The Effect of Shear Flexibility & Rotatory Inertia on Flexural Motion of Isotropic, ELASTIC PLATES Subjected to Transverse Impact Loadings

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CERTIFICATE

This thesis is being submitted under Regulation 16.4 of the Academic Regulations for Doctoral Programmes which allows a faculty member of the Institute to do Ph.D. research without the benefit of a Supervisor.

I hereby certify that the thesis entitled 'The Effect of Shear Flexibility and Rotatory Inertia on Flexural Motion of Isotropic, Elastic Plates Subjected to Transverse Impact Loadings', which I have submitted for award of Ph.D. degree of the Institute embodies my original work.

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ABSTRACT

An equation of motion for isotropic, elastic plates subjected to transverse loading has been developed. This equation includes the effects of shear flexibility and rotatory inertia terms. The approach has the flexibility of choosing the shape functions dependent on the actual problem in hand and is capable of including the additional terms due to the vertical compression also.

The Navier type analytical solution of a simply-supported plate is attempted. A problem of the vibration of a moderately thick concrete plate having thickness-width ratio of 0.1 is solved. The natural frequencies of the plate are obtained and tabulated. The load-deflection curves are plotted for different types of pulses. The contribution of rotatory inertia and shear deformation is then studied for a rectangular pulse. The results are also obtained for a steel plate.

The finite difference analog for the equation of motion of the plate is prepared with an object of tackling the different types of boundary and loading conditions. The analog is checked for the stability criterion and the results obtained for a simply supported rectangular plate are compared with the corresponding analytical solution. The method is then applied for solution of a square plate,

fixed on its boundaries. A plate subjected to a triangular pulse (i.e. blast wave) is also solved. The time-deflection curves are plotted for all these cases.

The solution for a circular plate, fixed on all the sides is then obtained by using the polar coordinate system.

The technique developed earlier is found to be applicable to this problem also.

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NOTATION

The following symbols are adopted in this thesis.

a	:	Length of a plate
a_{mn}	:	A constant used in the series solution.
a_{m1n1}	:	A particular coefficient of the series.
b	:	Width of a plate.
D	:	Flexural rigidity
$F(t)$	:	Function of time
$f(s)$, $f1(s)$	:	Laplace transforms of ϕ_{mn} and $F(t)$ respectively.
G	:	Shear Modulus.
H	:	Plate thickness.
i, j, k, m, n	:	Integers.
n	:	Dimensionless parameter used in the solution of circular plates.
K	:	Numerical factor taking into account the parabolic shape of the shear stress distribution.
M_x, M_y, M_z, M_{xy}	:	Moment components per unit length of the plate.
p	:	$\frac{m^2 \pi^2}{a^2} + \frac{n^2 \pi^2}{b^2}$
q	:	Transverse loading per unit area.
q_o	:	Maximum value of the uniformly distributed load acting on the plate in the transverse direction.

Q_x, Q_y :	Shear components per unit length of the plate.
t :	Time parameter.
t_o :	duration of application of the load.
u, v, w :	Displacement components along x,y,z directions of the plate respectively.
w_s :	Static deflection (cm).
$u1, v1, w1$:	The unknown displacement components of a plate separating the variation w.r.t. its thickness.
$\bar{u}1, \bar{v}1, \bar{w}1$:	Virtual displacements.
x, y, z :	The Cartesian coordinates used for the solution of a rectangular plate.
Y :	Young's Modulus.
$\Delta x, \Delta y, \Delta t$:	Finite-differences in the x,y,t directions respectively.
$E, F, I, M, M1, E_{N_S}, E_{N_R}$:	Constants appearing in the equation of motion relating the various plate properties.
A,A1,m4,m41,m42,m43,p4,q4:	Constants appearing in the transformed equation of motion.
a',a1,b',c,d1,d2,e,f, g,g2,h1,h2,h11,h21,h12,h22, m,n1,n2,p1,r,s,s1,A,R, 1	The constants appearing in the finite-difference analog of the equation of motion in Cartesian coordinates.

$a, a_o, a_{o1}, b_1, b_2, c, c_o$, $c_{o1}, d, d_o, d_{o1}, e, f_1, f_2$, $g, h_1, h_2, R_{11}, R_2, R_{21}$, R_3	:	The constants defined in the finite difference analog of the equation of motion in polar coordinates.
r, ϕ	:	The polar coordinates used for the solution of a circular plate.
$\Delta r, \Delta \phi$	:	Finite-differences in r, ϕ directions respectively.
ϕ_{mn}	:	Function of time.
ϕ_x, ϕ_y	:	Average rotations of the section x = constant, y = constant respectively.
λ_{mn}	:	Natural frequencies.
ψ	:	A function defining relationship between u1,v1.
ν	:	Poisson's ratio
ρ	:	Mass density per unit volume of the material.
$\epsilon_x, \epsilon_y, \epsilon_z$	:	Normal strain components.
$\sigma_x, \sigma_y, \sigma_z$	:	Normal stress components.
$\tau_{xy}, \tau_{yz}, \tau_{zx}$	:	Shear stress components
$\gamma_{xy}, \gamma_{yz}, \gamma_{zx}$	:	Shear strain components.
$\overline{\gamma}_x, \overline{\gamma}_y$	:	Average transverse shear strains for the section x = constant, y = constant respectively.
∇^2, ∇_r^2	:	Laplacian operators in Cartesian and Polar coordinates respectively.

CHAPTER-1

I N T R O D U C T I O N

1.1 GENERAL REMARKS:

Some systems are subjected to abrupt excitations. Physical examples are a punch press, the striking of a hammer, automotive travel on a rough road, the shooting of a gun, gust force on an airplane and the dropping of a package. The problem of dynamic response of plates also arises in Wing theory, in connection with turbulent boundary layers and very commonly in machine design (mechanical vibration). An abrupt excitation can cause strong transient vibrations which are important to analyse if the system is elastic. An engineer may be called upon to minimize the adverse effects of undesirable vibrations, or, at other times, to enhance desirable vibrations.

The study of a system subjected to the dynamic impulsive loading is considered to be of great importance, particularly when it is exposed to the nuclear weapons, the modern aircraft industry and the missile structures. For a pulse excitation, the steady state response is zero and one is only interested in the transient motion whose amplitude variation can be rapid.

In common engineering practice, it is recognised that a plate is simplest and the most common element in many structures. Many investigators have worked on the solutions of plate problems, but, its study for a rapidly applied force, displacement, or velocity excitation whose time duration is quite short and that has some large first derivative values (i.e., blasts, gas explosion, sonic booms and other accidental loads) is very much limited. Present investigation is an attempt in this direction.

The classical 'Thin-plate' (thickness-width ratio ≤ 0.1) theory fails to give reasonable values for the situations when the system under consideration involves the plates with higher thickness-width ratio. Examples of such systems are the Launching pad of a launching station, the concrete plug of an open caisson, or, some special purpose floor slabs. The present investigation makes an attempt in removing this limitation of thickness-width ratio by incorporating the rotatory inertia and shear deformation terms in the analysis.

It is easy to see that during vibration the elements of a plate perform not only a translatory motion but also rotate. The variable angle of rotation is equal to the slope of the deflection curve. This slope will obviously have the corresponding components of the angular velocity and the angular acceleration. Therefore, the inertial moment of the plate about an axis through its centre of mass and perpendicular to the respective plane will also come into picture. This moment is taken into account while writing the equations of dynamic equilibrium of a plate element. This contribution in the equation of motion is termed as the 'Rotatory Inertia'.

The classical theory is not entirely satisfactory for determining the transverse shears in a plate. Because of the fact that shear stresses $\tau_{xz} = \tau_{yz} = 0$ is the basic assumption of the classical theory, the final expression for the intensity of transverse shearing force 'Q_x' is a function of the poisson's ratio 'ν' while, the expression for the shear $V_y = Q_y + \frac{\partial M_{yx}}{\partial x}$ is not a function of the poisson's ratio. It is inconsistent with the equilibrium requirement for the strip of the plate. This fact was called to the attention of Langhaar[23] by M.C. Stippes.

The governing equations of the Reissner theory[13,14,15] for the bending of homogeneous isotropic plates included the expressions for the average rotations ϕ_x and ϕ_y of the section x = constant, and, y = constant respectively. The nature of the expressions indicated that the straight lines originally normal to the middle surface remained straight but not normal to the deformed middle surface. Further, the average rotation of a section was taken as the rotation in which normal remain perpendicular to the middle surface plus an additional rotation due to the 'Transverse Shear'. Referring to figure (1.1.1), the total rotation ϕ_x can be written as

$$\phi_x(x,y) = \frac{\partial w}{\partial x} + \bar{\gamma}_x(x,y) \qquad (1.1\text{-a})$$

where, $\bar{\gamma}_x$ denotes an average transverse shear strain for the section x = constant, and, w is the transverse, displacement.

Similarly, for the section y = constant,

$$\phi_y(x,y) = \frac{\partial w}{\partial y} + \bar{\gamma}_y(x,y) \qquad (1.1\text{-b})$$

The displacement at any point within the plate in the Reissner theory were computed on the basis that the total rotations ϕ_x and ϕ_y were small and that the transverse strain $e_z = \frac{\partial w}{\partial z}$ was negligible. In the present investigation, e_z is not assumed to be zero.

The dynamic loads are random in time and space or in time only. If the duration of load is short compared with the natural period of the structure, it is considered to be an 'Impulse', which represents the amount of energy that must be absorbed by the system.

A good example of dynamic loading condition with a short time history is the blast wave from a nearly instantaneous release of energy such as an explosion. Blast waves in an air media induce three separate and distinct auxiliary wave functions namely over pressure, dynamic pressure and reflection. The y can be considered as super-imposed dynamic loading conditions that play the major role in the design of structures[18]. The blast waves are approximated by triangular force-pulse with zero rise time (Fig.1.1.2.d). Similarly, the rectangular pulse (fig. 1.1.2.a), the sawtooth pulse (fig. 1.1.2.b) and the triangular pulse (fig. 1.1.2.c) are a few examples of such types of approximations.

The analysis of pulse and other discontinuous excitation problems cannot be easily done by classical methods; hence, the energy methods are usually employed for such problems. Since the advent of computer, numerical methods have developed very fast, which render handling of various initial, boundary and loading conditions easy.

1.2 HISTORICAL BACK GROUND:

The problem of flexural vibration of rectangular, single panelled plate of uniform thickness has been investigated by many investigators. Exact analytical solutions to plate problems (for three-dimensional case) have not been found, except for cases where two parallel edges of a uniform plate are freely supported. To analyze any other type of plates, approximate techniques have been employed. Of these approximate methods, the most widely used so far have been the Rayleigh, and, the Rayleigh-Ritz methods[57]. The success of these energy methods depends on the accurate initial assumption of the waveform of the vibrating plate under consideration.

Warburton (1954) obtained results for rectangular, isotropic, uniform plates with all the combinations of free, freely supported and clamped edges using the Rayleigh method, while Kanzawa and Kawai (1952) studied the orthotropic plate problem for several conditions by integral equation method. Huttington and Hoppmann (1958) used a Le'vy type approach and considered the case of orthotropic plates in which two of the opposite edges are simply supported. Hearmon and Rajappa (1959) gave frequency equations for orthotropic plates with simply supported, and, clamped edges.

Melosh (1963) used the principle of minimum potential energy to obtain the finite element formulation for complex structures. Gallagher (1963), Dawe (1965), Schmidt (1968), Edoardo (1969), Alferd and Charles (1970), Richard and Emmett (1971) made a significant contribution in the development of the finite-element method for the solution of plate problems.

Herrmann[17,32], Kurata and Ohannura[35], Raju[41] Deveral and Thorne[33] etc., have also done significant work towards the vibration study of the plate problems, but none of them has made any attempt to include either rotatory inertia or the shear deformation terms in their studies.

The first, successful attempt to include the effects of transverse shear deformation and normal pressure was made by Reissner[13,14,15]. The thick plate theory developed by him in 1945 is significant and is widely accepted. It is characterized by a sixth-order system of linear partial differential equations in terms of the transverse deflections, w, and the two shear stress resultants Q_x and Q_y. The nature of governing equations permits the specification of three boundary conditions on an edge and requires the evaluation of only two functions. Thus, the twisting moment M_{xy}, on edges which are free from shearing stresses τ_{xy}, is required to vanish.

Elements along these edges may then experience shear deformations parallel to the edges.

Analysis of the Reissner theory have been presented by Donnell, Druckar, and, Goodier[8]. Green[19] has shown that Reissner's equations can be conveniently derived from the stress equilibrium equations and the stress-strain relations. Salerno and Goldberg[51] applied Reissner's theory to the bending of simply supported rectangular plates subjected to a uniform transverse load. The edge boundary conditions were taken to be the vanishing of deflection, normal bending moment, and rotation of the edges in the direction of the edges.

Carley and Langhaar have also applied the Reissner's theory to the bending of a simply supported rectangular plate subjected to a uniform transverse load. In their studies, the third boundary condition was taken to be the vanishing of the edge twisting moment M_{xy}, that eliminated the presence of concentrated corner reactions which were necessary to preserve static equilibrium in the classical Kirchoff-Love theory. Frederick[16] presented solutions to the Reissner's thick-plate equations for problems concerning the bending of rectangular plates supported by an elastic foundation. He made a special mention about the need of three boundary conditions per edge. An attempt was made towards the study of the thickness-width parameter at which the inclusion of the effects of shear deformation and normal pressure could become important.

Discrete element analysis techniques including the effects of transverse shear deformation have been suggested in recent literature, but, have not been investigated in detail. Smith(55) has included the effects of transverse shear in the development of a stiffness matrix for a rectangular, moderately thick plate element. Love's theory for moderately thick plates was adopted for this formulation, because the stress resultants were expressed as distinct functions of the transverse displacement. Thus, additional degrees of freedom associated with shear deformation were not considered. The theory using seventh order displacement functions, which ensure inter-element compatibility and the generalized nodal displacements was applied to four examples. For the case of a square, simply supported plate with a uniformly distributed load and a thickness-width ratio of 0.1, the analysis gave an increase in the central deflection of 3.5 percent over the thin plate value. The Reissner's theory(51) gave an increase of 4.4 percent of this case.

Anderheggen[2] developed a finite-element procedure for solving plate bending problems based on a complementary energy or equilibrium approach. A two field formulation has been described in which the unknowns of the problem were both the stresses and displacement parameters. The displacement parameters were Lagrangian multipliers which provided approximate information on the deflected shape of the plate. Because both stresses and displacement parameters could be specified along a boundary, the anomalies found in the Kirchoff-Love theory such as 'Kirchoff-shear forces' and 'concentrated corner forces' were not present. However, the strain energy due to transverse shear deformation was not included in the development, and, the transverse shear forces Q_x and Q_y, were defined by the moment equilibrium equations. The procedure gave results that were in agreement with the classical theory for thin, uniformly loaded rectangular plates with various support conditions.

Herrmann[24] included the effects of transverse shear deformation by employing a mixed variational principle. The unknown nodal parameters were the transverse deflection w and the three moment resultants M_x, M_y and M_{xy}. The analysis when applied to a thick circular plate simply supported at its outer edge and subjected to a central hole, gave results in excellent agreement with an exact thick plate solution.

Clough and Fellippa(7) described a simple shear distortion mechanism which could be incorporated into an existing finite-element formulation without altering its basic structure. The mechanism is implemented by expressing the total rotation of a cross-section as the sum of mid surface plus a straight line rotation which represents a uniform shear strain through the thickness. The stiffness of a triangular sub-element is developed by considering two additional degrees of freedom due to shear distortion of each corner node. The shear deformation degrees of freedom are later suppressed by a static condensation procedure which is performed at the element level. Thus, the structure of the triangular sub-element is not altered by the inclusion of the shear deformation and it can be summed with similar sub-elements to form a quadrilateral element. The analysis has been applied to the bending of a square, simply supported plate subjected to a uniform load and to a central concentrated load. To test the shear deformation capability a moderately thick plate with a thickness-width ratio of 0.1 was chosen. The increase in central deflection over the classical thin plate value was found to be 10 percent for the plate with a uniform load and 234 percent for the plate with a central concentrated load. This is in disagreement with Reissner's theory, which gave a result of 4.4 percent for the plate with uniform load and with Smith's result of 3.7 percent for the plate with a central concentrated load.

A finite-element analysis for the bending of rectangular plates, with a transverse shear deformation capability has been developed by Pryor(6). The field equations of the Reissner theory are used as guidelines for formulating displacement functions associated with shear deformation degrees of freedom. The procedure is similar to that described by Clough and Fellippa(7) with certain fundamental differences such as the retention of the degrees of freedom associated with shearing deformations in the structural assemblage and the treatment of three boundary conditions on an edge. To test the capability of the analysis, results were compared with that of the Reissner theory for several example problems. Of particular interest is the ability of the finite-element analysis to satisfy the edge condition of twisting moment M_{xy}, and to accurately represent the distribution of shear stress resultants, Q_x and Q_y, along an edge. Results are in excellent agreement with those of the Reissner theory for maximum displacements and for distribution of stress resultants along supports. The added transverse shear degrees of freedom enable the element to satisfy boundary conditions that could not be handled by customary finite-element analysis.

An experimental verification of the shear effect was made by Goens[12]. The need to consider the shear deformation in the case of impact on a beam has been discussed by Flügge[62]. An interesting study of the contribution of rotatory inertia and shear deformation for beams appears in a recent book on 'vibration problems in Engineering', by Timoshenko, Young and Weaver[57].

Dubey[9,10] solved the complete equation of motion for a simply supported rectangular beam that included the shear deformation as well as the rotatory inertia terms. He used the finite-transform technique[54] for the solution. He developed a finite-difference analog for the same. He has suggested for a similar analysis to be performed for the plates.

Abott[36] has also stressed the utility of the finite-difference method. He found this method to be very successful in calculating the dynamic behaviour of a trolley wire overhead contact system for electric railways.

Extensive studies have been done in the field of vibration of plates. But surprisingly enough, except Mindlin(46,47), no body has made any attempt towards the inclusion of rotatory inertia along with the shear deformation in the study of vibration of plates. He deduced a two-dimensional theory of flexural motion of isotropic, elastic plates from the three-dimensional theory of elasticity. The theory included the effects of these terms in the same manner as Timoshenko's one dimensional theory of bars(57). Velocities of straight crested waves were computed and found to agree with those obtained from the three-dimensional theory. A uniqueness theory revealed that three edge conditions were required. The theory of flexural motions of elastic plates including these effects was extended to the crystal plates(47). The equations were solved approximately for the case of a rectangular plate excited by thickness-shear deformation parallel to one edge. Results of computation of resonant frequencies of rectangular AT-cut, quartz plates were shown and compared with experimental data.

Reismann[25] has presented a method for the solution of dynamic boundary value problems of elastic plates subjected to time-dependent normal surface loads and/or time-dependent boundary conditions. An explicit, exact solution of a ring plate, clamped at the outer boundary and subjected to a suddenly applied transverse shear force, has been presented. He plotted the ring plate frequency spectrum and the variation of the radial bending moment at its outer and inner edges versus time.

1.3 PRESENT INVESTIGATION:

Although Mindlin[46] has made a good attempt towards the inclusion of rotatory inertia and shear deformation terms in the equation of motion for a plate, he has not attempted its solution for the transverse deflections for any initial, or, boundary conditions for any type of the loading. Since the deflection is an important design criterion for an engineer, any attempt to evaluate it precisely will be a shot in his arm to produce efficient design. The present investigation aims at the detailed study of this particular aspect.

Based on the principle of virtual work, an equation of motion for a plate is developed herein. This approach has been illustrated by Vlasov and

Leont'ev(60) for the static analysis of plates. The Navier type analytical solution of the equation of motion for a rectangular plate is attempted. A list of natural frequencies is prepared for concrete and a steel plate. The deflections are studied when the plate is subjected to the forced vibrations (i;e., Rectangular, Sawtooth or Triangular pulses). The Mindlin's equation of motion is also solved and the results are compared with the results of the present equation of motion. A study of the contribution of shear deformation and rotatory inertia terms is also attempted.

A three-dimensional finite-difference analog is then developed. The results for a simply supported plate are compared with the analytical solution to check the accuracy of the finite-difference method. A rectangular and a square plate fixed on all the sides are then solved for a rectangular pulse. The plate is then solved for a triangular pulse with zero rise time. The governing differential equation of motion is then expressed in the polar coordinate system. A circular plate fixed on its boundaries is then solved.

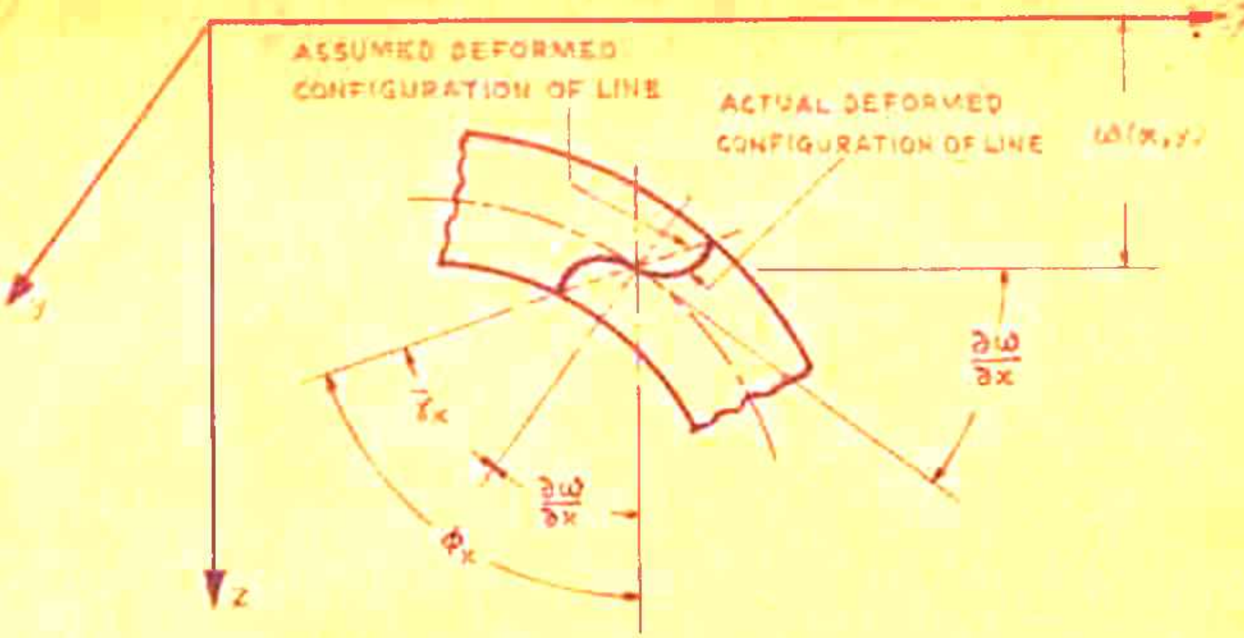
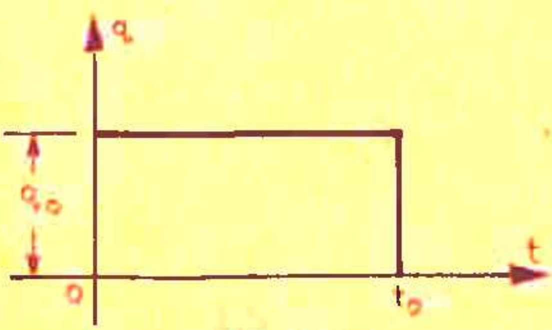
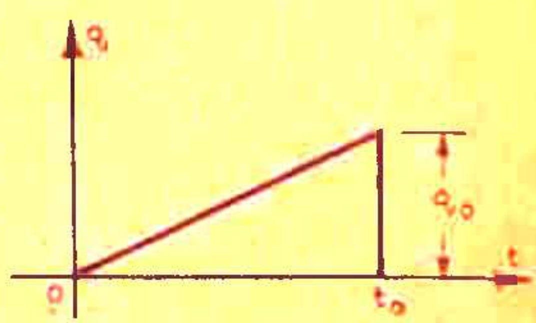


FIG. 1-1-1: DEFINITION OF ASSUMED DEFORMATION STATE FOR SECTION x = CONSTANT



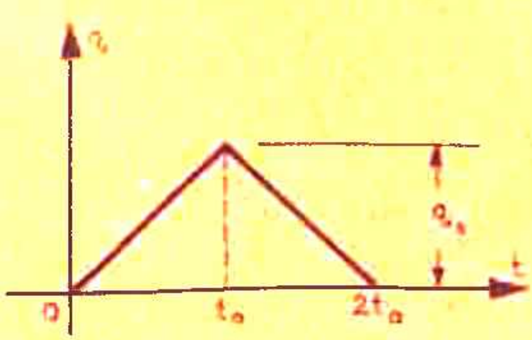
$q = q_0$ FOR $t \leq t_0$
$q = 0$ FOR $t > t_0$

FIG 1-1-2a: RECTANGULAR PULSE



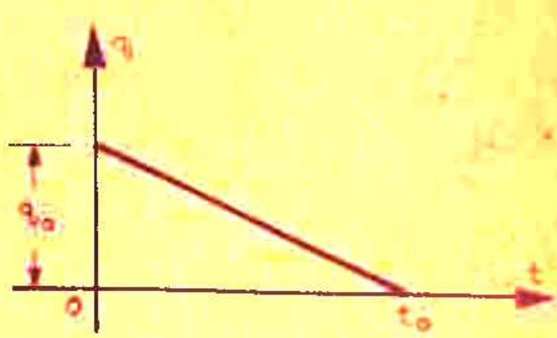
$q = q_0 . t/t_0$ FOR $t \leq t_0$
$q = 0$ FOR $t > t_0$

FIG. 1-1-2b: SAWTOOTH PULSE



$q = q_0 \cdot t/t_0$ FOR $t \leq t_0$
$q = q_0(2 - t/t_0)$ FOR $t_0 < t \leq 2t_0$.
$q = 0$ FOR $t \geq 2t_0$

FIG. 1-1-2c: TRIANGULAR PULSE



$q = q_0(1 - t/t_0)$ FOR $t \leq t_0$
$q = 0$ FOR $t > t_0$

FIG. 1-1-2d: TRIANGULAR PULSE WITH ZERO RISE TIME.

FIG. 1-1-2: VARIOUS TYPES OF LOAD PULSES

CHAPTER-2

DEVELOPMENT OF EQUATION OF MOTION

Based on the principle of virtual work, an equation of motion for a plate is developed herein. This approach has been illustrated by Vlasov and Leont'ev[60] for the static analysis of plates.

2.1 DEVELOPMENT OF THE EQUATION:

Consider a three-dimensional elastic plate of thickness H. Let the unknown displacement components of a point M(x,y,z,t) be u(x,y,z,t), v(x,y,z,t) and w(x,y,z,t). Here x,y,z represent the coordinate system and 't' is the time parameter. The displacements are considered to be positive when their directions coincide with the positive directions of the corresponding coordinate axes (Fig. 2.1.1). The unknown displacements are represented in the following manner:

$$
\begin{aligned}
u(x,y,z,t) &= u1(x,y,t).\ \phi_1(z) \\
v(x,y,z,t) &= v1(x,y,t).\ \phi_1(z) \\
w(x,y,z,t) &= w1(x,y,t).\ \phi_2(z)
\end{aligned}
\qquad (2.1.1.)
$$

The functions $\phi_1(z)$ and $\phi_2(z)$ determine the variation with thickness of the horizontal and vertical displacements respectively and are assumed to be known, linear functions; whereas u1(x,y,t), v1(x,y,t) and w1(x,y,t) are the dimensionless, unknown displacement components of the plate.

The plate is assumed to be perfectly elastic and is of the homogeneous and continuous material distributed over the volume so that the smallest element cut from the plate possesses the same specific physical properties of the plate. Further, it is assumed that the plate is Isotropic, i.e., its elastic properties are same in all the directions.

It is also assumed here that there are enough constraints to prevent the body from moving as a rigid body, so that, no displacements of particles of the body are possible without a deformation of it. The Hooke's law is applicable to the body; non-linear terms and the terms of higher order are neglected.

It should be emphasized at this stage that in the present analysis the cross-sections of the plate are not assumed to remain plane during bending. In other words, the 'shear deformations' are not neglected. Similarly, in contrast with the classical thin plate theory, 'Rotatory inertia' terms are also included.

The normal and shearing stresses in the case of a three-dimensional plate are given by:-

$$\sigma_x = \frac{2G}{(1-2\nu)} \left[(1-\nu) \frac{\partial u}{\partial x} + \nu \left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right]$$

$$\sigma_y = \frac{2G}{(1-2\nu)} \left[(1-\nu) \frac{\partial v}{\partial y} + \nu \left(\frac{\partial w}{\partial z} + \frac{\partial u}{\partial x} \right) \right]$$

$$\sigma_z = \frac{2G}{(1-2\nu)} \left[(1-\nu) \frac{\partial w}{\partial z} + \nu \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right]$$

$$\tau_{xy} = \tau_{yx} = G \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)$$

$$\tau_{yz} = \tau_{zy} = G \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)$$

$$\tau_{zx} = \tau_{xz} = G \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right)$$

...(2.1.2)

where, σ_x, σ_y, σ_z are the normal stress components

τ_{xy}, τ_{yz}, τ_{zx} are the shear stress components.

ν is the poisson's ratio.

G is the shear modulus defined as a function of the Young's modulus Y as

$$G = \frac{Y}{2(1+\nu)}$$

We also know that

$$\epsilon_x = \frac{\partial u}{\partial x}\ ; \quad \epsilon_y = \frac{\partial v}{\partial y}\ ; \quad \epsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\ ; \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\ ;$$

$$\gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

...(2.1.3)

Where,

$\epsilon_x, \epsilon_y, \epsilon_z$ are the normal strain components.

$\gamma_{xy}, \gamma_{yz}, \gamma_{zx}$ are the shear strain components.

Substituting Eqs.(2.1.1) in Eqs. (2.1.2), the following expressions are obtained:

$$\sigma_x = \frac{2G}{(1-2\nu)} \left[(1-\nu)\frac{\partial v1}{\partial y}\phi_1 + \nu\left(\frac{\partial v1}{\partial y}\phi_1 + w1.\phi_2'\right)\right]$$

$$\sigma_y = \frac{2G}{(1-2\nu)} \left[(1-\nu)\frac{\partial v1}{\partial y}\phi_1 + \nu\left(w1\ \phi_2' + \frac{\partial u1}{\partial x}\phi_1\right)\right]$$

$$\sigma_z = \frac{2G}{(1-2\nu)} \left[(1-\nu)w1.\phi_2' + \nu\left(\frac{\partial u1}{\partial x}\phi_1 + \frac{\partial v1}{\partial y}\phi_1\right)\right]$$

$$\tau_{xy} = \tau_{yx} = G\left[\frac{\partial u1}{\partial y}\phi_1 + \frac{\partial v1}{\partial x}\phi_1\right]$$

$$\tau_{yz} = \tau_{zy} = G \left[v1\phi_1' + \frac{\partial w1}{\partial y} \phi_2 \right]$$

$$\tau_{zx} = \tau_{xz} = G \left[u1\phi_1' + \frac{\partial w1}{\partial x} \phi_2 \right]$$

...(2.1.4)

Where, the primes denote the differentiation with respect to z.

