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THE CALCULUS OF VARIATIONS IN THE LARGE

BY

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FOREWORD

For several years the research of the writer has been oriented by a conception of what might be termed macro-analysis. It seems probable to the author that many of the objectively important problems in mathematical physics, geometry, and analysis cannot be solved without radical additions to the methods of what is now strictly regarded as pure analysis. Any problem which is non-linear in character, which involves more than one coordinate system or more than one variable, or whose structure is initially defined in the large, is likely to require considerations of topology and group theory in order to arrive at its meaning and its solution. In the solution of such problems classical analysis will frequently appear as an instrument in the small, integrated over the whole problem with the aid of group theory or topology. Such conceptions are not due to the author. It will be sufficient to say that Henri Poincaré was among the first to have a conscious theory of macro-analysis, and of all mathematicians was doubtless the one who most effectively put such a theory into practice.

The principal contribution of the author has been first to give an analysis in the large of a function f of m variables, and then to extend this analysis to functionals. The functionals chosen have been those of the Calculus of Variations. Although there are indications that further deep extensions to other functionals exist, such extensions are beyond the scope of these Lectures. Whereas the analogies between the theory of linear and quadratic forms and the theory of functionals have been well recognized since the work of Hilbert, the analogies in the large between functions and functionals here presented have not been so recognized, and the nature of the development of such analogies in many aspects has been most difficult.

The first four chapters of these Lectures deal with the theory in the small. They are concerned with the analogue for functionals, of the index of a critical point of the function f. Conjugate points, focal points, characteristic roots, the Poincaré rotation number, and the index of concavity of closed extremals are among the entities which serve to evaluate the index of a critical extremal, and which are unified by the theory of this index.

Chapter IV goes beyond the needs of the theory in the large in developing separation, comparison, and oscillation theorems in n-space. The most general algebraic form of linear, self-adjoint boundary conditions associated with the usual Jacobi differential equations is exposed in a parametric form in which only those coefficients appear which are arbitrary. The theory is sufficiently refined to specialize into a definite improvement upon the oscillation theorems of Bôcher [2] and Ettlinger [1, 2] in the 2-dimensional periodic case. Among other theorems, a necessary and sufficient condition for the existence of infinitely many characteristic roots in our self-adjoint boundary problems is established.

Except for a theorem on the order of vanishing of the determinant of a conjugate family, most of the work of the first four chapters can be readily extended to the Bolza form of the Lagrange problem if the proper assumptions as to "normalcy" are made.

Chapter V presents the general boundary problem in the large. It starts with a macroscopic definition of a Riemannian manifold R. The functional and boundary conditions on R are defined in parametric form, and in the large. A first problem which is solved concerns the invariantive or tensor definition of the indices of the preceding chapters. This aspect of the theory will be of interest to differential geometers. Chapter V treats the general accessory boundary problem in a way which is independent of the local coordinate systems employed. The author believes that this is the first general treatment of this character.

Chapter VI develops the theory of the critical points of a function of m variables in a manner which seems best adapted to an extension to the case of functionals. The analogous treatment for the case of functionals requires the development of the topology of the function space defined by a given boundary problem. For problems for which the end points are always distinct, the function space can be treated as in Chapter VII. The theory of the closed extremal in Chapter VIII requires a new approach to the topology of the corresponding function space. In particular homologies which are not defined by bounding are used here, and subgroups of substitutions of q points play an important rôle.

Chapter IX presents a solution of the Poincaré continuation problem which arose from Poincaré's study of Celestial Mechanics, Poincaré [2]. With Poincaré this problem reduced to the question of the existence and analytic continuation of a closed geodesic on a convex surface as the surface was varied analytically. Poincaré started with the principal ellipses on an ellipsoid. The validity of his reasoning has been questioned. In Chapter IX explicit objections are presented. The present writer enlarges the Poincaré continuation problem to mean the problem of finding those numerical invariants of critical sets of closed extremals, the possession of which is a guarantee of the continued existence and analytic variation of critical sets possessing the given numerical invariants as the basic Riemannian manifold is varied analytically. This theory is applied to show that on an m-ellipsoid with unequal axes the principal ellipses vary analytically into critical sets of geodesics with the same numerical invariants, as the m-ellipsoid is varied analytically through a 1-parameter family of closed manifolds.

The author takes occasion here to acknowledge his principal sources. First of all the author wishes to acknowledge his indebtedness to his colleague, Professor George D. Birkhoff, whose minimax principle, Birkhoff [1], was the original stimulus of the present investigations, and whose transformation theory of dynamics, though logically less closely related to these Lectures, has by virtue of its broad aims and accomplishments proved no less inspiring. The author's knowledge of the classical theory has been acquired largely from the treatises of Bolza and Hadamard, and from the works of Bliss whose papers on the n-dimensional theory have been particularly useful. In topology the author has

been fortunate in having the contemporary work of Veblen, Alexander, and Lefschetz to follow, and to have had their papers always at his disposal.

The bibliography at the end of the Lectures is not intended to be complete, but merely to list recent papers used by the author, or papers which may be regarded as related to the work of the author.

The author acknowledges the generous aid furnished him by the Milton Fund of Harvard University for the preparation of the manuscript. Dr. S. B. Myers and Dr. A. W. Tucker have been kind enough to read parts of the text and to offer valuable suggestions. Dr. Nancy Cole has greatly assisted both in the reading and in the preparation of the manuscript.

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Cambridge, Massachusetts.

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THE FIXED END POINT PROBLEM IN NON-PARAMETRIC FORM

The original plan of these Lectures was to start with a treatment of the problem under general end conditions. However the lack of a complete treatment in book form in English of the classical n-dimensional theory made it seem desirable to depart from this plan to the extent of giving an introductory chapter on the fixed end point problem. This chapter is an exposition of classical results treated for the most part by classical methods. Free use is made of the works of Bliss, Bolza, and Hadamard.

The Euler equations

1. Let it be recalled that a function F of n variables (w) is said to be of *class* C^m, $m \geqq 0$, in the variables (w) on a domain S, if F is continuous on S, together with all of its partial derivatives up to and including those of the mth order. A function $y(x)$ of a single variable x is said to be of class D^m, $m > 0$, on an interval $a \leqq x \leqq b$, if $y(x)$ is continuous on the interval, and if the interval can be divided into a finite set of subintervals on the closure of each of which $y(x)$ is of class C^m. The function $y(x)$ will be said to be of class D^0 on the interval $a \leqq x \leqq b$, if this interval can be divided into a finite set of subintervals on the interior of each of which $y(x)$ is of class C^0 and at the ends of which $y(x)$ possesses finite right and left hand limits.

Let

$$(x, y_1, \cdots, y_n) = (x, y)$$

be the rectangular coordinates of a point (x, y) in a euclidean space of $(n + 1)$ dimensions. Let R be an open region in the space (x, y). We shall consider a function

$$f(x, y_1, \cdots, y_n, p_1, \cdots, p_n) = f(x, y, p)$$

such that f_{y_i}, f_{p_i} and f are of class C^2 for (x, y) on R and for (p) unrestricted.

Let g be a curve in the region R of the form

$$y_i = \bar{y}_i(x) \qquad (i = 1, \cdots, n) \tag{1.1}$$

for x on the interval (a^1, a^2),

$$a^1 \leqq x \leqq a^2 \tag{1.2}$$

where the functions $\bar{y}_i(x)$ are of class D^1 on the interval (1.2). We term g a curve of class D^1.

In deriving the Euler equations we shall *admit* curves which have the following properties. They are of class D^1 on the interval (1.2), and join the end points of g on R. We shall consider the integral

$$J = \int_{a^1}^{a^2} f(x, y_1, \cdots, y_n, y'_1, \cdots, y'_n)\, dx$$

along these admissible curves and prove the following theorem.

THEOREM 1.1. *In order that the curve g afford a minimum to J relative to neighboring admissible curves it is necessary that g satisfy the conditions*

$$f_{p_i}(x, \bar{y}(x), \bar{y}'(x)) \equiv \int_{a^1}^{x} f_{y_i}(x, \bar{y}(x), \bar{y}'(x))\, dx + c_i \tag{1.3}$$

for x on its interval (1.2), *and for suitable constants c_i.*

In proving this theorem one considers a family $y_i = y_i(x, e)$ of admissible curves of the form

$$y_i(x, e) = \bar{y}_i(x) + e\eta_i(x) \qquad (i = 1, \cdots, n) \tag{1.4}$$

where e is a parameter near $e = 0$, $\eta_i(x)$ is of class D^1 on the interval (1.2) and vanishes at a^1 and a^2. For each value of e near $e = 0$ we thus obtain a value $J(e)$ of the integral J. Moreover if g affords a minimum to J relative to neighboring admissible curves it is necessary that

$$J'(0) = \int_{a^1}^{a^2} (\eta'_i f^0_{p_i} + \eta_i f^0_{y_i})\, dx = 0 \qquad (i = 1, \cdots, n) \tag{1.5}$$

where the superscript 0 indicates evaluation along g, that is for

$$(x, y, p) = (x, \bar{y}(x), \bar{y}'(x)).$$

Here and elsewhere we follow a convention of tensor analysis, that a repeated subscript or superscript i indicates a summation with respect to i. The right member of (1.5) is called the *first variation* of J along g. It is determined when g and the "variations" η_i are given.

The terms $\eta_i f^0_{y_i}$ in (1.5) can be integrated by parts, giving the result

$$J'(0) = \int_{a^1}^{a^2} \eta'_i [f^0_{p_i} - \int_{a^1}^{x} f^0_{y_i}\, dx]\, dx = 0, \tag{1.6}$$

the terms outside the integral having vanished since $\eta_i(a^1) = \eta_i(a^2) = 0$. The theorem will follow from (1.6) once we have proved the Du Bois-Reymond Lemma.

LEMMA 1.1. *If $\phi(x)$ is of class D^0 on (a^1, a^2) and*

$$\int_{a^1}^{a^2} \eta'(x)\phi(x)\, dx = 0 \tag{1.7}$$

for all functions $\eta(x)$ of class D^1 which vanish at a^1 and a^2, then $\phi(x)$ is constant on (a^1, a^2).

Let c be a constant such that

$$\int_{a^1}^{a^2} (\phi(x) - c)\, dx = 0.$$

The function

$$\eta(x) = \int_{a^1}^{x} (\phi(x) - c)\, dx$$

is then a function $\eta(x)$ of the type admitted in the lemma. For this function $\eta(x)$, (1.7) takes the form

$$0 = \int_{a^1}^{a^2} (\phi(x) - c)\phi(x)\, dx = \int_{a^1}^{a^2} (\phi(x) - c)^2\, dx,$$

from which it follows that $\phi(x) \equiv c$. The lemma is thereby proved.

Returning to (1.6) we take all the functions $\eta_i(x)$ identically zero except one, say $\eta_k(x)$. According to the lemma we can infer that the coefficient of η_k' in the integrand in (1.6) must be constant. This is true for $k = 1, 2, \cdots, n$. The theorem is thereby proved.

We state the following modification of Lemma 1.1 of use in a later chapter.

LEMMA 1.2. *If $\phi(x)$ and $\phi'(x)$ are of class D^0 on (a^1, a^2) and* (1.7) *holds for all functions $\eta(x)$ of class D^2 which vanish at a^1 and a^2, then $\phi(x)$ is constant on (a^1, a^2).*

On the basis of this lemma we could prove as above that a curve $y_i = \bar{y}_i(x)$ of class D^2 on (a^1, a^2) which affords a minimum to J relative to neighboring curves of class D^2 which join its end points, satisfies (1.3) as before.

We have the following consequences of the theorem.

Each segment of class C^1 of a minimizing curve g must satisfy the Euler equations

$$\frac{d}{dx} f_{p_i} - f_{y_i} \equiv 0 \qquad (i = 1, \cdots, n). \tag{1.8}$$

Again, at each corner $x = c$ on g, the right hand and left hand limits of f_{p_i} on g are equal, that is, on g

$$\Big[f_{p_i} \Big]_{c^-}^{c^+} = 0 \qquad (i = 1, \cdots, n). \tag{1.9}$$

These are the Weierstrass-Erdmann corner conditions.

Suppose g is of class C^1, satisfies (1.3), and that along g the determinant

$$| f_{p_i p_j} | \neq 0. \tag{1.10}$$

Then g is of class C^2 at least (Hilbert). The proof of this statement according to Mason and Bliss [1] is as follows. The n equations

$$f_{p_i}(x, \bar{y}(x), z) = \int_{a^1}^{x} f(x, \bar{y}(x), \bar{y}'(x))\, dx + c_i \tag{1.11}$$

can be regarded as determining n variables z_i as functions of x. They have the initial solution $z_i(x) = \bar{y}'_i(x)$ for each x. Upon taking account of (1.10) and employing the usual implicit function theorems one sees that the solution $z_i(x)$ must be of class C^1, and hence g of class C^2. We shall see later that g is of class C^3 provided (1.10) holds.

A curve of class D^1 satisfying (1.3) is called a discontinuous solution if it actually possesses a corner. The theory of discontinuous solutions received a great impetus from the dissertation of Carathéodory [1]. A bibliography for this field has been given by L. M. Graves [1]. Graves has also made many important contributions. Beyond using the Weierstrass-Erdmann corner conditions we shall not be concerned with discontinuous solutions.

A curve of class C^2 satisfying the Euler equations (1.8) will be called an *extremal*.

The existence of extremals

2. Suppose we have an extremal g of the form $y_i = \bar{y}_i(x)$ with x on the interval (a^1, a^2). To determine the extremals neighboring g it is useful to set

$$v_i = f_{p_i}(x, y, p) \qquad (i = 1, \cdots, n) \tag{2.1}$$

and in particular

$$\bar{v}_i(x) = f_{p_i}(x, \bar{y}(x), \bar{y}'(x)).$$

We term sets

$$(x, \bar{y}(x), \bar{y}'(x), \bar{v}(x))$$

sets (x, y, p, v) on g. We similarly define sets (x, y, p) or sets (x, y, v) on g.

We assume that the condition (1.10) holds along g. It follows that near sets (x, y, p, v) on g the relation (2.1) can be put in the form

$$p_i = p_i(x, y, v), \tag{2.2}$$

where the functions $p_i(x, y, v)$ are of class C^2 neighboring sets (x, y, v) on g. The Euler equations are then equivalent to the equations

$$\begin{aligned} \frac{dv_i}{dx} &= f_{y_i}(x, y, p(x, y, v)), \\ \frac{dy_i}{dx} &= p_i(x, y, v) \qquad (i = 1, \cdots, n), \end{aligned} \tag{2.3}$$

at least as far as extremals are concerned on which the sets (x, y, v) differ sufficiently little from similar sets on g.

According to the theory of differential equations, equations (2.3) have solutions of the form

$$\begin{aligned} y_i &= h_i(x, x^0, y^0, v^0), \\ v_i &= k_i(x, x^0, y^0, v^0) \qquad (i = 1, \cdots, n), \end{aligned}$$

which take on the values (y^0, v^0) when $x = x^0$, and for which the functions on the right are of class C^2 in their arguments for x on (a^1, a^2) or a slightly larger interval, and for (x^0, y^0, v^0) sufficiently near sets (x, y, v) on g. We now set

$$y_i(x, x^0, a, b) = h_i(x, x^0, a, f_p(x^0, a, b)). \tag{2.4}$$

We have in $y_i(x, x^0, a, b)$ the general solution of the Euler equations neighboring the solution g. The functions $y_i(x, x^0, a, b)$ are of class C^2 in their arguments for x on (a^1, a^2) or a slightly larger interval, and for (x^0, a, b) sufficiently near sets (x, y, y') on g. Moreover we have

$$\begin{aligned} a_i &\equiv y_i(x^0, x^0, a, b), \\ b_i &\equiv y_{ix}(x^0, x^0, a, b), \end{aligned} \tag{2.5}$$

for (x^0, a, b) near sets (x, y, y') on g. Reference to the second of equations (2.3) discloses the additional fact that the functions

$$h_{ix}(x, x^0, y^0, v^0)$$

and hence

$$y_{ix}(x, x^0, a, b)$$

are of class C^2 on the domain of their arguments.

The necessary conditions of Weierstrass and Legendre

3. The Weierstrass E-function is defined by the equation

$$E(x, y, p, q) = f(x, y, q) - f(x, y, p) - (q_i - p_i) f_{p_i}(x, y, p) \qquad (i = 1, \cdots, n).$$

We shall prove the following theorem (Weierstrass).

THEOREM 3.1. *If an arc g of class C^1 affords a minimum to J relative to all neighboring curves of class D^1 joining its end points, then*

$$E(x, y, y', q) \geqq 0, \tag{3.1}$$

for (x, y, y') on g and for any set (q).

Let g be represented as previously by $y_i = \bar{y}_i(x)$. Let (x^1, y^1) be any point of g. We treat the case where $x^1 > a^1$. The case $x^1 = a^1$ requires at most obvious changes.

Let $y_i = y_i(x)$ be a short arc γ of class C^1, which passes through (x^1, y^1) when $x = x^1$, and for which $y_i'(x^1) = q_i$, where q_i is arbitrarily prescribed. The curves defined for α constant by the family

$$y_i = y_i(x, \alpha) = \frac{(x - a^1)}{(\alpha - a^1)} [y_i(\alpha) - \bar{y}_i(\alpha)] + \bar{y}_i(x) \qquad (i = 1, \cdots, n), \tag{3.2}$$

reduce to g when $\alpha = x^1$, and in general join the initial point of g to the point

$$P_\alpha = (x, y) = (\alpha, y(\alpha))$$

on γ. We are here supposing that $\alpha \leqq x^1$, that $a^1 \leqq x \leqq \alpha$, and that α is taken near x^1.

We now evaluate J along the curve (3.2) leading from the initial point of g to P_α, and then along the curve $y_i = y_i(x)$ from P_α to (x^1, y^1). We have

$$J(\alpha) = \int_{a^1}^{\alpha} f(x, y(x, \alpha), y_x(x, \alpha))\, dx + \int_{\alpha}^{x^1} f(x, y(x), y'(x))\, dx.$$

If g is a minimizing arc, we must have $J'(x^1) \leqq 0$. Upon setting $p_i = \bar{y}_i'(x^1)$ we find that

$$J'(x^1) = f(x^1, y^1, p) - f(x^1, y^1, q) + \int_{a^1}^{x^1} \Big[y_{i\alpha} f_{y_i} + y_{i\alpha x} f_{p_i} \Big]^{\alpha = x^1} dx.$$

If we integrate the terms involving $y_{i\alpha x}$ by parts, and make use of the fact that g must satisfy the Euler equations, we find that

$$J'(x^1) = f(x^1, y^1, p) - f(x^1, y^1, q) + \Big[y_{i\alpha}(x, x^1)\, f_{p_i}(x, y(x, x^1), y_x(x, x^1)) \Big]_{x=a^1}^{x=x^1}.$$

From the identities

$$y_i(a^1, \alpha) \equiv \bar{y}_i(a^1), \qquad y_i(\alpha, \alpha) \equiv y_i(\alpha),$$

it follows, upon differentiating with respect to α and putting $\alpha = x^1$, that

$$y_{i\alpha}(a^1, x^1) = 0, \qquad p_i + y_{i\alpha}(x^1, x^1) = q_i.$$

Using these results we find that

$$J'(x^1) = f(x^1, y^1, p) - f(x^1, y^1, q) + (q_i - p_i)\, f_{p_i}(x^1, y^1, p).$$

The theorem follows from the condition $J'(x^1) \leqq 0$.

If the hypothesis of the theorem is modified by restricting the admissible curves to those on which (x, y, y') lies sufficiently near $(x, \bar{y}(x), \bar{y}'(x))$ on g, the minimum afforded by g is called a *weak* minimum. The minimum afforded by g in the theorem is called a *strong* minimum. For a weak minimum it is necessary that the condition (3.1) hold in its weak form; that is, for the sets (x, y, y', q) of which (x, y, y') is on g and (q) sufficiently near the set (y') on g. As a consequence of this weak condition one can derive the following condition, due in the plane to Legendre.

For a weak minimum it is necessary that

$$f_{p_i p_j}(x, \bar{y}(x), \bar{y}'(x))\, z_i z_j \geqq 0 \qquad (i, j = 1, \cdots, n), \tag{3.3}$$

for each point on g and every set (z).

One forms the function

$$\phi(e) = E(x, \bar{y}(x), \bar{y}'(x), \bar{y}'(x) + ez),$$

where the fourth set of arguments is the set

$$\bar{y}_i'(x) + ez_i \qquad (i = 1, \cdots, n).$$

One readily finds that $\phi''(0)$ equals the left member of (3.3). From the Weierstrass condition in the weak form it follows that $\phi(e)$ has a relative minimum when $e = 0$, and hence $\phi''(0) \geqq 0$.

Condition (3.3) is thereby proved.

The Jacobi condition

4. We suppose again that g is a minimizing arc of class C^1. We evaluate J along the family of curves (1.4) obtaining thereby a function $J(e)$. One readily finds that the so-called *second variation* takes the form

$$J''(0) = \int_{a^1}^{a^2} (f^0_{p_ip_j}\eta'_i\eta'_j + 2f^0_{p_iy_j}\eta'_i\eta_i + f^0_{y_iy_j}\eta_i\eta_j)\,dx$$

where the superscript zero indicates evaluation along g. One sets

$$f^0_{p_ip_j}\eta'_i\eta'_j + 2f^0_{p_iy_j}\eta'_i\eta_j + f^0_{y_iy_j}\eta_i\eta_j = 2\,\Omega(\eta, \eta')$$

and

$$I(\eta) = \int_{a^1}^{a^2} 2\,\Omega(\eta, \eta')\,dx.$$

For a minimizing arc we have $J'(0) = 0$ so that it is necessary that $J''(0) \geqq 0$. Zero is then necessarily a minimum value of the second variation. To discover the full consequences of this fact it is natural to consider the problem of minimizing the second variation among admissible functions (η) (see Bliss [4]) and in particular to consider the corresponding Euler equations

$$\frac{d}{dx}\Omega_{\eta'_i} - \Omega_{\eta_i} = 0 \qquad (i = 1, \cdots, n). \tag{4.1}$$

Equations (4.1) are termed the Jacobi equations. If formally expanded they are linear and homogeneous in the variables $\eta_i, \eta'_i, \eta''_i$. The determinant of the coefficients of η''_i is $|f_{p_ip_j}|$ evaluated along g. A point $x = c$ at which this determinant does not vanish will be termed a *non-singular* point of the Jacobi equations.

In the neighborhood of a non-singular point $x = \alpha$ one can infer that a solution of the Jacobi equations, if known to be of class C^1, is necessarily of class C^3, that all solutions are linearly dependent on $2n$ such solutions, and that a solution (η) which with (η') vanishes at α, vanishes identically neighboring α.

We shall prove the following lemma.

LEMMA. *If there exists a solution (η) of the Jacobi equations which is of class C^1 on the interval $a^1 \leqq x \leqq \alpha$, $\alpha > a^1$, and which vanishes at the ends of the interval, then*

$$\int_{a^1}^{\alpha} \Omega(\eta, \eta')\,dx = 0. \tag{4.2}$$

Because of the homogeneity of Ω in η_i and η'_i we can write

$$\int_{a^1}^{\alpha} 2\,\Omega\,dx = \int_{a^1}^{\alpha} (\eta_i\,\Omega_{\eta_i} + \eta'_i\,\Omega_{\eta'_i})\,dx.$$

If the terms involving η'_i be integrated by parts, we find that

$$\int_{a^1}^{\alpha} 2\,\Omega\,dx = \Big[\eta_i\,\Omega_{\eta'_i}\Big]_{a^1}^{\alpha} + \int_{a^1}^{\alpha} \eta_i\Big[\Omega_{\eta_i} - \frac{d}{dx}\Omega_{\eta'_i}\Big]\,dx.$$

The lemma follows at once.

Let (η) be a solution of the Jacobi equations of class C^2 on a closed interval of the x axis bounded by distinct points $x = c$ and $x = \alpha$, and suppose that (η) vanishes at the points $x = c$ and $x = \alpha$. If (η) does not then vanish identically between c and α neighboring $x = \alpha$, the point $x = \alpha$ on the x axis (or g) will be termed *conjugate* to the point $x = c$ on the x axis (or g).

The following theorem gives the Jacobi necessary condition.

THEOREM 4.1. *If g affords a weak minimum to J, no conjugate point of $x = a^1$ on the interval $a^1 < x < a^2$ can coincide with a point at which the Jacobi equations are non-singular.*

A proof of this theorem has been given by Bliss [4] essentially as follows.

Suppose the theorem false, and that there exists a solution (η) of the Jacobi equations which is of class C^2 on the interval $a^1 \leqq x \leqq \alpha$, $a^1 < \alpha < a^2$, which vanishes at $x = a^1$ and $x = \alpha$ without vanishing identically for $x < \alpha$ neighboring $x = \alpha$, where $x = \alpha$ is a point at which the Jacobi equations are non-singular. Let $\eta_i^*(x)$ be a set of functions equal to $\eta_i(x)$ on (a^1, α), and zero on the remainder of the interval (a^1, a^2). By virtue of the preceding lemma we see that $I(\eta^*) = 0$. But if g is a minimizing arc, as we are supposing, $I(\eta) \geqq 0$ for all admissible (η). Hence (η^*) affords a minimum for $I(\eta)$.

It follows that (η^*) must satisfy the Weierstrass-Erdmann corner conditions at $x = \alpha$; that is on (η^*) we must have

$$\Big[\Omega_{\eta'_i}\Big]_{\alpha^-}^{\alpha^+} = 0 \qquad (i = 1, \cdots, n). \tag{4.3}$$

If we make use of the fact that (η^*) vanishes at α, conditions (4.3) take the form

$$f_{p_i p_j}\eta'_j(\alpha) = 0 \qquad (i, j = 1, \cdots, n) \tag{4.4}$$

where the partial derivatives of f are evaluated at $x = \alpha$ on g. If the point α is non-singular, we see from (4.4) that (η') vanishes at α, and hence (η) vanishes identically near α. From this contradiction we infer the truth of the theorem.

Note. Strictly speaking the function Ω does not satisfy the requirements imposed on f in §1, since it is merely of class C^0 in x. But even with Ω of class C^0 in x, one sees that the proof of the Weierstrass-Erdmann corner conditions remains valid.

Conjugate points

5. It is necessary now to prepare for the sufficient conditions. To that end we shall obtain three representations of the conjugate points of a point $x = c$.

For the remainder of this chapter we shall suppose that we have given an extremal g along which

$$|f_{p_ip_j}| \neq 0 \qquad (i, j = 1, \cdots, n).$$

Let $\| v_{ij}(x, c) \|$ be an n-square matrix of functions of which the columns are solutions of the Jacobi equations for c constant, and which satisfy the initial conditions

$$(5.1) \qquad v_{ij}(c, c) = 0, \quad v_{ijx}(c, c) = \delta_i^j \qquad (i, j = 1, \cdots, n),$$

where δ_i^j is the Kronecker delta. Such solutions exist according to the general theory of differential equations. These n solutions are independent by virtue of (5.1). Set

$$(5.2) \qquad D(x, c) = | v_{ij}(x, c) |.$$

The determinant $D(x, c)$ vanishes at $x = c$ in a way which we shall now determine.

We can set

$$v_{ij}(x, c) = (x - c)a_{ij}(x, c) \qquad (i, j = 1, \cdots, n)$$

where

$$a_{ij}(x, c) = \int_0^1 v_{ijx}[c + t(x - c), c]\, dt.$$

We see that $a_{ij}(x, c)$ is continuous in x and c, and that

$$a_{ij}(c, c) = v_{ijx}(c, c) = \delta_i^j.$$

We thereby obtain a representation of $D(x, c)$ of the form

$$(5.3) \qquad D(x, c) = (x - c)^n A(x, c), \qquad A(c, c) = 1,$$

where $A(x, c)$ is continuous in x and c, for x and c on an interval

$$(5.4) \qquad a^1 - e < x < a^2 + e, \qquad e > 0,$$

slightly larger than (a^1, a^2).

We note the following:

The conjugate points of a point $x = c$ are the points $x \neq c$ at which $D(x, c) = 0$.

Let (v) be a proper linear combination of the columns of $D(x, c)$, that is, a linear combination in which the coefficients are not all null. If $D(\alpha, c) = 0$ with $\alpha \neq c$, there will exist a proper linear combination (v) of the columns of $D(x, c)$ which vanishes at $x = \alpha$. Moreover (v) is not identically zero near $x = c$, and hence not identically zero near $x = \alpha$. Thus $x = \alpha$ is a conjugate point of $x = c$.

Conversely if $x = \alpha$ is a conjugate point of $x = c$ there must be a solution of the Jacobi equations which is not identically zero, and which vanishes at $x = \alpha$ and $x = c$. But all solutions which vanish at $x = c$ are linear combinations of the columns of $D(x, c)$. Hence $D(\alpha, c) = 0$, and the statement in italics is proved.

Since $D(x, a^1)$ has an isolated zero at $x = a^1$ there will either be no conjugate point of $x = a^1$ on (a^1, a^2) or else there will exist a *first* such conjugate point $x = \alpha > a^1$.

The following fact will be used later. If α_0 is the first conjugate point of c_0, and α_0 and c_0 are both on the interval (5.4), the first conjugate point, if it exists, of a point c sufficiently near c_0 *will not precede* $\alpha_0 - e$, where e is an arbitrarily small positive constant. Upon referring to (5.3) we see for the case at hand that

$$A(x, c_0) \neq 0 \qquad (c_0 \leqq x < \alpha_0). \tag{5.5}$$

Our statement follows at once from the continuity of $A(x, c)$.

We now recall a principle discovered by Jacobi.

Let $h_i(x, \mu)$ be a one-parameter family of extremals which contains g for $\mu = \mu_0$, and for which the functions $h_i(x, \mu)$ are of class C^2 for μ near μ_0, and x on (5.4). According to Jacobi the functions

$$\eta_i(x) = h_{i\mu}(x, \mu_0) \qquad (i = 1, \cdots, n) \tag{5.6}$$

afford a solution of the Jacobi equations determined by g.

To prove this fact we note that

$$\frac{\partial}{\partial x} f_{p_i}[x, h(x, \mu), h_x(x, \mu)] - f_{y_i}[x, h(x, \mu), h_x(x, \mu)] \equiv 0.$$

If we differentiate the left members of these identities with respect to μ, interchange the order of differentiation with respect to x and μ, and set $\mu = \mu_0$, the resulting equations take the form of the Jacobi equations. One sees this the more readily if one first verifies the fact that

$$\frac{\partial}{\partial \mu} f_{p_i} = \Omega_{\eta_i'}(\eta, \eta'), \qquad \frac{\partial}{\partial \mu} f_{y_i} = \Omega_{\eta_i}(\eta, \eta'),$$

where (η) is given by (5.6) and the left members of these equations are evaluated for $\mu = \mu_0$.

We turn next to the family of extremals $y_i = y_i(x, x^0, a, b)$ of §2. We suppose that the extremal g is determined by the parameters $(x^0, a, b) = (c, \alpha, \beta)$. By the above principle of Jacobi the functions

$$\eta_i(x) = y_{ia_j}(x, c, \alpha, \beta) \qquad (i, j = 1, \cdots, n), \tag{5.7$'$}$$

as well as the functions

$$\eta_i(x) = y_{ib_j}(x, c, \alpha, \beta), \tag{5.7$''$}$$

afford solutions of the Jacobi equations for a fixed j. We here have $2n$ solutions. That these solutions are independent is readily proved. For upon suitably differentiating the members of (2.5) we find that

$$
\begin{aligned}
&y_{i\alpha_j}(c, c, \alpha, \beta) = \delta^j_i, \qquad y_{i\alpha_j x}(c, c, \alpha, \beta) = 0,\\
&y_{ib_j}(c, c, \alpha, \beta) = 0, \qquad y_{ib_j x}(c, c, \alpha, \beta) = \delta^j_i.
\end{aligned} \tag{5.8}
$$

We can now obtain another representation of the conjugate points of $x = c$ on (a^1, a^2). To that end consider the family of extremals

$$
y_i = \phi_i(x, b) = y_i(x, c, \alpha, b) \qquad (i = 1, \cdots, n), \tag{5.9}
$$

passing through the point on g at which $x = c$. The columns of the jacobian

$$
\frac{D(\phi_1, \cdots, \phi_n)}{D(b_1, \cdots, b_n)} = E(x), \qquad b_i = \bar{y}'_i(c), \tag{5.10}
$$

are solutions of the Jacobi equations. According to (5.8) they satisfy the same initial conditions at $x = c$ as do the corresponding columns of $\| v_{ij}(x, c) \|$. They are accordingly identical with these columns, so that $D(x, c)$ becomes identical with the jacobian (5.10). We therefore have the following lemma.

The conjugate points of $x = c$ on the x axis are the zeros $x \neq c$ of the jacobian $E(x)$.

A third representation of conjugate points is in terms of the so-called Mayer determinant. With its aid we shall prove a lemma used in later chapters.

Together with the matrix $\| v_{ij}(x, c) \|$ previously considered, we introduce here an n-square matrix $\| u_{ij}(x, c) \|$ whose columns represent solutions of the Jacobi equations which satisfy the conditions

$$
u_{ij}(c, c) = \delta^j_i, \qquad u_{ijx}(c, c) = 0 \qquad (i, j = 1, \cdots, n).
$$

We shall also consider a matrix $\| \eta_{ip}(x) \|$ of n rows and $2n$ columns, whose columns represent a set of $2n$ independent solutions of the Jacobi equations. The $2n$-square determinant

$$
\Delta(x, c) = \begin{vmatrix} \eta_{ip}(c) \\ \eta_{ip}(x) \end{vmatrix} \qquad (i = 1, \cdots, n;\ p = 1, \cdots, 2n) \tag{5.11}
$$

is called the *Mayer determinant.* We shall determine its relation to the determinant $D(x, c)$ previously considered.

We first verify the matrix identity

$$
\begin{Vmatrix} \eta_{ip}(c) \\ \eta_{ip}(x) \end{Vmatrix} \equiv \begin{Vmatrix} u_{ij}(c, c) & v_{ij}(c, c) \\ u_{ij}(x, c) & v_{ij}(x, c) \end{Vmatrix} \begin{Vmatrix} \eta_{jp}(c) \\ \eta'_{jp}(c) \end{Vmatrix}. \tag{5.12}
$$

This identity clearly holds if

$$
\eta_{ip}(x) \equiv u_{ij}(x, c)\eta_{jp}(c) + v_{ij}(x, c)\eta'_{jp}(c) \qquad (i, j = 1, \cdots, n;\ p = 1, \cdots, 2n). \tag{5.13}
$$

But (5.13) holds for $x = c$, and the equations obtained by differentiating (5.13) with respect to x hold for $x = c$. It follows that (5.13) holds identically. The identity (5.12) then follows.

We next observe that

$$W(c) = \begin{vmatrix} \eta_{ip}(c) \\ \eta'_{ip}(c) \end{vmatrix} \neq 0,$$

since the $2n$ columns of $\| \eta_{ip}(x) \|$ are independent. From (5.12) we thus obtain the important relation

$$\Delta(x, c) = D(x, c)W(c), \qquad W(c) \neq 0. \tag{5.14}$$

The zeros of $D(x, c)$ are thus the zeros of $\Delta(x, c) = 0$, so that we have a third representation of the points conjugate to $x = c$.

We shall now prove the following theorem.

THEOREM 5.1. *If the point $(\bar{x}^1, \bar{y}^1)$ on g is not conjugate to the point $(\bar{x}^2, \bar{y}^2)$ on g, then any two points (x^1, y^1) and (x^2, y^2) sufficiently near $(\bar{x}^1, \bar{y}^1)$ and $(\bar{x}^2, \bar{y}^2)$ respectively, can be joined by a unique extremal which may be represented in the form*

$$y_i = \phi_i(x, x^1, y^1, x^2, y^2) \qquad (i = 1, \cdots, n),$$

where the functions on the right are of class C^2 in their arguments, for x on an interval slightly larger than the interval $(\bar{x}^1, \bar{x}^2)$.

The family of extremals neighboring g has been represented in §2 in the form

$$y_i = y_i(x, x^0, a, b) \qquad (i = 1, \cdots, n).$$

Suppose that the set $(x^0, a, b) = (c, \alpha, \beta)$ determines g. To satisfy the theorem we seek to solve the equations

$$\begin{aligned} y_i^1 &= y_i(x^1, c, a, b), \\ y_i^2 &= y_i(x^2, c, a, b), \end{aligned} \tag{5.15}$$

for (a, b) as functions of (x^1, y^1, x^2, y^2). We have the initial solution

$$[x^1, y^1, x^2, y^2] = [\bar{x}^1, \bar{y}^1, \bar{x}^2, \bar{y}^2], \qquad [a, b] = [\alpha, \beta].$$

Moreover the jacobian of the right members of (5.15) with respect to the parameters (a, b), evaluated at the initial solution, is readily seen to be the Mayer determinant,

$$\Delta(\bar{x}^2, \bar{x}^1) \neq 0,$$

set up with the aid of the $2n$ independent solutions (5.7). It is not zero since $\bar{x}^2$ is not conjugate to $\bar{x}^1$.

We can accordingly solve equations (5.15) for (a, b) as functions

$$a_i(x^1, y^1, x^2, y^2), \qquad b_i(x^1, y^1, x^2, y^2)$$

of the coordinates of the given end points. The functions

$$\phi_i(x, x^1, y^1, x^2, y^2) = y_i[x, c, a(x^1, y^1, x^2, y^2), b(x^1, y^1, x^2, y^2)]$$

will satisfy the requirements of the lemma.

The Hilbert integral

6. Let there be given an n-parameter family of extremals of the form

$$y_i = y_i(x, \beta_1, \cdots, \beta_n) \qquad (i = 1, \cdots, n), \tag{6.1}$$

for which the functions $y_i(x, \beta)$ are of class C^2 in the variables (x, β) on some open region R in the (x, β) space, and for which the jacobian

$$\frac{D(y_1, \cdots, y_n)}{D(\beta_1, \cdots, \beta_n)} \neq 0$$

on R. If there is one and only one extremal of this family through each point (x, y) of an open region S of the (x, y) space, the family of extremals will be termed a *field* covering S. We suppose we have a field covering S.

The parameters (β) corresponding to each point (x, y) of S will be functions $\beta_i(x, y)$ of (x, y), of class C^2 on S. For (x, y) on S we set

$$p_i(x, y) = y_{ix}[x, \beta(x, y)].$$

The functions $p_i(x, y)$ are called the "slope functions" of the field. They define the direction of the extremal through (x, y).

The Hilbert integral is a line integral of the form

$$I^* = \int A(x, y)dx + B_i(x, y)dy_i = \int (f - p_i f_{p_i})dx + f_{p_i}dy_i$$

where p_i is to be set equal to $p_i(x, y)$. The expression used in the Hilbert integral arises naturally enough, as we shall see in Ch. II, in the condition,

$$(f - p_i f_{p_i})dx + f_{p_i}dy_i = 0 \qquad (i = 1, \cdots, n),$$

that the direction whose slopes are $p_i(x, y)$ cut the direction whose slopes are dy_i/dx transversally at (x, y). Transversality is the natural generalization of orthogonality.

As might be expected from this geometric setting, fields for which the Hilbert integral is independent of the path joining two points in S have a peculiar importance. They are called *Mayer fields*.

Let R' be the part of the space (x, β) that corresponds to S by virtue of (6.1). The integral I^* can equally well be represented by an integral on R', with dx and $d\beta_i$ as the independent differentials. Upon noting that

$$dy_i = y_{ix}dx + y_{i\beta_h}d\beta_h \qquad (i, h = 1, \cdots, n),$$

one obtains I^* in the form

$$I^* = \int C(x, \beta)dx + D_i(x, \beta)d\beta_i = \int f dx + f_{p_i}y_{i\beta_h}d\beta_h, \tag{6.2}$$

where y_i and p_i in f and f_{p_i}, are to be replaced by $y_i(x, \beta)$ and $y_{ix}(x, \beta)$.

If I^* is independent of the path on S, it will be independent of the path on R', and conversely. If R' is simply connected, the conditions that I^* be independent of the path joining two points take the form

$$\frac{\partial}{\partial \beta_h} C - \frac{\partial}{\partial x} D_h = 0,$$

$$\frac{\partial}{\partial \beta_h} D_k - \frac{\partial}{\partial \beta_k} D_h = 0 \qquad (h, k = 1, \cdots, n).$$

Upon setting $f_{p_i} = v_i(x, \beta)$ these conditions take the form

$$\frac{\partial}{\partial \beta_h}(f) - \frac{\partial}{\partial x}\left(v_i \frac{\partial y_i}{\partial \beta_h}\right) = 0, \tag{6.3}$$

$$\frac{\partial}{\partial \beta_h}\left(v_i \frac{\partial y_i}{\partial \beta_k}\right) - \frac{\partial}{\partial \beta_k}\left(v_i \frac{\partial y_i}{\partial \beta_h}\right) = 0 \qquad (i, h, k = 1, \cdots, n). \tag{6.3$'$}$$

Conditions (6.3) become

$$f_{y_i} \frac{\partial y_i}{\partial \beta_h} + v_i \frac{\partial^2 y_i}{\partial \beta_h \partial x} = v_i \frac{\partial^2 y_i}{\partial x \partial \beta_h} + \frac{\partial v_i}{\partial x} \frac{\partial y_i}{\partial \beta_h}.$$

Upon making use of the fact that

$$f_{y_i} \equiv \frac{\partial}{\partial x} f_{p_i} \equiv \frac{\partial v_i}{\partial x},$$

conditions (6.3) reduce to identities.

Conditions (6.3)$'$ are absent if $n = 1$. If $n > 1$, examples would show that they are not in general fulfilled.

Although the left members of (6.3)$'$ are not in general zero, we can prove that

$$\frac{\partial}{\partial \beta_h}\left(v_i \frac{\partial y_i}{\partial \beta_k}\right) - \frac{\partial}{\partial \beta_k}\left(v_i \frac{\partial y_i}{\partial \beta_h}\right) \equiv C, \tag{6.4}$$

where C is constant for each extremal, but may depend on (β). To establish (6.4) we evaluate the integral J along members of the field neighboring a particular member of the field, between $x = x_0$ and a variable x. We obtain thereby a function $J(x, \beta)$. Upon differentiating J under the integral sign, and integrating by parts in the usual way, we find that

$$\frac{\partial J}{\partial \beta_h} \equiv \left[v_i \frac{\partial y_i}{\partial \beta_h}\right]_{x_0}^{x}, \qquad \frac{\partial J}{\partial \beta_k} \equiv \left[v_i \frac{\partial y_i}{\partial \beta_k}\right]_{x_0}^{x}.$$

Upon differentiating the right hand member of the first of these identities with respect to β_k and the second with respect to β_h and equating the results, we find that

$$\left[\frac{\partial v_i}{\partial \beta_k} \frac{\partial y_i}{\partial \beta_h} + v_i \frac{\partial^2 y_i}{\partial \beta_k \partial \beta_h}\right]_{x_0}^{x} \equiv \left[\frac{\partial v_i}{\partial \beta_h} \frac{\partial y_i}{\partial \beta_k} + v_i \frac{\partial^2 y_i}{\partial \beta_h \partial \beta_k}\right]_{x_0}^{x}.$$

This reduces to

$$\left[\frac{\partial v_i}{\partial \beta_k}\frac{\partial y_i}{\partial \beta_h} - \frac{\partial v_i}{\partial \beta_h}\frac{\partial y_i}{\partial \beta_k}\right]^{x} \equiv \left[\frac{\partial v_i}{\partial \beta_k}\frac{\partial y_i}{\partial \beta_h} - \frac{\partial v_i}{\partial \beta_h}\frac{\partial y_i}{\partial \beta_k}\right]^{x_0},$$

and (6.4) is thereby proved.

Incidentally the proof of (6.4) does not depend at all upon the condition that there be but one extremal through each point of S.

We can now readily prove the following.

