Monotone Iterative Techniques for a Class of Nonlinear Boundary Value Problems

THESIS

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BY

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Under the Supervision of

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CERTIFICATE

This is to certify that the thesis entitled **"Monotone Iterative Techniques for a Class of Nonlinear Boundary Value Problems"** submitted by **Mr. Mandeep Singh, ID No. 2011PHXF034P** for award of Ph.D. Degree of the Institute embodies original work done by him under my supervision.

Signature of the Supervisor

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Dedicated to

my loving parents & my beloved nephew

Anshdeep

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Abstract

This thesis deals with second order nonlinear boundary value problems. We have considered both continuous as well as discrete boundary value problems. In continuous case, we examine both analytical and numerical methods, while in discrete case, we discuss analytical results. In analytical approach, monotone iterative methods are developed for nonlinear three point nonsingular/singular boundary value problems and nonlinear two point discrete boundary value problem, respectively. Under the existence of upper and lower solutions, we establish the analytical results for both cases. In numerical approach, we present two methods and solve the nonlinear two point singular boundary value problems, which arise in real life. We focus on variational iteration method (VIM), and homotopy perturbation method (HPM).

This thesis contains twelve chapters. It commences with introduction which is our chapter 1, then eleven chapters 2–12 and a bibliography section. In chapter 1, we discuss briefly about boundary value problems and show how the problems get more complicated when we deal with nonlinear three point boundary value problems. A brief introduction of monotone iterative method with upper and lower solutions are given. Further a survey of literature is given to provide a platform required for the forthcoming chapters. In chapters 2–5, nonlinear nonsingular boundary value problems are studied along with mixed type, Neumann type and Dirichlet type boundary conditions. In chapters 6–9, we consider the nonlinear singular boundary value problems with three point boundary conditions. In all cases, we make use of monotone iterative method with the support of upper and lower solutions to establish the existence results. Mostly, we prove existence results for two cases, i.e., when upper and lower solutions follow well order relation or reverse order relation. In chapter 10 & chapter 11, we study the numerical results for nonlinear two point singular boundary value problems by using variational iteration method (VIM) and homotopy perturbation method (HPM). Finally in chapter 12, by using the concept of monotone iterative method with upper and lower solutions, the existence results for nonlinear two point discrete boundary value problem are discussed.

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Nomenclature

$\in$	Belongs to
I	$[0,1]$
BVPs	Boundary value problems
SBVPs	Singular boundary value problems
$C(I)$	Set of continuous functions on I
$C^k(I)$	Set of functions with continuous kth derivative on I
L	Second order differential operator
$\mathbb{R}$	The set of real numbers
$\mathbb{R}\setminus\{0\}$	The set of real numbers except 0
$\mathbb{R}^+$	The set of non-negative real numbers
$\mathbb{N}$	The set of natural numbers
$G(t,s)$ or $G(x,t)$	Green's function
β_n or u_n	Upper solutions
α_n or v_n	Lower solutions
$\|\cdot\|_\infty$	Infinity norm
$J_\nu(x)$	Bessel function of first kind
$Y_\nu(x)$	Bessel function of second kind
$I_\nu(x)$	Modified Bessel function of first kind
$K_\nu(x)$	Modified Bessel function of first kind
$H_n(v_0,\cdots,v_n)$	He's polynomial
Δ	The forward difference operator

Chapter 1

Introduction

1.1 Introduction

The rate of change whether it is continuous or discrete is a natural phenomena of the universe, e.g., growth of plants, variation in investment market, population growth of species, or expansion of universe. According to domain either, we get a differential equation or difference equation and both are of great importance in pure as well as applied mathematics. Differential and difference equations are one of the most useable models which arise very frequently in various branches of modern science and engineering.

Boundary value problems are a combination of differential (or difference) equation and certain conditions defined at the boundary of the domain. These boundary conditions play a very crucial role, as minor modification in boundary condition can change the solution drastically.

In the present work, we have studied nonlinear boundary value problems for both cases, continuous as well as discrete. In continuous case, we discuss analytical and numerical solutions for three point and two point nonlinear boundary value problems, respectively. In discrete case, we deal with analytical results for a class of nonlinear two point discrete boundary value problems.

1.2 Boundary value problems (BVPs)

Consider the second order differential equations of the following form

$$y''(t) + f(t, y, y') = 0, \quad t \in [a, b]. \tag{1.1}$$

If the solution $y(t)$ of the differential equation (1.1) on the interval $[a,b]$, has a specific value and slope at any point $t_0 \in [a,b]$ given as

$$y(t_0) = Constant, \quad y'(t_0) = Constant, \tag{1.2}$$

then such problem is called Initial value problem. However, when the conditions are prescribed at the two endpoints of interval, then it is called two point boundary value problem. On the basis of boundary conditions we can classify the BVPs in the following way

$$\begin{aligned}
\text{Dirichlet or First kind} &: \quad y(a) = \xi_1, \quad y(b) = \xi_2, \\
\text{Neumann or Second kind} &: \quad y'(a) = \xi_1, \quad y'(b) = \xi_2, \\
\text{Robin or Mixed kind} &: \quad \alpha_1 y(a) + \alpha_2 y'(a) = \xi_1, \\
&\quad\;\; \beta_1 y(b) + \beta_2 y'(b) = \xi_2, \\
\text{Periodic} &: \quad y(a) = y(b), \quad y'(a) = y'(b).
\end{aligned}$$

In differential equation, the theory of linear/nonlinear (Nonsingular (Regular)/Singular) boundary value problems are of great importance. In comparison to initial value problem, the theory of boundary value problem is substantially more complicated because of its totally different nature of the underlying physical process. For boundary value problems, existence of one and only one solution on any sufficiently small interval $[a,b]$ is guaranteed, if $f(t,y,y')$ is continuous in (t,y,y') and satisfy the Lipschitz condition, which is defined as

$$\left| f(t,y,y') - f(t,x,x') \right| \leq K|y-x| + L|y'-x'|, \tag{1.3}$$

where K and L are two non negative Lipschitz constant, while the functions $f(t,y,y')$ and $f(t,x,x')$ are defined in the domain of f. However, in the case of large intervals, existence and uniqueness of the solution may fail, for both linear problem and nonlinear problem. We refer the work of Bailey et al. [18].

1.2.1 Existence and uniqueness : Linear BVPs

Consider the linear boundary value problem

$$y''(t) + y(t) = 0, \tag{1.4}$$

$$y(0) = 0, \tag{1.5}$$

$$y(b) = B. \tag{1.6}$$

Here $f(t,y,y') = y(t)$. It is clear that $f(t,y,y')$ is continuous and satisfies the Lipschitz condition (1.3) with $K = 1$, $L = 0$.

The general solution of the differential equation (1.4) is

$$y(t) = C_1 \cos t + C_2 \sin t.$$

Now making use of the boundary condition (1.5), we obtain

$$y(t) = C_2 \sin t, \quad 0 < t < b.$$

It is not always possible to choose such a value of C_2 so that boundary condition (1.6) holds. One can easily observe the following

(i) When $b \neq n\pi$, n being an integer, then we receive a unique value of C_2, such that $C_2 \sin b = B$, i.e., there exists a unique solution of the boundary value problem.

(ii) When $b = n\pi$ and $B \neq 0$, there is no solution.

(iii) When $B = 0$, every value of C_2 gives a solution, i.e., we receive an infinite number of solution.

From above discussion, it is clear that for a fixed a, the existence and uniqueness of linear problem fails on the interval $[a,b]$ for certain notable value of b.

Now we can see that even in the linear case a lot of problems occur due to the length of interval. Now one can imagine the degree of complexity, which might occur if the problem is nonlinear.

1.2.2 Existence and uniqueness : Nonlinear BVPs

Consider the nonlinear boundary value problem

$$y''(t) + |y(t)| = 0, \tag{1.7}$$

$$y(0) = 0, \tag{1.8}$$

$$y(b) = B. \tag{1.9}$$

Here $f(t,y,y') = |y|$ is continuous and satisfies the Lipschitz condition (1.3) with $K = 1$, $L = 0$.

We can rewrite the differential equation (1.7) in the following way

$$y''(t) - y(t) = 0, \qquad \text{when } y(t) \leq 0, \tag{1.10}$$

and

$$y''(t) + y(t) = 0, \qquad \text{when } y(t) \geq 0. \tag{1.11}$$

Equation (1.10) helps us in showing that a solution of differential equation (1.7) has at most

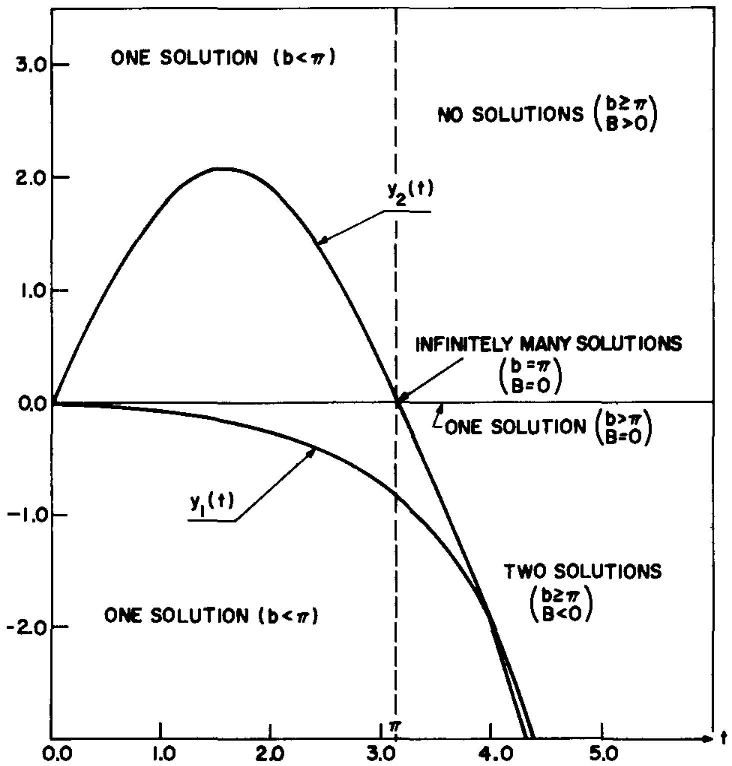

Fig. 1.1 The number of solutions of the boundary value problem (1.7)–(1.9) depends upon the magnitude of b and sign of B (see [18])

one zero if it has a negative slope at zero. Suppose $y(t_0) = 0$ and $y'(t_0) < 0$, then $y(t) < 0$ for all $t \in [t_0, t_1)$. If $t_1 = +\infty$, then $y(t)$ remains a solution of (1.10), at least until it has another zero, but it cannot have another zero since solutions of (1.10) have one zero at most. Now using the similar analysis, we can state that any solution $y(t)$ of (1.7), such that $y'(t_0) > 0$, has a second zero at $t_0 + \pi$.

Taking into the account of above discussion, it is clear that the nonlinear boundary value problem (1.7) may be solved by using the solution of (1.10) or (1.11).

Now, if $b < \pi$, we define $y(t)$ as

$$y(t) = \begin{cases} C\sin t, & \text{if } B > 0 \\ -C\sinh t, & \text{if } B < 0 \\ 0, & \text{if } B = 0. \end{cases} \tag{1.12}$$

where C is any nonnegative constant. Here $y(t)$ is a solution of (1.10) or (1.11) and hence, it is a solution of (1.7) and the length of interval is too small, for containing the second zero.

Further, making use of boundary condition (1.9), we get a unique number C for different sign of B, such that

$$C = \begin{cases} B[\sin b]^{-1}, & \text{if } B > 0 \\ -B[\sinh b]^{-1}, & \text{if } B < 0. \end{cases} \tag{1.13}$$

Hence, for $b < \pi$, the nonlinear boundary value problem has a unique solution.

If $b = \pi$, and $B = 0$, then equation (1.11) gives an infinite number of solution, while if $B > 0$, there is no solution at all.

If $B < 0$, then we get one and only one solution of boundary value problem defined as

$$y(t) = B[\sinh \pi]^{-1} \sinh t.$$

In case when $b > \pi$, the situation is quite interesting. For $B > 0$, all solutions with positive slope at $t = 0$, have a zero at π also, i.e., it must remain negative for $t > \pi$ at least until it has another zero. Hence, for $B > 0$ there is no solution at all.

For $B = 0$, we get a unique solution $y \equiv 0$, because there is no nontrivial solution, which has a zero at $t = 0$ and has a zero at $t > \pi$. Now, if $B < 0$, we receive exactly two solutions, satisfying equations (1.7), (1.8) and (1.9). One solution is defined as

$$y_1(t) = B[\sinh b]^{-1} \sinh t, \tag{1.14}$$

while other solution is apparent, as

$$y_2(t) = \begin{cases} C_1 \sin t, & \text{if } 0 \le t \le \pi \\ C_2[\sinh t - \tanh \pi \cosh t], & \text{if } \pi \le t \le b, \end{cases} \tag{1.15}$$

where, for $0 \le t \le \pi$, solution satisfies equation (1.11) and equation (1.10) for $\pi \le t \le b$ and has a zero at π. Now we choose C_2, in terms of B, such that boundary condition (1.9) at $t = b$ is satisfied, here we choose C_2 as

$$C_2 = B[\sinh b - \tanh \pi \cosh b]^{-1}. \tag{1.16}$$

We choose C_1 such that first derivative of $y(t)$ is continuous at $t = \pi$, to accomplish this, we choose

$$C_1 = -B[\sinh(b - \pi)]^{-1}. \tag{1.17}$$

Hence, we can rewrite the second solution of boundary value problem in the following form

$$y_2(t) = \begin{cases} -B[\sinh(b-\pi)]^{-1}\sin t, & \text{if } 0 \le t \le \pi \\ B[\sinh(b-\pi)]^{-1}\sinh(t-\pi), & \text{if } \pi \le t \le b. \end{cases} \tag{1.18}$$

To recapitulate the above argument, for $b < \pi$ the nonlinear boundary value problem (1.7)–(1.9) has unique solution for every value of B, while for $b \ge \pi$, the problem has either unique solution, no solution, or more than one solution, depending upon the value of B.

Now we can see that, when we shift from linear to nonlinear BVPs, the existence theory becomes more complicated. If it is a nonlinear BVPs, which is singular also, one can imagine how complicated the theory will be.

The theory of singular boundary value problems (SBVPs) is a more sophisticated and challenging. It always remains at the center of attraction for researcher due to behaviour of its solution in the neighbourhood of singular point, e.g., solution often becomes large in magnitude or experiences rapid change in magnitude or might be peculiar in some other manner. Thus, the study of the behaviour of physical systems, which are governed by SBVPs is essential. Also, the singular point of differential equation may also arise due to geometric singularities such as corners or sharp edges. Thus it is important and necessary to study the behaviour of solution most carefully.

In next section, we discuss about real life applications, which are modeled by nonlinear boundary value problems.

1.3 Real life applications

Nonlinear boundary value problems arise frequently in many branches of engineering, applied mathematics, astronomy, biological system, modern science etc. Few of them are briefly discussed in this section.

1.3.1 Oxygen diffusion in a spherical cell

In 1976, Lin [95] analyzed the oxygen diffusion in a spherical cell with the support of an oxygen uptake kinetics of the Michaelis-Menten type. In addition, he has used an unsteady state oxygen diffusion model. As the metabolic reaction in a cell are catalyzed by enzymes, he represented the oxygen uptake kinetics by the Michaelis-Menten equation

$$\text{oxygen uptake} = \frac{VP}{P + k_m},$$

where V is the maximum reaction rate, P the oxygen tension, and k_m, the Michaelis-Menten constant.

Mathematically, in a spherical cell, the unsteady state oxygen diffusion can be denoted by the following equation

$$\frac{\partial P}{\partial t} = D\left(\frac{\partial^2 P}{\partial r^2} + \frac{2}{r}\frac{\partial P}{\partial r}\right) - \frac{VP}{P + k_m}, \tag{1.19}$$

with initial and boundary conditions

$$t = 0; \qquad P = 0, \tag{1.20}$$

$$r = 0; \qquad \frac{\partial P}{\partial r} = 0, \tag{1.21}$$

$$r = r_0; \qquad D\frac{\partial P}{\partial r} = h(P_0 - P), \tag{1.22}$$

where D is the diffusion coefficient of oxygen in the protoplasm, r_0 the radius of cell, h the permeability of membrane, r the radial co-ordinate, and t the time. We can reduce, equation (1.19) and initial and boundary conditions (1.20)–(1.22) into dimensionless form for computational point of view, by introducing the following dimensionless variables and parameters

$$C = \frac{P}{P_0}, \quad \tau = \frac{tD}{r_0^2}, \quad R = \frac{r}{r_0}, \tag{1.23}$$

$$\alpha = \frac{V r_0^2}{P_0 D}, \quad K_m = \frac{k_m}{P_0}, \quad H = \frac{h r_0}{D}. \tag{1.24}$$

Equations (1.19) to (1.22) are then to transformed into

$$\frac{\partial C}{\partial \tau} = \frac{\partial^2 C}{\partial R^2} + \frac{2}{C}\frac{\partial C}{\partial R} - \frac{\alpha C}{C + K_m} \tag{1.25}$$

subject to

$$\tau = 0; \qquad C = 0, \tag{1.26}$$

$$R = 0; \qquad \frac{\partial C}{\partial R} = 0, \tag{1.27}$$

$$R = 1; \qquad D\frac{\partial C}{\partial R} = H(1 - C). \tag{1.28}$$

In 1978, McElwain [104] examined steady state, equations (1.25)–(1.28) given by,

$$\frac{d^2C}{dR^2} + \frac{2}{C}\frac{dC}{dR} - \frac{\alpha C}{C + K_m} = 0, \tag{1.29}$$

$$R = 0; \qquad \frac{dC}{dR} = 0, \tag{1.30}$$

$$R = 1; \qquad D\frac{dC}{dR} = H(1 - C). \tag{1.31}$$

1.3.2 Thermal explosion

In the theory of thermal explosions critical condition for inflammability is reached when the amount of heat developed by the chemical reaction is just equal to the amount lost to the surroundings.

Studies (see [31] and references there in) reveal that the loss of heat to the vessel walls, which must be in balance with the chemical heat liberation, takes place entirely by conduction inside the gas volume. For this limiting case of pure conduction the theory should yield the explosion limits from the kinetics of the reaction, the heat of reaction, the thermal conductivity of the gaseous mixture, and the form and dimensions of the vessel.

Chamber [31], considered the following equation, which relates the heat generated by the chemical reaction and that conducted away

$$\lambda \nabla^2 T = -QW, \tag{1.32}$$

where T is the gas temperature, Q the heat of reaction, λ the thermal conductivity, W the reaction velocity, and ∇^2 the Laplacian operator. P. L. Chamber considered this chemical reaction as monomolecular, and he further assumed that its velocity follows the Arrhenius law, i.e.,

$$W = ca \exp(-E/RT), \tag{1.33}$$

where c is the concentration of the reactant, a the frequency factor, and E the energy of activation of the reaction. Hence, from equation (1.32), we have

$$\nabla^2 T = -(Q/\lambda) ca \exp(-E/RT). \tag{1.34}$$

If $(T - T_0)$ is the maximum temperature increment, then

$$\theta = (E/RT_0{}^2)(T - T_0), \tag{1.35}$$

where T_0 is the temperature of walls of the vessel. Hence, equation (1.34) becomes

$$\nabla^2\theta = -\left[\frac{Q}{\lambda}\frac{E}{RT_0{}^2}ca\exp(-E/RT_0)\right]\exp(\theta). \tag{1.36}$$

Since the theory is concerned for geometries, where the conduction process depends on only one space coordinate (say x). If we replace, the space coordinate x by $z = x/r$, where r is the significant geometric dimension of the vessel, then we can get dimensionless Laplacian operator. There results then finally the Poisson-Boltzmann equation

$$\frac{d^2\theta}{dz} + \frac{k}{z}\frac{d\theta}{dz} = -\delta\exp\theta, \tag{1.37}$$

where

$$\delta = \left[\frac{Q}{\lambda}\frac{E}{RT_0{}^2}r^2ca\exp(-E/RT_0)\right]. \tag{1.38}$$

The Poisson-Boltzmann equation (1.37) has been used to obtain the explosion limits in

$(a.)$ an infinite plane-parallel vessel ($k = 0$),

$(b.)$ cylindrical vessel of length very much greater than its radius ($k = 1$),

$(c.)$ spherical vessel ($k = 2$).

The suitable boundary conditions are given as follows :
At centre of vessel,

$$z = 0, \quad \frac{d\theta}{dz} = 0, \quad \text{Owing to symmetry.} \tag{1.39}$$

At walls of vessel,

$$z = 1, \quad \theta = 0. \tag{1.40}$$

He has shown that the analytical result for $k = 1$, in terms of quadratures is possible and that for $k = 2$, the solution of the equation can be obtained in terms of a known tabulated function. Earlier, it was a general belief that the analytical solution can be obtained only for $k = 0$.

1.3.3 Shallow membrane cap

Baxley and Robinson [21] considered a shallow membrane cap which is rotationally symmetric in its undeformed state. When radial stress is applied on the boundary and a small uniform vertical pressure P is applied to the membrane, the shape that the cap takes, is described by a nonlinear model. In paper (see [21] and reference therein) authors show that under the assumptions of small strain and small constant vertical pressure, if the deformed membrane is rotationally symmetric, then the (rescaled) radial stress on a membrane whose undeformed profile is given in cylindrical coordinates by $z(r) = C(1-r^{\gamma}),\ \gamma > 1$, is determined by the following equation

$$r^2 S_r'' + 3rS_r' = \frac{\lambda^2 r^{2\gamma-2}}{2} + \frac{\beta \nu r^2}{S_r} - \frac{r^2}{8S_r^2}.$$

Here, the undeformed radius of the membrane is $r = 1$ and ν is the Poisson ratio while λ and β are positive constants depending on the pressure P, the thickness of the membrane, and Young's modulus. The boundary conditions are given as follows

$$S_r(1) = S > 0, \quad \text{if the stress } S \text{ at the boundary is specified,}$$

or,

$$b_0 S_r(1) + b_1 S_r'(1) = A > 0, \quad \text{if the radial displacement at the boundary is specified,}$$

where $b_0 > 0$ and $b_1 \geq 0$ and A is any real number. They also imposed a condition at the singular end ($r = 0$), and defined as

$$S_r \text{ bounded as } r \to 0^+.$$

1.3.4 Electrohydrodynamics

Keller [82] analyzed the equilibrium of a uniformly charged gas in a perfectly conducting container.

As equilibrium is the balance between the electric forces in the gas and the pressure forces. He observed that, in equilibrium their is a constant maxima for the density and pressure at the container surface.

The equilibrium condition, in terms of the pressure p, the mass density ρ, the charge density $a\rho$ and the electric field vector E, can be written as

$$\nabla p = a\rho E, \tag{1.41}$$

where constant a is the ratio of electric charge density to mass density in the gas or fluid and the charge (source of the field) is expressed by

$$\nabla \cdot E = 4\pi a\rho. \tag{1.42}$$

Making use of equations (1.41) and (1.42), he obtained

$$\nabla^2 v = f(v), \tag{1.43}$$

where $f(v) = 4\pi a^2 \rho[p(v)]$ is a nonnegative increasing function of v, here $v = \int_{p_0}^{p} \frac{dp}{\rho p}$.

For ideal gas, and $p_0 = 1$, equation (1.43) becomes

$$\nabla^2 u = e^u, \tag{1.44}$$

where

$$u = \frac{m}{RT} v + \log 4\pi \left[\frac{am}{RT}\right]^2,$$

and

$$v = \frac{RT}{m} \log p,$$

where T is the constant temperature, R is the gas constant, and m is the average mass of the molecules in the gas.

On the basis of geometry of the container (i.e., either it is sphere, a cylinder, or a pair of parallel planes), it may be assumed that the solution u of (1.44) is a function of one variable only. This variable which is denoted by r is the distance from the center of the sphere, from the axis of the cylinder, or from the median plane in the three, two or one dimensional cases, respectively. If $u = u(r)$ and n represents the dimension, then the equation (1.44) gives

$$u_{rr} + \frac{n-1}{r} u_r = e^u. \tag{1.45}$$

Regularity of u at the center of sphere or axis of the cylinder is needed that

$$u_r(0) = 0. \tag{1.46}$$

If equation (1.46) holds for $n = 1$, then the solution in that case will be symmetric in the median plane.

1.3.5 The distribution of heat sources in the human head

In 1975, Flesch [52] calculated the temperature distribution by considering two cases of heat generation rates. In which one was an explicit function of the radial distance from the centre and other an implicit function of the ambient temperature. He considered the following differential equation, which describes the study of the distribution of heat sources in human head

$$\frac{\partial^2\theta}{\partial r^2}+\frac{2}{\theta}\frac{\partial\theta}{\partial r}+\frac{q}{k}=0 \tag{1.47}$$

where q, θ, k and r are heat production rate per unit volume, absolute temperature, coefficient of thermal conductivity and radial coordinate of the sphere, respectively. He considered two sets of q as

$$q=q_1\cdot r^2, \quad \text{and} \quad q=q_2\cdot r^3.$$

Here q_1 and q_2 are constant.

In 1980, Gray [56] considered the spherically symmetrical equation of heat conduction

$$\frac{d^2\theta}{dr^2}+\frac{2}{\theta}\frac{d\theta}{dr}+\frac{q(\theta)}{k}=0, \tag{1.48}$$

subject to,

$$\theta(0) \ \text{finite}, \qquad -k\frac{d\theta}{dr}=\beta(\theta-\theta_a), \quad \text{at } r=R, \tag{1.49}$$

where β is a heat exchange coefficient, θ_a is the ambient temperature. He considered a different approximation for $q(\theta)$ as

$$q(\theta)=\alpha-N\theta,$$

where α and N are constants such that N is large subject to $q>0$. This law then can only hold over a limited range of temperature and would certainly not be applicable for temperatures in the hyperthermic region.

In 1981, Anderson and Arthurs [14] considered the nonlinearized form of $q(\theta)$ given by

$$q(\theta)=\alpha e^{-\frac{N\theta}{\alpha}}, \quad \alpha,\, N>0$$

compared with linearized model and discuss its significance over linear model. For smaller values of $\frac{N\theta}{\alpha}$, this $q(\theta)$ is similar to [56].

1.3.6 Astronomy

Chandrashekhar [32, Chapter IV : Polytropic and Isothermal Gas Spheres] in connection with the equilibrium of isothermal gas spheres derived the following Lane-Emden equation of index γ, where γ is a physical constant

$$\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{dy}{dx}\right) = -y^\gamma,$$
$$y'(0) = 0, \quad y(1) = B > 0.$$

1.4 Multi-point boundary value problems

As we have seen boundary conditions play pivot role in deciding existence and uniqueness of solutions for linear (or nonlinear) boundary value problems. The nature of two point nonlinear boundary value problem is totally different and complicated as compared to initial value problem. This complication will further increase if we have multi-point boundary conditions. That is the reason recently, multi-point boundary value problems have been center of attraction. Many physical phenomena can be modeled by ordinary differential equation with multi-point boundary conditions. The multi-point boundary conditions give a new edge, to the study of differential equations, with the presence of such boundary conditions, we can improve the qualitative and quantitative characteristics of the solution of differential equations.

Multi-point boundary value problems arise when the boundary conditions not only rely on the function values or its derivatives at end points, but also in the interior of the interval. We can have the following multi-point boundary conditions

$$y(a) = \sum_{i=1}^{m-2} \alpha_i y(\xi_i) + \gamma_1, \qquad y(b) = \sum_{j=1}^{n-2} \beta_j y(\eta_j) + \gamma_2,$$
$$y'(a) = \sum_{i=1}^{m-2} \alpha_i y'(\xi_i) + \gamma_1, \qquad y(b) = \sum_{j=1}^{n-2} \beta_j y(\eta_j) + \gamma_2,$$
$$y(a) = \sum_{i=1}^{m-2} \alpha_i y(\xi_i) + \gamma_1, \qquad y'(b) = \sum_{j=1}^{n-2} \beta_j y'(\eta_j) + \gamma_2,$$
$$y'(a) = \sum_{i=1}^{m-2} \alpha_i y'(\xi_i) + \gamma_1, \qquad y'(b) = \sum_{j=1}^{n-2} \beta_j y'(\eta_j) + \gamma_2,$$

where $\alpha_i,\ \xi_i,\ \beta_j,\ \eta_j,\ \gamma_1, \gamma_2\ \in \mathbb{R}$ and $\xi_i,\ \eta_j \in (a,b),\ (1 \le i \le m-2),\ (1 \le j \le n-2)$. Three point boundary value problem is a special case of multi-point boundary value problem, i.e., there is only one value of y or y' in the interior of the domain, which is connected to y or y' at the boundary.

Analytically, the study of three point boundary value problems is more interesting and challenging and quite different from two point boundary value problems.

1.4.1 Real life application

In this section, we discuss about some real life applications, which are modeled by three point boundary value problems.

1.4.1.1 Thermostat model

In 2000, Guidotti and Merino [58] discussed the thermostat model, by the following linear parabolic evolution equation

$$u_t - u_{xx} = 0, \qquad (x,t) \in (0,\pi) \times (0,\infty), \tag{1.50}$$

$$\frac{\partial u}{\partial \nu}(0,t) + \beta u(\pi,t) = 0, \qquad t \in (0,\infty), \tag{1.51}$$

$$\frac{\partial u}{\partial \nu}(\pi,t) = 0, \qquad t \in (0,\infty), \tag{1.52}$$

$$u(x,0) = u_0(x), \qquad x \in (0,\pi), \tag{1.53}$$

where boundary condition $\partial_\nu u(0,t) + \beta u(\pi,t) = 0$ is a variation of the classical local Robin condition $\partial_\nu u(0,t) + \beta u(0,t) = 0$.

In this model, temperature is measured by sensor at $x = \pi$. Making use of controller heat releases or extracts at $x = 0$, which is proportional to the temperature at $x = \pi$. Further, this model was studied by Infante and Webb [74] for nonlinear problems. They considered the following class of nonlinear three point boundary value problems

$$-u''(t) = f(t,u(t)), \quad t \in (0,1), \tag{1.54}$$

$$u'(0) = 0, \quad \beta u'(1) + u(\eta) = 0, \quad \eta \in [0,1], \tag{1.55}$$

where $\beta > 0$, and f is a non-negative function. The above three point BVP represents as the stationary state of a model for a heated bar, which is insulated at $t = 0$. With the support of controller adding or removing heat at the other end $t = 1$, depending on the temperature at $t = \eta$. A point sensor is placed at an arbitrary point $t = \eta$

1.4.1.2 Bridge design

Lazer and McKenna [90] observed that a linear model is insufficient to explain the large oscillatory behavior in suspension bridges. Also suspension bridges have other nonlinear behaviors such as traveling waves. If the roadbed of a suspension bridge is treated as a one-dimensional vibrating beam, the following equation is derived (see [90, Section 3])

$$u_{tt} + EIu_{xxxx} + \delta u_t = -ku^+ + W(x) + \varepsilon f(x,t), \tag{1.56}$$

$$u(0,t) = u(L,t) = u_{xx}(0,t) = u_{xx}(L,t) = 0. \tag{1.57}$$

Thus the suspension bridge is seen as a beam of length L, with hinged ends, whose downward deflection is measured by $u(x,t)$, with a small viscous damping term, subject to three separate forces; the stays, holding it up as nonlinear springs with spring constant k, the weight per unit length of the bridge $W(x)$ pushing it down, and the external forcing term $\varepsilon f(x,t)$. The loading $W(x)$ would usually be constant.

If W is replaced by the term $W(x) = W_0 \sin(\pi x/L)$, an error of magnitude around 10% is introduced in the loading and little less in the steady-state deflection. Second, if the forcing term is given by $f(x,t) = f(t)\sin(\pi x/L)$ and general solutions of (1.56)–(1.57), is of the form $u(x,t) = y(t)\sin(\pi x/L)$. These no-nodal solutions were the most commonly observed type for low velocities on the Tacoma Narrows Bridge. When this $u(x,t)$ is substituted into (1.56), this results into the differential equation

$$-y''(t) = f(t,y,y'), \tag{1.58}$$

where $f(t,y,y') = \delta y' + EI(\pi/L)^4 y + ky^+ - W_0 - \varepsilon f(t)$, where y^+ denotes y if y is non-negative, and zero if y is negative.

Large size bridges are sometimes contrived with multi-point supports, which gives rise to multi-point boundary conditions.

Zou et al. [168], discussed the bridge design model. They used a second order ordinary differential equation

$$u''(t) + f(t,u) = 0, \qquad 0 < t < 1, \tag{1.59}$$

where $u(t)$ denotes the displacement of the thread from the unloaded position.

They analyzed that generally small size bridges are designed with two supported points, which produces a standard two point boundary value conditions

$$u(0) = 0, \qquad u(1) = 0. \tag{1.60}$$

While, large size bridges are mainly designed with multi-point supports, which leads to multi-point boundary conditions. They examined that the two different types of boundary conditions can be set up at each end points. The position of the bridge at supporting points near $t = 0$ can be defined by the following boundary value condition

$$u(0) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i) + \lambda_1, \tag{1.61}$$

where $\xi_i \in (0,1),\ \ 1 \leq i \leq m-2$ and λ_1 is parameter. For controlling the angles of the bridge at supporting points near $x = 0$, they considered the following boundary value condition

$$u'(0) = \sum_{i=1}^{m-2} \alpha_i u'(\xi_i) + \lambda_1. \tag{1.62}$$

Similar situation holds near $t = 1$ and the multi point boundary conditions can be formulated as

$$u(1) = \sum_{i=1}^{m-2} \beta_i u(\xi_i) + \lambda_2, \tag{1.63}$$

or,

$$u'(1) = \sum_{i=1}^{m-2} \beta_i u'(\xi_i) + \lambda_2. \tag{1.64}$$

1.5 Existing techniques for nonlinear BVPs

For boundary value problems, we try to find out the solutions, mainly in two ways either analytically or numerically. Both techniques have its own importance and challenges. We can discuss the qualitative properties of the solutions with the help of analytical techniques, whereas numerical techniques assist us to solve more complicated problems, when analytical approach does not work.

In this thesis, we follow both approaches, i.e.,

- **Analytical approach :** We focus on existence and uniqueness of the solutions of nonlinear three point boundary value problems as well as two point discrete boundary value problem.
- **Numerical approach :** We deal with nonlinear two point singular boundary value problems.

1.5.1 Analytical study

The analytical study of boundary value problems has been discussed by several methods. Comprehensively, they can be divided into the following categories [149, 164]

1.5.1.1 Shooting method

Some special classes of second order singular boundary value problems have been studied successfully by shooting method like Negative Exponent Emden-Fowler boundary value problems [147]. This method is more productive if $f(t,y)$ is decreasing in y, while for other cases it often seems useless. Henderson et al. [70] have employed shooting method to obtain the solution of three point boundary value problem.

1.5.1.2 Nonlinear alternative

Leray and Schauder [91] in their celebrated paper introduced some "Nonlinear Alternative" theorems for compact maps (see [111]). These theorems have enhanced greatly the theory of ordinary differential equations. There are two major approaches to modern non-linear alternative theory

- Topological Degree Method: This is based on Degree theory [97].
- Topological Transverslity: This is based on Essential maps and was introduced by Granas [55].

For three point boundary value problems, Gupta [61], Ma [100], Liu [96], Infante and Webb [74] have discussed the existence and uniqueness of the solutions with the support of Topological methods.

Operator methods or topological methods are more relevant and has many advantages, when they deal with non-singular problems. But it still has some difficulties when treating singular problems.

1.5.1.3 Upper and lower solutions method

Upper and lower solutions method is one of the most substantial way for discussing the existence results for nonlinear problems. This technique covers a wide range of nonlinear boundary value problems.

Recently, there has been a lot of activity related to the theory of upper and lower solutions. It has been successfully coupled with other existing techniques (see [149]), e.g.,

- Topological Degree Theory (Duhoux [47]),
- Topological Transversality (Bobisud and O'Regan [24], Agarwal and O'Regan [7]),
- Monotone Iterative Method (Ladde et al. [89], Cherpion et al. [40], Coster and Habets [41]),

- Quasilinearization (O'Regan and El-Gebeily [112]).

In comparison to two point boundary value problems, lot of investigations are still pending related to second order three point boundary value problems. Zhang and Wang [165], Xian et al. [161] coupled the concept of upper and lower solutions with monotone iterative technique and fixed point index theory, respectively. While Henderson et al. [70], Guo and Ge [59], and Bao et al. [19], coupled it with shooting method, fixed point index method and fixed point theorem in a cone, respectively.

Zhang [164] shown that upper and lower solution technique is very promising technique as far as singular boundary value problems are concerned.

1.5.2 Numerical study

In last few decades, numerical methods for solving two point boundary value problem for ordinary differential equation have been discussed by several researcher. As we have already mentioned, nonlinear singular boundary value problems arise very commonly in various disciplines of applied mathematics and engineering such as, thermal explosion [31], electrohydrodynamics [82], physiological studies, oxygen diffusion in spherical cell [95, 104] and distribution of heat sources in the human head [14, 52, 56].

The numerical solutions of these singular boundary value problems have been discussed by several methods such as cubic spline and B-spline methods [39, 80, 132], mixed decomposition-spline methods (MDSM) [84], finite difference methods [36, 116, 121, 124], collocation methods [135]. Recently, some iterative methods like Adomian decomposition method (ADM), modified Adomian decomposition method (MADM) and homotopy analysis method (HAM) [1, 2, 43, 48], variational iteration method (VIM) [65, 79, 145], homotopy perturbation method (HPM) [64, 67] have been used to solve the singular boundary value problems.

1.6 Analytical approach

In this thesis, we discuss the analytical results for nonlinear three point boundary value problems and two point discrete boundary value problem. Here, we focus on upper and lower solutions method related to monotone iterative technique.

The method of upper and lower solution has a long history and some of its concepts can be traced back to Picard [131]. Dragoni [45] was first who established the notion of the method of lower and upper solutions for ordinary differential equations. He considered the second order boundary values problems

$$
\begin{aligned}
&u''(t) = f(t,u(t),u'(t)), \quad t \in [a,b] \equiv I,\\
&u(a) = A, \quad u(b) = B,
\end{aligned}
$$

for $f : I \times \mathbb{R}^2 \to \mathbb{R}$ being a continuous function and $A, B \in \mathbb{R}$.

A function $\alpha \in C^2(I)$ defines a lower solution if it satisfies the inequalities

$$
\begin{aligned}
&\alpha''(t) \geq f(t,\alpha(t),\alpha'(t)),\\
&\alpha(a) \leq A, \quad \alpha(b) \leq B.
\end{aligned}
$$

Similarly, a function $\beta \in C^2(I)$ is called an upper solution if it satisfies the reverse inequalities, i.e.,

$$
\begin{aligned}
&\beta''(t) \leq f(t,\beta(t),\beta'(t)),\\
&\beta(a) \geq A, \quad \beta(b) \geq B.
\end{aligned}
$$

For $\alpha \leq \beta$, existence of the solution of the above considered problem lying between α and β is proved.

Actually upper and lower solutions are treated as the bounds of the solution of the nonlinear BVP, and they ensure that there exists a solution of the problem lying between the lower and the upper solutions. It means, with the help of two well-ordered functions that satisfy some suitable inequalities, we can find a solution of nonlinear BVPs (see [26]).

Recently, several authors have been successfully employed these methods for different kinds of boundary value problems, e.g., first, second and higher order ordinary differential equations with different type of boundary conditions. Also partial differential equations of first and second order, have also been treated in the literature. For an overview of the method of lower and upper solutions of ordinary differential equations, we refer [26, 41].

In this thesis, we endeavour to combine fruitfully two basic techniques, namely, the method of upper and lower solutions and monotone iterative method, and investigate the existence of the solution of nonlinear three point boundary value problems. In recent years, this technique, i.e., monotone iterative method and upper and lower solutions have been studied successfully by several researchers. The popularity of this method is not only just for its constructive approaches, but also for the qualitative properties of solutions. The monotone sequences, which are governed by an iterative scheme also play a valuable role in the numerical treatment of various boundary value and initial value problems. That is why researchers have been using this technique to nonlinear regular as well as singular boundary values problems.

To summarize, the basic idea of monotone iterative method and upper-lower solutions, let us consider the following general second order nonlinear boundary value problem

$$\begin{aligned} -L[u] &= f(x,u), \qquad x \in [0,1], \\ B[u] &= A, \end{aligned}$$

where $u \in C^2(I)$, $I = [0,1]$, L is differential operator of second order and B is a boundary operator. Let there exist upper and lower solutions β and α in $C^2(I)$, such that $\alpha \leq \beta$ (or, $\alpha \geq \beta$ for reverse order case) such that

$$\begin{aligned} -L[\beta] &\geq f(x,\beta), \qquad x \in [0,1], \\ B[\beta] &\geq A, \end{aligned}$$

and α satisfies the reversed inequalities. Now we can generate two monotone sequences (α_n) and (β_n) from the iterative scheme

$$\begin{aligned} -L[u_{n+1}] - \lambda u_{n+1} &= f(x,u_n) - \lambda u_n, \qquad x \in [0,1], \\ B[u_{n+1}] &= A, \end{aligned}$$

where β and α are treated as initial guesses, and λ may be constant or function.

Making use of monotonicity of these sequences leads to the following relation

$$\alpha = \alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n \leq \alpha_{n+1} \cdots \leq \underline{u} \leq \bar{u} \leq \cdots \leq \beta_{n+1} \leq \beta_n \leq \cdots \leq \beta_1 \leq \beta_0 = \beta.$$

These sequences converge uniformly to a solution, say $\bar{u}$ and $\underline{u}$ such that

$$\lim_{n\to\infty} \beta_n = \bar{u}, \quad \text{and} \quad \lim_{n\to\infty} \alpha_n = \underline{u}$$

of nonlinear boundary value problem.

1.7 Literature review

We have already mentioned that, monotone iterative method associated with upper and lower solutions goes back at least to Picard [131]. He published this work in two "*Mémoires*", mainly, one for partial differential equation (PDE) [130] and other for ordinary differential

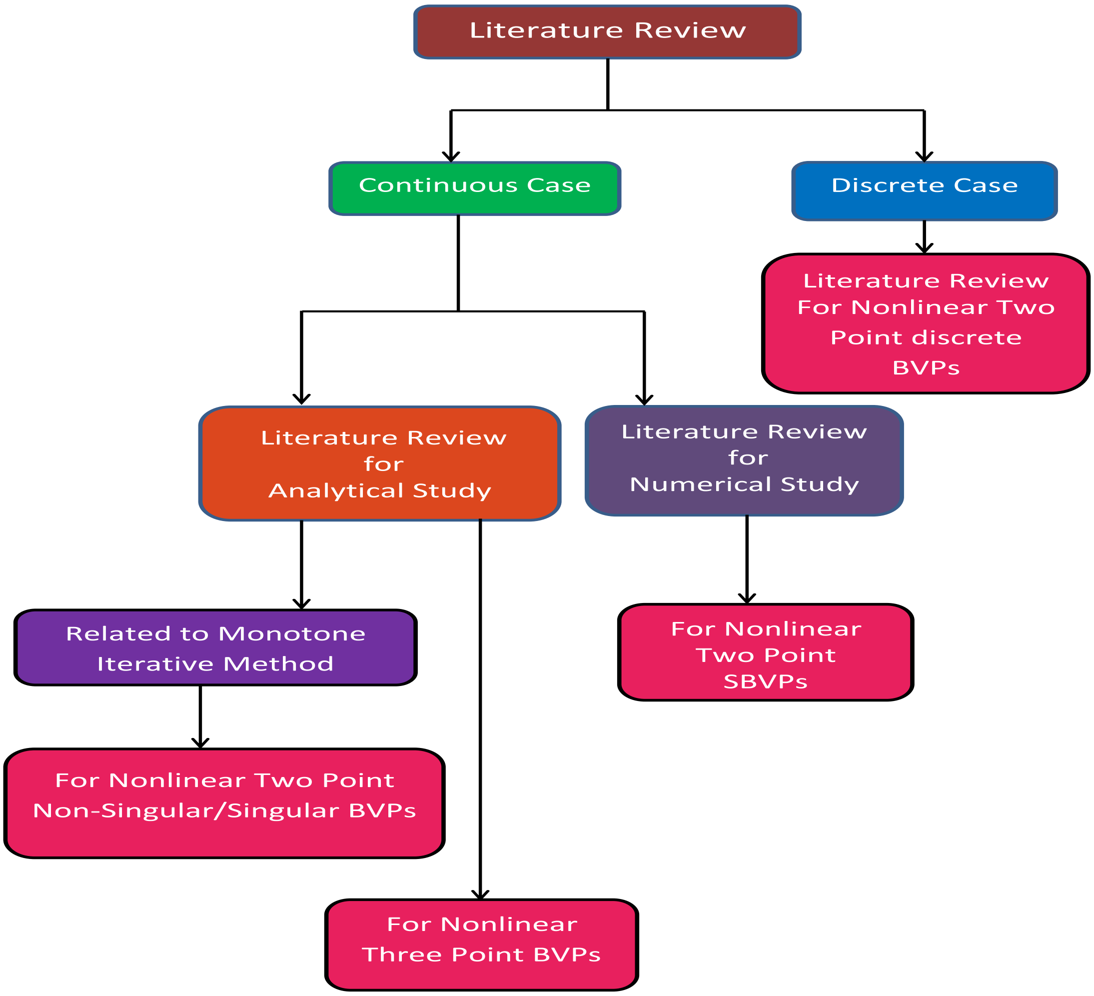

Fig. 1.2 A route map of literature review

equation (ODE) [131]. In both cases, the existence of a solution is guaranteed. He considered the following boundary value problem (BVP)

$$u'' + f(t,u) = 0, \quad u(a) = 0, \quad u(b) = 0, \tag{1.65}$$

assuming $u = 0$ is a solution and $f(t,u)$ is increasing, i.e., $f(t,0) = 0$. For nontrivial solution, he developed a convergent sequence of approximations $(\alpha_n)_n$ from the following scheme

$$-\alpha_n'' = f(t,\alpha_{n-1}) = 0, \quad \alpha_n(a) = 0, \quad \alpha_n(b) = 0. \tag{1.66}$$

Which gives a monotonic approximation

$$\alpha_0 \le \alpha_1 \le \alpha_2 \le \cdots, \tag{1.67}$$

that satisfies $\alpha_0 > 0$ on (a,b). Under some more assumptions, he proved the existence of a positive function α_0 such that

$$\alpha_0'' + f(t,\alpha_0) > 0, \quad \alpha_0(a) = 0, \quad \alpha_0(b) = 0. \tag{1.68}$$

Later on, such a function is referred as a lower solution and the method used by Picard is referred as the monotone iterative method.

Following, Chaplygin's work [33], with the support of upper and lower solutions technique, the Russian school has further studied and developed the monotone methods in a very precise way. In 1954, Babkin [16] considered the following approximation scheme, under the condition of upper β_0 and lower α_0 solutions with $\beta_0 \ge \alpha_0$, for the problem (1.65)

$$-\alpha_n'' + \lambda\alpha_n = f(t,\alpha_{n-1}) + \lambda\alpha_{n-1}, \quad \alpha_n(a) = 0,\ \alpha_n(b) = 0, \tag{1.69}$$

$$-\beta_n'' + \lambda\beta_n = f(t,\beta_{n-1}) + \lambda\beta_{n-1}, \quad \beta_n(a) = 0,\ \beta_n(b) = 0. \tag{1.70}$$

The assumption such that $f(t,u) + \lambda u$ is increasing in u, for some $\lambda > 0$, is the key to prove that the sequences $(\alpha_n)_n$ and $(\beta_n)_n$ are monotone and converge to the unique solution of (1.65).

In 1939, a notable result was given by Kantorovich [81]. He analyzed that the first monotone approximation scheme, which are mainly used for Cauchy problem as well as for other boundary value problem, has a common structure related to positive operators. Making use of this, he established an abstract formulation of the method, which was further developed by several authors [10, 11, 87, 88, 162].

Without reference of Russian school, Courant and Hilbert [42] described a monotone iterative scheme similar to Babkin's [16]. The main problem was to find appropriate conditions on the function f to apply the method. In 1968, one sided Lipschitz condition was introduced by Shampine [139],

$$f(x,v) - f(x,u) \le k(x)(u-v), \qquad \text{if} \quad u \ge v. \tag{1.71}$$

This condition unifies Courant and Hilbert [42] and other approaches in the literature (see [41]). In 1974, by assuming a Hölder condition on f, Amann [9] has generalized the one-sided Lipschitz condition, which is a particular case of the condition studied by Mlak

[106], for parabolic problem. Additionally, In 1978, Stuart [146] has considered that f is of bounded variation on compact intervals which gives $f = g - h$, where g and h are increasing functions. In 1992, Carl [30] has also used such assumption. In such a case, the approximation sequences are defined by the following equations

$$\begin{aligned} -\alpha_n'' + h(t,\alpha_n) &= g(t,\alpha_{n-1}), \qquad \alpha_n(a) = \alpha_n(b) = 0, \\ -\beta_n'' + h(t,\beta_n) &= g(t,\beta_{n-1}), \qquad \beta_n(a) = \beta_n(b) = 0. \end{aligned}$$

In general, this method gives implicit solutions, which reduces considerably the interest of the approach.

In 1964, a major result is established by Gendzhoyan [53]. He studied the problem, when nonlinearity depends on the derivative, i.e.,

$$u'' + f(t,u,u') = 0, \quad u(a) = 0, \quad u(b) = 0.$$

In the presence of lower and upper solution α_0 and β_0 such that $\beta_0 \geq \alpha_0$, the sequences of approximations $(\alpha_n)_n$ and $(\beta_n)_n$ are solutions of the following BVP,

$$\begin{gathered} -\alpha_n''(t) + l(t)\alpha_n' + k(t)\alpha_n = f(t,\alpha_{n-1},\alpha_{n-1}') + l(t)\alpha_{n-1}' + k(t)\alpha_{n-1}, \\ \alpha_n(a) = \alpha_n(b) = 0, \\ -\beta_n''(t) + l(t)\beta_n' + k(t)\beta_n = f(t,\beta_{n-1},\beta_{n-1}') + l(t)\beta_{n-1}' + k(t)\beta_{n-1}, \\ \beta_n(a) = \beta_n(b) = 0, \end{gathered}$$

where $k(t)$ and $l(t)$ are functions which depend on the nonlinear term f.

In 1977, Bernfeld and Chandra [23] studied the monotone iterative method for nonlinearities depending on derivative. The first approximations are defined as upper and lower solutions β_0 and $\alpha_0(\leq \beta_0)$ and other approximations are evaluated with the help of the following nonlinear problems

$$\begin{aligned} -\alpha_n''(t) + \lambda\alpha_n &= f(t,\alpha_{n-1},\alpha_n') + \lambda\alpha_{n-1}, \qquad \alpha_n(a) = \alpha_n(b) = 0, \\ -\beta_n''(t) + \lambda\beta_n &= f(t,\beta_{n-1},\beta_n') + \lambda\beta_{n-1}, \qquad \beta_n(a) = \beta_n(b) = 0, \end{aligned}$$

where λ is related to f. As right hand side of the above equations depends explicitly on α_n' and β_n' respectively, i.e., computation of the approximations is not explicit.

In 1986, Omari [109] proposed an alternative approach for the boundary value problem where nonlinearity depends on the derivative. The iterative process, for the Dirichlet problem, defined from the following piecewise BVP,

$$\begin{aligned}\alpha_n''(t) - k|\alpha_n' - \alpha_{n-1}'| - \lambda\alpha_n &= f(t, \alpha_{n-1}, \alpha_{n-1}') - \lambda\alpha_{n-1},\\ \alpha_n(a) = \alpha_n(b) &= 0,\\ \beta_n''(t) - k|\beta_n' - \beta_{n-1}'| - \lambda\beta_n &= f(t, \beta_{n-1}, \beta_{n-1}') - \lambda\beta_{n-1},\\ \beta_n(a) = \beta_n(b) &= 0.\end{aligned}$$

He also dealt with periodic and Neumann problems.

For the non well ordered case, i.e., $\alpha \geq \beta$, the first existence result was given by Amann et al. [12]. For monotone iterative method, first contribution was made by Omari and Trombetta [110]. Authors have considered in particular the periodic problem

$$-u''(t) + cu' + f(t,u) = 0, \quad u(a) = u(b), \quad u'(a) = u'(b),$$

and have proved the convergence of approximations $(\alpha_n)_n$ and $(\beta_n)_n$ defined as

$$\begin{aligned}-\alpha_n''(t) + c\alpha_n' + \lambda\alpha_n &= -f(t, \alpha_{n-1}) + \lambda\alpha_{n-1},\\ \alpha_n(a) = \alpha_n(b), &\qquad \alpha_n'(a) = \alpha_n'(b),\\ -\beta_n''(t) + c\beta_n' + \lambda\beta_n &= f(t, \beta_{n-1}) + \lambda\beta_{n-1},\\ \alpha_n(a) = \alpha_n(b), &\qquad \alpha_n'(a) = \alpha_n'(b).\end{aligned}$$

The existence of non well order (reverse order) case is due to the occurrence of anti-maximum principle, which is basically, result of the assumptions that the function $f(t,u) - \lambda u$ is non-decreasing in u for some $\lambda < 0$ and that this λ is such that the operator $-u'' + cu' + \lambda u$ is inverse negative on the space of periodic functions, i.e., anti maximum principle holds (see [41]). In 1996, Cabada and Sanchez [29] studied similar results for Neumann problem.

For the reverse order case, when f depends nonlinearly on u', Bellen [22], for periodic problem, Cabada et al. [27] and Cherpion et al. [40] for Neumann problems proved some important results.

All these discussion are for regular cases. Let us consider some important results for singular BVPs.

In 1952, Chambre [31] considered the singular boundary value problems (1.37)–(1.40) and discussed analytical solution for $k = 1$ in the terms of quadratures and for $k = 2$ in terms of known tabulated function. Probably this was the first result which started motivating researchers to explore analytically further possibilities in the singular boundary value problems. Next in 1956, Keller [82] established the existence results for singular boundary value problems with the use of monotone iterative methods.

Here, we discuss the results for two point singular boundary value problems, which are tackled with the support of monotone iterative method associated with upper and lower solutions. In 1975, Russell and Shampine [135] considered a special class of singular boundary value problems. Usually, such problems arise when partial differential equation are reduced to ordinary differential equation due to physical symmetry. They also state that if physical law is represented by the following equation

$$\Delta u(P) = f(P, u(P)), \tag{1.72}$$

and if one is interested in planar, cylindrical or spherical geometries, he is led to the differential equation

$$u''(x) + \frac{k}{x}u'(x) + f(x, u(x)) = 0, \tag{1.73}$$

with $k = 0,\ 1$ or $\ 2$, respectively. They considered the following boundary conditions,

$$u'(0) = 0, \quad (\text{or equivalently, } u(0) \text{ finite}), \quad u(b) = 0. \tag{1.74}$$

By using monotone iterative technique in the presence of upper and lower solutions, they established existence-uniqueness of the solutions of singular boundary value problems (1.73)–(1.74). They proposed the following iterative scheme

$$Lu_{n+1} = F(x, u_n(x)),$$

where,

$$\begin{aligned} Lu(x) &= -\left(u''(x) + \frac{k}{x}u'(x) - \lambda u(x)\right), \\ F(x, u(x)) &= f(x, u) + \lambda u. \end{aligned}$$

Here k satisfies the inequalities

$$\begin{aligned} k &> -\left(\frac{j_0}{b}\right)^2, \quad \text{if } k = 1, \\ k &> -\left(\frac{\pi}{b}\right)^2, \quad \text{if } k = 2, \end{aligned}$$

where j_0 is first positive zero of the Bessel function of first kind of order zero.

In 1987, Chawla and Shivkumar [38] have generalized the result of Russell and Shampine [135]. They examined the existence-uniqueness of the solution of the class of nonlinear two point singular boundary value problems

$$-\left(u''(x)+\frac{\alpha}{x}u'(x)\right)=f(x,u(x)),\quad 0<x<1,\ \alpha\geq 1$$
$$u'(0^+)=0,\quad u(1)=B.$$

They used the following iterative scheme

$$-\left(x^{\alpha}u'_{n+1}\right)'-x^{\alpha}u_*u_{n+1}=x^{\alpha}\left[f(x,u_n)-u_*u_n\right],\quad 0<x<1,\ \alpha\geq 1,$$
$$u'_{n+1}(0)=0,\quad u_{n+1}(1)=B,$$

where $u_*=\frac{\partial f}{\partial y}<k_1$, and k_1 is the first positive zero of $J_{\frac{\alpha-1}{2}}(\sqrt{k})$ and $k<k_1$.

In 1996 and 1997, Pandey [114, 115, 117], studied a more generalized class of singular differential equation

$$-(p(x)y')'=p(x)f(x,y),\quad 0<x\leq b, \tag{1.75}$$

where $p(x)$ satisfies the following conditions

(i) $p(x)>0$ on $(0,b)$.

(ii) $p(x)\in C^1(0,r)$ for some $r>b$.

(iii) $x\frac{p'(x)}{p(x)}$ is analytic in $\{z:|z|<r\}$, with Taylor series expansion

$$x\frac{p'(x)}{p(x)}=b_0+b_1x+b_2x^2+\cdots.$$

He imposed the following boundary conditions

(i) $y(0)=A,\quad y(b)=B$ when $\int_0^b\frac{dt}{p(t)}<\infty$ and $b_0\in[0,1)$ (see [114]).

(ii) $\lim_{x\to 0^+}y'(x)=0,\quad y(b)=B$ when $\int_0^b\frac{dt}{p(t)}<\infty$ and $b_0\in[0,1)$ (see [115]).

(iii) $\lim_{x\to 0^+}y'(x)=0,\quad y(b)=B$ when $\int_0^b\frac{dt}{p(t)}=\infty$ and $b_0\geq 1$ (see [117]).

He established existence of unique solutions of the considered problems by using eigenfunction expansion and monotone iterative technique with the support of upper and lower solutions.

Pandey and Verma [125–127] have analyzed the following class of singular differential equation

$$-(p(x)y')'=q(x)f(x,y),\quad 0<x\leq b, \tag{1.76}$$

where $p(x)$ satisfies the properties similar to [114, 115, 117], and $q(x)$ follows the following assumption

(i) $q(x) > 0$ on $(0,b)$.

(ii) $q(x) \in L^1(0,r)$ for some $r > b$.

(iii) $x^2 \frac{q(x)}{p(x)}$ is analytic in $\{z : |z| < r\}$, with Taylor series expansion

$$x\frac{q(x)}{p(x)} = c_0 + c_1 x + c_2 x^2 + \cdots .$$

They imposed the boundary conditions

(i) $y(0) = a, \quad \& \quad \alpha_1 y(b) + \beta_1 y'(b) = \gamma_1$ for $b_0 \in [0,1)$ (see [125]),

(ii) $y'(0) = 0, \quad \& \quad \alpha_1 y(b) + \beta_1 y'(b) = \gamma_1$ for $b_0 \geq 0$ (see [126]),

(iii) $\lim_{x\to 0^+} p(x)y'(x) = 0 \;\; \& \;\; \alpha_1 y(b) + \beta_1 p(b)y'(b) = \gamma_1$, for $b_0 > 0$ (see [127]),

where $a \geq 0,\ \alpha_1 > 0,\ \beta_1 \geq 0$ and γ_1 is finite. In these results [125–127], they established existence of unique solutions by using monotone iterative technique and eigen function expansion.

Recently, Verma [150], considered a class of nonlinear singular BVP

$$\begin{aligned} &-(x^\alpha y'(x))' + x^\alpha f(x,y(x),x^\alpha y'(x)) = 0, \quad 0 < x < 1, \;\; \alpha \geq 1, \\ &y'(0) = y'(1) = 0, \end{aligned}$$

where the source function $f(x,y(x),x^\alpha y'(x))$ is derivative dependent and boundary conditions are Neumann type. In this paper he used an iterative scheme which is as simple as possible from the computational point of view. He proposed the following iterative scheme

$$\begin{aligned} &-(x^\alpha y_n'(x))' + \lambda x^\alpha y_n'(x) = -x^\alpha f(x,y_{n-1}(x),x^\alpha y_{n-1}'(x)) + \lambda x^\alpha y_{n-1}'(x) \\ &y_n'(0) = y_n'(1) = 0 \end{aligned}$$

The work in this paper generalizes the work of Cherpion et al. [40] (for the non-singular case, $\alpha = 0$) to the singular case ($\alpha \geq 1$) and also generalizes the work of Chawla and ShivKumar [38] to derivative dependent source functions.

Further, Pandey and Verma [128, 129], Verma [151] and Verma and Agarwal [152], considered more generalized class of nonlinear singular boundary value problems and proved some important results.

Pandey and Verma [128, 129] considered the following class of singular differential equation

$$-(p(x)y'(x))' = q(x)f(x,y(x),py'(x)), \quad 0 < x \leq b, \tag{1.77}$$

where $p(x)$ and $q(x)$ satisfy the following assumptions

(i) $p(x) > 0$ on $(0,b)$, $p \in C[0,b) \bigcap C^1(0,b)$.

(ii) $q(x) > 0$ in $(0,b)$, $\int_0^b q(t)dt < \infty$.

Pandey and Verma [128], considered the boundary conditions

$$\lim_{x \to 0^+} p(x)y'(x) = 0 \;\;\&\;\; \alpha_1 y(b) + \beta_1 p(b)y'(b) = \gamma_1.$$

where $\alpha_1 > 0,\; \beta_1 \geq 0$ and γ_1 is any finite constant. While in [129], they considered boundary conditions for $\int_0^b \frac{dt}{p(t)} < \infty$ such as

$$y(0) = a, \;\;\&\;\; \alpha_1 y(b) + \beta_1 y'(b) = \gamma_1,$$

where $a \geq 0,\; \alpha_1 > 0,\; \beta_1 \geq 0$ and γ_1 is finite. With the support of monotone iterative method and upper and lower solutions method, they established existence of unique solutions of the considered problems for well order case

Verma [151] considered the following second order nonlinear singular boundary value problem

$$-(p(x)y'(x))' + p(x)f(x,y(x),py'(x)), \quad 0 < x < 1, \tag{1.78}$$

$$y'(0) = 0, \qquad y'(1) = 0, \tag{1.79}$$

where $f(x,y,py')$ is Lipschitz in py', and one sided Lipschitz in y, The functions $p(x)$ satisfies the following assumptions

(i) $p(0) = 0$ and $p > 1$ in $(0,1)$.

(ii) $p(x) \in C[0,1] \bigcap C^1(0,1)$.

(iii) for some $r > 1$, $\frac{xp'(x)}{p(x)}$ is analytic in $\{z : |z| < r\}$.

(iv) $\int_0^1 \frac{dt}{p(t)} = \infty$.

Verma and Agarwal, [152], considered the following nonlinear singular boundary value problem

$$-(p(x)y'(x))' + q(x)f(x,y(x),py'(x)), \qquad 0 < x < 1, \tag{1.80}$$

subject to the boundary conditions

$$\lim_{x\to 0} p(x)y'(x) = 0 \ \ \& \ \ \lim_{x\to 1} p(x)y'(x) = 0, \tag{1.81}$$

where $p(x)$ satisfies the assumptions similar to [151] (except (iv)), and $q(x)$ follows the following assumptions

(i) $q(x) > 0$ in $(0,1)$ and $q(x) \in C(0,1]$.

(ii) $\int_0^1 q(x)dx < \infty$.

(iii) $\lim_{x\to 0} \frac{q(x)}{p'(x)} \neq 0$.

(iv) $\int_0^1 \frac{1}{p(x)} \int_0^x q(s)dsdx < \infty$.

(v) $x^2 \frac{q(x)}{p(x)}$ is analytic in $\{z : |z| < r, r > 1\}$.

Verma [151] and Verma and Agarwal [152], established the existence results for nonlinear singular boundary value problems. They used the concept of monotone iterative method with upper and lower solutions and discussed the results for both well order and reverse order cases.

1.7.1 Three point boundary value problems

As compared to two point boundary value problems, multi-point boundary value problems do not have very vast literature. In 1918, Wilder [160] studied a differential system consisting of an ordinary differential equation with auxiliary conditions at more than two points. Since in this thesis we focus on three point BVPs, we will mention results only related to three point BVPs.

Neuberger [107], Zettl [163] and Loud [98] have discussed self adjoint boundary value problem with interior conditions. They analyzed their results, with the support of Green's functions, associated with non-homogeneous problems.

According to the literature, the main contribution to the theory of multi-point boundary value problem was made by II'in and Moiseev [73]. II'in and Moiseev [73] and Gupta [61] dealt with non-linear three point boundary value problems for ordinary differential equations. Since then, several researchers have been discussed various nonlinear multi-point boundary value problems by using Leray-Schauder continuation theorem, nonlinear alternative of Leray-Schauder, Coincidence degree theory or Fixed point theorem in cone (see [103] and references therein).

In 1971, Greguš et al. [57] established existence results of solution for the two point eigenvalue problem formed by the differential equation

$$y'' + [q(x;\lambda,\mu) + r(x)]y = 0, \quad x \in [a,c],$$

with three point boundary conditions

$$y(a) = y(b) = y(c) = 0, \quad b \in (a,c),$$

where λ and μ are the parameters, whose eigenvalues are sought.

In 1998, Ma [99] proved the existence of positive solutions for the following three point boundary value problem

$$u'' + a(t)f(u) = 0, \quad t \in (0,1), \tag{1.82}$$

$$u(0) = 0, \quad \alpha u(\eta) = u(1), \tag{1.83}$$

where $0 < \eta < 1$ and $0 < \alpha < \frac{1}{\eta}$. Here he used the fixed point theorem in cones.

Liu [96] used the Krasnoselskii's fixed point theorem in a cone. He studied the existence of single and multiple positive solutions to the second order differential equation (1.82), with three point boundary conditions

$$u'(0) = 0, \quad \beta u(\eta) = u(1), \quad \eta, \beta \in (0,1). \tag{1.84}$$

Infante and Webb [74] studied the nonlinear three point boundary value problem (1.54)–(1.55). They have discussed the existence of nontrivial solutions by using the theory of fixed point index.

In 2003, Ma [101] has proved several multiplicity results for the following three point boundary value problem,

$$u''(t) = f(t,u(t)), \quad t \in (0,1),$$
$$u(0) = 0, \quad u(1) = \alpha u(\eta),$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous, $\alpha \in (0,\infty)$ and $\beta \in (0,1)$. For this, the author developed the method of lower and upper solutions which are well ordered as well as reverse ordered.

