Chapter 2

A parameter-uniform numerical scheme for the parabolic singularly perturbed initial boundary value problems with large time delay

2.1 Introduction

The singularly perturbed partial differential equations (PDEs) model the physical problems for which the evolution depends on the present state of the system while the singularly perturbed delay partial differential equations (DPDEs) model physical problems for which the evolution not only depend on the present state of the system but also on the past history. The initial condition for PDEs is a function that is defined on the x-axis where $t = 0$ while in the case of DPDEs the initial condition is a function that is defined on an interval. The solution of the approximated PDE obtained by using Taylor's series expansion (the first terms of the Taylor series) in $u(x,t-\tau)$ in the given DPDE may behave quite differently from the solution of the actual DPDE. Thus the presence of small time delay may result in a large effect on the solution. So one should not ignore the lag effect and thus should not use differential equations model as a substitute for the DDEs model. For more details on this comment see Kuang [28] where he commented on the risk that researchers face if they ignore lags which they think are small. There are essential qualitative differences between DDEs and ODEs that make such a strategy risky. See Chapter 1 for more real life applications and different numerical schemes developed for these type of problems.

In this chapter, a numerical scheme for a class of singularly perturbed parabolic partial differential equation with the time delay on a rectangular domain in the x-t

plane is constructed. The presence of the perturbation parameter in the second-order space derivative gives rise to parabolic boundary layer(s) on one (or both) of the lateral side(s) of the rectangle. Thus the classical numerical methods on the uniform mesh are inadequate and fail to give good accuracy and results in large oscillations as the perturbation parameter approaches zero. To overcome this drawback a numerical method comprising the Crank-Nicolson scheme on a uniform mesh in temporal direction and a midpoint upwind finite difference scheme on a fitted piecewise-uniform mesh condensing in the boundary layer region is constructed. A priori explicit bounds on the solution of the problem and its derivatives which are useful for the error analysis of the numerical method are established. To establish the parameter-uniform convergence of the proposed method an extensive amount of analysis is carried out. It is shown that the proposed difference scheme is second-order accurate in the temporal direction and the first-order (up to a logarithmic factor) accurate in the spatial direction. To validate the theoretical results, the method is applied to two test problems. The performance of the method is demonstrated by calculating the maximum absolute errors and experimental orders of convergence. The numerical results show that the proposed method is simply applicable, accurate, efficient and robust.

The chapter is organized into the following structure. The detailed problem statement is given in Section 2.2. Some *a priori* estimates on the solution and its derivatives and some analytical results which are used in the convergence analysis are given in Section 2.3. The numerical method for the solution of singularly perturbed DPDEs based on the Crank-Nicolson is developed in Section 2.4. A brief convergence analysis of the proposed method is given in Section 2.5. In Section 2.6, numerical experiments are performed and a discussion on the results is given. Finally, some conclusions are drawn in Section 2.7.

2.2 Problem Statement: Preliminaries

We consider the following class of second-order singularly perturbed delay parabolic PDE with Dirichlet boundary conditions on the rectangle $D = \{(x,t) \in \Omega \times \Lambda = (0,1) \times (0,T]\}$ in the space-time plane

$$L_\varepsilon u(x,t) \equiv \frac{\partial u}{\partial t} - \varepsilon \frac{\partial^2 u}{\partial x^2} + a(x)\frac{\partial u}{\partial x} + b(x,t)u(x,t) = c(x,t)u(x,t-\tau) + f(x,t),\ (x,t) \in D, \tag{2.1a}$$

subject to the boundary conditions

$$u(x,t) = \phi_l(t),\ (x,t) \in \Gamma_l = \{0\} \times \Lambda = \{(0,t) :\ 0 < t \leq T\}, \tag{2.1b}$$

$$u(x,t) = \phi_r(t),\ (x,t) \in \Gamma_r = \{1\} \times \Lambda = \{(1,t) :\ 0 < t \leq T\}, \tag{2.1c}$$

and the interval condition

$$u(x,t) = \phi_b(x,t),\ (x,t) \in \Gamma_b = [0,1] \times [-\tau,0], \tag{2.1d}$$

where $\varepsilon \in (0,1]$ is a singular perturbation parameter, $\tau > 0$ represents the delay parameter, $a(x)$, $b(x,t)$, $c(x,t)$, $f(x,t)$, $\phi_l(t)$, $\phi_r(t)$ and $\phi_b(x,t)$ are sufficiently smooth and bounded functions and satisfy

$$a(x) \geq \alpha > 0,\ b(x,t) \geq \beta > 0,\ c(x,t) \leq \gamma < 0,\ \text{on } \bar{D}.$$

Denote the boundary by $\Gamma = \Gamma_l \cup \Gamma_r \cup \Gamma_b$. It is clear that the solution of (2.1a)-(2.1d) has a boundary layer of width $O(\varepsilon)$ on Γ_r. Also, the characteristics of the reduced problem of (2.1a)-(2.1d) are the vertical lines where x is a constant, thus the boundary layer arising in the solution is of parabolic type. In this study, our aim is to obtain and examine the approximate solution to observe the effect of the parameter ε on the boundary layer.

It is assumed that $T = k\tau$ for some positive integer k and the initial function $\phi_b(x,t)$ satisfies the following compatibility conditions at the corner points $(0,0)$ and $(1,0)$

$$\phi_b(0,0) = \phi_l(0), \quad \phi_b(1,0) = \phi_r(0),$$

and

$$\begin{aligned} \left.\frac{d\phi_l}{dt}\right|_{t=0} - \varepsilon \left.\frac{\partial^2 \phi_b}{\partial x^2}\right|_{(0,0)} + a(0) \left.\frac{\partial \phi_b}{\partial x}\right|_{x=0} + b(0,0)\phi_b(0,0) &= c(0,0)\phi_b(0,-\tau) \\ &\quad + f(0,0), \\ \left.\frac{d\phi_r}{dt}\right|_{t=0} - \varepsilon \left.\frac{\partial^2 \phi_b}{\partial x^2}\right|_{(1,0)} + a(1) \left.\frac{\partial \phi_b}{\partial x}\right|_{x=1} + b(1,0)\phi_b(1,0) &= c(1,0)\phi_b(1,-\tau) \\ &\quad + f(1,0). \end{aligned}$$

Under these assumptions and conditions, problem (2.1) has a unique solution [76].

However, only a few researchers focused on the numerical studies of singularly perturbed DPDEs. In most of the cases, the exact solutions of DDEs cannot be computed analytically, so the efficient numerical methods are needed to solve such equations. In literature, the DDEs have been reduced to ODEs where the coefficients depend on the delay by means of first-order accurate Taylor's series expansions of the terms that involve delay and the resulting ODEs have been solved either analytically when the coefficients of these equations are constant or numerically when they are not. Since Taylor's series expansion is valid only for the small shift, this approach fails in the case when the delay term is large. In this chapter, we provide a different approach which works very well in both the cases where the delay term is small or large. In this work, our aim is to provide an ε-uniform numerical method for the problem (2.1) with an appropriate piecewise-uniform mesh.

2.3 Some Analytical Results: A Priori Estimates

The operator L_ε in (2.1) satisfies the following maximum principle.

Lemma 2.3.1 (Maximum principle)**.** *Assume that* $\psi \in C^2(D) \cap C^0(\bar{D})$. *Suppose that* $L_\varepsilon \psi(x,t) \geq 0$ *for all* $(x,t) \in D$ *and* $\psi(x,t) \geq 0$ *for all* $(x,t) \in \Gamma$. *Then* $\psi(x,t) \geq 0$ *for all* $(x,t) \in \bar{D}$.

Proof. The result can easily be proved by contradiction. For, if there is $(\xi,\eta) \in \bar{D}$ such that $\psi(\xi,\eta) = \min_{(x,t)\in\bar{D}} \psi(x,t)$ and $\psi(\xi,\eta) < 0$. Then, we have $\psi_x|_{(\xi,\eta)} = \psi_t|_{(\xi,\eta)} = 0$ and $\psi_{xx}|_{(\xi,\eta)} \geq 0$ and thus $L_\varepsilon \psi|_{(\xi,\eta)} < 0$ which contradicts the given hypothesis and hence $\psi(x,t) \geq 0$ for all $(x,t) \in \bar{D}$. □

The following lemma shows the stability of the operator L_ε and the ε-uniform boundedness for the solution of (2.1) in the maximum norm.

Lemma 2.3.2. *The ε-uniform bound on the solution u of (2.1) is given by*

$$\|u\| \leq \|u\|_\Gamma + \frac{\|L_\varepsilon u\|}{\beta}.$$

Proof. For the barrier functions $\Psi(x,t) = \|u\|_\Gamma + \frac{\|L_\varepsilon u\|}{\beta} \pm u(x,t),\ (x,t) \in \bar{D}$, we have

$$\Psi(0,t) = \|u\|_\Gamma + \frac{\|L_\varepsilon u\|}{\beta} \pm u(0,t) \geq \|u\|_\Gamma \pm u(0,t) \geq 0,$$
$$\Psi(1,t) = \|u\|_\Gamma + \frac{\|L_\varepsilon u\|}{\beta} \pm u(1,t) \geq \|u\|_\Gamma \pm u(1,t) \geq 0.$$

Also, for $(x,t) \in \Gamma_b$

$$\Psi(x,t) = \|u\|_\Gamma + \frac{\|L_\varepsilon u\|}{\beta} \pm u(x,t) \geq \|u\|_\Gamma \pm u(x,t) \geq 0.$$

Furthermore, for all $(x,t) \in D$

$$\begin{aligned} L_\varepsilon \Psi(x,t) &= b\left[\|u\|_\Gamma + \frac{\|L_\varepsilon u\|}{\beta}\right] \pm L_\varepsilon u(x,t) \\ &\geq \beta\|u\|_\Gamma + \|L_\varepsilon u\| \pm L_\varepsilon u(x,t) \geq \|L_\varepsilon u\| \pm L_\varepsilon u(x,t) \geq 0. \end{aligned}$$

