Chapter 8

Conclusion and Future Scope

8.1 Summary of the Work Done

The present thesis contributes to the development of parameter-uniform layer resolving techniques, which are mainly concerned on the singularly perturbed parabolic boundary value problems of following six important classes

Class 1: Singularly perturbed parabolic differential-difference model with large delay in time

We consider the following class of second-order singularly perturbed delay parabolic PDE with Dirichlet boundary conditions on the rectangle $D = \{(x,t) \in \Omega \times \Lambda = (0,1) \times (0,T],$ where T is some fixed positive time$\}$ in the space-time plane:

$$L_\varepsilon u(x,t) \equiv \frac{\partial u}{\partial t} - \varepsilon \frac{\partial^2 u}{\partial x^2} + a(x)\frac{\partial u}{\partial x} + b(x,t)u = c(x,t)u(x,t-\tau) + f(x,t),\ (x,t) \in D. \tag{8.1a}$$

Equation (8.1a) is subject to the boundary conditions

$$u(x,t) = \phi_l(t),\ (x,t) \in \Gamma_l = \{0\} \times \Lambda = \{(0,t):\ 0 < t \leq T\}, \tag{8.1b}$$

$$u(x,t) = \phi_r(t),\ (x,t) \in \Gamma_r = \{1\} \times \Lambda = \{(1,t):\ 0 < t \leq T\}, \tag{8.1c}$$

and the interval condition

$$u(x,t) = \phi_b(x,t),\ (x,t) \in \Gamma_b = [0,1] \times [-\tau,0]. \tag{8.1d}$$

Here, $\varepsilon \in (0,1]$ is a perturbation parameter, $\tau > 0$ is a delay parameter, $a(x)$, $b(x,t)$, $c(x,t)$, $f(x,t)$, $\phi_l(t)$, $\phi_r(t)$ and $\phi_b(x,t)$ are sufficiently smooth and bounded functions, and satisfying

$$a(x) \geq \alpha > 0,\ b(x,t) \geq \beta > 0,\ c(x,t) \leq \gamma < 0, \text{ on } \bar{D}.$$

Class 2: Singularly perturbed parabolic differential-difference model with large delay in space with twin boundary layers and one interior layer

Let $D = \Omega \times \Lambda = (0,2) \times (0,T]$ be a rectangular domain in the space-time plane, where T is some fixed positive time. Consider the following class of second-order singularly perturbed parabolic PDEs on D:

$$Lu(x,t) \equiv u_t(x,t) - \varepsilon u_{xx}(x,t) + a(x)u(x,t) + b(x)u(x-1,t) = f(x,t),\ (x,t) \in D, \tag{8.2a}$$

subject to the initial condition

$$u(x,0) = u_0(x), \quad x \in \bar{\Omega}, \tag{8.2b}$$

and the interval and boundary conditions

$$u(x,t) = \phi(x,t),\ (x,t) \in D_L = \{(x,t) :\ -1 \leq x \leq 0;\ t \in \bar{\Lambda}\}, \tag{8.2c}$$

$$u(2,t) = \psi(t),\ (x,t) \in D_R = \{(2,t) :\ t \in \bar{\Lambda}\}, \tag{8.2d}$$

where $\varepsilon \in (0,1]$ is a small perturbation parameter. Again it is assumed that the functions $a(x), b(x), f(x,t), \phi(x,t), \psi(t)$ and $u_0(x)$ are sufficiently smooth, bounded, and independent of ε.

Class 3: Time dependent turning point model exhibiting twin boundary layers

Let $\Omega = (-1,1),\ \Lambda = (0,T], D = \Omega \times \Lambda, \Gamma_b = \{(x,0) : -1 \leq x \leq 1\}, \Gamma_l = \{(-1,t) : 0 \leq t \leq T\}, \Gamma_r = \{(1,t) : 0 \leq t \leq T\}$ and $\Gamma = \Gamma_l \cup \Gamma_b \cup \Gamma_r$. We consider the following singularly perturbed turning point BVP on the rectangular domain D.

$$L_\varepsilon y(x,t) \equiv -y_t + \varepsilon y_{xx} + a(x,t)y_x - b(x,t)y = f(x,t), \quad (x,t) \in D, \tag{8.3a}$$

$$y(x,0) = y_0(x), \quad x \in \Omega, \tag{8.3b}$$

$$y(-1,t) = \phi_l(t), \quad \text{on} \quad \Gamma_l, \tag{8.3c}$$

$$y(1,t) = \phi_r(t), \quad \text{on} \quad \Gamma_r. \tag{8.3d}$$

In order to ensure the existence of twin boundary layers in the solution, the problem (8.3) is considered under following assumptions

$$a(0,t) = 0, \quad a_x(0,t) < 0,\ 0 \le t \le T, \tag{8.4a}$$

$$|a(x,t)| \ge \alpha > 0, \quad 0 < \gamma \le |x| \le 1,\ 0 \le t \le T, \tag{8.4b}$$

$$b(x,t) \ge \beta > 0, \quad (x,t) \in D. \tag{8.4c}$$

To ensure that there is no other turning point in the region $[-1,1]$ it is assumed that

$$|a_x(x,t)| \ge \frac{|a_x(0,t)|}{2}, \quad (x,t) \in D. \tag{8.4d}$$

Class 4: Time-dependent multiple turning point model exhibiting single parabolic boundary layer