Zhang and Wang [165] discussed existence results, with the support of upper and lower solution method related to the monotone iterative technique, for a class of second order nonlinear three point singular boundary value problems of the form

$$-u''(t) = f(t, u(t)), \quad t \in (0,1), \tag{1.85}$$

$$u(0) = \xi, \quad u(1) - \lambda u(\delta) = \eta, \tag{1.86}$$

where $\delta \in (0,1)$, $\lambda > 0$ and $\xi, \eta \in \mathbb{R}$.

Xian et al.[161] considered the nonlinear differential equation (1.85), with boundary conditions $u(0) = 0, \quad u(1) - \alpha u(\eta) = 0$, where $\eta, \alpha \in (0,1)$, and $f \in C\left([0,1] \times \mathbb{R}^1, \mathbb{R}^1\right)$. Using fixed point index theory, they studied some multiplicity results for the solutions of three point boundary value problem for non well ordered upper and lower solutions.

In 2008, Li et al. [92] studied the existence and uniqueness of solutions of the second order nonlinear three point boundary value problem

$$u''(t) + f(t, u(t)) = 0, \quad t \in (0,1), \tag{1.87}$$

$$u'(0) = 0, \quad u(1) = \delta u(\eta), \tag{1.88}$$

where $0 < \eta < 1$, $\delta > 1$, $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$, with lower and upper solutions in reverse order. They obtained the sufficient conditions for the existence and uniqueness of solutions by using monotone iterative method. Li et al. [93] introduced a new concept related to upper and lower solutions and studied the existence and uniqueness of solutions of second order three point boundary value problems (1.87)–(1.88) with upper and lower solutions in reverse order.

In case, when f depends on u', we quote the work of Henderson et al. [70], Guo and Ge [59], and Bao et al. [19]. They used Shooting methods, Fixed point index method and Fixed point theorem in a cone, respectively.

Bao et al. [19] discussed the existence results for positive solutions of the following nonlinear three point boundary value problem

$$u''(t) + f(t, u(t), u'(t)) = 0, \quad t \in (0,1), \tag{1.89}$$

$$u(0) = 0, \quad u(1) = \gamma u(\eta), \tag{1.90}$$

where $f : C\left([0,1] \times \mathbb{R}^+ \times \mathbb{R}^1, \mathbb{R}\right)$ is continuous, $\alpha \in (0, \infty)$ and $\gamma, \eta \in (0,1)$. They used the concept of the fixed point index method under a non well ordered upper and lower solution condition.

1.8 Numerical approach

This thesis also deals with numerical solutions of nonlinear two point singular boundary value problems (SBVPs). Our main attention is on iterative methods, namely, variational iteration method (VIM), and homotopy perturbation method (HPM). There is an enormous literature related to numerical methods to solve nonlinear two point singular boundary value problems.

In 1970, Jamet [78] considered the following second order differential equation

$$Lu = \frac{d^2u}{dx^2} + f(x)\frac{du}{dx} - g(x)u = H(x), \quad 0 \le x \le 1, \tag{1.91}$$

where $f(x) \in C(0,1],\ \ f(x) \to \infty$, as $x \to 0,\ \ g(x), H(x) \in C[0,1]$ and $g(x) \ge 0$. On the basis of the rate of growth of $f(x)$ near the origin, he considered either the two point boundary conditions,

$$\begin{aligned} &u(0) = a, \quad u(1) = b, \\ &u \in C^2(0,1) \cap C[0,1], \end{aligned} \tag{1.92}$$

or the one point boundary condition

$$\begin{aligned} &u(1) = b, \quad u(x) \ \text{ is bounded at origin}, \\ &u \in C^2(0,1) \cap C[0,1] \cap B[0,1], \end{aligned} \tag{1.93}$$

Note that the operator L of (1.91) can be written as

$$Lu = \frac{1}{p(x)} \frac{d}{dx}\left[p(x)\frac{du}{dx}\right] - g(x)u, \tag{1.94}$$

where $p(x) = exp\left(-\int_x^1 f(t)dt\right)$. Here, he studied two finite difference schemes, a direct central difference analog of the equation (1.91) and a scheme obtained by differentiating the equation (1.94). He proved that the two point BVP (1.91)–(1.92) has unique solution for

$$f(x) < \frac{\sigma}{x}, \ \text{ for } x \text{ small},$$

where σ is a number, $0 < \sigma < 1$. Similarly, he discussed that if

$$f(x) > \frac{1}{x} - C, \ \text{ for } x \text{ small}, \ \ C \ge 0,$$

then one point BVP (1.91)–(1.92) has unique solution. He also shown that the solutions of both finite difference schemes converge to it. He also obtained a principal error function for the case $f(x) = \frac{\sigma}{x}$, $g(x) \equiv q$ (a constant) and illustrated a method of uniform error estimation. He shown the error to be $O(h^{1-\sigma})$.

In 1975, Russell and Shampine [135] have studied analytical results for singular boundary value problems (1.73)–(1.74) for the case $k = 1$ and $k = 2$. Additionally, they examined numerical solutions with three numerical techniques, namely, (i) Collocation method (ii) Finite difference (iii) Patch bases. These results modifid the result of Jamet [78]. The collocation method and singular spline for two-point singular boundary value problems have been discussed by Reddien [133] and Reddien and Schumaker [134]. They studied the existence, uniqueness and convergence rates of these methods.

The following class of singular boundary value problems are considered by several authors

$$(x^{\alpha}y')' = f(x,y), \quad 0 < x \leq 1, \quad y(0) = A, \quad y(1) = B, \tag{1.95}$$

and

$$\begin{aligned} &(+/-)x^{-\alpha}(x^{\alpha}y')' = f(x,y), \quad 0 < x \leq 1, \\ &y'(0) = 0, \text{ (or } \quad y(0) = \text{finite}), \quad y(1) = B, \end{aligned} \tag{1.96}$$

where A, B are finite constants. In 1982, Chawla and Katti [35] developed finite difference schemes for singular point boundary value problem (1.95). They assumed that the $\alpha \in (0,1)$ and source function $f(x,y)$ is continuous, $\frac{\partial f}{\partial y}$ exists, continuous and $\frac{\partial f}{\partial y} \geq 0$ in $\{(x,y) : [0;1] \times \mathbb{R}\}$. They established an identity based on non-uniform mesh, which gives various methods. They presented three methods for the singular boundary value problem (1.95) which are $O(h^2)$ convergent.

The authors ([34, 36]) constructed a new finite difference method for weakly singular two point boundary value problems (1.95). In papers [34] and [36] authors, shown that the method are based on uniform mesh and they provide $O(h^4)$ and $O(h^2)$ convergent approximation for all $\alpha \in [0,1)$, respectively.

In [37, 76, 136] two point singular boundary value problem (1.96) is considered. Chawla et al. [37] constructed a finite difference method based on uniform mesh and provides $O(h^2)$ convergence for all $\alpha \geq 1$, while, the construction of spline finite difference method for $\alpha = 1,\ 2$ or $\alpha \in (0,1)$ is discussed by Iyengar and Jain [76]. They established $O(h^2)$ convergence. For the same problem (1.96), Sakai and Usmani [136] have considered an application of simple non-polynomial splines and proved $O(h^2)$ convergence. They discussed two cases when $0 < \alpha < 1$ and $\alpha \geq 1$.

Jain and Jain [77], have derived three point finite difference method for singular boundary value problem

$$y'' + \frac{2}{x}y' = f(x,y), \;\; 0 < x \leq 1, \;\; y'(0) = 0, \quad y(1) = A.$$

In 2000, Guoqiang et al. [60] discussed three point finite difference approximation and a spline approximation for singular differential equation (1.96) with boundary conditions $y'(0^+) = 0,\ y(1) = A,\ \alpha \geq 1$. In 2002, a second order spline method for the singular differential equation (1.96) with boundary conditions $y'(0) = 0$ or $y(0) = a$, and $a_1 y(1) + b_1 y'(1) = c_1$, where $a_1 > 0,\ b_1 \geq 0$ has been derived by Pandey [119].

Let us consider a general class of singular boundary value problem

$$(p(x)y'(x))' = p(x)f(x,y(x)), \tag{1.97}$$

$$\lim_{x \to 0^+} py' = 0, \quad y(1) = A, \tag{1.98}$$

or, $$y'(0) = 0 \text{ or } y(0) = A, \quad ay(1) + by'(1) = c, \quad a > 0, \;\; b \geq 0. \tag{1.99}$$

Pandey [113, 116, 118] and Pandey and Singh [121] derived a finite difference method for singular boundary value problem (1.97)–(1.99), and extended various results existing in the literature. They established $O(h^2)$ convergence under quite general conditions of the functions $p(x)$ and $f(x,y)$.

Pandey and Singh [120, 122, 123] have described the finite difference method for the following singular boundary value problems

$$(p(x)y'(x))' = f(x,y(x)), \tag{1.100}$$

$$y'(0) = 0, \quad ay(1) + by'(1) = c, \quad a > 0, \;\; b \geq 0, \tag{1.101}$$

or, $$y(0) = A, \quad ay(1) + by'(1) = c, \quad a > 0, \quad b \geq 0. \tag{1.102}$$

The second order convergence of the methods have been established under quite general conditions on $p(x)$ and $f(x,y)$. These results extend the work of Chawla [34], Chawla and Katti [35] for a class of functions $p(x)$.

In 1996, Sen and Hossain [138] used a method, which is based on Newton's method. In order to linearize the following singular boundary value problem of the form

$$\sum_{k=0}^{2} (-1)^k \frac{d^k}{dx^k}\left[p_k(x)\frac{d^k u}{dx^k}\right] = F(x,u), \;\; 0 < x \leq 1, \tag{1.103}$$

they assumed that one or more coefficients of $p_k(x)$ can be infinite at $x = 0$, with boundary conditions

$$\sum_{k=0}^{2}(A_{l_k})\left(\frac{d^k u}{dx^k}\right)_{x\to 0}+\sum_{k=0}^{3}(B_{l_k})\left(\frac{d^k u}{dx^k}\right)_{x=1}=g_l, \qquad l=1,2,3,4. \tag{1.104}$$

They applied series expansion about a small neighbourhood of the point $x = 0$ and difference method in the rest of interval and presented consistency, stability and error estimates.

Making use of Green's functions and shooting method, Ha and Lee [63] discussed numerical solutions for the following two point singular boundary value problem

$$(p(x)u'(x))' + q(x)u(x) = -f(x), \quad a < x < b,$$
$$a_1u(a) + a_2u'(a) = 0, \quad b_1u(b) + b_2u'(b) = 0.$$

In 2003, El-Gabeily and Attili [49] proposed an iterative method, coupled with shooting for the following differential equation subject to certain boundary conditions,

$$-(py')' + qy = wf, \quad \text{on } J,$$

where $J = (a,b), \quad -\infty < a < b < \infty$.

In last few years, some iterative numerical methods like Adomain decomoposition method (ADM), modified Adomain decomoposition method (MADM), homotopy analysis method (HAM), variational iteration method (VIM) and Homotopy perturbation method (HPM) are developed (see [1, 2, 43, 48, 64, 65, 67, 79, 145]).

In 2008, Mittal and Nigam [105] discussed Adomain decomposition method for singular differential equation (1.100), with boundary conditions $y(0) = A, \quad y(1) = B$, or $y(0) = A, \ ay(1) + by'(1) = B$, where $p(x) = x^\gamma g(x), \ 0 \le \gamma < 1$ and $A, \ B$ are finite constant. In 2010, Khuri and Sayfy [84] discussed a new approach for the numerical solution of singular boundary value problems of the form

$$y'' + \frac{\alpha}{x}y' = f(x,y), \quad x \in [0,b], \ \alpha = 0, \ 1, \ 2,$$
$$y'(0) = 0, \ (\text{or}, y(0) = \eta), \quad a_1y(b) + a_2y'(b) = B,$$

where $a_1 > 0, \ a_2 \ge 0$, and B are finite constant. This approach based on a modified decomposition method in combination with the cubic B-spline collocation technique.

Lot of computations are required, when we use ADM or MADM, to solve the nonlinear singular boundary value problem (1.105)–(1.106). Basically, it requires computation of undetermined coefficients in a sequence of nonlinear algebraic or more difficult transcendental equations. Additionally, in some cases, it is not possible to determined the undetermined coefficients uniquely which may be the biggest demerit of these methods for solving nonlinear singular boundary value problem.

In 2014, Singh and Kumar [144] have presented an improved decomposition method (IDM) for singular boundary value problem

$$y'' + \frac{\alpha}{x}y' = f(x,y), \quad 0 < x \leq 1, \;\; \alpha \geq 1, \tag{1.105}$$

$$y'(0) = 0, \quad a_1 y(b) + a_2 y'(b) = B, \tag{1.106}$$

where $a_1 > 0$, $a_2 \geq 0$, and B are finite constant. This work is based on Green's function and the Adomian decomposition method (ADM).

Most recently, the variational iteration method and its modification have been studied extensively in the literature [65, 68, 79, 158]. The basic concept of this method is to construct a correction functional using a general Lagrange multiplier. We choose the multiplier in such a manner that its correction solution is improved with respect to the initial approximation or to the trial function. In 2010, Kanth and Aruna [79] used variational iteration method for the singular boundary value problem

$$y'' + \frac{\alpha}{x}y' + f(x,y) = 0, \quad x \in [0,1], \;\; \alpha \geq 1, \tag{1.107}$$

$$y(0) = A, \; (\text{or}, y'(0) = B), \quad y(1) = C \; (\text{or}, \; a_1 y(1) + a_2 y'(1) = b), \tag{1.108}$$

where A, B, C, a_1, a_2 and b are real constant. Wazwaz [158] has also explored variational iteration method for singular boundary value problem (1.105)–(1.106) for $\alpha = 0, 1, 2, 3$.

1.9 Discrete Boundary Value Problem

The boundary value problems in continuous case have been studied in great detail. However, the discrete analogue requires further exploration.In 1997, Agarwal and O'Regan [5] have presented two new existence results for a second order two point discrete boundary value problem

$$\Delta^2 y(i-1) + \mu f(i, y(i)) = 0, \quad i \in \mathbb{N}, \tag{1.109}$$

$$y(0) = 0, \quad y(T+1) = 0, \tag{1.110}$$

where $u \geq 0$, $T \in \{1, 2, 3, \cdots\}$, $N = \{1, 2 \cdots\}$ $N^+ = \{1, \cdots T+1\}$, $y : \mathbb{N}^+ \to \mathbb{R}$ and $f : N \times \mathbb{R} \to \mathbb{R}$ is continuous. One existence method is based on the notion of upper and lower

Fig. 1.3 A route map of thesis

solutions and the second result is based on the discrete Gelfand problem. In 1999, they [6] dealt with a nonlinear discrete boundary value problem (1.109)–(1.110), in which $\mu = 1$ and nonlinear source terms $f(i,u)$ may be singular at $u = 0$ point.

In 2003, Atici and Cabada [15] considered the following nonlinear periodic discrete boundary value problem

$$\begin{aligned} &-\Delta^2 y(n-1) + q(n)y(n) = f(n,y(n)), \;\; n \in [1,N], \\ &\;\; y(0) = y(N), \quad \Delta y(0) = \Delta y(N), \end{aligned}$$

where N is a fixed integer. They proved existence and uniqueness results for the solutions of considered problem by using an application of the Brower fixed point theorem and the properties of the Green's function. In 2002, Henderson and Thompson [71] coupled Brouwer degree theory with discrete upper and lower solutions and discussed the existence results of two point discrete boundary value problem.

In 1996, Zhuang et al. [166] derived existence result for two point discrete boundary value problem

$$\Delta^2 v_{k-1} + f(k,v_k) = 0, \quad k = 1,2,\cdots n, \tag{1.111}$$

$$v_k = 0 = v_{k+1}, \tag{1.112}$$

where $f(k,v)$ is a real function for $k = 1,2,\cdots,n$ and $v \in \mathbb{R}$. They obtained existence results by using monotone iterative method in the presence of upper and lower solution.

In 1998, Wang [156], considered the discrete boundary value problem

$$-\delta^2 u(t) + P_N f\left(\frac{t}{N}, u(t)\right) = 0, \quad t \in I_1^{N-1}, \tag{1.113}$$

$$u(0) = \alpha, \quad u(N) = \beta, \tag{1.114}$$

where $f : I_0^N \times \mathbb{R} \to \mathbb{R}$. He assumed that $I_1^{N-1} = \{1,2,\cdots,N-1\}$ and $I_0^N = I_1^{N-1} \bigcup \{0,N\}$ and for $u(t) : I_0^N \to \mathbb{R}$, he defined

$$\begin{aligned} \delta^2 u(t) &= u(t-1) - 2u(t) + u(t+1), \qquad t \in I_1^{N-1}, \\ P_N u(t) &= \frac{1}{12N^2}\left(u(t-1) + 10u(t) + u(t+1)\right), \quad t \in I_1^{N-1}. \end{aligned}$$

Basically, the boundary value problem (1.113)–(1.114) originate from the discretization of continuous boundary value problem $y''(x) = f(x,y(x)),\ 0 < x < 1,\ y(0) = \alpha,\ y(1) = \beta$ by using the fourth-order Numerov's method (see [156] and references therein). The existence results of the solutions of discrete boundary value problem (1.113)–(1.114) is obtained with the help of monotone iterative method in the presence of upper and lower solutions. In 1999, Wang and Agarwal [155] extended the results of [156] and proposed a monotone iterative method for a system of discrete boundary value problem.

In this thesis, we developed a monotone iterative method in the presence of upper and lower solutions, for two point discrete boundary value problems and discussed the existence results.

1.10 A survey of the contents of the thesis

A brief synopsis of the thesis is as fellows.

In chapter 2, we consider the following class of three point boundary value problem

$$\begin{aligned} &y''(t) + f(t,y) = 0, \quad 0 < t < 1, \\ &y'(0) = 0, \quad y(1) = \delta y(\eta), \end{aligned}$$

where $\delta > 0$, $0 < \eta < 1$, the source term $f(t,y)$ is Lipschitz and continuous. We use monotone iterative technique in the presence of upper and lower solution for both well order and reverse order case. Under some sufficient conditions we prove some existence results. We use examples and figures to demonstrate that monotone iterative method can efficiently be used for computation of solutions of nonlinear BVPs.

In chapter 3, we deal with derivative dependent nonlinear three point boundary value problem

$$\begin{aligned} &y'' + f(t,y,y') = 0,\ 0 < t < 1, \\ &y'(0) = 0,\ y'(1) = \delta y(\eta), \end{aligned}$$

where $\delta > 0$, $0 < \eta < 1$ and use monotone iterative technique to derive some sufficient conditions of existence. Examples are included to illustrate the effectiveness of the proposed results. We consider both well ordered and reverse ordered upper and lower solutions.

Chapter 4, deals with a class of nonlinear three point boundary value problems (BVPs) with Neumann type boundary conditions

$$\begin{aligned} &y''(t) + f(t,y,y') = 0, \quad 0 < t < 1, \\ &y'(0) = 0, \quad y'(1) = \delta y'(\eta), \end{aligned}$$

where $f \in C(I \times R, R)$, $I = [0,1]$, $0 < \eta < 1$, $0 < \delta < 1$. The source term (nonlinear term) depends on the solutions of the derivative, it is also Lipschitz and continuous. We use monotone iterative technique in the presence of upper and lower solutions for both well order and reverse order case. Under some sufficient conditions we prove some existence results. We also construct two examples to validate our results.

In chapter 5, we investigate the existence results for the following second order three point boundary value problem with Dirichlet type boundary conditions

$$\begin{aligned} &y''(t) + f(t,y,y') = 0, \quad 0 < t < 1, \\ &y(0) = 0, \quad y(1) = \delta y(\eta), \end{aligned}$$

where $f(I \times R, R)$, $I = [0,1]$, $0 < \eta < 1$, $\delta > 0$. We consider simple iterative scheme and develop a monotone iterative technique. Some examples are constructed to show the accuracy of the present method.

In chapter 6, we consider the following class of nonlinear three point singular boundary value problems (SBVPs)

$$\begin{aligned} &-y''(x) - \frac{2}{x}y'(x) = f(x,y), \quad 0 < x < 1, \\ &y'(0) = 0, \quad y(1) = \delta y(\eta), \end{aligned}$$

where $\delta > 0$ and $0 < \eta < 1$. We establish some maximum principles. Further using these maximum principles and monotone iterative technique in the presence of upper and lower solution we prove existence of solutions for the above class of nonlinear three point SBVPs. Here the nonlinear term is one sided Lipschitz continuous in its domain, also $x = 0$ is regular singular point of the above differential equation.

In chapter 7, we prove maximum and anti-maximum principle for the following differential inequalities,

$$\begin{aligned} &-(xy'(x))' - \lambda xy(x) \geq 0, \quad 0 < x < 1, \\ &y'(0) = 0, \quad y(1) - \delta y(\eta) \geq 0, \end{aligned}$$

where $\delta > 0$ and $0 < \eta < 1$ and use it to examine the existence of solutions of the following class of nonlinear three point singular boundary value problems (SBVPs)

$$\begin{aligned} &-y''(x) - \frac{1}{x}y'(x) = f(x,y), \quad 0 < x < 1, \\ &y'(0) = 0, \quad y(1) = \delta y(\eta). \end{aligned}$$

We use monotone iterative technique in the presence of upper and lower solutions which can be arranged in one way (well order) or the other (reverse order) and prove existence theorems. The point $x = 0$ is again a regular singular point of the differential equation.

In chapter 8, we consider a more generalize form of nonlinear three point singular boundary value problems (SBVPs). In this chapter, we analyze the existence of unique solutions of the following class of nonlinear three point singular boundary value problems (SBVPs),

$$\begin{aligned} -\left(x^{\alpha} y'(x)\right)' &= x^{\alpha} f(x,y), \quad 0 < x < 1, \\ y'(0) &= 0, \quad y(1) = \delta y(\eta), \end{aligned}$$

where $\delta > 0$, $0 < \eta < 1$ and $\alpha \geq 1$. This study shows some novel observations regarding the nature of the solution of the nonlinear three point SBVPs. We observe that when $sup\,(\partial f/\partial y) > 0$ for $\alpha \in \cup_{n\in\mathbb{N}}(4n-1, 4n+1)$ or $\alpha \in \{1,5,9,\cdots\}$ reverse ordered case occur. When $sup\,(\partial f/\partial y) > 0$ for $\alpha \in \cup_{n\in\mathbb{N}}(4n-3, 4n-1)$ or $\alpha \in \{3,7,11,\cdots\}$ and when $sup\,(\partial f/\partial y) < 0$ for all $\alpha \geq 1$ well order case occur. The results of chapter 6 and chapter 7 are particular cases of this chapter.

In chapter 9, we examine a class of nonlinear three point singular boundary value problems (SBVPs), when the nonlinearity depends upon derivative of the type

$$\begin{aligned} -y''(x) - \frac{1}{x} y'(x) &= f(x,y,xy'), \quad 0 < x < 1, \\ y'(0) &= 0, \quad y(1) = \delta y(\eta), \end{aligned}$$

where $\delta > 0$, $0 < \eta < 1$. We establish the maximum and anti-maximum principles for linear model and prove some inequalities based on Bessel and modified Bessel functions. Finally by using the Monotone iterative technique, we obtain the existence results for both well order and reverse order cases of upper and lower solutions. This chapter extends the results of chapter 7.

In chapter 10, we propose a modification to Quasi-Newton method and use it to solve a class of nonlinear two point singular boundary value problems (SBVPs)

$$-(x^{\alpha} y')' = x^{\alpha} f(x,y),\ x \in (0,1),\ y'(0) = 0,\ a_1 y(1) + b_1 y'(1) = c_1,$$

where $\alpha \geq 1$ and $a_1, b_1, c_1 \in \mathbb{R}$. We compute the relaxation parameter as a function of another variable and express it in terms of Bessel and modified Bessel functions. Since rate of convergence of solutions to the iterative scheme depends on relaxation parameter, we can have faster convergence. Some real life test examples validate our results very well.

In chapter 11, we propose an effective numerical technique for a class of nonlinear singular boundary value problems

$$-u'' - \frac{\alpha}{x}u' = f(x,u), \quad 0 < x < 1, \quad ' \equiv \frac{d}{dx}, \tag{1.115}$$

$$u'(0) = B, \quad a_1u(1) + b_1u'(1) = c_1, \tag{1.116}$$

where α, B, a_1, b_1, c_1 are real constants and $\alpha \geq 1$. We assume that $f(x,u)$ is continuous and Lipschitz continuous in a domain $D = \{(x,u) \in [0,1] \times \mathbb{R}\}$. Such nonlinear singular boundary value problems arise due to physical symmetry in chemistry and other branches. This technique is the combination of variational iteration and homotopy perturbation. It produces the approximate solution in the form of series, which is very handy from computational point of view. The effectiveness and accuracy of proposed method are revealed by some test examples.

In chapter 12, a monotone iterative method with the support of upper and lower solutions is proposed to solve nonlinear discrete boundary value problems

$$-\Delta^2 y(t-1) = f(t, y(t)), \qquad t \in [1,T],$$
$$y(0) = 0, \qquad y(T+1) = 0,$$

where T is a positive integer, $[1,T]$ is the set $\{1,2,\cdots,T\}$, $f : [0,T] \times R \to R$, and Δ is the forward difference operator. We establish existence results. Under some sufficient conditions, we establish maximum principle for linear discrete boundary value problem, which relies on Green's function and its constant sign. We then use it to establish existence of solution for the nonlinear discrete boundary value problem.

Chapter 2

Picard type iterative scheme for a class of nonlinear three point BVPs

2.1 Introduction

Consider the following nonlinear three point boundary value problem

$$y''(t) + f(t,y) = 0, \quad 0 < t < 1, \tag{2.1}$$

$$y'(0) = 0, \quad y(1) = \delta y(\eta), \tag{2.2}$$

where $f \in C(I \times \mathbb{R}, \mathbb{R})$, $I = [0,1]$, $0 < \eta < 1$, $\delta > 0$.

Li et. al. [92] studied the existence and uniqueness of solutions of second order three point BVP (2.1)–(2.2) with upper and lower solutions in the reversed order via the monotone iterative method in Banach space.

In this chapter, we establish some existence results for a class of nonlinear three point BVP (2.1)–(2.2). We allow $\sup\left(\frac{\partial f}{\partial y}\right)$ to take both negative and positive values. Our technique is based on Picard type iterative scheme given by

$$-y''_{n+1}(t) - \lambda y_{n+1}(t) = f(t, y_n(t)) - \lambda y_n(t),$$
$$y'_{n+1}(0) = 0, \quad y_{n+1}(1) = \delta y_{n+1}(\eta), \quad 0 < \eta < 1, \quad \delta > 0,$$

where $\lambda \in \mathbb{R} \setminus \{0\}$, $n \in \mathbb{N}$. This scheme is quite simple and efficient from computational point of view. We have considered both well order and reverse order cases.

This chapter is divided into four sections. In Section 2.2 and Section 2.3, we construct Green's function and establish maximum and anti-maximum principles, respectively. In Section 2.4, we generate monotone sequences by using results of Sections 2.2 and 2.3 with upper and lower solutions as initial iterates ordered in one way or the other. We prove our final results of existence. In Section 2.5, we show that the monotone iterative scheme is a powerful technique. For that by using iterative scheme proposed in this chapter we have computed the members of sequences in both cases (well order and non well order case). Finally, conclusions are drawn in Section 2.6.

2.2 Construction of Green's function

To investigate (2.1)–(2.2) we consider the following linear three point BVP

$$-y''(t)-\lambda y(t)=h(t), \quad 0<t<1, \tag{2.3}$$
$$y'(0)=0, \quad y(1)=\delta y(\eta)+b, \tag{2.4}$$

where $h\in C(I)$ and b is any constant. In this section we construct the Green's function. We divide it into two cases.

2.2.1 Case I: $\lambda>0$.

Let us assume

$$(H_0):\ 0<\lambda<\tfrac{\pi^2}{4}, \quad \sin\sqrt{\lambda}-\delta\sin\sqrt{\lambda}\eta\geq 0, \quad \delta\cos\sqrt{\lambda}\eta-\cos\sqrt{\lambda}>0.$$

There exists a range of $\lambda\in(0,\frac{\pi^2}{4})$, which supports (H_0) (see fig. 2.1).

Lemma 2.1. *The Green's function for the following linear three point BVP*

$$y''(t)+\lambda y(t)=0, \quad 0<t<1, \tag{2.5}$$
$$y'(0)=0, \quad y(1)=\delta y(\eta), \tag{2.6}$$

is given by

$$G(t,s)=\frac{1}{D_\lambda}\begin{cases}[\sin\sqrt{\lambda}(1-s)+\delta\sin\sqrt{\lambda}(s-\eta)]\cos\sqrt{\lambda}t, & 0\leq t\leq s\leq\eta,\\ \cos\sqrt{\lambda}s\ \sin\sqrt{\lambda}(1-t)+\delta\cos\sqrt{\lambda}s\ \sin\sqrt{\lambda}(t-\eta), & s\leq t, s\leq\eta,\\ \sin\sqrt{\lambda}(1-s)\cos\sqrt{\lambda}t, & t\leq s, \eta\leq s,\\ \cos\sqrt{\lambda}s\ \sin\sqrt{\lambda}(1-t)+\delta\cos\sqrt{\lambda}\eta\sin\sqrt{\lambda}(t-s), & \eta\leq s\leq t\leq 1,\end{cases}$$

where $D_\lambda=\sqrt{\lambda}(\delta\cos\sqrt{\lambda}\eta-\cos\sqrt{\lambda})$ *and if* H_0 *holds then* $G(t,s)\geq 0$.

Proof. We define the Green's function as given below

$$G(t,s) = \begin{cases} a_1 \cos\sqrt{\lambda}t + a_2 \sin\sqrt{\lambda}t, & 0 \le t \le s \le \eta, \\ a_3 \cos\sqrt{\lambda}t + a_4 \sin\sqrt{\lambda}t, & s \le t, s \le \eta, \\ a_5 \cos\sqrt{\lambda}t + a_6 \sin\sqrt{\lambda}t, & t \le s, \eta \le s, \\ a_7 \cos\sqrt{\lambda}t + a_8 \sin\sqrt{\lambda}t, & \eta \le s \le t \le 1. \end{cases}$$

Making use of continuity and jump of the Green's function, for any $s \in [0,\eta]$, we have the following system of equations

$$\begin{aligned} &a_1 \cos\sqrt{\lambda}s + a_2 \sin\sqrt{\lambda}s = a_3 \cos\sqrt{\lambda}s + a_4 \sin\sqrt{\lambda}s, \\ &(-\sqrt{\lambda}a_1 \sin\sqrt{\lambda}s + a_2\sqrt{\lambda}\cos\sqrt{\lambda}s) - (-\sqrt{\lambda}a_3 \sin\sqrt{\lambda}s + a_4\sqrt{\lambda}\cos\sqrt{\lambda}s) = -1, \end{aligned}$$

which gives

$$\begin{aligned} a_1 - a_3 &= \frac{1}{\sqrt{\lambda}}\sin\sqrt{\lambda}s, \\ a_2 - a_4 &= -\frac{1}{\sqrt{\lambda}}\cos\sqrt{\lambda}s. \end{aligned}$$

Now, using the boundary conditions, we have

$$\begin{aligned} &a_2 = 0, \\ &a_3 \cos\sqrt{\lambda} + a_4 \sin\sqrt{\lambda} = \delta(a_3 \cos\sqrt{\lambda}\eta + a_4 \sin\sqrt{\lambda}\eta). \end{aligned}$$

By solving above four equations, we get

$$\begin{aligned} a_1 &= \frac{\sin\sqrt{\lambda}(1-s) + \delta\sin\sqrt{\lambda}(s-\eta)}{\sqrt{\lambda}(\delta\cos\sqrt{\lambda}\eta - \cos\sqrt{\lambda})}, \\ a_2 &= 0, \\ a_3 &= \frac{\cos\sqrt{\lambda}s[\sin\sqrt{\lambda} - \delta\sin\sqrt{\lambda}\eta]}{\sqrt{\lambda}(\delta\cos\sqrt{\lambda}\eta - \cos\sqrt{\lambda})}, \\ a_4 &= \frac{1}{\sqrt{\lambda}}\cos\sqrt{\lambda}s. \end{aligned}$$

Similarly, for any $S \in [\eta,1]$, we have

$$\begin{aligned} &a_5 \cos\sqrt{\lambda}s + a_6 \sin\sqrt{\lambda}s = a_7 \cos\sqrt{\lambda}s + a_8 \sin\sqrt{\lambda}s, \\ &(-\sqrt{\lambda}a_5 \sin\sqrt{\lambda}s + a_6\sqrt{\lambda}\cos\sqrt{\lambda}s) - (-\sqrt{\lambda}a_7 \sin\sqrt{\lambda}s + a_8\sqrt{\lambda}\cos\sqrt{\lambda}s) = -1, \end{aligned}$$

which gives

$$
\begin{aligned}
a_5 - a_7 &= \frac{1}{\sqrt{\lambda}} \sin \sqrt{\lambda} s, \\
a_6 - a_8 &= -\frac{1}{\sqrt{\lambda}} \cos \sqrt{\lambda} s.
\end{aligned}
$$

By using the boundary conditions, we have

$$
\begin{aligned}
&a_6 = 0, \\
&a_7 \cos \sqrt{\lambda} + a_8 \sin \sqrt{\lambda} = \delta(a_5 \cos \sqrt{\lambda}\eta + a_6 \sin \sqrt{\lambda}\eta).
\end{aligned}
$$

Hence, we get

$$
\begin{aligned}
a_5 &= \frac{\sin \sqrt{\lambda}(1-s)}{\sqrt{\lambda}(\delta \cos \sqrt{\lambda}\eta - \cos \sqrt{\lambda})}, \\
a_6 &= 0, \\
a_7 &= \frac{\sin \sqrt{\lambda}\ \cos \sqrt{\lambda} s - \delta \sin \sqrt{\lambda} s\ \cos \sqrt{\lambda}\eta}{\sqrt{\lambda}(\delta \cos \sqrt{\lambda}\eta - \cos \sqrt{\lambda})}, \\
a_8 &= \frac{1}{\sqrt{\lambda}} \cos \sqrt{\lambda} s.
\end{aligned}
$$

This completes the construction of the Green's function $G(t,s)$.

When (H_0) holds, by using properties of *sine* and *cosine* it is easy to show that $G(t,s) \geq 0$, for any $s,t \in [0,1]$. ■

Lemma 2.2. *When $\lambda > 0$, let $y \in C^2(I)$ be a solution of boundary value problem (2.3)–(2.4). Then $y(t)$ is defined as,*

$$
y(t) = \frac{b \cos \sqrt{\lambda} t}{\cos \sqrt{\lambda} - \delta \cos \sqrt{\lambda}\eta} - \int_0^1 G(t,s)h(s)ds. \tag{2.7}
$$

Proof. Suppose $G(t,s)$ is the Green's function of the following homogeneous boundary value problem

$$
\begin{aligned}
&y''(t) + \lambda y(t) = 0, \quad 0 < t < 1, \\
&y'(0) = 0, \quad y(1) = \delta y(\eta),
\end{aligned}
$$

and $\bar{y}(t)$ satisfies the following linear boundary value problem

$$\begin{aligned} &-y''(t)-\lambda y(t)=0, \quad 0<t<1, \\ &y'(0)=0, \quad y(1)=\delta(\eta)+b. \end{aligned}$$

Then the linear boundary value problem (2.3)–(2.4) is equivalent to the integral equation

$$y(t)=\bar{y}(t)-\int_0^1 G(t,s)h(s)ds.$$

Suppose

$$\bar{y}(t)=c_1\cos\sqrt{\lambda}t+c_2\sin\sqrt{\lambda}t.$$

Making use of boundary conditions

$$\bar{y}'(0)=0, \text{ and } \bar{y}(1)=\delta\bar{y}(\eta)+b,$$

we get

$$\begin{aligned} c_1 &= \frac{b}{\cos\sqrt{\lambda}-\delta\cos\sqrt{\lambda}\eta}, \\ c_2 &= 0. \end{aligned}$$

Hence the linear three point boundary value problem (2.3)–(2.4) is equivalent to

$$y(t)=\frac{b\cos\sqrt{\lambda}t}{\cos\sqrt{\lambda}-\delta\cos\sqrt{\lambda}\eta}-\int_0^1 G(t,s)h(s)ds.$$

■

Remark 2.1. *Particularly, $y\in C^2(I)$ is a solution of the linear three boundary value problem (2.3)–(2.4) if and only if $y\in C(I)$ is a solution of the integral equation*

$$y(t)=\frac{b\cos\sqrt{\lambda}t}{\cos\sqrt{\lambda}-\delta\cos\sqrt{\lambda}\eta}-\int_0^1 G(t,s)h(s)ds.$$

2.2.2 Case II: $\lambda<0$.

Assume that

(H_0') $\lambda<0$, $\delta\cosh\sqrt{|\lambda|}\eta-\cosh\sqrt{|\lambda|}<0$, $\sinh\sqrt{|\lambda|}-\delta\sinh\sqrt{|\lambda|}\eta\geq 0$.

There exists a range of $\lambda < 0$, which supports (H_0') (see fig. 2.3).

Lemma 2.3. *The Green's function for the following linear three point BVP*

$$y''(t)+\lambda y(t)=0, \quad 0<t<1,$$
$$y'(0)=0, \quad y(1)=\delta y(\eta),$$

for $\lambda < 0$ is given by

$$G(t,s)=\frac{1}{D'_\lambda}\begin{cases} \Big[\sinh\sqrt{|\lambda|}(1-s) \\ \quad +\delta\sinh\sqrt{|\lambda|}(s-\eta)\Big]\cosh\sqrt{|\lambda|}t, & 0\le t\le s\le\eta \\ \cosh\sqrt{|\lambda|}s\ \sinh\sqrt{|\lambda|}(1-t) \\ \quad +\delta\cosh\sqrt{|\lambda|}s\ \sinh\sqrt{|\lambda|}(t-\eta), & s\le t, s\le\eta, \\ \sinh\sqrt{|\lambda|}(1-s)\cosh\sqrt{|\lambda|}t, & t\le s, \eta\le s, \\ \cosh\sqrt{|\lambda|}s\ \sinh\sqrt{|\lambda|}(1-t) \\ \quad +\delta\cosh\sqrt{|\lambda|}\eta\ \sin\sqrt{|\lambda|}(t-s), & \eta\le s\le t\le 1, \end{cases}$$

where $D'_\lambda = \sqrt{|\lambda|}(\delta\cosh\sqrt{|\lambda|}\eta-\cosh\sqrt{|\lambda|})$ and if H_0' holds then $G(t,s)\le 0$.

Proof. Proof is same as given in Lemma 2.1. ∎

Lemma 2.4. *When $\lambda < 0$, $y\in C^2(I)$ is a solution of boundary value problem (2.3)–(2.4) and is given by*

$$y(t)=\frac{b\cosh\sqrt{|\lambda|}t}{\cosh\sqrt{|\lambda|}-\delta\cosh\sqrt{|\lambda|}\eta}-\int_0^1 G(t,s)h(s)ds. \tag{2.8}$$

Proof. Proof is same as given in Lemma 2.2. ∎

2.3 Maximum and anti-maximum principle

Proposition 2.1. ***Anti-maximum principle***
Let $b\ge 0$, $h(t)\in C[0,1]$ is such that $h(t)\ge 0$ and (H_0) holds, then the solution of (2.3) and (2.4) is non-positive on I.

Proposition 2.2. ***Maximum principle***
Let $b\ge 0$, $h(t)\in C[0,1]$ is such that $h(t)\ge 0$, and (H_0') hold, then the solution of (2.3) and (2.4) is non-negative on I.

2.4 Nonlinear three point BVP

By using maximum and anti-maximum principles we develop theory which solves the three point nonlinear BVP (2.1)–(2.2). We divide it into the following two subsections.

2.4.1 Reverse ordered lower and upper solution

Theorem 2.1. *Let there exist α_0, β_0 in $C^2[0,1]$ such that $\beta_0 \le \alpha_0$ and satisfy*

$$-\beta_0''(t) \ge f(t,\beta_0), \quad 0<t<1, \quad \beta_0'(0)=0, \quad \beta_0(1) \ge \delta\beta_0(\eta), \tag{2.9}$$

and

$$-\alpha_0''(t) \le f(t,\alpha_0), \quad 0<t<1, \quad \alpha_0'(0)=0, \quad \alpha_0(1) \le \delta\alpha_0(\eta). \tag{2.10}$$

If $f: D \to \mathbb{R}$ is continuous on $D := \{(t,y) \in [0,1]\times\mathbb{R} : \beta_0 \le y \le \alpha_0\}$ and there exists $M>0$ such that for all $(t,y),(t,w) \in D$,

$$y \le w \Longrightarrow f(t,w) - f(t,y) \le M(w-y),$$

then the nonlinear three point boundary value problem (2.1)–(2.2) has at least one solution in the region D. Further if $\exists$ a constant λ such that $M-\lambda \le 0$ and (H_0) is satisfied and then the sequence (β_n) generated by

$$-y_{n+1}''(t) - \lambda y_{n+1} = F(t,y_n), \quad y_{n+1}'(0)=0, \quad y_{n+1}(1) = \delta y_{n+1}(\eta), \tag{2.11}$$

where $F(t,y_n) = f(t,y_n) - \lambda y_n$, with initial iterate β_0 converges monotonically and uniformly towards a solution $u(t)$ of (2.1)–(2.2). Similarly α_0 as an initial iterate leads to a non-increasing sequence (α_n) converging to a solution $v(t)$. Any solution $z(t)$ in D must satisfy

$$u(t) \le z(t) \le v(t).$$

Proof. From equation (2.9) and equation (2.11) (for $n=0$)

$$\begin{aligned} &-(\beta_0-\beta_1)'' - \lambda(\beta_0-\beta_1) \ge 0,\\ &(\beta_0-\beta_1)'(0)=0, \quad (\beta_0-\beta_1)(1) \ge \delta(\beta_0-\beta_1)(\eta). \end{aligned}$$

Since $h(t) \ge 0$ and $b \ge 0$, by using Proposition 2.1 we have $\beta_0 \le \beta_1$.

In view of $\lambda \geq M$, from equation (2.11) we get

$$-\beta_{n+1}''(t) \geq -(M-\lambda)(\beta_{n+1}-\beta_n)+f(t,\beta_{n+1})$$

and if $\beta_{n+1} \geq \beta_n$, then

$$-\beta_{n+1}''(t) \geq f(t,\beta_{n+1}), \quad \beta_{n+1}'(0)=0, \;\; \beta_{n+1}(1)=\delta\beta_{n+1}(\eta). \tag{2.12}$$

Since $\beta_0 \leq \beta_1$, then from equation (2.12) (for $n=0$) and (2.11) (for $n=1$) we get

$$\begin{aligned} &-(\beta_1-\beta_2)''-\lambda(\beta_1-\beta_2) \geq 0, \\ &(\beta_1-\beta_2)'(0)=0, \quad (\beta_1-\beta_2)(1) \geq \delta(\beta_1-\beta_2)(\eta), \end{aligned}$$

From Proposition 2.1 we have $\beta_1 \leq \beta_2$.

Now from equations (2.10) and (2.11) (for $n=0$)

$$\begin{aligned} &-(\beta_1-\alpha_0)''-\lambda(\beta_1-\alpha_0) \geq 0, \\ &(\beta_1-\alpha_0)'(0)=0 \quad (\beta_1-\alpha_0)(1) \geq \delta(\beta_1-\alpha_0)(\eta). \end{aligned}$$

Thus $\beta_1 \leq \alpha_0$ follows from proposition 2.1.

Now assuming $\beta_{n+1} \geq \beta_n$, $\beta_{n+1} \leq \alpha_0$, we show that $\beta_{n+2} \geq \beta_{n+1}$ and $\beta_{n+2} \leq \alpha_0$ for all n. From equation (2.11) (for $n+1$) and (2.12) we get

$$\begin{aligned} &-(\beta_{n+1}-\beta_{n+2})''-\lambda(\beta_{n+1}-\beta_{n+2}) \geq 0, \\ &(\beta_{n+1}-\beta_{n+2})'(0)=0, \quad (\beta_{n+1}-\beta_{n+2})(1) \geq \delta(\beta_{n+1}-\beta_{n+2})(\eta), \end{aligned}$$

and hence from Proposition 2.1 we have $\beta_{n+1} \leq \beta_{n+2}$.

From equation (2.11) (for $n+1$) and (2.10) we get,

$$\begin{aligned} &-(\beta_{n+2}-\alpha_0)''-\lambda(\beta_{n+2}-\alpha_0) \geq 0, \\ &(\beta_{n+2}-\alpha_0)'(0)=0, \quad (\beta_{n+2}-\alpha_0)(1) \geq \delta(\beta_{n+2}-\alpha_0)(\eta). \end{aligned}$$

Then from proposition 2.1, $\beta_{n+2} \leq \alpha_0$ and hence we have

$$\beta_1 \leq \beta_2 \leq \cdots \leq \beta_n \leq \beta_{n+1} \leq \cdots \leq \alpha_0,$$

and starting with α_0 it is easy to get

$$\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n \geq \alpha_{n+1} \geq \ldots \geq \beta_0.$$

Finally we show that $\beta_n \leq \alpha_n$ for all n. For this by assuming $\beta_n \leq \alpha_n$, we show that $\beta_{n+1} \leq \alpha_{n+1}$. From equation (2.11) it is easy to get

$$\begin{aligned} &-(\beta_{n+1}-\alpha_{n+1})'' - \lambda(\beta_{n+1}-\alpha_{n+1}) \geq 0, \\ &(\beta_{n+1}-\alpha_{n+1})'(0) = 0, \quad (\beta_{n+1}-\alpha_{n+1})(1) \geq \delta(\beta_{n+1}-\alpha_{n+1})(\eta). \end{aligned}$$

Hence from Proposition 2.1, $\beta_{n+1} \leq \alpha_{n+1}$. Thus we have

$$\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n \geq \alpha_{n+1} \geq \cdots \geq \beta_{n+1} \geq \beta_n \geq \cdots \geq \beta_2 \geq \beta_1 \geq \beta_0.$$

So the sequences (β_n) and (α_n) are monotonically non-decreasing and non-increasing, respectively and are bounded by β_0 and α_0. Hence by Dini's theorem they converges uniformly. Let $u(t) = \lim_{n\to\infty} \beta_n(t)$ and $v(t) = \lim_{n\to\infty} \alpha_n(t)$.

Using Lemma 2.2, the solution β_{n+1} of equation (2.11) is given by

$$\beta_{n+1} = \frac{b\cos\sqrt{\lambda}t}{\cos\sqrt{\lambda} - \delta\cos\sqrt{\lambda}\eta} - \int_0^1 G(t,s)(f(t,\beta_n) - \lambda\beta_n)ds.$$

Then by Lebesgue's dominated convergence theorem, taking the limit as n approaches to ∞, we get

$$u(t) = \frac{b\cos\sqrt{\lambda}t}{\cos\sqrt{\lambda} - \delta\cos\sqrt{\lambda}\eta} - \int_0^1 G(t,s)(f(t,u) - \lambda u)ds.$$

Which is the solution of boundary value problem (2.1)–(2.2).

Any solution $z(t)$ in D can play the role of $\beta_0(t)$, hence $z(t) \leq v(t)$ and similarly one concludes that $z(t) \geq u(t)$. ■

2.4.2 Well ordered lower and upper solution

Theorem 2.2. *Let there exist α_0, β_0 in $C^2[0,1]$ such that $\beta_0 \geq \alpha_0$ and satisfy*

$$-\beta_0''(t) \geq f(t,\beta_0), \quad 0 < t < 1, \quad \beta_0'(0) = 0, \quad \beta_0(1) \geq \delta\beta_0(\eta), \tag{2.13}$$

and

$$-\alpha_0''(t) \le f(t,\alpha_0), \quad 0<t<1, \quad \alpha_0'(0)=0, \quad \alpha_0(1) \le \delta\alpha_0(\eta). \tag{2.14}$$

If $f : D_0 \to \mathbb{R}$ *is continuous on* $D_0 := \{(t,y) \in [0,1]\times\mathbb{R} : \alpha_0 \le y \le \beta_0\}$ *and there exists* $M > 0$ *such that for all* $(t,\widetilde{y}),(t,\widetilde{w}) \in D_0$

$$\widetilde{y} \le \widetilde{w} \Longrightarrow f(t,\widetilde{w}) - f(t,\widetilde{y}) \ge -M(\widetilde{w}-\widetilde{y}),$$

then the nonlinear three point boundary value problem (2.1)–(2.2) has at least one solution in the region D_0*. If* $\exists$ *a constant* $\lambda < 0$*, such that* $\lambda + M \le 0$ *and* (H_0') *is satisfied then the sequence* (β_n) *generated by*

$$-y_{n+1}''(t) - \lambda y_{n+1} = F(t,y_n), \quad y_{n+1}'(0)=0, \quad y_{n+1}(1) = \delta y_{n+1}(\eta), \tag{2.15}$$

where $F(t,y_n) = f(t,y_n) - \lambda y_n$*, with initial iterate* β_0 *converges monotonically (non-increasing) and uniformly towards a solution* $\widetilde{u}(t)$ *of (2.1)–(2.2). Similarly* α_0 *as an initial iterate leads to a non-decreasing sequence* (α_n) *converging to a solution* $\widetilde{v}(t)$*. Any solution* $\widetilde{z}(t)$ *in* D_0 *must satisfy*

$$\widetilde{v}(t) \le \widetilde{z}(t) \le \widetilde{u}(t).$$

Proof. Proof follows from the analysis of Theorem 2.1. ■

2.5 Examples

To verify our results, we consider two examples and show that there exists at least one value of $\lambda \in \mathbb{R}\setminus\{0\}$ such that iterative scheme generates monotone sequences which converge to solutions of nonlinear problem. Thus these examples validate sufficient conditions derived in this chapter.

Example 2.1. *Consider the nonlinear three point boundary value problem*

$$-y''(t) = \frac{e^y}{32} - \frac{1}{64}, \quad 0<t<1, \tag{2.16}$$

$$y'(0)=0, \quad y(1) = 2y\left(\frac{1}{3}\right). \tag{2.17}$$

Here $f(t,y) = \frac{e^y}{32} - \frac{1}{64}$, $\delta = 2, \eta = \frac{1}{3}$. This problem has $\alpha_0 = 1$ and $\beta_0 = -1$ as lower and upper solution, i.e., it is non well ordered case. The nonlinear term is Lipschitz in y and continuous for all values of y, and Lipschitz constant M is $\frac{e}{32}$. For $0.0849463 \leq \lambda < \frac{\pi^2}{4}$, assumption (H_0) is true. To verify (H_0) in Figure 2.1 we plot inequalities assumed in (H_0). From Figure 2.3 to 2.7 we plot members of monotone sequences (β_n), (α_n) for different values of λ. Thus existence of a solution for the problem (2.16)–(2.17) is guaranteed.

Example 2.2. *Consider the nonlinear three point boundary value problem*

$$-y''(t) = \frac{1}{32}\left[\frac{e^2}{4} - \frac{\sin t}{4} - 2(y(t))^3\right], \tag{2.18}$$

$$y'(0) = 0, \qquad y(1) = \frac{1}{3}y\left(\frac{1}{2}\right). \tag{2.19}$$

Here $f(t,y) = \frac{1}{32}\left[\frac{e^2}{4} - \frac{\sin t}{4} - 2(y(t))^3\right]$, $\delta = \frac{1}{3}, \eta = \frac{1}{2}$. This problem has $\alpha_0 = -1$ and $\beta_0 = 1$ as lower and upper solution, i.e., it is well ordered case. The nonlinear term is Lipschitz in y and continuous for all values of y, and Lipschitz constant M is $\frac{3}{16}$. For $\lambda \leq -0.1875$, we can see that (H_0') is true. To verify (H_0') in Figure 2.2 we plot inequalities assumed in (H_0'). From Figure 2.4 to 2.8 we plot members of monotone sequences (β_n), (α_n) for different values of λ. Thus the problem (2.18)–(2.19) satisfies all the conditions of the Theorem 2.2, existence of a solution is guaranteed.

2.6 Conclusion

The monotone iterative technique coupled with upper and lower solutions is a powerful tool for computation of solutions of nonlinear three point boundary value problems. It proves existence of solutions analytically and gives us a tool so that numerical solutions can also be computed and then some real life problems, e.g., bridge design problem, thermostat problem etc can be solved. For $\lambda > 0$, we arrived at reverse order and $\lambda < 0$, we arrived well order case. We have plotted sequences for both $\lambda > 0$ and $\lambda < 0$. The plots are quite encouraging and will motivate researchers to explore further possibilities. Employing this technique Mathematica/Maple/MATLAB user friendly packages can be developed (see [28]).

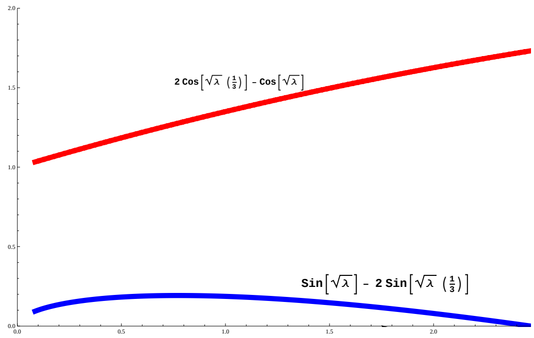

Fig. 2.1 Plot of (H_0) for example 2.1.

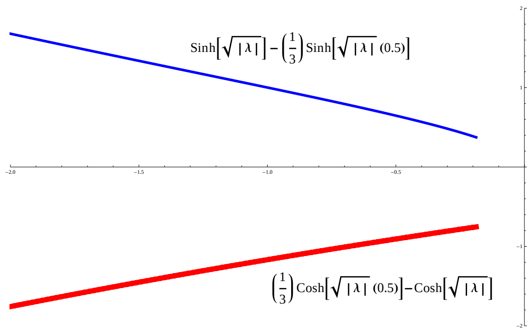

Fig. 2.2 Plot of (H_0') for example 2.2.

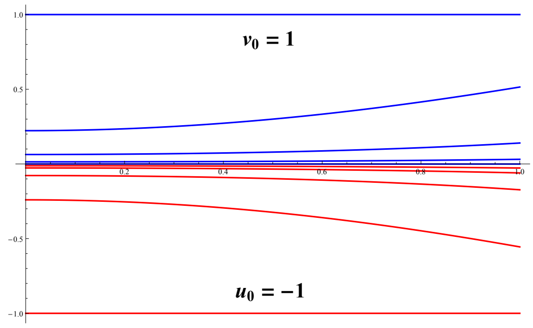

Fig. 2.3 Plot of upper (β_n) & lower (α_n) solutions for $n = 0, 1, 2, 3, 4$ and $\lambda = 0.9$.

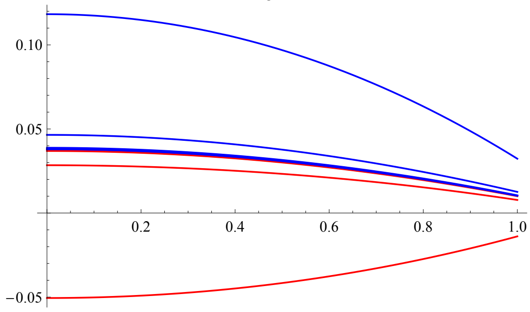

Fig. 2.4 Plot of upper (β_n) & lower (α_n) solutions for $n = 1, 2, 3, 4, 5$ and $\lambda = -0.2$.

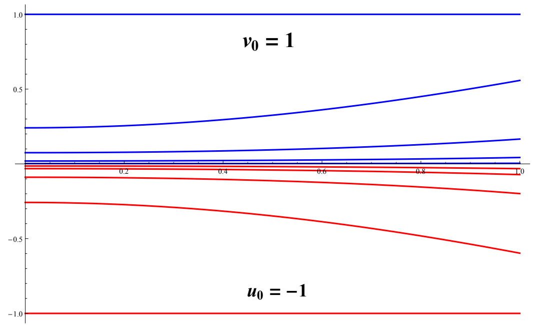

Fig. 2.5 Plot of upper (β_n) & lower (α_n) solutions for $n = 0, 1, 2, 3, 4$ and $\lambda = 1$.

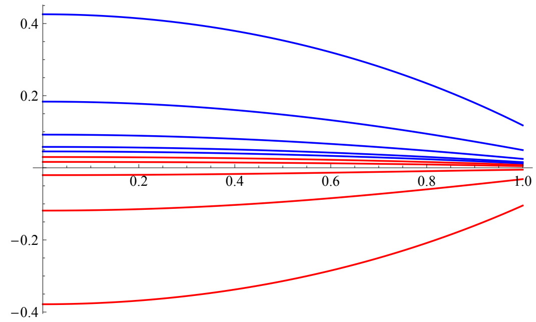

Fig. 2.6 Plot of upper (β_n) & lower (α_n) solutions for $n = 1, 2, 3, 4, 5$ and $\lambda = -1$.

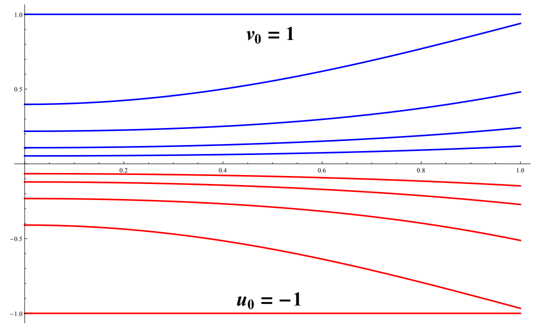

Fig. 2.7 Plot of upper (β_n) & lower (α_n) solutions for $n = 0, 1, 2, 3, 4$ and $\lambda = 2.3$.

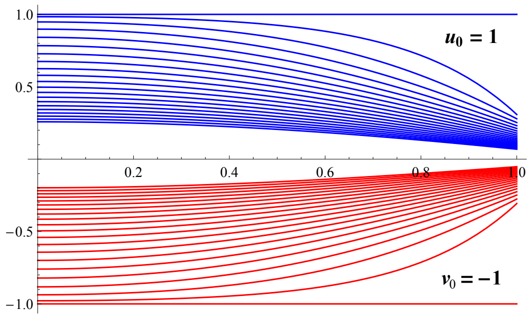

Fig. 2.8 Plot of upper (β_n) & lower (α_n) solutions for $n = 0, 1, 2, \cdots, 20$ and $\lambda = -20$.

Chapter 3

Nonlinear three point nonsingular BVPs with upper and lower solutions in reverse order : Derivative dependent case

3.1 Introduction

Consider the following class of nonlinear three point nonsingular BVP

$$y''(t) + f(t, y, y') = 0, \quad 0 < t < 1, \tag{3.1}$$

$$y'(0) = 0, \quad y'(1) = \delta y(\eta), \tag{3.2}$$

where $f \in C(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $I = [0, 1]$, $0 < \eta < 1$, $\delta > 0$. Here the source function f is derivative dependent.

In this chapter, we study existence of solutions of the second order nonlinear three point boundary value problem (3.1)–(3.2). We define the following iterative scheme

$$\begin{aligned} -y''_{n+1}(t) - \lambda y_{n+1} &= f(t, y_n(t), y'_n(t)) - \lambda y_n(t), \\ y'_{n+1}(0) = 0, \quad y'_{n+1}(1) &= \delta y_{n+1}(\eta), \end{aligned}$$

where $\lambda \in \mathbb{R} \setminus \{0\}$, $n \in \mathbb{N}$. Though source function is derivative dependent but to keep our iterative scheme as simple as possible we have not considered the derivative term in the iterative scheme. For $\lambda > 0$, we arrived at reverse order case and $\lambda < 0$ we arrived well order case.

This chapter is divided in six sections. In Sections 3.2–3.4, we discuss construction and positivity (negativity) of Green's function and some important results to be used in later sections. Section 3.5, we explore the nonlinear BVP and derive sufficient conditions which guarantee the existence of solutions of nonsingular nonlinear three point boundary value problems. In Section 3.6, two examples are constructed to validate our results. Finally, Conclusions are summarised in Section 3.7.

3.2 Construction of Green's function

To investigate (3.1) and (3.2), we consider the corresponding linear boundary value problem given by

$$-y''(t) - \lambda y(t) = h(t), \quad 0 < t < 1, \tag{3.3}$$

$$y'(0) = 0, \quad y'(1) = \delta y(\eta) + b, \tag{3.4}$$

where $h \in C(I)$ and b is any constant. In this section we prove some maximum and anti-maximum principles for the above linear problem and prove existence of some inequalities. We divide it in the following cases:

3.2.1 Case I: $\lambda > 0$

Let us assume

$(H_0): \ 0 < \lambda < \frac{\pi^2}{4}, \quad \sqrt{\lambda}\cos\sqrt{\lambda} - \delta\sin\sqrt{\lambda}\eta \geq 0.$

It is easy to see that (H_0) is satisfied, for some sub interval of $\lambda \in (0, \pi^2/4)$ (see fig. 3.1).

Lemma 3.1. *The Green's function for the following linear three point boundary value problem*

$$y''(t) + \lambda y(t) = 0, \quad 0 < t < 1, \tag{3.5}$$

$$y'(0) = 0, \quad y'(1) = \delta y(\eta), \tag{3.6}$$

for $\lambda > 0$ is given by

$$G(t,s) = \frac{1}{\sqrt{\lambda}(D_{\lambda_T})} \begin{cases} [\sqrt{\lambda}\cos\sqrt{\lambda}(1-s) + \delta\sin\sqrt{\lambda}(s-\eta)]\cos\sqrt{\lambda}t, & 0 \leq t \leq s \leq \eta, \\ \sqrt{\lambda}\cos\sqrt{\lambda}s\ \cos\sqrt{\lambda}(1-t) + \delta\cos\sqrt{\lambda}s\ \sin\sqrt{\lambda}(t-\eta), & s \leq t, s \leq \eta, \\ \sqrt{\lambda}\cos\sqrt{\lambda}(1-s)\cos\sqrt{\lambda}t, & t \leq s, \eta \leq s, \\ \sqrt{\lambda}\cos\sqrt{\lambda}s\ \cos\sqrt{\lambda}(1-t) + \delta\cos\sqrt{\lambda}\eta\ \sin\sqrt{\lambda}(t-s), & \eta \leq s \leq t \leq 1. \end{cases}$$

where $D_{\lambda_T} = \delta\cos\sqrt{\lambda}\eta + \sqrt{\lambda}\sin\sqrt{\lambda}$.

Proof. Define the Green's function as follows

$$G(t,s) = \begin{cases} a_1 \cos\sqrt{\lambda}t + a_2 \sin\sqrt{\lambda}t, & 0 \le t \le s \le \eta; \\ a_3 \cos\sqrt{\lambda}t + a_4 \sin\sqrt{\lambda}t, & s \le t, s \le \eta; \\ a_5 \cos\sqrt{\lambda}t + a_6 \sin\sqrt{\lambda}t, & t \le s, \eta \le s; \\ a_7 \cos\sqrt{\lambda}t + a_8 \sin\sqrt{\lambda}t, & \eta \le s \le t \le 1. \end{cases}$$

Using the properties of the Green's function, for any $s \in [0,\eta]$, we have

$$\begin{aligned} & a_1 \cos\sqrt{\lambda}s + a_2 \sin\sqrt{\lambda}s = a_3 \cos\sqrt{\lambda}s + a_4 \sin\sqrt{\lambda}s, \\ & (-a_1\sqrt{\lambda}\sin\sqrt{\lambda}s + a_2\sqrt{\lambda}\cos\sqrt{\lambda}s) - (-a_3\sqrt{\lambda}\sin\sqrt{\lambda}s + a_4\sqrt{\lambda}\cos\sqrt{\lambda}s) = -1, \end{aligned}$$

which gives

$$\begin{aligned} a_1 - a_3 &= \frac{1}{\sqrt{\lambda}} \sin\sqrt{\lambda}s, \\ a_2 - a_4 &= -\frac{1}{\sqrt{\lambda}} \cos\sqrt{\lambda}s. \end{aligned}$$

Using the boundary conditions, we have

$$\begin{aligned} & a_2 = 0, \\ & -a_3\sqrt{\lambda}\sin\sqrt{\lambda} + a_4\sqrt{\lambda}\cos\sqrt{\lambda} = \delta\left(a_3\cos\sqrt{\lambda}\eta + a_4\sin\sqrt{\lambda}\eta\right), \end{aligned}$$

By solving above equations, we get

$$\begin{aligned} a_1 &= \frac{\sqrt{\lambda}\cos\sqrt{\lambda}(1-s) + \delta\sin\sqrt{\lambda}(s-\eta)}{\sqrt{\lambda}(\delta\cos\sqrt{\lambda}\eta + \sqrt{\lambda}\sin\sqrt{\lambda})}, \\ a_2 &= 0, \\ a_3 &= \frac{\cos\sqrt{\lambda}s\left(\sqrt{\lambda}\cos\sqrt{\lambda} - \delta\sin\sqrt{\lambda}\eta\right)}{\sqrt{\lambda}(\delta\cos\sqrt{\lambda}\eta + \sqrt{\lambda}\sin\sqrt{\lambda})}, \\ a_4 &= \frac{1}{\sqrt{\lambda}}\cos\sqrt{\lambda}s. \end{aligned}$$

Similarly, for any $s \in [\eta, 1]$, we have

$$\begin{aligned} & a_5 \cos\sqrt{\lambda}s + a_6 \sin\sqrt{\lambda}s = a_7 \cos\sqrt{\lambda}s + a_8 \sin\sqrt{\lambda}s, \\ & \left(-a_5\sqrt{\lambda}\sin\sqrt{\lambda}s + a_6\sqrt{\lambda}\cos\sqrt{\lambda}s\right) - \left(-a_7\sqrt{\lambda}\sin\sqrt{\lambda}s + a_8\sqrt{\lambda}\cos\sqrt{\lambda}s\right) = -1, \end{aligned}$$

which gives

$$\begin{aligned}
a_5 - a_7 &= \frac{1}{\sqrt{\lambda}}\sin\sqrt{\lambda}s,\\
a_6 - a_8 &= -\frac{1}{\sqrt{\lambda}}\cos\sqrt{\lambda}s.
\end{aligned}$$

By using the boundary conditions, we have

$$\begin{aligned}
&a_6 = 0,\\
&-a_7\sqrt{\lambda}\sin\sqrt{\lambda} + a_8\sqrt{\lambda}\cos\sqrt{\lambda} = \delta\left(a_5\cos\sqrt{\lambda}\eta + a_6\sin\sqrt{\lambda}\eta\right).
\end{aligned}$$

This gives

$$\begin{aligned}
a_5 &= \frac{\sqrt{\lambda}(\cos\sqrt{\lambda}(1-s))}{\sqrt{\lambda}(\delta\cos\sqrt{\lambda}\eta + \sqrt{\lambda}\sin\sqrt{\lambda})},\\
a_6 &= 0,\\
a_7 &= \frac{\sqrt{\lambda}\cos\sqrt{\lambda}\cos\sqrt{\lambda}s - \delta\cos\sqrt{\lambda}\eta\sin\sqrt{\lambda}s}{\sqrt{\lambda}(\delta\cos\sqrt{\lambda}\eta + \sqrt{\lambda}\sin\sqrt{\lambda})},\\
a_8 &= \frac{1}{\sqrt{\lambda}}\cos\sqrt{\lambda}s.
\end{aligned}$$

Consequently, this completes the construction of the Green's functions. ■

Lemma 3.2. *Let $y \in C^2(I)$ be a solution of the boundary value problem (3.3) and (3.4) then*

$$y(t) = -\frac{b\cos\sqrt{\lambda}t}{\delta\cos\sqrt{\lambda}\eta + \sqrt{\lambda}\sin\sqrt{\lambda}} - \int_0^1 G(t,s)h(s)ds. \tag{3.7}$$

***Proof*.** Suppose $G(t,s)$ is the Green's function of

$$\begin{aligned}
&y''(t) + \lambda y(t) = 0, \quad 0 < t < 1,\\
&y'(0) = 0, \quad y'(1) = \delta y(\eta),
\end{aligned}$$

and $\bar{y}$ is solution of

$$\begin{aligned}
&-y''(t) - \lambda y(t) = 0, \quad 0 < t < 1,\\
&y'(0) = 0, \quad y'(1) = \delta y(\eta) + b,
\end{aligned}$$

then the boundary value problem (3.3) and (3.4) is equivalent to

$$y(t) = \bar{y} - \int_0^1 G(t,s)h(s)\,ds.$$

Suppose

$$\bar{y} = c_1 \cos\sqrt{\lambda}t + c_2 \sin\sqrt{\lambda}t.$$

Since

$$\bar{y}'(0) = (0), \quad \bar{y}'(1) = \delta\bar{y}(\eta) + b,$$

we get

$$c_1 = -\frac{b}{\delta\cos\sqrt{\lambda}\eta + \sqrt{\lambda}\sin\sqrt{\lambda}}, \qquad c_2 = 0.$$

Hence the solution of this boundary value problem (3.3) and (3.4) is given by

$$y(t) = -\frac{b\cos\sqrt{\lambda}t}{\delta\cos\sqrt{\lambda}\eta + \sqrt{\lambda}\sin\sqrt{\lambda}} - \int_0^1 G(t,s)h(s)ds.$$

Namely $y \in C^2(I)$ is a solution of the boundary value problem (3.3) and (3.4) if and only if $y \in C(I)$ is a solution of the integral equation

$$y(t) = -\frac{b\cos\sqrt{\lambda}t}{\delta\cos\sqrt{\lambda}\eta + \sqrt{\lambda}\sin\sqrt{\lambda}} - \int_0^1 G(t,s)h(s)ds.$$

■

3.2.2 Case II: $\lambda < 0$

Assume

(H_0') : $\lambda < 0$, $\sqrt{|\lambda|}\cosh\sqrt{|\lambda|} - \delta\sinh\sqrt{|\lambda|}\eta \geq 0$, $\delta\cosh\sqrt{|\lambda|}\eta - \sqrt{|\lambda|}\sinh\sqrt{|\lambda|} < 0$.