Therefore, by using maximum principle, we obtain the required result. □

Lemma 2.3.3. *The solution of the problem (2.1) satisfies the following estimate*

$$|u(x,t) - \phi_b(x,0)| \leq Ct, \quad (x,t) \in \bar{D}.$$

Proof. Using the compatibility conditions at the corner points an application of Lemma 2.3.1 on the barrier functions

$$\psi^\pm(x,t) = \begin{cases} Ct \pm (u(x,t) - \phi_b(x,0)), & 0 \leq t \leq T, \\ \pm(u(x,t) - \phi_b(x,t)), & -\tau \leq t \leq 0, \end{cases}$$

gives the required estimate. □

Lemma 2.3.4. *The solution of the problem (2.1) is bounded i.e., the solution $u(x,t)$ satisfies the following estimate*

$$|u(x,t)| \leq C, \quad (x,t) \in \bar{D}.$$

Proof. For all $(x,t) \in \bar{D}$, we have

$$|u(x,t)| = |u(x,t) - \phi_b(x,0) + \phi_b(x,0)| \leq |u(x,t) - \phi_b(x,0)| + |\phi_b(x,0)|.$$

Using Lemma 2.3.3 and the fact that $|\phi_b(x,0)|$ is bounded we get the required result. □

For the ε-uniform error estimate, we assume that the solution of (2.1) is more smooth than guaranteed by the result of above theorem. This can be done by imposing the stronger compatibility conditions at the corners. For sufficiently small $t \leq \tau$ assume that, for the data of problem (2.1), compatibility conditions are fulfilled, which ensure the required smoothness of u in a neighborhood of the corner points. Thus, we obtain

$$u \in C^{s,s/2}(\bar{D}^{\delta}), \tag{2.2}$$

where $\bar{D}^{\delta}$ is a small δ-neighborhood of the corner points and s is a parameter that ensures the required smoothness of the solution. The existence and uniqueness and the bounds on the derivatives of the solution of (2.1) are established in the following theorem:

Theorem 2.3.1. *Assume* $a \in C^{2+m}(\bar{\Omega}), b, c, f \in C^{2+m,1+m/2}(\bar{D}), \phi_l \in C^{2+m/2}([0,T])$, $\phi_b \in C^{4+m,2+m/2}(\Gamma_b), \phi_r \in C^{2+m/2}([0,T]), m \in (0,1)$. *Furthermore, assume that the compatibility conditions are satisfied at the corners. Then (2.1) has a unique solution* $u \in C^{4+m,2+m/2}(\bar{D})$. *Moreover, the derivatives of the solution u satisfy the following bounds*

$$\left|\frac{\partial^{i+j}u}{\partial x^i \partial t^j}\right| \leq C(1+\varepsilon^{-i}\exp(-\alpha(1-x)/\varepsilon)), \quad (x,t) \in \bar{D},$$

where i, j *are non-negative integers such that* $0 \leq i \leq 5$ *and* $0 \leq i+j \leq 5$.

Proof. For the proof of the existence and uniqueness, the readers may refer to [115]. The second part can be proved by transforming the variable x to the stretched variable $\zeta = x/\varepsilon$ and following the classical approach given in [76]. □

The bounds on the derivatives of the solution of (2.1) given in Theorem 2.3.1 are not adequate for the proof of the ε-uniform convergence of the proposed method. Therefore, the stronger bounds on these derivatives should be obtained by decomposing the solution into the smooth and the singular components. This decomposition method was initially given by Shishkin. Decompose the solution u into its smooth and singular components as follows

$$\underbrace{u(x,t)}_{\text{Solution}} = \underbrace{v(x,t)}_{\text{Smooth component}} + \underbrace{w(x,t)}_{\text{Singular component}} \quad (x,t) \in \bar{D},$$

where the smooth component $v(x,t)$ satisfies the non-homogeneous problem

$$L_\varepsilon v(x,t) = c(x,t)v(x,t-\tau) + f(x,t),\ (x,t) \in D, \tag{2.3a}$$

with the interval condition

$$v(x,t) = u(x,t),\ (x,t) \in \Gamma_b, \tag{2.3b}$$

and the boundary conditions

$$v(0,t) = u(0,t),\ v(1,t) = u(1,t),\ 0 \le t \le T. \tag{2.3c}$$

The smooth component $v(x,t)$ can be further decomposed by assuming necessary compatibility condition as

$$v(x,t) = v_0(x,t) + \varepsilon v_1(x,t) + \varepsilon^2 v_2(x,t), \quad (x,t) \in \bar{D},$$

where v_0 is the solution of reduced problem *i.e.,*

$$\frac{\partial v_0}{\partial t} + a(x)\frac{\partial v_0}{\partial x} + b(x,t)v_0(x,t) = c(x,t)v_0(x,t-\tau) + f(x,t), \quad (x,t) \in D,$$

with the interval conditions

$$v_0(x,t) = \phi_b(x,t), \quad (x,t) \in \Gamma_b, \quad v_0(x,t) = \phi_l(x,t), \quad (x,t) \in \Gamma_l,$$

and v_1 is the solution of the problem

$$\frac{\partial v_1}{\partial t} + a(x)\frac{\partial v_1}{\partial x} + b(x,t)v_1(x,t) = c(x,t)v_1(x,t-\tau) + \frac{\partial^2 v_0}{\partial x^2}, \quad (x,t) \in D,$$

with the interval conditions

$$v_1(x,t) = 0, \quad (x,t) \in \Gamma_b, \quad v_1(0,t) = 0, \quad t \in \bar{\Lambda},$$

and v_2 is the solution of the problem

$$L_\varepsilon v_2(x,t) = c(x,t)v_2(x,t-\tau) + \frac{\partial^2 v_1}{\partial x^2}, \quad (x,t) \in D,$$

with the interval conditions

$$v_2(x,t) = 0, \quad (x,t) \in \Gamma_b, \quad v_2(0,t) = v_2(1,t) = 0, \quad t \in \bar{\Lambda}.$$

Clearly, the smooth component $v(x,t)$ satisfies the following problem

$$L_\varepsilon v(x,t) = c(x,t)v(x,t-\tau) + f(x,t), \quad (x,t) \in D,$$

with the interval condition

$$v(x,t) = u(x,t), \quad (x,t) \in \Gamma_b,$$

and the boundary conditions

$$v(0,t) = u(0,t),\ v(1,t) = v_0(1,t) + \varepsilon v_1(1,t) + \varepsilon^2 v_2(1,t), \quad t \in \bar{\Lambda}.$$

On the other hand, the singular component satisfies the homogeneous problem

$$L_\varepsilon w(x,t) = c(x,t)w(x,t-\tau), \quad (x,t) \in D, \tag{2.4a}$$

with the interval condition

$$w(x,t) = 0, \quad (x,t) \in \Gamma_b, \tag{2.4b}$$

and the boundary conditions

$$w(0,t) = 0,\ w(1,t) = u(1,t) - v(1,t), \quad t \in \bar{\Lambda}. \tag{2.4c}$$

The bounds on the smooth and singular components are given by the following theorem

Theorem 2.3.2. *Assume that $a \in C^{4+m}(\bar{\Omega})$, $b,c,f \in C^{4+m,2+m/2}(\bar{D})$, $\phi_l \in C^{3+m/2}([0,T])$, $\phi_r \in C^{3+m/2}([0,T])$, $\phi_b \in C^{6+m,3+m/2}(\Gamma_b)$, $m \in (0,1)$, and let the condition (2.2), where $s = 6$, be satisfied. Then, for the positive integers i, j such that $0 \le i + 2j \le 4$,* *the mixed derivatives for v and w satisfy the following estimates*

$$\left|\frac{\partial^{i+j} v}{\partial x^i \partial t^j}\right| \le C(1+\varepsilon^{2-i}),\ (x,t)\in\bar{D},$$

$$\left|\frac{\partial^{i+j} w}{\partial x^i \partial t^j}\right| \le C\varepsilon^{-i}(\exp(-\alpha x/\varepsilon)+\exp(-\alpha(1-x)/\varepsilon)),\ (x,t)\in\bar{D}.$$

Proof. The proof can be done by following the approach given in [115]. □

2.4 Numerical Scheme

2.4.1 Semi-discretization for time variable

Based on the Crank-Nicolson method an implicit numerical scheme to solve (2.1) is introduced in this section. Since $u(x,t-\tau)$ term is there in our problem so in the difference scheme the point $t-\tau$ must coincide with a mesh point, to do this, we first divide the given interval $[-\tau,0]$ into m equal parts with spacing $\Delta t=\tau/m$ and use the same spacing for the interval $[0,T]$. Thus, the mesh for $[0,T]$ is defined as

$$\Lambda^M=\{t_j=j\Delta t:\ j=0,1,\dots,T/\Delta t\},$$

where we take T such that $M=T/\Delta t$ and so the mesh in the interval $[-\tau,T]$ is given by $t_j=j\Delta t,\ -m\le j\le M$. Thus the uniform meshes Λ^m_Γ and Λ^M with step-size Δt, with m and M mesh elements, are used on $[-\tau,0]$ and $[0,T]$, respectively. Introducing the operator $D_t^- v_i^j=\frac{v_i^j-v_i^{j-1}}{\Delta t}$ and the notation $v^{j+1/2}(x)=\frac{v^{j+1}(x)+v^j(x)}{2}$, we discretize (2.1) on $\Omega\times\Lambda^M$ as follows

$$\begin{aligned}
D_t^- U^{j+1}(x)-\varepsilon(U_{xx})^{j+1/2}+a(x)(U_x)^{j+1/2}+b^{j+1/2}(x)U^{j+1/2}&\\
=c^{j+1/2}(x)U^{j+1/2-m}(x)+f^{j+1/2}(x),\ x\in\Omega,\ 0\le j\le M-1,&\\
U^{j+1}(0)=\phi_l(t_{j+1}),\ U^{j+1}(1)=\phi_r(t_{j+1}),\quad 0\le j\le M-1,&\\
U^{j+1}(x)=\phi_b(x,t_{j+1}),\quad x\in\Omega,\ -(m+1)\le j\le -1,&
\end{aligned}$$

where $U^{j+1}(x)$ is the approximate solution of $u(x,t_{j+1})$ at $(j+1)$th time level. The above equations can be written in the following form

$$
\begin{aligned}
-\frac{\varepsilon}{2}(U_{xx})^{j+1}(x)+\frac{a(x)}{2}(U_x)^{j+1}(x)+\frac{d^{j+1/2}}{2}U^{j+1}(x) &= \frac{c^{j+1/2}(x)}{2}(U^{j-m+1}(x)\\
&\quad +U^{j-m}(x))+f^{j+1/2}(x)+\frac{\varepsilon}{2}(U_{xx})^{j}(x)-\frac{a(x)}{2}(U_x)^{j}(x)\\
&\quad -\frac{e^{j+1/2}(x)}{2}U^{j}(x), \quad x\in\Omega,\ 0\le j\le M-1,\\
U^{j+1}(0)=\phi_l(t_{j+1}),\ U^{j+1}(1)=\phi_r(t_{j+1}), &\quad 0\le j\le M-1,\\
U^{j+1}(x)=\phi_b(x,t_{j+1}), &\quad x\in\Omega,\ -(m+1)\le j\le -1.
\end{aligned}
$$