The family of extremals passing through the point $x = c$ on g, if represented in the form (5.9), *forms a Mayer field covering a neighborhood of any segment of g along which the jacobian of the family* (5.9) *does not vanish.*

All the conditions for a Mayer field are clearly satisfied except possibly the condition that the Hilbert integral be independent of the path. But turning to (6.4) we find that $C = 0$ for each extremal, as a consequence of the fact that $\phi(c, b) \equiv$ constant. Thus the integrability conditions (6.3)′ are satisfied, and we have a Mayer field.

Sufficient conditions

7. The following theorem is due to Weierstrass, at least if $n = 1$.

THEOREM 7.1. *Suppose g is an extremal of a Mayer field which covers a region S including g in its interior. If*

$$E(x, y, p(x, y), q) > 0$$

for (x, y) on S and any set $q_i \neq p_i(x, y)$, where $p_i(x, y)$ is the ith slope function of the field, then g affords a proper, strong minimum to J relative to all curves of class D^1 which join its end points on S.

Let γ be an admissible curve $y_i = y_i(x)$ joining the end points of g in S.

Corresponding to the given field we can set up the Hilbert integral I^* of §6. Reference to (6.2) shows that $J_g = I^*_g$. But since I^* is independent of the path joining its end points, $I^*_\gamma = I^*_\gamma$. Hence we can use the Hilbert integral to represent J_g as follows:

$$J_g = \int_\gamma (f - p_i f_{p_i})\, dx + f_{p_i} dy_i, \tag{7.1}$$

where $p_i = p_i(x, y)$. Using our representation $y_i(x)$ of γ, we have

$$J_g = \int_{a^1}^{a^2} [f - p_i f_{p_i} + f_{p_i} y'_i]\, dx \qquad (i = 1, \cdots, n),$$

where we understand that

$$y_i = y_i(x), \qquad y'_i = y'_i(x), \qquad p_i = p_i(x, y(x)).$$

We are thus lead to the Weierstrass formula

$$J_\gamma - J_g = \int_{a^1}^{a^2} E[x, y(x), p(x, y(x)), y'(x)]\, dx. \tag{7.2}$$

Hence

$$J_\gamma - J_g > 0, \tag{7.3}$$

unless $E \equiv 0$ in (7.2), that is, unless $y_i(x)$ satisfies the differential equations

$$\frac{dy_i}{dx} = p_i(x, y) \qquad (i = 1, \cdots, n). \tag{7.4}$$

But in such a case the uniqueness theorem of differential equation theory tells us that γ would coincide with the extremal of the field through its initial point, that is, with g. Thus (7.3) holds if γ is different from g, and the theorem is proved.

Before proceeding further it will be useful to enumerate certain conditions that occur frequently hereafter. In all these conditions we suppose that we have given an extremal g, defined by $y_i = \bar{y}_i(x)$, with x on the closed interval (a^1, a^2).

By the *Jacobi S-condition* will be understood the condition that there be no conjugate point of the initial point of g on g.

By the *Legendre S-condition* will be understood the condition that

$$f_{p_ip_j}(x, \bar{y}(x), \bar{y}'(x))z_iz_j > 0$$

for x on (a^1, a^2) and $(z) \neq (0)$.

By the *Weierstrass S-condition* will be understood the condition that

$$E(x, y, p, q) > 0$$

for all sets (x, y, p) sufficiently near sets (x, y, y') on g, and any set $(q) \neq (p)$.

The problem will be said to be *positively regular* in a region S of the (x, y) space, if

$$f_{p_ip_j}(x, y, p)\, z_iz_j > 0$$

for (x, y) on S, (p) unrestricted, and (z) any set not (0).

We come to the following theorem.

THEOREM 7.2. *In order that an extremal g afford a proper, strong minimum to J relative to neighboring curves of class D^1 which join its end points, it is sufficient that the Weierstrass, Legendre, and Jacobi S-conditions hold relative to g.*

A particular consequence of the Legendre S-condition, all that we use here, is that $|f_{p_ip_j}| \neq 0$ along g. With this condition satisfied the results on conjugate points in §5 apply. By virtue of the Jacobi S-condition we then know that the first conjugate point of $x = a^1$ on the x axis lies beyond $x = a^2$ or fails to exist.

According to the results of §5, the first conjugate point of a point $x = c$ prior to $x = a^1$, but sufficiently near $x = a^1$, will lie beyond $x = a^2$, or fail to exist.

Let $\bar{g}$ be the extremal obtained by extending g slightly. According to our final result in §6, the family of extremals passing through the point $x = c$ on $\bar{g}$, if properly represented, will form a Mayer field in a sufficiently small neighborhood of g.

The present theorem follows from Theorem 7.1.

If the condition of positive regularity holds for (x, y) near g, the Legendre S-condition certainly holds for g. The Weierstrass S-condition also holds relative to g. We see this upon using Taylor's formula which shows that

$$E(x, y, p, q) = \tfrac{1}{2}(q_i - p_i)(q_j - p_j)f_{p_ip_j}(x, y, p^*) \qquad (i, j = 1, \cdots, n),$$

where

$$p_i^* = p_i + \theta(q_i - p_i), \qquad (0 < \theta < 1).$$

CHAPTER II

GENERAL END CONDITIONS

To the reader the objective of the present chapter may appear to be the obtaining of necessary and sufficient conditions for a minimum under general end conditions, and such conditions *are* an immediate objective. But in reality steps are being taken towards a much larger goal.

Recall the analogy between function f and functional J, critical point and extremal, quadratic form and second variation, the topology of the domain of f and the topology of the domain of J. We here take the first step towards carrying out this analogy by assigning an index to an extremal, analogous to the index of a quadratic form. This index is the number of negative characteristic roots in a boundary problem associated with the extremal.

There remain for later chapters most important problems. What is the geometric significance of this index? Has it the property of invariance under changes of coordinate system, or otherwise put, can it be given an invariantive definition in a general parametric representation of the problem where overlapping coordinate systems are used? What relation does this index bear to other possible indices that could be assigned to the extremal in special cases? In particular what relation does it bear to conjugate points, focal points, the Poincaré rotation number, the order of concavity of a periodic extremal, or to other characteristic invariants of an extremal?

For the contemporary literature on the minimum problem under general end conditions the reader is referred to the papers by Bliss, Carathéodory, Myers, and to the Chicago Theses on the Calculus of Variations. The latter have appeared under the title *Contributions to the Calculus of Variations*, University of Chicago Press. Further references will be found in these theses. The preceding papers are primarily concerned with a minimum. The papers [8, 16] of Morse and [1] of Currier are concerned not only with minimizing extremals but also with the analytic and geometric characterization of extremals in general.

The end conditions

1. As in Chapter I we suppose that we have given an extremal g of the form

$$y_i = \bar{y}_i(x), \qquad a^1 \leqq x \leqq a^2 \qquad (i = 1, \cdots, n).$$

Points near the initial and final end points of g will be denoted respectively by

$$(x^s, y_1^s, \cdots, y_n^s) = (x^s, y^s) \qquad (s = 1, 2)$$

where $s = 2$ at the final end point and 1 at the initial end point.

A curve of class D^1 neighboring g will be termed admissible if its end points are given by the functions

$$x^s = x^s(\alpha_1, \cdots, \alpha_r), \qquad y_i^s = y_i^s(\alpha_1, \cdots, \alpha_r) \qquad (0 \leqq r \leqq 2n + 2), \tag{1.1}$$

for values of the parameters (α) near (0). For $r = 0$ the set (α) is vacuous, but it will be convenient to understand symbolically that

$$x^s(\alpha) = a^s, \qquad y_i^s(\alpha) = \bar{y}_i(a^s).$$

For $r > 0$ and for (α) near (0) we suppose that the functions in (1.1) are of class C^2 and that they give the end points of g when $(\alpha) = (0)$.

For $r > 0$ let $\theta(\alpha)$ be any function of (α) of class C^2. For $r = 0$, $\theta(\alpha)$ shall represent the symbol 0.

We seek the conditions under which g and the set $(\alpha) = (0)$, vacuous if $r = 0$, afford a minimum to the functional

$$J = \int_{x^1(\alpha)}^{x^2(\alpha)} f(x, y, y')dx + \theta(\alpha) \tag{1.2}$$

among sets (α) near (0), and admissible curves which join the end points $x^s(\alpha)$, $y_i^s(\alpha)$.

In the classical treatment of the problem with general end conditions, Bolza [2], Bliss [10], the end conditions have been given in the form of equations

$$\phi_p(x^s, y^s) = 0 \qquad (p = 0, 1, \cdots, m \leqq 2n + 2) \tag{1.3}$$

with the restriction that the functional matrix of the functions ϕ_p with respect to their arguments be of maximum rank. Bolza uses a function $g(x^s, y^s)$ in place of our function $\theta(\alpha)$. Conditions of the form (1.3) can be put in our form, but not always conversely, and the function $g(x^s, y^s)$ can be reduced to our more general $\theta(\alpha)$ by means of (1.1). See Osgood [1], p. 155.

But the real reasons we have chosen to represent our end conditions in parametric form are much deeper. The advantage of the parametric representation of surfaces over a representation of the form $\phi(x, y, z) = 0$ has long been clear. Corresponding advantages appear here when our end conditions are represented in the form (1.1). In particular the algebraic problem of setting up the second variation is much simpler and more symmetric. Moreover in the case where (x^1, y^1) is required by (1.1) to rest on an n-manifold M while (x^2, y^2) is fixed and J is the arc length, the part of the second variation which appears outside the integral sign *is* the second fundamental form of M except for a constant factor.

Our choice of end conditions in the form (1.1) was partly a matter of necessity. We shall presently deal with end conditions given in the large. If one recalls the fact that the only regular manifolds that can be represented by a single set of parameters in a regular way are those with Euler-Poincaré characteristics zero, one sees that for the purposes of analysis, geometric configurations must in general be represented by the aid of overlapping parametric systems. We must not only use parametric representations, but must consider transformations from one set of parameters to another.

An unexpected advantage of the form (1.1) was that it led to an algebraic representation of the most general set of self-adjoint boundary conditions associated with the Jacobi differential equations. As far as the author knows this is the first representation of these conditions which contains just the constants which are arbitrary. New numerical invariants of these boundary conditions thereby appear. In this way we are led to a natural and complete class of generalizations of the Sturm-Liouville separation, comparison, and oscillation theorems for the general self-adjoint system. (See Ch. IV.)

The transversality condition

2. Corresponding to the end conditions (1.1) our transversality condition is written formally as follows:

$$d\theta + \Big[(f - p_i f_{p_i})\, dx^s + f_{p_i} dy_i^s\Big]_{s=1}^{s=2} \equiv 0 \qquad (i = 1, \cdots, n), \tag{2.1}$$

where (x, y, p) is to be taken at the second end point of g when $s = 2$, and at the first end point of g when $s = 1$. If $r > 0$, $d\theta$, dx^s, and dy_i^s are to be expressed for $(\alpha) = (0)$ in terms of the differentials $d\alpha_h$, and (2.1) is to be understood as an identity in these differentials. If $r = 0$, we have $d\theta = dx^s = dy_i^s = 0$ so that (2.1) is automatically satisfied. In this section we suppose that g is of class C^1 and satisfies the Euler equations in the unexpanded form.

We shall prove the following theorem.

THEOREM 2.1. *A necessary condition that g afford a weak minimum to J relative to neighboring admissible curves of class C^1 is that it satisfy the transversality condition* (2.1).

The theorem is trivial in case $r = 0$. We suppose then that $r > 0$.

Points (x^s, y^s) near the end points of g can be joined by a curve of class C^1 neighboring g of the form

$$y_i = \bar{y}_i(x) + [y_i^1 - \bar{y}_i(x^1)] + \Big[(y_i^2 - \bar{y}_i(x^2)) - (y_i^1 - \bar{y}_i(x^1))\Big]\left(\frac{x - x^1}{x^2 - x^1}\right).$$

We are here supposing that the functions $\bar{y}_i(x)$ which define g have been extended as functions of class C^1 over an interval for x which includes the interval (a^1, a^2) in its interior. If we set the variables x^s, y^s respectively equal to the functions $x^s(\alpha)$, $y_i^s(\alpha)$, we obtain a family of admissible curves $y_i = \phi_i(x, \alpha)$ which join the end points $x^s(\alpha)$, $y_i^s(\alpha)$, and reduce to g for $(\alpha) = (0)$. Let $\alpha_h(e)$, $h = 1, \cdots, r$, be a set of functions of e of class C^1 for e near 0, with $\alpha_h(0) = 0$. Set

$$\phi_i(x, \alpha(e)) = y_i(x, e).$$

We have in $y_i = y_i(x, e)$ a one-parameter family of admissible curves satisfying the identity

$$y_i^s(\alpha(e)) \equiv y_i[x^s(\alpha(e)), e]. \tag{2.2}$$

We evaluate J along the curve of this family determined by the parameter e, setting $\theta = \theta(\alpha(e))$, and taking the limits of the integral as

$$x^s = x^s(\alpha(e)).$$

We thereby obtain a function $J(e)$ such that

$$J'(0) = \frac{d\theta}{de} + \left[f\frac{dx^s}{de} + f_{p_i}y_{ie}\right]_1^2. \tag{2.3}$$

If we differentiate the members of (2.2) with respect to e, we find that

$$\frac{dy_i^s}{de} = y_{ix}(x^s, e)\frac{dx^s}{de} + y_{ie}(x^s, e). \tag{2.4}$$

We now eliminate y_{ie} from (2.3) by means of (2.4), and recall that $J'(0) = 0$ for a minimizing arc g. We thus find that for $e = 0$

$$\frac{d\theta}{de} + \left[(f - y_{ix}f_{p_i})\frac{dx^s}{de} + f_{p_i}\frac{dy_i^s}{de}\right]_1^2 = 0.$$

The transversality condition follows and the theorem is proved

The second variation

3. We have already obtained a formula for the second variation in the case of fixed end points, that is, in the case $r = 0$. We now consider the case $r > 0$.

For $r > 0$, a set (α) in our end conditions (1.1) determines a pair of admissible end points. A set of functions,

$$\alpha_h = \alpha_h(e), \qquad \alpha_h(0) = 0, \tag{3.1}$$

will determine a set of such end points.

Suppose we have given such a set of functions $\alpha_h(e)$ of class C^2 for e near 0, and a one-parameter family of curves,

$$y_i = y_i[x, e], \tag{3.2}$$

joining the end points determined by $\alpha_h(e)$ and reducing to g for $e = 0$. We suppose that $y_i(x, e)$ is of class C^2 for e near 0. We are also supposing that

$$y_i[x^s(\alpha(e)), e] \equiv y_i^s[\alpha(e)] \qquad (i = 1, \cdots, n; s = 1, 2). \tag{3.3}$$

For each value of e near 0 we evaluate J along the corresponding curve (3.2), taking θ as $\theta(\alpha(e))$ and taking the limits x^s of the integral as $x^s(\alpha(e))$. We find thereby that

$$J'(e) = \frac{d\theta}{de} + \left[f\frac{dx^s}{de}\right]_1^2 + \int_{x^1}^{x^2}(y_{ie}f_{y_i} + y_{iex}f_{p_i})\,dx. \tag{3.4}$$

We shall obtain a formula for the second variation $J''(0)$. In it there appear the variations of y_i and α_h denoted by η_i and u_h respectively, and defined by the equations

$$\eta_i(x) = y_{ie}(x, 0) \qquad u_h = \alpha_h'(0) \qquad (i = 1, \cdots, n; h = 1, \cdots, r).$$

Before proceeding with the computation of $J''(0)$ it will be convenient to present two identities obtained by differentiating (3.3) with respect to e. Keeping the arguments as in (3.3) these identities are as follows:

$$y_{ie} + y_{ix}\frac{dx^s}{de} \equiv \frac{dy_i^s}{de}, \tag{3.5}$$

$$y_{iee} + 2y_{iex}\frac{dx^s}{de} + y_{ixx}\left(\frac{dx^s}{de}\right)^2 + y_{ix}\frac{d^2x^s}{de^2} \equiv \frac{d^2y_i^s}{de^2}. \tag{3.6}$$

We return now to $J'(e)$ and (3.4). Upon differentiating $J'(e)$ with respect to e and setting $e = 0$, we find that

$$\begin{aligned} J''(0) = \frac{d^2\theta}{de^2} &+ \left[f\frac{d^2x^s}{de^2} + f_x\left(\frac{dx^s}{de}\right)^2 + f_{y_i}\frac{dx^s}{de}\frac{dy_i^s}{de} + f_{p_i}\frac{dx^s}{de}\frac{dy_{ix}}{de}\right]_1^2 \\ &+ \left[\frac{dx^s}{de}(y_{ie}f_{y_i} + y_{ixe}f_{p_i})\right]_1^2 + \int_{a^1}^{a^2} 2\Omega(\eta, \eta')\,dx \\ &+ \frac{\partial}{\partial e}\int_{a^1}^{a^2}(y_{ie}f_{y_i}^0 + y_{ixe}f_{p_i}^0)\,dx \end{aligned} \tag{3.7}$$

where the superscript zero in the last term indicates evaluation for $e = 0$ *prior* to carrying out the operation $\partial/\partial e$. In carrying out this last operation we first integrate by parts and then differentiate. This last term then reduces to the following:

$$\left[f_{p_i}y_{iee}\right]_1^2 = \left[f_{p_i}\left\{\frac{d^2y_i^s}{de^2} - 2y_{iex}\frac{dx^s}{de} - y_{ixx}\left(\frac{dx^s}{de}\right)^2 - y_{ix}\frac{d^2x^s}{de^2}\right\}\right]_1^2 \qquad (e = 0), \tag{3.8}$$

where the expression on the right is obtained with the aid of (3.6). We next note that for $x = x^s(\alpha(e))$,

$$\frac{dy_{ix}}{de} = y_{ixe} + y_{ixx}\frac{dx^s}{de}, \tag{3.9}$$

and that we can write (3.5) in the form

$$y_{ie} = \frac{dy_i^s}{de} - y_{ix}\frac{dx^s}{de}. \tag{3.10}$$

We now make three replacements in (3.7). We replace the last general term in (3.7) by the right member of (3.8) and replace the left members of (3.9) and

(3.10), where they occur in (3.7), by the corresponding right members of (3.9) and (3.10) respectively. After these three replacements we find that

$$
\begin{aligned}
J''(0) = \Big[(f - y_{ix} f_{p_i}) \frac{d^2 x^s}{de^2} + (f_x - y_{ix} f_{y_i}) \left(\frac{dx^s}{de}\right)^2 + 2 f_{y_i} \frac{dx^s}{de} \frac{dy_i^s}{de} + f_{p_i} \frac{d^2 y_i^s}{de^2} \Big]_1^2 \\
+ \frac{d^2\theta}{de^2} + \int_{a^1}^{a^2} 2\Omega(\eta, \eta')\, dx,
\end{aligned} \tag{3.11}
$$

a first form for the second variation.

But the terms outside the integral in (3.11) can be reduced to a quadratic form in the variations $u_h = \alpha_h'(0)$. To that end it will be convenient to denote differentiation of x^s, y_i^s, and θ, with respect to α_h or α_k, by adding the subscript h or k. At $e = 0$ we find that

$$
\begin{aligned}
\frac{d^2 x^s}{de^2} &= x_{hk}^s u_h u_k + x_h^s \alpha_h''(0), \\
\frac{d^2 y_i^s}{de^2} &= y_{ihk}^s u_h u_k + y_{ih}^s \alpha_h''(0), \\
\left(\frac{dx^s}{de}\right)^2 &= x_h^s x_k^s u_h u_k, \qquad \frac{dx^s}{de}\frac{dy_i^s}{de} = x_h^s y_{ik}^s u_h u_k, \\
\frac{d^2\theta}{de^2} &= \theta_{hk} u_h u_k + \theta_h \alpha_h''(0).
\end{aligned} \tag{3.12}
$$

If the left members in (3.12) are replaced by the corresponding right members in (3.12), we find that (3.11) takes the form

$$
\begin{aligned}
J''(0) = b_{hk} u_h u_k + \theta_h \alpha_h''(0) + \Big[(f - \bar{y}_i' f_{p_i}) x_h^s + f_{p_i} y_{ih}^s \Big]_1^2 \alpha_h''(0) \\
+ \int_{a^1}^{a^2} 2\Omega(\eta, \eta')\, dx,
\end{aligned} \tag{3.13}
$$

where we have set

$$
\begin{aligned}
b_{hk} = \Big[(f - \bar{y}_i' f_{p_i}) x_{hk}^s + (f_x - \bar{y}_i' f_{y_i}) x_h^s x_k^s \\
+ f_{y_i}(x_h^s y_{ik}^s + x_k^s y_{ih}^s) + f_{p_i} y_{ihk}^s \Big]_1^2 + \theta_{hk},
\end{aligned} \tag{3.14}
$$

$$h, k = 1, \cdots, r > 0; \quad i = 1, \cdots, n; \quad s = 1, 2; \quad s \text{ not summed.}$$

These constants b_{hk} are fundamental. We note that $b_{hk} = b_{kh}$.

The formula for $J''(0)$ in (3.13) will be further simplified in case g is a minimizing arc, as we are assuming, by the fact that the coefficient of α_h'' is null. This follows from the transversality condition.

The variations η_i and u_h in (3.13) are not independent. In fact if we set $\eta_i(a^s) = \eta_i^s$, (3.10) leads to the relations (s not summed)

$$\eta_i^s = [y_{ih}^s(0) - \bar{y}_i'(a^s) x_h^s(0)] u_h.$$

For the purpose of reproducing these relations we write them in the form

$$
\begin{aligned}
(3.15)\qquad \eta_i^s &= c_{ih}^s u_h \qquad (i = 1, \cdots, n;\ h = 1, \cdots, r > 0),\\
c_{ih}^s &= y_{ih}^s(0) - \bar{y}_i'(a^s)x_h^s(0) \qquad (s = 1, 2;\ s \text{ not summed}).
\end{aligned}
$$

We summarize as follows:

THEOREM 3.1. *If g is an extremal which satisfies the transversality conditions, the second variation takes the form*

$$
(3.16)\qquad J''(0) = b_{hk}u_hu_k + \int_{a^1}^{a^2} 2\Omega(\eta, \eta')dx \qquad (h, k = 1, \cdots, r),
$$

where (η) and the r constants (u) are respectively the variations $y_{ie}(x, 0)$ and $\alpha_h'(0)$ (vacuous if $r = 0$), and satisfy the secondary end conditions,

$$
(3.17)\qquad \eta_i^s - c_{ih}^s u_h = 0 \qquad (i = 1, \cdots, n;\ h = 1, \cdots, r;\ s = 1, 2).
$$

For $r > 0$ the constants b_{hk} and c_{ih}^s are given by (3.14) and (3.15) respectively. For $r = 0$ they are not defined and disappear by convention from the preceding relations. For $r = 0$ the secondary end conditions (3.17) take the form

$$
\eta_i^s = 0 \qquad (i = 1, \cdots, n;\ s = 1, 2).
$$

Consider the case where J is the integral of the arc length and the end conditions require that the second end point be fixed while the first end point rests on a regular n-dimensional manifold M of the form

$$
x^1 = x^1(\alpha_1, \cdots, \alpha_n), \qquad y_i^1 = y_i^1(\alpha_1, \cdots, \alpha_n).
$$

One readily verifies the fact that the direction cosines of the tangent to g at its initial end point A are

$$
(3.18)\qquad f - p_if_{p_i}, \quad f_{p_1}, \cdots, f_{p_n}
$$

for (x, y, p) on g at A. The transversality conditions require that M cut g orthogonally at the point A. Referring to (3.14) we see that

$$
(3.19)\qquad b_{hk}u_hu_k = \Big[(f - p_if_{p_i})x_{hk}^1 + f_{p_i}y_{ihk}^1\Big]_1 u_hu_k.
$$

Bearing in mind that the direction (3.18) is normal to M at A, we see that the right member of (3.19) gives the terms of second order in the distance from the point $(\alpha) = (u)$ on M to the n-plane tangent to M at A, except for a factor $\pm 1/2$. Thus the form (3.19) is a second fundamental form of M at A. The implications of this fact both here and later could be pursued to advantage much further, but lack of space prevents such developments.

The accessory boundary problem

4. One can assign an index to a given quadratic form

$$
(4.0)\qquad a_{ij}z_iz_j \qquad (i, j = 1, \cdots, p)
$$

in the following way. Setting up the characteristic form

$$Q(z) = a_{ij}z_iz_j - \lambda z_iz_i, \tag{4.0$'$}$$

recall that the necessary conditions that Q have a minimum include the conditions

$$Q_{z_i}(z) = 2(a_{ij}z_j - \lambda z_i) = 0 \qquad (i = 1, \cdots, p). \tag{4.0$''$}$$

Numbers λ_k which with the sets $(z) \neq (0)$ satisfy (4.0)$''$ are called *characteristic* roots. The *index* of the form (4.0) can be defined as the number of negative characteristic roots (counted suitably if multiple).

Each of the above steps has its analogue in the theory of the second variation. *The analogue of the form* (4.0) *is the functional*

$$b_{hk}u_hu_k + \int_{a^1}^{a^2} 2\Omega(\eta, \eta')dx \qquad (h, k = 1, \cdots, r)$$

subject to the secondary end conditions

$$\eta_i^s - c_{ih}^s u_h = 0, \tag{4.1}$$

while the analogue of the characteristic form (4.0)$'$ *is the functional*

$$I(\eta, \lambda) = b_{hk}u_hu_k + \int_{a^1}^{a^2} (2\Omega(\eta, \eta') - \lambda\eta_i\eta_i)dx, \tag{4.2}$$

again subject to (4.1).

A set of n functions $\eta_i(x)$ and r constants (u) will be termed *admissible* if $\eta_i(x)$ is of class D^1 on (a^1, a^2) and if (η) with the r constants (u) satisfies the secondary end conditions (4.1).

The analogue of the conditions (4.0)$''$ is a set of necessary conditions that an admissible (η) with r constants (u) afford a minimum to $I(\eta, \lambda)$ relative to admissible sets (η) and (u), with λ fixed. These necessary conditions include the differential equations

$$\frac{d}{dx}\Omega_{\eta_i'} - \Omega_{\eta_i} + \lambda\eta_i = 0 \qquad (i = 1, \cdots, n),$$

and the transversality condition requiring that the condition

$$\Big[2\Omega_{\eta_i'}\, d\eta_i^s\Big]_1^2 + d[b_{hk}u_hu_k] \equiv 0 \qquad (r > 0) \tag{4.3}$$

be an identity in the differentials du_h. Upon using (4.1), (4.3) reduces to the conditions

$$\Big[2\Omega_{\eta_i'}c_{ih}^s\Big]_1^2 + b_{hk}u_k = 0 \qquad (h, k = 1, \cdots, r).$$

If we set

$$\zeta_i(x) = \Omega_{\eta_i'}[\eta(x), \eta'(x)], \qquad \zeta_i^s = \zeta_i(a^s),$$

(4.3) reduces to the conditions

$$(4.4) \qquad c^2_{ih}\zeta^2_i - c^1_{ih}\zeta^1_i + b_{hk}u_k = 0 \qquad (i = 1, \cdots, n; h, k = 1, \cdots, r > 0).$$

We term conditions (4.4) *the secondary transversality conditions.*

The analogue of the conditions (4.0)″ *is the set of conditions*

$$(4.5)' \qquad \frac{d}{dx}\Omega_{\eta'_i} - \Omega_{\eta_i} + \lambda\eta_i = 0 \qquad (i = 1, \cdots, n),$$

$$(4.5)'' \qquad \eta^s_i - c^s_{ih}u_h = 0,$$

$$(4.5)''' \qquad c^2_{ih}\zeta^2_i - c^1_{ih}\zeta^1_i + b_{hk}u_k = 0 \qquad (h, k = 1, \cdots, r),$$

defining what we call the accessory boundary problem associated with the extremal g.

For $r = 0$ the conditions (4.5)‴ disappear under our conventions, and the conditions (4.5)″ reduce to $\eta^s_i = 0$. For $r = 0$ we understand that the set (u) is empty.

We note that the boundary conditions in the accessory boundary problem are composed of the secondary end conditions and the secondary transversality conditions.

By a solution of the accessory boundary problem is meant a set of functions $\eta_i(x)$ which are of class C^2 on (a^1, a^2) and which with a constant λ, and r constants (u), satisfy the conditions (4.5) If $(\eta) \not\equiv (0)$, the solution is called a characteristic solution and λ a *characteristic* root. By the *index* of a characteristic root λ is meant the number of linearly independent characteristic solutions (η) corresponding to the root λ.

The final analysis of the index of the second variation subject to the secondary end conditions will be deferred to the next chapter. Under conditions of regularity to be given presently this *index* will be defined to be the number of negative characteristic roots, each counted a number of times equal to its index

The necessary condition on the characteristic roots

5. We shall now prove the following theorem. Its analogue in the theory of quadratic forms or of functions of several variables is clear.

THEOREM 5.1. *If an extremal g affords a weak minimum to J relative to neighboring admissible curves, there can exist no characteristic root* $\lambda < 0$.

In proving this theorem we shall make no assumption concerning the value of $|f_{p_ip_j}|$ along g.

We begin with the following lemma, $r \geqq 0$.

LEMMA. *If* (η) *is a characteristic solution satisfying* (4.5) *with r constants* (u) *and root* λ, *then for these constants* $I(\eta, \lambda) = 0$.

To prove the lemma we write I in the form

$$I(\eta, \lambda) = b_{hk}u_hu_k + \int_{a^1}^{a^2} (\eta_i\Omega_{\eta_i} + \eta'_i\Omega_{\eta'_i} - \lambda\eta_i\eta_i)dx,$$

and integrate the terms involving η'_i by parts, in the usual way. We find for the given (η), λ, and r constants (u), that

$$I(\eta, \lambda) = b_{hk}u_h u_k + \Big[\Omega_{\eta'_i}\eta_i^s\Big]_1^2. \tag{5.1}$$

If $r = 0$, the set (u) is empty, $\eta_i^s = 0$, and the lemma follows from (5.1). If $r > 0$, we multiply $(4.5)'''$ by u_h, sum with respect to h, and use $(4.5)''$. We thereby find that the right member of (5.1) is null.' The lemma is thereby proved. We return to the theorem.

Corresponding to the given characteristic solution (η) satisfying (4.5) with r constants (u), we shall exhibit an admissible family of curves $y_i = y_i(x, e)$, which is of the nature of the family (3.2), which reduces to g for $e = 0$, which satisfies the original end conditions with r parameters, $(\alpha) = (eu)$, and whose variations are $\eta_i(x)$. More precisely, $y_i(x, e)$ shall satisfy the identities

$$y_i^s(eu) \equiv y_i[x^s(eu), e] \qquad (s = 1, 2), \tag{5.2}$$

$$y_{ie}(x, 0) \equiv \eta_i(x). \tag{5.3}$$

For $r > 0$ such a family is given as follows:

$$\begin{aligned} y_i(x, e) \equiv \bar{y}_i(x) + e\eta_i(x) &- \{\bar{y}_i[x^2(eu)] - y_i^2(eu) + e\eta_i[x^2(eu)]\}\,\frac{x - x^1(eu)}{x^2(eu) - x^1(eu)} \\ &- \{\bar{y}_i[x^1(eu)] - y_i^1(eu) + e\eta_i[x^1(eu)]\}\,\frac{x - x^2(eu)}{x^1(eu) - x^2(eu)}. \end{aligned} \tag{5.4}$$

That this family reduces to g for $e = 0$ and satisfies (5.2) is verified by direct substitution. To verify (5.3), it is convenient first to observe that the brace

$$B = \{\bar{y}_i[x^s(eu)] - y_i^s(eu) + e\eta_i[x^s(eu)]\}$$

is zero for $e = 0$. Moreover at $e = 0$

$$\frac{dB}{de} = \{(\bar{y}'_i(a^s)x_h^s(0) - y_{ih}^s(0))u_h + \eta_i^s\} = 0,$$

as follows from (3.15). The identity (5.3) is now verified with ease.

For $r = 0$ we set $y_i(x, e) = \bar{y}_i(x) + e\eta_i(x)$.

For the family $y_i(x, e)$ so defined we know that

$$J''(0) = b_{hk}u_h u_k + \int_{a^1}^{a^2} 2\Omega(\eta, \eta')dx.$$

But by virtue of the lemma this becomes

$$J''(0) = \lambda \int_{a^1}^{a^2} \eta_i\eta_i dx$$

where λ is the characteristic root associated with $\eta_i(x)$. If $\lambda < 0$, $J''(0) < 0$. But this is impossible if g is a minimizing arc. Hence there can be no negative characteristic root and the theorem is proved.

The non-tangency hypothesis

6. Before proceeding to the sufficient conditions it is convenient to introduce an hypothesis which distinguishes a general case from a special case.

In ordinary problems involving transversality of a manifold to a given extremal it is generally customary to assume that the manifold is not tangent to the given extremal, or to insure this by other assumptions. There is here a corresponding assumption. In the case where the assumption is not made sufficient conditions involving the characteristic roots have been obtained by the author, Morse [8], and Myers [3], but the results are much simpler in case the assumption is made. Moreover simple examples in the plane will show the relative unimportance of the special case.

In the space of the $2n+2$ variables (x^s, y^s) consider the 2-dimensional manifold defined by the equations

$$y_i^s = \bar{y}_i(x^s) \qquad (s = 1, 2; i = 1, \cdots, n). \tag{6.1}$$

This manifold is essentially the arbitrary combination of a point of g near the final end of g with a point of g near the initial end of g. We call it the *extremal manifold.* The manifold

$$y_i^s = y_i^s(\alpha), \qquad x^s = x^s(\alpha)$$

in the same $(2n+2)$-space will be called the *terminal manifold.*

In case $r > 0$ we shall assume hereafter that the terminal manifold is regular, that is, that the functional matrix

$$\begin{Vmatrix} x_h^s(0) \\ y_{ih}^s(0) \end{Vmatrix} \qquad (s = 1, 2; h = 1, \cdots, r; i = 1, \cdots, n) \tag{6.2}$$

is of rank r.

If $r > 0$, our *non-tangency* condition is the condition that the extremal manifold and the terminal manifold possess no common tangent line at the point $(\alpha) = (0)$. If $r = 0$, we make the convention that the non-tangency condition is fulfilled.

We shall prove the following lemma.

LEMMA 6.1. *In case $r > 0$ a necessary and sufficient condition for the non-tangency condition to hold is that the matrix $\| c_{ih}^s \|$ of (3.15) be of rank r.*

A set of direction numbers of the tangents to the parametric curves on the terminal manifold at $(\alpha) = (0)$ is given by the r columns of the matrix (6.2). At the same point direction numbers of the tangents to the parametric curves on the extremal manifold are given by the two columns of the matrix

$$\begin{Vmatrix} 1 & 0 \\ 0 & 1 \\ \bar{y}_i'(a^1) & 0 \\ 0 & \bar{y}_i'(a^2) \end{Vmatrix} \tag{6.3}$$

The non-tangency condition implies that there is no linear relation between the columns of the matrices (6.2) and (6.3) which actually involves both matrices.

Suppose the non-tangency condition failed. There would then exist constants a_h not all zero, and constants k^s not both zero, such that

$$(6.4)' \qquad a_h x_h^s = k^s \qquad (h = 1, \cdots, r;\ s = 1, 2),$$

$$(6.4)'' \qquad a_h y_{ih}^s = k^s \bar{y}_i'(a^s) \qquad (i = 1, \cdots, n;\ s \text{ not summed}).$$

Upon eliminating k^s from these two sets of relations we find that

$$(6.5) \qquad a_h[y_{ih}^s - x_h^s \bar{y}_i(a^s)] = a_h c_{ih}^s = 0,$$

so that $\| c_{ih}^s \|$ could not be of rank r.

Conversely, suppose $\| c_{ih}^s \|$ were of rank less than r. Then relations (6.5) would hold with constants a_h not all zero. If, moreover, constants k^s are defined by the equations

$$(6.6) \qquad a_h x_h^s = k^s,$$

the relations (6.5) take the form (6.4)″. We thus have a relation between the columns of the two matrices (6.2) and (6.3) actually involving both matrices unless both constants $k^s = 0$. But both constants k^s can not be null because it would then follow from (6.6) and (6.4)″ that the matrix (6.2) would not be of rank r, contrary to hypothesis.

Thus if $\| c_{ih}^s \|$ were of rank less than r, the non-tangency condition would fail.

The lemma is thereby proved.

The variations (η) and r constants (u) appearing in the second variation satisfy the relations

$$(6.7) \qquad \eta_i^s - c_{ih}^s u_h = 0,$$

as we have seen. If $r > 0$ and the non-tangency condition holds, we can solve (6.7) for the variables (u) in terms of a suitable subset of r of the variations η_i^s. We thus have the following lemma.

Lemma 6.2. *If the non-tangency condition holds, the second variation can be written in the form*

$$(6.8) \qquad J''(0) = q(\eta) + \int_{a^1}^{a^2} 2\Omega(\eta, \eta')\, dx,$$

where $q(\eta)$ is a quadratic form in a suitable subset of r of the variations η_i^s.

In case $r = 0$ the form $q(\eta)$ disappears.

Another advantage of assuming that the non-tangency condition holds is that the accessory boundary conditions

$$(6.9) \qquad \eta_i^s - c_{ih}^s u_h = 0 \qquad (h, k = 1, \cdots, r;\ i = 1, \cdots, n;\ s = 1, 2),$$

$$(6.10) \qquad \zeta_i^2 c_{ih}^2 - \zeta_i^1 c_{ih}^1 + b_{hk} u_k = 0$$

can then be reduced to $2n$ linearly independent conditions

$$L_p(\eta, \zeta) = 0 \qquad (p = 1, \cdots, 2n) \tag{6.11}$$

on the variables η_i^s, ζ_i^s.

For we can eliminate the variables (u) from (6.9), leaving $2n-r$ ndependent linear conditions on the variables η_i^s. Upon replacing the variables (u) in (6.10) by linear combinations of the variables η_i^s obtained from (6.9), we obtain r more conditions on the variables η_i^s, ζ_i^s, independent of the $2n-r$ conditions on the variables η_i^s already obtained from (6.9). The $2n$ independent conditions (6.11) thus result. If $r = 0$, they are the conditions $\eta_i^s = 0$.

The form $Q(u, \lambda)$

7. For the remainder of this chapter we shall assume that g is an extremal satisfying the transversality and non-tangency conditions as well as the Legendre S-condition. As previously we suppose that g is given in the form $y_i = \bar{y}_i(x)$.

We here introduce the functional

$$J^\lambda = \theta(\alpha) + \int_{x^1(\alpha)}^{x^2(\alpha)} [f(x, y, y') - \frac{\lambda}{2} \sum_i (y_i - \bar{y}_i(x))^2]\, dx$$

subject to the end conditions (1.1). For each λ, g will still be an extremal and satisfy the transversality and Legendre S-conditions relative to J^λ.

The Jacobi equations are now the differential equations

$$\frac{d}{dx}\Omega_{\eta_i'} - \Omega_{\eta_i} + \lambda\eta_i = 0 \qquad (i = 1, \cdots, n), \tag{7.0}$$

where Ω is defined as previously. For each λ *conjugate points* are to be defined as previously in terms of solutions of (7.0).

We shall now define a quadratic form $Q(u, \lambda)$. We shall define Q only for the case in which $r > 0$ in the end conditions.

For $r > 0$ let $y_i(x, e)$ be any admissible family of curves of the same nature as the family (3.2), satisfying the end conditions for $\alpha_h = \alpha_h(e)$, and reducing to g for $e = 0$. The second variation here takes the form

$$\frac{d^2J^\lambda}{de^2} = b_{hk}u_h u_k + \int_{a^1}^{a^2} [2\Omega(\eta, \eta') - \lambda\eta_i\eta_i]\, dx = I(\eta, \lambda) \tag{7.1}$$

where η_i and u_h are respectively the variations $y_{ie}(x, 0)$ and $\alpha_h'(0)$, and the constants b_{hk} are defined as before. As previously we have

$$\eta_i^s = c_{ih}^s u_h \qquad (i = 1, \cdots, n;\ h = 1, \cdots, r;\ s = 1, 2). \tag{7.1$'$}$$

If for a given λ the end points of g are not conjugate, then the end points $x^s(\alpha)$, $y_i^s(\alpha)$ can be joined for each (α) sufficiently near (0), by an extremal of the form

$$y_i = \phi_i(x, \alpha), \qquad \phi_i(x, 0) = \bar{y}_i(x),$$

where, as we have seen in Theorem 5.1, Ch. I, the functions $\phi_i(x, \alpha)$ are of class C^2 in x and (α). We then let $J(\alpha, \lambda)$ denote the value of J^λ taken along the extremal determined by (α).

All of the first partial derivatives of $J(\alpha, \lambda)$ with respect to the variables (α) vanish when $(\alpha) = (0)$. This appears as an easy consequence of the transversality conditions.

The terms of the second order in $J(\alpha, \lambda)$, for λ constant, now come to the fore. They will be obtained by means of the relation

$$J_{\alpha_h \alpha_k}(0, \lambda) u_h u_k \equiv \left[\frac{d^2}{de^2} J(eu_1, \cdots, eu_r, \lambda) \right]_{e=0}, \tag{7.2}$$

where (u) is a set of r constants, and e is a parameter neighboring $e = 0$. To this end consider the family of extremals

$$y_i = y_i(x, e) = \phi_i(x, eu), \qquad \alpha_h = eu_h. \tag{7.3}$$

The right member of (7.2) is simply the second variation of J^λ for this family so that

$$J_{\alpha_h \alpha_k}(0, \lambda) u_h u_k = b_{hk} u_h u_k + \int_{a^1}^{a^2} [2\Omega(\eta, \eta') - \lambda \eta_i \eta_i] dx \tag{7.4}$$

where the set (u) is the set (u) used in (7.3), and $\eta_i(x) = y_{ie}(x, 0)$.

By the Jacobi principle the functions (η) appearing in (7.4) are solutions of (7.0) for the given λ, defining what it is convenient to call a *secondary extremal*. This secondary extremal satisfies the secondary end conditions (7.1)$'$ and is accordingly completely *determined* by the constants (u).

We summarize these results as follows:

Suppose the end points of g are not conjugate for a given λ. Let the value of J^λ, taken along the unique extremal joining the end points determined by (α) for (α) near (0), be denoted by $J(\alpha, \lambda)$. Upon setting

$$Q(u, \lambda) = J_{\alpha_h \alpha_k}(0, \lambda) u_h u_k \qquad (h, k = 1, \cdots, r > 0),$$

we find that

$$Q(u, \lambda) = b_{hk} u_h u_k + \int_{a^1}^{a^2} [2\Omega(\eta, \eta') - \lambda \eta_i \eta_i] dx, \tag{7.5}$$

where

$$\eta_i^s - c_{ih}^s u_h = 0 \qquad (i = 1, \cdots, n;\ h = 1, \cdots, r;\ s = 1, 2) \tag{7.6}$$

and (η) is on the unique secondary extremal joining the end points $(x, \eta) = (a^s, \eta^s)$.

We shall say that $I(\eta, \lambda)$ is *positive definite* for a given λ subject to (7.6), if it is positive for all curves (η) which are of class D^1, which satisfy (7.6), and on which $(\eta) \not\equiv (0)$.

We shall prove the following theorem.

THEOREM 7.1. *If the Legendre S-condition and the non-tangency condition hold, then $I(\eta, \lambda)$ is positive definite subject to* (7.6), *for $-\lambda$ sufficiently large, and $r \geqq 0$.*

According to Lemma 6.2, subject to (7.6) we have

$$I(\eta, \lambda) = q(\eta) + \int_{a^1}^{a^2} (2\Omega - \lambda\eta_i\eta_i)dx \tag{7.7}$$

where $q(\eta)$ is a quadratic form in the variables η_i^s.

Now any such form as $q(\eta)$ will satisfy a relation

$$q(\eta) \geqq -c[\eta_i^2\eta_i^2 + \eta_i^1\eta_i^1] \qquad (i = 1, \cdots, n) \tag{7.8}$$

provided c be a sufficiently large positive constant. If $h(x)$ is any function of x of class C^1 on (a^1, a^2), such that

$$h(a^1) = -1, \qquad h(a^2) = 1,$$

then (7.8) can be written in the form

$$q(\eta) \geqq -c\int_{a^1}^{a^2} \frac{d}{dx}[h(x)\eta_i\eta_i]dx, \tag{7.9}$$

where (η) represents any set of functions of x of class D^1 on (a^1, a^2).