This condition (H_0') is satisfied, for some values of $\lambda \in (-\infty, 0)$ (see fig. 3.2).

Lemma 3.3. *The Green's function for the following linear three point boundary value problem*

$$y''(t) + \lambda y(t) = 0, \quad 0 < t < 1,$$

$$y'(0) = 0, \quad y'(1) = \delta y(\eta),$$

for $\lambda < 0$ *is given by*

$$G(t,s) = \frac{1}{\sqrt{|\lambda|}(D_{\lambda_H})} \begin{cases} \Big[\sqrt{|\lambda|}\cosh\sqrt{|\lambda|}(1-s) \\ \quad -\delta\sinh\sqrt{|\lambda|}(\eta-s)\Big]\cosh\sqrt{|\lambda|}t, & 0 \le t \le s \le \eta, \\ \sqrt{|\lambda|}\cosh\sqrt{|\lambda|}s\cosh\sqrt{|\lambda|}(1-t) \\ \quad -\delta\cosh\sqrt{|\lambda|}s\sinh\sqrt{|\lambda|}(\eta-t), & s \le t, s \le \eta, \\ \sqrt{|\lambda|}\cosh\sqrt{|\lambda|}(1-s)\cosh\sqrt{|\lambda|}t, & t \le s, \eta \le s, \\ \sqrt{|\lambda|}\cosh\sqrt{|\lambda|}s\cosh\sqrt{|\lambda|}(1-t) \\ \quad +\delta\cosh\sqrt{|\lambda|}\eta\sinh\sqrt{|\lambda|}(t-s), & \eta \le s \le t \le 1. \end{cases}$$

where $D_{\lambda_H} = \delta\cosh\sqrt{|\lambda|}\eta - \sqrt{|\lambda|}\sinh\sqrt{|\lambda|}$.

Proof. The construction of Green's function is same as given in Lemma 3.1. ■

Lemma 3.4. *For* $\lambda < 0$, *a solution* $y \in C^2(I)$ *of the boundary value problem (3.3) and (3.4) is given by*

$$y(t) = -\frac{b\cosh\sqrt{|\lambda|}t}{\delta\cosh\sqrt{|\lambda|}\eta - \sqrt{|\lambda|}\sinh\sqrt{|\lambda|}} - \int_0^1 G(t,s)h(s)ds. \tag{3.8}$$

Proof. Proof is same as Lemma 3.2. ■

3.3 Maximum and anti-maximum principle

Proposition 3.1. (***Anti-maximum principle***)
Let (H_0) *holds,* $b \ge 0$ *and* $h(t) \in C[0,1]$ *is such that* $h(t) \ge 0$, *then the solution of (3.3) and (3.4) is non-positive for all* $t \in [0,1]$.

Proposition 3.2. (***Maximum principle***)
Let (H_0') *holds,* $b \ge 0$ *and* $h(t) \in C[0,1]$ *is such that* $h(t) \ge 0$, *then the solution of (3.3) and (3.4) is non-negative for all* $t \in [0,1]$.

3.4 Existence of some inequalities

Lemma 3.5. *Let $M,\ N \in \mathbb{R}^+$. If $\lambda \in (0, \frac{\pi^2}{4})$ is such that $M-\lambda \le 0$ and*

$$(M-\lambda)\cos\sqrt{\lambda} + N\sqrt{\lambda}\sin\sqrt{\lambda} \le 0,$$

then for all $t \in [0,1]$

$$(M-\lambda)\cos\sqrt{\lambda}t + N\sqrt{\lambda}\sin\sqrt{\lambda}t \le 0.$$

Proof. Using the properties of sin and cos, in the interval $(0, \frac{\pi^2}{4})$, we deduce that for all $t \in [0,1]$,

$$(M-\lambda)\cos\sqrt{\lambda}t + N\sqrt{\lambda}\sin\sqrt{\lambda}t \le (M-\lambda)\cos\sqrt{\lambda} + N\sqrt{\lambda}\sin\sqrt{\lambda} \le 0.$$

■

Lemma 3.6. *Let $M,\ N \in \mathbb{R}^+$. If $\lambda < 0$ is such that $M+\lambda \le 0$, and*

$$\lambda \le -M - \frac{N^2}{2} - \frac{N}{2}\sqrt{N^2+4M},$$

then for all $t \in [0,1]$,

$$(M+\lambda)\cosh\sqrt{|\lambda|}t + N\sqrt{|\lambda|}\sinh\sqrt{|\lambda|}t \le 0.$$

Proof. We have

$$(M+\lambda)\cosh\sqrt{|\lambda|}t + N\sqrt{|\lambda|}\sinh\sqrt{|\lambda|}t \le [(M+\lambda)+N\sqrt{|\lambda|}]\cosh\sqrt{|\lambda|}t.$$

The right hand side of above the inequality will be non-positive for all $t \in [0,1]$ if

$$[(M+\lambda)+N\sqrt{|\lambda|}] \le 0.$$

The above inequality is satisfied provided

$$\lambda \le -M - \frac{N^2}{2} - \frac{N}{2}\sqrt{N^2+4M}.$$

This completes the lemma. ■

Lemma 3.7. *Suppose (H_0) holds; then*

(i) $G(t,s) \ge 0$,

(ii) $\frac{\partial G(t,s)}{\partial t} \leq 0$, *and*

(iii) $(M-\lambda)G(t,s) - N\ (sign\ y')\frac{\partial G(t,s)}{\partial t} \leq 0$, *whenever we have* $M-\lambda \leq 0$

for any $t,s \in [0,1]$ *and* $t \neq s$.

Proof. From (H_0), we get that $G \geq 0$. Using the fact that $G(t,s)$ is the solution of (3.5) and (3.6) it is easy to verify that, $\frac{\partial G(t,s)}{\partial t} \leq 0$, for $t \neq s$. Form (iii), we arrive at two inequalities one of which is true. So it is sufficient to prove

$$(M-\lambda)G(t,s) - N\frac{\partial G(t,s)}{\partial t} \leq 0. \tag{3.9}$$

Substituting $G(t,s)$ from Lemma 3.1 and by using (H_0) and Lemma 3.5 it is easy to prove that the inequality (3.9) holds, for $t,s \in [0,1]$ and $t \neq s$. ∎

Lemma 3.8. *Suppose* (H_0') *holds; then for any* $t,\ s \in [0,1]$

(i) $G(t,s) \leq 0$,

(ii) $\frac{\partial G(t,s)}{\partial t} \leq 0$, *and*

(iii) $(M+\lambda)G + N\ (sign\ y')\frac{\partial G(t,s)}{\partial t} \geq 0$ *whenever we have* $M+\lambda-N\lambda \leq 0$,

for any $t,s \in [0,1]$ *and* $t \neq s$.

Proof. The first two parts (i) and (ii) can be proved by similar to Lemma 3.7. For part (iii), it is sufficient to prove that for all $t,\ s \in [0,1]$ and $t \neq s$,

$$(M+\lambda)G + N\frac{\partial G(t,s)}{\partial t} \geq 0.$$

As $G(t,s)$ is the Green's function of (3.3) and (3.4), it is easy to see that $\frac{\partial G(t,s)}{\partial t} \geq (-\lambda\ G(t,s))$, for $t \neq s$. Which along with condition $M+\lambda-N\lambda \leq 0$ gives $(M+\lambda)G + N\frac{\partial G(t,s)}{\partial t} \geq (M+\lambda)G - N\lambda\ G \geq 0$. This completes the proof. ∎

Remark 3.1. *The condition (iii) of Lemma 3.8 gives a bound on N. In fact, if N verifies* $M+\lambda-N\lambda \leq 0$, *we have*

$$N \leq 1 + \frac{M}{\lambda} < 1.$$

3.5 Nonlinear BVP

3.5.1 Reverse ordered lower and upper solutions

We define upper and lower solutions by some suitable differential inequalities.

Definition 3.1. *A function $\alpha_0 \in C^2[0,1]$ is a lower solution of the boundary value problem (3.1) and (3.2) if*

$$\begin{aligned} -\alpha_0''(t) &\le f(t,\alpha_0,\alpha_0'), \quad 0<t<1,\\ \alpha_0'(0)=0, &\quad \alpha_0'(1) \le \delta\alpha_0(\eta). \end{aligned}$$

Definition 3.2. *A function $\beta_0 \in C^2[0,1]$ is an upper solution of the boundary value problem (3.1) and (3.2) if*

$$\begin{aligned} -\beta_0''(t) &\ge f(t,\beta_0,\beta_0'), \quad 0<t<1,\\ \beta_0'(0)=0, &\quad \beta_0'(1) \ge \delta\beta_0(\eta). \end{aligned}$$

In this chapter, we consider the sequences $(\alpha_n)_n$ and $(\beta_n)_n$ defined by the following equations

$$-\alpha_{n+1}''(t)-\lambda\alpha_{n+1} = f(t,\alpha_n,\alpha_n')-\lambda\alpha_n, \tag{3.10}$$
$$\alpha_{n+1}'(0)=0, \quad \alpha_{n+1}'(1)=\delta\alpha_{n+1}(\eta), \tag{3.11}$$

$$-\beta_{n+1}''(t)-\lambda\beta_{n+1} = f(t,\beta_n,\beta_n')-\lambda\beta_n, \tag{3.12}$$
$$\beta_{n+1}'(0)=0, \quad \beta_{n+1}'(1)=\delta\beta_{n+1}(\eta), \tag{3.13}$$

where $\lambda \in \mathbb{R}\setminus\{0\}$.

In this section we develop theory which proves existence of a range R_λ for the value of λ and enables us to choose at least one value of $\lambda \in R_\lambda$ such that the sequences $(\alpha_n)_n$ and $(\beta_n)_n$ converge monotonically to solutions of (3.1) and (3.2). We arrive at the following result (Theorem 3.1).

Theorem 3.1. *Let (H_0) be true. Further assume that*

(H_1) *there exist α_0 and $\beta_0 \in C^2[0,1]$ as lower and upper solutions of (3.1) and (3.2) such that $\alpha_0 \ge \beta_0$ for all $t \in [0,1]$;*

(H_2) *the function* $f : D \to \mathbb{R}$ *is continuous on*

$$D := \{(t,u,v) \in [0,1] \times \mathbb{R}^2 : \beta_0(t) \le u \le \alpha_0(t)\};$$

(H_3) *there exists* $M > 0$ *such that for all* $(t,u_1,v),(t,u_2,v) \in D$

$$u_1 \le u_2 \Longrightarrow f(t,u_2,v) - f(t,u_1,v) \le M(u_2 - u_1);$$

(H_4) *there exists* $N > 0$ *such that for all* $(t,u,v_1),(t,u,v_2) \in D$

$$|f(t,u,v_2) - f(t,u,v_1)| \le N|v_2 - v_1|;$$

(H_5) *for all* $(t,u,v) \in D, |f(t,u,v)| \le \varphi(|v|)$;
where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *is continuous and satisfies*

$$\max_{t\in[0,1]} \alpha_0 - \min_{t\in[0,1]} \beta_0 \le \int_{l_0}^{\infty} \frac{s\,ds}{\varphi(s)},$$

where $l_0 = [2|\Omega_0|]$ *and* $\Omega_0 = \max\{|\alpha_0(t)|_\infty, |\beta_0(t)|_\infty\}$.
Let $\lambda \in (0, \frac{\pi^2}{4})$ *be such that* $M - \lambda \le 0$*, and*

$$(M-\lambda)\cos\sqrt{\lambda} + N\sqrt{\lambda}\sin\sqrt{\lambda} \le 0,$$

and for all $t \in [0,1]$

$$f(t,\beta_0(t),\beta_0'(t)) - f(t,\alpha_0(t),\alpha_0'(t)) - \lambda(\beta_0 - \alpha_0) \ge 0.$$

Then the sequences (α_n) *and* (β_n) *defined by (3.10), (3.11) and (3.12), (3.13) converges uniformly in* $C^1([0,1])$ *to solution* v *and* u *of nonlinear boundary value problem (3.1) and (3.2), such that for all* $t \in [0,1]$,

$$\beta_0(t) \le u \le v \le \alpha_0(t).$$

The proof of above theorem requires several results given below.

Lemma 3.9. *Let* $0 < \lambda < \frac{\pi^2}{4}$*. If* α_n *is a lower solution of (3.1) and (3.2),* α_{n+1} *is defined by (3.10) and (3.11), then* $\alpha_{n+1} \le \alpha_n$.

Proof. Since $y(t) = \alpha_{n+1} - \alpha_n$ satisfies the linear boundary value problem (3.3) and (3.4) with $h(t) \ge 0$, and $b \ge 0$. Hence the result can be concluded by Proposition 3.1. ■

Proposition 3.3. *Assume* $(H_0),(H_1),(H_2),(H_3),(H_4)$ *are valid and let* $0<\lambda<\frac{\pi^2}{4}$ *be such that* $M-\lambda\le 0$ *and* $(M-\lambda)\cos\sqrt{\lambda}+N\sqrt{\lambda}\sin\sqrt{\lambda}\le 0$ *then the function* α_n *defined recursively by (3.10) and (3.11) are such that for all* $n\in\mathbb{N}$

(i) α_n *is a lower solution of equations (3.1) and (3.2);*

(ii) $\alpha_{n+1}\le\alpha_n$.

Proof. The proof relies on mathematical induction.
Step 1 : By using Lemma 3.9, we can prove that the claim holds for $n=0$.
Step 2 : Suppose claims are true for $n-1$, then we will show that they are true for n. Let α_{n-1} be a lower solution of (3.1) and (3.2) and $\alpha_n\le\alpha_{n-1}$. Let $y=\alpha_n-\alpha_{n-1}$. We have

$$\begin{aligned}-\alpha_n''-f(t,\alpha_n,\alpha_n') &= -f(t,\alpha_n,\alpha_n')+f(t,\alpha_{n-1},\alpha_{n-1}')+\lambda(\alpha_n-\alpha_{n-1}),\\ &\le M(\alpha_{n-1}-\alpha_n)+N|\alpha_n'-\alpha_{n-1}'|+\lambda(\alpha_n-\alpha_{n-1})\\ &= (\lambda-M)y+N\,(sign\ y')y'.\end{aligned}$$

Let $(\lambda-M)y+N\,(sign\ y')y'=g$. For proving the claim, we have to show that $g\le 0$. Since y satisfies

$$\begin{aligned}-y''-\lambda y &= \alpha_{n-1}''+f(t,\alpha_{n-1},\alpha_{n-1}')\ge 0\\ y'(0)=0,&\quad y'(1)\ge\delta y(\eta),\end{aligned}$$

we can write y as given in Lemma 3.2 with $h(t)=\alpha_{n-1}''+f(t,\alpha_{n-1},\alpha_{n-1}')\ge 0$. Thus to show $g\le 0$, it is sufficient to prove

$$(M-\lambda)\cos\sqrt{\lambda}t+N\sqrt{\lambda}\sin\sqrt{\lambda}t\le 0,$$

and

$$(M-\lambda)G(t,s)-N\frac{\partial G(t,s)}{\partial t}\le 0,\quad t\ne s,$$

for all $t\in[0,1]$. Which is easily obtained by using the Lemma 3.5 and Lemma 3.7. Thus we deduce that $\alpha_{n+1}\le\alpha_n$ ■

In the same way, we can prove the following result.

Proposition 3.4. *Assume* $(H_0),(H_1),(H_2),(H_3),(H_4)$ *are valid and let* $0<\lambda<\frac{\pi^2}{4}$ *be such that* $M-\lambda\le 0$ *and* $(M-\lambda)\cos\sqrt{\lambda}+N\sqrt{\lambda}\sin\sqrt{\lambda}\le 0$, *then the function* β_n *defined recursively by (3.12) and (3.13) are such that for all* $n\in\mathbb{N}$

(i) β_n *is an upper solution of equations (3.1) and (3.2);*

(ii) $\beta_{n+1} \geq \beta_n$.

Proposition 3.5. *Assume* $(H_0),(H_1),(H_2),(H_3),(H_4)$ *are valid and let* $0 < \lambda < \frac{\pi^2}{4}$ *be such that* $M - \lambda \leq 0$ *and* $(M-\lambda)\cos\sqrt{\lambda} + N\sqrt{\lambda}\sin\sqrt{\lambda} \leq 0$, *and for all* $t \in [0,1]$

$$f(t,\beta_0(t),\beta_0'(t)) - f(t,\alpha_0(t),\alpha_0'(t)) - \lambda(\beta_0 - \alpha_0) \geq 0,$$

then for all $n \in \mathbb{N}$, *the functions* α_n *and* β_n *defined by (3.10), (3.11) and (3.12), (3.13) satisfy* $\alpha_n \geq \beta_n$.

Proof. We define

$$h_i(t) = f(t,\beta_i,\beta_i') - f(t,\alpha_i,\alpha_i') - \lambda(\beta_i - \alpha_i), \quad \text{for all } \ i \in \mathbb{N}.$$

We analyze that, for all $i \in \mathbb{N}$, $y_i := \beta_i - \alpha_i$ satisfies

$$-y_i'' - \lambda y_i = f(t,\beta_{i-1},\beta_{i-1}') - f(t,\alpha_{i-1},\alpha_{i-1}') - \lambda(\beta_{i-1} - \alpha_{i-1}) = h_{i-1}.$$

Claim 1. $\alpha_1 \geq \beta_1$.

As $h_0 \geq 0$, the function $y_1 = \beta_1 - \alpha_1$ is a solution of (3.3) and (3.4) with $h(t) = h_0(t) \geq 0$ and

$$(\beta_1 - \alpha_1)'(0) = \beta_1'(0) - \alpha_1'(0) = 0, \quad (\beta_1 - \alpha_1)'(1) = \beta_1(1) - \alpha_1(1) = \delta(\beta - \alpha)(\eta).$$

As $h_0 \geq 0$ and $b = 0$, by Proposition 3.1, $y_1(t) \leq 0$, *i.e.*, $\alpha_1 \geq \beta_1$.

Claim 2. Let $n \geq 2$. If $h_{n-2} \geq 0$ and $\alpha_{n-1} \geq \beta_{n-1}$, then $h_{n-1} \geq 0$ and $\alpha_n \geq \beta_n$. First we will prove that, for all $t \in [0,1]$, the function h_{n-1} is non-negative, as we have

$$\begin{aligned} h_{n-1} &= f(t,\beta_{n-1},\beta_{n-1}') - f(t,\alpha_{n-1},\alpha_{n-1}') - \lambda(\beta_{n-1} - \alpha_{n-1}) \\ &\geq -M(\alpha_{n-1} - \beta_{n-1}) - N|\alpha_{n-1}' - \beta_{n-1}')| - \lambda(\beta_{n-1} - \alpha_{n-1}) \\ &= -\left[(M-\lambda)y_{n-1} + N\ (sign\ y_{n-1}')y_{n-1}'\right]. \end{aligned}$$

Since, y_{n-1} is a solution of (3.3) and (3.4) with $h(t) = h_{n-2}(t) \geq 0$, $b = 0$. Hence following the analysis similar to the proof of Proposition 3.3, we show that h_{n-1} is nonnegative. Since $y_n'(0) = 0$ and $y_n'(1) = \delta y(\eta)$, i.e., $b = 0$, we deduce from Proposition 3.1 that y_n is non-positive, i.e., $\alpha_n \geq \beta_n$. ■

Lemma 3.10. *If* $f(t,y,y')$ *satisfies* (H_5) *then there exists* $R > 0$ *such that any solution of*

$$-y''(t) \geq f(t,y,y'), \quad 0 < t < 1, \tag{3.14}$$

$$y'(0) = 0, \quad y'(1) \geq \delta y(\eta), \tag{3.15}$$

with $y \in [\beta_0(t), \alpha_0(t)]$ *for all* $t \in [0,1]$ *satisfies* $\|y'\|_\infty \leq R$.

Proof. We can divide the proof in three parts.

Case :(i) If solution is not monotone throughout the interval, then we consider the interval $(t_0, t] \subset (0,1)$ such that $y'(t_0) = 0$ and $y'(t) > 0$ for $t > t_0$. Using $|f| \leq \varphi$ and integrating (3.14) from t_0 to t we get

$$\int_0^{y'} \frac{s\,ds}{\varphi(s)} \leq \max_{t\in[0,1]} \alpha_0 - \min_{t\in[0,1]} \beta_0.$$

From (H_5) we can choose $R > 0$ such that

$$\int_0^{y'} \frac{s\,ds}{\varphi(s)} \leq \max_{t\in[0,1]} \alpha_0 - \min_{t\in[0,1]} \beta_0 \leq \int_{l_0}^{R} \frac{s\,ds}{\varphi(s)} \leq \int_0^{R} \frac{s\,ds}{\varphi(s)}.$$

Which gives

$$y'(t) \leq R.$$

Now consider the case in which $y'(t) < 0$ for $t < t_0$, $y'(t_0) = 0$, and proceeding in the similar way we get

$$-y'(t) \leq R,$$

and the result follows.

Case :(ii) If y is monotonically decreasing in $(0,1)$, that is $y'(t) < 0$ in $t \in (0,1]$ then by Mean value theorem there exists a point $\tau \in (0,1)$ such that

$$-y'(\tau) \leq 2|\Omega_0|,$$

where $\Omega_0 = \max\{|\alpha_0(t)|_\infty, |\beta_0(t)|_\infty\}$. Now, using $|f| \leq \varphi$ and integrating (3.14) from t to τ we get

$$\int_0^{-y'} \frac{s\,ds}{\varphi(s)} \leq \max_{t\in[0,1]} \alpha_0 - \min_{t\in[0,1]} \beta_0.$$

From (H_5) we can choose R, such that

$$\int_0^{-y'} \frac{s\,ds}{\varphi(s)} \leq \max_{t\in[0,1]} \alpha_0 - \min_{t\in[0,1]} \beta_0 \leq \int_0^{R} \frac{s\,ds}{\varphi(s)}.$$

Which gives $-y' \leq R$.

Case :(iii) If y is monotonically increasing in $(0,1)$, that is $y'(t) > 0$ in $t \in (0,1]$. Then similar to case (ii) we conclude that $y' \leq R$. ■

Lemma 3.11. *If $f(t,y,y')$ satisfies (H_5), then there exists $R > 0$ such that any solution of*

$$-y''(t) \leq f(t,y,y'), \quad 0 < t < 1, \tag{3.16}$$

$$y'(0) = 0, \quad y'(1) \leq \delta y(\eta), \tag{3.17}$$

with $y \in [\beta_0(t), \alpha_0(t)]$ for all $t \in [0,1]$ satisfies $\|y'\|_\infty \leq R$.

***Proof*.** Proof follows from the analysis of Lemma 3.10 ■

Now we complete the proof of Theorem 3.1.

***Proof of Theorem 3.1*.** Proposition 3.3, 3.4 and 3.5 together give rise to the following

$$\alpha_0 \geq \alpha_1 \geq \cdots \geq \alpha_n \geq \cdots \geq \beta_n \geq \cdots \geq \beta_1 \geq \beta_0, \tag{3.18}$$

where the sequences $(\alpha_n)_n$ and $(\beta_n)_n$ are defined, respectively by (3.10), (3.11) and (3.12), (3.13).

It is clear that the sequences $(\alpha_n)_n$ and $(\beta_n)_n$ are monotonic in nature and bounded. Hence they converge pointwise to functions

$$v(t) = \lim_{n\to\infty} \alpha_n(t) \text{ and } u(t) = \lim_{n\to\infty} \beta_n(t)$$

which are such that for all $n, \alpha_n \geq v \geq u \geq \beta_n$.

Using iterative scheme (3.10)–(3.11) along with inequality (3.18) and Lemma 3.11, we prove that the sequence $(\alpha_n)_n$ is equibounded and equicontinuous in $C^1([0,1])$ i.e., any subsequence $(\alpha_{n_k})_k$ of $(\alpha_n)_n$ is also equibounded and equicontinuous in $C^1([0,1])$. Now, we deduce from Arzela-Ascoli theorem that there exists a subsequence $(\alpha_{n_{k_j}})_j$ of $(\alpha_{n_k})_k$ which converges uniformly in $C^1([0,1])$.

Making use of uniqueness of the limit and monotonicity of the sequence $(\alpha_n)_n$, we have $\alpha_n \to v$ in $C^1([0,1])$. As any subsequence of $(\alpha_n)_n$ contains a subsequence $(\alpha_{n_{k_j}})_j$ which converge in $C^1([0,1])$ to v it follows that $\alpha_n \to v$ in $C^1([0,1])$. In the same manner, using iterative scheme (3.12)–(3.13) along with inequality (3.18), and Lemma 3.10 we prove that $(\beta_n)_n$ converges uniformly to u in $C^1([0,1])$.
Claim 2. The functions u and v are solution of (3.1) and (3.2).

As the derivative is a closed operator, going to the limit in (3.10), (3.11) and (3.12), (3.13), it is straightforward to see that u and v are solution of (3.1) and (3.2). ■

3.5.2 Well ordered lower and upper solutions

In this section, we consider the case when $\lambda < 0$ and the upper and lower solutions are well ordered. In that case, we have to consider the opposite one sided Lipschitz condition on $f(t,.,v)$. We state main result as Theorem 3.2 and then all the supporting results without proof. Proof of all the results given below will follow similar to the case of upper and lower solutions in reverse order.

Theorem 3.2. *Let (H_0') be true. Further assume that*

(H_1') *there exist α_0 and $\beta_0 \in C^2[0,1]$ as lower and upper solutions of (3.1) and (3.2) such that $\alpha_0 \leq \beta_0$ for all $t \in [0,1]$;*

(H_2') *the function $f : D \to \mathbb{R}$ is continuous on*

$$D := \{(t,u,v) \in [0,1] \times \mathbb{R}^2 : \alpha_0(t) \leq u \leq \beta_0(t)\};$$

(H_3') *there exists $M > 0$ such that for all $(t,u_1,v),(t,u_2,v) \in D$,*

$$u_1 \leq u_2 \Longrightarrow f(t,u_2,v) - f(t,u_1,v) \geq -M(u_2 - u_1);$$

(H_4') *there exists $N > 0$ such that for all $(t,u,v_1),(t,u,v_2) \in D$,*

$$|f(t,u,v_2) - f(t,u,v_1)| \leq N|v_2 - v_1|;$$

(H_5') *for all $(t,u,v) \in D, |f(t,u,v)| \leq \varphi(|v|)$; where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and satisfies*

$$\max_{t\in[0,1]} \beta_0 - \min_{t\in[0,1]} \alpha_0 \leq \int_{l_0}^{\infty} \frac{s\,ds}{\varphi(s)},$$

where $l_0 = [2|\Omega_0|]$ and $\Omega_0 = \max\{|\alpha_0(t)|_\infty, |\beta_0(t)|_\infty\}$.

Let $\lambda < 0$ be such that $\lambda \leq \min\left\{-M, -\frac{M}{1-N}, -M - \frac{N^2}{2} - \frac{N}{2}\sqrt{N^2+4M}\right\}$, and for all $t \in [0,1]$

$$f(t,\beta_0(t),\beta_0'(t)) - f(t,\alpha_0(t),\alpha_0'(t)) - \lambda(\beta_0 - \alpha_0) \geq 0.$$

Then the sequences (α_n) and (β_n) defined by (3.10), (3.11) and (3.12), (3.13) converges uniformly in $C^1([0,1])$ to solution v and u of (3.1) and (3.2), such that for all $t \in [0,1]$, $\alpha_0(t) \leq v \leq u \leq \beta_0(t)$.

Lemma 3.12. *Let $\lambda < 0$. If α_n is a lower solution of (3.1) and (3.2), α_{n+1} is defined by (3.10) and (3.11), then $\alpha_{n+1} \geq \alpha_n$.*

Proposition 3.6. *Assume $(H_0'), (H_1'), (H_2'), (H_3'), (H_4')$ are valid and let $\lambda < 0$ be such that*

$$\lambda \leq \min\left\{-M, -\frac{M}{1-N}, -M - \frac{N^2}{2} - \frac{N}{2}\sqrt{N^2+4M}\right\},$$

then the function α_n defined recursively by (3.10) and (3.11) are such that for all $n \in \mathbb{N}$

(i) *α_n is a lower solution of equations (3.1) and (3.2);*

(ii) *$\alpha_{n+1} \geq \alpha_n$.*

Lemma 3.13. *Let $\lambda < 0$. If β_n is an upper solution of (3.1) and (3.2) and β_{n+1} is defined by (3.12) and (3.13), then $\beta_{n+1} \leq \beta_n$.*

Proposition 3.7. *Assume $(H_0'), (H_1'), (H_2'), (H_3'), (H_4')$ are valid and let $\lambda < 0$ be such that*

$$\lambda \leq \min\left\{-M, -\frac{M}{1-N}, -M - \frac{N^2}{2} - \frac{N}{2}\sqrt{N^2+4M}\right\},$$

then the function β_n defined recursively by (3.12) and (3.13) are such that for all $n \in \mathbb{N}$

(i) *β_n is an upper solution of equations (3.1) and (3.2);*

(ii) *$\beta_{n+1} \leq \beta_n$.*

Proposition 3.8. *Assume $(H_0'), (H_1'), (H_2'), (H_3'), (H_4')$ are valid and let $\lambda < 0$ be such that*

$$\lambda \leq \min\left\{-M, -\frac{M}{1-N}, -M - \frac{N^2}{2} - \frac{N}{2}\sqrt{N^2+4M}\right\},$$

and for all $t \in [0,1]$

$$f(t, \beta_0(t), \beta_0'(t)) - f(t, \alpha_0(t), \alpha_0'(t)) - \lambda(\beta_0 - \alpha_0) \geq 0.$$

Then for all $n \in \mathbb{N}$, the functions α_n and β_n defined (3.10), (3.11) and (3.12), (3.13) satisfy $\alpha_n \leq \beta_n$.

Lemma 3.14. *If $f(t,y,y')$ satisfies (H_5') then there exists $R > 0$ such that any solution of*

$$-y''(t) \geq f(t,y,y'), \quad 0 < t < 1, \tag{3.19}$$

$$y'(0) = 0, \quad y'(1) \geq \delta y(\eta), \tag{3.20}$$

with $y \in [\alpha_0(t), \beta_0(t)]$ *for all* $t \in [0,1]$ *satisfies* $\|y'\|_\infty \leq R$.

Lemma 3.15. *If* $f(t,y,y')$ *satisfies* (H_5') *then there exists* $R > 0$ *such that any solution of*

$$-y''(t) \leq f(t,y,y'), \quad 0 < t < 1, \tag{3.21}$$

$$y'(0) = 0, \quad y'(1) \leq \delta y(\eta), \tag{3.22}$$

with $y \in [\alpha_0(t), \beta_0(t)]$ *for all* $t \in [0,1]$ *satisfies* $\|y'\|_\infty \leq R$.

3.6 Examples

To verify our results, we consider examples and show that it is possible to choose a value of $\lambda \in R_\lambda$ so that iterative scheme generates monotone sequences which converge to a solution of nonlinear problem. Thus these examples validate sufficient conditions derived in this chapter. It is also shown that it not difficult to validate the conditions used in the present work, which guarantees the existence of solutions.

Example 3.1. *Consider the nonlinear three point boundary value problem*

$$-y''(t) = \frac{2e^y - e^{y'}}{64}, \quad 0 < t < 1, \tag{3.23}$$

$$y'(0) = 0, \quad y'(1) = 2y(0.2). \tag{3.24}$$

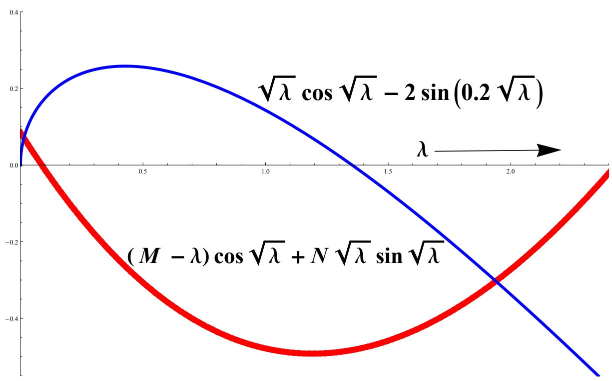

Fig. 3.1 Plots of (H_0) and $(M-\lambda)\cos\sqrt{\lambda} + N\sqrt{\lambda}\sin\sqrt{\lambda} \leq 0$ for example 3.1 .

Here $f(t,y,y') = \frac{2e^y - e^{y'}}{64}$, $\delta = 2, \eta = \frac{1}{5}$. This problem has $\alpha_0 = 1$ and $\beta_0 = -1$ as lower and upper solutions, i.e., it is non well ordered case. The nonlinear term is Lipschitz in both

y and y' and continuous for all values of y and y'. It is easy to see that Nagumo condition is given by

$$|f(t,u,v)| \leq \frac{2e+e^{|v|}}{64},$$

i.e., $\varphi = \frac{2e+e^{|v|}}{64}$. Using Lemma 3.10 we can compute bound for y', i.e., $|y'| \leq \frac{1}{4\sqrt{2}}$, i.e., $R = \frac{1}{4\sqrt{2}}$. The Lipschitz constants are calculated as $M = 0.0849463$ and $N = 0.0186463$. Now we can find out a subinterval $R_\lambda = (\xi_1, \xi_2)$ of $\left(0, \frac{\pi^2}{4}\right)$ such that for all $\lambda \in (\xi_1, \xi_2)$ the conditions $\lambda \geq M = 0.0849463$, $(M-\lambda)\cos\sqrt{\lambda} + N\sqrt{\lambda}\sin\sqrt{\lambda} \leq 0$, $\sqrt{\lambda}\cos\sqrt{\lambda} - \delta\sin\sqrt{\lambda}\eta \geq 0$ and $f(t,\beta_0,\beta_0') - f(t,\alpha_0,\alpha_0') - \lambda(\beta_0 - \alpha_0) \geq 0$ are true. The inequality $f(t,\beta_0,\beta_0') - f(t,\alpha_0,\alpha_0') - \lambda(\beta_0 - \alpha_0) \geq 0$ gives $\lambda \geq 0.036725$.

Inequalities $(M-\lambda)\cos\sqrt{\lambda} + N\sqrt{\lambda}\sin\sqrt{\lambda} \leq 0$ and $\sqrt{\lambda}\cos\sqrt{\lambda} - \delta\sin\sqrt{\lambda}\eta \geq 0$ are nonlinear in nature and not easy to compute the bound for λ. Using Mathematica we can easily plot (see Fig. 3.1) both of them and see existence of the interval (ξ_1, ξ_2). It can be seen that if value of δ is increased then the length of the interval (ξ_1, ξ_2) decreases. Thus from the sufficient conditions derived for reversed order upper and lower solutions it is guaranteed that solution of the nonlinear three point boundary value problem (3.23) and (3.24) exists.

Example 3.2. *Consider the nonlinear three point boundary value problem*

$$-y''(t) = \frac{[y'(t)]^2}{36} - 5y(t) - \frac{\sin t}{4}, \tag{3.25}$$

$$y'(0) = 0, \qquad y'(1) = \left(\frac{1}{2}\right) y\left(\frac{1}{2}\right). \tag{3.26}$$

Here $f(t,y,y') = \frac{[y'(t)]^2}{36} - 5y(t) - \frac{\sin t}{4}$, $\delta = \frac{1}{2}, \eta = \frac{1}{2}$.

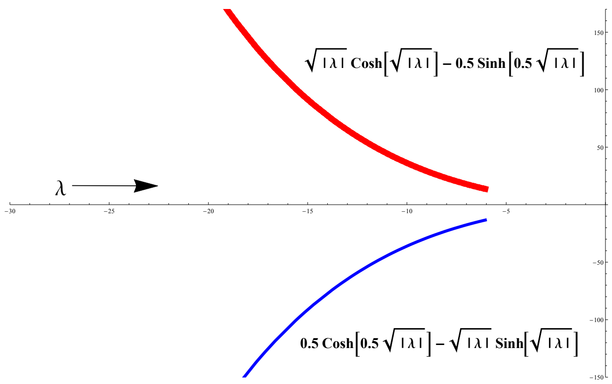

Fig. 3.2 Plots of (H_0') for example 3.2.

This problem has $\alpha_0 = -(t^2 + \frac{1}{2})$ and $\beta_0 = 0$ as lower and upper solutions, i.e., it is well ordered case. The nonlinear term is Lipschitz in both y and y' and continuous for all values of y and y'. It is easy to see that Nagumo condition is given by

$$|f(t,u,v)| \leq \frac{31}{4} + \frac{|v|^2}{36},$$

i.e., $\varphi = \frac{31}{4} + \frac{|v|^2}{36}$. Using Lemma 3.14 we can compute easily that $|y'| \leq 3e^{\frac{1}{24}}$, i.e., $R = 3e^{\frac{1}{24}}$. The Lipschitz constants are calculated as $M = 5$ and $N = \frac{1}{6}e^{\frac{1}{24}}$. Now we can find out a range for $\lambda < 0$ such that

$$\lambda \leq \min\left\{-M, -\frac{M}{1-N}, -M - \frac{N^2}{2} - \frac{N}{2}\sqrt{N^2+4M}\right\}.$$

Which gives us $\lambda \leq -6.05149$. For $\lambda < -6.05149$, it is verified that the inequalities assumed in H_0' is true (see Fig. 3.2).

Thus the sufficient conditions derived for well order upper and lower solutions guarantee that solution of the nonlinear three point boundary value problem (3.25) and (3.26) exists.

3.7 Conclusion

This chapter is devoted to the study of nonlinear three point BVPs, in which source function f is dependent on derivative. We have constructed Green's function and proved that they are of constant signs. Anti maximum and maximum principles are also established. Making use of iterative scheme in the presence of upper and lower solutions, two monotone sequences are generated. For $\lambda > 0$, we get reverse order case and we arrive well order case for $\lambda < 0$. Under some sufficient conditions, we have established the existence results which are verified with the help of two examples.

Chapter 4

Existence results for nonlinear three point BVPs with Neumann type boundary conditions

4.1 Introduction

This chapter also deals with existence of solutions for a class of nonlinear nonsingular differential equation with derivative dependent source function. Here, we impose Neumann type boundary conditions and focus on the following class of nonlinear three point nonsingular BVP

$$y''(t) + f(t, y, y') = 0, \quad 0 < t < 1, \tag{4.1}$$

$$y'(0) = 0, \quad y'(1) = \delta y'(\eta), \tag{4.2}$$

where $f \in C(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $I = [0, 1]$, $0 < \eta < 1$, $0 < \delta < 1$. Similar to chapter 3, here again we consider an iterative scheme which is simple from computational point of view. We arrive at both well order and reverse order cases.

This chapter is organized in seven sections. In Section 4.2, we discuss the corresponding linear case and construct Green's function. In Sections 4.3 and 4.4, we discuss some important lemmas and maximum and anti-maximum principles. In Section 4.5, we derive sufficient conditions which guarantee the existence of solutions of nonlinear three point nonsingular BVP for both case; i.e., when upper and lower solutions are well ordered and also when reverse ordered. In Section 4.6, two examples are constructed to validate our results. Finally, conclusion are given in Section 4.7.

4.2 Linear case and Green's function

This section deals with linear three point BVPs with Neumann type boundary conditions.

4.2.1 Construction of the Green's function

Consider the corresponding linear three point BVP given by

$$Ly \equiv -y''(t) - \lambda y(t) = h(t), \quad 0 < t < 1, \tag{4.3}$$
$$y'(0) = 0, \quad y'(1) = \delta y'(\eta) + b, \tag{4.4}$$

where $h \in C(I)$ and b is any constant. Based on the sign of λ, we can divide the construction of Green's function into two cases. In one case $\lambda > 0$, we get Green's function in terms of trigonometric functions (cos and sin). In the case when $\lambda < 0$, we get Green's function in terms of hyperbolic functions (cosh and sinh).

4.2.1.1 Case I: $\lambda > 0$

Let us assume

(H_0) : $\lambda \in (0, \pi^2/4)$, $\sin(\sqrt{\lambda}) - \delta \sin(\eta\sqrt{\lambda}) > 0$ and $\cos(\sqrt{\lambda}) - \delta \cos(\eta\sqrt{\lambda}) \geq 0$.

(H_0) is satisfied for some sub interval of $\lambda \in (0, \pi^2/4)$ (see fig. 4.1)

Lemma 4.1. *The Green's function of the linear three point BVP*

$$y''(t) + \lambda y(t) = 0, \quad 0 < t < 1, \tag{4.5}$$
$$y'(0) = 0, \quad y'(1) = \delta y'(\eta), \tag{4.6}$$

is

$$G(t,s) = \begin{cases} \frac{\cos(\sqrt{\lambda}t)(\cos(\sqrt{\lambda}(s-1)) - \delta\cos(\sqrt{\lambda}(s-\eta)))}{\sqrt{\lambda}(\sin(\sqrt{\lambda}) - \delta\sin(\eta\sqrt{\lambda}))}, & 0 \leq t \leq s \leq \eta, \\ \frac{\cos(\sqrt{\lambda}s)(\cos(\sqrt{\lambda}(t-1)) - \delta\cos(\sqrt{\lambda}(t-\eta)))}{\sqrt{\lambda}(\sin(\sqrt{\lambda}) - \delta\sin(\eta\sqrt{\lambda}))}, & s \leq t,\ s \leq \eta, \\ \frac{\cos(\sqrt{\lambda}(s-1))\cos(\sqrt{\lambda}t)}{\sqrt{\lambda}(\sin(\sqrt{\lambda}) - \delta\sin(\eta\sqrt{\lambda}))}, & t \leq s,\ \eta \leq s, \\ \frac{\delta\sin(\eta\sqrt{\lambda})\sin(\sqrt{\lambda}(s-t)) + \cos(\sqrt{\lambda}s)\cos(\sqrt{\lambda}(1-t))}{\sqrt{\lambda}(\sin(\sqrt{\lambda}) - \delta\sin(\eta\sqrt{\lambda}))}, & \eta \leq s \leq t \leq 1. \end{cases}$$

Proof. We define the Green's function as given below

$$G(t,s)=\begin{cases} a_1\cos\sqrt{\lambda}t+a_2\sin\sqrt{\lambda}t, & 0\le t\le s\le\eta;\\ a_3\cos\sqrt{\lambda}t+a_4\sin\sqrt{\lambda}t, & s\le t,\ s\le\eta;\\ a_5\cos\sqrt{\lambda}t+a_6\sin\sqrt{\lambda}t, & t\le s,\ \eta\le s;\\ a_7\cos\sqrt{\lambda}t+a_8\sin\sqrt{\lambda}t, & \eta\le s\le t\le 1.\end{cases}$$

Using the continuity and jump discontinuity of the Green's function, for any $s\in[0,\eta]$, we arrive at

$$\begin{aligned}&a_1\cos\sqrt{\lambda}s+a_2\sin\sqrt{\lambda}s=a_3\cos\sqrt{\lambda}s+a_4\sin\sqrt{\lambda}s,\\ &\left(-a_1\sqrt{\lambda}\sin\sqrt{\lambda}s+a_2\sqrt{\lambda}\cos\sqrt{\lambda}s\right)-\left(-a_3\sqrt{\lambda}\sin\sqrt{\lambda}s+a_4\sqrt{\lambda}\cos\sqrt{\lambda}s\right)=-1.\end{aligned}$$

Thus

$$\begin{aligned}a_1-a_3&=\frac{1}{\sqrt{\lambda}}\sin\sqrt{\lambda}s,\\ a_2-a_4&=-\frac{1}{\sqrt{\lambda}}\cos\sqrt{\lambda}s.\end{aligned}$$

Making use of the boundary conditions, we get

$$\begin{aligned}&a_2=0,\\ &-a_3\sqrt{\lambda}\sin\sqrt{\lambda}+a_4\sqrt{\lambda}\cos\sqrt{\lambda}=\delta\left(-a_3\sqrt{\lambda}\sin\sqrt{\lambda}\eta+a_4\sqrt{\lambda}\cos\sqrt{\lambda}\eta\right).\end{aligned}$$

This results into

$$\begin{aligned}a_1&=\frac{\cos\left(\sqrt{\lambda}(s-1)\right)-\delta\cos\left(\sqrt{\lambda}(s-\eta)\right)}{\sqrt{\lambda}\left(\sin\left(\sqrt{\lambda}\right)-\delta\sin\left(\eta\sqrt{\lambda}\right)\right)},\\ a_2&=0,\\ a_3&=\frac{\cos\left(\sqrt{\lambda}s\right)\left(\cos\left(\sqrt{\lambda}\right)-\delta\cos\left(\eta\sqrt{\lambda}\right)\right)}{\sqrt{\lambda}\left(\sin\left(\sqrt{\lambda}\right)-\delta\sin\left(\eta\sqrt{\lambda}\right)\right)},\\ a_4&=\frac{\cos\left(\sqrt{\lambda}s\right)}{\sqrt{\lambda}}.\end{aligned}$$

Similarly for any $s \in [\eta, 1]$, we have

$$a_5 \cos\sqrt{\lambda}s + a_6 \sin\sqrt{\lambda}s = a_7 \cos\sqrt{\lambda}s + a_8 \sin\sqrt{\lambda}s,$$
$$\left(-a_5\sqrt{\lambda}\sin\sqrt{\lambda}s + a_6\sqrt{\lambda}\cos\sqrt{\lambda}s\right) - \left(-a_7\sqrt{\lambda}\sin\sqrt{\lambda}s + a_8\sqrt{\lambda}\cos\sqrt{\lambda}s\right) = -1,$$

which gives

$$a_5 - a_7 = \frac{1}{\sqrt{\lambda}}\sin\sqrt{\lambda}s,$$
$$a_6 - a_8 = -\frac{1}{\sqrt{\lambda}}\cos\sqrt{\lambda}s.$$

By using the boundary conditions, we get

$$a_6 = 0,$$
$$-a_7\sqrt{\lambda}\sin\sqrt{\lambda} + a_8\sqrt{\lambda}\cos\sqrt{\lambda} = \delta\left(-a_5\sqrt{\lambda}\sin\sqrt{\lambda}\eta + a_6\sqrt{\lambda}\cos\sqrt{\lambda}\eta\right).$$

Hence,

$$a_5 = \frac{\cos\left(\sqrt{\lambda}(s-1)\right)}{\sqrt{\lambda}\left(\sin\left(\sqrt{\lambda}\right) - \delta\sin\left(\eta\sqrt{\lambda}\right)\right)},$$
$$a_6 = 0,$$
$$a_7 = \frac{\delta\sin\left(\eta\sqrt{\lambda}\right)\sin\left(\sqrt{\lambda}s\right) + \cos\left(\sqrt{\lambda}\right)\cos\left(\sqrt{\lambda}s\right)}{\sqrt{\lambda}\left(\sin\left(\sqrt{\lambda}\right) - \delta\sin\left(\eta\sqrt{\lambda}\right)\right)},$$
$$a_8 = \frac{\cos\left(\sqrt{\lambda}s\right)}{\sqrt{\lambda}}.$$

This completes the construction of Green's function. ■

Lemma 4.2. *Let $\lambda > 0$. If $y \in C^2(I)$ is the solution of the three point BVP* (4.3) *and* (4.4)*, then it can be expressed as*

$$y(t) = \frac{b\cos\sqrt{\lambda}t}{\sqrt{\lambda}(\delta\sin\sqrt{\lambda}\eta - \sin\sqrt{\lambda})} - \int_0^1 G(t,s)h(s)ds. \tag{4.7}$$

Proof. See the proof of Lemma 3.2 of chapter 3. ■

4.2.1.2 Case II : $\lambda < 0$

Let us assume

(H_0') $\lambda < 0$, $\sinh\sqrt{|\lambda|} - \delta\sinh\sqrt{|\lambda|}\eta > 0$ and $\delta\cosh\sqrt{|\lambda|}\eta - \cosh\sqrt{|\lambda|} \leq 0$.

The assumption (H_0') is satisfied for some values of $\lambda \in (-\infty, 0)$ (see fig. 4.2).

Lemma 4.3. *The Green's function of the linear three point BVP*

$$\begin{aligned} y''(t) + \lambda y(t) = 0, \quad 0 < t < 1, \\ y'(0) = 0, \quad y'(1) = \delta y'(\eta), \end{aligned}$$

is

$$G(t,s) = \begin{cases} \frac{\cosh(\sqrt{|\lambda|}t)(\delta\cosh(\sqrt{|\lambda|}(s-\eta)) - \cosh(\sqrt{|\lambda|}(s-1)))}{\sqrt{|\lambda|}(\sinh(\sqrt{|\lambda|}) - \delta\sinh(\eta\sqrt{|\lambda|}))}, & 0 \leq t \leq s \leq \eta, \\ \frac{\cosh(\sqrt{|\lambda|}s)(\delta\cosh(\sqrt{|\lambda|}(t-\eta)) - \cosh(\sqrt{|\lambda|}(t-1)))}{\sqrt{|\lambda|}(\sinh(\sqrt{|\lambda|}) - \delta\sinh(\eta\sqrt{|\lambda|}))}, & s \leq t,\ s \leq \eta, \\ -\frac{\cosh(\sqrt{|\lambda|}(s-1))\cosh(\sqrt{|\lambda|}t)}{\sqrt{|\lambda|}(\sinh(\sqrt{|\lambda|}) - \delta\sinh(\eta\sqrt{|\lambda|}))}, & t \leq s,\ \eta \leq s, \\ \frac{\delta\sinh(\eta\sqrt{|\lambda|})\sinh(\sqrt{|\lambda|}(s-t)) - \cosh(\sqrt{|\lambda|}s)\cosh(\sqrt{|\lambda|}(1-t))}{\sqrt{|\lambda|}(\sinh(\sqrt{|\lambda|}) - \delta\sinh(\eta\sqrt{|\lambda|}))}, & \eta \leq s \leq t \leq 1. \end{cases}$$

Proof. The construction of Green's function is same as given in Lemma 4.1. ■

Lemma 4.4. *Let $\lambda < 0$. If $y \in C^2(I)$ is a solution of the three point BVP* (4.3) *and* (4.4)*, then it is given by*

$$y(t) = \frac{b\cosh\sqrt{|\lambda|}t}{\sqrt{|\lambda|}(\sinh(\sqrt{|\lambda|}) - \delta\sinh(\eta\sqrt{|\lambda|}))} - \int_0^1 G(t,s)h(s)ds. \quad (4.8)$$

Proof. Proof follows from the analysis of Lemma 3.2 of chapter 3. ■

4.3 Some inequalities

Lemma 4.5. *Let $\lambda \in (0, \pi^2/4)$ and $\lambda - M \geq 0$. Further if*

$$(\lambda - M)\cos\sqrt{\lambda} - N\sqrt{\lambda}\sin\sqrt{\lambda} \geq 0,$$

then for all $t \in [0,1]$

$$(\lambda - M)\cos\sqrt{\lambda}t - N\sqrt{\lambda}\sin\sqrt{\lambda}t \geq 0,$$

where $M,N \in \mathbb{R}^+$.

Proof. Using monotonicity of sin and cos, we derive that for all $t \in [0,1]$,

$$(\lambda - M)\cos\sqrt{\lambda}t - N\sqrt{\lambda}\sin\sqrt{\lambda}t \geq (\lambda - M)\cos\sqrt{\lambda} - N\sqrt{\lambda}\sin\sqrt{\lambda} \geq 0.$$

Which completes the proof. ■

Lemma 4.6. *If* $\lambda < 0$ *is such that* $M + \lambda \leq 0$*, and*

$$\lambda \leq -M - \frac{N^2}{2} - \frac{N}{2}\sqrt{N^2 + 4M},$$

then for all $t \in [0,1]$*,*

$$(M+\lambda)\cosh\sqrt{|\lambda|}t + N\sqrt{|\lambda|}\sinh\sqrt{|\lambda|}t \leq 0,$$

where $M,N \in \mathbb{R}^+$.

Proof. As

$$(M+\lambda)\cosh\sqrt{|\lambda|}t + N\sqrt{|\lambda|}\sinh\sqrt{|\lambda|}t \leq \left[(M+\lambda) + N\sqrt{|\lambda|}\right]\cosh\sqrt{|\lambda|}t.$$

We will have $\left[(M+\lambda) + N\sqrt{|\lambda|}\right]\cosh\sqrt{|\lambda|}t \leq 0$ for all $t \in [0,1]$ if

$$\left[(M+\lambda) + N\sqrt{|\lambda|}\right] \leq 0.$$

The above inequality will be satisfied if

$$\lambda \leq -M - \frac{N^2}{2} - \frac{N}{2}\sqrt{N^2 + 4M}.$$

This completes the proof. ■

Lemma 4.7. *Let* (H_0) *be satisfied. Then*

(i) $G(t,s) \geq 0$*,*

(ii) $\frac{\partial G(t,s)}{\partial t} \leq 0$ *and*

(iii) $(\lambda - M)G(t,s) + N(\operatorname{sign} y')\frac{\partial G(t,s)}{\partial t} \geq 0$

for any $t,s \in [0,1]$ *such that* $t \neq s$*, and* $M,\ N \in \mathbb{R}^+$.

Proof. The conditions assumed in (H_0) ensure that $G(t,s) \geq 0$. Since $G(t,s)$ is the solution of (4.5)–(4.6), we deduce that $\frac{\partial G(t,s)}{\partial t} \leq 0$ for $t \neq s$. For part (iii), it will be sufficient to prove

$$(\lambda - M)G + N\frac{\partial G(t,s)}{\partial t} \geq 0, \tag{4.9}$$

as

$$(\lambda - M)G - N\frac{\partial G(t,s)}{\partial t} \geq 0,$$

is true. Putting the value of $G(t,s)$ and $\frac{\partial G(t,s)}{\partial t} \leq 0$ for $t \neq s$ in

$$(\lambda - M)G + N\frac{\partial G(t,s)}{\partial t},$$

and by Lemma 4.5, the inequality (4.9) is true. ∎

Lemma 4.8. *Assume (H_0'). Then for any $t,s \in [0,1]$ and $t \neq s$, we have*

(i) $G(t,s) \leq 0$,

(ii) $\frac{\partial G(t,s)}{\partial t} \leq 0$ *and*

(iii) $(M+\lambda)G(t,s) + N(\operatorname{sign} y')\frac{\partial G(t,s)}{\partial t} \geq 0$ *whenever we have* $M + \lambda - N\lambda \leq 0$

where $M,N \in \mathbb{R}^+$.

Proof. Parts (i) and (ii) follow the analysis of Lemma 4.7. For part (iii), it will be sufficient to prove that for all $t,s \in [0,1]$ and $t \neq s$

$$(M+\lambda)G(t,s) + N\frac{\partial G(t,s)}{\partial t} \geq 0.$$

Since $G(t,s)$ is the the Green function for (4.3)–(4.4), we have

$$\frac{\partial G(t,s)}{\partial t} \geq -\lambda G(t,s), \quad t \neq s.$$

The above inequality along with condition $M + \lambda - N\lambda \leq 0$ gives

$$(M+\lambda)G(t,s) + N\frac{\partial G(t,s)}{\partial t} \geq (M+\lambda - N\lambda)G(t,s) \geq 0.$$

∎

4.4 Anti-maximum and maximum principle

Proposition 4.1. *Let $b \geq 0$, $h(t) \in C[0,1]$ be such that $h(t) \geq 0$, and (H_0) holds. Then the solution of* (4.3) *and* (4.4) *is non-positive.*

Proposition 4.2. *Let $b \geq 0$, $h(t) \in C[0,1]$ be such that $h(t) \geq 0$ and (H_0') holds. Then the solution of* (4.3) *and* (4.4) *is non-negative.*

4.5 Existence results for nonlinear three point BVP

In this section, we prove two existence results for the nonlinear three point BVP with Neumann type boundary condition. On the basis of the order of upper and lower solutions, we divide this section into the following subsections.

4.5.1 Reverse ordered case

Definition 4.1. The functions $\alpha_0, \beta_0 \in C^2[0,1]$ are called lower and upper solutions for the class of three point BVP (4.1)–(4.2) if they satisfy the following inequalities:

$$\begin{aligned} -\alpha_0''(t) &\leq f(t, \alpha_0, \alpha_0'), \quad 0 < t < 1, \\ \alpha_0'(0) = 0, &\quad \alpha_0'(1) \leq \delta \alpha_0'(\eta), \end{aligned}$$

and

$$\begin{aligned} -\beta_0''(t) &\geq f(t, \beta_0, \beta_0'), \quad 0 < t < 1, \\ \beta_0'(0) = 0, &\quad \beta_0'(1) \geq \delta \beta_0'(\eta). \end{aligned}$$

The sequences $(\alpha_n)_n$ and $(\beta_n)_n$ are defined by the following iterative schemes

$$-\alpha_{n+1}''(t) - \lambda \alpha_{n+1}(t) = f(t, \alpha_n, \alpha_n') - \lambda \alpha_n, \tag{4.10}$$

$$\alpha_{n+1}'(0) = 0, \quad \alpha_{n+1}'(1) = \delta \alpha_{n+1}'(\eta), \tag{4.11}$$

$$-\beta_{n+1}''(t) - \lambda \beta_{n+1}(t) = f(t, \beta_n, \beta_n') - \lambda \beta_n, \tag{4.12}$$

$$\beta_{n+1}'(0) = 0, \quad \beta_{n+1}'(1) = \delta \beta_{n+1}'(\eta), \tag{4.13}$$

where $\lambda \in \mathbb{R} \setminus \{0\}$.

Theorem 4.1. *Assume that (H_0) and following hypothesis holds :*

(H_1) *there exist* α_0 *and* $\beta_0 \in C^2[0,1]$ *as lower and upper solutions of* (4.1) *and* (4.2) *such that* $\alpha_0 \geq \beta_0$ *for all* $t \in [0,1]$*;*

(H_2) *the function* $f : D \to \mathbb{R}$ *is continuous on* $D := \{(t,u,v) \in [0,1] \times \mathbb{R}^2 : \beta_0(t) \leq u \leq \alpha_0(t)\}$*;*

(H_3) *there exists* $M > 0$ *such that for all* $(t,u_1,v),(t,u_2,v) \in D$*,*

$$u_1 \leq u_2 \to f(t,u_2,v) - f(t,u_1,v) \leq M(u_2 - u_1);$$

(H_4) *there exists* $N > 0$ *such that for all* $(t,u,v_1),(t,u,v_2) \in D$*,*

$$|f(t,u,v_2) - f(t,u,v_1)| \leq N|v_2 - v_1|;$$

(H_5) *for all* $(t,u,v) \in D$*,* $|f(t,u,v)| \leq \varphi(|v|)$*, such that* $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *is continuous and satisfies*

$$\max_{t\in[0,1]} \alpha_0 - \min_{t\in[0,1]} \beta_0 \leq \int_{l_0}^{\infty} \frac{s\,ds}{\varphi(s)},$$

where $l_0 = [2|\Omega_0|]$ *and* $\Omega_0 = \max\{|\alpha_0(t)|_\infty, |\beta_0(t)|_\infty\}$.

Let $\lambda > 0$ *such that* $\lambda - M \geq 0$*,*

$$(\lambda - M)\cos\sqrt{\lambda} - N\sqrt{\lambda}\sin\sqrt{\lambda} \geq 0,$$

and for all $t \in [0,1]$

$$f(t,\beta_0(t),\beta_0'(t)) - f(t,\alpha_0(t),\alpha_0'(t)) - \lambda(\beta_0 - \alpha_0) \geq 0.$$

Then the sequences (α_n) *and* (β_n) *defined by* (4.10)*,* (4.11) *and* (4.12)*,* (4.13) *converge uniformly in* $C^1([0,1])$ *to the solutions* v *and* u *of the nonlinear boundary-value problem* (4.1) *and* (4.2)*, such that for all* $t \in [0,1]$

$$\beta_0(t) \leq u \leq v \leq \alpha_0(t).$$

The proof of the above theorem can be divided into several small results stated as follows.

Lemma 4.9. *If* α_n *is a lower solution of* (4.1) *and* (4.2)*,* α_{n+1} *is defined by* (4.10) *and* (4.11) *where* $\lambda \in (0,\pi^2/4)$*, then* $\alpha_{n+1} \leq \alpha_n$*.*

Proof. Since $y(t) = \alpha_{n+1} - \alpha_n$ satisfies $Ly \geq 0$, (4.4) with $b \geq 0$, the result can be concluded by Proposition 4.1. ■

Proposition 4.3. *Let* (H_0)–(H_4) *hold and there exists* $\lambda \in (0, \pi^2/4)$ *such that* $\lambda - M \geq 0$ *and* $(\lambda - M)\cos\sqrt{\lambda} - N\sqrt{\lambda}\sin\sqrt{\lambda} \geq 0$*, then the function* α_n *defined by* (4.10) *and* (4.11) *are such that for all* $n \in \mathbb{N}$*,*

(i) α_n *is a lower solution of* (4.1)–(4.2)*; and*

(ii) $\alpha_{n+1} \leq \alpha_n$.

Proof. We prove it by induction.

By Lemma 4.9, the claim holds for $n = 0$.

Let α_{n-1} is a lower solution of (4.1) and (4.2) and $\alpha_n \leq \alpha_{n-1}$. Let $y = \alpha_n - \alpha_{n-1}$. Then we have

$$-\alpha_n'' - f(t, \alpha_n, \alpha_n') \leq (\lambda - M)y + N(\operatorname{sign} y')y'.$$

Let $(\lambda - M)y + N(\operatorname{sign} y')y' = g$. Now to show α_n is a lower solution we have to show that $g \leq 0$. Since y is given by Lemma 4.2 with $h(t) = \alpha_{n-1}'' + f(t, \alpha_{n-1}, \alpha_{n-1}') \geq 0$. Thus to show $g \leq 0$, it is enough to prove that

$$(\lambda - M)\cos\sqrt{\lambda}t - N\sqrt{\lambda}\sin\sqrt{\lambda}t \geq 0,$$
$$(\lambda - M)G(t,s) + N\frac{\partial G(t,s)}{\partial t} \geq 0,$$

for all $t, s \in [0,1]$ and $t \neq s$. Lemma 4.5 and Lemma 4.7 verify the existence of above two inequalities. Thus $\alpha_{n+1} \geq \alpha_n$. ■

Similarly we can prove the following result.

Proposition 4.4. *Let* (H_0)–(H_4) *be true and there exists* $\lambda \in (0, \pi^2/4)$ *such that* $\lambda - M \geq 0$ *and* $(\lambda - M)\cos\sqrt{\lambda} - N\sqrt{\lambda}\sin\sqrt{\lambda} \geq 0$*. Then the function* β_n *defined by* (4.12)–(4.13) *are such that for all* $n \in \mathbb{N}$

(i) β_n *is an upper solution of* (4.1)–(4.2)*;*

(ii) $\beta_{n+1} \geq \beta_n$.

Proposition 4.5. *Let* (H_0)–(H_4) *be true and there exists* $\lambda \in (0, \pi^2/4)$ *such that* $\lambda - M \geq 0$ *and*

$$(\lambda - M)\cos\sqrt{\lambda} - N\sqrt{\lambda}\sin\sqrt{\lambda} \geq 0,$$

and for all $t \in [0,1]$,

$$f(t, \beta_0(t), \beta_0'(t)) - f(t, \alpha_0(t), \alpha_0'(t)) - \lambda(\beta_0 - \alpha_0) \geq 0.$$

Then for all $n \in \mathbb{N}$, the functions α_n and β_n defined by (4.10)–(4.11) *and* (4.12)–(4.13) *satisfy* $\alpha_n \geq \beta_n$.