In the operator form, the above equations can be written as

$$
\begin{cases}
\hat{L}_\varepsilon U^{j+1}(x)=g(x,t_{j+1}), & x\in\Omega,\ 0\le j\le M-1,\\
U^{j+1}(0)=\phi_l(t_{j+1}),\ U^{j+1}(1)=\phi_r(t_{j+1}), & 0\le j\le M-1,\\
U^{j+1}(x)=\phi_b(x,t_{j+1}), & x\in\Omega,\ -(m+1)\le j\le -1.
\end{cases} \tag{2.5}
$$

where $g(x,t_{j+1})=\frac{c^{j+1/2}(x)}{2}(U^{j-m+1}(x)+U^{j-m}(x))+f^{j+1/2}(x)+\frac{\varepsilon}{2}(U_{xx})^{j}(x)$ $-\frac{a(x)}{2}(U_x)^{j}(x)-\frac{e^{j+1/2}(x)}{2}U^{j}(x)$, $d^{j+1/2}(x)=b^{j+1/2}(x)+\frac{2}{\Delta t}$, $e^{j+1/2}(x)=b^{j+1/2}(x)-\frac{2}{\Delta t}$ and the operator $\hat{L}_\varepsilon$ is defined as

$$
\hat{L}_\varepsilon U^{j+1}(x)\equiv -\frac{\varepsilon}{2}(U_{xx})^{j+1}(x)+\frac{a(x)}{2}(U_x)^{j+1}(x)+\frac{d^{j+1/2}(x)}{2}U^{j+1}(x). \tag{2.6}
$$

The finite difference operator $\hat{L}_\varepsilon$ defined in (2.6) satisfies the following maximum principle.

Lemma 2.4.1 (Maximum Principle)**.** *Assume that* $\psi^{j+1}(0)\ge 0, \psi^{j+1}(1)\ge 0$ *and* $\hat{L}_\varepsilon\psi^{j+1}(x)\ge 0$ *for all* $x\in\Omega$ *then* $\psi^{j+1}(x)\ge 0$ *for all* $x\in\bar{\Omega}$.

Proof. Suppose there exists $p\in\Omega$ such that $\psi^{j+1}(p)=\min_{x\in\Omega}\psi^{j+1}(x)<0$. It follows that $(\psi^{j+1})_x(p)=0$ and $(\psi^{j+1})_{xx}(p)\ge 0$. Then, we have

$$
\begin{aligned}
\hat{L}_\varepsilon\psi^{j+1}(p) &= -\frac{\varepsilon}{2}(\psi^{j+1})_{xx}(p)+\frac{a(x)}{2}(\psi^{j+1})_x(p)+\frac{d^{j+1/2}(p)}{2}\psi^{j+1}(p)\\
&\le \frac{d^{j+1/2}(p)}{2}\psi^{j+1}(p)<0,
\end{aligned}
$$

as $d^{j+1/2}(p) = b^{j+1/2}(p) + \frac{2}{\Delta t} \geq \beta + \frac{2}{\Delta t} > 0$. Which contradicts the given hypothesis and hence $\psi^{j+1}(x) \geq 0$ for all $x \in \bar{\Omega}$. □

The local truncation error e_{j+1} of the temporal semi-discretization *i.e.,* for equation (2.5) is given by $e_{j+1} = U^{j+1}(x) - u(x, t_{j+1})$. The estimate for e_{j+1} is given by the following lemma.

Lemma 2.4.2. *The local error estimate in the temporal direction is given by*

$$\|e_{j+1}\| \leq C(\Delta t)^3.$$

Proof. Writing $u(x, t_{j+1}) = u(x, t_{j+1/2} + \Delta t/2)$ and $u(x, t_j) = u(x, t_{j+1/2} - \Delta t/2)$ and using Taylor's series expansion up to $O((\Delta t)^3)$ the proof can easily be done by applying the above maximum principle. □

The local truncation error e_{j+1} measures the contribution of each time step to the global error of the time semi-discretization given by $E_j = \sum_{k=1}^{j} e_k$. The estimate for E_j is given by the following theorem.

Theorem 2.4.1. *The global error estimate at t_j is given by*

$$\|E_j\| \leq C(\Delta t)^2, \quad j \leq T/\Delta t.$$

Therefore, the time semi-discretization process is uniformly convergent of order two.

Proof. The global error estimate at the $(j+1)$th time step is given by

$$\begin{aligned}
\|E_j\| &= \left\| \sum_{k=1}^{j} e_k \right\|, \quad j \leq \frac{T}{\Delta t} \\
&\leq \|e_1\| + \|e_2\| + \cdots + \|e_j\|.
\end{aligned}$$

Using the estimate given in Lemma 2.4.2, we obtain

$$\begin{aligned}
\|E_j\| &\leq Cj(\Delta t)^3 \\
&= C(j\Delta t)(\Delta t)^2 \\
&\leq CT(\Delta t)^2 \quad \text{as } j\Delta t \leq T \\
&= C(\Delta t)^2.
\end{aligned}$$

□

The asymptotic behavior, with respect to ε, of the solution $U^{j+1}(x)$ of (2.5) and its derivatives, with respect to x are given by the following theorem.

Theorem 2.4.2. *The solution $U^{j+1}(x)$ of (2.5) and its derivatives satisfy the following estimates*

$$\left|\frac{d^k U^{j+1}(x)}{dx^k}\right| \leq C(1+\varepsilon^{-k}\exp(-\alpha(1-x)/\varepsilon)),\ k=0,1,2,3.$$

Proof. Using the smoothness of f and U^{j+1}, an application of the maximum principle for $\hat{L}_\varepsilon$ gives $|U^{j+1}| \leq C$. Considering the appropriate auxiliary boundary value problem the results for the derivatives can be obtained by following the technique given in [116]. □

The stronger bounds can be obtained by decomposing the solution of (2.5) into regular and singular components as

$$U^{j+1}(x) = V^{j+1}(x) + W^{j+1}(x), \quad x \in \bar{\Omega},$$

where the smooth component $V^{j+1}(x)$ satisfies the non-homogeneous problem

$$\hat{L}_\varepsilon V^{j+1}(x) = g(x,t_{j+1}), \quad x \in \bar{\Omega},$$

with the boundary condition

$$V^{j+1}(0) = U^{j+1}(0),$$

and the singular component $W^{j+1}(x)$ satisfies the homogeneous problem

$$\hat{L}_\varepsilon W^{j+1}(x) = 0, \quad x \in \bar{\Omega},$$

with the boundary conditions

$$W^{j+1}(0) = 0,\ W^{j+1}(1) = U^{j+1}(1) - V^{j+1}(1).$$

The following theorem gives the bounds for the smooth and the singular components and their derivatives.

Theorem 2.4.3. *The smooth component $V^{j+1}(x)$ and its derivatives satisfy the following bounds*

$$\left|\frac{d^k V^{j+1}(x)}{dx^k}\right| \leq C(1+\varepsilon^{2-k}),\ k=0,1,2,$$

and the singular component $W^{j+1}(x)$ and its derivatives satisfy the following bounds

$$\left|\frac{d^k W^{j+1}(x)}{dx^k}\right| \leq C\varepsilon^{-k}\exp(-\alpha(1-x)/\varepsilon),\ k=0,1,2,3.$$

2.4.2 The spatial discretization

Since there is a boundary layer at the right side of the domain, so in the space direction, the fitted piecewise-uniform mesh is constructed by dividing $\bar{\Omega}$ into two subintervals $\bar{\Omega}=\bar{\Omega}_1\cup\bar{\Omega}_2$, where $\Omega_1=(0,1-\sigma)$ and $\Omega_2=(1-\sigma,1)$, where σ is such that

$$\sigma=\min\{0.5,\sigma_0\varepsilon\ln N\},$$

where $\sigma_0\geq 1/\alpha$. A piecewise-uniform mesh Ω^N is thus obtained by placing a uniform mesh with $N/2$ mesh elements on both Ω_1 and Ω_2 respectively. Thus the fitted piecewise-uniform mesh $\Omega^N=\{x_i\}_{i=0}^N$ that discretizes the interval $[0,1]$ with N piecewise-uniform subintervals is defined as

$$x_i=\begin{cases}\frac{2(1-\sigma)}{N}i, & \text{if}\quad i=0,1,\ldots,N/2,\\ 1-\sigma+\frac{2\sigma}{N}\left(i-\frac{N}{2}\right), & \text{if}\quad i=N/2+1,\ldots,N,\end{cases}$$

with piecewise-uniform mesh spacing defined as

$$h_i=x_i-x_{i-1}=\begin{cases}\frac{2(1-\sigma)}{N}, & \text{if}\quad i=1,2,\ldots,N/2,\\ \frac{2\sigma}{N}, & \text{if}\quad i=N/2+1,\ldots,N.\end{cases}$$

The fitted-piecewise-uniform meshes $D^{N,M}$ on D and $\Gamma_b^{N,m}$ on Γ_b are then defined as the tensor products

$$D^{N,M}=\Omega^N\times\Lambda^M,\ \Gamma_b^{N,m}=\Omega^N\times\Lambda_\Gamma^m.$$

The boundary points $\Gamma^{N,M}$ on $D^{N,M}$ are given by $\Gamma^{N,M} = \bar{D}^{N,M} \cap \Gamma$. Further, we denote $\Lambda_\Gamma = (-\tau, 0)$, $\Gamma_l^M = \Gamma_b^{N,m} \cap \Gamma_l$ and $\Gamma_r^M = \Gamma_b^{N,m} \cap \Gamma_r$.