In order to determine the displacement components u1,v1 and w1; a cut from the plate of an elementary column of height H and sides dx = 1, dy = 1 (Figs. 2.1.1, 2.1.2) is made. This column possesses three degrees of freedom in three directions at a particular time t.

The generalized equilibrium conditions of the elementary column considered as virtual displacements can, therefore, be written as follows:

$$\int \frac{\partial \sigma_x}{\partial x} \phi_1 . \, dz - \int \tau_{xz} . \, \phi_1' \, dz + \int \frac{\partial \tau_{xy}}{\partial y} \phi_1 dz - \int \rho \frac{\partial^2 u1}{\partial t^2} \phi_1^2 \, dz = 0$$

$$\int \frac{\partial \sigma_y}{\partial y} \phi_1 dz - \int \tau_{yz} \phi_1' dz + \int \frac{\partial \tau_{yx}}{\partial x} \phi_1 \, dz - \int \rho \frac{\partial^2 v1}{\partial t^2} \phi_1^2 \, dz = 0$$

$$\int \frac{\partial \tau_{zx}}{\partial x} \phi_2 \, dz - \int \sigma_z \, \phi_2' \, dz + \int \frac{\partial \tau_{xy}}{\partial y} \phi_2 \, dz$$

$$- \int \rho \frac{\partial^2 w1}{\partial t^2} \phi_2^2 \, dz + \int q \, \phi_2 \, dz = 0$$

...(2.1.5)

Where, ρ is the mass density per unit volume of the material and $q = q(x,y,t)$ is the transverse loading per unit area.

Eqns. (2.1.5) state that the total work done by all the external and internal forces acting over the elementary column over the virtual displacements equals zero. The terms $\int \rho \frac{\partial^2 u1}{\partial t^2} \phi_1^2 \, dz$ and $\int \rho \frac{\partial^2 v1}{\partial t^2} \phi_1^2 \, dz$ appearing here are due to the rotatory inertia.

It may be noted that,

$\bar{u}(x,y,z,t) = \phi_1(z)$; $\bar{v}(x,y,z,t) = \phi_1(z)$;

$\bar{w}(x,y,z,t) = \phi_2(z)$

for unit virtual displacements,

$\overline{u1}(x,y,t) = 1$; $\overline{v1}(x,y,t) = 1$,

$\overline{w1}(x,y,t) = 1$.

Substituting Eqns. (2.1.4) in Eqns.(2.1.5), the following system of partial differential equations for the functions u1, v1 and w1 are obtained -

$$\int \frac{2G}{(1-2\nu)} \left[(1-\nu) \frac{\partial^2 u1}{\partial x^2} \phi_1 + \nu \left(\frac{\partial^2 v1}{\partial x \partial y} \phi_1 + \frac{\partial w1}{\partial x} \phi_2^1 \right) \right] \phi_1 dz$$

$$- \int G(u1\phi_1^1 + \frac{\partial w1}{\partial x} \phi_2) \phi_1^1 \, dz + \int G(\frac{\partial^2 u1}{\partial y^2} \phi_1^2 + \frac{\partial^2 v1}{\partial x \partial y} \phi_1^2) dz$$

$$- \int \rho \frac{\partial^2 u1}{\partial t^2} \phi_1^2 \, dz = 0$$

$$\int \frac{2G}{(1-2\nu)} \left[(1-\nu) \frac{\partial^2 v1}{\partial y^2} \phi_1 + \nu \left(\frac{\partial w1}{\partial y} \phi_2^1 + \frac{\partial^2 u1}{\partial x \partial y} \phi_1 \right) \right] \phi_1 dz$$

$$- \int G(v1\phi_1^1 + \frac{\partial w1}{\partial y} \phi_2) \phi_1^1 dz$$

$$+ \int G(\frac{\partial^2 u1}{\partial x \partial y}\phi_1^2 + \frac{\partial^2 v1}{\partial x^2} \phi_1^2) dz - \int \rho \frac{\partial^2 v1}{\partial t^2} \phi_1^2 \, dz = 0$$

$$\int G(\frac{\partial u1}{\partial x} \phi_1^1 + \frac{\partial^2 w1}{\partial x^2} \phi_2)\phi_2 \, dz - \int \frac{2G}{(1-2\nu)} \left[(1-\nu) w1 \, \phi_2^1 \right.$$

$$\left. + \nu \left(\frac{\partial u1}{\partial x} \phi_1 + \frac{\partial v1}{\partial y} \phi_1 \right) \right] \phi_2^1 \, dz$$

$$+ \int G(\frac{\partial v1}{\partial y} \phi_1^1 + \frac{\partial^2 w1}{\partial y^2} \phi_2)\phi_2 \, dz$$

$$- \int \rho \frac{\partial^2 w1}{\partial t^2} \phi_2^2 \, dz + \int q.\phi_2 \, dz = 0$$

...(2.1.6)

The functions $\phi_1(z)$ and $\phi_2(z)$ are assumed to be linearly independent, and, their selection must actually be based on an experimental law. The elastic plate is considered to be a sufficiently thick slab, capable of sustaining normal and tangential loads. The solution for a thick isotropic plate is approximated from the viewpoint of the theory of elasticity(51). In the present study, even the simplest model prescribed by the functions ϕ_1 and ϕ_2 is expected to be closer to the reality. For a plate resting on its boundaries for any type of boundary conditions, these functions can be assumed to be (Fig. 2.1.3) -

$$\phi_1(z) = \frac{H-2z}{2}$$

$$\phi_2(z) = 1.$$

Integration of the functions within the limits of the plate thickness yields:

$$\int_0^H (\phi_1^2)\, dz = \int_0^H \left(\frac{H-2z}{2}\right)^2 dz = \frac{H^3}{12}$$

$$\int_0^H (\phi_1')^2 dz = \int_0^H (-1)^2\, dz = H$$

$$\int_0^H \phi_1 \phi_2'\, dz = 0$$

$$\int_0^H (\phi_2')^2 \, dz = 0$$

$$\int_0^H (\phi_2)^2 \, dz = H$$

$$\int_0^H \phi_2 \, \phi_1' \, dz = -H$$

It is assumed that no body forces act on the plate and that a vertical surface load q(x,y,t) is applied to the plate. Note that $\int q(x,y,t)\, \phi_2(0)\, dz = q.$

Thus, substituting above relations in Eqn. (2.1.6) : -

$$\frac{2G}{(1-2\nu)} \left[(1-\nu) \frac{\partial^2 u1}{\partial x^2} \frac{H^3}{12} + \nu \left(\frac{\partial^2 v1}{\partial x \partial y} \cdot \frac{H^3}{12} \right) \right]$$

$$- G(u1.H - \frac{\partial w1}{\partial x} .H) + G\left(\frac{\partial^2 u1}{\partial y^2} + \frac{\partial^2 v1}{\partial x \partial y}\right) \frac{H^3}{12}$$

$$- \frac{\rho H^3}{12} \cdot \frac{\partial^2 u1}{\partial t^2} = 0$$

$$\frac{2G}{(1-2\nu)} \left[(1-\nu) \frac{\partial^2 v1}{\partial y^2} \cdot \frac{H^3}{12} + \nu \left(\frac{\partial^2 u1}{\partial x \partial y} \cdot \frac{H^3}{12} \right) \right]$$

$$-GH(v1 - \frac{\partial w1}{\partial y}) + G\left(\frac{\partial^2 u1}{\partial x \partial y} + \frac{\partial^2 v1}{\partial x^2}\right) \frac{H^3}{12}$$

$$- \frac{\rho H^3}{12} \cdot \frac{\partial^2 v1}{\partial t^2} = 0$$

$$G\left[\frac{\partial u1}{\partial x}(-H) + \frac{\partial^2 w1}{\partial x^2} H\right] + GH\left(\frac{\partial^2 w1}{\partial y^2} - \frac{\partial v1}{\partial y}\right)$$

$$- \rho H. \frac{\partial^2 w1}{\partial t^2} + q = 0$$

$$\ldots(2.1.7)$$

or,

$$\frac{D(1-\nu)}{(1-2\nu)}\left[(1-\nu)\frac{\partial^2 u1}{\partial x^2} + \nu\frac{\partial^2 v1}{\partial x\,\partial y}\right] + GH\left(-u1 + \frac{\partial w1}{\partial x}\right)$$

$$+ D\frac{(1-\nu)}{2}\left[\frac{\partial^2 u1}{\partial y^2} + \frac{\partial^2 v1}{\partial x\,\partial y}\right] - \rho\frac{H^3}{12}\cdot\frac{\partial^2 u1}{\partial t^2} = 0 \quad ..(i)$$

$$\frac{D(1-\nu)}{(1-2\nu)}\left[(1-\nu)\frac{\partial^2 v1}{\partial y^2} + \nu\frac{\partial^2 u1}{\partial x\,\partial y}\right] + GH\left(-v1 + \frac{\partial w1}{\partial y}\right)$$

$$+ \frac{D(1-\nu)}{2}\left[\frac{\partial^2 u1}{\partial x\,\partial y} + \frac{\partial^2 v1}{\partial x^2}\right] - \frac{\rho H^3}{12}\cdot\frac{\partial^2 v1}{\partial t^2} = 0 \quad ..(ii)$$

$$GH\left[\left(\frac{\partial^2 w1}{\partial x^2} - \frac{\partial u1}{\partial x}\right) + \left(\frac{\partial^2 w1}{\partial y^2} - \frac{\partial v1}{\partial y}\right)\right]$$

$$- \rho H\frac{\partial^2 w1}{\partial t^2} + q = 0 \quad \ldots(iii)$$

$$\ldots(2.1.8)$$

Where, $D = \frac{EH^3}{12(1-\nu^2)}$ is the flexural rigidity of the plate. The terms $\left[GH(-u1 + \frac{\partial w1}{\partial x})\right]$ and $\left[GH(-v1 + \frac{\partial w1}{\partial y})\right]$ in the above equations are due to the influence of shear deformation, while, the terms $(\frac{\rho H^3}{12} \cdot \frac{\partial^2 u1}{\partial t^2})$ and $(\frac{\rho H^3}{12} \cdot \frac{\partial^2 v1}{\partial t^2})$ are evolved because of the effect of rotatory inertia.

Differentiating Eqn. (2.1.8(i)) w.r.t. x and Eqn. (2.1.8(ii)) w.r.t. y,

$$\frac{D(1-\nu)}{(1-2\nu)}\left[(1-\nu)\frac{\partial^3 u1}{\partial x^3} + \nu\frac{\partial^3 v1}{\partial x^2 \partial y}\right] - GH\left[\frac{\partial u1}{\partial x} - \frac{\partial^2 w1}{\partial x^2}\right]$$

$$+ \frac{D(1-\nu)}{2}\left[\frac{\partial^3 u1}{\partial x \partial y^2} + \frac{\partial^3 v1}{\partial x^2 \partial y}\right] - \frac{\rho H^3}{12} \cdot \frac{\partial^3 u1}{\partial x \partial t^2} = 0 \quad ..(2.1.8.ia)$$

$$\frac{D(1-\nu)}{(1-2\nu)}\left[(1-\nu)\frac{\partial^3 v1}{\partial y^3} + \nu\frac{\partial^3 u1}{\partial x^2 \partial y}\right] - GH\left[\frac{\partial v1}{\partial y} - \frac{\partial^2 w1}{\partial y^2}\right]$$

$$+ \frac{D(1-\nu)}{2}\left[\frac{\partial^3 u1}{\partial x \partial y^2} + \frac{\partial^3 v1}{\partial x^2 \partial y}\right] - \frac{\rho H^3}{12} \cdot \frac{\partial^3 v1}{\partial y \partial t^2} = 0 \quad ..(2.1.8.iia)$$

Adding (2.1.8.ia) and (2.1.8.iia),

$$\frac{D(1-\nu)}{(1-2\nu)}\left[(1-\nu)\frac{\partial^3 u1}{\partial x^3} + \nu\frac{\partial}{\partial x} \cdot \frac{\partial^2 u1}{\partial y^2} + (1-2\nu)\frac{\partial}{\partial x} \cdot \frac{\partial^2 u1}{\partial y^2}\right]$$

$$+ \frac{D(1-\nu)}{(1-2\nu)} \left[\frac{\partial}{\partial y} \cdot \frac{\partial^2 v1}{\partial x^2} + (1-\nu) \frac{\partial}{\partial y} \cdot \frac{\partial^2 v1}{\partial y^2} (1-2\nu) \frac{\partial}{\partial y} \frac{\partial^2 v1}{\partial x^2} \right]$$

$$- GH (\psi - \nabla^2 w1) - \frac{\rho H^3}{12} \frac{\partial^2 \psi}{\partial t^2} = 0 \quad \ldots (2.1.8a)$$

Where, $\psi = \frac{\partial u1}{\partial x} + \frac{\partial v1}{\partial y}$

Introducing the operator $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ in Eqn.(2.1.8a),

$$\frac{D(1-\nu)^2}{(1-2\nu)} \psi - GH \psi - \frac{\rho H^3}{12} \frac{\partial^2 \psi}{\partial t^2} = - GH \nabla^2 w1 \quad \ldots \quad (2.1.9)$$

From Eqn. (2.1.8.i)

$$\psi = \nabla^2 w1 + \frac{q}{GH} - \frac{\rho}{G} \cdot \frac{\partial^2 w1}{\partial t^2} \qquad (2.1.10)$$

Substituting the value of ψ in Eqn.(2.1.9), the following equation is obtained:-

$$\frac{D(1-\nu)^2}{(1-2\nu)} \left[w1 - \frac{\rho}{G} \frac{\partial^2 w1}{\partial t^2} + \frac{q}{GH} \right]$$

$$- GH \left[\nabla^2 w1 - \frac{\rho}{G} \cdot \frac{\partial^2 w1}{\partial t^2} + \frac{q}{GH} \right]$$

$$- \frac{\rho H^3}{12} \cdot \frac{\partial^2}{\partial t^2} \left[\nabla^2 w1 - \frac{\rho}{G} \cdot \frac{\partial^2 w1}{\partial t^2} + \frac{q}{GH} \right] = - GH \nabla^2 w1$$

OR,

$$\left[D\nabla^2\nabla^2 - \left[\frac{\rho H^3}{12}\frac{(1-2\nu)}{(1-\nu)^2} + \frac{\rho D}{G}\right]\frac{\partial^2}{\partial t^2}\nabla^2 + \frac{\rho H(1-2\nu)}{(1-\nu)^2}\frac{\partial^2}{\partial t^2} + \frac{\rho^2 H^3}{12G}\frac{(1-2\nu)}{(1-\nu)^2}\frac{\partial^4}{\partial t^4}\right] w1$$

$$= \left[\frac{(1-2\nu)}{(1-\nu)^2} - \frac{D}{GH}\nabla^2 + \frac{\rho H^2}{12G}\frac{(1-2\nu)}{(1-\nu)^2}\frac{\partial^2}{\partial t^2}\right] q \quad \ldots(2.1.11)$$

The Eqn.(2.1.11) is the present equation of motion, as compared to the Mindlin's equation of motion given below;-

$$\left[D\nabla^2\nabla^2 - \left(\frac{\rho H^3}{12} + \frac{\rho D}{kG}\right)\frac{\partial^2}{\partial t^2}\nabla^2 + \rho H\frac{\partial^2}{\partial t^2} + \frac{\rho^2 H^3}{12Gk}\frac{\partial^4}{\partial t^4}\right] w1$$

$$= \left[1 - \frac{D}{kGH}\nabla^2 + \frac{\rho H^2}{12kG}\frac{\partial^2}{\partial t^2}\right] q \quad \ldots(2.1.12)$$

Where, k was the numerical factor that took into account the parabolic shape of the shear stress distribution.

2.2 CONTRIBUTION OF DIFFERENT TYPES OF EFFECTS:

If the rotatory inertia terms in Eqn.(2.1.8) are neglected, the following equation of motion (instead of Eqn.(2.1.11) is obtained:

$$D \nabla^4 w1 - \frac{\rho D}{G} \frac{\partial^2}{\partial t^2} \nabla^2 w1 + \frac{\rho H(1-2\nu)}{(1-\nu)^2} \frac{\partial^2 w1}{\partial t^2}$$

$$= \frac{(1-2\nu)}{(1-\nu)^2} q - \frac{D}{GH} \nabla^2 q \qquad (2.2.1)$$

If the shear deformation terms are neglected, the following equation of motion is obtained:

$$D \nabla^4 w1 - \frac{\rho H^3}{12} \frac{(1-2\nu)}{(1-\nu)^2} \frac{\partial^2}{\partial t^2} \nabla^2 w1 + \frac{\rho H(1-2\nu)}{(1-\nu)^2} \frac{\partial^2 w1}{\partial t^2}$$

$$= \frac{(1-2\nu)}{(1-\nu)^2} q \qquad (2.2.2)$$

Similarly, if both the shear deformation and the rotatory inertia terms are neglected,

$$D \nabla^4 w1 + \frac{\rho H(1-2\nu)}{(1-\nu)^2} \frac{\partial^2 w1}{\partial t^2} = \frac{1-2\nu}{(1-\nu)^2} q \qquad (2.2.3)$$

This equation is similar to the equation of motion of the classical thin plate theory, according to which,

$$D \nabla^4 w1 + \rho H \frac{\partial^2 w1}{\partial t^2} = q \qquad (2.2.4)$$

2.3 THICK PLATE RESTING ON A RIGID FOUNDATION:

If the plate under consideration is resting on a rigid foundation, the function $\phi_2(z)$ as described above will not work. Instead, a function $\phi_R(z) = \frac{H-z}{H}$ (Fig. 2.3.1)is expected to be closer to the reality. Hence,

$$\int_0^H (\phi_R^1)^2 \, dz = \int_0^H \left(-\frac{1}{H}\right)^2 dz = \frac{1}{H}$$

$$\int_0^H (\phi_R)^2 \, dz = \int_0^H \frac{H^2+z^2-2zH}{H^2} \, dz = \frac{H}{3}$$

$$\int_0^H \phi_1 \cdot \phi_R^1 \, dz = \int_0^H \left(\frac{H-2z}{2}\right)\left(-\frac{1}{H}\right) dz = 0$$

$$\int_0^H \phi_R \, \phi_1^1 \, dz = -\frac{H}{2}$$

Substituting these expressions in Eq.(2.1.6):-

$$\frac{D(1-\nu)}{(1-2\nu)}\left[(1-\nu)\frac{\partial^2 u1}{\partial x^2} + \nu\frac{\partial^2 v1}{\partial x \partial y}\right] - GH\left(u1 - \frac{1}{2}\frac{\partial w1}{\partial x}\right)$$

$$+ \frac{D(1-\nu)}{2}\left(\frac{\partial^2 u1}{\partial x^2} + \frac{\partial^2 v1}{\partial x \partial y}\right)$$

$$- \frac{\rho H^3}{12} \cdot \frac{\partial^2 u1}{\partial t^2} = 0 \quad \ldots. \text{ (i)}$$

$$\frac{D(1-\nu)}{(1-2\nu)}\left[(1-\nu)\frac{\partial^2 v1}{\partial y^2} + \frac{\partial^2 u1}{\partial x \partial y}\right] - GH\left(v1 - \frac{1}{2} \cdot \frac{\partial w1}{\partial y}\right)$$

$$+ \frac{D(1-\nu)}{2}\left[\frac{\partial^2 u1}{\partial x \partial y} + \frac{\partial^2 v1}{\partial x^2}\right] - \frac{\rho H^3}{12} \cdot \frac{\partial^2 v1}{\partial t^2} = 0 \quad \ldots \text{(ii)}$$

$$GH\left(-\frac{1}{2}\frac{\partial u1}{\partial x} + \frac{1}{3}\frac{\partial^2 w1}{\partial x^2}\right) - \frac{2G}{H}\frac{(1-\nu)}{(1-2\nu)} w1$$

$$+ GH\left(-\frac{1}{2}\frac{\partial v1}{\partial y} + \frac{1}{3}\frac{\partial^2 w1}{\partial y^2}\right) - \frac{\rho H}{3}\frac{\partial^2 w1}{\partial t^2} + q = 0 \ \ldots(iii)$$

$$\ldots\ (2.3.1)$$

Differenting Eqn.(2.3.1(i)) w.r.t. x and Eqn. (2.3.1(ii)) w.r.t. y, and, adding -

$$\frac{D(1-\nu)^2}{(1-2\nu)}\nabla^2\psi - GH\,\psi - \frac{\rho H^3}{12}\frac{\partial^2\psi}{\partial t^2} = -\frac{1}{2}\,GH\,\nabla^2 w1 \quad (2.3.2)$$

where, $\psi = \frac{\partial u1}{\partial x} + \frac{\partial v1}{\partial y}$ and $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

From Eqn. (2.3.1);

$$\psi = \left[\frac{GH}{3}\nabla^2 w1 - \frac{2G}{H}\frac{(1-\nu)}{(1-2\nu)} w1 - \frac{\rho H}{3}\frac{\partial^2 w1}{\partial t^2} + q\right]\frac{2}{GH} \quad (2.3.3)$$

Substituting Eqn. (2.3.3) in Eqn. (2.3.2), the following equation of motion is obtained:-

$$\left\{\frac{2D}{3}\frac{(1-\nu)^2}{(1-2\nu)} - \left(\frac{4D}{H^2}\frac{(1-\nu)^3}{(1-2\nu)^2} + \frac{2G}{3} - \frac{GH}{2}\right)\nabla^2\right.$$

$$+ \frac{2G}{H}\frac{(1-\nu)}{(1-2\nu)} - \left(\frac{2}{3}\frac{\rho D}{G}\frac{(1-\nu)^2}{(1-2\nu)} + \frac{\rho H^3}{18}\right)\frac{\partial^2}{\partial t^2}\nabla^2$$

$$\left. + \frac{\rho H}{3}\frac{(2-3\nu)}{(1-2\nu)}\frac{\partial^2}{\partial t^2} + \frac{\rho^2 H^3}{18G}\frac{\partial^4}{\partial t^4}\right\} w1$$

$$= \left[1 + \frac{\rho H^2}{6G}\frac{\partial^2}{\partial t^2} - \frac{2D}{GH}\frac{(1-\nu)^2}{(1-2\nu)}\nabla^2\right] q \qquad (2.3.4)$$

Where, the term $\left[\frac{4D}{H^2}\frac{(1-\nu)^3}{(1-2\nu)^2} + \frac{2G}{3} - \frac{GH}{2}\nabla^2\right]w1$

is due to the vertical compression. Thus, in contrast to Mindlin's, the present approach is capable of including these additional terms also.

2.4 EXPRESSIONS FOR MOMENTS AND SHEAR COMPONENTS:

An exact solution of the governing plate equation (Eqn. 2.1.11) must simultaneously satisfy the differential equation and the boundary conditions of any given plate problem. Bending moment, torsional moment and the transverse shear are the various force components to be considered, particularly when the static boundary conditions are required. The expressions for these components are obtained as follows:-

$$M_x = \int_0^H \sigma_x . z \, dz = \int_0^H \frac{2G}{(1-2\nu)}\left[(1-\nu)\frac{\partial u}{\partial x} + \nu\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] z.dz$$

$$= \int_0^H \frac{2G}{(1-2\nu)}\left[(1-\nu)\frac{\partial u1}{\partial x}\phi_1 + \nu\left(\frac{\partial v1}{\partial y}\phi_1 + w1\,\phi_2'\right)\right] z.dz$$

$$= -\frac{GH^3}{6(1-2\nu)}\left[(1-\nu)\frac{\partial u1}{\partial x} + \nu\frac{\partial v1}{\partial y}\right]$$

$$= \frac{D(\nu-1)}{(1-2\ )}\left[(1-\nu)\frac{\partial u1}{\partial x} + \nu\frac{\partial v1}{\partial y}\right] \qquad (2.4.1)$$

$$M_y = \int_0^H \frac{2G}{(1-2\nu)} \left[(1-\nu)\frac{\partial v}{\partial y} + \nu\left(\frac{\partial w}{\partial z} + \frac{\partial u}{\partial x}\right)\right] z.dz$$

$$= \frac{D(\nu-1)}{(1-2\nu)} \left[(1-\nu)\frac{\partial v1}{\partial y} + \nu\frac{\partial u1}{\partial x}\right] \qquad (2.4.2)$$

$$M_z = \int_0^H \sigma_z.z\ dz = \frac{2G}{(1-2\nu)} \int_0^H (1-\nu)\ w1.\phi_2^1 + \nu\left(\frac{\partial u1}{\partial x}\phi_1 + \frac{\partial v1}{\partial y}\phi_1\right)\Big| z.dz$$

$$= 2G\frac{\nu}{(1-2\nu)}\left(-\frac{H^3}{12}\right)\left[\frac{\partial u1}{\partial x} + \frac{\partial v1}{\partial y}\right] \qquad (2.4.3)$$

$$M_{xy} = \int_0^H \tau_{xy}.z.dz = \int_0^H G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)z.dz$$

$$= \frac{D(\nu-1)}{2}\left[\frac{\partial u1}{\partial y} + \frac{\partial v1}{\partial x}\right] \qquad (2.4.4)$$

$$Q_x = \int_0^H \tau_{xz}.dz = G\int_0^H \left(u1\ \phi_1^1 + \frac{\partial w1}{\partial x}\right) dz$$

$$= GH\left(\frac{\partial w1}{\partial x} - u1\right) \qquad (2.4.5)$$

$$Q_y = \int_0^H \tau_{yz}.dz = G\int_0^H \left(v1\ \phi_1^1 + \frac{\partial w1}{\partial y}\phi_2\right) dz$$

$$= GH\left(\frac{\partial w1}{\partial y} - v1\right) \qquad (2.4.6)$$

The components M_x, M_y, M_z, M_{xy} in the above expressions are the moment components per unit length of the plate, while, Q_x and Q_y are the shear components per unit length of the plate.

Substituting equations (2.4.1) to (2.4.3) in Eqn. (2.1.8), the following set of equations is obtained: -

$$- \frac{\partial M_x}{\partial x} - \frac{\partial M_{xy}}{\partial y} = - Q_x + \frac{\rho H^3}{12} \cdot \frac{\partial^2 u1}{\partial t^2} \quad \ldots(i)$$

$$- \frac{\partial M_{xy}}{\partial x} - \frac{\partial M_y}{\partial y} = - Q_y + \frac{\rho H^3}{12} \cdot \frac{\partial^2 v1}{\partial t^2} \quad \ldots(ii)$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = - q + \rho H \frac{\partial^2 w1}{\partial t^2} \quad \ldots(iii)$$

$$\ldots(2.4.7)$$

From equations (2.4.1) to (2.4.3), (2.4.5) and (2.4.6) the following equation is obtained:

$$M_x - \frac{\nu H^2}{6(1-2\nu)} \left[\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} \right] - \frac{H^2}{6} \cdot \frac{\partial Q_x}{\partial x}$$

$$= - \frac{D(1-\nu)}{(1-2\nu)} \left[(1-\nu) \frac{\partial^2 w1}{\partial x^2} + \nu \frac{\partial^2 w1}{\partial y^2} \right] \quad (2.4.8)$$

From eqn. (2.4.7(iii)) and Eqn. (2.4.8),

$$M_x - \frac{\nu H^2}{6(1-2\nu)} \left[-q + \rho H \frac{\partial^2 w1}{\partial t^2} \right] - \frac{H^2}{6} \cdot \frac{\partial Q_x}{\partial x}$$

$$= - \frac{D(1-\nu)}{(1-2\nu)} \left[(1-\nu) \frac{\partial^2 w1}{\partial x^2} + \nu \frac{\partial^2 w1}{\partial y^2} \right] \quad (2.4.9)$$

Similarly,

$$M_y - \frac{\nu H^2}{6(1-2\nu)} \left| -q + \rho H \frac{\partial^2 w1}{\partial t^2} \right| - \frac{H^2}{6} \cdot \frac{\partial Q_y}{\partial y}$$

$$= - \frac{D(1-\nu)}{(1-2\nu)} \left[(1-\nu) \frac{\partial^2 w1}{\partial y^2} + \nu \frac{\partial^2 w1}{\partial x^2} \right] \qquad (2.4.10)$$

and

$$M_{xy} - \frac{H^2}{12} \left[\frac{\partial Q_x}{\partial y} + \frac{\partial Q_y}{\partial x} \right] = - D(1-\nu) \frac{\partial^2 w1}{\partial x \partial y} \qquad (2.4.11)$$

The above expressions are written in a form similar to that given by Reissner(13). These equations can be used for the stress analysis of the plate.

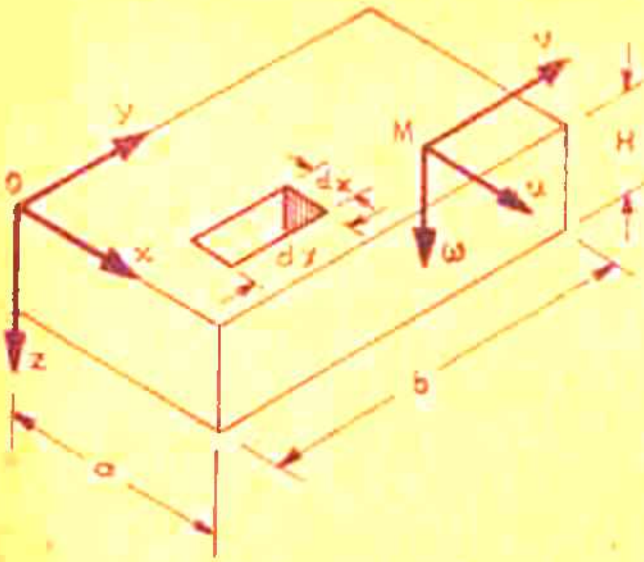


FIG. 2-1-1: A-THREE-DIMENSIONAL ELASTIC PLATE OF THICKNESS H.

