coarse grade of emery or carborundum. For grinding silica glass or porcelain, the abrasive should be free from all magnetic particles, since some particles will lodge in the pores and will be removed with great difficulty. As the grinding progresses, finer grades of abrasive should be used, the last grinding being with a very fine abrasive. The surface should not be polished, as the wire tends to slip on a polished surface.

Recently some experiments have been made by Moon on lapping a thread in a cylinder. This will have very great advantages, as the irregularities in winding can be greatly reduced or possibly eliminated. The results are very encouraging, but the method has not yet been used with a form for a large solenoid.

86. Wire for a Solenoid.—The most suitable material for the wire of a solenoid is hard-drawn copper.[1] Selected commercial wire may be used, but there are some advantages in procuring soft-drawn wire several sizes larger than desired and drawing it to the required size through sapphire or agate dies. (Diamond dies do not always give a good surface.) The drawing is carried out in a lathe. A large wooden drum is mounted in a lathe, with the die mounted on the carriage. Enough wire is pulled through the die by hand so that it can be attached to the drum. As the drum rotates, the carriage is advanced by the lathe screw so that a single layer of wire is wound on the drum. Another drum is substituted in the lathe, and a smaller die replaces the one previously used. The wire is then drawn from the first drum to the second. This process is continued until wire of nearly the correct size and hardness is obtained. The wire should be uniform and free from kinks.

87. Winding of a Solenoid.—The winding of the wire onto the cylindrical form (see Fig. 38) is carried out in a precision lathe, with the procedure much the same as for drawing the wire. The cylinder must be so carefully mounted that at each end the axis of rotation and the axis of the cylinder are coincident within a few hundredths of a millimeter. Distortion of the cylinder is minimized by applying the force necessary to clamp it to the sup-

[1] Recently oxygen-free copper wire has become available. This may give better wire than is now obtainable. The National Physical Laboratory has found silver wire unsatisfactory.

porting arbor through end disks which compress the cylinder in the axial direction. Radial forces are undesirable.

The die through which the wire is drawn is mounted on the lathe carriage so that its plane is at a slight angle with the direction of the wire as it leaves the die. This angle is adjusted until the wire as it leaves the die has a tendency to form into a coil having a diameter about that of the cylinder. As the wire is drawn directly through the die onto the cylinder, there is no tendency for the wire to roll out of place as is the case when a straight wire is bent around a cylinder. To improve the stability

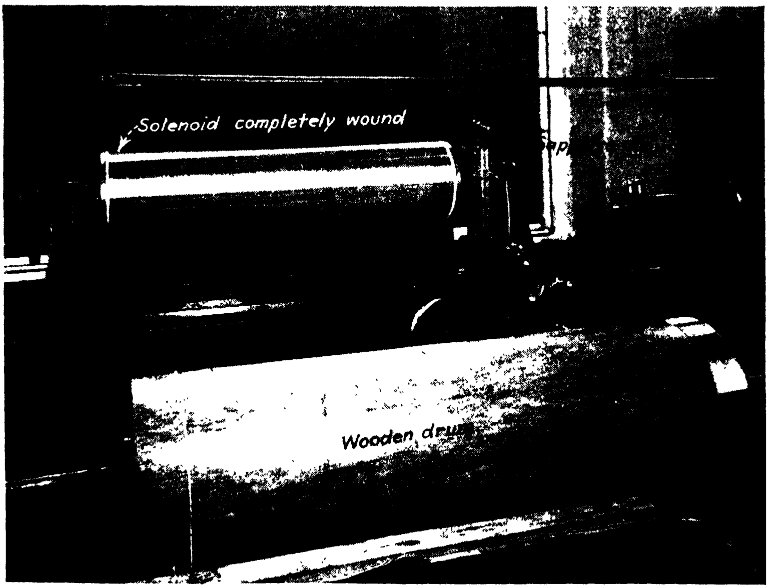

FIG. 38.—Setup for winding a solenoid.

of the wire on the cylinder, the under side of the wire is coated with a thin layer of shellac or other quick-drying adhesive. This coating is accomplished by means of a small hard-rubber roller, the bottom of which passes through the adhesive mixture, while the top touches the under side of the wire with sufficient force to produce a rotation of the roller. By regulating the consistency of the adhesive, a suitable amount may be transferred to the wire.

The ends of the winding may be securely held by attaching them to copper blocks set into the cylinder. These blocks should be so inserted in the cylinder that their upper faces are nearly coincident with the surface of the cylinder. The wire may be attached by clamping or by soldering. The winding may not be uniform for a few turns at the ends, so that the end blocks should not be used as the electrical terminals of the winding.

88. Construction of a Mutual Inductor Consisting of Concentric and Coaxial Solenoids.—Given two well-constructed solenoids having linear dimensions in the ratio of about 2 to 1, an absolute mutual inductor can be constructed by mounting them so that they are concentric and coaxial and by attaching suitable leads to each solenoid. It may not be feasible by mechanical measurements alone to place the two solenoids exactly concentric and coaxial. Hence one of the solenoids should be so mounted that its position can be adjusted. The following suggested mounting, in which the axes are horizontal, is only one of a number that might be employed.

The inner solenoid is mounted as shown in Fig. 39 on a long piece of plate glass which projects from each end of the large cylinder and which sets on pillars that are supported by the top of a pier. The outer solenoid is attached to a square of plate glass that is carried on three supports resting on the top of the pier. The three supports act in a hole-slot-plane system constructed in the glass and permit the five motions necessary to adjust the coils so that they will be coaxial and concentric. The support which is associated with the hole consists of a vertical screw mounted in a double slide, one slide moving in the direction of the axis of the coils, the other at right angles to it. The support associated with the slot consists of a vertical screw and a slide which moves in a direction perpendicular to the axis of the coils. The support associated with the plane consists of a vertical screw only. All six screws in the supports have graduated heads, so that their motions can be measured.

The five adjustments which are required to make the outer solenoid coaxial and concentric with the inner solenoid can be made as follows:

1. Translation axially: axial screw of hole support only.
2. Translation vertical: all three vertical screws turned same amount in the *same* direction.
3. Translation horizontally perpendicular to axis: screws of hole and slot in slides perpendicular to axis turned equal amounts in *same* direction.

4. Rotation about vertical axis through center: screws of hole and slot in slides perpendicular to axis turned equal amounts but in *opposite* directions.
5. Rotation about that horizontal axis through center which is perpendicular to axis of cylinder: vertical screws of slot and plane turned equal amounts in one direction, and screw of hole turned an equal amount in *opposite* directions.

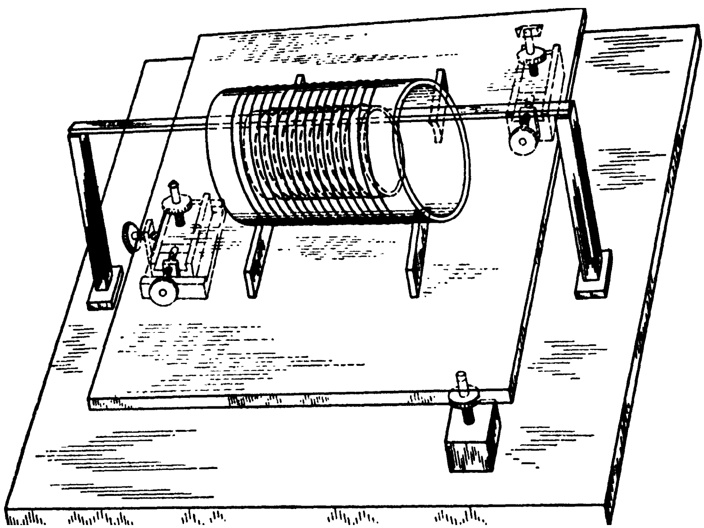

FIG. 39.—A hole-slot-plane mounting for adjusting two solenoids so that they are concentric and coaxial. The hole is at the center of the left side of the plate glass that supports the outer solenoid, the slot is at the back of the right side, and the plane is at the front of the right side.

To use the mounting to adjust the cylinders so that they are concentric and coaxial, some method of accurately measuring the change in mutual inductance produced by changes in position must be provided. For a given rotation or translation, the mutual inductance is determined for several positions in a region near that location where the mutual inductance goes through a maximum or minimum. The solenoid is then placed in the position that gives maximum or minimum, and the same procedure is employed to find the correct position with respect to each of the four other motions. After adjustments have been made for the five motions, the procedure should be repeated a second and even a third time to ensure that in making the later adjustments the first have not been affected.

For each translation or rotation the mutual inductance is either a maximum or a minimum when the position is that desired. A consideration of the direction and space variation of the magnetic induction produced by a current in the outer solenoid at a point near its center will show whether a maximum or a minimum is to be expected for a given translation or rotation. Near the

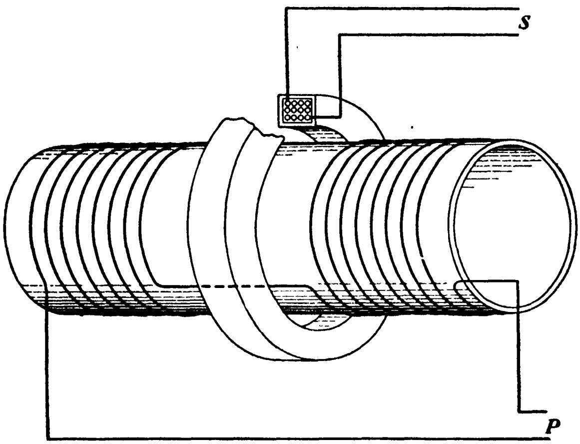

FIG. 39a.—Diagram of a Campbell mutual inductor.

center the direction of the magnetic induction is in the direction of the axis; the decrease in its value along a line parallel to the axis is more rapid as the end plane is approached; the increase in its value in the radial direction is more rapid than the first power of the radius. It follows that (1) for translations perpendicular to the axis, the mutual inductance is a minimum when the two solenoids are coaxial; (2) for a translation along the axis, the mutual inductance is a maximum when the solenoids are concentric; and (3) for a rotation, the mutual inductance is a maximum when the solenoids are coaxial.

The leads are attached by soldering near each end a short wire which extends radially outward. One end of the short wire is first formed to fit the wire on the solenoid. The end is then tinned, and, while maintained hot, is pressed against the proper place on the wire of the solenoid and worked until a firm joint is obtained. Wires parallel to the axis are attached to the short radial wires.

89. Construction of a Campbell Mutual Inductor.—A Campbell mutual inductor consists of a long solenoid, the center of which is lacking, and a multiple-layer coil of large diameter which is concentric and coaxial with the solenoid. A diagram showing the construction is given in Fig. 39*a*. The solenoidal part may be wound as one single solenoid reaching from end to end of the form. At two points near the center of the winding, copper blocks for holding the winding should be inserted in the form, these blocks being similar to those used at the ends of the winding. After the wire is fastened to these blocks, the center portion is removed.

The multiple-layer coil is wound in the rectangular groove of a large ring, which is usually of the same material as the form for the solenoid. The wire must be insulated, the most suitable insulation being enamel. Only ordinary precautions are required in winding this multiple-layer coil.

90. Construction of a Self Inductor from a Single-layer Solenoid.—A single-layer solenoid, the construction of which has been already described, only requires leads to become a self inductor. Each lead is attached several turns from an end of the solenoidal winding, the two leads being on the same generator of the cylinder. The leads are soldered to the winding as already explained, and extend radially outward for the same distance at each end, usually 1 cm. At the outer end of each radial lead, a wire is soldered to extend, in the direction of a generator, to within ½ cm of the center plane of the solenoid, where each wire bends at right angles and is carried in a radial direction to its point of attachment to the outside circuit. The arrangement of leads has been shown in Fig. 35.

CHAPTER XVI

ABSOLUTE MEASUREMENT OF RESISTANCE; SURVEY OF METHODS

The measurement of resistance in terms of length and time can be accomplished by a number of different methods. A classification of all methods which have been proposed is given in Table X. Following each specific method is given the name of the man who proposed it.

A brief consideration will be given to the methods in order to indicate those which are capable of giving results of high precision. In the following chapter a few selected methods will be considered in detail.

TABLE X.—METHODS FOR ABSOLUTE MEASUREMENT OF THE OHM

A. Calorimetric method (Joule).

B. Methods involving an induced electromotive force.

- I. Relative motion of a coil and magnet.
  - 1. Damping of a magnet (W. Weber).
  - 2. Rotation of a magnet (Lippman).
  - 3. Dropping of a magnet (Mengarine).
- II. Rotation of a coil in the earth's magnetic field (the earth inductor).
  - 1. Earth inductor with rotation through 180° (W. Weber).
  - 2. Earth inductor with uniform rotation.
    - *a*. Earth inductor with tangent galvanometer (W. Weber).
    - *b*. Combined earth inductor and tangent galvanometer (the revolving coil of the B.A. committee) (Lord Kelvin).
- III. Nonuniform motion of a conductor in the magnetic field of a current.
  - 1. Damping of a vibrating coil (Nettleton and Lewellyn).
  - 2. Displacement of a coil (Kirchhoff).
- IV. Uniform motion of a conductor in the magnetic field of a current (generator with air-cored magnets).
  - 1. Commutating generator.
    - *a*. Average value of generated electromotive force (Rosa).
    - *b*. Maximum value of generated electromotive force (Lippman).
  - 2. Homopolar generator (Lorenz apparatus) (Lorenz).
- V. Varying currents in a mutual inductance.
  - 1. Transient currents (Rowland).

TABLE X.—METHODS FOR ABSOLUTE MEASUREMENT OF THE OHM.—(*Continued*)

2. Commutated currents.
   a. Sudden reversal of current (Roiti).
   b. Step-by-step reversal of current (Wenner).
3. Sinusoidal currents.
   a. Intermediary capacitance (Campbell).
   b. Two mutual inductances in series (Campbell).
   c. Two mutual inductances with two-phase measured currents (Campbell).
   d. Two mutual inductances with two-phase balanced currents (Wenner).

VI. Varying currents in a self inductance.
1. Transient currents (Maxwell).
2. Commutated currents (Curtis).
3. Sinusoidal currents.
   a. Intermediary capacitance calibrated by a commutator bridge (Rosa).
   b. Intermediary capacitance calibrated by a resonance bridge (Grüeneisen and Giebe).

91. Calorimetric Method.—The calorimetric method for the absolute measurement of resistance requires the measurement of the heat liberated in a resistance by a known current in a known time. The equation is

$$HJ = I^2Rt$$

where H = heat liberated in time t.
J = mechanical equivalent of heat.
I = current flowing through the resistance R.

Not only is H difficult to measure, but J is a constant which has never been accurately determined by thermomechanical measurements. Hence the method cannot be considered an accurate one for the absolute measurement of resistance.

92. Methods Involving an Induced Electromotive Force.—Concerning the six classes of methods which involve an induced electromotive force, some entire classes have features which prevent them from giving precise results. All the methods of Class I require a knowledge of the properties of a magnet, which cannot be measured precisely. In Class II there is only one method—the revolving coil—in which the value of the earth's magnetic field is not required. The methods of Class III, involving the nonuniform motion of a conductor, do not appear to be adaptable to precise measurements and will not be described.

However, the uniform motion of a conductor, Class IV, gives rise to important methods, one of which, the Lorenz method, is perhaps the best known of all the methods for the absolute measurement of resistance. Likewise Classes V and VI, involving varying currents in mutual and self inductances, contain important methods.

93. Method of the Revolving Coil.—The method of the revolving coil (No. II, 2, *b*) was proposed by Lord Kelvin and carried out by Maxwell, Steward, and Jenkin as a means of establishing

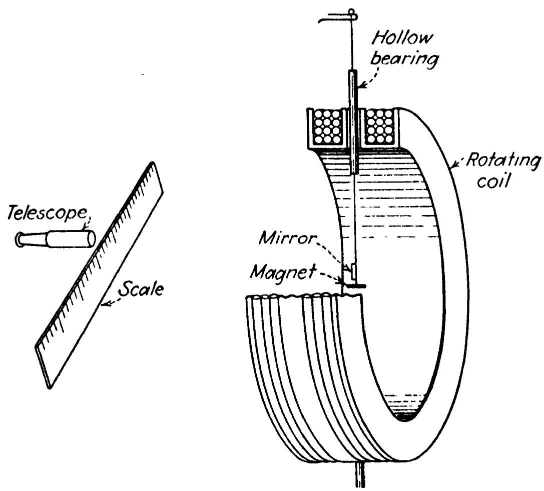

FIG. 40.—Diagram of the rotating-coil method for the absolute measurement of resistance.

the B.A. unit of resistance. It is not capable of giving high precision, but is included because of its simplicity and its historical importance.

The apparatus shown in Fig. 40 consists of a coil of wire separated into two parts and arranged to rotate around a hollow vertical axis, through which extends the suspension for supporting a magnet at the center of the coil. In addition to the apparatus shown in the figure, provision must be made for rotating the coil at a uniform angular velocity ω. The angular deflection φ of the magnet is read by the telescope and scale. An alternating electromotive force, which depends on the intensity of the horizontal component of the earth's field, is induced in the rotating coil. This electromotive force causes an alternating current to flow in the coil, the magnitude of which depends on the resistance and inductance of the coil. This current produces at the center of the coil a horizontal magnetic field which at any instant can be resolved into two components, one in the direction of the earth's field, the other at right angles to it. If the period of the magnet is long relative to the time of a rotation of the coil, the magnet will not follow the changes in the magnetic field produced by the rotating coil, but will be deflected from its normal position in the earth's field by an angle which depends both on the horizontal intensity of the earth's field and on the integral value of the magnetic field produced by the current. The cotangent of this angle of deflection is the ratio of the integral of the total horizontal field in the direction of the earth's field to the integral of the horizontal field at right angles. But since the current and hence the magnetic field produced by it depend on the intensity of the earth's field, the direction of the resulting magnetic field is independent of the value of the intensity of the earth's field, so that the angular deflection of the magnet does not depend on the strength of the earth's field.

A complete theory of the rotating coil method was given by Maxwell.[1] The following derivation of the formula for computing the resistance makes use of present-day nomenclature in alternating-current theory.

Let A = mean radius of coil.
N = number of turns on coil.
R = resistance of coil.
L = inductance of coil.
ω = angular velocity of coil.
H = horizontal component of the magnetic induction of the earth's field.

The time t will be counted from the instant that the plane of the coil coincides with the direction of the horizontal component of the earth's magnetic field. The angle through which the coil has turned in time t is ωt; the horizontal magnetic flux through the coil at this time is $\pi A^2 H \sin \omega t$; and the rate of change of flux is $\pi\omega A^2 H \cos \omega t$. Hence the electromotive force induced in the coil at instant t is

[1] Maxwell, "Electricity and Magnetism," 3d ed., Art. 764. Originally published in Reports of British Association for 1863, p. 111.

$$e = \pi\omega A^2HN \cos \omega t = E \cos \omega t \tag{1}$$

This electromotive force causes a current i to flow, such that

$$i = I \cos (\omega t + \alpha) \tag{2}$$

where $$I = \frac{E}{\sqrt{R^2 + \omega^2 L^2}} \quad \text{and} \quad \tan \alpha = \frac{\omega L}{R} \tag{3}$$

The magnetic field H_i at the center of the coil caused by the current is

$$H_i = \frac{2\pi i N}{A} \tag{4}$$

Substituting values from Eqs. (1), (2), and (3),

$$H_i = \frac{2\pi^2\omega AHN^2 \cos (\omega t + \alpha)}{\sqrt{R^2 + \omega^2 L^2}} \tag{5}$$

$$= H_m \cos (\omega t + \alpha) \tag{6}$$

where H_m = maximum field produced by the current at the center of the coil.

The direction of H_i is perpendicular to the plane of the coil. Since H_i is a vector, it may be resolved into two component parts, one H_e in the direction of the earth's field, and the other H_p perpendicular to the earth's field,

$$H_e = -H_i \sin \omega t = -H_m \sin \omega t \cos (\omega t + \alpha) \tag{7}$$

$$H_p = -H_i \cos \omega t = -H_m \cos \omega t \cos (\omega t + \alpha) \tag{8}$$

Four limitations will now be assumed applicable to the apparatus:

1. That the period and damping of the magnet are such that its deflection is sensibly constant. This can be accomplished by having the period long relative to the time of rotation of the coil and the damping large. In this case the deflection is dependent only on the average value of the magnetic fields.
2. That the magnet is so short that the field at the poles is the same as at the center of the coil.
3. That the torsion and friction of the suspension of the magnet do not influence the deflection.
4. That the magnetic field of the magnet does not induce an appreciable electromotive force in the rotating coil.

With these limitations, the angular deflection σ of the magnet is given by the equation

$$\tan \sigma = \frac{\frac{2}{T}\int_0^{\frac{T}{2}} H_p \, dt}{H + \frac{2}{T}\int_0^{\frac{T}{2}} H_e \, dt} \tag{9}$$

The integrals are taken to give the average value of the components of the magnetic field. The limits are taken at the beginning and end of a half period of the rotation of the coil, since this time corresponds to a complete period of the magnetic components.

Carrying out the integrations by substituting the values of H_p and H_e, and simplifying by inserting the values of cos α and sin α,

$$\tan \sigma = \frac{\pi^2 \omega A H N^2 R}{H(R^2 + \omega^2 L^2) - \pi^2 A H N^2 \omega^2 L} \tag{10}$$

Since H is a factor in both the numerator and denominator, it disappears from the equation. Rearranging,

$$R^2 - R\pi^2 \omega A N^2 \cot \sigma + \omega^2 L(L - \pi^2 A N^2) = 0 \tag{11}$$

This is a quadratic equation in R. All the quantities except R and L are mechanical, and L can be computed from the mechanical quantities. Hence R can be determined in terms of mechanical quantities. If L is small so that the terms containing it may be neglected, the equation becomes

$$R = \pi^2 \omega A N^2 \cot \sigma \tag{12}$$

There are several difficulties connected with this method, One is the difficulty of reading the deflection with accuracy. especially as there is always some vibration. Another is the necessity of transferring the resistance of the coil, which must rotate in air, to a standard resistor. The method is not considered capable of giving precise results.

94. Generator Methods.—In Class IV are grouped methods which make use of the principle of the electric generator in which a conductor has uniform motion in a magnetic field. In order that the magnetic field may be accurately known, it must be produced by a current only, no magnets or magnetizable material being in the neighborhood. Two forms of generators have been used: the commutating generator and the homopolar generator. The first has never been given a satisfactory experimental trial; the second has been used more often than any other method.

95. Average Electromotive Force of a Commutating Generator. In order to explain the principles of a commutating generator as used to determine the absolute value of a resistance, consider a coil placed inside a very long horizontal solenoid and mounted to rotate about a vertical axis. Let the mean radius of the coil be A, its angular velocity ω, and the number of turns N. Let the pitch of the winding on the solenoid be p and the current through it I. Then the magnetic intensity H and the magnetic induction B at a point near the center of the solenoid is given by the equation

$$H = B = \frac{4\pi I}{p} \tag{13}$$

If time t is counted from the instant that the plane of the coil coincides with the axis of the solenoid, the magnetic flux φ through the coil at any instant t is

$$\varphi = \pi A^2 B \sin \omega t = \frac{4\pi^2 A^2 I}{p} \sin \omega t \tag{14}$$

and the electromotive force e at that instant is

$$e = N \frac{d\varphi}{dt} = \frac{4\pi^2 A^2 N I \omega}{p} \cos \omega t = E \cos \omega t \tag{15}$$

where N = number of turns of wire in the coil. In order to employ the average value of this electromotive force, the coil is connected to an external circuit through a commutator,[1] which is so arranged that the current is reversed each time that the current becomes zero, so that in the external circuit the current is always in one direction. If the coil and external circuit together have a resistance R and an inductance L, the instantaneous current i is given by the equation

[1] Method proposed by Rosa, *Bull. B. S.*, **5**, 499 (1909).

$$i = \frac{E}{\sqrt{R^2 + \omega^2 L^2}} \cos (\omega t + \alpha) \tag{16}$$

By integrating Eq. (16) over a half cycle beginning when $\omega t + \alpha = \pi/2$, and multiplying the result by $2/T$, the average value of the current, I_{ave}, is given by the equation

$$I_{ave} = \frac{2E}{\pi\sqrt{R^2 + \omega^2 L^2}} = \frac{8\pi A^2 N I \omega}{p\sqrt{R^2 + \omega^2 L^2}} \tag{17}$$

where the value of E is obtained from Eq. (15).

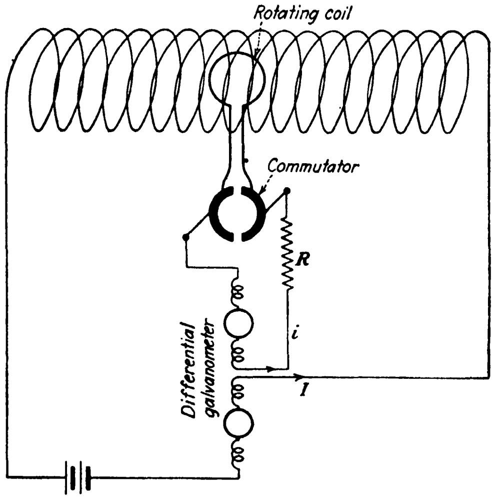

FIG. 40a.—Diagram of a method for the absolute measurement of resistance which employs the average electromotive force of a commutating generator.

If this current flows through one coil of a ballistic differential galvanometer, as shown in Fig. 40a, and the current I in the solenoid through the other coil of the galvanometer, and if R is adjusted until the deflection is zero, then $I = I_{ave}$, so that

$$R = \omega\sqrt{\frac{64\pi^2 A^4 N^2}{p^2} - L^2} \tag{18}$$

Since the mutual inductance $\overline{M}$ between the solenoid and the coil at the instant that their axes coincide is

$$\overline{M} = \frac{4\pi^2 A^2 N}{p} \tag{19}$$

it follows that

$$R = \omega\sqrt{\frac{4\overline{M}^2}{\pi^2} - L^2} \tag{20}$$

This method has never been given a complete experimental test, but several difficulties are apparent: *viz.*, the mean diameter of a revolving coil must be determined; the commutation must be at the exact instant when the current is zero; the period of the galvanometer must be so long relative to the period of rotation that a satisfactory balance can be obtained; the self inductance of the coil circuit must be determined, which circuit includes the portion in the galvanometer; the resistance evaluated includes the copper of the coil and galvanometer; and the earth's magnetic field induces an electromotive force which must be taken into account. On the other hand, the method has some advantages: the induced electromotive force may be made large so that thermal electromotive forces are not troublesome; the sensitivity is ample, and the equation for computation of the resistance is simple. Until a complete experimental test has been carried through, the accuracy that can be obtained by it can only be estimated. In the opinion of the author, other methods are more promising.

96. Maximum Electromotive Force of a Commutating Generator.—The maximum electromotive force generated in the rotating coil of a commutating generator can be balanced against the fall in potential over a resistance R in series with the solenoid, which forms the field coil of the generator, as shown in Fig. 41. Since in Eq. (15), for the electromotive force of a generator of this type, the maximum occurs when $\cos \omega t = 1$, the condition for a balance is

$$RI = \frac{4\pi^2 A^2 N I \omega}{p} = \overline{M} I \omega \tag{21}$$

so that

$$R = \omega \overline{M} \tag{22}$$

where all the symbols have the same meaning as in the preceding section.

This method has several advantages: a standard resistance is measured; no current is drawn from the coil, so that a sensitive galvanometer can be employed; the induced electromotive force can be so high that thermal electromotive forces are negligible. On the other hand, there are a number of disadvantages: the radius of a rotating coil must be accurately determined; the electromotive force induced in the coil by its rotation in the earth's field may be several per cent of the total electromotive force and must be eliminated or a correction applied therefor; the time during which the coil circuit can be connected to the outside circuit is so small that the sensitivity is greatly reduced. To show the importance of the last disadvantage, if the coil is to be connected to the external circuit only when the electromotive force is within one part in a million of its maximum value, then it can be connected only 5 ten-thousandths of the time. Stated

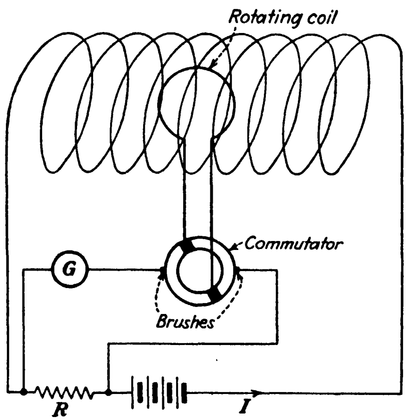

FIG. 41.—Diagram of a method for the absolute measurement of resistance which employs the maximum electromotive force of a generator.

otherwise, if the coil makes 100 revolutions per second, with two connections per revolution, each connection will be for 2½ microseconds. With a commutator 15 cm in diameter, the commutator bars can be only 0.1 mm in width even if the brushes make a line contact on the commutator.

97. The Homopolar Generator; Lorenz Apparatus.—The homopolar generator was first proposed by Lorenz as a method for the absolute measurement of resistance, so that this type of generator is often called a *Lorenz apparatus*. It has been used by several observers and is considered by many students in this field the most useful method that has been proposed for the absolute determination of the ohm. It is so simple and direct that some may have been blind to its inherent weaknesses.

The principle can be illustrated by considering that a rotating disk has its axis coincident with the axis of an infinitely long solenoid in which a current I is flowing. If the winding of the solenoid has a pitch p, the magnetic induction B inside the solenoid has the same value as the magnetic intensity H, so that

$$B = H = \frac{4\pi I}{p} \tag{23}$$

If the radius of the disk is A and its time of rotation T, then the electromotive force induced in each radial element is

$$e = \frac{\pi A^2 B}{T} = \frac{4\pi^2 A^2 I}{pT} \tag{24}$$

If the drop in potential over a resistance R, connected in series with the solenoid, is made equal to this induced electromotive force,

$$R = \frac{4\pi^2 A^2}{pT} = \frac{M}{T} \tag{25}$$

where M = mutual inductance between the solenoid and a circle that coincides with the circumference of the disk.