Introducing the operators

$$D_x^- v_i^j = \frac{v_i^j - v_{i-1}^j}{h_i},\ D_x^+ v_i^j = \frac{v_{i+1}^j - v_i^j}{h_{i+1}},\ \delta_x^2 v_i^j = \frac{(D_x^+ - D_x^-) v_i^j}{{}_i},$$

where ${}_i = \frac{h_i + h_{i+1}}{2}$ and the notation $v_{i-1/2}^j = \frac{v_{i-1}^j + v_i^j}{2}$, we fully discretize (2.5) on $D^{N,M}$ as follows

$$\begin{cases} L_{mp}^N \tilde{U}(x_i) = \tilde{g}(x_{i-1/2}), \quad x_i \in \Omega^N, \\ \tilde{U}(1) = \phi_l(t_{j+1}), \quad \tilde{U}(N) = \phi_r(t_{j+1}), \quad 0 \le j \le M-1, \\ \tilde{U}(x_i)|_{t=0} = \phi_b(x, 0), \quad x_i \in \Omega^N, \end{cases} \tag{2.7}$$

where $\tilde{U}(x_i)$ is the approximation of $U^{j+1}(x_i)$ that is $\tilde{U}(x_i) \approx U^{j+1}(x_i) \approx U(x_i, t_{j+1})$, the midpoint upwind operator L_{mp}^N is defined as

$$L_{mp}^N \tilde{U} :\equiv -\frac{\varepsilon}{2} \delta_x^2 \tilde{U} + \frac{a(x_{i-1/2})}{2} D^- \tilde{U} + \frac{d^{j+1/2}(x_{i-1/2})}{2} \tilde{U}.$$

The function $\tilde{g}(x_{i-1/2})$ is given by

$$\begin{aligned} \tilde{g}(x_{i-1/2}) =& \frac{c^{j+1/2}(x_{i-1/2})}{2} (U^{j-m+1}(x_i) + U^{j-m}(x_i)) + f^{j+1/2}(x_{i-1/2}) \\ &+ \frac{\varepsilon}{2} \delta_x^2 U_N^j(x_i) - \frac{a(x_{i-1/2})}{2} D^- U_N^j(x_i) - \frac{e^{j+1/2}(x_{i-1/2})}{2} U_N^j(x_i), \end{aligned}$$

where $U_N^j(x_i)$ is the approximation of U^j at x_i at j^{th} level. The value of the function at $x_{i-1/2}$ is the average of the values at x_{i-1} and x_i *i.e.,* $a(x_{i-1/2}) = \frac{a(x_{i-1}) + a(x_i)}{2}$ etc.

2.5 Parameter Uniform Convergence Analysis

The finite difference operator L_{mp}^N defined in (2.7) satisfies the following discrete maximum principle.

Lemma 2.5.1 (Discrete Maximum Principle)**.** *Assume that* $\tilde{\psi}(x_0) \ge 0$, $\tilde{\psi}(x_N) \ge 0$ *and* $L_{mp}^N \tilde{\psi}(x_i) \ge 0$ *for all* $x_i \in \Omega^N$ *then* $\tilde{\psi}(x_i) \ge 0$ *for all* $x_i \in \Omega^N$.

Proof. Assume there exists $p_i \in \Omega^N$ such that $\tilde{\psi}(p_i) = \min_{x\in\Omega^N} \tilde{\psi}(x_i)$ and $\tilde{\psi}(p_i) < 0$. Then, we have

$$\begin{aligned}
L^N_{mp}\tilde{\psi}(p_i) &= -\frac{\varepsilon}{2}\delta_x^2\tilde{\psi}(p_i) + \frac{a(x_{i-1/2})}{2}D^-\tilde{\psi}(p_i) + \frac{d^{j+1/2}(x_{i-1/2})}{2}\tilde{\psi}(p_i) \\
&= -\frac{\varepsilon}{2_i}\left(\frac{\tilde{\psi}(p_{i+1}) - \tilde{\psi}(p_i)}{h_{i+1}} - \frac{\tilde{\psi}(p_i) - \tilde{\psi}(p_{i-1})}{h_i}\right) \\
&+ \frac{a(x_{i-1/2})}{2}\left(\frac{\tilde{\psi}(p_i) - \tilde{\psi}(p_{i-1})}{h_i}\right) + \frac{d^{j+1/2}(x_{i-1/2})}{2}\tilde{\psi}(p_i) \\
&< 0,
\end{aligned}$$

which contradicts the given hypothesis $L^N_{mp}\tilde{\psi}(x_i) \geq 0$ for all $x_i \in \Omega^N$ and hence $\tilde{\psi}(x_i) \geq 0$ for all $x_i \in \Omega^N$. □

Lemma 2.5.2. *Let $\tilde{\psi}(x)$ be any mesh function on Ω^N such that $\tilde{\psi}(x_0) = \tilde{\psi}(x_N) = 0$, then*

$$|\tilde{\psi}(x_i)| \leq \max_{x_i\in\Omega^N} |L^N_{mp}\tilde{\psi}(x_i)|, \; x_i \in \Omega^N.$$

Proof. Consider the barrier functions $\Psi^{\pm}(x_i) = \max_{x_i\in\Omega^N} |L^N_{mp}\tilde{\psi}(x_i)| \pm \tilde{\psi}(x_i)$. Then

$$\begin{aligned}
\Psi^{\pm}(0) &= \max_{x_i\in\Omega^N} |L^N_{mp}\tilde{\psi}(x_i)| \pm \tilde{\psi}(0) = \max_{x_i\in\Omega^N} |L^N_{mp}\tilde{\psi}(x_i)| \geq 0, \\
\Psi^{\pm}(1) &= \max_{x_i\in\Omega^N} |L^N_{mp}\tilde{\psi}(x_i)| \pm \tilde{\psi}(1) = \max_{x_i\in\Omega^N} |L^N_{mp}\tilde{\psi}(x_i)| \geq 0,
\end{aligned}$$

Also,

$$\begin{aligned}
L^N_{mp}\Psi^{\pm}(x_i) &= L^N_{mp}\left[\max_{x_i\in\Omega^N} |L^N_{mp}\tilde{\psi}(x_i)| \pm \tilde{\psi}(x_i)\right] \\
&= \frac{d^{j+1/2}(x_{i-1/2})}{2} \max_{x_i\in\Omega^N} |L^N_{mp}\tilde{\psi}(x_i)| \pm L^N_{mp}\tilde{\psi}(x_i) \\
&= \frac{1}{2}\left(b^{j+1/2}(x_{i-1/2}) + \frac{2}{\Delta t}\right) \max_{x_i\in\Omega^N} |L^N_{mp}\tilde{\psi}(x_i)| \pm L^N_{mp}\tilde{\psi}(x_i) \\
&\geq \left(\frac{\beta}{2} + \frac{1}{\Delta t}\right) \max_{x_i\in\Omega^N} |L^N_{mp}\tilde{\psi}(x_i)| \pm L^N_{mp}\tilde{\psi}(x_i) \\
&\geq \max_{x_i\in\Omega^N} |L^N_{mp}\tilde{\psi}(x_i)| \pm L^N_{mp}\tilde{\psi}(x_i) \\
&\geq 0.
\end{aligned}$$

An application of Lemma 2.5.1 yields $\Psi^{\pm}(x_i) \geq 0, \ \forall\, x_i \in \Omega^N$ and thus the result is obtained. □

Theorem 2.5.1 (Error in the spatial direction)**.** *Let U^{j+1} be the solution of the problem (2.5) after temporal discretization and $\tilde{U}$ be the solution of (2.7) after the full discretization. Then, the error estimate is given by*

$$|\tilde{U}(x_i) - U^{j+1}(x_i)| \leq CN^{-1}(\ln N)^2, \quad x_i \in \Omega^N.$$

Proof. Let $\bar{D}^N_{k\tau} = \Omega^N \times \Lambda_k^{N_\tau}$, where $\Omega_k^{N_\tau}$ be a uniform mesh with $N_\tau(=m)$ mesh elements used in $[(k-1)\tau, k\tau]$ for some positive integer k such that $k \leq T/\tau$. Also, denote by $U^{j+1}_{k\tau}(x_i)$ the numerical approximation of $u(x_i, t_{j+1})$ on $\bar{D}^N_{k\tau}$. We'll prove the theorem by induction on the subintervals $[(k-1)\tau, k\tau]$ for k such that $k \leq T/\tau$. For the first interval $[0, \tau]$ the right-hand side of (2.1) becomes $f(x,t) - c(x,t)\phi_b(x, t-\tau)$ that is independent of ε and thus following the classical approach (see for example [117]), we obtain

$$|\tilde{U}(x_i) - U^{j+1}(x_i)|_{\bar{D}^N_{k\tau}} \leq CN^{-1}. \tag{2.8}$$

Now since the delay term $u(x, t-\tau)$ depends on ε, so the above result cannot be applied on the intervals $[(k-1)\tau, k\tau],\ k \geq 2$. Thus, we give a detailed proof in a different way by considering the estimate of smooth and singular components separately. First, we obtain the estimate on the interval $[\tau, 2\tau]$. The solution $\tilde{U}(x_i)$ of (2.7) can be decomposed into the smooth and singular components in an analogous manner as for the solution $U^{j+1}(x_i)$ of the problem (2.5). Thus

$$\tilde{U}(x_i) = \tilde{V}(x_i) + \tilde{W}(x_i),$$

where the smooth component $\tilde{V}(x_i)$ is the solution to the following inhomogeneous problem

$$\begin{aligned} L^N_{mp}\tilde{V}(x_i) &= \tilde{g}(x_{i-1/2}) \quad x_i \in \Omega^N, \\ \tilde{V}(0) = V^{j+1}(0), &\quad \tilde{V}(N) = V^{j+1}(N),\ m \leq j \leq 2m, \end{aligned}$$

and the singular component $\tilde{W}(x_i)$ is the solution to the following homogeneous problem

$$\begin{aligned} &L_{mp}^N W(x_i) = 0,\ x_i \in \Omega^N, \\ &\quad \tilde{W}(0) = W^{j+1}(0), \quad \tilde{W}(N) = W^{j+1}(N),\ m \le j \le 2m. \end{aligned}$$