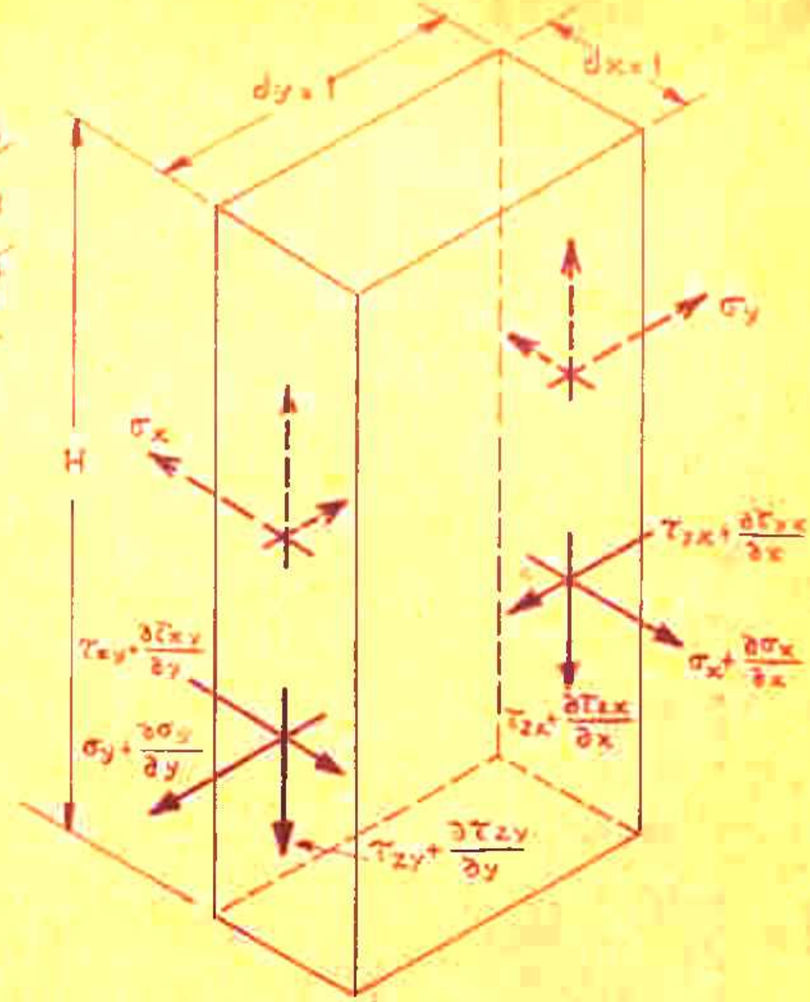


FIG. 2-1-2: AN ELEMENTRY COLUMN OF HEIGHT H

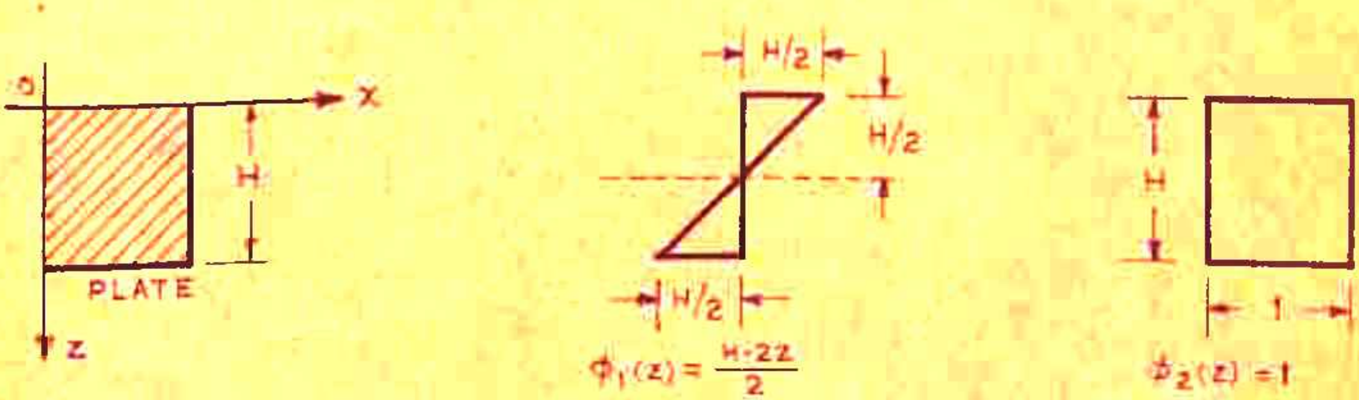


FIG. 2-1-3: THE ASSUMED FUNCTIONS DEFINING THE VARIATION WITH HEIGHT.

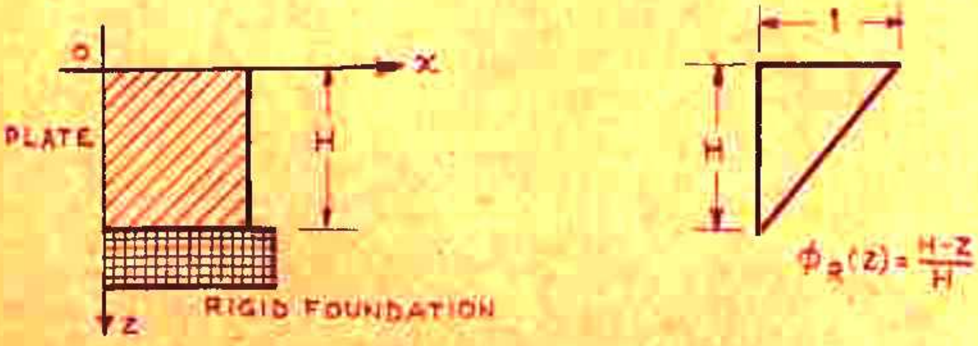


FIG. 2-3-1: THE ASSUMED FUNCTION FOR A PLATE RESTING ON A RIGID FOUNDATION.

CHAPTER-3

ANALYTICAL SOLUTION

3.1 GENERAL REMARKS:

The Navier type solution of a plate simply supported on all the edges is attempted. The Eqn.(2.1.11) is written in the following form:

$$\left[D \nabla^2\nabla^2 - E \frac{\partial^2}{\partial t^2} \nabla^2 + F \frac{\partial^2}{\partial t^2} + I \frac{\partial^4}{\partial t^4}\right] w1$$

$$= \left[M1 + M \frac{\partial^2}{\partial t^2} - N \nabla^2\right] q \qquad (3.1.1)$$

where,

$$E = \frac{\rho H^3}{12} \frac{(1-2\ \ )}{(1-\ \ )^2} + \frac{\rho D}{G}$$

$$F = \frac{\rho H(1-2\nu)}{(1-\nu)^2}$$

$$I = \frac{\rho^2 H^3(1-2\nu)}{12G(1-\nu)^2}$$

$$M1 = \frac{1-2\nu}{(1-\nu)^2}$$

$$M = \frac{\rho H^2}{12G} \frac{(1-2\nu)}{(1-\nu)^2}$$

$$N = \frac{D}{GH}$$

(3.1.2)

The solution of Eqn. (3.1.11) is attempted by the help of following boundary conditions:-

(i) The plate is simply supported on all the sides, thus,

at $x = 0, a$ $w1$, M_x are zero

at $y = 0, b$ $w1$, M_y are zero

(ii) The faces of the plate are taken free from the tangential tractions. Thus,

$$G\left[-v1 + \frac{\partial w1}{\partial y}\right] = G\left[-u1 + \frac{\partial w1}{\partial x}\right] = 0$$

Hence, from (i) and (ii),

$$w1(x,0) = w1\ (x,b) = w1(0,y) = w1(a,y) = 0$$

$$-D\left[(1-\nu)\frac{\partial^2 w1}{\partial x^2} + \nu\frac{\partial^2 w1}{\partial y^2}\right]_{x=0,a} = -D\left[(1-\nu)\frac{\partial^2 w1}{\partial y^2} + \nu\frac{\partial^2 w1}{\partial x^2}\right]_{y=0,b} = 0$$

(3.1.3)

For these boundary conditions, it is seen that, whatever functions of $w1$ may be, it can always be represented within the limits of the rectangle by double series

$$w1 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \phi_{mn} \cdot \mathrm{Sin}\,\frac{m\pi x}{a} \cdot \mathrm{Sin}\,\frac{n\pi y}{b} \qquad (3.1.4)$$

Where, the sine functions, $\mathrm{Sin}\,\frac{m\pi x}{a}$ and $S\,\frac{n\pi y}{n}$, are the 'orthogonal functions' and ϕ_{mn} is a function of time. m,n are the integers.

Performing the various differentials on the Eqn. (3.1.4), following expressions are obtained:

$$\frac{\partial^2 w1}{\partial x^2} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \phi_{mn} \left(- \frac{m^2 \pi^2}{a^2} \right). \text{Sin} \frac{m\pi x}{a} . \text{Sin} \frac{n\pi y}{b}$$

$$\frac{\partial^2 w1}{\partial y^2} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \phi_{mn} \left(- \frac{n^2 \pi^2}{b^2} \right). \text{Sin} \frac{m\pi x}{a} . \text{Sin} \frac{n\pi y}{b}$$

$$\frac{\partial^4 w1}{\partial x^4} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \phi_{mn} . \frac{m^4 \pi^4}{a^4} . \text{Sin} \frac{m\pi x}{a} . \text{Sin} \frac{n\pi y}{b}$$

$$\frac{\partial^4 w1}{\partial y^4} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \phi_{mn} . \frac{n^4 \pi^4}{b^4} . \text{Sin} \frac{m\pi x}{a} . \text{Sin} \frac{n\pi y}{b}$$

$$\frac{\partial^4 w1}{\partial x^2 \partial y^2} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \phi_{mn} . \left(- \frac{m^2 \pi^2}{a^2} \right) \left(- \frac{n^2 \pi^2}{b^2} \right). \text{Sin} \frac{m\pi x}{a} . \text{Sin} \frac{n\pi y}{b}$$

$$\frac{\partial^4 w1}{\partial x^2 \partial t^2} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \ddot{\phi}_{mn} \left(- \frac{m^2 \pi^2}{a^2} \right). \text{Sin} \frac{m\pi x}{a} . \text{Sin} \frac{n\pi y}{b}$$

$$\frac{\partial^4 w1}{\partial y^2 \partial t^2} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \ddot{\phi}_{mn} \left(- \frac{n^2 \pi^2}{b^2} \right). \text{Sin} \frac{m\pi x}{a} . \text{Sin} \frac{n\pi y}{b}$$

$$\frac{\partial^2 w1}{\partial t^2} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \ddot{\phi}_{mn} \text{ Sin} \frac{m\pi x}{a} . \text{Sin} \frac{n\pi y}{b}$$

$$\frac{\partial^4 w1}{\partial t^2} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \ddot{\ddot{\phi}}_{mn} \text{ Sin} \frac{m\pi x}{a} \text{ Sin} \frac{n\pi y}{b}$$

Where dot represents differentiation of ϕ with respect to time. Substituting all these derivatives in the equation of motion (Eqn.3.1.1):

$$D\left[\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\phi_{mn}\frac{m^4\pi^4}{a^4}\,\text{Sin}\,\frac{m\pi x}{a}\;\text{Sin}\,\frac{n\pi y}{b}\right.$$

$$+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\phi_{mn}\frac{n^4\pi^4}{b^4}\,.\,\text{Sin}\,\frac{m\pi x}{a}\,.\,\text{Sin}\,\frac{n\pi y}{b}$$

$$\left.+2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\phi_{mn}\,.\,\frac{m^2\pi^2}{a^2}\;\frac{n^2\pi^2}{b^2}\,.\text{Sin}\,\frac{m\pi x}{a}.\;\text{Sin}\,\frac{n\pi y}{b}\right]$$

$$-E\left[\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\ddot{\phi}_{mn}\left(-\frac{m^2\pi^2}{a^2}\right).\,\text{Sin}\,\frac{m\pi x}{a}.\,\text{Sin}\,\frac{n\pi y}{b}\right.$$

$$\left.+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\ddot{\phi}_{mn}\left(-\frac{n^2\pi^2}{b^2}\right)\,\text{Sin}\,\frac{m\pi x}{a}.\,\text{Sin}\,\frac{n\pi y}{b}\right]$$

$$+F\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\ddot{\phi}_{mn}\;\text{Sin}\,\frac{m\pi x}{a}\;\text{Sin}\,\frac{n\pi y}{b}$$

$$+I\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\ddot{\ddot{\phi}}_{mn}\;\text{Sin}\,\frac{m\pi x}{a}.\;\text{Sin}\,\frac{n\pi y}{b}$$

$$=\left[M1+M\frac{\partial^2}{\partial t^2}-N\nabla^2\right]q \qquad (3.1.5)$$

The above equation is now solved for the required set of the Initial conditions.

3.2 FREE VIBRATIONS:

From the definition of free vibrations, it is known that no loading exists on the plate ,i.e.,

$q = 0$. Eqn.(3.1.5) is thus modified as follows:

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} I \ddddot{\phi}_{mn} + \left[E\left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2} \right) + F \right] \ddot{\phi}_{mn}$$

$$+ \left[D\left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2} \right)^2 \right] \phi_{mn}$$

$$\sin \frac{m\pi x}{a} \quad \sin \frac{n\pi y}{b} = 0 \qquad (3.2.1)$$

Multiplying both the sides of Eqn.(3.2.1) by $\sin\frac{m'\pi x}{a}.dx.\ \sin \frac{n'\pi y}{b}\ dy$ and integrating in the limits from 0 to a, and from 0 to b:-

$$\frac{ab}{4}\left\{ I.\ddddot{\phi}_{mn} + \left[E\left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2} \right) + F \right] \ddot{\phi}_{mn} + \left[D\left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2} \right)^2 \right] \phi_{mn} \right\} = 0 \qquad (3.2.2)$$

Putting $p = \frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}$,

$$I.\ddddot{\phi}_{mn} + (Ep + F)\ \ddot{\phi}_{mn} + D\ p^2\ \phi_{mn} = 0 \qquad (3.2.3)$$

For the case of harmonic motion $\phi_{mn} = e^{i\lambda_{mn} t}$

Thus,

$$I.\lambda_{mn}^4 - (Ep + F)\lambda_{mn}^2 + Dp^2 = 0 \qquad (3.2.4)$$

Solving Eqn.(3.2.4), the following expression for various natural frequencies λ_{mn} is thus obtained:

$$\lambda_{mn} = \pm \sqrt{\frac{(Ep+F) \pm \sqrt{(Ep+F)^2 - 4I\,Dp^2}}{2I}} \qquad (3.2.5)$$

The natural frequencies will obviously be different if shear deformation and/or rotatory inertia terms are neglected.

If rotatory inertia is neglected, the Eqns. (3.2.4), (3.2.5) are modified as:

$$Dp^2 - (E_{NR}\cdot p + F)\,\lambda_{mn}^2 = 0 \qquad (3.2.4a)$$

and,

$$\lambda_{mn} = \pm \sqrt{\frac{Dp^2}{E_{NR}\cdot p+F}} \qquad (3.2.5a)$$

Where, $E_{NR} = \frac{\rho D}{G}$

If shear deformation is neglected,

$$Dp^2 - (E_{NS}\cdot p+F)\,\lambda_{mn}^2 = 0 \qquad (3.2.4b)$$

and

$$\lambda_{mn} = \pm \sqrt{\frac{Dp^2}{E_{NS}\cdot p+F}} \qquad (3.2.5b)$$

where,

$$E_{NS} = \frac{\rho H^3}{12}\;\frac{(1-2\nu)}{(1-\nu)^2}$$

If both the shear deformation and rotatory inertia terms are neglected,

$$Dp^2 - F\lambda_{mn}^2 = 0 \qquad (3.2.4c)$$

and

$$\lambda_{mn} = \pm \sqrt{\frac{Dp^2}{F}} \qquad (3.2.5c)$$

An attempt is made herein to study the effect of rotatory inertia and shear deformation terms on the various natural frequencies of a moderately thick steel and a concrete plate. The effect of these terms is also studied in the equation of motion proposed by Mindlin[45].

3.3 FORCED VIBRATIONS:

The loading function $q(x,y,t)$ can be expressed as

$$q(x,y,t) = q(x,y).\ F(t) \qquad (3.3.1)$$

where $F(t)$ = function of time

$q(x,y)$ = loading function of x,y only.

Let $$q(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn}.\ \text{Sin}\,\frac{m\pi x}{a}\ .\ \text{Sin}\,\frac{n\pi y}{b} \qquad (3.3.2)$$

where, a_{mn} is a constant.

To calculate any particular coefficient a_{m1n1} of this series, multiply both the sides of Eqn. (3.3.2) by Sin $\frac{n1\pi y}{b}$ by and integrate from 0 to b, as follows:

$$\int_0^b q(x,y).\ \text{Sin}\ \frac{n1\pi y}{b}\ dy = \frac{b}{2} \sum_{m=1}^{\infty} a_{mn1}\ \text{Sin}\ \frac{m\pi x}{a} \quad \text{..(3.3.3)}$$

Multiplying both the sides of above equation by Sin $\frac{m1\pi x}{a}$ dx and integrating from 0 to a;

$$\int_0^a \int_0^b q(x,y).\ \text{Sin}\ \frac{m1\pi x}{a}\ \text{Sin}\ \frac{n1\pi y}{b}\ dxdy = \frac{ab}{4}\ a_{m1n1} \quad (3.3.4)$$

Performing the integration indicated in Eqn. (3.3.4) for a given load distribution, the coefficients a_{m1n1} are found. The given load is represented as a sum of partial sinusoidal loadings(50,58). The deflection produced by each partial loading is being calculated and the summation of such terms is the total deflection. Hence, the equation of motion Eqn.(3.1.1) becomes :

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[I.\ddot{\phi}_{mn} + (Ep+F)\dot{\phi}_{mn} + Dp^2\phi_{mn} \right] \text{Sin}\ \frac{m\pi x}{a}\ .\text{Sin}\ \frac{n\pi y}{b}$$

$$= \left[M1 + M\frac{\partial^2}{\partial t^2} - N\nabla^2 \right] q(x,y)\ F(t) \quad \text{---}\ (3.3.5)$$

The above equation is solved here for the uniformly loaded plate, i.e., $q(x,y) = q_o$. For this loading, value of constant a_{m1n1} is given by:

$$a_{m1n1} = \frac{4}{ab} \int_0^a \int_0^b q_o . \text{Sin} \frac{m1\pi x}{a} . \text{Sin} \frac{n1\pi y}{b} \, dxdy$$

$$= \frac{16 \, q_o}{\pi^2 mn} \qquad (3.3.6)$$

Thus,

$$q(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{16 \, q_o}{\pi^2 mn} . \text{Sin} \frac{m\pi x}{a} \text{Sin} \frac{n\pi y}{b} \qquad (3.3.7)$$

Substituting the value of $q(x,y)$ in Eqn.(3.3.5)

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[I.\ddot{\phi}_{mn} + (Ep+F)\ddot{\phi}_{mn} + Dp^2 \phi_{mn} \right] \text{Sin} \frac{m\pi x}{a} \text{Sin} \frac{n\pi y}{b}$$

$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{16 \, q_o}{\pi^2 mn} \left[M1 \; F(t) + M\ddot{F}(t) \right] \text{Sin} \frac{m\pi x}{a} . \text{Sin} \frac{n\pi y}{b}$$

(3.3.8)

Multiplying both the sides of Eqn.(3.3.8) by $\text{Sin} \frac{m1\pi x}{a} dx . \text{Sin} \frac{n1\pi y}{b} dy$ and integrating from 0 to a from 0 to b, the following equation is obtained:

$$I.\ddddot{\phi}_{mn} + (Ep+F)\ddot{\phi}_{mn} + Dp^2\phi_{mn} = \frac{16\,q_o}{\pi^2 mn}\left[M\ddot{F}(t) + M1F(t)\right]$$

(3.3.9)

The solution of Eqn.(3.3.9) is attempted for the following initial conditions:

Initial displacement, velocity, acceleration and Jerk are assumed to be zero. Thus,

$$\phi(x,y,o) = \dot{\phi}(x,y,o) = \ddot{\phi}(x,y,0) = \dddot{\phi}(x,y,o) = 0$$

(3.3.10)

Taking Laplace transform of both the sides of Eqn.(3.3.9):

$$I\left[S^4.f(S) - S^3\phi(o) - S^2\dot{\phi}(o) - S\ddot{\phi}(o) - \dddot{\phi}(o)\right]$$
$$+ (Ep+F)\left[S^2f(S) - S\phi(o) - \dot{\phi}(o)\right] + Dp^2.f(S)$$
$$= \frac{16q_o}{\pi^2 mn}\left\{M\left[S^2 f_1(S) - S\,F(o) - \dot{F}(o)\right] + M1.f_1(S)\right\}$$

(3.3.11)

Where $f(S) = L(\phi_{mn})$ and $f_1(S) = L[F(t)]$.

Introducing the initial conditions Eqn.(3.3.10) in the transformed Eqn. (3.3.11):-

$$f(S)\ [IS^4+(Ep+F)S^2+Dp^2] = \frac{16q_o}{\pi^2 mn}\ [(MS^2+M1)f_1(S)$$

$$-MS\ F(o) - M\dot{F}(o)] \quad ..(3.3.12)$$

Eqn.(3.3.12) is now solved for different types of pulses, as follows:

3.3.A <u>RECTANGULAR PULSE</u>:

A rectangular pulse of magnitude q_o acts on the surface of the plate for a duration to .
For this pulse,

$$F(t) = 1 \qquad \text{for } t \leq to$$
$$F(t) = 0 \qquad \text{for } t > to \qquad ..(3.3.A1)$$

Taking the Laplace transform of this function,

$$L\ [F(t)] = L\ [u(t) - u(t-to)] = \frac{1}{S}\ [1- e^{-to.S}]$$

(3.3.A2)

Thus, from Eqn.(3.3.12),

$$f(S)\ [I.S^4 + (Ep+F)S^2 + Dp^2] = \frac{16q_o}{\pi^2 mn}\ [\frac{1}{S} - \frac{e^{-to.S}}{S}]$$

Or,

$$f(S) = \frac{16q_o\ M1}{\pi^2 mn} \quad \frac{\left(\frac{1}{S} - \frac{e^{-to.S}}{S}\right)}{(IS^4+(Ep+F)S^2 + Dp^2)}$$

Or,

$$f(S) = \frac{16q_o.M1}{\pi^2 mnI} \quad \frac{\left(\frac{1}{S} - \frac{e^{-to.S}}{S}\right)}{(S^2+\frac{Ep+F}{2I})^2 - \left[(\frac{Ep+F}{2I})^2 - \frac{Dp^2}{I}\right]}$$

(3.3.A3)

Let;

$$p_4^2 = \left(\frac{Ep+F}{2I}\right)^2 - \frac{Ap^2}{I}$$

$$m_4^2 = \frac{Ep+F}{2I} - p_4$$

$$q_4^2 = \frac{Ep+F}{2I} + p_4 .$$

Eqn.(3.3.A3) is thus written as

$$f(S) = \frac{16q_oM1}{\pi^2 mnI} \quad \frac{\left[\frac{1}{S} - \frac{e^{-to.S}}{S}\right]}{(S^2+m_4^2)\ (S^2+q_4^2)}$$

Or,

$$f(S) = \frac{16q_oM1}{\pi^2 mnI(q_4^2-m_4^2)} \left[\frac{1}{S^2+m_4^2} - \frac{1}{S^2+q_4^2}\right] \left[\frac{1}{S} - \frac{e^{-to.S}}{S}\right]$$

Or,

$$f(S) = \frac{16q_o M1}{\pi^2 mmI(q_4^2-m_4^2)} \left[\frac{1}{S(S^2+m_4^2)} - \frac{1}{S(S^2+q_4^2)} \right.$$

$$\left. + \frac{e^{-to.S}}{S(S^2+m_4^2)} - \frac{e^{-to.S}}{S(S^2+q_4^2)} \right] \quad (3.3.A4)$$

Use of partial fractions gives

$$\frac{1}{S(S^2+m_4^2)} = \frac{1}{m_4^2} \left[\frac{1}{S} - \frac{S}{S^2+m_4^2} \right]$$

and

..(3.3.A5)

$$\frac{1}{S(S^2+q_4^2)} = \frac{1}{q_4^2} \left[\frac{1}{S} - \frac{S}{S^2+q_4^2} \right]$$

Thus, from Eqn.(3.3.A4)

$$f(S) = A \left\{ \frac{1}{m_4^2} \left(\frac{1}{S} - \frac{S}{S^2+m_4^2} \right) - \frac{1}{q_4^2} \left(\frac{1}{S} - \frac{S}{S^2+q_4^2} \right) \right.$$

$$\left. + \frac{e^{-to.S}}{m_4^2} \left(\frac{1}{S} - \frac{S}{S^2+m_4^2} \right) - \frac{e^{-to.S}}{q_4^2} \left(\frac{1}{S} - \frac{S}{S^2+m_4^2} \right) \right\}$$

(3.3.A6)

Where,

$$A = \frac{16q_o.M1}{\pi^2 mmI(q_4^2 - m_4^2)}$$

Taking Inverse Laplace transform of Eqn.(3.3.A6), and, making use of the second shifting theorem;

$$\phi_{mn} = A\left\{\frac{1}{m_4^2}(1-\text{Cos}\,m_4.t) - \frac{1}{q_4^2}(1-\text{Cos}\,q_4.t) + \frac{1}{m_4^2}\left[1-\text{Cos}\,m_4(t-to).u(t-to)\right] - \frac{1}{q_4^2}\left[1-\text{Cos}\,q_4(t-to).u(t-to)\right]\right\} \quad (3.3.A7)$$

Note the discontinuity in the response that is being caused because of the discontinuity in the applied force at $t = to$.

For $t \leq to$, the Eqn. (3.3.A7) gives

$$\phi_{mn} = A\left[\frac{1}{m_4^2}(1-\text{Cos}\,m_4.t) - \frac{1}{q_4^2}(1-\text{Cos}\,q_4.t)\right] \quad (3.3.A8)$$

while, for $t > to$

$$\phi_{mn} = A\left\{\frac{1}{m_4^2}\left[\text{Cos}\,m_4(t-to) - \text{Cos}\,m_4.t\right] - \frac{1}{q_4^2}\left[\text{Cos}\,q_4(t-t_o) - \text{Cos}\,q_4.t\right]\right\} \quad (3.3.A9)$$

Thus, the transverse displacements are determined as:

$$w(x,y,z,t) = w1(x,y,t).\phi_2(z)$$

$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \phi_{mn}.\text{Sin}\,\frac{m\pi x}{a}\,.\,\text{Sin}\,\frac{n\pi y}{b}\,\phi_2(z) \qquad (3.3.A10)$$

Eqn.(3.3.A10) sums up the various terms and thus gives the total deflection for a rectangular pulse of magnitude q_o and duration to.

3.3 AN <u>Rotatory Inertia and/or Shear Deformation Neglected</u>:

Eqn.(3.1.4) satisfies all the boundary conditions (Eqn. 3.1.3) and also the equations 2.13 to 2.15 (Chapter 2). Only two initial conditions are now required for the complete solution. The initial displacement and velocity are assumed to be zero.

If rotatory inertia is neglected,

$$(E_{NR}p+F)\ddot{\phi}_{mn} + Dp^2\phi_{mn} = \frac{16q_o.M1}{\pi^2 mn} F(t) \qquad (3.3.AN1)$$

Taking Laplace transform and its Inverse, following expressions are now obtained for t $\leq$ to

$$\phi_{mn} = A1(1-\text{Cos}m_{41}.t) \qquad (3.3.AN2)$$

for $t > to$

$$\phi_{mn} = A1 \left[\text{Cos } m_{41}(t-to) - \text{Cos } m_{41}.t \right] \quad (3.3.AN3)$$

Where,

$$m_{41}^2 = \frac{Dp^2}{E_{NR}+F}$$

$$A1 = \frac{16q_o.M1}{\pi^2 mn\ Dp^2} \quad ..(3.3.AN4)$$

If shear deformation is neglected, Eqn.(3.3.AN1) is modified as:

$$(E_{NS}p + F)\ddot{\phi}_{mn} + Dp^2\ \phi_{mn} = \frac{16q_o M1}{\pi^2 mn} .F(t) \quad (3.3.AN5)$$

Thus, for $t \leq to$

$$\phi_{mn} = A1 \left[1-\text{Cos}(m_{42}.t)\right] \quad (3.3.AN6)$$

and for $t > to$,

$$\phi_{mn} = A1(\text{Cos } m_{42}(t-to) - \text{Cos } m_{42}t\ ) \quad (3.3.AN7)$$

Where,

$$m_{42}^2 = \frac{Dp^2}{E_{NS}.p+F}$$

Similarly, if both the rotatory inertia and shear deformation terms are neglected:

$$\phi'_{mn} = A1(1-\text{Cos } m_{43}.t) \qquad (3.3.AN8)$$

$$t \leq to$$

$$= A1(\text{Cos } m_{43}(t-to) - \text{Cos } m_{43}.t) \qquad (3.3.AN9)$$

$$t > to$$

where,

$$m_{43}^2 = \frac{Dp^2}{F}$$

The transverse displacements are then determined by Eqn.(3.3.A10).

3.3B <u>TRIANGULAR PULSE</u>:

For a triangular pulse of magnitude q_o and duration 2 to,

$$F(t) = t/to \qquad \text{for } t \leq to$$

$$= 2 - {}^{t}/to \qquad \text{for } to \leq t \leq 2to$$

$$= 0 \qquad \text{for } t > 2to$$

(3.3.B1)

Taking the Leplace transform of this function

$$f_1(S) = \frac{1}{to.S^2}\left[1 + e^{-2to.S} - 2e^{-to.S}\right] \qquad (3.3.B2)$$

Thus,

$$A.f_1(S)\left[\frac{1}{S^2+m_4^2}-\frac{1}{S^2+q_4^2}\right]=\frac{A}{to.S^2(S^2+m_4^2)}-\frac{A}{to.S^2(S^2+q_4^2)}$$

$$+\frac{A.e^{-2to.S}}{toS^2(S^2+m_4^2)}-\frac{A.e^{-2to.S}}{toS^2(S^2+q_4^2)}-\frac{A.2.e^{-toS}}{toS^2(S^2+m_4^2)}$$

$$+\frac{A.2.e^{-toS}}{to.S^2(S^2+q_4^2)} \qquad (3.3.B3)$$

Or,

$$A.f_1(S)\left[\frac{1}{S^2+m_4^2}-\frac{1}{S^2+q_4^2}\right]=\left[\frac{A}{to.m_4^2}\left(\frac{1}{S^2}-\frac{1}{S^2+m_4^2}\right)\right.$$

$$-\frac{A}{to.q_4^2}\left(\frac{1}{S^2}-\frac{1}{S^2+q_4^2}\right)+\frac{A.e^{-2to.S}}{to.m_4^2}\left(\frac{1}{S^2}-\frac{1}{S^2+m_4^2}\right)$$

$$-\frac{A.e^{-2to.S}}{to\ q_4^2}\left(\frac{1}{S^2}-\frac{1}{S^2+q_4^2}\right)-\frac{A.2.e^{-to.S}}{to.m_4^2}\left(\frac{1}{S^2}-\frac{1}{S^2+m_4^2}\right)$$

$$\left.+\frac{A.2e^{-to.S}}{to.q_4^2}\left(\frac{1}{S^2}-\frac{1}{S^2+q_4^2}\right)\right] \qquad (3.3.B4)$$

Taking Inverse Laplace transform of the above equation;

$$\phi_{mn}=\frac{A}{to.m_4^2}\left[t-\frac{Sin(m_4.t)}{m_4}\right]-\frac{A}{to.q_4^2}\left[t-\frac{Sin\ q_4.t}{q_4}\right]$$

$$+ \left\{ \frac{A}{to.m_4^2} \left[(t-2to) - \mathrm{Sin}\ \frac{m_4(t-2to)}{m_4} \right] - \frac{A}{to.q_4^2} \right.$$

$$\left. \left[(t-2to) - \frac{\mathrm{Sin} q_4(t-2to)}{q_4} \right] \right\} u(t-2to)$$

$$+ \left\{ - \frac{2A}{to.m_4^2} \left[(t-to) - \frac{\mathrm{Sin}\ m_4(t-to)}{m_4} \right] \right.$$

$$\left. + \frac{2A}{to.q_4^2} \left[(t-to) - \frac{\mathrm{Sin}\ q_4(t-to)}{q_4} \right] \right\} u(t-to)$$

(3.3.B5)

Note the discontinuity in the response at two points which is due to the discontinuity in the applied force at $t = to$ and at $t = 2to$.