***Proof*.** We define

$$h_i(t) = f(t,\beta_i,\beta_i') - f(t,\alpha_i,\alpha_i') - \lambda(\beta_i - \alpha_i), \quad \text{for all } i \in \mathbb{N}.$$

Now, for all $i \in \mathbb{N}$, $y_i := \beta_i - \alpha_i$ satisfies

$$-y_i'' - \lambda y_i = f(t,\beta_{i-1},\beta_{i-1}') - f(t,\alpha_{i-1},\alpha_{i-1}') - \lambda(\beta_{i-1} - \alpha_{i-1}) = h_{i-1}.$$

Claim 1. $\alpha_1 \geq \beta_1$. The function $y_1 = \beta_1 - \alpha_1$ is a solution of (4.3)–(4.4) with $h(t) = h_0(t) \geq 0$ and $b = 0$, by Proposition 4.1, $y_1(t) \leq 0$; i.e., $\alpha_1 \geq \beta_1$.

Claim 2. Let $n \geq 2$. If $h_{n-2} \geq 0$ and $\alpha_{n-1} \geq \beta_{n-1}$, then $h_{n-1} \geq 0$ and $\alpha_n \geq \beta_n$. First we will prove that, for all $t \in [0,1]$, the function h_{n-1} is non-negative, as we have

$$h_{n-1} \geq -[(\lambda - M)y_{n-1} + N(\text{sign}\, y_{n-1}')y_{n-1}'].$$

Since y_{n-1} is a solution of (4.3)–(4.4) with $h(t) = h_{n-2}(t) \geq 0$, $b = 0$. Hence we can proceed similar to the proof of Proposition 4.3 to show that $h_{n-1} \geq 0$. Now $y_n'(0) = 0$ and $y_n'(1) = \delta y'(\eta)$, i.e., $b = 0$, we deduce from Proposition 4.1 that $y_n \leq 0$, i.e., $\alpha_n \geq \beta_n$. ■

Lemma 4.10. *If $f(t,y,y')$ satisfies (H_5), then there exists $R > 0$ such that any solution of the differential inequality*

$$-y''(t) \geq f(t,y,y'), \quad 0 < t < 1, \tag{4.14}$$

$$y'(0) = 0, \quad y'(1) \geq \delta y'(\eta), \tag{4.15}$$

with $y \in [\beta_0(t), \alpha_0(t)]$ for all $t \in [0,1]$ satisfies $\|y'\|_\infty \leq R$.

Lemma 4.11. *If $f(t,y,y')$ satisfies (H_5), then there exists $R > 0$ such that any solution of the differential inequality*

$$-y''(t) \leq f(t,y,y'), \quad 0 < t < 1, \tag{4.16}$$

$$y'(0) = 0, \quad y'(1) \leq \delta y'(\eta) \tag{4.17}$$

with $y \in [\beta_0(t), \alpha_0(t)]$ for all $t \in [0,1]$ satisfies $\|y'\|_\infty \leq R$.

The proof of above two Lemmas are similar to the proof of Lemma 3.10 of Chapter 3 (Priory bound). Now we complete the proof of Theorem 4.1.

Proof of Theorem 4.1. The proof is same as proof of Theorem 3.1 of chapter 3. ■

4.5.2 Well ordered case

We state our main result as Theorem 4.2. Proof here is similar to non well ordered case, so we skip.

Theorem 4.2. *Assume* (H_0') *and the following hypothesis hold:*

(H_1') *there exist* $\alpha_0, \beta_0 \in C^2[0,1]$ *as lower and upper solutions of* (4.1)–(4.2) *such that* $\alpha_0 \le \beta_0$ *for all* $t \in [0,1]$*;*

(H_2') *the function* $f : \widetilde{D} \to \mathbb{R}$ *is continuous on* $\widetilde{D} := \{(t,u,v) \in [0,1] \times \mathbb{R}^2 : \alpha_0(t) \le u \le \beta_0(t)\}$*;*

(H_3') *there exists* $M > 0$ *such that for all* $(t,u_1,v),(t,u_2,v) \in \widetilde{D}$,

$$u_1 \le u_2 \to f(t,u_2,v) - f(t,u_1,v) \ge -M(u_2 - u_1);$$

(H_4') *there exists* $N > 0$ *such that for all* $(t,u,v_1),(t,u,v_2) \in \widetilde{D}$,

$$|f(t,u,v_2) - f(t,u,v_1)| \le N|v_2 - v_1|;$$

(H_5') *for all* $(t,u,v) \in \widetilde{D}, |f(t,u,v)| \le \varphi(|v|)$*; where* $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *is continuous and satisfies*

$$\max_{t\in[0,1]} \beta_0 - \min_{t\in[0,1]} \alpha_0 \le \int_{l_0}^{\infty} \frac{s\,ds}{\varphi(s)},$$

where $l_0 = [2|\Omega_0|]$ *and* $\Omega_0 = \max\{|\alpha_0(t)|_\infty, |\beta_0(t)|_\infty\}$.

Let $\lambda < 0$ *be such that* $\lambda \le \min\{-M, -\frac{M}{1-N}, -M - \frac{N^2}{2} - \frac{N}{2}\sqrt{N^2+4M}\}$*, and for all* $t \in [0,1]$,

$$f(t,\beta_0(t),\beta_0'(t)) - f(t,\alpha_0(t),\alpha_0'(t)) - \lambda(\beta_0 - \alpha_0) \ge 0$$

then the sequences (α_n) *and* (β_n) *defined by* (4.10)–(4.11) *and* (4.12)–(4.13) *converge uniformly in* $C^1([0,1])$ *to solution* v *and* u *of* (4.1)–(4.2)*, such that for all* $t \in [0,1]$,

$$\alpha_0(t) \le v \le u \le \beta_0(t).$$

4.6 Examples

In this section we consider two examples and verify that conditions derived in this chapter can actually be verified and existence of solutions is guaranteed.

Example 4.1. *Consider the nonlinear three point BVP given by*

$$-y''(t) = \frac{10y^3 - 9e^{y'}}{90}, \quad 0 < t < 1, \tag{4.18}$$

$$y'(0) = 0, \quad y'(1) = 0.6y'(0.9), \tag{4.19}$$

where $f(t,y,y') = \frac{10y^3 - 9e^{y'}}{90}$, $\delta = 0.6$, $\eta = 0.9$. *Here* $\alpha_0 = 1$ *and* $\beta_0 = -1$ *are lower and upper solutions, respectively. It is a non well ordered case.*

The priory bound can be computed as follows. $\varphi = (10 + 9e^{|v|})/90$. $|y'| \leq \sqrt{\frac{1}{5}}$; i.e., $R = \sqrt{1/5}$. The Lipschitz constants are computed as $M = 1/3$ and $N = e^R/10 = 0.156395$.

The inequality $f(t,\beta_0(t),\beta_0'(t)) - f(t,\alpha_0(t),\alpha_0'(t)) - \lambda(\beta_0 - \alpha_0) \geq 0$ is satisfied when $\lambda \geq 0.11111$. Now we can find out a subinterval $(0.44, 1.8) \subset (0.11111, 2.4674)$ (approx) such that $(\lambda - M)\cos\sqrt{\lambda} - N\sqrt{\lambda}\sin\sqrt{\lambda} \geq 0$ and inequalities in (H_0) are satisfied (cf. Figure 4.1).

Thus it is guaranteed that $\exists$ at least one $\lambda \in (0.44, 1.8)$ such that sequences generated by iterative scheme converge uniformly to a solution of the nonlinear three point boundary value problem (4.18) and (4.19).

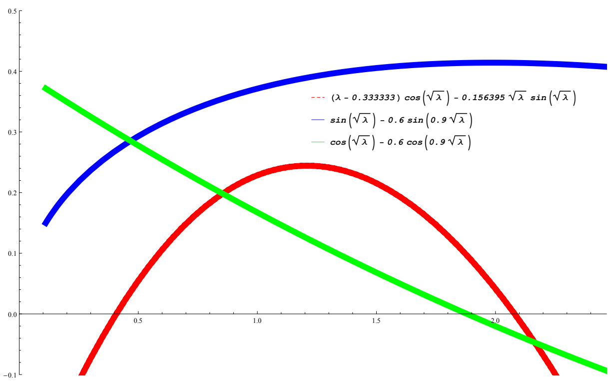

Fig. 4.1 Plots of (H_0) and $(\lambda - M)\cos\sqrt{\lambda} - N\sin\sqrt{\lambda} \geq 0$ for example 4.1.

Example 4.2. *Consider the nonlinear three point BVP*

$$-y''(t) = \frac{[y'(t)]^2}{60} - 5y(t) - \frac{e^2}{18}, \tag{4.20}$$

$$y'(0) = 0, \quad y'(1) = 0.7y'(0.5), \tag{4.21}$$

where $f(t,y,y') = \frac{[y'(t)]^2}{60} - 5y(t) - \frac{e^2}{18}$, $\delta = 7/10$, $\eta = \frac{1}{2}$.

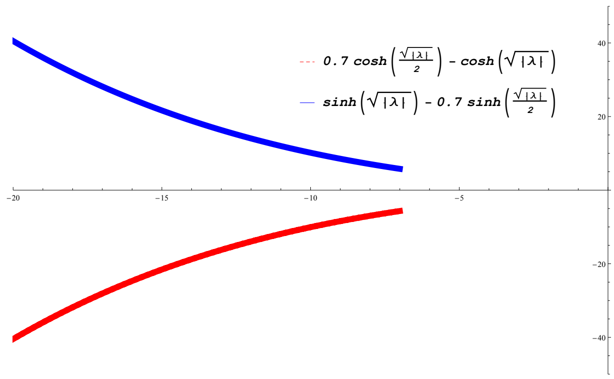

Fig. 4.2 Plots of (H_0') for example 4.2.

This example has $\alpha_0 = -(t^2 + \frac{1}{2})$ and $\beta_0 = (t^2 + \frac{1}{2})$ as lower and upper solutions, respectively. It means, we are in well ordered case. The priory bound can be computed as follows. $\varphi = \frac{15}{2} + \frac{e^2}{18} + \frac{|v|^2}{60}$, $|y'| \le 2e^{\frac{3}{32}}$; i.e., $R = 3e^{\frac{1}{20}}$. The Lipschitz constants are given by $M = 5$ and $N = R/30 = 0.105127$.

Now we can find out at least one $\lambda < 0$ such that when

$$\lambda \le \min\left\{-M, -\frac{M}{1-N}, -M - \frac{N^2}{2} - \frac{N}{2}\sqrt{N^2 + 4M}\right\},$$

(H_0') is satisfied (cf. Figure 4.2), and we get two monotonic sequences.

Thus for any $\lambda < -5.58739$, the sequences of solutions of the iterative scheme converge uniformly to the solutions of the nonlinear three point boundary value problem (4.20) and (4.21).

4.7 Conclusion

In this chapter we have established some existence results for nonlinear nonsingular derivative dependent differential equation subject to three point Neumann type boundary conditions. We arrive at both reverse order case and well order case. We have also shown that for both positive side and negative side of λ axis, there exists a range of λ for which the proposed iterative scheme gives us uniformly convergent sequences which converges to a solution of the nonlinear problem.

Chapter 5

Existence results for nonlinear three point BVPs with Dirichlet type boundary conditions

5.1 Introduction

In this chapter we present existence results for second order nonlinear three point boundary value problem with derivative dependent source function subject to Dirichlet type boundary conditions

$$y''(t) + f(t,y,y') = 0, \quad 0 < t < 1, \tag{5.1}$$

$$y(0) = 0, \quad y(1) = \delta y(\eta), \tag{5.2}$$

where $f \in C(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R}),\ I = [0,1],\ 0 < \eta < 1,\ \delta > 0$. Bao et al. [19] discussed the existence results for positive solutions of three point boundary value problem (5.1)–(5.2) for $0 < \delta < 1$. They used fixed point index method under a non-well-ordered upper and lower solutions condition.

The result of this chapter is an improvement over a recent result due to Bao et al. [19]. They assume two conditions $f(t,0,0) = 0$ and $yf(t,y,y') \geq 0$ for $y \geq 0$. Consider $-y'' = h(t) + y$ which is linear but $f(t,0,0) \neq 0$. So $f(t,0,0) = 0$ fails. Another simple example is $-y'' = \sin y$. Here $y \sin y$ will change its sign for $y \geq 0$, so the condition $yf(t,y,y') \geq 0$ for $y \geq 0$ fails. But for both these problems the results of this chapter are applicable.

Here we are looking for a simple monotone iterative scheme and propose the following

$$-y''_{n+1} - \lambda y_{n+1} = f(t, y_n, y'_n) - \lambda y_n, \tag{5.3}$$

$$y_{n+1}(0) = 0, \quad y_{n+1}(1) = \delta y_{n+1}(\eta). \tag{5.4}$$

We have considered “λ” as a constant.

Cherpion et al. [40, Section 5.4] stated that (5.3)–(5.4) with constant λ do not work. Also they [40, Remark 5.4] stated that due to lack of uniform anti-maximum principle it seems impossible to develop monotone iterative technique for reverse ordered upper and lower solution.

Remark 5.1. *In this chapter, we have shown that even with constant λ monotone sequences can be generated. Though Remark by Cherpion et al. [40, Remark 5.4] appears to be true for three point BVP and we also observe that uniform anti-maximum principle does not exist.*

5.2 The linear case

Here we consider the corresponding linear three point BVP. We prove maximum principle and also prove existence of some differential inequalities. Consider the corresponding nonhomogeneous linear three point BVP

$$Ly \equiv -y''(t) - \lambda y(t) = h(t), \quad 0 < t < 1, \tag{5.5}$$

$$y(0) = 0, \quad y(1) = \delta y(\eta) + b, \tag{5.6}$$

where $h \in C(I)$, & b any constant.
Case I: $\lambda > 0$. Let us assume

(H_0) $\quad 0 < \lambda < \frac{\pi^2}{4}, \quad \cos\sqrt{\lambda} - \delta\cos\sqrt{\lambda}\eta \leq 0, \quad \delta\sin\sqrt{\lambda}\eta - \sin\sqrt{\lambda} < 0.$

There exists a range of λ, for which (H_0) holds (see figure 5.1).

Lemma 5.1. *The Green's function for the three point BVP, $Ly = 0$, $y(0) = 0$, $y(1) = \delta y(\eta)$ for $\lambda > 0$, is*

$$G(t,s) = k_1 \begin{cases} [\sin\sqrt{\lambda}(1-s) - \delta\sin\sqrt{\lambda}(\eta - s)]\sin\sqrt{\lambda}t, & 0 \leq t \leq s \leq \eta, \\ \sin\sqrt{\lambda}s\ [\sin\sqrt{\lambda}(1-t) - \delta\sin\sqrt{\lambda}(\eta - t)], & s \leq t, s \leq \eta, \\ \sin\sqrt{\lambda}(1-s)\sin\sqrt{\lambda}t, & t \leq s, \eta \leq s, \\ \delta\sin\sqrt{\lambda}\eta\ \sin\sqrt{\lambda}(t-s) + \sin\sqrt{\lambda}s\ \sin\sqrt{\lambda}(1-t), & \eta \leq s \leq t \leq 1, \end{cases}$$

where $k_1 = \frac{1}{\sqrt{\lambda}(\delta\sin\sqrt{\lambda}\eta - \sin\sqrt{\lambda})}$. If (H_0) is true then $G(t,s) \leq 0$.

***Proof*.** The Green's function for the three point BVP, $Ly = 0$, $y(0) = 0$, $y(1) = \delta y(\eta)$ for $\lambda > 0$, is defined as

$$G(t,s) = \begin{cases} a_1 \cos\sqrt{\lambda}t + a_2 \sin\sqrt{\lambda}t, & 0 \le t \le s \le \eta, \\ a_3 \cos\sqrt{\lambda}t + a_4 \sin\sqrt{\lambda}t, & s \le t, s \le \eta, \\ a_5 \cos\sqrt{\lambda}t + a_6 \sin\sqrt{\lambda}t, & t \le s, \eta \le s, \\ a_7 \cos\sqrt{\lambda}t + a_8 \sin\sqrt{\lambda}t, & \eta \le s \le t \le 1. \end{cases}$$

The unknown variables a_1, a_2, a_3 and a_4 are computed with the help of the definition of Green's function. For any $s \in [0,\eta]$, from continuity and jump of $G(t,s)$, we get

$$\begin{aligned} &a_1 \cos\sqrt{\lambda}s + a_2 \sin\sqrt{\lambda}s = a_3 \cos\sqrt{\lambda}s + a_4 \sin\sqrt{\lambda}s, \\ &(-\sqrt{\lambda}a_1 \sin\sqrt{\lambda}s + a_2\sqrt{\lambda}\cos\sqrt{\lambda}s) - (-\sqrt{\lambda}a_3 \sin\sqrt{\lambda}s + a_4\sqrt{\lambda}\cos\sqrt{\lambda}s) = -1. \end{aligned}$$

Thus,

$$\begin{aligned} a_1 - a_3 &= \frac{1}{\sqrt{\lambda}}\sin\sqrt{\lambda}s, \\ a_2 - a_4 &= -\frac{1}{\sqrt{\lambda}}\cos\sqrt{\lambda}s. \end{aligned}$$

Then by using the three point boundary value condition, we have

$$\begin{aligned} &a_1 = 0, \\ &a_3 \cos\sqrt{\lambda} + a_4 \sin\sqrt{\lambda} = \delta(a_3 \cos\sqrt{\lambda}\eta + a_4 \sin\sqrt{\lambda}\eta). \end{aligned}$$

The values of a_1, a_2, a_3 and a_4 are given by

$$\begin{aligned} a_1 &= 0, \\ a_2 &= \frac{\sin\sqrt{\lambda}(1-s) - \delta\sin\sqrt{\lambda}(\eta - s)}{\sqrt{\lambda}(\delta\sin\sqrt{\lambda}\eta - \sin\sqrt{\lambda})}, \\ a_3 &= -\frac{1}{\sqrt{\lambda}}\sin\sqrt{\lambda}s, \\ a_4 &= -\frac{\sin\sqrt{\lambda}s(\cos\sqrt{\lambda} - \delta\cos\sqrt{\lambda}\eta)}{\sqrt{\lambda}(\delta\sin\sqrt{\lambda}\eta - \sin\sqrt{\lambda})}. \end{aligned}$$

Similarly, for any $s \in [\eta, 1]$, we have

$$a_5 = 0,$$

$$a_6 = \frac{\sin\sqrt{\lambda}(1-s)}{\sqrt{\lambda}(\delta \sin\sqrt{\lambda}\eta - \sin\sqrt{\lambda})},$$
$$a_7 = -\frac{1}{\sqrt{\lambda}}\sin\sqrt{\lambda}s,$$
$$a_8 = \frac{\sin\sqrt{\lambda}(1-s)}{\sqrt{\lambda}(\delta \sin\sqrt{\lambda}\eta - \sin\sqrt{\lambda})} + \frac{1}{\sqrt{\lambda}}\cos\sqrt{\lambda}s.$$

Thus the construction of $G(t,s)$ is complete.

We can easily prove that the constant sign of Green's function will be non-positive when (H_0) holds. ■

Lemma 5.2. *Let $y \in C^2(I)$ be a solution of three point BVP (5.5)–(5.6), then*

$$y(t) = \frac{b\sin\sqrt{\lambda}t}{\sin\sqrt{\lambda} - \delta\sin\sqrt{\lambda}\eta} - \int_0^1 G(t,s)h(s)ds. \tag{5.7}$$

Proof. The three point nonhomogeneous linear boundary value problem (5.5)–(5.6) is equivalent to

$$y(t) = \bar{y} - \int_0^1 G(t,s)h(s)\,ds,$$

where $\bar{y}$ is the solution of

$$Ly = 0, \quad y(0) = 0, \quad y(1) = \delta y(\eta) + b,$$

and $G(t,s)$ is the solution of

$$Ly = 0, \quad y(0) = 0, \quad y(1) = \delta y(\eta).$$

Suppose

$$\bar{y} = c_1\cos\sqrt{\lambda}t + c_2\sin\sqrt{\lambda}t,$$

then by using the boundary value conditions

$$\bar{y}(0) = 0, \quad \bar{y}(0) = \delta\bar{y}(\eta) + b,$$

we get

$$\bar{y} = \frac{b\sin\sqrt{\lambda}t}{\sin\sqrt{\lambda} - \delta\sin\sqrt{\lambda}\eta}.$$

Hence the boundary value problem (5.5)–(5.6) is equivalent to

$$y(t) = \frac{b\sin\sqrt{\lambda}t}{\sin\sqrt{\lambda} - \delta\sin\sqrt{\lambda}\eta} - \int_0^1 G(t,s)h(s)ds.$$

■

Case II: $\lambda < 0$**.** Let us assume

(H_0') $\lambda < 0$, $\cosh\sqrt{|\lambda|} - \delta\cosh\sqrt{|\lambda|}\eta \geq 0$ and $\delta\sinh\sqrt{|\lambda|}\eta - \sinh\sqrt{|\lambda|} < 0$.

There exists a range of $\lambda < 0$, for which (H_0') holds (see figure 5.2).

Lemma 5.3. *The Green's function of the three point BVP,* $Ly = 0$, $y(0) = 0$, $y(1) = \delta y(\eta)$ *for* $\lambda < 0$ *is*

$$G(t,s) = k_2 \begin{cases} [\sinh\sqrt{|\lambda|}(1-s) - \delta\sinh\sqrt{|\lambda|}(\eta-s)]\sinh\sqrt{|\lambda|}t, & 0 \leq t \leq s \leq \eta, \\ \sinh\sqrt{|\lambda|}s\,[\sinh\sqrt{|\lambda|}(1-t) - \delta\sinh\sqrt{|\lambda|}(\eta-t)], & s \leq t, s \leq \eta, \\ \sinh\sqrt{|\lambda|}(1-s)\sinh\sqrt{|\lambda|}t, & t \leq s, \eta \leq s, \\ \delta\sinh\sqrt{|\lambda|}\eta\ \sinh\sqrt{|\lambda|}(t-s) & \\ +\sinh\sqrt{|\lambda|}s\sinh\sqrt{|\lambda|}(1-t), & \eta \leq s \leq t \leq 1, \end{cases}$$

where $k_2 = \frac{1}{\sqrt{|\lambda|}(\delta\sinh\sqrt{|\lambda|}\eta - \sinh\sqrt{|\lambda|})}$. *If* (H_0') *is true then* $G(t,s) \leq 0$.

Lemma 5.4. *Let* $y \in C^2(I)$ *be a solution of three point BVP (5.5)–(5.6). Then* $y(t)$ *is given by the following equation*

$$y(t) = \frac{b\sinh\sqrt{|\lambda|}t}{\sinh\sqrt{|\lambda|} - \delta\sinh\sqrt{|\lambda|}\eta} - \int_0^1 G(t,s)h(s)ds. \tag{5.8}$$

5.2.1 Existence of some differential inequalities

In this section we prove existence of some differential inequalities which govern the range of λ and if these inequalities are true, the solutions generated by iterative scheme are monotonic.

Lemma 5.5. *Let* $M \in \mathbb{R}^+$ *and* $N : [0,1] \to [0,\infty)$ *such that* $N(0) = 0$, $N'(t) \geq 0$. *If* $0 < \lambda < \frac{\pi^2}{4}$ *is such that* $\lambda - M \leq 0$ *and*

(*i*) *if* $(\lambda - M)\cos\sqrt{\lambda} + N(t)\sqrt{\lambda}\sin\sqrt{\lambda} \leq 0$, *then for all* $t \in [0,1]$

$$(\lambda - M)\cos\sqrt{\lambda}t + N(t)\sqrt{\lambda}\sin\sqrt{\lambda}t \leq 0.$$

(ii) *If* $\lambda + \sup N'(t) \leq M$*, then for all* $t \in [0,1]$

$$(\lambda - M)\sin\sqrt{\lambda}t + N(t)\sqrt{\lambda}\cos\sqrt{\lambda}t \leq 0.$$

Proof. The function

$$(\lambda - M)\cos\sqrt{\lambda}t + N(t)\sqrt{\lambda}\sin\sqrt{\lambda}t$$

is non-decreasing for all $t \in [0,1]$ and satisfy the following inequality,

$$(\lambda - M)\cos\sqrt{\lambda}t + N(t)\sqrt{\lambda}\sin\sqrt{\lambda}t \leq (\lambda - M)\cos\sqrt{\lambda} + N(t)\sqrt{\lambda}\sin\sqrt{\lambda} \leq 0.$$

By using the assumptions (i) is easily verified.

Using the properties of *sine*, *cosine* and assumptions, we can easily see that for all $t \in [0,1]$,

$$(\lambda - M)\sin\sqrt{\lambda}t + N(t)\sqrt{\lambda}\cos\sqrt{\lambda}t \leq 0.$$

Hence (ii) is verified. ■

Lemma 5.6. *Let* $M \in \mathbb{R}^+$ *and* $N : [0,1] \to [0,\infty)$ *such that* $N(0) = 0$*. If* $\lambda < 0$ *is such that* $M + \lambda \leq 0$*, and*

(i) *if* $[(M+\lambda) + N\sqrt{|\lambda|}] \leq 0$*, then for all* $t \in [0,1]$*,*

$$(M+\lambda)\cosh\sqrt{|\lambda|}t + N(t)\sqrt{|\lambda|}\sinh\sqrt{|\lambda|}t \leq 0.$$

(ii) *If* $(M+\lambda) + N'(t) + N(t)\sqrt{|\lambda|} \leq 0$*, then for all* $t \in [0,1]$*,*

$$(M+\lambda)\sinh\sqrt{|\lambda|}t + N(t)\sqrt{|\lambda|}\cosh\sqrt{|\lambda|}t \leq 0.$$

Proof. As

$$(M+\lambda)\cosh\sqrt{|\lambda|}t + N(t)\sqrt{|\lambda|}\sinh\sqrt{|\lambda|}t \leq [(M+\lambda) + N(t)\sqrt{|\lambda|}]\cosh\sqrt{|\lambda|}t.$$

The right hand side of the above inequality will be non-positive for all $t \in [0,1]$ if

$$\left[(M+\lambda) + N(t)\sqrt{|\lambda|}\right] \leq 0.$$

This completes the part (i) of Lemma.

Using the assumptions and the properties of sinh and cosh, we can easily see that for all $t \in [0,1]$

$$(M+\lambda)\sinh\sqrt{|\lambda|}t + N(t)\sqrt{|\lambda|}\cosh\sqrt{|\lambda|}t$$

is a non-increasing function, which proves part (ii). ■

Lemma 5.7. *Let (H_0) be true. If $y(t)$ is the solution of (5.5)–(5.6) then we have*

(*i*) $G(t,s) \leq 0$ *and*

(*ii*) $(\lambda - M)G(t,s) + N(t)(sign\ y')\frac{\partial G(t,s)}{\partial t} \geq 0$

for any $t,s \in [0,1]$ and $t \neq s$.

Proof. The condition (H_0) guarantees that $G(t,s) \leq 0$. Putting the value of $G(t,s)$ and $\frac{\partial G(t,s)}{\partial t}$ for $t \neq s$ in

$$(\lambda - M)G(t,s) + N(t)(sign\ y')\frac{\partial G(t,s)}{\partial t},$$

and using the Lemma 5.5, we can prove that

$$(\lambda - M)G(t,s) + N(t)(sign\ y')\frac{\partial G(t,s)}{\partial t} \geq 0, \quad \forall\, s,\ t \in [0,1] \ \text{and} \quad s \neq t.$$

■

Lemma 5.8. *Let (H_0') be true. Let $y(t)$ be the solution of (5.5)–(5.6) then we have*

(*i*) $G(t,s) \leq 0$, *for any $t,s \in [0,1]$,*

(*ii*) $(M+\lambda)G(t,s) + N(t)(sign\ y')\frac{\partial G(t,s)}{\partial t} \geq 0$,

for any $t,s \in [0,1]$ and $t \neq s$.

Proof. By Lemma 5.6 and analysis similar to proof of Lemma 5.7, completes the proof of this lemma. ■

5.2.2 Maximum principle

By using the constant sign of Green's function the following two results can be concluded easily.

Proposition 5.1. *Suppose that (H_0) holds. Let $y(t)$ be the solution of (5.5)–(5.6) and if $b \geq 0$, $h(t) \in C[0,1]$ be such that $h(t) \geq 0$, then $y(t) \geq 0$.*

Proposition 5.2. *Suppose that (H_0') holds. Let $y(t)$ be the solution of (5.5)–(5.6) and if $b \geq 0$, $h(t) \in C[0,1]$ be such that $h(t) \geq 0$, then $y(t) \geq 0$.*

5.3 Nonlinear three point BVP

In this section we consider the nonlinear three point BVP. We show that it is possible to find out a range of $\lambda \in \mathbb{R}\setminus\{0\}$ on λ axis so that the iterative scheme (5.3)–(5.4) generates monotone sequences. Which finally proves existence of solutions for nonlinear three point BVP (5.1)–(5.2).

We define lower solution and upper solution represented by the functions $\alpha_0(t)$ and $\beta_0(t)$, respectively, such that $\alpha_0 \leq \beta_0$.

Definition 5.1. *Let $\alpha_0, \beta_0 \in C^2[0,1]$. Then $\alpha_0(t)$ and $\beta_0(t)$ are called lower solution and upper solution of the nonlinear three point BVP (5.1)–(5.2), respectively if they satisfy*

$$\begin{aligned} -\alpha_0''(t) &\leq f(t,\alpha_0,\alpha_0'), \quad 0<t<1,\\ \alpha_0(0)=0, &\quad \alpha_0(1)\leq \delta\alpha_0(\eta), \end{aligned}$$

and

$$\begin{aligned} -\beta_0''(t) &\geq f(t,\beta_0,\beta_0'), \quad 0<t<1,\\ \beta_0(0)=0, &\quad \beta_0(1)\geq \delta\beta_0(\eta). \end{aligned}$$

Our proof is based on uniform convergence of the sequences and for that we use Arzela-Ascoli theorem. To implement this we need equicontinuity and equiboundedness of (y_n) and (y_n'). Equicontinuity and equiboundedness of y_n and y_n' can be proved by continuity of the Green's function and continuity of the solution on $[0,1]$ and continuity of the nonlinear term $f(t,y,y')$. Equiboundedness of (y_n') is established by the following two lemmas.

5.3.1 Priori bound

(H_P) Let $|f(t,u,v)| \leq \varphi(|v|)$ for all $(t,u,v)\in D$. Assume that

$$\varphi : \mathbb{R}^+ \to \mathbb{R}^+$$

is continuous and satisfies $\max\limits_{t\in[0,1]} \beta_0 - \min\limits_{t\in[0,1]} \alpha_0 \leq \int_{l_0}^{\infty} \frac{s\,ds}{\varphi(s)}$. Here $l_0 = |\Omega_0|$ and $\Omega_0 = \max\{|\alpha_0(t)|_\infty, |\beta_0(t)|_\infty\}$.

Lemma 5.9. *Assume that $f(t,y,y')$ satisfies (H_P). Then there exists $R>0$ such that any solution of*

$$-y''(t) \geq f(t,y,y'), \quad 0<t<1, \tag{5.9}$$

$$y(0) = 0, \quad y(1) \geq \delta y(\eta), \tag{5.10}$$

with $y \in [\alpha_0(t), \beta_0(t)]$ *for all* $t \in [0,1]$ *satisfies* $\|y'\|_\infty \leq R$.

Proof. The proof can be divided in two parts.
Case : (i) If solution is not monotone in $[0,1]$, then consider the interval $(t_0,t] \subset (0,1)$ such that $y'(t_0) = 0$ and $y'(t) > 0$ for $t > t_0$. Integrating (5.9) from t_0 to t we get

$$\int_0^{y'} \frac{s\,ds}{\varphi(s)} \leq \max_{t\in[0,1]} \beta_0 - \min_{t\in[0,1]} \alpha_0.$$

From (H_P) we can choose $R > 0$ such that

$$\int_0^{y'} \frac{s\,ds}{\varphi(s)} \leq \max_{t\in[0,1]} \beta_0 - \min_{t\in[0,1]} \alpha_0 \leq \int_{l_0}^{R} \frac{s\,ds}{\varphi(s)} \leq \int_0^{R} \frac{s\,ds}{\varphi(s)},$$

which gives

$$y'(t) \leq R.$$

Now we consider the case in which $y'(t) < 0$ for $t < t_0$, $y'(t_0) = 0$, and proceeding in the similar way we get

$$-y'(t) \leq R,$$

and the result follows.
Case : (ii) If y is monotonically decreasing in $(0,1)$, that is $y'(t) < 0$ in $t \in (0,1]$ then by mean value theorem there exists a point $\tau \in (0,1)$ such that

$$-y'(\tau) \leq |\Omega_0|,$$

where $\Omega_0 = \max\{|\alpha_0(t)|_\infty, |\beta_0(t)|_\infty\}$.

Now, integrating (5.9) from t to τ, using (H_P) we can choose R, such that $-y' \leq R$.

Similarly if y is monotonically increasing in $(0,1)$, that is $y'(t) > 0$ in $t \in (0,1]$ proof can be completed as above. ■

Lemma 5.10. *If* $f(t,y,y')$ *satisfies* (H_P)*, then there exists* $R > 0$ *such that any solution of*

$$-y''(t) \leq f(t,y,y'), \quad 0 < t < 1, \tag{5.11}$$

$$y(0) = 0, \quad y(1) \leq \delta y(\eta), \tag{5.12}$$

with $y \in [\alpha_0(t), \beta_0(t)]$ *for all* $t \in [0,1]$ *satisfies* $\|y'\|_\infty \leq R$.

Proof. Proof follows from the analysis of Lemma 5.9. ■

Now we state the existence Theorem 5.1 (for $\lambda > 0$) and Theorem 5.2 (for $\lambda < 0$) which are the main results of this chapter.

Theorem 5.1. *Let (H_0) be true. Further assume that*

(H_1) *there exist α_0 and $\beta_0 \in C^2[0,1]$ as lower and upper solutions of (5.1)–(5.2) such that $\alpha_0 \leq \beta_0$, for all $t \in [0,1]$;*

(H_2) *the function $f : D \to \mathbb{R}$ is continuous on*

$$D := \{(t,u,v) \in [0,1] \times \mathbb{R}^2 : \alpha_0(t) \leq u \leq \beta_0(t)\};$$

(H_3) *there exists $M > 0$ such that for all $(t,u_1,v),(t,u_2,v) \in D$;*

$$u_1 \leq u_2 \Longrightarrow f(t,u_2,v) - f(t,u_1,v) \geq M(u_2 - u_1);$$

(H_4) *there exists $N : [0,1] \to [0,\infty)$ such that $N(0) = 0$, $N'(t) \geq 0$ and for all $(t,u,v_1),(t,u,v_2) \in D$*

$$|f(t,u,v_2) - f(t,u,v_1)| \leq N(t)|v_2 - v_1|.$$

(H_5) *Let $\lambda > 0$ be such that $\lambda - M \leq 0$, $(\lambda - M)\cos\sqrt{\lambda} + N(t)\sqrt{\lambda}\sin\sqrt{\lambda} \leq 0$ and $\lambda + \sup N'(t) \leq M$, and for all $t \in [0,1]$*

$$f(t,\beta_0(t),\beta_0'(t)) - f(t,\alpha_0(t),\alpha_0'(t)) - \lambda(\beta_0 - \alpha_0) \geq 0.$$

Then the sequences (α_n) and (β_n) defined by (5.3)–(5.4) converge uniformly in $C^1([0,1])$ to solution v and u of (5.1)–(5.2), such that for all $t \in [0,1]$,

$$\alpha_0 \leq v \leq u \leq \beta_0.$$

Proof. With the help of assumptions (H_1), (H_2), (H_3), (H_4) and (H_5), we conclude that

$$\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n \leq \cdots \leq \beta_n \leq \cdots \leq \beta_2 \leq \beta_1 \leq \beta_0. \tag{5.13}$$

It is clear that the sequences $(\alpha_n)_n$ and $(\beta_n)_n$ are monotonic and bounded. Hence they converge to the functions $u(t)$ and $v(t)$ (say), respectively, which are such that for all n, $\alpha_n \leq v \leq u \leq \beta_n$.

By using the equations (5.3)–(5.4), inequality (5.13) and Lemma 5.9, 5.10, we prove that the sequences $(\beta_n)_n$ and $(\alpha_n)_n$ are equibounded and equicontinuous in $C^1([0,1])$, i.e., any subsequence of $(\beta_n)_n$ and $(\alpha_n)_n$ are also equibounded and equicontinuous in $C^1([0,1])$. Now by using Arzela-Ascoli theorem, we conclude that the subsequences of $(\beta_n)_n$ and $(\alpha_n)_n$ contain a subsequence which converge uniformly in $C^1([0,1])$.

By uniqueness of the limit and monotonicity of the sequences $(\alpha_n)_n$ and $(\beta_n)_n$, we have $\alpha_n \to v$ and $\beta_n \to u$.

We write the solution of iterative scheme (5.3)–(5.4) for both (α_n) and (β_n) by using Lemma 5.2, where $h(t)$ is in terms of nonlinear term f. Now by using uniform convergence, one can easily conclude the existence of the solution of nonlinear three point BVP. This completes the proof. ∎

Theorem 5.2. *Let (H_0'), (H_1), (H_2) and (H_4) be true. Further assume that*

(H_1') *there exists $M > 0$ such that for all $(t,u_1,v),(t,u_2,v) \in D$,*

$$u_1 \le u_2 \Longrightarrow f(t,u_2,v) - f(t,u_1,v) \ge -M(u_2 - u_1).$$

(H_2') *Let $\lambda < 0$ be such that $M + \lambda \le 0$, $(M+\lambda) + N'(t) + N(t)\sqrt{|\lambda|} \le 0$, $[(M+\lambda) + N\sqrt{|\lambda|}] \le 0$ and for all $t \in [0,1]$,*

$$f(t,\beta_0(t),\beta_0'(t)) - f(t,\alpha_0(t),\alpha_0'(t)) - \lambda(\beta_0 - \alpha_0) \ge 0.$$

Then the sequences (α_n) and (β_n) defined by (5.3)–(5.4) converge monotonically in $C^1([0,1])$ to solution v and u of (5.1)–(5.2), such that for all $t \in [0,1]$, $\alpha_0 \le v \le u \le \beta_0$.

Proof. Proof is same as Theorem 5.1. ∎

5.4 Examples

To verify our results, we consider two examples for both $\lambda > 0$, $\lambda < 0$ and show that it is possible to compute a range of λ so that iterative scheme generates monotone sequences which converge to the solution of nonlinear problem.

Example 5.1. *Consider the nonlinear three point BVP*

$$-y''(t) = \left(\frac{9e^y + te^{y'}}{15}\right), \quad 0 < t < 1, \tag{5.14}$$

$$y(0) = 0, \quad y(1) = 0.95y(0.2). \tag{5.15}$$

This problem has $\alpha_0 = 0$ and $\beta_0 = 3\left(t - \frac{t^2}{2}\right)$ as lower and upper solutions, respectively. The nonlinear term is Lipschitz in both y and y' and continuous for all values of y and y'. It is easy to see that Nagumo condition is given by

$$|f(t,u,v)| \leq \frac{9}{15}e^{\frac{3}{2}} + \frac{1}{15}e^{v},$$

i.e., $\varphi = \frac{9}{15}e^{\frac{3}{2}} + \frac{1}{15}e^{v}$. Using Lemma 5.9 we can compute easily that $|y'| \leq \sqrt{\frac{1}{10}}$, i.e., $R = 0.316228$. The Lipschitz constants M and $N(t)$ are computed as $M = \frac{3}{5}$ and $N(t) = \frac{t}{15}e^{R}$, respectively. In Figures 5.1, 5.4, 5.6, 5.8, 5.10 we discuss constant sign of some inequalities and describe monotonic behavior of solutions. In Figure 5.1 we have verified that it is possible to get a range of λ such that (H_0) is true. In Figures 5.4, 5.6, 5.8, 5.10 we have shown that for different values of $\lambda \in [0.15, 0.49]$ monotonic sequences are obtained and both converge to a solution of nonlinear problem (5.14)–(5.15). In this range all the inequalities are also true which are required to generate monotonic sequences. The range $[0.15, 0.49]$ is not sharp and is based on computations done in Mathematica 10. In Figure 5.3 we have shown that if λ is not in the range $[0.15, 0.49]$ then monotonicity is lost.

Example 5.2. *Consider the nonlinear three point BVP*

$$-y''(t) = \frac{(e^t - 1)}{36}\left[(y'(t))^2 - y(t) - \frac{\cos t}{4}\right], \quad 0 < t < 1, \tag{5.16}$$

$$y(0) = 0, \quad y(1) = 0.5y(0.5). \tag{5.17}$$

Here $f(t,y,y') = \frac{(e^t-1)}{36}\left[(y'(t))^2 - y(t) - \frac{\cos t}{4}\right]$, $\delta = 0.5, \eta = 0.5$. This problem has $\alpha_0 = (\frac{t^2}{4} - t)$ and $\beta_0 = \frac{t}{2}$ as lower and upper solutions, respectively. The nonlinear term is Lipschitz in both y and y' and continuous for all values of y and y'. It is easy to see that Nagumo condition is given by

$$|f(t,u,v)| \leq 0.0477301(|v|^2 + 1),$$

i.e., $\varphi = 0.0477301(|v|^2 + 1)$. Using Lemma 5.10 we can compute bound for y', i.e., $|y'| \leq \frac{3}{4}e^{\frac{5}{4}(0.0477301)}$, i.e., $R = 0.796109$. The Lipschitz constants are $M = 0.0477301$ and $N(t) = \frac{(e^t-1)}{36}(1.59222)$.

In Figure 5.2 we have verified that for $\lambda < -1$, (H_0') is true. In Figures 5.5, 5.7, 5.9, 5.11, we describe monotonic behavior of the sequences. For $\lambda < -1$ all the inequalities required are also valid. The upper bound for λ is not sharp and is based on computations done in Mathematica 10. In Figures 5.5, 5.7, 5.9, 5.11 we have shown that for different values of λ, monotonic sequences are obtained and both converge to a solution of nonlinear problem

(5.16)–(5.17). Here it is also visible from the Figure 5.11 that sequence thus obtained are uniformly convergent.

5.5 Conclusion

In this chapter we have considered an iterative scheme which is simple enough for computational point of view. We did not consider λ as function of t. The method developed in this chapter can be coded to generate a user friendly package which can be efficiently used to compute solutions of the nonlinear three point BVP whose close form solutions is not known.

We have constructed two examples one for each case $\lambda > 0$ and $\lambda < 0$ and show that derived sufficient conditions can generate solutions for a class of nonlinear three point BVPs. Mainly it is initial iterates (upper and lower solutions) choice of which matters and success of the method depends on them. If initial iterates are chosen properly then it is guaranteed that sequences will converge to the solutions of the nonlinear BVP. In Figure 5.3 we also observe that if λ does not belong to the range sequences are not monotone.

We also observe that Remark 5.4 of Cherpion et al. [40] seems to be true even in case of three point BVP with Dirichlet type boundary condition.

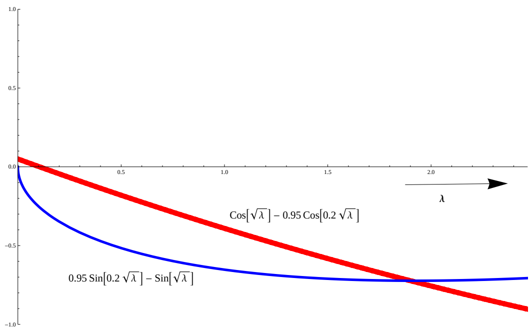

Fig. 5.1 Plot of (H_0) for example 5.1.

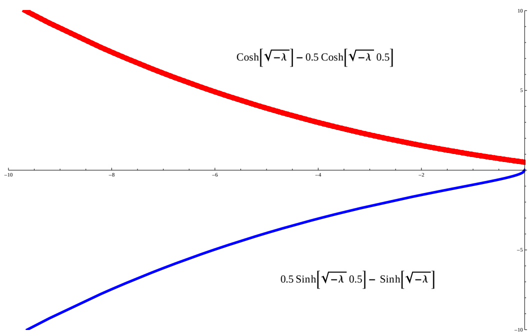

Fig. 5.2 Plot of (H_0') for example 5.2.

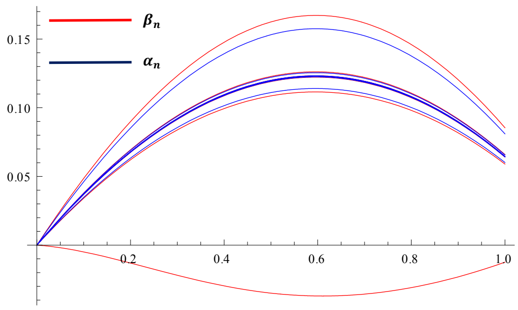

Fig. 5.3 Non-Monotonicity for $\lambda = 2$.

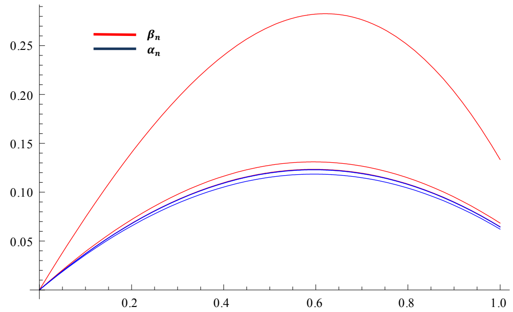

Fig. 5.4 Plot of upper (β_n) & lower (α_n) solutions for $\lambda = 0.4$ and $n = 1, 2, 3$.

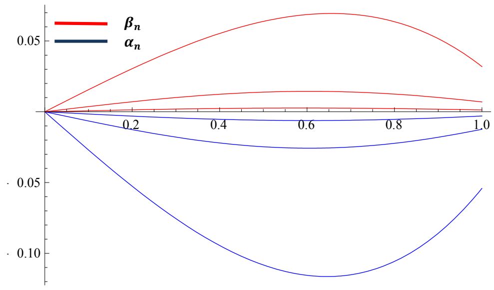

Fig. 5.5 Plot of upper (β_n) & lower (α_n) solutions for $\lambda = -2$ and $n = 1, 2, 3$.

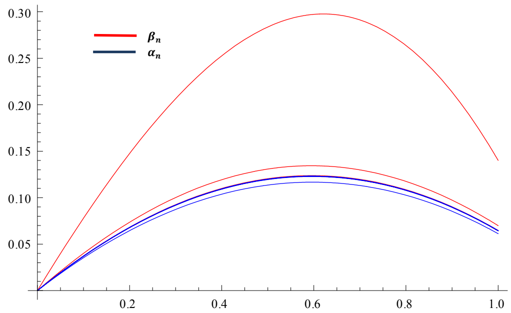

Fig. 5.6 Plot of upper (β_n) & lower (α_n) solutions for $\lambda = 0.3$ and $n = 1, 2, \cdots, 10$.

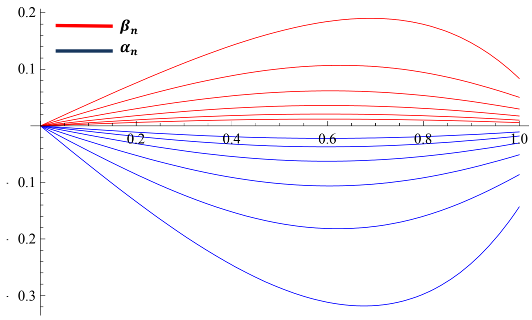

Fig. 5.7 Plot of upper (β_n) & lower (α_n) solutions for $\lambda = -10$ and $n = 1, 2, \cdots, 6$.

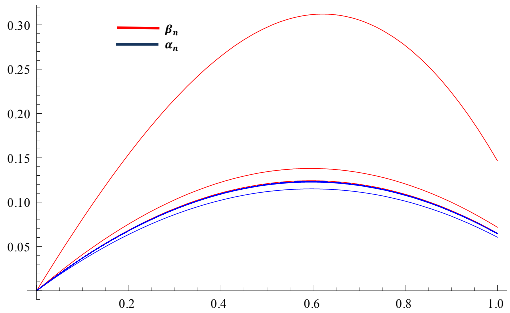

Fig. 5.8 Plot of upper (β_n) & lower (α_n) solutions for $\lambda = 0.2$ and $n = 1, 2, \cdots, 20$.

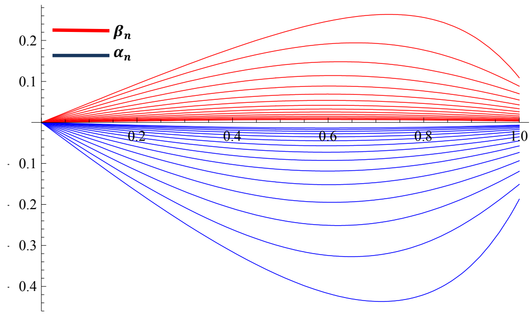

Fig. 5.9 Plot of upper (β_n) & lower (α_n) solutions for $\lambda = -25$ and $n = 1, 2, \cdots, 15$.

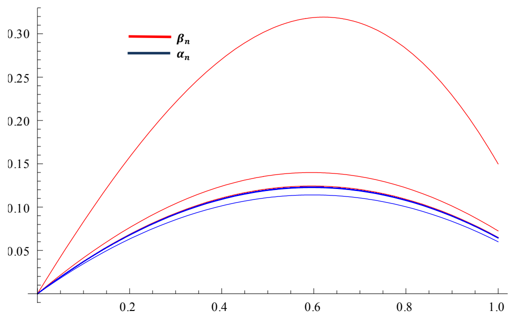

Fig. 5.10 Plot of upper (β_n) & lower (α_n) solutions for $\lambda = 0.15$ and $n = 1, 2, \cdots, 30$.

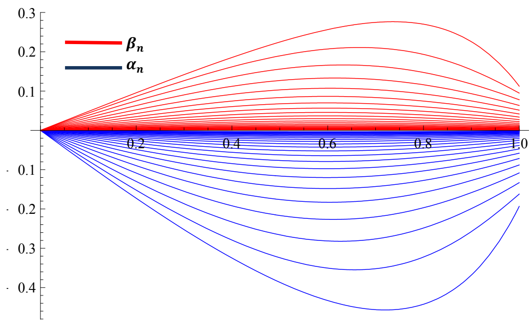

Fig. 5.11 Plot of upper (β_n) & lower (α_n) solutions for $\lambda = -30$ and $n = 1, 2, \cdots, 25$.

Chapter 6

Nonlinear three point singular boundary value problems arising due to spherical symmetry

6.1 Introduction

Singular differential equations arise in several real life problems [21, 31, 32, 44, 46, 82], and the behavior of a physical system modeled by differential equation frequently is most interesting in the neighborhood of a singular point [25]. Many problems in applied mathematics and engineering lead to singular boundary value problems of the form

$$-y'' - \frac{\alpha}{x}y' = f(x,y), \quad 0 < x < 1, \tag{6.1}$$

$$y'(0) = 0, \quad y(1) = A, \tag{6.2}$$

where A is a finite constant and $\alpha \geq 1$. Existence and uniqueness of solutions of (6.1)–(6.2) has been studied by several researchers, e.g., [31, 38, 82, 135, 150, 152, 153].

Recently lot of activity is noted on the upper and lower solution techniques. Zhang [164] in his work justified that this technique is most promising specially for singular boundary value problems.

Three point variation of the two point SBVPs (6.1)–(6.2) in spherical symmetry can be written as

$$-y''(x) - \frac{2}{x}y'(x) = f(x,y), \quad 0 < x < 1, \tag{6.3}$$

$$y'(0) = 0, \quad y(1) = \delta y(\eta), \tag{6.4}$$

where $f(I\times\mathbb{R},\mathbb{R})$, $I=[0,1]$, $0<\eta<1$, $\delta>0$. The singular three point BVP (6.3)–(6.4) are motivated by the mathematical model of heat generated in a chemical reaction ([31]) and equilibrium of charged gas in a spherical shaped container ([82]). Equations (6.3)–(6.4) model the thermal balance ([31]) between the heat generated by the chemical reaction and that conducted away in spherical vessel. The boundary condition $y(1)=\delta y(\eta)$ represents the relation between temperature on the outer surface and a surface of a sphere concentric with the vessel and radius less than container. Similarly equilibrium of a charged gas in a spherical container ([82]) can be extended for three point boundary value problems of the type (6.3)–(6.4).

Lots of results are available based on different analytical techniques for nonlinear three point BVPs [4, 17, 57, 62, 70, 92, 102, 108, 141, 142]. But when existing theory is applied to nonlinear three point SBVPs lot of complications arise and in this thesis we have made an honest effort to address some of these issues. In this chapter we consider nonlinear three point SBVP (6.3)–(6.4) which represents some physical phenomenon occurring in spherical geometry. We use monotone iterative technique which is analytical but computational in nature. It is not easy to establish maximum principle for the corresponding linear case for three point BVPs. As to achieve that we need to validate some inequalities which are nonlinear in nature.

In this chapter we propose the following iterative scheme which is similar to the one considered in [38] and [135]

$$\left.\begin{aligned}&-y''_{n+1}(x)-\frac{2}{x}y'_{n+1}(x)-\lambda y_{n+1}(x)=f(x,y_n)-\lambda y_n(x), \qquad \lambda\in\mathbb{R}\setminus\{0\},\\ &y'_{n+1}(0)=0,\quad y_{n+1}(1)=\delta y_{n+1}(\eta).\end{aligned}\right\}\tag{6.5}$$

We allow $\sup\left(\frac{\partial f}{\partial y}\right)$ to take both negative and positive values.

Under quite general conditions we show that a range for values of λ on both side of real line can be found so that the above iterative scheme produces convergent monotonic sequences which are solutions of the iterative scheme. These sequences converge uniformly to the solution of the nonlinear three point singular boundary value problem (6.3)–(6.4). To start the iteration and to produce monotonic sequences we need some initial guess in terms of the differential inequalities. These inequalities provide initial guess as well as upper and lower bound for above discussed sequences of solutions. Due to lack of anti-maximum principle we do not arrive at reverse order case, but we get well order case for both $\lambda>0$ and $\lambda<0$.

This chapter is organized in following sections. In Section 6.2, we use Lommel's transformation to find out two linearly independent solutions in terms of spherical Bessel functions. Using these two linearly independent solutions Green's function is constructed in Section 6.3 and Section 6.4 states maximum principle. Finally all these results are used to establish existence theorems (See Section 6.5). The sufficient conditions derived in this chapter are verified for 4 examples.

6.2 Lommel's transformation

This section is devoted to the corresponding linear case of the nonlinear three point SBVP (6.3)–(6.4). We consider the following class of three point linear SBVP,

$$-(x^2y'(x))' - \lambda x^2 y(x) = x^2 h(x), \quad 0 < x < 1, \tag{6.6}$$

$$y'(0) = 0, \quad y(1) = \delta y(\eta) + b, \tag{6.7}$$

where $h \in C(I)$ and b is any constant.

The corresponding homogeneous system is given by

$$-(x^2y'(x))' - \lambda x^2 y(x) = 0, \quad 0 < x < 1, \tag{6.8}$$

$$y'(0) = 0, \quad y(1) = \delta y(\eta). \tag{6.9}$$

Consider the differential equation (6.8) written in the form

$$x^2 y''(x) + 2xy'(x) + \lambda x^2 y(x) = 0. \tag{6.10}$$

Using Lommel's transformation (§cf [38, 51])

$$z = x\sqrt{\lambda}, \qquad w = x^{\frac{1}{2}} y(x), \tag{6.11}$$

the standard Bessel's equation

$$z^2 \frac{d^2w}{dz^2} + z\frac{dw}{dz} + (z^2 - \nu^2)w = 0, \tag{6.12}$$

is transformed into (6.10). Now, if $w_1(z)$ and $w_2(z)$ are two linearly independent solutions of Bessel's equation (6.12), then the two linearly independent solutions of (6.10) are given by

$$y_1(x) = x^{-\frac{1}{2}} w_1\left(x\sqrt{\lambda}\right), \quad y_2(x) = x^{-\frac{1}{2}} w_2\left(x\sqrt{\lambda}\right). \tag{6.13}$$

Hence the two linearly independent solutions of (6.10) can be obtained in terms of $w_1(z)$ and $w_2(z)$. A solution of (6.12) which leads to say y_1 bounded in the neighborhood of the origin is $w_1 = J_{\frac{1}{2}}(z)$. Hence a solution of (6.10) which remains bounded in the neighborhood of the origin (except for a multiplicative constant) denoted as $y_1(x,\lambda)$ is given by

$$y_1(x,\lambda) = \begin{cases} x^{-\frac{1}{2}} J_{\frac{1}{2}}\left(x\sqrt{\lambda}\right), & \text{if } \lambda > 0; \\ (ix)^{-\frac{1}{2}} J_{\frac{1}{2}}\left(ix\sqrt{|\lambda|}\right), & \text{if } \lambda < 0. \end{cases} \tag{6.14}$$

6.3 Green's function

In this section we construct Green's function. We divide it into two cases.

6.3.1 Case I: $\lambda > 0$.

Let us assume

$(H_0):$ $0 < \lambda \leq j^2_{-\frac{1}{2},1}$, $0 < \delta < 1$, $\eta \cos\sqrt{\lambda} - \delta \cos\eta\sqrt{\lambda} \leq 0$, $\eta \sin\sqrt{\lambda} - \delta \sin\eta\sqrt{\lambda} > 0$ where $j_{-\frac{1}{2},1}$ is the first positive zero of $J_{-\frac{1}{2}}(x)$.

There exists a range of $\lambda > 0$ such that (H_0) is true (see figures 6.1 & 6.2).

Lemma 6.1. *The Green's function for the following linear three point SBVP*

$$(x^2 y'(x))' + \lambda x^2 y(x) = 0, \quad 0 < x < 1, \tag{6.15}$$

$$y'(0) = 0, \quad y(1) = \delta y(\eta), \tag{6.16}$$

is given by

$$G(x,t) = \begin{cases} \dfrac{\sin\left(x\sqrt{\lambda}\right)\left(\eta\sin\left(\sqrt{\lambda}(t-1)\right) - \delta\sin\left(\sqrt{\lambda}(t-\eta)\right)\right)}{x\,t\,\sqrt{\lambda}\,\left(\eta\sin\left(\sqrt{\lambda}\right) - \delta\sin\left(\eta\sqrt{\lambda}\right)\right)}, & 0 \leq x \leq t \leq \eta; \\ \dfrac{\sin\left(t\sqrt{\lambda}\right)\left(\eta\sin\left(\sqrt{\lambda}(x-1)\right) - \delta\sin\left(\sqrt{\lambda}(x-\eta)\right)\right)}{x\,t\,\sqrt{\lambda}\,\left(\eta\sin\left(\sqrt{\lambda}\right) - \delta\sin\left(\eta\sqrt{\lambda}\right)\right)}, & t \leq x,\ t \leq \eta; \\ \dfrac{\eta\sin\left(\sqrt{\lambda}(t-1)\right)\sin\left(x\sqrt{\lambda}\right)}{x\,t\,\sqrt{\lambda}\left(\eta\sin\left(\sqrt{\lambda}\right) - \delta\sin\left(\eta\sqrt{\lambda}\right)\right)}, & x \leq t,\ \eta \leq t; \\ \dfrac{\left(\delta\sin\left(\eta\sqrt{\lambda}\right)\sin\left(\sqrt{\lambda}(t-x)\right) + \eta\sin\left(t\sqrt{\lambda}\right)\sin\left(\sqrt{\lambda}(x-1)\right)\right)}{x\,t\,\sqrt{\lambda}\,\left(\eta\sin\left(\sqrt{\lambda}\right) - \delta\sin\left(\eta\sqrt{\lambda}\right)\right)}, & \eta \leq t \leq x \leq 1, \end{cases}$$

and if (H_0) holds then $G(x,t) \leq 0$.

Proof. Define the Green's function by the following equations

$$G(x,t)=\begin{cases} a_1\frac{1}{\sqrt{x}}J_{\frac{1}{2}}\left(x\sqrt{\lambda}\right)+a_2\frac{1}{\sqrt{x}}J_{-\frac{1}{2}}\left(x\sqrt{\lambda}\right), & 0\le x\le t\le\eta;\\ a_3\frac{1}{\sqrt{x}}J_{\frac{1}{2}}\left(x\sqrt{\lambda}\right)+a_4\frac{1}{\sqrt{x}}J_{-\frac{1}{2}}\left(x\sqrt{\lambda}\right), & t\le x,\ t\le\eta;\\ a_5\frac{1}{\sqrt{x}}J_{\frac{1}{2}}\left(x\sqrt{\lambda}\right)+a_6\frac{1}{\sqrt{x}}J_{-\frac{1}{2}}\left(x\sqrt{\lambda}\right), & x\le t,\ \eta\le t;\\ a_7\frac{1}{\sqrt{x}}J_{\frac{1}{2}}\left(x\sqrt{\lambda}\right)+a_8\frac{1}{\sqrt{x}}J_{-\frac{1}{2}}\left(x\sqrt{\lambda}\right), & \eta\le t\le x\le 1. \end{cases}$$

According to the definition and properties of the Green's function, for any $t\in[0,\eta]$, we have

$$a_1\frac{1}{\sqrt{t}}J_{\frac{1}{2}}\left(t\sqrt{\lambda}\right)+a_2\frac{1}{\sqrt{t}}J_{-\frac{1}{2}}\left(t\sqrt{\lambda}\right)=a_3\frac{1}{\sqrt{t}}J_{\frac{1}{2}}\left(t\sqrt{\lambda}\right)+a_4\frac{1}{\sqrt{t}}J_{-\frac{1}{2}}\left(t\sqrt{\lambda}\right),$$

$$\begin{aligned}&\left(-a_1\sqrt{\lambda}\frac{1}{\sqrt{t}}J_{\frac{3}{2}}\left(t\sqrt{\lambda}\right)+a_2\sqrt{\lambda}\frac{1}{\sqrt{t}}J_{-\frac{3}{2}}\left(t\sqrt{\lambda}\right)\right)\\ &\qquad-\left(-a_3\sqrt{\lambda}\frac{1}{\sqrt{t}}J_{\frac{3}{2}}\left(t\sqrt{\lambda}\right)+a_4\sqrt{\lambda}\frac{1}{\sqrt{t}}J_{-\frac{3}{2}}\left(t\sqrt{\lambda}\right)\right)=-\frac{1}{t^2},\end{aligned}$$

and thus

$$\begin{aligned}a_1-a_3&=-\frac{\pi J_{-\frac{1}{2}}\left(t\sqrt{\lambda}\right)}{2\sqrt{t}},\\ a_2-a_4&=\frac{\pi J_{\frac{1}{2}}\left(t\sqrt{\lambda}\right)}{2\sqrt{t}}.\end{aligned}$$

Using the boundary conditions, we have

$$\begin{aligned}&a_2=0,\\ &a_3J_{\frac{1}{2}}\left(\sqrt{\lambda}\right)+a_4J_{-\frac{1}{2}}\left(\sqrt{\lambda}\right)=\delta\left(a_3\frac{1}{\sqrt{\eta}}J_{\frac{1}{2}}\left(\eta\sqrt{\lambda}\right)+a_4\frac{1}{\sqrt{\eta}}J_{-\frac{1}{2}}\left(\eta\sqrt{\lambda}\right)\right).\end{aligned}$$

Therefore

$$\begin{aligned}a_1&=\frac{\sqrt{\frac{\pi}{2}}\left(\eta\sin\left((t-1)\sqrt{\lambda}\right)-\delta\sin\left(\sqrt{\lambda}(t-\eta)\right)\right)}{t\sqrt[4]{\lambda}\left(\eta\sin\left(\sqrt{\lambda}\right)-\delta\sin\left(\eta\sqrt{\lambda}\right)\right)},\\ a_2&=0,\end{aligned}$$

$$a_3 = \frac{\sqrt{\frac{\pi}{2}}\ \sin\left(t\sqrt{\lambda}\right)\left(\eta\cos\left(\sqrt{\lambda}\right)-\delta\cos\left(\eta\sqrt{\lambda}\right)\right)}{t\ \sqrt[4]{\lambda}\left(\eta\sin\left(\sqrt{\lambda}\right)-\delta\sin\left(\eta\sqrt{\lambda}\right)\right)},$$
$$a_4 = -\frac{\sqrt{\frac{\pi}{2}}\ \sin\left(t\sqrt{\lambda}\right)}{t\ \sqrt[4]{\lambda}}.$$

For any $t \in [\eta, 1]$, we have

$$a_5\frac{1}{\sqrt{t}}J_{\frac{1}{2}}\left(t\sqrt{\lambda}\right)+a_6\frac{1}{\sqrt{t}}J_{-\frac{1}{2}}\left(t\sqrt{\lambda}\right)=a_7\frac{1}{\sqrt{t}}J_{\frac{1}{2}}\left(t\sqrt{\lambda}\right)+a_8\frac{1}{\sqrt{t}}J_{-\frac{1}{2}}\left(t\sqrt{\lambda}\right),$$

$$\left(-a_5\sqrt{\lambda}\frac{1}{\sqrt{t}}J_{\frac{3}{2}}\left(t\sqrt{\lambda}\right)+a_6\sqrt{\lambda}\frac{1}{\sqrt{t}}J_{-\frac{3}{2}}\left(t\sqrt{\lambda}\right)\right)$$
$$-\left(-a_7\sqrt{\lambda}\frac{1}{\sqrt{t}}J_{\frac{3}{2}}\left(t\sqrt{\lambda}\right)+a_8\sqrt{\lambda}\frac{1}{\sqrt{t}}J_{-\frac{3}{2}}\left(t\sqrt{\lambda}\right)\right)=-\frac{1}{t^2},$$

and hence

$$a_5-a_7=-\frac{\pi J_{-\frac{1}{2}}\left(t\sqrt{\lambda}\right)}{2\sqrt{t}},$$
$$a_6-a_8=\frac{\pi J_{\frac{1}{2}}\left(t\sqrt{\lambda}\right)}{2\sqrt{t}}.$$

By using the boundary conditions, we have

$$a_6=0,$$
$$a_7J_{\frac{1}{2}}\left(\sqrt{\lambda}\right)+a_8J_{-\frac{1}{2}}\left(\sqrt{\lambda}\right)=\delta\left(a_5\frac{1}{\sqrt{\eta}}J_{\frac{1}{2}}\left(\eta\sqrt{\lambda}\right)+a_6\frac{1}{\sqrt{\eta}}J_{-\frac{1}{2}}\left(\eta\sqrt{\lambda}\right)\right).$$

Thus

$$a_5=\frac{\sqrt{\frac{\pi}{2}}\ \eta\ \sin\left(\sqrt{\lambda}(t-1)\right)}{t\ \sqrt[4]{\lambda}\left(\eta\sin\left(\sqrt{\lambda}\right)-\delta\sin\left(\eta\sqrt{\lambda}\right)\right)},$$
$$a_6=0,$$
$$a_7=\frac{\sqrt{\frac{\pi}{2}}\ \left(\eta\cos\left(\sqrt{\lambda}\right)\sin\left(\sqrt{\lambda}t\right)-\delta\sin\left(\eta\sqrt{\lambda}\right)\cos\left(\sqrt{\lambda}t\right)\right)}{t\ \sqrt[4]{\lambda}\left(\eta\sin\left(\sqrt{\lambda}\right)-\delta\sin\left(\eta\sqrt{\lambda}\right)\right)},$$

$$a_8 = -\frac{\sqrt{\frac{\pi}{2}}\ \sin\left(\sqrt{\lambda}t\right)}{t\ \sqrt[4]{\lambda}},$$

which completes the construction of Green's function. Using (H_0), we get that $G(x,t) \leq 0$. ■

Lemma 6.2. *Let $y \in C^2(I)$ be a solution of nonhomogeneous linear three point SBVP (6.6)–(6.7) then*

$$y(x) = \frac{b\ \eta\ \sin\left(x\sqrt{\lambda}\right)}{x\left(\eta\ \sin\left(\sqrt{\lambda}\right) - \delta \sin\left(\eta\sqrt{\lambda}\right)\right)} - \int_0^1 t^2 G(x,t)h(t)dt. \tag{6.17}$$

Proof. Suppose $G(x,t)$ is the Green's function of

$$(x^2y'(x))' + \lambda x^2 y(x) = 0, \quad 0 < x < 1,$$
$$y'(0) = 0, \quad y(1) = \delta y(\eta),$$

and $\bar{y}$ is solution of

$$(x^2y'(x))' + \lambda x^2 y(x) = 0, \quad 0 < x < 1,$$
$$y'(0) = 0, \quad y(1) = \delta(\eta) + b,$$

then the boundary value problem (6.6)–(6.7) is equivalent to

$$y(t) = \bar{y} - \int_0^1 t^2 G(x,t)h(t)dt.$$

Suppose

$$\bar{y} = c_1 \frac{1}{\sqrt{x}} J_{\frac{1}{2}}\left(x\sqrt{\lambda}\right) + c_2 \frac{1}{\sqrt{x}} J_{-\frac{1}{2}}\left(x\sqrt{\lambda}\right).$$

Since

$$\bar{y}'(0) = 0, \text{ and } \bar{y}(1) = \delta \bar{y}(\eta) + b,$$

we get

$$c_1 = \frac{b}{J_{\frac{1}{2}}\left(\sqrt{\lambda}\right) - \frac{\delta}{\sqrt{\eta}} J_{\frac{1}{2}}\left(\eta\sqrt{\lambda}\right)},$$

$$c_2 = 0.$$

Namely $y \in C^2(I)$ is a solution of the boundary value problem (6.6)–(6.7) if and only if $y \in C(I)$ is a solution of the integral equation

$$y(x) = \frac{b\,\eta\,\sin\left(x\sqrt{\lambda}\right)}{x\left(\eta\sin\left(\sqrt{\lambda}\right) - \delta\sin\left(\eta\sqrt{\lambda}\right)\right)} - \int_0^1 t^2 G(x,t)h(t)dt.$$

■

6.3.2 Case II: $\lambda < 0$.

Assume that

(H_0') : $\lambda < 0$, $\delta > 0$, $\eta\cosh\left(\sqrt{|\lambda|}\right) - \delta\cosh\left(\eta\sqrt{|\lambda|}\right) \geq 0$, and $\eta\sinh\left(\sqrt{|\lambda|}\right)$
$- \delta\sinh\left(\eta\sqrt{|\lambda|}\right) > 0$.