From (7.9) we see that

$$I(\eta, \lambda) \geqq \int_{a^1}^{a^2} [2\Omega - \lambda\eta_i\eta_i - c\frac{d}{dx}(h\eta_i\eta_i)]dx.$$

Under the integral sign we have a symmetric quadratic form

$$H(\eta_1', \cdots, \eta_n', \eta_1, \cdots, \eta_n). \tag{7.10}$$

We shall use the Kronecker rule for determining the index of the form H. To that end we set $A_0 = 1$, and A_k equal to the determinant of the form obtained by setting the last $2n-k$ of the variables equal to zero in H.

According to Kronecker the index of the form H is the number of changes of sign in the sequence $A_0, A_1, \cdots, A_{2n}$, if the form is regularly arranged. See Dickson [1], p. 81. If one notes that the terms in H due to the introduction of $q(\eta)$ under the integral sign do not involve any terms quadratic in η_i', one sees as a consequence of the Legendre S-condition, that the numbers $A_0, A_1, \cdots, A_n$ are all positive. Moreover the remaining A_k's all become positive for $-\lambda$ sufficiently large. Hence for $-\lambda$ sufficiently large, the form H is positive definite. Hence $I(\eta, \lambda)$ is positive definite subject to (7.6) and the theorem is proved.

We shall verify the following corollary.

COROLLARY. *For $r > 0$ in the end conditions and for $-\lambda$ sufficiently large, the form $Q(u, \lambda)$ exists and is positive definite.*

For $-\lambda$ sufficiently large the integral

$$\int_{a^1}^{a^2} [2\Omega(\eta, \eta') - \lambda\eta_i\eta_i]dx$$

is positive except when $(\eta) \equiv (0)$. It follows that the end points of g cannot then be conjugate. Hence the construction by which the function $J(\alpha, \lambda)$ was set up is valid. The form $Q(u, \lambda)$ then exists, and by virtue of the preceding theorem and (7.5) it must be positive definite for $-\lambda$ sufficiently large.

Sufficient conditions

8. We continue with the hypotheses made in the first paragraph of the last section.

We shall prove the following lemma.

LEMMA 8.1. *If for a given λ there are no pairs of conjugate points on g, if the Legendre S-condition holds, and in case $r > 0$ if $Q(u, \lambda)$ is positive definite, then subject to the secondary end conditions*

$$\eta_i^s - c_{ih}^s u_h = 0 \qquad (h = 1, \cdots, r \geqq 0), \tag{8.0}$$

the functional $I(\eta, \lambda)$ is positive definite.

In the problem of minimizing $I(\eta, \lambda)$ certain facts should be observed.

The corresponding Jacobi equations will be the same regardless of what particular secondary extremal γ is regarded as the extremal g. For a fixed λ the distribution of conjugate points on (a^1, a^2) will then be independent of γ. Further a field of secondary extremals which covers a segment (a, b) of the x axis can be extended so as to cover a region in the (x, η) space which includes all points between the n-planes $x = a$ and $x = b$, as follows from the fact that the coordinates η_i on extremals of such a field can all be multiplied by an arbitrary constant $k \neq 0$, and still form a field of extremals.

We also note that the Legendre S-condition for g is the condition of positive regularity for $I(\eta, \lambda)$ for all points (x, η) between the planes $x = a^1$ and $x = a^2$ inclusive, thus giving the strongest sufficient condition in its category for I.

To turn to the lemma we see that each secondary extremal defined on (a^1, a^2) gives a proper minimum relative to admissible curves which join its end points. The lemma is accordingly true if $r = 0$.

If $r > 0$, the end points of $\eta_i(x)$ are determined by constants (u) in (8.0), and we have

$$I(\eta, \lambda) \geqq Q(u, \lambda),$$

the equality sign holding only when (η) is a secondary extremal. Since $Q(u, \lambda)$ is positive definite, I is positive except when $(\eta) \equiv (0)$. Thus I is positive subject to (8.0), provided $(\eta) \not\equiv (0)$.

The lemma is thereby proved.

We shall now prove the following lemma, $r \geqq 0$.

LEMMA 8.2. *If all characteristic roots are positive, then for $\lambda \leqq 0$ and sets (η) subject to the secondary end conditions, $I(\eta, \lambda)$ is positive definite.*

We are assuming that the transversality and non-tangency conditions hold for g, as well as the Legendre S-condition. If the lemma were false, there would be a least upper bound $\lambda_0 \leqq 0$ of the values of λ for which $I(\eta, \lambda)$ is positive definite. We would then have

$$I(\eta, \lambda_0) \geqq 0 \tag{8.1}$$

for all admissible curves (η) satisfying (8.0).

I say that the equality must be excluded in (8.1) for any admissible curve $(\bar{\eta}) \not\equiv (0)$, satisfying (8.0). For if $I(\bar{\eta}, \lambda_0) = 0$, $(\bar{\eta})$ would afford a minimum to $I(\eta, \lambda_0)$ among admissible curves satisfying (8.0). It would follow that $(\bar{\eta})$ could have no corners, would satisfy the secondary transversality conditions, and hence be a characteristic solution. Hence λ_0 would be a characteristic root contrary to our hypothesis that there is no characteristic root $\lambda_0 \leqq 0$.

We conclude that $I(\eta, \lambda_0) > 0$ unless $(\eta) \equiv (0)$.

Thus the segment (a^1, a^2) of the x axis affords a proper minimum to $I(\eta, \lambda_0)$ relative to admissible curves (η) satisfying (8.0). According to the Jacobi necessary condition there can be no conjugate points of $x = a^1$ on the open interval (a^1, a^2), for $\lambda = \lambda_0$. Since $I(\eta, \lambda_0)$ is positive definite, $x = a^2$ cannot be conjugate to $x = a^1$.

Not only will there be no conjugate point of $x = a^1$ on (a^1, a^2) for $\lambda = \lambda_0$, but it also follows from our representations of conjugate points in §5, Ch. I, that there will be no conjugate point of $x = a^1$ on (a^1, a^2), for λ slightly in excess of λ_0. Hence, in case $r = 0$, λ_0 cannot be the least upper bound of the values of λ for which $I(\eta, \lambda)$ is positive definite.

We turn now to the case $r > 0$. Since $x = a^1$ is not conjugate to $x = a^2$ for $\lambda = \lambda_0$, and hence for λ sufficiently near λ_0, our construction of $Q(u, \lambda)$ is valid for λ sufficiently near λ_0. From (7.5) we next see that $Q(u, \lambda_0)$ must be positive definite. It will accordingly be positive definite for values of λ slightly in excess of λ_0. We can then infer from Lemma 8.1 that $I(\eta, \lambda)$ is positive definite for λ slightly in excess of λ_0. But this is contrary to the choice of λ_0. The lemma is thereby established in case $r > 0$.

The proof is complete.

We now state the basic sufficiency theorem of this chapter, $r \geqq 0$.

THEOREM 8.1. *In order that an extremal g afford a proper, strong, relative minimum in our problem it is sufficient that it satisfy the transversality and non-tangency conditions, the Legendre and Weierstrass S-conditions, and that all characteristic roots be positive.*

We first consider the case $r > 0$.

It follows from Lemma 8.2 that the end points of (a^1, a^2) are never conjugate for $\lambda \leqq 0$. Accordingly the form $Q(u, \lambda)$ exists for each $\lambda \leqq 0$, and turning to Lemma 8.2 again, we see that $Q(u, \lambda)$ is positive definite for each such λ.

Let $J(\alpha, 0)$ be represented by means of Taylor's formula with the remainder as a term of the second order in the variables (α). This remainder is approximated by the positive definite form $Q(\alpha, 0)/2$ in such a fashion that we can be assured that

$$J(\alpha, 0) > J(0, 0) \tag{8.2}$$

for all sets $(\alpha) \neq (0)$ sufficiently near (0).

Now if the Legendre S-condition holds along g and there are no pairs of conjugate points on g, one sees readily from the form of these conditions that they also hold when g is replaced by a neighboring extremal segment g_α with end points determined by a set (α) sufficiently near (0). Moreover the field of extremals which was used to prove that g afforded a minimum to J in the fixed end point problem can now be similarly defined for each extremal g_α. To that end we take a family F_α of extremals issuing from the point on g_α at which $x = a^1 - e$, where e is a small positive constant. For e sufficiently small, for (α) sufficiently near (0), and for a set of initial slopes (p) sufficiently near those on g_α at $x = a^1 - e$, it is seen that each family F_α will form a field covering a neighborhood N of g *independent* of F_α.

Each extremal g_α for which (α) is sufficiently near zero will then afford a minimum to J relative to admissible curves γ which join its end points and lie on a sufficiently small neighborhood $N' \subset N$ of g. Thus on N'

$$J_\gamma \geqq J(\alpha, 0), \tag{8.3}$$

the equality holding only when $\gamma = g_\alpha$. From (8.2) and (8.3) we see that

$$J_\gamma \geqq J(0, 0),$$

the equality holding only if γ is an extremal g_α and g_α is the extremal g.

The theorem is thereby proved in case $r > 0$.

In case $r = 0$ we see from Lemma 8.2 that $I(\eta, 0)$ is positive definite. It follows that there is no conjugate point of $x = a^1$ on (a^1, a^2) for $\lambda = 0$. The theorem then follows from the sufficiency theorem of Ch. I.

Since the Legendre S-condition entails the Weierstrass S-condition in its weak form, we have the following corollary of the theorem.

COROLLARY. *The conditions of the theorem, omitting the Weierstrass S-condition are sufficient for g to afford a proper, weak, relative minimum to J*

We also note the following:

In the theorem the condition that all characteristic roots be positive can be replaced by the condition that the second variation be positive definite subject to the secondary end conditions.

The proof, in Lemma 8.2, that $I(\eta, 0)$ is positive definite subject to (8.0), leads with obvious changes to a proof of the following theorem.

THEOREM 8.2. *If g satisfies the transversality and non-tangency conditions, if the Legendre S-condition holds, and if $\lambda = 0$ is the smallest characteristic root, then the second variation will be positive for all admissible sets $(\eta) \not\equiv (0)$ except for those characteristic solutions for which $\lambda = 0$.*

Our theorems take a particularly simple form for the case of *periodic extremals.* Here we suppose that the integrand f and the extremal g have a period ω in x and that $a^2 - a^1 = \omega$. We compare g with the neighboring curves of class D^1 whose end points are *congruent*, i.e., whose y-coordinates at $x = a^1$ and $x = a^2$ are the same. We can take these common y-coordinates as the end parameters (α). Thus the end conditions take the form

$$y_i^s = \alpha_i, \qquad x^s = a^s \qquad (i = 1, \cdots, n; s = 1, 2).$$

We also suppose that $\theta(\alpha) \equiv 0$.

From (3.12) we see that $b_{hk} = 0$. The accessory boundary conditions take the form

$$\zeta_i^1 = \zeta_i^2, \qquad \eta_i^1 = \eta_i^2 \qquad (i = 1, \cdots, n).$$

Any non-null periodic solution of the accessory boundary problem is then a characteristic solution.

The transversality conditions are automatically fulfilled, as well as the non-tangency condition.

We have the following theorems.

THEOREM 8.3. *In order that a periodic extremal g afford a weak minimum to J relative to neighboring curves of class D^1 joining congruent points, it is necessary that the accessory differential equations for $\lambda < 0$ have no periodic solutions $(\eta) \not\equiv (0)$.*

THEOREM 8.4. *In order that a periodic extremal g afford a proper, strong minimum to J relative to neighboring curves of class D^1 joining congruent points, it is sufficient that the Legendre and Weierstrass S-conditions be satisfied, and that the accessory differential equations for $\lambda \leqq 0$ have no periodic solutions $(\eta) \not\equiv (0)$.*

The importance of the study of the relations between the calculus of variations and the theory of characteristic roots in the associated linear boundary problems has been revealed in many significant ways by Hilbert and Courant [1] in their well known treatise on mathematical physics. In $(n + 1)$-space with the general end conditions in non-parametric form, Cope [1] first obtained the necessary condition on the characteristic roots. See also Bliss [10]. Sufficiency conditions involving characteristic roots in the general problem in the Bolza form of the Lagrange problem were first established by the author, Morse [8, 14, 15, 16]. The sufficiency conditions of the present chapter and their proof can also be readily adapted to the Lagrange problem for the case of an extremal that is identically normal.

In this connection the author has recently obtained what is believed to be the first proof of the following theorem. An extremal in the Lagrange problem for which the first multiplier can be taken as unity will afford a minimum in the fixed end point problem, provided the usual Jacobi, Legendre and Weierstrass sufficient conditions hold. The hypotheses of this theorem admit cases where the family of extremals through a point fail in general to form a field. The proof will be published in the Transactions of the American Mathematical Society.

CHAPTER III

THE INDEX FORM

In this chapter we shall deal with the functional

$$J = \theta(\alpha) + \int_{x^1}^{x^2} f(x, y, y')dx,$$

subject to the end conditions

$$(0.1) \qquad x^s = x^s(\alpha), \qquad y_i^s = y_i^s(\alpha) \qquad (i = 1, \cdots, n; s = 1, 2),$$

as described in §1 of the last chapter. Except for a temporary diversion in §4, where we shall establish a necessary condition, we shall assume that we have an extremal g satisfying the transversality and non-tangency conditions, as well as the Legendre S-condition.

We define the index v of such an extremal to be the number of negative characteristic roots λ in the accessory boundary problem, counting each root a number of times equal to its index.

Such an index of g may also be regarded as the index of the quadratic functional given by the second variation subject to the secondary end conditions

$$(0.2) \qquad \eta_i^s - c_{ih}^s u_h = 0 \qquad (i = 1, \cdots, n; h = 1, \cdots, r; s = 1, 2)$$

of the preceding chapter, thus generalizing the notion of the index of a quadratic form. We shall show that this index is finite. With the extremal g we shall associate an ordinary quadratic form Q to be called the *index form.* The index of Q will turn out to be the index of g. This index form is the key to all subsequent analysis in the small.

We use the index form to treat the problems with one or two end manifolds and the problem with periodic end conditions. New invariants are introduced and results of generality and refinement are obtained. See Morse [3, 5, 7, 10, 16, 17].

Definition of the index form

1. We begin with the following lemma.

LEMMA. *A decrease of λ never causes a decrease of the distance from a point $x = c$ to the first following conjugate point.*

Suppose $x = c_1$ is the first conjugate point following $x = c$ for $\lambda = \lambda_1$. Let c_0 be a value of x such that

$$c < c_0 < c_1.$$

According to the sufficient conditions in the fixed end point theory, the integral

$$\int_{c}^{c_0} [2\Omega(\eta, \eta') - \lambda_1 \eta_i \eta_i] dx$$

will be positive on all curves $(\eta) \not\equiv (0)$, of class D^1, vanishing at c and c_0. If now λ_1 be replaced by a smaller constant λ_0, the integral will be positive as before. Hence no point c_0 between c and c_1 can be conjugate to $x = c$ for $\lambda = \lambda_0$, and the lemma is proved.

For any fixed value of λ there will exist a positive lower bound $d(\lambda)$ of the distances between pairs of conjugate points on (a^1, a^2). This follows from the representation of conjugate points by means of the zeros of the function $A(x, c)$ of (5.3), Ch. I. By virtue of the preceding lemma a lower bound $d(\lambda_0)$ will serve as a similar lower bound for all smaller values of λ.

With this understood let

$$x = a_0, \cdots, x = a_{p+1} \qquad (a_0 = a^1;\ a_{p+1} = a^2) \tag{1.1}$$

be a set of points on the x axis, arranged in the order of their subscripts, and chosen so as to divide (a^1, a^2) into segments of lengths less than $d(\lambda)$. Let us cut across g at the point at which $x = a_q$ by an n-dimensional manifold M_q of the form

$$x = X^q(\beta), \qquad y_i = Y_i^q(\beta) \qquad (q = 1, \cdots, p), \tag{1.2}$$

where the functions $X^q(\beta)$ and $Y_i^q(\beta)$ are of class C^2 in the parameters

$$(\beta) = (\beta_1, \beta_2, \cdots, \beta_n)$$

for (β) near (0). We suppose that M_q intersects g at $x = a_q$ when $(\beta) = (0)$, but is not tangent to g. We suppose further that the representation of M_q at the point $(\beta) = (0)$ is *regular*; that is, not all of the jacobians of n of the functions X^q, Y_i^q with respect to the n parameters (β) are zero at $(\beta) = (0)$. We term the manifolds M_q the *intermediate* manifolds.

Let

$$P_1, \cdots, P_p \tag{1.3}$$

be a set of points neighboring g on the respective intermediate manifolds M_q. Let (v) be a set of $pn + r = \delta$ variables of which the first r shall equal the r parameters (α). The next n (the first n if $r = 0$) shall equal the parameters (β) of the point P_1 on M_1, the next n those of the point P_2 on M_2, the next a similar set for P_3, and so on to P_p. If $r > 0$, a set (α) determines the end points

$$(x^1(\alpha), y^1(\alpha)) = P_0,$$

$$(x^2(\alpha), y^2(\alpha)) = P_{p+1}.$$

The complete set (v) *determines* the points

$$P_0, \cdots, P_{p+1}, \tag{1.4}$$

and for points (1.4) sufficiently near g is uniquely determined by these points (1.4). For $(v) = (0)$ these points lie on g.

If the points (1.4) are sufficiently near g they can be successively joined by curves which are extremals for the given λ. Denote the resulting broken extremal by E^0. We shall say that (v) *determines* the above broken extremal E^0. The value of J^λ taken along E^0 will be denoted by $J(v, \lambda)$.

We shall term $J(v, \lambda)$ an index function belonging to g, to J^λ, and to the given end conditions.

With the aid of the Euler equations one sees that the first partial derivatives of $J(v, \lambda)$ with respect to the variables $v_{r+1}, \cdots, v_\delta$ are all zero for $(v) = (0)$, and with the aid of the transversality conditions, that the partial derivatives with respect to the variables $v_1, \cdots, v_r$, $(r > 0)$ are also zero for $(v) = (0)$. Thus $J(v, \lambda)$ has a critical point with respect to the variables (v) when $(v) = (0)$.

We turn now to the terms of the second order of $J(v, \lambda)$. They are obtained by means of an identity in the variables $(z) = (z_1, \cdots, z_\delta)$, namely

$$J_{v_\alpha v_\beta}(0, \lambda) z_\alpha z_\beta \equiv \frac{d^2}{de^2} J(ez_1, \cdots, ez_\delta; \lambda) \qquad (\alpha, \beta = 1, \cdots, \delta; e = 0), \tag{1.5}$$

where e is to be set equal to zero after the differentiation.

By the index form associated with g, with the given end conditions and intermediate manifolds, we mean the form

$$Q(z, \lambda) = J_{v_\alpha v_\beta}(0, \lambda) z_\alpha z_\beta \qquad (\alpha, \beta = 1, \cdots, \delta).$$

The following theorem contains a special application of the theory of the index form. Its proof is practically identical with the proof of that part of Theorem 8.1 of Ch. II which begins with the paragraph containing (8.2).

THEOREM 1.1. *In order that an extremal g and set $(\alpha) = (0)$ afford a proper strong minimum to J relative to neighboring admissible curves satisfying the end conditions, it is sufficient that g satisfy the transversality conditions, that the Weierstrass and Legendre S-conditions hold along g, and that the index form $Q(z, 0)$ be positive definite.*

To obtain a representation of the index form in terms of the second variation we consider the family of broken extremals E^0 which are "determined" by sets

$$(v) = (ez_1, \cdots, ez_\delta)$$

in which (z) is held fast and e allowed to vary near $e = 0$. We represent this family in the form

$$y_i = y_i(x, e) \qquad (i = 1, \cdots, n). \tag{1.6}$$

The functions $y_i(x, e)$ will be of class C^2 for e near 0 and x on the component extremals of E^0 between successive points (1.4).

To obtain a more explicit representation of the end points of the extremal segments which make up the extremal E^0, it is convenient to use an alternative notation for the variables (z), namely

$$(z_1, \cdots, z_\delta) = (u_1, \cdots, u_r; z_1^1, \cdots, z_n^1, \cdots, z_1^p, \cdots, z_n^p).$$

With this understood the successive end points of the component extremals of E^0 are seen to be respectively the points

$$\begin{array}{lll} x^1(eu_1, \cdots, eu_r), & y_i^1(eu_1, \cdots, eu_r), & \\ X^q(ez_1^q, \cdots, ez_n^q) & Y_i^q(ez_1^q, \cdots, ez_n^q) & (q = 1, \cdots, p), \\ x^2(eu_1, \cdots, eu_r), & y_i^2(eu_1, \cdots, eu_r). & \end{array}$$

The reader should here recall, in case $r = 0$, that the symbols $x^s(\alpha)$, $y_i^s(\alpha)$ are used formally for the end points of g.

We turn to our formula for b_{hk} in (3.14), Ch. II. For $r > 0$ and s not summed, we set

$$\beta_{hk}^s = [(f - p_i f_{r_i})x_{hk}^s + (f_x - p_i f_{y_i})x_h^s x_k^s + f_{y_i}(x_h^s y_{ik}^s + x_k^s y_{ih}^s) + f_{p_i} y_{ihk}^s], \tag{1.7}$$

$$h, k = 1, \cdots, r; \qquad i = 1, \cdots, n; \qquad s = 1, 2; \qquad (\alpha) = (0);$$

for (x, y, p) on g at $x = a^s$. Let

$$B_{hk}^q \qquad (h, k = 1, \cdots, n; q = 1, \cdots, p)$$

denote an expression similar to β_{hk}^s in (1.7), replacing the derivatives of $x^s(\alpha)$ and $y_i^s(\alpha)$ in (1.7) by the derivatives of $X^q(\beta)$ and $Y_i^q(\beta)$ with respect to the variables (β), taking (x, y, p) on g at $x = a_q$, and setting $(\beta) = (0)$.

To obtain the second variation of J^λ relative to the family (1.6), suppose for the moment that $\theta(\alpha) \equiv 0$. Let J_q^λ denote the value of J^λ taken along the qth extremal segment of the broken extremal (1.6) determined by e. For $e = 0$ we have

$$\frac{d^2 J_1^\lambda}{de^2} = B_{ij}^1 z_i^1 z_j^1 - \beta_{hk}^1 u_h u_k + \int_{a_0}^{a_1} (2\Omega - \lambda \eta_i \eta_i) dx$$

$$\frac{d^2 J_2^\lambda}{de^2} = B_{ij}^2 z_i^2 z_j^2 - B_{ij}^1 z_i^1 z_j^1 + \int_{a_1}^{a_2} (2\Omega - \lambda \eta_i \eta_i) dx,$$

. . . .

$$\frac{d^2 J_{p+1}^\lambda}{de^2} = \beta_{hk}^2 u_h u_k - B_{ij}^p z_i^p z_j^p + \int_{a_p}^{a_{p+1}} (2\Omega - \lambda \eta_i \eta_i) dx.$$

$$(i, j = 1, \cdots, n; h, k = 1, \cdots, r),$$

We now restore the function,

$$\theta(\alpha) = \theta(eu_1, \cdots, eu_r),$$

and combine these results in the formula

$$\frac{d^2J^\lambda}{de^2} = b_{hk}u_hu_k + \int_{a^1}^{a^2} (2\Omega - \lambda\eta_i\eta_i)dx \qquad (h, k = 1, \cdots, r),$$

where b_{hk} is given by (3.14) in Ch. II in case $r > 0$, and is non-existent in case $r = 0$.

Here $\eta_i(x) = y_{ie}(x, 0)$. Accordingly the functions $\eta_i(x)$ define a broken secondary extremal E with end points and corners respectively at the points

$$x = a_0, \cdots, x = a_{p+1}.$$

This broken secondary extremal is uniquely determined by the set (z).

In fact as we have seen in §3, Ch. II (s not summed)

$$\eta_i(a^s) = [y^s_{ih}(0) - x^s_h(0)\bar{y}'_i(a^s)]u_h + 0 \qquad (h = 1, \cdots, r) \tag{1.8}$$

where the terms in u_h are non-existent if $r = 0$. Similarly for each point $x = a_q$ on g we have (q not summed)

$$\eta_i(a_q) = [Y^q_{ik}(0) - X^q_k(0)\bar{y}'_i(a_q)]z^q_k \qquad (k = 1, \cdots, n). \tag{1.8$'$}$$

For future reference we write (1.8) and (1.8)′ respectively in the forms

$$\begin{aligned} \eta^s_i - c^s_{ih}u_h &= 0 \qquad (h = 1, \cdots, r),\\ \eta_i(a_q) - C^q_{ik}z^q_k &= 0 \qquad (k = 1, \cdots, n), \end{aligned} \tag{1.8$''$}$$

where the constants c^s_{ih} and C^q_{ik} are the coefficients of u_h and z^q_k in (1.8) and (1.8)′ respectively.

It follows from the non-tangency hypothesis, as we have seen in Ch. II, that the rank of the matrix $\| c^s_{ih} \|$ is r if $r > 0$. Similarly it follows from the fact that the intermediate manifolds M_q are not tangent to g that the rank of the matrix $\| C^q_{ik} \|$ is n. That is for each intermediate manifold

$$| C^q_{ik} | \neq 0 \qquad (i, k = 1, \cdots, n). \tag{1.9}$$

We draw the following conclusions.

By virtue of the relations (1.8)″ an admissible broken secondary extremal (η) determines and is determined by a unique set (z).

THEOREM 1.2. *The index form*

$$Q(z, \lambda) = J_{v_\alpha v_\beta}(0, \lambda)z_\alpha z_\beta \qquad (\alpha, \beta = 1, \cdots, \delta)$$

is given by the formula

$$Q(z, \lambda) = b_{hk}u_hu_k + \int_{a^1}^{a^2} [2\Omega(\eta, \eta') - \lambda\eta_i\eta_i]dx \qquad (h, k = 1, \cdots, r), \tag{1.10}$$

where

$$(z_1, \cdots, z_\delta) = (u_1, \cdots, u_r, z_1^1, \cdots, z_n^1, \cdots, z_1^p, \cdots, z_n^p)$$

and (η) *is taken along the broken secondary extremal determined by* (z) *in* (1.8)$''$.

We now define what we shall term the *special index form.*

A particular choice of the "intermediate" manifolds is the set of n-planes

$$x = a_1, \cdots, x = a_p. \tag{1.11}$$

A special choice of the parameters (β) on the n-plane $x = a_q$ will be the following:

$$\beta_i = y_i - \bar{y}_i(a_q) \qquad (i = 1, \cdots, n).$$

The relations (1.8)$''$ determining the broken secondary extremal in (1.10) now consist of the secondary end conditions

$$\eta_i^s - c_{ih}^s u_h = 0 \qquad (h = 1, \cdots, r; i = 1, \cdots, n) \tag{1.12}$$

and the intermediate conditions

$$\eta_i(a_q) = z_i^q \qquad (q = 1, \cdots, p). \tag{1.13}$$

An index form set up in this manner will be called a special index form.

Properties of the index form

2. We begin with the following theorem.

THEOREM 2.1. *The form* $Q(z, \lambda)$ *is singular if and only if* λ *is a characteristic root.*

The conditions that the form $Q(z, \lambda)$ be singular are that the linear equations

$$Q_{z_\alpha} = 0 \qquad (\alpha = 1, \cdots, \delta) \tag{2.0}$$

have at least one solution $(z) \neq (0)$. If such a solution (z) is given, we shall show that the broken secondary extremal E determined by (z) affords a characteristic solution.

Of the conditions (2.0) the first r conditions, taken with the side conditions

$$\eta_i^s - c_{ih}^s z_h = 0 \qquad (h = 1, \cdots, r), \tag{2.1}$$

lead to the conditions

$$Q_{z_h} = 2b_{hk}z_k + 2\left[\Omega_{\eta_i'} \frac{\partial \eta_i^s}{\partial z_h}\right]_1^2 = 0 \qquad (h, k = 1, \cdots, r). \tag{2.2}$$

Conditions (2.2) may be written in the form

$$c_{ih}^2 \zeta_i^2 - c_{ih}^1 \zeta_i^1 + b_{hk} z_h = 0, \tag{2.2$'$}$$

a form identical with the secondary transversality conditions. In satisfying (2.2)$'$ and (2.1), E satisfies all the boundary conditions that a characteristic solution must satisfy.

Let us now examine the geometric meaning of the next n conditions

$$Q_{z_\alpha} = 0 \qquad (\alpha = r + 1, \cdots, r + n).$$

These conditions are associated with the corner of E at $x = a_1$. We have found it convenient to set

$$(z_{r+1}, \cdots, z_{r+n}) = (z_1^1, \cdots, z_n^1),$$

and with this notation in mind we find that

$$Q_{z_i^1} = 2\left[\Omega_{\eta_j'}(\eta, \eta') \frac{\partial \eta_j(a_1)}{\partial z_i^1}\right]_{a_1^+}^{a_1^-} = 0 \qquad (i, j = 1, \cdots, n) \tag{2.3}$$

where (η) is taken on the broken secondary extremal determined by (z). From (1.8)″ and (1.9) we see that

$$\left|\frac{\partial \eta_j(a_1)}{\partial z_i^1}\right| = \left|C_{ij}^1\right| \neq 0 \qquad (i, j = 1, \cdots, n).$$

From (2.3) we can accordingly conclude that

$$\left[\Omega_{\eta_j'}\right]_{a_1^+}^{a_1^-} = 0 \qquad (j = 1, \cdots, n). \tag{2.4}$$

Conditions (2.4) reduce to the conditions

$$f_{p_i p_j}\left[\eta_i'(a_1)\right]_{a_1^+}^{a_1^-} = 0 \qquad (i, j = 1, \cdots, n), \tag{2.5}$$

where (x, y, p) is taken at $x = a_1$ on g. From these conditions we conclude that

$$\left[\eta_i'(a_1)\right]_{a_1^+}^{a_1^-} = 0 \qquad (i = 1, \cdots, n). \tag{2.6}$$

Thus the conditions (2.3) imply that E has no corner at $x = a_1$. Similarly the remaining conditions imply that E has no corner at the remaining points

$$x = a_2, \cdots, x = a_p.$$

Thus E has no corners at all and satisfies (2.1) and (2.2)′. It is not identical with the x axis since $(z) \neq (0)$. It is accordingly a characteristic solution. Thus if $Q(z, \lambda)$ is singular, λ is a characteristic root.

Conversely let there be given a characteristic solution (η) satisfying conditions (4.5) of Ch. II with a constant λ, and with r constants (u). Let (z) be the set which determines this secondary extremal (η), that is, the set (z) which satisfies (1.8)″ with (η). The first r constants in (z) will necessarily be the r constants (u). Conditions (2.2)′, and hence (2.2), are satisfied since (η) is a characteristic solution. The first r conditions in (2.0) then follow. All conditions such as (2.3) are satisfied because of the absence of corners on the secondary extremal (η). Hence all conditions (2.0) are satisfied. Moreover $(z) \neq (0)$ since $(\eta) \not\equiv (0)$.

Thus $Q(z, \lambda)$ is singular when λ is a characteristic root and the proof is complete.

For a given λ it is clear that linearly independent secondary extremals that satisfy the secondary end conditions (2.1) will "determine" and "be determined by" linearly independent sets (z). Since the *nullity* of the form Q is the number of linearly independent solutions (z) of the equations (2.0), we have the following theorem.

THEOREM 2.2. *If λ is a characteristic root, the nullity of the form $Q(z, \lambda)$ equals the index of the root λ.*

The reader should understand that the number of "intermediate manifolds" M_q which it is necessary to use to set up the index form $Q(z, \lambda)$ depends upon λ. But a construction valid for a particular $\lambda = \lambda_0$, is also valid for all values of $\lambda < \lambda_0$. Whenever we compare index forms for two different values of λ we shall always understand that they are set up with the aid of common intermediate manifolds M_q.

We come to the following lemma.

LEMMA 2.1. *The form $Q(z, \lambda)$ has the property that*

$$Q(z, \lambda') < Q(z, \lambda'') \tag{2.7}$$

provided $(z) \neq (0)$ and $\lambda'' < \lambda'$.

Let (η) represent the broken secondary extremal E determined by (z) when $\lambda = \lambda''$. From (1.10) we have

$$I(\eta, \lambda') - Q(z, \lambda'') = (\lambda'' - \lambda') \int_{a^1}^{a^2} \eta_i \eta_i dx. \tag{2.8}$$

From (2.8) we see that

$$I(\eta, \lambda') < Q(z, \lambda'') \qquad (z) \neq (0).$$

But from the minimizing properties of the component extremal arcs of broken secondary extremals,

$$Q(z, \lambda') \leqq I(\eta, \lambda').$$

The lemma follows from the last two inequalities.

By the *sum* of a number of sets (z) will be meant the set (z) obtained by adding sets (z) as if they were vectors. By a *critical* set (z) with characteristic root λ will be understood a set $(z) \neq (0)$ at which all the partial derivatives of $Q(z, \lambda)$ with respect to the variables (z) vanish.

We shall now prove the following lemma.

LEMMA 2.2. *The index form $Q(z, \lambda)$ is negative if evaluated for a sum $(z) \neq (0)$ of a finite number of critical sets with distinct characteristic roots each less than λ.*

Let (z) be the sum. Let λ' be the largest of the characteristic roots and (z') the corresponding critical set. Let (z'') be the sum of the remaining critical sets so that $(z) = (z') + (z'')$.

From the preceding lemma we have

$$Q(z, \lambda) < Q(z, \lambda') \qquad (\lambda' < \lambda), \tag{2.9}$$

and this inequality proves the lemma if there is but one critical set in the sum, since the right hand form is then zero.

Now, as a matter of the algebra of quadratic forms,

$$Q(z, \lambda') = Q(z', \lambda') + z''_\alpha Q_{z_\alpha}(z', \lambda') + Q(z'', \lambda') \qquad (\alpha = 1, \cdots, \delta). \tag{2.10}$$

But since (z') is a critical set for $\lambda = \lambda'$, this equality reduces to

$$Q(z, \lambda') = Q(z'', \lambda'). \tag{2.11}$$

We now use mathematical induction, assuming the lemma true for a sum involving one less critical set than the original sum. The right hand form is as a consequence negative. The lemma then follows from (2.9).

LEMMA 2.3. *The sets (z) in any finite ensemble of critical sets (z) with distinct characteristic roots, are linearly independent.*

Suppose there were such a linear dependence. Let (z) be the linear combination which is zero. We can regard (z) as a sum of critical sets with distinct characteristic roots. Let (z') and (z'') now be defined as in the preceding lemma; $(z') \neq (0)$, and hence $(z'') \neq (0)$. Equations (2.10) and (2.11) hold as before. But the left member of (2.11) is zero since $(z) = (0)$, while the right member is negative by virtue of the preceding lemma. From this contradiction we infer the truth of the lemma.

For a fixed number δ of variables (z) there cannot be more than δ sets (z) which are independent. From this fact and the preceding lemma we deduce the following.

If the index form for λ^0 involves δ variables (z), there can be at most δ characteristic roots less than λ^0.

From the lemma it also follows that the members of any finite set of characteristic solutions (η) with distinct roots λ are linearly independent.

We now come to a fundamental theorem, Morse [16].

THEOREM 2.3. *The index of the form $Q(z, \lambda^*)$ equals the number h of characteristic roots less than λ^*, counting each root a number of times equal to its index.*

To prove the theorem we recall that $Q(z, \lambda)$ will be positive definite for λ sufficiently large and negative. If λ now be increased, Q will remain nonsingular except when λ passes through a characteristic root λ_1. According to Theorem 2.2 the index q_1 of such a root equals the nullity of the form Q when $\lambda = \lambda_1$. As λ increases through λ_1, it follows from the theory of quadratic forms that the index of Q changes by at most q_1. Thus the index of $Q(z, \lambda^*)$ is at most h, the sum of the indices of roots $\lambda < \lambda^*$.

Corresponding to each characteristic root $\lambda < \lambda^*$ of index q, there are q linearly independent critical sets (z). According to Lemma 2.2 these sets will make

$Q(z, \lambda^*)$ negative, as will any linear combination of them not null. But according to Lemma 2.3 the members of any finite ensemble of critical sets with distinct characteristic roots will be independent. Thus there are h critical sets which are independent and possess roots $\lambda < \lambda^*$. These h critical sets regarded as points (z), taken with the point $(z) = (0)$, determine an h-plane in the space (z). On this h-plane $Q(z, \lambda^*)$ is negative definite.

It will follow from Lemma 7.1 in §7 that the index of $Q(z, \lambda^*)$ is at least h. But we have seen that it is at most h. Thus the index of $Q(z, \lambda^*)$ is exactly h and the theorem is proved.

We have the following corollary of Theorems 2.2 and 2.3.

COROLLARY. *The index and nullity of the index form are independent of the number, distribution, and parametric representation of the intermediate manifolds used to define this form, provided the intersections of these intermediate manifolds with g divide g into sufficiently small segments, and provided each intermediate manifold is regularly represented and not tangent to g.*

We shall subsequently make almost exclusive use of the so-called *special index form,* defined at the end of §1. The above corollary justifies our use of the special index form. We have set up the index form in its more general form in order that we might later establish the geometric invariance of its index.

Conjugate families

3. We shall now obtain certain properties of the differential equations

$$\frac{d}{dx}\Omega_{\eta_i'} - \Omega_{\eta_i} + \lambda_i\eta_i = 0 \qquad (i = 1, \cdots, n). \tag{3.1}$$

Corresponding to each solution $\eta_i(x)$ (class C^2), we set

$$\zeta_i(x) = \Omega_{\eta_i'}(\eta, \eta'). \tag{3.2}$$

From the fact that $|f_{p_ip_j}| \neq 0$ along g it follows that each set (x, η, ζ) in (3.2) uniquely determines a set (x, η, η') and conversely. Accordingly two solutions for which the sets (η, ζ) are the same at a point $x = c$ are identical. If $\eta_i(x)$ is a solution of (3.1) and $\zeta_i(x)$ is given by (3.2) it will at times be convenient to speak of the set $\eta_i(x), \zeta_i(x)$ as a solution of (3.1).

If (η, ζ) and $(\bar\eta, \bar\zeta)$ represent two solutions of (3.1), one has the relation

$$\eta_i(x)\bar\zeta_i(x) - \zeta_i(x)\bar\eta_i(x) \equiv \text{constant}. \tag{3.3}$$

In fact the x-derivative of the left member of (3.3) is identically zero, as follows with the aid of (3.1). If the constant in (3.3) is zero, then following von Escherich one terms the two solutions *conjugate.* See Bolza [1], p. 626.

A first fact to be noted is that if a system S of k independent solutions of (3.1) *are mutually conjugate, then k is at most n.*

Suppose $k > n$. Let

$$a = \left\| \begin{matrix} \eta_{ij}(x) \\ \zeta_{ij}(x) \end{matrix} \right\| \qquad (i = 1, \cdots, n; j = 1, \cdots, n) \tag{3.4}$$

be a matrix of n solutions of which the jth column gives the jth solution (η, ζ) in the set S. The matrix $\boldsymbol{a}$ is of rank n for every x. In fact, if there were a linear relation between its columns for $x = a$, that linear relation would hold identically, since the vanishing of a solution (η, ζ) at $x = a$ implies the identical vanishing of the solution (η, ζ). Now every solution (η, ζ) of the system S satisfies the relations

$$\eta_{ij}(a)\zeta_i(a) - \zeta_{ij}(a)\eta_i(a) = 0 \qquad (i, j = 1, \cdots, n) \tag{3.5}$$

where a is any particular value of x. We have here n equations in $2n$ variables $\eta_i(a)$, $\zeta_i(a)$. Since the rank of $\boldsymbol{a}$ is n, the variables $\eta_i(a)$, $\zeta_i(a)$ are linearly dependent on any n independent solutions of (3.5), in particular upon the columns of $\boldsymbol{a}$ at $x = a$.

It follows that the solution $\eta_i(x)$, $\zeta_i(x)$ is dependent upon the columns of $\boldsymbol{a}$, contrary to the supposition that we had $k > n$ independent solutions. The proof is now complete.

A system of n linearly independent mutually conjugate solutions will be called a *conjugate base*. The set of all solutions linearly dependent on the solutions of a conjugate base will be called a *conjugate family*. If the columns of the matrix (3.4) represent the solutions of a given base, the determinant

$$D(x) = | \eta_{ij}(x) |$$

will be called the *determinant* of that base. It is readily seen that the determinants of two different bases of the same conjugate family are non-zero constant multiples of one another.

If $D(x)$ vanishes to the rth order at $x = a$, then $x = a$ will be called a *focal point of the rth order* of the given family. It is conceivable that $D(x)$ might vanish to any order at a point $x = a$, in fact might vanish identically. The facts here are given in the following theorem, Morse [13].

THEOREM 3.1. *If $D(x)$ is the determinant of a conjugate base, the order of its vanishing at $x = a$ equals the nullity s of the determinant $D(a)$.*

We suppose that $s > 0$ and $a = 0$.

If s is the nullity of $D(0)$, there exist just s linearly independent solutions of the conjugate family which vanish at $x = 0$. Let us take a new conjugate base $\| y_{ij}(x) \|$ in which the first s columns are these solutions which vanish at $x = 0$. If $s < n$, the last $n - s$ columns of $\| y_{ij}(0) \|$ will be of rank $n - s$, for otherwise there would exist additional independent solutions of the family vanishing at $x = 0$, dependent on these last $n - s$ columns. From each of the first s columns we can factor out an x and so write

$$D(x) = x^s E(x),$$

where $E(x)$ is continuous in x for x near 0.

(a). I say that $E(0) \neq 0$.

To prove (a) we note that $E(0)$ is the determinant obtained from $| y_{ij}(x) |$ by differentiating the first s columns and then putting $x = 0$ in all columns. If $s = n$ and $E(0) = 0$, one could find a non-trivial linear combination of the columns of $E(0)$ which would be zero, and which would equal the derivatives $\eta_i'(0)$ of the corresponding linear combination $\eta_i(x)$ of the columns of $\| y_{ij}(x) \|$. We would then have

$$\eta_i'(0) = \eta_i(0) = 0 \qquad (i = 1, \cdots, n),$$

so that $\eta_i(x) \equiv 0$, contrary to the fact that the columns of $\| y_{ij}(x) \|$ are linearly independent.

Suppose then that $s < n$ and $E(0) = 0$. Set

$$u_i(x) = c_j y_{ij}(x) \qquad (i = 1, \cdots, n; j = 1, \cdots, s),$$

$$z_i(x) = c_h y_{ih}(x) \qquad (i = 1, \cdots, n; h = s + 1, \cdots, n).$$

Since $E(0)$ is zero we can determine constants $c_1, \cdots, c_n$, not all zero, such that

$$u_i'(0) = z_i(0) \qquad (i = 1, \cdots, n) \tag{3.6}$$

as follows from the form of $E(0)$.

I say that $[z(0)] \neq [0]$. For otherwise it would follow that $c_{s+1} = \cdots = c_n = 0$, since the rank of the last $n - s$ columns of $E(0)$ is $n - s$. Hence the remaining constants $c_1, \cdots, c_s$ could not all be zero. Hence $(u) \not\equiv (0)$. But from the definition of $u_i(x)$ and from (3.6) respectively, we see that if $[z(0)] = [0]$,

$$u_i(0) = 0, \qquad u_i'(0) = 0 \qquad (i = 1, \cdots, n),$$

so that $u_i(x) \equiv 0$. From this contradiction we infer that $[z(0)] \neq [0]$.

To return to the proof of (a) we note that $u_i(x)$ and $z_i(x)$ are conjugate solutions. At $x = 0$ the condition that these solutions be conjugate reduces to

$$f_{p_i p_k} u_k'(0) z_i(0) = 0 \qquad (i, k = 1, \cdots, n) \tag{3.7}$$

where (x, y, p) is taken at $x = 0$ on g. With the aid of (3.6), condition (3.7) becomes

$$f_{p_i p_k} z_k(0) z_i(0) = 0. \tag{3.8}$$

But $(z) \neq (0)$ so that (3.8) contradicts the Legendre S-condition.