For $t \leq to$, Eqn. (3.3.B5) gives

$$\phi_{mn} = \frac{A}{to.m_4^2} \left(t - \frac{\mathrm{Sin}\ m_4.t}{m_4} \right) - \frac{A}{to.q_4^2} \left(t - \frac{\mathrm{Sin}\ q_4.t}{q_4} \right)$$

(3.3.B6)

For $to \leq t \leq 2to$,

$$\phi_{mn} = \frac{A}{to.m_4^2} \left(t - \frac{\mathrm{Sin}\ m_4.t}{m_4} \right) - \frac{A}{to.\ q_4^2} \left(t - \frac{\mathrm{Sin}\ q_4.t}{q_4} \right)$$

$$- \frac{2A}{to.m_4^2} \left[(t-to) - \frac{\mathrm{Sin}\ m_4(t-to)}{m_4} \right]$$

$$+ \frac{2A}{to.q_4^2} \left[(t-to) - \frac{\mathrm{Sin}\ q_4(t-to)}{q_4} \right] \qquad (3.3.B7)$$

and for $t > 2to$,

$$\phi_{mn} = \frac{A}{to.m_4^2} \left[- \frac{\sin m_4.t}{m_4} + \frac{2\sin m_4(t-to)}{m_4} - \frac{\sin m_4(t-2to)}{m_4} \right]$$

$$+ \frac{A}{to.q_4^2} \left(-2 \frac{\sin q_4(t-to)}{q_4} + \frac{\sin q_4.t}{q_4} + \frac{\sin q_4(t-to)}{q_4} \right)$$

(3.3.B8)

Thus, the transverse displacements are determined by Eqn.(3.3.A10).

3.3C SAWTOOTH PULSE:

For a Sawtooth pulse of magnitude q_o and duration to,

$$F(t) = t/to \quad \text{for} \quad t \leq to$$
$$= 0 \quad \text{for} \quad t > to \qquad ..(3.3.C1)$$

Taking the Laplace transform of this function

$$f_1(S) = \frac{1}{to.S^2} \left[1-(to.S + 1)\, e^{-to.S} \right] (3.3.C2)$$

Thus,

$$A.f_1(S) \left[\frac{1}{S^2+m_4^2} - \frac{1}{S^2+q_4^2} \right] = \frac{A}{to.m_4^2} \left[\frac{1}{S^2} - \frac{1}{S^2+m_4^2} \right]$$

$$- \frac{A}{to.q_4^2}\left[\frac{1}{S^2} - \frac{1}{S^2+q_4^2}\right] - \frac{A.e^{-to.S}}{to.m_4^2}\left(\frac{1}{S^2} - \frac{1}{S^2+m_4^2}\right)$$

$$+ \frac{A.e^{-to.S}}{to.q_4^2}\left(\frac{1}{S^2} - \frac{1}{S^2+q_4^2}\right) - \frac{A.e^{-to.S}}{m_4^2}\left(\frac{1}{S} - \frac{S}{S^2+m_4^2}\right)$$

$$+ \frac{A.e^{-to.S}}{q_4^2}\left(\frac{1}{S} - \frac{S}{S^2+q_4^2}\right) \qquad (3.3.C3)$$

Taking Inverse Laplace transform,

$$\phi_{mn} = \frac{A}{to.m_4^2}\left(t - \frac{\mathrm{Sin}\ m_4.t}{m_4}\right) - \frac{A}{to.q_4^2}\left(t - \frac{\mathrm{Sin}\ q_4.t}{q_4}\right)$$

$$- \frac{A}{to.m_4^2}\left[(t-to) - \frac{\mathrm{Sin}\ m_4(t-to)}{m_4}\right] u(t-to)$$

$$+ \frac{A}{to.q_4^2}\left[(t-to) - \frac{\mathrm{Sin}\ q_4(t-to)}{q_4}\right].u(t-to)$$

$$- \frac{A}{m_4^2}\left[1-\mathrm{Cos}\ m_4(t-to)\right] u(t-to) + \frac{A}{q_4^2}\left[1-\mathrm{Cos} q_4(t-to)\right]$$

$$. u(t-to) \qquad (3.3.C4)$$

Note the discontinuity at $t = to$. Thus for

$t \leq to$;

$$\phi_{mn} = A\left[\frac{1}{to.m_4^2}\left(t - \frac{\mathrm{Sin}\ m_4.t}{m_4}\right) - \frac{1}{to.q_4^2}\left(t - \frac{\mathrm{Sin}\ q_4.t}{q_4}\right)\right]$$

$$(3.3.C5)$$

for $t > to$;

$$\phi_{mn} = A\left\{ \frac{1}{to.m_4^2}\left(t - \frac{\sin m_4.t}{m_4}\right) - \frac{1}{to.q_4^2}\left(t - \frac{\sin q_4.t}{q_4}\right) - \frac{1}{to.m_4^2}\left[(t-to) - \frac{\sin m_4(t-to)}{m_4}\right] + \frac{1}{to.q_4^2}\left[(t-to) - \frac{\sin q_4(t-to)}{q_4}\right] - \frac{1}{m_4^2}\left(1-\cos m_4(t-to)\right) + \frac{1}{q_4^2}\left(1-\cos q_4(t-to)\right)\right\}$$

(3.3.06)

The transverse displacements are then determined by Eqn.(3.3.A10).

CHAPTER-4

DEVELOPMENT OF FINITE DIFFERENCE ANALOG

4.1 GENERAL REMARKS:

The equation of motion (Eqn. 2.1.11) has been solved analytically (Chapter 3). The major difficulty in solving the boundary value problem governed by the equation consists in finding a proper combination of the displacement components that would exhibit the property of orthogonality which is essential in obtaining a solution. For this reason, even the analytical solution of a fixed beam has not been found yet[10].

For a plate simply supported on all the sides, the boundary conditions are homogeneous and the loading can be expressed mathematically. Thus, the orthogonal 'sine' functions could be used.

The numerical solutions are expected to be adaptable to different types of boundary, initial, or, loading conditions. The finite-difference and the finite-element methods look to be the two most powerful methods. The finite-element approach is viewed as minimization of a functional without reference to the differential equations, while in the finite-difference approach, the governing differential equations are approximated without reference to the functionals. What happens in a finite-element is determined entirely by nodal displacements, while in a finite-difference mesh, there are nodes outside each element.

Available information suggests that there are types of problems to which the finite-difference method is better suited than the finite-element method and vice versa. It appears that neither method will wholly supplant the other. For a given number of degrees of freedom, both appear capable of about the same accuracy.

Less computer time and storage capacity may be needed to generate the solution by the finite-difference method. The solution of the present equation of motion (Eqn. 2.1.11) is now attempted by this method.

4.2 FINITE DIFFERENCE FORM OF THE EQUATION OF MOTION:

Finite-difference analog to Eqn. (2.1.11) is developed from standard central difference operators. Every standard book on the finite-difference methods, for instance, 'The Computing Methods - Vol. II', by Berezin and Zhidkov[3] contains these operators.

The operators required by the present equation of motion are:

$$\frac{\partial^2 w1}{\partial x^2} = \frac{1}{(\Delta x)^2}\Big[w1(x-1,y,t+1) - 2w1(x,y,t+1) + w1(x+1,y,t+1)\Big]$$

$$\frac{\partial^4 w1}{\partial x^4} = \frac{1}{(\Delta x)^4}\Big[w1(x-2,y,t+1)-4w1(x-1,y,t+1)+6w1(x,y,t+1) -4w1(x+1,y,t+1) + w1(x+2,y,t+1)\Big]$$

$$\frac{\partial^2 w1}{\partial t^2} = \frac{1}{(\Delta t)^2}\Big[w1(x,y,t-1)-2w1(x,y,t) + w1(x,y,t+1)\Big]$$

$$\frac{\partial^4 w1}{\partial t^4} = \frac{1}{(\Delta t)^4}\Big[w1(x,y,t-2)-4w1(x,y,t-1)+6w1(x,y,t) -4w1(x,y,t+1) +w1(x,y,t+2)\Big]$$

$$\frac{\partial^4 w1}{\partial x^2\,\partial y^2} = \frac{1}{(\Delta x)^2(\Delta y)^2}\Big[w1(x-1,y-1,t+1)-2w1(x,y-1,t+1) +w1(x+1,y-1,t+1)-2w1(x-1,y,t+1) +4w1(x,y,t+1)-2w1(x+1,y,t+1) +w1(x-1,y+1,t+1)-2w1(x,y+1,t+1) +w1(x+1,y+1,t+1)\Big]$$

$$\frac{\partial^4 w1}{\partial y^4} = \frac{1}{(\Delta y)^4}\Big[w1(x,y-2,t+1)-4w1(x,y-1,t+1)+6w1(x,y,t+1) -4w1(x,y+1,t+1)+w1(x,y+2,t+1)\Big]$$

$$\frac{\partial^4 w1}{\partial x^2\,\partial t^2} = \frac{1}{(\Delta x)^2(\Delta t)^2}\Big[w1(x-1,y,t-1)-2w1(x,y,t-1)+w1(x+1,y,t-1) -2w1(x-1,y,t)+4w1(x,y,t)-2w1(x+1,y,t) +w1(x-1,y,t+1)-2w1(x,y,t+1)+w1(x+1,y,t+1)\Big]$$

$$\frac{\partial^4 w1}{\partial y^2 \partial t^2} = \frac{1}{(\Delta y)^2 (\Delta t)^2} [w1(x,y-1,t-1) - 2w1(x,y,t-1) + w1(x,y+1,t-1)$$
$$-2w1(x,y-1,t) + 4w1(x,y,t) - 2w1(x,y+1,t)$$
$$+w1(x,y-1,t+1) - 2w1(x,y,t+1) + w1(x,y+1,t+1)]$$

$$\frac{\partial^2 q}{\partial t^2} = \frac{1}{(\Delta t)^2} [q(x,y,t-1) - 2q(x,y,t) + q(x,y,t+1)]$$

$$\frac{\partial^2 q}{\partial x^2} = \frac{1}{(\Delta x)^2} [q(x-1,y,t) - 2q(x,y,t) + q(x+1,y,t)]$$

$$\frac{\partial^2 q}{\partial y^2} = \frac{1}{(\Delta y)^2} [q(x,y-1,t) - 2q(x,y,t) + q(x,y+1,t)]$$

Where, Δx, Δy, Δt are the respective finite-differences in the x,y and t directions.

Introducing the above expressions in the equation of motion (Eqn.3.1.1):

$$D \Big\{ \frac{1}{(\Delta x)^4} [w1(x-2,y,t+1) - 4w1(x-1,y,t+1) + 6w1(x,y,t+1)$$
$$- 4w1(x+1,y,t+1) + w1(x+2,y,t+1)]$$
$$+ \frac{1}{(\Delta y)^4} [w1(x,y-2,t+1) - 4w1(x,y-1,t+1) + 6w1(x,y,t+1)$$
$$- 4w1(x,y+1,t+1) + w1(x,y+2,t+1)]$$
$$+ \frac{2}{(\Delta x)^2 (\Delta y)^2} [w1(x-1,y-1,t+1) - 2w1(x,y-1,t+1)$$
$$+ w1(x+1,y-1,t+1) - 2w1(x-1,y,t+1) + 4w1(x,y,t+1)$$

$$- 2w1(x+1,y,t+1) + w1(x-1,y+1,t+1) - 2w1(x,y+1,t+1)$$
$$+ w1(x+1,y+1,t+1)\Big]\Big\}$$

$$- E\Big\{\frac{1}{(\Delta x)^2(\Delta t)^2}\Big[w1(x-1,y,t-1) - 2w1(x,y,t-1) + w1(x+1,y,t-1)$$
$$- 2w1(x-1,y,t) + 4w1(x,y,t) - 2w1(x+1,y,t) + w1(x-1,y,t+1)$$
$$- 2w1(x,y,t+1) + w1(x+1,y,t+1)\Big]$$
$$+ \frac{1}{(\Delta y)^2(\Delta t)^2}\Big[w1(x,y-1,t-1) - 2w1(x,y,t-1)$$
$$+ w1(x,y+1,t-1) - 2w1(x,y-1,t) + 4w1(x,y,t) - 2w1(x,y+1,t$$
$$+ w1(x,y-1,t+1) - 2w1(x,y,t+1) + w1(x,y+1,t+1)\Big]\Big\}$$

$$+ \frac{F}{(\Delta t)^2}\Big[w1(x,y,t-1) - 2w1(x,y,t) + w1(x,y,t+1)\Big]$$

$$+ \frac{I}{(\Delta t)^4}\Big[w1(x,y,t-2) - 4w1(x,y,t-1) + 6w1(x,y,t)$$
$$- 4w1(x,y,t+1) + w1(x,y,t+2)\Big]$$

$$= M1.q(x,y,t) + \frac{M}{(\Delta t)^2}\Big[q(x,y,t-1) - 2q(x,y,t)$$
$$+ q(x,y,t+1)\Big] - N\Big\{\frac{1}{(\Delta x)^2}\Big[q(x-1,y,t) - 2q(x,y,t)$$
$$+ q(x+1,y,t)\Big] + \frac{1}{(\Delta y)^2}\Big[q(x,y-1,t) - 2q(x,y,t)$$
$$+ q(x,y,+1,t)\Big]\Big\}$$

...(4.2.1)

Rearranging the terms of Eqn.(4.2.1):

$$a' w1(x,y,t) + b'[w1(x-1,y,t) + w1(x+1,y,t)] + f[w1(x,y-1,t)$$
$$+ w1(x,y+1,t)] + c[w1(x-2,y,t+1) + w1(x+2,y,t+1)]$$
$$+ e[w1(x,y-2,t+1) + w1(x,y+2,t+1)] + n1[w1(x-1,y,t+1)$$
$$+ w1(x+1,y,t+1)] + n2[w1(x,y-1,t+1) + w1(x,y+1,t+1)]$$
$$+ h2.w1(x,y,t+1) + m[w1(x-1,y-1,t+1) + w1(x+1,y-1,t+1)$$
$$+ w1(x-1,y+1,t+1) + w1(x+1,y+1,t+1)] + d1[w1(x-1,y,t-1)$$
$$+ w1(x+1,y,t-1)] + d2[w1(x,y-1,t-1) + w1(x,y+1,t-1)]$$
$$+ h1.w1(x,y,t-1) + g[w1(x,y,t+2) + w1(x,y,t-2)]$$
$$= p1.q(x,y,t) + r[q(x,y,t-1) + q(x,y,t+1)]$$
$$+ s[q(x-1,y,t) + q(x,y,t) + q(x+1,y,t)]$$
$$+ s1[q(x,y-1,t) + q(x,y+1,t)]$$

...(4.2.2)

Where,

$$a' = -\frac{4E}{(\Delta x)^2(\Delta t)^2} - \frac{4E}{(\Delta y)^2(\Delta t)^2} - \frac{2F}{(\Delta t)^2} + \frac{6I}{(\Delta t)^4}$$

$$b' = \frac{2E}{(\Delta x)^2\ (\Delta t)^2}$$

$$f = \frac{2E}{(\Delta y)^2(\Delta t)^2}$$

$$c = \frac{D}{(\Delta x)^4}$$

$$e = \frac{D}{(\Delta y)^4}$$

$$n1 = -\frac{4D}{(\Delta x)^4} - \frac{4D}{(\Delta x)^2(\Delta y)^2} - \frac{E}{(\Delta x)^2(\Delta t)^2}$$

$$n2 = -\frac{4D}{(\Delta y)^4} - \frac{4D}{(\Delta x)^2(\Delta y)^2} - \frac{E}{(\Delta y)^2(\Delta t)^2}$$

$$h1 = \frac{2E}{(\Delta x)^2(\Delta t)^2} + \frac{2E}{(\Delta y)^2\ (\Delta t)^2} + \frac{F}{(\Delta t)^2} - \frac{4I}{(\Delta t)^4}$$

$$h2 = \frac{6D}{(\Delta x)^4} + \frac{6D}{(\Delta y)^4} + \frac{8D}{(\Delta x)^2(\Delta y)^2} + \frac{2E}{(\Delta x)^2(\Delta t)^2} + \frac{2E}{(\Delta y)^2(\Delta t)^2}$$

$$+ \frac{F}{(\Delta t)^2} - \frac{4I}{(\Delta t)^4}$$

$$d1 = -\frac{E}{(\Delta x)^2(\Delta t)^2}$$

$$d2 = -\frac{E}{(\Delta y)^2(\Delta t)^2}$$

$$m = \frac{2D}{(\Delta x)^2(\Delta y)^2}$$

$$g = \frac{I}{(\Delta t)^4}$$

$$p1 = M1 + \frac{2N}{(\Delta x)^2} + \frac{2N}{(\Delta y)^2} - \frac{2M}{(\Delta t)^2}$$

$$s = -\frac{N}{(\Delta x)^4}$$

$$r = \frac{M}{(\Delta t)^2}$$

$$s1 = -\frac{N}{(\Delta y)^2}$$

...(4.2.3)

It is to be noted that the finite-difference approach is basically one of developing an algorithm or recurrence formula, which predicts the deflected position of each node at some future time t+2, based upon the knowledge of deflections at times t+1, t, t-1 and t-2. The Eqn.(4.2.2) is re-written in a form suitable for the programming:

$$\begin{aligned} w1(x,y,t+2) = \frac{1}{g} \Big\{ & p1.q(x,y,t)+r\left[q(x,y,t-1) + q(x,y,t+1)\right] \\ & + s\left[q(x-1,y,t) + q(x+1,y,t)\right]+s1\left[q(x,y-1,t)\right. \\ & \left. + q(x,y+1,t)\right] - q.w1(x,y,t-2)-a'.w1(x,y,t) \\ & - b'\left[w1(x-1,y,t) +w1(x+1,y,t)\right] \\ & - f\left[w1(x,y-1,t) + w1(x,y+1,t)\right] - c\left[w1(x-2,y,t+1)\right. \\ & \left. + w1(x+2,y,t+1)\right] - e\left[w1(x,y-2,t+1)\right. \\ & \left. + w1(x,y+2,t+1)\right] - n1\left[w1(x-1,y,t+1)\right. \\ & \left. + w1(x,y+2,t+1)\right] - n2\left[w1(x,y-1,t+1)\right. \end{aligned}$$

$$+ w1(x,y+1,t+1)\Big] - h2.w1(x,y,t+1)$$

$$- m\Big[w1(x-1,y-1,t+1) + w1(x+1,y-1,t+1)$$

$$+ w1(x-1,y+1,t+1) + w1(x+1,y+1,t+1)\Big]$$

$$- d1\Big[w1(x-1,y,t-1) + w1(x+1,y,t-1)\Big]$$

$$- d2\Big[w1(x,y-1,t-1) + w1(x,y+1,t-1)\Big]$$

$$- h1.w1(x,y,t-1)\Big\} \qquad \ldots(4.2.4)$$

The Eqn.(4.24) is represented by Molecule I (Fig. 4.2.1). If the values of w1 at times t-2, t-1, t and t+1 are known, the values can be obtained at time t+2, in an explicit manner.

4.3 INITIAL CONDITIONS:

Since the values of w1 at time t+1 are usually not defined by the initial conditions, the Eqn.(4.2.4) cannot be used to start the solution.

The solution is attempted here for the following initial conditions:-

(i) Deflections w1 are assumed to be zero for $t \leq 0$.

(ii) At the time of application of load, the rate of change of acceleration is assumed to be constant. Therefore, for the first time increment, second partial derivative of acceleration with respect to time is zero, i.e.,

$$\frac{\partial^4 w1}{\partial t^4} = \frac{1}{(\Delta t)^4} \left[w1(x,y,t-2) - 4w1(x,y,t-1) + 6w1(x,y,t) - 4w1(x,y,t+1) + w1(x,y,t+2) \right] \qquad (4.3.1)$$

Hence, for the first time increment, the Eqn.(4.2.4) is modified as:

$$\begin{aligned} w1(x,y,t+1) = \frac{1}{h21} \Big\{ & p1.q(x,y,t) + r\left[q(x,y,t-1)+q(x,y,t+1)\right] \\ & + s\left[q(x-1,y,t)+q(x+1,y,t)\right] + s1\left[q(x,y-1,t)\right. \\ & \left. + q(x,y+1,t)\right] - a1.w1(x,y,t) - b'\left[w1(x-1,y,t)\right. \\ & \left. + w1(x+1,y,t)\right] - f\left[w1(x,y-1,t)\right. \\ & \left. + w1(x,y+1,t)\right] - c\left[w1(x-2,y,t+1)\right. \\ & \left. + w1(x+2,y,t+1)\right] - e\left[w1(x,y-2,t+1)\right. \\ & \left. + w1(x,y+2,t+1)\right] - n1\left[w1(x-1,y,t+1)\right. \\ & \left. + w1(x+1,y,t+1)\right] - n2\left[w1(x,y-1,t+1)\right. \\ & \left. + w1(x,y+1,t+1)\right] - h11.w1(x,y,t-1) \\ & - d1\left[w1(x-1,y,t-1) + w1(x+1,y,t-1)\right] \\ & - d2\left[w1(x,y-1,t-1) + w1(x,y+1,t-1)\right] \Big\} \end{aligned}$$

(4.3.2)

where,

$$a1 = \alpha' - \frac{6I}{(\Delta t)^4}$$

$$h11 = h1 + \frac{4I}{(\Delta t)^4}$$

$$h21 = h2 + \frac{4I}{(\Delta t)^4}$$

(4.3.3)

Eqn.(4.3.2) is represented by the Molecule II (Fig. 4.3.1). It can be solved implicitly. Knowing the values of w1 at times t-1 and t, the values are determined at time t+1.

After the first time increment, the second partial derivative of acceleration with respect to time is constant. So, for the second time increment, the Eqn.(4.2.4) is modified as:

$$\begin{aligned} w1(x,y,t+1) = \frac{1}{h22} \Big\{ & p1.q(x,y,t) + r\big[q(x,y,t-1) + q(x,y,t+1)\big] \\ & + s\big[q(x-1,y,t) + q(x+1,y,t)\big] + s1\big[q(x,y-1,t) \\ & + q(x,y+1,t)\big] - a2.w1(x,y,t) - b'\big[w1(x-1,y,t) \\ & + w1(x+1,y,t)\big] - f\big[w1(x,y-1,t) + w1(x,y+1,t)\big] \\ & - c\big[w1(x-2,y,t+1) + w1(x+2,y,t+1)\big] \end{aligned}$$

$$- e\,[w1(x,y-2,t+1) + w1(x,y+2,t+1)]$$
$$- n1\,[w1(x-1,y,t+1) + w1(x+1,y,t+1)]$$
$$- n2\,[w1(x,y-1,t+1) + w1(x,y+1,t+1)]$$
$$- m\,[w1(x-1,y-1,t+1) + w1(x+1,y-1,t+1)$$
$$+ w1(x-1,y+1,t+1) + w1(x+1,y+1,t+1)] - d1\,[w1(x-1,y,t-1)$$
$$+ w1(x+1,y,t-1)] - d2\,[w1(x,y-1,t-1) + w1(x,y+1,t-1)]$$
$$- h12.w1(x,y,t-1) - g2.w1(x,y,t-2) - g.w1(x,y,t-3)\}$$

(4.3.4)

where,

$$h12 = h1 + \frac{I}{(\Delta t)^4}; \qquad h22 = h2 + \frac{6I}{(\Delta t)^4}$$

$$a_2 = a' - \frac{4I}{(\Delta t)^4}; \qquad g_2 = g - \frac{4I}{(\Delta t)^4}$$

(4.3.5)

Eqn.(4.3.4) is represented by Molecule III (Fig.4.3.2). It can be solved implicitly. Knowning the values of w1 at times t-3, t-2, t-1 and t; the values are determined at time t+1.

4.4 BOUNDARY CONDITIONS:

Eqn.s(4.2.4),(4.3.2) and (4.3.4) are solvable for any type of the boundary conditions. For a plate simply supported on all the four sides (Fig. 4.4.1);

(i) Deflections are zero at $x = 0$, a and at $y = 0, b$. Thus,

$$w1(o,y,t) = w1(a,y,t) = w1(x,o,t) = w1(x,b,t) = 0 \tag{4.4.1}$$

(ii) Bending moments $(M_x)_{x=o,a} = (M_y)_{y=o,b} = 0$

Thus for $x = o,a$;

$$\frac{(1-\nu)}{(\Delta x)^2}\left[w1(x-1,y,t) - 2w1(x,y,t) + w1(x+1,y,t\right]$$

$$= \frac{-\nu}{(\Delta y)^2}\left[w1(x,y-1,t)-2w1(x,y,t)+w1(x,y+1,t)\right] \tag{4.4.2}$$

Substituting Eqn.(4.4.1) into Eqn.(4.4.2);

$$\left.\begin{aligned} w1(-1,y,t) &= -w1(1,y,t) \\ w1(a+1,y,t) &= -w1(a-1,y,t) \end{aligned}\right| \tag{4.4.3}$$

Similarly for $y = o,b$;

$$
\left.\begin{aligned}
w1(x,-1,t) &= -\ w1(x,1,t) \\
w1(x,b+1,t) &= -\ w1(x,b-1,t)
\end{aligned}\right| \qquad (4.4.4)
$$

Eqns.(4.4.1),(4.4.3) and (4.4.4) are the required boundary conditions for the complete solution of a simply supported rectangular plate.

4.5 STABILITY AND CONVERGENCE:

Since, in a finite-difference solution, it is possible for the higher harmonics to introduce errors which grow exponentially and thus cause the calculated deflections to become unbounded as the time approaches infinity, the solution might become unstable. Detailed studies of a limited number of partial differential equations indicates that stability implies convergence. The case of free vibrations is used here to get an approximate idea about the time increment required for a stable solution. Eqn.(2.2.3) can be written in the following form:

$$
\begin{aligned}
R^2 \Big\{ & w1(x-2,y,t) + w1(x+2,y,t) + w1(x,y-2,t) + w1(x,y+2,t) \\
& + 20.w1(x,y,t) - 8\left[w1(x-1,y,t) + w1(x+1,y,t)\right. \\
& \left. + w1(x,y+1,t) + w1(x,y-1,t)\right] + 2\left[w1(x-1,y+1,t)\right. \\
& \left. + w1(x+1,y+1,t) + w1(x-1,y-1,t) + w1(x+1,y-1,t)\right]\Big\} \\
& + \left[w1(x,y,t-1) - 2w1(x,y,t) + w1(x,y,t+1)\right] = 0
\end{aligned}
$$

(4.5.1)

$$\text{where,}\quad R^2 = \frac{D(1-\nu)^2}{(1-2\nu)} \cdot \frac{(\Delta t)^2}{(\Delta 1)^4}$$

$$\Delta 1 = \frac{\Delta x}{1} = \frac{\Delta y}{1} \qquad (4.5.2)$$

Two initial conditions and eight boundary conditions are prescribed. At $t = t$ and thereafter, the only unknown is $w1(x,y,t+1)$. To investigate stability, a series solution of Eqn.(4.5.1) is assumed to be

$$w1(x,y,t) = A.e^{k\phi} . \text{Sin}(i.\alpha_m). \text{Sin}(j.\beta_n) \qquad (4.5.3)$$

where, A is a constant and $i = 0,1,2,...M$; $j = 0,1,2,....N.$ and $k = 2,3,... \infty.$

Substituting Eqn.(4.5.3) into Eqn.(4.5.1);

$$e^{2\phi} + e^{\phi}\left\{4.R^2\left[(\cos\alpha_m-2)^2 + (\text{Cos}\beta_n-2)^2 - 4+2\text{Cos}\alpha_m.\text{Cos}\beta_n\right] - 2\right\} + 1 = 0. \qquad (4.5.4)$$

For a rectangular plate with M by N elements and hinged supports along the edges, the deflections must satisfy Eqn.s(4.4.1,4.4.4,4.4.5). These boundary conditions are satisfied for

$$\alpha_m = \frac{m}{M}\pi \qquad m = 1,2,\ldots,M-1$$

$$\text{and} \quad \beta_n = \frac{n}{N}\pi \qquad n = 1,2,\ldots,N-1 \qquad (4.5.5.)$$

Thus, Eqn.(4.5.3) becomes;

$$w1(x,y,t) = \sum_{m=1}^{M-1} \sum_{n=1}^{N-1} \text{Sin}\left(\frac{im\pi}{M}\right).\ \text{Sin}\left(\frac{jn\pi}{N}\right) \left[c_1 e^{\phi_1 . k} + c_2 . e^{\phi_2 . k}\right] \qquad (4.5.6)$$

where, c_1 and c_2 are the constants.

For stability;

$$|e^{\phi_1}| = |e^{\phi_2}| \leq 1 \qquad (4.5.7)$$

An examination of Eqn.(4.5.4) shows that Eqn.(4.5.7) is satisfied if the discriminant.

$$\left\{4R^2\left[(\text{Cos}\alpha_m-2)^2 + (\text{Cos}\beta_n-2)^2-4-2\,\text{Cos}\alpha_m.\text{Cos}\beta_n\right]-2\right\}^2-4 \leq 0 \qquad (4.5.8)$$

In the limit as $m \to M$ and $n \to N$; Eqn.(4.5.8) is satisfied if $R^2 \leq \frac{1}{16}$.