The simple homopolar generator just described has several limitations. The earth's magnetic field causes troublesome corrections; the induced electromotive force is small so that thermal electromotive forces may introduce errors; the diameter of the disk must be measured while rotating; and electrical connections must be made to the center and circumference of the disk.

Smith's modification of the Lorenz apparatus has overcome several of the limitations of the simple apparatus. The modified apparatus gave results more precise than any which had been obtained previous to that determination. Since it is one of the outstanding methods, it will be described in detail in the next chapter.

98. Varying Currents in a Mutual Inductance.—In Class V are grouped those methods in which the value of a resistance is determined by comparing the fall in potential produced by a current in this resistance with the electromotive force induced in the secondary of a standard mutual inductance when the current in the primary changes at a known rate. The standard mutual inductor is of such form that its inductance can be computed from its mechanical dimensions. Since the induced electromotive force cannot be constant for any appreciable length of time, some special method must be devised for comparing the electromotive force induced in the secondary of the mutual inductance with the potential drop in the resistance.

A number of methods have been proposed, but only one, an alternating-current method, has actually been used to obtain a result of high precision, although three others have sufficient

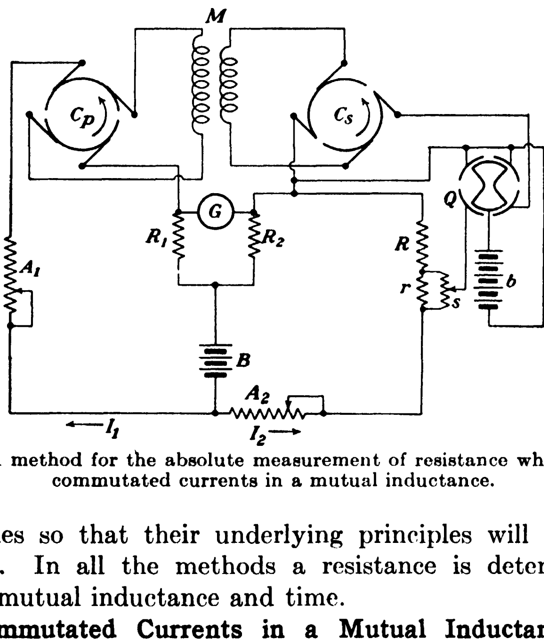

FIG. 42.—A method for the absolute measurement of resistance which employs commutated currents in a mutual inductance.

possibilities so that their underlying principles will be herein described. In all the methods a resistance is determined in terms of mutual inductance and time.

99. Commutated Currents in a Mutual Inductance.—The principle involved in using commutated currents in a mutual inductance to make an absolute measurement of resistance is to balance the integral of the induced electromotive force in the secondary of a mutual inductance, when the primary current is reversed a given number of times per second, against the fall of potential produced by the primary current in a known resistance. In Fig. 42 is a diagram of circuits which might be used for this purpose. The two commutators C_p and C_s, which are on a shaft that rotates at a known speed, are so adjusted that each reverses its connections when the other is about midway between its reversals. The current in the primary is supplied by the battery B, which also supplies current to the auxiliary circuit containing the standard resistors R_2 and R. With the commutators stationary, the current in R_1 or R_2 is adjusted by A_1 or A_2 until the fall of potential over R_1 is the same as over R_2. The commutators are then run at a speed to make N reversals per second of the current in the primary of M. The secondary of the mutual inductor is connected through the commutator C_s to opposite quadrants of an electrometer which has its needle charged by an independent battery b. The two remaining quadrants are connected to the standard resistor R which can be varied by the slide wire. The slide wire is adjusted until there is no deflection of the electrometer. Then the integral of the induced electromotive force is equal to the fall of potential in R (which includes a part of r that is determined from the slide-wire setting).

When the currents in the primary and auxiliary circuits have been adjusted, the equation connecting currents and resistances is

$$I_1R_1 = I_2R_2 \tag{26}$$

The instantaneous electromotive force e_s, induced in the secondary, is given by the equation

$$e_s = M\frac{di_1}{dt} \qquad \text{or} \qquad e_s\,dt = M\,di_1 \tag{27}$$

The average value of the electromotive force is obtained by integrating Eq. (27) from $t = 0$ when $i_1 = -I_1$ to $t = T$ when $i_1 = +I_1$ and dividing the integral by $1/T$. Hence,

$$\frac{1}{T}\int_0^T e_s\,dt = \frac{M}{T}\int_{-I_1}^{+I_1} di_1 = \frac{2MI_1}{T} \tag{28}$$

But this average value is balanced by the potential drop RI_2. Hence

$$RI_2 = \frac{2MI_1}{T} \tag{29}$$

Eliminating the currents by means of Eq. (26) and noting that $1/T = N$

$$R = \frac{2MNR_2}{R_1} \tag{30}$$

A number of difficulties will have to be overcome before this method will give results of precision. The electrometer must have extreme sensitivity and may not accurately integrate the electromotive force. The commutator must have a uniform speed, and there must not be any chattering of the brushes. The capacitance between parts of the apparatus may influence the final result. The above are some of the obvious difficulties. An experimental determination will probably disclose others. However, the author is of the opinion that the underlying principle can be used to develop a method that will give a precise result.

100. Sinusoidal Current in a Mutual Inductance Combined with an Intermediary Capacitance.—The value of a resistance can be determined in terms of a mutual inductance and time by employing an intermediary capacitance. The mutual inductance has such form that its value in absolute henrys can be computed from its mechanical dimensions. The value can also be measured in terms of time and the laboratory unit of resistance. The difference in the two values of the inductance results from the difference between the laboratory unit of resistance and the absolute unit. Then a correction can be applied to the laboratory unit of resistance to make it agree with the absolute unit.

The procedures for measuring the dimensions of the inductor and for computing the absolute value of the inductance have already been described. The procedure for measuring the inductance in terms of time and the laboratory unit of resistance combines two methods, each of which has been described. The first is the alternating-current bridge for measuring a mutual inductance in terms of resistance and capacitance (Sec. 76). The second is the Maxwell bridge for measuring a capacitance in terms of resistance and time (Sec. 77). A setup which uses these two methods is shown in Fig. 43. On the right is the alternating-current bridge which is balanced by adjusting the resistance Q' in series with the secondary of the inductance and the capacitance in parallel with the resistance R'. The primary of the inductance is so connected in the supply circuit that the electromotive force induced in the Q arm by the mutual inductance is in the opposite direction to that induced by the self inductance of the secondary. The resistances of the bridge arms are so chosen that the capacitance C_M required to balance the bridge is small. The variable capacitor C_V may have sufficient capacitance, or, as shown in the diagram, a small fixed capacitance C_a may also be required, in which case $C_M = C_a + C_V$. The primary is then removed from the supply circuit; the larger fixed capacitance C_b is substituted for C_a; and the bridge is again balanced by increasing C_V by ΔC_V, so that

$$C_L = C_b + C_V + \Delta C_V$$

The capacitance is then switched to the Maxwell bridge on the left, and the values of C_L and C_M are determined in terms of resistance and time.

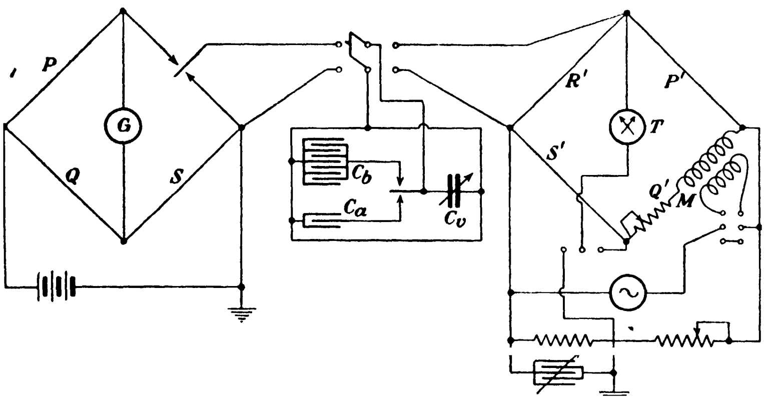

FIG. 43.—Setup for the absolute measurement of resistance using alternating current in a mutual inductance combined with an intermediary capacitance.

From the measurements with the alternating-current bridge,

$$M = (C_L - C_M)\frac{P'R'S'}{R' + S'} \tag{31}$$

But by the Maxwell bridge,

$$C_L = \frac{QF_L}{nPS_L} \tag{32}$$

and

$$C_M = \frac{QF_m}{nPS_m} \tag{33}$$

where n, the number of charges and discharges per second, and P/Q, the ratio of the resistances in two arms, are the same in both measurements. Substituting Eqs. (32) and (33) in Eq. (31),

$$M = \frac{Q}{nP}\left(\frac{F_L}{S_L} - \frac{F_m}{S_m}\right)\frac{P'R'S'}{R' + S'} \tag{34}$$

Each of the resistances must be evaluated in terms of one resistance which is taken as the standard of the laboratory. The value of this standard in laboratory units is R, and in absolute units is R_a, which is the quantity to be determined. Let $R/R_a = q$ or $R = qR_a$. Then $Q = qQ_a$, $P = qP_a$, etc., for all the resistances employed in the measurement. Substituting these values in Eq. (34) and noting that, when all the resistances are in absolute units, the mutual inductance computed by Eq. (34) is the inductance M_a in absolute units,

$$M = qM_a \tag{35}$$

But M_a has already been determined by computing its value from the dimensions of the inductor, so that

$$q = \frac{M}{M_a} \tag{36}$$

and

$$R_a = \frac{R}{q} \tag{37}$$

The above method is quite simple in operation, although two separate sets of observations are required. Moreover, for the measurement of mutual inductance, the self inductance of one coil is important in each of the two required bridge balances and is eliminated only by combining the results of two observations, so that any drift during the measurements would introduce an error. Air condensers must be employed, but they are not very stable, so that the entire set of observations must be made in a relatively short time.

One source of error is the capacitance between the primary and secondary. By making measurements at two or more low frequencies, an extrapolation to zero frequency can be made, since as a first approximation the effect of capacitance increases as the square of the frequency. For this extrapolation, frequencies below 100 cycles per second should be used. It has been shown (page 120) that this will give the inductance at zero frequency only when the capacitance is small.

The method requires a number of resistances of quite different values, so that their calibration presents some difficulties. To illustrate a method of overcoming this difficulty, a possible set of data will be considered. Assume that an inductor is constructed having a mutual inductance of 10 millihenrys. In the alternating-current bridge, a self inductor is connected to the secondary so that the sum of its inductance and that of the secondary is 25 millihenrys. Also assume that an air condenser having a maximum capacity of 0.1 microfarad is available. Then when measuring only the self inductance, a balance can be obtained with 500 ohms in each arm of the bridge. When the secondary is connected in the circuit supplying the bridge, the capacitance must be reduced to 0.02 microfarad. Tabulating the bridge constants:

$$L = 25 \text{ millihenrys}$$
$$M = 10 \text{ millihenrys}$$
$$C_L = 0.1 \text{ microfarad}$$
$$C_M = 0.02 \text{ microfarad}$$
$$P' = R' = S' = 500 \text{ ohms}$$

In measuring the capacitances with the Maxwell bridge,

$$Q = P = 100 \text{ ohms}$$
$$n = 100 \text{ charges and discharges per second}$$
$$S_L = 100\,000 \text{ ohms}$$
$$S_m = 500\,000 \text{ ohms}$$

The 500-ohm coils of the alternating-current bridge can be calibrated by comparing with two 1000-ohm standards in parallel by means of a Wheatstone bridge. In calibrating the resistances of the Maxwell bridge, resistances of 100 000 ohms and 500 000 ohms should be available to substitute for the capacitor and contactor (see page 127). Standards of this value are difficult to maintain, so that some arrangement should be made for comparing these resistances with standards of lower value. For the 100 000-ohm resistance, a suitable method is to use a resistance box with ten 10 000-ohm coils which can be connected in parallel for comparison with a 1000-ohm standard and in series for calibrating the bridge. For the 500 000 ohms, the first five coils in a subdivided megohm box may be used and each calibrated against the above 100 000 ohms. All the resistances required in the above are referred to 1000-ohm standards, of which one had a known value in laboratory ohms.

101. Alternating Currents with Two Mutual Inductances in Series.—A method for the absolute measurement of resistance which employs two mutual inductances in series has been proposed by Campbell.[1] The diagram of connections is shown in Fig. 44. An alternating current, in which the effective value of the fundamental component is I_1, exists in the primary of the mutual inductance M_1 and in the resistance R. The junction point between M_1 and R is maintained at the potential of the earth by means of a third arm grounded at the proper point. The secondary of M_1 is connected to the primary of M_2, and the resistances r and r_0 and a variable self inductance L_0 are inserted in the circuit. The current in this circuit is I_2. The electromotive force induced in the secondary of M_2 is balanced against the fall in potential of I_1 in R plus that of I_2 in r. This balance is obtained by varying L_0 and r_0 until there is no deflection of the vibration galvanometer G, which is tuned to the frequency of the fundamental component of the alternating current.

To derive the equations representing the conditions at balance, let E_2 and E_3 be the effective values of the fundamental components of the electromotive forces induced in the secondaries of M_1 and M_2, respectively. Then for low frequencies,

$$E_2 = \pm j\omega M_1 I_1 = (R_c + j\omega L_c)I_2 \qquad (38)$$

$$E_3 = \pm j\omega M_2 I_2 = (R + j\omega l_R)I_1 + (r + j\omega l_r)I_2 \qquad (39)$$

where R_c and L_c = total resistance and total inductance, respectively, of the circuit in which there is the current I_2.

l_R and l_r = self inductances of the resistances R and r.

Eliminating I_1 and I_2, separating into real and imaginary parts, rearranging, and using the negative sign for M_1 and the positive for M_2,

[1] *Proc. Roy. Soc.* (*London*), **107**, 310 (1925).

$$RR_c = \omega^2(M_1M_2 - M_1l_r + L_cl_R) \quad (40)$$
$$RL_c = M_1r - l_RR_c \quad (41)$$

The inductances l_r and l_R are small so that as a first approximation

$$RR_c = \omega^2M_1M_2 \quad (42)$$
$$RL_c = M_1r \quad (43)$$

These equations are useful in deciding on the values of the resistances, inductances, and frequency to be used.

In deciding on apparatus for this method, consideration must first be given to the mutual inductances. Absolute standards having 10-millihenry mutual inductance[1] have been constructed. Two working standards of this value should be made, so that each can be compared with the absolute standard. In order to reduce

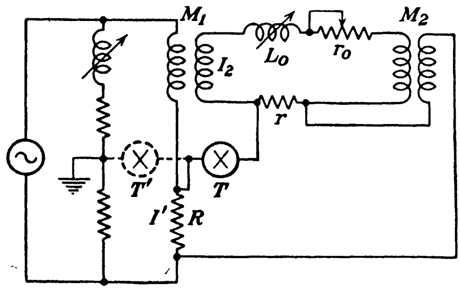

FIG. 44.—Setup for the absolute measurement of resistance using alternating currents with two mutual inductances in series.

the capacitance between windings, the primary and secondary should be well separated, causing their resistances and self inductances to be relatively large. Reasonable values to expect would be 10 ohms each for the resistances of the primary and secondary and 20 mh each for their self inductances. Since R_c must be more than 20 ohms and L_c more than 40 ohms, and since these quantities can be increased by r_0 and L_0, reasonable values to use are R_c = 25 ohms and L_c = 50 millihenrys. Using these values, with M_1 and M_2 as 10 millihenrys each, and a frequency of 25 cycles per second, Eqs. (5) and (6) give values of R and r as 0.1 ohm and 0.5 ohm, respectively. If the frequency is 100 cycles per second and the other constants are as above, the values of R

[1] The National Physical Laboratory of England has described three having this value, of which two were constructed for Japan. See Dye, *N.P.L., Coll. Researches*, **21**, 1 (1929).

and r are 1.6 ohms and 8 ohms, respectively. These are not convenient values to compare with standards. However, if r_0 is increased so that $R_c = 40$ ohms, then R and r are 1 ohm and 5 ohms, respectively.

In the method of balancing by varying r_0 and L_0, a change in one variable does not require a change in the other, as can be seen from Eqs. (42) and (43). This condition is very favorable for quickly and accurately bringing the current through the vibration galvanometer G to zero.

The value of R/R_c can be obtained by measurements with a Wheatstone bridge. The value of RR_c can be obtained from Eq. (40), giving a result in terms of the square of a frequency and the product of two mutual inductances, each of which has been compared with a standard mutual inductance, the value of which has been computed from the measured dimensions of the inductor. By eliminating R_c, R is determined in absolute measure.

In Eq. (40) for computing RR_c, the two correction terms L_cl_R and $-M_1l_r$ must be evaluated. To make these terms cancel, the two resistances R and r should be so constructed that their inductances are proportional to their resistances. This can be accomplished by winding them bifilarly from wire of the same size, or by constructing them from strip, the material for the two resistors being of the same width and thickness. For each resistance, the proper length of strip is selected, then folded in the center and bound together with thin and uniform insulation between the two halves. With such coils

$$\frac{l_r}{l_R} = \frac{r}{R} \tag{44}$$

Substituting the value of r from Eq. (44) in Eq. (41), clearing of fractions, and dividing by R,

$$M_1l_r - L_cl_R = l_R^2\frac{R_c}{R} \tag{45}$$

Substituting Eq. (46) in Eq. (40), placing $R_c = KR$, and solving for R,

$$R = \omega\sqrt{\frac{M_1M_2}{K} - l_R^2} \tag{46}$$

The method has several advantages. Except for the standard of mutual inductance, the apparatus required is to be found in many laboratories. The manipulation for obtaining a balance is direct and simple provided an alternating current of constant frequency is available. The formula for computing the resistance is devoid of troublesome correction terms.

Some disadvantages of the method are apparent. The alternating current must have a very small amount of harmonics. The frequency of the alternating current must remain very constant, even from cycle to cycle; otherwise a balance cannot be obtained. The resistance R_c, mostly of copper and very inductive, must be accurately compared with a standard resistance. These disadvantages will be discussed in some detail.

If there are any harmonics in the current I_1, there will be a current through the galvanometer when a balance has been obtained for the fundamental. Even a very selective vibration galvanometer has an appreciable sensitivity to the harmonics, so that, with harmonics present in the current, a perfect balance cannot be obtained. Even with the best sources of alternating current now available, harmonics are present to some extent. Electrical filters are not feasible for the low frequencies required by this method, but mechanical filters have been constructed for low frequencies, which offer a high impedance to all frequencies except in a very narrow band. Such a filter inserted in the galvanometer circuit may aid in obtaining a balance. The discussion of these filters[1] is outside the range of this work.

It is difficult to obtain by a rotating machine a frequency which is constant to one part in a million over each cycle. However, by means of a vibrating crystal or a vibrating tuning fork, this constancy can be attained, but the wave form of the alternating current which is produced by the use of either contains a large percentage of harmonics. The wave form can be improved by filters, but it is doubtful whether a sufficiently pure sine wave can be obtained.

The resistance R_c is eliminated by assuming that its value when compared with R by direct current is the same as when it is used in the alternating-current measurement. This assumption is difficult to test at any one frequency, but is probably valid if

[1] The theory is treated in an article by Kock in *Z. angew. Math. Mech.*, **14**, 173 (1934).

results at two frequencies agree. The comparison of the two resistances R_c and R with an accuracy of one part in a million is difficult on account of the large amount of copper in R_c. If only one-third of the resistance is copper, a change in its temperature of 0.001°C will produce a change of one part in a million in the resistance of R_c. Moreover, such a resistance might have a very large load coefficient, which would be very difficult to determine and apply.

102. Mutual Inductances in a Two-phase Circuit.—Of those methods which employ mutual inductances in a two-phase, alternating-current circuit, only that one[1] will be described in which the instantaneous electromotive force induced in either

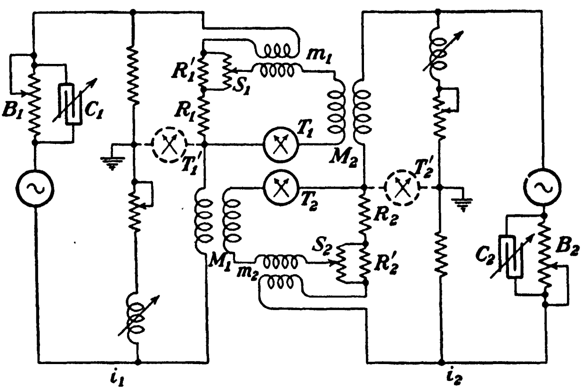

FIG. 45.—Setup for the absolute measurement of resistance using two mutual inductances in a balanced, two-phase circuit.

secondary is balanced against the fall in potential produced by a current in a resistance. The diagram of connections for this method, which will be called the balanced two-phase method, is shown in Fig. 45. The circuit of each phase is exactly similar to that of the opposite phase, so that only that of phase I will be described. One phase of a two-phase generator supplies current to the circuit in which a resistance B_1, shunted by a capacitance C_1, regulates the magnitude and phase of the current in the resistance R_1 and the primaries of the mutual inductances M_1 and m_1. A third arm maintains the junction between R_1 and M_1 at earth potential. The drop in potential over the resistance R_1 is

[1] The method was suggested by Wenner, Weibel, and Silsbee, *Bull. B. S.*, **12**, 11 (1915).

balanced against the electromotive forces induced in the secondaries of M_2 and m_1, the balance being indicated by the vibration galvanometer T_1 and obtained by varying R_1 and m_1. The variation of R_1 is secured by shunting a small portion of R_1' by a slide wire S_1. The mutual inductance m_1 is a small variable which may have either positive or negative values. The mutual inductance M_1 is a standard of fixed value.

The equations, when balances have been obtained for the two circuits which contain the galvanometers T_1 and T_2, can be derived by equating the electromotive forces and potential drops in each circuit. Let the instantaneous current i_1 in the resistance R_1 be represented by the equation

$$i_1 = I_1 \cos \omega t = \text{real part of } I_1\epsilon^{j\omega t} \qquad (47)$$

where ω = angular velocity of the alternating current.
I_1 = maximum value of alternating current.

Then if the current in phase II differs in phase from that in phase I by $\frac{\pi}{2} + \eta$ radians, where η is a small angle that may be positive or negative, the instantaneous current i_2 is represented by the equation

$$i_2 = I_2 \cos\left(\omega t + \frac{\pi}{2} + \eta\right) = \text{the real part of } I_2\epsilon^{j\left(\omega t + \frac{\pi}{2} + \eta\right)}$$

The resistance R_1 has an inductance l_1, and R_2 an inductance l_2. In the circuit containing the galvanometer T_1,

$$I_1[R_1 + j\omega(l_1 + m_1)]\epsilon^{j\omega t} = jI_2M_2\omega\epsilon^{j\left(\omega t + \frac{\pi}{2} + \eta\right)} \qquad (48)$$

In the circuit containing T_2,

$$I_2[R_2 + j\omega(l_2 + m_2)]\epsilon^{j\left(\omega t + \frac{\pi}{2} + \eta\right)} = jI_1M_1\omega\epsilon^{j\omega t} \qquad (49)$$

Multiplying Eq. (49) by $\epsilon^{-j\left(\frac{\pi}{2} + \eta\right)}$, removing the factor $\epsilon^{-j\omega t}$ from both equations, noting that $\epsilon^{j\pi/2} = j$ and $\epsilon^{-j\pi/2} = -j$, and expanding $\epsilon^{j\eta}$ to $\cos\eta + j\sin\eta$ and $\epsilon^{-j\eta}$ to $\cos\eta - j\sin\eta$, Eqs. (48) and (49) become

$$I_1[R_1 + j\omega(l_1 + m_1)] = -I_2M_2\omega(\cos\eta + j\sin\eta) \qquad (50)$$
$$I_2[R_2 + j\omega(l_2 + m_2)] = I_1M_1\omega(\cos\eta - j\sin\eta) \qquad (51)$$

Separating into real and imaginary parts, the real parts are

$$I_1R_1 = -I_2M_2\omega \cos \eta \qquad (52)$$
$$I_2R_2 = I_1M_1\omega \cos \eta \qquad (53)$$

and the imaginary are

$$I_1(l_1 + m_1) = -I_2M_2 \sin \eta \qquad (54)$$
$$I_2(l_2 + m_2) = -I_1M_1 \sin \eta \qquad (55)$$

Equation (52) shows that a balance can be obtained only if M_2 is negative, *i.e.*, if the connections to either the primary or the secondary of M_2 are reversed from those to M_1. Also if η is positive, m_2 must be negative to satisfy Eq. (55). Multiplying Eqs. (52) and (53), and dividing the resulting equation by the common factor I_1I_2, there is obtained, since M_2 is negative, the equation

$$R_1R_2 = \omega^2M_1M_2 \cos^2 \eta \qquad (56)$$

The value of η can be changed by varying C_1. When C_1 has such a value that $\eta = 0$, the settings of R_1 and R_2 will be a maximum as shown by Eqs. (52) and (53), provided the currents remain constant. But if the currents vary, Eq. (56) shows that R_1R_2 is a maximum when $\eta = 0$. In practice, R_1R_2 can be plotted against C_1, giving a parabola from which the value of C_1, corresponding to the maximum settings of R_1 and R_2, can be determined. With this value of C_1, the circuits are balanced, giving the settings on R_1 and R_2 corresponding to $\cos \eta = 1$. The ratio of R_1 to R_2 can be accurately determined by means of a Wheatstone bridge. Let this ratio be k so that $R_1 = kR_2$. Substituting in Eq. (56) and noting that $\cos \eta = 1$,

$$R_2 = \omega\sqrt{\frac{M_1M_2}{k}} \qquad (57)$$

If the frequency is accurately known, and if both M_1 and M_2 have been compared with an absolute standard of mutual inductance, the absolute value of R_2 can be computed.

The mutual inductances M_1 and M_2 should each have the same value as the absolute standard with which they are to be compared. Each should have a small capacitance between windings, and the distributed capacitance in both the secondary and primary should be small. The comparison with the absolute standard should be made at two or three low frequencies so that an interpolation can be made to zero frequency.

As an example, consider that $M_1 = M_2 = 10$ millihenrys and that k is approximately unity. Then, for a frequency of 25 cycles per second, R_1 and R_2 must each be about 1.57 ohms. All these values are of convenient size. Sufficient sensitivity can be obtained with currents of reasonable value.

The method has the advantage that the resistance of a standard resistor is measured directly in terms of a frequency and a mutual inductance. In this respect it is superior to the mutual-inductance method previously described. However it has the other disadvantage of that method, *viz.*, the necessity of having a sinusoidal alternating current of constant frequency, and the additional disadvantage of requiring a two-phase current with the complication of adjusting two vibration galvanometers simultaneously. As already described under the preceding method, the production of a suitable alternating current for measurements of the highest precision has not been accomplished. If that can be done, the method offers great promise.

103. Self-inductance Methods.—Of the self-inductance methods which have been proposed for the absolute measurement of resistance, only the one that makes use of an intermediary capacitance need be considered. This method, which has many points in common with the mutual-inductance method that uses an intermediary capacitance, makes use of a self inductor of such form (usually a single-layer solenoid) that its self inductance can be computed in absolute units from the mechanical dimensions of the inductor. The inductance can then be measured in terms of the laboratory units of resistance and time. Assuming the unit of time to be correct, a comparison of the two values of inductance will give the correction that must be applied to the laboratory unit of resistance to make it agree with the absolute unit. As this method has been used to give results of high precision, it will be described in detail in the next chapter.

CHAPTER XVII

ABSOLUTE MEASUREMENT OF RESISTANCE—DETAILS OF PRECISION METHODS

Details have been published of only two methods for the absolute measurement of resistance which have actually been used to obtain results of high precision. These methods are the homopolar generator (Lorenz apparatus), described by F. E. Smith[1] and the self-inductance method with intermediary

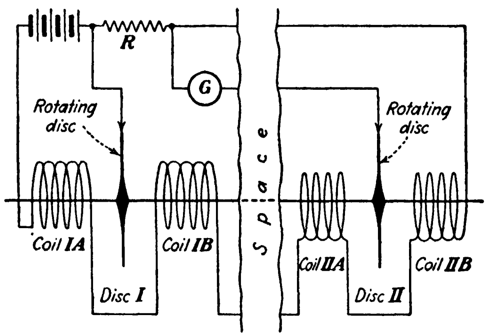

FIG. 46.—Diagram of the Lorenz apparatus as modified by F. E. Smith.

capacitance as used both by Grüneisen and Giebe[2] and by Curtis, Moon, and Sparks.[3] Both these methods will be carefully described, especial attention being paid in each case to the difficulties involved in making a determination.

104. Description of the Homopolar Generator.—A diagram of the homopolar generator (modified Lorenz apparatus) as used by Smith is shown in Fig. 46. On a long shaft two disks are mounted so that they can rotate about their common axis. In Smith's apparatus each disk was about 53.5 cm in diameter, and the distance between them was 167 cm. Two pairs of short solenoids are so mounted that the axis of each coincides with the axis

[1] *N.P.L., Coll. Researches*, **11**, 209 (1914).
[2] *Ann. phys.*, **368**, 179 (1920).
[3] *J. Research N.B.S.*, **16**, 1 (1936).

of the disks. A pair of solenoids is associated with each disk, one on each side such that their magnetic fields are in the same direction over the entire surface of the disk. However, the direction of the current in one pair is opposite to that in the other pair so that the magnetic field at one disk is opposite to that at the other disk. When the disks rotate, the direction of the electromotive force induced in one disk is from the center to circumference, in the other from circumference to center. Since the two disks are connected by an electrical conductor, the induced electromotive forces produce a difference of potential between the two circumferences, the value of which depends on the total number of tubes of magnetic induction passing through each disk and on the speed of rotation. But the number of tubes of induction passing through a disk as a result of unit current in one of the solenoids is the same as the mutual inductance between the solenoid and the circumference of the disk, and the total number of tubes through the disk when there is unit current in all the four solenoids is the sum of the mutual inductances (taken with proper sign) between the circumference and the four solenoids.

To derive the equation for the resistance, let

E = potential difference between the two circumferences.
I = current in the solenoids.
M_1 = sum of the four mutual inductances between the circumference of disk I and each of the four solenoids.
M_2 = similar sum for disk II.
T = time of revolution of the disk.