Thus $E(0) \neq 0$, and the order of $D(x)$ at $x = 0$ equals the nullity s of $D(0)$.

COROLLARY. *The zeros of the determinant of a conjugate base are isolated and have at most the order n.*

We can now describe the most general conjugate family. According to the preceding corollary one can always choose a point c at which a determinant of a base of the family is not zero. One can then always choose a new base such that

$$\eta_{ij}(c) = \delta_i^j \qquad (i, j = 1, \cdots, n) \tag{3.9}$$

where δ_i^j is the Kronecker delta. Such a base will be called *unitary* at $x = c$. Let

$$\| \zeta_{ij}(x) \|$$

be the matrix of the corresponding functions $\zeta_i(x)$. In order that the hth and kth columns of this new base be conjugate, it is necessary and sufficient that at $x = c$

$$\eta_{ik}\zeta_{ih} - \eta_{ih}\zeta_{ik} = 0 \qquad (i, h, k = 1, \cdots, n). \tag{3.10}$$

Upon making use of (3.9) we find that (3.10) reduces to the conditions

$$\zeta_{kh}(c) = \zeta_{hk}(c). \tag{3.11}$$

We have thereby proved that the most general conjugate family F without focal point at $x = c$ possesses a base satisfying the conditions

$$\eta_{ij}(c) = \delta_i^j, \qquad \zeta_{ij}(c) = \zeta_{ji}(c), \tag{3.12}$$

where the values $\zeta_{ij}(c)$ are arbitrary except for the condition of symmetry.

We shall term the elements $\zeta_{ij}(c)$ in (3.12) the *canonical constants* of the family F at $x = a$. By virtue of (3.12) these constants uniquely determine the family F.

Necessary conditions, one end point variable

4. We shall here consider the case where the second end point is fixed, while the first end point rests upon a manifold M, given by the equations

$$x = x^1(\alpha_1, \cdots, \alpha_r), \qquad y_i = y_i^1(\alpha_1, \cdots, \alpha_r) \qquad (0 \leqq r \leqq n) \tag{4.1}$$

For $r = 0$ the set (α) is empty, but as previously we understand symbolically that for $r = 0$,

$$x^1(\alpha) = \alpha^1, \qquad y_i^1(\alpha) = \bar{y}_i(a^1).$$

For $r > 0$ we suppose that the functions in (4.1) are of class C^2 for (α) near (0), and that for $(\alpha) = (0)$ they give the first end point of g. We suppose $\theta(\alpha)$ of class C^2, and define admissible curves and the problem of minimizing J as in Ch. II.

In case $r > 0$ a minimizing arc g must satisfy the transversality condition

$$d\theta - (f - p_i f_{p_i})dx^1 - f_{p_i} dy_i^1 \equiv 0$$

regarded as an identity in the differentials $d\alpha_h$ where (x, y, p) is taken at $x = a^1$ on g. If $r = 0$ the transversality condition is automatically satisfied. We suppose that g satisfies the transversality condition.

We now turn to the second variation.

As in §3, Ch. II, we suppose that we have a family of admissible curves

$$y_i = y_i(x, e)$$

of the nature of the family (3.2) of Ch. II joining end points determined by parameters $\alpha_h = \alpha_h(e)$ (vacuous if $r = 0$). We suppose that the family gives g when $e = 0$. According to Theorem 3.1 of Ch. II the second variation will take the form

$$J''(0) = b_{hk}u_h u_k + \int_{a^1}^{a^2} 2\Omega(\eta, \eta')dx = I^*(\eta, u), \tag{4.2}$$

where (η) and the r constants (u) are respectively the variations of $y_i(x, e)$ and $\alpha_h(e)$ (if $r > 0$) for $e = 0$, and satisfy the end conditions

$$\eta_i^1 - c_{ih}^1 u_h = 0 \qquad (i = 1, \cdots, n; h = 1, \cdots, r) \tag{4.3}$$

where

$$c_{ih}^1 = y_{ih}^1(0) - \bar{y}_i'(a^1)x_h^1(0). \tag{4.4}$$

The constants b_{hk} are given in (3.14), Ch. II.

In addition to the accessory boundary problem previously defined we here consider a problem to be called the focal boundary problem.

The focal boundary problem shall be defined by the following differential equations and boundary conditions

$$\frac{d}{dx}\Omega_{\eta'_i} - \Omega_{\eta_i} = 0 \qquad (i = 1, \cdots, n), \tag{4.5)$'$}$$

$$\eta_i^1 - c_{ih}^1 u_h = 0, \tag{4.5)$''$}$$

$$c_{ih}^1 \zeta_i^1 - b_{hk}u_k = 0 \qquad (h, k = 1, \cdots, r). \tag{4.5)$'''$}$$

Let $x = \alpha$ be a point on the x axis distinct from $x = a^1$. Let (η) be a solution of the focal boundary problem which vanishes at $x = \alpha$ and which is of class C^2 on the closed interval bounded by $x = a^1$ and $x = \alpha$. If (η) is not identically zero between a^1 and α neighboring $x = \alpha$, the point $x = \alpha$ on g will be termed a *focal* point of M on g. We extend this definition, including $x = a^1$ as a focal point of M on g in case there exists a solution of the focal boundary problem which vanishes at $x = a^1$, and which is of class C^2 but not identically zero neighboring $x = a^1$.

We shall now derive a necessary condition analogous to the Jacobi condition. In deriving it we do not assume that the end manifold M is regular, nor do we need the special assumptions which are made at this stage from the point of view of the envelope theory.

Theorem 4.1. *If g affords a weak minimum to J in the one-variable end point problem, there exists no focal point of the end manifold M at a point $x = c$ on g for which $a^1 < c < a^2$, and at which the Jacobi equations are non-singular.*

We have already proved this theorem in case $r = 0$, that is in case the end points are fixed. Suppose then that $r > 0$.

Suppose the theorem is false, and that there exists a set $(\bar{\eta})$ which satisfies the focal boundary problem with the constants $(u) = (u^0)$, and which vanishes at $x = c$ without being identically zero near $x = c$. Formula (5.4) of Ch. II with $(\eta) = (\bar{\eta})$ therein and $a^2 = c$, will give us a family of admissible curves $y_i(x, e)$ joining the points determined by $\alpha_h = eu_h^0$ on the end manifold, to the point P on g at which $x = c$. We extend this family from P to the second end point of g by following along g.

For the extended family the second variation will take the form (4.2), where $(u) = (u^0)$ and (η) defines the curve (λ):

$$(\lambda) \qquad \begin{aligned} \eta_i(x) &\equiv \bar{\eta}_i(x) && (a^1 \leqq x \leqq c), \\ \eta_i(x) &\equiv 0 && (c \leqq x \leqq a^2). \end{aligned}$$

Upon integrating the second variation by parts and using the focal boundary conditions one finds that $I^*(\eta, u^0) = 0$.

For $(u) = (u^0)$ the curve (λ) must afford a minimum to $I^*(\eta, u^0)$ relative to neighboring curves of class D^2 which join the same end points. For in the contrary case there would exist a curve (η^*) of class D^2, joining the end points of (λ), and such that

$$I^*(\eta^*, u^0) < 0. \tag{4.6}$$

One could then use (5.4) in Ch. II to set up a family of admissible curves $y_i(x, e)$ of class D^2 for which $(\eta) = (\eta^*)$ and

$$J''(0) = I^*(\eta^*, u^0) < 0. \tag{4.7}$$

In verifying (4.7) one naturally breaks J up into a sum of integrals between the corners of the curves $y_i(x, e)$. That no contribution to the terms outside the integral in the second variation is made at the corners is readily seen upon using (3.11) of Ch. II between corners and summing. But in case g is a minimizing arc, as we are assuming, (4.7) is impossible.

The curve (λ) must then afford a minimum to $I^*(\eta, u^0)$ in the fixed end point problem relative to curves of class D^2. Hence (λ) must satisfy the Weierstrass-Erdmann corner condition at $x = c$ in accordance with the remarks following Lemma 1.2, Ch. I. Exactly as in the proof of the Jacobi necessary condition we now conclude that $\bar{\eta}_i \equiv 0$ near $x = c$, and from this contradiction we infer the truth of the theorem.

Focal points

5. Focal points may be regarded as generalizations of centers of principal normal curvature of a surface. As such they have obvious geometric content.

Their theory also serves to unite such diverse elements as conjugate points, characteristic roots, and the conjugate families of von Escherich. Moreover we shall see in Ch. IV that the theory of focal points is identical with the theory of ordinary self-adjoint boundary problems with conditions at one end alone.

We now return to the assumptions of §§1, 2, and 3, namely that g be an extremal satisfying the Legendre S-condition and the transversality conditions. We also assume that the representation of the end manifold M is regular ($r > 0$), and that M is not tangent to g.

From the fact that M is regular and not tangent to g it follows that $| c^1_{ih} |$ in (4.5) is of rank r. Hence the parameters (u) can be eliminated from the conditions (4.5)″ and (4.5)‴ yielding n linearly independent homogeneous conditions on the $2n$ variables η^1_i, ζ^1_i alone. There accordingly exist n independent solutions of the focal boundary problem (4.5) upon which all other solutions are linearly dependent. Let the columns of a matrix

$$|| \eta_{ij}(x) || \qquad (i, j = 1, \cdots, n)$$

represent n such solutions. We continue with a proof of the following statement.

Any two solutions of the focal boundary problem are mutually conjugate.

Suppose that (η, ζ) and $(\bar{\eta}, \bar{\zeta})$ represent two solutions of the focal boundary problem (4.5), satisfying the boundary conditions of (4.5) with the r constants (u) and $(\bar{u})$ respectively. If $r > 0$ we multiply the members of (4.5)‴ by $\bar{u}_h$ and sum. We thereby find that

$$\bar{\eta}^1_i \zeta^1_i - b_{hk} u_k \bar{u}_h = 0.$$

Upon interchanging the rôles of the two solutions it appears that

$$\eta^1_i \bar{\zeta}^1_i - b_{hk} \bar{u}_k u_h = 0.$$

Upon recalling that $b_{hk} = b_{kh}$ we see that

$$\bar{\eta}^1_i \zeta^1_i - \eta^1_i \bar{\zeta}^1_i = 0, \tag{5.0}$$

so that if $r > 0$, any two solutions of a focal boundary problem are mutually conjugate. In case $r = 0$, $\eta^1_i = \bar{\eta}^1_i = 0$ and (5.0) is again satisfied.

The statement in italics is thereby proved.

The columns of the matrix $|| \eta_{ij}(x) ||$ accordingly form the base of a conjugate family. We shall call $| \eta_{ij}(x) |$ a *focal determinant* corresponding to M and g. The zeros of $| \eta_{ij}(x) |$ will be used to define the *focal points* of M on g. According to the theory of conjugate families in §3 the zeros of $| \eta_{ij}(x) |$ are isolated. As we have seen in §3 a zero $x = c$ of $| \eta_{ij}(x) |$ possesses an order h equal to the number μ of linearly independent solutions of the focal boundary problem which vanish at $x = c$. We term this number μ the *index* of the focal point $x = c$. In extending the present theory to the Lagrange problem, the index of c would be defined as μ, not h. The equality $\mu = h$ does not necessarily hold in the Lagrange problem.

We shall now give a geometric interpretation of focal points in line with their classical definition. For the purposes of this interpretation we need to assume that the functions $x^1(\alpha)$, $y_h^1(\alpha)$ and $\theta(\alpha)$ are of class C^3. When this interpretation is completed we shall return to the assumption that these functions are of class C^2.

In case g satisfies the transversality conditions determined by M, as we are assuming it does, the manifold M is said to *cut* the extremal g *transversally*. The following facts flow readily from Theorem 15.1, Ch. V. There exists a family of extremals which are cut transversally by M at points near g, and which are representable in the form

$$y_i = \phi_i[x, \mu_1, \cdots, \mu_n] \tag{5.1}$$

where the functions ϕ_i are of class C^2 in x and (μ) for (μ) near (0), and give g for $(\mu) = (0)$. Moreover

$$\begin{aligned} \bar{y}_i(a^1) &\equiv \phi_i[a^1, \mu_1, \cdots, \mu_n] & (r = 0), \\ y_i^1(\alpha) &\equiv \phi_i[x^1(\alpha), \alpha_1, \cdots, \alpha_r, \mu_{r+1}, \cdots, \mu_n] & (r > 0). \end{aligned} \tag{5.2}$$

Finally the representation is such that the jacobian

$$D(x) = \frac{D(\phi_1, \cdots, \phi_n)}{D(\mu_1, \cdots, \mu_n)}, \qquad (\mu) = (0),$$

has at most an isolated zero at $x = a^1$.

LEMMA 5.1. *The columns of the determinant $D(x)$ satisfy the focal boundary problem.*

The lemma is true if $r = 0$ as we have already seen. We suppose then that $r > 0$.

The conditions that the extremals of the family be cut transversally by M at the point (α) on M may be given the form

$$(f - p_i f_{p_i})x_h^1 + f_{p_i} y_{ih}^1 - \theta_h = 0 \qquad (i = 1, \cdots, n;\ h = 1, \cdots, r) \tag{5.3}$$

where (x, y, p) is taken at the point (α) on M on any one of the extremals of the family issuing from that point with (μ) near (0).

Let (u) be an arbitrary set of n constants and e a parameter neighboring 0 Consider the one-parameter family of extremals

$$y_i = y_i(x, e) = \phi_i(x, eu_1, \cdots, eu_n). \tag{5.4}$$

If in (5.3) we set

$$\begin{aligned} (\alpha) &= (eu_1, \cdots, eu_r), \\ x &= x^1(eu_1, \cdots, eu_r), \\ y_i &= y_i[x^1(eu_1, \cdots, eu_r), e], \\ p_i &= y_{ix}[x^1(eu_1, \cdots, eu_r), e], \end{aligned}$$

then (5.3) reduces to a set of r identities in e. We shall differentiate these identities with respect to e and set $e = 0$. In so doing ambiguity will be avoided if we set

$$f_{p_i}[x, y(x, e), y_x(x, e)] \equiv F_i(x, e).$$

Differentiating (5.3) with respect to e, we find that for $h, k = 1, \cdots, r$,

$$\begin{aligned}(5.5)\qquad &(f - p_i f_{p_i})x^1_{hk}u_k + f_x x^1_h x^1_k u_k + f_{y_i} x^1_h y^1_{ik} u_k + \left\{f_{p_i}\frac{dp_i}{de}x^1_h\right\} \\ &- (p_i F_{ix} x^1_h x^1_k u_k + p_i F_{ie} x^1_h) - \left\{f_{p_i}\frac{dp_i}{de}x^1_h\right\} \\ &+ F_{ix} y^1_{ih} x^1_k u_k + F_{ie} y^1_{ih} + f_{p_i} y^1_{ihk} u_k - \theta_{hk} u_k = 0.\end{aligned}$$

In this result we first cancel the two braces. We then set $\eta_i(x) = y_{ie}(x, 0)$ and let $\zeta_i(x)$ denote the corresponding function $\Omega_{\eta'_i}$. We note that

$$F_{ie}(x, 0) = \zeta_i(x).$$

By virtue of the Euler equations we can also set

$$(5.6)\qquad F_{ix} = f_{y_i}.$$

With these simplifications (5.5) takes the form

$$\begin{aligned}(5.7)\qquad &[(f - p_i f_{p_i})x^1_{hk} + (f_x - p_i f_{y_i})x^1_h x^1_k + f_{y_i}(x^1_h y^1_{ik} + y^1_{ih} x^1_k) + f_{p_i} y^1_{ihk}]u_k \\ &+ \zeta^1_i (y^1_{ih} - p_i x^1_h) - \theta_{hk} u_k = 0.\end{aligned}$$

Upon referring to (4.4) and (4.5) we find that (5.7) takes the form

$$(5.8)\qquad c^1_{ih}\zeta^1_i - b_{hk}u_k = 0 \qquad (h, k = 1, \cdots, r)$$

where the constants b_{hk} and c^1_{ih} are those in (4.5).

On the other hand we have the identity

$$y^1_i(eu_1, \cdots, eu_r) \equiv y_i[x^1(eu_1, \cdots, eu_r), e],$$

differentiation of which with respect to e leads to the relation

$$y^1_{ih}u_h = y_{ix}x^1_h u_h + y_{ie}.$$

Upon putting $e = 0$ in this relation and recalling the definition in (4.4) of the constants c^1_{ih} we see that the variations $\eta_i(x) = y_{ie}(x, 0)$ satisfy the relations

$$(5.9)\qquad \eta^1_i - c^1_{ih}u_h = 0 \qquad (h = 1, \cdots, r).$$

Thus the variations $\eta_i(x)$ and corresponding set $\zeta_i(x)$ satisfy (5.8) and (5.9) combined.

To come to the lemma let $(\eta)_p$ be the pth column of the determinant $D(x)$ and let $(\zeta)_p$ be the corresponding set (ζ). We see that the variations $(\eta)_p$ are precisely the variations $y_{ie}(x, 0)$ of the family (5.4) when the constant $u_p = 1$ in

(5.4) and the remaining $n - 1$ constants (u) in (5.4) are null. With these constants (u), $(\eta)_p$ and $(\zeta)_p$ satisfy (5.8) and (5.9), and the lemma is proved.

We are thus led to the following theorem.

THEOREM 5.1. *The solutions of the focal boundary problem form a conjugate family F for which the columns of the jacobian $D(x)$ form a conjugate base. The jacobian $D(x)$ is thus a focal determinant belonging to M and g.*

The theorem follows at once from the preceding lemma if the columns of $D(x)$ are independent. But it is known that $D(x)$ has at most an isolated zero at $x = a^1$, so that its columns must be independent and the theorem is proved.

The index of g in terms of focal points

6. We continue with end conditions of the form

$$
\begin{aligned}
x^1 &= x^1(\alpha_1, \cdots, \alpha_r), & y_i^1 &= y_i^1(\alpha_1, \cdots, \alpha_r) & (0 \leqq r \leqq n),\\
x^2 &= a^2, & y_i^2 &= \bar{y}_i(a^2),
\end{aligned}
\tag{6.0}
$$

where the functions involved are of class C^2. In case $r > 0$ the end manifold $x^1(\alpha)$, $y_i^1(\alpha)$ is to be regularly represented and not tangent to g. We again consider the functional

$$J^\lambda = \theta(\alpha) + \int_{x^1(\alpha)}^{a^2} \Big[f(x, y, y') - \frac{\lambda}{2} \sum_i (y_i - \bar{y}_i(x))^2\Big]dx.$$

Corresponding to J^λ, the end conditions (6.0) and the extremal g, we now set up the "special index form" $Q(z, \lambda)$ defined at the end of §1. According to Theorem 1.2,

$$Q(z, \lambda) = b_{hk}u_h u_k + \int_{a^1}^{a^2} [2\Omega(\eta, \eta') - \lambda\eta_i\eta_i]dx \qquad (h, k = 1, \cdots, r) \tag{6.1}$$

where

$$(z_1, \cdots, z_\delta) = (u_1, \cdots, u_r, z_1^1, \cdots, z_n^1, \cdots, z_1^p, \cdots, z_n^p)$$

and (η) in (6.1) lies on the broken secondary extremal whose end points are given by the secondary end conditions

$$
\begin{aligned}
&\eta_i^1 - c_{ih}^1 u_h = 0 \qquad (i = 1, \cdots, n;\ h = 1, \cdots, r),\\
&\eta_i^2 = 0,
\end{aligned}
\tag{6.2}
$$

and whose corners lie at the successive points

$$
\begin{aligned}
x &= a_1, & \eta_i(a_1) &= z_i^1,\\
&\cdots\cdots & &\cdots\cdots\\
x &= a_p, & \eta_i(a_p) &= z_i^p.
\end{aligned}
\tag{6.3}
$$

We begin with the following theorem.

THEOREM 6.1. *The index form $Q(z, 0)$ is singular if and only if the second end point A^2 of g is a focal point of the end manifold M. If $Q(z, 0)$ is singular, its nullity equals the index of A^2 as a focal point of M.*

To prove the theorem we note that the addition of the conditions

$$\eta_i^2 = 0 \qquad (i = 1, \cdots, n)$$

to our focal boundary problem (4.5) gives a problem B, identical for $\lambda = 0$ with the accessory boundary problem B_λ corresponding to g and to the present end conditions. Now a necessary and sufficient condition that A^2 be a focal point of M is that the problem B possess a solution not identically (0). In such a case the index of A^2 as a focal point of M will equal the index of $\lambda = 0$ as a characteristic root of B_λ, as follows from the definitions of these indices.

The theorem now follows from Theorems 2.1 and 2.2.

Subject to our secondary end conditions at $x = a^1$, namely

$$\eta_i^1 - c_{ih}^1 u_h = 0 \qquad (h = 1, \cdots, r \geqq 0), \tag{6.4}$$

we can write

$$b_{hk}u_h u_k = a_{ij}\eta_i^1\eta_j^1, \qquad a_{ij} = a_{ji} \qquad (i, j = 1, \cdots, n),$$

for suitable choices of the constants a_{ij}. We then have

$$Q(z, 0) = a_{ij}\eta_i^1\eta_j^1 + \int_{a^1}^{a^2} 2\Omega(\eta, \eta')dx \tag{6.5}$$

subject to (6.4). We can now prove the following lemma.

LEMMA 6.1. *The functional*

$$a_{ij}\eta_i^1\eta_j^1 + \int_{a^1}^{b} 2\Omega(\eta, \eta')dx \qquad (b > a^1; i, j = 1, \cdots, n) \tag{6.6}$$

taken over the interval (a^1, b) and subject to the conditions

$$\eta_i(b) = 0 \qquad (i = 1, \cdots, n) \tag{6.7}$$

will be positive definite provided the point $x = b$ is sufficiently near the point $x = a^1$.

We first choose a constant λ^* so large and negative that the functional

$$a_{ij}\eta_i^1\eta_j^1 + \int_{a^1}^{a^2} (2\Omega(\eta, \eta') - \lambda\eta_i\eta_i)dx \qquad (i, j = 1, \cdots, n)$$

taken over the interval (a^1, a^2) is positive definite subject to the conditions $\eta_i(a^2) = 0$, provided $\lambda \leqq \lambda^*$. For this choice of λ the functional

$$a_{ij}\eta_i^1\eta_j^1 + \int_{a^1}^{b} (2\Omega(\eta, \eta') - \lambda\eta_i\eta_i)dx \qquad (i, j = 1, \cdots, n) \tag{6.8}$$

taken over the interval (a^1, b) and subject to the conditions $\eta_i(b) = 0$, will be positive definite for any choice of b such that $a^1 < b \leqq a^2$.

We shall now choose the constant b so as to satisfy the lemma.

The problem of minimizing the functional (6.6) subject to the conditions $\eta_i(b) = 0$ may be regarded as the problem of minimizing the functional

$$a_{ij}u_iu_j + \int_{a^1}^{b} 2\Omega(\eta, \eta')dx, \tag{6.9}$$

subject to the end conditions

$$\eta_i^1 = u_i, \qquad \eta_i^2 = \eta_i(b) = 0 \qquad (i = 1, \cdots, n). \tag{6.10}$$

The corresponding accessory boundary problem will then take the form

$$\frac{d}{dx}\Omega_{\eta_i'} - \Omega_{\eta_i} + \lambda\eta_i = 0, \tag{6.11)$'$}$$

$$\eta_i^1 = u_i, \qquad \zeta_i^1 = a_{ij}u_j \qquad (i, j = 1, \cdots, n), \tag{6.11)$''$}$$

$$\eta_i^2 = 0. \tag{6.11)$'''$}$$

All solutions of (6.11)$'$ which satisfy (6.11)$''$ are linearly dependent on the columns of a matrix

$$\left\| \begin{matrix} \eta_{ij}(x, \lambda) \\ \zeta_{ij}(x, \lambda) \end{matrix} \right\| \qquad (i, j = 1, \cdots, n)$$

of n solutions of (6.11)$'$ which satisfy the initial conditions

$$\eta_{ij}(a^1, \lambda) = \delta_i^j,$$

$$\zeta_{ij}(a^1, \lambda) = a_{ij} \qquad (i, j = 1, \cdots, n).$$

The determinant $D(x, \lambda) = |\eta_{ij}(x, \lambda)|$ is continuous in x and λ. Moreover $D(a^1, \lambda) \equiv 1$. Hence for a closed interval for λ, such as the interval $\lambda^* \leqq \lambda \leqq 0$, there will exist a constant $b > a^1$ differing from a^1 by so little that

$$D(x, \lambda) \neq 0 \qquad (a^1 \leqq x \leqq b).$$

It is now easy to prove that the lemma holds for this choice of b.

In the problem (6.11) there can be no characteristic root less than λ^*, by virtue of the choice of λ^*. Nor can there be any characteristic root λ for which

$$\lambda^* \leqq \lambda \leqq 0,$$

since that would imply that $D(b, \lambda) = 0$ contrary to the choice of b. Thus the problem (6.11) possesses no characteristic roots $\lambda \leqq 0$. It follows from Lemma 8.2 of Ch. II that the functional (6.6) is positive definite subject to (6.7), for the above choice of b.

The lemma is thereby proved.

The following lemma is a first step towards determining the index of the special form $Q(z, 0)$ in terms of focal points of the end manifold M.

LEMMA 6.2. *The index of $Q(z, 0)$ is at most equal to the sum of the indices of the focal points of M on g between the end points of g.*

The set (z) "determines" a broken secondary extremal, the successive ends of whose segments lie on the n-planes

$$x = a_0, \cdots, x = a_{p+1} \qquad (a_0 = a^1, a_{p+1} = a^2). \tag{6.12}$$

For simplicity let us suppose $a^1 = 0$. Let a^2 now be decreased to the constant b of Lemma 6.1, holding $a^1 = 0$, and decreasing the remaining x coordinates $x = a_q$ in the same ratio as a^2. For this choice of the n-planes (6.12) we suppose $Q(z, 0)$ defined and evaluated as before. For this choice of a^2 the form $Q(z, 0)$ will be positive definite.

Now let a^2 increase, the constants $a_1, \cdots, a_p$ increasing in the same ratio as a^2, and a^1 remaining null. If a^2 thereby coincides with the coordinate $x = c$ of a focal point of M, the nullity of $Q(z, 0)$ will equal the index k of the focal point. But as a^2 increases the coefficients of the form $Q(z, 0)$ vary continuously. It follows from the theory of characteristic roots of quadratic forms that the index of $Q(z, 0)$ will increase by at most k as a^2 increases through c. The index of $Q(z, 0)$ will not otherwise change. Hence as a^2 increases from b to its original value the index of $Q(z, 0)$ will increase by at most the sum of the indices of the focal points of M on g between the end points of g.

The lemma is thereby proved.

Any curve $\eta_i(x)$ which is of class D^1 on (a^1, a^2) and satisfies the conditions

$$\eta_i^1 - c_{ih}^1 u_h = 0, \qquad \eta_i^2 = 0 \qquad (h = 1, \cdots, r),$$

with a set of r constants $u_1, \cdots, u_r$ will be said to *determine* a set

$$(z_1, \cdots, z_\delta) = (u_1, \cdots, u_r, z_1^1, \cdots, z_n^1, \cdots, z_1^p, \cdots, z_n^p)$$

in which the constants z_i^q are given by (6.3).

We now come to the basic theorem.

THEOREM 6.2. *The index of the form $Q(z, 0)$ equals the sum of the indices of the focal points of M on g between the the end points of g.*

Suppose the focal points of M on g between the end points of g have x coordinates

$$b_1 < b_2 < \cdots < b_\sigma,$$

and that their respective indices are

$$r_1, r_2, \cdots, r_\sigma.$$

Now the index of the form $Q(z, 0)$ is independent of the number of intermediate n-planes (6.12) with which one cuts across the x axis provided only that these n-planes divide (a^1, a^2) into sufficiently small segments. We can therefore suppose the n-planes $x = a_q$ in (6.12) so placed as to separate the focal points from one another, and so placed that no n-plane $x = a_q$ passes through a focal point, $(q = 1, \ldots, p)$.

According to Lemma 6.2 the index v of $Q(z, 0)$ is such that

$$v \leqq r_1 + r_2 + \cdots + r_\sigma. \tag{6.13}$$

We shall prove that (6.13) is an equality.

Corresponding to the focal point at $x = b_i$ there are r_i linearly independent secondary extremals

$$h_j^i \qquad (j = 1, \cdots, r_i; i = 1, \cdots, \sigma) \tag{6.14}$$

which represent solutions of the focal boundary problem and which vanish at $x = b_i$. From the curves (6.14) for each value of i we now form r_i new curves

$$g_j^i \qquad (j = 1, \cdots, r_i; i = 1, \cdots, \sigma) \tag{6.15}$$

which are identical with the curves (6.14) on the interval (a^1, b_i), and are identical with the x axis on the interval (b_i, a^2). Let

$$(z)_j^i \qquad (j = 1, \cdots, r_i; i = 1, \cdots, \sigma) \tag{6.16}$$

be the set (z) "determined" by the curve g_j^i. Concerning the sets (6.15) and (6.16) we shall prove the following:

(a). *The* $r_1 + r_2 + \cdots + r_\sigma$ *sets* (z) *in* (6.16) *are linearly independent.*

(b). *If* (η) *is taken on any linear combination of the curves* (6.15), $I(\eta, 0) = 0$.

(c). *For any linear combination* $(z) \neq (0)$ *of the sets* (6.16), $Q(z, 0) < 0$.

We shall first prove (a).

Suppose there were a non-trivial linear relation between the sets (z) in (6.16). Let (η) represent the corresponding linear combination of the curves (6.15). We see that (η) vanishes at each of the points

$$x = a_0, \cdots, x = a_{p+1} \qquad (a_0 = a^1; a_{p+1} = a^2). \tag{6.17}$$

Moreover if $x = a_k$ is the last point of the set (6.17) preceding $x = b_\sigma$ we see that on the interval (a_k, b_σ), (η) represents a secondary extremal (without corner) vanishing at a_k and b_σ. Hence $(\eta) \equiv (0)$ on (a_k, b_σ).

Now the only curves of the set (6.15) not identical with the x axis on (a_k, b_σ) are the curves of the set (6.15) for which $i = \sigma$, and these curves were chosen linearly independent. It follows that (η) can involve none of the curves (6.15) for which $i = \sigma$. One can now prove in a similar manner that (η) can involve none of the curves (6.15) for which $i = \sigma - 1$, and so on down to the curves for which $i = 1$. Thus (η) can involve none of the curves (6.15). From this contradiction we infer the truth of (a).

To prove (b) we represent $I(\eta, 0)$ in the form

$$I(\eta, 0) = b_{hk}u_h u_k + \int_{a^1}^{a^2} 2\Omega(\eta, \eta')dx \tag{6.18}$$

as in Ch. II, (4.2). If (η) is any linear combination of the curves (6.15) and (ζ) represents the set of corresponding functions ζ_i, integration by parts in (6.18) leads to the result

$$I(\eta, 0) = b_{hk}u_h u_k + \left[\eta_i^s \zeta_i^s\right]_1^2 + \sum_j \left[\eta_i \zeta_i\right]_{b_j^+}^{b_j^-} \qquad (j = 1, \cdots, \sigma).$$

If we make use of the fact that (η) satisfies the secondary end and transversality conditions, we find that

$$I(\eta, 0) = \sum_j \left[\eta_i \zeta_i\right]_{b_j^+}^{b_j^-}. \tag{6.19}$$

Now in the neighborhood of $x = b_j$ we can write

$$\eta_i(x) = w_i(x) + v_i(x),$$

where $v_i(x)$ represents a secondary extremal without corner at $x = b_j$, while $w_i(x)$ represents a broken secondary extremal for which

$$w_i(x) \equiv 0, \qquad x \geqq b_j \qquad (i = 1, \cdots, n).$$

Hence the term in (6.19) involving b_j reduces to

$$\left[v_i \Omega_{\eta_i'}(w, w')\right]^{b_j^-}. \tag{6.20}$$

We note finally that the secondary extremals (v) and (w), for $x \leqq b_j$, are the continuations of solutions of the focal boundary problem and hence mutually conjugate. We see then that the term in (6.20) equals

$$\left[w_i \Omega_{\eta_i'}(v, v')\right]^{b_j^-} = 0.$$

Thus $I(\eta, 0) = 0$ and (b) is proved.

To prove (c) let (z) be any linear combination of the sets (6.16), not (0). Let (η) represent the corresponding linear combination of the curves (6.15), and $(\bar{\eta})$ the broken secondary extremal determined by (z). According to (b), $I(\eta, 0) = 0$. Now the corners of the curve (η) lie on the n-planes $x = b_j$ while the corners of the curve $(\bar{\eta})$ lie on the n-planes $x = a_q$. Hence there will be some extremal segment of $(\bar{\eta})$ which joins the end points of a portion γ of (η), which portion γ is not an extremal segment. Hence

$$I(\bar{\eta}, 0) < I(\eta, 0) = 0.$$

But

$$Q(z, 0) = I(\bar{\eta}, 0).$$

Hence $Q(z, 0) < 0$ and (c) is proved.

To prove the theorem we note that the set of all linear combinations (z) of the sets (6.16) may be regarded as a set of points on an

$$r_1 + r_2 + \cdots + r_\sigma \tag{6.21}$$

plane through the origin in the space of the points (z). On this plane $Q(z, 0)$ is negative definite. According to Lemma 7.1 the index of $Q(z, 0)$ must be at least the sum (6.21). The theorem now follows from Lemma 6.2.

We have the following remarkable corollary of Theorems 2.3 and 6.2. In it focal points and characteristic roots are counted a number of times equal to their respective indices.

COROLLARY 6.1. *The number of focal points of the end manifold which lie between the end points of g equals the number of negative characteristic roots in the corresponding accessory boundary problem.*

We also note the following corollary.

COROLLARY 6.2. *The number of conjugate points of an end point of g between the end points of g equals the number of negative characteristic roots in the boundary problem*

$$\frac{d}{dx}\Omega_{\eta_i'} - \Omega_{\eta_i} + \lambda\eta_i = 0,$$

$$\eta_i(a^1) = \eta_i(a^2) = 0 \qquad (i = 1, \cdots, n).$$

With the aid of this corollary it is easy to prove that there are infinitely many positive characteristic roots in any accessory boundary problem (Morse [16]). We shall take this up in Ch. IV in a broader setting.

The two preceding theorems taken with Theorem 1.1 give us the following.

THEOREM 6.3. *In order that an extremal g afford a minimum to J in our one-variable end point problem, it is sufficient that the end manifold cut g transversally without being tangent to g, that the Legendre and Weierstrass S-conditions hold along g, and that there be no focal points of the end manifold for which $a^1 < x \leqq a^2$.*

Hahn [1] and Rozenberg [1] have made effective use of broken extremals with one intermediate vertex. They have studied the minimum problem when the end points of the extremal g are conjugate. They have also determined the nullity of the corresponding index form in the case where there is one vertex and the end points are fixed. The first n conjugate points are interpreted in terms of classes of broken extremals for which $J > J_g$.

Certain lemmas on quadratic forms

7. The following lemmas on quadratic forms will be extremely useful. The quadratic forms involved are assumed to be symmetric.

LEMMA 7.1. (a) *A necessary and sufficient condition that the index of a quadratic form $Q(z)$ be at least h is that $Q(z)$ be negative definite on some h-plane π through the* *origin in the space* (z). (b) *A necessary and sufficient condition that the index plus the nullity of* $Q(z)$ *be at least* k *is that* $Q(z)$ *be negative semi-definite on some* k*-plane through the origin.*

If the index of Q is p, the form can be carried by a real linear, non-singular transformation into the form

$$-y_1^2 - \cdots - y_p^2 + y_{p+1}^2 + \cdots + y_m^2 \qquad (m \leqq \mu), \tag{7.0}$$

where μ is the number of variables (z). Suppose that Q is negative definite on the h-plane π. Now the $(\mu - p)$-plane

$$y_1 = \cdots = y_p = 0$$

intersects the image of the h-plane π in the space (y) in a hyperplane π' of dimensionality at least $h - p$. If $p < h$, π' would be more than a point, and it would follow from (7.0) that Q would not be negative on π'. From this contradiction we infer that the index p is at least h. On the other hand Q is negative definite on the p-plane

$$y_{p+1} = \cdots = y_\mu = 0,$$

where p is the index of Q. Thus (a) is proved.

The proof of (b) is not essentially different and will be omitted.

Our second lemma is the following. Cf. Hilbert and Courant [1]; also Morse [16], p. 544.

Lemma 7.2. *Let* $Q(z)$ *be a quadratic form in* μ *variables* (z). *Let* $Q_1(v)$ *be the form obtained by evaluating* $Q(z)$ *on a* $(\mu - \rho)$*-plane*

$$z_i = a_{ij}v_j \qquad (i = 1, \cdots, \mu; j = 1, \cdots, \mu - \rho). \tag{7.1}$$

If the index of Q *is* k, *the index* k_1 *of* Q_1 *lies between* k *and* $k - \rho$ *inclusive.*

If k is the index of Q, there will be a k-plane π which passes through the origin in the space (z), on which Q is negative definite. The intersection of π with the $(\mu - \rho)$-plane (7.1) will be a hyperplane π' of dimensionality at least $k - \rho$. For sets $(v) \neq (0)$ corresponding to sets (z) on π', $Q_1(v) < 0$. We must then have $k_1 \geqq k - \rho$.

Since $Q_1(v)$ has the index k_1, there exists a k_1-plane π_1 in the space (v) on which $Q_1(v)$ is negative definite. When (v) is on π_1 the points (z) given by (7.1) will lie on a k_1-plane $\bar{\pi}_1$. On $\bar{\pi}_1$, $Q(z)$ will be negative definite. Hence $k \geqq k_1$.

The lemma is thereby proved.

Lemma 7.3. *Let* $Q'(z)$ *and* $Q''(z)$ *be two quadratic forms in* μ *variables* (z) *such that*

$$Q'(z) = Q''(z) + D(z). \tag{7.2}$$

If the indices of Q', Q'', D *and* $-D$ *are respectively* v', v'', N *and* P, *then*

$$v'' - P \leqq v' \leqq v'' + N. \tag{7.3}$$

The form Q'' will be negative definite on a v''-plane π passing through the origin in the space (z). There will exist a similar $(\mu - P)$-plane π_1 on which $D \leqq 0$. Now π and π_1 will intersect in a hyperplane π_2 of dimensionality at least $v'' - P$. (We understand that $v'' - P$ may be negative, and that π_2 then reduces to the 0-plane $(z) = (0)$.) We see then that $Q'(z)$ will be negative definite on π_2. Hence $v' \geqq v'' - P$. Upon transposing $D(z)$ to the other side of (7.2), we see that $v'' \geqq v' - N$.

Relations (7.3) are thereby proved.

LEMMA 7.4. *Let $L(v, w)$ be a quadratic form in the variables*

$$(v_1, \cdots, v_r), \qquad (w_1, \cdots, w_q),$$

such that $L(v, 0)$ is non-singular. After a suitable non-singular linear transformation from the variables (v, w) to the variables (p, w), $L(v, w)$ will assume the form

$$L(v, w) \equiv L(p, 0) + H(w),$$

where $H(w)$ can be obtained from $L(v, w)$ by eliminating the variables (v) by means of the r equations

$$L_{v_i}(v, w) = 0 \qquad (i = 1, \cdots, r). \tag{7.4}$$

Suppose that

$$L(v, 0) \equiv a_{ij}v_iv_j, \qquad a_{ij} = a_{ji}.$$

Subject the variables (v, w) to the non-singular transformation to variables (p, w) determined by setting

$$\begin{aligned} L_{v_i}(v, w) &= 2(a_{i1}p_1 + \cdots + a_{ir}p_r) \qquad (i = 1, \cdots, r), \\ w_k &= w_k \qquad (k = 1, \cdots, q). \end{aligned} \tag{7.5}$$

One can solve equations (7.5) for the variables (v) as linear functions of the variables (p) and (w), since $L(v, 0)$ is non-singular, and since $| a_{ij} |$ is accordingly not zero. Under (7.5), $L(v, w)$ will take the form

$$L(v, w) = \alpha_{ij}p_ip_j + 2\beta_{ik}p_iw_k + \gamma_{hk}w_hw_k \quad (i, j = 1, \cdots, r; h, k = 1, \cdots, q), \tag{7.6}$$

where $\alpha_{ij} = \alpha_{ji}$ and $\gamma_{hk} = \gamma_{kh}$.

Suppose now that $(\bar{p}, \bar{w})$ is a second set of variables (p, w) corresponding under the transformation (7.5) to variables $(\bar{v}, \bar{w})$. If we set up the bilinear form with the symmetric matrix belonging to $L(v, w)$, we have

$$\bar{v}_iL_{v_i}(v, w) + \bar{w}_hL_{w_h}(v, w) \equiv 2\alpha_{ij}p_i\bar{p}_j + 2\beta_{ik}p_i\bar{w}_k + 2\beta_{ik}\bar{p}_iw_k + 2\gamma_{hk}w_h\bar{w}_k \tag{7.7}$$

subject to (7.5). Consistent with (7.5) we now set

$$L_{v_i}(v, w) = 0, \qquad (p) = (0), \qquad (\bar{w}) = (0) \qquad (i = 1, \cdots, r),$$

keeping (w) and $(\bar{p})$ arbitrary. We then see from (7.7) that

$$2\beta_{ik}\bar{p}_i w_k \equiv 0$$

so that $\beta_{ik} = 0$.

We can thus write (7.6) in the form

$$L(v, w) \equiv L(p, 0) + \gamma_{hk}w_h w_k \tag{7.8}$$

subject to (7.5). If we now reduce $L(v, w)$ in (7.8) to the form $H(w)$ by means of the conditions (7.4), we must also set $(p) = (0)$ in (7.8) since (7.8) is subject to (7.5). We thus obtain the identity

$$H(w) \equiv \gamma_{hk}w_h w_k \tag{7.9}$$

from (7.8). Thus

$$L(v, w) = L(p, 0) + H(w) \tag{7.10}$$

subject to (7.5), and the lemma is proved.

The preceding lemma will be applied in the following form.

Lemma 7.5. *Let $L(v, w)$ be a quadratic form in the variables (v) and (w) such that $L(v, 0)$ is non-singular, and let $H(w)$ be the quadratic form obtained from $L(v, w)$ upon eliminating the variables (v) by means of the r equations*

$$L_{v_i}(v, w) = 0 \qquad (i = 1, \cdots, r).$$

Then the nullity of the form $L(v, w)$ will equal the nullity of $H(w)$, and the index of $L(v, w)$ will equal the sum of the indices of the forms $L(v, 0)$ and $H(w)$.

Two end manifolds

8. We have already treated this case under the general theory. We shall here seek such conditions for a minimum as can be given in terms of the focal points of the end manifolds together with the usual transversality, Weierstrass, and Legendre conditions. This problem has been treated by Bliss when $n = 1$. See Bolza [1], p. 328. For $n > 1$ the results now available, as will be seen, are scarcely predictable from the results for $n = 1$. The results as here derived depend upon a use of the index form and a preliminary theory of focal points of one manifold. Such a theory was given by the author in the Annalen, Morse [10]. With the aid of these results Dr. A. E. Currier, in a Harvard Thesis, 1930, obtained necessary and sufficient conditions for a minimum. His paper in the Transactions, Currier [1], modifies his earlier treatment and treats the parametric case. He restricts himself to the case of n-dimensional end manifolds. In the present section the author treats the case of general end manifolds in a new manner.

We suppose that the end manifolds M^1 and M^2 are given respectively in the forms

$$x^1 = x^1(\alpha_1^1, \cdots, \alpha_{r_1}^1), \qquad y_i^1 = y_i^1(\alpha_1^1, \cdots, \alpha_{r_1}^1) \qquad (0 < r_1 \leqq n), \tag{8.1}$$

$$x^2 = x^2(\alpha_1^2, \cdots, \alpha_{r_2}^2), \qquad y_i^2 = y_i^2(\alpha_1^2, \cdots, \alpha_{r_2}^2) \qquad (0 < r_2 \leqq n), \tag{8.2}$$

where the functions involved are of class C^2 for (α^s) near (0), and yield the end points of g for $(\alpha^1) = (0)$ and $(\alpha^2) = (0)$ respectively. We suppose that these end manifolds are regular, and cut g transversally at the respective end points A^1 and A^2 of g, without being tangent to g. We suppose g extended at either end so as to give an open extremal segment $\bar{g}$ containing g in its interior. Along $\bar{g}$ we suppose that the Legendre S-condition holds. For simplicity we suppose that $\theta \equiv 0$ in J.