Thus, for a stable explicit solution; the value of

$$\frac{D(1-\nu)^2}{(1-2\nu)} \quad \frac{(\Delta t)^2}{(\Delta 1)^4} \leq \frac{1}{16} \qquad (4.5.9)$$

4.6 SOLUTION STEPS:

The solution of a plate problem is now obtained in the following manner:-

1. The dimensions and the material properties of the plate are defined.
2. A suitable mesh size is chosen and the various constants required by the equation of motion are calculated.
3. The loading and the initial conditions are then defined.
4. A suitable time increment is chosen to satisfy the stability requirements.
5. By the help of Molecule II and the eight boundary conditions, the deflections are obtained in an implicit manner for the first time increment.
6. By the help of Molecule III and the same eight boundary conditions, the deflections are obtained for the second time increment.
7. For the remaining time increments, the results are obtained in an explicit manner. Molecule I is now used to obtain the solution upto the desired time steps. It is to be noted that, in contrast to the analytical solution, the results are now obtained for a future time, only

if the deflections at the previous time steps are already known.

4.7 RECTANGULAR PLATE WITH FIXED EDGES:

If the plate is fixed along all the four sides Fig. (4.4.2), the following boundary conditions have to be satisfied:-

$$(w1)_{x=o,a} = 0; \qquad \left(\frac{\partial w1}{\partial x}\right)_{x=o,a} = 0$$
$$(w1)_{y=o,b} = 0; \qquad \left(\frac{\partial w1}{\partial y}\right)_{y=o,b} = 0 \qquad (4.7.1)$$

Thus,

$$w1(o,y,t) = w1(a,y,t) = w1(x,o,t) = w1(x,b,t) = 0$$
$$w1(-1,y,t) = w1(1,y,t)$$
$$w1(a+1,y,t) = w1(a-1,y,t)$$
$$w1(x,-1,t) = w1(x,1,t)$$
$$w1(x,b+1,t) = w1(x,b-1,t) \qquad (4.7.2)$$

The steps required for the solution of a fixed plate are same as indicated in Art.4.6 for a plate simply supported on all the sides.

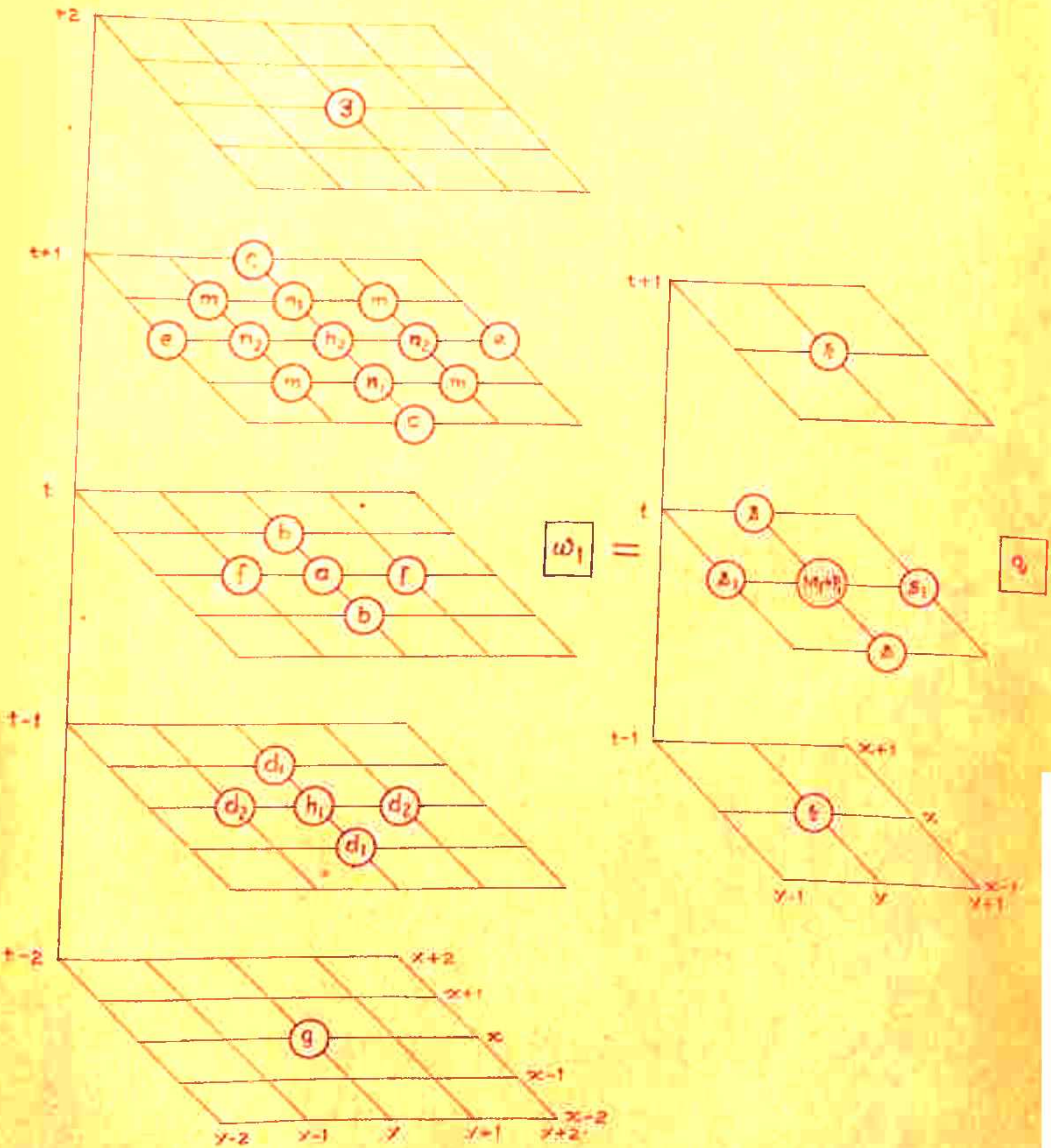


FIG. 4·11: MOLECULE '1'

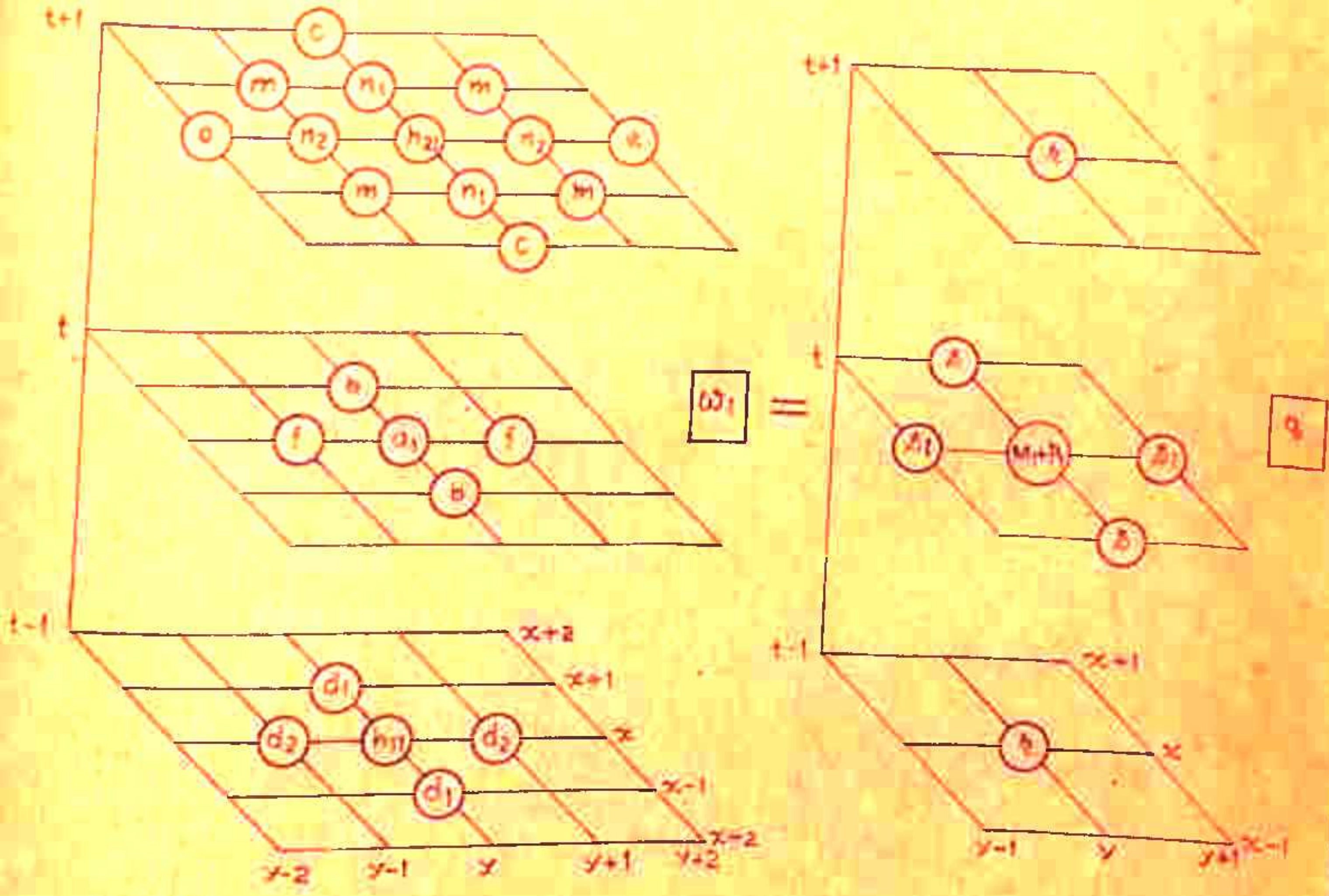


FIG. 4-21 : MOLECULE '2'

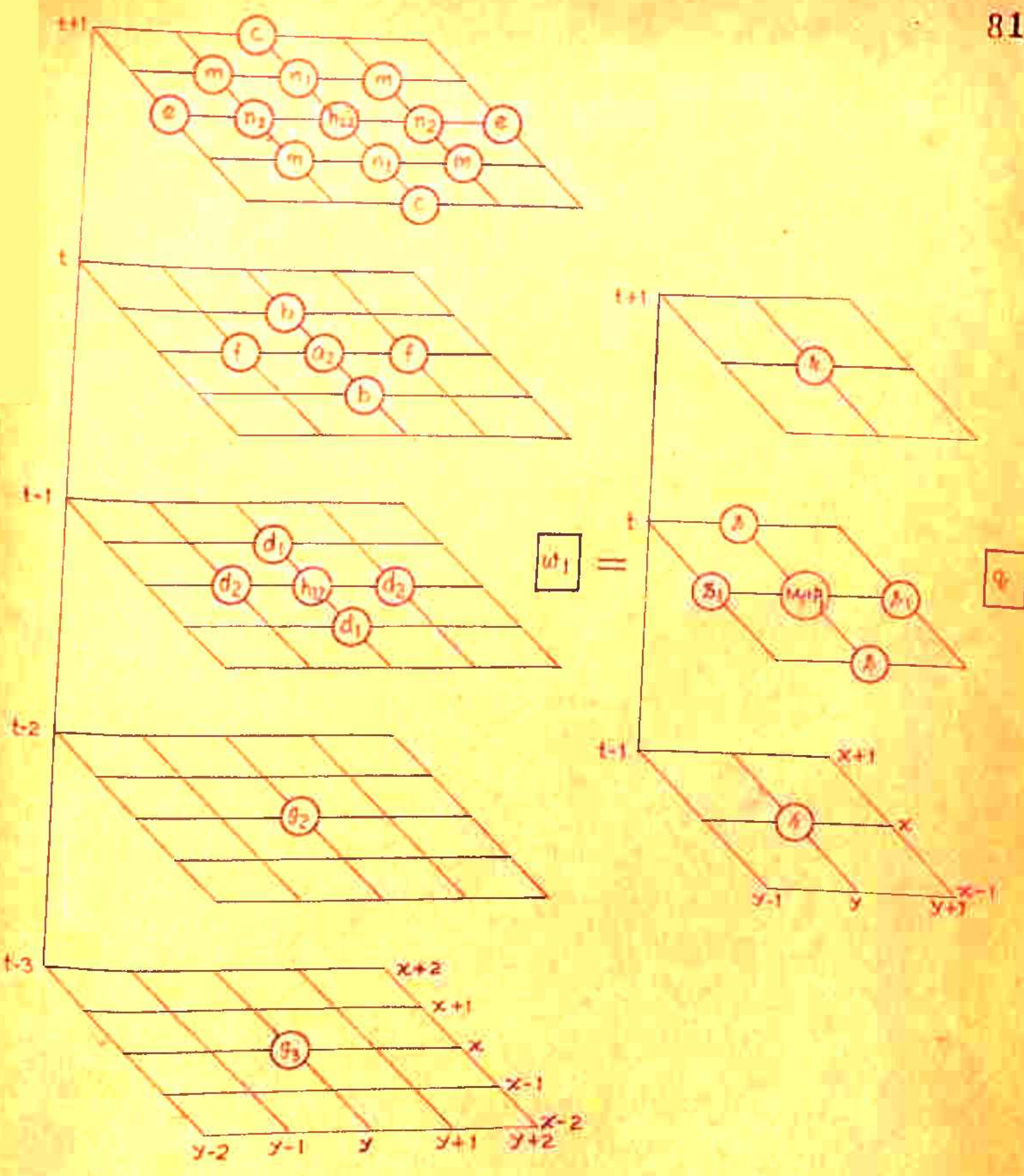


FIG. 4.2.2: MOLECULE '3'

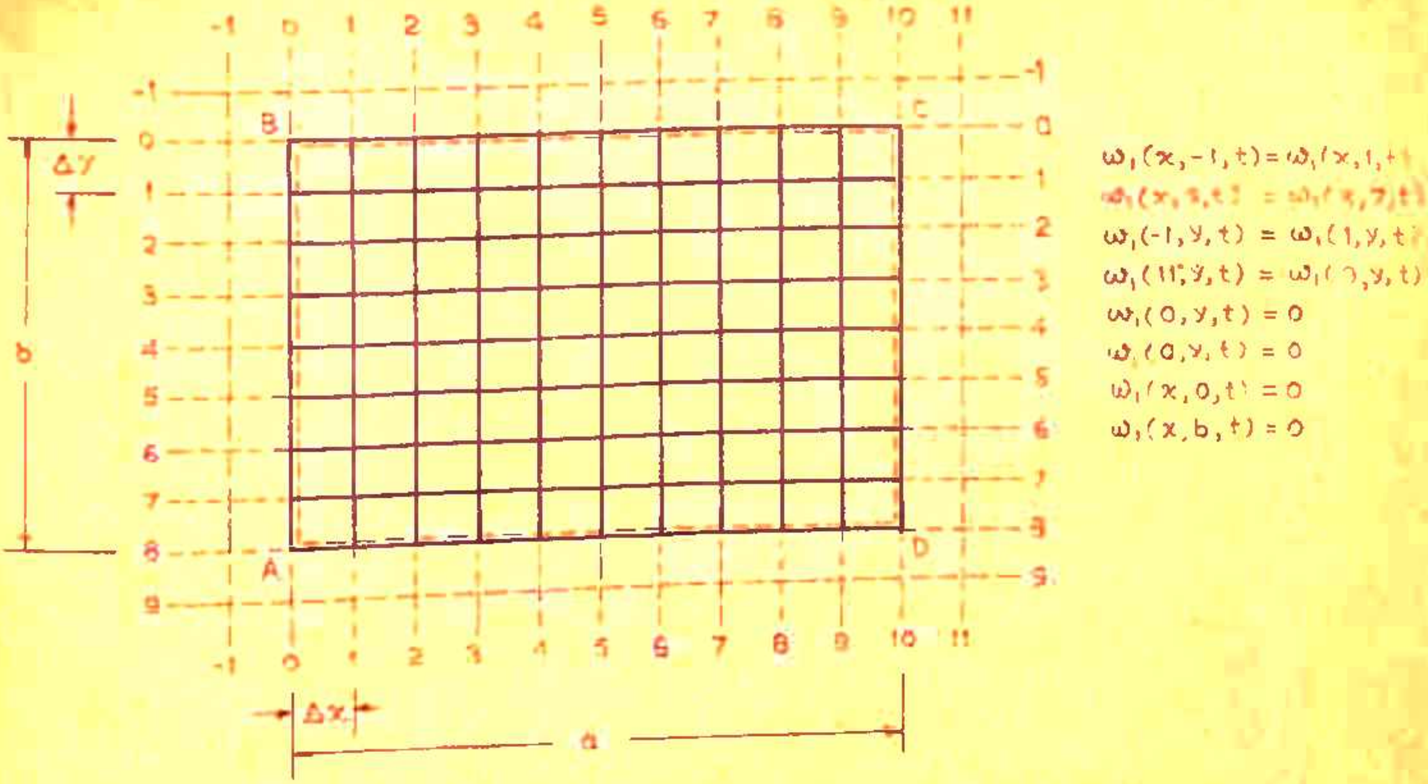


FIG. 4-4·1 : MESH USED FOR THE PLATE SIMPLY SUPPORTED ON ALL SIDES

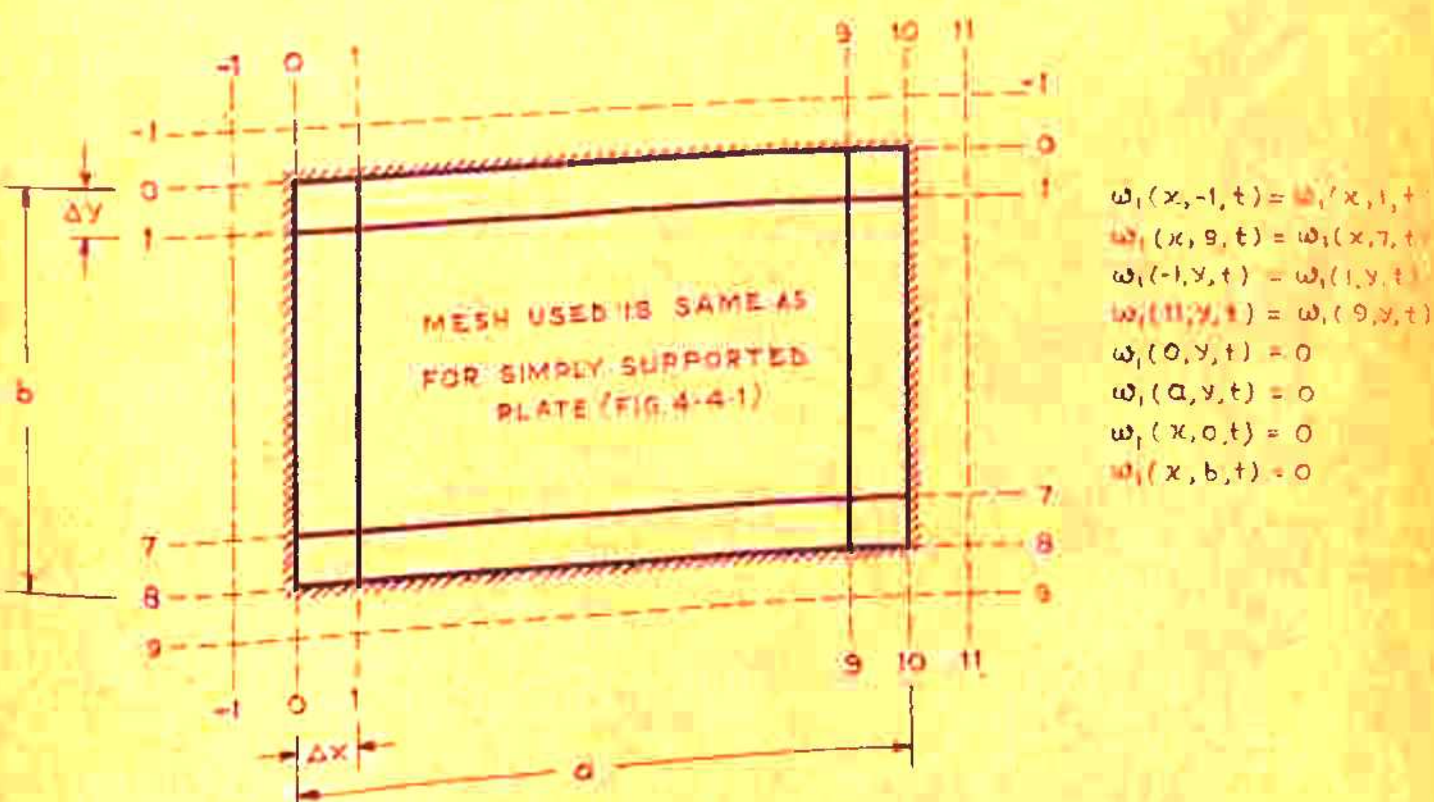


FIG. 4-4·2 : PLATE FIXED ON ALL FOUR SIDES.

CHAPTER-5

CIRCULAR PLATES

5.1 GENERAL REMARKS:

The Eqn.(2.1.11) has been solved numerically (Chapter 4). The Cartesian coordinates were used to obtain the desired results. When circular plates are analyzed, it is convenient to express the governing differential equation of motion in a polar coordinate system. This can be accomplished by coordinate transformation. An alternative approach, based on the equilibrium condition of an infinitesimal small plate element[50] is analogous to the derivation given in Art. 2.1(Chapter-2).

If the coordinate transformation technique is used, the following geometric relationships between the Cartesian coordinates (x,y) and polar coordinates (r,ϕ) are applicable (Fig. 5.1.1):

$$x = r \cos\phi$$
$$y = r \sin\phi$$
$$r = \sqrt{x^2 + y^2} \qquad (5.1.1)$$
$$\phi = \tan^{-1} \frac{y}{x}$$

Since x is a function of r and ϕ, the derivatives of $w1(r,\phi,t)$ with respect to x are transformed into derivatives with respect to r and ϕ. Thus,

$$\frac{\partial w1}{\partial x} = \frac{\partial w1}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial w1}{\partial \phi} \cdot \frac{\partial \phi}{\partial x} \tag{5.1.2}$$

Using Eqn.(5.1.1),

$$\frac{\partial r}{\partial \phi} = \operatorname{Cos}\phi$$

$$\frac{\partial \phi}{\partial x} = -\frac{1}{r} \cdot \operatorname{Sin}\phi \tag{5.1.3}$$

Therefore,

$$\frac{\partial w1}{\partial x} = \operatorname{Cos}\phi \cdot \frac{\partial w1}{\partial r} - \frac{1}{r} \cdot \operatorname{Sin}\phi \frac{\partial w1}{\partial \phi} \tag{5.1.4}$$

From which,

$$\frac{\partial^2 w1}{\partial x^2} = \operatorname{Cos}^2\phi \frac{\partial^2 w1}{\partial r^2} + \frac{1}{r^2} \operatorname{Sin}^2\phi \frac{\partial^2 w1}{\partial \phi^2} + \frac{1}{r} \operatorname{Sin}^2\phi \frac{\partial w1}{\partial r} - \frac{1}{r} \operatorname{Sin}2\phi \frac{\partial^2 w1}{\partial r \partial \phi} + \frac{1}{r^2} \operatorname{Sin}2\phi \frac{\partial w}{\partial \phi} \tag{5.1.5}$$

In a similar manner,

$$\frac{\partial w1}{\partial y} = \operatorname{Sin}\phi \frac{\partial w1}{\partial r} + \frac{1}{r} \operatorname{Cos}\phi \frac{\partial w1}{\partial \phi} \tag{5.1.6}$$

$$\frac{\partial^2 w1}{\partial y^2} = \operatorname{Sin}^2\phi \frac{\partial^2 w1}{\partial r^2} + \frac{1}{r^2} \operatorname{Cos}^2\phi \frac{\partial^2 w1}{\partial \phi^2} + \frac{1}{r}\operatorname{Cos}^2\phi \frac{\partial w1}{\partial r} + \frac{1}{r} \operatorname{Sin}2\phi \frac{\partial^2 w1}{\partial r \partial \phi} - \frac{1}{r^2} \operatorname{Sin}2\phi \frac{\partial w1}{\partial \phi} \tag{5.1.7}$$

and,

$$\frac{\partial^2 w1}{\partial x\,\partial y} = \frac{1}{2}.\,Sin2\phi\,\frac{\partial^2 w1}{\partial r^2} - \frac{1}{r^2}\,Cos2\phi\,\frac{\partial w1}{\partial \phi} - \frac{1}{2r^2}\,Sin2\phi\,\frac{\partial^2 w1}{\partial \phi}$$

$$- \frac{1}{2r}\,Sin2\phi.\,\frac{\partial w1}{\partial r} + \frac{1}{r}\,Cos2\phi\,\frac{\partial^2 w1}{\partial r\,\partial \phi} \qquad (5.1.8)$$

The Laplace operator becomes:

$$\nabla_r^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{1}{r}\frac{\partial}{\partial r} \qquad (5.1.9)$$

Thus, the equation of motion of a plate takes the following form:

$$\left\{ D\,\nabla_r^2\,\nabla_r^2 - \left[\frac{\rho H^3}{12}\,\frac{(1-2\nu)}{(1-\nu)^2} + \frac{\rho D}{G}\right]\frac{\partial^2}{\partial t^2}\nabla_r^2 + \frac{\rho H(1-2\nu)}{(1-\nu)^2}\,\frac{\partial^2}{\partial t^2} + \frac{\rho^2 H^3}{12G}\,\frac{(1-2\nu)}{(1-\nu)^2}\,\frac{\partial^4}{\partial t^4}\right\} w1$$

$$= \left[\frac{1-2\nu}{(1-\nu)^2}\,\frac{D}{GH}\,\nabla_r^2 + \frac{\rho H^2}{12G}\,\frac{(1-2\nu)}{(1-\nu)^2}\,\frac{\partial^2}{\partial t^2}\right] q \qquad (5.1.10)$$

where, $w1 = w1(r,\phi,t)$

$q = q\,(r,\phi,t)$

If the plate is under the action of lateral loads, which are radially symmetric with respect to the origin and the supports have the same type of symmetry, w1 will be independent of ϕ. Thus, the Laplacian operator becomes

$$\nabla_r^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} \qquad (5.1.10a)$$

Consequently, the equation of motion of the circular plate has the following form:

$$D\left[\frac{\partial^4 w1}{\partial r^4} + \frac{2}{r}\frac{\partial^3 w1}{\partial r^3} - \frac{1}{r^2}\frac{\partial^2 w1}{\partial r^2} + \frac{1}{r^3}\frac{\partial w1}{\partial r}\right]$$

$$+ \frac{\rho H(1-2\nu)}{(1-\nu)^2}\frac{\partial^2 w1}{\partial t^2} - \left[\frac{\rho H^3}{12}\frac{(1-2\nu)}{(1-\nu)^2} + \frac{\rho D}{G}\right]$$

$$\left[\frac{\partial^4 w1}{\partial r^2 \partial t^2} + \frac{1}{r}\frac{\partial^3 w1}{\partial r \partial t^2}\right] + \frac{\rho^2 H^3}{12G}\frac{(1-2\nu)}{(1-\nu)^2}\frac{\partial^4 w1}{\partial t^4}$$

$$= \left\{\frac{1-2\nu}{(1-\nu)^2} - \frac{D}{GH}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) + \frac{\rho H^2}{12G}\frac{(1-2\nu)}{(1-\nu)^2}\frac{\partial^2}{\partial t^2}\right\} q$$

(5.1.11)

where,

$$w1 = w1\,(r, t)$$

$$q = q(r, t)$$

5.2 FINITE DIFFERENCE ANALOG OF EQUATION OF MOTION:

The operators required by Eqn.(5.1.11) are:

$$\frac{\partial^2 w1}{\partial r^2} = \frac{1}{(\Delta r)^2}\left[w1(r-1,t+1) - 2w1(r,t+1) + w1(r+1,t+1) \right]$$

$$\frac{\partial^4 w1}{\partial r^4} = \frac{1}{(\Delta r)^4}\left[w1(r-2,t+1) - 4w1(r-1,t+1) + 6w1(r,t+1) - 4w1(r+1,t+1) + w1(r+2,t+1) \right]$$

$$\frac{\partial^2 w1}{\partial t^2} = \frac{1}{(\Delta t)^2}\left[w1(r,t-1) - 2w1(r,t) + w1(r,t+1) \right]$$

$$\frac{\partial^4 w1}{\partial t^4} = \frac{1}{(\Delta t)^4}\left[w1(r,t-2) - 4w1(r,t-1) + 6w1(r,t) - 4w1(r,t+1) + w1(r,t+2) \right]$$

$$\frac{1}{r^2}\frac{\partial w1}{\partial r} = \frac{1}{(\Delta r)^4}\left[-\frac{1}{2n^3} w1(r-1,t+1) + \frac{1}{2n^3} w1(r+1,t+1) \right]$$

$$\frac{2}{r}\frac{\partial^3 w1}{\partial r^3} = \frac{1}{(\Delta r)^4}\left[-\frac{1}{n} w1(r-2,t+1) + \frac{2}{n} w1(r-1,t+1) - \frac{2}{n} w1(r+1,t+1) + \frac{1}{n} w1(r+2,t+1) \right]$$

$$\frac{\partial w1}{\partial r} = \frac{1}{(\Delta r)}\left[-\frac{1}{2} w1(r-1,t+1) + \frac{1}{2} w1(r+1,t+1) \right]$$

$$\frac{\partial^4 w1}{\partial r^2 \partial t^2} = \frac{1}{(\Delta r)^2(\Delta t)^2}\Big[w1(r-1,t-1) - 2w1(r,t-1) + w1(r+1,t-1) - 2w1(r-1,t) + 4w1(r,t) - 2w1(r+1,t) + w1(r-1,t+1) - 2w1(r,t+1) + w1(r+1,t+1) \Big]$$

$$\frac{\partial^3 w1}{\partial t^2 \partial r} = \frac{1}{2(\Delta t)^2(\Delta r)} \Big[w1(r+1,t-1) - 2w1(r+1,t) + w1(r+1,t+1) - w1(r-1,t-1) + 2w1(r-1,t) - w1(r-1,t+1) \Big]$$

where,

$$n = \frac{r}{\Delta r}$$

Substituting above expressions in Eqn.(5.1.11);

$$\frac{D}{(\Delta r)^4} \Big[w1(r-2,t+1) - 4w1(r-1,t+1) + 6w1(r,t+1) - 4w1(r+1,t+1) + w1(r+2,t+1) + \frac{1}{n}.w1(r-2,t+1) + \frac{2}{n}.w1(r-1,t+1) - \frac{2}{n}.w1(r+1,t+1) + \frac{1}{n}.w1(r+2,t+1) + \frac{1}{2n^3}.w1(r-1,t+1) + \frac{1}{2n^3}.w1(r+1,t+1) \Big] - \Big[\frac{\rho H^3}{12} \frac{(1-2\nu)}{(1-\nu)^2} + \frac{\rho D}{G} \Big]$$

$$\frac{1}{(\Delta r)^2(\Delta t)^2} \Big[w1(r-1,t-1) - 2w1(r,t-1) + w1(r+1,t-1) - 2w1(r-1,t) + 4w1(r,t) - 2w1(r+1,t) + w1(r-1,t+1) - 2w1(r,t+1) + w1(r+1,t+1) \Big] + \frac{1}{(\Delta r)^2(\Delta t)^2}$$

$$\Big[-\frac{1}{2n}\big(w1(r-1,t+1) - 2w1(r-1,t) + w1(r-1,t-1)\big) + \frac{1}{2n}\big(w1(r+1,t-1) - 2w1(r+1,t) + w1(r+1,t+1)\big) \Big]$$

$$+ \frac{\rho H(1-2\nu)}{(\Delta t)^2(1-\nu)^2} \Big[w1(r,t-1) - 2w1(r,t) + w1(r,t+1) \Big]$$

$$+ \frac{\rho^2 H^3}{12G} \frac{(1-2\nu)}{(1-\nu)^2(\Delta t)^4} \Big[w1(r,t-2) - 4w1(r,t-1) + 6w1(r,t) - 4w1(r,t+1) + w1(r,t+2) \Big]$$

$$= \frac{(1-2\nu)}{(1-\nu)^2} q(r,t) - \frac{D}{GH(\Delta r)^2} \Big[q(r-1,t) - 2q(r,t) + q(r+1,t) \Big]$$

$$+ \frac{1}{2n(\Delta r)^2} \left[q(r+1,t) - q(r-1,t) \right]$$

$$+ \frac{\rho H^2}{12G} \frac{(1-2\nu)}{(1-\nu)^2(\Delta t)^2} \left[q(r,t-1) - 2q(r,t) + q(r,t+1) \right]$$