Then
$$E = (M_1 + M_2)\frac{I}{T}$$

If this potential difference is made equal to the fall in potential when the current I flows through a standard resistance, R, then

$$IR = (M_1 + M_2)\frac{I}{T}$$
or
$$R = (M_1 + M_2)/T$$

In the Smith apparatus the mutual inductance between the circumference of a disk and its associated solenoids was about two hundred times greater than between the same circumference and solenoids associated with the other disk. Hence, if the first inductance is to be determined with an accuracy of one part in a million, the second should be evaluated to two parts in ten thousand.

The two solenoids associated with a disk are placed symmetrically with respect to the disk. By this arrangement the lines of magnetic induction resulting from the current in the two coils are parallel to the common axis at all points in the center plane of the disk, and hence a slight displacement of the disk in the axial direction will make only an exceedingly small change in the number of tubes of induction passing through the disk. Smith showed that, if a disk of infinitesimal thickness were displaced by ¼ mm from the symmetrical position, the change in the mutual inductance would be less than two parts in a million. Hence, with a disk having a rim with a thickness of ½ mm, the position of the contact brush cannot affect the result by more than two parts in a million.

Each of the lines of magnetic induction of a single solenoid has a point where its direction is perpendicular to the axis. Hence in every plane perpendicular to the axis of the solenoid at a point beyond the end of the solenoid there is a circle which has its center in the axis and which has the property that at every point on its circumference the lines of magnetic induction are radial. If the disk is of such size that the center of its edge coincides with the circle indicated above, then a slight change in size will produce only an infinitesimal change in the number of tubes of induction through the disk. In Smith's apparatus, a change in diameter of 0.2 mm produced a change in mutual inductance of only four parts in a million.

Combining the results of the last two paragraphs, it appears that current in the two solenoids associated with a disk produces a magnetic field which not only has zero value at the edge of the disk but also has both its axial and radial derivatives zero at that point. Both these later conditions are very important, since the disk must have finite thickness and since the diameter of a rapidly rotating disk cannot be accurately determined.

105. Computation of the Constants of a Lorenz Apparatus.— The computation of the mutual inductance between a solenoid and the circumference of one of the disks can best be accomplished by considering that this is the difference of two inductances: one between the circumference and the actual solenoid plus an extension to the plane of the disk, and the other between the circumference and the extension only. Each of these computations can be made by the formulas, which are given in Chapter XIV, for the mutual inductance between a solenoid and a coaxial circle in the plane of one end. For computing the mutual inductance between the circumference of a disk and its associated solenoids, the elliptic integral formula should be employed. For computing the mutual inductance between the circumference of a disk and the solenoids associated with the second disk, the series formula is more convenient than the elliptic integral formula and is sufficiently accurate.

If all four solenoids have the same dimensions and are similarly placed relative to their respective disks, assumed to have the same diameter, only six inductances need be computed instead of the 16, which are required when all solenoids are different. However, slight differences often exist, and the calculation can often be simplified by making use of the partial derivatives of the formula for the mutual inductance. These are not given, but can readily be obtained.

106. A Suggested Addition to Smith's Apparatus.—While each disk is so designed relative to the two solenoids associated with it that a slight variation either in the axial position of the brush contact or in the diameter of the disk will produce only a slight effect in the induced electromotive force, the same condition does not hold between the one disk and the coils associated with the other disk. In Smith's apparatus, the displacement of the brush-contact circle by 0.25 mm from the center of the disk produced a change of only one part in a million in the total induced electromotive force as a result of the coils associated with the disk, but a change of nearly twelve parts in a million as a result of the coils associated with the second disk. This variation produced a considerable uncertainty in the result, as the exact position of a brush on the disk could not be determined. This uncertainty can be nearly or quite eliminated by placing four additional coils as shown in Fig. 47. The coils are identical with those used in the apparatus, are placed coaxial with them, and are spaced at distances the same as the principal coils of the apparatus. The two pairs of coils which are symmetrical with respect to a disk produce a magnetic field in which the lines of magnetic induction are parallel to the axis of the coils at all points in the plane that is midway between them. Thus at the disk the lines of induction are so nearly parallel that the induced electromotive force is practically the same in all elements of the disk. The pair of coils farthest from each disk produces a difference in the induced electromotive force between the two edges of the disk which is only one-sixteenth of that which occurred without the additional coils.

The uncertainty in the radius of the disk was probably about 0.01 mm in Smith's apparatus. (The expansion caused by the rotation was 0.04 mm.) Because of this uncertainty in the radius, there was an uncertainty in the computed inductance between the circumference of a disk and the pair of coils associated with the other disk, such that the uncertainty in the total

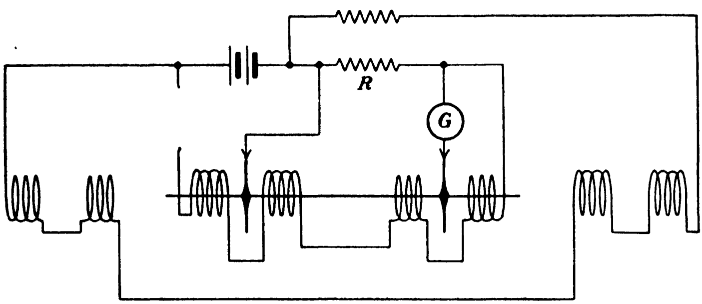

FIG. 47.—A suggested addition to the Lorenz apparatus of Smith.

inductance was about one part in a million. With the additional coils this would be doubled.

The additional coils would not need to be made with as much care as the main coils. For example, the diameter of the main coils of Smith's apparatus had to be known to 0.1 micron in order that the inductance could be computed to one part in a million, while, if the additional coils were known only to 10 microns, the accuracy would be the same.

107. Division of the Disks into Segments.—In order to increase the induced electromotive force, Smith divided each disk into ten equal segments. Each segment was insulated from the others and connected to the corresponding segment on the other disk by an insulated wire which ran through the hollow shaft connecting the two disks. The symmetry of the tubes of magnetic induction about the axis shows that the path taken by the wire connecting two segments does not affect the induced electromotive force. Five brushes were used on each disk, spaced 72° apart, so that no segment would ever be in contact with more than one brush. By connecting brush 1 of the second disk to brush 2 of the first disk, etc., the potential difference between brush 1 of disk 1 and brush 5 of disk 2 is five times that between any pair of brushes.

While the division of the disks into segments was made primarily for the purpose of increasing the induced electromotive force in the hope that thermal electromotive forces would be less troublesome, little direct advantage resulted, for the thermal electromotive forces increased about as rapidly as the induced electromotive force.

108. Adjustment of Coils in Smith's Apparatus.—In Smith's apparatus, each of the four coils must be adjusted to be coaxial with the rotating disks, and all must be at the same distance from the disks. If a single brush is used on an undivided disk, the axial adjustments must be accurately made. However, with a segmented disk, using several brushes with the circuits connected either in series or parallel, the coaxial adjustment need not be made with great precision. Smith showed experimentally that with five brushes a displacement of a coil from the coaxial position by 0.8 mm in a direction perpendicular to the axis produced a change in the average induced electromotive force of only one part in a million. Also a rotation so that the axis of a coil made an angle of 10′ with the axis of the disk produced a like change in electromotive force. Hence with the segmented disk and several brushes, the coils could easily be set to their coaxial positions by mechanical measurements. In the placing of the coils along the axis, it is more important to have all of them at the same distance from the disks than to have them at any predetermined distance. This can be done by balancing the induced electromotive forces in the two disks. First place coil I*A* at the position decided upon in designing the apparatus, using mechanical measurements. Then connect coils I*A* and II*B* to a battery, and the brushes to a galvanometer as shown in Fig. 48. The direction of the current through the coils is such that the induced electromotive forces in the two disks oppose each other. Coil II*B* is moved in the axial direction until the galvanometer deflection is zero. The same procedure is then followed with coils 1*B* and II*A*. Finally all four coils are connected in series so that the magnetic fields of 1*A* and I*B* are in opposite directions, as also are those of II*A* and II*B*. Now the galvanometer deflection is made zero by moving I*B* and II*A* by equal amounts but in opposite directions. Then all the coils are at the same distance as coil 1*A*.

109. Accuracy Required in Measuring Dimensions.—The accuracy required is not the same for the four different dimensions involved in the computation of the mutual inductance between one of the coils and the circumference of the disk. The general principle for determining the effect of an error in one dimension on the computed inductance is given in Appendix II. These principles have not been completely applied to an apparatus of the form used by Smith However, there is sufficient informa-

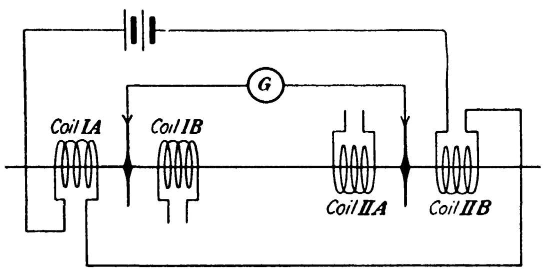

FIG. 48.—Connections used in adjusting the coils of the Lorenz apparatus.

tion to warrant some statement concerning the required accuracy in the mechanical measurements.

1. The diameter of the coil must be measured with nearly twice the relative accuracy required for the mutual inductance.

2. The diameter of the disk is relatively unimportant, since the apparatus was designed so that changes in diameter of the disk would produce only second-order effects in the mutual inductance.

3. The length of the coils must be measured to about one-tenth of the relative accuracy required for the mutual inductance.

4. The distance from the disk to the nearer end of a coil must be determined with about the same accuracy as required for the mutual inductance. In practice, the distance between the two coils associated with a disk is measured, the disk being midway between the coils.

5. The distance between the two disks must be known with about one-fortieth of the accuracy required for the mutual inductance.

Assuming that each dimension must not introduce an error greater than one part in a million in the mutual inductance M, the precision with which each of the dimensions of Smith's apparatus must be measured is as follows:

Diameter of coil, a, 0.1 micron.

Diameter of disk, A, 38 microns.

Length of coil, l, 2 microns.

Distance between the two coils associated with one disk, x, 0.3 micron.

Distance between disks, z, 0.1 mm or 100 microns.

It has already been shown that the disk need not be centered between the two coils closer than a few tenths of a millimeter.

110. Measurement of Dimensions of Smith's Apparatus.—Methods of measuring the length and diameter of a coil have already been described. The diameters of the disks can be measured with a machinist's micrometer. The distance between disks can be measured with a bar calibrated to half millimeters. The measurement of the distance between the two coils requires special description. To facilitate this measurement, Smith had lines parallel to the end of the coil ruled on metal plugs inserted near each end of each coil. The apparatus was assembled, the electrical measurements were made, and for each pair of coils the distances between the two plugs on the left-hand ends of the coils and between the two plugs on the right-hand ends were determined. The apparatus was then taken apart, and each coil was turned end for end. Then the apparatus was reassembled with the coils spaced so that the induced electromotive force was the same as in the previous assembly, and the distances between the plugs were again measured.

The distance from the disk to the end of the coil can be obtained from the two sets of measurements and the lengths of the coils. For deriving the equation, the following nomenclature will be used (see Fig. 49):

l_a and l_b = lengths of coils a and b, respectively, both of which have previously been measured.

δ_1 = distance from plug a_1 to center of adjacent lead of coil IA.

δ_2 = distance from plug a_2 to center of adjacent lead of coil IA.

ξ_1 = distance from plug b_1 to center of adjacent lead of coil I*B*.

ξ_2 = distance from plug b_2 to center of adjacent lead of coil I*B*.

x_d and x_r = distance between adjacent ends of coils for direct and reversed positions, respectively.

a_1b_1 and a_2b_2 = measured distances between the indicated plugs for *direct* positions of coils.

$a_1'b_1'$ and $a_2'b_2'$ = measured distances between plugs for *reversed* positions of coils.

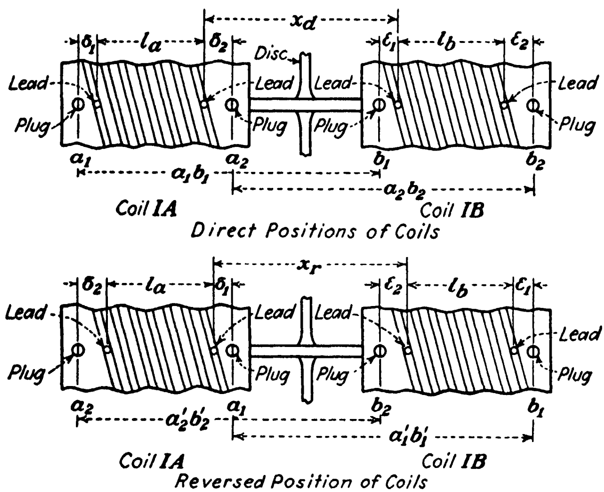

FIG. 49.—Diagram to show the method used by Smith to determine the distance between the coils of the Lorenz apparatus.

In the direct position

$$a_1b_1 = \delta_1 + l_a + x_d - \xi_1 \tag{1}$$
$$a_2b_2 = -\delta_2 + x_d + l_b + \xi_2 \tag{2}$$

After a series of electrical measurements with the coils in the direct position, each coil is turned end for end and placed at approximately the same distance from the disk by the method already described. (Coils 3 and 4 should not be changed until 1 and 2 are set at the correct distance.) Designating the new distances by primed letters,

$$a_2'b_2' = \delta_2 + l_a + x_r - \xi_2 \tag{3}$$
$$a_1'b_1' = -\delta_1 + l_b + x_r + \xi_1 \tag{4}$$

Combining Eq. (1) with Eq. (4) and Eq. (2) with Eq. (3), there results the equations

$$a_1b_1 + a_1'b_1' = l_a + l_b + x_d + x_r \tag{5}$$
$$a_2b_2 + a_2'b_2' = l_a + l_b + x_d + x_r \tag{6}$$

Solving Eq. (5),

$$x_d + x_r = a_1b_1 + a_1'b_1' - (l_a + l_b) \tag{7}$$

Also, from Eq. (6),

$$x_d + x_r = a_2b_2 + a_2'b_2' - (l_a + l_b) \tag{8}$$

But $x_d + x_r$ is four times the distance from the end of the coil to the disk. Since $x_d = x_r$, the sum of Eqs. (7) and (8) gives the result

$$x_d = \frac{(a_1b_1 + a_1'b_1' + a_2b_2 + a_2'b_2')}{4} - \frac{(l_a + l_b)}{2} \tag{9}$$

This method of determining x_d has the advantage that the distances a_1b_1, etc., which must be measured in the assembled apparatus, are all of approximately the same length, so that a single calibration of a length standard suffices for all these measurements. However, there is the disadvantage that a rather short distance is determined as the difference of two distances, both of which must be known to the same absolute accuracy as is required in the smaller distance. Also the value which is finally determined as the distance from the end of the coil to the disk, and which is used in computing M, results from measurements at two positions of the coils, but the electrical measurements are also made with the coils in both positions so that any slight difference between x_d and x_r does not appreciably affect the result.

In addition to the above dimensions, the distance between each disk and the coils associated with the other disk is required. A measurement of the distance between any set of plugs in coils 1 and 2 with any set in 3 and 4 gives data which, when combined with that already taken, supply all the required distances. This distance is needed with an accuracy of about 50 microns, which is readily obtained.

111. Effect of Constructional Imperfections.—The construction of any piece of apparatus for precision measurements involves so many difficult operations that measurable imperfections may be discovered in the completed apparatus. In Smith's modification of the Lorenz apparatus, the design was such that certain parts did not have to be made with extreme precision. This especially applied to the rotating disks, where neither the diameter nor the endwise motion needed especial attention. However, the imperfections of the coils were such that corrections were applied on account of them. The forms on which the coils were wound were slightly conical and their cross sections were ellipses. Also the pitch of the winding was not strictly uniform. In deriving the formulas for computing the corrections to the inductance resulting from these imperfections, the following assumptions were made: (1) that the mutual inductance between a circle and a conical coil is the same as between a circle and a coil made from a series of circular hoops, each having a diameter equal to the mean diameter of the cone at that point; (2) that the mutual inductance between a circle and a loop of wire which is nearly circular is the same as when the loop is replaced by a circle having the same average diameter; (3) that the average deviation of a turn from its correct position produces the same effect on the mutual inductance between the turn and the circumference of the disk as is produced by the variable deviation of the turn. The computed corrections based on these assumptions will give a more nearly correct result for the mutual inductance than would be obtained without applying them. However, the experimental conditions do not coincide exactly with these assumptions. Only when the imperfections are so small that the correction terms are negligible is the computed mutual inductance certainly correct.

Smith's coils were so nearly circular that no correction for lack of circularity was required. The corrections to the computed inductance of each of the four coils for the conicality of the coil and the irregularities in winding are given in Table XI.

The correction for conicality is the change produced by changing the coil end for end. The correction for irregularities in winding depends on which end is nearer to the disk. For all of Smith's coils, when one end was toward the disk, the mutual inductance was increased; with the opposite end toward the disk, the inductance was decreased. Hence the two columns are headed "Positive" and "Negative." This table shows that coils with smaller imperfections would probably give a more accurate result.

TABLE XI.—COMPUTED CORRECTION FOR SMITH'S COILS

Coil	Correction for conicality	Correction for irregularities in winding	
		Positive	Negative
1	50×10^{-6}	17×10^{-6}	12×10^{-6}
2	36	33	29
3	21	18	25
4	12	16	51

112. The Electrical Measurement.—The electrical measurement consists of determining when the induced electromotive force in the disks is equal to the fall in potential over the resistance. By adjusting the speed of the disk, the electromotive force can be altered until the galvanometer (see Fig. 42) shows no deflection. This would be very simple if it were not for the thermal electromotive forces which are generated at the brush contacts. To eliminate the effects of these, Smith reversed the current at 15-second intervals and adjusted the speed until there was no change in galvanometer deflection on reversal. If the thermal electromotive forces remained constant for 30 seconds, this procedure would completely eliminate their effect. By using brushes in the shape of a violin bow and lubricating them freely with kerosene, Smith reduced the thermal electromotive forces until the change in 30 seconds was seldom more than a few hundredths of a microvolt. By making a large number of reversals, the probable error of an observation was reduced to a few parts in a million.

113. Measurement of the Speed of the Disk.—In Smith's apparatus, the speed was measured on a chronograph drum connected by a worm gear to the shaft of the driving motor. The method of controlling and measuring the speed was adapted from a method used by Rosa and Dorsey.[1] As more precise methods of controlling and measuring the speed are now available, this method will not be described.

[1] *Bull. B.S.*, **3**, 561 (1907).

A suggested method of controlling the speed uses the precise frequencies that are now available. These are furnished by vibrating crystals, which often remain constant to a part in ten million for several weeks. From a crystal vibrating at 100 000 cycles per second, frequencies of 10 000, 1000, and 100 cycles per second may be available. If three synchronous motors designed for these three frequencies are mounted on one shaft, the 100-cycle motor can easily be started and will supply the necessary power, while the higher frequency motors will prevent hunting by the low-frequency motor. This method has been used in some commercial appliances, but no data are available as to the accuracy obtainable. This gives but a single speed, so that a balance must be obtained by adjusting the resistance.

114. Accuracy Obtainable with a Lorenz Apparatus.—Smith estimated that the error in his determination did not exceed twenty parts in a million. A consideration of the results obtained when using the brushes in series and in parallel and when running the disks at different speeds indicates that this is a reasonable estimate. However, there are no data to show the effect of coil imperfections, and these may have introduced an appreciable error. Two independent sets of coils should be employed to give results of the highest accuracy. By suitable design and construction, an apparatus might be produced which would give results accurate to a few parts in a million.

115. Description of the Self-inductance Method with an Intermediary Capacitance.—The self-inductance method with an intermediary capacitance determines the correction to the laboratory ohm to make it agree with the absolute ohm, instead of determining the absolute value of a particular standard as is done by the Lorenz apparatus. To obtain this correction, a self inductor is so constructed that its inductance can be computed from its measured dimensions, giving the value of its inductance L_a in absolute henrys. The inductance is also measured in terms of time and the laboratory unit of resistance, usually international ohms, giving a value L_i. Then for any laboratory standard having a resistance R_i, the value in absolute ohms R_a is determined from the ratio

$$\frac{R_a}{R_i} = \frac{L_a}{L_i}. \tag{10}$$

Methods for constructing the inductor, for measuring its mechanical dimensions, and for computing its inductance in absolute henrys have already been described. The method of measuring the self inductance in terms of the laboratory units of resistance and time makes use of two bridges which have also been previously described in detail, but an abridged description will be given here so that the special features required in applying them to this method can be more readily considered.

The bridges are shown in Fig. 50. At the right is an alternating-current bridge by which the inductance can be measured in terms of resistance and capacitance by means of an alternating current of low frequency. At the left is Maxwell's absolute

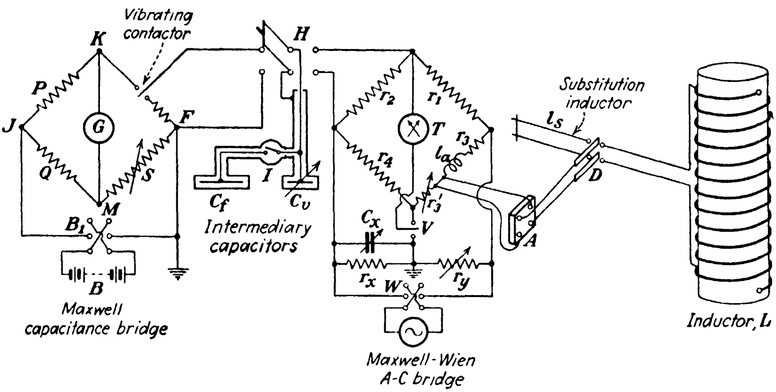

FIG. 50.—Setup for the absolute measurement of resistance using alternating current in a self inductance combined with an intermediary capacitance.

capacitance bridge for measuring the capacitance in terms of resistance and time. By means of the switch H, the capacitance can be connected in either bridge as desired.

In the alternating-current bridge, the inductor L and the substitution inductor l_s have very nearly the same resistance, so that on removing one and inserting the other by means of the switch D the balance is restored by removing the capacitance C_f by the switch I and changing C_v to C_v', with such slight changes in the resistance of the Q_1 arm that the inductance is not changed thereby. For frequencies so low that the capacitance of the coil does not influence the result, the difference between the inductance is given by the equation (see page 116 for derivation)

$$L - l_s = (C_f + C_v - C_v')r_1r_4 \qquad (11)$$

where r_1 and r_4 are the resistances of the arms as shown on the diagram.

In the Maxwell bridge, the values of $C_f + C_v$ and C_v' as used in the alternating-current bridge are determined in terms of resistance and time. The addition of a small capacitance, such as the capacitance of the leads and commutator, does not affect the result, as only the difference between the two measured values is required. An air or vacuum capacitor must be used for both C_f and C_v, as these are the only kinds of capacitor in which the capacitance does not depend on the method of measurement. The equation for capacitance is (see page 125 for derivation)

$$C_f + C_v = \frac{Q}{nPS}\left\{\frac{1 - \dfrac{Q^2}{(P+Q+G)(S+Q+B)}}{\left[1 + \dfrac{BQ}{P(S+Q+B)}\right]\left[1 + \dfrac{GQ}{S(P+Q+G)}\right]}\right\}$$
$$= \frac{QF}{nPS} \qquad (12)$$

where n is the number of charges and discharges of the condenser per second, and the resistances are as indicated on the diagram. The correction factor F must be carefully evaluated. The computation should be made by the exact formula given in Eq. (12) unless it can be shown that one of the approximate formulas given on pages 125 and 126 is sufficiently accurate.

Also

$$C_v' = \frac{Q'F'}{nP'S'} \qquad (13)$$

where Q', P', and S' are different values of the resistances and F' is a different value of the correction factor. In case C_v' is small (1000 micromicrofarads or less), the correction factor F' will generally be so near unity that its computation is not necessary. Substituting the values of $C_f + C_v$ and C_v' in Eq. (11), the equation becomes

$$L - l_s = \frac{r_1 r_4}{n}\left(\frac{QF}{PS} - \frac{Q'F'}{P'S'}\right) \qquad (14)$$

The ratio of this measured inductance to the computed inductance is equal to the ratio of the value of a resistance standard in laboratory ohms to its value in absolute ohms.

In an actual experiment, all the resistances are compared by means of a Wheatstone bridge with a single standard resistance, the value of which is R_i, in terms of the laboratory unit (often international ohms). Substituting the resistance values in Eq. (14), there results an equation of the form

$$(L - l_s)_i = \frac{KR_i}{n} \tag{15}$$

where $(L - l_s)_i$ is the measured value of the difference of the two inductances, and K is a numerical constant which depends only on the ratios of the resistances used in the bridges, and hence is independent of the resistance unit used in comparing them. If the value of the standard resistance in absolute ohms is R_a, and if this value had been used in making the measurements, the equation for inductance would have been

$$(L - l_s)_a = \frac{KR_a}{n} \tag{16}$$

But $(L - l_s)_a$ is the same as the inductance which has been computed from the measured mechanical dimensions of the inductors. By dividing Eq. (16) by Eq. (15),

$$\frac{(L - l_s)_a}{(L - l_s)_i} = \frac{R_a}{R_i} \tag{17}$$

This is a proof of Eq. (10).

116. Numerical Example.—An example of the constants that have been satisfactorily used in applying this method are the following:

Computed inductance L of single-layer solenoid, 39.90210 millihenrys.

Inductance l_s of substitution inductor, 0.00048 millihenry.

Inductance l_0 of auxiliary coil in Q arm to insure that the the inductance in this arm is sufficient to make a readable value for C_v', about 0.1 millihenry.

The air condenser to be used, C_f, had a capacitance of about 0.25 microfarad. Hence, for measuring $L + l_0$,

$$r_1 r_4 = 160\,000 \text{ (ohm)}^2$$

Or if r_1 and r_4 are to have the same value,

$$r_1 = r_4 = 400 \text{ ohms}$$

When the inductance L is disconnected and the substitution inductance l_s is inserted in its place, $l_0 + l_s$ is 0.105 millihenry. The bridge is balanced when the resistances of the arms are unchanged and the capacitance is about 650 micromicrofarads. The value of l_0 may be changed somewhat until C_v and C_v' have convenient values.

In measuring the capacitance of 0.25 microfarad by the Maxwell absolute bridge, if the number of charges and discharges per second is 100 and if $P = Q = 100$ ohms, S will equal about 40 000 ohms. To obtain an accuracy of one part in a million, the reading of S must be made to 0.04 ohm, so that a resistance box in this arm with a hundredth-ohm dial is necessary. The value of F, if $G = 10$ ohms and $B = 1$ ohm, is 0.998669. To obtain an accuracy of one part in a million in F, B must be measured so that its value is known to 0.04 ohm, G to 0.2 ohm, Q to 0.1 ohm, P to 0.2 ohm, and S to 30 ohms. The value of Q/PS is determined by substituting a calibrated 40 000-ohm resistance (designated as R) in place of the commutator. The calibration of this resistance with the required accuracy will be considered later.

In measuring the capacitance of 650 micromicrofarads when the number of charges and discharges per second is 100, PS/Q must be about 15 megohms. It may not be feasible to measure this by substituting a calibrated resistance of 15 megohms in place of the commutator. In such a case, the value of each resistance may be separately determined with the required accuracy. Convenient values are $Q = 10$ ohms, $P = 10\,000$ ohms, and $S = 15\,000$. Since the capacitance is required to 0.2 micromicrofarad, each resistance must be measured to about one part in ten thousand. The value of F is so nearly unity that no correction is required.

In calibrating the resistances of the alternating-current and Maxwell bridges, a convenient method is to refer all resistance measurements to a single 100-ohm standard, the value of which is known in terms of the resistance unit of the laboratory. This standard and at least three additional standards of the same nominal value are kept in an oil bath at a constant temperature. A Wheatstone bridge is arranged to intercompare these coils and to compare them with the resistances to be calibrated. The resistances r_1 and r_4 of the alternating-current bridge can be directly compared with the four 100-ohm standards connected in series. To obtain a calibration for the 40 000 ohms used to determine the value of PS/Q in the capacitance bridge, it is convenient to have a box with ten 4000-ohm coils which can be connected either in series or in parallel. When in parallel, the box may be directly compared with the four 100-ohm standards in series. The value in series, if the coils differ in value by less than 0.1 per cent, is 100 times the value in parallel with an accuracy of a part in a million. The calibration of the resistances used in measuring the smaller capacitance must be made often enough to insure that their values are always known with an accuracy of one part in ten thousand.

117. Time Required for an Electrical Measurement.—In making a determination such as outlined above, all the electrical observations for an individual determination must be made within a few hours, since neither air capacitors nor resistance boxes are sufficiently stable to ensure that their values do not change from day to day by more than one part in a million. The standard inductor and the standard resistance are depended upon to remain constant during a series of measurements which may last weeks or years. Hence the procedure must be so carefully considered that all the necessary observations for a complete determination can be made in such a short time that the resistances and capacitances do not change by an appreciable amount. Whenever possible, it is desirable to make two or more sets of measurements in one day to detect any drift that may occur.

118. The Assumptions Involved.—The important assumptions involved in the self-inductance method are: (1) that the inductor is so perfectly constructed that its inductance can be accurately computed from its dimensions; (2) that the resistors r_1 and r_4 have the same resistance when used in the alternating-current bridge as when calibrated by direct current; (3) that the capacitors have the same capacitance in the alternating-current bridge as when calibrated by a charge and discharge method; and (4) that the galvanometer in the Maxwell bridge correctly integrates the current through it. Besides these assumptions there are many precautions that must be observed. The accuracy of the final result depends upon the skill with which the assumptions are met and all difficulties in manipulation overcome.