The focal boundary problem (4.5) corresponding to the end manifold M^1 will have boundary conditions of the form

$$\begin{aligned} &\eta_i^1 - c_{ih}^1 u_h^1 = 0, \\ &c_{ih}^1 \zeta_i^1 + \beta_{hk}^1 u_k^1 = 0 \qquad (h, k = 1, \cdots, r_1), \end{aligned} \tag{8.3}$$

where β_{hk}^1 can be obtained from (1.7) and c_{ih}^1 from (1.8)″ upon setting $(\alpha^1) = (\alpha)$ in the representation of M^1. If $x = a^2$ be regarded as the initial end point of an extremal segment to the right of $x = a^2$ (that is with $x \geqq a^2$), the focal boundary problem corresponding to M^2 will have boundary conditions of the form

$$\begin{aligned} &\eta_i^2 - c_{ih}^2 u_h^2 = 0, \\ &c_{ih}^2 \zeta_i^2 + \beta_{hk}^2 u_k^2 = 0 \qquad (h, k = 1, \cdots, r_2), \end{aligned} \tag{8.4}$$

where β_{hk}^2 may be obtained from (1.7) and c_{ih}^2 from (1.8)″ upon setting $(\alpha^2) = (\alpha)$ in the representation of M^2.

Let F_1 and F_2 be the conjugate families of secondary extremals satisfying the conditions (8.3) and (8.4) respectively. Let $x = c$ be any point which is not a focal point of M^1 or M^2. Let

$$\zeta_{ij}^1(c), \qquad \zeta_{ij}^2(c) \qquad (i, j = 1, \cdots, n),$$

be respectively the two sets of symmetric "canonical constants" $\zeta_{ij}(c)$ of §3 which determine the families F_1 and F_2 at $x = c$.

If g is a minimizing arc in the problem with end conditions (8.1) and (8.2), no point $x = c$ on g between A^1 and A^2 can be a focal point either of M^1 or of M^2. In addition to this fact we have the following theorem.

THEOREM 8.1. *In order that g afford a weak relative minimum to J in the problem with two end manifolds it is necessary that*

$$D(w) = (\zeta_{ij}^1(c) - \zeta_{ij}^2(c))w_i w_j \geqq 0 \qquad (i, j = 1, \cdots, n) \tag{8.5}$$

for any set (w) and for any point $x = c$ on g between A^1 and A^2.

This theorem will be shown to be a consequence of the fact that the special index form $Q(z, 0)$ of §1, corresponding to the present problem, cannot be negative if g is a minimizing arc.

In setting up this index form after the manner of §1 we take the parameters (α) of §1 as the parameters

$$(\alpha) = (\alpha^1_1, \cdots, \alpha^1_{r_1}, \alpha^2_1, \cdots, \alpha^2_{r_2}). \tag{8.6}$$

If g is a minimizing arc there can be no conjugate point of A^1 or A^2 on g between A^1 and A^2. Hence in defining $Q(z, 0)$ we need at most one "intermediate" n-plane, and this we take as the n-plane $x = c$. The special index form $Q(z, 0)$ can now be defined as in §1. In it we shall set

$$(z) = (u^1_1, \cdots, u^1_{r_1}, u^2_1, \cdots, u^2_{r_2}, w_1, \cdots, w_n) \tag{8.7}$$

putting

$$Q(z, 0) = L(u^1, u^2, w). \tag{8.8}$$

We see then from (1.10) that for $h, k = 1, \cdots, r_2$ and $\mu, \nu = 1, \cdots, r_1$,

$$L(u^1, u^2, w) = \beta^2_{hk}u^2_h u^2_k - \beta^1_{\mu\nu}u^1_\mu u^1_\nu + \int_{a^1}^{a^2} 2\Omega(\eta, \eta')dx. \tag{8.9}$$

Here (η) lies on a broken secondary extremal E with a corner at most at $x = c$. The equations by means of which E is determined are given in (1.8)″. They are as follows:

$$\begin{cases} \eta^1_i = c^1_{ih}u^1_h & (h = 1, \cdots, r_1), \\ \eta^2_i = c^2_{ik}u^2_k & (k = 1, \cdots, r_2); \end{cases} \tag{8.10}$$

$$\eta_i(c) = w_i \qquad (i = 1, \cdots, n).$$

We shall apply Lemma 7.5 to $L(u^1, u^2, w)$ to show that the index of L equals the sum of the indices of the forms

$$L(u^1, u^2, 0), \qquad H(w), \tag{8.11}$$

where $H(w)$ is the form obtained from L upon eliminating the variables (u^1, u^2) by means of the conditions

$$\frac{\partial L}{\partial u^1_h} = 0 \qquad (h = 1, \cdots, r_1),$$
$$\frac{\partial L}{\partial u^2_k} = 0 \qquad (k = 1, \cdots, r_2). \tag{8.12}$$

As a condition precedent to the application of Lemma 7.5 we should know that the form $L(u^1, u^2, 0)$ is non-singular. To that end we first note that

$$L(u^1, u^2, 0) = L(u^1, 0, 0) + L(0, u^2, 0). \tag{8.13}$$

Now the form $L(u^1, 0, 0)$ is the index form associated with the one-variable end point problem when the end manifold is M^1 and the fixed end point is the point $x = c$ on g, and no intermediate manifolds are employed. Since $x = c$ is not a focal point of M^1, the form $L(u^1, 0, 0)$ is non-singular as affirmed in Theorem 6.1. If we interchange the order of the end points by making a transformation $\bar{x} = -x$, we see in a similar manner that the form $L(0, u^2, 0)$ is non-singular. *Thus* $L(u^1, u^2, 0)$ *is non-singular and Lemma* 7.5 *is applicable.*

We return to the theorem and note that if g is a minimizing arc the index form $L(u^1, u^2, w)$ must have the index zero. Since Lemma 7.5 is applicable we can infer that it is *necessary* that $H(w) \geqq 0$.

We shall complete the proof by establishing the identity

$$H(w) \equiv (\zeta^1_{ij}(c) - \zeta^2_{ij}(c))w_i w_j. \tag{8.14}$$

Upon using (8.9) and (8.10) we find that the conditions (8.12) can be given the respective forms

$$\begin{aligned} c^1_{ih}\zeta^1_i + \beta^1_{hk}u^1_k &= 0 \qquad (h, k = 1, \cdots, r_1), \\ c^2_{ih}\zeta^2_i + \beta^2_{hk}u^2_k &= 0 \qquad (h, k = 1, \cdots, r_2). \end{aligned} \tag{8.15}$$

Since $L(u^1, u^2, 0)$ is non-singular the conditions (8.12) determine the variables u^1_k, u^2_k as linear functions $u^1_k(w)$, $u^2_k(w)$ of the variables (w). For these variables $u^1_k(w)$, $u^2_k(w)$ the variables η^s_i can be taken so as to satisfy (8.10). Thus for a given set (w), the conditions (8.10) and (8.12) can be satisfied simultaneously. Since (8.15) then holds on the corresponding broken extremal E_w, we see that the two segments of E_w belong respectively to the two families F_1 and F_2.

Now a member (η) of the family F_1 which satisfies the conditions

$$\eta_i(c) = w_i \qquad (i = 1, \cdots, n) \tag{8.16}$$

determines a set $\zeta_i(x)$ for which

$$\zeta_i(c) = \zeta^1_{ij}(c)w_j, \qquad (\eta) \text{ in } F_1. \tag{8.17}$$

A member of the family F_2 which satisfies (8.16) determines a set $\zeta_i(x)$ for which

$$\zeta_i(c) = \zeta^2_{ij}(c)w_j, \qquad (\eta) \text{ in } F_2. \tag{8.18}$$

We shall represent $H(w)$ by means of the right member of (8.9) noting that (η) therein satisfies (8.10) and (8.15). If we then integrate by parts over the intervals (a^1, c) and (c, a^2) respectively, we find that for the set $\eta_i(x)$ in (8.9) and the corresponding set $\zeta_i(x)$

$$H(w) = \Big[\zeta_i\eta_i\Big]_{c^+}^{c^-}.$$

But we are concerned with a broken secondary extremal E_w whose two component extremals satisfy (8.17) and (8.18) respectively, as well as (8.16). Hence

$$H(w) = [\zeta^1_{ij}(c) - \zeta^2_{ij}(c)]w_i w_j,$$

as was to be proved.

We have previously noted that it is necessary that $H(w) \geqq 0$ if g is a minimizing arc. The theorem follows directly.

We can now prove the complementary theorem.

THEOREM 8.2. *In order that g afford a proper, strong, relative minimum to J it is sufficient that the end manifolds M^1 and M^2 cut g transversally without being tangent to g, that the Weierstrass and Legendre S-conditions hold along g, that there be no focal points of M^1 or M^2 on g between M^1 and M^2, and that the form $D(w)$ of Theorem 8.1 be positive definite.*

We first note that there can be no conjugate point of A^1 or A^2 on g at a point $x = c$ between A^1 and A^2. For under the conditions of the theorem a segment of g between $x = a^1$ and $x = x_0$, with $c < x_0 < a^2$, will afford a minimum to J in the problem with one end point variable on M^1 and the other fixed at $x = x_0$ on g, as we have seen in Theorem 6.3. By the Jacobi necessary condition in the fixed end point problem no such conjugate point as $x = c$ can then exist.

We can now set up the index form $L(u^1, u^2, w)$ as in the preceding proof with an intermediate n-plane $x = c$. We note that the forms $L(u^1, 0, 0)$ and $L(0, u^2, 0)$, interpreted as index forms as in the preceding proof, must be positive definite, since there are no focal points of M^1 for which $a^1 < x \leqq c$, or of M^2 for which $c \leqq x < a^2$. Moreover we have seen in (8.14) that $H(w) \equiv D(w)$, so that $H(w)$ is positive definite, as well as $L(u^1, u^2, 0)$.

It follows from Lemma 7.5 that $L(u^1, u^2, w)$ is positive definite. The theorem follows from Theorem 1.1.

From this point on we shall assume that there are no focal points of M^1 or M^2 on g between A^1 and A^2. In counting focal points we adhere to the convention that a focal point is to be counted a number of times equal to its index. Moreover we shall say that a point at which $x > x_0$ lies to the *right* of a point at which $x = x_0$.

We then come to the following lemma.

LEMMA 8.1. *If the focal points of M^1 and M^2 for which $x \geqq a^2$ are respectively counted in the order of increasing x, then a necessary condition that*

$$(\zeta^1_{ij}(c) - \zeta^2_{ij}(c))w_iw_j \geqq 0 \qquad (a^1 < c < a^2), \tag{8.19}$$

is that the kth focal point of M^1 on $\bar{g}$ lie to the right of, or coincide with, the kth focal point of M^2.

Let $x = \bar{c}$ be a point on $\bar{g}$ for which $\bar{c} > a^2$. Consider the functional

$$J^1 = \frac{\zeta^1_{ij}(c)\alpha_i\alpha_j}{2} + \int_c^{\bar{c}} \Omega(y, y')dx \qquad (a^1 < c < a^2 < \bar{c}),$$

taken along curves of class D^1 whose first end point lies on the n-plane $x = c$ at the point

$$x = c, \qquad y_i = \alpha_i \qquad (i = 1, \cdots, n) \tag{8.20}$$

and the other end point is found at the point

$$x = \bar{c}, \qquad y_i = 0. \tag{8.21}$$

Consider also a second functional

$$J^2 = \frac{\zeta^2_{ij}(c)\alpha_i\alpha_j}{2} + \int_c^{\bar{c}} \Omega(y, y')dx$$

subject to the same end conditions. The x axis between $x = c$ and $x = \bar{c}$ inclusive, will be an extremal g^0 relative to both functionals and will be cut transversally by the n-plane $x = c$. The focal boundary problem (4.5) corresponding to these two functionals, to the extremal g^0, and to the end manifold (8.20), will possess boundary conditions of the respective forms

$$\eta_i(c) = u_i, \qquad \zeta_i(c) = \zeta^1_{ij}(c)u_j \qquad (i = 1, \cdots, n), \tag{8.22}$$

$$\eta_i(c) = u_i, \qquad \zeta_i(c) = \zeta^2_{ij}(c)u_j. \tag{8.23}$$

But these two focal boundary problems are seen to define precisely the conjugate families F_1 and F_2. Thus the focal points on the x axis, of the n-plane $x = c$, relative to the functionals J^1 and J^2 have the same x coordinates and indices as the focal points of M^1 and M^2 relative to J on the extremal $\bar{g}$.

Relative to J^1 and J^2, the extremal g^0, and the end conditions (8.20) and (8.21), we now set up special index forms $Q^1(z, 0)$ and $Q^2(z, 0)$ respectively, using the same intermediate n-planes in the two cases. We then have two formulas

$$Q^s(z, 0) = \zeta^s_{ij}(c)u_iu_j + \int_c^{\bar{c}} 2\Omega(\eta, \eta')dx \qquad (i, j = 1, \cdots, n;\ s = 1, 2), \tag{8.24}$$

where we set the first n variables in the set (z) equal to n variables (u), and where (η) is taken on the broken secondary extremal determined by (z). From (8.24) we see that

$$Q^1(z, 0) - Q^2(z, 0) = [\zeta^1_{ij}(c) - \zeta^2_{ij}(c)]u_iu_j. \tag{8.25}$$

From (8.19) we see then that

$$Q^2(z, 0) \leqq Q^1(z, 0).$$

The index of $Q^2(z, 0)$ must then be at least as great as that of $Q^1(z, 0)$. This means that the conjugate family F_2 must have at least as many focal points between $x = c$ and $x = \bar{c}$ as does F_1.

The lemma follows directly.

From this lemma and from Theorem 8.1 we infer the following.

THEOREM 8.3. *In order that g afford a weak relative minimum to J it is necessary that there be no focal points of M^1 or of M^2 on g between M^1 and M^2, and that the kth focal point of M^1 on $\bar{g}$ to the right of A^1 lie to the right of or coincide with the kth focal point of M^2 to the right of A^2.*

We shall now prove the following lemma.

LEMMA 8.2. *In order that the difference form*

$$D(w) = [\zeta^1_{ij}(c) - \zeta^2_{ij}(c)]w_iw_j \qquad (a^1 < c < a^2)$$

of Theorem 8.1 *be positive definite it is sufficient that there be no focal points of M^1 or of M^2 on $\bar{g}$ between M^1 and M^2, and that on some segment of $\bar{g}$ for which $c < x < \bar{c}$ there be n more focal points of M^2 than of M^1.*

We consider again the forms $Q^1(z, 0)$ and $Q^2(z, 0)$ set up in the proof of Lemma 8.1. We write the relation (8.25) in the form

$$Q^1(z, 0) = Q^2(z, 0) + D(w),$$

where (w) gives the first n of the variables (z). Let the indices of $-D$, Q^1, and Q^2 be respectively P, v', and v''. According to Lemma 7.3,

$$P \geqq v'' - v'. \tag{8.26}$$

Now if there are n more focal points of M^2 than of M^1 for which $c < x < \bar{c}$, we must have $v'' - v' = n$. We then see from (8.26) that

$$P = n.$$

Hence $D(w)$ is positive definite, and the lemma is proved.

We see incidentally from (8.26) that $v'' - v'$ can be at most n, that is there can be *at most* n more focal points of M^2 than of M^1 for which $a^1 < c < x < \bar{c}$.

From this lemma and from Theorem 8.2 we obtain the final result.

THEOREM 8.4. *In order that g afford a proper, strong, relative minimum to J it is sufficient that the end manifolds M^1 and M^2 cut g transversally without being tangent to g, that the Legendre and Weierstrass S-conditions hold along g, that there be no focal points of M^1 or M^2 on g between M^1 and M^2, and that on some closed extremal extension of g on which the Legendre S-condition holds there exist a segment $a^2 \leqq x < \bar{c}$ on which there are n more focal points of M^2 than of M^1.*

The rôles of M^1 and M^2 can be interchanged in an obvious manner.

Periodic extremals, a necessary condition

9. In the following three sections we shall suppose that the integrand $f(x, y, y')$ as well as the functions $\bar{y}_i(x)$ and $\bar{y}'_i(x)$ have a period ω in x. For simplicity we set $\theta(\alpha) = 0$ and take a^1 as 0 and a^2 as ω.

Our end conditions here have the form

$$\begin{aligned} x^1 &= 0, \qquad y^1_i = \alpha_i, \\ x^2 &= \omega, \qquad y^2_i = \alpha_i \end{aligned} \qquad (i = 1, \cdots, n). \tag{9.0}$$

The corresponding secondary end conditions become

$$\begin{aligned} x^1 &= 0, \qquad \eta^1_i = u_i, \\ x^2 &= \omega, \qquad \eta^2_i = u_i \end{aligned} \qquad (i = 1, \cdots, n), \tag{9.1}$$

while the secondary transversality conditions reduce to the conditions

$$\zeta_i^1 - \zeta_i^2 = 0.$$

The second variation $I(\eta, \lambda)$ of J^λ takes the form

$$I(\eta, \lambda) = \int_0^\omega [2\Omega(\eta, \eta') - \lambda\eta_i\eta_i]dx. \tag{9.2}$$

Two points in the (x, y) space whose x coordinates differ by ω and whose coordinates (y) are the same will be called *congruent* points. We make a similar convention for the space (x, η).

If g affords a minimum to J relative to neighboring curves of class D^1 that join congruent end points, it is necessary that there be no periodic solutions of the accessory differential equations for which $\lambda < 0$ and $(\eta) \not\equiv (0)$, as we have already seen. Moreover if x_0 is any value of x, it is also necessary that there be no conjugate point of x between x_0 and $x_0 + \omega$. The following theorem contains still another necessary condition.

THEOREM 9.1. *If g affords a weak minimum to J relative to neighboring admissible curves joining congruent end points, it is necessary that*

$$\eta_i(x_0 + \omega)\zeta_i(x_0 + \omega) - \eta_i(x_0)\zeta_i(x_0) \geqq 0 \tag{9.3}$$

for every solution of the Jacobi equations which is of class C^1 and joins congruent points on the n-planes $x = x_0$ and $x = x_0 + \omega$ respectively.

Suppose that $\eta_i(x)$ is a solution of the Jacobi equations of the nature described in the lemma. Regard this solution $\eta_i(x)$ as defined merely on the interval $(x_0, x_0 + \omega)$. Let the functions $\eta_i(x)$ now be defined at all remaining points x by the condition that $\eta_i(x)$ have the period ω. The curves

$$y_i(x, e) = \bar{y}_i(x) + e\eta_i(x) \qquad (0 \leqq x \leqq \omega)$$

will then form a family of admissible curves joining congruent points in the (x, y) space. For this family the second variation of J integrated by parts in the usual way will reduce to the left member of (9.3), at least if $x_0 = 0$. If $x_0 \neq 0$ an obvious use of the periodicity of $\eta_i(x)$ leads to the same result.

But for a minimizing arc g, $J''(0)$ cannot be negative. Hence (9.3) must hold and the theorem is proved.

The order of concavity

10. We continue with the periodic extremal g of the preceding section. Along g we now assume that the Legendre S-condition holds.

We have already determined the index of a periodic extremal in terms of the characteristic roots of the accessory boundary problem. In the next section we shall give another mode of evaluation of this index in terms of conjugate points and a new numerical invariant. This new invariant will now be defined.

For each value of λ near 0, let

$$\| p_{ij}(x, \lambda) \|, \qquad \| q_{ij}(x, \lambda) \| \qquad (i, j = 1, \cdots, n) \tag{10.1}$$

be respectively n-square matrices whose columns are solutions of the accessory differential equations set up for (9.2). Let

$$\| \zeta^p_{ij}(x, \lambda) \|, \qquad \| \zeta^q_{ij}(x, \lambda) \|$$

be respectively the matrices of the corresponding sets ζ_i. We now suppose that these solutions satisfy the initial conditions

$$\left\| \begin{matrix} p_{ij}(0, \lambda), & q_{ij}(0, \lambda) \\ \zeta^p_{ij}(0, \lambda), & \zeta^q_{ij}(0, \lambda) \end{matrix} \right\| = \left\| \begin{matrix} \delta^i_i & 0 \\ 0 & \delta^j_i \end{matrix} \right\|. \tag{10.2}$$

Now for a given λ any secondary extremal can be given the form

$$\eta_i(x) = b_j p_{ij}(x, \lambda) + c_j q_{ij}(x, \lambda) \tag{10.3}$$

where the b_j's and c_j's are constants. One sees that a necessary and sufficient condition that some solution be periodic and not identically zero is that

$$\left| \begin{matrix} p_{ij}(\omega, \lambda) - \delta^j_i, & q_{ij}(\omega, \lambda) \\ \zeta^p_{ij}(\omega, \lambda), & \zeta^q_{ij}(\omega, \lambda) - \delta^j_i \end{matrix} \right| = 0. \tag{10.4}$$

If the condition (10.4) holds for $\lambda = 0$, we term g *degenerate*. We shall assume throughout this section that g is non-degenerate.

Let F_λ be the family of those secondary extremals which join congruent points on the n-planes $x = 0$ and $x = \omega$ for the given λ. We seek a base for the family F_λ. We shall restrict ourselves to values of λ near 0. The conditions that (η) in (10.3) define an extremal of F_λ are that

$$b_j p_{ij}(\omega, \lambda) + c_j q_{ij}(\omega, \lambda) - b_i = 0. \tag{10.5}$$

The matrix of the coefficients of the constants (b) and (c) is

$$\| p_{ij}(\omega, \lambda) - \delta^j_i, \qquad q_{ij}(\omega, \lambda) \|. \tag{10.6}$$

By virtue of our assumption that g is non-degenerate this matrix will be of rank n for $\lambda = 0$, and hence of rank n for λ sufficiently near 0.

For λ sufficiently near 0 all solutions (b, c) of (10.5) will be linearly dependent on n particular independent solutions of (10.5), and these solutions can be so chosen as to vary continuously with λ. By virtue of (10.3) these n particular solutions of (10.5) will define n particular independent solutions of the Jacobi equations upon which all solutions in the family F_λ will be dependent. We represent these solutions by the columns of the matrix

$$\| z_{ij}(x, \lambda) \| \tag{10.7}$$

and let

$$\| \zeta^z_{ij}(x, \lambda) \| \tag{10.7)'$$

be the matrix of the corresponding sets ζ_i.

Members of the family F_λ can be represented in the form

$$\eta_i = z_{ik}(x, \lambda)w_k, \qquad \zeta_i = \zeta^z_{ih}(x, \lambda)w_h \qquad (i, h, k = 1, \cdots, n), \tag{10.8}$$

where the w_j's are constants. If the second variation $I(\eta, \lambda)$ be taken along the curve (10.8), we find that

$$I(\eta, \lambda) = \Big[\eta_i\zeta_i\Big]_0^\omega. \tag{10.9}$$

If one uses (10.8), the right member of (10.9) reduces to a quadratic form

$$a_{hk}(\lambda)w_hw_k = D(w, \lambda)$$

in which

$$a_{hk}(\lambda) = z_{ik}(0, \lambda)\Big[\zeta^z_{ih}(x, \lambda)\Big]_0^\omega \qquad (h, k = 1, \cdots, n). \tag{10.10}$$

We shall term $D(w, \lambda)$ the *general difference form* corresponding to the segment $(0, \omega)$ of the x axis. It is defined only for λ near 0. We shall establish three properties of this form.

I. *The form $D(w, \lambda)$ is symmetric.*

To see this recall that the hth and kth columns of the base (10.7) satisfy the conditions

$$z_{ih}\zeta^z_{ik} - z_{ik}\zeta^z_{ih} \equiv \text{constant}$$

identically in x for each λ. Upon successively substituting $x = 0$ and $x = \omega$ in this identity one finds that $a_{hk} = a_{kh}$ as required.

For a particular value of λ, say λ^0, it may be possible to set up a special base

$$\| z^0_{ij}(x) \|, \qquad \| \zeta^0_{ij}(x) \|, \tag{10.11}$$

for the family of secondary extremals F_{λ^0} using some special definition not applicable for all values of λ near 0. For such a special base the form

$$D^0(w) = a^0_{\mu\nu}\, w_\mu w_\nu \qquad (\mu, \nu = 1, \cdots, n),$$

in which

$$a^0_{\mu\nu} = z^0_{i\mu}(0)\Big[\zeta^0_{i\nu}(x)\Big]_0^\omega,$$

will be called the corresponding *special difference form.*

We shall now prove the following.

II. *For $\lambda = \lambda^0$ the index of the general difference form $D(w, \lambda^0)$ equals the index of any special difference form $D^0(w)$ set up for $\lambda = \lambda^0$ corresponding to a special choice of base for the family F_{λ^0}.*

Between our bases we necessarily have a relation for $\lambda = \lambda^0$ of the form,

$$z_{ik} \equiv z^0_{i\mu} c_{\mu k} \qquad (i, \mu, k = 1, \cdots, n),$$

$$\zeta^z_{ih} \equiv \zeta^0_{i\nu} c_{\nu h} \qquad (\nu, h = 1, \cdots, n),$$

where $\| c_{\mu k} \|$ is a non-singular n-square matrix of constants. If we make use of the definitions of a_{hk} and $a^0_{\mu\nu}$ we find that

$$a_{hk}(\lambda^0) = c_{\nu h} a^0_{\mu\nu} c_{\mu k}.$$

According to the theory of quadratic forms the indices of the forms $D(w, \lambda^0)$ and $D^0(w)$ must then have the same values, and II is proved.

III. *The nullity of the general difference form $D(w, 0)$, evaluated for $\lambda = 0$, equals the index of $x = \omega$ as a conjugate point of $x = 0$.*

To establish this fact we first note that

$$| \zeta^z_{ih}(\omega, 0) - \zeta^z_{ih}(0, 0) | \neq 0.$$

Otherwise one could readily obtain a periodic solution of the Jacobi equations not $(\eta) \equiv (0)$. If we now turn to the definition of $a_{hk}(0)$ in (10.10), we see that the nullity of $| a_{hk}(0) |$ equals the nullity of $| z_{ik}(0, 0) |$. The latter nullity is seen to be equal to the number of independent solutions of the Jacobi equations which vanish at $x = 0$ and $x = \omega$, that is, the index of $x = \omega$ as a conjugate point of $x = 0$.

Statement III is thereby established.

We shall term the index of $D(w, 0)$ the order of concavity of the segment $(0, \omega)$ of the x axis.

The justification of this definition will appear later. In it we have associated the form $D(w, 0)$ with a particular segment $(0, \omega)$ of the x axis. This is necessary. In fact if one should change the origin to some other point $x = x_0$, the index of the new form $D(w, 0)$ would not necessarily be the same as that of the old, as simple examples would show.

If $x = 0$ and $x = \omega$ are not conjugate for $\lambda = \lambda^0$ a *special difference form* $D^0(w)$ can be set up as follows. As the base (10.11) we can take a set of solutions of the Jacobi equations such that

$$z^0_{ij}(\omega) = z^0_{ij}(0) = \delta^j_i \qquad (i, j = 1, \cdots, n). \tag{10.12}$$

The corresponding "special difference form" $D^0(w)$ then reduces to the form

$$D^0(w) = w_i w_j \Big[\zeta^0_{ij}(x) \Big]_0^\omega. \tag{10.13}$$

We shall use this form in the next section.

The index of a periodic extremal

11. We continue with our study of a non-degenerate periodic extremal along which the Legendre S-condition holds. We set up the special index form $Q(z, \lambda)$ of §1 corresponding to end conditions of the form (9.0).

We find that

$$Q(z, \lambda) = \int_0^\omega [2\Omega(\eta, \eta') - \lambda\eta_i\eta_i]dx \tag{11.1}$$

where (η) is the broken secondary extremal "determined" by (z).

We shall now prove the following theorem.

THEOREM 11.1. *If g is non-degenerate, its index will equal the number of conjugate points of $x = 0$ on the interval $0 < x \leqq \omega$, plus the order of concavity of the segment $(0, \omega)$ of the x axis.* (Morse [7, 17].)

We distinguish between two cases.

Case I. The points $x = 0$ and $x = \omega$ are not conjugate. In this case the special difference form $D^0(w)$ can be set up for $\lambda = 0$ as at the end of §10. By virtue of §10, II, the order of concavity of $(0, \omega)$ will then equal the index of $D^0(w)$.

We shall base our proof of the theorem under Case I upon Lemma 7.5. To apply this lemma it will be convenient to denote the first n of the variables (z) in $Q(z, \lambda)$ by $(w_1, \cdots, w_n)$ and the remaining $\delta - n$ variables (z) by $(v_1, \cdots, v_{\delta - n})$. We then write $Q(z, \lambda)$ as a form

$$Q(z, \lambda) = L^\lambda(w, v).$$

We note that the form $L^0(0, v)$ is non-singular since $L^0(0, v)$ is the special index form associated with the fixed end point problem, and since the end points of g are not conjugate. According to Lemma 7.5 the index of $Q(z, 0)$ will equal the index of $L^0(0, v)$ plus the index of a form $H(w)$ obtained from $L^0(w, v)$ by eliminating the variables (v) by means of the conditions

$$\frac{\partial L^0(w, v)}{\partial v_j} = 0 \qquad (j = 1, \cdots, \delta - n). \tag{11.2}$$

To interpret the conditions (11.2) we turn to (11.1). In (11.1), (η) and (ζ) must be taken on the broken secondary extremal E determined by $(z) = (w, v)$. With the aid of (11.1) one sees that the conditions (11.2) reduce to a set of n conditions of the form

$$\Big[\zeta_i(x)\Big]_{a_q^+}^{a_q^-} = 0, \qquad (i = 1, \cdots, n), \tag{11.3}$$

one set at each corner $x = a_q$ of E. The conditions (11.3) and hence (11.2) imply the absence of corners on E.

We wish to determine the index of $H(w)$. Now subject to (11.2), $H(w) = Q(z, 0)$ by definition of $H(w)$. From (11.3) we see in addition that subject to (11.2)

$$H(w) = Q(z, 0) = \Big[\zeta_i(x)\eta_i(x)\Big]_0^\omega. \tag{11.4}$$

Here (η) and (ζ) are on the secondary extremal determined by (z). But by virtue of (11.2), (z) is determined by (w), so that (η) and (ζ) in (11.4) must be on the secondary extremal E_w which joins the points

$$x = 0, \qquad \eta_i = w_i,$$

$$x = \omega, \qquad \eta_i = w_i.$$

But the functions (η), (ζ) on E_w can be represented in terms of the special base defined at the end of §10 as follows:

$$\begin{aligned} \eta_i &= z^0_{ik}(x)w_k, \\ \zeta_i &= \zeta^0_{ih}(x)w_h \qquad (i, h, k = 1, \cdots, n). \end{aligned} \tag{11.5}$$

We see then that $H(w)$ in (11.4) reduces to the difference form

$$H(w) = w_h w_k \Big[\zeta^0_{hk}(x) \Big]_0^\omega = D^0(w)$$

of (10.13). Thus the index of $H(w)$ equals that of $D^0(w)$.

According to Lemma 7.5 the index of $Q(z, 0)$ equals the index of $L^0(0, v)$ plus the index of $H(w)$. But the index of $L^0(0, v)$ is the number of conjugate points of $x = 0$ on the interval $0 < x \leqq \omega$, and the index of $H(w)$ is the index of $D^0(w)$, that is, the order of concavity of the segment $(0, \omega)$. The theorem is accordingly proved in Case I.

Case II. The point $x = \omega$ is a conjugate point of $x = 0$ of index ρ $(\lambda = 0)$. This case can be treated as a limiting case of the preceding.

For $\lambda \neq 0$ but sufficiently near 0, $x = \omega$ will not be conjugate to $x = 0$, for otherwise the characteristic roots in the fixed end point problem would not be isolated. Let σ be the number of conjugate points of $x = 0$ preceding $x = \omega$ for $\lambda = 0$. For $\lambda < 0$, but sufficiently near 0, there will be σ conjugate points of $x = 0$ preceding $x = \omega$, while for $\lambda > 0$ there will be $\sigma + \rho$ such conjugate points. This follows from the fact that the number of conjugate points of $x = 0$ on the interval $0 < x < \omega$ for a given λ equals the number of characteristic roots less than λ in the fixed end point problem.

Let u^-, u^0, and u^+ be respectively the indices of the general difference form $D(w, \lambda)$ of §10 for $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$, with λ near 0. For $\lambda > 0$ but sufficiently near zero, g regarded as a periodic extremal of J^λ comes under Case I. For Case I the theorem is already established. Hence if τ is the index of $Q(z, \lambda)$ we have

$$\tau = \sigma + \rho + u^+. \tag{11.6}$$

Similarly for $\lambda < 0$ (τ is unchanged)

$$\tau = \sigma + u^-.$$

Hence

$$\rho = u^- - u^+. \tag{11.7}$$

But according to III in §10, the nullity of $D(w, 0)$ is ρ, while (11.7) tells us that the index of $D(w, \lambda)$ decreases also by ρ as λ increases through 0. It follows from the theory of characteristic roots of quadratic forms that

$$u^0 = u^+.$$

Hence (11.6) gives the result

$$\tau = \sigma + \rho + u^0.$$

The theorem is thereby proved in Case II.

We shall make use of Theorem 11.1 to obtain sufficient conditions for a minimum. It will be illuminating however first to note the following special necessary conditions. If g affords a minimum to J, it is necessary that the form $Q(z, 0)$ have the index zero. If then g is non-degenerate and affords a minimum to J, it follows from Theorem 11.1 that $x = 0$ *cannot be conjugate* to $x = \omega$ and that the order of concavity of $(0, \omega)$ must be zero.

The way is thus prepared for the following corollary of Theorem 11.1.

Corollary. *In order that a non-degenerate periodic extremal g afford a proper, strong, relative minimum to J, it is sufficient that the Legendre and Weierstrass S-conditions hold along g, that there be no conjugate point of $x = 0$ on the interval $0 < x \leqq \omega$, and that the order of concavity of the interval $(0, \omega)$ be zero.*

The case $n = 1$. In this case we can obtain a very explicit determination of the order of concavity of the segment $(0, \omega)$. For $n = 1$ there is but one variable y_i or p_i so that subscripts can be dropped.

There are two cases according as $x = 0$ is or is not conjugate to $x = \omega$.

Case I. Suppose first that $x = 0$ is not conjugate to $x = \omega$. The special base $\| z^0_{ij}(x) \|$ defined at the end of §10 reduces here to a single solution $\eta(x)$ such that

$$\eta(\omega) = \eta(0) = 1.$$

If $\zeta(x)$ is the corresponding function $\Omega_{\eta'}$, the special difference form (10.13) reduces to the form

$$(\zeta(\omega) - \zeta(0))w_1^2.$$

But

$$\zeta(\omega) - \zeta(0) = f_{pp}(\eta'(\omega) - \eta'(0))$$

where f_{pp} is evaluated at $x = 0$ on g. The order of concavity of $(0, \omega)$ is thus 1 or 0 according as

$$\eta'(\omega) < \eta'(0)$$

or

$$\eta'(\omega) > \eta'(0).$$

In the first case we say that the segment $(0, \omega)$ is relatively *concave*, in the second relatively *convex* (Morse [3] p. 239).

Case II. There remains the special case in which $x = 0$ is conjugate to $x = \omega$. According to III, §10, the nullity of $D(w, 0)$ will then be 1. Hence $D(w, 0) \equiv 0$ and the order of concavity is zero.

The index of a non-degenerate periodic extremal g in the plane can accordingly be evaluated as follows: $n = 1$.

(A). *Let m be the number of conjugate points of $x = 0$ on the interval $0 < x < \omega$. If $x = 0$ is not conjugate to $x = \omega$, the index of g is m or $m + 1$ according as $(0, \omega)$ is relatively convex or concave. If $x = 0$ is conjugate to $x = \omega$, the index of g is $m + 1$.*

We shall use the preceding to establish a result of importance in Ch. IX.

(B). *Let g be a non-degenerate periodic extremal on which a point x is never conjugate to the point $x + \omega$, and on which there are m conjugate points of the point $x = 0$ on the interval $0 < x < \omega$ with $m > 0$. The index of g will then be m or $m + 1$ according as m is odd or even.*

Since the point $x = 0$ is not conjugate to the point $x = \omega$ there exists an secondary extremal E on which

$$\eta(0) = \eta(\omega) = 1.$$

Let E' be a secondary extremal obtained from E by replacing each point (x, η) on E by the point $(x + \omega, \eta)$. We see that E and E' intersect at $(\omega, 1)$. Moreover E and E' will not be tangent at $(\omega, 1)$, since E would then represent a periodic extremal contrary to the hypothesis that g is non-degenerate. As x increases through ω, E thus crosses E' at $(\omega, 1)$. We see that the segment $(0, \omega)$ is relatively concave, or relatively convex, according as E enters or does not enter the region between E' and the x axis when E crosses E' at the point $(\omega, 1)$, with increasing x.

To prove the theorem we have merely to show that the segment $(0, \omega)$ is relatively convex if m is odd and relatively concave if m is even.

We consider the case where m is odd, say $m = 2r - 1$ with $r > 0$. In this case we shall prove that as x increases through ω, the extremal E cannot enter the region between E' and the x axis at the point $(\omega, 1)$.

If we use the Sturm Separation Theorem, and compare E with a secondary extremal which vanishes at $x = 0$, but on which η is not identically null, we see that on E, η must vanish $2r$ times on the interval $0 < x < \omega$.

Let $x = a$ be the first zero of η on E following $x = 0$. The first conjugate point of $x = 0$, following $x = 0$, must follow $x = a$, as the Sturm Separation Theorem shows. Hence the first conjugate point of $x = \omega$ following $x = \omega$ must follow $x = a + \omega$. By virtue of the Sturm Separation Theorem, E cannot then intersect E' on the interval

$$\omega < x \leqq a + \omega. \tag{11.8}$$

Moreover E cannot intersect the x axis on the interval (11.8). For that would mean that the point $x = a$ had $2r$ conjugate points on the interval

$$a < x \leqq a + \omega. \tag{11.9}$$

But the point $x = 0$ has $2r - 1$ conjugate points on the interval $0 < x < \omega$ and upon continuously varying a from 0 to its given value, the number of conjugate points on the interval (11.9) would remain $2r - 1$ since no point x is conjugate to the corresponding point $x + \omega$.

Thus E can intersect neither E' nor the x axis on the interval (11.8). It follows that E cannot enter the region between E' and the x axis at the point $(\omega, 1)$. Hence if m is odd the segment $(0, \omega)$ is convex and the index is m.

The proof of the lemma in the case m is even is similar.

For the general theory of the minimizing periodic extremal prior to the work of the author, the reader is referred first to Hadamard [1], p. 432. The Poincaré necessary condition that there be no pair of conjugate points on a minimizing periodic extremal is here derived together with other conditions bearing on a minimum in the plane. Carathéodory [2] has considered periodic extremals in n-space. Among other results he has shown that the Poincaré necessary condition does not hold. Hedlund [1] has shown that the Poincaré condition does not hold even for surfaces in the non-orientable case. Further references to papers on periodic extremals will be given in Ch. IX in connection with the theory in the large.

CHAPTER IV

SELF-ADJOINT SYSTEMS†

That the calculus of variations had much to do with the theory of separation, comparison, and oscillation theorems was evident even in the papers of Sturm. Certain aspects of this fact have been strikingly brought out by Hilbert and Courant [1]. But the nature of the results so far obtained calls for the setting up of a general framework and theory for such problems. The present chapter aims at such a theory. Although the results are confined to the case of a system of second-order self-adjoint differential equations with self-adjoint boundary conditions, yet they are capable of a much broader development. In particular one could consider such systems of second-order and linear differential equations as appear in the accessory differential equations of a Lagrange problem (Morse [16]). In particular by a reduction to a Lagrange problem the baffling case of the general even-order, self-adjoint, ordinary differential equation can be successfully treated. (Results not yet published.)

Starting with a new parametric representation of self-adjoint boundary conditions, comparison theorems are classified in a general way and new numerical invariants are introduced. A mode of proof of the existence of characteristic roots is developed which for the case at hand is more powerful than any hitherto developed. In particular one may recall that the methods of integral equations depend in general upon the fact that the parameter enters linearly and analytically. Such restrictions are unnecessary here. Missing oscillation theorems for general boundary conditions are here obtained. Cf. Hickson [1]. Finally the theory of boundary problems, self-adjoint at one end point, is shown to be identical with the theory of focal points, thus giving this class of problems a geometric setting.

Since these Lectures were given, Dr. Kuen-Sen Hu [1] has generalized the results previously published by the author, Morse [10, 16], to a form of the Bolza problem with somewhat less restrictive hypotheses. In the present chapter we make use of a classification of separation, comparison, and oscillation theorems which enables us to go deeper into the questions involved. The generalization of our theorems to the Lagrange problem under suitable normalcy conditions is obvious.

Among the earlier papers one may refer to Bliss [9], Plancherel [1], Richardson [1].

Self-adjoint differential equations

1. Consider a system of n differential equations of the form

$$L_i(\eta) = A_{ij}\eta_j'' + B_{ij}\eta_j' + C_{ij}\eta_j = 0 \qquad (i, j = 1, \cdots, n) \tag{1.0}$$

†This chapter can be omitted by the reader interested chiefly in the theory in the large.

where x is the independent variable and A_{ij}, B_{ij} and C_{ij} are continuous functions of x on the interval $a^1 \leqq x \leqq a^2$. If A_{ij} is of class C^1, these differential equations can be written in infinitely many ways in the form

$$L_i(\eta) = \frac{d}{dx}(a_{ij}\eta'_j + b_{ij}\eta_j) - (c_{ij}\eta'_j + d_{ij}\eta_j) = 0 \tag{1.1}$$

where a_{ij} and b_{ij} are of class C^1 and c_{ij} and d_{ij} of class C^0 in x. The system $L_i(\eta)$ will be unchanged as differential conditions if we replace d_{ij}, c_{ij}, and b_{ij} respectively by

$$\begin{aligned} &d_{ij}(x) + g'_{ij}(x), \\ &c_{ij}(x) + g_{ij}(x), \\ &b_{ij}(x) + g_{ij}(x), \end{aligned} \tag{1.2}$$

where $g_{ij}(x)$ is an arbitrary function of x of class C^1. We term such changes *admissible modifications* of (1.1). In particular we can use this arbitrariness of the coefficients to make $d_{ij}(x)$ an arbitrary set of symmetric elements

$$d_{ij}(x) = d_{ji}(x) \tag{1.3}$$

of class C^1. We can then still add an arbitrary constant to c_{ij} and b_{ij}.

We shall now use a definition of self-adjointness which will not require the assumption of further differentiability of the coefficients in (1.1). To that end let

$$M(u, v, u', v') = \alpha_{ij}u_iv_j + \beta_{ij}u_iv'_j - \gamma_{ij}u'_iv_j \qquad (i, j = 1, \cdots, n) \tag{1.4}$$

be a bilinear form in which the coefficients α_{ij}, β_{ij}, and γ_{ij} are continuously differentiable in x. We shall say that the system (1.1) is *self-adjoint* if there exists a bilinear form M such that the condition (Davis, D. R., [1])

$$u_iL_i(v) - v_iL_i(u) = \frac{d}{dx}M(u, v, u', v') \tag{1.5}$$

when expanded is a formal identity in the variables (u, v, u', v', u'', v'') and x. We shall prove the following lemma.

LEMMA 1.1. *In case the system* (1.1) *is self-adjoint, then after an admissible modification, equations* (1.1) *will assume a form in which*

$$a_{ij} \equiv a_{ji}, \qquad c_{ij} \equiv b_{ji}, \qquad d_{ij} \equiv d_{ji}, \tag{1.6}$$

where these functions are of class C^1 in x.