(5.2.1)

Or,

$$w1(r-2,t+1) \left[\frac{D}{(\Delta r)^4}(1-\frac{1}{n}) \right] + w1(r-1,t+1) \left[\frac{-4D}{(\Delta r)^4} + \frac{2}{n}\frac{D}{(\Delta r)^4} \right.$$

$$\left. - \frac{D}{n^2(\Delta r)^4} - \frac{D}{2n^3(\Delta r)^4} - \frac{E}{(\Delta r)^2(\Delta t)^2} + \frac{1}{2n}\frac{E}{(\Delta r)^2(\Delta t)^2} \right]$$

$$+ w1(r,t+1) \left[\frac{6D}{(\Delta r)^4} + \frac{2}{n^2}\frac{D}{(\Delta r)^4} + \frac{2E}{(\Delta r)^2(\Delta t)^2} \right.$$

$$\left. + \frac{F}{(\Delta t)^2} - \frac{4I}{(\Delta t)^4} \right] + w1(r+1,t+1) \left[\frac{-4D}{(\Delta r)^4} - \frac{2}{n}\frac{D}{(\Delta r)^4} \right.$$

$$\left. - \frac{D}{n^2(\Delta r)^4} + \frac{D}{2n^3(\Delta r)^4} - \frac{E}{(\Delta r)^2(\Delta t)^2} - \frac{1}{2n} \cdot \frac{E}{(\Delta r)^2(\Delta t)^2} \right]$$

$$+ w1(r+2,t+1) \left[\frac{D}{(\Delta r)^4}(1+\frac{1}{n}) \right] + w1(r-1,t+1)$$

$$\left[\frac{-E}{(\Delta r)^2(\Delta t)^2}(1-\frac{1}{2n}) \right] + w1(r+1,t-1)$$

$$\left[\frac{-E}{(\Delta r)^2(\Delta t)^2}(1+\frac{1}{2n}) \right] + w1(r,t-1) \left[\frac{2E}{(\Delta r)^2(\Delta t)^2} \right.$$

$$\left. + \frac{F}{(\Delta t)^2} - \frac{4I}{(\Delta t)^4} \right] + w1(r-1,t) \left[\frac{2E}{(\Delta r)^2(\Delta t)^2} - \frac{E}{n(\Delta r)^2(\Delta t)^2} - \right.$$

$$+ w1(r,t) \left[\frac{-4E}{(\Delta r)^2(\Delta t)^2} - \frac{2F}{(\Delta t)^2} + \frac{6I}{(\Delta t)^4} \right]$$

$$+ w1(r+1,t) \left[\frac{2E}{(\Delta r)^2(\Delta t)^2} \left(1 + \frac{1}{2n} \right) \right]$$

$$+ (w1(r,t-2) + w1(r,t+2)) \frac{I}{(\Delta t)^4}$$

$$= \quad M1.q(r,t) - \frac{N}{(\Delta r)^2} \left[(1 - \frac{1}{2n})q(r-1,t) + (1+ \frac{1}{2n})q(r+1,t) \right.$$

$$\left. - 2q(r,t) \right] + \frac{M}{(\Delta t)^2} \left[q(r,t-1) - 2q(r,t) + q(r,t+1) \right]$$

(5.2.2.)

Or,

$$f_2.w1(r-2,t+1) + d_1.w1(r-1,t+1) + d.w1(r,t+1)$$

$$+ e.w1(r+1,t+1) + f_1.w1(r+2,t+1) + h_1.w1(r-1,t-1)$$

$$+ h_2.w1(r+1,t-1) + c.w1(r,t-1) + b_1.w1(r-1,t) + a.w1(r,t)$$

$$+ g \left[w1(r,t-2) + w1(r,t+2) \right] + b_2.w1(r+1,t)$$

$$= R_3.q(r,t) - R_{11}q(r-1,t) - R_{21}.q(r+1,t)$$

$$+ R_2 \left[q(r,t+1) + q(r,t-1) \right]$$

(5.2.3)

Where,

$$a = - \frac{4E}{(\Delta r)^2(\Delta t)^2} - \frac{2F}{(\Delta t)^2} + \frac{6I}{(\Delta t)^4}$$

$$b_1 = \frac{E}{(\Delta r)^2 (\Delta t)^2} \left(2 - \frac{1}{n} \right)$$

$$b_2 = \frac{E}{(\Delta r)^2 (\Delta t)^2} \left(2 + \frac{1}{n} \right)$$

$$g = \frac{I}{(\Delta t)^4}$$

$$c = \frac{2E}{(\Delta t)^2 (\Delta r)^2} + \frac{F}{(\Delta t)^2} - \frac{4I}{(\Delta t)^4}$$

$$d = \frac{6D}{(\Delta r)^4} + \frac{2}{n^2} \frac{D}{(\Delta r)^4} + \frac{2E}{(\Delta r)^2 (\Delta t)^2} + \frac{F}{(\Delta t)^2} - \frac{4I}{(\Delta t)^4}$$

$$e = - \frac{4D}{(\Delta r)^4} - \frac{2}{n} \frac{D}{(\Delta r)^4} - \frac{D}{n^2 (\Delta r)^4} + \frac{D}{2n^3 (\Delta r)^4} - \frac{E}{(\Delta r)^2 (\Delta t)^2}$$

$$- \frac{E}{2n (\Delta r)^2 (\Delta t)^2}$$

$$f_1 = \frac{D}{(\Delta r)^4} \left(1 + \frac{1}{n} \right)$$

$$f_2 = \frac{D}{(\Delta r)^4} \left(1 - \frac{1}{n} \right)$$

$$h_1 = \frac{E}{(\Delta r)^2 (\Delta t)^2} \left(\frac{1}{2n} - 1 \right)$$

$$h_2 = - \frac{E}{(\Delta r)^2 (\Delta t)^2} \left(\frac{1}{2n} + 1 \right)$$

$$d_1 = - \frac{4D}{(\Delta r)^4} + \frac{2}{n} \frac{D}{(\Delta r)^4} - \frac{D}{n^2 (\Delta r)^4} \left(1 + \frac{1}{2n} \right)$$

$$- \frac{E}{(\Delta r)^2 (\Delta t)^2} \left(1 - \frac{1}{2n} \right)$$

$$R_2 = \frac{N}{(\Delta t)^2}$$

$$R_3 = M1 - 2R_2 + 2.\frac{N}{(\Delta r)^2}$$

$$R_{11} = \frac{N}{(\Delta r)^2}\left(1 - \frac{1}{2n}\right)$$

$$R_{21} = \frac{N}{(\Delta r)^2}\left(1 + \frac{1}{2n}\right)$$

Rewriting Eqn. (5.2.3) in a form suitable for programming:

$$\begin{aligned} w1(r,t+2) = \frac{1}{g}\Big[&- g.w1(r,t-2) - c.w1(r,t-1) - h_2.w1(r+1,t-1) \\ &- h_1.w1(r-1,t-1) - b_1.w1(r-1,t) - a.w1(r,t) - b_2.w1(r+1,t) \\ &- f_2.w1(r-2,t+1) - d_1.w1(r-1,t+1) - d.w1(r,t+1) \\ &- e.w1(r+1,t+1) - f_1.w1(r+2,t+1) + R_3.q(r,t) - R_{11}.q(r-1,t) \\ &- R_{21}.q(r+1,t) + R_2.q(r,t+1) + R_2.q(r,t-1) \Big] \end{aligned} \quad (5.2.5)$$

Thus, knowing the values of w1 at times t-2, t-1 and t+1, the values at time t+2 can be calculated in an explicit manner.

5.3 INITIAL CONDITIONS:

Eqn.(5.2.5) is solvable if the values of w1 at time t+1 are known. But, usually the initial conditions available are not of this kind. In the present investigation, the initial conditions are assumed as under:

(i) Deflections are zero for $t \leq 0$.

(ii) At the time of application of load, the rate of change of acceleration is constant, so, the second partial derivative of acceleration with respect to time is zero, i.e.,

$$\frac{\partial^4 w1}{\partial t^4} = \frac{1}{(\Delta t)^4} \Big[w1(r,t-2) - 4w1(r,t-1) + 6w1(r,t) - 4w1(r,t+1) + w1(r,t+2) \Big] = 0 \tag{5.3.1}$$

Hence, for the first time increment;

$$\begin{aligned} w1(r,t+1) = \frac{1}{do} \Big\{ & R_3 . q(r,t) + R_2 \left[q(r,t+1) + q(r,t-1) \right] \\ & - R_{11} . q(r-1,t) - R_{21} . q(r+1,t) - f_2 . w1(r+2,t+1) \\ & - d_1 . w1(r-1,t+1) - e . w1(r+1,t+1) - f_1 . w1(r+2,t+1) \\ & - h_1 . w1(r-1,t-1) - h_2 . w1(r+1,t-1) - co . w1(r,t-1) \\ & - b_1 . w1(r-1,t) - ao . w1(r,t) - b_2 . w1(r+1,t) \Big\} \end{aligned} \tag{5.3.2}$$

where,

$$ao = a - \frac{6I}{(\Delta t)^4}$$

$$do = d + \frac{4I}{(\Delta t)^4}$$

$$co = c + \frac{4I}{(\Delta t)^4}$$

(5.3.3)

Eqn.(5.3.2) is solvable in an implicit manner. Knowing the values of w1 at times t-1 and t, its value can be obtained at time t+1.

After the first time increment, the second partial derivative of acceleration with respect to time is constant, hence,

$$\begin{aligned} w1(r,t+1) = \frac{1}{do1} \Big\{ & R_3.q(r,t) + R_2 \left[q(r,t+1) + q(r,t-1) \right] \\ & - R_{11}.q(r-1,t) - R_{21}.q(r+1,t) - f_2.w1(r-2,t+1) \\ & - d_1.w1(r-1,t+1) - e.w1(r+1,t+1) - f_1.w1(r+2,t+1) \\ & - h_1.w1(r-1,t-1) - h_2.w1(r+1,t-1) - co1.w1(r,t-1) \\ & - b_1.w1(r-1,t) - ao1.w1(r,t) - b_2.w1(r+1,t) \\ & - g_2.w1(r,t-2) - g_3.w1(r,t-3) \Big\} \end{aligned}$$

(5.3.4)

where,

$$do1 = d + \frac{6I}{(\Delta t)^4}$$

$$ao1 = a - \frac{4I}{(\Delta t)^4}$$

$$co1 = c + \frac{I}{(\Delta t)^4}$$

(5.3.5)

Eqn. (5.3.4) is solved in an implicit manner. Knowing the values of w1 at times t-3, t-2, t-1 and t, the values are obtained at time t+1.

5.4 BOUNDARY CONDITIONS:

A circular plate, fixed on its periphery (Fig.5.4.1) is solved here. The boundary conditions for such a plate are:

(i) Deflections are zero at r = ro,

$$w1(ro,t) = 0; \qquad (5.4.1)$$

(ii) The slope $\left(\frac{\partial w1}{\partial r}\right)_{r=ro} = 0.$

$$w1(r+1,t) = w1(r-1,t) \qquad (5.4.2)$$

(iii) The slope $\left(\frac{\partial w1}{\partial r}\right)_{r=0} = 0$

$$w1(-1,t) = w1(1,t) \qquad (5.4.3)$$

$$w1(o,t) = w1(1,t) \tag{5.44}$$

Eqns.(5.4.1) to (5.4.4.) are the additional equations required (along with Eqns.(5.2.5),(5.3.2) and (5.3.4)) for the complete solution of a circular plate, fixed on its periphery.

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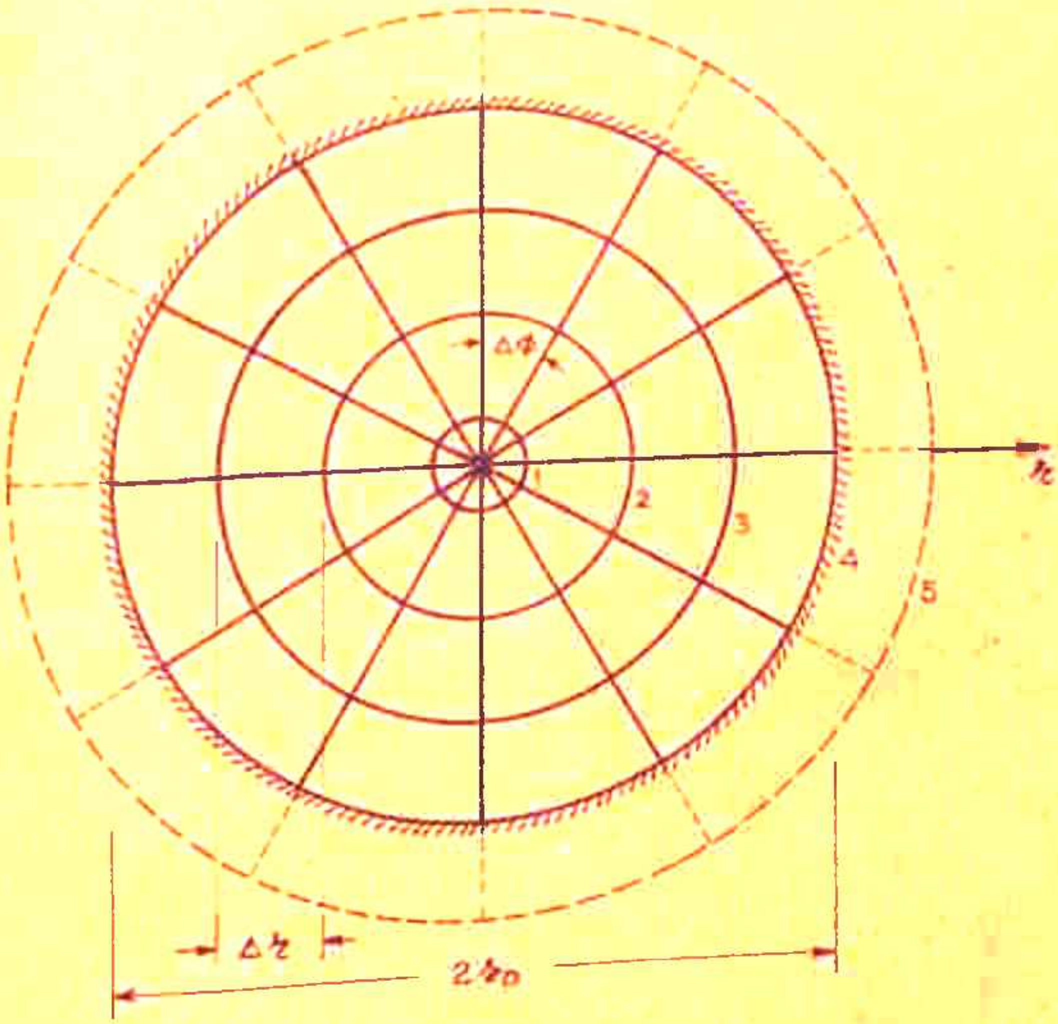

$$\omega_1(5,t) = \omega_1(3,t)$$

$$\omega_1(4,t) = 0$$

$$\omega_1(0,t) = \omega_1(1,t)$$

$$\omega_1(-1,t) = \omega_1(1,t)$$

FIG. 5-4·1: CIRCULAR PLATE FIXED ON ITS PERIPHERY.

CHAPTER-6

RESULTS AND DISCUSSIONS

A rectangular plate having thickness-width ratio of 0.1 is solved with the following data:-

Length of plate	=	200 cm.
Width of plate	=	160 cm.
Thickness of plate	=	16.0 cm.
Loading intensity	=	1.0 kg/cm^2
Duration of application of load	=	0.2 sec.

For a concrete plate;

Modulus of elasticity	=	1.4×10^5 kg/cm^2
Poisson's ratio	=	0.15
Unit weight	=	2.4 gm/cm^3

For a steel plate;

Modulus of elasticity	=	2.0×10^6 kg/cm^2
Poisson's ratio	=	0.30
Unit weight	=	7.6 gm/cm^3

An IBM 1130 computer available in the Birla Institute of Technology and Science is used for the solution of free as well as the forced vibration problems.

6.1 FREE VIBRATIONS:

The natural frequencies of a simply supported rectangular plate are obtained for m,n = 1 to 7. The results are tabulated in ascending order for first twenty frequencies of the system.

The results of Author's equation of motion (Eqn.3.1.1) for a concrete plate are listed in table (6.1.1). Col (i) indicates the results if rotatory inertia terms are neglected. Col (ii) represents the results if shear deformation terms are neglected. The frequencies of Col (iii) are obtained if both these effects are neglected. Col (iva) indicates the results if both the effects are included. In col (ivb) the results of maximum natural frequencies are listed. Similar results are obtained for a steel plate and are listed in table (6.1.2).

Similarly, table (6.1.3) and (6.1.4) represent the results of the Mindlin's equation for a concrete and a steel plate respectively.

Study of these tables reveals that the effect of neglecting rotatory inertia terms is to reduce the natural frequencies of the system, while, the effect of neglecting shear deformation terms is to increase its natural frequencies. If both the effects are neglected, an increase in the natural frequencies is observed.

The natural period of the simply supported concrete plate in the present investigation is 0.279 sec., while, it is 0.1175 sec. for a steel plate of the same dimensions. The solution of Mindlin's equation indicated the natural period of 0.282 sec. for concrete plate, and, 0.13 sec. for the steel plate.

Thus, the present equation of motion gives lower value of natural period as compared to the Mindlin's equation.

If the rotatory inertia and shear deformation terms in the Author's equation of motion are neglected, the natural period is 0.2715 sec. for the concrete plate, and, 0.114 sec. for the steel plate. Thus, there appears to be about 3 percent reduction in the values of the natural period.

A critical study of the tables (6.1.1.) to (6.1.4) indicates that, while the effects of shear deformation and rotatory inertia terms are almost negligible for the lowest mode, these effects are appreciable for higher modes. ~~lower modes, the difference in the values goes on propagating very fast for the higher modes.~~ Apart from this, uncertainty as to the precise nature of the restraint exerted by the supports, as well as the influence of damping, may be expected to have a much greater effect on the higher modes. Thus, the correction for these terms should not remain merely an academic interest.

The classical thin plate theory that ignores the shear deformation and rotatory inertia terms is not expected to give accurate values of the frequencies of higher modes. The inclusion of these terms should give a better insight of the design problem in hand of an engineer. Also, a correct estimate of natural frequencies must suggest for improved techniques for the control of noise and vibrations. The Author, therefore, strongly recommends the inclusion of these terms in the analysis of plates.

6.2 FORCED VIBRATIONS:

The analytical solution of a simply supported rectangular steel plate is obtained for a rectangular pulse. The central deflections obtained from the Eqn.(3.3.A.10) are tabulated (table 6.2.1). The values of ϕ_{mn} required for the solution of Eqn.(3.3.A10) are calculated by the help of Eqns.(3.3.A8) and (3.3.A9). Col (i) in table (6.2.1) lists the values of deflections if first eleven terms of the series are considered, while, Col (ii) lists the corresponding deflections of the first fifteen terms of the series. Thus the series solution of the present equation, like that of the thin plate solution, converges very fast. Hence, it is sufficient to consider the first eleven terms of the series only.

The solution of Mindlin's equation of motion (Eqn.2.1.12) is obtained for a simply supported rectangular steel plate. The deflection at the centre of plate is maximum, hence, is of an interest for a designer. These deflections are plotted in Fig. (6.2.1) for three different types of pulses. The deflections are observed to be maximum for a rectangular pulse. The figure represents the ratio of dynamic v/s static deflections against the ratio of time of interest v/s the duration of application of the load. The static deflection calculated by the finite-element method for the plate under consideration is 0.5266×10^{-2} cm., while, the maximum deflection for the first peak in the case of rectangular pulse is 1.05×10^{-2} cm. For the triangular and sawtooth pulses, the maximum central deflection in the first peak is only 0.65×10^{-2} cm.

Fig. (6.2.2) to Fig. (6.2.4) represent the central deflections obtained by the solution of Author's equation (Eqn. 2.1.11) compared to the solution of Mindlin's equation (Eqn.2.1.12) for various pulses. It is observed that the Author's equation gives lower values of deflections as compared to Mindlin's. The difference in the first peak value is about 14.3 percent in the case of rectangular pulse (Fig. 6.2.2) while, it is of the order of 20 percent in the case of triangular and sawtooth pulses (Figs. 6.2.3, 6.2.4).

Figs. (6.2.5), (6.2.6) represent the central deflections for the rectangular and triangular pulses respectively, for a similar concrete plate. The Author's equation is still observed to give lower values as compared to the Mindlin's equation, but, the difference is very small. It is about 2.8 percent in the case of rectangular pulse and 1 percent in the case of triangular pulse. Fig. (6.2.7) indicates the trend of deflections upto a time of 2.0 seconds for a triangular pulse.

Fig. (6.2.8) represents the contribution of rotatory inertia and/or shear deformation terms in the Author's equation of motion. Table (6.2.2) also lists the Ist and IInd peak values for the sake of comparison. There is not much of a difference in the first peak value of the deflection. But, the trend of values in the deflections from the second peak onwards is entirely different because of the inclusion of these terms in the equation of motion. The difference in the value of second peak is of the order of about 30 to 35 percent (Table 6.2.2).

6.3 NUMERICAL SOLUTION:

The solution of a rectangular concrete plate is now attempted for a rectangular pulse. The size of mesh used is $\Delta x = \Delta y$ = 20 cm. The time increment Δt is governed by the stability crieterian (Eqn. 4.5.9), according to which $\Delta t \leq$ 0.00756 seconds. The solution is obtained for Δt = 0.0004, 0.0003 and 0.0002 seconds also. Fig. (6.3.1) shows the variation of deflections as a function of time. The solution looks to be stable and convergent for a time interval of 0.0003 seconds. The central deflection for first peak is now observed to be 0.056 cm for a plate fixed on all the sides.

Fig. (6.3.2) compares the results of finite-difference solution for a simply-supported plate against the analytical solution. The difference in the first peak value is only about 3.9 percent.

Fig. (6.3.3) represents the results for a square plate (140 cm x 140 cm) fixed on all the sides. The first peak value for the central deflection now is 0.025 cm for a 16 cm thick plate, while, its value is observed to be 0.17 cm for an 8 cm thick plate.

The plate, simply supported on all sides is also solved for a triangular pulse with zero rise time. The results are plotted in figure (6.3.4).

A circular concrete plate, fixed on its periphery is also solved with the following data:

Radius	=	70 cm.
Thickness	=	16 cm.
Modulus of Elasticity	=	1.4×10^5 kg/cm^2
Poisson's ratio	=	0.15
Unit weight	=	2.4 gm/cm^3

The solution for the plate problem is obtained by including both rotatory inertia as well as the shear deformation terms. The solution is found to be stable and convergent for a time increment of 0.0002 seconds. Solution is also obtained without the inclusion of these terms. Fig.(6.3.5) represents the relative deflection time curves at r = 30 cm. The results for r = 50 cm. are also plotted (Fig. 6.3.6). About 25 percent increase in the deflections is observed because of the shear deformation and the rotatory inertia terms.

6.4 CONCLUSIONS:

While the effects of shear deformation and rotatory inertia terms is almost negligible for lower modes, the difference in the values goes on propagating very fast for the higher modes.

The present equation of motion gives lower value of natural period as compared to the Mindlin's equation. The present equation also gives lower bound values of deflections than that of the Mindlin's equation.

For a simply-supported rectangular plate, the contribution of shear deformation and rotatory inertia terms is significant from the second peak onwards. The contribution of shear deformation and rotatory inertia is found to be very significant even for the first peak value in the case of a circular plate fixed on all its periphery.

The finite-difference analog prepared here gives results that are in close agreement with the corresponding analytical solution of the plate problem. The method is found to be very good for determining the time-displacement curves, which are expected to increase the intuitional understanding of the dynamic behaviour of the system.

6.5 RECOMMENDATIONS FOR FUTURE WORK:

It may be noted that in the present equation of motion the effect of damping has been neglected, since, it has been found to be of little importance in the problem of structural dynamics[39]. One is usually interested in the first peak value of deflection and not in a continuous state of vibration. Anyhow, the effect of damping may be expected to have a much greater influence on the higher modes; particularly for the design of vibration elements used to control the electronic circuits. So, further study is recommended in this direction.

The selection of functions $\phi_1(z)$ and $\phi_2(z)$ must actually be based on an experimental law. Thus, if some work is done in this direction, the present solution might possibly be improved.

Available information suggests that there are types of problems to which the finite-differences are better suited than the finite elements and vice versa. It is recommended that the solution of the present problem be attempted by the finite-element method.

Further work is also recommended for the plates resting on an elastic foundation.

Plates of variable thickness, Annular plates, plates subjected to the concentrated loads, etc., are seen in many practical problems. The present work can be extended to handle such situations.

A plate is the basic element and is liable to be very much affected due to an earthquake. The effect of such an event is strongly recommended to be studied, since, the contribution of these terms is expected to be very high, and thus, is expected to give a better insight of the design problem in hand of an engineer.

TABLE 6.1.1

Natural Frequencies For Various Modes of Vibrations

(Rad/sec)

Author's Equation — Concrete Plate

Sr. No.	λ_{mn}	Neglecting Rotatory inertia (i)	Neglecting shear deformation (ii)	Neglecting both Rotatory Inertia and Shear deformation. (iii)	Including both the effects $(\lambda)_{min}$ (iva)	Including both the effects $(\lambda_{mn})_{max}$ (ivb)
1.	λ_{11}	20.9680	22.7845	22.9378	22.4322	1115.06
2.	λ_{21}	41.6819	49.0780	49.7918	47.5064	1142.82
3.	λ_{12}	52.0173	63.6923	64.8972	61.1098	11157.93
4.	λ_{22}	68.6392	89.3701	91.7513	84.4958	1183.98
5.	λ_{13}	91.820	129.779	134.829	120.120	1223.87
6.	λ_{23}	104.660	154.500	161.683	141.293	1257.69
7.	λ_{14}	134.463	218.266	232.735	278.474	1404.12
8.	λ_{24}	144.526	241.776	259.589	212.976	1328.99
9.	λ_{15}	177.642	325.870	358.613	278.474	1404.12
10.	λ_{25}	185.768	348.009	385.467	295.253	1423.50
11.	λ_{71}	204.856	402.195	452.602	335.630	1470.35
12.	λ_{72}	216.027	435.259	494.562	359.836	1498.58
13.	λ_{16}	220.638	449.186	512.464	369.943	1510.40
14.	λ_{26}	227.397	469.880	539.318	384.872	1527.89
15.	λ_{73}	233.572	489.071	564.494	398.623	1544.05
16.	λ_{74}	256.281	561.873	662.399	450.061	1604.77
17.	λ_{17}	263.284	584.987	694.289	466.173	6223.89
18.	λ_{27}	269.049	604.233	721.143	479.517	1639.76
19.	λ_{75}	282.977	651.510	788.278	512.038	1678.58
20.	λ_{76}	312.669	755.650	942.199	582.543	1763.38

TABLE 6.1.2

Natural Frequencies for Various Modes of Vibrations (Rad/sec.)