The assumption of a perfect inductor can never be fully realized. For a solenoid such as is usually used as a standard of self inductance, the important considerations are that the form on which the wire is wound shall be a circular cylinder, that the wires shall be round, that the pitch shall be uniform, and that the number of turns shall be an integer. The construction of a circular cylinder can be satisfactorily accomplished as has been described. The making of round wire is more difficult. Commercial wire is usually somewhat elliptical, and generally has striae on its surface. Satisfactory wire can be made by drawing copper wire through a sapphire die which has the hole drilled in the direction of the principal axis of the crystal. There is no difficulty in making the number of turns sufficiently near an integer. The variations in pitch are appreciable in all the solenoids that have been described in the literature. All these have depended on the accuracy of the screw of a precision lathe and the faithfulness with which the lathe carriage follows the motion of the nut of the screw. The variations can be detected by the naked eye in some of the best solenoids. The effect of variations in pitch usually introduces an uncertainty of ten or even twenty parts in a million in the computed inductance. It is hoped that Moon's method, to which reference has previously been made, will appreciably reduce this uncertainty.

The second assumption of the method is that the resistors of the alternating-current bridge have the same resistance in the bridge as when calibrated by direct current. In most commercial resistors there is sufficient dielectric loss in the insulating material between the wires to lower appreciably the alternating-current resistance. Organic materials which absorb moisture, such as silk, cotton, and shellac, are quite unsatisfactory as the insulation for such coils. A change in resistance has been observed with a bare manganin wire wound in a screw thread on a porcelain spool when the humidity of the surrounding air was above normal but still low enough so that surface leakage did not cause the change. Grüeneisen and Giebe used bare wire wound on mica cards. Curtis, Moon, and Sparks used bare wire wound on a tube of Pyrex glass. Both appeared to be satisfactory, but farther experimental work is desirable.

The third assumption is that the capacitors have the same capacitance when used in the alternating-current bridge as when calibrated by a charge and discharge method. It is well known that this condition is fulfilled only with air and vacuum capacitors.

But these capacitors must have some solid insulating material to maintain the distance between the plates. Probably the most satisfactory material for this purpose is fused quartz, because of its small dielectric absorption. A satisfactory test of the quality of a capacitor is to measure the capacitance with two different times of charge and discharge, say 100 and 200 per second. If the measured capacitance is the same under these two conditions, there is little probability that it is different with alternating current of low frequency.

The fourth assumption is that the galvanometer in the Maxwell bridge correctly integrates the current. This has been discussed on page 123 in connection with the Maxwell bridge, but it is so fundamental to this method for the absolute measurement of resistance that some additional discussion will be given. The galvanometer should be designed for this particular application. The natural period must be long so that the motion during a period of the vibrating contactor will be small. The magnet of the galvanometer must have a high coercive force so that the magnetic field of the large current in the galvanometer coil, sometimes as much as an ampere for a few millionths of a second, will not produce a significant change in the magnetic field of the magnet. Eddy currents in the pole pieces of the magnet should be avoided. The resistance of the galvanometer coil should be low to improve the accuracy of the correction factor, and the external critical damping resistance should be about 200 ohms to facilitate the operation of the bridge under normal conditions. Every galvanometer should be tested by the method shown in Fig. 29 to ensure that it integrates correctly. Even then two or more galvanometers having somewhat different constants should be used in the course of a series of measurements.

119. Necessary Precautions.—One of the precautions that must be observed is to place the inductor so far from the bridge that the mutual inductances between the various parts of the bridge and the inductor are small. The effect of small inductances can be eliminated by using a reversing switch A (see Fig. 50) in the leads to the inductor. The average value of the readings of the variable capacitor for the two positions of A gives the reading that would be obtained were no mutual inductances present. However, this reversal does not eliminate the effect of a mutual inductance between the solenoid and the part of the leads between A and D. This must be accomplished by placing the two leads in such a position that the induced electromotive force is the same in each and in opposite directions. If, in the diagram, the axis of the coil, its leads, and the substitution inductor all lie in the plane of the paper, the leads AD are perpendicular to the plane of the paper. The reversing switch A also serves to eliminate the effect of any extraneous electromotive force that is induced in the inductor, whether this is caused by the current in the circuit connected to the bridge or by the current in some independent circuit in which the frequency is the same as in the measuring circuit.

The reversing switch W is useful in providing two pairs of readings, each pair of which has the same mean value as the other pair, but the individual readings are different so that the observer is not prejudiced when making the second set of readings. To illustrate, suppose that there is a transformer between the alternator and the reversing switch W, and that the magnetic field from this transformer induces a small electromotive force in the inductor. Also suppose that there is a small mutual inductance between l_a and the inductor. Then the four readings of C_v corresponding to the four possible combinations of positions of the reversing switches A and W will all be different. However, the average of the two for one position of W will be the same as that for the other position.

The inductor must be at some distance from any magnetic material. The table on which the inductor is placed must be free from any iron nails or screws. The position when the electrical measurements are made must be near the center of a room, away from iron piping, reinforcing, and window weights. Even the nails in the floor may influence the measured inductance. With care, the inductor may be kept at a sufficient distance from such materials, but in all cases consideration must be given to the permeability of the form on which the wire of the inductor is wound. Forms made of such materials as porcelain, glass, and fused quartz may affect the inductance by several parts in a million.

The fixed capacitances C_f and C_v and the disconnecting switch I must be completely shielded if the bridges are to be grounded as shown. In the capacitance bridge, the shield is at earth potential, but this is not the case with the alternating-current bridge. Hence only by complete shielding is the capacitance difference the same in the Maxwell bridge as in the alternating-current bridge.

The detector T must be so placed that it is not affected by the magnetic field of the inductor or by that of any other part of the alternating-current circuit. The effect of a magnetic field on the detector can be determined by observing its deflection when it is disconnected from the bridge. If there is a deflection when the switch W is open, then the magnetic field results from some outside apparatus. On the other hand, if there is a deflection only when the switch W is closed, the magnetic field probably results from the inductor. In the latter case, the relative positions of the inductor and detector should be changed to make the deflection zero, although if the deflection is small, its effect can be eliminated by making an additional set of readings in which the galvanometer leads are reversed.

120. Accuracy Attainable with the Self-inductance Method.— An accuracy of a few parts in a million should be possible with the self-inductance method. A solenoid can be more accurately constructed than any other form of inductor, but the imperfections of most solenoids introduce an uncertainty of ten or twenty parts in a million in the computed inductance. The measurements of the dimensions can be made with an accuracy that would introduce an error of only one or two parts in a million in the computed inductance of a perfect inductor. The formula for computing the inductance neglects only terms of so little importance that their omission does not appreciably influence the result. The one uncertainty is concerned with the distribution of current in the cross section of the wire, but this is less than two parts in a million in most cases. The electrical measurements are difficult, but with sufficient care can be made with an accuracy of a few parts in a million. By making a number of measurements, the average should be in error by only one or two parts in a million.

From the above analysis, it appears that the greatest source of error is in the construction of the solenoid. Curtis, Moon, and Sparks estimate that the error in their result may be twenty parts in a million, resulting largely from imperfections in the inductors. The next improvement will necessarily be concerned with the construction of a more perfect solenoid.

CHAPTER XVIII

ABSOLUTE MEASUREMENT OF CURRENT; SURVEY OF METHODS

The value of a current is determined in c.g.s. electromagnetic units either from the magnetic field produced at a point by the current or from the force exerted on a conductor carrying the current when the conductor is in a magnetic field of known strength. The first principle is employed in the tangent galvanometer and the sine galvanometer; the second in the current balance and the electrodynamometer. Each of these classes of instruments will be described and the accuracy with which a current can be measured by them will be discussed.

121. The Tangent Galvanometer.—The tangent galvanometer consists of a small magnet, free to rotate in a horizontal plane, which is placed at the center of a circular coil of wire. The cross-sectional dimensions of the coil must be small relative to its radius, and its plane must be vertical and coincident with the magnetic meridian. When a current I flows in the coil, the magnetic field at the center is

$$H_i = \frac{2\pi nI}{r} \tag{1}$$

and its direction is perpendicular to the horizontal component H_e of the earth's magnetic field. The magnet places itself in the direction of the resulting magnetic field, which is the vector sum of H_e and H_i and hence makes an angle θ with the direction of the magnetic meridian such that $\tan \theta = H_i/H_e$. Substituting the value of H_i and solving,

$$I = \frac{rH_e}{2\pi n} \tan \theta \tag{2}$$

For a precise determination, corrections must be applied for the cross section of the coil and for the finite length of the needle. Also the coil must be accurately placed in the magnetic meridian and the needle accurately located at the center of the coil.

The method is not suitable for the precise measurement of current since the variability of the earth's magnetic field prevents its accurate measurement in c.g.s units. Haga and Boerema[1] claimed an accuracy of 100 parts in a million, which is probably the limit of accuracy to be attained by this method. The method is, however, of great historical importance, since for nearly a half century following the first description[2] in 1837, it was in regular use in all electrical laboratories. It was discarded when laboratories found that the magnetic field from commercial circuits affected its readings. The tangent galvanometer is not now used either as a laboratory instrument for measuring current or as an instrument of precision.

122. The Sine Galvanometer.—The sine galvanometer is constructed like the tangent galvanometer except that the coil is so mounted that it can be rotated about a vertical axis. With no current in the coil, the plane of the coil and the axis of the magnet are placed in the magnetic meridian. With a current I in the coil, the coil is turned around a vertical axis by an angle φ until the plane of the coil and the axis of the magnet again coincide. The direction of the resulting magnetic field, obtained by vector addition, is such that

$$\sin \varphi = \frac{H_i}{H_e} \tag{3}$$

so that

$$I = \frac{rH_e}{2\pi n} \sin \varphi \tag{4}$$

The sine galvanometer has all the defects of the tangent galvanometer. Its only advantage is that it gives a larger deflection of the magnet for a given current. As the same result can be obtained by increasing the number of turns of the tangent galvanometer, the sine galvanometer has never been extensively used.

123. The Absolute Electrodynamometer.—The absolute electrodynamometer is an instrument by which the absolute value of a current that flows through two coils can be determined from the measured torque between them when their dimensions and relative positions are known. Two different methods for measuring the torque have been employed: one by observation of the

[1] *Proc. Roy. Sci. Acad. Amsterdam*, 587 (1910).

[2] Pouillet, *Pogg. Ann.*, **42**, 281 (1837).

torsion produced in a fine wire which has previously been calibrated to determine the torque required to twist it through unit angle; the other by comparison with a gravitational force using either a balance with a beam of known length or a bifilar suspension.

The measurement of torque by the torsion of a fine wire was used by Guthe.[1] His method of calibrating the wire did not assure a result of greater accuracy than 100 parts in a million. There seem to be some fundamental reasons why this can never be made a precise method of measuring torque. An important reason is the very large temperature coefficient of the torsion modulus in all the materials that might be used. This varies from about 120 parts per million per degree centigrade for platinum and quartz to about 500 for copper and brass. Another reason is the apparent necessity of calibrating the wire under conditions which are quite different from those used in making the measurement. The calibration is accomplished by using the wire in a torsional pendulum, in which the wire supports a mass of known moment of inertia. From the period of the pendulum and the moment of inertia, the torque required to turn the wire through unit angle can be computed. In the measurement of current, the moving coil of the electrodynamometer is suspended by the wire. The zero position of the coil is determined with no current. Then, with current through both of the coils of the electrodynamometer (which deflects the moving coil), the upper end of the wire is turned to bring the moving coil to its zero position. The torque is determined from the angle through which the upper end of the wire is turned and the torque per unit angle (which was obtained from the torsion pendulum). In this measurement, the angle through which the upper end is turned must be large if it is to be measured accurately, but in the calibration by the torsion pendulum the angle must be small to ensure isochronous vibrations. As no wire is perfectly elastic, this difference of angle limits the accuracy of a determination of torque by this method.

A bifilar suspension was used by Shaw.[2] He found that the most accurate method of determining its constant was by observing the period of a torsional pendulum in which the suspension

[1] *Bull. B. S.*, **2**, 33 (1906).

[2] *Roy. Soc. Phil. Trans.*, **214A**, 147 (1914).

supported a mass of known moment of inertia. This is the same method that was used by Guthe for calibrating a wire. The accuracy was about the same with the bifilar suspension as with the wire.

The determination of the torque between coils by means of a balance with arms of known length has been used by Pellat.[1] In this method, the smaller coil is attached to the beam of a balance with its axis vertical and its center near the center of inertia of the beam. Then the torque between this coil and the large fixed coil within which the balance is placed can be balanced against the torque produced by a weight of known mass placed on one of the pans of the balance. This mechanical torque is equal to the product obtained by multiplying the distance between the center knife edge and the one supporting the pan by the mass of the weight and by the acceleration of gravity. All these factors are capable of accurate measurement.

Two types of coils have been used in absolute electrodynamometers—solenoids and coils of square cross section. Coils of square cross section have not been developed to a point where their diameters can be accurately determined, so they cannot be used to give a precise result. The use of a Helmholz pair of coils, either for the fixed coil or for the moving coil or for both, does not remove the requirement for an accurate measurement of the ratios of the dimensions.

The dimensions of a single-layer solenoid can be accurately measured. Hence an absolute electrodynamometer, in which the coils are single-layer solenoids and the torque is measured by a balance having a beam of known length, should be capable of giving a result of high precision. The great difficulty is in measuring the small torque. No experimental determination by this method has yet been made. However, it will be discussed in detail in the next chapter, which deals with precision methods.

124. The Current Balance.—The current balance is an instrument for obtaining an absolute value of a current by measuring the electromagnetic force between coils of known dimensions and positions. A coil is attached, with its axis vertical, to one pan of a balance, and a coaxial fixed coil is placed several centimeters above (or below) the moving coil. The electromagnetic force between these coils, when the same current is in both, is

[1] *Bull. Soc. Int. Elec.*, (2) 8, 573 (1908).

balanced by a weight placed on one of the pans of the balance. The force for unit current can be computed from the dimensions and positions of the two coils. The measured force (the mass of the balancing weight multiplied by gravity) is divided by the computed force per unit current to give the square of the absolute value of the current in the coils. Two different kinds of coils have been employed, *viz.*, single-layer solenoids and multiple-layer coils of square cross section. With both kinds the electromagnetic force is small relative to the force exerted by gravity on the mass of the coil attached to the pan of the balance, so that a balance of high sensitivity must be employed if the electromagnetic force is to be accurately measured.

Coils of square cross section can be employed in a current balance, since the ratio of the two radii is the important constant that is required to compute the current and this ratio can be determined electrically. If the two coils are about the same size, their cross-sectional dimensions and the distance between them must be accurately known, and these quantities cannot be measured with precision. However, if the diameter of the moving coil is about half that of the fixed coil, as in the *Rayleigh current balance*, the importance of the cross-sectional dimensions is so greatly decreased that they can be measured with sufficient accuracy, and the distance between them for maximum force per unit current can be satisfactorily computed from the ratio of the radii. Hence the Rayleigh balance is capable of giving a result of high accuracy. It will be described in detail in the next chapter.

If both the fixed and moving coils are single-layer solenoids, the instrument is called an *Ayrton-Jones current balance.* The dimensions of the solenoids can be accurately measured, and, by using two fixed coils with their ends close together, the position of the moving coil for maximum force per unit current can be accurately computed. This method is also capable of giving a precise result, and so will be carefully described under precision methods.

CHAPTER XIX

ABSOLUTE MEASUREMENT OF CURRENT; PRECISION METHODS

The survey of the last chapter has shown that there are three methods for the absolute measurement of current which are capable of giving results of precision, *viz.*, the Rayleigh current balance, the Ayrton-Jones current balance, and the electrodynamometer balance. These will be described in detail, in order that the strong and weak points of the different methods can be discussed. As the electrodynamometer balance has not yet received an experimental test, its description will necessarily be less complete than the others.

125. The Rayleigh Current Balance.—The Rayleigh current balance[1] consists essentially of two coils of square cross section, the smaller of which is suspended, with its plane horizontal, from the pan of a balance and the larger of which is placed coaxial with the smaller and at such a distance that the electromagnetic force between the coils is a maximum. In order to increase the force for a given current, each coil consists of a large number of turns. To increase the force still more, two large fixed coils are usually employed, one being placed above and the other below the small moving coil, the two being so connected to the electrical circuit that the electromagnetic force is double that of a single fixed coil.

A diagram showing the arrangement of the important mechanical features of the Rayleigh current balance is given in Fig. 51. The fixed coils and the pillar of the balance are fastened to a marble slab. The fixed coils are so spaced that, when the moving coil is coaxial and midway between them, the electromagnetic force on the moving coil produced by a given current is the maximum that can be obtained by any spacing. The moving coil is

[1] The first complete description is in a paper by Lord Rayleigh and Mrs. Sedgwick, *Phil. Trans.*, **175**, 411 (1884). The most recent publication on this type of balance is by Curtis and Curtis, *B.S. J. Research*, **12**, 665 (1934).

suspended from the right-hand pan of the balance, and a suitable counterweight is placed on the left-hand pan. On the right-hand pan is shown a standard weight, which is placed on the pan when the electromagnetic force is upward. The swings of the balance are observed by a telescope and scale.

An observation to determine the value of a current consists of two weighings. In the first, the standard weight is removed, the current is given such directions in the three coils that the

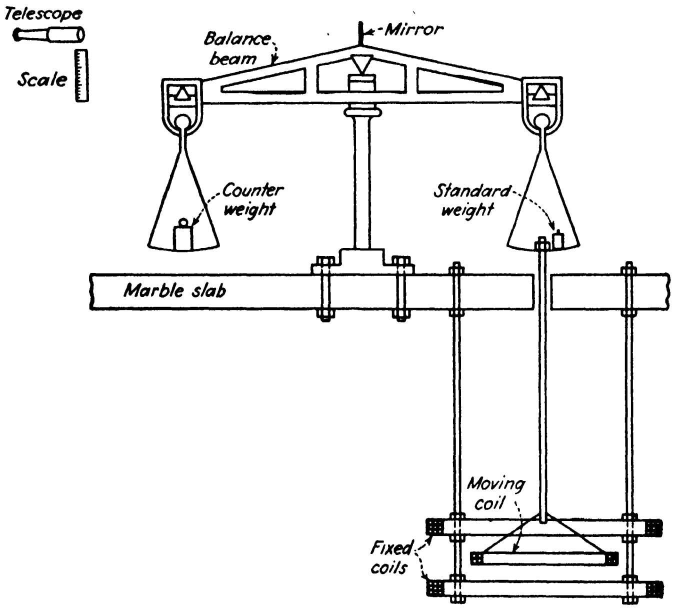

FIG. 51.—Diagram to show the mechanical features of a Rayleigh current balance.

electromagnetic force on the moving coil is downward, and the counterweight is adjusted until the middle of the swings of the beam is very near its rest position. The second weighing is then made without arresting the balance; the standard weight is placed on the pan and simultaneously the direction of the current in the fixed coils is reversed, so that the electromagnetic force is upward. The standard weight should be of such value that the gravimetric force on it very nearly equals twice the electromagnetic force on the moving coil, in which case the middle of the swings will be nearly the same in the two weighings.

If they are exactly the same, the equation connecting the electromagnetic and gravimetric forces is

$$2FI^2 = Mg$$

where F = the electromagnetic force in dynes per unit current which is exerted on the moving coil and which is computed from the dimensions of the coils.
I = the current in c.g.s. units.
M = mass of the standard weight, in grams.
g = value of gravity at the pan of the balance.

A formula for determining the accuracy with which the various quantities must be known to obtain a desired accuracy in the current is obtained by taking the logarithm of the above equation and then writing the differentials of the resulting equation. The resulting equation is

$$2\frac{dI}{I} = \frac{dM}{M} + \frac{dg}{g} - \frac{dF}{F}$$

Hence an error of two parts in a million in either M, g, or F will cause an error of only one part in a million in the current I.

A diagram of a balance used by Curtis and Curtis[1] is shown in Fig. 52. The operator is outside the balance room and makes the observations through a window. From the operator's position there extends a rod which is connected both to a reversing switch and to a weight lifter, so that the operator can simultaneously reverse the current in the fixed coils and add (or remove) the standard weight to the pan of the balance. The swings of the balance can be controlled by an air jet which the operator may produce under one pan of the balance. The heat generated in the fixed coils is removed by circulating water at a definite temperature. There are flowmeters for measuring the rate of flow of the water and resistance thermometers for measuring its temperature. The entire balance with its case is mounted on the slab which supports the fixed coils. The position of the balance can be adjusted, by means of slides not shown in the diagram, to make the moving coil coaxial with the fixed coils.

The leads for carrying the current to and from the moving coil must be carefully designed and constructed. They must carry a

[1] *B.S. J. Research*, **12**, 665 (1934).

rather large current, yet be so flexible as not to affect materially the sensitivity of the balance. Also they must be so located that, when the current is flowing, the force between them and the fixed coils is small. A satisfactory arrangement is to stretch two sets of very fine wires (diameter about 0.025 mm) from a metallic

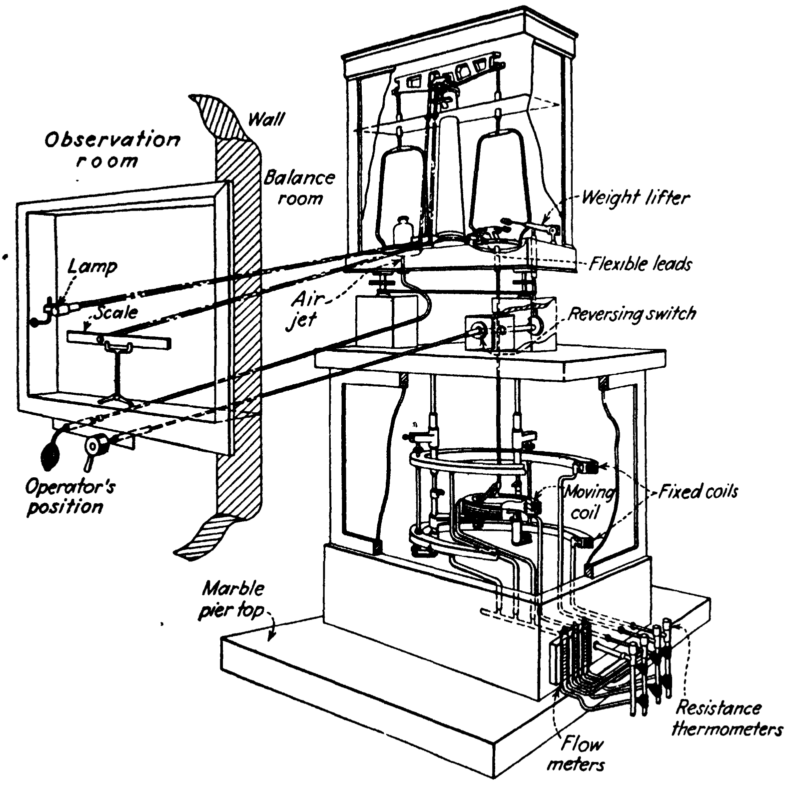

FIG. 52.—Diagram of the Rayleigh current balance as used by Curtis and Curtis.

strip attached to the pan of the balance to a metallic strip in the balance case. Each set should have the same number of wires (30 to 50), and one set should be 1 or 2 cm above the other. A diagram is shown in Fig. 53.

126. Manipulation of the Balance.—The manipulation of the balance requires extreme care on account of the very high precision with which the weighings must be made. Since the mass of the moving coil is generally 100 or 200 times the mass of the standard weight, the balance must be capable of weighing to about one part in 100 million, if the electromagnetic force is to be determined to two parts in a million, and must be operated as though weighings were being made with that accuracy. Moreover these weighings must be made with the moving coil heated by the current through it, so that convection currents are set up in the surrounding air.

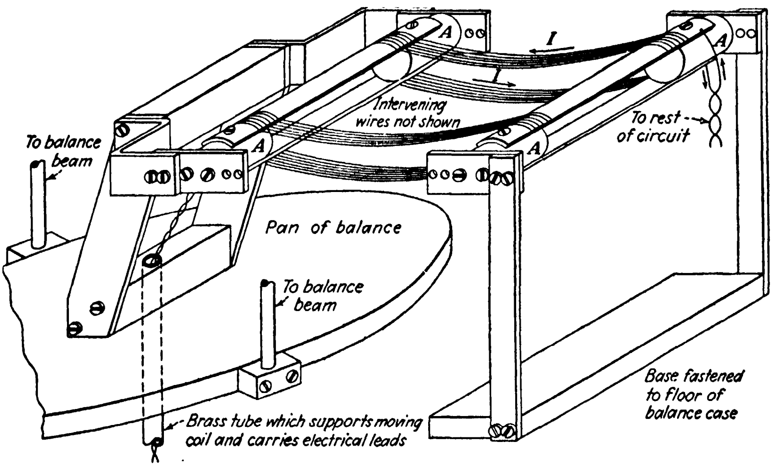

FIG. 53.—Diagram of the leads to the moving coil of a current balance.

To make weighings to the required accuracy, the following precautions must be observed:

1. The balance must not be arrested when the current in the fixed coils is reversed with the coincident addition or removal of the weight. This might not be the case with a perfect balance, but the degree of perfection that would be required is greater than can be attained. The difficulty of making a sufficiently perfect balance is shown by observing that an unexpected change in the length of one arm of 1/100 micron (one fiftieth of a wave length of light) would introduce an error of one part in a million in the electromagnetic force.

2. The density of the counterweights should be the same as the average density of the suspended coil to avoid changes in the rest point of the balance as a result of changes in barometric pressure.

3. The current in the moving coil should be held constant for several hours before an observation is started in order that the convection currents in the air will be stable.

4. The temperature of the balance beam must not vary by an appreciable amount during a weighing. The constancy required depends on the coefficient of expansion of the beam. With a brass beam having a coefficient of expansion of 18×10^{-6} per degree centigrade, if the difference in temperature between the two halves of the beam changes by 0.0005°C, the apparent weight will change by one in 10^8. With a beam of quartz glass having a coefficient of 0.26×10^{-6} per degree centigrade, a change in temperature of 0.04°C would be required to produce a like result.

To aid in keeping the temperature of the beam constant, the balance should be placed in a room without outside windows and with sufficient thermal insulation so that the temperature change is slow. Moreover the operator should not be in the same room as the balance, since radiation from his body may influence the temperature of the balance beam. The optical system for observing the swings of the balance should be so arranged that the two halves of the balance beam will receive the same amount of heat from the light beam.

The swings of the balance should be observed by a telescope and scale or a lamp and scale. In either case readings can be made to about 0.1 mm. A suitable amplitude of swing can be obtained by a jet of air against the bottom of one of the pans. The jet can be produced by a rubber bulb and transmitted from the operator to the balance case by a tube. As there is always some damping of the swings, three readings should be made on one side of the rest position and two on the other. The scale should be numbered from one end, so that the rest position for any set of swings is one-half the sum of the average reading at the two ends of the swing.

In operating the balance, the current in the coils is held constant, and the rest position is determined with the standard weight removed from the pan. The current in the fixed coils is then reversed, and the weight is simultaneously added to the pan, the entire operation being carried out so that no arrestment of the balance is necessary. Another rest position is determined as soon as the swings can be adjusted to a convenient amplitude.

The current in the fixed coils is again reversed, the weight is removed, and a third rest position is determined. If conditions have not changed during the observations, the third reading will agree with the first, and no further readings need be taken. In that case, the difference between the first and second rest positions, divided by the sensitivity of the balance, is the fraction of a milligram that must be added to or subtracted from the mass of the weight to determine the mass that counterbalances the electromagnetic force between the coils. However, conditions are almost always changing by enough to affect the rest position. Hence six or more rest points, with the standard weight alternately on and off the pan, are generally taken. The data can be most readily reduced by employing a graphical

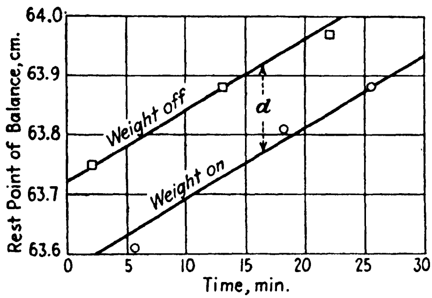

FIG. 54.—Typical curves showing the change-of-rest point with time during a set of observations with a Rayleigh current balance.

method, which is illustrated in Fig. 54. In recording the data, the time is noted at which each rest position is determined. A plot is then made with the rest positions as ordinates and the times at which they were observed as abscissas. If the conditions that affect the rest position have changed uniformly during the time required for the six observations, the plotted points will lie on two parallel lines, as shown in the figure, on one line when the weight is on, on the other when it is off. The vertical distance *d* between the lines corresponds to the change in rest position with the weight on and with the weight off. The amount to be added to or subtracted from the mass of the weight is the distance *d* divided by the sensitivity of the balance.

127. The Electrical Circuit of the Current Balance.—The electrical circuit of the current balance is shown in its simplest form by the diagram of Fig. 55. The source of the current is the lead storage battery B. In the circuit are the four-terminal standard resistor R, the fixed and the moving coils, the reversing switch S, and the adjustable resistor for regulating the value of the current. The current must be regulated until the fall in potential over the resistance R is just equal to the electromotive

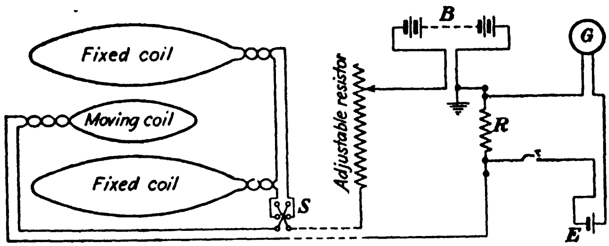

FIG. 55.—The electrical connections of the Rayleigh current balance.

force E of the standard cell. This equality is indicated by a zero deflection of the galvanometer G. The reversing switch S serves to reverse the current in the two fixed coils without reversing the current in the moving coil, so that in one position the electromagnetic force on the moving coil is upward and in the other downward. A simple switch, like the one shown in the diagram, opens the circuit very suddenly, so that rather high electromotive forces are induced in the fixed coils, thus endangering the insulation. Hence a switch is generally employed which reduces the current to zero in several steps. A rotary switch with two steps is shown diagrammatically in Fig. 56. The adjustable resistor must permit very small changes in resistance. If the total resistance of the circuit is 100 ohms, the regulation of the current to one part in a million requires that resistance steps of 0.0001 ohm shall be provided in the adjustable resistor. But the contact resistance of the brushes of a resistance box may vary by nearly this amount. Hence the small variations of resistances are best accomplished by varying a rather large

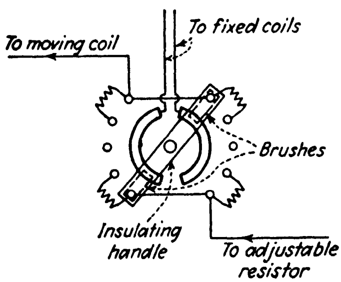

FIG. 56.—Diagram of a rotary reversing switch for a Rayleigh current balance. The brushes must be so wide that each one is always in contact with a stud.

resistance which shunts a much smaller one. A resistance box using this principle was described on page 89. Under favorable conditions the current may be kept constant to one part in a million by changing the adjustable resistance two or three times a minute.