After a suitable modification of (1.1) we will have $d_{ij} = d_{ji}$ where d_{ij} is of class

C^1 in x. Upon then equating coefficients of corresponding terms in (1.5) we find that

$$
\begin{aligned}
\beta_{ij} &\equiv \gamma_{ij}, \\
a_{ij} &\equiv \beta_{ij}, \\
a_{ij} &\equiv \beta_{ji}, \\
b_{ij} - c_{ij} &\equiv \alpha_{ij}, \\
c_{ji} - b_{ji} &\equiv \alpha_{ji}, \\
b'_{ij} - b'_{ji} &\equiv \alpha'_{ij}.
\end{aligned} \tag{1.7}
$$

From these conditions we see that

$$
\begin{aligned}
a_{ij} &\equiv a_{ji}, \\
\alpha_{ij} &\equiv -\alpha_{ji}, \\
b_{ij} - b_{ji} &= \alpha_{ij} + e_{ij},
\end{aligned} \tag{1.8}
$$

where e_{ij} is a constant of integration. From (1.8) we find that $e_{ii} = 0$. For $i > j$ we now add a constant to b_{ij} so chosen as to make $e_{ij} = 0$ in (1.8), adding the same constant to c_{ij}, as is admissible. From (1.8) we see that $e_{ji} = -e_{ij}$, so that $e_{ij} = 0$ without exception. The fourth and fifth conditions in (1.7) taken with (1.8) now show that

$$
c_{ij} \equiv b_{ji}. \tag{1.9}
$$

We have chosen d_{ij} so as to be of class C^1 while a_{ij} and b_{ij} were so given. The modified coefficients b_{ij} will still be of class C^1, as will c_{ij}, by virtue of (1.9). The proof of the lemma is now complete.

We are thus led to the following theorem.

THEOREM 1.1. *A necessary and sufficient condition that the equations* (1.1) *be self-adjoint is that after a suitable admissible modification, equations* (1.1) *take the form*

$$
L_i(\eta) = \frac{d}{dx}(R_{ij}\eta'_j + Q_{ij}\eta_j) - (Q_{ji}\eta'_j + P_{ij}\eta_j) = 0 \tag{1.10}
$$

where P_{ij}, Q_{ij}, and R_{ij} are of class C^1 in x and

$$
R_{ij}(x) \equiv R_{ji}(x), \qquad P_{ij}(x) \equiv P_{ji}(x).
$$

That the condition of the theorem is necessary has already been proved. That it is sufficient is readily seen upon taking the bilinear form M as the form

$$
u_i(R_{ij}v'_j + Q_{ij}v_j) - v_i(R_{ij}u'_j + Q_{ij}u_j).
$$

Thus in case the equations (1.1) are self-adjoint, they are the Euler equations of the integral

$$2\int_{a^1}^{a^2} \Omega(\eta, \eta')dx$$

where

$$2\Omega = R_{ij}\eta'_i\eta'_j + 2Q_{ij}\eta'_i\eta_j + P_{ij}\eta_i\eta_j. \tag{1.11}$$

We shall term Ω a *differential form* corresponding to the equations (1.1).

A representation of self-adjoint boundary conditions

2. Before defining self-adjoint boundary conditions it will be convenient to define *adjointness relative to a bilinear form.*

Let $P(u, v)$ be a bilinear form in m variables (u) and m variables (v). Suppose the matrix of coefficients in $P(u, v)$ is of rank m. Let there be given p homogeneous independent linear forms

$$U_1, \cdots, U_p \qquad (0 < p < m) \tag{2.0}$$

in the variables (u) together with $m - p$ homogeneous independent linear forms

$$V_1, \cdots, V_{m-p} \tag{2.1}$$

in the variables (v). The conditions

$$U_i = 0 \qquad (i = 1, \cdots, p) \tag{2.2}$$

will be said to be *adjoint* to the conditions

$$V_j = 0 \qquad (j = 1, \cdots, m - p) \tag{2.3}$$

relative to the form $P(u, v)$ if $P(u, v)$ vanishes whenever its variables are subjected to the conditions (2.2) and (2.3).

If the conditions (2.2) are given, a corresponding set of adjoint conditions can be obtained as follows (Bôcher [2]). To the forms (2.0) one adjoins $m - p$ other forms $U_{p+1}, \cdots, U_m$ of such a nature that the forms

$$U_1, \cdots, U_m \tag{2.4}$$

are independent. According to the theory of bilinear forms there will then exist m independent homogeneous linear forms V_i^* in the variables (v) such that

$$P(u, v) \equiv U_1V_m^* + \cdots + U_mV_1^*. \tag{2.5}$$

The conditions

$$V_j^* = 0 \qquad (j = 1, \cdots, m - p) \tag{2.6}$$

are clearly adjoint to the conditions (2.2).

Any other conditions (2.3) *adjoint to the conditions* (2.2) *will be shown to be equivalent to the conditions* (2.6) *in the sense that a set* (v) *which satisfies* (2.3) *will satisfy* (2.6) *and vice-versa.*

To prove this suppose the conditions (2.3) adjoint to (2.2), and that (v) satisfies (2.3). Let V_k^* be one of the forms in (2.6). Choose a set (u) such that each of the m forms U_i in (2.4) is null except the one which multiplies V_k^* in (2.5). For this choice of (u) and (v) the form (2.5) must vanish according to our definition of adjoint conditions. We conclude that $V_k^* = 0$ for our choice of (v). We have thereby proved that a set (v) which satisfies (2.3) also satisfies (2.6).

Conversely it now follows from the fact just proved and the fact that the forms (2.3) and (2.6) are respectively independent, that a set (v) which satisfies (2.6) satisfies (2.3).

It will be convenient to represent the conditions (2.2) and (2.3) by means of linear conditions involving auxiliary parameters. Such sets of conditions will be regarded as adjoint if as conditions on (u) and (v) they are respectively equivalent to adjoint conditions of the forms (2.2) and (2.3).

A set of conditions will be termed *self-adjoint* if equivalent to its adjoint system.

We return now to a set of self-adjoint differential equations of the form

$$L_i(\eta) = \frac{d}{dx}\Omega_{\eta_i'} - \Omega_{\eta_i} = 0 \qquad (i = 1, \cdots, n) \tag{2.7}$$

where Ω is given by (1.11). We shall assume that the system (2.7) is positive regular, that is, that

$$R_{ij}(x)w_iw_j > 0 \qquad (i, j = 1, \cdots, n) \tag{2.8}$$

for any set of constants $(w) \neq (0)$, and for x on (a^1, a^2).

As previously, we set

$$\zeta_i = \Omega_{\eta_i'}(\eta, \eta'), \tag{2.9}$$

regarding this as a transformation from the variables (η, η') to variables (η, ζ). We shall also use variables $(\bar\eta, \bar\eta')$ and a corresponding set

$$\bar\zeta_i = \Omega_{\eta_i'}(\bar\eta, \bar\eta'). \tag{2.10}$$

Subject to (2.9) and (2.10) the Green's formula takes the form

$$\int_{a^1}^{a^2} (\eta_i L_i(\bar\eta) - L_i(\eta)\bar\eta_i)dx = \Big[\eta_i\bar\zeta_i - \zeta_i\bar\eta_i\Big]_{a^1}^{a^2}. \tag{2.11}$$

We set

$$\Big[\eta_i\bar\zeta_i - \zeta_i\bar\eta_i\Big]_{a^1}^{a^2} = P(\eta, \zeta; \bar\eta, \bar\zeta) \tag{2.12}$$

regarding this as a bilinear form in the two sets of $4n$ variables

$$(\eta_i^s, \zeta_i^s) \qquad (\bar\eta_i^s, \bar\zeta_i^s) \qquad (s = 1, 2; i = 1, \cdots, n).$$

To define our boundary conditions we let η and ζ be matrices each consisting of one column, and containing respectively the elements

$$\eta^1_1, \cdots, \eta^1_n, \quad \eta^2_1, \cdots, \quad \eta^2_n, \tag{2.13}$$

$$\zeta^1_1, \cdots, \zeta^1_n, -\zeta^2_1, \cdots, -\zeta^2_n. \tag{2.14}$$

Let p and q be matrices of $2n$ columns and ρ rows, $0 < \rho < 4n$, such that the matrix $\| p, q \|$ is of rank ρ. *The general boundary problem will now be given the form*

$$\frac{d}{dx}\Omega_{\eta'_i} - \Omega_{\eta_i} = 0 \qquad (i = 1, \cdots, n),$$

$$p\eta = q\zeta. \tag{2.15}$$

Conditions (2.15) require that the variables of η and ζ satisfy ρ linear, homogeneous, independent equations. By the boundary conditions *adjoint* to (2.15) will be meant the conditions adjoint to (2.15) relative to the bilinear form $P(\eta, \zeta; \bar{\eta}, \bar{\zeta})$. The conditions adjoint to (2.15) then require that the variables of η and ζ satisfy $4n - \rho$ linear, homogeneous, independent equations. These adjoint conditions may be given parametrically as stated in the following lemma.

LEMMA 2.1. *The conditions adjoint to the conditions* (2.15) *can be represented in matrix notation in the form*

$$\bar{\eta} = q^*v, \qquad \bar{\zeta} = p^*v, \tag{2.16}$$

where v is a column of ρ parameters $(v_1, \cdots, v_\rho)$ and where p^* and q^* are the matrices conjugate to p and q.

The conditions (2.16) are equivalent to $4n - \rho$ independent linear relations among the elements of $\bar{\eta}$ and $\bar{\zeta}$, as one sees upon eliminating the parameters (v). To prove that the conditions (2.16) are adjoint to the conditions (2.15) we have merely to show that the form $P(\eta, \zeta; \bar{\eta}, \bar{\zeta}) = 0$, subject to (2.15) and (2.16). But we have the following matrix formula for P,—the addition of an asterisk to a matrix shall indicate the conjugate of the matrix,—

$$\| P(\eta, \zeta; \bar{\eta}, \bar{\zeta}) \| = \bar{\eta}^*\zeta - \bar{\zeta}^*\eta.$$

Subject to (2.16) we find that

$$\| P \| = v^*q\zeta - v^*p\eta,$$

and subject to (2.15) this is seen to be null.

The lemma is thereby proved.

We continue with the following lemma.

LEMMA 2.2. *In order that the conditions* (2.15) *be self-adjoint it is necessary and sufficient that $\rho = 2n$, and that the matrix pq^* be symmetric.*

It is clearly necessary that $\rho = 2n$.

If $\rho = 2n$, a necessary and sufficient condition that the system (2.15) be self-adjoint is that the matrix equation

$$p\bar{\eta} = q\bar{\zeta}$$

be satisfied by all sets (η, ζ) which are given by (2.16). This gives the condition

$$pq^*v = qp^*v. \tag{2.17}$$

Now (2.17) holds for every set (v) if and only if pq^* is symmetric, and the lemma is proved.

The following theorem gives a new and basic representation of self-adjoint boundary conditions. In it there appear just the coefficients which are arbitrary. It gives the second precise link between the theory of self-adjoint boundary conditions and the theory in the preceding chapters.

THEOREM 2.1. *Any set of self-adjoint boundary conditions can be given the form*

$$\eta - cu = 0, \tag{2.18}'$$

$$c^*\zeta - bu = 0, \tag{2.18}''$$

where u is a column of r parameters with $0 \leqq r \leqq 2n$, c a matrix of rank r of r columns and $2n$ rows, and b a symmetric matrix of r rows and columns. Conversely any set of conditions of this form is self-adjoint.

We shall first prove that any set of self-adjoint conditions of the form (2.15) can be given the form (2.18). In the conditions (2.15) suppose q has the rank r (possibly zero). Without loss of generality we can suppose the conditions (2.15) are replaced by an equivalent set in which the last $2n - r$ rows of q are null. According to Lemma 2.1 self-adjoint conditions of the form (2.15) are equivalent to conditions of the form

$$\eta = q^*v, \qquad \zeta = p^*v, \tag{2.19}$$

where v is a column of $2n$ parameters. But if the members of the second matrix equation in (2.19) are multiplied on the left by q and the relation $qp^* = pq^*$ used, (2.19) yields the conditions

$$\eta = q^*v, \qquad q\zeta = pq^*v. \tag{2.20}$$

Thus self-adjoint conditions (2.15) lead to conditions (2.20). But the conditions (2.20) lead back to the conditions (2.15) as one sees upon replacing q^*v by η in (2.20). Thus conditions (2.20) are equivalent to (2.15) if (2.15) is self-adjoint.

To reduce conditions (2.20) to the required form recall that pq^* is symmetric. Moreover its elements are seen to be null except for a matrix b of elements in its first r rows and columns. If we let c denote the matrix of elements in the first r columns of q^*, and take (u) as the first r of the parameters (v), (2.20) takes the form (2.18) as desired.

Conversely conditions of the form (2.18) are always self-adjoint. To prove this we note that the elimination of the parameters (u) will yield $2n$ linearly independent linear conditions on the elements of $\boldsymbol{\eta}$ and ζ. To complete the proof it will be sufficient to show that the bilinear form

$$(2.21) \qquad \| P(\eta, \zeta, \bar{\eta}, \bar{\zeta}) \| = \bar{\boldsymbol{\eta}}^* \zeta - \bar{\zeta}^* \boldsymbol{\eta}$$

is null, subject to (2.18) and to the corresponding conditions

$$(2.22) \qquad \bar{\boldsymbol{\eta}} - c\bar{u} = 0, \qquad c^*\bar{\zeta} - b\bar{u} = 0,$$

where $\bar{u}$ is a column of r parameters. Subject to these conditions the form (2.21) becomes

$$\bar{u}^*c^*\zeta - \bar{\zeta}^*cu,$$

and upon using (2.22) and (2.18) again, the form finally reduces to

$$\bar{u}^*bu - \bar{u}^*b^*u \equiv 0.$$

The proof of the theorem is now complete.

We shall now drop the matrix notation and represent our self-adjoint boundary conditions (2.18) in the form

$$(2.23)' \qquad \eta_i^s - c_{ih}^s u_h = 0 \qquad (s = 1, 2; i = 1, \cdots, n),$$

$$(2.23)'' \qquad c_{ih}^2 \zeta_i^2 - c_{ih}^1 \zeta_i^1 + b_{hk} u_k = 0 \qquad (h, k = 1, \cdots, r),$$

where $\| c_{ih}^s \|$ is a matrix of rank r and $\| b_{hk} \|$ is symmetric. We shall term (2.23)′ the *accessory end-plane* π_r in the space of the $2n$ variables η_i^s regarding the variables (u) as parameters. We include the 0-plane $\eta_i^s = 0$ as a special case, calling it the null end-plane. The symmetric quadratic form $b_{hk}u_hu_k$ will be called the *accessory end-form*. Its value will be regarded as a function of the point on π_r represented by (u).

We see that the general self-adjoint boundary problem with differential equations in the form

$$\frac{d}{dx} \Omega_{\eta_i'} - \Omega_{\eta_i} = 0 \qquad (i = 1, \cdots, n),$$

and boundary conditions in the form (2.23), is uniquely determined by giving an accessory end-plane π_r,

$$(2.24) \qquad \eta_i^s - c_{ih}^s u_h = 0 \qquad (s = 1, 2; h = 1, \cdots, r; 0 \leqq r \leqq 2n)$$

in which the matrix $\| c_{ih}^s \|$ is of rank r, an accessory end-form

$$(2.25) \qquad b_{hk}u_hu_k \qquad (h, k = 1, \cdots, r)$$

in which the coefficients are symmetric, and a differential form

$$(2.26) \qquad \Omega(\eta, \eta')$$

as previously described.

The accessory end-plane and end-form are geometric invariants in the following sense.

A necessary and sufficient condition that two sets of self-adjoint boundary conditions of the form (2.23) *be equivalent is that their accessory end-planes π_r consist of the same points $\boldsymbol{\eta}$ in the space of the $2n$ variables $\boldsymbol{\eta}$ and that their accessory end-forms be numerically equal for values of their parameters which determine the same point on π_r.*

If two sets of equivalent conditions (2.23)′ are given, I say that their accessory end-planes consist of the same points $\boldsymbol{\eta}$. In fact in the space of the $4n$ variables $\boldsymbol{\eta}$ and $\boldsymbol{\zeta}$ the given boundary conditions define a $2n$-plane π_{2n} obviously independent of the parametric representation of the conditions. I say that the accessory end-plane π_r is the orthogonal projection in the space $\boldsymbol{\eta}$, $\boldsymbol{\zeta}$ of π_{2n} on the coordinate $2n$-plane of the variables $\boldsymbol{\eta}$. This appears at once from the form of (2.23). Hence if the boundary conditions are equivalent, there can be but one end-plane π_r.

To turn to the accessory end-forms, suppose that we have given a set of conditions (2.23), and a second and equivalent set of conditions (2.23)′ with accessory end-form

$$\bar{b}_{hk}\bar{u}_h\bar{u}_k.$$

Let (u) be any set of r parameters. Corresponding to (u) there exists a unique point $\boldsymbol{\eta}$ on π_r which satisfies (2.23)′, and at least one set $\boldsymbol{\zeta}$ which then satisfies (2.23)″. Upon multiplying the hth condition in (2.23)″ by u_h and summing, using (2.23)′, we find that

$$\eta_i^1\zeta_i^1 - \eta_i^2\zeta_i^2 = b_{hk}u_hu_k. \tag{2.27}$$

On the other hand this same set $\boldsymbol{\eta}$, $\boldsymbol{\zeta}$ must satisfy the equivalent conditions (2.23)′ with a set $(\bar{u})$, and we must have

$$\eta_i^1\zeta_i^1 - \eta_i^2\zeta_i^2 = \bar{b}_{hk}\bar{u}_h\bar{u}_k.$$

Thus for sets (u) and $(\bar{u})$ which determine the same point $\boldsymbol{\eta}$ on π_r, we have

$$\bar{b}_{hk}\bar{u}_h\bar{u}_k = b_{hk}u_hu_k.$$

The conditions of the theorem are accordingly necessary.

To prove the conditions sufficient suppose that we have given a non-singular (in case $r > 0$) linear transformation

$$u_h = a_{hp}\bar{u}_p \qquad (h, p = 1, \cdots, r) \tag{2.28}$$

from parameters $(\bar{u})$ to parameters (u). Suppose also that an accessory end-plane π_r is represented in two ways,

$$\eta_i^s = c_{ih}^s u_h, \qquad \eta_i^s = \bar{c}_{ih}^s\bar{u}_h,$$

where

$$\bar{c}_{ip}^s = c_{ih}^s a_{hp} \qquad (i = 1, \cdots, n; h, p = 1, \cdots, r),$$

and that we have two accessory end-forms such that

$$\bar{b}_{pq}\bar{u}_p\bar{u}_q = b_{hk}u_hu_k$$

subject to (2.28), where

$$\bar{b}_{pq} = a_{hp}b_{hk}a_{kq} \qquad (h, k, p, q = 1, \cdots, r).$$

To prove the conditions of the theorem sufficient we have merely to prove that the conditions (2.23) are equivalent to the conditions

$$(2.29) \qquad \begin{aligned} \eta_i^s - \bar{c}_{ip}^s\bar{u}_p &= 0 \qquad (i = 1, \cdots, n; s = 1, 2), \\ \bar{c}_{ip}^1\zeta_i^1 - \bar{c}_{ip}^2\zeta_i^2 &= \bar{b}_{pq}\bar{u}_p \qquad (p, q = 1, \cdots, r). \end{aligned}$$

To prove this statement we first observe that the conditions (2.23) are equivalent to the conditions obtained by replacing u_k by $a_{kq}\bar{u}_q$, that is, to the conditions

$$(2.30) \qquad \begin{aligned} \eta_i^s - \bar{c}_{ip}^s\bar{u}_p &= 0, \\ c_{ih}^1\zeta_i^1 - c_{ih}^2\zeta_i^2 &= b_{hk}a_{kq}\bar{u}_q. \end{aligned}$$

But if the hth condition in (2.30) is multiplied by a_{hp} and the resulting equations summed with respect to h, we obtain the conditions

$$\bar{c}_{ip}^1\zeta_i^1 - \bar{c}_{ip}^2\zeta_i^2 = \bar{b}_{pq}\bar{u}_q.$$

Thus the satisfaction of the conditions (2.23) by variables η, ζ entails the satisfaction of the conditions (2.29) by the same variables. Similarly the conditions (2.29) lead to conditions (2.23). Hence the two sets of conditions are equivalent.

The proof of the theorem is now complete.

It follows from the preceding that we have three numerical invariants associated with each set of self-adjoint boundary conditions, namely, the dimension r of the accessory end-plane and the index and nullity of its accessory end-form. These numerical invariants together with similar invariants associated with two problems are fundamental in what follows.

Boundary problems involving a parameter

3. We shall consider a boundary value problem B involving a parameter σ in such a manner that for each value of σ there is defined a self-adjoint boundary problem of the sort already defined.

The differential form shall be

$$(3.1) \qquad 2\omega(\eta, \eta', \sigma) = P_{ij}(x, \sigma)\eta_i\eta_j + 2Q_{ij}(x, \sigma)\eta'_i\eta_j + R_{ij}(x, \sigma)\eta'_i\eta'_j \qquad (i, j = 1, \cdots, n)$$

and the accessory end form

$$(3.2) \qquad b_{hk}(\sigma)u_hu_k \qquad (h, k = 1, \cdots, r; 0 \leqq r \leqq 2n),$$

while the accessory end-plane shall be

$$\eta_i^s - c_{ih}^s u_h = 0 \qquad (s = 1, 2), \tag{3.3}$$

where c_{ih}^s is independent of σ.

We shall consider the functional

$$I(\eta, \sigma) = b_{hk}(\sigma)u_h u_k + \int_{a^1}^{a^2} 2\omega(\eta, \eta', \sigma)dx$$

evaluated for functions $\eta_i(x)$ of class D^1 subject to (3.3), and admit problems B satisfying the five following hypotheses.

A.1. For any real number σ and for x on the interval (a^1, a^2) the functions

$$P_{ij},\ Q_{ij},\ R_{ij},\ \frac{\partial R_{ij}}{\partial x},\ \frac{\partial Q_{ij}}{\partial x},\ b_{hk} \tag{3.4}$$

shall be continuous.

A.2. The matrix of elements c_{ih}^s shall be of rank r, and the matrix of elements b_{hk} symmetric.

A.3. For each value of σ and of x on (a^1, a^2) and set $(w) \neq (0)$,

$$R_{ij}(x, \sigma)w_i w_j > 0.$$

A.4. For $-\sigma$ sufficiently large the functional $I(\eta, \sigma)$ shall be positive definite.

A.5. The accessory form shall decrease monotonically as σ increases. It may in particular be independent of σ. For $(\eta) \neq (0)$ and x fixed, $\omega(\eta, \eta', \sigma)$ shall definitely decrease as σ increases.

By a *characteristic solution* of B is meant a set of functions $\eta_i(x)$ of class C^2 in x which with constants (u) and σ satisfies the boundary problem B, but which is not identically null. The corresponding constant σ is called a *characteristic root*. By the *index* of a characteristic root σ is meant the number of independent characteristic solutions corresponding to that root. In *counting* characteristic roots each root will always be taken a number of times equal to its index.

Corresponding to the functional

$$J^\sigma = \tfrac{1}{2}b_{hk}(\sigma)\alpha_h\alpha_k + \int_{a^1}^{a^2} \omega(y, y', \sigma)dx$$

subject to the end conditions

$$y_i^s - c_{ih}^s\alpha_h = 0,$$

for each value of σ we can set up a special index form $Q(z, \sigma)$, just as the special index form $Q(z, \lambda)$ was set up at the end of §1, Ch. III, for each value of λ. We shall then have

$$Q(z, \sigma) = b_{hk}(\sigma)u_h u_k + \int_{a^1}^{a^2} 2\omega(\eta, \eta', \sigma)dx,$$

where (η) is the secondary broken extremal E "determined" in B by the set (z) for the given σ. More explicitly if we set

$$(z) = (u_1, \cdots, u_r, z_1^1, \cdots, z_n^1, \cdots, z_1^p, \cdots, z_n^p),$$

the end points of E are given by the conditions

$$\eta_i^s - c_{ih}^s u_h = 0, \qquad x = a^s, \tag{3.5}$$

and the intermediate corners or vertices by the conditions

$$\eta_i(a_q) = z_i^q \qquad (q = 1, \cdots, p)$$

where $x = a_q$ is the qth "intermediate" n-plane.

Essentially as in §2, Ch. III, we can prove the following theorem.

THEOREM 3.1. *The form $Q(z, \sigma)$ is singular if and only if σ is a characteristic root. If σ is a characteristic root, the nullity of $Q(z, \sigma)$ equals the index of σ, and the index of $Q(z, \sigma)$ equals the number of characteristic roots less than σ.*

In reviewing the proof in Ch. III we call attention to the fact that the right member of (2.8) will here be replaced by the expression

$$[b_{hk}(\sigma') - b_{hk}(\sigma'')]u_h u_k + \int_{a^1}^{a^2} [2\omega(\eta, \eta', \sigma') - 2\omega(\eta, \eta', \sigma'')]dx.$$

Moreover for $\sigma'' < \sigma'$ and $(\eta) \not\equiv (0)$ this functional is negative, as follows from Hypothesis A.5. One continues as before. We recall that Hypothesis A.4 affirms that for $-\sigma$ sufficiently large $I(\eta, \sigma)$ is positive definite. This was proved in the earlier case, but is an hypothesis here.

As a corollary of the theorem we see that the number of characteristic roots less than a given constant is finite, and that accordingly the characteristic roots are isolated.

We shall arrange the characteristic roots in a sequence

$$\sigma_0 \leqq \sigma_1 \leqq \sigma_2 \leqq \sigma_3 \leqq \cdots \tag{3.6}$$

in which each root appears a number of times equal to its index. The number of roots may be either finite or infinite as examples will show.

We shall say that the problem B depends *continuously* on a parameter α for α near α_0 if the functions in (3.4) are continuous in x, σ, and α, and if the remaining Hypotheses A are satisfied for each value of α.

We shall prove the following theorem.

THEOREM 3.2. *If the kth characteristic root in the problem B exists for $\alpha = \alpha_0$, it exists and varies continuously with α for α sufficiently near α_0.*

For $\alpha = \alpha_0$ let σ' and σ'' be two constants respectively less and greater than σ_k, so chosen as to separate σ_k from the characteristic roots not equal to σ_k. Designate the roots equal to σ_k when $\alpha = \alpha_0$, including c_k, by

$$\sigma_h^0, \cdots, \sigma_{h+v}^0. \tag{3.7}$$

The forms $Q(z, \sigma')$ and $Q(z, \sigma'')$ set up for $\alpha = \alpha_0$ will be non-singular and possess indices respectively equal to h and $h + v + 1$.

If now the parameters α be continuously varied, the coefficients in these forms will vary continuously. For a sufficiently small variation of α they will remain non-singular and hence unchanged in index. After such a variation there will then still be $v + 1$ roots $\sigma_h, \cdots, \sigma_{h+v}$ between σ' and σ''. Since σ' and σ'' can be initially taken as near as we please to σ_k^0 we see that σ_k will vary continuously with α as stated.

The theorem is thereby proved.

Comparison of problems with different boundary conditions

4. We shall now compare two problems B and B_1 with a differential form $\omega(\eta, \eta', \sigma)$ in common. The accessory end-planes in B and B_1 will be denoted respectively by π_r and π_{r_1} where r and r_1 are the dimensions of these end-planes. Let π_r and π_{r_1} be respectively represented by means of parameters (u) and (u^1). Let the corresponding accessory end-forms be

$$b_{hk}(\sigma)u_hu_k \qquad (h, k = 1, \cdots, r) \tag{4.0}$$

and

$$b^1_{pq}(\sigma)u^1_pu^1_q \qquad (p, q = 1, \cdots, r_1). \tag{4.1}$$

If π_r and π_{r_1} are identical and identically represented with $(u) = (u^1)$, we shall call

$$d(u, \sigma) = [b^1_{hk}(\sigma) - b_{hk}(\sigma)]u_hu_k \tag{4.2}$$

the *difference form* corresponding to B_1 and B.

If on the other hand π_{r_1} is a section of π_r, and if

$$b^1_{pq}(\sigma)u^1_pu^1_q = b_{hk}(\sigma)u_hu_k$$

when (u^1) and (u) determine the same point on π_{r_1}, then B_1 will be called a *sub-problem* of B.

We shall now give three comparison theorems, one of each of the following types:

I. *A comparison of a problem with a sub-problem.*

II. *A comparison of two problems with a common accessory end-plane.*

III. *A comparison of two general problems.*

In all three cases we suppose the problems have a common differential form.

THEOREM 4.1. *Let there be given a problem B and sub-problem B_1 with accessory end-planes π_r and π_{r_1} respectively. If v and v_1 are respectively the numbers of characteristic roots in B and B_1 less than a given constant σ, then*

$$v - (r - r_1) \leqq v_1 \leqq v. \tag{4.3}$$

To prove the theorem we refer π_r to parameters (u) in such a manner that π_{r_1} is obtainable from π_r by setting the last $r - r_1$ of the parameters (u) equal to zero. For this choice of end parameters let $Q(z, \sigma)$ and $Q_1(z, \sigma)$ be the special index forms corresponding respectively to B and B_1, using the same intermediate n-planes. The form $Q_1(z, \sigma)$ can be obtained from $Q(z, \sigma)$ upon setting

$$z_{r_1+1} = \cdots = z_r = 0$$

and renumbering the remaining variables. According to Lemma 7.2 of Ch. III the index of $Q_1(z, \sigma)$ must then lie between v and $v - (r - r_1)$ inclusive.

The theorem is accordingly proved.

We shall prove the following corollary.

COROLLARY. *The number of roots in B on any open or closed, finite interval of the σ axis differs from the corresponding number for a sub-problem B_1 by at most $r - r_1$.*

Let v' and v'' be respectively the numbers of characteristic roots less than σ' and σ'' in B with $\sigma' < \sigma''$. Let v_1' and v_1'' be the corresponding numbers for B_1. By virtue of (4.3) there exist integers m' and m'' such that

$$v' = v_1' + m', \qquad 0 \leqq m' \leqq r - r_1,$$

$$v'' = v_1'' + m'', \qquad 0 \leqq m'' \leqq r - r_1,$$

so that

$$v'' - v' = v_1'' - v_1' + m'' - m', \qquad | m'' - m' | \leqq r - r_1.$$

But $v'' - v'$ and $v_1'' - v_1'$ are respectively the numbers of roots in B and B_1 on the interval

$$\sigma' \leqq \sigma < \sigma'', \tag{4.4}$$

so that the corollary is proved for intervals of the type (4.4).

Now corresponding to any finite interval whatsoever there exists a closely approximating interval of the form (4.4) containing the same roots of B and B_1. The corollary is accordingly true in general.

We term a problem with end conditions $\eta_i^s = 0$ a problem *with null end-plane* or null end points. Every problem B possesses a sub-problem with null end points. Of all sub-problems of B the problem with null end points possesses the minimum number of roots less than a constant σ^*. In a problem with r end parameters there will be at most r more roots less than σ^* than appear in the corresponding problem with null end points.

Our second theorem is the following:

THEOREM 4.2. *Suppose B_1 and B have a common accessory r-plane and common differential form. Let $d(u, \sigma)$ be a corresponding difference form (4.2), and let $N(\sigma)$ and $P(\sigma)$ be respectively the indices of $d(u, \sigma)$ and $-d(u, \sigma)$. If $v(\sigma)$ and $v_1(\sigma)$ are respectively the numbers of characteristic roots less than σ in B and B_1, then*

$$v(\sigma) - P(\sigma) \leqq v_1(\sigma) \leqq v(\sigma) + N(\sigma). \tag{4.5}$$

We suppose the end conditions in B and B_1 represented in terms of common parameters (u) and let $Q(z, \sigma)$ and $Q_1(z, \sigma)$ then be the corresponding special index forms set up with common intermediate n-planes. We have

$$Q_1(z, \sigma) - Q(z, \sigma) \equiv d(u, \sigma)$$

where the first r of the z's are given by (u). It then follows from Lemma 7.3 of Ch. III that (4.5) holds as stated.

Suppose the accessory end-planes π_r and π_{r_1} of the two problems B and B_1 intersect in a t-plane π_t. Let B_1^t and B^t be respectively the sub-problems of B_1 and B for which π_t is the accessory end-plane. Let $\delta(u, \sigma)$ be a corresponding difference form for B_1^t and B^t. The form $\delta(u, \sigma)$ will be called a *maximal difference form* for B_1 and B. We denote the index of $\delta(u, \sigma)$ by $N(\sigma)$ and that of $-\delta(u, \sigma)$ by $P(\sigma)$.

Our third theorem can now be stated as follows:

THEOREM 4.3. *Let there be given two problems B_1 and B with common differential form, and with accessory end-planes π_r and π_{r_1} intersecting in a t-plane π_t. Let $\delta(u, \sigma)$ be a corresponding maximal difference form with its indices $N(\sigma)$ and $P(\sigma)$. If $v(\sigma)$ and $v_1(\sigma)$ are respectively the numbers of roots of B and B_1 less than the constant σ, then*

$$v(\sigma) - P(\sigma) - r + t \leqq v_1(\sigma) \leqq v(\sigma) + N(\sigma) + r_1 - t. \tag{4.6}$$

Let $h_1(\sigma)$ and $h(\sigma)$ be respectively the numbers of characteristic roots less than σ in B_1^t and B^t. According to Theorem 4.2 there exists an integer $q(\sigma)$ such that

$$h_1(\sigma) = h(\sigma) + q(\sigma), \qquad -P(\sigma) \leqq q(\sigma) \leqq N(\sigma). \tag{4.7}$$

But according to Theorem 4.1 we have

$$\begin{aligned} v_1(\sigma) &= h_1(\sigma) + d_1, \qquad 0 \leqq d_1 \leqq r_1 - t, \\ v(\sigma) &= h(\sigma) + d, \qquad 0 \leqq d \leqq r - t. \end{aligned} \tag{4.8}$$

From (4.7) and (4.8) we find that

$$v_1(\sigma) - v(\sigma) = q(\sigma) + d_1 - d,$$

from which we see that

$$v(\sigma) + q(\sigma) - d \leqq v_1(\sigma) \leqq v(\sigma) + q(\sigma) + d_1.$$

The required inequalities now follow from the inequalities in (4.7) and (4.8) limiting q, d, and d_1.

Theorem 4.3 reduces to Theorem 4.1 when $r_1 = t$ and $P = N = 0$. It reduces to Theorem 4.2 when $r = r_1 = t$.

COROLLARY. *The number of characteristic roots on any finite interval of the σ axis differs by at most $2n$ for any two problems with common differential form and with end conditions which are independent of σ.*

To prove this corollary it will be sufficient to prove that the extreme members of the inequalities (4.6) differ by at most $2n$.

This difference is

$$N + P + r + r_1 - 2t \leqq r + r_1 - t \tag{4.9}$$

since $N + P \leqq t$. But since π_t is the intersection of π_r and π_{r_1} we must have

$$r + r_1 - 2n \leqq t$$

or

$$r + r_1 - t \leqq 2n.$$

The corollary now follows from (4.9).

A general oscillation theorem

5. Let $\Delta(x, \sigma)$ be an n-square determinant of elements $\eta_{ij}(x, \sigma)$ whose columns are solutions of the Jacobi equations such that

$$\eta_{ij}(a^1, \sigma) = 0, \qquad \eta'_{ij}(a^1, \sigma) = \delta_i^j \qquad (i, j = 1, \cdots, n).$$

Recall that $\Delta(x, \sigma)$ vanishes at each conjugate point $x = c$ of $x = a^1$ to an order equal to the index of $x = c$ as a conjugate point. A zero $x = c$ of $\Delta(x, \sigma)$ will be counted a number of times equal to its index.

We have already seen that $\Delta(x, \sigma)$ vanishes on the interval $a^1 < x < a^2$ a number of times exactly equal to the number of characteristic roots less than σ in the problem with null end points. This is a first oscillation theorem. But by virtue of Theorem 4.1 we can compare the number $v(\sigma)$ of characteristic roots less than σ in any problem B with the number less than σ in the corresponding null end point problem. We thereby obtain the following general oscillation theorem.

THEOREM 5.1. *If r is the dimension of the accessory end-plane of a problem B and $v(\sigma)$ is the number of characteristic roots in B less than σ, then $\Delta(x, \sigma)$ vanishes on the interval $a^1 < x < a^2$ at least $v(\sigma) - r$ times and at most $v(\sigma)$ times.*

The case $n = 1$ has been extensively treated by various mathematicians. See Ince [1], p. 247. In spite of this fact it is possible to use the preceding theorem to obtain narrower limits on the number of zeros of characteristic solutions than have been obtained before.

In case $n = 1$ we first note that the only possible values of r, the dimension of the accessory end-plane, are $r = 0, 1$ or 2. The determinant $\Delta(x, \sigma)$ reduces to a solution $\eta(x)$ of the Jacobi equations for which

$$\eta(a^1) = 0, \qquad \eta'(a^1) = 1.$$

Let σ^* be a characteristic root in a problem B. The root σ^* can be simple or double. We suppose σ^* represented by σ_p in case it is simple, and by $\sigma_p = \sigma_{p-1}$ in case it is double. In either case we understand that σ_p is the $(p + 1)$st root, counting roots according to their indices. Let τ represent the integer 1 or 0 according as $x = a^2$ is or is not a conjugate point of a^1. We shall prove that $\Delta(x, \sigma^*)$ vanishes at least μ times on the interval

$$a^1 < x < a^2, \tag{5.1}$$

where

$$\mu = p + 1 - r - \tau. \tag{5.2}$$

Let B^0 be the boundary problem with null end points. Let σ^0 be a constant which separates σ^* from any characteristic root in B or B^0 for which $\sigma > \sigma^*$. According to the preceding theorem $\Delta(x, \sigma^0)$ vanishes at least

$$v(\sigma^0) - r = p + 1 - r$$

times on the interval (5.1). Hence $\Delta(x, \sigma^*)$ vanishes at least $p + 1 - r - \tau$ times on (5.1) as one sees upon continuously decreasing σ from σ^0 to σ^*.

We can now establish the following theorem.

I. *In case $n = 1$, a characteristic solution η^* corresponding to a simple root σ_p vanishes at least $p - 1$ times and at most $p + 1$ times on* (5.1). *A characteristic solution η^* corresponding to a double root $\sigma_p = \sigma_{p-1}$ vanishes either $p - 1$ or p times on* (5.1).

Observe that η^* vanishes on (5.1) at least as many times as $\Delta(x, \sigma)$, that is at least $p + 1 - r - \tau$ times. If $r = 0$ or 1, or if $r = 2$ and $\tau = 0$, this is at least $p - 1$ times.

There remains the case $r = 2$, $\tau = 1$. In this case we note that η^* cannot vanish at a^1 and a^2, because we see then that the parameters (u) in the boundary conditions would be null, and hence $\zeta^1 = \zeta^2 = 0$ for η^*. We would then have $\eta^* \equiv 0$ which is impossible. Nor can η^* vanish at a^1 or a^2 alone, since the hypothesis that $\tau = 1$ would then make η^* vanish at both a^1 and a^2. The solution η^* must then vanish on (5.1) once more than $\Delta(x, \sigma^*)$, or at least $p - 1$ times as stated.

On the other hand $\Delta(x, \sigma^*)$ vanishes at most p times on (5.1) in case σ_p is a simple root, as follows from Theorem 5.1. Hence in case σ_p is a simple root η^* vanishes at most $p + 1$ times on (5.1). A similar use of the theorem in case $\sigma_p = \sigma_{p-1}$ is a double root shows that η^* vanishes at most p times on (5.1).

Thus statement I is proved.

Note that the spread between $p - 1$ and $p + 1$ is 3. Compare Ettlinger [1, 2] where the difference between the limits is 5.

Bôcher [2] has treated the periodic case at length. Ince [1] has summarized Bôcher's results in a theorem on p. 247, loc. cit. In this theorem the existence of infinitely many positive characteristic roots is affirmed. This result will follow from our general existence theorem to be established in a later section. The theorem of Bôcher also asserts that each characteristic solution vanishes an even number of times. This follows from the periodicity of the boundary conditions. The principal part of Bôcher's theorem is a special case of the following:

II. *In case* $n = r = 1$, *a characteristic solution corresponding to a simple root* σ_p *vanishes either* p *or* $p + 1$ *times on the interval* $a^1 \leqq x < a^2$, *while a characteristic solution corresponding to a double root* $\sigma_p = \sigma_{p-1}$ *vanishes exactly* p *times on* $a^1 \leqq x < a^2$.

To prove this statement recall that $\Delta(x, \sigma^*)$ and hence η^* will vanish at least $p + 1 - r - \tau$ times on (5.1). In case $r = 1$ and $\tau = 0$, this is at least p times. Hence in this case η^* vanishes at least p times on (5.1). In case $r = \tau = 1$, $\Delta(x, \sigma^*)$ vanishes at least $p - 1$ times on (5.1) and at least $p + 1$ times on $a^1 \leqq x \leqq a^2$. Hence in this case η^* vanishes at least p times on $a^1 \leqq x < a^2$.

The remaining facts in II follow from I in case η^* does not vanish at a^1. In case η^* vanishes at a^1 its zeros are those of $\Delta(x, \sigma^*)$. According to the relations between conjugate points and characteristic roots in the problem with null end points we see that in this case η^* vanishes exactly p times on (5.1) in case σ^* is a simple root, and exactly $p - 1$ times in case σ^* is a double root. Hence in case η^* vanishes at a^1 it vanishes $p + 1$ times on $a^1 \leqq x < a^2$ if σ^* is a simple root and p times if σ^* is a double root.

The proof of II is complete.

The existence of characteristic roots

6. We begin with the following lemma.

LEMMA 6.1. *If the x-coordinate* $x_k(\sigma)$ *of the kth conjugate point of* $x = a^1$ *exists for* $\sigma = \sigma^*$, *it exists and decreases as* σ *increases from* σ^* *neighboring* σ^*.

Let $x_k(\sigma^*) = a''$. Let the index form $Q(z, \sigma)$ be set up for the problem with null end points at $x = a^1$ and $x = a''$, and with σ near σ^*. The index plus the nullity of $Q(z, \sigma^*)$ will be at least k. Accordingly $Q(z, \sigma^*)$ will be non-positive on a k-plane π_k through the origin in the space (z). But

$$Q(z, \sigma) = \int_{a^1}^{a''} 2\omega(\eta, \eta', \sigma)dx,$$

where (η) represents the broken secondary extremal determined by (z).

Suppose now that σ is slightly larger than σ^*. Recall that $\omega(\eta,\eta',\sigma)$ is assumed to be a decreasing function of σ for $(\eta) \neq (0)$. We see then that for $\sigma > \sigma^*$, $Q(z, \sigma) < 0$ on π_k, and that accordingly the index of $Q(z, \sigma)$ will be at least k. Hence the kth conjugate point of $x = a^1$ must precede a'' for $\sigma > \sigma^*$, and the lemma is proved.

Let the segment of the x axis

$$a^1 \leqq x \leqq a^2$$

be denoted by γ. We state the following theorem.

THEOREM 6.1. *A necessary and sufficient condition that there exist an infinite number of characteristic roots in an admissible boundary problem B is that there exist a point P on γ with the following property. Corresponding to any segement $\bar{\gamma}$ of γ which gives a neighborhood of P on γ there exists an admissible curve λ which joins P to a point $Q \neq P$ on $\bar{\gamma}$, along which $(\eta) \not\equiv (0)$ and on which*

$$\int_\lambda \omega(\eta, \eta', \sigma)dx \leqq 0 \tag{6.1}$$

for all values of σ exceeding a sufficiently large positive constant σ_λ.

Recall that any two points on the x axis which are conjugate can be joined by a secondary extremal L along which $(\eta) \not\equiv (0)$ and (6.1) holds.

Suppose the condition of the theorem were not necessary. It would then follow from the preceding lemma that with each point P on γ there could be associated an open interval I which contained P in its interior (or on its boundary if P is an end point of γ) and which contained no conjugate point of P, no matter how large σ might be. The whole segment γ could then be covered by a finite set of such intervals I. But according to a separation theorem to be established in §8 there can be at most n conjugate points of $x = a^1$ on each open interval I, and hence at most a finite number N of conjugate points of $x = a^1$ on γ, where N is independent of σ.