There exists a range of $\lambda < 0$ for which (H_0') is true (see figures 6.3 & 6.5).

Lemma 6.3. *The Green's function for the following linear three point SBVP*

$$(x^2y'(x))' + \lambda x^2 y(x) = 0, \quad 0 < x < 1,$$
$$y'(0) = 0, \quad y(1) = \delta y(\eta),$$

for $\lambda < 0$ is given by

$$G(x,t) = \begin{cases} \dfrac{\sinh\left(\sqrt{|\lambda|}x\right)\left(\eta\sinh\left(\sqrt{|\lambda|}(t-1)\right) - \delta\sinh\left(\sqrt{|\lambda|}(t-\eta)\right)\right)}{x\,t\,\sqrt{|\lambda|}\left(\eta\sinh\left(\sqrt{|\lambda|}\right) - \delta\sinh\left(\eta\sqrt{|\lambda|}\right)\right)}, & 0 \leq x \leq t \leq \eta; \\ \dfrac{\sinh\left(\sqrt{|\lambda|}\,t\right)\left(\eta\sinh\left(\sqrt{|\lambda|}(x-1)\right) - \delta\sinh\left(\sqrt{|\lambda|}(x-\eta)\right)\right)}{x\,t\,\sqrt{|\lambda|}\left(\eta\sinh\left(\sqrt{|\lambda|}\right) - \delta\sinh\left(\eta\sqrt{|\lambda|}\right)\right)}, & t \leq x, t \leq \eta; \\ \dfrac{\eta\sinh\left(\sqrt{|\lambda|}(t-1)\right)\sinh\left(\sqrt{|\lambda|}x\right)}{x\,t\,\sqrt{|\lambda|}\left(\eta\sinh\left(\sqrt{|\lambda|}\right) - \delta\sinh\left(\eta\sqrt{|\lambda|}\right)\right)}, & x \leq t, \eta \leq t; \\ \dfrac{\left(\delta\sinh\left(\eta\sqrt{|\lambda|}\right)\sinh\left(\sqrt{|\lambda|}(t-x)\right) + \eta\sinh\left(t\sqrt{|\lambda|}\right)\sinh\left((x-1)\sqrt{|\lambda|}\right)\right)}{x\,t\,\sqrt{|\lambda|}\left(\eta\sinh\left(\sqrt{|\lambda|}\right) - \delta\sinh\left(\eta\sqrt{|\lambda|}\right)\right)}, & \eta \leq t \leq x \leq 1. \end{cases}$$

and if $\left(H_0'\right)$ holds then $G(x,t) \leq 0$.

Proof. Proof is same as given in Lemma 6.1. ■

Lemma 6.4. *Let $y \in C^2(I)$ be a solution of nonhomogeneous linear three point SBVP (6.6)–(6.7) then*

$$y(x) = \frac{b\,\eta \sinh\left(x\sqrt{|\lambda|}\right)}{x\left(\eta \sinh\left(\sqrt{|\lambda|}\right) - \delta \sinh\left(\eta\sqrt{|\lambda|}\right)\right)} - \int_0^1 t^2 G(x,t)h(t)dt. \tag{6.18}$$

Proof. Proof is same as given in Lemma 6.2. ■

6.4 Maximum principle

We require two results. They are as follows.

Proposition 6.1. *Let (H_0) holds, $b \geq 0$ and $h(x) \in C[0,1]$ is such that $h(x) \geq 0$, then $y(x)$ is non-negative for all $x \in [0,1]$.*

Proposition 6.2. *Let (H_0') holds, $b \geq 0$ and $h(x) \in C[0,1]$ is such that $h(x) \geq 0$, then $y(x)$ is non-negative for all $x \in [0,1]$.*

6.5 Nonlinear three point SBVP

In this section, we develop the theory of monotone iterative method for nonlinear three point SBVPs. We divide it into the following two subsections.

6.5.1 Case I: When $\lambda > 0$

Theorem 6.1. *Let there exist α_0, β_0 in $C^2[0,1]$ such that $\beta_0 \geq \alpha_0$ and satisfy*

$$-(x^2\beta_0'(x))' \geq x^2 f(x,\beta_0), \quad 0 < x < 1, \quad \beta_0'(0) = 0, \quad \beta_0(1) \geq \delta\beta_0(\eta), \tag{6.19}$$

and

$$-(x^2\alpha_0'(x))' \leq x^2 f(x,\alpha_0), \quad 0 < x < 1, \quad \alpha_0'(0) = 0, \quad \alpha_0(1) \leq \delta\alpha_0(\eta). \tag{6.20}$$

If $f : D \to \mathbb{R}$ is continuous on $D := \{(x,y) \in [0,1]\times\mathbb{R} : \alpha_0 \leq y \leq \beta_0\}$ and there exists $M > 0$ such that for all $(x,y),(x,w) \in D$

$$y \leq w \Longrightarrow f(x,w) - f(x,y) \geq M(w-y),$$

then the nonlinear three point SBVP (6.3)–(6.4) has at least one solution in the region D. If $\exists$ *a constant* λ *such that* $M-\lambda \geq 0$ *and* (H_0) *is satisfied then the sequence* (β_n) *generated by*

$$-(x^2 y'_{n+1}(x))' - \lambda x^2 y_{n+1} = x^2 F(x, y_n), \quad y'_{n+1}(0) = 0, \quad y_{n+1}(1) = \delta y_{n+1}(\eta), \qquad (6.21)$$

where $F(x,y) = f(x,y) - \lambda y$, *with initial iterate* β_0 *converges monotonically (non-increasing) and uniformly towards a solution* $\widetilde{\beta}(x)$ *of (6.3)–(6.4). Similarly* α_0 *as an initial iterate leads to a non-decreasing sequence* (α_n) *converging to a solution* $\widetilde{\alpha}(x)$. *Any solution* $z(x)$ *in D must satisfy*

$$\widetilde{\alpha}(x) \leq z(x) \leq \widetilde{\beta}(x).$$

Proof. From equation (6.19) and equation (6.21) (for $n = 0$)

$$\begin{aligned} &-(x^2(\beta_0 - \beta_1)'(x))' - \lambda x^2(\beta_0 - \beta_1) \geq 0, \\ &(\beta_0 - \beta_1)'(0) = 0, \quad (\beta_0 - \beta_1)(1) \geq \delta(\beta_0 - \beta_1)(\eta). \end{aligned}$$

Since $h(x) \geq 0$ and $b \geq 0$, by using Proposition 6.1 we have $\beta_0 \geq \beta_1$.

In view of $M - \lambda \geq 0$, from equation (6.21) we get

$$-(x^2 \beta'_{n+1}(x))' \geq x^2 \left[(M-\lambda)(\beta_n - \beta_{n+1}) + f(x, \beta_{n+1})\right]$$

and if $(\beta_n \geq \beta_{n+1})$, then

$$-(x^2 \beta'_{n+1}(x))' \geq x^2 f(x, \beta_{n+1}); \quad \beta'_{n+1}(0) = 0, \ \beta_{n+1}(1) = \delta \beta_{n+1}(\eta). \qquad (6.22)$$

Since $\beta_0 \geq \beta_1$, then from equation (6.22) (for $n = 0$) and (6.21) (for $n = 1$) we get

$$\begin{aligned} &-(x^2(\beta_1 - \beta_2)'(x))' - \lambda x^2(\beta_1 - \beta_2) \geq 0, \\ &(\beta_1 - \beta_2)'(0) = 0, \quad (\beta_1 - \beta_2)(1) \geq \delta(\beta_1 - \beta_2)(\eta). \end{aligned}$$

From Proposition 6.1 we have $\beta_1 \geq \beta_2$.

Now from equations (6.20) and (6.21) (for $n = 0$)

$$\begin{aligned} &-(x^2(\beta_1 - \alpha_0)'(x))' - \lambda x^2(\beta_1 - \alpha_0) \geq 0, \\ &(\beta_1 - \alpha_0)'(0) = 0 \quad (\beta_1 - \alpha_0)(1) \geq \delta((\beta_1 - \alpha_0)(\eta). \end{aligned}$$

Thus $\beta_1 \geq \alpha_0$ follows from proposition 6.1.

Now assuming $\beta_n \geq \beta_{n+1}$, $\beta_{n+1} \geq \alpha_0$, we show that $\beta_{n+1} \geq \beta_{n+2}$ and $\beta_{n+2} \geq \alpha_0$ for all n. From equation (6.21) (for $n+1$) and (6.22) we get

$$\begin{aligned} &-(x^2(\beta_{n+1}-\beta_{n+2})'(x))' - \lambda x^2(\beta_{n+1}-\beta_{n+2}) \geq 0, \\ &(\beta_{n+1}-\beta_{n+2})'(0) = 0, \quad (\beta_{n+1}-\beta_{n+2})(1) \geq \delta(\beta_{n+1}-\beta_{n+2})(\eta), \end{aligned}$$

and hence from Proposition 6.1 we have $\beta_{n+1} \geq \beta_{n+2}$.

From equation (6.21) (for $n+1$) and (6.20) we get,

$$\begin{aligned} &-(x^2(\beta_{n+2}-\alpha_0)'(x))' - x^2\lambda(\beta_{n+2}-\alpha_0) \geq 0, \\ &(\beta_{n+2}-\alpha_0)'(0) = 0, \quad (\beta_{n+2}-\alpha_0)(1) \geq \delta(\beta_{n+2}-\alpha_0)(\eta). \end{aligned}$$

Then from proposition 6.1, $\beta_{n+2} \geq \alpha_0$ and hence we have

$$\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n \geq \beta_{n+1} \geq \cdots \geq \alpha_0$$

and starting with α_0 it is easy to get

$$\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n \leq \alpha_{n+1} \leq \cdots \leq \beta_0.$$

Finally we show that $\beta_n \geq \alpha_n$ for all n. For this by assuming $\beta_n \geq \alpha_n$, we show that $\beta_{n+1} \geq \alpha_{n+1}$. From equation (6.21) it is easy to get

$$\begin{aligned} &-(x^2(\beta_{n+1}-\alpha_{n+1})'(x))' - \lambda x^2(\beta_{n+1}-\alpha_{n+1}) \geq 0, \\ &(\beta_{n+1}-\alpha_{n+1})'(0) = 0, \quad (\beta_{n+1}-\alpha_{n+1})(1) \geq \delta(\beta_{n+1}-\alpha_{n+1})(\eta). \end{aligned}$$

Hence from Proposition 6.1, $\beta_{n+1} \geq \alpha_{n+1}$. Thus we have

$$\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n \leq \alpha_{n+1} \leq \cdots \leq \beta_{n+1} \leq \beta_n \leq \cdots \leq \beta_2 \leq \beta_1 \leq \beta_0.$$

So the sequences (β_n) and (α_n) are monotonically non-increasing and non-decreasing, respectively and are bounded by β_0 and α_0. Hence by Dini's theorem they converges uniformly. Let $\widetilde{\beta}(x) = \lim_{n\to\infty} \beta_n(x)$ and $\widetilde{\alpha}(x) = \lim_{n\to\infty} \alpha_n(x)$.

Using Lemma 6.2, the solution β_{n+1} of (6.21) is given by

$$\beta_{n+1} = \frac{b\,\eta\,\sin\left(x\sqrt{\lambda}\right)}{x\left(\eta\sin\left(\sqrt{\lambda}\right) - \delta\sin\left(\eta\sqrt{\lambda}\right)\right)} - \int_0^1 G(x,t)t^2(f(t,\beta_n) - \lambda\beta_n)dt.$$

Then by Lebesgue's dominated convergence theorem, taking the limit as $n \to \infty$, we get

$$\widetilde{\beta}(x) = \frac{b\,\eta\,\sin\left(x\sqrt{\lambda}\right)}{x\left(\eta\sin\left(\sqrt{\lambda}\right) - \delta\sin\left(\eta\sqrt{\lambda}\right)\right)} - \int_0^1 G(x,t)t^2(f(t,\widetilde{\beta}) - \lambda\widetilde{\beta})dt.$$

Which is the solution of boundary value problem (6.3)–(6.4). The same is true for (α_n).

Any solution $z(x)$ in D can play the role of $\beta_0(x)$, hence $z(x) \geq \widetilde{\alpha}(x)$ and similarly one concludes that $z(x) \leq \widetilde{\beta}(x)$. ∎

6.5.2 Case II: When $\lambda < 0$

Theorem 6.2. *Let there exist α_0, β_0 in $C^2[0,1]$ such that $\beta_0 \geq \alpha_0$ and satisfy*

$$-(x^2\beta_0'(x))' \geq x^2 f(x,\beta_0), \quad 0 < x < 1, \quad \beta_0'(0) = 0, \quad \beta_0(1) \geq \delta\beta_0(\eta), \tag{6.23}$$

and

$$-(x^2\alpha_0'(x))' \leq x^2 f(x,\alpha_0), \quad 0 < x < 1, \quad \alpha_0'(0) = 0, \quad \alpha_0(1) \leq \delta\alpha_0(\eta). \tag{6.24}$$

If $f : \widetilde{D} \to \mathbb{R}$ is continuous on $\widetilde{D} := \{(x,y) \in [0,1]\times\mathbb{R} : \alpha_0 \leq y \leq \beta_0\}$ and there exists $M > 0$ such that for all $(x,\widetilde{y}),(x,\widetilde{w}) \in \widetilde{D}$

$$\widetilde{y} \leq \widetilde{w} \Longrightarrow f(x,\widetilde{w}) - f(x,\widetilde{y}) \geq -M(\widetilde{w} - \widetilde{y})$$

then the nonlinear three point SBVP (6.3)–(6.4) has at least one solution in the region $\widetilde{D}$. If $\exists$ a constant λ such that $M + \lambda \leq 0$ and (H_0') is satisfied then the sequence (β_n) generated by

$$-(x^2y_{n+1}'(x))' - \lambda x^2 y_{n+1} = x^2F(x,y_n), \quad y_{n+1}'(0) = 0, \quad y_{n+1}(1) = \delta y_{n+1}(\eta), \tag{6.25}$$

where $F(x,y) = f(x,y) - \lambda y$, with initial iterate β_0 converges monotonically (non-increasing) and uniformly towards a solution $\beta(x)$ of (6.3)–(6.4). Similarly using α_0 as an initial iterate leads to a non-decreasing sequence (α_n) converging to a solution $\alpha(x)$. Any solution $Z(x)$ in $\widetilde{D}$ must satisfy

$$\alpha(x) \leq Z(x) \leq \beta(x).$$

Proof. Proof follows from the analysis of Theorem 6.1. ∎

6.6 Examples

With the help of following examples, we verify our results and show that it is possible to choose a value of "λ" so that iterative scheme generates monotone sequences which converge to solution of nonlinear singular problem. Thus these examples validate sufficient conditions derived in the Theorem 6.1 and Theorem 6.2.

Example 6.1. *Consider the nonlinear three point SBVP*

$$-y''(x) - \frac{2}{x}y'(x) = \frac{3}{4}e^{y(x)}, \tag{6.26}$$

$$y'(0) = 0, \qquad y(1) = \frac{2}{5}y\left(\frac{1}{2}\right). \tag{6.27}$$

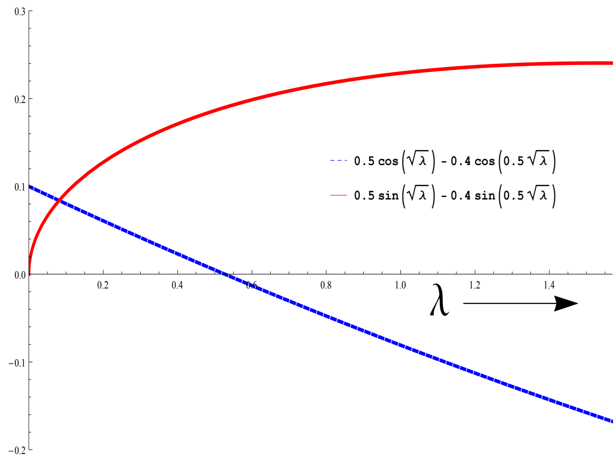

Fig. 6.1 Plot of (H_0) for example 6.1.

Here $f(x,y) = \frac{3}{4}e^y$, $\delta = \frac{2}{5}$, $\eta = \frac{1}{2}$. This problem has $\alpha_0 = 0$ and $\beta_0 = \frac{2-x^2}{3}$ as lower and upper solutions, and it is well ordered case. The nonlinear term is Lipschitz in y and continuous for all value of y, and Lipschitz constant M is $\frac{3}{4}$. Now we find out a subinterval $R_\lambda = (\xi_1, \xi_2)$ of $(0,\ j^2_{-\frac{1}{2},1})$ such that the conditions $M - \lambda \geq 0$ and (H_0) (See Figure 6.1) are true. Thus existence of at least one solution is guaranteed.

Example 6.2. *Consider the nonlinear three point SBVP*

$$-y''(x) - \frac{2}{x}y'(x) = y(x) + 1, \tag{6.28}$$

$$y'(0) = 0, \qquad y(1) = \frac{1}{2}y\left(\frac{3}{10}\right). \tag{6.29}$$

Here $f(x,y) = y+1$, $\delta = \frac{1}{2}$, $\eta = \frac{3}{10}$. This problem has $\alpha_0 = 0$ and $\beta_0 = 2 - x^2$ as lower and upper solutions, and this is a well ordered case. The source term is linear, Lipschitz in y and continuous for all value of y, and Lipschitz constant M is 1. Now we find out a subinterval $R_\lambda = (\xi_1, \xi_2)$ of $(0,\ j^2_{-\frac{1}{2},1})$ such that the conditions $M - \lambda \geq 0$ and (H_0) (See Figure 6.2) are true. Thus existence of at least one solution is guaranteed.

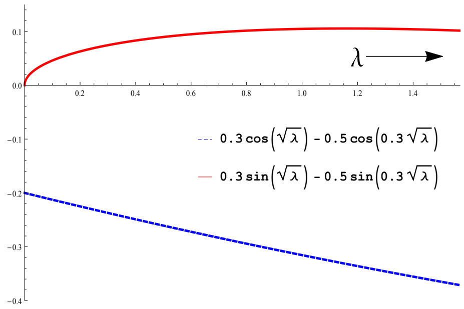

Fig. 6.2 Plot of (H_0) for example 6.2.

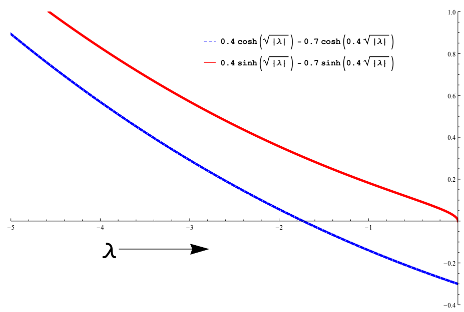

Fig. 6.3 Plot of (H_0') for example 6.3.

Example 6.3. *Consider the nonlinear three point SBVP*

$$-y''(x) - \frac{2}{x}y'(x) = \frac{1}{36}\left[\frac{e^2}{5} - 2(y(x))^3\right], \tag{6.30}$$

$$y'(0) = 0, \qquad y(1) = \frac{7}{10}y\left(\frac{2}{5}\right). \tag{6.31}$$

Here $f(x,y) = \frac{1}{36}\left[\frac{e^2}{5} - 2(y(x))^3\right]$, $\delta = \frac{7}{10}, \eta = \frac{2}{5}$. This problem has $\alpha_0 = -1$ and $\beta_0 = 1$ as lower and upper solutions, and this is a well ordered case. The nonlinear term is Lipschitz in y and continuous for all value of y, and Lipschitz constant M is $\frac{1}{6}$. For some λ less than $\left(-\frac{1}{6}\right)$, (H_0') (See Figure 6.3) will be true. Using Mathematica 10 and iterative scheme (6.5) we compute upper and lower solutions (See Figure 6.4). Thus existence of at least one solution is guaranteed.

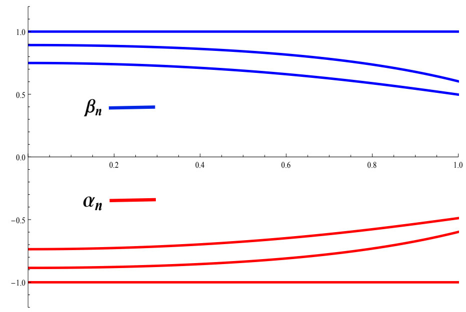

Fig. 6.4 Plot of upper (β_n) & lower (α_n) solutions for $n = 0(1)2$ and $\lambda = -10$.

Example 6.4. *Consider the nonlinear three point SBVP*

$$-y''(x) - \frac{2}{x}y'(x) = 1 - 2y(x), \tag{6.32}$$

$$y'(0) = 0, \qquad y(1) = \frac{1}{10}y\left(\frac{2}{5}\right). \tag{6.33}$$

Here $f(x,y) = 1 - 2y$, $\delta = \frac{1}{10}$, $\eta = \frac{2}{5}$. This problem has $\alpha_0 = -1$ and $\beta_0 = 1$ as lower and upper solutions, and this is a well ordered case. The nonlinear term is Lipschitz in y and continuous for all value of y, and Lipschitz constant M is 2. For $\lambda < -2$ we can see that (H_0') (See Figure 6.5) will be true. Using Mathematica 10 and iterative scheme (6.5) we compute upper and lower solutions (See Figure 6.6). These sequences converge uniformly to a solution of the problem (6.32)–(6.33).

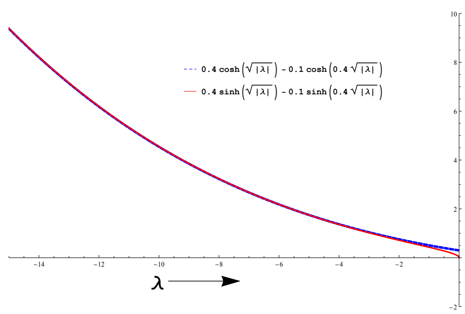

Fig. 6.5 Plot of (H_0') for example 6.4.

6.7 Conclusion

In this chapter we establish existence of solutions for a class of nonlinear singular three point boundary value problems. The BVPs of this kind can be considered as generalizations of two point SBVPs in spherical symmetry, e.g., [31], [82].We allow the Lipschitz constant to take both positive and negative values. Due to lack of uniform anti-maximum principle reversed ordered upper and lower solutions case is not observed. We have used Mathematica to plot solutions for $\frac{\partial f}{\partial y} < 0$ but the same could not be achieved for $\frac{\partial f}{\partial y} > 0$. The work in this paper can further be generalized to a class of singular nonlinear differential equations, e.g.,

$$-(py')' = qf(x, y, py'),\ 0 < x < 1,\ p(0) = 0,$$

subject to different kind of multi point boundary conditions, which depend on the nature of p, q and f.

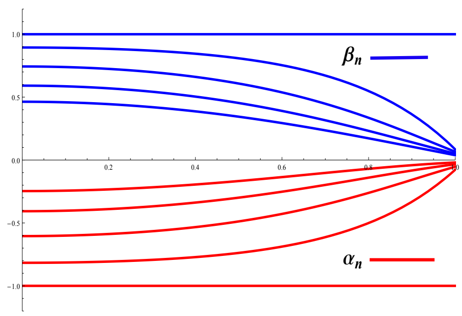

Fig. 6.6 Plot of upper (β_n) & lower (α_n) solutions for $n = 0(1)4$ and $\lambda = -24$.

Chapter 7

Nonlinear three point singular boundary value problems arising due to cylindrical symmetry

7.1 Introduction

Consider the following nonlinear two point singular boundary value problems (SBVPs)

$$-y''-\frac{\alpha}{x}y'=f(x,y),\quad 0<x<1, \tag{7.1}$$

$$y'(0)=0,\quad y(1)=A, \tag{7.2}$$

In chapter 6, we explored the three point variation of (7.1)–(7.2) for $\alpha=2$ and we could get only well order case. In this chapter, we consider the case when $\alpha=1$. Three point variation of the two point SBVP (7.1)–(7.2) for $\alpha=1$, can be written as

$$-y''(x)-\frac{1}{x}y'(x)=f(x,y),\quad 0<x<1, \tag{7.3}$$

$$y'(0)=0,\quad y(1)=\delta y(\eta), \tag{7.4}$$

where $f:\ I\times\ \mathbb{R}\to\ \mathbb{R}, I=[0,1],\ 0<\eta<1,\ \delta>0$. Equations (7.3)–(7.4) model thermal balance ([31]) between heat generated by the chemical reaction and that conducted away in cylindrical vessel. In this case we consider cylindrical vessel and there is another concentric cylinder inside the cylindrical vessel which we can use to monitor the temperature inside the vessel. The boundary condition at $x=1$ is the temperature at the walls of outer cylinder which is related to the temperature at walls at $x=\eta$ of an interior cylinder by $y(1)=\delta y(\eta)$.

This model can help us to maintain the required temperature interior to the vessel which is otherwise not possible.

There are lot of results available when upper solution β_0 and lower solution α_0 are well ordered, i.e., $\alpha_0 \leq \beta_0$. But situation is quite different when upper and lower solutions occur in the reverse order, that is $\alpha_0 \geq \beta_0$, and lot of exploration is still needed. When upper and lower solutions are in reverse order some results are available for regular three point BVPs ([92, 141, 142] and the references there in). There are lots of differences between regular and singular differential equations and hence lot of complications arise when existing theory is applied to three point SBVPs.

In this chapter we consider nonlinear three point SBVP (7.3)–(7.4) and use monotone iterative technique in the the presence of upper and lower solutions. We establish maximum and anti-maximum principles for the corresponding linear case for three point SBVPs.

We propose the following iterative scheme

$$\left.\begin{aligned} &-y''_{n+1}(x)-\frac{1}{x}y'_{n+1}(x)-\lambda y_{n+1}(x)=f(x,y_n)-\lambda y_n(x), \quad \lambda\in\mathbb{R}\setminus\{0\},\\ &y'_{n+1}(0)=0, \quad y_{n+1}(1)=\delta y_{n+1}(\eta), \end{aligned}\right\} \tag{7.5}$$

where sup $\left(\frac{\partial f}{\partial y}\right)$ allowed to take both negative and positive values.

Under quite general conditions we show that a range for values of λ on both side of real line can be found so that the above iterative scheme produces convergent monotonic sequences which are solutions of the iterative scheme. These sequences converge uniformly to the solution of the nonlinear three point SBVP (7.3)–(7.4). To start the iteration and to produce monotonic sequences we need some initial guess in terms of differential inequalities. These inequalities provide initial guess as well as upper and lower bound for above discussed sequences of solutions. For $\lambda > 0$, we get reverse order case and for $\lambda < 0$, we arrive at well order case.

This chapter is organized in the following sections. In Section 7.2, we use Lommel's transformation to find out two linearly independent solutions in terms of Bessel function $(J_0,\ Y_0)$ and modified Bessel functions $(I_0,\ K_0)$. Using these two linearly independent solutions Green's function is constructed in Section 7.3 and Section 7.4 state maximum and anti-maximum principles. Finally all these results are used to establish two existence theorems for nonlinear three point SBVPs in Section 7.5. The sufficient conditions derived in this chapter are verified for two examples (see Section 7.6). In Section 7.7, we summarize the conclusions.

7.2 The Linear case

This section is devoted to a corresponding linear case of the nonlinear three point SBVP (7.3)–(7.4). We consider the following class of three point linear SBVP,

$$-(xy'(x))' - \lambda xy(x) = xh(x), \quad 0 < x < 1, \tag{7.6}$$

$$y'(0) = 0, \quad y(1) = \delta y(\eta) + b, \tag{7.7}$$

where $h \in C(I)$ and b is any constant.

The corresponding homogeneous system is given by

$$-(xy'(x))' - \lambda xy(x) = 0, \quad 0 < x < 1, \tag{7.8}$$

$$y'(0) = 0, \quad y(1) = \delta y(\eta). \tag{7.9}$$

The differential equation (7.8) can be written in the following form

$$xy''(x) + y'(x) + \lambda xy(x) = 0. \tag{7.10}$$

Using Lommel's transformation (§cf [38, 51])

$$z = x\sqrt{\lambda}, \qquad w = y(x), \tag{7.11}$$

the Bessel's equation

$$z^2\frac{d^2w}{dz^2} + z\frac{dw}{dz} + z^2\, w = 0, \tag{7.12}$$

is transformed into (7.10). Now, if $w_1(z)$ and $w_2(z)$ are two linearly independent solutions of Bessel's equation (7.12), then the two linearly independent solutions of (7.10) are given by

$$y_1(x) = w_1\left(x\sqrt{\lambda}\right), \quad y_2(x) = w_2\left(x\sqrt{\lambda}\right). \tag{7.13}$$

Hence the two linearly independent solutions of (7.10) can be obtained in terms of $w_1(z)$ and $w_2(z)$.

A solution of (7.12) which is bounded in the neighborhood of the origin is $w_1 = J_0(z)$ (for $\lambda > 0$) and $w_1 = I_0(z)$ (for $\lambda < 0$). Hence a solution of (7.10) which remains bounded in the neighborhood of the origin (except for a multiplicative constant) denoted as $y_1(x,\lambda)$ is given by

$$y_1(x,\lambda) = \begin{cases} J_0\left(x\sqrt{\lambda}\right), & \text{if } \lambda > 0; \\ I_0\left(x\sqrt{|\lambda|}\right), & \text{if } \lambda < 0. \end{cases} \tag{7.14}$$

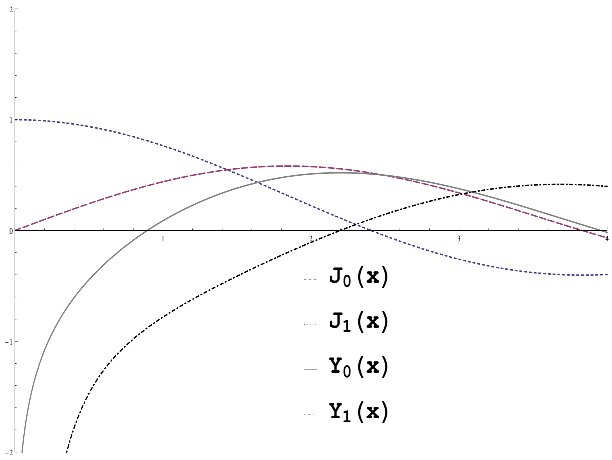

Fig. 7.1 Plot of Bessel functions $J_0(x),\ J_1(x),\ Y_0(x),\ Y_1(x)$.

Remark 7.1. *Let $y_{0,1},\ y_{1,1},\ j_{0,1}$ and $j_{1,1}$ be the first positive zeros of $Y_0(x)$, $Y_1(x)$, $J_0(x)$ and $J_1(x)$ respectively then $y_{0,1} < y_{1,1} < j_{0,1} < j_{1,1}$.*

7.3 Green's function for linear three point SBVPs

In this section we construct Green's function. We divide it into two cases.

7.3.1 Case I: $\lambda > 0$.

Let us assume

$$(H_0):\ 0<\lambda<\ y_{0,1}^2,\quad \delta\, Y_0\left(\eta\sqrt{\lambda}\right) - Y_0\left(\sqrt{\lambda}\right) \le 0,\quad J_0\left(\sqrt{\lambda}\right) - \delta J_0\left(\eta\sqrt{\lambda}\right) < 0.$$

There exists a range of $\lambda > 0$ such that (H_0) is true (See figure 7.2).

Lemma 7.1. *For $0<\lambda<\ y_{0,1}^2$,*

$$J_0\left(x\sqrt{\lambda}\right) Y_0\left(t\sqrt{\lambda}\right) - Y_0\left(x\sqrt{\lambda}\right) J_0\left(t\sqrt{\lambda}\right) \le 0,\ \ 0 \le t,\ x \le 1$$

such that $t \le x$ and x is fixed.

Proof. Let $x = x_0 \in [0,1]$ be fixed. Let

$$\widetilde{F}(x_0,t) = J_0\left(x_0\sqrt{\lambda}\right)Y_0\left(t\sqrt{\lambda}\right) - Y_0\left(x_0\sqrt{\lambda}\right)J_0\left(t\sqrt{\lambda}\right).$$

Using the properties of $J_0\left(t\sqrt{\lambda}\right), Y_0\left(t\sqrt{\lambda}\right)$ for $t \le x_0$ and $0 < \lambda < y_{0,1}^2$, we get that $\widetilde{F}(x_0,t)$ is an increasing function of t. At $t = x_0$, $\widetilde{F}(x_0,x_0) = 0$. Thus $\widetilde{F}(x_0,t) \le 0, \quad \forall\ t \le x_0$. But x_0 can take any value in $[0,1]$ therefore $\widetilde{F}(x,t) \le 0 \quad \forall\ t \le x$. ■

Lemma 7.2. *The Green's function for the following linear three point SBVP*

$$(xy'(x))' + \lambda xy(x) = 0, \quad 0 < x < 1, \tag{7.15}$$
$$y'(0) = 0, \quad y(1) = \delta y(\eta), \tag{7.16}$$

is given by

$$G(x,t) = \frac{1}{D_\lambda}\begin{cases} \pi J_0\left(x\sqrt{\lambda}\right)\left(J_0\left(t\sqrt{\lambda}\right)\left(\delta Y_0\left(\eta\sqrt{\lambda}\right) - Y_0\left(\sqrt{\lambda}\right)\right)\right. \\ \left.+Y_0\left(t\sqrt{\lambda}\right)\left(J_0\left(\sqrt{\lambda}\right) - \delta J_0\left(\eta\sqrt{\lambda}\right)\right)\right), & 0 \le x \le t \le \eta; \\ \pi J_0\left(t\sqrt{\lambda}\right)\left(J_0\left(x\sqrt{\lambda}\right)\left(\delta Y_0\left(\eta\sqrt{\lambda}\right) - Y_0\left(\sqrt{\lambda}\right)\right)\right. \\ \left.+Y_0\left(x\sqrt{\lambda}\right)\left(J_0\left(\sqrt{\lambda}\right) - \delta J_0\left(\eta\sqrt{\lambda}\right)\right)\right), & t \le x,\ t \le \eta; \\ \pi J_0\left(x\sqrt{\lambda}\right)\left(J_0\left(\sqrt{\lambda}\right)Y_0\left(t\sqrt{\lambda}\right) - Y_0\left(\sqrt{\lambda}\right)J_0\left(t\sqrt{\lambda}\right)\right), & x \le t,\ \eta \le t; \\ \pi\left(J_0\left(x\sqrt{\lambda}\right)\left(\delta J_0\left(\eta\sqrt{\lambda}\right)Y_0\left(t\sqrt{\lambda}\right) - Y_0\left(\sqrt{\lambda}\right)J_0\left(t\sqrt{\lambda}\right)\right)\right. \\ \left.+\left(J_0\left(\sqrt{\lambda}\right) - \delta J_0\left(\eta\sqrt{\lambda}\right)\right)\left(J_0\left(t\sqrt{\lambda}\right)Y_0\left(x\sqrt{\lambda}\right)\right)\right), & \eta \le t \le x \le 1, \end{cases}$$

where $D_\lambda = 2\left(J_0\left(\sqrt{\lambda}\right) - \delta J_0\left(\eta\sqrt{\lambda}\right)\right)$ *and if* (H_0) *holds then* $G(x,t) \ge 0$.

Proof. We define Green's function as given below

$$G(x,t) = \begin{cases} a_1\, J_0\left(x\sqrt{\lambda}\right) + a_2\, Y_0\left(x\sqrt{\lambda}\right), & 0 \le x \le t \le \eta; \\ a_3\, J_0\left(x\sqrt{\lambda}\right) + a_4\, Y_0\left(x\sqrt{\lambda}\right), & t \le x,\ t \le \eta; \\ a_5\, J_0\left(x\sqrt{\lambda}\right) + a_6\, Y_0\left(x\sqrt{\lambda}\right), & x \le t,\ \eta \le t; \\ a_7\, J_0\left(x\sqrt{\lambda}\right) + a_8\, Y_0\left(x\sqrt{\lambda}\right), & \eta \le t \le x \le 1. \end{cases}$$

Using the properties of the Green's function, for any $t \in [0,\eta]$, we have

$$a_1 J_0\left(t\sqrt{\lambda}\right) + a_2 Y_0\left(t\sqrt{\lambda}\right) = a_3 J_0\left(t\sqrt{\lambda}\right) + a_4 Y_0\left(t\sqrt{\lambda}\right),$$
$$\left(-a_1\sqrt{\lambda}\, J_1\left(t\sqrt{\lambda}\right) - a_2\sqrt{\lambda}\, Y_1\left(t\sqrt{\lambda}\right)\right) - \left(-a_3\sqrt{\lambda}\, J_1\left(t\sqrt{\lambda}\right) - a_4\sqrt{\lambda}\, Y_1\left(t\sqrt{\lambda}\right)\right) = -\frac{1}{t},$$

which gives

$$a_1 - a_3 = \frac{1}{2}\pi Y_0\left(t\sqrt{\lambda}\right),$$
$$a_2 - a_4 = -\frac{1}{2}\pi J_0\left(t\sqrt{\lambda}\right).$$

Using the boundary conditions, we have

$$a_2 = 0,$$
$$a_3\, J_0\left(\sqrt{\lambda}\right) + a_4\, Y_0\left(\sqrt{\lambda}\right) = \delta\left(a_3\, J_0\left(\eta\sqrt{\lambda}\right) + a_4\, Y_0\left(\eta\sqrt{\lambda}\right)\right).$$

Using above four equation, we get

$$a_1 = \frac{\pi\left(J_0\left(t\sqrt{\lambda}\right)\left(\delta Y_0\left(\eta\sqrt{\lambda}\right) - Y_0\left(\sqrt{\lambda}\right)\right) + Y_0\left(t\sqrt{\lambda}\right)\left(J_0\left(\sqrt{\lambda}\right) - \delta J_0\left(\eta\sqrt{\lambda}\right)\right)\right)}{2\left(J_0\left(\sqrt{\lambda}\right) - \delta J_0\left(\eta\sqrt{\lambda}\right)\right)},$$
$$a_2 = 0,$$
$$a_3 = \frac{\pi J_0\left(t\sqrt{\lambda}\right)\left(\delta Y_0\left(\eta\sqrt{\lambda}\right) - Y_0\left(\sqrt{\lambda}\right)\right)}{2\left(J_0\left(\sqrt{\lambda}\right) - \delta J_0\left(\eta\sqrt{\lambda}\right)\right)},$$
$$a_4 = \frac{1}{2}\pi J_0\left(t\sqrt{\lambda}\right).$$

Similarly for any $t \in [\eta, 1]$, we have

$$a_5 J_0\left(t\sqrt{\lambda}\right) + a_6 Y_0\left(t\sqrt{\lambda}\right) = a_7 J_0\left(t\sqrt{\lambda}\right) + a_8 Y_0\left(t\sqrt{\lambda}\right),$$
$$\left(-a_5\sqrt{\lambda}\, J_1\left(t\sqrt{\lambda}\right) - a_6\sqrt{\lambda}\, Y_1\left(t\sqrt{\lambda}\right)\right) - \left(-a_7\sqrt{\lambda}\, J_1\left(t\sqrt{\lambda}\right) - a_8\sqrt{\lambda}\, Y_1\left(t\sqrt{\lambda}\right)\right) = -\frac{1}{t},$$

which gives

$$a_5 - a_7 = \frac{1}{2}\pi Y_0\left(t\sqrt{\lambda}\right),$$
$$a_6 - a_8 = -\frac{1}{2}\pi J_0\left(t\sqrt{\lambda}\right).$$

By using the boundary conditions in $[\eta, 1]$ we have

$$a_6 = 0,$$
$$a_7\, J_0\left(\sqrt{\lambda}\right) + a_8\, Y_0\left(\sqrt{\lambda}\right) = \delta\left(a_5\, J_0\left(\eta\sqrt{\lambda}\right) + a_6\, Y_0\left(\eta\sqrt{\lambda}\right)\right).$$

By above four equations, we get

$$a_5 = \frac{\pi\left(J_0\left(\sqrt{\lambda}\right)Y_0\left(t\sqrt{\lambda}\right) - Y_0\left(\sqrt{\lambda}\right)J_0\left(t\sqrt{\lambda}\right)\right)}{2\left(J_0\left(\sqrt{\lambda}\right) - \delta J_0\left(\eta\sqrt{\lambda}\right)\right)},$$
$$a_6 = 0,$$
$$a_7 = \frac{\pi\left(\delta J_0\left(\eta\sqrt{\lambda}\right)Y_0\left(t\sqrt{\lambda}\right) - Y_0\left(\sqrt{\lambda}\right)J_0\left(t\sqrt{\lambda}\right)\right)}{2\left(J_0\left(\sqrt{\lambda}\right) - \delta J_0\left(\eta\sqrt{\lambda}\right)\right)},$$
$$a_8 = \frac{1}{2}\pi J_0\left(t\sqrt{\lambda}\right).$$

This completes the construction of Green's function.

Now by Lemma 7.1, $G(x,t) \geq 0$. ■

Lemma 7.3. *Let $y \in C^2(I)$ be a solution of nonhomogeneous linear three point SBVPs* (7.6)–(7.7) *then*

$$y(x) = \frac{b\, J_0\left(x\sqrt{\lambda}\right)}{J_0\left(\sqrt{\lambda}\right) - \delta J_0\left(\eta\sqrt{\lambda}\right)} - \int_0^1 t\, G(x,t)h(t)dt. \tag{7.17}$$

Proof. Suppose $G(x,t)$ is the Green's function of

$$(xy'(x))' + \lambda xy(x) = 0, \quad 0 < x < 1,$$
$$y'(0) = 0, \quad y(1) = \delta y(\eta),$$

and $\bar{y}(x)$ is solution of

$$(xy'(x))' + \lambda xy(x) = 0, \quad 0 < x < 1,$$
$$y'(0) = 0, \quad y(1) = \delta(\eta) + b,$$

then the boundary value problem (7.6)–(7.7) is equivalent to

$$y(x) = \bar{y}(x) - \int_0^1 t\, G(x,t)h(t)dt.$$

Suppose

$$\bar{y}(x) = c_1\, J_0\left(x\sqrt{\lambda}\right) + c_2\, Y_0\left(x\sqrt{\lambda}\right).$$

Since

$$\bar{y}'(0) = 0, \text{ and } \bar{y}(1) = \delta \bar{y}(\eta) + b,$$

we get

$$c_1 = \frac{b}{J_0\left(\sqrt{\lambda}\right) - \delta J_0\left(\eta\sqrt{\lambda}\right)},$$
$$c_2 = 0.$$

Hence the three point linear SBVP (7.6)–(7.7) is equivalent to

$$y(x) = \frac{b\, J_0\left(x\sqrt{\lambda}\right)}{J_0\left(\sqrt{\lambda}\right) - \delta J_0\left(\eta\sqrt{\lambda}\right)} - \int_0^1 t\, G(x,t)h(t)dt.$$

■

7.3.2 Case II: $\lambda < 0$.

Assume that

$$(H_0') : \ \lambda < 0, \ \delta\, K_0\left(\eta\sqrt{|\lambda|}\right) - \ K_0\left(\sqrt{|\lambda|}\right) \geq 0, \ I_0\left(\sqrt{|\lambda|}\right) - \delta\, I_0\left(\eta\sqrt{|\lambda|}\right) > 0.$$

There exists a range of $\lambda < 0$, such that (H_0') holds (See figure 7.3).

Lemma 7.4. *For sufficiently small $\lambda < 0$*

$$I_0\left(t\sqrt{|\lambda|}\right) K_0\left(x\sqrt{|\lambda|}\right) - I_0\left(x\sqrt{|\lambda|}\right) K_0\left(t\sqrt{|\lambda|}\right) \leq 0, \quad 0 \leq t,\ x \leq 1,$$

such that $t \leq x$ and x is fixed.

Proof. Proof follows by arguments similar to Lemma 7.1. ■

Lemma 7.5. *The Green's function for the following linear three point SBVP*

$$(xy'(x))' + \lambda xy(x) = 0, \quad 0 < x < 1,$$
$$y'(0) = 0, \quad y(1) = \delta y(\eta),$$

for $\lambda < 0$ is given by

$$G(x,t) = \frac{1}{D'_\lambda}$$
$$\times \begin{cases} I_0\left(x\sqrt{|\lambda|}\right)\left(K_0\left(t\sqrt{|\lambda|}\right)\left(\delta I_0\left(\eta\sqrt{|\lambda|}\right) - I_0\left(\sqrt{|\lambda|}\right)\right)\right. \\ \left. +I_0\left(t\sqrt{|\lambda|}\right)\left(K_0\left(\sqrt{|\lambda|}\right) - \delta K_0\left(\eta\sqrt{|\lambda|}\right)\right)\right), & 0 \le x \le t \le \eta; \\ I_0\left(t\sqrt{|\lambda|}\right)\left(I_0\left(x\sqrt{|\lambda|}\right)\left(K_0\left(\sqrt{|\lambda|}\right) - \delta K_0\left(\eta\sqrt{|\lambda|}\right)\right)\right. \\ \left. -K_0\left(x\sqrt{|\lambda|}\right)\left(I_0\left(\sqrt{|\lambda|}\right) - \delta I_0\left(\eta\sqrt{|\lambda|}\right)\right)\right), & t \le x, t \le \eta; \\ I_0\left(x\sqrt{|\lambda|}\right)\left(K_0\left(\sqrt{|\lambda|}\right) I_0\left(t\sqrt{|\lambda|}\right) - I_0\left(\sqrt{|\lambda|}\right) K_0\left(t\sqrt{|\lambda|}\right)\right), & x \le t, \eta \le t; \\ I_0\left(x\sqrt{|\lambda|}\right)\left(K_0\left(\sqrt{|\lambda|}\right) I_0\left(t\sqrt{|\lambda|}\right) - \delta I_0\left(\eta\sqrt{|\lambda|}\right) K_0\left(t\sqrt{|\lambda|}\right)\right) \\ -\left(I_0\left(\sqrt{|\lambda|}\right) - \delta I_0\left(\eta\sqrt{|\lambda|}\right)\right)\left(I_0\left(t\sqrt{|\lambda|}\right) K_0\left(x\sqrt{|\lambda|}\right)\right), & \eta \le t \le x \le 1. \end{cases}$$

where $D'_\lambda = I_0\left(\sqrt{|\lambda|}\right) - \delta I_0\left(\eta\sqrt{|\lambda|}\right)$ *and if* (H'_0) *holds then* $G(x,t) \le 0$.

Proof. The construction of Green's function is same as given in Lemma 7.2. Using Lemma 7.4 and (H'_0) we get that $G(x,t) \le 0$. ■

Lemma 7.6. *Let* $y \in C^2(I)$ *be a solution of nonhomogeneous linear three point SBVP* (7.6)–(7.7) *then*

$$y(x) = \frac{b\, I_0\left(x\sqrt{|\lambda|}\right)}{I_0\left(\sqrt{|\lambda|}\right) - \delta I_0\left(\eta\sqrt{|\lambda|}\right)} - \int_0^1 t\, G(x,t) h(t) dt. \tag{7.18}$$

Proof. Proof is same as given in Lemma 7.3. ■

7.4 Anti-maximum and maximum principles

Using the positivity and negativity of Green's function, we established anti-maximum and maximum principle. Which help us to prove the monotonicity of the sequences.

Proposition 7.1. (***Anti-maximum principle***)
Let $b \ge 0$, $h(x) \in C[0,1]$ *is such that* $h(x) \ge 0$, *and* (H_0) *holds, then the solution of* (7.6) *and* (7.7) *is non-positive.*

Proposition 7.2. (***Maximum principle***)
Let $b \ge 0$, $h(x) \in C[0,1]$ *is such that* $h(x) \ge 0$, *and* (H'_0) *holds, then the solution of* (7.6) *and* (7.7) *is non-negative.*

7.5 The Nonlinear SBVP

Based on maximum and anti-maximum principle, we establish the existence of solution of nonlinear three point SBVP and divide. We divide it into the two cases.

7.5.1 Reverse ordered lower and upper solutions

Theorem 7.1. *Let there exist α_0, β_0 in $C^2[0,1]$, such that $\beta_0 \leq \alpha_0$ and satisfy*

$$-(x\beta_0'(x))' \geq xf(x,\beta_0), \quad 0<x<1; \quad \beta_0'(0)=0, \quad \beta_0(1) \geq \delta\beta_0(\eta), \tag{7.19}$$

and

$$-(x\alpha_0'(x))' \leq xf(x,\alpha_0), \quad 0<x<1; \quad \alpha_0'(0)=0, \quad \alpha_0(1) \leq \delta\alpha_0(\eta). \tag{7.20}$$

If $f : D \to \mathbb{R}$ is continuous on $D := \{(x,y) \in [0,1]\times\mathbb{R} : \beta_0 \leq y \leq \alpha_0\}$ and there exists $M > 0$ such that for all $(x,y),(x,w) \in D$

$$y \leq w \Longrightarrow f(x,w) - f(x,y) \leq M(w-y), \tag{7.21}$$

then the nonlinear three point SBVP (7.3)–(7.4) *has at least one solution in the region D. If $\exists$ a constant λ such that $M-\lambda \leq 0$ and (H_0) is satisfied then the sequence (β_n) generated by*

$$-(xy_{n+1}'(x))' - \lambda xy_{n+1} = xF(x,y_n), \quad y_{n+1}'(0)=0, \quad y_{n+1}(1) = \delta y_{n+1}(\eta), \tag{7.22}$$

where $F(x,y_n) = f(x,y_n) - \lambda y_n$, with initial iterate β_0 converges monotonically (non-decreasing) and uniformly towards a solution $\widetilde{\beta}(x)$ of (7.3)–(7.4)*. Similarly α_0 as an initial iterate leads to a non-increasing sequence (α_n) converging to a solution $\widetilde{\alpha}(x)$. Any solution $z(x)$ in D must satisfy*

$$\widetilde{\beta}(x) \leq z(x) \leq \widetilde{\alpha}(x).$$

Proof. From equation (7.19) and equation (7.22) (for $n=0$)

$$\begin{aligned}
&-(x(\beta_0-\beta_1)'(x))' - \lambda x(\beta_0-\beta_1) \geq 0,\\
&(\beta_0-\beta_1)'(0)=0, \quad (\beta_0-\beta_1)(1) \geq \delta(\beta_0-\beta_1)(\eta).
\end{aligned}$$

Since $h(x) \geq 0$ and $b \geq 0$, by using Proposition 7.1 we have $\beta_0 \leq \beta_1$.

In view of $M-\lambda \leq 0$, from equations (7.22) and (7.21), we get

$$-\,(x\beta_{n+1}'(x))' \geq x[-(M-\lambda)(\beta_{n+1}-\beta_n)+f(x,\beta_{n+1})]$$

and if $(\beta_n \leq \beta_{n+1})$, then

$$-(x\beta_{n+1}'(x))' \geq xf(t,\beta_{n+1}); \quad \beta_{n+1}'(0)=0, \quad \beta_{n+1}(1)=\delta\beta_{n+1}(\eta). \tag{7.23}$$

Since $\beta_0 \leq \beta_1$, then from equation (7.23) (for $n=0$) and (7.22) (for $n=1$) we get

$$\begin{aligned}&-\,(x(\beta_1-\beta_2)'(x))'-\lambda x(\beta_1-\beta_2) \geq 0,\\ &(\beta_1-\beta_2)'(0)=0, \quad (\beta_1-\beta_2)(1) \geq \delta(\beta_1-\beta_2)(\eta),\end{aligned}$$

From Proposition 7.1 we have $\beta_1 \leq \beta_2$.

Now from equations (7.20) and (7.22) (for $n=0$)

$$\begin{aligned}&-\,(x(\beta_1-\alpha_0)'(x))'-\lambda x(\beta_1-\alpha_0) \geq 0,\\ &(\beta_1-\alpha_0)'(0)=0 \quad (\beta_1-\alpha_0)(1) \geq \delta((\beta_1-\alpha_0)(\eta).\end{aligned}$$

Thus $\beta_1 \leq \alpha_0$ follows from proposition 7.1.

Now assuming $\beta_n \leq \beta_{n+1}$, $\beta_{n+1} \leq \alpha_0$, we show that $\beta_{n+1} \leq \beta_{n+2}$ and $\beta_{n+2} \leq \alpha_0$ for all n. From equations (7.22) (for $n+1$) and (7.23) we get

$$\begin{aligned}&-\,(x(\beta_{n+1}-\beta_{n+2})'(x))'-\lambda x(\beta_{n+1}-\beta_{n+2}) \geq 0,\\ &(\beta_{n+1}-\beta_{n+2})'(0)=0, \quad (\beta_{n+1}-\beta_{n+2})(1) \geq \delta(\beta_{n+1}-\beta_{n+2})(\eta),\end{aligned}$$

and hence from Proposition 7.1 we have $\beta_{n+1} \leq \beta_{n+2}$.

From equation (7.22) (for $n+1$) and (7.20) we get,

$$\begin{aligned}&-\,(x(\beta_{n+2}-\alpha_0)'(x))'-x\lambda(\beta_{n+2}-\alpha_0) \geq 0,\\ &(\beta_{n+2}-\alpha_0)'(0)=0, \quad (\beta_{n+2}-\alpha_0)(1) \geq \delta(\beta_{n+2}-\alpha_0)(\eta).\end{aligned}$$

Then from proposition 7.1, $\beta_{n+2} \leq \alpha_0$ and hence we have

$$\beta_1 \leq \beta_2 \leq \cdots \leq \beta_n \leq \beta_{n+1} \leq \cdots \leq \alpha_0$$

and starting with α_0 it is easy to get

$$\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n \geq \alpha_{n+1} \geq \cdots \geq \beta_0.$$

Finally we show that $\beta_n \leq \alpha_n$ for all n. For this by assuming $\beta_n \leq \alpha_n$, we show that $\beta_{n+1} \leq \alpha_{n+1}$. From equation (7.22) it is easy to get

$$\begin{aligned}
&-(x(\beta_{n+1}-\alpha_{n+1})'(x))' - \lambda x(\beta_{n+1}-\alpha_{n+1}) \geq 0,\\
&(\beta_{n+1}-\alpha_{n+1})'(0) = 0, \quad (\beta_{n+1}-\alpha_{n+1})(1) \geq \delta(\beta_{n+1}-\alpha_{n+1})(\eta).
\end{aligned}$$

Hence from Proposition 7.1, $\beta_{n+1} \leq \alpha_{n+1}$. Thus we have

$$\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n \geq \alpha_{n+1} \geq \cdots \geq \beta_{n+1} \geq \beta_n \geq \cdots \geq \beta_2 \geq \beta_1 \geq \beta_0.$$

So the sequences (β_n) and (α_n) are monotonically non-decreasing and non-increasing, respectively and are bounded by β_0 and α_0. Hence by Dini's theorem they converges uniformly. Let $\widetilde{\beta}(x) = \lim_{n\to\infty} \beta_n(x)$ and $\widetilde{\alpha}(x) = \lim_{n\to\infty} \alpha_n(x)$.

Using Lemma 7.3, the solution β_{n+1} of equation (7.22) is given by

$$\beta_{n+1} = \frac{b\, J_0\left(x\sqrt{\lambda}\right)}{J_0\left(\sqrt{\lambda}\right) - \delta J_0\left(\eta\sqrt{\lambda}\right)} - \int_0^1 t\, G(x,t)(f(t,\beta_n) - \lambda\beta_n)dt.$$

Then by Lebesgue's dominated convergence theorem, taking the limit as $n \to \infty$, we get

$$\widetilde{\beta}(x) = \frac{b\, J_0\left(x\sqrt{\lambda}\right)}{J_0\left(\sqrt{\lambda}\right) - \delta J_0\left(\eta\sqrt{\lambda}\right)} - \int_0^1 t\, G(x,t)(f(t,\widetilde{\beta}) - \lambda\widetilde{\beta})dt,$$

which is the solution of boundary value problem (7.3)–(7.4). Similar equation can be defined for the sequence of lower solutions also.

Any solution $z(x)$ in D can play the role of $\beta_0(x)$, hence $z(x) \leq \widetilde{\alpha}(x)$ and similarly one concludes that $z(x) \geq \widetilde{\beta}(x)$. ■

7.5.2 Well-ordered lower and upper solutions

Theorem 7.2. *Let there exist α_0, β_0 in $C^2[0,1]$ such that $\beta_0 \geq \alpha_0$ and satisfy*

$$-(x\beta_0'(x))' \geq xf(x,\beta_0), \quad 0<x<1; \quad \beta_0'(0) = 0, \quad \beta_0(1) \geq \delta\beta_0(\eta), \tag{7.24}$$

and

$$-(x\alpha_0'(x))' \le xf(x,\alpha_0), \quad 0<x<1; \quad \alpha_0'(0)=0, \quad \alpha_0(1) \le \delta\alpha_0(\eta). \tag{7.25}$$

If $f:\widetilde{D}_0 \to R$ *is continuous on* $\widetilde{D}_0 := \{(x,y) \in [0,1]\times\mathbb{R} : \alpha_0 \le y \le \beta_0\}$ *and there exists* $M>0$ *such that for all* $(x,\widetilde{y}),(x,\widetilde{w}) \in \widetilde{D}_0$

$$\widetilde{y} \le \widetilde{w} \Longrightarrow f(x,\widetilde{w}) - f(x,\widetilde{y}) \ge -M(\widetilde{w}-\widetilde{y}) \tag{7.26}$$

then the nonlinear three point SBVP (7.3)–(7.4) *has at least one solution in the region* $\widetilde{D}$. *If* $\exists$ *a constant* λ *such that* $\lambda<0$, $M+\lambda \le 0$ *and* (H_0') *is satisfied then the sequence* (β_n) *generated by*

$$-(xy_{n+1}'(x))' - \lambda x y_{n+1} = xF(x,y_n), \quad y_{n+1}'(0)=0, \quad y_{n+1}(1)=\delta y_{n+1}(\eta), \tag{7.27}$$

where $F(x,y_n)=f(x,y_n)-\lambda y_n$, *with initial iterate* β_0 *converges monotonically (non-increasing) and uniformly towards a solution* $\beta(x)$ *of* (7.3)–(7.4). *Similarly* α_0 *as an initial iterate leads to a non-decreasing sequence* (α_n) *converging to a solution* $\alpha(x)$. *Any solution* $Z(x)$ *in* $\widetilde{D}$ *must satisfy*

$$\alpha(x) \le Z(x) \le \beta(x).$$

Proof. Proof follows from the analysis of Theorem 7.1. ■

7.6 Examples

In this section we apply our results to the following examples and show that there exists at least one value of $\lambda \in \mathbb{R}\setminus\{0\}$ such that iterative scheme generates monotone sequences which converge to solutions of nonlinear problem.

Thus these examples validate sufficient conditions derived in Theorem 7.1 and Theorem 7.2.

Example 7.1. *Consider the nonlinear three point SBVP*

$$-y''(x) - \frac{1}{x}y'(x) = \frac{e^{y(x)}-1}{64}, \tag{7.28}$$

$$y'(0)=0, \quad y(1)=3y\left(\frac{1}{2}\right). \tag{7.29}$$

Here $f(x,y) = \frac{e^{y(x)}-1}{64}$, $\delta = 3$, $\eta = \frac{1}{2}$. This problem has $\alpha_0 = 1$ and $\beta_0 = -1$ as lower and upper solutions, i.e., it is non well ordered case. The nonlinear term is Lipschitz in y, continuous for all value of y, and Lipschitz constant M is $\frac{e}{64}$. Now we can find out a subinterval $R_\lambda = (\xi_1, \xi_2)$ of $(\frac{e}{64}, y_{0,1}^2)$ such that the conditions $M - \lambda \leq 0$ and (H_0) are true (See Figure 7.2). Hence there exists a solution of the problem (7.28)–(7.29) in the region $D := \{(x,y) \in [0,1] \times \mathbb{R} : \beta_0 \leq y \leq \alpha_0\}$.

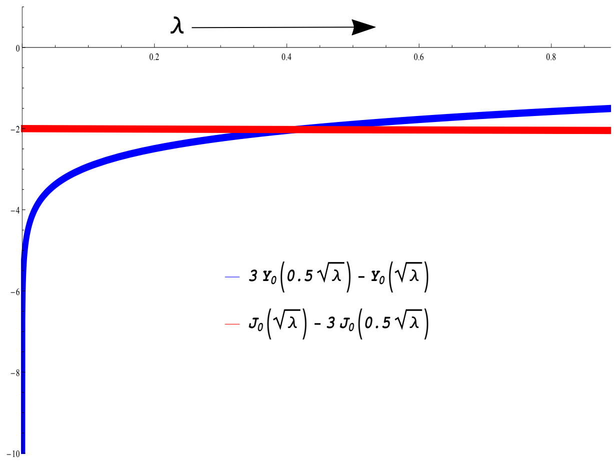

Fig. 7.2 Plot of (H_0) for example 7.1.

Example 7.2. *Consider the nonlinear three point SBVP*

$$-y''(x) - \frac{1}{x}y'(x) = \frac{e^x - 3(y(x))^3}{96}, \tag{7.30}$$

$$y'(0) = 0, \quad y(1) = \frac{1}{2}y\left(\frac{1}{3}\right). \tag{7.31}$$

Here $f(x,y) = \frac{e^x-3(y(x))^3}{96}$, $\delta = \frac{1}{2}, \eta = \frac{1}{3}$. This problem has $\alpha_0 = -1$ and $\beta_0 = 1$ as lower and upper solutions, respectively. This is a well ordered case. The nonlinear term is Lipschitz in y, continuous for all value of y and Lipschitz constant M is $\frac{3}{32}$. For some λ less than $\left(-\frac{3}{32}\right)$, (H_0') will be true (See Figure 7.3). Hence there exists a solution of the problem (7.30)–(7.31) in the region $\widetilde{D}_0 := \{(x,y) \in [0,1] \times \mathbb{R} : \alpha_0 \leq y \leq \beta_0\}$.

7.7 Conclusion

In this chapter we deal with existence of solution for nonlinear singular differential equation $-y''(x) - \frac{1}{x}y'(x) = f(x,y)$ on $0 < x < 1$ subject to $y'(0) = 0$, $y(1) = \delta y(\eta)$. We have

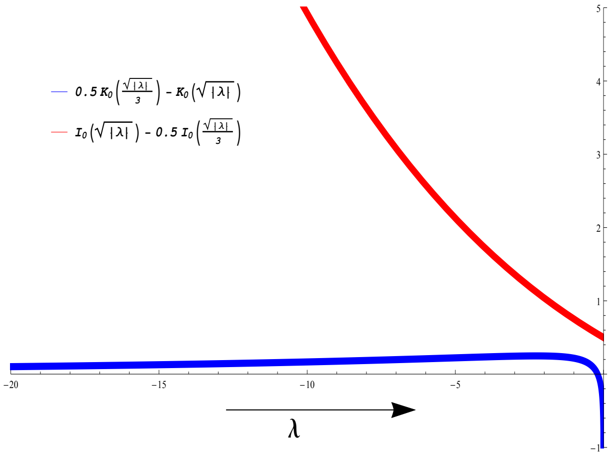

Fig. 7.3 Plot of (H_0') for example 7.2

computed a range of $\lambda \in R \setminus \{0\}$ such that the iterative scheme gives uniformly convergent sequence. The limit of this sequence is nothing but a solution of the nonlinear problem. Region of multiple solutions is also determined.

Chapter 8

Nonlinear three point SBVPs : A classification

8.1 Introduction

In this chapter we generalize the result of chapter 6 & 7 and classify well order and reverse order cases for different values of α. For this, we consider the following nonlinear three point SBVP

$$-(x^\alpha y'(x))' = x^\alpha f(x,y), \quad 0 < x < 1, \tag{8.1}$$

$$y'(0) = 0, \quad y(1) = \delta y(\eta), \tag{8.2}$$

where $f : I \times \mathbb{R} \to \mathbb{R}$, $I = [0,1]$, $0 < \eta < 1$, $\delta > 0$ and $\alpha \geq 1$.