The thermal electromotive forces that are present in the circuit which includes the coils have no effect on the result, since they only add to, or subtract from, the voltage of the battery. However, a thermal electromotive force in the loop which includes the standard cell would directly affect the value of the current in terms of the laboratory units of resistance and electromotive force. Hence, so far as possible, either this loop should be of copper or the parts that are not of copper should be maintained at a constant temperature. All keys, binding posts, galvanometer suspensions, etc., should be of copper. The standard resistor R and the standard cell E should be kept in constant-temperature baths so that the two terminals of each will be at the same temperature. As a test of the thermal electromotive forces in the loop, the standard cell may be replaced by a copper coil having about the same resistance as the cell. Then with the battery circuit open, the deflection of the galvanometer, with the key in the loop closed, divided by the voltage sensitivity of the galvanometer, will give the thermal electromotive force in the loop. This should be less than 1 microvolt.

The circuit should be grounded at the battery terminal next to the standard resistor as shown in the figure. All the other portions of the circuit should be well insulated from the earth. A small leakage current from the battery to earth will not influence the result, but may affect the steadiness of the current. The remainder of the circuit should have an insulation resistance to earth of at least 100 megohms.

128. Computation of the Maximum Force per Unit Current.—The computation of the maximum force between a fixed and a moving coil when each is carrying unit current can be most readily accomplished by first computing the maximum force between two filaments located at the centers of the coils and then modifying this to allow for the finite cross sections of the coils.

The force per unit current between two filaments which coincide with the circumferences of two coaxial circles that have different diameters is a maximum for a definite axial distance between the filaments. This is shown by the plot in Fig. 57. Since the maximum force is desired, the value of the axial distance need not be known accurately because at that point a change in axial distance produces only a second-order change in the force.

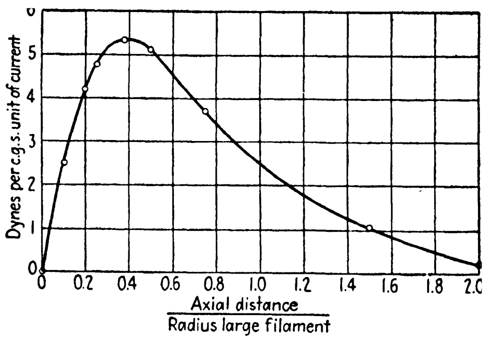

FIG. 57.—The electromagnetic force per c.g.s. unit of current between two coaxial circular coils of a single turn as a function of the axial distance between them when the diameter of one coil is twice that of the other.

Let a_1 = radius of the larger circle.
a_2 = radius of the smaller circle.
z_m = axial distance between the filaments for maximum force.
$\alpha = a_2/a_1$.
$y_m = z_m/a_1$.

Then

$$y_m = 0.5 - \tfrac{9}{20}\alpha^2 - \tfrac{1}{16}\alpha^4 \tag{1}$$

provided

$$0 < \alpha < 0.75$$

This is an empirical equation but, so long as α is less than 0.75, it is sufficiently accurate to permit the calculation of the maximum force with an accuracy of one part in a million. The next step is the calculation of an auxiliary constant k, which is given by the equation

$$k^2 = \frac{4\alpha}{(1+\alpha)^2 + y_m^2} \tag{2}$$

The value of the maximum force F_m is then given by the equation

$$F_m = \frac{\pi k y_m}{\sqrt{\alpha}}\left(\frac{2-k^2}{1-k^2}E - 2K\right) \tag{3}$$

where E and K are elliptic integrals of the first and second kind, respectively, to the modulus k. If the elliptic integrals are to be computed by the arithmetico-geometrical means as given in Appendix I, only the series of a's, b's, and c's need be developed. Substituting in Eq. (3) the values of K and E as given by these series, there results the equation

$$F_m = \frac{\pi^2 k y_m}{4a_n(1 - k^2)\sqrt{\alpha}}[k^4 - 2(2 - k^2)(c_1^2 + 2c_2^2 + 4c_3^2 + \cdots)] \quad (4)$$

In this formula a_n refers to the nth term of the a series which is given in Appendix I. Where high precision is required, computing by Eq. (4) is generally easier than obtaining values for use in Eq. (3) by interpolating in a table of elliptic integrals.

The value of y_m enters as a factor, so that one would expect that it would need to be known to one part in a million if F_m is to have the same accuracy. However, y_m enters in several of the other terms, so that a small change in y_m is compensated by changes in the other terms. Hence the value of y_m should be used as though it were known to one part in a million, but, if a change as large as 1000 parts in a million is made before the computation is started, the result will not be affected.

In comparing the computations on coils which are nearly alike, the following formulas are useful:

$$\frac{\Delta F_m}{F_m} = \epsilon\frac{\Delta\alpha}{\alpha} \quad (5)$$

where

$$\epsilon = \frac{1 - \alpha^2}{2y_m^2} = \frac{1 - \alpha^2}{2(\frac{1}{2} - \frac{9}{20}\alpha^2 - \frac{1}{16}\alpha^4)^2} \quad (6)$$

and

$$\frac{\Delta F_m}{F_m} = \gamma\left(\frac{\Delta y_m}{y_m}\right)^2 \quad (7)$$

where

$$\gamma = -\frac{1}{2}\frac{3(1 - \alpha^2)^2 - 2y_m^2(1 + \alpha^2)}{(1 - \alpha^2)^2 + 2y_m^2(1 + \alpha^2) + y_m^4} \quad (8)$$

$$= -\left(0.5 + \frac{1 - \alpha}{3}\right) \text{ approx.} \quad (9)$$

The approximation was obtained empirically, but is within 5 per cent for values of α between 0.1 and 0.9, which is generally as precise as is needed.

The maximum force $\mathfrak{F}_m$ between two circular coils having rectangular cross sections and carrying unit current in each turn is, to a first approximation, equal to the maximum force F_m

between two filaments carrying unit current and located at the geometric centers of the cross sections of the coils multiplied by the product of the number of turns in one coil and the number in the second. A second approximation can be obtained by adding a function which includes, in addition to the ratio of the radii of the circular filaments, the ratio of each cross-sectional dimension to the radius of its filament. A further refinement takes account of the fact that, for coils of finite cross section, the axial distance for maximum force may not be the same as the corresponding axial distance for the filaments located at the centers of their cross sections. A formula which takes all these factors into account has been derived[1] on the assumption that the coils were composed of a number of circular hoops of insulated wire arranged within a rectangle of dimensions $2b$ and $2c$, all the n hoops being cut by a radial plane in which the current is transferred from hoop to hoop and in which the current enters and leaves the coil. There is the same current in each hoop, and the magnetic field is symmetrical around the axis of the coil. The equation which includes fourth-order corrections is

$$\mathfrak{F}_m = n_1 n_2 F_m \left[1 + \Delta_2 + \Delta_4 + \frac{(x\Delta'_2)^2}{2\lambda_2} \right] \tag{10}$$

In this formula the symbols have the following significance:

n_1 and n_2 = number of turns of wire in the larger coil and the smaller coil, respectively.

F_m = maximum force for unit current in two filaments located at the centers of the cross sections of the coils [see Eq. (3) or its equivalent, Eq. (4)].

a_1 and a_2 = mean radii of the larger coil and the smaller coil, respectively.

b_1 and b_2 = one-half the axial width of the larger coil and the smaller coil, respectively.

c_1 and c_2 = one-half the radial depth of the larger coil and the smaller coil, respectively.

$$\alpha = \frac{a_2}{a_1}$$

$$A^2 = a_1^2 + a_2^2 \tag{11}$$

[1] C. Snow, The Attraction between Coils in the Rayleigh Current Balance, *B.S. J. Research*, **11**, 681 (1933).

$$\beta = \frac{1 - \alpha^2}{1 + \alpha^2} \tag{12}$$

$$y_m = 0.5 - \tfrac{9}{20}\,\alpha^2 - \tfrac{1}{16}\,\alpha^4 \qquad \text{if } 0 < \alpha < 0.75 \tag{13}$$

$$x = \frac{y_m}{\sqrt{1 + \alpha^2}} \tag{14}$$

$$\lambda_2 = \frac{3\beta^2 - 2x^2}{\beta^2 + 2x^2 + x^4} \tag{15}$$

$$\lambda_3 = \frac{4x^2 + \lambda_2(11x^4 + 10x^2 - \beta^2)}{\beta^2 + 2x^2 + x^4} \tag{16}$$

$$\lambda_4 = \frac{16x^2\lambda_3(x^2 + 1) - 3x^2\lambda_2(23x^2 + 8)}{\beta^2 + 2x^2 + x^4} \tag{17}$$

$$B_1 = \frac{c_1^2}{a_1^2}[5(b_1^2 + b_2^2 - c_2^2) - 3c_1^2] - \frac{c_2^2}{a_2^2}[5(b_1^2 + b_2^2 - c_1^2) - 3c_2^2] \tag{18}$$

$$B_2 = \frac{c_1}{a_1^2}[5(b_1^2 + b_2^2 - c_2^2) - 3c_1^2] + \frac{c_2^2}{a_2^2}[5(b_1^2 + b_2^2 - c_1^2) - 3c_2^2] \tag{19}$$

$$B_3 = \frac{c_1^2}{a_1^2}[9c_1^2 + 15c_2^2 - 10(b_1^2 + b_2^2)] + \frac{c_2^2}{a_2^2}[9c_2^2 + 15c_1^2 - 10(b_1^2 + b_2^2)] \tag{20}$$

$$B_4 = \frac{1}{A^2}[10(b_1^2c_1^2 + b_2^2c_2^2) - 10(b_1^2 - c_1^2)(b_2^2 - c_2^2) - 3(b_1^4 + c_1^4 + b_2^4 + c_2^4)] \tag{21}$$

$$\Delta_2 = \frac{1}{12x^2}\left[\left(\frac{c_2^2}{a_2^2} - \frac{c_1^2}{a_1^2}\right)\beta + \frac{2(c_1^2 - b_1^2 - c_2^2 - b_2^2)\lambda_2}{A^2}\right] \tag{22}$$

$$\Delta_4 = \frac{1}{360A^2x^4}\left\{B_4\lambda_4(\beta B_1 - x^2B_2)\lambda_3 + \left[3\beta B_1 - x^2B_3 - 5x^4\left(\frac{c_1c_2A}{a_1a_2}\right)^2\right]\lambda_2 - 6\beta B_1\right\} \tag{23}$$

$$x\Delta_2' = \frac{1}{12x^2}\left\{2\beta\left(\frac{c_1^2}{a_1^2} - \frac{c_2^2}{a_2^2}\right) + \left[(x^2 - \beta)\frac{c_1^2}{a_1^2} + (x^2 + \beta)\frac{c_2^2}{a_2^2}\right]\lambda_2 + \frac{2(b_1^2 - c_1^2 + b_2^2 - c_2^2)}{A^2}\lambda_3\right\} \tag{24}$$

129. Measurement of the Ratio of the Radii.—The measurement of the ratio of the mean radii of the coils of a current balance must be made with higher accuracy than can be obtained by mechanical measurements. The method generally employed deduces the ratio from the ratio of the currents required to produce equal magnetic intensities at their centers. In this method the two coils are mounted so that they are concentric and coplanar, with their common plane vertical and coincident with the magnetic meridian. A small magnet is suspended by a fiber so that its center is at the common center of the coils. Currents are sent through the coils in opposite directions, and their magnitudes are adjusted until the magnet has the same

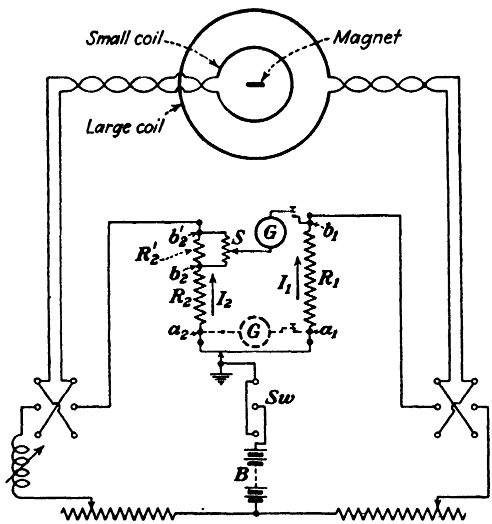

FIG. 58.—The electrical connections for measuring the ratio of the radii of two coils.

deflection as with no current in the coils. Then the ratio of the product of the current and the number of turns for one coil to the like product for the other coil is very nearly equal to the ratio of the mean radii of the coils. The number of turns is determined when the coils are constructed; the ratio of the currents is measured when they are adjusted to produce no deflection of the magnet.

A diagram of an electrical circuit which is suitable for measuring the ratio of the radii is shown in Fig. 58. Current from the battery B divides at a point above the switch S_w, the current I_1 flowing through the resistance R_1, the large coil, the reversing switch, and a variable resistance; the current I_2

flows through the resistance R_2, the small coil, the reversing switch, a variable inductance, and a variable resistance. The three switches should be mechanically connected and the blades so arranged that the switch S_w always opens before the other two open and closes after they close. In this way the currents I_1 and I_2 are started and stopped simultaneously. By adjusting the inductance until the time constants of the two circuits are the same, the reversal of the currents produces only a small deflection of the magnet.

The standard resistances R_1 and R_2 have both current and potential terminals. The connections from S_w to the resistors are adjusted until, with I_1 and I_2 at their nominal values, a_1 and a_2 are at the same potential. The galvanometer is then connected to the potential terminal b_2 and to the contact on the slide wire S. The standard resistance in the small coil circuit has three potential terminals; the resistance between the terminals a_2 and b_2 is R_2 and that between b_2 and b_2' is R_2'. The ratio R_2'/R_2 should be about 1/1000. The slide wire S, having 1000 divisions and a total resistance at least 10 000 times as large as R_2', is connected across the terminals b_2 and b_2'. The galvanometer is balanced by moving the contactor along the slide wire. The number of divisions between b_2 and the position of the slide-wire contact when the galvanometer deflection is zero, is designated by x. The resistance of a slide-wire division is z and the resistances of the connections from the slide wire to b_1 and b_1' are az and bz, respectively.

$$\frac{I_1}{I_2} = \frac{R_2}{R_1}\left[1 + \frac{R_2'(a + x)}{R_2\left(a + b + 1000 + \frac{R_2'}{z}\right)}\right] \tag{25}$$

With the values of resistance as indicated, R_2'/z does not influence the result by one part in a million and hence can be neglected. The equation for α, the ratio of the radii of the coils, is

$$\alpha = \frac{a_2}{a_1} = \frac{n_2 I_2}{n_1 I_1} \tag{26}$$

Combining Eqs. (25) and (26), and neglecting small terms,

$$\alpha = \frac{n_2}{n_1}\frac{R_1}{R_2}\left[1 - \frac{R_2'(a + x)}{R_2(a + b + 1000)}\right] \tag{27}$$

The earth's field seldom remains constant for more than a few minutes at a time, so that readings must be made rapidly. A suitable procedure will be outlined after describing the method of mounting the coils and magnet.

The coils must be provided with a mounting which will allow each to be adjusted so that its center coincides with the center of the magnet and its plane is vertical and in the magnetic meridian. The magnet must be very small (1 or 2 mm long), must be attached to a small galvanometer mirror, and must be suspended by a very fine quartz fiber. It must be inside a nonmagnetic tube to shield it from air currents and it should have attached to it the wing of an insect to provide air damping. To place each coil in its proper position relative to the magnet, it may be necessary to give the coil translations along three perpendicular axes and rotations about two of these axes. The correct position cannot be judged with sufficient accuracy by mechanical measurements, but can be obtained by a series of determinations of the ratio of the currents that produce no deflection of the magnet. The ratio of the currents is measured in the same way as indicated above, but the purpose is now to place the coils concentric and coplanar. After this is accomplished, the same kind of measurement is made to determine the ratio of the radii.

130. Adjustment of the Coils.—A suitable series of measurements for placing the coils so that they are coplanar and concentric with the center of the magnet at their common center is as follows: Determine the slide-wire reading for zero deflection of the galvanometer when the ratio of the currents is such as to cause zero deflection of the magnet. Displace the small coil a measured amount in a direction perpendicular to its plane and redetermine the slide-wire reading. Continue displacing the coil and making slide-wire readings until a sufficient number of observations have been obtained for plotting a curve with positions of the coil as abscissas and slide-wire readings as ordinates. If the coil has been displaced so that its plane has been on both sides of the magnet, the curve will indicate the position of the coil at which the slide-wire reading is a *maximum*. The coil is then set at the position corresponding to the maximum slide-wire reading. The small coil is then displaced along a horizontal line in its plane, the slide-wire readings are determined at measured positions, a curve is plotted, and the coil is placed at the position indicated by the *minimum* of the curve. The small coil is then displaced along a vertical line in its plane and a procedure similar to the preceding is followed. In this way the center of the small coil is made to coincide approximately with the center of the magnet. Then the large coil is placed concentric with the magnet in the same manner as outlined above for the small coil. With the large coil, however, the displacements perpendicular to the plane of the coil give a curve of slide-wire readings which has a minimum, and displacements in the plane of the coil give a curve which has a maximum.

The next procedure is to rotate one coil about that horizontal line in its plane which passes through its center and similarly the other coil, and to determine in each case when the coil is vertical by the maximum or minimum of the slide-wire reading. Finally each coil must be rotated about a vertical axis to bring its plane into the magnetic meridian. However, it is more important that the planes of the coils should exactly coincide than that they should lie in the magnetic meridian. Hence the plane of the small coil is made approximately coincident with the meridian by determining the position at which the reversal of a small current in it will give equal and opposite deflections of the magnet. The large coil is then rotated about a vertical axis to make its plane coincide very accurately with the plane of the small coil. In this case the maximum-minimum method used in the preceding adjustments cannot be applied since the magnet may not lie exactly in the plane of the small coil. A suitable procedure[1] is to determine the setting of the large coil at which the ratio of the currents is the same when the magnet is deflected several centiradians from its normal position by means of a stationary magnet as when it is in its normal position in the plane of the small coil. The most convenient form of stationary magnet is an air-cored coil of wire in which there is a current. With direct and reversed current in this auxiliary coil, equal and opposite deflections of the magnet are produced. Only when the large and small coils are coplanar is the ratio of their currents independent of the amount and direction of the current in the auxiliary coil.

[1] The theory of this method is given by Curtis and Curtis, *op. cit.*, 699.

After a complete set of observations has been made, another set is made starting with the coils in the positions obtained in the first set. This is necessary since the different adjustments are not entirely independent of one another. If any adjustment in the second set is different from that in the first set, a third set must be made. This is continued until the position of the coil at the end of the set is not appreciably different from that at the beginning.

A suitable procedure in making a set of observations is the following: The battery switch is momentarily opened to permit the reading of the zero position of the magnetometer; with the switch closed, the current in one of the coils is varied until the reading of the magnetometer is the same as the zero position; and as nearly simultaneously as possible with the varying of the current, the slide wire is adjusted to produce no deflection of the galvanometer. After the slide-wire reading is recorded, the current is reversed and the process repeated. A skilled observer will make a set of readings in about ten seconds. Except during magnetic storms, the earth's magnetic field will not change appreciably in this time. However, in regions where large amounts of direct current are used, changes in the magnetic field may occur even in this short time, so that such regions are not suitable for observations with the magnetometer.

When the coils have been accurately adjusted to be concentric and coplanar, with the center of the magnet at their common center, the ratio of the radii is determined by the method described above. However, in determining the ratio of the radii from the ratio of the currents, several corrections must be applied which do not need to be considered in making the adjustments of the coils. These will now be described.

131. Temperature Coefficients of Expansion of the Coils.—The temperature coefficient of expansion of a coil is the constant by which a given change in the temperature of the coil must be multiplied to give the percentage change in any one of its linear dimensions, usually its radius. In measuring the temperature coefficient, the same change in temperature must occur in both the winding and the form, though the two are not necessarily at the same temperature. Hence, in measuring the ratio of the radii for determining the temperature coefficient, the current in the coils must be approximately the same in all the measurements.

The temperature coefficient of expansion of one coil can be determined by measuring, at different temperatures of this coil, the ratio of its radii to that of another coil kept at a constant temperature. The experimental arrangement for varying the temperature will depend on the design of the coils. If the large coil is water-cooled, its temperature can be changed by varying the temperature of the circulating water, without affecting the temperature of the small coil. Then the temperature of the small coil can be changed by varying the temperature of the room, while the temperature of the large coil is held constant by the circulating water. The temperature of each can best be determined by measuring its resistance. In making a determination of the temperature, the resistance of a coil should be measured with the same current as was employed in determining the ratio of the radii. Since the temperature is required only to 0.05°C, the resistance need be measured only to one part in ten thousand.

In comparing the results on the ratio of the radii of two coils, it is necessary to reduce the different determinations to some standard temperature s. The formula for this is

$$\alpha_s = \alpha_m[1 - \tau_2(t_2 - s) + \tau_1(t_1 - s)] \tag{28}$$

where α_s = ratio of the radii at the standard temperature.
α_m = measured ratio of the radii.
τ_1 = temperature coefficient of the large coil of which the measured temperature is t_1.
τ_2 = temperature coefficient of the small coil of which the measured temperature is t_2.

In securing the data for determining τ_1 and τ_2, measurements at the standard temperature s may not be feasible. In such a case, measurements can be made at two convenient temperatures for one coil, say t_1 and t_1' for the larger coil, while the temperature of the smaller coil is held constant. The two ratios of radii are designated as α_m and α_m'. Substituting the result of each measurement in Eq. (28) and subtracting the two resulting equations,

$$\alpha_m - \alpha_m' = \alpha_s\tau_1(t_1' - t_1) \tag{29}$$

Since α_s need not be known accurately, α_m can be substituted for it. Hence

$$\tau_1 = \frac{\alpha_m - \alpha_m'}{\alpha_m(t_1' - t_1)} \tag{30}$$

In a similar manner the value of τ_2 can be determined.

132. Load Coefficients of Expansion of the Coils.—The load coefficient of expansion of a coil is the constant by which the load, in watts, must be multiplied to give the correction to the radius which results from the load. The necessity for such a correction can be understood by considering the effect of a difference in temperature between the winding of the coil and the coil form when only the temperature of the winding is measured. In a water-cooled coil, the heat generated by the current must flow from the winding through the form, so that the form is at a lower temperature than the winding, the temperature difference being proportional to the amount of heat per second (load in watts) generated in the winding. If the radius of the coil is influenced by the temperature of the form as well as by that of the winding, the load coefficient gives the amount that one watt expended in the coil causes the radius, as determined from the temperature of the coil, to be less than would have been the case if the winding and form had been at the same temperature.

If the coil is so wound that it is not constrained by the form, the radius is determined by the temperature of the winding only, so that the coil does not have a load coefficient. However, if the coil is wound under tension (as it usually is), it will have a load coefficient, since the radius is determined in part by the temperature of the form. Since the measured temperature is higher than the mean temperature, the correction for load is always negative.

The relationship between the radius of a coil a_s at a standard temperature s, with zero load in the winding, and the radius a at a temperature t, with a load p in the winding, can be expressed by the equation

$$a = a_s[1 + \tau(t - s) - \psi p] \tag{31}$$

where τ = temperature coefficient of expansion.
ψ = load coefficient of expansion.

Hence for the ratio of the radii of two coils,

$$\alpha_s = \alpha_m[1 + \tau_1(t_1 - s) - \tau_2(t_2 - s) - \psi_1 p_1 + \psi_2 p_2] \tag{32}$$

where α_s = ratio of the radii at the standard temperature s and zero load.

α_m = measured ratio of the radii when the large coil has a measured temperature t_1 and a load p_1 and the small coil has a measured temperature t_2 and a load p_2.

To determine the load coefficient of a coil, some setup must be arranged in which the load on one coil can be changed without changing the load on the other. A simple setup can be made when the coil is made of two nearly identical windings which

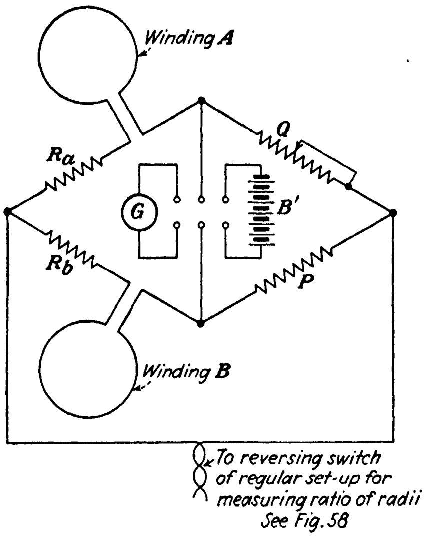

FIG. 59.—Auxiliary electric circuit used in measuring the load coefficient of a coil.

are normally connected in parallel. The diagram for a portion of this setup is given in Fig. 59, which is an addition to the setup for measuring the ratio of the radii given in Fig. 58. The two windings, which are actually adjacent turns in the coil, are shown as separate windings in the diagram. When the battery B' is disconnected, the windings have the same current as normally used in measuring the ratio of the radii, the resistances P, Q, R_a, and R_b merely serving as a part of the resistance of the circuit. With the normal current in the coils, with the resistances R_a and R_b equal, and with the galvanometer G connected, Q is adjusted until there is no deflection of the galvanometer. The main battery B is then disconnected and the auxiliary battery B' is connected in place of the galvanometer. As the bridge has already been balanced, the two corners of the bridge connected to the reversing switch are at the same potential, so that B' does not cause a current in the second coil, but equal and opposite currents in windings A and B. If these conditions are not exactly fulfilled, there may be a slight deflection of the magnet as a result of the current from B', but this will not introduce an error if the deflected position is taken as zero in determining the ratio of the radii by means of current from the main battery B. When both batteries are connected, the current from the battery B' is, in one winding (say A), added to the current from battery B, while in the other winding the two currents are subtracted. Hence the heat dissipated in one winding is much larger than in the other. When the currents are constant, the potential drops are measured over R_a, R_b, winding A, and winding B. From these measurements and the resistances under standard conditions, the temperatures of the windings and the load in watts on each winding can be determined. If the two windings are closely related thermally, their temperatures will be the same. The load can be computed for each winding. The total load for the coil is the sum of the loads on the two windings.

The data for the load coefficient are obtained after the coils have been adjusted so that they are concentric and coplanar, with the magnet at their common center. The batteries (both well insulated) must be connected and approximate settings made. Final settings cannot be made until the currents have been constant for 1 to 2 hours to allow the temperatures of the coils to become stable.

The method as given is readily applicable to a large coil, since the normal load in measuring the ratio of the radii is small and a large increase is permissible. With a small coil this is not usually the case. For such a coil, the best procedure is to obtain two values of the ratio of the radii corresponding to two different values of the current in it. From the two ratios of the radii, the two temperature coefficients, and the load coefficient of the large coil, the load coefficient of the small coil can be computed.

To obtain the load coefficient of a large coil, the ratios of currents are determined for loads p_1 and p_1' on the large coil and for a constant load p_2 on the small coil. Multiplying the ratios of currents by the ratio of the number of turns gives α_m and α_m' as the measured ratios of the radii. The temperatures of the large coil are t_1 and t_1' at the time of the two measurements, and those of the small coil are t_2 and t_2'. Substituting each set of data in Eq. (32) for the load coefficient, eliminating the load coefficient of the small coil, ψ_2, between the two simultaneous equations, and solving for ψ_1, the resulting equation is

$$\psi_1 = \frac{1}{p_1' - p_1}\left[\frac{\alpha_m' - \alpha_m}{\alpha_s} + \tau_2(t_2 - t_2') - \tau_1(t_1 - t_1')\right] \quad (33)$$

Since α_s need not be known accurately, the value of α_m may be used in place of α_s.

133. Correction for the Sectional Dimensions of the Coils.—The correction for the sectional dimensions of the coils arises from the fact that the magnetic field at the center of a coil carrying a current is not the same as if the current were concentrated in a filament located at the center of the cross section of the coil. The equation for the magnetic intensity at the center of a coil can be determined from the equation for the magnetic intensity of a filament by extending the latter (by means of Taylor's theorem for two variables) to make it apply to any point in the cross section of the winding, then integrating over the cross section. Carrying out this integration, the magnetic intensity at the center of a coil of rectangular cross section is given by the equation

$$H = \frac{2\pi nI}{a}\left(1 - \frac{b^2}{2a^2} + \frac{c^2}{3a^2} + \frac{3b^4}{8a^4} + \frac{c^4}{5a^4} - \frac{b^2c^2}{a^4} \cdots\right) \quad (34)$$
$$= \frac{2\pi nI}{a}(1 - \Delta)$$

where n = number of turns in the coil.
I = current in each turn of the coil.
$2b$ = axial dimension of the cross section of the coil.
$2c$ = radial dimension of the cross section.
a = mean radius of the coil.
Δ = correction for reducing the magnetic field of a coil of rectangular cross section to that of a filament having a radius equal to the mean radius of the coil.