Author's Equation

Steel Plate

Sr. No.	λ_{mn}	Neglecting Rotatory Inertia (i)	Neglecting Shear deformation (ii)	Neglecting both the effects (iii)	Including both the effects. $(\lambda_{mn})_{min.}$ (iva)	$(\lambda_{mn})_{max.}$ (ivb)
1.	λ_{11}	48.5559	54.6419	55.0095	53.4343	2242.71
2.	λ_{21}	93.9524	117.698	119.410	112.373	2314.61
3.	λ_{12}	115.938	152.747	155.636	144.042	2353.53
4.	λ_{22}	150.628	214.327	220.038	198.023	2420.34
5.	λ_{13}	198.021	311.236	323.348	279.265	2522.01
6.	λ_{23}	223.933	370.522	387.750	327.050	2582.44
7.	λ_{14}	283.489	523.447	558.145	444.789	2733.28
8.	λ_{24}	303.474	579.829	622.546	486.485	2787.37
9.	λ_{15}	369.003	781.503	860.027	629.557	2975.56
10.	λ_{25}	385.046	834.597	924.428	665.868	3023.96
11.	λ_{71}	422.707	964.546	1085.43	752.755	3140.80
12.	λ_{72}	444.737	1043.83	1186.05	804.532	3211.11
13.	λ_{16}	453.831	1077.23	1228.99	826.087	3240.53
14.	λ_{26}	467.158	1126.86	1293.39	857.855	3284.04
15.	λ_{73}	479.334	1172.89	1353.77	887.053	3324.20
16.	λ_{74}	524.119	1347.48	1588.56	995.722	3475.04
17.	λ_{17}	537.934	1402.91	1665.04	1029.59	3522.49
18.	λ_{27}	549.307	1449.07	1729.44	1057.59	3561.87
19.	λ_{75}	576.790	1562.45	1890.44	1125.63	3658.13
20.	λ_{76}	635.418	1812.20	2259.41	1272.24	3868.26

TABLE 6.1.3

NATURAL FREQUENCIES FOR VARIOUS MODES OF VIBRATIONS(Rad/Sec.):

Mindlin's Equation

Concrete Plate

Sr. No.	λ_{mn}	Neglecting Rotatory Inertia (i)	Neglecting shear deformation (ii)	Neglecting both the effects (iii)	Including both the effects $(\lambda_{mn})_{min}$ (iva)	Including both the effects $(\lambda_{mn})_{max}$ (ivb)
1.	λ_{11}	20.9737	22.4269	22.5778	22.0244	1020.47
2.	λ_{21}	42.2638	48.3077	49.0104	46.5225	1048.59
3.	λ_{12}	53.0606	62.6927	63.8787	59.7646	1063.87
4.	λ_{22}	70.6234	87.9676	90.3113	82.4570	1090.16
5.	λ_{13}	95.4422	127.742	132.713	116.866	1130.32
6.	λ_{23}	109.316	152.075	159.146	137.235	1154.26
7.	λ_{14}	141.760	214.841	229.082	187.791	1214.20
8.	λ_{24}	152.772	237.982	255.515	205.808	1235.74
9.	λ_{15}	189.139	320.756	352.985	268.025	1310.85
10.	λ_{25}	198.084	342.547	379.418	283.905	1330.20
11.	λ_{71}	219.117	395.883	445.499	322.034	1376.95
12.	λ_{72}	231.436	428.428	486.800	344.837	1405.511
13.	λ_{16}	236.524	442.136	504.422	354.348	1416.89
14.	λ_{26}	243.981	462.505	530.854	368.382	1434.33
15.	λ_{73}	250.795	481.396	555.635	381.299	1450.43
16.	λ_{74}	275.860	553.055	652.004	429.515	151.093
17.	λ_{17}	283.590	575.806	682.392	444.588	1529.98
18.	λ_{27}	289.954	594.750	709.825	457.062	1545.79
19.	λ_{75}	305.327	641.285	775.906	487.424	1584.44
20.	λ_{76}	338.098	743.791	927.343	553.086	1668.86

TABLE 6.1.4

NATURAL FREQUENCIES FOR VARIOUS MODES OF VIBRATIONS(Rad/sec):

Mindlin's Equation

Steel Plate

Sr. No.	λ_{mn}	Neglecting Rotatory Inertia	Neglecting shear deformation	Neglecting both the effects	Including both the effects	
					$(\lambda_{mn})_{min}$	$(\lambda_{mn})_{max}$
		(i)	(ii)	(iii)	(iva)	(ivb)
1.	λ_{11}	44.5476	47.6343	49.7015	48.2945	2046.39
2.	λ_{21}	89.7673	102.604	107.888	101.621	2111.03
3.	λ_{12}	112.699	113.158	140.619	130.291	2146.03
4.	λ_{22}	150.002	186.841	198.806	179.185	2206.13
5.	λ_{13}	202.717	271.321	292.148	252.827	2297.63
6.	λ_{23}	232.184	323.004	350.335	296.172	2352.03
7.	λ_{14}	301.086	456.317	504.288	403.046	2487.86
8.	λ_{24}	324.485	505.469	562.476	440.918	2536.57
9.	λ_{15}	401.729	681.279	777.041	570.953	2706.11
10.	λ_{25}	420.727	727.564	835.228	603.975	2749.71
11.	λ_{71}	465.401	840.848	980.696	683.015	2854.99
12.	λ_{72}	491.566	909.971	1071.61	730.132	2918.35
13.	λ_{16}	502.372	939.087	1110.40	749.751	2944.86
14.	λ_{26}	518.211	982.351	1168.59	778.670	2984.08
15.	λ_{73}	532.683	1022.47	1223.14	805.252	3020.27
16.	λ_{74}	585.920	1174.67	1435.28	904.213	3156.22
17.	λ_{17}	602.339	1222.99	1504.38	935.072	3199.00
18.	λ_{27}	615.855	1263.23	1562.56	960.580	3234.49
19.	λ_{75}	648.509	1362.07	1708.03	1022.57	3321.27
20.	λ_{76}	718.114	1579.79	2041.40	1156.20	3510.70

TABLE 6.2.1

S.S. Rectangular Steel Plate Subjected to a Rectangular Pulse:

Sr.No.	Time sec.	Central deflections (cm.) x 10^{-2} (i) 1st 11 terms of series	(ii) 1st 15 terms of series.
1.	0.003	0.003680	0.003687
2.	0.006	0.014763	0.014775
3.	0.009	0.033445	0.033446
4.	0.012	0.062305	0.062296
5.	0.015	0.105802	0.105803
6.	0.018	0.165221	0.165232
7.	0.021	0.235820	0.235822
8.	0.024	0.312489	0.312481
9.	0.027	0.391581	0.391587
10.	0.030	0.467784	0.467804
11.	0.033	0.573713	0.543727
12.	0.036	0.620611	0.620608
13.	0.039	0.694248	0.694240

TABLE 6.2.2

Central Deflections of a S.S. Rectangular Concrete Plate:

Rectangular pulse of magnitude q_o and duration to.

Sr. No.		1s Peak		2nd Peak	
		Deflection (cm.)	Percentage difference as compared to (1)	Deflection (Cm.)	Percentage difference as compare to (1)
1.	Shear deformation and Rotatory Inertia Included	0.164793		0.122389	
2.	Shear deformation neglected	0.160223	2.74	0.081311	33.4
3.	Rotatory Inertia neglected	0.159922	2.8	0.080953	33.6
4.	Classical Thin plate Theory	0.159345	3.29	0.080591	34.4

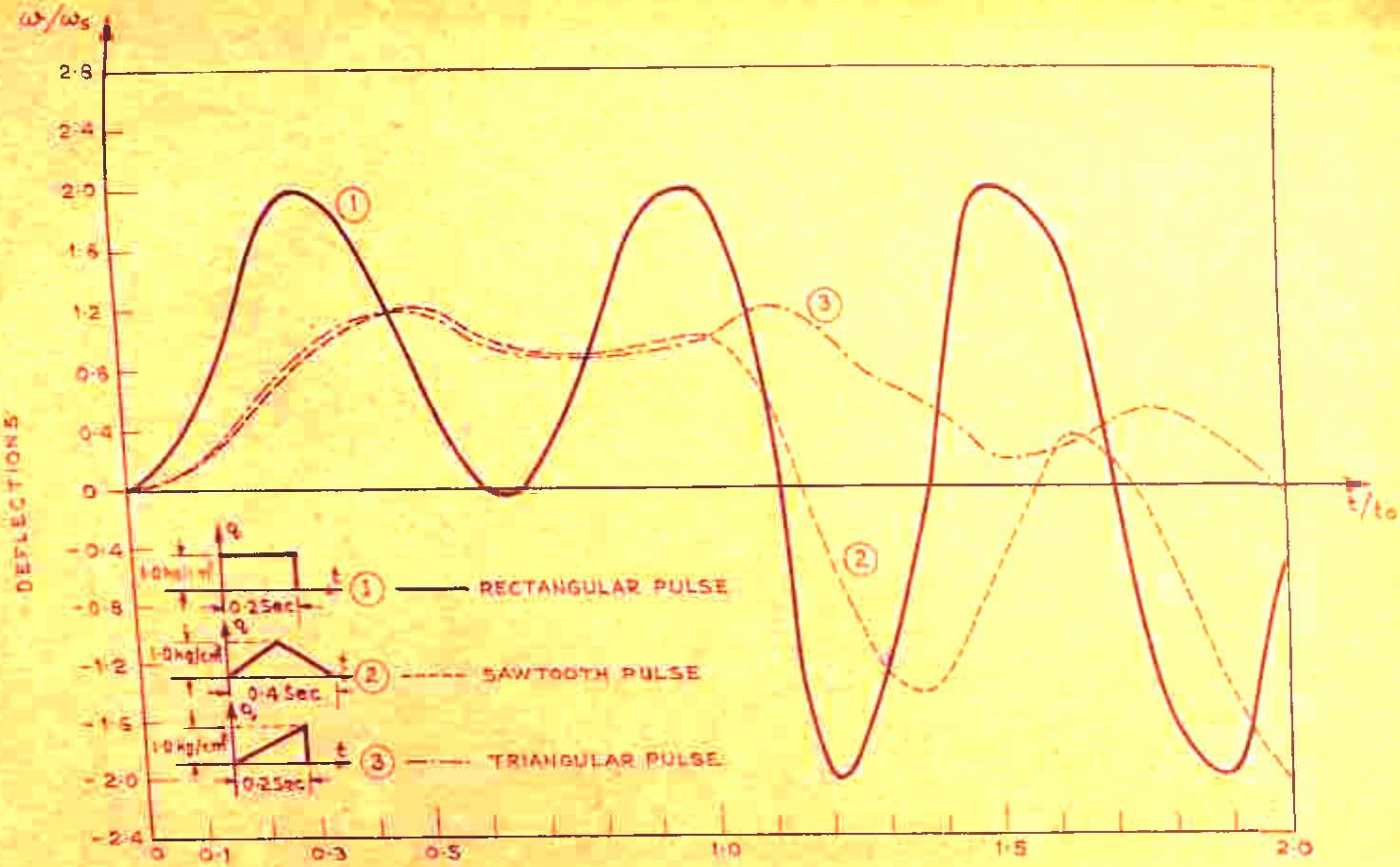

FIG. 6·2·1 : ANALYTICAL SOLUTION OF S.S. RECTANGULAR STEEL PLATE
MINDLIN'S EQUATION
$W_s = 0{\cdot}6265 \times 10^{-2}$ cm
$t_0 = 0{\cdot}2$ sec

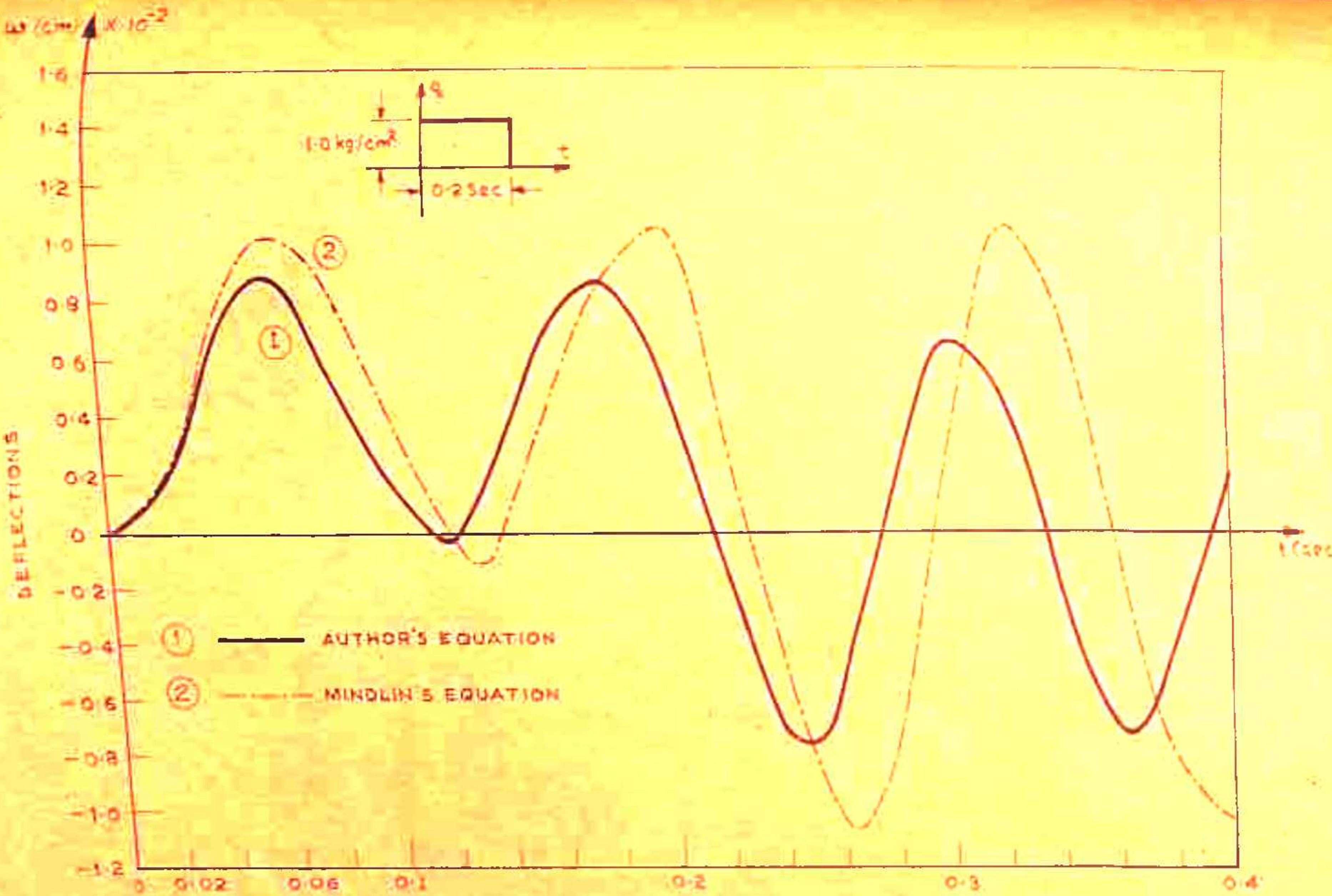

FIG. 5-2-2: CENTRAL DEFLECTIONS OF AUTHOR'S V/S MINDLIN'S EQUATION. (RECTANGULAR PULSE ON A S.S. STEEL PLATE)

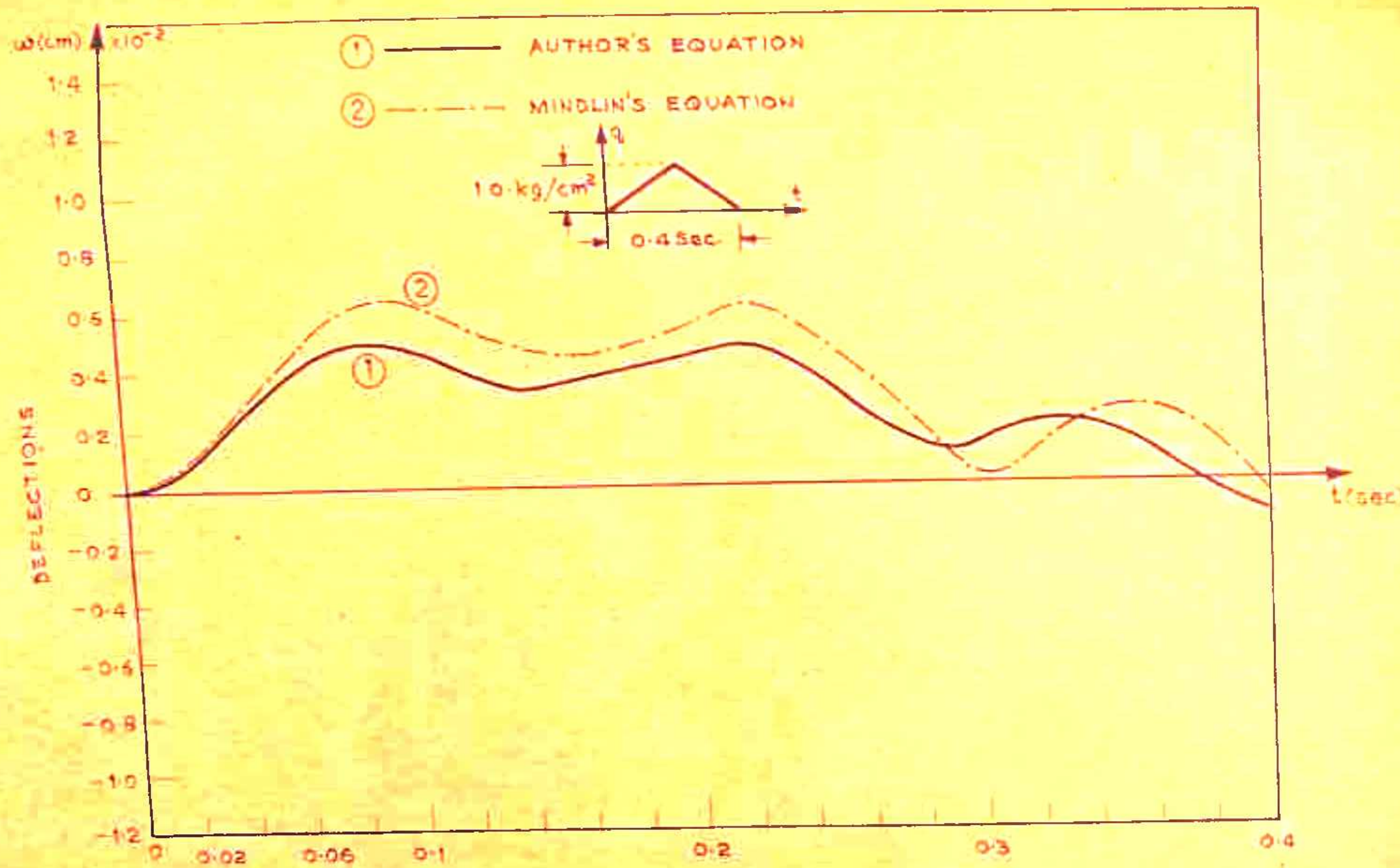

FIG. 6·2·3. CENTRAL DEFLECTIONS OF AUTHOR'S V/S MINDLIN'S EQUATION
(TRIANGULAR PULSE ON A S.S STEEL PLATE)

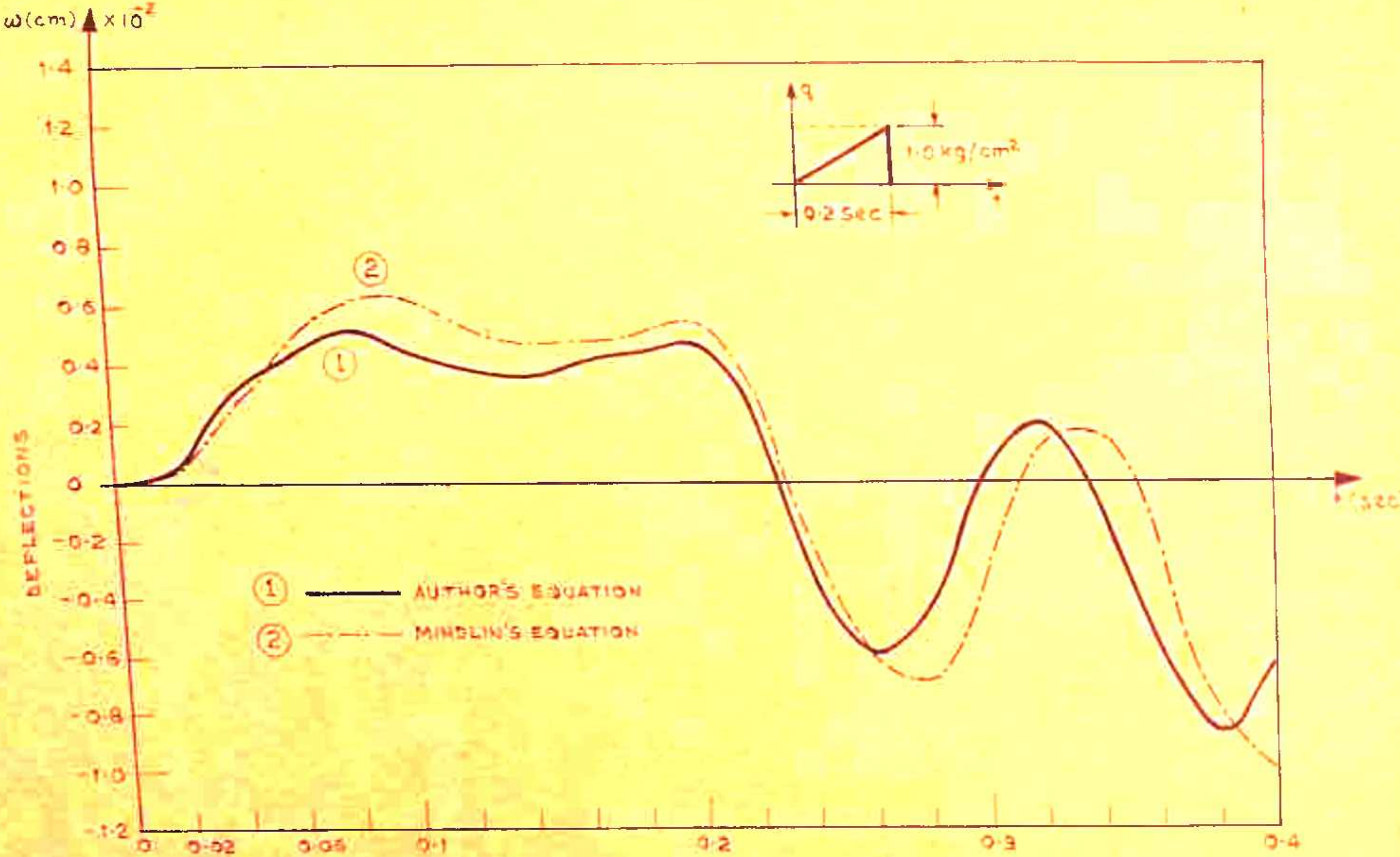

FIG 6-2-4 : CENTRAL DEFLECTIONS OF AUTHORS V/S MINDLIN'S EQUATION (SAWTOOTH PULSE ON A S.S. STEEL PLATE)

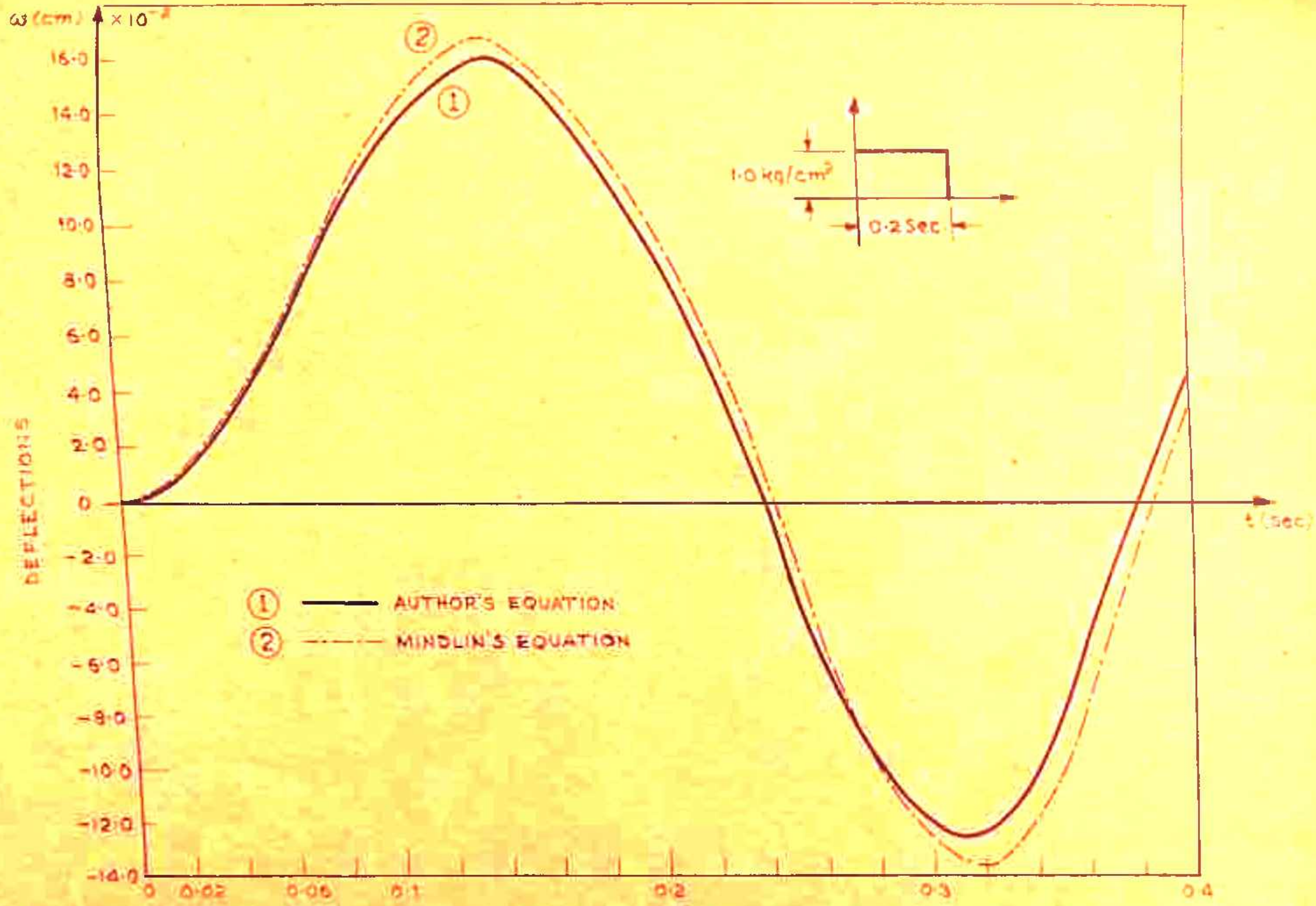

FIG 6·25: CENTRAL DEFLECTIONS OF AUTHOR'S V/S MINDLIN'S EQUATION (RECTANGULAR PULSE ON A S. S. CONCRETE PLATE)

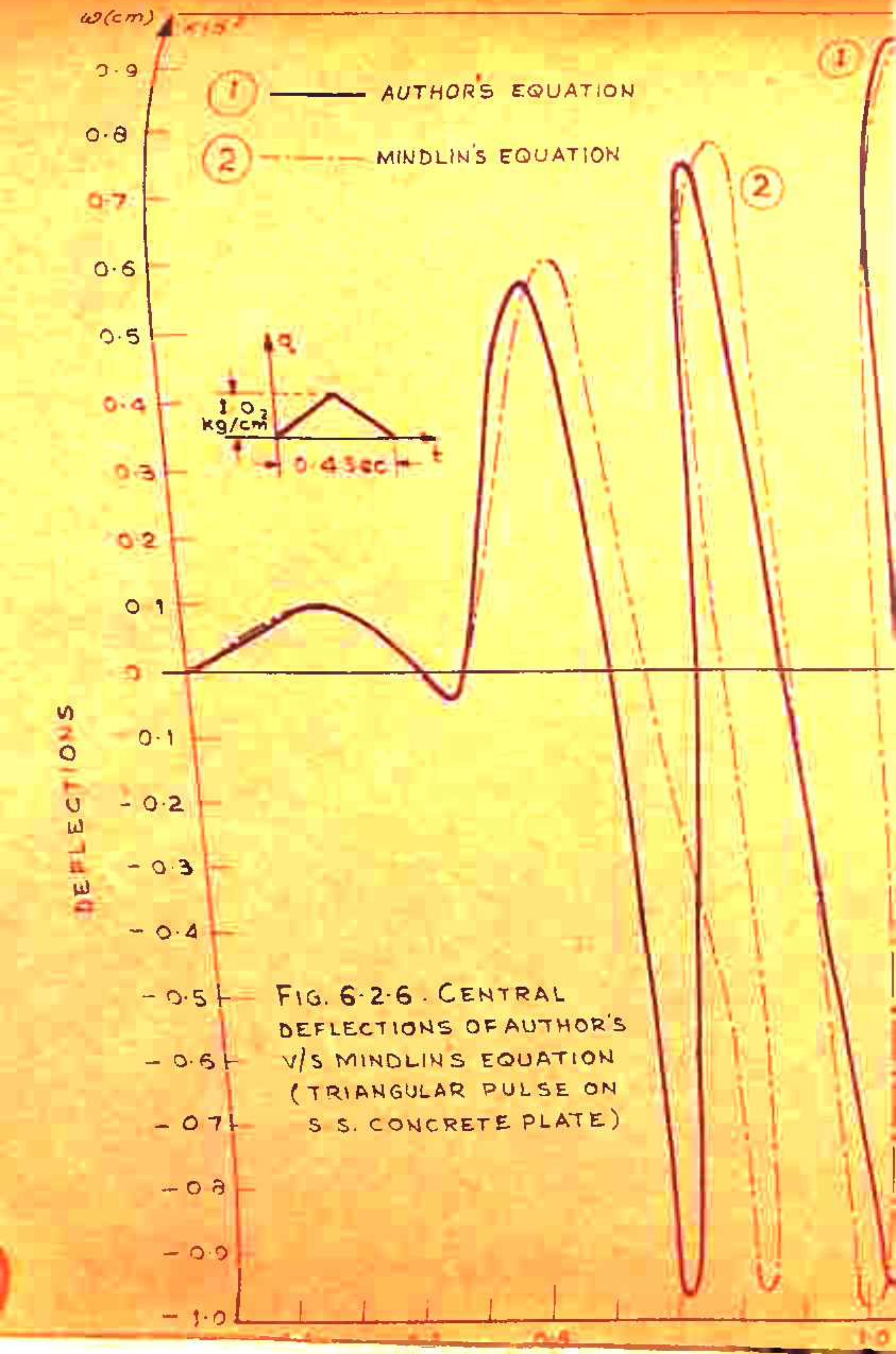

FIG. 6·2·6. CENTRAL DEFLECTIONS OF AUTHOR'S V/S MINDLINS EQUATION (TRIANGULAR PULSE ON S.S. CONCRETE PLATE)