But this is impossible. For we are assuming that there are infinitely many characteristic roots σ in the problem B, so that there will necessarily be infinitely many characteristic roots in the problem with null end points. For σ sufficiently large there must then be arbitrarily many conjugate points of $x = a^1$ on γ, in particular there must be more than N such conjugate points. The condition of the theorem is accordingly necessary.

We shall now prove the condition sufficient.

Suppose the condition of the theorem is satisfied by a point P. One sees that one at least of the two sides of P on γ must have the property that the condition of the theorem is satisfied by curves λ whose end points Q all lie on that side of P. Suppose the side preceding P has this property.

For σ sufficiently large, say $\sigma > \sigma^1$, the first conjugate point $x_1(\sigma)$ of a^1 following a^1 cannot follow P. For otherwise for $\sigma > \sigma^1$ there could be no conjugate point of P between P and $x = a^1$ as follows from our separation theorem, Theorem 8.3. In such a case (6.1) could not be satisfied for Q between $x = a^1$ and P.

As σ increases, $x_1(\sigma)$ will decrease in accordance with the preceding lemma. For $\sigma > \sigma^1$ and sufficiently large, say $\sigma > \sigma^2$, the second conjugate point $x_2(\sigma)$ of a^1 cannot follow P for the reasons cited in the case of $x_1(\sigma)$. Reasoning thus, one sees that in general for σ sufficiently large there must be arbitrarily many conjugate points of $x = a^1$ preceding P. It follows from the oscillation theorem of the preceding section that there must be infinitely many characteristic roots in B.

The condition of the theorem is accordingly sufficient and the proof is complete. We note the following corollary of the theorem.

COROLLARY. *A sufficient condition that there be infinitely many positive characteristic roots in B is that $\omega(\eta, \eta', \sigma)$ involve σ only in terms of the form*

$$- \sigma\, a_{ij}(x)\, \eta_i \eta_j \tag{6.2}$$

and that the form (6.2) *be positive definite for each x on (a^1, a^2) and for $\sigma < 0$.*

To show the power of the preceding methods we shall briefly indicate an important extension of this corollary.

If Hypotheses A.1, A.2 *and* A.3 *are satisfied, but the form* (6.2) *is assumed positive for $\sigma < 0$ merely for one point x_0 and one set $(\eta^0) \neq (0)$, there still exist infinitely many characteristic roots greater than any constant σ^*.*

The special index form $Q(z, \sigma)$ can still be set up for each σ as before. For any finite range of values of σ the same set of intermediate n-planes can be used. But here the number of characteristic roots θ on a given interval $\sigma' \leqq \sigma < \sigma''$ may very well be infinite

As in the earlier case the nullity of the form $Q(z, \sigma)$ equals the index ρ of σ as a characteristic root, and in case σ^* is an isolated root the index of $Q(z, \sigma)$ can change by at most ρ as σ passes through σ^*. But in the present case, a priori at least, the index may increase, or decrease, or remain constant. Let v'' be the index of $Q(z, \sigma'')$ and v' the index of $Q(z, \sigma')$. If θ is finite, then upon varying σ from σ' to σ'' inclusive, we see that

$$\theta \geqq | v'' - v' |. \tag{6.3}$$

Previously (6.3) was an equality. As it stands (6.3) can still be used to prove the existence of infinitely many characteristic roots.

Let I be an interval which contains x_0 in its interior and is so small that for the given (η^0),

$$a_{ij}(x)\, \eta_i^0 \eta_j^0 > 0$$

on I. One sees readily that there must be a conjugate point of $x = a^1$ on each of any set of distinct subintervals of I, for σ sufficiently large and positive (Morse [16], p. 543). But as previously and with no alteration in the proof, the index of $Q(z, \sigma)$ will differ by at most r from the number of conjugate points of $x = a^1$ on $a^1 < x < a^2$ for the given σ. If for a given σ', σ'' is chosen sufficiently large, the change in index of $Q(z, \sigma)$ as σ increases from σ' to σ'' must be arbitrarily great, so that from (6.3) we see that θ must be arbitrarily great.

There must then be arbitrarily many characteristic roots greater than σ', and the statement in italics is proved.

With this digression we return to Hypotheses A.

Comparison of problems possessing different forms ω

7. In this section we shall consider two problems B and B' satisfying Hypotheses A and possessing a common accessory end-plane π_r. We suppose further

that B and B' possess *infinitely many characteristic roots.* If one wishes one can drop this last assumption, adding the qualification "if σ_h exists" to each statement about σ_h.

We suppose the common accessory end-plane is represented in terms of the same parameters (u) in both problems. The accessory end-form of B' minus that of B will be denoted by

$$\Delta b_{hk}(\sigma)u_h u_k \qquad (h, k = 1, \cdots, r) \tag{7.0}$$

and the differential form of B' minus that of B as given in §3 will be denoted by

$$\Delta\omega(\eta, \eta', \sigma). \tag{7.1}$$

The $(k + 1)$st characteristic roots of B and B' will be respectively denoted by σ_k and σ_k'. By the *difference problem D* corresponding to B' and B in the order written, will be understood the problem in which the differential form is the form (7.1), the accessory end-plane is the end-plane π_r, and the accessory end-form is the form (7.0).

The case in which $\Delta\omega \equiv 0$ has been treated in part in §5. We shall here consider the case where the difference problem satisfies Hypotheses A, and prove the following theorem.

THEOREM 7.1. *If the difference problem D corresponding to problems B' and B satisfies Hypotheses* A *and possesses ρ_h characteristic roots less than σ_h, then*

$$\sigma'_{h+\rho_h} \geqq \sigma_h \qquad (h = 0, 1, \cdots). \tag{7.2}$$

Let the special index forms corresponding to B', B, and D, set up with the same intermediate n-planes, be denoted by Q', Q, and Q^0, respectively. We have

$$Q'(z,\sigma) = b_{hk}(\sigma)u_h u_k + \int_{a^1}^{a^2} 2\omega(\eta,\eta',\sigma)dx + \Delta b_{hk}(\sigma)u_h u_k + \int_{a^1}^{a^2} 2\Delta\omega(\eta,\eta',\sigma)dx \tag{7.3}$$

where (η) is determined by (z) in the problem B'. From (7.3) we see that

$$Q'(z, \sigma) \geqq Q(z, \sigma) + Q^0(z, \sigma). \tag{7.4}$$

Let k' and k be respectively the numbers of roots less than σ_h in B' and B. It follows from (7.4) that for $\sigma = \sigma_h$ the index of the form on the right of (7.4) is at least k'. But it follows from Lemma 7.3 of Ch. III that the index of a sum of two quadratic forms is at most the sum of the indices of the two forms. Thus

$$k' \leqq k + \rho_h. \tag{7.5}$$

If (7.2) were false and

$$\sigma'_{h+\rho_h} < \sigma_h,$$

we would have

$$k' \geqq h + \rho_h + 1, \qquad k \leqq h$$

leading to a contradiction of (7.5). Thus (7.2) holds as written.

(a). *We shall now prove that the equality can hold in* (7.2) *for a given h only if B, B', and D have in common at least one characteristic solution with root $\sigma = \sigma_h$.*

We suppose then that

$$\sigma'_{h+\rho_h} = \sigma_h$$

for a given h. For this h there will exist an $(h + \rho_h + 1)$-plane π' through the origin in the space (z) such that

$$Q'(z, \sigma_h) \leqq 0 \qquad \text{(on } \pi'), \tag{7.6}$$

since the index plus the nullity of the form (7.6) is at least $h + \rho_h + 1$. If μ is the number of the variables (z), there will exist a $(\mu - h)$-plane π through the origin such that

$$Q(z, \sigma_h) \geqq 0 \qquad \text{(on } \pi). \tag{7.7}$$

Furthermore π' and π can and will be so chosen that on them the forms in (7.6) and (7.7) respectively will be zero only if (z) is a critical point of these forms.

Now π' and π will intersect in a hyperplane π^0 of dimensionality at least

$$(h + \rho_h + 1) + (\mu - h) - \mu = \rho_h + 1.$$

From (7.4) we see that

$$Q^0(z, \sigma_h) \leqq 0 \qquad \text{(on } \pi^0). \tag{7.8}$$

But since the form (7.8) has the index ρ_h there must be a straight line λ on π^0 on which the form (7.8) vanishes. A comparison of (7.4), (7.6) and (7.7) shows that the forms (7.6) and (7.7) also vanish on λ. Hence each point of λ must be a critical point of the forms (7.6) and (7.7).

We see then that the curve $\eta_i^0(x)$ determined in B' for $\sigma = \sigma_h$ by a point

$$(z^0) \neq (0)$$

on λ, will represent a characteristic solution in B'. For such a point (7.4) will be an equality. But turning to (7.3) we see that (7.4) can be an equality only if the curve $\eta_i^0(x)$ is a secondary extremal in B and D as well as in B'. Since (z^0) is a critical point of the forms (7.6) and (7.8), the curve $\eta_i^0(x)$ must be a characteristic solution in B and D as well as B'.

The statement in italics is thereby proven.

We have the following corollary.

COROLLARY. *For the condition*

$$\sigma'_h \geqq \sigma_h \tag{7.9}$$

to hold it is sufficient that the difference problem D satisfy Hypotheses A and possess no characteristic root less than σ_h.

With the aid of (7.3) one immediately verifies the truth of the following generalization of the Sturm-Liouville comparison theorems.

In order that (7.9) *hold it is sufficient that*

$$\Delta b_{hk}(\sigma)u_h u_k + \int_{a^1}^{a^2} 2\Delta\omega(\eta, \eta', \sigma)dx \geqq 0 \tag{7.10}$$

for all curves (η) *of class* D^1 *and sets* (u) *which satisfy the secondary end conditions with* (η). *In order that* $\sigma_h' > \sigma_h$ *it is sufficient to exclude the equality in* (7.10) *for* $(\eta) \not\equiv (0)$.

Boundary conditions at one end alone

8. We return now to the differential form $\Omega(\eta, \eta')$ of §1 and corresponding Euler equations

$$\frac{d}{dx}\Omega_{\eta_i'} - \Omega_{\eta_i} = 0 \qquad (i = 1, \cdots, n) \tag{8.1}$$

not involving a parameter σ assuming that $R_{ij}(x)w_i w_j > 0$ for $(w) \neq 0$. Our boundary conditions shall be conditions at $x = a^1$ of the form

$$p_{ij}\eta_j^1 = q_{ij}\zeta_j^1 \qquad (i, j = 1, \cdots, n), \tag{8.2}$$

where the coefficients p_{ij} and q_{ij} are constants and the matrix $\| p_{ij}, q_{ij} \|$ has the rank n. We say that the conditions (8.2) are *self-adjoint* at $x = a^1$ if subject to (8.2) and to the conditions

$$p_{ij}\bar{\eta}_j^1 = q_{ij}\bar{\zeta}_j^1, \tag{8.2$'$}$$

the bilinear form

$$\eta_i^1\bar{\zeta}_i^1 - \zeta_i^1\bar{\eta}_i^1 = 0. \tag{8.3}$$

We *assume* that the conditions (8.2) are self-adjoint.

We see that a necessary and sufficient condition that the conditions (8.2) be self-adjoint at $x = a^1$ is that conditions (8.2) together with the n conditions $\eta_i^2 = 0$ form a system self-adjoint in the sense of §2. It follows from the results of §2 that the conditions (8.2) can be given the form

$$\eta_i^1 - c_{ih}u_h = 0 \qquad (i = 1, \cdots, n;\ h = 1, \cdots, r;\ 0 \leqq r \leqq n), \tag{8.4$'$}$$

$$c_{ih}\zeta_i^1 - b_{hk}u_k = 0 \qquad (h, k = 1, \cdots, r), \tag{8.4$''$}$$

where $\| c_{ih} \|$ is a matrix of constants of rank r and $\| b_{hk} \|$ a symmetric matrix of constants.

We shall give two additional interpretations of the conditions (8.2) of which the first is in terms of transversality.

The end conditions (8.4)′ require that the initial point (x, η) lie on an r-plane L_r. Conditions (8.4) applied to an extremal satisfying (8.1) require that this extremal cut L_r transversally relative to the functional

$$J = \tfrac{1}{2}b_{hk}u_h u_k + \int_{a^1}^{a^2} \Omega(\eta, \eta')dx. \tag{8.5}$$

We take this statement as a convention when $r = 0$.

We can give a second interpretation of the conditions (8.2) in terms of the conjugate families of von Escherich defined in §3, Ch. III. We begin by choosing a base $\boldsymbol{b}$ of n independent sets (η^1, ζ^1) which satisfy (8.4) with parameters (u), and upon which all other sets which satisfy (8.4) are linearly dependent. Let $\| \eta_{ij}(x) \|$ be an n-square matrix whose columns represent the extremals (η) determined at $x = a^1$ by the members of the base $\boldsymbol{b}$. The family of extremals cutting L_r transversally relative to (8.5) can be represented in the form

$$\eta_i(x) = \eta_{ij}(x)v_j \qquad (i, j = 1, \cdots, n) \tag{8.6}$$

where (v) is a set of n constants which serve as parameters of the family. I say that any two members of the family (8.6) satisfy the identity

$$\eta_i(x)\bar{\zeta}_i(x) - \zeta_i(x)\bar{\eta}_i(x) \equiv 0 \qquad (i = 1, \cdots, n). \tag{8.7}$$

In fact the left member of (8.7) is known to be identically constant, and this constant must be zero since the conditions (8.2) and (8.2)′ entail the satisfaction of (8.3). Thus the conditions (8.4) define a family of extremals conjugate in the sense of von Escherich.

Conversely the members of any conjugate family F satisfy conditions of the form (8.4). For the initial values (η^1, ζ^1) of members of F depend linearly upon the initial values of members of the base used to define the family F. We see then that these initial values (η^1, ζ^1) of members of F must satisfy n independent linear conditions L. Now (8.7) will be satisfied identically by any two members of F and in particular will be satisfied at $x = a^1$. The conditions L are then self-adjoint at $x = a^1$ in accordance with our definition of self-adjointness at $x = a^1$, and can accordingly be put in the form (8.4).

We summarize in the following theorem.

THEOREM 8.1. *The conditions* (8.4) *have the following three interpretations.* I. *They have the form of the most general boundary conditions which are self-adjoint at* $x = a^1$. II. *They define the general conjugate family of extremals of the differential equations* (8.1). III. *They define the n-parameter family of extremals which are cut transversally by the r-plane* (8.4)′ *relative to the functional* (8.5).

Recall that the focal points of the conjugate family F are defined as the points $x = c$ at which the determinant $| \eta_{ij}(x) |$ of a base of the family vanishes. Each focal point $x = c$ of F is assigned an index equal to the number of independent solutions of the family which vanish at $x = c$, and each focal point is counted a number of times equal to its index. If a focal point $x = c$ has the index n, then the focal points of F other than $x = c$ may be regarded as the conjugate points of $x = c$.

We shall say that the given boundary problem depends continuously on a parameter μ, if the coefficients in $\Omega(\eta, \eta')$ together with the derivatives R'_{ij} and Q'_{ij} are continuous in x and μ, while the coefficients c_{ih} and b_{hk} are continuous in μ. We suppose that the matrix $\| c_{ih} \|$ remains constantly of rank r and the matrix $\| b_{hk} \|$ remains symmetric. Let the boundary problem thereby defined be denoted by β_μ. We suppose that μ is confined to values near $\mu = 0$.

We shall prove the following theorem.

THEOREM 8.2. *If the kth focal point following $x = a^1$ of the conjugate family F defined by β_μ exists for $\mu = 0$, it exists and varies continuously with μ for μ sufficiently near* 0.

Let $x = c$ be the kth focal point of F following $x = a^1$ when $\mu = 0$. Let $x = c'$ and $x = c''$ be two points on the x axis separating $x = c$ from the other focal points of F. We suppose $c' < c''$. To the end conditions of the form (8.4)$'$ in β_μ we adjoin the conditions $\eta^2_i = 0$, obtaining thus the conditions

$$
\begin{aligned}
&\eta^1_i - c_{ih}(\mu)u_h = 0 \qquad (i = 1, \cdots, n;\ h = 1, \cdots, r),\\
&\eta^2_i = 0.
\end{aligned}
\tag{8.8}
$$

Corresponding to these end conditions and to the functional J in (8.5), here depending on μ, we set up the special index form of §1, Ch. III. We take a^2 successively as c' and c'', and let the corresponding index forms be denoted by $H'(z, \mu)$ and $H''(z, \mu)$.

The forms $H'(z, 0)$ and $H''(z, 0)$ are non-singular since c' and c'' are not focal points of a^1. For a sufficiently small variation of μ they will remain non-singular and their indices remain constant. But their indices are respectively the numbers of focal points preceding c' and c'' and following $x = a^1$. Thus the kth focal point of $x = a^1$ must lie between $x = c'$ and $x = c''$ for μ sufficiently near $\mu = 0$. The theorem follows from the fact that c' and c'' can be taken arbitrarily close to $x = c$.

We shall now establish a theorem on the interrelations of the focal points of any two conjugate families (Morse [10]).

THEOREM 8.3. *If two conjugate families F and F^0 have ρ linearly independent solutions in common, then the number of focal points of F on any interval γ (open or closed) differs from the corresponding number for F^0 by at most $n - \rho$.*

Let a^1 and a^2 be so chosen that a^1 is not a focal point of F or F^0, and that the interval $a^1 < x < a^2$ includes just the focal points of F and F^0 on γ. Since $x = a^1$ is not a focal point of F, at a^1 the members of F will satisfy conditions of the form

$$
\eta^1_i = u_i, \qquad \zeta^1_i = \zeta_{ij}u_j \qquad (i, j = 1, \cdots, n),
\tag{8.9}
$$

where $\zeta_{ij} = \zeta_{ji}$. Similarly at a^1 the members of F^0 will satisfy conditions of the form

$$\eta_i^1 = u_i, \qquad \zeta_i^1 = \zeta_{ij}^0 u_j, \tag{8.10}$$

where $\zeta_{ij}^0 = \zeta_{ji}^0$. The quadratic form

$$\Delta(u) = (\zeta_{ij}^0 - \zeta_{ij})u_i u_j \qquad (i, j = 1, \cdots, n) \tag{8.11}$$

will be called the *difference form* at $x = a^1$ corresponding to F^0 and F. Now a necessary and sufficient condition that the members of F and F^0 determined by a set (u) be identical is that

$$(\zeta_{ij}^0 - \zeta_{ij})u_j = 0 \qquad (i, j = 1, \cdots, n).$$

We see then that the number of independent solutions common to F and F^0 equals the nullity of the difference form $\Delta(u)$.

We now adjoin the condition $\eta_i^2 = 0$ to the conditions (8.9) and (8.10) thereby obtaining two new boundary problems B and B^0. Corresponding to B and B^0 and the functional (8.5) we set up the special index forms $Q(z)$ and $Q^0(z)$ respectively as in Ch. III ($\lambda = 0$). We use the same intermediate n-planes in B and B^0. We then have

$$Q^0(z) - Q(z) = \Delta(u) \tag{8.12}$$

where the variables (u) equal the first n of the variables (z). It follows from Lemma 7.2 of Ch. III that if v and v^0 are respectively the indices of Q and Q^0, and N and P are respectively the indices of $\Delta(u)$ and $-\Delta(u)$, then

$$v - P \leqq v^0 \leqq v + N.$$

Hence

$$| v^0 - v | \leqq P + N = n - \rho.$$

The theorem now follows from the fact that the indices of Q and Q^0 are respectively the numbers of focal points of F and F^0 on $a^1 < x < a^2$.

The two preceding theorems enable us to prove the following:

THEOREM 8.4. *The kth conjugate point of a point $x = c$ following $x = c$ advances or regresses continuously with $x = c$ as long as it remains on the interval on which the problem is defined.*

Choose $x = a^1$ as a point preceding $x = c$ and not conjugate to $x = c$. The conjugate family F_c consisting of the solutions of (8.5) which vanish at $x = c$, will satisfy conditions of the form

$$\eta_i^1 = u_i \qquad (i, j = 1, \cdots, n),$$
$$\zeta_i^1 = \zeta_{ij} u_j \qquad (\zeta_{ij} = \zeta_{ji}),$$

at c, where the coefficients ζ_{ij} will be continuous in c at least for a sufficiently small variation of c. That the kth conjugate point $x_k(c)$ of $x = c$ varies continuously with $x = c$ now follows from Theorem 8.2.

Suppose next that c increases from c_0. Then $x_k(c) \neq x_k(c_0)$, at least after a sufficiently small increase of c from c_0. For otherwise there would be infinitely many conjugate points of $x_k(c_0)$ near $x = c_0$. Now there are at most $k - 1$ focal points of F_{c_0} on the interval

$$c_0 < x < x_k(c_0). \tag{8.13}$$

If $x_k(c)$ decreased as c increased, at least $n + k$ focal points of F_c would thereby appear on the interval (8.13) contrary to Theorem 8.2. Hence $x_k(c)$ increases with c. It follows that $x_k(c)$ must decrease as c decreases and the theorem is proved.

We shall give a typical comparison theorem.

THEOREM 8.5. *Let F and F^0 be two conjugate families for which $x = a^1$ is not a focal point. If the difference form*

$$\Delta(u) = (\zeta^0_{ij} - \zeta_{ij})u_iu_j \qquad (i, j = 1, \cdots, n)$$

of (8.11) *is positive definite, the kth focal point of F^0 following $x = a^1$ will be preceded by the kth focal point of F.*

We use the notation of the proof of Theorem 8.3 taking a^2 as the kth focal point of F^0. We are led to the relation

$$Q^0(z) - \Delta(u) = Q(z) \tag{8.14}$$

of (8.12).

If there are h focal points of F^0 on the interval $a^1 < x \leqq a^2$, $Q^0(z)$ will be negative semi-definite on an h-plane π_h through the origin. Moreover π_h can be so chosen that $Q^0(z) = 0$ on π_h only if (z) is a critical point (z^0) of $Q^0(z)$. But such a critical point (z^0) determines a solution (η) of (8.1) satisfying the conditions (8.10). If $(z^0) \neq (0)$, the corresponding set

$$(u_1, \cdots, u_n) = (z^0_1, \cdots, z^0_n)$$

cannot be null in (8.10) since (η) would be identically null. Thus if $Q^0(z^0) = 0$ on π_h at a point $(z^0) \neq (0)$, the corresponding set (u) is not null. It follows from (8.14) that $Q(z)$ is negative definite on π_h. The index of $Q(z)$ is then at least h, and since $h \geqq k$ the kth focal point of F must precede the kth focal point of F^0.

CHAPTER V

THE FUNCTIONAL ON A RIEMANNIAN SPACE

In classical problems in parametric form the domain of the variables is usually taken as a region in a euclidean space. A more general domain is a so-called Riemannian space with a metric defined by a positive definite quadratic form

$$ds^2 = g_{ij}(x)dx^i dx^j \qquad (i, j = 1, \cdots, m). \tag{0.1}$$

We shall prefer a Riemannian space for two principal reasons. In the first place a Riemannian space presents a suitable medium for a treatment of the numerous invariants of the functional and for the presentation of the principal hypotheses. In the second place the "Jacobi least action integral" cannot be adequately treated otherwise, at least in the large. For at the present time adequate answers cannot be given to questions concerning the possibility of embedding Riemannian manifolds in the large in euclidean spaces of high dimension.

A novel feature of this chapter is the invariantive formulation of the accessory boundary problem. From the point of view of tensor analysis and Riemannian geometry, entities may be regarded as geometrical if defined by invariants or the vanishing of the components of tensors, because such entities are then independent of the coordinate systems employed. From this point of view characteristic roots as defined in this chapter are *geometric* entities. The classical definitions of such roots do not afford roots of this character and considerable care is required in the modification of the classical definitions. One calls an entity *restrictedly topological* if it can be defined by means which would be purely topological except for analytical restrictions on the functions employed. It will appear in Ch. VII that the number of negative characteristic roots is restrictedly topological, at least in the non-degenerate case. This fact would be extremely significant if one were to develop the present theory purely from the point of view of abstract spaces, as presumably will be done shortly.

As a matter of detail we call attention to the considerable simplification in the classical minimum theory due to the author's elimination of Behaghel's formula. See Bliss [3]. We also give a proof of the existence of families of extremals cut transversally by a manifold of *any* dimension. In general this chapter completes the basic theory in the small.

A Riemannian space in the large

1. Riemannian spaces as ordinarily defined are local affairs. It is necessary for us to add topological structure in the large. To that end we suppose that we have given an ordinary m-dimensional simplicial circuit K in an auxiliary euclidean space on which a neighborhood of each point is well defined. See Lefschetz [1], Veblen [1]. Our Riemannian space R will now be defined as

follows. Its points and their neighborhoods shall be the one-to-one images of the respective points and their neighborhoods on K. Moreover K shall be a manifold in the sense that a neighborhood of each of its points shall be homeomorphic with a neighborhood of a point (x) in a euclidean m-space of coordinates $(x) = (x^1, \cdots, x^m)$. With at least one such representation of a neighborhood of a point of R there shall be associated a positive definite form such as (0.1), defining a metric for the neighborhood. We suppose that the coefficients $g_{ij}(x)$ are of class C^3. We term the coordinates (x) *admissible*. We also admit any other set of local coordinates (z) obtainable from admissible coordinates (x) by a transformation of the form

$$z^i = z^i(x) \tag{1.1}$$

in which the functions $z^i(x)$ are of class C^4 and possess a non-vanishing jacobian. We also require that any two coordinate systems (x) and (z) which admissibly represent a neighborhood of the same point P on R be related as in (1.1). In transforming our problem to non-parametric form neighboring a given extremal we shall find it necessary to *admit* transformations merely of class C^3 and to term the resulting coordinates *specially admissible.*

A set of points of R will be said to form a regular r-manifold on R of class C^n if the images of its points in any admissible coordinate system (x) are locally representable in the form

$$x^i = x^i(u_1, \cdots, u_r),$$

where the functions $x^i(u)$ are of class C^n in the parameters (u) and the functional matrix of the functions $x^i(u)$ is of rank r. By a *regular arc g* of class C^n we shall mean a closed segment of a regular 1-dimensional manifold of class C^n. By a curve of class D^1 we shall mean a finite continuous succession of regular arcs of class C^1.

We shall now prove the following theorem.

THEOREM 1.1. *Let g be a simple regular arc of class C^4 along which t is the arc length. A neighborhood of g can then be admissibly represented as a whole by a neighborhood of the x^m axis in a euclidean space (x) in such a manner that g corresponds to the x^m axis with $t = x^m$.*

The theorem is true for a segment of g sufficiently near any point P of g. For if (z) is an admissible coordinate system neighboring P, any segment of g sufficiently near P can be represented as stated in the theorem in the form $z^i = \varphi^i(t)$. One at least of the derivatives $\dot\varphi^i(t) \neq 0$, say $\dot\varphi^m \neq 0$. The transformation

$$\begin{aligned} z^i &= x^i + \varphi^i(x^m) \qquad (i = 1, \cdots, m-1), \\ z^m &= \varphi^m(x^m) \end{aligned} \tag{1.2}$$

is admissible and affords the desired local coordinate system (x). This leads us to the following lemma.

LEMMA. *Let there be given two overlapping segments g_1 and g_2 of g, with g_2 extending beyond g_1 and g_1 commencing prior to g_2. If the theorem is true for g_1 and g_2 separately, it is true for the arc $g_1 + g_2$ into which g_1 and g_2 combine.*

For simplicity suppose that $t = 0$ is an inner point of both g_1 and g_2. Suppose that the coordinate system (x) represents g_2 as in the theorem, with $t = x^m$ along g_2, and that the coordinate system (z) similarly represents g_1 with $t = z^m$ along g_1. Suppose that $(x) = (z) = (0)$ when $t = 0$. Since both coordinate systems are admissible neighboring the point $t = 0$ on g, they are there related by a transformation of the form

$$z^i = a^i_j x^j + \eta^i(x) \qquad (i, j = 1, \cdots, m) \tag{1.3}$$

where a^i_j is a constant, $| a^i_j | \neq 0$, and $\eta^i(x)$ is a function of class C^4 with a null differential at $(x) = (0)$. Since $t = z^m = x^m$ along g near the point $t = 0$, we see that $a^m_m \equiv 1$. Without loss of generality we can suppose that a^i_j equals the Kronecker δ^i_j since we could bring this about by replacing the coordinate system (x) by the coordinate system

$$\bar{x}^i = a^i_j x^j.$$

Suppose then that (1.3) takes the form

$$z^i = x^i + \eta^i(x). \tag{1.4}$$

Let e be a small positive constant and $h(t)$ a function of class C^4 in absolute value less than 1 and such that

$$h(t) \equiv 1, \qquad t \leqq 0,$$

$$h(t) \equiv 0, \qquad t \geqq e.$$

The transformation (1.3) is valid neighboring $t = 0$ on g_1 and g_2. If e is sufficiently small, the transformation

$$z^i = x^i + h(x^m)\eta^i(x) \qquad (i = 1, \cdots, m) \tag{1.5}$$

is defined neighboring the whole of g_2. It is identical with the transformation (1.4) for $x^m < 0$, and reduces to the identity for $x^m > e$. The coordinate system (z) can now be regarded as representing the neighborhoods of g_1 and g_2 combined. Neighboring the points of g_1 for which $t < 0$, (z) shall represent R as before. Neighboring points of g_2 for which $t \geqq 0$, (z) shall now be the representation obtained from the given representation (x) by the transformation (1.5). Along $g_1 + g_2$ we see that $z^m = t$. One also sees that the jacobian of the transformation (1.5) is not null on g_2 if e is sufficiently small.

To prove the theorem we cover the whole of g by a finite ordered set of local coordinate systems each of the required nature neighboring the portion of g covered, and excepting the first, each overlapping its predecessor neighboring some point of g. The theorem follows upon making a finite number of applications of the preceding lemma.

We shall prove the following theorem.

THEOREM 1.2. *There exists a non-singular transformation of class C^3 of the coordinate system of Theorem 1.1 into coordinates (x) in which g again corresponds to the x^m axis, while x^m equals the arc length t along g and $g_{ij}(x) = \delta_j^i$ along g.*

Let the coordinates of Theorem 1.1 be denoted by (z). Along g then $z^m = t$. Let $a_{ij}(z^m)$ be the value of $g_{ij}(z)$ at the point $t = z^m$ on g. We make the transformation

$$\begin{aligned} x^i &= z^i & (i = 1, \cdots, m-1), \\ x^m &= a_{mj}(z^m)z^j & (j = 1, \cdots, m), \end{aligned} \tag{1.6}$$

generalizing the reduction of quadratic forms due to Lagrange. See Bôcher [1], p. 131. We note that $a_{mm} \equiv 1$, since $t = z^m$ along g. We then see that along g the differentials of (x) and (z) are transformed in the same manner as (x) and (z), and that accordingly (1.6) carries the basic form ds^2 into one in which along g, dx^m appears only in the form $(dx^m)^2$. As in the Lagrange reduction we turn to the residual form in the variables $dx^1, \cdots, dx^{m-1}$. By making transformations similar to (1.6), this residual form can be reduced along g to a form in which the squares of the differentials alone appear, multiplied by positive functions of z^m. A further obvious transformation will make these coefficients unity along g. Thus ds^2 will take the required form. Moreover after each transformation $t = x^m$ as desired, and the theorem is proved.

The coordinate system of Theorem 1.2 will be termed normal. This system is specially admissible.

Up to this point we have supposed that the arc g is without multiple points. In case g has multiple points we divide it into a finite number of consecutive segments so small that each arc of g which is composed of three successive segments is without multiple points. According to the preceding theorems, each of these segments is interior to a normal coordinate system. We restrict these coordinate systems to neighborhoods N of the segments so small that no system has any points in common with the second following or second preceding system. We now define a new Riemannian manifold covering R on which each of the above neighborhoods N is to be regarded as distinct from each of the other neighborhoods save the ones immediately following and preceding N. On the resulting manifold $\bar{R}$ the arc g is without multiple points and possesses a neighborhood coverable by a single normal coordinate system.

The definition of a Riemannian manifold here used was introduced in brief in 1929 in Morse [4], p. 166. Veblen and Whitehead (see Veblen [2, 3]) have presented a general axiomatic basis for differential geometry. The manifolds which we employ come under those of Veblen and Whitehead, although the definition adopted in this chapter is perhaps simpler for our purposes.

Basic tensors

2. With each admissible coordinate system (x) we suppose that we have given a function

$$F(x, r) = F(x^1, \cdots, x^m, r^1, \cdots, r^m) \qquad (r) \neq (0)$$

serving to define an integral

$$\int F(x, \dot{x})dt \tag{2.0}$$

in that system, where $\dot{x}^i$ represents the derivative of x^i with respect to the parameter t. When the variables (x) are subjected to the transformation

$$z^i = z^i(x), \tag{2.1}$$

we understand that the variables (r) are subjected to the transformation

$$\sigma^h = \frac{\partial z^h}{\partial x^i} r^i \qquad (h, i = 1, \cdots, m). \tag{2.2}$$

That is, we suppose that (r) is transformed as a contravariant tensor or vector. The function $F(x, r)$ is then to be replaced by a function $Q(z, \sigma)$ such that

$$Q(z, \sigma) = F(x, r), \qquad (\sigma) \neq (0) \tag{2.3}$$

subject to (2.1) and (2.2). Our integral then takes the form

$$\int Q(z, \dot{z})dt.$$

For at least one coordinate system neighboring each point of R (and consequently for all such coordinate systems) we assume that the corresponding integrand is positive and of class C^3 for $(r) \neq (0)$. We also assume that $F(x, r)$ is positive homogeneous of order 1 in the variables (r). That is we assume that

$$F(x, kr) \equiv kF(x, r), \qquad (r) \neq (0), \tag{2.4}$$

for all positive numbers k. Upon differentiating the identity (2.4) with respect to k and r^j we find that

$$r^i F_{r^i r^j} \equiv 0 \qquad (i, j = 1, \cdots, m), \tag{2.5}$$

so that

$$| F_{r^i r^j} | \equiv 0,$$

an important peculiarity of the parametric form.

We shall now consider certain tensors and invariants which enter into the theory. See Eisenhart [1].

Upon differentiating (2.3) with respect to r^i, with σ^h subject to (2.2), we find that

$$Q_{\sigma^h} \frac{\partial z^h}{\partial x^i} = F_{r^i}. \tag{2.6}$$

Thus F_{r^i} is a covariant vector. It follows that

$$F_{r^i}dx^i \tag{2.7}$$

is an invariant. The expression (2.7) enters into the transversality conditions and into the Hilbert integral. If (r) and (σ) are contravariant tensors, the Weierstrass E-function

$$E(x, r, \sigma) = F(x, \sigma) - \sigma^i F_{r^i}(x, r) \tag{2.8}$$

is another invariant. The expression

$$F_{r^i r^j}\lambda^i\lambda^j$$

is an invariant provided (λ) is a contravariant vector, since $F_{r^i r^j}$ is seen to be a covariant tensor of the second order. One also verifies the fact that

$$\frac{d}{dt}F_{r^i} - F_{x^i} = \frac{\partial z^k}{\partial x_i}\left[\frac{d}{dt}Q_{\sigma k} - Q_{zk}\right].$$

We consider the bordered determinant

$$\begin{vmatrix} F_{r^i r^j} & u_i \\ v_j & 0 \end{vmatrix} = B. \tag{2.9}$$

We see that

$$\begin{vmatrix} F_{r^i r^j} & u_i \\ v_j & 0 \end{vmatrix} = -A^{ij}u_i v_j \tag{2.10}$$

where A^{ij} is the cofactor of $F_{r^i r^j}$ in the determinant of these elements. But if the first m rows of B are multiplied respectively by $r^1, \cdots, r^m$, added, and then substituted for the kth row, the elements in the resulting kth row will all be zero by virtue of (2.5), provided

$$r^i u_i = 0.$$

Regarding B as a polynomial in u_i and v_j we see that $r^i u_i$ must be a factor of B, at least if $r^k \neq 0$. But we are assuming that $(r) \neq (0)$, so that at least one of the variables $r^k \neq 0$. By operating upon the columns of B in a similar manner we see that $r^j v_j$ is also a factor of B. Thus we have the relation

$$A^{ij}u_i v_j \equiv F_1(x, r)\,[r^i u_i]\,[r^j v_j] \tag{2.11}$$

where F_1 is a factor of proportionality. If we let $(u) = (v) = (r)$, we see that

$$F_1(x, r) = \frac{A^{ij}r^i r^j}{(r^i r^i)^2}$$

so that F_1 is continuous in (x) and (r) for $(r) \neq (0)$.

Upon equating the coefficients of $u_i v_j$ in the two members of (2.11), we find that

$$A^{ij} = F_1 r^i r^j, \tag{2.12}$$

relations which have been used by Weierstrass to define F_1 when $m = 2$, and by others when $m > 2$. See Hadamard [1], p. 95, and Bliss [3].

If (u) and (v) are transformed as covariant vectors into vectors (u') and (v'), it follows from the theory of adjoint quadratic forms (see Bôcher [1], p. 160) that

$$\begin{vmatrix} F_{r^i r^j} & u_i \\ v_j & 0 \end{vmatrix} = c^2 \begin{vmatrix} Q_{\sigma^i \sigma^j} & u'_i \\ v'_j & 0 \end{vmatrix}$$

where

$$c = \left| \frac{\partial z^i}{\partial x^j} \right| .$$

From (2.11) we then see that

$$F_1(x, r) = c^2 Q_1(z, \sigma)$$

where Q_1 is formed from Q in the system (z, σ) as was F_1 from F in the system (x, r). See Bolza [1], p. 346.

The necessary conditions of Euler, Weierstrass, and Legendre

3. Suppose that we have given a simple, regular, sensed curve g of class C^1 joining two points A^1 and A^2 on R. We admit sensed curves of class D^1 joining the end points of g in the order A^1 and A^2, and denote the value of our integral along such curves by J. We state the following theorem.

THEOREM 3.1. *If g affords a weak relative minimum to J, it is necessary that*

$$\frac{d}{dt} F_{r^i} - F_{x^i} \equiv 0 \qquad (i = 1, \cdots, m) \tag{3.1}$$

along g in each coordinate system (x) in which g enters.

To prove this theorem we turn to a particular coordinate system (x) in which g enters, and consider the problem of minimizing J as an integral in non-parametric form in the space of coordinates $(t, x^1, \cdots, x^m)$. If

$$x^i = \gamma^i(t) \qquad (t^1 \leqq t \leqq t^2),$$

is a representation of class C^1 of an arc of g in the system (x), these same equations define an arc $\bar{g}$ in the space $(t, x^1, \cdots, x^m)$. If g is a minimizing arc in the space (x), $\bar{g}$ will afford at least a weak minimum in the corresponding non-parametric fixed end point problem in the space (t, x). Conditions (3.1) then follow from the non-parametric theory.

A regular arc which is of class C^2 and satisfies (3.1) will be called an *extremal*. The condition of Weierstrass is as follows.

THEOREM 3.2. *If g affords a strong, relative minimum to J, it is necessary that*

$$E(x, \dot{x}, \sigma) \geqq 0$$

for $(x, \dot{x})$ on g and for any non-null vector (σ).

To prove this condition we again operate in a single coordinate system (x). Suppose the parameter t on g so chosen that $t = 1$ specifies a prescribed point $P \neq A^1$ on g, and the points $0 \leqq t \leqq 1$ all lie on g in the coordinate system (x). Suppose that g is cut at the point $t = 1$ by a regular curve $x^i = x^i(\alpha)$ of class C^1 for $\alpha \leqq 1$ and near 1, and such that

$$\gamma^i(1) = x^i(1)$$

where $x^i = \gamma^i(t)$ is our representation of g. We evaluate J along the curve, α = const., of the family,

$$x^i(t, \alpha) = \gamma^i(t \cdot \alpha) + (x^i(\alpha) - \gamma^i(\alpha))t \qquad (0 \leqq t \leqq 1),$$

passing from the point $t = 0$ to the point $t = 1$. To this we add the value of J along the curve $x^i = x^i(\alpha)$ passing from a point $\alpha < 1$ to the point $\alpha = 1$. We see that

$$x^i(t, 1) \equiv \gamma^i(t),$$

$$x^i(0, \alpha) \equiv \gamma^i(0),$$

$$x^i(1, \alpha) \equiv x^i(\alpha),$$

and with the aid of these identities we find for the function $J(\alpha)$ that

$$J'(1) \equiv \dot{x}^i(1)\, F_{r^i}[\gamma(1), \dot{\gamma}(1)] - F[\gamma(1), \dot{x}(1)].$$

But for a minimizing arc it is necessary that $J'(1) \leqq 0$. We set $\dot{x}^i(1) = \sigma^i$ and observe that (σ) can be taken as an arbitrary non-null vector. The condition $J'(1) \leqq 0$ reduces to the condition of the theorem in case $P \neq A^1$.

The continuity of E insures the truth of the theorem in the case $P = A^1$ as well.

As a corollary of the Weierstrass condition we have the following analogue of the Legendre condition.

COROLLARY. *For a weak minimum it is necessary that*

$$F_{r^i r^j}(x, \dot{x})\lambda^i\lambda^j \geqq 0$$

for $(x, \dot{x})$ on g and for any vector (λ).

To prove this corollary we consider the function

$$\varphi(e) = E(x(t), \dot{x}(t), \dot{x}(t) + e\lambda)$$

where $x^i = x^i(t)$ defines g. Observe that $\varphi(e)$ has a minimum zero, when $e = 0$, by virtue of the Weierstrass condition. Hence $\varphi''(0) \geqq 0$. But a simple computation shows that $\varphi''(0)$ is the left member of (3.2), and condition (3.2) is established.

Extremals

4. We shall continue by obtaining a general representation of extremals. To that end we first verify the fact that the relations

$$\dot{x}^i\left(\frac{d}{dt} F_{r^i} - F_{x^i}\right) \equiv 0 \qquad (r^i = \dot{x}^i) \tag{4.1}$$

hold identically along any regular curve $x^i = x^i(t)$ of class C^2. These identities are a consequence of (2.5) and the identities

$$r^i F_{r^i x^j} \equiv F_{x^j} \qquad (i, j = 1, \cdots, m)$$

derived from (2.4) by differentiation with respect to k and x^j.

We set

$$\varphi(x, r) \equiv g_{ij}(x) r^i r^j$$

and consider the differential equations

$$\frac{d}{dt} F_{r^i} - F_{x^i} = 0, \tag{4.2$'$}$$

$$\varphi(x, \dot{x}) = 1, \tag{4.2$''$}$$

of which (4.2)″ requires t to be the arc length. Let $\lambda(t)$ be an unknown function of t and consider the system

$$\frac{d}{dt}(F_{r^i} + \lambda\varphi_{r^i}) - (F_{x^i} + \lambda\varphi_{x^i}) = 0, \tag{4.3$'$}$$

$$\varphi(x, \dot{x}) = 1. \tag{4.3$''$}$$

Upon differentiating (4.2)″ with respect to t and making use of the homogeneity of $\varphi(x, r)$ in the variables (r), one finds that

$$\dot{x}^i\left(\frac{d}{dt} \varphi_{r^i} - \varphi_{x^i}\right) = 0 \qquad (r^i = \dot{x}^i). \tag{4.4}$$

Upon multiplying the ith condition in (4.3)′ by $\dot{x}^i$, summing, and making use of (4.1) and (4.4), we find that

$$\dot{x}^i \varphi_{r^i} \frac{d\lambda}{dt} = 2\frac{d\lambda}{dt} = 0.$$

Hence for any solution $x^i(t)$, $\lambda(t)$, of (4.3) for which $\lambda = 0$ initially, we must have $\lambda \equiv 0$. The functions $x^i(t)$ will then define a solution of (4.2).

Suppose that we have a solution g of (4.2) in the form

$$x^i = \gamma^i(t) \qquad (i = 1, \cdots, m)$$

where the functions $\gamma^i(t)$ are of class C^1 on an interval (t^1, t^2). It will be convenient to set

$$v_i^0(t) = F_{r^i}(\gamma(t), \dot\gamma(t))$$

and to term $\gamma^i(t)$, $\dot\gamma^i(t)$, $v_i^0(t)$ sets x^i, r^i, v_i on g. We assume that $F_1 \neq 0$ for (x, r) on g. From (2.10) and (2.11) it then follows that

$$\begin{vmatrix} F_{r^ir^j} & \varphi_{r^i} \\ \varphi_{r^j} & 0 \end{vmatrix} \neq 0 \tag{4.5}$$

on g.