Given that $f(x,y)$ is continuous and Lipschitz continuous in its domain, we propose the following monotone iterative scheme for the nonlinear three point SBVP (8.1)–(8.2),

$$\left.\begin{aligned} & -y''_{n+1}(x) - \frac{\alpha}{x} y'_{n+1}(x) - \lambda y_{n+1}(x) = f(x,y_n) - \lambda y_n(x), \quad \lambda \in \mathbb{R} \setminus \{0\}, \\ & y'_{n+1}(0) = 0, \quad y_{n+1}(1) = \delta y_{n+1}(\eta), \end{aligned}\right\} \tag{8.3}$$

and prove that solution exists and belongs to the class $C[0,1] \cap C^2[0,1]$.

Definition 8.1. *If the functions $u_0, v_0 \in C^2[0,1]$ are defined as*

$$-(x^\alpha u'_0(x))' \geq x^\alpha f(x,u_0), \quad 0 < x < 1; \tag{8.4}$$

$$u'_0(0) = 0, \quad u_0(1) \geq \delta u_0(\eta), \tag{8.5}$$

and

$$-(x^{\alpha}v_0'(x))' \leq x^{\alpha}f(x,v_0), \quad 0<x<1; \tag{8.6}$$
$$v_0'(0)=0, \quad v_0(1) \leq \delta v_0(\eta), \tag{8.7}$$

then u_0 and v_0 are called upper and lower solutions of the nonlinear three point SBVP (8.1)–(8.2), respectively. If $v_0 \leq u_0$, we say it is well order case and if $v_0 \geq u_0$, it is known as reverse order case.

We observe that depending on the values of α, we can classify well ordered and reversed order cases. The classification, we deduce does not exist in the literature to the best of our knowledge. These results also generalize some existing result [92, 142, 143, 154]. Also the purpose of this chapter is to prove existence of unique solution for a class of nonlinear three point SBVP (8.1)–(8.2).

This chapter is organized in several sections. In Section 8.2, we use Lommel's transformation to find out two linearly independent solutions in the terms of Bessel functions. Using these two linearly independent solutions Green's functions are constructed for different classes of α (See figure 8.1) in Section 8.3 and Section 8.4 states maximum and anti-maximum principles. Finally in Section 8.5 all these results are used to establish some existence and uniqueness theorems. The sufficient conditions derived in this chapter are verified for certain values of α which belongs to different classes of α in Section 8.6. Finally, the conclusions based on the observation are given in Section 8.7

8.2 The linear three point SBVP

The linear SBVP corresponding to the nonlinear three point SBVP (8.1)–(8.2) is studied in this section. We consider the following inhomogeneous class of three point linear SBVP,

$$-(x^{\alpha}y'(x))' - \lambda x^{\alpha}y(x) = x^{\alpha}h(x), \quad 0<x<1, \tag{8.8}$$
$$y'(0)=0, \quad y(1)=\delta y(\eta)+b, \tag{8.9}$$

where $h \in C(I)$ and b is any constant. To solve the inhomogeneous system (8.8)–(8.9), we consider the corresponding homogeneous system

$$-(x^{\alpha}y'(x))' - \lambda x^{\alpha}y(x) = 0, \quad 0<x<1, \tag{8.10}$$
$$y'(0)=0, \quad y(1)=\delta y(\eta). \tag{8.11}$$

Using Lommel's transformation (§cf [38, 51]) $z = \beta\zeta^\gamma$, $w = \zeta^{-a}v(\zeta)$, the standard Bessel's equation (8.12) is transformed into (8.13)

$$z^2\frac{d^2w}{dz^2} + z\frac{dw}{dz} + (z^2 - \nu^2)w = 0, \tag{8.12}$$

$$\zeta^2\frac{d^2v}{d\zeta^2} + \zeta(1-2a)\frac{dv}{d\zeta} + \left[(\beta\gamma\zeta^\gamma)^2 + (a^2 - \nu^2\gamma^2)\right]v = 0. \tag{8.13}$$

Now, by Lommel's Transformation the two linearly independent solutions of (8.13) are given by

$$v_1(x) = \zeta^a w_1(\beta\zeta^\gamma), \quad v_2(x) = \zeta^a w_2(\beta\zeta^\gamma), \tag{8.14}$$

where $w_1(z)$ and $w_2(z)$ are two linearly independent solutions of Bessel's equation (8.12). Now, if we set $\nu = a = \frac{1-\alpha}{2}$, $\gamma = 1$, $\beta^2 = \lambda$, then (8.13) reduces to (8.10) and hence we obtain the two linearly independent solutions of (8.10) in terms of $w_1(z)$ and $w_2(z)$. A solution of (8.10) which is bounded in the neighborhood of the origin (except for a multiplicative constant) given by $x^\nu J_{-\nu}\left(x\sqrt{\lambda}\right)$, if $\lambda > 0$ and $x^\nu I_{-\nu}\left(x\sqrt{|\lambda|}\right)$, if $\lambda < 0$.

Note 8.1. *Here $J_{-\nu}$, Y_ν are Bessel functions of first and second kind and $I_{-\nu}$ and K_ν are Modified Bessel functions of first and second kind.*

8.3 Green's function

The Green's function is constructed in terms of Bessel functions and sign of Bessel functions and hence sign of Green's function depend on values of λ and α. Hence we divide this section into the following subsections.

8.3.1 Case I: When $\lambda > 0$ and $\alpha \notin \{1,3,5,\cdots\}$.

Suppose that

$$\begin{aligned}(H_0):\ & 0 < \lambda < j_{\nu,1}^2, \quad 0 < \delta < 1, \quad \delta\eta^\nu J_\nu\left(\eta\sqrt{\lambda}\right) - J_\nu\left(\sqrt{\lambda}\right) \geq 0, \quad \text{and} \\ & J_{-\nu}\left(\sqrt{\lambda}\right) - \delta\eta^\nu J_{-\nu}\left(\eta\sqrt{\lambda}\right) > 0, \quad \text{for} \quad \alpha \in \textstyle\bigcup_{n\in N}(4n-3, 4n-1), \quad \text{and}\end{aligned}$$

$$\begin{aligned}(H_1):\ & 0 < \lambda < j_{\nu,1}^2, \quad \delta \geq 1, \quad \delta\eta^\nu J_\nu\left(\eta\sqrt{\lambda}\right) - J_\nu\left(\sqrt{\lambda}\right) \leq 0 \quad \text{and} \\ & J_{-\nu}\left(\sqrt{\lambda}\right) - \delta\eta^\nu J_{-\nu}\left(\eta\sqrt{\lambda}\right) < 0, \quad \text{for} \quad \alpha \in \textstyle\bigcup_{n\in N}(4n-1, 4n+1),\end{aligned}$$

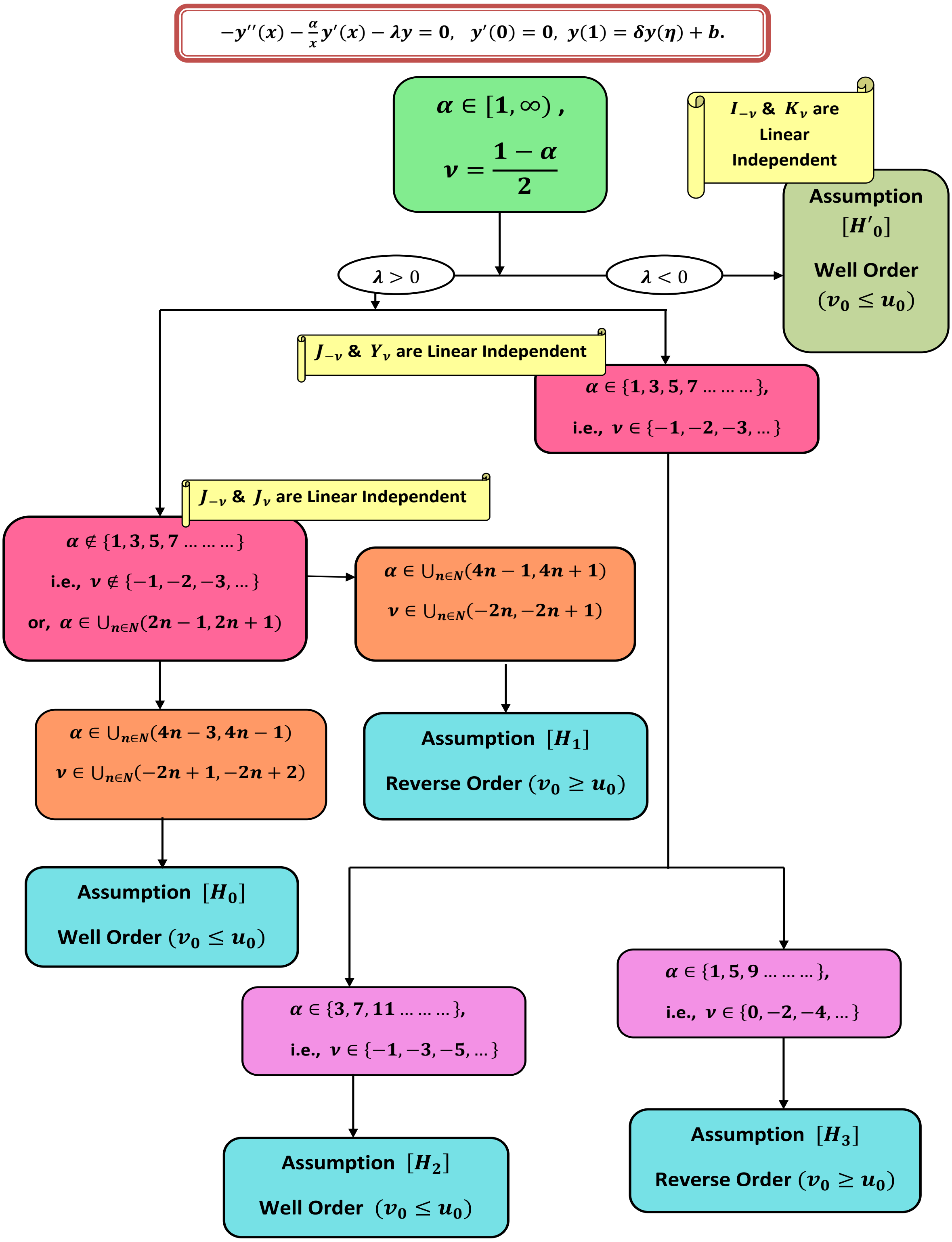

Fig. 8.1 Classification of well order and reverse order cases for different values of α

where $j^2_{\nu,1}$ is the first zero of $J_\nu(x)$.

For a range of $\lambda > 0$, (H_0) and (H_1) are true, where $\alpha \in \bigcup_{n\in N}(4n-3,4n-1)$ and $\alpha \in \bigcup_{n\in N}(4n-1,4n+1)$, respectively.

Next two lemmas help us to fix the sign of Green's function.

Lemma 8.1. *For* $0 < \lambda < j^2_{\nu,1}$, *the Bessel functions of first kind* (J_ν *and* $J_{-\nu}$) *satisfy the following inequality*

$$r^\nu\left(J_{-\nu}\left(s\sqrt{\lambda}\right)J_\nu\left(r\sqrt{\lambda}\right)-J_\nu\left(s\sqrt{\lambda}\right)J_{-\nu}\left(r\sqrt{\lambda}\right)\right)\geq 0,\ \ 0<r\leq s\leq 1,$$

where $\nu = \frac{1-\alpha}{2}$ *and* $\alpha \in \bigcup_{n\in N}(4n-3,4n-1)$.

***Proof*.** Suppose

$$\widetilde{\Phi}(s,r) = r^\nu\left(J_{-\nu}\left(s\sqrt{\lambda}\right)J_\nu\left(r\sqrt{\lambda}\right)-J_\nu\left(s\sqrt{\lambda}\right)J_{-\nu}\left(r\sqrt{\lambda}\right)\right),$$

and let $s = s_0 \in [0,1]$ be fixed. As

$$\begin{aligned}J_{-\nu}\left(s_0\sqrt{\lambda}\right)J_{-1+\nu}\left(r\sqrt{\lambda}\right)+J_\nu\left(s_0\sqrt{\lambda}\right)J_{1-\nu}\left(r\sqrt{\lambda}\right)&\\ \leq J_{-\nu}\left(s_0\sqrt{\lambda}\right)J_{-1+\nu}\left(s_0\sqrt{\lambda}\right)+J_\nu\left(s_0\sqrt{\lambda}\right)J_{1-\nu}\left(x_0\sqrt{\lambda}\right)&\\ = \frac{2\sin\nu\pi}{\pi s_0\sqrt{\lambda}}\leq 0,&\end{aligned}$$

for $r \leq s_0$. Now making use of the above inequality, we deduce that $\widetilde{\Phi}(s_0,t)$ is a non-increasing function of r. As $\widetilde{\Phi}(s_0,s_0) = 0$ at $r = s_0$, which implies that $\widetilde{\Phi}(s_0,r) \geq 0,\ \ \forall\ r \leq s_0$. But as s_0 takes any value in $[0,1]$ therefore $\widetilde{\Phi}(s,r) \geq 0,\ \ \forall\ \ 0 < r \leq s \leq 1.$ ∎

Lemma 8.2. *For* $0 < \lambda < j^2_{\nu,1}$, *the Bessel functions of first kind* (J_ν *and* $J_{-\nu}$) *satisfy the following inequality*

$$r^\nu\left(J_{-\nu}\left(s\sqrt{\lambda}\right)J_\nu\left(r\sqrt{\lambda}\right)-J_\nu\left(s\sqrt{\lambda}\right)J_{-\nu}\left(r\sqrt{\lambda}\right)\right)\leq 0,\ \ 0<r\leq s\leq 1,$$

where $\nu = \frac{1-\alpha}{2}$ *and* $\alpha \in \bigcup_{n\in N}(4n-1,4n+1)$.

Proof. Proof follows from the analysis similar to Lemma 8.1, and the inequality

$$\begin{aligned}
J_{-\nu}\left(s_0\sqrt{\lambda}\right)J_{-1+\nu}\left(r\sqrt{\lambda}\right)+J_{\nu}\left(s_0\sqrt{\lambda}\right)J_{1-\nu}\left(r\sqrt{\lambda}\right)& \\
\geq\; & J_{-\nu}\left(s_0\sqrt{\lambda}\right)J_{-1+\nu}\left(s_0\sqrt{\lambda}\right)+J_{\nu}\left(s_0\sqrt{\lambda}\right)J_{1-\nu}\left(s_0\sqrt{\lambda}\right)\\
=\; & \frac{2\sin\nu\pi}{\pi s_0\sqrt{\lambda}}\geq 0,
\end{aligned}$$

for $r \leq s_0$. ■

Lemma 8.3. *For the linear three point SBVPs (8.10)–(8.11), where $\alpha \notin \{1,3,5,\cdots\}$, the Green's function is given by*

$$G(x,t)=\begin{cases} G_1(x,t), & 0\leq x\leq t\leq \eta;\\ G_2(x,t), & t\leq x,\; t\leq \eta;\\ G_3(x,t), & x\leq t,\; \eta\leq t;\\ G_4(x,t), & \eta\leq t\leq x\leq 1,\end{cases}$$

where

$$\begin{aligned}
G_1(x,t)=&\frac{\pi\csc(\pi\nu)t^{\nu}x^{\nu}J_{-\nu}\left(x\sqrt{\lambda}\right)}{2\left(J_{-\nu}\left(\sqrt{\lambda}\right)-\delta\eta^{\nu}J_{-\nu}\left(\eta\sqrt{\lambda}\right)\right)}\Big[\delta\eta^{\nu}\left(J_{\nu}\left(\eta\sqrt{\lambda}\right)J_{-\nu}\left(t\sqrt{\lambda}\right)\right.\\
&\left.-J_{-\nu}\left(\eta\sqrt{\lambda}\right)J_{\nu}\left(t\sqrt{\lambda}\right)\right)+\left(J_{-\nu}\left(\sqrt{\lambda}\right)J_{\nu}\left(t\sqrt{\lambda}\right)-J_{\nu}\left(\sqrt{\lambda}\right)J_{-\nu}\left(t\sqrt{\lambda}\right)\right)\Big],\\
G_2(x,t)=&\frac{1}{2}\pi\csc(\pi\nu)t^{\nu}x^{\nu}J_{-\nu}\left(t\sqrt{\lambda}\right)\\
&\left(\frac{J_{-\nu}\left(x\sqrt{\lambda}\right)\left(\delta\eta^{\nu}J_{\nu}\left(\eta\sqrt{\lambda}\right)-J_{\nu}\left(\sqrt{\lambda}\right)\right)}{J_{-\nu}\left(\sqrt{\lambda}\right)-\delta\eta^{\nu}J_{-\nu}\left(\eta\sqrt{\lambda}\right)}+J_{\nu}\left(x\sqrt{\lambda}\right)\right),\\
G_3(x,t)=&\frac{\pi\csc(\pi\nu)t^{\nu}x^{\nu}\left(J_{-\nu}\left(\sqrt{\lambda}\right)J_{\nu}\left(t\sqrt{\lambda}\right)-J_{\nu}\left(\sqrt{\lambda}\right)J_{-\nu}\left(t\sqrt{\lambda}\right)\right)J_{-\nu}\left(x\sqrt{\lambda}\right)}{2\left(J_{-\nu}\left(\sqrt{\lambda}\right)-\delta\eta^{\nu}J_{-\nu}\left(\eta\sqrt{\lambda}\right)\right)},\\
G_4(x,t)=&\frac{1}{2}\pi\csc(\pi\nu)t^{\nu}x^{\nu}\left(\frac{J_{-\nu}\left(x\sqrt{\lambda}\right)\left(\delta\eta^{\nu}J_{-\nu}\left(\eta\sqrt{\lambda}\right)J_{\nu}\left(t\sqrt{\lambda}\right)-J_{\nu}\left(\sqrt{\lambda}\right)J_{-\nu}\left(t\sqrt{\lambda}\right)\right)}{J_{-\nu}\left(\sqrt{\lambda}\right)-\delta\eta^{\nu}J_{-\nu}\left(\eta\sqrt{\lambda}\right)}\right.\\
&\left.+J_{-\nu}\left(t\sqrt{\lambda}\right)J_{\nu}\left(x\sqrt{\lambda}\right)\right).
\end{aligned}$$

If H_0 (or H_1) holds then $G(x,t)\leq 0$ (or $G(x,t)\geq 0$).

Proof. We define the Green's function as follows

$$G(x,t)=\begin{cases} a_1\, x^{\nu}J_{-\nu}\left(x\sqrt{\lambda}\right)+a_2\, x^{\nu}J_{\nu}\left(x\sqrt{\lambda}\right), & 0\le x\le t\le\eta;\\ a_3\, x^{\nu}J_{-\nu}\left(x\sqrt{\lambda}\right)+a_4\, x^{\nu}J_{\nu}\left(x\sqrt{\lambda}\right), & t\le x,\ t\le\eta;\\ a_5\, x^{\nu}J_{-\nu}\left(x\sqrt{\lambda}\right)+a_6\, x^{\nu}J_{\nu}\left(x\sqrt{\lambda}\right), & x\le t,\ \eta\le t;\\ a_7\, x^{\nu}J_{-\nu}\left(x\sqrt{\lambda}\right)+a_8\, x^{\nu}J_{\nu}\left(x\sqrt{\lambda}\right), & \eta\le t\le x\le 1. \end{cases}$$

Using the continuity and jump discontinuity of the Green's function, for any $t\in[0,\eta]$, we get

$$\begin{aligned} &a_1t^{\nu}J_{-\nu}\left(t\sqrt{\lambda}\right)+a_2t^{\nu}J_{\nu}\left(t\sqrt{\lambda}\right)=a_3t^{\nu}J_{-\nu}\left(t\sqrt{\lambda}\right)+a_4t^{\nu}J_{\nu}\left(t\sqrt{\lambda}\right),\\ &-a_1t^{\nu}\sqrt{\lambda}J_{1-\nu}\left(t\sqrt{\lambda}\right)+a_2t^{\nu}\sqrt{\lambda}J_{\nu-1}\left(t\sqrt{\lambda}\right)\\ &\qquad +a_3t^{\nu}\sqrt{\lambda}J_{1-\nu}\left(t\sqrt{\lambda}\right)-a_4t^{\nu}\sqrt{\lambda}J_{\nu-1}\left(t\sqrt{\lambda}\right)=-\frac{1}{t^{1-2\nu}}, \end{aligned}$$

and boundary conditions, we have the following system of equations

$$A_0X_0=B_0,$$

where

$$A_0=\begin{pmatrix} 1 & 0 & -1 & 0\\ 0 & 1 & 0 & -1\\ 0 & 1 & 0 & 0\\ 0 & 0 & J_{-\nu}\left(\sqrt{\lambda}\right)-\delta\eta^{\nu}J_{-\nu}\left(\eta\sqrt{\lambda}\right) & J_{\nu}\left(\sqrt{\lambda}\right)-\delta\eta^{\nu}J_{\nu}\left(\eta\sqrt{\lambda}\right) \end{pmatrix},$$

$$X_0=\begin{pmatrix} a_1\\ a_2\\ a_3\\ a_4 \end{pmatrix},\qquad B_0=\begin{pmatrix} \frac{\pi t^{\nu}J_{\nu}\left(t\sqrt{\lambda}\right)}{2\sin\nu\pi}\\ -\frac{\pi t^{\nu}J_{-\nu}\left(t\sqrt{\lambda}\right)}{2\sin\nu\pi}\\ 0\\ 0 \end{pmatrix}.$$

Solution of above system gives,

$$\begin{aligned} a_1=&\frac{\pi\csc(\pi\nu)t^{\nu}}{2\left(J_{-\nu}\left(\sqrt{\lambda}\right)-\delta\eta^{\nu}J_{-\nu}\left(\eta\sqrt{\lambda}\right)\right)}\Big[J_{-\nu}\left(t\sqrt{\lambda}\right)\left(\delta\eta^{\nu}J_{\nu}\left(\eta\sqrt{\lambda}\right)-J_{\nu}\left(\sqrt{\lambda}\right)\right)\\ &+J_{\nu}\left(t\sqrt{\lambda}\right)\left(J_{-\nu}\left(\sqrt{\lambda}\right)-\delta\eta^{\nu}J_{-\nu}\left(\eta\sqrt{\lambda}\right)\right)\Big], \end{aligned}$$

$$
\begin{aligned}
a_2 =&0,\\
a_3 =&\frac{\pi\csc(\pi\nu)t^{\nu}J_{-\nu}\left(t\sqrt{\lambda}\right)\left(\delta\eta^{\nu}J_{\nu}\left(\eta\sqrt{\lambda}\right)-J_{\nu}\left(\sqrt{\lambda}\right)\right)}{2\left(J_{-\nu}\left(\sqrt{\lambda}\right)-\delta\eta^{\nu}J_{-\nu}\left(\eta\sqrt{\lambda}\right)\right)},\\
a_4 =&\frac{1}{2}\pi\csc(\pi\nu)t^{\nu}J_{-\nu}\left(t\sqrt{\lambda}\right).
\end{aligned}
$$

Similarly for any $t\in[\eta,1]$, we have

$$
a_5t^{\nu}J_{-\nu}\left(t\sqrt{\lambda}\right)+a_6t^{\nu}J_{\nu}\left(t\sqrt{\lambda}\right)=a_7t^{\nu}J_{-\nu}\left(t\sqrt{\lambda}\right)+a_8t^{\nu}J_{\nu}\left(t\sqrt{\lambda}\right),
$$

and

$$
\begin{aligned}
&-a_5t^{\nu}\sqrt{\lambda}J_{1-\nu}\left(t\sqrt{\lambda}\right)+a_6t^{\nu}\sqrt{\lambda}J_{\nu-1}\left(t\sqrt{\lambda}\right)\\
&\qquad+a_7t^{\nu}\sqrt{\lambda}J_{1-\nu}\left(t\sqrt{\lambda}\right)-a_8t^{\nu}\sqrt{\lambda}J_{\nu-1}\left(t\sqrt{\lambda}\right)=-\frac{1}{t^{1-2\nu}}.
\end{aligned}
$$

The above two equations and boundary conditions in $[\eta,1]$ gives

$$
A_1X_1=B_1,
$$

where

$$
A_1=\begin{pmatrix}
1 & 0 & -1 & 0\\
0 & 1 & 0 & -1\\
0 & 1 & 0 & 0\\
\delta\eta^{\nu}J_{-\nu}\left(\eta\sqrt{\lambda}\right) & \delta\eta^{\nu}J_{\nu}\left(\eta\sqrt{\lambda}\right) & -J_{-\nu}\left(\sqrt{\lambda}\right) & -J_{\nu}\left(\sqrt{\lambda}\right)
\end{pmatrix},
$$

$$
X_1=\begin{pmatrix} a_5\\ a_6\\ a_7\\ a_8\end{pmatrix},\qquad
B_1=\begin{pmatrix}\frac{\pi t^{\nu}J_{\nu}\left(t\sqrt{\lambda}\right)}{2\sin\nu\pi}\\ -\frac{\pi t^{\nu}J_{-\nu}\left(t\sqrt{\lambda}\right)}{2\sin\nu\pi}\\ 0\\ 0\end{pmatrix}.
$$

By above four equations we have

$$
a_5=\frac{\pi\csc(\pi\nu)t^{\nu}\left(J_{-\nu}\left(\sqrt{\lambda}\right)J_{\nu}\left(t\sqrt{\lambda}\right)-J_{\nu}\left(\sqrt{\lambda}\right)J_{-\nu}\left(t\sqrt{\lambda}\right)\right)}{2\left(J_{-\nu}\left(\sqrt{\lambda}\right)-\delta\eta^{\nu}J_{-\nu}\left(\eta\sqrt{\lambda}\right)\right)},
$$

$$a_6 = 0,$$
$$a_7 = \frac{\pi \csc(\pi\nu) t^{\nu} \left(\delta\eta^{\nu} J_{-\nu}\left(\eta\sqrt{\lambda}\right) J_{\nu}\left(t\sqrt{\lambda}\right) - J_{\nu}\left(\sqrt{\lambda}\right) J_{-\nu}\left(t\sqrt{\lambda}\right)\right)}{2\left(J_{-\nu}\left(\sqrt{\lambda}\right) - \delta\eta^{\nu} J_{-\nu}\left(\eta\sqrt{\lambda}\right)\right)},$$
$$a_8 = \frac{1}{2}\pi \csc(\pi\nu) t^{\nu} J_{-\nu}\left(t\sqrt{\lambda}\right).$$

This completes the construction of Green's function. Using (H_0) (or H_1) and Lemma 8.1 (or Lemma 8.2) we get that $G(x,t) \le 0$ (or $G(x,t) \ge 0$). ■

8.3.2 Case II: When $\lambda > 0$ and $\alpha \in \{1,3,5,\cdots\}$.

Suppose that

$$(H_2): \; 0 < \lambda < y_{\nu,1}^2, \quad 0 < \delta < 1, \quad \delta\eta^{\nu} Y_{\nu}\left(\eta\sqrt{\lambda}\right) - Y_{\nu}\left(\sqrt{\lambda}\right) \ge 0,$$
$$J_{-\nu}\left(\sqrt{\lambda}\right) - \delta\eta^{\nu} J_{-\nu}\left(\eta\sqrt{\lambda}\right) > 0, \text{ for } \alpha \in \{3,7,11,\cdots\}, \text{ and}$$

$$(H_3): \; 0 < \lambda < y_{\nu,1}^2, \quad \delta \ge 1, \quad \delta\eta^{\nu} Y_{\nu}\left(\eta\sqrt{\lambda}\right) - Y_{\nu}\left(\sqrt{\lambda}\right) \le 0,$$
$$J_{-\nu}\left(\sqrt{\lambda}\right) - \delta\eta^{\nu} J_{-\nu}\left(\eta\sqrt{\lambda}\right) < 0, \text{ for } \alpha \in \{1,5,9,\cdots\},$$

where $y_{\nu,1}^2$ is the first zero of $Y_{\nu}(x)$. For a given range of λ, the assumptions (H_2) and (H_3) hold, where $\alpha \in \{3,7,11,\cdots\}$ and $\alpha \in \{1,5,9,\cdots\}$, respectively.

Lemma 8.4. *For $0 < \lambda < y_{\nu,1}^2$, the Bessel functions of first and second kind ($J_{-\nu}$ and Y_{ν}) satisfy the following inequality*

$$r^{\nu}\left(J_{-\nu}\left(s\sqrt{\lambda}\right) Y_{\nu}\left(r\sqrt{\lambda}\right) - Y_{\nu}\left(s\sqrt{\lambda}\right) J_{-\nu}\left(r\sqrt{\lambda}\right)\right) \ge 0, \;\; 0 < r \le s \le 1,$$

where $\nu = \frac{1-\alpha}{2}$ and $\alpha \in \{3,7,11,\cdots\}$.

Proof. Suppose

$$\widetilde{F}(s,r) = r^{\nu}\left(J_{-\nu}\left(s\sqrt{\lambda}\right) Y_{\nu}\left(r\sqrt{\lambda}\right) - Y_{\nu}\left(s\sqrt{\lambda}\right) J_{-\nu}\left(r\sqrt{\lambda}\right)\right),$$

and let $s = s_0 \in [0,1]$ be fixed.

Now as $J_{-\nu}\left(s_0\sqrt{\lambda}\right) \ge J_{1-\nu}\left(r\sqrt{\lambda}\right)$ and $-Y_{\nu}\left(s_0\sqrt{\lambda}\right) \ge Y_{-1+\nu}\left(r\sqrt{\lambda}\right)$ for $r \le s_0$, when $0 < \lambda \le y_{\nu,1}^2$, where $\alpha \in \{3,7,11,\cdots\}$. So with the help of these inequalities we see that $\widetilde{F}(s_0,r)$ is an non-increasing function of r. As $\widetilde{F}(s_0,s_0) = 0$ at $r = s_0$, which implies that $\widetilde{F}(s_0,r) \ge 0, \;\; \forall \; r \le s_0$. But as s_0 takes any value in $[0,1]$ therefore $\widetilde{F}(s,r) \ge 0, \;\; \forall \; 0 < r \le s \le 1$. ■

Lemma 8.5. *For $0<\lambda< y_{\nu,1}^2$, the Bessel functions of first and second kind ($J_{-\nu}$ and Y_ν) satisfy the following inequality*

$$r^\nu\left(J_{-\nu}\left(s\sqrt{\lambda}\right)Y_\nu\left(r\sqrt{\lambda}\right)-Y_\nu\left(s\sqrt{\lambda}\right)J_{-\nu}\left(r\sqrt{\lambda}\right)\right)\leq 0,\;\; 0<r\leq s\leq 1,$$

where $\nu=\frac{1-\alpha}{2}$ and $\alpha\in\{1,5,9,\cdots\}$.

Proof. By using the inequalities

$$J_{-\nu}\left(s_0\sqrt{\lambda}\right)\geq J_{1-\nu}\left(r\sqrt{\lambda}\right)\quad\text{and}\quad Y_\nu\left(s_0\sqrt{\lambda}\right)\geq -Y_{-1+\nu}\left(r\sqrt{\lambda}\right)$$

for $r\leq s_0$, when $0<\lambda\leq y_{\nu,1}^2$, where $\alpha\in\{1,5,9,\cdots\}$, we can prove this lemma as we did in Lemma 8.4. ■

Lemma 8.6. *For the linear three point SBVPs (8.10)–(8.11), where $\alpha\in\{1,3,5,\cdots\}$, the Green's function is given by*

$$G(x,t)=\begin{cases} G_1(x,t), & 0\leq x\leq t\leq\eta;\\ G_2(x,t), & t\leq x,\ t\leq\eta;\\ G_3(x,t), & x\leq t,\ \eta\leq t;\\ G_4(x,t), & \eta\leq t\leq x\leq 1,\end{cases}$$

where

$$\begin{aligned}G_1(x,t)=&\frac{\pi\sec(\pi\nu)t^\nu x^\nu J_{-\nu}\left(x\sqrt{\lambda}\right)}{2\left(J_{-\nu}\left(\sqrt{\lambda}\right)-\delta\eta^\nu J_{-\nu}\left(\eta\sqrt{\lambda}\right)\right)}\left[J_{-\nu}\left(t\sqrt{\lambda}\right)\left(\delta\eta^\nu Y_\nu\left(\eta\sqrt{\lambda}\right)-Y_\nu\left(\sqrt{\lambda}\right)\right)\right.\\ &\left.+Y_\nu\left(t\sqrt{\lambda}\right)\left(J_{-\nu}\left(\sqrt{\lambda}\right)-\delta\eta^\nu J_{-\nu}\left(\eta\sqrt{\lambda}\right)\right)\right],\end{aligned}$$

$$\begin{aligned}G_2(x,t)=&\frac{1}{2}\pi\sec(\pi\nu)t^\nu x^\nu J_{-\nu}\left(t\sqrt{\lambda}\right)\\ &\left(\frac{J_{-\nu}\left(x\sqrt{\lambda}\right)\left(\delta\eta^\nu Y_\nu\left(\eta\sqrt{\lambda}\right)-Y_\nu\left(\sqrt{\lambda}\right)\right)}{J_{-\nu}\left(\sqrt{\lambda}\right)-\delta\eta^\nu J_{-\nu}\left(\eta\sqrt{\lambda}\right)}+Y_\nu\left(x\sqrt{\lambda}\right)\right),\end{aligned}$$

$$G_3(x,t)=\frac{\pi\sec(\pi\nu)t^\nu x^\nu J_{-\nu}\left(x\sqrt{\lambda}\right)\left(J_{-\nu}\left(\sqrt{\lambda}\right)Y_\nu\left(t\sqrt{\lambda}\right)-Y_\nu\left(\sqrt{\lambda}\right)J_{-\nu}\left(t\sqrt{\lambda}\right)\right)}{2\left(J_{-\nu}\left(\sqrt{\lambda}\right)-\delta\eta^\nu J_{-\nu}\left(\eta\sqrt{\lambda}\right)\right)},$$

$$G_4(x,t) = \frac{1}{2}\pi\sec(\pi\nu)t^\nu x^\nu \left(\frac{J_{-\nu}\left(x\sqrt{\lambda}\right)\left(\delta\eta^\nu J_{-\nu}\left(\eta\sqrt{\lambda}\right)Y_\nu\left(t\sqrt{\lambda}\right) - Y_\nu\left(\sqrt{\lambda}\right)J_{-\nu}\left(t\sqrt{\lambda}\right)\right)}{J_{-\nu}\left(\sqrt{\lambda}\right) - \delta\eta^\nu J_{-\nu}\left(\eta\sqrt{\lambda}\right)} \right.$$
$$\left. + J_{-\nu}\left(t\sqrt{\lambda}\right)Y_\nu\left(x\sqrt{\lambda}\right)\right).$$

If H_2 (or H_3) holds then $G(x,t) \leq 0$ (or $G(x,t) \geq 0$).

Proof. The construction of Green's function follows the analysis similar to Lemma 8.3. Using the assumption (H_2) (or H_3) and Lemma 8.4 (or Lemma 8.5) we get that $G(x,t) \leq 0$ (or $G(x,t) \geq 0$). ∎

Now we state Lemmas 8.7, 8.9 and 8.10 and we omit proof for brevity.

Lemma 8.7. *If $y \in C^2(I)$ is a solution of inhomogeneous linear three point SBVPs (8.8)–(8.9), then*

$$y(x) = \frac{b\, x^\nu J_{-\nu}\left(x\sqrt{\lambda}\right)}{J_{-\nu}\left(\sqrt{\lambda}\right) - \delta\eta^\nu J_{-\nu}\left(\eta\sqrt{\lambda}\right)} - \int_0^1 t^\alpha\, G(x,t)h(t)dt. \tag{8.15}$$

8.3.3 Case III: When $\lambda < 0$.

Suppose that

(H_0') : $\delta > 0,\ K_\nu\left(\sqrt{|\lambda|}\right) - \delta\eta^\nu K_\nu\left(\eta\sqrt{|\lambda|}\right) \leq 0$ and $I_{-\nu}\left(\sqrt{|\lambda|}\right) - \delta\eta^\nu I_{-\nu}\left(\eta\sqrt{|\lambda|}\right) > 0$ for $\alpha \in [1,\infty)$.

Lemma 8.8. *For $\lambda < 0$, the modified Bessel functions of first and second kind ($I_{-\nu}$ and K_ν) satisfy the following inequality*

$$r^\nu\left(K_\nu\left(s\sqrt{|\lambda|}\right)I_{-\nu}\left(r\sqrt{|\lambda|}\right) - I_{-\nu}\left(s\sqrt{|\lambda|}\right)K_\nu\left(r\sqrt{|\lambda|}\right)\right) \leq 0, \quad 0 < r \leq s \leq 1$$

where $\nu = \frac{1-\alpha}{2}$ and $\alpha \in [1,\infty)$.

Proof. Suppose

$$\widetilde{F_1}(s,r) = r^\nu\left(K_\nu\left(s\sqrt{|\lambda|}\right)I_{-\nu}\left(r\sqrt{|\lambda|}\right) - I_{-\nu}\left(s\sqrt{|\lambda|}\right)K_\nu\left(r\sqrt{|\lambda|}\right)\right),$$

and further assume that $s = s_0 \in [0,1]$ be fixed. The function $\widetilde{F_1}(s_0,r)$ is non-decreasing in r for all $\alpha \in [1,\infty)$. At $r = s_0$, $\widetilde{F_1}(s_0,s_0) = 0$, i.e., $\widetilde{F_1}(s_0,r) \leq 0,\ \forall\ r \leq s_0$. But as s_0 takes any value in $[0,1]$ therefore $\widetilde{F}(s,r) \leq 0,\ \forall\ 0 < r \leq s \leq 1$. ∎

Lemma 8.9. *For the following linear three point SBVPs (8.10)–(8.11), where $\alpha \in [1,\infty)$, the Green's function is given by*

$$G(x,t) = \begin{cases} G_1(x,t), & 0 \le x \le t \le \eta; \\ G_2(x,t), & t \le x,\ t \le \eta; \\ G_3(x,t), & x \le t,\ \eta \le t; \\ G_4(x,t), & \eta \le t \le x \le 1, \end{cases}$$

where

$$G_1(x,t) = \frac{t^\nu x^\nu I_{-\nu}\left(x\sqrt{|\lambda|}\right)}{I_{-\nu}\left(\sqrt{|\lambda|}\right) - \delta\eta^\nu I_{-\nu}\left(\eta\sqrt{|\lambda|}\right)} \Big[I_{-\nu}\left(t\sqrt{|\lambda|}\right)\left(K_\nu\left(\sqrt{|\lambda|}\right) - \delta\eta^\nu K_\nu\left(\eta\sqrt{|\lambda|}\right)\right) - K_\nu\left(t\sqrt{|\lambda|}\right)\left(I_{-\nu}\left(\sqrt{|\lambda|}\right) - \delta\eta^\nu I_{-\nu}\left(\eta\sqrt{|\lambda|}\right)\right)\Big],$$

$$G_2(x,t) = \frac{t^\nu x^\nu I_{-\nu}\left(t\sqrt{|\lambda|}\right)}{I_{-\nu}\left(\sqrt{|\lambda|}\right) - \delta\eta^\nu I_{-\nu}\left(\eta\sqrt{|\lambda|}\right)} \Big[I_{-\nu}\left(x\sqrt{|\lambda|}\right)\left(K_\nu\left(\sqrt{|\lambda|}\right) - \delta\eta^\nu K_\nu\left(\eta\sqrt{|\lambda|}\right)\right) - K_\nu\left(x\sqrt{|\lambda|}\right)\left(I_{-\nu}\left(\sqrt{|\lambda|}\right) - \delta\eta^\nu I_{-\nu}\left(\eta\sqrt{|\lambda|}\right)\right)\Big],$$

$$G_3(x,t) = \frac{t^\nu x^\nu I_{-\nu}\left(x\sqrt{|\lambda|}\right)\left(K_\nu\left(\sqrt{|\lambda|}\right) I_{-\nu}\left(t\sqrt{|\lambda|}\right) - I_{-\nu}\left(\sqrt{|\lambda|}\right) K_\nu\left(t\sqrt{|\lambda|}\right)\right)}{I_{-\nu}\left(\sqrt{|\lambda|}\right) - \delta\eta^\nu I_{-\nu}\left(\eta\sqrt{|\lambda|}\right)},$$

$$G_4(x,t) = \frac{t^\nu x^\nu}{I_{-\nu}\left(\sqrt{|\lambda|}\right) - \delta\eta^\nu I_{-\nu}\left(\eta\sqrt{|\lambda|}\right)} \Big[I_{-\nu}\left(x\sqrt{|\lambda|}\right)\left(K_\nu\left(\sqrt{|\lambda|}\right) I_{-\nu}\left(t\sqrt{|\lambda|}\right) - \delta\eta^\nu I_{-\nu}\left(\eta\sqrt{|\lambda|}\right) K_\nu\left(t\sqrt{|\lambda|}\right)\right) - I_{-\nu}\left(t\sqrt{|\lambda|}\right) K_\nu\left(x\sqrt{|\lambda|}\right)\left(I_{-\nu}\left(\sqrt{|\lambda|}\right) - \delta\eta^\nu I_{-\nu}\left(\eta\sqrt{|\lambda|}\right)\right)\Big],$$

and if (H_0') holds then $G(x,t) \le 0$.

Lemma 8.10. *If $y \in C^2(I)$ is a solution of inhomogeneous linear three point SBVPs (8.8)–(8.9) then*

$$y(x) = \frac{b\, x^\nu I_{-\nu}\left(x\sqrt{|\lambda|}\right)}{I_{-\nu}\left(\sqrt{|\lambda|}\right) - \delta\eta^\nu I_{-\nu}\left(\eta\sqrt{|\lambda|}\right)} - \int_0^1 t^\alpha\, G(x,t) h(t)dt.$$

8.4 Maximum and anti-maximum principles

The constant sign of Green's function produces the following anti-maximum and maximum principles.

Proposition 8.1. (***Anti-maximum principle***)
Assume $\lambda > 0$ *and* (H_1) *or* (H_3) *holds, and* $y \in C^2(I)$ *satisfies*

$$\begin{aligned} &-(x^{\alpha}y'(x))' - \lambda x^{\alpha}y(x) \geq 0, \quad 0 < x < 1, \\ &y'(0) = 0, \quad y(1) \geq \delta y(\eta). \end{aligned}$$

Then $y(x) \leq 0$, $\forall x \in [0,1]$.

Proposition 8.2. (***Maximum principle***)

(Max_1) *Assume* $\lambda > 0$ *and* (H_0) *or* (H_2) *holds, and* $y \in C^2(I)$ *satisfies*

$$\begin{aligned} &-(x^{\alpha}y'(x))' - \lambda x^{\alpha}y(x) \geq 0, \quad 0 < x < 1, \\ &y'(0) = 0, \quad y(1) \geq \delta y(\eta). \end{aligned}$$

Then $y(x) \geq 0$, $\forall x \in [0,1]$.

(Max_2) *Assume* $\lambda < 0$, (H_0') *holds and* $y \in C^2(I)$ *satisfies*

$$\begin{aligned} &-(x^{\alpha}y'(x))' - \lambda x^{\alpha}y(x) \geq 0, \quad 0 < x < 1, \\ &y'(0) = 0, \quad y(1) \geq \delta y(\eta). \end{aligned}$$

Then $y(x) \geq 0$, $\forall x \in [0,1]$.

8.5 Existence results

On the basis of anti-maximum and maximum principles, we divide this section into the following two subsections.

8.5.1 Reverse ordered case

Theorem 8.1. *Assume that*

(R_1) *there exist* u_0, v_0 *in* $C^2[0,1]$ *such that* $u_0 \leq v_0$, *where* u_0 *satisfies (8.4)–(8.5) and* v_0 *satisfies (8.6)–(8.7);*

(R_2) *the function* $f: D \to \mathbb{R}$ *is continuous on* $D := \{(x,y) \in [0,1] \times \mathbb{R} : u_0 \le y \le v_0\}$*;*

(R_3) *there exists* $M_1 > 0$ *such that for all* $(x,y),(x,w) \in D$*,*

$$y \le w \Longrightarrow f(x,w) - f(x,y) \le M_1(w-y);$$

(R_4) *there exists a constant* $\lambda > 0$ *such that* $M_1 - \lambda \le 0$ *and* H_1 *or* H_3 *holds.*

Then the nonlinear three point SBVP (8.1)–(8.2) has at least one solution in the region D. Sequence (u_n) *generated by equation (8.3), with initial iterate* u_0 *converges monotonically (non-decreasing) and uniformly towards a solution* $u(x)$ *of (8.1)–(8.2). Similarly* v_0 *as an initial iterate leads to a non-increasing sequences* (v_n) *converging to a solution* $v(x)$*. Any solution* $z(x)$ *in D satisfies*

$$u(x) \le z(x) \le v(x).$$

Proof. By using analysis similar to the the proof of Theorem 7.1 of chapter 7, we have

$$u_0 \le u_1 \le u_2 \le \ldots \le u_n \le u_{n+1} \le \ldots \le v_{n+1} \le v_n \le \ldots \le v_2 \le v_1 \le v_0.$$

So the sequence (u_n) is monotonically non-decreasing and bounded above by v_0, similarly (v_n) is non-increasing and bounded below by u_0. Hence by Dini's theorem they converge uniformly. Let $u(x) = \lim_{n\to\infty} u_n(x)$ and $v(x) = \lim_{n\to\infty} v_n(x)$.

Now by using Lemma 8.7, the solution u_{n+1} of (8.3) is given by

$$u_{n+1} = \frac{b\, x^\nu J_{-\nu}\left(x\sqrt{\lambda}\right)}{J_{-\nu}\left(\sqrt{\lambda}\right) - \delta\eta^\nu J_{-\nu}\left(\eta\sqrt{\lambda}\right)} - \int_0^1 G(x,t)t^\alpha(f(t,u_n) - \lambda u_n)dt.$$

Now by Lebesgue's dominated convergence theorem, we get

$$u(x) = \frac{b\, x^\nu J_{-\nu}\left(x\sqrt{\lambda}\right)}{J_{-\nu}\left(\sqrt{\lambda}\right) - \delta\eta^\nu J_{-\nu}\left(\eta\sqrt{\lambda}\right)} - \int_0^1 G(x,t)t^\alpha(f(t,u) - \lambda u)dt.$$

Which is the solution of nonlinear three point SBVP (8.1)–(8.2). The same is true for (α_n). It is easy to see that $u(x) \le z(x) \le v(x)$. ■

8.5.2 Well ordered case

Based on the sign of λ, we prove two Existence theorems; Theorem 8.2 and Theorem 8.3. The proof of these theorems are similar to the proof of Theorem 8.1.

Theorem 8.2. *Assume that*

(W_1) *there exist u_0, v_0 in $C^2[0,1]$ such that $v_0 \le u_0$, where u_0 satisfies (8.4)–(8.5) and v_0 satisfies (8.6)–(8.7);*

(W_2) *the function $f : D_1 \to \mathbb{R}$ is continuous on $D_1 := \{(x,y) \in [0,1] \times \mathbb{R} : v_0 \le y \le u_0\}$;*

(W_3) *there exists $M_2 > 0$ such that for all $(x,y),(x,w) \in D_1$,*

$$y \le w \Longrightarrow f(x,w) - f(x,y) \ge M_2(w-y);$$

(W_4) *there exists a constant $\lambda > 0$ such that $M_2 - \lambda \ge 0$ and H_0 or H_2 holds.*

Then the nonlinear three point SBVP (8.1)–(8.2) has at least one solution in the region D_1. Sequence (u_n) generated by equation (8.3), with initial iterate u_0 converges monotonically (non-increasing) and uniformly towards a solution $u(x)$ of (8.1)–(8.2). Similarly v_0 as an initial iterate leads to a non-decreasing sequence (v_n) converging to a solution $v(x)$. Any solution $z(x)$ in D_1 satisfies

$$v(x) \le z(x) \le u(x).$$

Theorem 8.3. *Assume that*

(W_1') *there exist u_0, v_0 in $C^2[0,1]$ such that $v_0 \le u_0$, where u_0 satisfies (8.4)–(8.5) and v_0 satisfies (8.6)–(8.7);*

(W_2') *the function $f : D_2 \to \mathbb{R}$ is continuous on $D_2 := \{(x,y) \in [0,1] \times \mathbb{R} : v_0 \le y \le u_0\}$;*

(W_3') *there exists $M_3 > 0$ such that for all $(x,\widetilde{y}),(x,\widetilde{w}) \in D_2$,*

$$\widetilde{y} \le \widetilde{w} \Longrightarrow f(x,\widetilde{w}) - f(x,\widetilde{y}) \ge -M_3(\widetilde{w}-\widetilde{y});$$

(W_4') *there exists a constant $\lambda < 0$ such that $M_3 + \lambda \le 0$ and (H_0') holds.*

Then the nonlinear three point SBVP (8.1)–(8.2) has at least one solution in the region D_2. Sequence (u_n) generated by equation (8.3), with initial iterate u_0 converges monotonically (non-increasing) and uniformly towards a solution $u(x)$ of (8.1)–(8.2). Similarly v_0 as an initial iterate leads to a non-decreasing sequence (v_n) converging to a solution $v(x)$. Any solution $z(x)$ in D_2 satisfies

$$v(x) \le z(x) \le u(x).$$

8.5.3 Uniqueness

Theorem 8.4. *Let $f(x,y)$ be continuous on D (or D_1 or D_2) and there exists a constant $M_\lambda > 0$ such that*

$$f(x,u) - f(x,v) \le M_\lambda (u - v),$$

and $M_\lambda < \lambda_1$, where $\lambda_1 \le \min\left\{ j^2_{\nu,1}, y^2_{\nu,1}, i^2_{-\nu,1}, k^2_{\nu,1} \right\}$. Then the nonlinear three point SBVP (8.1)–(8.2) has a unique solution.

***Proof*.** Suppose $u(x)$ and $v(x)$ are any two solutions of (8.1)–(8.2) then we get

$$\begin{aligned} &-(x^\alpha (u-v)')' = x^\alpha \left[f(x,u) - f(x,v) \right], \\ &\quad (u-v)'(0) = 0, \quad (u-v)(1) = \delta(u-v)(\eta), \end{aligned}$$

which gives

$$\begin{aligned} &-(x^\alpha (u-v)')' - M_\lambda x^\alpha (u-v) \le 0, \\ &\quad (u-v)'(0) = 0, \quad (u-v)(1) = \delta(u-v)(\eta). \end{aligned}$$

By the maximum and anti-maximum principles, whenever $M_\lambda < \lambda_1$, we get $u - v \le 0$ or $u - v \ge 0$, (i.e., $u \le v$ or $u \ge v$) for different class of α. Similarly by changing the role of u and v, we get $u \ge v$ or $u \le v$. Hence $u \equiv v$. Therefore the solution of (8.1)–(8.2) is unique. ■

8.6 Examples

We give some examples and validate the assumptions which guarantee the existence results which is derived in the Theorem 8.1, Theorem 8.2 and Theorem 8.3.

8.6.1 Reverse ordered upper and lower solution

The following example validate the result of Theorem 8.1, and gives a range of λ for which we can generate two monotone sequences which converge to the solution of nonlinear SBVP.

Example 8.1. *Consider the nonlinear three point SBVP*

$$-y''(x) - \frac{\alpha}{x} y'(x) = \frac{\alpha\,(e^y - 1) - x}{4}, \tag{8.16}$$

$$y'(0) = 0, \qquad y(1) = 3y\left(\frac{1}{7}\right), \tag{8.17}$$

where $f(x,y) = \frac{\alpha\ (e^y-1)-x}{4}$, $\delta = 3$, $\eta = \frac{1}{7}$ and α satisfies (H_1) or (H_3). In this problem we choose lower and upper solutions as $v_0 = 1$ and $u_0 = -1$, where $v_0 \geq u_0$, i.e., it is reverse ordered case. Here nonlinear term satisfies all assumptions for Theorem 8.1 and Lipschitz constant M_1 is $\frac{e\alpha}{4}$. Now we can find out a range for $\lambda \in (\frac{e\alpha}{4},\ j^2_{\nu,1})$ such that (H_1) or (H_3) is true. Hence there exists a unique solution of the SBVP (8.16)–(8.17) in domain $D := \{(x,y) \in [0,1] \times \mathbb{R} : u_0 \leq y \leq v_0\}$.

8.6.2 Well ordered upper and lower solutions

The following examples validate the results of Theorem 8.2 and Theorem 8.3. On the basis of sign of "λ", we divide this subsection into the following two parts.

8.6.2.1 When $\lambda > 0$

Example 8.2. *Consider the nonlinear three point SBVP*

$$-y''(x) - \frac{\alpha}{x}y'(x) = de^y, \tag{8.18}$$

$$y'(0) = 0, \qquad y(1) = \frac{1}{2}y\left(\frac{1}{3}\right), \tag{8.19}$$

where $f(x,y) = de^y$, $d = \frac{2(1+\alpha)e^{-\frac{2}{3}}}{3}$ $\delta = \frac{1}{2}$, $\eta = \frac{1}{3}$ and α satisfies (H_0) or (H_2). Here we choose lower and upper solutions as $v_0 = 0$ and $u_0 = \frac{2-x^2}{3}$, where $v_0 \leq u_0$, i.e., it is well ordered case. The nonlinear term satisfies all assumptions for Theorem 8.2 and Lipschitz constant M_2 is d. Now we can find out a range for $\lambda > 0$ such that the conditions $M_2 - \lambda \geq 0$, (H_0) or (H_2) are true. Hence existence of a unique solution is guaranteed in domain $D_1 := \{(x,y) \in [0,1] \times \mathbb{R} : v_0 \leq y \leq u_0\}$.

8.6.2.2 When $\lambda < 0$

Example 8.3. *Consider the nonlinear three point SBVP*

$$-y''(x) - \frac{\alpha}{x}y'(x) = 1 - 2y^3, \tag{8.20}$$

$$y'(0) = 0, \qquad y(1) = \frac{1}{3}y\left(\frac{1}{4}\right). \tag{8.21}$$

Here $f(x,y) = 1 - 2y^3$, $\delta = \frac{1}{3}$, $\eta = \frac{1}{4}$ and α satisfies (H_0'). In this problem we choose lower and upper solutions as $v_0 = -1$ and $u_0 = 1$, where $v_0 \leq u_0$, i.e., it is well ordered case. The nonlinear term satisfies all assumptions of Theorem 8.3, and Lipschitz constant M_3 is

6. Now we can find out a range for $\lambda < 0$ such that the conditions $M_3 + \lambda \leq 0$ and (H_0') are true. Hence a unique solution exists in domain $D_2 := \{(x,y) \in [0,1] \times \mathbb{R} : v_0 \leq y \leq u_0\}$.

Example 8.4. *Consider the nonlinear three point SBVP*

$$-y''(x) - \frac{3}{x}y'(x) = 1 - 7y^2, \tag{8.22}$$

$$y'(0) = 0, \qquad y(1) = 2.2y\left(\frac{1}{5}\right). \tag{8.23}$$

Here $f(x,y) = 1 - 7y^3$, $\delta = 2.2$, $\eta = \frac{1}{5}$ and α satisfies (H_0'). In this problem we choose lower and upper solutions as $v_0 = 0$ and $u_0 = \frac{7}{4} + \frac{5}{2}x^2$ i.e., it is well ordered case. The nonlinear term satisfies all assumptions for Theorem 8.3, and Lipschitz constant M_3 is $\frac{119}{2}$. Now we can find out a range for $\lambda < 0$ such that the conditions $M_3 + \lambda \leq 0$ and (H_0') are true. Thus unique solution exists in domain $D_2 := \{(x,y) \in [0,1] \times \mathbb{R} : v_0 \leq y \leq u_0\}$.

8.7 Conclusion

In this chapter we have established existence of a unique solution of nonlinear three point SBVPs $-(x^\alpha y'(x))' = x^\alpha f(x,y), 0 < x < 1, y'(0) = 0, y(1) = \delta y(\eta)$, where $\delta > 0, 0 < \eta < 1$ and $\alpha \geq 1$. We observe that when $sup\,(\partial f/\partial y) > 0$ for $\alpha \in \cup_{n\in\mathbb{N}}(4n-1, 4n+1)$ or $\alpha \in \{1,5,9,\cdots\}$ reverse ordered case occur. When $sup\,(\partial f/\partial y) > 0$ for $\alpha \in \cup_{n\in\mathbb{N}}(4n-3, 4n-1)$ or $\alpha \in \{3,7,11,\cdots\}$ and when $sup\,(\partial f/\partial y) < 0$ for all $\alpha \geq 1$ well order case occur. This classification does not exist in the literature to the best of our knowledge.

Chapter 9

Nonlinear three point SBVPs with derivative dependent source term

9.1 Introduction

The appropriate equation for the thermal balance between the heat generated by the chemical reaction and that conducted away can be written as

$$\nabla^2 u(P) = f(P, u(P), du(P)/dP). \tag{9.1}$$

Due to geometric similarity, we arrive at the following nonlinear singular differential equation

$$-y''(x) - \frac{\gamma}{x} y'(x) = f(x, y, x^\gamma y'), \quad 0 < x < 1, \tag{9.2}$$

where γ corresponds to geometry of the vessel under consideration. In this work we consider the case when $\gamma = 1$, i.e., the reaction is taking place in cylindrical vessel whose length is much larger than the radius. Thus we have the following singular differential equation

$$-y''(x) - \frac{1}{x} y'(x) = f(x, y, xy'), \quad 0 < x < 1. \tag{9.3}$$

Chamber [31] considered the case when $f(x, y, xy') = e^y$. His model was based on Arrhenius law.

The work in this chapter generalizes the results of chapter 7. We consider the following three point boundary conditions,

$$y'(0) = 0, \quad y(1) = \delta y(\eta), \tag{9.4}$$

where $\delta > 0$ and $0 < \eta < 1$. Similar to chapter 7 in this chapter also we arrive at both reverse order and well order cases.

We prove some inequalities based on Bessel and modified Bessel functions and establish the existence results for (9.3)–(9.4) in a region $D := \{(x,u,w) \in [0,1]\times\mathbb{R}^2 : \beta_0(x) \le u \le \alpha_0(x)\}$ or $\widetilde{D} := \{(x,u,w) \in [0,1]\times\mathbb{R}^2 : \alpha_0(x) \le u \le \beta_0(x)\}$ by using the monotone iterative method with upper and lower solutions that are reverse ordered and well ordered. The functions $\beta_0(x)$ and $\alpha_0(x)$ are called upper and lower solutions of nonlinear three point SBVP, (9.3)–(9.4). The function $\beta_0(x)$ satisfies the differential inequalities

$$-(x\beta_0'(x))' \ge xf(x,\beta_0,x\beta_0'), \qquad \beta_0'(0) = 0, \quad \beta_0(1) \ge \delta\beta_0(\eta),$$

and the function $\alpha_0(x)$ satisfies the reverse differential inequalities. We further assume that

(F_1) the function $f : D$ (or $\widetilde{D}$) $\to \mathbb{R}$ is continuous on D (or $\widetilde{D}$);

(F_2) for all $(x,u_1,w),(x,u_2,w) \in D$ (or $\widetilde{D}$),

(a) when $\lambda > 0$, there exists a constant $M_1 > 0$ in the region D such that

$$u_1 \le u_2 \Longrightarrow f(x,u_2,w) - f(x,u_1,w) \le M_1(u_2 - u_1);$$

(b) when $\lambda < 0$, there exists a constant $M_2 > 0$ in the region $\widetilde{D}$ such that

$$u_1 \le u_2 \Longrightarrow f(x,u_2,w) - f(x,u_1,w) \ge -M_2(u_2 - u_1);$$

(F_3) there exists $N > 0$ such that for all $(x,u,w_1),(x,u,w_2) \in D$ (or $\widetilde{D}$),

$$|f(x,u,w_2) - f(x,u,w_1)| \le N|w_2 - w_1|.$$

We consider the following monotone iterative scheme for nonlinear three point SBVP (9.3)–(9.4),

$$\left.\begin{aligned} &-y_{n+1}''(x) - \frac{1}{x}y_{n+1}'(x) - \lambda y_{n+1}(x) = f(x,y_n,xy_n') - \lambda y_n(x),\\ &y_{n+1}'(0) = 0, \quad y_{n+1}(1) = \delta y_{n+1}(\eta), \end{aligned}\right\} \tag{9.5}$$

where $\lambda \in \mathbb{R}\setminus\{0\}$ and $f(x,y,xy')$ satisfies (F_1), (F_2) and (F_3).

9.2 Linear case and Green's function

Corresponding linear case of (9.3)–(9.4) is same as linear case of chapter 7, where $h(t) = f(t,y,xy') - \lambda y$ and assumptions (H_0) and (H_0') are chosen same as of chapter 7, so maximum and anti-maximum principles of chapter 7 are directly applicable.

9.3 Some inequalities and existence results

We prove some inequalities based upon properties of Bessel and modified Bessel function and establish the existence results for both cases, i.e., when upper and lower solutions are well ordered or in reverse order. We divide this section into the following two subsections.

9.3.1 Reverse ordered lower and upper solutions $(\alpha_0 \geq \beta_0)$

Lemma 9.1. *If* $0 < \lambda < y_{0,1}^2$*, then the Bessel functions* J_0 *and* J_1 *satisfy the following inequality*

$$(\lambda - M_1)J_0(x\sqrt{\lambda}) - Nx\sqrt{\lambda}J_1(x\sqrt{\lambda}) \geq 0,$$

for all $x \in [0,1]$*, whenever*

$$\lambda \geq M_1 + \frac{N^2}{2} + \frac{N}{2}\sqrt{N^2 + 4M_1}\ ,$$

such that $M_1, N \in \mathbb{R}^+$.

Proof. When $0 < \lambda < y_{0,1}^2$, the Bessel functions satisfy the inequality $J_0(x\sqrt{\lambda}) \geq J_1(x\sqrt{\lambda})$, for all $x \in [0,1]$, which gives us

$$(\lambda - M_1)J_0(x\sqrt{\lambda}) - Nx\sqrt{\lambda}J_1(x\sqrt{\lambda}) \geq \Big((\lambda - M_1) - N\sqrt{\lambda}\Big)J_0(x\sqrt{\lambda}).$$

Now right hand side will be positive provided $\Big((\lambda - M_1) - N\sqrt{\lambda}\Big) \geq 0$, which gives $\lambda \geq M_1 + \frac{N^2}{2} + \frac{N}{2}\sqrt{N^2 + 4M_1}$. Hence the lemma. ∎

Remark 9.1. *It is clear that* $G(x,t) \geq 0$*, for all* $x,t \in [0,1]$*, when* (H_0) *holds. As* $G(x,t)$ *satisfies*

$$\begin{aligned} &-(xG'(x))' - \lambda xG(x) = 0, \quad 0 < x < 1,\\ &G'(0) = 0, \quad G(1) = \delta G(\eta), \end{aligned}$$

we deduce that $G'(x,t) \leq 0$ *and* $xG'(x,t) \geq \frac{\lambda}{\lambda-1}G(x,t)$ *for* $\lambda < 1$ *and for all* $x,\ t \in [0,1]$, *such that* $x \neq t$.

Lemma 9.2. *Suppose* (H_0) *holds and such that* $1 > \lambda \geq M_1$ *then for all* $x,t \in [0,1]$, *we have the inequality*

$$(\lambda - M_1)G(x,t) + N\,x\,(sign\ y')\frac{\partial G(x,t)}{\partial x} \geq 0,$$

whenever $(\lambda - M_1) - N\frac{\lambda}{1-\lambda} \geq 0$ *and* $M_1, N \in \mathbb{R}^+$ *and* $x \neq t$.

Proof. From the above Remark 9.1, it is clear that to prove the above inequality, it is sufficient to prove $(\lambda - M_1)G(x,t) + N\,x\,\frac{\partial G(x,t)}{\partial x} \geq 0$, for all $x,\ t \in [0,1]$ and $x \neq t$. Now again by using Remark 9.1, we write

$$(\lambda - M_1)G(x,t) + N\,x\,\frac{\partial G(x,t)}{\partial x} \geq \left((\lambda - M_1) - N\frac{\lambda}{1-\lambda}\right)G(x,t).$$

Now if $(\lambda - M_1) - N\frac{\lambda}{1-\lambda} \geq 0$, then right hand side will be positive. This completes the proof of lemma. ∎

Remark 9.2. *If* $y_n = \alpha_{n+1} - \alpha_n$, *and* f *is defined in domain D, then we observe that*

$$-(xy_n')' - \lambda x y_n = xf(x,\alpha_n,x\alpha_n') + (x\alpha_n')', \tag{9.6}$$

$$y_n'(0) = 0, \quad y_n(1) = \delta y_n(\eta), \tag{9.7}$$

and if we assume that α_n *is lower solution of (9.3)–(9.4), then (9.6)–(9.7) are reduced to the following SBVP*

$$-(xy_n')' - \lambda x y_n = xf(x,\alpha_n,x\alpha_n') + (x\alpha_n')' \geq 0,$$
$$y_n'(0) = 0, \quad y_n(1) \geq \delta y_n(\eta).$$

Finally, by using the Proposition 7.1, we get $y_n \leq 0$, *i.e.,* $\alpha_{n+1} \leq \alpha_n$. *Similarly we can get* $\beta_{n+1} \geq \beta_n$, *where* β_n *is an upper solution of (9.3)–(9.4).*

Proposition 9.1. *Suppose* (H_0) *holds, the source function* f *satisfies* (F_1), (F_2) *and* (F_3) *and there exists* $0 < \max\left\{M_1, M_1 + \frac{N^2}{2} + \frac{N}{2}\sqrt{N^2+4M_1}\right\} \leq \lambda < 1$, *such that* $(\lambda - M_1) - N\frac{\lambda}{1-\lambda} \geq 0$ *is valid. Then the functions* α_n *and* β_n, *satisfy the following relations*

(a) $\alpha_{n+1} \leq \alpha_n$, *for all* $n \in \mathbb{N}$, *where* α_n *is a lower solution of (9.3)–(9.4),*

(b) $\beta_{n+1} \geq \beta_n$, *for all* $n \in \mathbb{N}$, *where* β_n *is an upper solution of (9.3)–(9.4),* *and α_n, β_n are defined recursively by (9.5).*

Proof. The above claim is proved by using the principle of Mathematical Induction. Claim (a) holds for $n=0$, i.e., $\alpha_1 \leq \alpha_0$ (see Remark (9.2)). Now suppose that claim is true for $n-1$, i.e., $\alpha_n \leq \alpha_{n-1}$, where α_{n-1} is lower solution of (9.3)–(9.4), and we will show that the claim is true for n.