The error $\tilde{U}(x_i) - U^{j+1}(x_i)$ which is the sum of the errors $\tilde{V}(x_i) - V^{j+1}(x_i)$ and $\tilde{W}(x_i) - W^{j+1}(x_i)$ is obtained by estimating the errors in smooth and singular components separately. The error in smooth component is obtained in a classical way. We have

$$\begin{aligned} L_{mp}^N(V^{j+1}(x_i) - \tilde{V}(x_i)) &= L_{mp}^N V^{j+1}(x_i) - L_{mp}^N \tilde{V}(x_i) \\ &= L_{mp}^N V^{j+1}(x_i) - \tilde{g}(x_{i-1/2}) \\ &= L_{mp}^N V^{j+1}(x_i) - \hat{L}_\varepsilon V^{j+1}(x_{i-1/2}) \\ &= (L_\varepsilon - L_\varepsilon^N) v_{j+1}(x_i). \end{aligned}$$

The classical estimates (refer to [117]) at each point $x_i \in \Omega^N$ are given by

$$|L_{mp}^N(V^{j+1}(x_i) - \tilde{V}(x_i))| \le CN^{-1}(\varepsilon + N^{-1}).$$

An application of Lemma 2.5.1, yields

$$|V^{j+1}(x_i) - \tilde{V}(x_i)| \le CN^{-1}(\varepsilon + N^{-1}), \quad x_i \in \Omega^N. \tag{2.9}$$

The error $W^{j+1}(x_i) - \tilde{W}(x_i)$ in the singular component depends on the transition parameter value *i.e.,* whether $\tau = 1/2$ or $\tau = \sigma_0 \varepsilon \ln N$. In the first case when $\tau = 1/2$ ($\sigma_0 \varepsilon \ln N > 1/2$) the mesh is uniform. The argument we used for smooth component yields

$$L_{mp}^N(W^{j+1}(x_i) - \tilde{W}(x_i)) = C(x_{i+1} - x_{i-1}) \left(\varepsilon \left| \frac{d^3 W^{j+1}}{dx^3} \right| + \left| \frac{d^2 W^{j+1}}{dx^2} \right| \right).$$

The use of Theorem 2.4.3 for the estimates of $\left| \frac{d^2 W^{j+1}}{dx^2} \right|$ and $\left| \frac{d^3 W^{j+1}}{dx^3} \right|$ and the fact $x_{i+1} - x_{i-1} \le 2N^{-1}$ lead to

$$L_{mp}^N(W^{j+1}(x_i) - \tilde{W}(x_i)) \le CN^{-1} \varepsilon^{-2}.$$

Now since $\sigma_0\varepsilon\ln N > 1/2$ so $\varepsilon^{-1} \leq 2\sigma_0\ln N$, so

$$L_{mp}^N(W^{j+1}(x_i) - \tilde{W}(x_i)) \leq CN^{-1}(\ln N)^2.$$

An application of Lemma 2.5.2 yields

$$|W^{j+1}(x_i) - \tilde{W}(x_i)| \leq CN^{-1}(\ln N)^2.$$

In the second case, when $\sigma = \sigma_0\varepsilon\ln N$ ($\sigma_0\varepsilon\ln N < 1/2$) *i.e.,* when the mesh is piecewise-uniform with the mesh spacing $2(1-\sigma)/N$ in the subinterval $[0, 1-\sigma)$ and $2\sigma/N$ in the subinterval $[1-\sigma, 1]$). We find the errors in $[0, 1-\sigma)$ and $[1-\sigma, 1]$ separately. In the first interval $[0, 1-\sigma)$, Theorem 2.4.3 yields

$$|W^{j+1}(x)| \leq C\exp(-\alpha(1-x)/\varepsilon) \leq C\exp(-\alpha\sigma/\varepsilon), \quad 0 \leq i \leq N/2.$$

In this case $\sigma = \sigma_0\varepsilon\ln N \geq \varepsilon\ln N/\alpha$ and so the above inequality yields

$$|W^{j+1}(x_i)| \leq CN^{-1}, \quad 0 \leq i \leq N/2. \tag{2.10}$$

The similar bound on $\tilde{W}(x_i)$ can be obtained by constructing an appropriate mesh function $\mathscr{W}(x_i)$ and following the approach given in [118]. Thus,

$$|\tilde{W}(x_i)| \leq CN^{-1}, \quad 0 \leq i \leq N/2. \tag{2.11}$$

The triangle inequality gives

$$|W^{j+1}(x_i) - \tilde{W}(x_i)| \leq CN^{-1}, \quad 0 \leq i \leq N/2. \tag{2.12}$$

Now by using the classical argument on the interval $[1-\sigma, 1]$, we obtain

$$|L_{mp}^N(W^{j+1}(x_i) - \tilde{W}(x_i))| \leq C\varepsilon^{-2}(x_{i+1} - x_{i-1}), \quad N/2+1 \leq i \leq N-1.$$

Since in the second interval the mesh spacing is $2\sigma/N$, therefore $x_{i+1} - x_{i-1} = 4\sigma/N$ and thus

$$|L_{mp}^N(W^{j+1}(x_i) - \tilde{W}(x_i))| \leq C\varepsilon^{-2}\sigma N^{-1}, \quad N/2+1 \leq i \leq N-1. \tag{2.13}$$

Also,

$$|\tilde{W}(x_N) - W^{j+1}(1)| = 0, \tag{2.14}$$

and using the inequality (2.12), we obtain

$$|\tilde{W}(x_{N/2}) - W^{j+1}(x_{N/2})| \leq CN^{-1}. \tag{2.15}$$

Now, for the suitable choices of C_1 and C_2, introduce the barrier function

$$\Phi_i = (x_i - (1-\sigma))C_1\varepsilon^{-2}\sigma N^{-1} + C_2N^{-1}, \quad N/2 \leq i \leq N,$$

and the mesh function

$$\Psi_i^{\pm} = \Phi_i \pm (\tilde{W}(x_i) - W^{j+1}(x_i)), \quad N/2 \leq i \leq N.$$

Then, we have

$$\begin{aligned}
\Psi_{N/2}^{\pm} &= \Phi_{N/2} \pm (\tilde{W}(x_{N/2}) - W^{j+1}(x_{N/2})) \\
&\geq (x_{N/2} - (1-\sigma))C_1\varepsilon^{-2}\sigma N^{-1} + C_2N^{-1} \pm (\mp CN^{-1}), \text{ using (2.15)} \\
&= C_2N^{-1} \pm (\mp CN^{-1}), \text{ as } x_{N/2} = 1-\sigma \\
&= (C_2 \mp C)N^{-1}.
\end{aligned}$$

Choose C_2 such that $C_2 \mp C \geq 0$ and so $\Psi_{N/2}^{\pm} \geq 0$. Now

$$\begin{aligned}
\Psi_N^{\pm} &= \Phi_N \pm (\tilde{W}(x_N) - W^{j+1}(x_N)) \\
&= (x_N - (1-\sigma))C_1\varepsilon^{-2}\sigma N^{-1} + C_2N^{-1}, \text{ using (2.14)} \\
&= \sigma C_1\varepsilon^{-2}\sigma N^{-1} + C_2N^{-1}, \text{ as } x_N = 1 \\
&= C_1\sigma_0^2 N^{-1}(\ln N)^2 + C_2N^{-1} \\
&\geq 0.
\end{aligned}$$

Also,

$$\begin{aligned}
L^N_{mp}\Psi^{\pm}_i &= L^N_{mp}[\Phi_i \pm (\tilde{W}(x_i) - W^{j+1}(x_i))] \\
&= L^N_{mp}\Phi_i \pm L^N_{mp}(\tilde{W}(x_i) - W^{j+1}(x_i)) \\
&\geq \left(\beta + \frac{2}{\Delta t}\right)((x_i - (1-\sigma))C_1\varepsilon^{-2}\sigma N^{-1} + C_2N^{-1}) \pm L^N_{mp}(\tilde{W}(x_i) - W^{j+1}(x_i)) \\
&\geq (x_i - (1-\sigma))C_1\varepsilon^{-2}\sigma N^{-1} + C_2N^{-1} \pm (\mp C\varepsilon^{-2}\sigma N^{-1}), \text{ using (2.13)} \\
&= (x_i - \sigma)C_1\varepsilon^{-2}\sigma N^{-1} + C_2N^{-1} \pm (\mp C\varepsilon^{-2}\sigma N^{-1}) \\
&= ((x_i - \sigma)C_1 \mp C)\varepsilon^{-2}\sigma N^{-1} + C_2N^{-1}.
\end{aligned}$$

Since, $x_i - \sigma \geq 0$ so we can choose C_1 such that $(x_i - \sigma)C_1 \mp C \geq 0$ and so

$$L^N_{mp}\Psi^{\pm}_i \geq 0.$$

Thus by using the discrete maximum principle on the interval $[1-\sigma, 1]$, we obtain

$$\Psi^{\pm}_i \geq 0, \quad N/2 \leq i \leq N,$$

and so for $N/2 \leq i \leq N$,

$$\begin{aligned}
|\tilde{W}(x_i) - W^{j+1}(x_i)| &\leq \Phi_i \\
&= (x_i - (1-\sigma))C_1\varepsilon^{-2}\sigma N^{-1} + C_2N^{-1} \\
&\leq C_1\varepsilon^{-2}\sigma^2 N^{-1} + C_2N^{-1}.
\end{aligned}$$