In deriving the equation the assumptions are made that the current distribution is uniform and that the total current is equal to the number of turns times the current in each turn.[1] Equating the equations for the magnetic fields of the two coils and solving for the ratio of the radii, the following equation is obtained:

$$\alpha = \frac{a_2}{a_1} = \frac{n_2 I_2}{n_1 I_1}(1 + \Delta_1 - \Delta_2)$$
$$= \frac{n_2 I_2}{n_1 I_1}\left\{1 + \left[1 - \frac{1}{a_1^2}\left(\frac{c_1^2}{3} - \frac{b_1^2}{2}\right)\right]\left[\frac{1}{a_2^2}\left(\frac{c_2^2}{3} - \frac{b_2^2}{2}\right) - \frac{1}{a_1^2}\left(\frac{c_1^2}{3} - \frac{b_1^2}{2}\right)\right]\right.$$
$$\left. + \frac{1}{a_2^4}\left(\frac{c_2^4}{5} - b_2^2 c_2^2 + \frac{3b_2^4}{8}\right) - \frac{1}{a_1^4}\left(\frac{c_1^4}{5} - b_1^2 c_1^2 + \frac{3b_1^4}{8}\right) + \cdots \right\} \quad (35)$$

where α = ratio of the mean radii of the coils.
a_1 and a_2 = mean radii of the large coil and the small coil, respectively.
n_1 and n_2 = number of turns in the large coil and the small coil, respectively.
I_1 and I_2 = current in each turn of the large coil and the small coil, respectively.
Δ_1 and Δ_2 = correction to the magnetic intensity at the center of the large coil and the small coil, respectively.
b_1 and b_2 = one-half the axial breadth of the large coil and the small coil, respectively.
c_1 and c_2 = one-half the radial depth of the large coil and the small coil, respectively.

The indicated derivation has assumed that the current is uniformly distributed in the cross section of each coil. However, it can be shown that, if the turns are equally spaced in the axial direction and also in the radial direction, but not necessarily the same in the two directions, the formula as given is sufficiently accurate. In such a case the axial breadth, $2b$, of a coil is equal to the number of turns per layer times the distance between the centers of adjacent turns, while the radial depth is the number of layers times the distance between the centers of adjacent layers. As it is impossible to construct a coil in which the spacings of the turns and layers are uniform and precisely measurable, the coils should be of such size that the correction for cross section is as small as is consistent with other requirements.

[1] The equation was derived by Rayleigh, "Scientific Papers," Vol. II, p. 291. Also given by Curtis and Curtis, *B.S. J. Research*, **12**, 665 (1934).

134. Correction for the Length of the Magnet.—The observed ratio of the radii must be corrected for the effective length of the magnet which is used to indicate when the magnetic fields of the two coils are identical. The sum of the magnetic fields at the poles of the magnet is zero when the currents in the coils do not cause a deflection of the magnet, but, on account of the differences in their gradients, the sum of the fields at the common center of the coils is not exactly zero, as postulated in deriving the formula for the ratio of the radii. The correction to the ratio of the radii for the fact that the resulting field is zero at the poles of the magnet and not at the center of the coils depends on the square of the effective length of the magnet and on its orientation with respect to the plane of the coils. If the magnet lies in the plane of the coils, the correction to the ratio of the radii, α, for the effective length $2r$ of the magnet is given by the equation

$$\alpha = \frac{a_2}{a_1} = \frac{n_2 I_2}{n_1 I_1}\left[1 + \frac{3r^2}{4a_2^2}(1 - \alpha^2) + \cdots\right] \tag{36}$$

The effective length of a magnet is approximately equal to five-sixth of the actual length in the direction of magnetization. As it is desirable to have the correction small, the length of the magnet should not exceed 1 or 2 mm.

135. Design of the Coils for a Rayleigh Current Balance.—The coils for a Rayleigh current balance must meet certain requirements in order that a current can be measured by the balance with an accuracy of one part in a million. These are:

1. The electromagnetic force on the suspended coil, when the current in the coils is that normally carried, must be at least 250 000 times the sensitivity of the balance to be employed. Then the reversal of the current in the fixed coils will give a force which can be measured to one part in 500 000, which is sufficient to determine the current to one part in a million.

2. The coils must be of such radii that the ratio of the radii can be determined to one part in a million. One of the limiting conditions is the correction for the effective length of the magnet used in measuring the ratio of the radii. As the effective length of the magnet $2r$ cannot, for mechanical reasons, be made much less than 1 mm and cannot be measured closer than 1 per cent,

the radius of the suspended coil, a_2, must be so large that the correction term for the length of the magnet will not introduce an error of one part in a million in the ratio of the radii. Hence a_2 must be at least 10 cm.

3. The ratio of the radii of the coils should be between 0.3 and 0.7. If it is less than 0.3, the force is too small; if it is greater than 0.7, the correct position of the suspended coil cannot be determined with sufficient accuracy.

4. The cross sections of the coils should be so chosen that the effect, on the computed force, of errors in determining the cross sections will be small. With coils of square cross section, a given error (in microns) in measuring the radial depth c_1 of the large coil will have the same effect on the computed force as a like error in measuring the radial depth c_2 of the small coil, when c_1 is ten or twenty times as large as c_2. A similar statement can be made regarding the axial widths b_1 and b_2. This would seem to indicate that the cross section of the large coil should be much larger than that of the small coil. However, decreasing the cross section of the small coil decreases the magnetic field that can be produced at its center by a current in it, thus limiting the sensitivity that can be obtained in measuring the ratio of the radii. A design must compromise between these two requirements: one that the cross section of the large coil should be much larger than that of the small coil in order to make the correction terms small; the other that the cross sections should be proportional to the radii in order that the ratio of the radii can be readily measured. A reasonable ratio of the cross-sectional dimensions is for c_1 and b_1 to be 2 or 3 times c_2 and b_2, respectively.

5. The size of wire in the coils must be such that the current is readily obtained from the batteries available and is of convenient size to measure in terms of the units of electromotive force and resistance of the laboratory. A convenient current to measure is 1 ampere. Assume that the mean radius of the moving coil is 10 cm and that both its cross-sectional dimensions are 1 cm. Also assume that the radius of each fixed coil is 20 cm and that both cross-sectional dimensions are 4 cm. If the same size wire is used in all the coils, the resistance of each fixed coil will be 32 times the resistance of the moving coil. Hence, if the wire size is chosen to make the resistance of the small coil 1 ohm,

the resistance of all the coils in series is 65 ohms. If the coils are wound with a single wire, the diameter will be about 0.9 mm exclusive of enamel insulation, and about 0.65 mm if two wires are wound side by side and the two windings are connected in parallel in normal use.

The current in the coils develops considerable heat which must be continuously transferred to outside bodies. If all the heat were radiated, and if the surface of the coils radiated heat as a black body, the stationary temperature of the moving coil when carrying 1 ampere would be about 4°C, and that of the fixed coils

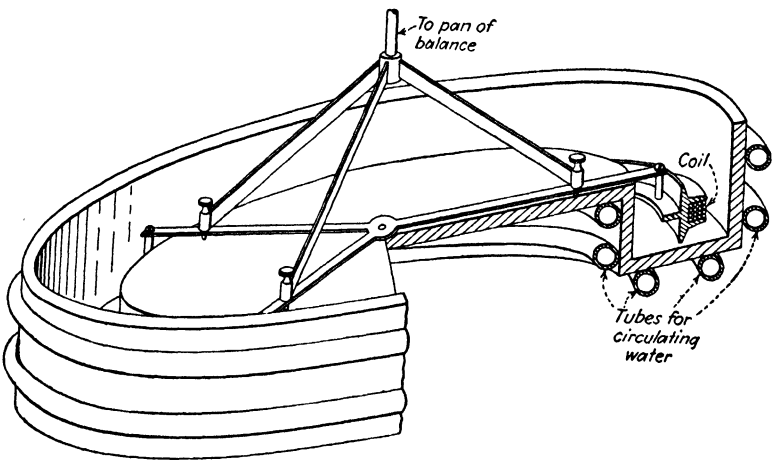

FIG. 60.—Water jacket for removing the heat generated in the moving coil of a Rayleigh current balance.

would be about 8°C, above room temperature. But so large a temperature difference will produce convection currents in the air which are quite undesirable in the neighborhood of a balance. To avoid this condition and to decrease the temperature correction, the fixed coils may be maintained near room temperature by a circulation of water through channels in the forms. The moving coil cannot be cooled in this manner, but the temperature can be regulated by surrounding the coil with a jacket that can be cooled by circulating water. The jacket should be of such form that the cooling is largely by radiation, since air currents are often unsteady, thus interfering with the weighing. A suitable type of jacket is shown in Fig. 60.

136. Adjustment of the Coils in the Rayleigh Current Balance. The coils, when used in the current balance, should be adjusted so that they are coaxial and their planes are horizontal, and so that the distance between the planes of the fixed coils is such that each exerts on the moving coil a maximim force for unit current when the moving coil is midway between them. The planes of the fixed coils can be adjusted to a horizontal position with sufficient accuracy by means of a sensitive level. An error of 2′ will not appreciably affect the result. The moving coil may be compared with a horizontal surface by some method such as the use of a machinist's surface gage placed on the lower fixed coil, or it may be adjusted by observing the force per unit current at different angular positions. An error of 2′ is allowable. The remaining adjustments can best be made in connection with measurements of the force per unit current.

To determine the correct distance between the fixed coils so that each exerts a maximim force per unit current on the moving coil, the force on the moving coil is measured when the current has the same direction in each of the two fixed coils, so that one fixed coil exerts an upward force on the moving coil, the other fixed coil a downward force. This resultant force is very small when the moving coil is midway between the fixed coils (zero if the two fixed coils are exactly alike). If the fixed coils are at the correct distance apart,[1] the force is the same in positions somewhat above or below the mid-plane. The distance between the two fixed coils is adjusted until this condition is fulfilled. For each adjustment of the distance, the force is measured at two or more positions of the moving coil near the mid-plane, the different positions of the moving coil being obtained by adding or removing small weights from the pan of the balance. At each position of the moving coil the rest point is first determined with the current in one direction in the fixed coils, then with it reversed, the current in the moving coil remaining in the same direction for the two measurements. From the change in the rest point on the reversal of the current and the sensitivity of the balance, the force is determined. For each adjustment, a graph, made by taking the force as ordinate and the vertical position of the moving coil as abscissa, gives a

[1] For proof of this statement, see Curtis and Curtis, *B.S. J. Research*, **12**, 687 (1934).

straight line, the slope of which depends on the amount that the distance between the fixed coils is in error. When the fixed coils are the correct distance apart, the line is horizontal.

To adjust the fixed coils so that they are coaxial, they may be so mounted that one of them can be moved in its plane by small and measurable amounts. Also the moving coil (or the fixed coil combination) must be capable of a measurable translation in its plane. Then with the current in the fixed coils in the same direction, and the moving coil midway between them, the small force on the moving coil is measured for two positions, say in an east-west line. If the two forces are different, the fixed coil is moved a small amount in an east-west line and the force is again measured at two positions as above. This is continued until the force is the same at the two positions (zero if the two coils are identical). The same procedure is then followed for displacements in a north-south line. When these adjustments have been completed and checked, the fixed coils are coaxial and the correct distance apart to exert maximum force on the moving coil.

The above description of the adjustments of the fixed coils may give the impression that a perfect set of adjustments is required. Such is not the case, for, if there is only a slight error in any of the adjustments, a correction[1] for the lack of adjustment can readily be obtained. In fact, if the mechanical construction places the coils very nearly coaxial, no provision need be made for moving one coil in its plane, as a correction can be applied for any slight lack of coaxiality.

To determine the position at which the moving coil is coaxial with the fixed coils and midway between them, the force on the moving coil for a constant current in the coils is measured at a number of positions very near the correct position. In these measurements the current is in opposite directions in the fixed coils so that the force on the moving coil is nearly the maximum that can be obtained for that current. First the force is measured for three or more positions in a vertical line. A vertical displacement is produced by adding or removing a small weight from one pan of the balance. The distance between any two positions of the moving coil is determined from the rest points

[1] The method of determining the value of this correction is described by Curtis and Curtis, *op. cit.*, 686 *et seq.*

of the balance and the magnification of the optical system. If the forces are plotted as ordinates and the rest points as abscissas, the points lie on a parabola having its axis parallel to the Y axis. The rest point corresponding to the vertex of the parabola indicates the position of the moving coil at which it is midway between the two fixed coils. The data can be treated algebraically to determine this position, but the graphical method is much simpler. The forces and positions are plotted as described above, and a parabola with its axis parallel to the Y axis is drawn through the plotted points. From this curve the mid-position is estimated. Then the force is measured at this position, and at two other positions equally displaced above and below the position for maximum force. If the mid-position has been correctly estimated, the last two values of the force will be the same. If the last two values are not the same, another curve is plotted from which a new estimate is made of the mid-position of the coil.

When the moving coil has been placed in the mid-plane between the fixed coils, a series of measurements are made to determine the position at which it is coaxial with the fixed coils. First three or more determinations of the force are made at different positions, along an east-west line, in which either the moving coil or the pair of fixed coils can be displaced. The forces and positions are then plotted as described above, the only difference being that in this case the force is a minimum when the axis of the moving coil is in the north-south plane through the common axis of the fixed coils. When the moving coil is placed with its axis in this plane, it is displaced in a north-south line and the position for minimum force is determined. At this last position the moving coil is coaxial with the fixed coils and is midway between them.

137. Determination of the Absolute Value of the Current.—When the coils have been adjusted by the methods just described, a series of observations are made to determine the absolute value of the current. The current is maintained constant and a measurement is made of the change in force on the moving coil when the current is reversed in the fixed coils. Let M represent the mass in grams which must be placed on the pan of the balance to counteract the change in the force produced by the reversal of the current. Equating the forces

$$Mg = 2(\mathfrak{F}_1 + \mathfrak{F}_2)\left(\frac{I_a}{10}\right)^2 \tag{37}$$

where g = acceleration of gravity at the pan of the balance in cm/sec^2.
M = mass in grams.
$\mathfrak{F}_1$ = computed force (dynes) per unit current between the moving coil and one fixed coil.
$\mathfrak{F}_2$ = computed force (dynes) per unit current between the moving coil and the other fixed coil.
I_a = current in absolute amperes.

Solving for the current,

$$I_a = \frac{10\sqrt{Mg}}{\sqrt{2(\mathfrak{F}_1 + \mathfrak{F}_2)}} \tag{38}$$

In computing $\mathfrak{F}_1$ and $\mathfrak{F}_2$ by Eq. (10), that ratio of the radii of the coils must be used which corresponds to the actual temperatures and loads at the time of measuring the force. The force exerted by the mass M must be corrected for the buoyant effect of the atmosphere. The downward force of the mass is

$$Mg - Mg\frac{\rho_a}{\rho_s} = Mg\left(1 - \frac{\rho_a}{\rho_s}\right) \tag{39}$$

where ρ_a = density of air.
ρ_s = density of the weights.

The density of air has been discussed under the subject of weighing. The correction is decreased as the density of the weight which measures the force is increased. For a weight of platinum-iridium, the correction for density is about 57 parts per million at sea level, while for an aluminum weight the correction is about 450 parts per million and varies several parts per million with the normal variations in the barometric pressure.

The value of gravity, g, must be known at the pan of the balance with an accuracy of two parts per million if the current is to be measured with an accuracy of one part in a million. This requires that a gravity determination be made within a few hundred meters of the place where the weighings are made since local anomalies may cause a variation of several parts in a hundred thousand from the values computed by the formula which involves the latitude and sea level. For changes in elevation where the surrounding terrain is reasonably flat and the ground has normal density, the change in g may be computed by the formulas

$$g_H = g_s - 0.0003086H$$
$$g_d = g_s - 0.0000851d$$

where g_H = value of gravity at a height H, in meters, above the surface.

g_d = value of gravity at a depth d, in meters, below the surface.

g_s = value of gravity at the surface.

While the values of $\mathfrak{F}_1$ and $\mathfrak{F}_2$ must be determined for the particular temperature and load which existed at the time of the weighing, yet the complete computation of the maximum force per unit current for either $\mathfrak{F}_1$ or $\mathfrak{F}_2$ need be made only for one temperature and no load. Then for any other temperature and load, the correction may be computed for each by the formula

$$\mathfrak{F} = \mathfrak{F}_s[1 + \epsilon(\tau_2\Delta t_2 - \tau_1\Delta t_1 - \psi_2 p_2 + \psi_1 p_1)] \tag{40}$$

where $\mathfrak{F}$ = force under the conditions prevailing at the time of measurement.

$\mathfrak{F}_s$ = computed force at the standard temperature and zero load.

Δt_1 and Δt_2 = increases in temperature, above that taken as standard, of the large and small coils, respectively.

τ_1 and τ_2 = temperature coefficients of expansion of the large and small coils, respectively.

p_1 and p_2 = loads, in watts, on the large and small coils, respectively.

ψ_1 and ψ_2 = load coefficients of the large and small coils, respectively.

ϵ = constant which depends only on the ratio of the radii of the coils and which is given with sufficient approximation by the equation

$$\epsilon = \frac{1 - \alpha^2}{2y_m{}^2} = \frac{1 - \alpha^2}{2(1/2 - 9/20\alpha^2 - 1/16\alpha^4)^2} \tag{41}$$

A correction to the measured force must be made on account of the force exerted on the moving coil by the current in the leads of the fixed coils, as well as that exerted on the leads of the moving coil by the current in the fixed coils. Both of these forces may be determined experimentally. First the leads of the fixed coils are disconnected *at the coils* and the ends are short-circuited. Then the force between the moving coil and the fixed coil leads is measured, using as large a current as the moving coil can safely carry. After the fixed coils are again connected, the moving coil is similarly removed from the circuit and a measurement made of the force on the leads of the moving coil exerted by a current in the fixed coils.

The corrections for leads will be small if the leads have been suitably made. Two types are satisfactory: twisted leads and concentric leads. Twisted leads are made by twisting uniformly two wires coated with a thin insulation such as enamel. Before use, the insulation resistance should be measured. Concentric leads are made by placing an insulated wire inside a small copper tube. The wire should fill the tube, since the external magnetic field is zero only if the wire is coaxial with the tube.

The force per unit current has been computed on the assumption that all magnetic bodies are so far away that they do not affect the force. The force which a magnetic body exerts on the moving coil varies at least as the fourth power of the distance, perhaps as the fifth or sixth power. Hence even a large magnet need not be very far from the moving coil before the force it exerts is negligibly small. Magnetic masses more than 3 m from the balance can ordinarily be ignored.

If the earth's magnetic field is strictly uniform, it does not exert a force on the moving coil. If it is nonuniform, there may be a vertical force; if the balance is manipulated by reversing the current in the fixed coils, the force of the earth's field on the moving coil merely shifts the zero of the balance and does not introduce an error in the result.

138. Comparison of Measured Current with Laboratory Standards.—The current which is measured in the current balance must also be measured in terms of the standards of resistance and electromotive force of the laboratory. In Fig. 55 on page 230, which shows the circuit of the current balance, the current is measured in terms of the standard resistance R and the standard cell having an electromotive force E. The ratio E/R gives the value of the current in laboratory units, designated as I_l to distinguish it from the value I_a in absolute units which is determined by the current balance. Since

$$I_l \text{ in laboratory units} = I_a \text{ in absolute units,}$$
$$1 \text{ laboratory ampere} = I_a/I_l \text{ absolute amperes.}$$

139. The Ayrton-Jones Current Balance.—The Ayrton-Jones current balance has many points of similarity with the Rayleigh current balance. The essential difference is in the form of the coils between which the electromagnetic force is exerted. In the Ayrton-Jones balance[1] all the coils are single-layer solenoids. A moving solenoid is attached to the pan of a balance, and this is inside a fixed solenoid, the winding of which is divided at its center. The two halves of the winding are connected as shown in Fig. 61, so that the current in the lower half is in a direction opposite to that in the upper half. The length of the moving solenoid is generally about equal to one-half the fixed solenoid. The force on the moving solenoid is a maximum when its mid-plane coincides with the mid-plane of the fixed solenoid.

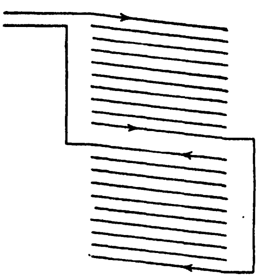

FIG. 61.—Direction of the current in the two halves of the fixed solenoid of an Ayrton-Jones current balance.

From this description it is seen that the moving solenoid of the Ayrton-Jones balance corresponds to the moving coil of the Rayleigh balance. Also the upper and lower halves of the fixed solenoid of the Ayrton-Jones balance correspond to the upper and lower coils of the Rayleigh balance. However, the position of the moving solenoid for maximum electromagnetic force between it and the two halves of the fixed solenoid is not the position for maximum force between it and either half alone, whereas in the Rayleigh balance the moving coil is normally in the position for maximum force between it and each fixed coil.

[1] For a description of this balance as used at the National Physical Laboratory, see Ayrton, Mather, and Smith, A New Current Weigher, *N.P.L. Coll. Researches*, **4**, 1 (1908).

In the N.P.L. form of the Ayrton-Jones current balance, a moving solenoid is attached to each pan of the balance and a fixed solenoid is associated with each moving solenoid. The advantages and disadvantages of this arrangement will be discussed later. A photograph of the balance as used at the National Physical Laboratory is reproduced in Fig. 62.

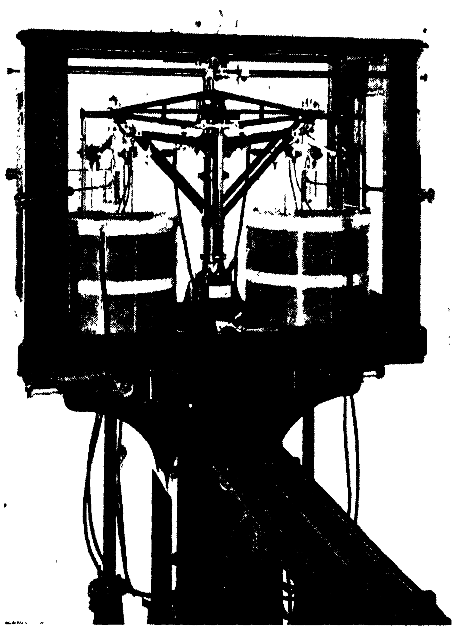

FIG. 62.—Photograph of the Ayrton-Jones current balance at the National Physical Laboratory.

Those features of this balance which are the same as those of the Rayleigh balance will not be again described. As the fundamental difference in the two balances is the shape of the coils, the description of the Ayrton-Jones balance will be largely concerned with the coils and their use in the balance. A description has already been given of a method of constructing a solenoid.

140. Computation of the Force per Unit Current.—The force per unit current F between the moving solenoid and the upper (or lower) half of the fixed solenoid can be computed by the formula

$$F = \frac{i_f i_m(M_1 - M_2)}{p_m} \tag{42}$$

where i_f = current in the fixed solenoid.

i_m = current in the moving solenoid.

p_m = pitch of the moving solenoid.

M_1 = mutual inductance between the upper half of the fixed solenoid and a circle at that end of the moving solenoid which is *within* this half.

M_2 = mutual inductance between the upper half of the fixed solenoid and a circle at that end of the moving solenoid which is *outside* this half.

This formula assumes that the moving solenoid is equivalent to a current sheet having a length equal to the product of the pitch by the number of turns. Hence the ends of the current sheet are circles which are coaxial with the solenoid and which have a diameter equal to the mean diameter of the moving solenoid. The axial distance between these circles is equal to the pitch of the solenoid multiplied by the number of turns. These circles are the ones used in computing M_1 and M_2 in Eq. (42). The solenoids are adjusted experimentally to be coaxial so that M_1 and M_2 can be computed by the formula for the mutual inductance between a solenoid and a coaxial circle.

In applying the formula, the mid-plane of the moving solenoid is taken as coincident with the plane which is midway between the two halves of the fixed solenoid because in this position the force exerted by the fixed solenoid on the moving solenoid is a maximum. The force of one-half of the fixed solenoid on the moving solenoid can then be computed, using the following dimensions:

l_1 = maximum length of one-half of the fixed solenoid (product of the pitch of the winding by the number of turns).

$2l_m$ = maximum length of moving solenoid (product of the pitch of the winding by the number of turns).

$2g$ = shortest distance between the adjacent turns of the two halves of the fixed solenoid.

The factor (2) has been introduced in this and the preceding term to simplify later expressions.

To compute M_1, the upper half of the fixed solenoid is considered as divided into two solenoids by a plane perpendicular to its axis and passing through the circle which determines the upper end of the moving solenoid. Then the length of the two parts of the fixed solenoid are $l_m - g$ and $l_1 - (l_m - g)$. The mutual inductance between a circle, having a diameter of the moving solenoid and lying in the end plane of a helix which has the diameter of the fixed solenoid, and one of the above lengths is computed and added to the mutual inductance obtained by computing with the second length, other dimensions being the same. The sum is the value of M_1.

To compute M_2, the upper half of the fixed solenoid is assumed to be extended until the plane which is perpendicular to its axis and which contains the circle at the end of the moving solenoid also contains the end point of the extended portion. The extended portion then has a length $l_m + g$, and the combined actual and extended solenoid has a length $l_1 + l_m + g$. The mutual inductance between an end circle of the moving solenoid and the extended solenoid is computed and subtracted from the mutual inductance between the end circle and the combined solenoids. This difference gives the value of M_2.

The formula for the mutual inductance M between a helix and a coaxial circle lying in the end plane of the helix, as given in Sec. 83, is equivalent to the following formula:

$$M = \frac{\pi z}{p}\left[\sqrt{4z^2 + (H + D)^2}(K - E) + \frac{(H - D)^2}{\sqrt{4z^2 + (H + D)^2}}(K - \Pi)\right] \quad (43)$$

where z = length of helix.

p = pitch of helix.

H = mean diameter of helix.

D = diameter of circle which is coaxial with the helix and which lies in a plane that is perpendicular to the axis of the helix and passes through the extreme point of the helix (the axis of the wire on the solenoid).

K, E = complete elliptic integrals of the first and second kinds to modulus k.

Π = complete elliptic integral of the third kind to modulus k and parameter $-c^2$.

$$k^2 = \frac{4HD}{4z^2 + (H + D)^2}$$

$$c^2 = \frac{4HD}{(H + D)^2}$$

This formula employs the complete elliptic integral of the third kind instead of the incomplete elliptical integrals of the first and second kinds as given in the earlier chapter. This nomenclature corresponds with that employed in recent publications on the Ayrton-Jones balance. This formula must be used for computing four different mutual inductances, all having the same dimensions except the lengths of the helices. The common dimensions are:

H = mean diameter of the fixed solenoid.

D = diameter of the circle (mean diameter of the moving solenoid).

p_m = pitch of the moving solenoid.

p_f = pitch of the fixed solenoid.

The different lengths and the corresponding mutual inductances are:

Length (z in formula)	Designation of Mutual Inductance
$l_m - g$	M_w
$l_1 - (l_m - g)$	M_x
$l_m + g$	M_y
$l_1 + l_m + g$	M_z

Then $$M_1 - M_2 = M_w + M_x + M_y - M_z \tag{44}$$

In practice l_1 is approximately equal to $2l_m$ and g is small, so that the values of M_w, M_x, M_y are nearly equal.

The force per unit current as computed by inserting the value of $M_1 - M_2$ from Eq. (44) in Eq. (42) gives only the force between a helical fixed coil and a cylindrical current sheet. In an actual balance the following forces must be considered:

1. The force per unit current between two helices is completely represented by adding to Eq. (42) relatively small correction terms, one of which depends on the cosine of the azimuthal

angle, as measured from the common axis, between the initial points of the windings on the two helices. By making the azimuthal angle 90°, this term becomes zero.

2. The force per unit current between two helices also contains a term which depends on the axial components of the windings and is independent of the azimuthal angle. Snow has developed the following formula (as yet unpublished), by which this axial force F_a can be computed.

$$F_a = \frac{8}{\pi}\left(\int_0^{l_m-g} Q dz + \int_0^{l_1-(l_m-g)} Q dz + \int_0^{l_m+g} Q dz - \int_0^{l_1+l_m+g} Q dz\right) \tag{45}$$

where all the dimensions have the same significance as in the preceding equation and

$$Q = \frac{4K}{\sqrt{4z^2 + (H + D)^2}}$$

when K = complete elliptic integral of the first kind to modulus k.

$$k^2 = \frac{4HD}{4z^2 + (H + D)^2}$$

The integrals cannot be evaluated by ordinary analytic methods. Their values can be obtained for any given set of dimensions by plotting Q as a function of z and obtaining each integral as the area under the curve between the two limits of the integration. An evaluation by Snow for the N.P.L. current balance gave F_a = 1.0 dyne per unit current when the total force was 95 000 dynes per unit current.

3. The force per unit current in the axial direction between the axial part of each lead of the moving solenoid and the axial part of each lead of the fixed solenoid can be computed by the formula

$$\begin{aligned} F_{11} = \ln\,[q + m + \sqrt{r^2 + (q + m)^2}] \\ + \ln\,[q - l + \sqrt{r^2 + (q - l)^2}] \\ - \ln\,[m + q - l + \sqrt{r^2 + (m + q - l)^2}] \\ - \ln\,[q + \sqrt{r^2 + q^2}] \end{aligned} \tag{46}$$

where l and m = lengths of the two leads.
q = axial distance between the lower ends of the two leads.
r = perpendicular distance between the leads.

When the currents are in the same direction, the direction of the force is such that it tends to make the line joining the centers of the wires perpendicular to the wires. When the currents are in opposite directions, the direction of the force is such that it tends to increase the distance between the centers.

4. The force between the radial parts of the leads is zero when the azimuthal angle between the windings is 90°.

In computing the inductances for the second fixed solenoid and the circles at the ends of the moving solenoid, a variation formula is often useful. This formula is

$$\frac{dM}{M} = q\frac{dH}{H} + r\frac{dD}{D} + s\frac{dz}{z} \tag{47}$$

where

$$q = \frac{\frac{2H^2}{D+H}[K(H+D) - \Pi(H-D)]}{K(2H^2+2D^2+4z^2) - E[(H+D)^2+4z^2] - \Pi(H-D)^2} \tag{48}$$

$$r = \frac{\frac{2D^2}{D+H}[K(H+D) + \Pi(H-D)]}{K(2H^2+2D^2+4z^2) - E[(H+D)^2+4z^2] - \Pi(H-D)^2} \tag{49}$$

$$s = -1 + \frac{K(2H^2+2D^2+8z^2) - 2E[(H+D)^2+4z^2]}{K(2H^2+2D^2+4z^2) - E[(H+D)^2+4z^2] - \Pi(H-D)^2} \tag{50}$$

In Appendix II it is shown that $q + r + s = 1$. The constants for the variation formula are easily computed since the complete elliptic integrals K, E, and Π to modulus k and parameter $-c^2$ have already been computed in determining the value of M. Moreover, as the variation formula should never be used if the variation of any of the dimensions is greater than 100 times the required accuracy in M, the computation of q, r, and s may be made with a slide rule.