Let $\Omega = (0,1)$, $\Lambda = (0,T]$, $\mathscr{D} = \Omega \times \Lambda$, with boundary $\Gamma = \Gamma_l \cup \Gamma_b \cup \Gamma_r$, where $\Gamma_l = \{(0,t) \mid 0 \le t \le T\}$, $\Gamma_b = \{(x,0) \mid 0 \le x \le 1\}$ and $\Gamma_r = \{(1,t) \mid 0 \le t \le T\}$ are the left, bottom, and the right boundaries of $\mathscr{D}$. We consider the following problem

$$L\psi(x,t) \equiv -\psi_t + \varepsilon\psi_{xx} + a(x,t)\psi_x - b(x,t)\psi = f(x,t), \quad (x,t) \in \mathscr{D}, \tag{8.5a}$$

$$\psi(x,0) = \psi_b(x), \quad x \in \bar{\Omega}, \tag{8.5b}$$

$$\psi(0,t) = \psi_l(t) \text{ on } \Gamma_l, \tag{8.5c}$$

$$\psi(1,t) = \psi_r(t) \text{ on } \Gamma_r, \tag{8.5d}$$

where $0 < \varepsilon \ll 1$ is a diffusion parameter. The following assumptions are made which ensure that the problem (8.5) has a unique solution.

- The functions $a(x,t),\ b(x,t),\ f(x,t)$ in $\bar{\mathscr{D}}$ and $\psi_l(t),\ \psi_r(t),\ \psi_b(x)$ on Γ are smooth enough and bounded.
- $a(x,t) = a^*(x,t)x^p,\ p \ge 1$ where $a^*(x,t)$ is smooth and satisfies $a^*(x,t) \le \alpha < 0,\ (x,t) \in \bar{\mathscr{D}}$.
- $b(x,t) \ge \beta > 0,\ (x,t) \in \bar{\mathscr{D}}$.
- The initial function satisfies the compatibility conditions.

Class 5: Two parameter singularly perturbed parabolic model with two boundary layers

Consider the rectangular domain $R = \Omega \times \Lambda$ where $\Omega = (0,1)$, $\Lambda = (0,T]$. The boundary of the domain is $\Gamma = \Gamma_l \cup \Gamma_b \cup \Gamma_r$, where $\Gamma_l = \{(0,t) \mid 0 \le t \le T\}$, $\Gamma_b = \{(x,0) \mid 0 \le x \le 1\}$ and $\Gamma_r = \{(1,t) \mid 0 \le t \le T\}$ are the left, bottom, and the right boundaries of R. We consider the following two-parameter singularly perturbed parabolic boundary value problem on the rectangular domain R

$$L\psi(x,t) \equiv -\frac{\partial \psi}{\partial t} + \varepsilon_1 \frac{\partial^2 \psi}{\partial x^2} + \varepsilon_2 a(x,t)\frac{\partial \psi}{\partial x} - b(x,t)\psi = f(x,t), \quad (x,t) \in R, \tag{8.6a}$$

$$\psi(x,0) = \psi_b(x) \text{ on } \Gamma_b, \tag{8.6b}$$

$$\psi(0,t) = \psi_l(t) \text{ on } \Gamma_l, \tag{8.6c}$$

$$\psi(1,t) = \psi_r(t) \text{ on } \Gamma_r, \tag{8.6d}$$

where ε_1, ε_2 are two small parameters lying in $(0,1]$.

Class 6: Two parameter singularly perturbed parabolic problems with discontinuous convection term coefficient and source term

Consider the following two-parameter parabolic singularly perturbed boundary value problem defined on $D = \Omega \times \Lambda = (0,1) \times (0,T]$

$$L\psi(x,t) \equiv -\psi_t + \varepsilon_1 \psi_{xx} + \varepsilon_2 a(x,t)\psi_x - b(x,t)\psi = f(x,t), \quad (x,t) \in D^- \cup D^+, \tag{8.7a}$$

$$\psi(0,t) = \psi_l(t) \text{ on } \Gamma_l, \quad \psi(1,t) = \psi_r(t) \text{ on } \Gamma_r, \tag{8.7b}$$

$$\psi(x,0) = \psi_b(x) \text{ on } \Gamma_b, \tag{8.7c}$$

where $D^- = \Omega^- \times \Lambda$, $D^+ = \Omega^+ \times \Lambda$, $\Omega^- = (0,e)$, $\Omega^+ = (e,1)$, and $e \in \Omega$ is the point of discontinuity of $a(x,t)$ and $f(x,t)$. The boundary of the domain D is defined as $\Gamma = \Gamma_b \cup \Gamma_l \cup \Gamma_r$, where $\Gamma_l = \{(0,t) \mid 0 \le t \le T\}, \Gamma_r = \{(1,t) \mid 0 \le t \le T\}$ and $\Gamma_b = \{(x,0) \mid 0 \le x \le 1\}$. Here $0 < \varepsilon_1, \varepsilon_2 \ll 1$ are the diffusion and convection parameters respectively.