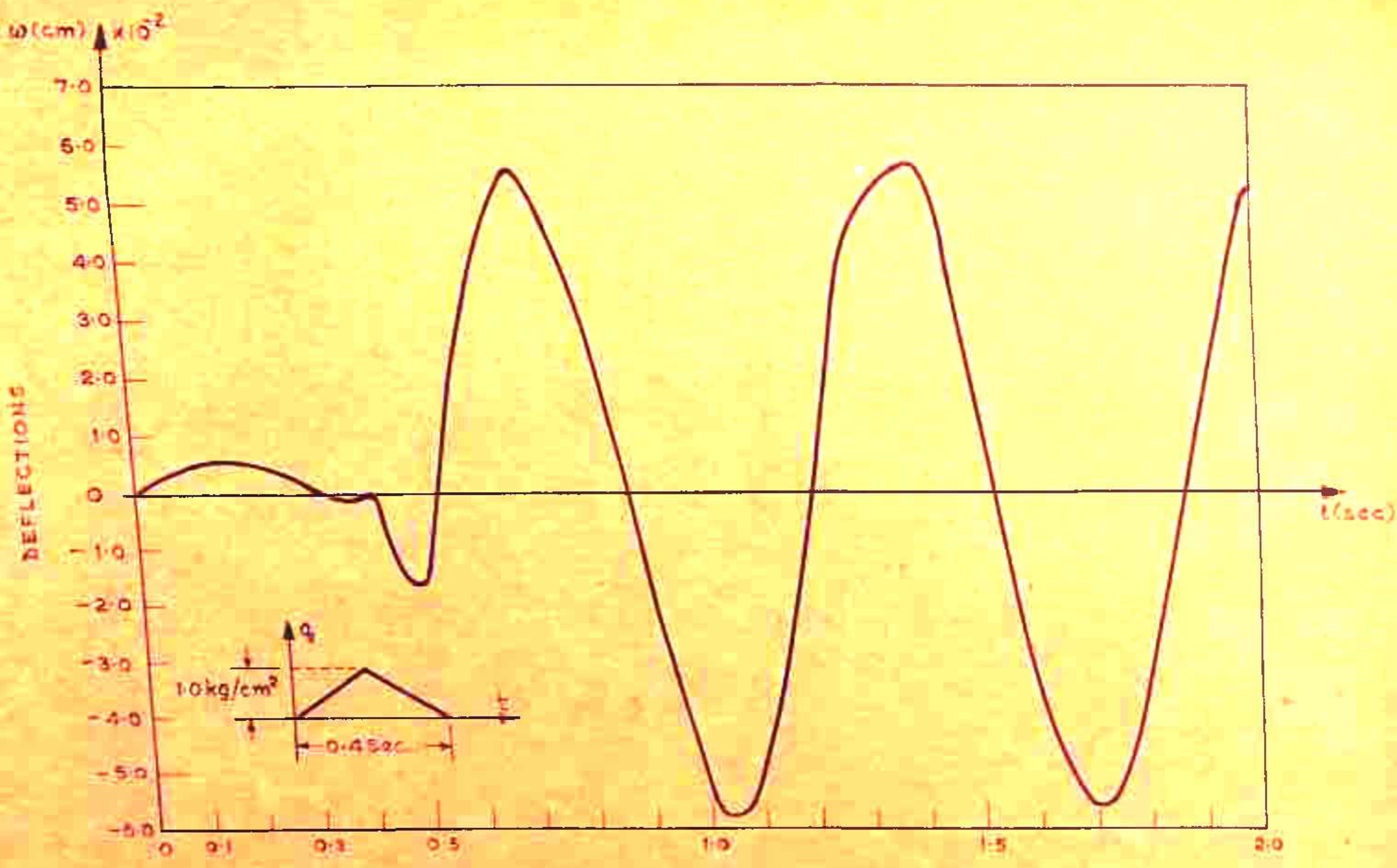

FIG 6·27 CENTRAL DEFLECTIONS OF AUTHOR'S EQUATION (TRIANGULAR PULSE ON A S.S. STEEL PLATE)

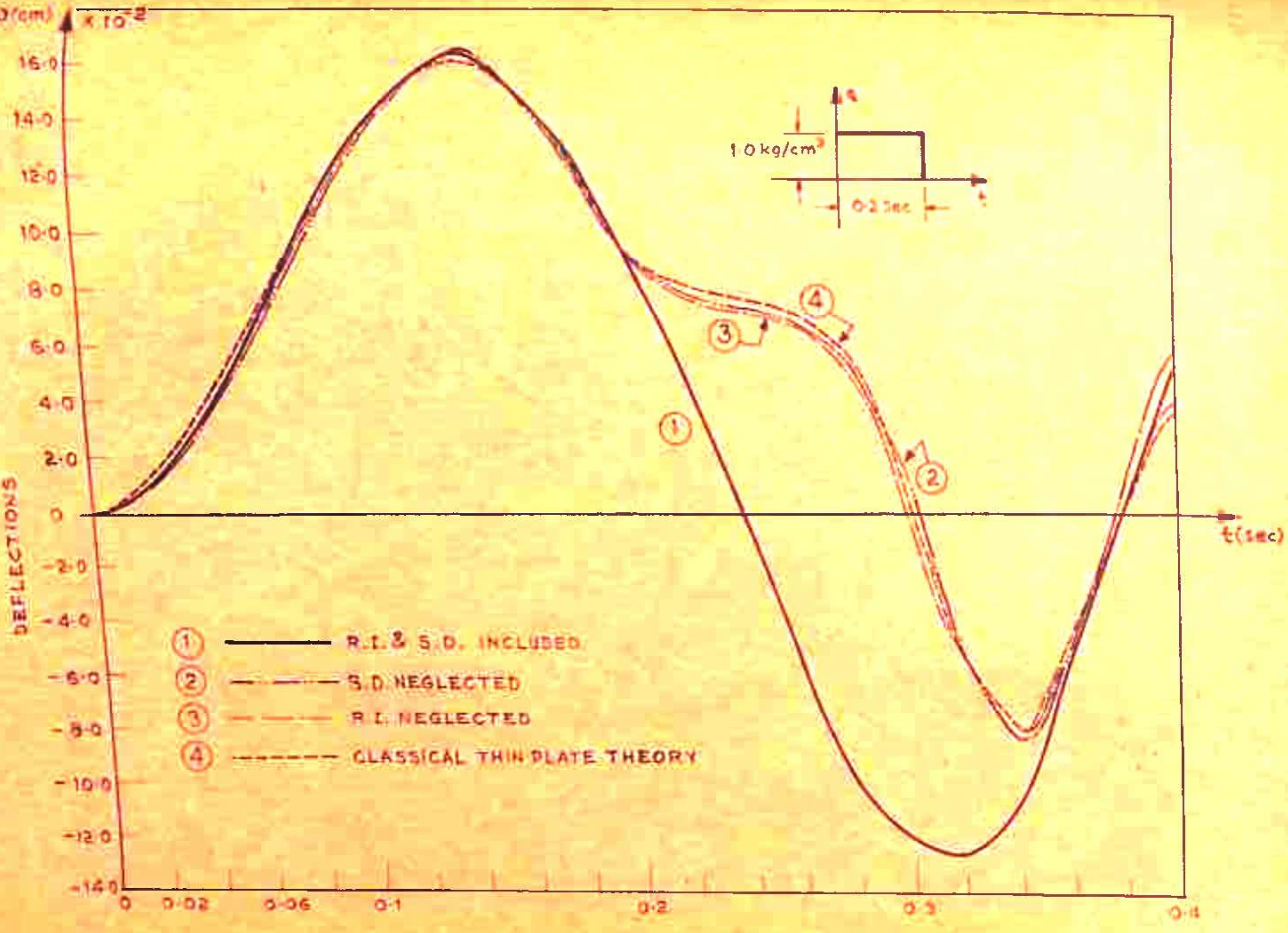

FIG. 6·2·8 : CONTRIBUTION OF SHEAR DEFORMATION AND ROTATORY INERTIA TERMS (S.S. RECTANGULAR CONCRETE PLATE)

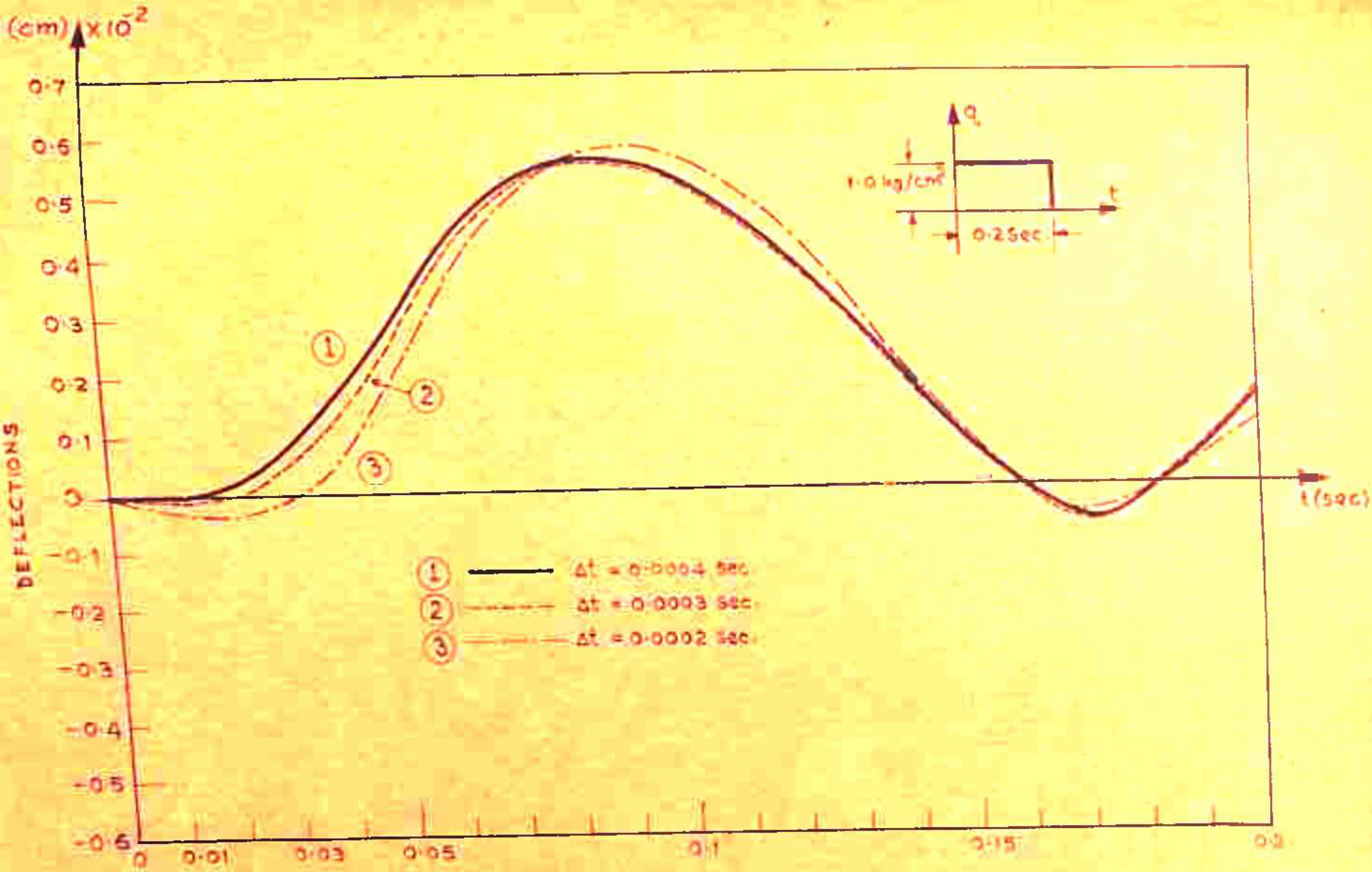

FIG. 6-3.1: 'TIME INCREMENT' USED FOR A STABLE SOLUTION (RECTANGULAR PULSE ON A CONCRETE PLATE FIXED ON ALL THE SIDES)

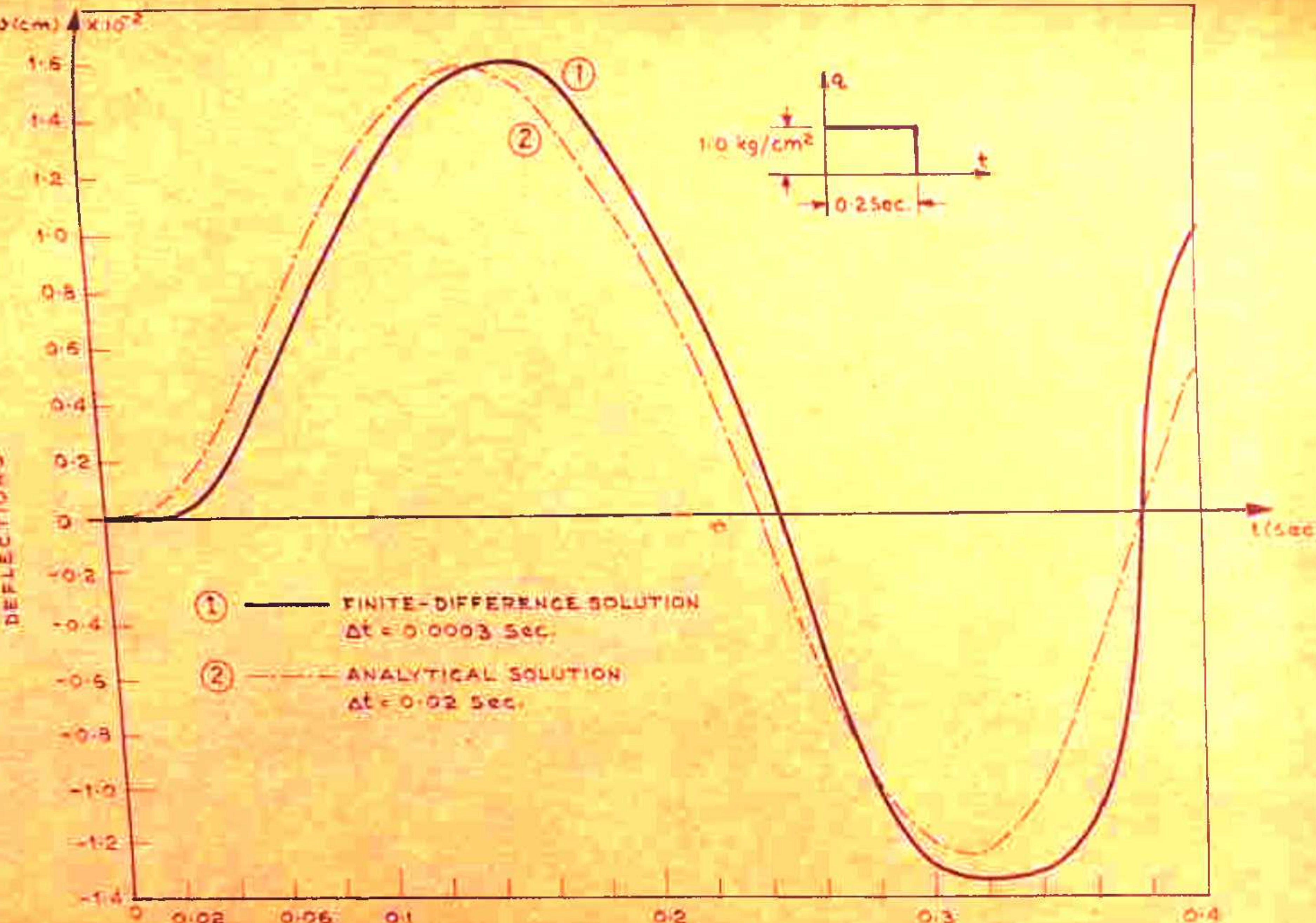

FIG. 6·3·2 : FINITE DIFFERENCE V/S ANALYTICAL SOLUTION (RECTANGULAR PULSE ON A S.S. RECTANGULAR CONCRETE PLATE)

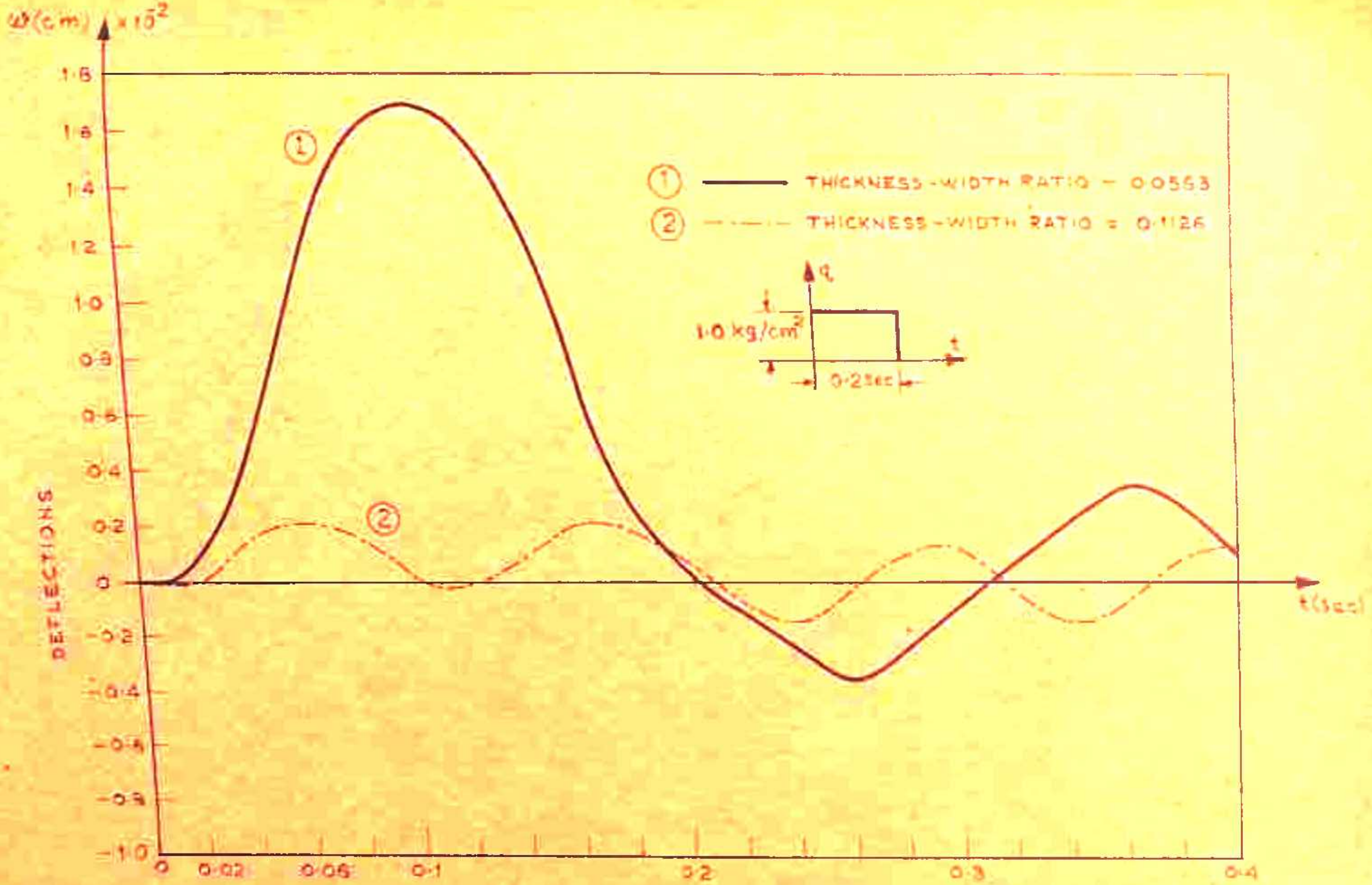

FIG 6-33 EFFECT OF THICKNESS-WIDTH RATIO ON DEFLECTIONS (RECTANGULAR PULSE ON A FIXED, SQUARE, CONCRETE PLATE)

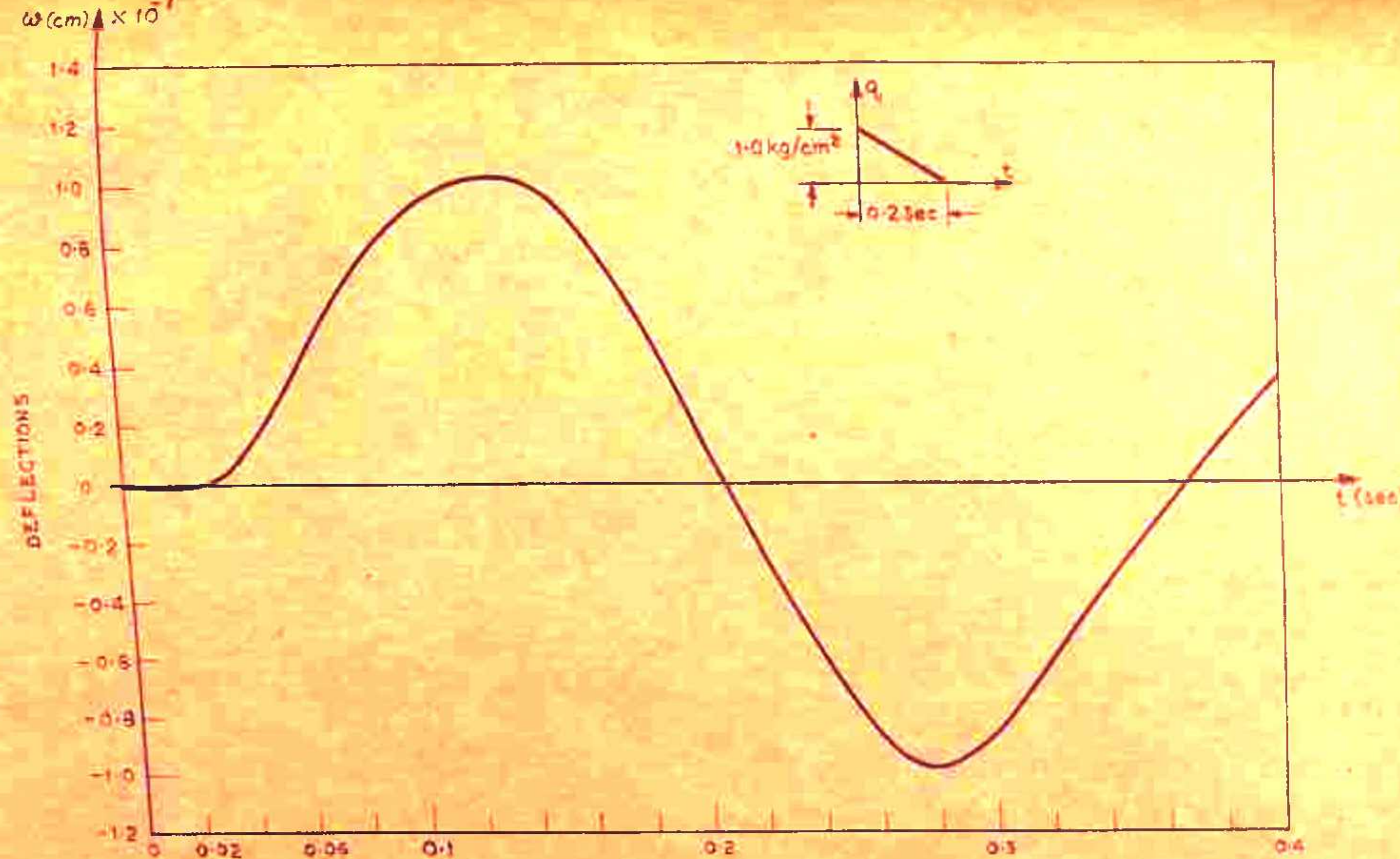

FIG. 5-3-4: FINITE-DIFFERENCE SOLUTION FOR A S.S. RECTANGULAR CONCRETE PLATE SUBJECTED TO A TRIANGULAR PULSE.

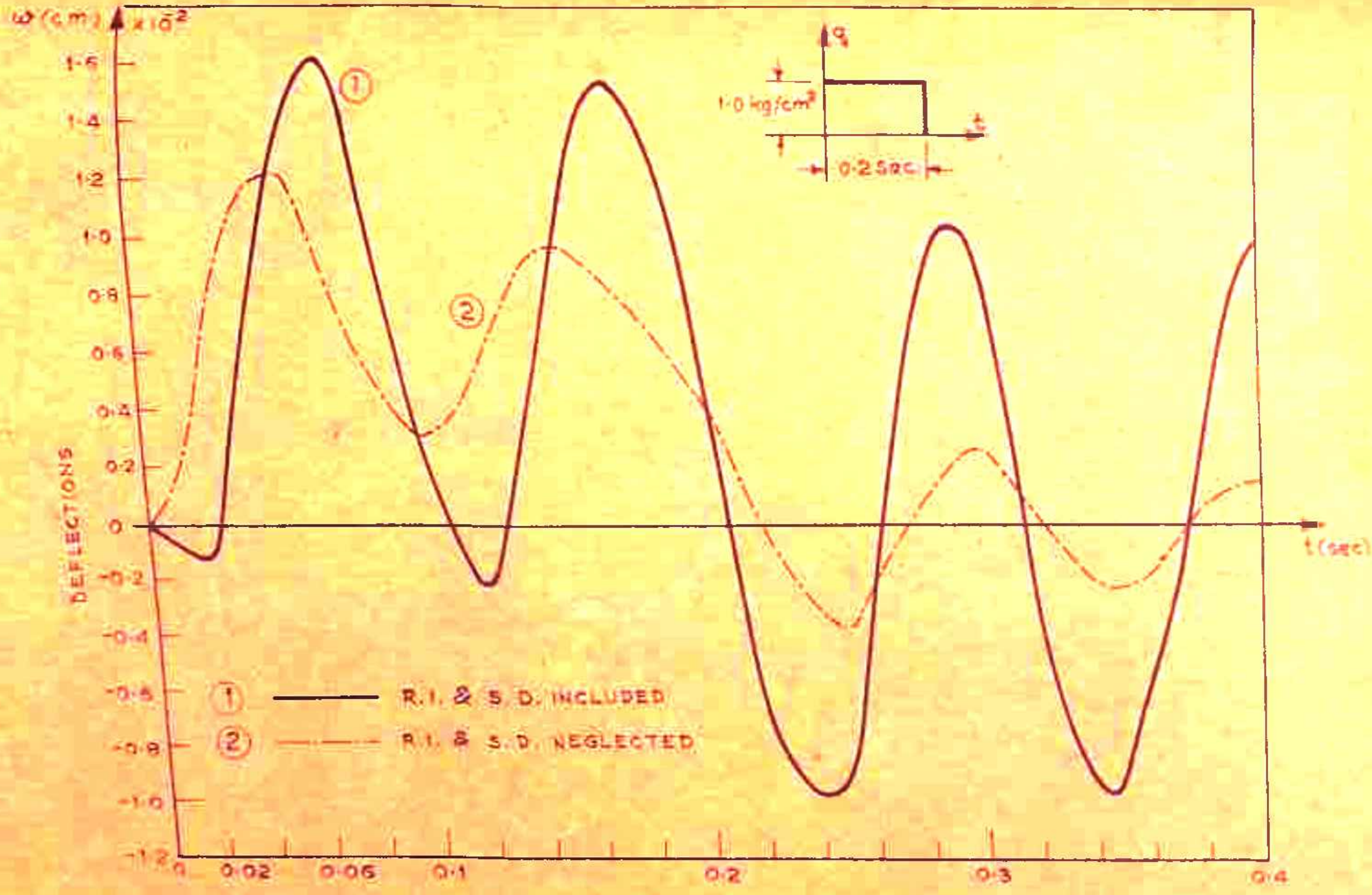

FIG. 6·3·5 TIME-DEFLECTION CURVE OF PLATE AT λ = 30·0 cm (FIXED-CIRCULAR CONCRETE PLATE SUBJECTED TO A RECTANGULAR PULSE)

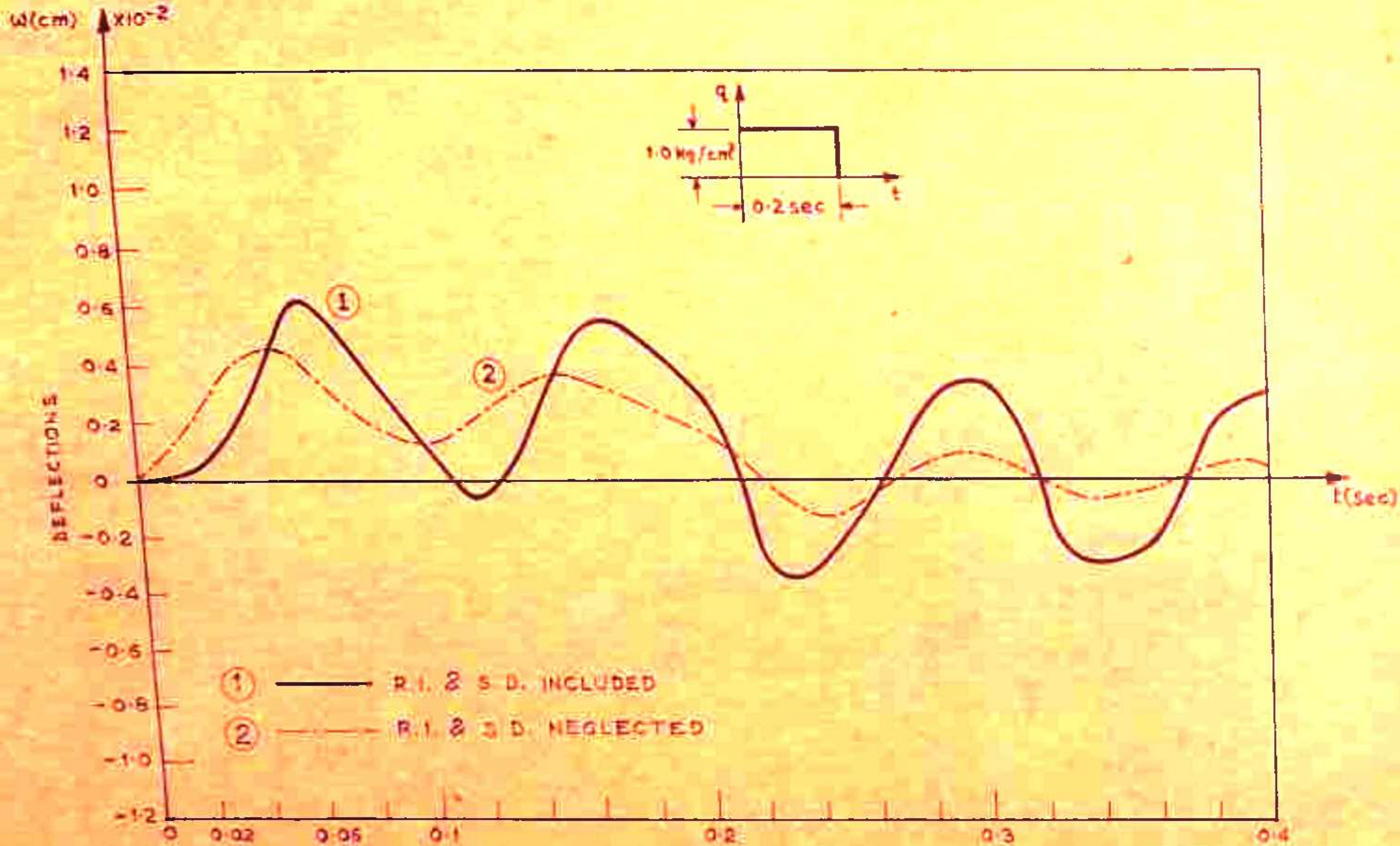

FIG. 5-3.6: TIME-DEFLECTION CURVE OF PLATE AT h = 50 cm. (FIXED CIRCULAR CONCRETE PLATE SUBJECTED TO A RECTANGULAR PULSE)

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