141. Effect of Errors in the Measured Dimensions.—In the Ayrton-Jones balance, the effect of an error in the measurement of each dimension on the final result must be considered, since each dimension is separately measured. The formula for computing the effect of errors in the measured dimensions is obtained by differentiating the equation for the force between the solenoids. Before differentiating, Eq. (42) for the force should be rewritten in a more convenient form. The same current i is in the fixed and moving solenoids. The pitch of the moving solenoid p_m is replaced by the quotient obtained by dividing the length of the solenoid $2l_m$ by the number of turns, N_m. Using this nomenclature, the equation for the force F is

$$F = \frac{i^2 N_m (M_1 - M_2)}{2l_m} \tag{51}$$

In deriving this equation, N_m was considered as an integer and will be considered as a constant in making the differentiation. Taking the logarithm of both members of Eq. (51), differentiating, and rearranging, there results the equation

$$\frac{di}{i} = \frac{1}{2}\frac{dF}{F} + \frac{1}{2}\frac{dl_m}{l_m} - \frac{1}{2}\frac{d(M_1 - M_2)}{M_1 - M_2} \tag{52}$$

Before this equation can be applied to determine the effect of the dimensions, an expression must be obtained for

$$\frac{d(M_1 - M_2)}{(M_1 - M_2)}$$

in terms of the measured constants. But from the variation equation just given for M, together with the lengths of the different solenoids, a formula can be written for the increment of each inductance that must be computed. The formula for M_w is

$$dM_w = q_w M_w \frac{dH}{H} + r_w M_w \frac{dD}{D} + s_w M_w \frac{l_m}{l_m - g}\frac{dl_m}{l_m} - s_w M_w \frac{g}{l_m - g}\frac{dg}{g} \tag{53}$$

There are similar equations for dM_x, dM_y, and dM_z. Since

$$M_1 - M_2 = M_w + M_x + M_y - M_z,$$
$$d(M_1 - M_2) = dM_w + dM_x + dM_y - dM_z \tag{54}$$

Adding the increment equations and dividing by $M_1 - M_2$,

$$\frac{d(M_1 - M_2)}{M_1 - M_2} = \frac{q_wM_w + q_xM_x + q_yM_y - q_zM_z}{M_1 - M_2}\frac{dH}{H} + \frac{r_wM_w + r_xM_x + r_yM_y - r_zM_z}{M_1 - M_2}\frac{dD}{D}$$
$$+ \left(\frac{s_wM_w}{l_m - g} - \frac{s_xM_x}{l_1 + g - l_m} + \frac{s_yM_y}{l_m + g} - \frac{s_zM_z}{l_1 + g + l_m}\right)\frac{l_m}{M_1 - M_2}\frac{dl_m}{l_m}$$
$$- \left(\frac{s_wM_w}{l_m - g} - \frac{s_xM_x}{l_1 + g - l_m} - \frac{s_yM_y}{l_m + g} + \frac{s_zM_z}{l_1 + g + l_m}\right)\frac{g}{M_1 - M_2}\frac{dg}{g}$$
$$+ \left(\frac{s_xM_x}{l_1 + g - l_m} - \frac{s_zM_z}{l_1 + g + l_m}\right)\frac{l_1}{M_1 - M_2}\frac{dl_1}{l_1} \quad (55)$$

Since all the mutual inductances can be computed from the dimensions, the value of $d(M_1 - M_2)/(M_1 - M_2)$ is given in terms of the measured dimensions. This value can be substituted in the equation for di/i to give an equation for determining the effect of an error in each of the measured quantities on the value of the current.

From the equation it follows that, to determine the current to one part in a million, the force must be known to two parts in a million. No simple generalization can be made concerning the other quantities. As a concrete example, the numerical coefficients will be computed for a fixed and a moving solenoid of the Ayrton-Jones balance of the National Physical Laboratory. The measured dimensions are:

$$H = 33.00 \text{ cm}$$
$$D = 20.36 \text{ cm}$$
$$l_m = 6.49 \text{ cm}$$
$$l_1 = 12.70 \text{ cm}$$
$$g = 0.14 \text{ cm}$$

The computed inductances and constants are:

$M_w = 5860\text{m}\mu\text{h}$	$q_w = -1.05$	$r_w = 2.26$	$s_w = -0.21$
$M_x = 5860$	$q_x = -1.05$	$r_x = 2.26$	$s_x = -0.21$
$M_y = 6063$	$q_y = -1.03$	$r_y = 2.25$	$s_y = -0.22$
$M_z = 11\,293$	$q_z = -0.41$	$r_z = 2.05$	$s_z = -0.64$
$M_1 - M_2 = 6490$			

$$\frac{di}{i} = 0.5\frac{dF}{F} - 1.1\frac{dH}{H} + 1.3\frac{dD}{D} + 0.31\frac{dl_m}{l_m} - 0.004\frac{dg}{g} - 0.4\frac{dl_1}{l_1} \quad (56)$$

From this equation can be computed the error in each of the measured dimensions which will introduce an error of one part in a million in the absolute value of the current. The results are given in the following table:

TABLE XII

Dimension	Error in Microns That Produces an Error of One Part in a Million in the Current
H	0.3
D	0.15
$2l_m$	0.4
g	0.4
l_1	0.4

142. Temperature Effects.—Various temperature effects result from the heating of the solenoids by the measuring current. No balance of the Ayrton-Jones type has yet been constructed in which the solenoids are cooled by the circulation of a liquid. Hence all the heat generated must either be dissipated by convection and radiation, or absorbed in the solenoid, thus raising its temperature. The rise in temperature affects the dimensions of the solenoids, but on account of the temperature gradients the amount of the change is very difficult to determine. The magnitude of the error that will be introduced by the uncertainty in the temperature of a solenoid depends on the thickness of the forms, on its thermal conductivity, on its expansivity, on its thermal capacity, and on its elastic properties. However, the variation equation given in the preceding section for the N.P.L. balance shows that, if the two solenoids expand by the same relative amount, the effect on the current is small. In the N.P.L. balance as used in 1908 all weighings were made within 20 minutes after closing the circuit, and the assumption is made that the dimensions did not change in that interval.

The convection currents of air which flow past the moving solenoid may cause an unsteadiness in the rest point of the balance. The lower end of the fixed solenoid may be closed to prevent air currents from flowing past the moving solenoid, but this produces a greater rise in temperature of the moving solenoid. At the National Physical Laboratory the fixed solenoid was not closed and air currents gave no trouble during the first 20 minutes.

143. Design of Solenoids for an Ayrton-Jones Current Balance. The solenoids for an Ayrton-Jones current balance must be of such size that their dimensions can be accurately measured and the force between them when they are carrying their maximum current is large enough to be accurately determined. For design purposes, it is convenient to express the mutual inductances which appear in the formula for the force by the first term of the series formulas by which they can be computed.[1] Also the assumption will be made that $l_1 = 2l_m$ and that $g = 0$. Then

$$F = i^2 \frac{\pi^2 D^2 N_1 N_m}{l_m} \left[\frac{3}{\sqrt{H^2 + 4l_m{}^2}} - \frac{1}{\sqrt{H^2 + 4(3l_m)^2}} \right] \quad (57)$$

From this equation, it can be shown that, if the length and diameter of all the solenoids are increased by the same factor, while the pitch of the winding remains constant, the force per unit current is increased as the square of this factor. From this it follows that the solenoids should be as large as can be conveniently handled, but beyond a reasonable size there is no advantage, since the sensitivity of a balance on which a large load can be weighed is less than that of a balance which is designed for a smaller load.

The selection of the ratio of length to diameter of the fixed solenoid is a matter for judgment, as no formula for determining the most satisfactory ratio has been developed. It would seem desirable to have the length about equal to the diameter. However, if this ratio is maintained for the fixed solenoid, the length of the moving solenoid will probably be nearly twice its diameter. This will make a very heavy weight to be supported on the pan of the balance. Hence a shorter length is more likely to be chosen.

144. Adjustment of the Solenoids in the Ayrton-Jones Current Balance.—The solenoids of the Ayrton-Jones current balance require adjustments similar to those of the coils of the Rayleigh balance. However, the two halves of a fixed solenoid are generally made on a single form, so that they are coaxial and at a fixed

[1] The series formula was developed by Rosa, *Bull. B. S.*, **3**, 209 (1907). The result obtained from using the first term only will not be in error by more than 10 per cent unless the length of the fixed solenoid is less than the diameter.

distance apart. By means of levels, the axes of all the solenoids can be made vertical. The adjustment of the solenoids, so that they are coaxial and so that the center of the moving solenoid is in the plane which is midway between the two halves of the fixed solenoid, is accomplished by determining the force per unit current for a number of different positions of the fixed solenoid (or the moving solenoid) and finding the position such that any horizontal displacement increases the force per unit current and any vertical displacement decreases the force per unit current. The procedure is exactly the same as for the Rayleigh balance. Also, when this position is found, the method of determining the absolute value of the current and of comparing it with the laboratory standards is the same as described under the Rayleigh balance.

145. A Double Balance of the Ayrton-Jones Type.—The Ayrton-Jones balance at the National Physical Laboratory has a moving solenoid attached to each pan of the balance and a fixed solenoid for each moving solenoid. This constitutes a double balance and gives a force nearly double that of a single balance having the same dimensions of the solenoids, without increasing the weight on the knife edges of the balance. Another advantage is the greater stability caused by the symmetrical heating. This is particularly important in this type of balance since it is always operated during the interval when the temperatures of the different parts are rapidly changing. When a single set of coils was used in the N.P.L. balance, the drift in the rest point with normal current was more than 0.1 division for each swing of the balance, while an accuracy of determination of the rest point of nearly 0.01 division was required to give an accuracy of one part in a million in the measured current. With the double balance it is necessary to determine the force between each moving solenoid and the fixed solenoid associated with the opposite moving solenoid. This may be 1 per cent of the total force on a moving solenoid. It can be evaluated experimentally by making two weighings with the same value of the current, in the second of which the current is reversed in only one moving solenoid and its associated fixed solenoid. To obtain the equation for eliminating the force between opposite solenoids, let D be the force per unit current between a moving solenoid and its associated fixed solenoid and S be the force per unit current between a moving solenoid and the opposite fixed solenoid. If the currents in the two moving solenoids are in the same direction and the currents in the parts of the fixed solenoids are in the directions to give maximum force on the moving solenoid, the total force per unit current, F_S, is $2D + 2S$. Then if the current is reversed in one moving solenoid and in its associated fixed solenoids, the force F_D is $2D - 2S$. Writing the equations and adding,

$$4D = F_S + F_D$$

Hence the effect of the force between a moving solenoid and the opposite fixed solenoids is entirely eliminated.

146. The Electrodynamometer Balance.—The electrodynamometer balance consists of a large fixed coil having its axis horizontal and a small turning coil having its axis vertical. The small coil is within, and concentric with, the fixed coil and is mounted on the beam of a balance whose center knife edge lies in the plane which is perpendicular to the axis of the fixed coil

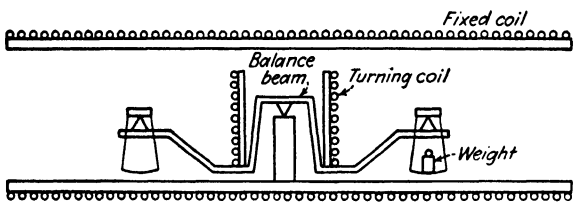

FIG. 63.—Diagram of the electrodynamometer balance.

and which contains the centers of the coils. In Fig. 63 is given a diagram of a balance of this type, in which each of the coils is a single-layer solenoid and the beam of the balance has a shape resembling the letter *W*.

The fixed solenoid must have a large diameter since the balance and turning solenoid have to be mounted inside it, and it must be long so that the torque can be accurately computed. The turning solenoid must be as large as can be used inside the fixed solenoid. The most suitable shape is a cylinder having its length equal to its diameter.

147. Computation of the Torque per Unit Current in the Electrodynamometer Balance.—The torque per unit current between the two concentric solenoids with their axes perpendicular is given by the formula[1]

$$T = \frac{2\pi^2 N_1 N_2 H_2^2}{\sqrt{H_1^2 + z_1^2}}\left(1 \pm \frac{\rho_2^2}{4H_2^2}\right)(1 + S) \qquad (58)$$

where T = torque per unit current.
N_1 = number of turns on the fixed solenoid.
H_1 = mean diameter of fixed solenoid.
z_1 = length of fixed solenoid (product of pitch and number of turns).
N_2 = number of turns on the turning solenoid.
H_2 = mean diameter of turning solenoid.
z_2 = length of turning solenoid (product of pitch and number of turns).
ρ_2 = diameter of wire of turning solenoid.
S = correction term which can be computed by the formula

$$S = \sin^2\alpha_1 \sum_{s=1}^{\infty} (-1)^{s+1}\left(\frac{r_2}{r_1}\right)^{2s}\left(\frac{2s+1}{2}\right)F(\alpha_1,\alpha_2) \qquad (59)$$

$$F(\alpha_1,\alpha_2) = \left[\frac{1\cdot 3\cdot 5\cdots(2s-1)}{2\cdot 4\cdot 6\cdots 2s}\right]C_s(\alpha_1)C_{s+1}(\alpha_2) \qquad (59a)$$

$$r_1 = \sqrt{H_1^2 + z_1^2}$$
$$r_2 = \sqrt{H_2^2 + z_2^2}$$
$$\sin\alpha_1 = \frac{H_1}{r_1}$$
$$\sin\alpha_2 = \frac{H_2}{r_2}$$

$$C_1(\alpha) = 1$$
$$C_2(\alpha) = 1 - \tfrac{7}{4}\sin^2\alpha$$
$$C_3(\alpha) = 1 - \tfrac{9}{2}\sin^2\alpha + \tfrac{33}{8}\sin^4\alpha$$
$$C_4(\alpha) = 1 - \tfrac{33}{4}\sin^2\alpha + \tfrac{143}{8}\sin^4\alpha - \tfrac{715}{64}\sin^6\alpha$$

$$C_s(\alpha) = 1 + \sum_{n=1}^{n=s-1}\left[\frac{(1-s)(2-s)\cdots(n-s)}{2\cdot 3\cdots(n+1)}\right]\left[\frac{2s+3}{2}\cdot\frac{2s+5}{4}\cdots\frac{2s+2n+1}{2n}\right]\sin^{2n}\alpha \qquad (60)$$

[1] This formula was developed by Snow, *B. S. J. Research*, **1**, 685 (1928).

Equation (60) is the same as Eq. (51) of Chapter XIV on page 144. If the fixed coil is long, the computation of S is not difficult since only one or two terms in the series for S are required to give the torque to one part in a million.

The plus and minus sign results from the uncertainty in the distribution of current in the cross section of the wire on the turning solenoid. If the current is assumed to be uniformly distributed over the cross section—the so-called uniform distribution—the derivation of the formula gives the term as $1 + \frac{\rho^2}{4H_2^2}$. If the current density is assumed to be greater on the inner part of the turns, varying inversely as the length (and hence resistance) of the filaments into which the wire may be divided—the so-called natural distribution—the derivation gives the term as $1 - \frac{\rho^2}{4H^2}$. There is no direct experimental method of deciding between these assumptions.

148. Effect of Errors in the Mechanical Measurements.—The equation for computing the current is obtained by equating the mechanical torque and the electrical torque. Hence

$$mgb = \frac{2\pi^2 I^2 N_1 N_2 H_2^2}{\sqrt{H_1^2 + z_1^2}}\left(1 \pm \frac{\rho_2^2}{4H_2^2}\right)(1 + S) \tag{61}$$

where m = mass required to balance the electrical torque.
b = one-half the length of balance beam.
g = acceleration of gravity.

This equation can be readily solved to give the value of the current I in terms of the mechanical units.

In determining the effect of errors, the terms in parentheses may be taken as equal to unity. Then, taking the logarithm of both sides of the equation and the derivative of the resulting equation, there is obtained the equation

$$\frac{\delta I}{I} = \frac{\delta m}{2m} + \frac{\delta g}{2g} + \frac{\delta b}{2b} - \frac{\delta H_2}{H_2} + \frac{H_1^2}{(H_1^2 + z_1^2)}\frac{\delta H_1}{2H_1} + \frac{z_1^2}{(H_1^2 + z_1^2)}\frac{\delta z_1}{2z_1} \tag{62}$$

The effect of errors in N_1 and N_2 which amount to only a part of a turn cannot be determined from this equation. In the case of N_2, the effect of an error will depend on the place at which the leads to the turning solenoid are attached. If they lie near the vertical plane which contains the axis of rotation of the solenoid, an error will have very little effect, since the conductor at this point does not contribute to the torque. In the case of N_1, a small error in attaching leads so that the number of turns is not exactly an integer will have very little effect on the torque since the fraction of a turn is so far from the turning solenoid. A much larger error is likely to be introduced by irregularities in the winding of this solenoid. The formula for the torque was derived on the assumption that the winding was uniform, and hence the formula cannot be made to show the effect of irregularities of winding.

The equation shows that a relative error in H_2 will produce the same relative error in the current, while for m, g, and b the relative error in the current is only one-half the relative error in the quantities. For H_1 and z_1 the relative error in the current is less than one-half the error in the quantities, the exact amount for each depending on the dimensions of the solenoid. To indicate the accuracy with which the various measurements must be made, the following table has been constructed using the dimensions of an electrodynamometer which has been considered by the author.

TABLE XIII.—THE REQUIRED ACCURACY IN AN ELECTRODYNAMOMETER BALANCE WHICH WILL DETERMINE THE ABSOLUTE VALUE OF THE CURRENT TO ONE PART IN A MILLION

Quantity measured	Symbol	Value of quantity	Required accuracy, p.p.m.	Absolute accuracy required
Mass	m	1 g	2	0.002 mg
Gravity	g	980 cm/sec²	2	0.002 cm/sec²
Length, balance arm	b	25 cm	2	0.5 micron
Diameter, turning coil	H_2	10 cm	1	0.1 micron
Diameter, fixed coil	H_1	30 cm	25	7.5 microns
Length, fixed coil	z_1	100 cm	2.2	2.2 microns

The determination of some of the quantities with the required accuracy is very difficult. However, the difficulties do not seem to be insuperable.

149. Operation of the Electrodynamometer.—The operation of the electrodynamometer consists of determining the mass required on the balance pan to balance the torque produced by the current in the solenoids. The current is held constant and the auxiliary weights are added to one of the balance pans until the scale reading is zero. The current in the *fixed* solenoid is then reversed, and a weight is added simultaneously to one pan of the balance so that the scale reading remains zero. If the mass of the added weight is m_1 and the length of the balance arm to which the weight is applied is b_1, then

$$m_1 g b_1 = 2TI^2 \tag{63}$$

where T = computed torque per unit current.

The current in the *turning* solenoid is then reversed, the weight removed, and the balance again brought to the zero position by means of the auxiliary weights. Then the current in the fixed solenoid is reversed and a weight is added to the opposite pan of the balance. If the mass of the weight is m_2 and the length of the balance arm is b_2,

$$m_2 g b_2 = 2TI^2 \tag{64}$$

Combining these equations and solving for I^2,

$$I^2 = \frac{g(b_1 + b_2)}{2T} \cdot \frac{m_1 m_2}{m_1 + m_2} \tag{65}$$

This procedure eliminates any effect of an external magnetic field provided it is constant during a pair of observations. Also, by using the weight first on one pan and then on the other, the equation for the current requires only the total length of the balance beam, which is $b_1 + b_2$. This is a very great advantage since the distance between the two end knife edges which are both pointed in the same direction can be measured with much greater accuracy than the distance from the center knife edge to one of the end knife edges, in which case one points in one direction and the other in the opposite direction.

It is important that the weights shall be of a material having a susceptibility very near zero, since they are used in a magnetic field. As there is no material which has exactly zero susceptibility, two weights should be provided, one from a slightly paramagnetic material, the other from a slightly diamagnetic material. If the results by the two weights agree, it can be assumed that no error results from the susceptibilities of the materials.

When the turning solenoid is moved so that its center occupies different positions along a line parallel to the axis of the fixed solenoid, the torque per unit current is a maximum when its center is in the plane that passes through the center of the fixed solenoid and is perpendicular to its axis. If the center of the turning solenoid is moved in this plane, the torque per unit current is a minimum when the solenoids are concentric. Hence with suitable methods of moving the turning solenoid, it can be experimentally centered. Also this same method of determining the position at which the torque is a maximum may be used to level the fixed solenoid and to place the axis of the turning solenoid in a vertical position. As a result the solenoids can be completely adjusted by determining the maximum or minimum torque corresponding to different translations and rotations.

150. Probable Accuracy Attainable with an Electrodynamometer Balance.—There does not appear to be any fundamental principle of the electrodynamometer that prevents it from being used to make an accurate determination of the absolute value of the ampere. However, the method has some features that will make such a determination more difficult than one with a current balance. The number of mechanical dimensions that must be accurately determined is large and some of them are difficult to measure. Perhaps the most difficult is the length of the balance beam. This will require a special comparator, and at best involves assumptions concerning the distortion of a knife edge under load. Another feature that offers some difficulty is the necessity of making the weighing in the limited space within the large solenoid while there is heat developed in that space by the current in the turning solenoid. Fortunately the heating will be symmetrical with respect to the balance beam, so that air currents may not appreciably affect the weighing. The amount by which the temperature will affect the length of the balance beam will depend on the coefficient of expansion of the material of the beam. If fused quartz is used for the beam and for the form of the turning solenoid, then a change of 2° in temperature will affect the result by only one part in a million. However, quartz in dry air retains electrostatic charges for some time, so that it may be necessary to ionize the air by means of some radioactive material.

The one difficulty that may limit the accuracy of this method is the uncertainty concerning the distribution of current in the cross section of the wire of the turning solenoid. Even in a well-designed instrument which has the same current in both solenoids, this may introduce an uncertainty of twenty parts in a million in the torque or ten parts in a million in the current. It may be possible to design an instrument in which the size of wire on the turning solenoid is so small that the current distribution becomes unimportant. This will probably require that the current in the turning solenoid shall be much smaller than that in the fixed solenoid. This will require a special electrical setup to measure the ratio of the currents in the two solenoids. If this can be accomplished, the method can probably be used to give results of the highest accuracy.

CHAPTER XX

THE ABSOLUTE MEASUREMENT OF ELECTROMOTIVE FORCE

A precise measurement of the absolute value of an electromotive force has never been made. There are two different types of apparatus by which such a measurement could be made: one is the electromagnetic type, the other the electrostatic type.

151. Electromagnetic Apparatus for the Absolute Measurement of Electromotive Force.—The electromagnetic type of apparatus for the absolute measurement of electromotive force is the equivalent of a homopolar generator in which the generated electromotive force can be computed from the rate of cutting of tubes of magnetic induction. If the Lorenz apparatus were used to measure the electromotive force generated by it, it would be an apparatus of this type. However, the computation of the electromotive force by any apparatus of this type requires a knowledge of the absolute value of the current that is employed to produce the magnetic field. Hence the apparatus is more suitable for the absolute measurement of resistance, and has always been so used. Also the generated electromotive force is so small that it cannot readily be compared with a standard cell.

152. Electrostatic Apparatus for the Absolute Measurement of Electromotive Force.—Electrostatic apparatus for the absolute measurement of electromotive force is usually an electrometer of such form that its constant can be computed from its dimensions. The form that has usually been adopted is the Kelvin absolute electrometer, in which a circular metallic plate, surrounded by a guard ring and suspended from the pan of a balance, is at a measured distance from a large fixed plate of metal. In all the instruments that have been described, the force has been too small to allow precise measurement. However, present-day technique might permit the operation of the entire apparatus in a vacuum, so that much higher voltages could be employed. It is therefore useful to inquire whether precise results might be obtained with reasonable dimensions of the apparatus.

The electrostatic force of attraction F between a disk of radius R, which is surrounded by a guard ring, and a plate which is parallel to the face of the disk at a distance h from it, when a potential difference V is maintained between the disk with its guard ring and the plate, is given by the equation

$$F = \frac{R^2V^2}{8h^2} \tag{1}$$

This force is balanced by the mass m, acted on by gravity. Assuming values which can be accurately measured, *e.g.*, $m = 1g$, $R = 10$ cm, $h = 1$ cm, the value of V is 8.8 electrostatic units of voltage or 2600 volts. This is a high voltage, but one that can be impressed on a condenser with 1-cm spacing, even in air. Hence the method appears possible, although many experimental and theoretical difficulties will doubtless be encountered.

The absolute value of an electromotive force as obtained by an electrostatic method is in electrostatic units of voltage. To change these to electromagnetic units requires a knowledge of the ratio of the electromagnetic to the electrostatic unit. This ratio is not known more accurately than 100 parts in a million. Before any electrostatic method can give precise results in electromagnetic units, a new determination of the ratio of the units is necessary.

CHAPTER XXI

RATIO OF THE ELECTROMAGNETIC UNIT TO THE ELECTROSTATIC UNIT OF ELECTRICITY

The ratio of the electromagnetic unit to the electrostatic unit of electricity can be most accurately determined[1] by measuring the capacitance, in electromagnetic units, of a capacitor whose capacitance in electrostatic units can be computed from its linear dimensions and the dielectric constant of the medium between the plates.[2] There are three forms of capacitors that have been used, *viz.*, two concentric spheres, two coaxial cylinders with guard cylinders at each end of one cylinder, and two plane parallel plates with a guard ring on one plate. The concentric spheres have the disadvantage that the lead to the inner sphere must pass through the outer sphere, thus introducing into the electromagnetic capacitance a condition which cannot be included in the electrostatic capacitance. This difficulty is not present in either the coaxial cylinders or the parallel plates.

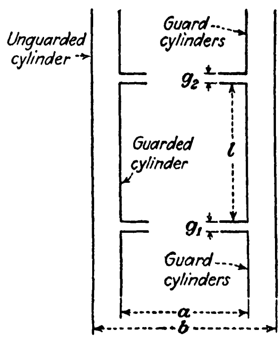

FIG. 64.—Diagram to show a longitudinal cross section of a cylindrical air capacitor.

153. An Air Capacitor Consisting of Two Coaxial Cylinders.—An air capacitor consisting of two coaxial cylinders with guard cylinders at each end can have such dimensions that both its electrostatic and its electromagnetic capacitance can be accurately determined. The correct position of the two electrodes can be determined by electrical measurements and the necessary measurements of dimensions can be accurately made. A diagram

[1] For a discussion of the methods that have been proposed, see Rosa and Dorsey, *Bull. B.S.*, **3**, 605 (1907).

[2] The most accurate determination as yet made, was carried out by Rosa and Dorsey, *Bull. B.S.*, **3**, 433 (1907). They claim an accuracy of 100 parts in a million.

showing the essential features of such a capacitance is given in Fig. 64.

The formula for computing the capacitance of a cylindrical air capacitor is

$$C_s = \frac{k\left(l + \frac{g_1}{2} + \frac{g_2}{2}\right)}{2 \ln\left(\frac{b}{a}\right)} \tag{1}$$

where C_s = capacitance in electrostatic units.
k = dielectric constant of air.
l = length of the guarded cylinder.
b = inside diameter of the outside (unguarded) cylinder.
a = outside diameter of the inside cylinders.
g_1 = width of lower gap.
g_2 = width of upper gap.

The assumptions made in deriving the above formula are: (1) the guard and guarded cylinders have the same diameter; (2) all the cylinders are coaxial; (3) the width of the gap is so small that second-order correction terms are negligible; (4) the guard cylinders are so long that increasing their length would not affect the charge on the guarded cylinder.

1. The guard and guarded cylinders must have the same diameter at adjacent ends to a very high order of accuracy. A correction formula to account for a difference in diameter could be developed, but is unnecessary if the construction is sufficiently precise. The guard cylinders should be made as an integral part of the guarded cylinder, so that the difference in diameter of adjacent ends will be only a few hundredths of a micron.

2. The guard and guarded cylinders must be coaxial to ensure that the charge density is uniform at the end of the guarded cylinder. In other words, the guard cylinders completely fulfill their function only when they are coaxial. This adjustment must be accomplished mechanically. The outer cylinder must also be coaxial with the others in order that the distance between cylinders will be the same at all points. The adjustment to make the outer cylinder coaxial with the inner one can be made by electrical measurements, as will be described later.

3. In the formula, the effective length of the guarded cylinder is the actual length plus one-half the width of each guard gap.[1] This effective length has been derived by assuming that the guarded cylinder has flat metallic ends and that the end of each guard cylinder which is next to the guarded cylinder is also flat. When the gaps are very narrow, the ends of the cylinders do not need to be completely closed, but the flat portion must extend 1 or 2 cm from the edge.

4. The minimum length of guard cylinders can be computed by determining the length required in order that the charge density on the inner end may be the same as that on the infinitely long cylinder. The charge density for unit potential difference between coaxial and concentric cylinders is given approximately at any distance from the ends by the formula

$$Q_0 = \frac{1}{4\pi p}\left[1 + \frac{1}{2\pi}e^{-(2\pi n+1)}\right] \tag{2}$$

where Q_0 = charge density for unit potential difference.
p = difference in radii of the two cylinders.
x = distance from one end.
$n = x/p$.

The exponential term becomes less than one part in a million if the numerical value of the exponent is greater than 14, which is the case if n is greater than 2. Hence the length of each guard cylinder must be at least twice the distance between the cylindrical surfaces.

154. Measurement of the Capacitance of a Cylindrical Air Capacitor in Electromagnetic Units.—The measurement of the capacitance is generally made by means of a double Maxwell bridge, which is shown diagramatically in Fig. 65. The equations for this bridge have not been developed so completely as those for the single Maxwell bridge described in a previous chapter. However, as the capacitance to be measured is always small, it is improbable that the correction terms which a complete analysis might disclose would appreciably alter the result.