Since $\sigma = \sigma_0\varepsilon \ln N$, so

$$|\tilde{W}(x_i) - W^{j+1}(x_i)| \leq CN^{-1}(\ln N)^2. \tag{2.16}$$

Combining the error estimates for the singular components in the subintervals $[0, 1 - \sigma)$ and $[1-\sigma, 1]$ given by the inequalities (2.12) and (2.16), we obtain

$$|\tilde{W}(x_i) - W^{j+1}(x_i)| \leq CN^{-1}(\ln N)^2, \quad 0 \leq i \leq N. \tag{2.17}$$

On combining the estimates for smooth and singular components on $\bar{D}^N_{2\tau}$ given by the inequalities (2.9) and (2.17), the triangle inequality yields

$$|\tilde{U}(x_i) - U^{j+1}(x_i)| \leq CN^{-1}(\ln N)^2, \quad 0 \leq i \leq N. \tag{2.18}$$

Following the same approach, similar results can be obtained for $\bar{D}^N_{k\tau}$ for all k satisfying $k \leq T/\tau$ by induction on k. □

Theorem 2.5.2 (Error in the fully discrete scheme)**.** *Let $u(x,t)$ be the solution of the problem (2.1) and $\tilde{U}(x_i)$ be the approximation to the solution $u(x_i,t_j)$ of the fully discretized scheme given by (2.7). Then, the error estimate for the totally discrete scheme is given by*

$$|\tilde{U}(x_i) - u(x_i,t_{j+1})| \leq C((\Delta t)^2 + N^{-1}(\ln N)^2), \quad 1 \leq i \leq N-1, \ 0 \leq j \leq M-1.$$

Proof. The proof easily follows by combining the estimates given in Theorem 2.4.1 and Theorem 2.5.1. □

2.6 Numerical Illustrations

Example 2.6.1. Consider the following singularly perturbed delay parabolic initial boundary value problem

$$\begin{aligned}
\frac{\partial u(x,t)}{\partial t} - \varepsilon \frac{\partial^2 u(x,t)}{\partial x^2} + (2-x^2)\frac{\partial u(x,t)}{\partial x} + xu(x,t) &= -u(x,t-\tau) \\
&\quad + 10t^2 \exp(-t)x(1-x), \ (x,t) \in (0,1) \times (0,2], \\
u(x,t) = 0, \ (x,t) \in [0,1] \times [-1,0], \\
u(0,t) = 0, \ u(1,t) = 0, \ t \in [0,2].
\end{aligned}$$

Example 2.6.2. Consider the following singularly perturbed delay parabolic initial boundary value problem on $D = (0,1) \times (0,2]$

$$
\begin{aligned}
&\frac{\partial u(x,t)}{\partial t} - \varepsilon \frac{\partial^2 u(x,t)}{\partial x^2} + (2-x^2)\frac{\partial u(x,t)}{\partial x} + (x+1)(t+1)u(x,t) = -u(x,t-\tau) \\
&\qquad + 10t^2 \exp(-t) x(1-x), \\
&u(x,t) = 0, \ (x,t) \in [0,1] \times [-1,0], \\
&u(0,t) = 0, \ u(1,t) = 0, \ t \in [0,2].
\end{aligned}
$$

Since the exact solutions of the test problems are not known, the accuracy of the numerical results obtained by the proposed method is computed by using the double mesh principle. Let $\tilde{U}^{2N,2M}(x_i)$ be the numerical solution obtained on the fine mesh $D^{2N,2M} = \Omega^{2N} \times \Lambda^{2M}$ with $2N$ mesh intervals in the spatial direction and $2M$ mesh intervals in the temporal direction. The mesh $D^{2N,2M}$ is obtained by the mesh $D^{N,M}$ by inserting N and M more points in the spatial and temporal directions, respectively, by selecting the midpoints of all $\{(x_i,t_j)\}$ *i.e.,* $(x_{i+1/2},t_j) = \frac{(x_i,t_j)+(x_{i+1},t_j)}{2}$ for $j = 0,1,2,\ldots,M$. Note that in this way the transition parameter will be same as in the original mesh. Then, for each value of ε, the maximum pointwise error is estimated by

$$
E_{\varepsilon}^{N,M} = \max_{t_j} \left(\max_{i} |\tilde{U}^{2N,2M}(x_i,t_j) - \tilde{U}^{N,M}(x_i,t_j)| \right),
$$

and the corresponding order of convergence is calculated by

$$
p_{\varepsilon}^{N,M} = \log_2 \left(\frac{E_{\varepsilon}^{N,M}}{E_{\varepsilon}^{2N,2M}} \right).
$$

The ε-uniform error is calculated by using

$$
E^{N,M} = \max_{\varepsilon} E_{\varepsilon}^{N,M},
$$

and the ε-uniform order of convergence is calculated as

$$
p^{N,M} = \log_2 \left(\frac{E^{N,M}}{E^{2N,2M}} \right).
$$

Table 2.1: $E_\varepsilon^{N,M}$, $E^{N,M}$, $p_\varepsilon^{N,M}$ and $p^{N,M}$ for Example 2.6.1.

	N					
ε	16	32	64	128	256	512
2^{0}	2.29e-03	1.31e-03	6.99e-04	3.61e-04	1.83e-04	9.23e-05
	0.8058	0.9062	0.9533	0.9802	0.9874	
2^{-4}	1.18e-02	9.18e-03	6.04e-03	3.61e-03	2.05e-03	1.13e-03
	0.3622	0.6039	0.7425	0.8164	0.8593	
2^{-8}	2.78e-02	1.39e-02	6.30e-03	2.68e-03	1.35e-03	6.63e-04
	1.0000	1.1417	1.2331	0.9893	1.0259	
2^{-12}	3.36e-02	1.81e-02	9.13e-03	4.49e-03	2.17e-03	1.03e-03
	0.8925	0.9873	1.0239	1.0490	1.0751	
2^{-16}	3.40e-02	1.84e-02	9.36e-03	4.66e-03	2.30e-03	1.14e-03
	0.8858	0.9751	1.0062	1.0187	1.0126	
2^{-20}	3.41e-02	1.84e-02	9.38e-03	4.67e-03	2.31e-03	1.15e-03
	0.8901	0.9720	1.0062	1.0155	1.0063	
2^{-24}	3.41e-02	1.84e-02	9.38e-03	4.67e-03	2.31e-03	1.15e-03
	0.8901	0.9720	1.0062	1.0155	1.0063	
2^{-28}	3.41e-02	1.84e-02	9.38e-03	4.67e-03	2.31e-03	1.15e-03
	0.8901	0.9720	1.0062	1.0155	1.0063	
2^{-32}	3.41e-02	1.84e-02	9.38e-03	4.67e-03	2.31e-03	1.15e-03
	0.8901	0.9720	1.0062	1.0155	1.0063	
$E^{N,M}$	3.41e-02	1.84e-02	9.38e-03	4.67e-03	2.31e-03	1.15e-03
$p^{N,M}$	0.8901	0.9720	1.0062	1.0155	1.0063	

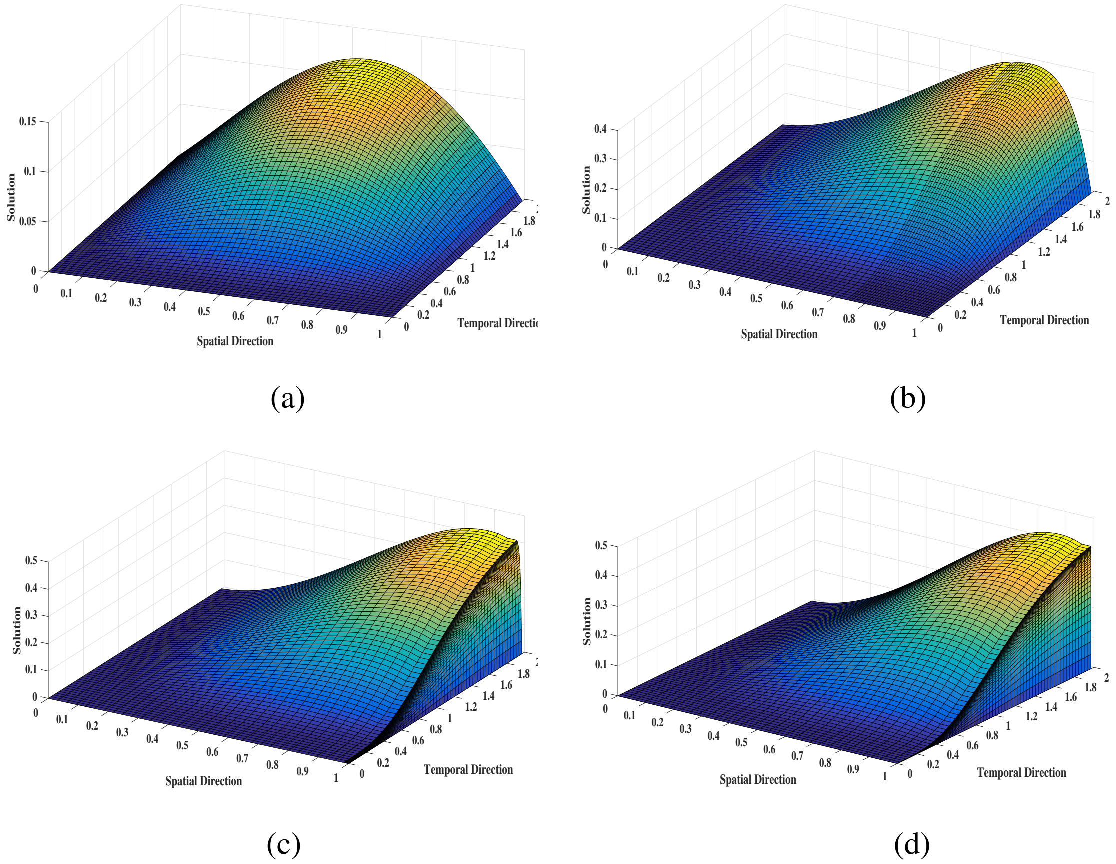

Figure 2.1: Numerical solution profiles for Example 2.6.1 for different values of ε (a) $\varepsilon = 1$ (b) $\varepsilon = 2^{-4}$ (c) $\varepsilon = 2^{-8}$ and (d) $\varepsilon = 2^{-12}$.