Let $y = \alpha_n - \alpha_{n-1}$, then it is clear that y satisfies

$$\begin{aligned} &-(xy')' - \lambda xy = (x\alpha'_{n-1})' + xf(x, \alpha_{n-1}, x\alpha'_{n-1}) \geq 0, \\ &\quad y'(0) = 0, \quad y(1) \geq \delta y(\eta). \end{aligned}$$

To show that $\alpha_{n+1} \leq \alpha_n$, we have to prove that α_n is a lower solution of (9.3)–(9.4), i.e.,

$$-(x\alpha'_n)' - xf(x, \alpha_n, x\alpha'_n) \leq x\left[(\lambda - M_1)y + N(sign\ y')xy'\right],$$

where right hand side should be negative. Now, by using equation (7.17) it is sufficient to prove

$$\begin{aligned} &(\lambda - M_1)J_0(x\sqrt{\lambda}) - Nx\sqrt{\lambda}J_1(x\sqrt{\lambda}) \geq 0, \\ &(\lambda - M_1)G(x,t) + N\,x\,(sign\ y')\frac{\partial G(x,t)}{\partial x} \geq 0, \quad x \neq t, \end{aligned}$$

for all $x, t \in [0,1]$. Which are true by Lemma 9.1 and Lemma 9.2. Hence $\alpha_{n+1} \leq \alpha_n$. Using similar analysis we can prove the claim (b). Hence $\beta_{n+1} \geq \beta_n$. ■

Proposition 9.2. *Suppose (H_0) holds, the source term f satisfies (F_1), (F_2) and (F_3) and there exists $0 < \max\left\{M_1, M_1 + \frac{N^2}{2} + \frac{N}{2}\sqrt{N^2 + 4M_1}\right\} \leq \lambda < 1$ such that $(\lambda - M_1) - N\frac{\lambda}{1-\lambda} \geq 0$ and for all $x \in [0,1]$*

$$f(x, \beta_0(x), x\beta'_0(x)) - f(x, \alpha_0(x), x\alpha'_0(x)) - \lambda(\beta_0 - \alpha_0) \geq 0,$$

is valid. Then for all $n \in \mathbb{N}$, the functions α_n and β_n defined by (9.5), satisfy $\alpha_n \geq \beta_n$.

Proof. Suppose $y_i = \beta_i - \alpha_i$, it is clear that y_i satisfies the singular differential equation

$$\left.\begin{aligned} -(xy'_i)' - x\lambda y_i = &\ x\left[f(x, \beta_{i-1}(x), x\beta'_{i-1}(x))\right. \\ &\left. - f(x, \alpha_{i-1}(x), x\alpha'_{i-1}(x)) - \lambda(\beta_{i-1} - \alpha_{i-1})\right], \\ = &\ x[h_{i-1}], \end{aligned}\right\} \tag{9.8}$$

where $h_{i-1} = f(x, \beta_{i-1}(x), x\beta'_{i-1}(x)) - f(x, \alpha_{i-1}(x), x\alpha'_{i-1}(x)) - \lambda(\beta_{i-1} - \alpha_{i-1})$.

To prove $\beta_i \leq \alpha_i$, for all $i \in \mathbb{N}$, we have to show that $h_{i-1} \geq 0$, for all $i \in \mathbb{N}$. We use Mathematical Induction. For $i = 1$, the equation (9.8) is reduced into

$$\begin{aligned} -(xy_1')' - x\lambda y_1 &= x\left[f(x,\beta_0(x),x\beta_0'(x)) - f(x,\alpha_0(x),x\alpha_0'(x)) - \lambda(\beta_0 - \alpha_0)\right], \\ &= x[h_0], \end{aligned}$$

by using the conditions (F_2) and (F_3), we can easily show that $h_0 \geq 0$, and $y_1'(0) = 0$, $y_1(1) = \delta y_1(\eta)$. Using Proposition 7.1, we deduce that $y_1 \leq 0$, i.e., $\beta_1 \leq \alpha_1$. Now suppose $h_{n-2} \geq 0$ and $\beta_{n-1} \leq \alpha_{n-1}$, and we have to prove that $h_{n-1} \geq 0$ and $\beta_n \leq \alpha_n$.

As

$$\begin{aligned} h_{n-1} &= f(x,\beta_{n-1}(x),x\beta_{n-1}'(x)) - f(x,\alpha_{n-1}(x),x\alpha_{n-1}'(x)) - \lambda(\beta_{n-1} - \alpha_{n-1}), \\ &= -\left[(\lambda - M_1)y_{n-1} + N(sign\, y_{n-1}')xy_{n-1}'\right], \end{aligned}$$

where $y_{n-1} = \beta_{n-1} - \alpha_{n-1}$ is the solutions of nonhomogeneous linear BVP (7.6)–(7.7), with $h_{n-2} \geq 0$ and $y_{n-1}'(0) = 0$, $y_{n-1}(1) = \delta y_{n-1}(\eta)$. We follow the same analysis as we did in Proposition 9.1 and we have $h_{n-1} \geq 0$, and $y_n'(0) = 0$, $y_n(1) = \delta y_n(\eta)$. By using Proposition 7.1, we deduce that $y_n \leq 0$, i.e., $\alpha_n \geq \beta_n$. ■

9.3.1.1 Priori's bound

Lemma 9.3. *If $f(x,y,xy')$ satisfies*

(F_R) *For all $(x,y,xy') \in D$, $|f(x,y,xy')| \leq \varphi(|xy'|)$; where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and satisfies*

$$\frac{1}{2} < \int_{l_0}^{\infty} \frac{ds}{\varphi(s)},$$

where $l_0 = 2\sup|x\Omega_0(x)|$ and $\Omega_0 = \max\{|\alpha_0(t)|_\infty, |\beta_0(t)|_\infty\}$. Then there exists $R > 0$ such that any solution of

$$-(xy'(x))' \geq xf(x,y,xy'), \quad 0 < x < 1, \tag{9.9}$$

$$y'(0) = 0, \quad y(1) \geq \delta y(\eta), \tag{9.10}$$

with $y \in [\beta_0(x), \alpha_0(x)]$ satisfies $\|xy'\|_\infty \leq R$.

Proof. We can divide this proof into following three cases:

Case : (i) Suppose that the nature of the solution of nonlinear three point SBVP (9.3)–(9.4) is non monotone throughout the interval. First we consider the interval $(x_0, x] \in [0,1]$, and assume that the slope of the solution at point x_0 is zero, and $y'(x) > 0$, for all $x > x_0$.

Integrating the equation (9.9) from x_0 to x, we get

$$\int_0^{xy'} \frac{ds}{\varphi(s)} \leq \frac{1}{2}.$$

We choose R such that

$$\int_0^{xy'} \frac{ds}{\varphi(s)} \leq \frac{1}{2} < \int_{l_0}^{R} \frac{ds}{\varphi(s)} \leq \int_0^{R} \frac{ds}{\varphi(s)}.$$

This gives $xy'(x) \leq R$.

Now suppose that the slope of the solution at point x_0 is zero, and $y'(x) < 0$, for all $x < x_0$. Following the same analysis (as we did above), we get $-xy'(x) \leq R$.

Case : (ii) Suppose the nature of the solution of nonlinear three point SBVP (9.3)–(9.4) is monotonically increasing throughout the interval, i.e., $y'(x) > 0$ in $(0,1)$, then (by using Mean value Theorem) $\exists$ a $\tau \in (0,1)$, such that

$$y'(\tau) = \frac{y(1) - y(0)}{1 - 0} \leq 2|\Omega_0|,$$

where $\Omega_0 = \max\{|\alpha_0(t)|_\infty, |\beta_0(t)|_\infty\}$.

Integrating equation (9.9) from τ to x and then using the assumption (F_R) we get,

$$\int_0^{xy'} \frac{ds}{\varphi(s)} \leq \frac{1}{2} + \int_0^{l_0} \frac{ds}{\varphi(s)} < \int_0^{R} \frac{ds}{\varphi(s)},$$

which gives $xy'(x) \leq R$.

Similarly, when y, i.e., the solution of nonlinear three point SBVP (9.3)–(9.4) is monotonically decreasing throughout the interval, then we get $-xy'(x) \leq R$. ■

Similarly we can prove the following result.

Lemma 9.4. *If $f(x,y,xy')$ satisfies (F_R), then there exists $R > 0$ such that any solution of*

$$-(xy'(x))' \leq xf(x,y,xy'), \quad 0 < x < 1, \tag{9.11}$$

$$y'(0) = 0, \quad y(1) \leq \delta y(\eta), \tag{9.12}$$

with $y \in [\beta_0(x), \alpha_0(x)]$ satisfies $\|xy'\|_\infty \leq R$.

Theorem 9.1. *Suppose (H_0) holds, the source term $f(x,y,xy')$ satisfies (F_1), (F_2) and (F_3) and there exists $\lambda > 0$ such that $1 > \lambda \geq \max\left\{M_1, M_1 + \frac{N^2}{2} + \frac{N}{2}\sqrt{N^2 + 4M_1}\right\}$ and $(\lambda -$*

$M_1) - N\frac{\lambda}{1-\lambda} \geq 0$, *and for all* $x \in [0,1]$

$$f(x,\beta_0(x),x\beta_0'(x)) - f(x,\alpha_0(x),x\alpha_0'(x)) - \lambda(\beta_0 - \alpha_0) \geq 0,$$

is valid, then the sequences (α_n) *and* (β_n) *defined by (9.5), starting with* α_0 *and* β_0 *as initial guesses, converge uniformly in* $C^1([0,1])$ *to solution* v *and* u *of nonlinear BVP (9.3)–(9.4), such that for all* $x \in [0,1]$ $\beta_0 \leq u \leq v \leq \alpha_0$. *Any solution* $z(x)$ *of (9.3)–(9.4) in* D *satisfies* $u(x) \leq z(x) \leq v(x)$.

Proof. We can easily show that

$$\alpha_0 \geq \alpha_1 \geq \cdots \geq \alpha_n \geq \cdots \geq \beta_n \geq \cdots \geq \beta_1 \geq \beta_0, \tag{9.13}$$

with the help of Proposition 9.1 and 9.2, it is clear that (α_n) and (β_n) are monotone and bounded. Now by using Dini's theorem these sequences converge uniformly. Suppose $\alpha_n \to v$ and $\beta_n \to u$. By using Priori bound and (F_1), we can find that the sequences $(x\alpha_n')$ and $(x\beta_n')$ are equibounded and equicontinuous in $C^1([0,1])$, i.e., there exist uniformly convergent subsequences $(x\alpha_{nk}')$ and $(x\beta_{nk}')$ in $C^1([0,1])$ (Arzela-Ascoli Theorem). It is easy to check that $x\alpha_n' \longrightarrow xv'$ and $x\beta_n' \longrightarrow xu'$, whenever $\alpha_n \to v$ and $\beta_n \longrightarrow u$.

As equation (7.17) represents the solution of (9.5) with $h(x) = f(x,y_n,xy_n) - \lambda y_n$. By taking limit as $n \to \infty$ on both sides of (7.17), we get that v and u are solutions of nonlinear three point SBVPs (9.3)–(9.4). Any solution $z(x)$ in D plays the role of α_0, i.e., $z(x) \leq v(x)$. Similarly we get $z(x) \geq u(x)$. ∎

9.3.2 Well ordered lower and upper solutions $(\alpha_0 \leq \beta_0)$

Lemma 9.5. *Let* $\lambda < 0$, *then Modified Bessel functions* I_0 *and* I_1 *satisfy the following inequality*

$$(\lambda + M_2)I_0(x\sqrt{|\lambda|}) + Nx\sqrt{|\lambda|}I_1(x\sqrt{|\lambda|}) \leq 0,$$

for all $x \in [0,1]$ *if* λ *satisfies*

$$\lambda \leq -M_2 - \frac{N^2}{2} - \frac{N}{2}\sqrt{N^2 + 4M_2}\,,$$

where $M_2, N \in \mathbb{R}^+$.

Proof. When $\lambda < 0$, the Modified Bessel's function I_0 and I_1 satisfy the inequality

$$I_0(x\sqrt{|\lambda|}) \geq I_1(x\sqrt{|\lambda|}),$$

for all $x \in [0,1]$, which gives

$$(\lambda+M_2)I_0(x\sqrt{|\lambda|})+Nx\sqrt{|\lambda|}I_1(x\sqrt{|\lambda|}) \leq \Big((M_2+\lambda)+N\sqrt{|\lambda|}\Big)I_0(x\sqrt{|\lambda|}).$$

For the required inequality, we must have $(M_2+\lambda)+N\sqrt{|\lambda|} \leq 0$, i.e.,

$$\lambda \leq -M_2 - \frac{N^2}{2} - \frac{N}{2}\sqrt{N^2+4M_2}.$$

■

Remark 9.3. *By argument, similar to Remark 9.1, we get* $G'(x,t) \leq 0$ *and* $-xG'(x,t) \leq \lambda G(x,t)$ *for* $x \neq t$.

Lemma 9.6. *Suppose* (H_0') *holds and* $\lambda < 0$ *such that* $\lambda + M_2 \leq 0$, *then for all* $x,t \in [0,1]$, *we have the inequality*

$$(\lambda+M_2)G(x,t)+N\,x\,(sign\;y')\frac{\partial G(x,t)}{\partial x} \geq 0, \quad (x \neq t),$$

whenever $(\lambda+M_2)-N\lambda \leq 0$ *such that* $M_2,N \in \mathbb{R}^+$.

Proof. See the proof of Lemma 9.2, with Remark 9.3. ■

Remark 9.4. *By argument, similar to Remark 9.2, we show that* $\alpha_{n+1} \geq \alpha_n$ *and* $\beta_{n+1} \leq \beta_n$, *in* $\widetilde{D}$.

Proposition 9.3. *Suppose* (H_0') *holds,* f *satisfies* (F_1), (F_2) *and* (F_3) *and there exists* $\lambda < 0$ *such that* $\lambda \leq \min\left\{-M_2, -M_2 - \frac{N^2}{2} - \frac{N}{2}\sqrt{N^2+4M_2}, -\frac{M_2}{1-N}\right\}$, *then the functions* α_n *and* β_n, *satisfy the following relations*

(*a*) $\alpha_{n+1} \geq \alpha_n$, *for all* $n \in \mathbb{N}$, *where* α_n *is lower solution of (9.3)–(9.4),*

(*b*) $\beta_{n+1} \leq \beta_n$, *for all* $n \in \mathbb{N}$, *where* β_n *is an upper solution of (9.3)–(9.4),*

where α_n *and* β_n *are defined recursively by (9.5).*

Proof. See the proof of Proposition 9.1 with Lemma 9.5, Lemma 9.6 and Remark 9.4. ■

Proposition 9.4. *Suppose* (H_0') *holds, the source term* f *satisfies* (F_1), (F_2) *and* (F_3) *and* $\lambda < 0$ *such that* $\lambda \leq \min\{-M_2, -M_2 - \frac{N^2}{2} - \frac{N}{2}\sqrt{N^2+4M_2}, -\frac{M_2}{1-N}\}$, *and for all* $x \in [0,1]$

$$f(x,\beta_0(x),x\beta_0'(x)) - f(x,\alpha_0(x),x\alpha_0'(x)) - \lambda(\beta_0-\alpha_0) \geq 0,$$

is valid. Then for all $n \in \mathbb{N}$, *the functions* α_n *and* β_n *defined by (9.5), satisfy* $\alpha_n \leq \beta_n$.

Proof. Proof is similar to the proof of Proposition 9.2. ■

Lemma 9.7. *If $f(x,y,xy')$ satisfies*

(F_W) *For all $(x,y,xy') \in \widetilde{D}$, $|f(x,y,xy')| \leq \varphi(|xy'|)$; where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and satisfies*

$$\frac{1}{2} < \int_{l_0}^{\infty} \frac{ds}{\varphi(s)},$$

where $l_0 = 2\sup|x\Omega_0(x)|$ and $\Omega_0 = \max\{|\alpha_0(t)|_\infty, |\beta_0(t)|_\infty\}$. Then there exists $R > 0$ such that any solution of

$$-(xy'(x))' \geq xf(x,y,xy'), \quad 0 < x < 1, \tag{9.14}$$

$$y'(0) = 0, \quad y(1) \geq \delta y(\eta), \tag{9.15}$$

with $y \in [\alpha_0(x), \beta_0(x)]$ satisfies $\|xy'\|_\infty \leq R$.

Lemma 9.8. *If $f(x,y,xy')$ satisfies (F_W), then there exists $R > 0$ such that any solution of*

$$-(xy'(x))' \leq xf(x,y,xy'), \quad 0 < x < 1, \tag{9.16}$$

$$y'(0) = 0, \quad y(1) \leq \delta y(\eta), \tag{9.17}$$

with $y \in [\alpha_0(x), \beta_0(x)]$ satisfies $\|xy'\|_\infty \leq R$.

Theorem 9.2. *Suppose (H_0') holds, the source term $f(x,y,xy')$ satisfies (F_1), (F_2) and (F_3) and $\lambda < 0$ be such that $\lambda \leq \min\left\{-M_2, -M_2 - \frac{N^2}{2} - \frac{N}{2}\sqrt{N^2+4M_2}, -\frac{M_2}{1-N}\right\}$, and for all $x \in [0,1]$*

$$f(x,\beta_0(x),x\beta_0'(x)) - f(x,\alpha_0(x),x\alpha_0'(x)) - \lambda(\beta_0 - \alpha_0) \geq 0,$$

is valid, then the sequences (α_n) and (β_n) defined by (9.5), starting with α_0 and β_0 as initial guesses, converge uniformly in $C^1([0,1])$ to solution v and u of nonlinear BVP (9.3)–(9.4), such that for all $x \in [0,1]$ $\alpha_0 \leq v \leq u \leq \beta_0$. Any solution $z(x)$ of (9.3)–(9.4) in $\widetilde{D}$ satisfy $v(x) \leq z(x) \leq u(x)$.

Proof. The proof of this Theorem follows same analysis as we did in Theorem 9.1. ■

9.4 Examples

Numerical Examples are discussed in this section which help us to validate our analytical results, and show that $\exists$ a $\lambda \in \mathbb{R} \setminus \{0\}$ which satisfies the sufficient conditions of Theorem

9.1, and Theorem 9.2. For convenience let

$$\chi = \frac{f(x,\beta_0(x),x\beta_0'(x)) - f(x,\alpha_0(x),x\alpha_0'(x))}{(\beta_0-\alpha_0)}.$$

Example 9.1. *Consider the nonlinear three point SBVP*

$$-y''(x) - \frac{1}{x}y'(x) = \frac{(y(x))^3}{80} + \frac{xy'}{7} + \frac{\sin x}{160}, \tag{9.18}$$

$$y'(0) = 0, \quad y(1) = 3y\left(\frac{1}{4}\right). \tag{9.19}$$

Here solution of nonlinear three point SBVP (9.18)–(9.19) has $\alpha_0 = 1$ and $\beta_0 = -1$ as lower and upper bounds, respectively. This is a reverse ordered case. The nonlinear sources term satisfies the conditions (F_1), (F_2) and (F_3) in domain D. Here Lipschitz constants are computed as $M_1 = \frac{3}{80}$ and $N = \frac{1}{7}$. From figure 9.1 it is clear that we can find out a range of $\lambda > 0$ such that

$$\max\left\{M_1,\ \max\{\chi\},\ M_1 + \frac{N^2}{2} + \frac{N}{2}\sqrt{N^2+4M_1}\right\} < \lambda < y_{0,1}^2,$$

i.e., $0.0771902 \le \lambda < 1$. So that (H_0), and $(\lambda - M_1) - N\frac{\lambda}{1-\lambda} \ge 0$ are satisfied. Thus Theorem 9.1 is applicable here and there exists a solution of (9.18)–(9.19).

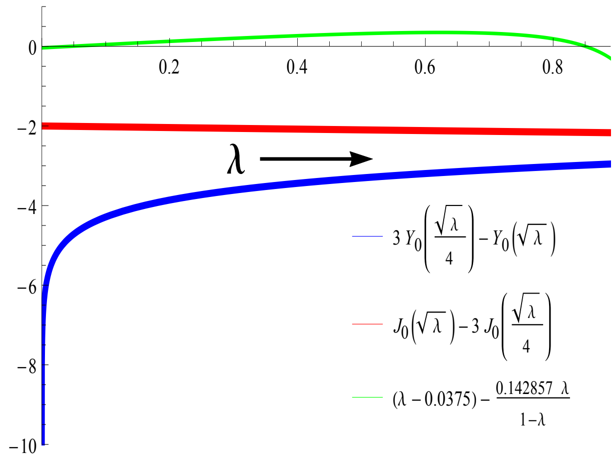

Fig. 9.1 Plot of (H_0) and $(\lambda - M_1) - N\frac{\lambda}{1-\lambda}$ for example 9.1.

Example 9.2. *Consider the nonlinear three point SBVP*

$$-y''(x) - \frac{1}{x}y'(x) = \frac{(e^x)}{100} - \frac{y^3}{30} + \frac{xy'}{5}, \tag{9.20}$$

$$y'(0) = 0, \quad y(1) = 0.6y\left(\frac{2}{5}\right). \tag{9.21}$$

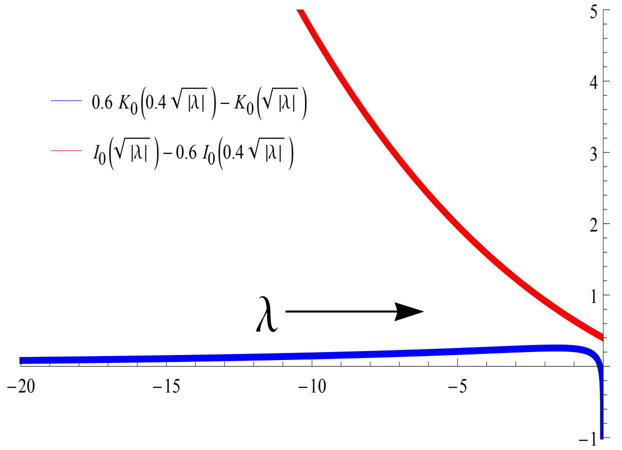

Fig. 9.2 Plot of (H_0') for example 9.2.

Here solution of nonlinear three point SBVP (9.20)–(9.21) has $\alpha_0 = -1$ and $\beta_0 = 1$ as lower and upper bounds, respectively. This is a well ordered case. The nonlinear sources term satisfies the conditions (F_1), (F_2) and (F_3) in domain $\widetilde{D}$. Here Lipschitz constants are computed as $M_2 = \frac{1}{10}$ and $N = \frac{1}{5}$. From figure 9.2, it is clear that we can find out a range of $\lambda < 0$ such that

$$\lambda \leq \min\left\{-M_2, \ -M_2 - \frac{N^2}{2} - \frac{N}{2}\sqrt{N^2 + 4M_2}, \ -\frac{M_2}{1-N}, \ \min\{\chi\}\right\},$$

i.e., $\lambda \leq -0.186332$. So that (H_0'), is satisfied. Thus Theorem 9.2 is applicable here and existence of solution of (9.20)–(9.21) is guaranteed.

9.5 Conclusion

In this chapter we generalize the result of chapter 7 and extend it for the case when source function is derivative dependent, i.e., $f(x, y, xy')$. Two examples validate the results of this chapter.

Chapter 10

Quasi-Newton iteration method and numerical solution of nonlinear SBVPs

10.1 Introduction

In this chapter we compute relaxation parameter for Quasi-Newton's method (see [65, 68, 69, 75, 79, 85, 86, 137, 145, 158, 167]) and find the approximate solutions of the following class of nonlinear SBVPs

$$-(x^{\alpha}y')' = x^{\alpha}f(x,y), \quad 0<x<1, \quad ' \equiv \frac{d}{dx}, \tag{10.1}$$

$$y'(0) = A, \quad a_1y(1)+b_1y'(1) = c_1, \tag{10.2}$$

where α, A, a_1, b_1, c_1 are real constants and $\alpha \geq 1$. Here we assume that $f(x,y)$ is continuous and Lipschitz continuous in $D = \{(x,y) \in [0,1] \times \mathbb{R}\}$.

As far as analytical results are considered enormous literature are available for two point SBVPs (see [38, 50, 125, 126, 135]). Russell and Shampine [135] showed that the above class have unique solution for $\alpha = 1$ if $K < j_0^2$, where j_0 is the first positive zero of Bessel function $J_0(x)$, for $\alpha = 2$, the problem has unique solution if $K < \pi^2$, here K is Lipschitz constant. Chawla and Shivakumar [38] have shown that the SBVP (10.1)–(10.2) has unique solution for all α, if $K = \frac{\partial f}{\partial y} < K_1^2$, where K_1 is the first zero of $J_{\left(\frac{\alpha-1}{2}\right)}(\sqrt{K})$. El-Gebeily and Boumenir [50] have shown that the problem has a unique solution for certain boundary conditions under the assumption that the range of $\frac{\partial f}{\partial y}$ has empty intersection with the closure of the spectrum of the singular differential operator, where f denotes the nonlinearity. Pandey and Verma [125, 126] generalized some of these results for a general class of SBVPs.

The numerical solutions of these SBVP have been discussed by several methods such as cubic spline and B-spline methods [39, 80, 132], mixed decomposition-spline method (MDSM) [84], finite difference method [36, 116, 121, 124], which has been very popular and have several advantages, but needed a lot of computational work.

Iterative methods are preferred over other numerical methods as less computational work is needed and they provide highly accurate approximations or even exact solutions. Recently, researchers have used Adomian decomposition method (ADM), modified Adomian decomposition method (MDM) and Homotopy analysis method (HAM) [1, 2, 43, 48] for non-linear SBVPs.

Literature shows that Quasi-Newton methods also referred as Varitional iteration methods (VIMs) (see [65, 68, 69, 75, 79, 85, 86, 137, 145, 158, 167]) are very efficient for solving nonlinear differential equations. Kanth and Aruna [79] and Wazwaz [158] considered class of SBVPs (10.1)–(10.2) and discussed certain aspects of iterative scheme referred as variational method [86].

In this chapter we propose a modification to Quasi-Newton method and use it to solve a class of nonlinear two point SBVPs (10.1)–(10.2). We generalize the relaxation parameter (λ) and compute it as a function of the variable (ω). The relaxation parameter (λ) is expressed in terms of Bessel and modified Bessel functions. When $\omega = 0$ our results will coincide with the results of Kanth and Aruna [79] and Wazwaz [158]. For positive values of ω our scheme converges faster. We allow $\frac{\partial f}{\partial y}$ to take both positive and negative values.

We have organized this chapter into the following sections. In Section 10.2 we discuss the basic idea of Quasi-Newton iteration method and its convergence and in Section 10.3 we verify our results with suitable test examples. Finally, conclusion are given in Section 10.4.

10.2 The basic idea of Quasi-Newton iteration method

Roots of nonlinear equation $\phi(x) = 0$ can be computed by Newton's method given by

$$x_{n+1} = x_n - \frac{\phi(x_n)}{\phi'(x_n)}.$$

If we replace $\frac{1}{\phi'(x_n)}$ by an approximation (say λ) the resulting method

$$x_{n+1} = x_n - \lambda \phi(x_n)$$

is then referred as Quasi-Newton iteration method. This approximation λ is referred as relaxation parameter. Since x_n is an approximated root $\phi(x_n) \neq 0$, so we look for an optimal value of λ so that the difference $x_{n+1} - x_n$ which is equal to $= \lambda\phi(x_n)$ is minimized. Using this optimal value of λ we generate a sequence (x_n) which converges to a root of the nonlinear equation.

The solution of the differential equation (10.1) is a zero of (10.1). So the above ideas can be used to compute the solution of the (10.1). In next paragraph we discuss some of the preliminary results which uses the above concepts.

Schunk [137] used these concepts, to calculate the bending of cylindrical panels, but his work went unnoticed. Zhukov [167] used this method for thin rectangular slabs. The method was strengthened by Kirichenko and Krys'ko [86], they considered a class of equations which were described by positive definite operators. Inokuti et al. [75] referred the relaxation parameter as Lagrange's parameter and solved the nonlinear equations, which may involve algebraic, differential, integral, or finite difference operators. He [65, 68, 69] has popularized this method and after this several author started referring this method as He's variation iteration method. In this chapter, we consider the following non-linear differential equation

$$L(y) + N(y) = g(x), \tag{10.3}$$

where L, is linear operators and N is non-linear operator, respectively and $g(x)$ is the source term. Using a modified correction functional as suggested by Soltani and Shirzadi [145] for (10.3), we can write the following iterative scheme for $n \geq 0$

$$y_{n+1}(x) = y_n + \int_0^x \lambda\,[L(y_n(t)) - L_1(y_n(t)) + N(\tilde{y}_n(t)) + L_1(\tilde{y}_n(t)) - g(t)]\,dt, \tag{10.4}$$

where L_1 is a linear operator and λ is the relaxation parameter and it is identified optimally with the help of variational theory, $\tilde{y}_n$ is treated as restricted variation, i.e., $\delta\tilde{y}_n = 0$.

10.2.1 Relaxation parameter in terms of Bessel functions

For the nonlinear SBVPs (10.1) we define the following iterative scheme

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda\left(-\ddot{y}_n(t) - \frac{\alpha}{t}\dot{y}_n(t) - \omega y_n(t) - \tilde{f}(t,y_n) + \omega\tilde{y}_n(t)\right)dt, \tag{10.5}$$

where $\dot{} \equiv \frac{d}{dt}$. When $\omega = 0$ this scheme is same as considered in [79, 158]. By taking the variation on both sides of (10.5),

$$\delta y_{n+1}(x) = \delta y_n(x) + \delta\int_0^x \lambda\left(-\ddot{y}_n(t) - \frac{\alpha}{t}\dot{y}_n(t) - \omega y_n(t) - \tilde{f}(t,y_n) + \omega\tilde{y}_n(t)\right)dt,$$

we get

$$\delta y_{n+1}(x) = \delta y_n(x) + \delta \int_0^x \lambda \left(-\ddot{y}_n(t) - \frac{\alpha}{t}\dot{y}_n(t) - \omega y_n(t)\right) dt, \text{ where } \delta \tilde{y}_n = 0.$$

Integrating by parts, we get

$$\delta y_{n+1}(x) = \left(1 + \lambda_x(x) - \frac{\alpha\lambda(x)}{x}\right)\delta y_n(x) \\ - \delta\lambda(x) y_n'(x) - \int_0^x \left(\lambda_{tt} - \alpha\frac{(t\lambda_t - \lambda)}{t^2} + \omega\lambda\right)\delta y_n(t)dt = 0.$$

Hence, we get

$$1 + \lambda_x(x) - \frac{\alpha\lambda(x)}{x} = 0, \tag{10.6}$$

$$\lambda(x) = 0, \tag{10.7}$$

$$-\lambda_{tt}(t) + \alpha\frac{(t\lambda_t(t) - \lambda(t))}{t^2} - \omega\lambda(t) = 0. \tag{10.8}$$

We can write (10.8) as follows

$$t^2\lambda_{tt} - t\alpha\lambda_t + (\alpha + t^2\omega)\lambda = 0. \tag{10.9}$$

The standard Bessel's equation

$$z^2\frac{d^2w}{dz^2} + z\frac{dw}{dz} + (z^2 - \nu^2)w = 0, \tag{10.10}$$

is transformed (Lommel's transformations $z = \beta\zeta^\gamma$, $w = \zeta^{-a}v(\zeta)$) into (10.11)

$$\zeta^2\frac{d^2v}{d\zeta^2} + \zeta(1-2a)\frac{dv}{d\zeta} + \left[(\beta\gamma\zeta^\gamma)^2 + (a^2 - \nu^2\gamma^2)\right]v = 0. \tag{10.11}$$

Now, if we set $\nu = \frac{(1-\alpha)}{2}$, $a = \frac{(1+\alpha)}{2}$, $\gamma = 1$, $\beta^2 = \omega$, $\zeta = t$ then (10.11) is reduced into (10.9). The transformed Bessel's equation (10.11) has two linearly independent solutions, which are defined as

$$v_1(\zeta) = \zeta^a w_1(\beta\zeta^\gamma), \quad v_2(\zeta) = \zeta^a w_2(\beta\zeta^\gamma), \tag{10.12}$$

where $w_1(z)$ and $w_2(z)$ are two linearly independent solutions of the Bessel's equation (10.10). Hence, we obtain two linearly independent solutions of (10.9) in terms of $w_1(z)$ and $w_2(z)$.

The bounded solution of (10.9) is given by $t^{\nu+\alpha}J_{-\nu}\left(t\sqrt{\omega}\right)$, if $\omega > 0$ and $t^{\nu+\alpha}I_{-\nu}\left(t\sqrt{\omega}\right)$, if $\omega < 0$, where $J_{-\nu}$ and $Y_{-\nu}$ are Bessel functions of first and second kind, respectively, and $I_{-\nu}$ and K_{ν} are modified Bessel functions of first and second kind, respectively.

By using the conditions (10.6) and (10.7), we obtain the optimal values of the relaxation parameter. For $\omega > 0$ we get

$$\lambda(t) = \frac{\pi x t^{\nu} t^{\alpha}}{2x^{\nu}x^{\alpha}}\left[J_{-\nu}\left(t\sqrt{\omega}\right)Y_{-\nu}\left(x\sqrt{\omega}\right) - J_{-\nu}\left(x\sqrt{\omega}\right)Y_{-\nu}\left(t\sqrt{\omega}\right)\right], \tag{10.13}$$

and similarly for $\omega < 0$, we get

$$\lambda = \frac{t^{\alpha}t^{\nu}\ x}{x^{\nu}\ x^{\alpha}}\left[I_{-\nu}\left(x\sqrt{|\omega|}\right)K_{\nu}\left(t\sqrt{|\omega|}\right) - I_{-\nu}\left(t\sqrt{|\omega|}\right)K_{\nu}\left(x\sqrt{|\omega|}\right)\right]. \tag{10.14}$$

The successive approximation y_{n+1}, $n \geq 0$ can be computed from the correctional functional (10.5) and the sequence $(y_n(x))$ converges uniformly (will be proved in the next section) to the exact solution (say $y(x)$) of the nonlinear SBVP (10.1) where the initial approximation y_0 may be chosen so that it satisfies at least the initial or boundary conditions.

10.2.2 Convergence analysis

To prove that the limit of the sequence $(y_n(x))$ obtained from (10.5), will converge to the solutions of (10.1)–(10.2), we have to prove that the sequence is convergent. It is clear that

$$y_0(x) + \sum_{i=1}^{n}(y_i - y_{i-1}) = y_n(x) \tag{10.15}$$

is n^{th} partial sum of the infinite series

$$y_0(x) + \sum_{i=1}^{\infty}(y_i - y_{i-1}). \tag{10.16}$$

Therefore to prove that the sequence $(y_n(x))$ converges (uniformly) it is enough to prove that (10.16) converges (uniformly).

Theorem 10.1. *If $\omega > 0$, and for any $n = 0, 1, 2, \cdots$, $y_n(x) \in C^2[0,1]$, and further assume that there exists $N > 0$ such that for all $f(x,u)$, $f(x,v) \in D$*

$$|f(x,u) - f(x,v)| \leq N|u - v|, \tag{10.17}$$

where $D = \{(x,y) \in [0,1] \times \mathbb{R}\}$. Then the sequence defined by (10.15), will converge uniformly to the exact solution of nonlinear SBVP (10.1)–(10.2).

Proof. As from equation (10.5) (for $n = 0$),

$$y_1(x) = y_0(x) - \int_0^x \lambda \left(\ddot{y}_0(t) + \frac{\alpha}{t}\dot{y}_0(t) + \tilde{f}(t,y_0)\right) dt, \quad n \geq 0.$$

Integrating by parts and by using equations (10.6)–(10.8) on the right hand side of above equations, we get

$$|y_1(x) - y_0(x)| = \left| -\int_0^x \left(\left(-\lambda_t(t) + \frac{\alpha\lambda(t)}{t}\right)\dot{y}_0(t) + \lambda\tilde{f}(t,y_0)\right) dt \right| \tag{10.18}$$

$$\leq \int_0^x \left(\left|-\lambda_t(t) + \frac{\alpha\lambda(t)}{t}\right| |\dot{y}_0(t)| + \left|\lambda\tilde{f}(t,y_0)\right|\right) dt. \tag{10.19}$$

Now, from equation (10.5) (for $n = 1$), and by similar analysis, we get

$$|y_2(x) - y_1(x)| = \left| \int_0^x \left(\left(-\lambda_{tt}(t) + \alpha\frac{(t\lambda_t(t) - \lambda)}{t^2}\right)(y_1(t) - y_0(t)) \right.\right.$$
$$\left.\left. -\lambda\left(\tilde{f}(t,y_1) - \tilde{f}(t,y_0)\right)\right) dt \right|, \tag{10.20}$$

$$|y_2(x) - y_1(x)| \leq \left| \int_0^x \lambda\left(\omega(y_1(t) - y_0(t)) - \left(\tilde{f}(t,y_1) - \tilde{f}(t,y_0)\right)\right) dt \right| \tag{10.21}$$

$$\leq \int_0^x |\lambda| \left(|\omega| \, |(y_1(t) - y_0(t))| + \left|\left(\tilde{f}(t,y_1) - \tilde{f}(t,y_0)\right)\right|\right) dt. \tag{10.22}$$

Further, by using Lipschitz condition, we get

$$|y_2(x) - y_1(x)| \leq \int_0^x |\lambda| \left(|(\omega + N)| \, |(y_1(t) - y_0(t))|\right) dt, \tag{10.23}$$

where N is Lipschitz constant. In general, we have

$$|y_{n+1}(x) - y_n(x)| \leq \int_0^x |\lambda| \left(|(\omega + N)| \, |(y_n(t) - y_{n-1}(t))|\right) dt. \tag{10.24}$$

Using series expansion of $J_{-\nu}$, $Y_{-\nu}$, $I_{-\nu}$ and K_ν and Appendix A, we can easily conclude

$$\left|\frac{\lambda}{t}\right| \quad \& \quad \lambda_t$$

are bounded for all $t \leq x \leq 1$ and $\alpha \geq 1$. So we define

$$(M_1)_\infty = \sup\left\{\left|-\lambda_t(t) + \frac{\alpha\lambda(t)}{t}\right| |\dot{y}_0(t)| + \left|\lambda \tilde{f}(t, y_0)\right|\right\}, \tag{10.25}$$

$$(M_2)_\infty = \sup\{|\lambda|\,|(\omega + N)|\}. \tag{10.26}$$

Consider

$$M = \max\{(M_1)_\infty, (M_2)_\infty\}. \tag{10.27}$$

From equations (10.19), (10.25) and (10.27), we get

$$|y_1(x) - y_0(x)| \le \int_0^x (M_1)_\infty dt \le \int_0^x M dt = Mx. \tag{10.28}$$

Similarly from equation (10.23), (10.26) and (10.27)

$$|y_2(x) - y_1(x)| \le \int_0^x (M_2)_\infty |y_1(t) - y_0(t)| dt \le \int_0^x M \times Mt dt = \frac{M^2x^2}{2!}. \tag{10.29}$$

In general

$$\begin{aligned} |y_{n+1}(x) - y_n(x)| &\le \int_0^x (M_2)_\infty |y_n(t) - y_{n-1}(t)| dt \\ &\le \int_0^x M \times \frac{M^n t^n}{n!} dt = \frac{M^{n+1}x^{n+1}}{(n+1)!}, \quad \forall x \in [0,1]. \end{aligned} \tag{10.30}$$

As the series $\sum_{n=0}^{\infty} \frac{M^{n+1}x^{n+1}}{(n+1)!}$ is convergent, $\forall x \in [0,1]$. Therefore the series defined by (10.16)

$$|y_0(x)| + \sum_{i=1}^{\infty} |(y_i(x) - y_{i-1}(x))| \le |y_0(x)| + \sum_{n=0}^{\infty} \frac{M^n x^n}{(n)!}, \tag{10.31}$$

is absolutely convergent, i.e., the sequence of partial sums is convergent for $x \in [0,1]$. Hence by the Weierstrass M-Test

$$|y_0(x)| + \sum_{i=1}^{\infty} |(y_i(x) - y_{i-1}(x))|,$$

converges uniformly $\forall x \in [0,1]$. ■

Similarly, from the convergence analysis for $\omega < 0$, we arrive at the following theorem.

Theorem 10.2. *If $\omega < 0$, and for any $n \in 0,1,2,\cdots$, $y_n(x) \in C^2[0,1]$. Further let us assume that there exists $N > 0$ such that for all $f(x,u)$, $f(x,v) \in D$*

$$|f(x,u) - f(x,v)| \leq N|u-v|. \tag{10.32}$$

Then the sequence defined by (10.15), will converge uniformly to the exact solutions of nonlinear SBVP (10.1)–(10.2).

10.3 Examples

In this section we consider four examples and illustrate that our modified version of Quasi-Newton's method gives better results than that of [79, 158]. In the limiting case $\omega \to 0$ the numerical and analytical results are exactly same as result of [79, 158].

Example 10.1. *Consider the linear singular two point boundary value problem*

$$-y''(x) - \frac{1}{x}y'(x) = y(x) - \frac{5}{4} - \frac{x^2}{16}, \qquad 0 < x < 1, \tag{10.33}$$

$$y'(0) = 0, \quad y(1) = \frac{17}{16}. \tag{10.34}$$

The exact solution of this problem is $y(x) = 1 + \frac{x^2}{16}$. Here $\frac{\partial f}{\partial y} > 0$. Now by using the equation (10.5) we get

$$y_{n+1}(x) = y_n(x) - \int_0^x \lambda \left(\ddot{y}_n(t) + \frac{1}{t}\dot{y}_n(t) + y_n(t) - \frac{5}{4} - \frac{t^2}{16} \right) dt. \tag{10.35}$$

Here λ is given by (10.13). Using the equation (10.35) with initial approximation $y_0(x) = a$, we get the following successive approximations

$$\begin{aligned}
y_0(x) =& a, \\
y_1(x) =& \frac{((4a-5)\omega+1)J_0\left(\sqrt{\omega}x\right)}{4\omega^2} - \frac{16a - x^2 - 20}{16\omega} + a - \frac{1}{4\omega^2}, \\
y_2(x) =& a - \frac{\omega - 1}{2\omega^3} - \frac{1}{4\omega^2} - \frac{(\omega-1)\left(16a - x^2 - 20\right)}{16\omega^2} \\
& - \frac{16a - x^2 - 20}{16\omega} + \frac{\pi(\omega-1)x^2(4a\omega - 5\omega + 1)J_1\left(\sqrt{\omega}x\right)^2 Y_0\left(\sqrt{\omega}x\right)}{16\omega^2} \\
& + J_0\left(\sqrt{\omega}x\right) \left(\frac{\pi(1-\omega)x^2(4a\omega - 5\omega + 1)J_1\left(\sqrt{\omega}x\right) Y_1\left(\sqrt{\omega}x\right)}{16\omega^2} \right.
\end{aligned}$$

$$+\frac{4a\omega^2-2a\omega-5\omega^2+4\omega-1}{2\omega^3}\Bigg),$$

$\vdots$

Table 10.1 Solution (y_1) of example 10.1 for different values of ω.

x/y_1	$\omega=0$ ([79])	$\omega=0.1$	$\omega=0.2$	$\omega=0.3$	$\omega=0.6$	$\omega=0.72$	$\omega=0.9$	Exact
0	0.994792	0.995335	0.995874	0.996407	0.997976	0.998591	0.999501	1
0.1	0.99543	0.995972	0.996509	0.997041	0.998606	0.99922	1.00013	1.000625
0.2	0.99735	0.997887	0.99842	0.998947	1.0005	1.00111	1.00201	1.0025
0.3	1.00057	1.00109	1.00162	1.00213	1.00366	1.00426	1.00514	1.005625
0.4	1.0051	1.00561	1.00612	1.00662	1.0081	1.00868	1.00953	1.01
0.5	1.01099	1.01147	1.01195	1.01243	1.01382	1.01437	1.01518	1.015625
0.6	1.01827	1.01871	1.01915	1.01958	1.02086	1.02136	1.0221	1.0225
0.7	1.02699	1.02737	1.02775	1.02812	1.02922	1.02965	1.03028	1.030625
0.8	1.03723	1.03752	1.0378	1.03809	1.03893	1.03925	1.03974	1.04
0.9	1.04903	1.0492	1.04937	1.04953	1.05001	1.0502	1.05047	1.050625
1	1.0625	1.0625	1.0625	1.0625	1.0625	1.0625	1.0625	1.0625

Table 10.2 Solution (y_2) of example 10.1 for different values of ω.

x/y_2	$\omega=0$ ([79])	$\omega=0.1$	$\omega=0.2$	$\omega=0.3$	$\omega=0.6$	$\omega=0.72$	$\omega=0.9$	Exact
0	1.00014	1.00011	1.00009	1.00007	1.00002	1.00001	1	1
0.1	1.00077	1.00074	1.00071	1.00069	1.00065	1.00064	1.00063	1.000625
0.2	1.00264	1.00261	1.00259	1.00257	1.00252	1.00251	1.0025	1.0025
0.3	1.00576	1.00574	1.00571	1.00569	1.00565	1.00564	1.00563	1.005625
0.4	1.01014	1.01011	1.01009	1.01007	1.01002	1.01001	1.01	1.01
0.5	1.01576	1.01573	1.01571	1.01569	1.01565	1.01564	1.01563	1.015625
0.6	1.02262	1.0226	1.02258	1.02256	1.02252	1.02251	1.0225	1.0225
0.7	1.03074	1.03072	1.0307	1.03068	1.03064	1.03063	1.03063	1.030625
0.8	1.04009	1.04007	1.04006	1.04004	1.04001	1.04001	1.04	1.04
0.9	1.05068	1.05067	1.05066	1.05065	1.05063	1.05063	1.05063	1.050625
1	1.0625	1.0625	1.0625	1.0625	1.0625	1.0625	1.0625	1.0625

Now we find the values of a by imposing the boundary condition $y(1)=\frac{17}{16}$ on the above approximations for different values of ω. The solutions at different space points are displayed in tables 10.1 and 10.2.

Example 10.2. *Consider the following nonlinear two-point SBVP ($\alpha = 2$ and $f(x,y) = y^{\gamma}$), derived by Chandrasekhar [32] where γ is a physical constant, in connection with the equilibrium of isothermal gas spheres. We consider the case of $\gamma = 5$.*

$$-y''(x) - \frac{2}{x}y'(x) = y^5, \qquad 0 < x < 1, \tag{10.36}$$

$$y'(0) = 0, \quad y(1) = \sqrt{\frac{3}{4}}. \tag{10.37}$$

The exact solution of this problem is $y(x) = (1 + \frac{x^2}{3})^{-\frac{1}{2}}$. Here $\frac{\partial f}{\partial y} > 0$. Now by using the equation (10.5), we get

$$y_{n+1}(x) = y_n(x) - \int_0^x \lambda \left(\ddot{y}_n(t) + \frac{1}{t}\dot{y}_n(t) + (y_n(t))^5 \right) dt, \tag{10.38}$$

where λ is defined by (10.13). Using the equation (10.38) with initial approximation $y_0(x) = a$, we get the following successive approximations (to save some space we do not write $y_2(x)$)

$$\begin{aligned} y_0(x) =& a, \\ y_1(x) =& a - \frac{a^5 \left(\sqrt{\omega}x - \sin\left(\sqrt{\omega}x \right) \right)}{\omega^{3/2} x}, \\ & \vdots \end{aligned}$$

Now we find the values of a by imposing the boundary condition $y(1) = \sqrt{\frac{3}{4}}$ on the above approximations for different values of ω. The solutions at different space points are displayed in table 10.3.

Example 10.3. *Consider the nonlinear two point SBVP [84],*

$$-y''(x) - \frac{1}{x}y'(x) = e^y, \qquad 0 < x < 1, \tag{10.39}$$

$$y'(0) = 0, \quad y(1) = 0. \tag{10.40}$$

The exact solution of this problem is $y(x) = 2\ \ln\left(\frac{C+1}{C\,x^2+1} \right)$, where $C = 3 - 2\sqrt{2}$. Here $\frac{\partial f}{\partial y} > 0$. Now by using the equations (10.5), we get

$$y_{n+1}(x) = y_n(x) - \int_0^x \lambda \left(\ddot{y}_n(t) + \frac{1}{t}\dot{y}_n(t) + e^{y_n(t)} \right) dt. \tag{10.41}$$

Table 10.3 Solution (y_2) of example 10.2 for different values of ω.

x/y_2	$\omega = 0$ ([79])	$\omega = 0.1$	$\omega = 0.2$	$\omega = 1$	$\omega = 2$	$\omega = 2.3$	Exact
0	0.993678	0.993989	0.994293	0.996453	0.998454	0.9989	1
0.1	0.992067	0.992376	0.992677	0.994819	0.996804	0.997247	0.998337488
0.2	0.987282	0.987583	0.987878	0.989967	0.991904	0.992336	0.993399268
0.3	0.979461	0.97975	0.980032	0.982038	0.983896	0.98431	0.985329278
0.4	0.968827	0.969099	0.969366	0.971256	0.973006	0.973397	0.974354704
0.5	0.955679	0.95593	0.956176	0.95792	0.959533	0.959892	0.960768923
0.6	0.940377	0.940602	0.940822	0.942382	0.94382	0.94414	0.944911183
0.7	0.923325	0.923517	0.923704	0.925027	0.926243	0.926512	0.927145541
0.8	0.904958	0.905104	0.905248	0.906258	0.90718	0.907382	0.907841299
0.9	0.885714	0.885799	0.885883	0.886468	0.886997	0.887112	0.887356509
1	0.866025	0.866025	0.866025	0.866025	0.866025	0.866025	0.866025404

Table 10.4 Solution (y_1) of example 10.3 for different values of ω.

x/y_1	$\omega = 0$	$\omega = 0.2$	$\omega = 0.4$	$\omega = 0.6$	$\omega = 0.7$	$\omega = 0.78$	Exact
0	0.357403	0.350549	0.343887	0.337407	0.334233	0.331724	0.316694
0.1	0.353829	0.347	0.340362	0.333905	0.330742	0.328242	0.313266
0.2	0.343107	0.336358	0.329797	0.323415	0.320288	0.317817	0.303015
0.3	0.325237	0.318639	0.312224	0.305984	0.302927	0.30051	0.286047
0.4	0.300218	0.293869	0.287696	0.28169	0.278748	0.276422	0.262531
0.5	0.268052	0.262086	0.256285	0.250643	0.247879	0.245694	0.232697
0.6	0.228738	0.223337	0.218087	0.212982	0.210481	0.208504	0.196827
0.7	0.182276	0.177681	0.173216	0.168875	0.16675	0.16507	0.155248
0.8	0.128665	0.125185	0.121805	0.118521	0.116914	0.115645	0.108323
0.9	0.0679066	0.0659284	0.0640089	0.0621457	0.0612344	0.0605148	0.0564386
1	-5.55112E-17	-1.11022E-16	1.11022E-16	-5.55112E-17	5.55112E-17	0	0

Here λ is defined by (10.13). Using the equation (10.41) with initial approximation $y_0(x) = a$, we get the successive approximations. Since expressions are lengthy we are not mentioning these here. We find the values of a by imposing the boundary condition $y(1) = 0$ on the above approximations for different values of ω. The solutions at different space points are displayed in table 10.4.

Example 10.4. [1] *Consider the following nonlinear two-point SBVP which occurs in diffusion problems with Michaelis-Menten kinetics ([13]),*

$$y''(x) + \frac{2}{x}y'(x) = \frac{ny}{y+k}, \quad 0 < x < 1, \tag{10.42}$$

$$y'(0) = 0, \quad 5y(1) + y'(1) = 5, \tag{10.43}$$

where $n = 0.76129$ *and* $k = 0.03119$.

Here $\frac{\partial f}{\partial y} < 0$. Now by using the equations (10.5), we get

$$y_{n+1}(x) = y_n(x) - \int_0^x \lambda \left(\ddot{y}_n(t) + \frac{2}{t}\dot{y}_n(t) - \frac{0.76129\, y}{y + 0.03119} \right) dt, \tag{10.44}$$

where λ is defined by (10.14). Using the equation (10.44) with initial approximation $y_0(x) = a$, we get the following successive approximations

$$\begin{aligned} y_0(x) =& a, \\ y_1(x) =& a - \frac{\frac{0.76129ax}{\omega} - \frac{0.76129a\left(\sinh\left(\sqrt{\omega}x\right)\right)}{\omega^{3/2}}}{ax + 0.03119x}, \\ &\vdots \end{aligned}$$

Now we compute the values of a by using the boundary condition $y(1) = 0$ on the above approximations for different values of ω. The solutions at different space points are displayed in table 10.5. The exact solution of this problem is not available, so making use of absolute residual error, we show the efficiency of our technique and show how well the approximate solution satisfies nonlinear SBVP (10.42)–(10.43)

$$R_\omega = \left| y''(x) + \frac{2}{x}y'(x) - \frac{ny}{y+k} \right|,$$

where $n = 0.76129$ and $k = 0.03119$.

10.4 Conclusion

In this chapter we have shown that introduction of parameter ω (correction term) in iterative scheme greatly influence the convergence of the solution. In Tables 10.1, 10.2, 10.3 and 10.4,

[1] Exact solution is not known.

Table 10.5 Solution (y_1) of example 10.4 for different values of ω.

x/y_1	$\omega = -3$	$\omega = -1$	$\omega = -0.5$	$\omega = -0.1$	$\omega = 0$	$\omega = 0$ ([79])
0	0.793101	0.817568	0.823268	0.827713	0.828808	0.828808024
0.1	0.794324	0.818791	0.824491	0.828935	0.830031	0.830030824
0.2	0.798014	0.822467	0.828163	0.832605	0.833699	0.833699224
0.3	0.804238	0.828618	0.834295	0.838722	0.839813	0.839813223
0.4	0.813109	0.83728	0.842906	0.847292	0.848373	0.848372822
0.5	0.824787	0.848507	0.854022	0.858319	0.859378	0.85937802
0.6	0.839486	0.862366	0.867676	0.871811	0.872829	0.872828818
0.7	0.857475	0.87894	0.883909	0.887774	0.888725	0.888725216
0.8	0.879086	0.89833	0.902769	0.906219	0.907067	0.907067213
0.9	0.90472	0.920653	0.924315	0.927156	0.927855	0.92785481
1	0.934858	0.946046	0.948611	0.9506	0.951088	0.951088007

Table 10.6 Absolute residual errors of example 10.4 for different values of ω.

x/R_ω	R_{-3}	R_{-1}	$R_{0.5}$	$R_{0.1}$	R_0
0	0	0	0	0	0
0.1	0.00362525	0.00118256	0.000571692	8.29781E-05	3.92021E-05
0.2	0.0145672	0.00473802	0.00228916	0.00033263	0.000156143
0.3	0.0330256	0.0106898	0.00515958	0.000751083	0.000348856
0.4	0.0593387	0.0190771	0.00919487	0.0013418	0.000614153
0.5	0.0939897	0.0299551	0.0144116	0.00210944	0.000947756
0.6	0.137618	0.0433951	0.020831	0.00305974	0.00134446
0.7	0.191033	0.0594853	0.0284788	0.0041993	0.00179834
0.8	0.255232	0.078331	0.0373853	0.00553538	0.00230293
0.9	0.331422	0.100055	0.0475849	0.00707574	0.00285147
1	0.421043	0.1248	0.0591165	0.00882843	0.00343707

we have shown that when $\omega = 0$ our results are same as results in [79, 158] and for $\omega > 0$ the results are improved and are getting closer and closer to exact solutions. In tables 10.1 and 10.2 we have taken values of ω up to 0.72 which is less than square of first positive zeros of respective Bessel functions (see [135]) and we have also taken value of $\omega = 0.9$ which is greater that square of first positive zeros of respective Bessel functions (see [50]). In table 10.5 due the absence of exact results we compare the results with the results given in [79]. This table also shows that when value of ω is increasing the results are better (see Table 10.6).

Chapter 11

VIM coupled with HPM for a class of nonlinear SBVPs

11.1 Introduction

We consider the following class of nonlinear two point singular boundary value problems (SBVPs)

$$\left.\begin{aligned} & -u'' - \frac{\alpha}{x}u' = f(x,u), \quad 0 < x < 1, \quad ' \equiv \frac{d}{dx}, \\ & u'(0) = B, \quad a_1 u(1) + b_1 u'(1) = c_1, \end{aligned}\right\} \tag{11.1}$$

where α, B, a_1, b_1, c_1 are real constants and $\alpha \geq 1$. We assume that $f(x,u)$ is continuous and Lipschitz in $D = \{(x,u) \in [0,1] \times \mathbb{R}\}$. Extensive literature is available for both analytical ([38, 125, 126, 135, 140] and the references therein) and numerical results ([1, 3, 36, 48, 116, 144, 148, 157] and the references therein).

The prime motive of this chapter is to derive an effective numerical technique for a class of SBVP (11.1). The proposed technique is based on the concept of variational iteration method (VIM) [65, 69, 79, 158, 159] coupled with homotopy perturbation method (HPM) [66, 67]. It gives approximate solution in the form of a series. To increase the accuracy of the solution obtained by our technique we can compute more number of terms which is otherwise difficult. The convergence analysis and the error estimate of the proposed method are also discussed.

In VIM [79], an iterative scheme for nonlinear SBVPs

$$L(u) + N(u) = g(x) \tag{11.2}$$

is defined as

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(t)\,[Lu_n(t) + Nu_n(t) - g(t)]\,dt$$

where L is a linear differential operator, N is a nonlinear operator, and $g(x)$ is an inhomogeneous term. It is easy to see that we will get the best solution if

$$\int_0^x \lambda(t)\,[Lu_n(t) + Nu_n(t) - g(t)]\,dt,$$

is minimized. For minimization we use homotopy perturbation method [67].

11.2 Homotopy perturbation method (HPM)

Actually, homotopy perturbation method (HPM) is combination of homotopy analysis and perturbation method, which mainly removes the restrictions on small parameter for perturbation methods. Homotopy plays an important role in differential topology, which is basically used to solve the nonlinear algebraic equations. In this analysis, a homotopy $\mathscr{H} : [0,1] \times \mathbb{R} \to \mathbb{R}$ (see [94] and the references there in)

$$\mathscr{H}(x,p) = pf(x) + (1-p)(x-a) = 0,\ x \in \mathbb{R},\ p \in [0,1],$$

is constructed for nonlinear algebraic equation $f(x) = 0$, where p is an imbedding parameter and $x - a = 0$ is a simple algebraic equation. It is clear that, when we vary p from 0 to 1, the homotopy $\mathscr{H}(x,p)$ is varied from, $(x-a)$ to $f(x)$, i.e., at $p = 1$, we get the solution of nonlinear algebraic equation $f(x) = 0$. This process is called *deformation*, and we say that $(x-a)$ & $f(x)$ are homotopic.

By using the homotopy analysis [94] and elimination of small parameter ([67] and the references there in), He [67] proposed a new perturbation method for nonlinear differential equations

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{11.3}$$

with boundary conditions

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma,$$

here $A \equiv L + N$ is a general differential operator, where L and N are linear and nonlinear differential operators, respectively. B is a boundary operator, $f(r)$ is a known analytic function and Γ is the boundary of the domain Ω. So, equation (11.3) can be written as

$$L(u) + N(u) - f(r) = 0,\ \ r \in \Omega.$$

Now we employ the ideas of homotopy analysis and construct the homotopy $v(r,p):\Omega\times[0,1]\to\mathbb{R}$, which satisfies

$$H(v,r,p)=(1-p)[L(v)-L(u_0)]+p[A(v)-f(r)]=0,\ \ p\in[0,1],\ r\in\Omega,$$

which is equivalent to

$$H(v,r,p)=L(v)-L(u_0)+pL(u_0)+p[N(v)-f(r)]=0, \tag{11.4}$$

where $p\in[0,1]$ is an embedding parameter, and u_0 is an initial guess of (11.3), satisfying the initial conditions. From (11.4), we have

$$H(v,r,0)=L(v)-L(u_0)=0, \tag{11.5}$$

$$H(v,r,1)=L(v)+N(v)-f(r)=0. \tag{11.6}$$

It is clear from (11.5) and (11.6), that $L(v)-L(u_0)$ and $L(v)+N(v)-f(r)$ are homotopic.

Now, we introduce perturbation and take p as a small parameter. We expand the solution of equation (11.4) as a power series of p given by

$$v=v_0+pv_1+p^2v_2+\cdots, \tag{11.7}$$

where $v_i,\ i=0,1,2,\cdots$ are unknowns to be determined. At $p=1$, we obtain the approximate solution of equation (11.3) given by

$$v(r,1)=u(r)=v_0+v_1+v_2+\cdots.$$

Now, we write the nonlinear term in integral powers of parameter p given as

$$N(v)=\sum_{i=0}^{\infty}p^iH_i=H_0+pH_1+p^2H_2+\cdots, \tag{11.8}$$

where H_n's are defined as

$$H_n(v_0,\cdots,v_n)=\frac{1}{n!}\frac{\partial^n}{\partial p^n}N\left(\sum_{i=0}^{n}p^iv_i\right)\Bigg|_{p=0},\ \ n=0,1,2\cdots. \tag{11.9}$$

In literature H_n's are also known as He's polynomial [54].

Finally, we substitute (11.7) and (11.8) into equation (11.4), collect coefficients of different powers of p and equating them to zero, we get

$$\begin{aligned}
&p^0 : L(v_0) - L(u_0) = 0,\\
&p^1 : L(v_1) + L(u_0) + H_0 - f(r) = 0,\\
&p^2 : L(v_2) + H_1 = 0,\\
&p^3 : L(v_3) + H_2 = 0,\\
&\vdots\\
&p^{n+1} : L(v_{n+1}) + H_n = 0,\\
&\vdots
\end{aligned}$$

Now, using the above system of equations we compute $v_i,\ i = 0, 1, 2, \cdots$ and $\sum_{i=0}^{\infty} v_i$ to get the solution $v(r, 1) = u(r)$ of the nonlinear equation (11.3).

11.2.1 Variational iteration method (VIM)

The iterative scheme for nonlinear SBVP (11.1) is given by (see [65])

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda \left(-\ddot{u}_n(t) - \frac{\alpha}{t}\dot{u}_n(t) - \tilde{f}(t, u_n) \right) dt, \tag{11.10}$$

where $\dot{} \equiv \frac{d}{dt}$.

Following the analysis of [79], we arrive at stationary conditions given by

$$1 + \lambda_x(x) - \frac{\alpha\lambda(x)}{x} = 0, \tag{11.11}$$

$$\lambda(x) = 0, \tag{11.12}$$

$$-\lambda_{tt}(t) + \alpha\frac{(t\lambda_t(t) - \lambda(t))}{t^2} = 0. \tag{11.13}$$

By using the stationary conditions, the value of the Lagrange multipliers can be easily obtained. It is given as follows

$$\lambda(t) = \begin{cases} \dfrac{t^\alpha}{x^{\alpha-1}(1-\alpha)} - \dfrac{t}{(1-\alpha)}, & \forall\ \alpha > 1, \\ \lim\limits_{\alpha\to 1}\left[\dfrac{t^\alpha}{x^{\alpha-1}(1-\alpha)} - \dfrac{t}{(1-\alpha)}\right], & \text{for } \alpha = 1. \end{cases}$$

11.3 HPM & VIM

In this section we combine the two techniques HPM and VIM together and solve a class of nonlinear two point singular boundary value problems. Here we construct homotopy with the help of VIM. We follow an intuitive route [72] for the construction of the homotopy.

To get the solution of the nonlinear SBVP (11.1), we couple the concept of HPM with VIM, i.e., we consider the following homotopy (Appendix B) for equation (11.10)

$$H(x,v,p) = (1-p)[u_0 - v] + p\int_0^x \lambda(t)[Lv(t) + Nv(t) - g(t)]dt = 0, \tag{11.14}$$

where $p \in [0,1]$ is the embedding parameter, and $u_0(x)$ is the initial guess satisfying the initial conditions. It follows from (11.14) that

$$H(x,v,0) = v - u_0 = 0, \tag{11.15}$$
$$H(x,v,1) = \int_0^x \lambda(t)[Lv(t) + Nv(t) - g(t)]dt = 0. \tag{11.16}$$

As embedding parameter p is varied from 0 to 1, $v(p,x)$ changes from $u_0(x)$ to the best approximation of equation (11.10).

We expand $v(p,x)$ in a power series of p, where we take p as a small parameter,

$$v(p,x) = \sum_{i=0}^{\infty} p^i v_i. \tag{11.17}$$

At $p = 1$, we get the best solution of nonlinear differential equation (11.2),

$$v(1,x) = u(x) = \sum_{i=0}^{\infty} v_i. \tag{11.18}$$

Now using He's polynomials, we decompose the nonlinear term (see equation (11.8)), i.e.,

$$N(v) = \sum_{i=0}^{\infty} p^i H_i. \tag{11.19}$$

Substituting (11.17) and (11.19) into (11.14), and comparing the coefficients of same powers of p we get

$$p^0 : v_0 = u_0(x)$$
$$p^1 : v_1 = \int_0^x \lambda(t)[Lv_0 + H_0 - g(t)]dt,$$

$$
\begin{aligned}
&p^2: v_2 = v_1 + \int_0^x \lambda(t)[Lv_1 + H_1]dt,\\
&\vdots\\
&p^{n+1}: v_{n+1} = v_n + \int_0^x \lambda(t)[Lv_n + H_n]dt,\\
&\vdots
\end{aligned}
$$

We solve these set of equations to obtain the series solution

$$
u = \lim_{n\to\infty} u_n = \sum_{i=0}^{\infty} v_i \tag{11.20}
$$

where

$$
u_n = \sum_{i=0}^{n} v_i.
$$

Additionally, for the nonlinear SBVP (11.1), we choose $u_0 = A$, where $A = \sum_{i=0}^{\infty} A_i p^i$, where p is a small parameter. Making use of this initial approximation, we can write the equation (11.14) as

$$
(1-p)[A-v] + p\int_0^x \lambda(t)[Lv(t) + Nv(t) - g(t)]dt = 0,
$$

where $A = \sum_{i=0}^{\infty} A_i p^i$ and $v = \sum_{i=0}^{\infty} v_i p^i$.

Now by collecting the coefficients of different powers of p and equate them to zero, we get

$$
\left.
\begin{aligned}
&p^0: \quad v_0 = u_0\\
&p^1: \quad v_1 = v_0 + (A_1 - A_0) + \int_0^x \lambda(t)[Lv_0(t) + H_0(t) - g(t)]dt,\\
&p^2: \quad v_2 = v_1 + (A_2 - A_1) + \int_0^x \lambda(t)[Lv_1(t) + H_1(t)]dt,\\
&\qquad\qquad \vdots\\
&p^{n+1}: \quad v_{n+1} = v_n + (A_{n+1} - A_n) + \int_0^x \lambda(t)[Lv_n(t) + H_n(t)]dt,\\
&\qquad\qquad \vdots
\end{aligned}
\right\} \tag{11.21}
$$

We use equations labled as equation (11.21) to compute our solution.