[1] The formula is correct only if the guard gap is very narrow. A more exact formula for the effective length l_e is

$$l_e = l + \frac{g_1}{2} + \frac{g_2}{2} - \frac{g_1^2 + g_2^2}{\pi(b - a)} + \cdots$$

The resistances of the two parts of the bridge should be identical arm for arm so that the potentials of the guard and of the guarded cylinders are the same at the beginning and end of the charging cycle. The accuracy with which the adjustment of S_1 must follow that of S can be determined experimentally for any capacitor under investigation. The resistance G_1 can be a manganin resistance of value G, since neither the inductance of a suitable galvanometer nor, if the contactor is properly adjusted, the motion of its coil influences the result. The value of Q should be as small as is consistent with the required sensitivity.

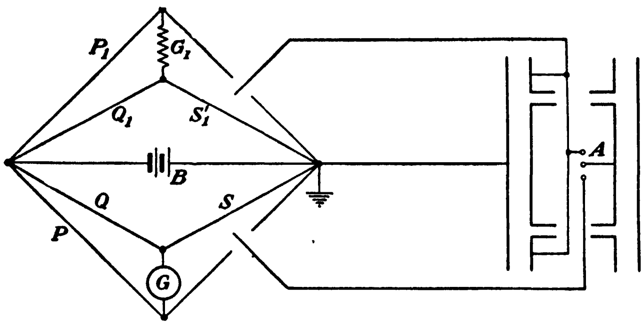

FIG. 65.—Diagram of a double Maxwell bridge connected to measure the capacitance of a cylindrical capacitor. The switch A is used to connect the guard cylinder to the guarded cylinder when measuring the capacitance of the leads.

The capacitance C is computed in electromagnetic units by the formula

$$C = \frac{QF}{nPS} \tag{3}$$

where

$$F = 1 - \frac{Q^2}{Q'S'} - \frac{QG}{Q'S} - \frac{QB}{PS'}$$
$$Q' = Q + P + G$$
$$S' = S + Q + B$$

If all resistances are in ohms and n is given in vibrations per second, the capacitance is in farads. The capacitance in c.g.s. electromagnetic units, C_m, is obtained by multiplying the value in farads by 10^{-9}. The value of v, the ratio of the electrostatic unit of electricity to the electromagnetic unit, is given by the equation

$$v^2 = \frac{C_s}{C_m} \tag{4}$$

where C_s = computed value of the capacitance in electrostatic units.

C_m = measured value in c.g.s. electromagnetic units.

155. Adjustments of the Cylinders.—The outer cylinder of a cylindrical capacitor can be adjusted to be coaxial with the guarded cylinder by determining the position at which the capacitance is a minimum. The guard and guarded cylinders must first be made coaxial by mechanical measurements. When the axis is to be vertical, a method of mounting the outer cylinder so that it may be readily adjusted is by means of a double gimbal. When suitably constructed, the gimbal will allow translation along and rotation around two horizontal axes. With suitable scales for measuring the motions, the position at which the capacitance is a minimum can be determined. This corresponds to the coaxial position.

156. Design of a Cylindrical Capacitor.—A cylindrical air capacitor for determining the ratio of the electric units should be so designed that both the electrostatic and the electromagnetic capacitances can be accurately determined. For readily estimating the effect of the different dimensions on the capacitance, an approximate formula for the capacitance can be developed from Eq. (1). Let $a = b - 2t$, where t is the radial distance between the outer and inner cylinders. Substituting in Eq. (1), expanding the logarithm, and neglecting all small quantities,

$$C_s = \frac{Klb}{4t} \tag{5}$$

Hence the electrostatic capacitance depends on the length of the guarded cylinder, on the distance between the inner and outer cylinders, and on the radius of the outer cylinder. Of these dimensions the one which is most difficult to measure is the distance between the cylinders, since this must be kept small in order that the electromagnetic capacitance will be large enough to measure. A suitable set of dimensions would be:

Length of guarded cylinder, 50 cm.

Length of each guard cylinder, 10 cm.

Outer diameter of guard and guarded cylinders, 30 cm.

Inner diameter of outer cylinder, 32 cm.
Width of guard gap, 0.01 cm.
Diameter of hole in closed ends of guarded and guard cylinders, 20 cm.

Since the distance between the two cylindrical surfaces would be only 1 cm, which would have to be measured to 0.01 micron to give an accuracy of one part in a million in the computed capacitance, both cylinders would have to be finished as the best optical surfaces.

The capacitance of such a condenser is about 385 electrostatic units (430 micromicrofarads). The measurement in electromagnetic units of this small capacitance can probably be made with an accuracy of a few parts in a million. The electrostatic capacitance can be determined with about the same accuracy.

157. An Air Capacitor Consisting of Parallel Plates.—An air capacitor consisting of two circular plates which have their plane surfaces parallel can be accurately constructed and its

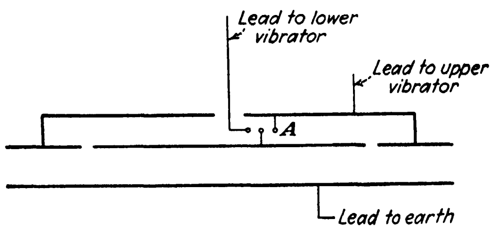

FIG. 66.—Diagram of a section of a parallel-plate capacitor. The connections of the leads to the double Maxwell bridge are indicated.

capacitance can be precisely computed. One plate consists of a central guarded plate which is insulated from the surrounding guard plate. The opposite plate is a single piece. A diagram is shown in Fig. 66, with the guarded plate completely shielded.

The formula for computing the capacitance in electrostatic units is

$$C_s = \frac{k\left(r + \frac{g}{2}\right)^2}{4d} \tag{6}$$

where C_s = capacitance in electrostatic units.
k = dielectric constant of the medium between the plates.
r = radius of guarded plate.

g = width of gap between guard and guarded plates.
d = distance between guarded and opposite plate.

The assumptions made in deriving the formula are: (1) the plates are plane and strictly parallel; (2) the guard and guarded plates are coplanar; (3) the width of the gap is so small that second-order correction terms are negligible; (4) the guard ring is so wide that the distribution of the charge on the guarded plate is uniform.

1. The plates must be plane and parallel to a high order of accuracy. The testing for planeness can be made by an optical flat. It is desirable that the deviation from planeness of each plate shall not exceed one-fiftieth of a wave length of light. The parallelism should be such that the maximum difference in distance between the plates does not exceed 0.1 micron. Formulas have been developed for plates that are not exactly parallel, but should not be required.

2. The guard and guarded plates must be strictly coplanar in order that the distribution of charge on the guarded plate shall be uniform. To accomplish this purpose, the guard plate must be an accurate extension of the guarded plate.

3. The conditions which apply to the gap are the same as those for a cylindrical condenser.

4. The minimum width of guard ring can be computed by the same formula as for the minimum length of a guard cylinder. However, the guard ring must also extend to cover the back of the guarded plate, as shown in the figure, but the position of the portion back of the plate is immaterial.

The measurement of the capacitance is made by the double Maxwell bridge already described. As all the adjustments of this capacitor are generally made mechanically, only one capacitance must be measured. However, the lead capacitance must be determined with the same absolute accuracy as the capacitance itself. In the figure, a method is shown for connecting the leads so that the lead capacitance can be determined.

158. Design of a Parallel-plate Capacitor.—The design of a parallel-plate air capacitor for the determination of the ratio of the electric units must be such that both the electrostatic and the electromagnetic capacitances can be accurately measured. For an accurate determination of the electrostatic capacitance, the dimensions must be so large that they can be accurately measured, but not so large as to prevent an accurate facing of the plates. The following dimensions seem reasonable:

Diameter of guarded plate, 50 cm.
Width of gap, 0.02 cm.
Width of guard ring, 5 cm.
Thickness of guard and guarded plates at the gap, 1 cm.
Diameter of opposite plate, 60 cm.
Distance from the guard and guarded plates to the opposite plate, 1 cm.

With these dimensions, the capacitance is about 156 electrostatic units or about 173 micromicrofarads. This is an exceedingly small capacitance to measure with an accuracy of one part in a million. The difficulty can be appreciated by considering the change in height of a lead wire that would produce a change of one part in a million in the capacitance. If the height of the wire is 1 m and its length is 10 cm, a change in height of 0.1 mm will change the capacitance by 10^{-4} micromicrofarads.

159. Dielectric Constant of Air.—The dielectric constant of air at normal temperature and pressure is not known to one part in a million. The most probable value is 1.000590 with an uncertainty of 2 or 3 in the last significant figure. The value of $(k - 1)$ is directly proportional to the pressure and inversely proportional to the absolute temperature. Hence an increase in atmospheric pressure of 8 mm of mercury, or a decrease in temperature of 0.5°C will produce an increase of one part in a million in k.

160. Temperature Effects in an Air Capacitor.—The effect of temperature changes on the capacitance of an air capacitor can be estimated from the properties of the materials used in the construction. In any capacitor in which all the materials used in its construction have identical coefficients of linear expansion, the relative change in capacitance per degree is the same as the common coefficient of linear expansion. Such a condition may very nearly exist in a cylindrical capacitor, since the outer and inner cylinders should not be constrained. There is, however, an additional temperature effect resulting from the change in width of the gap, which in turn is caused by the expansion of the insulating material. In general, this latter will be small, so that only the expansion of the cylindrical electrodes need be discussed. Since materials suitable for electrodes have coefficients of linear expansion varying from ten to twenty parts per million per degree centigrade, the temperature must be measured or controlled to 0.1°C or 0.05°C. In addition, temperature gradients must be very small. If the outer cylinder *only* should change in temperature, the distance between cylinders, and hence the capacitance, would change much more than when both cylinders change by the same amount.

A change in temperature of a parallel-plate capacitor changes the areas of the plates and the distance between them. The change in area can be computed from the coefficient of linear expansion of the material of the guarded plate. The distance between the plates is often determined by some insulating material, frequently by three small cylinders of equal length which act as spacers between the guard plate and the opposite plate. If the spacers have a linear expansion which is double that of the plates, the temperature coefficient of capacitance is zero. This condition can be approached, as by steel (11×10^{-6}) and fluorspar (19.5×10^{-6}) or by nickel (13.8×10^{-6}) and Iceland spar cut parallel to axis (26.3×10^{-6}). However, in such a capacitor the temperature gradient must be very small, which can be accomplished only by keeping the instrument at a very constant temperature.

161. Systematic Errors Inherent in the Method.—An important systematic error in the capacitance method of obtaining the ratio of the electrical units is the error in the value of the resistance unit. The method assumes that the resistance unit of the laboratory where the measurements are made is the c.g.s. unit. However, the equation for computing the ratio contains only the square root of the resistance, so that the error in the ratio is only one-half the error in the resistance unit.

Other systematic errors are those connected with the dielectric constant and susceptibility of air. The former has just been discussed. The latter is of the order of 10^{-8} and so is entirely negligible.

162. The Velocity of Light.—The velocity of light in a vacuum is, according to the electromagnetic theory, equal to the ratio of the electromagnetic unit of electricity to the electrostatic unit. This equality has been generally accepted for the last half century, but has been experimentally verified only to an accuracy of about 100 parts in a million. Recently there have been several new determinations of the velocity of light, but the last determination of the ratio of the units was completed more than twenty-five years ago and claimed an accuracy of only 100 parts in a million. With apparatus that can now be constructed, a result accurate to ten or twenty parts in a million should be attainable.

APPENDIX I

COMPUTATION OF ELLIPTIC INTEGRALS

The elliptic integrals are of three kinds, designated as $F(k,\varphi)$; $E(k,\varphi)$; and $\Pi(n,k,\varphi)$ for the first, second, and third kinds, respectively. The integrals defining these are

$$F(k,\varphi) = \int_0^\varphi \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}$$

$$E(k,\varphi) = \int_0^\varphi \sqrt{1 - k^2 \sin^2 \theta}\, d\theta$$

$$\Pi(n,k,\varphi) = \int_0^\varphi \frac{d\theta}{(1 + n \sin^2 \theta)\sqrt{1 - k^2 \sin^2 \theta}}$$

In these integrals k is called the modulus, φ the amplitude, and n the parameter. If the upper limit of integration is $\pi/2$, so that $\varphi = \pi/2$, the integrals are called *complete elliptic integrals* and have been given special designations; thus

$$F\left(k, \frac{\pi}{2}\right) = K \text{ (or sometimes } F\text{)}$$

$$E\left(k, \frac{\pi}{2}\right) = E$$

$$\Pi\left(n,k, \frac{\pi}{2}\right) = \Pi$$

If $\varphi < \pi/2$, the integrals are called *incomplete elliptic integrals.*

Each of the elliptic integrals can be expanded into a series of ascending powers of $k^2 \sin^2 \theta$ and integrated term by term. However, the convergence of the resulting series is so slow that this is seldom a practical method of determining their numerical value. The complete integrals of the first and second kinds for values of k from 0 to 1 are given in various tables,[1] but the

[1] The most recent tables are by Nagaoka and Sakurai, *Sci. Pap. Inst. Phys. Chem. Research,* Table 1, 1922. In this table values of K and E are given for 1000 values of k^2 equally spaced between 0 and 1. These are more convenient than the tables of Legendre published in 1825, in which K and E are given for values of α, where $\alpha = \sin^{-1} k$.

computation is not difficult and is necessary when an accuracy greater than that given in the tables is required. The incomplete integrals are also tabulated[1] but the labor involved in making a double interpolation in the tables is nearly as great as making a computation.

Integrals of the first and second kinds can be computed[2] by constructing a table of values as given in Table XIV.

The method of computing the elliptic integrals by this table is not satisfactory if $b_0 < 0.1$. However, cases arising in absolute electrical measurements in which $b_0 < 0.1$ are so rare that it is not necessary to outline here the method by which they can be computed.

The incomplete integral of the third kind has not arisen in any formula and so will not be treated. The complete integral of the third kind can be expressed in terms of the complete and incomplete integrals of the first and second kind, the formula depending on the value of the parameter n relative to the modulus k. The only case arising in the formulas given in this treatise is when n is negative and has a value between $-k^2$ and -1. For convenience let $c^2 = -n$ so that c is a positive number, less than unity. Then if $\tan \varphi = \sqrt{\dfrac{1 - c^2}{c^2 - k^2}}$ and $k' = \sqrt{1 - k^2}$,

$$\Pi = K + \frac{c}{\sqrt{(1 - c^2)(c^2 - k^2)}}\left[\frac{\pi}{2} + (K - E)F(k',\varphi) + K \cdot E(k',\varphi)\right]$$

As n is a negative quantity, having its absolute value between 1 and k^2, all the square roots are real quantities. Also it should be noted that the complete integrals K and E are to modulus k, whereas the incomplete integrals, $F(k',\varphi)$ and $E(k',\varphi)$, are to modulus k' and amplitude φ determined by the relation given above. Hence, to compute an elliptic integral of the third kind, two tables of values are required. The first is the $a_0 = 1$, $b_0 = k'$, $c_0 = k$ as shown, but in this table the ψ's do not need to be determined since only complete integrals are required. The second table, complementary to the first, starts with $a_0' = 1$,

[1] Published by Legendre in 1825. A photographic reproduction was published by Emde in 1931, and also by Cambridge University Press in 1934.

[2] For the theory of the method here given and its application to all possible cases, see King, "The Direct Numerical Calculation of Elliptic Functions and Integrals," Cambridge University Press, 1924.

TABLE XIV—TABLE FOR COMPUTING THE COMPLETE AND INCOMPLETE ELLIPTIC INTEGRALS OF THE FIRST AND SECOND KINDS

$$
\begin{array}{llll}
a_0 = 1 & b_0 = \sqrt{1 - k^2} = k' & c_0 = k & \psi_0 = \varphi \\
a_1 = \tfrac{1}{2}(a_0 + b_0) & b_1 = \sqrt{a_0 b_0} & c_1 = \tfrac{1}{2}(a_0 - b_0) & \psi_1 = \left[\psi_0 + \tan^{-1}\left(\frac{b_0}{a_0}\tan\psi_0\right)\right] \\
a_2 = \tfrac{1}{2}(a_1 + b_1) & b_2 = \sqrt{a_1 b_1} & c_2 = \tfrac{1}{2}(a_1 - b_1) & \psi_2 = \left[\psi_1 + \tan^{-1}\left(\frac{b_1}{a_1}\tan\psi_1\right)\right] \\
\cdots & \cdots & \cdots & \cdots \\
a_n = \tfrac{1}{2}(a_{n-1} + b_{n-1}) & b_n = \sqrt{a_{n-1} b_{n-1}} & c_n = \tfrac{1}{2}(a_{n-1} - b_{n-1}) & \psi_n = \left[\psi_{n-1} + \tan^{-1}\left(\frac{b_{n-1}}{a_{n-1}}\tan\psi_{n-1}\right)\right]
\end{array}
$$

Of all the angles that can be represented by $\tan^{-1}\left(\frac{b_{n-1}}{a_{n-1}}\tan\psi_{n-1}\right)$, the one to be chosen is the one that is smaller than ψ_{n-1} by an amount less than $\pi/2$ radians.

This table is continued until $a_n = b_n$, so that $c_{n+1} = 0$ and $\psi_{n+1}/2^{n+1} = \psi_n/2^n$. These conditions are reached with very few terms, usually $a_3 = b_3$, unless b_0 is very small. The elliptic integrals are given by the equations

$$K = \frac{\pi}{2a_n}$$

$$F(k,\varphi) = \frac{\psi_n}{a_n 2^n}$$

$$E = K[1 - \tfrac{1}{2}(c_0^2 + 2c_1^2 + 4c_2^2 + \cdots + 2^n c_n^2 + \cdots)]$$

$$E(k,\varphi) = \left(\frac{E}{K}\right)F(k,\varphi) + c_1 \sin^2\psi_1 + c_2 \sin^2\psi_2 + \cdots + c_n \sin\psi_n + \cdots$$

$b_0' = k$, $c_0' = k'$, $\psi_0' = \tan^{-1}\sqrt{\frac{1-c^2}{c^2-k^2}}$ and is continued in the same manner as shown in Table XIV.

As an example, compute the complete elliptic integrals[1] to modulus $k = 0.7866814$ and parameter $n = -0.9438574$. Tabulating these constants and those directly derived from them,

$$\begin{aligned}
k &= 0.7866814 \\
k^2 &= 0.6188676 \\
k'^2 &= 1 - k^2 = 0.3811324 \\
k' &= 0.6173590 \\
c^2 &= 0.9438574 \\
c &= 0.9715233 \\
c'^2 &= 1 - c^2 = 0.0561426 \\
c' &= 0.2369443 \\
c^2 - k^2 &= 0.3249898 \\
\sqrt{c^2 - k^2} &= 0.5700709 \\
\varphi &= \tan^{-1}\frac{c'}{\sqrt{c^2-k^2}} = 22^\circ\ 34'\ 10''
\end{aligned}$$

To obtain K and E, construct the set of values for a_n, b_n, and c_n as shown in Table XIV. Since only the complete integrals are to be computed, values of ψ are not required. The values are:

$a_0 = 1$	$b_0 = 0.6173590$	$c_0 = 0.7866814$
$a_1 = 0.8086795$	$b_1 = 0.7857220$	$c_1 = 0.1913205$
$a_2 = 0.7972008$	$b_2 = 0.7971180$	$c_2 = 0.0114788$
$a_3 = 0.7971594$	$b_3 = 0.7971594$	$c_3 = 0.0000414$

Then

$$K = \frac{\pi}{2a_3} = 1.970492$$

and

$$E = K[1 - \tfrac{1}{2}(c_0^2 + 2c_1^2 + 4c_2^2 + \cdots)]$$

$$E = K\left[1 - \frac{1}{2}\begin{array}{|l|}0.6188676\\0.0732069\\0.0005271\\\hline 0.6926016\end{array}\right]$$

$$E = 1.288109$$

In order to obtain Π, it is necessary to compute the complete and incomplete elliptic integrals of both the first and second

[1] Constants used by Smith in computing M_{θ_2}. *N.P.L. Coll. Researches*, **4**, 1 (1908).

kinds to modulus k'. The incomplete integrals are to amplitude φ. For this computation, the following set of values is required:

$$\begin{array}{lll} a_0' = 1 & b_0' = k = 0.7866814 & c_0' = k' = 0.6173590 \\ a_1' = 0.8933407 & b_1' = 0.8869504 & c_1' = 0.1066593 \\ a_2' = 0.8901456 & b_2' = 0.8901399 & c_2' = 0.0031950 \\ a_3' = 0.8901428 & b_3' = 0.8901427 & c_3' = 0.0000028 \end{array}$$

$$\psi_0' = \varphi = 22^\circ\ 34'\ 10.0''$$

$$\frac{\psi_1'}{2} = 20^\circ\ 20'\ 16.3''$$

$$\frac{\psi_2'}{4} = 20^\circ\ 17'\ 13.8''$$

$$\frac{\psi_3'}{8} = 20^\circ\ 17'\ 13.6''$$

Then
$$K(k') = \frac{\pi}{2a_3'} = 1.764657$$

$$F(k',\varphi) = \frac{\psi_3' \text{ radians}}{8a_3'} = \frac{20^\circ\ 17'\ 13.6''}{180^\circ} \cdot \frac{\pi}{a_3'} = 0.3977754$$

$$E(k') = K(k')[1 - \tfrac{1}{2}(c_0'^2 + 2c_1'^2 + 4c_2'^2 + \cdots)]$$

$$= K(k')\left[1 - \frac{1}{2}\left|\begin{array}{l} 0.3811324 \\ 0.0227524 \\ 0.0000409 \\ \hline 0.4039257 \end{array}\right.\right]$$

$$= 1.408262$$

$$E(k',\varphi) = \frac{E(k')}{K(k')}F(k',\varphi) + c_1' \sin \psi_1' + c_2' \sin \psi_2' + c_3' \sin \psi_3' + \cdots$$

$$= \begin{array}{l} 0.3174395 \\ 0.0695181 \\ 0.0031570 \\ 0.0000008 \\ \hline \end{array}$$

$$= 0.3901154$$

$$\Pi(k,n) = K + \frac{c}{c'\sqrt{c^2 - k^2}}\left[\frac{\pi}{2} + (K - E)F(k',\varphi') - K \cdot E(k',\varphi')\right]$$

$$= K + 7.192470\left|\begin{array}{l} +1.5707963 \\ +0.2714352 \\ -0.7687191 \\ \hline 1.0735124 \end{array}\right.$$

$$= 9.691697$$

APPENDIX II

DETERMINATION OF THE EFFECT OF AN ERROR IN A MEASURED DIMENSION ON A COMPUTED INDUCTANCE AND ON A COMPUTED ELECTRODYNAMIC FORCE

Any equation for the computation of the inductance of an inductor from its dimensions can be written in the form

$$M = f(a,b,c,\text{ etc.}) \tag{1}$$

where M = inductance (either self or mutual).
a, b, c, etc. = the measured dimensions.
f = a homogeneous function of the first degree.

The differential of this function is

$$dM = \frac{\partial f}{\partial a}da + \frac{\partial f}{\partial b}db + \frac{\partial f}{\partial c}dc + \cdots \tag{2}$$

Dividing by M, replacing f by its identity M, and multiplying the successive terms of the second member by a/a, b/b, c/c, etc.,

$$\frac{dM}{M} = \frac{a}{M}\frac{\partial M}{\partial a}\cdot\frac{da}{a} + \frac{b}{M}\frac{\partial M}{\partial b}\cdot\frac{db}{b} + \frac{c}{M}\frac{\partial M}{\partial c}\cdot\frac{dc}{c} + \cdots \tag{3}$$

This equation is very useful in determining the effect on the computed inductance of an error in any one of the measured dimensions on the inductance. The coefficients of da/a, db/b, dc/c, etc., can all be computed from the equation for M, since it is always possible to obtain the partial derivatives.

As an example of the application of this formula, assume that

$$M = a^2 f(b,c) \tag{4}$$

where b and c are constants.

Then
$$\frac{dM}{da} = 2af(b,c) \tag{5}$$

and
$$\frac{a}{M}\frac{dM}{da} = 2$$

so that

$$\frac{dM}{M} = 2\frac{da}{a} \tag{6}$$

Hence an error of one part in a million in a (expressed mathematically as $da/a = 10^{-6}$) will introduce an error of two parts in a million in M.

The determination of the coefficients in Eq. (3) is often laborious, and a check is useful. This can be obtained by employing Euler's theorem for homogeneous functions. Applying this theorem to Eq. (1), which is homogeneous and of the first degree,

$$a\frac{\partial M}{\partial a} + b\frac{\partial M}{\partial b} + c\frac{\partial M}{\partial c} + \cdots = M \tag{7}$$

Dividing by M,

$$\frac{a}{M}\frac{\partial M}{\partial a} + \frac{b}{M}\frac{\partial M}{\partial b} + \frac{c}{M}\frac{\partial M}{\partial c} + \cdots = 1 \tag{8}$$

The terms in Eq. (8) are the coefficients of the terms in Eq. (3). Hence, in any equation which expresses the relative error in the inductance as the sums of the relative errors of the dimensions, the sum of the coefficients of the terms which give the dimensional errors is unity. This summation to unity applies whether the coefficients are expressed algebraically or numerically.

Similarly the electrodynamic force F between conductors carrying unit current can be represented by the equation

$$F = \varphi(a,b,c, \text{etc.}) \tag{9}$$

where φ is a homogeneous function of zero degree of the dimensions a,b,c, etc. By taking the differential and dividing by F,

$$\frac{dF}{F} = \frac{a}{F}\frac{\partial F}{\partial a}\frac{da}{a} + \frac{b}{F}\frac{\partial F}{\partial b}\frac{db}{b} + \frac{c}{F}\frac{\partial F}{\partial c}\frac{dc}{c} + \cdots \tag{10}$$

Applying Euler's theorem to Eq. (9),

$$a\frac{\partial F}{\partial a} + b\frac{\partial F}{\partial b} + c\frac{\partial F}{\partial c} + \cdots = 0$$

Hence the sum of the coefficients of da/a, db/b, dc/c, etc., is zero.

APPENDIX III

RESULTS OF ABSOLUTE DETERMINATIONS OF ELECTRICAL UNITS

Absolute determinations of the ohm and ampere are now (1936) in progress in the national laboratories of England, Germany, Japan, and the United States. The only published descriptions of this experimental work are those[1] from the National Bureau of Standards of the United States. However, brief statements of preliminary values have been published by other laboratories.

The latest values of the ohm (March, 1936) and the method used in obtaining them are given below:

1 N.B.S. international ohm = 1.00045 absolute ohms (self inductance with intermediary capacitance).

1 N.P.L. international ohm[2] = 1.00049 absolute ohms (Lorenz method).

1 N.P.L. international ohm[2] = 1.00047 absolute ohms (mutual inductance with intermediary capacitance).

1 E.T.L. international ohm[3] = 1.00046 absolute ohms (same as preceding).

The abbreviations are: N.B.S. for the National Bureau of Standards of the United States, N.P.L. for the National Physical Laboratory of England, and E.T.L. for the Electrotechnical Laboratory of Japan.

In order to compare the results of different laboratories, a selection must be made of a working unit of resistance in terms of which all the different results can be expressed. The International Bureau of Weights and Measures (abbreviated B.I.P.M. for Bureau International des Poids et Mesures) made in 1935 an intercomparison of resistance standards furnished them by the

[1] Curtis and Curtis, An Absolute Determination of the Ampere, *B.S. J. Research*, **12**, 665 (1934). Curtis, Moon, and Sparks, An Absolute Determination of the Ohm, *J. Research N.B.S.*, **16**, 1 (1936).

[2] Annual Report of the National Physical Laboratory for 1934, p. 57.

[3] *Procès-verbaux, Comité International des Poids et Mesures*, **16**, 28 (1933).

six recognized national laboratories. By means of the results of this intercomparison and the value of each standard in terms of the unit of the laboratory which furnished it, the value of the resistance unit in each of the national laboratories was determined in terms of the mean of all the laboratories. The values[1] as given below are adjusted to be correct as of Mar. 15, 1935.

Country	Values of the Resistance Units of the Different National Laboratories in Terms of the Mean of All Laboratories
England	0.999996_4
France	1.000000*
Germany	1.000009_8
Japan	0.999988_8
Russia	1.000010_6
United States	0.999994_5

* At the time of this intercomparison France changed its unit to make it agree with the mean of the five other countries.

The results of the absolute determinations can be expressed in terms of the mean international ohm, giving the following values:

1 mean international ohm = 1.00046 absolute ohms (United States)
= 1.00049 } absolute ohms (England)
= 1.00047 }
= 1.00047 absolute ohms (Japan)

There have been published only two recent values of the absolute ampere. These are:

1 N.B.S. international ampere = 0.99993 absolute ampere (Rayleigh balance[2]).

1 N.P.L. international ampere = 0.99988 absolute ampere (Ayrton-Jones balance[3]).

In order to intercompare these values, it is necessary to establish a common unit of electromotive force, as well as to make use of the common unit of resistance. The International

[1] *Procès-verbaux, International Committee of Weights and Measures*, **17**, 290 (1935).

[2] *B.S. J. Research*, **12**, 665 (1934).

[3] Annual Report of the National Physical Laboratory for 1934, p. 57.

Bureau[1] has determined as of Dec. 10, 1934, the values of the units of electromotive force of the national laboratories in terms of the mean of the six laboratories. These are:

Country	Values of the Electromotive Force Units of the Different National Laboratories in Terms of the Mean of All Laboratories
England	1.000005
France	1.000000*
Germany	0.999996
Japan	0.999998
Russia	1.000013
United States	0.999986

* The unit of electromotive force of France was changed to agree with the mean of the other laboratories.

The absolute values of the ampere in terms of the mean units of resistance and electromotive force are:

1 mean international ampere = 0.99994 absolute amperes (United States).
= 0.99987 absolute amperes (England).

The International Committee of Weights and Measures has been given authority over the electrical units by a treaty signed by practically all civilized nations. This committee has decided to change from the international electrical units to the absolute units on Jan. 1, 1940. The national laboratories are attempting to increase the accuracy of absolute electrical measurements so that the differences in the results of the different laboratories will be only one or two parts in a hundred thousand.

[1] *Procès-verbaux, International Committee of Weights and Measures,* **17, 300** (1935).

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