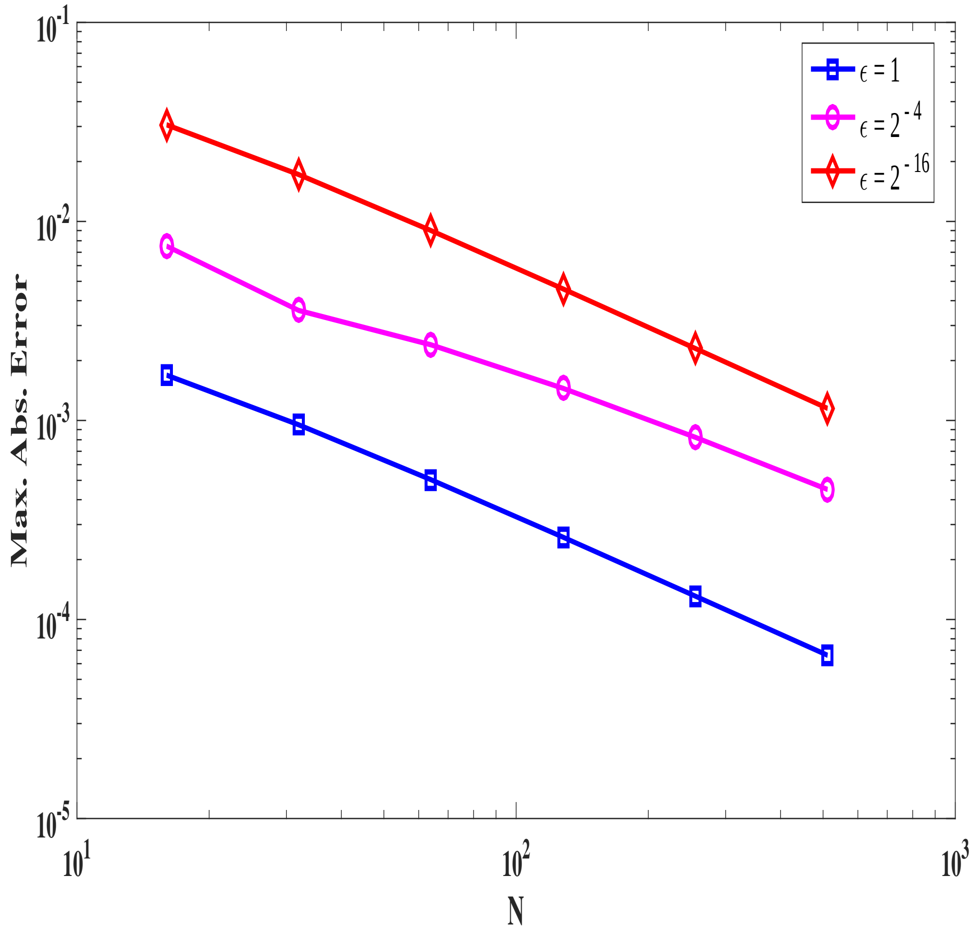

Figure 2.2: The log-log error plot for Example 2.6.1.

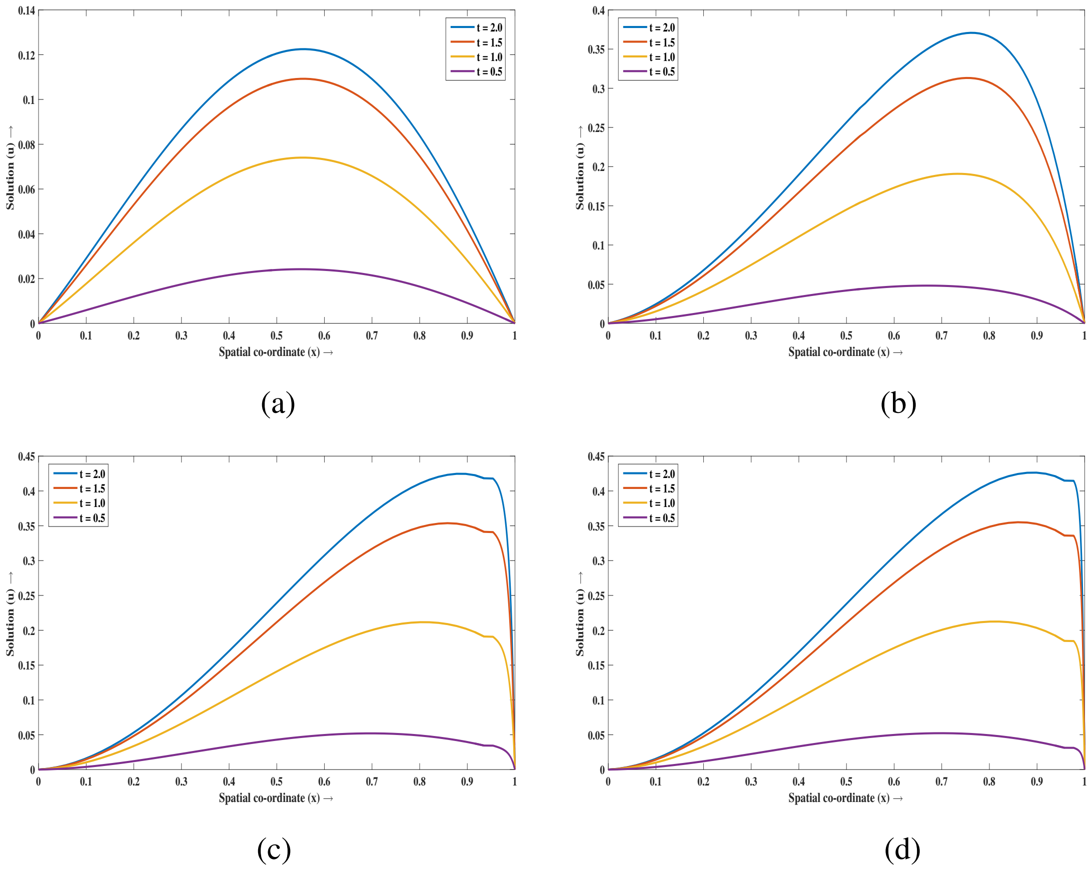

Figure 2.3: Numerical solution for Example 2.6.1 for different values of ε and t (a) $\varepsilon = 1$ (b) $\varepsilon = 0.1$ (c) $\varepsilon = 0.01$ and (d) $\varepsilon = 0.005$.

Table 2.2: $E_\varepsilon^{N,M}$, $E^{N,M}$, $p_\varepsilon^{N,M}$ and $p^{N,M}$ for Example 2.6.2.

	N					
ε	16	32	64	128	256	512
2^{0}	1.69e-03	9.51e-04	5.04e-04	2.59e-04	1.31e-04	6.61e-05
	0.8295	0.9160	0.9605	0.9834	0.9868	
2^{-4}	7.52e-03	3.57e-03	2.40e-03	1.45e-03	8.23e-04	4.50e-04
	1.0748	0.5729	0.7270	0.8171	0.8710	
2^{-8}	2.59e-02	1.40e-02	6.97e-03	3.34e-03	1.59e-03	7.88e-04
	0.8875	1.0062	1.0613	1.0708	1.0128	
2^{-12}	3.03e-02	1.69e-02	8.85e-03	4.48e-03	2.23e-03	1.10e-03
	0.8423	0.9333	0.9822	1.0065	1.0195	
2^{-16}	3.06e-02	1.72e-02	8.99e-03	4.58e-03	2.30e-03	1.15e-03
	0.8311	0.9360	0.9730	0.9937	1.0000	
2^{-20}	3.06e-02	1.72e-02	9.00e-03	4.58e-03	2.30e-03	1.15e-03
	0.8311	0.9344	0.9746	0.9937	1.0000	
2^{-24}	3.06e-02	1.72e-02	9.00e-03	4.58e-03	2.30e-03	1.15e-03
	0.8311	0.9344	0.9746	0.9937	1.0000	
2^{-28}	3.06e-02	1.72e-02	9.00e-03	4.58e-03	2.30e-03	1.15e-03
	0.8311	0.9344	0.9746	0.9937	1.0000	
2^{-32}	3.06e-02	1.72e-02	9.00e-03	4.58e-03	2.30e-03	1.15e-03
	0.8311	0.9344	0.9746	0.9937	1.0000	
$E^{N,M}$	3.06e-02	1.72e-02	9.00e-03	4.58e-03	2.30e-03	1.15e-03
$p^{N,M}$	0.8311	0.9344	0.9746	0.9937	1.0000	