The model problem 2 is parabolic reaction-diffusion type and all other model problems are parabolic convection-diffusion type. It is well known that the classical numerical methods are unstable and fail to give the accurate results for singular perturbation problems. These type of problems have a narrow region where the solution of a differential equation changes rapidly, called boundary layer, and width of boundary layer approaches 0 as $\varepsilon \to 0$. Therefore, it is important to develop numerical methods for these problems, whose accuracy is independent of the perturbation parameter ε. In this thesis, we have used fitted-mesh methods which use classical schemes on specially designed meshes.

The main results and some important observations of this thesis are highlighted below.

In Chapter 2, a uniformly convergent numerical scheme for a class of singularly perturbed parabolic partial differential equation with the time delay (Class 1) on a rectangular domain in the x-t plane is proposed and analyzed which is applicable for delay that is either $o(\varepsilon)$ or $O(\varepsilon)$. This scheme involves a numerical method comprising the finite difference method consisting of a midpoint upwind finite difference scheme on a fitted piecewise-uniform mesh condensing in the boundary layer region. It is shown that the discrete solution obtained by this scheme is second-order accurate in the temporal direction and the first-order (up to a logarithmic factor) accurate in the spatial direction.

Then, in Chapter 3, Crank-Nicolson difference formula (on a uniform mesh) is used in time to semi-discretize the given PDE, and then the standard finite difference scheme (on a piecewise-uniform mesh) is used for the system of ordinary differential equations obtained in the semi-discretization for the case when the delay of unit length is present in the spatial direction (Class 2) which is more challenging because in addition to boundary layers, an interior layer occurs due to the delay term. The solution of these type of problems, in general, exhibits twin boundary layers (due to the presence of the perturbation parameter) and an interior layer (due to the presence of the large delay parameter in the reaction term). It is found that the method is ε-uniformly convergent of order two in the temporal direction and almost first-order in the spatial direction.

In Chapter 4, we considered a time-dependent singularly perturbed turning point differential equation of convection diffusion type (Class 3). To resolve the boundary, a fitted-mesh is constructed and the cubic B-spline basis functions on this mesh are used to discretize the given equation. Theoretical error bounds are given for the analytic solution and its derivatives. We proved first-order accuracy in the temporal direction and the second-order accuracy (up to a logarithm factor) in the spatial direction. In Chapter 5, we have considered the case of multiple boundary turning point (Class 4) at left end point of the spatial direction. We have proposed Crank-Nicolson difference formula (on a uniform mesh) in time to semi-discretize the given PDE, and then the standard finite difference scheme (on a piecewise-uniform mesh) is used for the system of ordinary differential equations obtained in the semi-discretization. The proposed scheme is proved to be parameter-uniform convergent of $\mathscr{O}((\Delta t)^2 + N^{-1}\ln N)$.

We have considered an important class of time-dependent two-parameter singularly perturbed boundary value problems (Class 5) in Chapter 6. A parameter-uniform implicit scheme is developed for two different cases: Case I. $\varepsilon_1/\varepsilon_2^2 \to 0$ as $\varepsilon_2 \to 0$, and Case II. $\varepsilon_2^2/\varepsilon_1 \to 0$ as $\varepsilon_1 \to 0$. The finite difference scheme on a predefined Shishkin mesh is used to discretize the system of ordinary differential equations in the spatial direction obtained by means of the Crank-Nicolson scheme on an equidistant mesh in the temporal direction. Through rigorous analysis, the theoretical results for two different cases are proved which show that the method is convergent irrespective to the size of the parameters $\varepsilon_1, \varepsilon_2$. The order of accuracy in the first and second cases are shown $\mathscr{O}((\Delta t)^2 + N^{-1}(\ln N)^2)$ and $\mathscr{O}((\Delta t)^2 + N^{-2}(\ln N)^2)$ respectively and efficiency of the method is proved by several numerical experiments. Then, we have generalized this numerical scheme on the time-dependent singularly perturbed two-parameter boundary value problems having discontinuous convection coefficient (Class 6) and source term whose solution exhibits dual boundary layers and an interior layer for both the cases.

8.2 Future Scope

Based on these approaches, one can extend the work in the following directions:

1. Following the approach of chapters (2 and 3), one can extend these works for higher dimensional singularly perturbed differential-difference equations.

2. The work can be extended for solving non-linear singularly perturbed boundary value problems.

3. The work can be extended for solving the system of singularly perturbed problems.