11.4 Accuracy and efficiency

The accuracy and efficiency of proposed technique are discussed in this section. Here, we study the existence and uniqueness of the solution of SBVP (11.1) and examine the convergence analysis and error estimate for the proposed technique.

We consider the norm

$$\|u\| = \max_{0 \le x \le 1} |u(x)|, \quad u \in \mathbb{X},$$

where $\mathbb{X} = C[0,1]$ is a Banach space.

Further assume that there exists $N_0 > 0$ such that for all $f(x,y)$, $f(x,z) \in D$

$$|f(x,y) - f(x,z)| \le N_0 |y - z|, \tag{11.22}$$

where $D = \{(x,y) \in [0,1] \times \mathbb{R}\}$.

11.4.1 Existence and uniqueness of solutions

Theorem 11.1. *The nonlinear singular boundary value problem (11.1) where $f(x,u)$ satisfies the Lipschitz condition (11.22) and $N_0 < 2(1+\alpha)$, has a unique solution.*

Proof. Let y_1 and y_2 be two distinct solutions of nonlinear SBVP (11.1), so they will satisfy the equation (11.16) , i.e.,

$$\int_0^x \lambda(t)[y_1''(t) + \frac{\alpha}{t} y_1'(t) + f(t,y_1)]dt = 0, \tag{11.23}$$

where $L = -\frac{d^2}{dt^2} - \frac{\alpha}{t}\frac{d}{dt}$, $N = -f(t,\_)$ and $g(t) = 0$. Similarly we can define it for y_2.

Now, integration by part and stationary conditions (11.11)–(11.13), yield

$$y_1(x) = v_0 - \int_0^x \lambda(t) f(t,y_1) dt. \tag{11.24}$$

Similarly,

$$y_2(x) = v_0 - \int_0^x \lambda(t) f(t,y_2) dt. \tag{11.25}$$

Making use of equations (11.24)–(11.25), we get

$$\begin{aligned} |y_1 - y_2| &= \left| \int_0^x \lambda(t) \left[f(t,y_1) - f(t,y_2) \right] dt \right|, \\ \max_{0 \le x \le 1} |y_1 - y_2| &= \max_{0 \le x \le 1} \left| \int_0^x \lambda(t) \left[f(t,y_1) - f(t,y_2) \right] dt \right| \end{aligned}$$

$$
\begin{aligned}
&= \max_{0\le t\le 1} |f(t,y_1)-f(t,y_2)| \max_{0\le x\le 1}\left|\int_0^x \lambda(t)dt\right|,\\
\|y_1-y_2\| &\le N_0\|y_1-y_2\| \max_{0\le x\le 1}\left|\frac{x^2}{2+2\alpha}\right|.
\end{aligned}
$$

Hence, we have

$$
\|y_1-y_2\| \le \gamma \|y_1-y_2\|,
$$

where $\gamma = \frac{N_0}{2+2\alpha} < 1$. This gives that $y_1 = y_2$. Hence, the theorem is proved. ■

11.4.2 Convergence analysis

Now, to show the convergence of proposed technique, we use equations (11.10) and stationary conditions (11.11)–(11.13), and deduce,

$$
u_{n+1} = u_n - \int_0^x \left(\left(-\lambda_t(t) + \frac{\alpha\lambda(t)}{t}\right)\dot{u}_n(t) + \lambda \tilde{f}(t,u_n)\right) dt. \tag{11.26}
$$

Similarly, we have

$$
u_n = u_{n-1} - \int_0^x \left(\left(-\lambda_t(t) + \frac{\alpha\lambda(t)}{t}\right)\dot{u}_{n-1}(t) + \lambda \tilde{f}(t,u_{n-1})\right) dt. \tag{11.27}
$$

Now,

$$
\begin{aligned}
|u_{n+1}-u_n| &= \left|\int_0^x \left(\left(-\lambda_{tt}(t) + \alpha\frac{(t\lambda_t(t)-\lambda)}{t^2}\right)(u_n-u_{n-1})\right.\right.\\
&\qquad \left.\left. + \lambda(t)\left(\tilde{f}(t,u_n)-\tilde{f}(t,u_{n-1})\right)\right) dt\right|\\
&= \left|\int_0^x \left(\lambda(t)\left(\tilde{f}(t,u_n)-\tilde{f}(t,u_{n-1})\right)\right) dt\right|,
\end{aligned}
$$

or

$$
\begin{aligned}
\max_{0\le x\le 1}|u_{n+1}-u_n)| &= \max_{0\le x\le 1}\left|\int_0^x \left(\lambda(t)\left(\tilde{f}(t,u_n)-\tilde{f}(t,u_{n-1})\right)\right) dt\right|\\
&\le \max_{0\le t\le 1}|\tilde{f}(t,u_n)-\tilde{f}(t,u_{n-1})| \max_{0\le x\le 1}\left|\int_0^x \lambda(t)dt\right|.
\end{aligned}
$$

As f satisfies the Lipschitz condition, so we get

$$
\|u_{n+1}-u_n\| \le N_0 \max_{0\le t\le 1}|u_n-u_{n-1}| \max_{0\le x\le 1}\left|\frac{x^2}{2+2\alpha}\right|
$$

$$\leq \gamma \|u_n - u_{n-1}\|.$$

Hence, we have

$$\|u_{n+1} - u_n\| \leq \gamma \|u_n - u_{n-1}\|, \tag{11.28}$$

where $\gamma < 1$.

Theorem 11.2. *Let $v_n(x), u_n(x) \in \mathbb{X}$ and further we assume that $\|v_0\|$ is a finite, then we have $\|v_{n+1}\| \leq \gamma \|v_n\|, \quad \gamma < 1$, for $n = 0, 1, 2, \cdots$, and the sequence $(u_n = \sum_{i=0}^{n} v_i)$ converges to the solution of SBVP (11.1).*

Proof. As (u_n) is the sequence of partial sum of the series (11.20), i.e.,

$$\begin{aligned}
&u_1 = v_0 + v_1, \\
&u_2 = v_0 + v_1 + v_2, \\
&\vdots \\
&u_n = v_0 + v_1 + v_2 + \cdots + v_n, \\
&\vdots
\end{aligned}$$

which gives

$$v_{n+1} = u_{n+1} - u_n, \quad n = 1, 2, 3, \cdots.$$

Now with the help of (11.28), we can write

$$\|v_{n+1}\| = \|u_{n+1} - u_n\| \leq \gamma \|u_n - u_{n-1}\| = \gamma \|v_n\|.$$

Hence, we obtain

$$\|u_{n+1} - u_n\| = \|v_{n+1}\| \leq \gamma \|v_n\| \leq \gamma^2 \|v_{n-1}\| \leq \cdots \leq \gamma^{n+1} \|v_0\|.$$

To show the convergence of the sequence (u_n), we use Cauchy criterion

$$\begin{aligned}
\|u_n - u_m\| &= \|(u_n - u_{n-1}) + (u_{n-1} - u_{n-2}) + \cdots + (u_{m+1} - u_m)\| \\
&\leq \|(u_n - u_{n-1})\| + \|(u_{n-1} - u_{n-2})\| + \cdots + \|(u_{m+1} - u_m)\| \\
&\leq \gamma^n \|v_0\| + \gamma^{n-1} \|v_0\| + \cdots + \gamma^{m+1} \|v_0\| \\
&\leq \gamma^{m+1} \left[1 + \gamma + \gamma^2 + \cdots + \gamma^{n-m-1}\right] \|v_0\| \\
&\leq \frac{\gamma^{m+1} (1 - \gamma^{n-m})}{1 - \gamma} \|v_0\|.
\end{aligned}$$

As $0 < \gamma < 1$, we have

$$\|u_n - u_m\| \leq \frac{\gamma^{m+1}}{1-\gamma} \|v_0\|. \tag{11.29}$$

Now taking limit $m \to \infty$, we get

$$\|u_n - u_m\| \to 0.$$

Thus, the sequence (u_n) is a Cauchy sequence in Banach space $\mathbb{X}$, so the series $\sum_{i=0}^{n} v_i$ is convergent. ∎

11.4.3 Error estimate

Theorem 11.3. *The maximum absolute truncation error in the computation of the series solution (11.20) of SBVP (11.1) is given by*

$$\max_{0 \leq x \leq 1} \left| u(x) - \sum_{i=0}^{m} v_i \right| \leq \frac{\gamma^{m+1}}{1-\gamma} \|v_0\|.$$

Proof. From inequality (11.29), we have

$$\|u_n - u_m\| \leq \frac{\gamma^{m+1}}{1-\gamma} \|v_0\|,$$

where $n \geq m$. If we fix m and varies $n \to \infty$, then we get

$$\max_{0 \leq x \leq 1} \left| u(x) - \sum_{i=0}^{m} v_i \right| \leq \frac{\gamma^{m+1}}{1-\gamma} \|v_0\|.$$

This completes the proof. ∎

11.5 Real life examples

11.5.1 Problem 1 : Equilibrium of isothermal gas sphere

Chandrashekhar [32] derived the following nonlinear two point SBVP ($\alpha = 2$ and $f(x,u) = u^{\gamma}$), where γ is a physical constant. Here we consider the case, when $\gamma = 5$.

$$-u''(x) - \frac{2}{x}u'(x) = u^5, \quad 0 < x < 1, \tag{11.30a}$$

$$u'(0) = 0, \quad u(1) = \sqrt{\frac{3}{4}}. \tag{11.30b}$$

The exact solution of this problem is $u(x) = (1+\frac{x^2}{3})^{-\frac{1}{2}}$.

Using (11.21) and (11.30), we obtain the values of $v_1, v_2, \cdots$ as

$$\left.\begin{aligned}
v_0 &= A_0, \\
v_1 &= A_1 - \frac{1}{6}A_0^5x^2, \\
v_2 &= \frac{1}{24}A_0^9x^4 - \frac{5}{6}A_1A_0^4x^2 + A_2, \\
v_3 &= -\frac{5}{432}A_0^{13}x^6 + \frac{3}{8}A_1A_0^8x^4 - \frac{5}{6}A_2A_0^4x^2 - \frac{5}{3}A_1^2A_0^3x^2 + A_3, \\
v_4 &= \frac{35A_0^{17}x^8}{10368} - \frac{65}{432}A_1A_0^{12}x^6 + \frac{3}{8}A_2A_0^8x^4 + \frac{3}{2}A_1^2A_0^7x^4 \\
&\quad - \frac{5}{6}A_3A_0^4x^2 - \frac{10}{3}A_1A_2A_0^3x^2 - \frac{5}{3}A_1^3A_0^2x^2 + A_4, \\
v_5 &= -\frac{7A_0^{21}x^{10}}{6912} + \frac{595A_1A_0^{16}x^8}{10368} - \frac{65}{432}A_2A_0^{12}x^6 - \frac{65}{72}A_1^2A_0^{11}x^6 \\
&\quad \frac{3}{8}A_3A_0^8x^4 + 3A_1A_2A_0^7x^4 + \frac{7}{2}A_1^3A_0^6x^4 - \frac{5}{6}A_4A_0^4x^2 - \frac{5}{3}A_2^2A_0^3x^2 \\
&\quad - \frac{10}{3}A_1A_3A_0^3x^2 - 5A_1^2A_2A_0^2x^2 - \frac{5}{6}A_1^4A_0x^2 + A_5, \\
&\quad \vdots
\end{aligned}\right\} \quad (11.31)$$

We have also computed other components, but due to lack of space we have not listed.

Using boundary conditions of (11.30), we get the values of A_i (Table 11.1).

Table 11.1 Numerical values of A_i, where $i = 0, 1, 2, \cdots$

A_0=0.866025,	A_5=0.00329187,	A_{10}=0.000291152
A_1=0.0811899,	A_6=0.00191103,	A_{11}=0.000190205
A_2=0.0266404,	A_7=0.00115173,	A_{12}=0.000125714
A_3=0.0117741,	A_8=0.000713747,	A_{13}=0.00008391
A_4=0.00597904,	A_9=0.000451962.	A_{14}=0.0000564804.

Hence, by using (11.31) and Table 11.1, we can write an approximate series solutions containing 15-terms i.e., $u = \sum_{i=0}^{14} v_i$.

In Table 11.2, we show the efficiency of this numerical technique. Here, we have discussed the approximated series solutions containing respectively 6, 10, 12 terms and their corresponding absolute errors, which shows a systematical decline in absolute errors. In Table 11.3 we compare our numerical results with the result of [79].

Table 11.2 Numerical solutions and absolute errors for problem (11.30).

x	Approximations			Exact Solution	Absolute Error		
	u_6	u_{10}	u_{12}		a_6	a_{10}	a_{12}
0	0.996812	0.99942	0.999736	1	0.003188	0.00058	0.000264
0.1	0.995216	0.997771	0.99808	0.998337	0.003121	0.000566	0.000257
0.2	0.990472	0.992869	0.993159	0.993399	0.002927	0.00053	0.00024
0.3	0.982703	0.984856	0.985115	0.985329	0.002626	0.000473	0.000214
0.4	0.972106	0.973952	0.974172	0.974355	0.002249	0.000403	0.000183
0.5	0.95894	0.960444	0.960622	0.960769	0.001829	0.000325	0.000147
0.6	0.943513	0.944664	0.9448	0.944911	0.001398	0.000247	0.000111
0.7	0.926161	0.926972	0.927067	0.927146	0.000985	0.000174	7.9E-05
0.8	0.907234	0.907735	0.907793	0.907841	0.000607	0.000106	4.8E-05
0.9	0.887079	0.887308	0.887335	0.887357	0.000278	4.9E-05	2.2E-05
1	0.866025	0.866025	0.866025	0.866025	0	0	0

Table 11.3 Numerical solutions and absolute errors for problem (11.30).

x	Approximations			Exact Solution	Absolute Error		
	Proposed method (u_{14})	Solution (y_2) in [79]	Solution (y_3) in [79]		u_{14}	for y_2 [79]	for y_3 [79]
0	0.999877	0.993678	1.000392358	1	0.000123	0.006322	0.000392358
0.1	0.998217	0.992067	0.998726589	0.998337	0.00012	0.00627	0.000389589
0.2	0.993287	0.987282	0.993778768	0.993399	0.000112	0.006117	0.000379768
0.3	0.985229	0.979461	0.985693317	0.985329	1E-04	0.005868	0.000364317
0.4	0.97427	0.968827	0.97469805	0.974355	8.5E-05	0.005528	0.00034305
0.5	0.9607	0.955679	0.961086726	0.960769	6.9E-05	0.00509	0.000317726
0.6	0.944859	0.940377	0.945197991	0.944911	5.2E-05	0.004534	0.000286991
0.7	0.927109	0.923325	0.927393267	0.927146	3.7E-05	0.003821	0.000247267
0.8	0.907819	0.904958	0.908035953	0.907841	2.2E-05	0.002883	0.000194953
0.9	0.887346	0.885714	0.887473778	0.887357	1.1E-05	0.001643	0.000116778
1	0.866025	0.866025	0.866025404	0.866025	0	0	4.03784E-07

11.5.2 Problem 2 : Thermal explosion in cylindrical vessel

Chamber [31] derived the following nonlinear two point SBVP, which arises in the thermal explosion

$$-u''(x) - \frac{1}{x}u'(x) = e^u, \quad 0 < x < 1, \tag{11.32a}$$

$$u'(0) = 0, \quad u(1) = 0. \tag{11.32b}$$

The exact solution of this SBVP is $u(x) = 2\ \ln\left(\frac{C+1}{C\,x^2+1}\right)$, where $C = 3 - 2\sqrt{2}$.

Table 11.4 Numerical values of A_i, where $i = 0, 1, 2, \cdots$

A_0=0	A_5=0.00153809
A_1=0.25	A_6=0.000587463
A_2=0.046875	A_7=0.000233786
A_3=0.0130208	A_8=0.000095889
A_4=0.00427246	

By employing the equations (11.21) and (11.32), we obtain the components $\{v_i\}$ of the series solutions of SBVP as

$$\left.\begin{aligned}
&v_0 = A_0,\\
&v_1 = A_1 - \frac{1}{4}x^2 e^{A_0},\\
&v_2 = -\frac{1}{4}x^2 e^{A_0} + \frac{1}{64}x^2\left(e^{A_0}\left(e^{A_0}x^2+16\right) - 16e^{A_0}A_1\right) + A_2,\\
&v_3 = -\frac{1}{4}x^2 e^{A_0} + \frac{1}{64}x^2\left(e^{A_0}\left(e^{A_0}x^2+16\right) - 16e^{A_0}A_1\right)\\
&\quad -\frac{e^{A_0}x^2}{2304}\Big[\Big(x^2e^{A_0}\left(x^2\left(2e^{A_0}+e^{A_0}\right)+36\right)\\
&\qquad +36A_1\left(x^2\left(-\left(e^{A_0}+e^{A_0}\right)\right)+8A_1-16\right)+576A_2\Big)\Big]+A_3,\\
&v_4 = -\frac{1}{4}x^2 e^{A_0} + \frac{1}{64}x^2\left(e^{A_0}\left(e^{A_0}x^2+16\right) - 16e^{A_0}A_1\right)\\
&\quad -\frac{e^{A_0}x^2}{2304}\Big[\Big(x^2e^{A_0}\left(x^2\left(2e^{A_0}+e^{A_0}\right)+36\right)\\
&\qquad +36A_1\left(x^2\left(-\left(e^{A_0}+e^{A_0}\right)\right)+8A_1-16\right)+576A_2\Big)\Big]\\
&\qquad -\frac{e^{A_0}x^2}{147456}\Big(-6x^6e^{3A_0} - x^4e^{2A_0}\left(11e^{A_0}x^2 - 128A_1+128\right)\\
&-x^2e^{A_0}\left(e^{2A_0}x^4 - 64e^{A_0}(6A_1-1)x^2 + 1152((A_1-2)A_1+2A_2)\right)\\
&\quad +64\Big(e^{2A_0}A_1x^4 - 18e^{A_0}(A_1(3A_1-2)+2A_2)x^2\\
&+96(-6A_2+A_1((A_1-3)A_1+6A_2)+6A_3)\Big)\Big)+A_4,\\
&\quad\vdots
\end{aligned}\right\}\qquad (11.33)$$

Making use of boundary conditions of (11.32), we obtain the values of $A_0, A_1, A_2, \cdots$ (See Table 11.4).

Hence, by using (11.33) and Table 11.4, we can write an approximate series solutions of SBVP (11.32), containing 9-terms i.e., $u = \sum_{i=0}^{8} v_i$.

In Table 11.5, we have discussed the approximated series solution (containing different terms) for SBVP (11.32) and their corresponding absolute errors, which shows a systematical decline in absolute errors.

11.5.3 Problem 3 : Thermal distribution in the human head

Duggan and Goodman [46] derived the following nonlinear two point SBVP which describes the thermal distribution profile in the human head

$$-u''(x)-\frac{2}{x}u'(x)=e^{-u},\quad 0<x<1, \tag{11.34a}$$

$$u'(0)=0,\quad 2u(1)+u'(1)=0. \tag{11.34b}$$

By employing the equations (11.21) and (11.34), we obtain the components $\{v_i\}$ of the series solutions of SBVP as

Table 11.5 Numerical solutions and absolute errors for (11.32).

	Approximations			Exact	Absolute Error		
x	u_4	u_6	u_8	Solution	a_4	a_6	a_8
0	0.314168	0.316294	0.316624	0.316694	0.002526	0.0004	7E-05
0.1	0.310782	0.312872	0.313196	0.313266	0.002484	0.000394	7E-05
0.2	0.300656	0.302642	0.30295	0.303015	0.002359	0.000373	6.5E-05
0.3	0.283886	0.285707	0.285987	0.286047	0.002161	0.00034	6E-05
0.4	0.260628	0.262232	0.262479	0.262531	0.001903	0.000299	5.2E-05
0.5	0.231095	0.232446	0.232653	0.232697	0.001602	0.000251	4.4E-05
0.6	0.195552	0.196628	0.196792	0.196827	0.001275	0.000199	3.5E-05
0.7	0.15431	0.155103	0.155223	0.155248	0.000938	0.000145	2.5E-05
0.8	0.107716	0.108229	0.108306	0.108323	0.000607	9.4E-05	1.7E-05
0.9	0.0561473	0.0563936	0.0564308	0.0564386	0.0002913	4.5E-05	7.8E-06
1	-8.32667E-17	4.16334E-17	7.1205E-17	0	8.32667E-17	4.16334E-17	7.1205E-17.

$$\left.\begin{aligned}
&v_0=A_0,\\
&v_1=A_1-\frac{1}{6}e^{-A_0}x^2,\\
&v_2=-\frac{1}{120}e^{-2A_0}x^4+\frac{1}{6}e^{-A_0}A_1x^2+A_2,\\
&v_3=-\frac{1}{120}e^{-2A_0}x^4+\frac{1}{6}e^{-A_0}A_1x^2\\
&-\frac{e^{-3A_0}\left(-63e^{A_0}\left(2A_1+1\right)x^4+630e^{2A_0}\left(A_1\left(A_1+2\right)-2A_2\right)x^2+4x^6\right)}{7560}+A_3,
\end{aligned}\right\} \tag{11.35a}$$

$$\left.\begin{aligned}
v_4 &= -\frac{1}{120}e^{-2A_0}x^4+\frac{1}{6}e^{-A_0}A_1x^2\\
&-\frac{e^{-3A_0}\left(-63e^{A_0}\left(2A_1+1\right)x^4+630e^{2A_0}\left(A_1\left(A_1+2\right)-2A_2\right)x^2+4x^6\right)}{7560}\\
&-\frac{e^{-4A_0}}{22680x}\Big(-12e^{A_0}\left(3A_1+1\right)x^7+378e^{2A_0}\left(A_1^2+A_1-A_2\right)x^5\\
&-630e^{3A_0}\left(-6A_2+A_1\left(A_1\left(A_1+3\right)-6A_2\right)+6A_3\right)x^3+\frac{61x^9}{72}\Big)+A_4,\\
&\vdots
\end{aligned}\right\} \quad (11.35b)$$

Making use of boundary conditions of (11.34), we obtain the values of $A_0, A_1, A_2, \cdots$ (See Table 11.6). Hence, by using (11.35) and Table 11.6, we can write an approximate series

Table 11.6 Numerical values of A_i, where $i = 0, 1, 2, \cdots$.

A_0= 0	A_4= -0.0145847	A_8= -0.0011053
A_1= 0.333333	A_5= 0.00712792	A_9= 0.000627972
A_2= -0.0861111	A_6= -0.00369133	A_{10}= -0.000363262
A_3= 0.032672	A_7= 0.00199019	

solutions of SBVP (11.34), containing 11-terms, i.e., $u = \sum_{i=0}^{10} v_i$.

To check the efficiency of our technique for this problem we use absolute residual error because exact solution is not available. The absolute residual error measures that how well the approximate solution satisfies nonlinear SBVP (11.34).

$$R_n = \left| -u_n''(x) - \frac{2}{x}u_n'(x) - e^{-u_n} \right|, \quad 0 < x < 1.$$

Table 11.7 shows the numerical values of residual error R_n, $n = 7, 8, 10$ and their systematical decay. We also compare our results with the results in [116] and [46].

11.5.4 Problem 4 : Rotationally symmetric shallow membrane cap

The following nonlinear two point SBVP arises in the study of radial stress on a rotationally symmetric shallow membrane cap [20, 44]

$$-u''(x) - \frac{3}{x}u'(x) = \frac{1}{8u^2} - \frac{1}{2}, \quad 0 < x < 1, \tag{11.36a}$$

$$u'(0) = 0, \quad u(1) = 1. \tag{11.36b}$$

Table 11.7 Numerical solutions and absolute residual errors for (11.34).

x	Approximations u_7	u_8	u_{10}	Solution in [116]	Solution in [46]	Absolute Residual Error R_7	R_8	R_{10}
0	0.270736	0.269631	0.269896	–	0.270350067	0.00419401	0.00239874	0.000822617
0.1	0.269457	0.268362	0.268624	0.268756903	0.269077693	0.0041414	0.00236739	0.000811194
0.2	0.265615	0.264548	0.264804	0.26493282	0.265254341	0.0039873	0.00227573	0.000777891
0.3	0.259194	0.258171	0.258416	0.258539792	0.25886127	0.0037425	0.00213063	0.000725465
0.4	0.250163	0.249202	0.249432	0.249548183	0.249867127	0.00342364	0.00194262	0.00065808
0.5	0.238484	0.237596	0.237809	0.237915891	0.238227763	0.00305148	0.00172462	0.000580749
0.6	0.224104	0.223298	0.223491	0.22358771	0.223885976	0.00264868	0.00149053	0.000498695
0.7	0.206954	0.206237	0.206408	0.206494486	0.20677117	0.00223761	0.0012537	0.000416765
0.8	0.186953	0.186327	0.186477	0.186552018	0.18679895	0.00183824	0.00102571	0.00033896
0.9	0.164001	0.163468	0.163596	0.163659686	0.163870628	0.00146662	0.000815495	0.000268165
1	0.137982	0.13754	0.137646	0.137698751	0.137872638	0.0011339	0.000628885	0.000206077

Table 11.8 Numerical solutions and absolute residual errors for (11.36).

x	u_6	Solution in [79]	Absolute Residual Error R_6	R_7	R_8	R_9
0	0.954135	0.952148432	1.70634E-09	2.13697E-10	1.90742E-11	3.86774E-13
0.1	0.954589	0.95263173	1.52099E-09	2.08909E-10	1.79321E-11	3.28515E-13
0.2	0.95595	0.954081048	1.02873E-09	1.93842E-10	1.48054E-11	1.81022E-13
0.3	0.95822	0.956494659	3.95944E-10	1.6746E-10	1.05031E-11	1.04916E-14
0.4	0.961403	0.959869678	1.74014E-10	1.30779E-10	6.09157E-12	1.14464E-13
0.5	0.965503	0.964202058	5.21019E-10	8.86357E-11	2.53958E-12	1.58318E-13
0.6	0.970526	0.969486581	5.89803E-10	4.90143E-11	3.81084E-13	1.3195E-13
0.7	0.976479	0.975716845	4.44409E-10	1.9698E-11	4.40037E-13	7.64944E-14
0.8	0.983369	0.982885249	2.24681E-10	4.11388E-12	4.29934E-13	2.99205E-14
0.9	0.991206	0.990982981	6.17023E-11	4.23439E-13	1.74638E-13	6.60583E-15
1	1	1	0	0	0	5.55112E-17

Table 11.9 Numerical values of A_i, where $i = 0, 1, 2, \cdots$

A_0=1	A_4=1.13249$\times10^{-7}$	A_8=9.7943$\times10^{-12}$
A_1=-0.046875	A_5=-6.18833$\times10^{-8}$	A_9=5.23386$\times10^{-13}$
A_2=0.000976563	A_6=-3.29419$\times10^{-9}$	
A_3=0.0000336965	A_7=1.10069 $\times10^{-12}$	

By employing the equations (11.21) and (11.36), we obtain the components $\{v_i\}$ of the series solutions of SBVP as

$$\left.\begin{aligned} v_0 &= A_0, \\ v_1 &= -\frac{x^2}{64A_0^2} + A_1 + \frac{x^2}{16}, \end{aligned}\right\} \tag{11.37a}$$

$$\left.\begin{aligned}
v_2 &= \frac{x^4}{1536A_0^3} - \frac{x^4}{6144A_0^5} + \frac{A_1x^2}{32A_0^3} + A_2,\\
v_3 &= -\frac{x^6}{32768A_0^4} + \frac{11x^6}{589824A_0^6} - \frac{13x^6}{4718592A_0^8} + \frac{5A_1x^4}{6144A_0^6} - \frac{A_1x^4}{512A_0^4}\\
&\quad + \frac{A_2x^2}{32A_0^3} - \frac{3A_1^2x^2}{64A_0^4} + A_3,\\
v_4 &= -\frac{x^6}{32768A_0^4} + \frac{11x^6}{589824A_0^6} - \frac{13x^6}{4718592A_0^8} + \frac{5A_1x^4}{6144A_0^6}\\
&- \frac{A_1x^4}{512A_0^4} + \frac{A_2x^2}{32A_0^3} - \frac{3A_1^2x^2}{64A_0^4} + \frac{x^2}{1509949440A_0^{11}}\left(8A_0^2\left(520A_0x^4 + 4160A_1x^4\right.\right.\\
&-320A_0^3x^2\left(480A_1 - 480A_2 + 11x^2\right) - 6A_0^2\left(320A_1x^2\left(240A_1 + 11x^2\right) + 51x^6\right)\\
&+5760A_0^5\left(64A_1\left(24A_1 + x^2\right) - 64A_2\left(48A_1 + x^2\right) + x^4\right)\\
&+288A_0^4\left(80A_1\left(32A_1\left(16A_1 + x^2\right) + x^4\right) + x^6\right)\\
&\left.\left.-5898240A_0^6(A_2 - A_3) + 101x^6\right) - 85x^6\right) + A_4,\\
&\vdots
\end{aligned}\right\} \quad (11.37b)$$

Making use of boundary conditions of (11.36), we obtain the values of $A_0, A_1, A_2, \cdots$ (See Table 11.9).

Hence, by using (11.37) and Table 11.9, we can write an approximate series solutions of SBVP (11.36), containing 10-terms, i.e., $u = \sum_{i=0}^{9} v_i$. Similar to above problem, exact solution of this problem (11.36) is also not known. So, again we check the efficiency of our technique with the use of absolute residual error.

$$R_n = \left| -u_n''(x) - \frac{3}{x}u_n'(x) - \frac{1}{8u_n^2} + \frac{1}{2} \right|, \quad 0 < x < 1.$$

Table 11.8 shows the numerical values of residual error R_n, $n = 6,7,8,9$ and their systematical decay. Also we compare our result with the result of [79].

11.6 Conclusion

In this chapter, we have applied proposed homotopy perturbation method coupled with variational iteration method to nonlinear singular boundary value problems arising in science and engineering. The proposed method is convergent and provides us approximate solutions which are very close to exact solution or best solution, known so far. This method can be preferred over finite difference method as it does not require matrix inversion. Using absolute and residual errors, we show the computational power of proposed method.

Chapter 12

Monotone iterative technique for nonlinear discrete BVPs

12.1 Introduction

The main aspire of this chapter is to develop monotone iterative technique for the following discrete boundary value problem

$$-\Delta^2 y(t-1) = f(t, y(t)), \qquad t \in [1, T], \tag{12.1}$$
$$y(0) = 0, \qquad y(T+1) = 0, \tag{12.2}$$

where T is a positive integer, $[1, T]$ is the discrete interval $\{1, 2, \cdots, T\}$, $f : [0, T] \times \mathbb{R} \to \mathbb{R}$, and Δ is the forward difference operator. Here $f(t, y)$ is defined for all t in $[1, T]$ and for all real numbers y. Also $f(t, y)$ satisfies growth conditions with respect to y known as one sided Lipschitz condition given as

$$y \leq w \Longrightarrow f(t, w) - f(t, y) \geq M(w - y).$$

We assume that $f(t, y)$ is continuous in y for each $t \in [1, T]$. Agarwal et al. [8] used critical point theory and discussed the existence result for the same nonlinear discrete boundary value problem (12.1)–(12.2).

We introduce monotone iterative scheme for nonlinear discrete boundary value problem (12.1)–(12.2) defined as,

$$-\Delta^2 y_{n+1}(t-1) - \lambda y_{n+1}(t) = f(t, y_n(t)) - \lambda y_n(t), \quad y_{n+1}(0) = 0, \quad y_{n+1}(T+1) = 0, \tag{12.3}$$

where $\lambda \in \mathbb{R} \setminus \{0\}$. This technique is a discrete version of Picard type monotone iterative technique [131], (see [41] and references therein). For discrete boundary value problems, we could observe only few references [155, 156, 166].

We establish maximum principle for the corresponding linear discrete boundary value problem. We construct Green's function and prove that it is of constant sign for linear discrete boundary value problems (see [83]).

We associate the concept of upper and lower solutions with monotone iterative technique and establish a existence result for nonlinear discrete boundary value problem. This existence result reveals that the upper and lower solutions are treated as initial values for monotone iterative technique, which generate monotonically convergent sequences. Furthermore we obtain the existence uniqueness result for nonlinear discrete boundary value problem (12.1)–(12.2).

12.2 Linear discrete BVP

To explore the existence results for nonlinear discrete boundary value problem (12.1)–(12.2), we consider the following non-homogeneous linear discrete boundary value problem

$$-Ly \equiv -\Delta^2 y(t-1) - \lambda y(t) = h(t), \qquad t \in [1,T], \tag{12.4}$$

$$y(0) = 0, \qquad y(T+1) = B, \tag{12.5}$$

where B is any arbitrary constant. The corresponding homogeneous discrete boundary value problem will be

$$\Delta^2 y(t-1) + \lambda y(t) = 0, \qquad t \in [1,T], \tag{12.6}$$

$$y(0) = 0, \qquad y(T+1) = 0. \tag{12.7}$$

Solving non-homogeneous discrete boundary value problem (12.4)–(12.5) is equivalently to find a y, such that

$$y(t) = \bar{y} - \sum_{s=1}^{T} G(t,s)h(s) \tag{12.8}$$

where $\bar{y}$ is the solution of homogeneous difference equations (12.6), with non-homogeneous boundary conditions (12.5) and $G(t,s)$ is the Green's function of (12.6)–(12.7). Here, we discuss the solution of nonhomogeneous discrete boundary value problem (12.4)–(12.5). We divide it into the following cases.

Remark 12.1. *The characteristic equation for* (12.6) *is*

$$m^2 + (\lambda - 2)m + 1 = 0.$$

If $|\lambda - 2| < 2$, *and* $\cos\theta = \frac{(2-\lambda)}{2}$, *then*

$$m = \cos\theta \pm i\sin\theta = e^{\pm i\theta}.$$

Therefore a general solution of equation (12.6) *is*

$$y(t) = c_1 \cos(\theta t) + c_2 \sin(\theta t).$$

By using equation (12.7)*, we have*

$$y(0) = c_1 = 0, \quad y(T+1) = c_2 \sin((T+1)\theta) = 0,$$

which gives

$$\theta = \frac{n\pi}{T+1}.$$

Thus all the eigenvalues are given by,

$$\lambda_n = 2 - 2\cos\left(\frac{n\pi}{T+1}\right), \quad n = 1, 2, 3 \cdots T.$$

12.2.0.1 Case I: $|\lambda - 2| < 0$

Lemma 12.1. *The Green's function* $G(t,s)$ *of the discrete boundary value problem (12.6)–(12.7), for* $0 < \lambda < \lambda_1$, *is given by*

$$G(t,s) = \begin{cases} \frac{-\sin\theta(T+1-s)\sin\theta t}{\sin\theta\sin\theta(T+1)}, & 0 \le t \le s; \\ \\ \frac{-\sin\theta(T+1-s)\sin\theta t}{\sin\theta\sin\theta(T+1)} + \frac{\sin\theta(t-s)}{\sin\theta}, & s \le t \le T+1; \end{cases} \tag{12.9}$$

where $\theta = \tan^{-1}\left(\frac{\sqrt{4-(\lambda-2)^2}}{(2-\lambda)}\right)$*, where* λ_1 *(see Remark 12.1) is the first eigenvalue of (12.6)–(12.7).*

Proof. We define the Green's function as given below

$$G(x,t) = \begin{cases} u(t,s), & 0 \le t \le s \le T+1; \\ v(t,s), & 0 \le s \le t \le T+1; \end{cases}$$

where $u(\cdot,s)$ is the solution of the following discrete boundary value problem for each fixed $s\in[1,T]$

$$Lu=0, \tag{12.10}$$
$$u(0,s)=0, \tag{12.11}$$
$$u(T+1,s)=-y(T+1,s), \tag{12.12}$$

and $v(t,s):=u(t,s)+y(t,s)$, where $y(\cdot,\cdot)$ is the Cauchy function for $Ly=0, \forall\, t,s\in[1,T]$. For each $s\in[1,T]$, $v(\cdot,s)$ is a solution of $Ly=0$ satisfying the boundary condition $y(T+1,s)=0$.

The Cauchy function for $Ly=0$ is given by

$$y(t,s)=\frac{\sin\theta(t-s)}{\sin\theta}, \tag{12.13}$$

where $\theta=\tan^{-1}\left(\frac{\sqrt{4-(\lambda-2)^2}}{(2-\lambda)}\right)$. Now from equations (12.10)–(12.12), we get

$$u(t,s)=\frac{-\sin\theta(T+1-s)\sin\theta t}{\sin\theta\sin\theta(T+1)}, \tag{12.14}$$

for each fixed $s\in[1,T]$.

Next

$$v(t,s)=u(t,s)+y(t,s) \tag{12.15}$$
$$=\frac{-\sin\theta(T+1-s)\sin\theta t}{\sin\theta\sin\theta(T+1)}+\frac{\sin\theta(t-s)}{\sin\theta}. \tag{12.16}$$

This completes the construction of Green's function. ■

Lemma 12.2. *Let y be a solution of non-homogeneous discrete boundary value problem (12.4)–(12.5), then*

$$y(t)=\frac{B\sin\theta t}{\sin\theta(T+1)}-\sum_{s=1}^{T}G(t,s)h(s). \tag{12.17}$$

Proof. Suppose $\bar{y}$ is the solution of $Ly=0, \quad t\in[1,T]$, subject to $y(0)=0, \ y(T+1)=B$, and $G(t,s)$ is the solution of homogeneous discrete boundary value problem (12.6)–(12.7). Then the discrete boundary value problem (12.4)–(12.5) is analogous to

$$y(t)=\bar{y}-\sum_{s=1}^{T}G(t,s)h(s).$$

The general solution of difference equation $Ly = 0$ is given by

$$\bar{y}(t) = c_1 \cos \theta t + c_2 \sin \theta t,$$

where $\theta = \tan^{-1}\left(\frac{\sqrt{4-(\lambda-2)^2}}{(2-\lambda)}\right)$.

Since

$$\bar{y}(0) = 0, \text{ and } \bar{y}(T+1) = B,$$

we get

$$c_1 = 0,$$
$$c_2 = \frac{B}{\sin \theta (T+1)}.$$

Hence the discrete boundary value problem (12.4)–(12.5) is equivalent to

$$y(t) = \frac{B \sin \theta t}{\sin \theta (T+1)} - \sum_{s=1}^{T} G(t,s)h(s).$$

■

Here we state four Lemmas 12.3, 12.4, 12.5 and 12.6 without proof. Proof of these are similar to the case I

12.2.0.2 Case II: $\lambda < 0$

Lemma 12.3. *The Green's function $G(t,s)$ of the discrete boundary value problem (12.6)–(12.7) for $\lambda < 0$, is given by*

$$G(t,s) = \begin{cases} \frac{1}{(\alpha-\beta)}\left(\alpha^{T+1-s} - \beta^{T+1-s}\right)\frac{(\beta^t-\alpha^t)}{(\alpha^{T+1}-\beta^{T+1})}, & 0 \le t \le s; \\ \\ \frac{1}{(\alpha-\beta)}\left(\alpha^{T+1-s} - \beta^{T+1-s}\right)\frac{(\beta^t-\alpha^t)}{(\alpha^{T+1}-\beta^{T+1})} + \frac{(\alpha^{t-s}-\beta^{t-s})}{(\alpha-\beta)}, & s \le t \le T+1; \end{cases} \tag{12.18}$$

where $\alpha = \frac{(2-\lambda)+\sqrt{(\lambda-2)^2-4}}{2}$ and $\beta = \frac{(2-\lambda)-\sqrt{(\lambda-2)^2-4}}{2}$.

Lemma 12.4. *Let y be the solution of non-homogeneous difference equation (12.4)–(12.5), then*

$$y(t) = \frac{(\alpha^t - \beta^t)B}{(\alpha^{T+1} - \beta^{T+1})} - \sum_{s=1}^{T} G(t,s)h(s). \tag{12.19}$$

12.2.0.3 Case III: $\lambda = 0$

Lemma 12.5. *The Green's function $G(t,s)$ of the discrete boundary value problem (12.6)–(12.7) for $\lambda = 0$, is given by*

$$G(t,s) = \begin{cases} \frac{t(s-(T+1))}{T+1}, & 0 \le t \le s; \\ \\ \frac{s(t-(T+1))}{T+1}, & s \le t \le T+1. \end{cases} \tag{12.20}$$

Lemma 12.6. *Let y be the solution of non-homogeneous difference equation (12.4)–(12.5), then*

$$y(t) = \frac{B}{T+1} - \sum_{s=1}^{T} G(t,s)h(s). \tag{12.21}$$

Remark 12.2. *Using Lemma 12.2, Lemma 12.4 and Lemma 12.6 the solution of non-homogeneous linear discrete boundary value problems (12.4)–(12.5) can be written as*

$$y(t) = B\psi(t) - \sum_{s=1}^{T} G(t,s)h(s). \tag{12.22}$$

where $\psi(t)$ is defined as $\frac{\sin\theta t}{\sin\theta(T+1)}$ or $\frac{(\alpha^t-\beta^t)}{(\alpha^{T+1}-\beta^{T+1})}$ or $\frac{B}{T+1}$ and $G(t,s)$ is defined by (12.9) *or* (12.18) *or* (12.20)*, respectively.*

Remark 12.3. *For each fixed value of $s \in [1,T]$, $u(\cdot,s)$ is the solution of (12.10) which satisfies the discrete boundary conditions $u(0,s) = 0$ and $u(T+1,s) = -y(T+1,s)$, where $y(\cdot,\cdot)$ is a Cauchy function and satisfies $y(s,s) = 0$, $y(s+1,s) > 0$. As difference equation (12.10) disconjugate on $[0,T+1]$, i.e,*

$$u(t,s) < 0,$$

for $t \in [1,T]$. Also $v(\cdot,s)$ is a solution of $Ly = 0$ satisfying the boundary condition $v(T+1,s) = 0$, and

$$v(s,s) = u(s,s) + y(s,s) = u(s,s) < 0,$$

then

$$v(t,s) < 0,$$

for $t \in [1,T]$. Therefore

$$G(t,s) < 0,$$

for $t,s \in [1,T]$.

12.3 Maximum principle

Proposition 12.1. *If y satisfies the non-homogeneous linear discrete boundary value problem*

$$\begin{aligned} -\Delta^2 y(t-1) - \lambda y(t) = h(t), \quad & t \in [1,T], \\ y(0) = 0, \quad y(T+1) = B, & \end{aligned}$$

with $h(t) \geq 0$ and $B \geq 0$, then $y(t) \geq 0$ for all $t \in [1,T]$ and $\lambda < \lambda_1$.

Proof. The proof is an immediate consequences of Remarks 12.2, 12.3. ■

12.4 Nonlinear discrete BVP

In this section, we examine the existence results for nonlinear discrete boundary value problem, with the support of monotone iterative method and upper and lower solutions of the nonlinear discrete boundary value problem.

Let us first define the bounds of the solution of the nonlinear discrete boundary value problems

Definition 12.1. *A function $\beta_0(t)$ is an upper solution of nonlinear discrete boundary value problem (12.1)–(12.2) if it satisfies*

$$-\Delta^2 \beta_0(t-1) \geq f(t, \beta_0(t)), \quad t \in [1,T], \quad \beta_0(0) = 0, \quad \beta_0(T+1) \geq 0. \tag{12.23}$$

Definition 12.2. *A function $\alpha_0(t)$ is a lower solution of nonlinear discrete boundary value problem (12.1)–(12.2) if it satisfies*

$$-\Delta^2 \alpha_0(t-1) \leq f(t, \alpha_0(t)), \quad t \in [1,T], \quad \alpha_0(0) = 0, \quad \alpha_0(T+1) \leq 0. \tag{12.24}$$

Theorem 12.1. *If $f : D_0 \to R$ is continuous on $D_0 := \{(t,y) \in [0,T+1] \times R : \alpha_0 \leq y \leq \beta_0\}$ in y for each t and there exists a constant $M > 0$ such that for all $(t,y),(t,w) \in D_0$*

$$y \leq w \Longrightarrow f(t,w) - f(t,y) \geq M(w-y), \tag{12.25}$$

then the region D_0, contains at least one solution of the nonlinear discrete boundary value problem (12.1)–(12.2). If a constant $\lambda \leq M$ is chosen such that $\lambda < \lambda_1$ then the sequence (β_n) generated by

$$-\Delta^2 y_{n+1}(t-1) - \lambda y_{n+1}(t) = F(t, y_n(t)), \quad y_{n+1}(0) = 0, \quad y_{n+1}(T+1) = 0, \tag{12.26}$$

where $F(x,y_n(t)) = f(t,y_n(t)) - \lambda y_n$, converges monotonically (non-increasing) and uniformly towards a solution $\widetilde{\beta}(t)$ of (12.1)–(12.2). Similarly α_0 leads to a non-decreasing sequence (α_n) converging to a solution $\widetilde{\alpha}(t)$. Any solution $z(t)$ in D_0 must satisfy

$$\widetilde{\alpha}(t) \leq z(t) \leq \widetilde{\beta}(t).$$

Proof. Making the use of equations (12.23) and (12.26) (for $n = 0$)

$$-\Delta^2(\beta_0-\beta_1)(t-1)-\lambda(\beta_0-\beta_1)(t) \geq 0, \quad (\beta_0-\beta_1)=0, \quad (\beta_0-\beta_1)(T+1) \geq 0. \quad (12.27)$$

As $(\beta_0 - \beta_1)$ satisfies the above equation (12.27), with $h(t) \geq 0$, and $B \geq 0$, then by the Proposition 12.1, we have $\beta_0 \geq \beta_1$.

As $M - \lambda \geq 0$, using the equations (12.25) and (12.26), we have

$$-\Delta^2\beta_{n+1}(t-1) \geq (M-\lambda)(\beta_n - \beta_{n+1})(t) + f(t,\beta_{n+1}(t)),$$

and if $\beta_n - \beta_{n+1} \geq 0$, then

$$-\Delta^2\beta_{n+1}(t-1) \geq f(t,\beta_{n+1}(t)), \quad \beta_{n+1}(0) = 0, \quad \beta_{n+1}(T+1) = 0. \quad (12.28)$$

Since $\beta_0 \geq \beta_1$, then by making the use of equations (12.28) (for $n = 0$) and (12.26) (for $n = 1$) we get

$$-\Delta^2(\beta_1 - \beta_2)(t-1) - \lambda(\beta_1 - \beta_2)(t) \geq 0,$$
$$(\beta_1 - \beta_2)(0) = 0, \quad (\beta_1 - \beta_2)(T+1) \geq 0.$$

In the view of Proposition 12.1, we have $\beta_1 \geq \beta_2$.

Now with the support of equations (12.24) and (12.26) (for $n = 0$), we get

$$-\Delta^2(\beta_1 - \alpha_0)(t-1) - \lambda(\beta_1 - \alpha_0)(t) \geq 0,$$
$$(\beta_1 - \alpha_0)(0) = 0, \quad (\beta_1 - \alpha_0)(T+1) \geq 0,$$

which gives $\beta_1 \geq \alpha_0$, (Proposition 12.1).

To use mathematical induction, we assume that $\beta_{n+1} \leq \beta_n$, $\alpha_0 \leq \beta_{n+1}$ and show that $\beta_{n+2} \leq \beta_{n+1}$ and $\alpha_0 \leq \beta_{n+2}$ for all n. Now making the use of equations (12.26) (for $n+1$) and (12.28)

$$-\Delta^2(\beta_{n+1} - \beta_{n+2})(t-1) - \lambda(\beta_{n+1} - \beta_{n+2})(t) \geq 0,$$

$$(\beta_{n+1}-\beta_{n+2})(0)=0, \quad (\beta_{n+1}-\beta_{n+2})(T+1)\geq 0,$$

we have $\beta_{n+1}\leq\beta_n$ (Propositions 12.1).

From equations (12.26) (for $n+1$) and (12.24)

$$\begin{aligned} &-\Delta^2(\beta_{n+2}-\alpha_0)(t-1)-\lambda(\beta_{n+2}-\alpha_0)(t)\geq 0,\\ &(\beta_{n+2}-\alpha_0)(0)=0, \quad (\beta_{n+2}-\alpha_0)(T+1)\geq 0. \end{aligned}$$

Thus we have $\alpha_0\leq\beta_{n+2}$ (Proposition 12.1) and hence we have

$$\alpha_0\leq\ldots\leq\beta_{n+1}\leq\beta_n\leq\ldots\leq\beta_2\leq\beta_1\leq\beta_0,$$

and if we choose α_0 as an initial iterate, then we easily get

$$\alpha_0\leq\alpha_1\leq\alpha_2\leq\ldots\leq\alpha_n\leq\alpha_{n+1}\leq\ldots\leq\beta_0.$$

Finally we prove that $\alpha_n\leq\beta_n$ for all n. For this by assuming $\alpha_n\leq\beta_n$, we show that $\beta_{n+1}\geq\alpha_{n+1}$. From equation (12.26) it is easy to get

$$\begin{aligned} &-\Delta^2(\beta_{n+1}-\alpha_{n+1})(t-1)-\lambda(\beta_{n+1}-\alpha_{n+1})(t)\geq 0,\\ &(\beta_{n+1}-\alpha_{n+1})(0)=0, \quad (\beta_{n+1}-\alpha_{n+1})(T+1)\geq 0. \end{aligned}$$

Hence by Proposition 12.1, $\beta_{n+1}\geq\alpha_{n+1}$. Thus we have

$$\alpha_0\leq\alpha_1\leq\alpha_2\leq\ldots\leq\alpha_n\leq\alpha_{n+1}\leq\ldots\leq\beta_{n+1}\leq\beta_n\leq\ldots\leq\beta_2\leq\beta_1\leq\beta_0.$$

So the sequences (β_n) and (α_n) are monotonically non-increasing and non-decreasing, respectively and are bounded by β_0 and α_0. Hence they converge uniformly. Let $\beta(t)=\lim\limits_{n\to\infty}\beta_n(t)$ and $\alpha(t)=\lim\limits_{n\to\infty}\alpha_n(t)$.

The solution β_{n+1} of equation (12.26) is given by (Remark 12.3).

$$\beta_{n+1}=B\psi(t)-\sum_{s=1}^{T}G(t,s)(f(t,\beta_n(t))-\lambda\beta_n).$$

Now as $n\to\infty$, we get

$$\widetilde{\beta}(t)=B\psi(t)-\sum_{s=1}^{T}G(t,s)(f(t,\widetilde{\beta}(t))-\lambda\widetilde{\beta}(t)).$$

Which is the solution of boundary value problem (12.1)–(12.2).

It is clear that any arbitrary solution $z(t)$ can be treated as upper solution $\beta_0(t)$, i.e., we get $z(t) \geq \alpha_0(t)$, similarly one concludes that $z(t) \leq \beta_0(t)$. ■

Theorem 12.2. *Let $f(t,y)$ is continuous in y for each t in $[1,T]$ and there is a constant $M > 0$ such that*

$$f(t,w) - f(t,y) \geq M(w-y), \tag{12.29}$$

and $M < \lambda_1$. Then the nonlinear discrete boundary value problem (12.1)–(12.2) *has unique solution.*

Proof. Suppose $u(t)$ and $v(t)$ be any two solutions of (12.1)–(12.2) then we get

$$\begin{aligned} -\Delta^2(u-v)(t-1) &= [f(t,u(t)) - f(t,v(t))], \\ (u-v)(0) = 0, &\quad (u-v)(T+1) = 0, \end{aligned}$$

which gives

$$\begin{aligned} -\Delta^2(u-v)(t-1) - M(u-v)(t-1) &\geq 0, \\ (u-v)(0) = 0, &\quad (u-v)(T+1) = 0. \end{aligned}$$

By the maximum (for, $B = 0$), whenever $M < \lambda_1$, we get $u - v \geq 0$ (i.e., $u \geq v$). Similarly by changing the role of u and v, we get $u \leq v$. Hence $u \equiv v$. Therefore the solution of the (12.1)–(12.2) is unique. ■

12.5 Examples

Example 12.1. *Consider the nonlinear discrete boundary value problems*

$$-\Delta^2 y(t-1) = \frac{e^{y(t)}}{e^{(T+1)^2}}, \quad t \in [1,T], \tag{12.30}$$

$$y(0) = 0, \;\; y(T+1) = 0. \tag{12.31}$$

Here, $\alpha_0 = 0$ and $\beta_0 = (T+1)t - \frac{t^2}{2}$ are defined as lower and upper solutions of the solution of nonlinear discrete boundary value problem (12.30)–(12.31), respectively. The nonlinear source term is continuous for all values of $y(t)$ and satisfies one sided Lipschitz condition, with constant $M = \frac{1}{e^{(T+1)^2}}$. By Theorem 12.1 and Theorem 12.2, discrete BVP (12.30)–(12.31) has a unique solution.

Example 12.2. *Consider the nonlinear discrete boundary value problems*

$$-\Delta^2 y(t-1) = e^t - e^{y(t)}, \quad t \in [1,T], \tag{12.32}$$

$$y(0) = 0, \;\; y(T+1) = 0. \tag{12.33}$$

Here, $\alpha_0 = 0$ and $\beta_0 = t$ are defined as lower and upper solutions of the solution of nonlinear discrete boundary value problem (12.32)–(12.33), respectively. The nonlinear source term is continuous for all values of $y(t)$ and satisfies one sided Lipschitz condition, with constant $M = e^{(T+1)}$. By Theorem 12.1 and Theorem 12.2, discrete BVP (12.32)–(12.33) has a unique solution

12.6 Conclusion

In this chapter have established existence of a unique solution for a class of nonsingular difference equation subject to Dirichlet type boundary conditions. Though the results are simple but new and may lead some new development in near future, related to theory of difference equations.

Conclusions and future scope of work

The present thesis is devoted to the study of nonlinear nonsingular / singular boundary value problems. We have considered some continuous cases and a discrete case. In the continuous case we consider nonlinear three point nonsingular / singular BVPs and two point SBVPs. While, in the discrete case, we consider a nonlinear nonsingular discrete boundary value problem.

We develop monotone iterative method for both well ordered and reverse ordered upper and lower solutions for the following class of differential and difference equations, with suitable boundary conditions.

- Nonlinear three point nonsingular boundary value problem when source function is independent from derivative with mixed type boundary condition.

- Nonlinear three point nonsingular boundary value problem when source function depends on derivative, with Dirichlet, Neumann and mixed type boundary conditions.

- Nonlinear three point singular boundary value problems of the form

$$-y'' - \frac{\alpha}{x}y' = f(x,y), \quad 0 < x < 1, \tag{12.34}$$

  with suitable boundary conditions, for $\alpha \geq 1$.

- Three point nonlinear singular boundary value problem (12.34) for $\alpha = 1$, when source function depends on derivative,

- Nonlinear two point discrete boundary value problem.

In all above cases under certain sufficient conditions we construct Green's function, establish maximum and anti maximum principles and hence establish existence results.

For the numerical solutions of nonlinear singular two point boundary value problems we use variational iteration method and homotopy perturbation method. Convergence of both the methods is established and results are validated by solving several real life problems.

On the basis of work done in this thesis we can consider the following as future scope of this work

- In this thesis we have developed monotone iterative method (MIT) associated with upper and lower solutions for three point nonlinear boundary value problems, which can be extended for four point or multi-point nonlinear nonsingular/singular BVPs.
- Monotone iterative method can be generalized for nonlinear discrete singular boundary value problems.
- Some new numerical methods can be proposed for three point and multi point boundary value problems.

Appendix A

VIM : Lagrange multipliers

For $\omega > 0$, the Lagrange multipliers is

$$\lambda(t) = \frac{\pi x t^{\nu} t^{\alpha}}{2x^{\nu} x^{\alpha}} \left[\left(J_{-\nu}\left(t\sqrt{\omega}\right) Y_{-\nu}\left(x\sqrt{\omega}\right) - J_{-\nu}\left(x\sqrt{\omega}\right) Y_{-\nu}\left(t\sqrt{\omega}\right) \right) \right],$$

where

$$J_{-\nu}\left(t\sqrt{\omega}\right) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{t\sqrt{\omega}}{2}\right)^{-\nu+2m}}{m!\,\Gamma(m-\nu+1)},$$

$$J_{\nu}\left(t\sqrt{\omega}\right) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{t\sqrt{\omega}}{2}\right)^{\nu+2m}}{m!\,\Gamma(m+\nu+1)},$$

and

$$Y_{-\nu}\left(t\sqrt{\omega}\right) = \frac{2}{\pi} J_{-\nu}\left(t\sqrt{\omega}\right) \left(\ln \frac{t\sqrt{\omega}}{2} + \gamma \right) - \frac{1}{\pi} \sum_{m=0}^{-\nu-1} \frac{(-\nu-m-1)!}{m!} \left(\frac{t\sqrt{\omega}}{2} \right)^{2m+\nu} + \frac{1}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^{m-1} \left[\left(1+\frac{1}{2}+\cdots+\frac{1}{m}\right) + \left(1+\frac{1}{2}+\cdots+\frac{1}{m-\nu}\right) \right]}{m!\,(m-\nu)!} \left(\frac{t\sqrt{\omega}}{2} \right)^{2m-\nu}.$$

We have also a relations

$$Y_{-\nu}\left(t\sqrt{\omega}\right) = \frac{J_{\nu}\left(t\sqrt{\omega}\right) - \cos\nu\pi J_{-\nu}\left(t\sqrt{\omega}\right)}{\sin\nu\pi}.$$

i.e.,

$$\lambda(t) = \frac{\pi x t^{\alpha}}{2\,x^{\alpha}} \left[\left(\frac{2}{\pi} \left(\sum_{m=0}^{\infty} \frac{(-1)^m x^{-2\nu+2m} \left(\frac{\sqrt{\omega}}{2}\right)^{-\nu+2m}}{m!\,\Gamma(m-\nu+1)} \right) \left(\ln\left(\frac{x\sqrt{\omega}}{2}\right) + \gamma \right) \right.\right.$$
$$- \frac{1}{\pi} \sum_{m=0}^{-\nu-1} \frac{(-\nu-m-1)!}{m!} (x)^{2m} \left(\frac{\sqrt{\omega}}{2}\right)^{2m+\nu}$$
$$+ \frac{1}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^{m-1} \left[\left(1+\frac{1}{2}+\cdots+\frac{1}{m}\right) + \left(1+\frac{1}{2}+\cdots+\frac{1}{m-\nu}\right) \right]}{m!\,(m-\nu)!} (x)^{2m-2\nu} \left(\frac{\sqrt{\omega}}{2}\right)^{2m-\nu}$$
$$\left.\left. + \left(\sum_{m=0}^{\infty} \frac{(-1)^m x^{-2\nu+2m} \left(\frac{\sqrt{\omega}}{2}\right)^{-\nu+2m}}{m!\,\Gamma(m-\nu+1)} \right) \cot \nu\pi \right) \sum_{m=0}^{\infty} \frac{(-1)^m t^{2m} \left(\frac{\sqrt{\omega}}{2}\right)^{-\nu+2m}}{m!\,\Gamma(m-\nu+1)} \right]$$
$$- \frac{\pi x}{2\,x^{\alpha}} \left[\left(\sum_{m=0}^{\infty} \frac{(-1)^m x^{-2\nu+2m} \left(\frac{\sqrt{\omega}}{2}\right)^{-\nu+2m}}{m!\,\Gamma(m-\nu+1)} \right) \csc \nu\pi \sum_{m=0}^{\infty} \frac{(-1)^m t^{2m+1} \left(\frac{\sqrt{\omega}}{2}\right)^{\nu+2m}}{m!\,\Gamma(m+\nu+1)} \right].$$

Now for $\omega < 0$, the Lagrange multipliers is

$$\lambda(t) = \frac{t^{\alpha} t^{\nu}\, x}{x^{\nu}\, x^{\alpha}} \left[I_{-\nu}\left(x\sqrt{|\omega|}\right) K_{\nu}\left(t\sqrt{|\omega|}\right) - I_{-\nu}\left(t\sqrt{|\omega|}\right) K_{\nu}\left(x\sqrt{|\omega|}\right) \right],$$

where

$$I_{\nu}\left(x\sqrt{|\omega|}\right) = \sum_{m=0}^{\infty} \frac{\left(\frac{x\sqrt{|\omega|}}{2}\right)^{2m+\nu}}{m!\,\Gamma(m+\nu+1)},$$

$$I_{-\nu}\left(x\sqrt{|\omega|}\right) = \sum_{m=0}^{\infty} \frac{\left(\frac{x\sqrt{|\omega|}}{2}\right)^{2m-\nu}}{m!\,\Gamma(m-\nu+1)},$$

and

$$K_{\nu}\left(x\sqrt{|\omega|}\right) = \frac{\pi}{2\sin\nu\pi} \left[I_{-\nu}\left(x\sqrt{|\omega|}\right) - I_{\nu}\left(x\sqrt{|\omega|}\right) \right].$$

$$\lambda(t) = \frac{\pi\, t^{\alpha}}{2x^{\alpha}\ \sin\nu\pi}\left[\sum_{m=0}^{\infty}\frac{(t)^{2m}\left(\frac{\sqrt{|\omega|}}{2}\right)^{2m-\nu}}{m!\,\Gamma(m-\nu+1)}\left(\sum_{m=0}^{\infty}\frac{(x)^{2m+\alpha}\left(\frac{\sqrt{|\omega|}}{2}\right)^{2m-\nu}}{m!\,\Gamma(m-\nu+1)}\right.\right.$$

$$\left.\left. - \left(\sum_{m=0}^{\infty}\frac{(x)^{2m+\alpha}\left(\frac{\sqrt{|\omega|}}{2}\right)^{2m-\nu}}{m!\,\Gamma(m-\nu+1)} - \sum_{m=0}^{\infty}\frac{(x)^{2m+1}\left(\frac{\sqrt{|\omega|}}{2}\right)^{2m+\nu}}{m!\,\Gamma(m+\nu+1)}\right)\right)\right]$$

$$- \frac{\pi}{2\sin\nu\pi}\left[\sum_{m=0}^{\infty}\frac{(x)^{2m}\left(\frac{\sqrt{|\omega|}}{2}\right)^{2m-\nu}}{m!\,\Gamma(m-\nu+1)}\sum_{m=0}^{\infty}\frac{(t)^{2m+1}\left(\frac{\sqrt{|\omega|}}{2}\right)^{2m+\nu}}{m!\,\Gamma(m+\nu+1)}\right].$$

Appendix B

VIM and HPM

Construction of homotopy for the nonlinear SBVP (11.1) is discussed in this section.

We define the $(n+1)^{\text{th}}$ approximate solution for SBVP (11.1) as (see Section 11.2.1)

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(t)[Lu_n(t) + Nu_n(t) - g(t)]dt, \tag{B.1}$$

where $Lu_n = -u_n'' - \frac{\alpha}{t}u_n'$, $Nu_n = -f(t,u_n)$ and $g(t) = 0$.

We introduce $v = \sum_{i=0}^{\infty} p^i v_i$, $N(v) = \sum_{i=0}^{\infty} p^i H_i$ and the n$^{\text{th}}$ approximate solution $u_n = \sum_{i=0}^{n} v_i$. Also note that $N(v_0) = H_0, N(v_0 + v_1) = H_0 + H_1$ and $N\left(\sum_{i=0}^{n} v_i\right) = \sum_{i=0}^{n} H_i$.

Substituting these values into (B.1), we get

$$\left(\sum_{i=0}^{n+1} v_i\right) = \left(\sum_{i=0}^{n} v_i\right) + \int_0^x \lambda(t)\left[L\left(\sum_{i=0}^{n} v_i\right)(t) + N\left(\sum_{i=0}^{n} v_i\right)(t) - g(t)\right]dt.$$

Now, after solving the above equation for different values of n, we get

$$\begin{aligned}
v_0 &= u_0 \\
v_1 &= \int_0^x \lambda(t)[Lv_0(t) + H_0(t) - g(t)]dt, \\
v_2 &= v_1 + \int_0^x \lambda(t)[Lv_1(t) + H_1(t)]dt, \\
&\vdots \\
v_{n+1} &= v_n + \int_0^x \lambda(t)[Lv_n(t) + H_n(t)]dt, \\
&\vdots
\end{aligned}$$

Which yields

$$\begin{aligned}
v(x,p) = &\sum_{i=0}^{\infty} p^i v_i = v_0 + pv_1 + p^2 v_2 + \cdots + p^n v_n + \cdots \\
= &v_0 + \left[p\int_0^x \lambda(t)[Lv_0 + H_0 - g(t)]dt \right] + \left[p^2 v_1 + p\int_0^x \lambda(t)[Lv_1(t)p + H_1(t)p]dt \right] \\
&+ \left[p^3 v_2 + p\int_0^x \lambda(t)[Lv_2(t)p^2 + H_2(t)p^2]dt \right] + \\
&\cdots + \left[p^{n+1} v_n + p\int_0^x \lambda(t)[Lv_n(t)p^n + H_n(t)p^n]dt \right] + \cdots .
\end{aligned}$$

That gives

$$(1-p)[u_0 - v] + p\int_0^x \lambda(t)[Lv(t) + Nv(t) - g(t)]dt = 0.$$

Hence, we get the following homotopy

$$H(x,v,p) = (1-p)[u_0 - v] + p\int_0^x \lambda(t)[Lv(t) + Nv(t) - g(t)]dt = 0.$$

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Publications

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1. Mandeep Singh, Amit K. Verma, Picard type iterative scheme with initial iterates in reverse order for a class of nonlinear three point BVPs. *International Journal of Differential Equations*, Volume 2013, year 2013, Article ID 728149, 6 pages.

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1. Mandeep Singh and Amit K. Verma, Nonlinear three point Singular BVPs : A Classification, *arXiv:1508.07408 [math.CA], Classical Analysis and ODEs, 08/2015.*

Communicated

1. Amit K. Verma and Mandeep Singh, On Quasi-Newton iteration method and numerical solution of nonlinear singular boundary value problems. Under review.

2. Mandeep Singh and Amit K. Verma, Monotone iterative technique for nonlinear discrete boundary value problems. Under review.

Paper presented in International Conference

1. Amit K. Verma and Mandeep Singh, New existence results for nonlinear three point singular boundary value problems arising due to spherical symmetry, *International Conference on Advances in Mathematics – 2015 (ICAMS-2015)*, March 19-21, 2015, Khalsa College, Patiala, Punjab, India.

2. Amit K. Verma and Mandeep Singh, Existence results for a class of nonlinear three point singular boundary value problems, *South Asian Mathematics Research Scholar Meet*, May 8-9, 2015, South Asian University, Akbar Bhawan Chanakyapuri, New Delhi- 110021.

3. Amit K. Verma and Mandeep Singh, Classification of well order and reverse order cases for a class of nonlinear three point BVPs, *International Conference on Recent Advances in Mathematical Biology, Analysis and Applications-ICMBAA-2015*, June 4-6, 2015, Department of Applied Mathematics, Aligarh Muslim University, Aligarh 202 002, India.

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