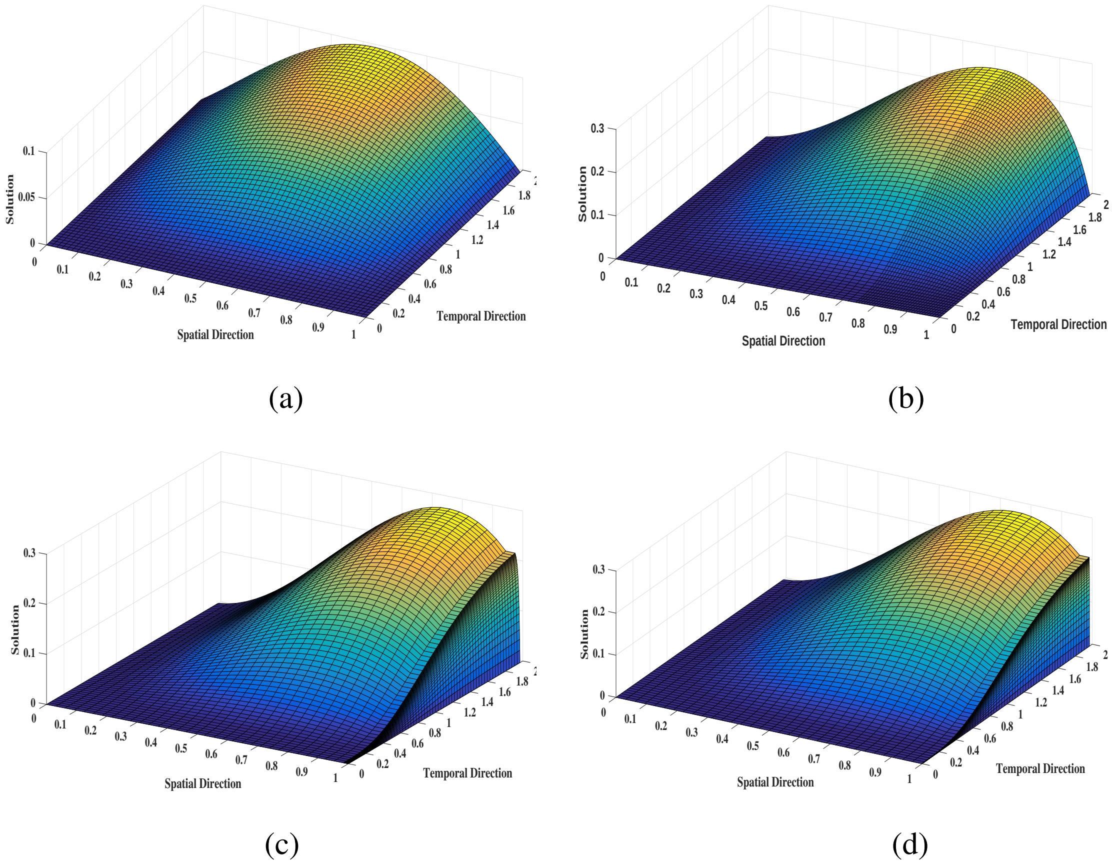

Figure 2.4: Numerical solution profiles for Example 2.6.2 for different values of ε (a) $\varepsilon = 1$ (b) $\varepsilon = 2^{-4}$ (c) $\varepsilon = 2^{-8}$ and (d) $\varepsilon = 2^{-12}$.

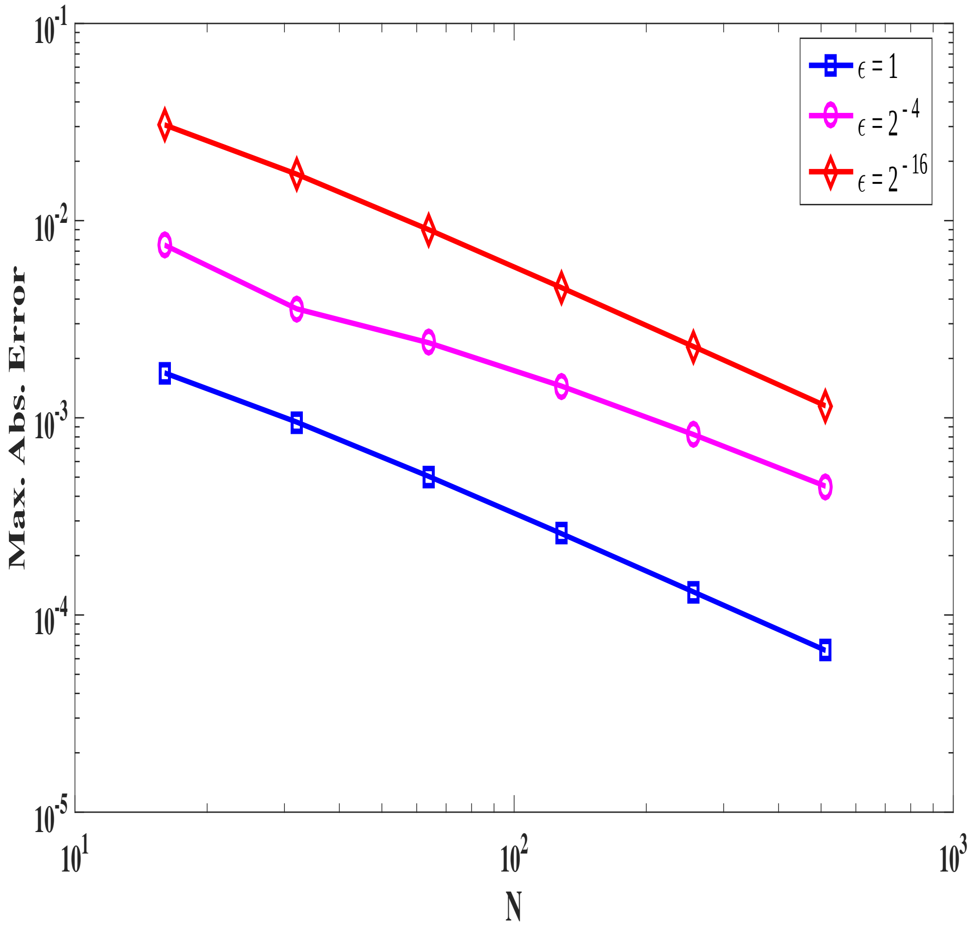

Figure 2.5: The log-log error plot for Example 2.6.2.

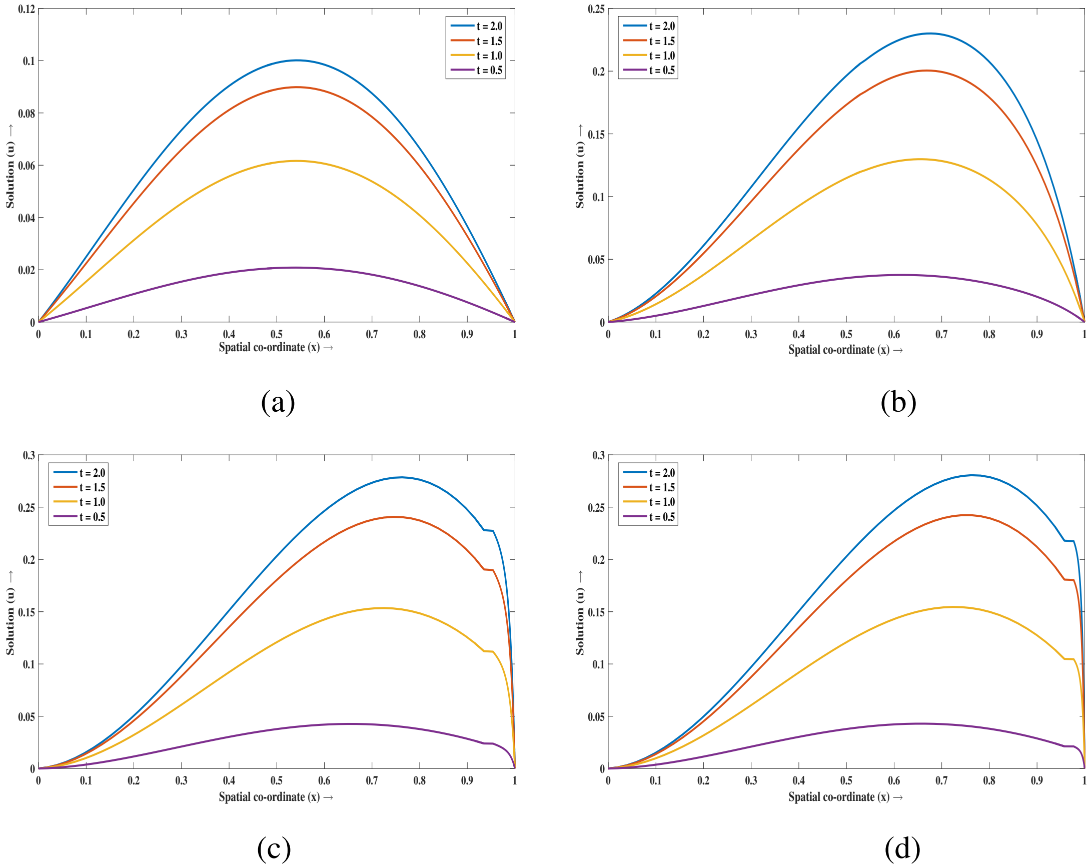

Figure 2.6: Numerical solution for Example 2.6.2 for different values of ε and t (a) $\varepsilon = 1$ (b) $\varepsilon = 0.1$ (c) $\varepsilon = 0.01$ and (d) $\varepsilon = 0.005$.

The computed maximum pointwise errors $E_\varepsilon^{N,M}$, the ε-uniform errors $E^{N,M}$, the orders of convergence $p_\varepsilon^{N,M}$ and the corresponding ε-uniform orders of convergence $p^{N,M}$ obtained by the proposed scheme are listed in Tables 2.1 and 2.2. From these results, we clearly see that the convergence is independent of the diffusion parameter ε and is according to Theorem 2.5.2. All the computations have been done by taking $\sigma_0 = 1$, and the results shown in Tables 2.1 and 2.2 are obtained by taking $M = N$.

To visualize the appearance of the boundary layers in the solutions and to show the effect of the parameter ε on the boundary layer width, the surface plots (Figs. 2.1 and 2.4) have been plotted for both the examples. These two graphs (Figs. 2.1 and 2.4) are plotted by taking $M = N = 64$. Figs. 2.3 and 2.6 provide the solution for Example 2.6.1 and 2.6.2 for different values of time t. These two graphs (Figs. 2.3 and 2.6) are plotted by taking $M = N = 100$. The maximum pointwise errors for the solutions are also plotted on a log-log scale in Figures 2.2 and 2.5. From these figures, one can easily observe the ε-uniform convergence.

2.7 Conclusion

A robust implicit unconditionally stable numerical method on a fitted piecewise-uniform mesh condensing in the boundary layer region is constructed for solving a class of singularly perturbed parabolic partial differential equations with time delay. The mesh is constructed in such a way that the point $t-\tau$ coincides with one of the collocation points. The method is shown second-order accurate in time and almost first-order accurate in space. The method can be extended to equations involving delays as well, where the delays may be constant, time-dependent or random. The proposed method can also be extended to the singularly perturbed parabolic problems exhibiting parabolic boundary layers in the neighborhood of both left and right part of the given domain as ε approaches 0. However, in this case, the solutions have, in general, a different kind of layer than the layers examined in this chapter. Test examples are presented which numerically validate the theoretical result. In Figs. 2.1 and 2.4 the numerical results are plotted in order to show the physical phenomenon of the given problems. The numerical results presented in Tables 2.1 and 2.2 show the convergence of the proposed method independent of ε.