To solve the equations (4.3) we set

$$\begin{aligned} v_i &= F_{r^i}(x, r) + \lambda\varphi_{r^i}(x, r), \\ 1 &= \varphi(x, r). \end{aligned} \tag{4.6}$$

By virtue of (4.5), equations (4.6) have unique solutions

$$\begin{aligned} r^i &= r^i(x, v), \\ \lambda &= \lambda(x, v), \end{aligned} \tag{4.7}$$

of class C^2 in (x) and (v) for (x, r, v) near sets on g, and λ near zero. The system (4.3) can then be given the form

$$\begin{aligned} \frac{dx^i}{dt} &= r^i(x, v), \\ \frac{dv_i}{dt} &= q_i(x, v), \qquad \lambda = \lambda(x, v), \end{aligned} \tag{4.8}$$

where

$$q_i \equiv F_{x^i}[x, r(x, v)] + \lambda(x, v)\varphi_{x^i}[x, r(x, v)].$$

Equations (4.8) have solutions of the form

$$\begin{aligned} x^i &= h^i(t, t_0, x_0, v_0), \\ v_i &= k_i(t, t_0, x_0, v_0), \end{aligned} \tag{4.9}$$

which take on the values (x_0, v_0) when $t = t_0$, and for which the functions h^i and k_i are of class C^2 in their arguments for t and t_0 on the interval (t^1, t^2) and (t_0, x_0, v_0) sufficiently near sets (t, x, v) on g.

We do not wish the general solution of (4.3) but only those solutions which are solutions of (4.2), and are solutions for which $\lambda = 0$ initially. We obtain these solutions from (4.9) by setting

$$v_{i0} = F_{r^i}(x_0, r_0) \tag{4.10}$$

in (4.9), since (4.10) taken with (4.6) is easily seen to imply the condition that $\lambda(x_0, v_0) = 0$. Our general solution of (4.2) thus takes the form

$$x^i = x^i(t, t_0, x_0, r_0) \tag{4.11}$$

where the functions on the right are of class C^2 in their arguments for t as before and (t_0, x_0, r_0) sufficiently near sets $(t, x, \dot{x})$ on g. Moreover we have

$$x_0^i = x^i(t_0, t_0, x_0, r_0), \qquad \varphi(x_0, r_0) = 1,$$
$$r_0^i = x_t^i(t_0, t_0, x_0, r_0).$$

Reference to the first of equations (4.8) discloses the additional fact that the functions k_i and hence the functions x_t^i are of class C^2 in their arguments.

Suppose that (z) is a coordinate system overlapping the system (x). In the system (z) suppose there is given a family of extremals neighboring g, with t as the arc length, depending on certain parameters (a). To *continue* this family in the system (x) we understand that a point $t = t_1$ is selected on g which lies in both the systems (x) and (z). By means of the transformation between the two coordinate systems the values of the variables $(x, \dot{x})$ at the point $t = t_1$ on the extremal determined by (a) can be expressed as functions $x_1^i(a), r_1^i(a)$. We then regard the family

$$x^i = x^i(t, t_1, x_1(a), r_1(a)) = \varphi^i(t, a)$$

as a *continuation* in the system (x) of the family of extremals given in (z). It is clear that the functions $\varphi^i(t, a)$ will be independent of the particular point $t = t_1$ on g used to define them.

Conjugate points

5. We shall define the conjugate points of a point $t = t_1$ on an extremal g. Let (ρ) be the unit contravariant vector which gives the direction of g at $t = t_1$. Suppose the point $t = t_1$ on g is interior to a coordinate system (x). In the system (x) let the components r^i of the unit vectors neighboring (ρ) be regularly represented as functions $r^i(u)$ of class C^2 of $n = m - 1$ parameters (u). Suppose that (ρ) corresponds to $(u) = (0)$. In the system (x) the extremals issuing from the point

$$x_1^i = \gamma^i(t_1)$$

on g with directions neighboring (ρ) can be represented in terms of the functions in (4.11) in the form

$$x^i = x^i(t, t_1, \gamma(t_1), r(u)) = \varphi^i(t, u). \tag{5.1}$$

The jacobian

$$M(t, t_1) = \frac{D(\varphi^1, \cdots, \varphi^m)}{D(t, u_1, \cdots, u_n)}, \qquad (u) = (0), \tag{5.2}$$

vanishes at $t = t_1$. We can factor $t - t_1$ out of each of its last n columns, and for t neighboring $t = t_1$ represent this jacobian in the form

$$M(t, t_1) = (t - t_1)^n N(t, t_1) \qquad (n = m - 1) \tag{5.3}$$

where $N(t, t_1)$ is continuous in t and t_1 and

$$N(t, t_1) = | \rho^i, \ r^i_{u_j}(0) | \qquad (i = 1, \cdots, m; j = 1, \cdots, n). \tag{5.4}$$

The last n columns of this determinant give n independent vectors, since the vectors $r^i(u)$ are regularly represented. Moreover these columns represent vectors orthogonal to the vector (ρ) as one sees upon differentiating the identity

$$g_{ij}(x_1)r^i(u)r^j(u) \equiv 1 \qquad (i, j = 1, \cdots, m)$$

with respect to u_k and setting $(u) = (0)$. Thus $N(t_1, t_1) \neq 0$. Hence near $t = t_1$ the determinant $M(t, t_1)$ vanishes at most at $t = t_1$.

If the family (5.1) is "continued" into an overlapping coordinate system (z), one obtains a new local representation of these extremals and a new jacobian

$$\frac{D(z^1, \cdots, z^m)}{D(t, u_1, \cdots, u_n)}, \qquad (u) = (0),$$

which we call a *continuation* of the original jacobian. On any common segment of g these jacobians will vanish simultaneously and to the same orders in t.

By the conjugate points of the point $t = t_1$ on g we mean the points $t \neq t_1$ on g at which the jacobian $M(t, t_1)$ or its successive continuations vanish.

Let g be an extremal on which the arc length t increases from t^1 to t^2 inclusive. To show that the conjugate points of $t = t^1$ are isolated we need to represent our functional in non-parametric form. To that end we refer the neighborhood of g to normal coordinates (x) as in §1, with g corresponding to the x^m axis and $t = x^m$ along g. We set

$$(x^1, \cdots, x^m) = (y_1, \cdots, y_n, x) \qquad (n = m - 1)$$

and

$$f(x, y_1, \cdots, y_n, p_1, \cdots, p_n) = F(y_1, \cdots, y_n, x, p_1, \cdots, p_n, 1) \tag{5.5}$$

where $F(x, \dot{x})$ is the integrand in the normal system (x). For any admissible curves neighboring g for which $\dot{x}^m > 0$ our functional J takes the form

$$J = \int_{t^1}^{t^2} f(x, y, y')dx \qquad (t^1 \leqq x \leqq t^2).$$

We assume that the *Legendre S-condition holds along g*, that is that

$$F_{r^i r^j}(x, \dot{x})\lambda^i\lambda^j > 0 \qquad (i, j = 1, \cdots, m) \tag{5.6}$$

for $(x, \dot{x})$ on g and for any unit vector (λ) not $\pm(\dot{x})$. For the present system of coordinates we see that

$$F_{r^m r^j}(x, \dot{x}) = 0 \qquad (j = 1, \cdots, m)$$

along g, as follows from (2.5). Moreover we see from (5.5) that

$$F_{r^i r^j} = f_{p_i p_j} \qquad (i, j = 1, \cdots, m-1)$$

so that (5.6) takes the form

$$f_{p_i p_j} z^i z^j > 0 \qquad (i, j = 1, \cdots, m-1)$$

for sets $(z) \neq (0)$. Thus the Legendre S-condition holds along g in the non-parametric problem.

As in (5.1) let the extremals through the point $t = t^1$ on g be represented in the form

$$x^i = \varphi^i(t, u) \qquad (t^1 \leqq t \leqq t^2)$$

with $(u) = (0)$ corresponding to g. Since

$$\varphi^m_t(t, 0) = 1 \neq 0,$$

we can take x^m as a parameter instead of t, and so represent these extremals in the form

$$y_i = y_i(x, u).$$

One sees that for $t = x$,

$$\frac{D(\varphi^1, \cdots, \varphi^m)}{D(t, u_1, \cdots, u_n)} \equiv \frac{D(y_1, \cdots, y_n)}{D(u_1, \cdots, u_n)}, \qquad (u) = (0). \tag{5.7}$$

This is an identity in $t = x$. Now the first determinant in (5.7) vanishes near $t = t^1$, at most at $t = t^1$. The right hand determinant certainly does not then vanish identically near $t = t^1$. But its columns represent a base for a conjugate family of solutions of the Jacobi equations in non-parametric form. We draw the following conclusions from the theory of conjugate systems. Cf. Ch. III, §3.

If the Legendre S-condition holds along g, the conjugate points of a given point are isolated, and the jacobian $M(t, t_1)$ and its continuations, defining these conjugate points by their zeros, vanish at most to the order $m - 1$ in t.

The Hilbert integral

6. Let Λ be an n-parameter family of extremals represented in terms of the arc length t and $m - 1$ parameters (β). We suppose that t ranges over an interval $t^1 \leqq t \leqq t^2$ and that (β) is a point in an open simply-connected region in a euclidean n-space. Locally we suppose that the points on Λ are representable in the form

$$x^i = x^i(t, \beta), \tag{6.1}$$

where the functions $x^i(t, \beta)$ are of class C^2 in their arguments for (t, β) near some particular set (t^0, β^0), and where

$$\frac{D(x^1, \cdots, x^m)}{D(t, \beta_1, \cdots, \beta_n)} \neq 0 \qquad (n = m - 1).$$

In the large we assume that the totality of points on Λ forms a one-to-one continuous image on R of the complete product domain (t, β). We say then that the extremals Λ form a *field* S_1.

Locally the parameters (t, β) corresponding to a point (x) of S_1 will be functions

$$t(x), \qquad \beta_h(x) \qquad\qquad (h = 1, \cdots, n),$$

of class C^2. We set

$$r^i(x) = x^i_t(t(x), \beta(x)).$$

Locally the Hilbert integral then has the form

$$\int F_{r^i}(x, r(x))dx^i.$$

Essentially as in Ch. I, §6, we could prove the following:

A necessary and sufficient condition that the Hilbert integral be independent of the path on the field S_1 is that on this field

$$\frac{\partial}{\partial\beta_h}\left(F_{r^i}\frac{\partial x^i}{\partial\beta_k}\right) \equiv \frac{\partial}{\partial\beta_k}\left(F_{r^i}\frac{\partial x^i}{\partial\beta_h}\right) \qquad\qquad (h, k = 1, \cdots, n).$$

In particular the family of extremals (5.1) *passing through the point $t = t^1$ on g and neighboring g, forms such a field neighboring any segment of g which is simple and closed in the point set sense, on which $t \neq t^1$, and on which there is no conjugate point of $t = t^1$.*

A field S_1 on which the Hilbert integral is independent of the path will be called a *Mayer field*.

Sufficiency theorems

7. We begin by enumerating certain conditions which appear in subsequent theorems. In all of these conditions we suppose that we have an extremal g on which the arc length t increases from t^1 to t^2 inclusive, and which is locally representable in the form $x^i = \gamma^i(t)$. No generality is lost by assuming that g is without multiple points, for in the case of multiple points we have seen in §1 that we could cover the neighborhood of g by a new Riemannian space N in which g would be replaced by a covering extremal without multiple points.

By the *Jacobi S-condition* is meant the condition that there be no conjugate point of the initial point of g on g.

By the *Legendre S-condition* on g is meant the condition that

$$(7.1) \qquad F_{r^ir^j}(x, \dot{x})\lambda^i\lambda^j > 0 \qquad\qquad (i, j = 1, \cdots, m)$$

for $(x, \dot{x})$ on g and (λ) any unit vector not $\pm (\dot{x})$.

By the *Weierstrass S-condition* on g is meant the condition that

$$(7.2) \qquad E(x, \dot{x}, \sigma) > 0$$

for $(x, \dot{x})$ on g and (σ) any unit vector not $(\dot{x})$.

The problem will be said to be *positive regular* on a domain S of R if for each local representation of points (x) of S

$$F_{r^ir^j}(x, r)\lambda^i\lambda^j > 0, \tag{7.3}$$

for (x) on S and for arbitrary unit vectors (r) and (λ) of which (r) is not $\pm (\lambda)$.

We begin with the following lemma.

LEMMA 7.1. *If $F_1 \neq 0$ along g and the Weierstrass S-condition holds along g, then in each local system*

$$E(x, r, \sigma) > 0 \tag{7.4}$$

for all sets (x, r, σ) in which (r) and (σ) are unequal unit contravariant vectors and the set (x, r) is in a sufficiently small neighborhood of the sets $(x, \dot{x})$ on g.

Without loss of generality we can suppose that g is covered by a single coordinate system (x). Let the condition (7.1) with the equality added be denoted by (7.1)$'$. We note first that (7.1)$'$ must hold as a consequence of the Weierstrass S-condition. But from the condition that $F_1 \neq 0$ along g it follows that $| F_{r^ir^j} |$ has the rank $m - 1$, as one verifies from (2.12). From this fact and (7.1)$'$ it follows that (7.1) holds.

Now the roots ρ of the characteristic equation

$$| F_{r^ir^j} - \rho\delta_i^j | = 0 \tag{7.5}$$

with (x, r) on g, are all positive by virtue of (7.1), save one which is null, corresponding to the fact that $| F_{r^ir^j} |$ is always null. But from the continuity of these roots we see that the same state persists for (x, r) sufficiently near sets on g. Consequently (7.3) must hold for sets (x, r) sufficiently near sets on g and (λ) any direction different from $\pm (r)$. We turn to the definition of the Weierstrass E-function and use Taylor's formula to represent E as a function of (σ), with (r) as the point of expansion. We see that (7.4) must hold when (r) and (σ) are unequal unit vectors and (x, r, σ) is on a sufficiently small (open) neighborhood N of sets $(x, \dot{x}, \dot{x})$ on g.

But sets $(x, \dot{x}, \sigma)$ not on N, for which (σ) is a unit vector and $(x, \dot{x})$ on g form a closed ensemble for which E is bounded away from zero by virtue of (7.2). Hence for sets (x, r, σ) not on N for which (r) and (σ) are unit vectors, and (x, r) within a sufficiently small neighborhood N_1 of sets $(x, \dot{x})$ on g, E will still be positive.

The lemma holds as stated for (x, r) on N_1.

We come to the following theorem.

THEOREM 7.1. *In order that the extremal g afford a proper, strong minimum to J relative to neighboring curves of class D^1 which join g's end points, it is sufficient that the Weierstrass and Jacobi S-conditions hold and $F_1 \neq 0$ along g.*

By virtue of the preceding lemma the condition (7.4) holds in an easily applied form. To obtain a Mayer field including g we make use of the identity of conjugate points in the parametric theory with those in the non-parametric theory, and infer that the extremals issuing from a point on g prior to g's initial point A^1, but sufficiently near A^1, form a Mayer field covering g. From here on the proof is essentially the same as the proof of Theorems 7.1 and 7.2 of Ch. I.

We shall establish the following corollary of the theorem.

COROLLARY. *In order that g afford a proper, strong minimum to J it is sufficient that the Jacobi S-condition hold along g, and that the problem be positive regular along g.*

We shall show that the hypotheses of the theorem are fulfilled under the conditions of the corollary. In particular under the condition of positive regularity $| F_{r_i r_j} |$ must be of rank $m - 1$ for (x, r) on g, and hence $F_1 \neq 0$ on g.

To deduce the Weierstrass S-condition from the regularity condition we turn to the function $E(x, r, \sigma)$ and let (r) and (σ) be any two unit contravariant vectors such that (r) is not $+(\sigma)$. A use of Taylor's formula as described in the proof of Lemma 7.1 now shows that $E(x, r, \sigma) > 0$ for (r) not $-(\sigma)$. This is a consequence of the regularity condition. For $(r) = -(\sigma)$, Taylor's formula is not applicable since F is not defined for $(r) = (0)$ and such a point might enter in the application. But this difficulty is easily met. One verifies the fact that the identity

$$E(x, r, -r) = E(x, \rho, -r) + E(x, \rho, r) \tag{7.6}$$

is valid for any two unit vectors (ρ) and (r). If (ρ) is now chosen different from $\pm (r)$, the right member of (7.6) is positive by virtue of (7.3). Hence

$$E(x, r, -r) > 0$$

for (x) on g. Thus the Weierstrass S-condition on g is implied by the condition of positive regularity along g.

The corollary follows from the theorem.

Note. In order to meet the difficulty that Taylor's formula could not be applied for $(r) = -(\sigma)$, a formula known as Behaghel's formula has been introduced. In the light of the above treatment this formula is no longer necessary.

The Jacobi equations in tensor form

8. Let g be an extremal locally representable in a coordinate system (x) in the form

$$x^i = \gamma^i(t) \tag{8.1}$$

where t is the arc length along g. Corresponding to g we set

$$2\omega(\eta, \dot\eta) = F_{r_i r_j}\dot\eta^i\dot\eta^j + 2F_{r_i x_j}\dot\eta^i\eta^j + F_{x_i x_j}\eta^i\eta^j \qquad (i, j = 1, \cdots, m)$$

where the arguments (x, r) in the partial derivatives of F are taken on g. Let $x^i = x^i(t, e)$ be a family of curves joining the points t^1 and t^2 on g in the system

(x) and reducing to g for $e = 0$. Suppose that the functions $x^i(t, e)$ are of class C^2 for t on (t^1, t^2) and e near 0. The second variation takes the form

$$J''(0) = \int_{t^1}^{t^2} 2\omega(\eta, \dot{\eta})dt \qquad [\eta^i = x^i_e(t, 0)]. \tag{8.2}$$

If we change from coordinates (x) to coordinates (z), we naturally understand that η^i shall be replaced by the variation $\eta_0^i(t)$ along g of z^i with respect to e. The variation (η) is then transformed as a contravariant vector. Thus

$$\eta^i(t) = \frac{\partial x^i}{\partial z^j}\eta_0^j(t) \qquad (i, j = 1, \cdots, m)$$

where the partial derivatives of (x) are evaluated at the point t on g. In terms of the integrand $Q(z, \dot{z})$ replacing $F(x, \dot{x})$ in the system (z), we set

$$2\omega^0(\eta_0, \dot{\eta}_0) = Q_{\sigma^i\sigma^j}\dot{\eta}_0^i\dot{\eta}_0^j + 2Q_{\sigma^i z^j}\dot{\eta}_0^i\eta_0^j + Q_{z^i z^j}\eta_0^i\eta_0^j$$

evaluating the partial derivatives of Q along g as before. If $\eta^i(t)$ and $\eta_0^i(t)$ are components of class C^2 of the same contravariant vector given respectively in the systems (x) and (z), I say that

$$\frac{d}{dt}\omega_{\dot{\eta}^i} - \omega_{\eta^i} \equiv \frac{\partial z^j}{\partial x^i}\left(\frac{d}{dt}\omega^0_{\dot{\eta}_0^j} - \omega^0_{\eta_0^j}\right) \tag{8.3}$$

where the partial derivatives of z^j are evaluated at the point t on g.

To establish (8.3) consider the family of curves

$$x^i = x^i(t, e) = \gamma^i(t) + e\eta^i(t),$$

and let $z^i(t, e)$ represent the same family of curves in the system (z). As we have seen in §2, we have

$$\frac{d}{dt}F_{r^i} - F_{x^i} \equiv \frac{\partial z^k}{\partial x^i}\left[\frac{d}{dt}Q_{\sigma^k} - Q_{z^k}\right] \tag{8.4}$$

where we understand that

$$x^i = x^i(t, e), \qquad r^i = x^i_t(t, e), \qquad z^i = z^i(t, e), \qquad \sigma^i = z^i_t(t, e).$$

Equations (8.3) follow from (8.4) upon differentiating (8.4) with respect to e and setting $e = 0$.

Thus the operator

$$L_i(\eta) = \frac{d}{dt}\omega_{\dot{\eta}^i} - \omega_{\eta^i}$$

is a covariant vector provided g is an extremal.

The Jacobi equations are not all independent. In fact they satisfy the relation

$$\dot{\gamma}^i L_i(\eta) \equiv 0, \tag{8.5}$$

an identity in t for all sets (η) of class C^2 in t.

To prove (8.5) we make use of the previously established identity

$$\dot{x}^i\left(\frac{d}{dt}F_{r^i} - F_{x^i}\right) \equiv 0 \qquad (r^i = \dot{x}^i). \tag{8.6}$$

In particular we set

$$x^i = \gamma^i(t) + e\eta^i(t),$$

$$\dot{x}^i = \dot{\gamma}^i(t) + e\dot{\eta}^i(t),$$

whereupon (8.6) becomes an identity in t and e. Upon differentiating this identity with respect to e and setting $e = 0$, (8.5) results as stated.

We come to the question of the solutions of the equations $L_i(\eta) = 0$. The determinant of the coefficients of the variables $\ddot{\eta}^i$ in $L_i(\eta)$ is $| F_{r^ir^j} |$, and is therefore null. To meet the difficulty which thereby arises we replace the equations $L_i(\eta) = 0$ by the system

$$L_i(\eta) = 0, \tag{8.7)$'$}$$

$$\frac{d^2}{dt^2}(g_{ij}\dot{\gamma}^i\eta^j) = 0 \qquad (i, j = 1, \cdots, m). \tag{8.7)$''$}$$

The parenthesis in (8.7)″ is an invariant which we denote by η^T. It is the algebraic value of the projection of the vector (η) on the tangent to g at the point t. From (8.7)″ we see that

$$\eta^T = at + b,$$

where a and b are constants.

To solve the system (8.7) we introduce the auxiliary system

$$\begin{aligned} &L_i(\eta) + \mu g_{ij}\dot{\gamma}^j = 0, \\ &\frac{d^2}{dt^2}(g_{ij}\dot{\gamma}^i\eta^j) = 0. \end{aligned} \tag{8.8}$$

The determinant of the coefficients of the variables $\ddot{\eta}^i$ and μ in (8.8) is seen to be

$$\begin{vmatrix} F_{r^ir^j} & g_{ij}\dot{\gamma}^j \\ g_{ij}\dot{\gamma}^i & 0 \end{vmatrix} \equiv -F_1(\gamma, \dot{\gamma}) \neq 0.$$

Use has thereby been made of (2.11). We can solve the system (8.8) for the variables $\ddot{\eta}^i$ and μ in terms of the remaining variables $(t, \eta, \dot{\eta})$ in (8.8). But upon using (8.5) we see that $\mu \equiv 0$ in solutions of (8.8), so that (8.8) may be regarded as identical with (8.7). Accordingly (8.7) can be put in the form

$$\ddot{\eta}^i = M^i(t, \eta, \dot{\eta})$$

where $M(t, \eta, \dot{\eta})$ is linear and homogeneous in the variables η^i and $\dot{\eta}^i$.

In the conditions (8.7), $L_i(\eta)$ is a covariant and the parenthesis in (8.7)″ is an invariant. By the *Jacobi equations in tensor form* along g we understand a set of conditions of the form (8.7) for each local coordinate system (x) into which g enters. By a *solution* of these equations we mean a contravariant vector defined along g with a representation $\eta^i(t)$ of class C^2 in each coordinate system in which g enters, satisfying the corresponding system (8.7). The *identity* of two solutions is conditioned then merely by the identity of the contravariant vectors which define these solutions. A set of solutions are *dependent* if their representatives $\eta^i(t)$ in each coordinate system are dependent. It is clear that dependence in one system necessitates dependence in all.

With this understood we state the following theorem.

THEOREM 8.1. *A point $t = t''$ on g is conjugate to a point $t = t'$ on g if and only if there is a solution of the Jacobi equations in tensor form which is not identically null and which vanishes at t' and t''. Moreover the number of independent solutions vanishing at t' and t'' equals the corresponding number in a non-parametric representation of the problem in normal coordinates.*

To prove the theorem we refer the neighborhood of g to the normal coordinates of §1. The extremal g is thereby represented by the x^m axis and $g_{ij} = \delta^i_j$ along g. Along g we have

$$\eta^T = g_{ij}\dot{\gamma}^i\eta^j = \eta^m$$

so that the condition (8.7)″ here implies that

$$\eta^m = at + b.$$

Accordingly a solution (η) of (8.7) which vanishes twice must here be such that $\eta^m \equiv 0$. Moreover reference to (8.5) shows that the condition $L_m(\eta) = 0$ is always satisfied by sets (η) of class C^2 so that it may be discarded. Accordingly for solutions of (8.7) which vanish twice (8.7) reduces to the conditions

$$L_i(\eta) = 0, \qquad \eta^m = 0 \qquad (i = 1, \cdots, m-1). \tag{8.9}$$

Suppose the problem is now put into non-parametric form as in §5 with $f(x, y, y')$ as the integrand, and

$$\frac{d}{dx}\Omega_{\eta'^i} - \Omega_{\eta^i} = 0 \qquad (i = 1, \cdots, m-1), \tag{8.10}$$

the corresponding Jacobi equations set up for the x axis as an extremal with dependent variables

$$\eta^1, \cdots, \eta^n \qquad (n = m-1). \tag{8.11}$$

Using (5.5), we verify the fact that if $t = x$, the conditions (8.9), in so far as they bear on the variables (8.11), are identical with the conditions (8.10). The theorem follows directly.

The general end conditions

9. We suppose that we have given an extremal g on which the arc length t increases from t^1 to t^2 inclusive. Points near the initial and final end points of g will be denoted respectively by

$$(x^{11}, \cdots, x^{m1}) \qquad\qquad (x^{12}, \cdots, x^{m2}).$$

A curve of class D^1 neighboring g will be termed admissible if its end points are given by functions

$$x^{is} = x^{is}(\alpha^1, \cdots, \alpha^r) \qquad (0 \leqq r \leqq 2m;\ s = 1, 2) \tag{9.1}$$

for values of the parameters (α) near (0). For $r = 0$ we understand that the functions on the right symbolize the end points of g. For $r > 0$ and for (α) near (0) we suppose the functions in (9.1) are of class C^2 and that they give the end points of g when $(\alpha) = (0)$.

For $r > 0$ let $\theta(\alpha)$ be any function of (α) of class C^2. For $r = 0$, $\theta(\alpha)$ shall represent the symbol 0.

Our general functional now has the form

$$J = \theta(\alpha) + \int F(x, \dot{x})dt$$

where the integral is to be evaluated along admissible curves with end points determined by the set (α).

Our transversality condition here takes the form

$$d\theta + \Big[F_{r^i}dx^{is}\Big]_{s=1}^{s=2}, \qquad (\alpha) = (0), \tag{9.2}$$

where (x, r) is to be taken on g at the respective ends of g. The differentials $d\theta$ and dx^{is} are to be expressed in terms of the differentials $d\alpha^h$, and (9.2) regarded as an identity in these differentials. We shall now prove the following:

A necessary condition that g afford a weak minimum to J relative to neighboring admissible curves of class C^1 is that it satisfy the transversality condition (9.2).

The end conditions impose no restrictions on the end values of the parameter t. In particular we will certainly still have a minimum if we restrict ourselves to admissible curves for which the end values of t are t^1 and t^2. If we regard the problem as one in the space of the variables $(t, x^1, \cdots, x^m)$ in non-parametric form, the above transversality condition follows from the corresponding condition in non-parametric form.

The second variation

10. We have already obtained a formula for the second variation in the case of fixed end points. We consider the case $r > 0$. We again suppose the neighborhood of g covered by a single coordinate system (x).

Suppose that we have given a set of functions $\alpha^h(e)$ of class C^2 for e near 0, and a 1-parameter family of curves

$$x^i = x^i(t, e) \qquad (t^1 \leqq t \leqq t^2)$$

such that $x^i(t, e)$ is of class C^2 for t on its interval and e near 0, and such that

$$x^i(t^s, e) \equiv x^{is}(\alpha(e)), \tag{10.1}$$

$$x^i(t, 0) \equiv \gamma^i(t), \tag{10.2}$$

where $\gamma^i(t)$ defines g. For each value of e near 0 we evaluate $J = J(e)$ along the corresponding curve (10.1), taking θ as $\theta(\alpha(e))$. We readily find that

$$J''(0) = \frac{d^2\theta}{de^2} + \Big[F_{r^i}x^i_{ee}\Big]_1^2 + \int_{t^1}^{t^2} 2\omega(\eta, \dot\eta)dt,$$

where $\eta^i = x^i_e(t, 0)$, $(\alpha) = (0)$, and (x, r) in F_{r^i} is to be taken on g at the respective ends of g.

We indicate differentiation of the functions $x^{is}(\alpha)$ and $\theta(\alpha)$ with respect to α^h or α^k and evaluation at $(\alpha) = (0)$, by subscripts h or k respectively. If we set $\alpha^{h\prime}(0) = u^h$, the second variation takes the form

$$J''(0) = \beta_{hk}u^hu^k + \int_{t^1}^{t^2} 2\omega(\eta, \dot\eta)dt \qquad (h, k = 1, \cdots, r) \tag{10.3}$$

where

$$\beta_{hk} = \theta_{hk}(0) + \Big[F_{r^i}x^{is}_{hk}(0)\Big]_1^2. \tag{10.4}$$

Moreover if we differentiate (10.1) with respect to e and set $e = 0$, we find that

$$\eta^{is} - x^{is}_h(0)u^h = 0 \qquad (i = 1, \cdots, m;\ h = 1, \cdots, r;\ s = 1, 2), \tag{10.5}$$

where the superscript s on (η) indicates evaluation at $t = t^s$.

As in the non-parametric theory in Ch. II, §1, so here we are led to consider the functional (10.3) subject to (10.5). We term the conditions (10.5) the *secondary end conditions*. If a curve $\eta^i(t)$ of class C^1 and set (u) satisfy (10.5) and afford a minimum to the second variation among curves of class D^1 and sets (u) which satisfy (10.5), it is necessary that $\eta^i(t)$ satisfy the Jacobi conditions $L_i(\eta) = 0$ and a counterpart of the transversality conditions of (9.2). If we set

$$\zeta_i = \omega_{\dot\eta^i}(\eta, \dot\eta),$$

these transversality conditions take the form

$$\zeta^2_i x^{i2}_h - \zeta^1_i x^{i1}_h + \beta_{hk}u^k = 0 \qquad (i = 1, \cdots, m;\ h, k = 1, \cdots, r) \tag{10.6}$$

where ζ^s_i stands for the value of ζ_i when $t = t^s$.

We term (10.6) the *secondary transversality conditions.*

The accessory problem in tensor form

11. In order to define the accessory problem in tensor form we introduce certain new tensors. To that end we let $x^i = \gamma^i(t)$ represent the extremal g as previously, with t the arc length. Let η^i be a contravariant vector at the point

t on g. The covariant components η_i of this vector and of the vector $\dot{\gamma}^i$ have the respective forms

$$g_{ij}\eta^j, \qquad g_{ij}\dot{\gamma}^j.$$

Let us resolve (η) into components tangent and orthogonal to g respectively. The algebraic value of the component of (η) tangent to g at the point t is the invariant

$$\eta^T = g_{pq}\eta^p\dot{\gamma}^q \qquad (p, q = 1, \cdots, m). \tag{11.0}$$

The covariant vector projection of (η) on the tangent to g at the point t is then

$$\eta_i^T = \eta^T g_{ij}\dot{\gamma}^j.$$

Let (η^N) represent the covariant component of (η) orthogonal to g. We have

$$\eta_i^N = \eta_i - \eta_i^T.$$

Combining the preceding results we obtain the formula

$$\eta_i^N = g_{ij}\eta^j - (g_{ij}\dot{\gamma}^j)(g_{pq}\dot{\gamma}^q\eta^p) \tag{11.1}$$

giving η_i^N as a linear function of the variables η^i.

Our accessory problem is now formally defined by the conditions

$$\ddot{\eta}^T = 0, \tag{11.2a}$$

$$L_i(\eta) + \lambda\eta_i^N = 0 \qquad (i = 1, \cdots, m), \tag{11.2b}$$

$$\eta^{is} - x_k^{is}u^k = 0 \qquad (s = 1, 2), \tag{11.2c}$$

$$x_h^{i2}\zeta_i^2 - x_h^{i1}\zeta_i^1 + \beta_{hk}u^k = 0 \qquad (h, k = 1, \cdots, r), \tag{11.2d}$$

where η^T and η_i^N are given by (11.0) and (11.1) respectively.

The conditions of this problem are well defined and self-consistent if they are associated with a single coordinate system (x) covering the whole neighborhood of g. They are also well-defined if different coordinate systems are used. For the left member of (11.2a) is an invariant, the left members of (11.2b) define a covariant vector, those of (11.2c) a contravariant vector, and finally the left members of (11.2d) are invariants, subject to (11.2c), as we shall see.

To that end we write (11.2d) more fully in the form

$$(x_h^{i2}\zeta_i^2 + F_{ri}^2 x_{hk}^{i2}u^k) - (x_h^{i1}\zeta_i^1 + F_{ri}^1 x_{hk}^{i1}u^k) + \theta_{hk}u^k = 0. \tag{11.3}$$

The term $\theta_{hk}u^k$ is clearly an invariant. We shall prove that the second parenthesis is an invariant subject to (11.2c). That the first parenthesis is also an invariant subject to (11.2c) will follow similarly.

The statement that the parentheses in (11.3) are invariants is not yet well-defined in that we have not yet stated how ζ_i is to be transformed. To come to this point let η^i be the components of a contravariant vector associated with the point t on g in a system (x). Let η_0^i be the components of the same vector in a system (z). We have

$$\eta_0^i = \frac{\partial z^i}{\partial x^j}\eta^j \tag{11.4}$$

where the partial derivatives are evaluated at the point t on g. When η^i and $\dot{\eta}^i$ are formally given at a point t on g, we understand that $\dot{\eta}_0^i$ is then defined in the system (z) by the equations

$$\dot{\eta}_0^i = \frac{\partial z^i}{\partial x^j}\dot{\eta}^j + \frac{d}{dt}\left(\frac{\partial z^i}{\partial x^j}\right)\eta^j \tag{11.5}$$

where the coefficients of $\dot{\eta}^j$ and η^j are taken along g. This is consistent with the behavior of actual variations. With this understood the variables ζ_i and ζ_{i0} in the systems (x) and (z) respectively are defined by the formulas

$$\zeta_i = \omega_{\dot{\eta}^i}(\eta, \dot{\eta}), \qquad \zeta_{i0} = \omega^0_{\dot{\eta}_0^i}(\eta_0, \dot{\eta}_0),$$

where ω and ω^0 have been defined in §8.

While the preceding modes of transforming (η), $(\dot{\eta})$, and (ζ) into (η_0), $(\dot{\eta}_0)$, and (ζ_0) are consistent with the transformations of these entities if they *are actual variations* derived from some admissible family of curves, the following statements and their proofs are free from the necessity of setting up such an admissible family. To carry this idea through one must always understand that (η_0), $(\dot{\eta}_0)$, and (ζ_0) are *formally* defined as above in terms of (η) and $(\dot{\eta})$, and are not necessarily derived from variations.

We shall prove that the parentheses in (11.3) *are invariant subject to* (11.2c).

To that end suppose that η^i, $\dot{\eta}^i$, and (u) are given at $t = t^1$ on g, with (η) and (u) subject to (11.2c). Let the formulas

$$r^i(e) = \dot{\gamma}^i(t^1) + e\dot{\eta}^i \tag{11.6}$$

define a contravariant vector r^i in the system (x) at the point $x^{i1}(eu)$. Let $z^{is} = z^{is}(\alpha)$ be the representation of the end conditions in the system (z). Let $\sigma^i(e)$ be the contravariant components in the system (z), of the vector $r^i(e)$. As we have seen previously, we have the identity

$$F_{r^i}(x^1(eu), r(e))x_h^{i1}(eu) \equiv Q_{\sigma^i}(z^1(eu), \sigma(e))z_h^{i1}(eu) \qquad (h = 1, \cdots, r). \tag{11.7}$$

That the second parenthesis in (11.3) is an invariant subject to (11.2c) will follow upon differentiating (11.7) with respect to e and setting $e = 0$. To that end we need to establish the formulas

$$\frac{d}{de}F_{r^i}[x^1(eu), r(e)] = \omega_{\dot{\eta}^i}(\eta, \dot{\eta}) \qquad (e = 0), \tag{11.8}$$

$$\frac{d}{de}Q_{\sigma^i}[z^1(eu), \sigma(e)] = \omega^0_{\dot{\eta}_0^i}(\eta_0, \dot{\eta}_0) \qquad (e = 0), \tag{11.9}$$

where η_0^i and $\dot{\eta}_0^i$ are given by (11.4) and (11.5) in terms of η^i and $\dot{\eta}^i$.

To establish (11.8) we note that

$$\frac{d}{de} F_{r^i} = F_{r^i r^j} \frac{dr^j}{de} + F_{r^i x^j} x_h^{j1} u^h \qquad (e = 0).$$

Use of (11.6) and (11.2c) then shows that

$$\frac{d}{de} F_{r^i} = F_{r^i r^j} \dot{\eta}^j + F_{r^i x^j} \eta^j = \omega_{\dot{\eta}^i}(\eta, \dot{\eta}),$$

as desired.

To establish (11.9) recall that at the point $z^i = z^{i1}(eu)$

$$\sigma^i \equiv \frac{\partial z^i}{\partial x^j} r^j(e) \tag{11.10}$$

by definition of $\sigma^i(e)$. Upon differentiating (11.10) with respect to e and setting $e = 0$ we find that

$$\frac{d\sigma^i}{de} = \frac{\partial z^i}{\partial x^j} \frac{dr^j}{de} + r^j \frac{\partial^2 z^i}{\partial x^j \partial x^p} \frac{dx^{p1}}{de} \qquad (e = 0,\ t = t^1). \tag{11.11}$$

But in (11.11) when $e = 0$ and $t = t^1$,

$$\frac{dx^{p1}}{de}(eu) = x_h^{p1} u^h = \eta^p, \tag{11.12}$$

since (η) is subject to (11.2c). Upon making use of (11.6), (11.11) takes the form

$$\frac{d\sigma^i}{de} = \frac{\partial z^i}{\partial x^j} \dot{\eta}^j + \frac{d}{dt}\left(\frac{\partial z^i}{\partial x^p}\right) \eta^p \qquad (e = 0,\ t = t^1) \tag{11.13}$$

where until after the differentiation the parenthesis is taken at the point t on g. Referring to (11.5) we thus see that when $e = 0$

$$\frac{d\sigma^i}{de} = \dot{\eta}_0^i, \tag{11.14}$$

a formula of use in the proof of (11.9).

To complete the proof of (11.9) we note that

$$\frac{dz^i}{de} = \frac{\partial z^i}{\partial x^p} \frac{dx^p}{de},$$

and upon referring to (11.12) and (11.4) we see that

$$\frac{dz^i}{de} = \eta_0^i \qquad (e = 0). \tag{11.15}$$

Formula (11.9) now follows from (11.14) and (11.15).

To establish the invariance of the second parenthesis in (11.3) we differentiate (11.7) with respect to e and set $e = 0$. Formulae (11.8) and (11.9) lead us to ζ^1_i and ζ^1_{i0} respectively, and we find that when $e = 0$

$$\zeta^1_i x^{i1}_h + F^1_{ri} x^{i1}_{hk} u^k = \zeta^1_{i0} z^{i1}_h + Q^1_{\sigma i} z^{i1}_{hk} u^k$$

as desired.

The invariance of the first parenthesis in (11.3) subject to (11.2c) follows similarly.

By a *solution* of the accessory problem in tensor form we mean a contravariant vector μ defined at each point t of g and possessing components of class C^2 in each local coordinate system into which g enters. The components of μ in a given coordinate system shall satisfy the conditions (11.2a) and (11.2b) corresponding to this system. The components of μ in any two systems covering the neighborhoods of the respective end points of g must satisfy the conditions (11.2c) and (11.2d), corresponding to these coordinate systems.

We observe that the operators

$$H_i(\eta, \lambda) = L_i(\eta) + \lambda \eta^N_i \qquad (i = 1, \cdots, m)$$

have the property already established for $L_i(\eta)$ in §8, that the relation

$$\dot{\gamma}^i H_i(\eta, \lambda) \equiv 0$$

is an identity in t for every set (η) of class C^2. Proceeding as in the treatment of the equations $L_i(\eta) = 0$ in §8 we can now show that if $F_1 \neq 0$ along g, (11.2a) and (11.2b) can be put in the form

$$\ddot{\eta}^i = M^i(t, \eta, \dot{\eta}, \lambda) \qquad (i = 1, \cdots, m)$$

where M^i is linear and homogeneous in the variables (η) and $(\dot{\eta})$, with coefficients which are of class C^1 in t and λ.

A. W. Tucker [2] has taken up the question of the invariance of the left number of (11.2d) from a more general point of view. He has introduced a process of generalized covariant differentiation appropriate to the problem and has given a new and elegant proof of the invariance in question.

The non-tangency condition

12. We here introduce the analogue of the non-tangency condition of §6, Ch. II. We note its invariant character and find an adequate mode of representing it. We shall make important use of it.

The set of points in the space of the $2m$ variables

$$(x^{11}, \cdots, x^{m1}, x^{12}, \cdots, x^{m2})$$

which are given by the equations

$$x^{i1} = \gamma^i(t), \qquad x^{i2} = \gamma^i(\tau),$$

for t and τ near t^1 and t^2 respectively define a regular 2-manifold which we term the *extremal manifold.* The r-dimensional manifold $x^{is} = x^{is}(\alpha)$ will be called the *terminal manifold.* The terminal manifold and the extremal manifold intersect in the point determined by $(\alpha) = (0)$.

We shall assume that the terminal manifold is regular $(r > 0)$.

Our non-tangency condition $(r > 0)$ *is the condition that the terminal and extremal manifolds have no common tangent line. In case* $r = 0$ *we understand that the non-tangency condition is always fufilled.*

One readily sees that a necessary and sufficient condition that the non-tangency condition hold is that the matrix

$$\left\| \begin{matrix} x_h^{i1}(0) & \dot\gamma^i(t^1) & 0 \\ x_h^{i2}(0) & 0 & \dot\gamma^i(t^2) \end{matrix} \right\| \qquad (i = 1, \cdots, m;\ h = 1, \cdots, r) \tag{12.0}$$

of $r + 2$ columns and $2m$ rows be of rank $r + 2$.

We consider the class of variations locally of the form

$$\eta^i = \rho(t)\dot\gamma^i(t), \tag{12.1}$$

where $\rho(t)$ is a function of class C^2 in t, and $\gamma^i(t)$ represents g. We call these variations *tangential variations.* We shall show that the tangential variations are solutions of the equations $L_i(\eta) = 0$. In fact for values of a constant e sufficiently near zero the functions

$$x^i = \gamma^i(t + e\rho(t))$$

afford admissible representations of g, and must accordingly satisfy the Euler equations

$$\frac{d}{dt} F_{r^i} - F_{x^i} = 0. \tag{12.2}$$

Upon differentiating (12.2) with respect to e and setting $e = 0$, one finds that (η) in (12.1) satisfies the equations $L_i(\eta) = 0$ as stated.

These tangential variations are also solutions of the conditions (11.2b), namely

$$H_i(\eta, \lambda) = L_i(\eta) + \lambda\eta_i^N = 0.$$

In fact for a tangential variation (η) the corresponding vector η_i^N is null as one can verify from (11.1). In order that a tangential variation be a solution of (11.2a), that is, $\ddot\eta^T \equiv 0$, it is necessary and sufficient that $\ddot\rho(t) \equiv 0$. Thus tangential variations of the form

$$\eta^i = (at + b)\dot\gamma^i(t) \qquad (a, b \text{ constant})$$

are solutions of (11.2a) and (11.2b), and these are the only solutions of the form (12.1).

When we come to conditions (11.2c) we have the following lemma.

LEMMA 12.1. *If the non-tangency condition holds, there are no non-null tangential variations which are solutions of conditions* (11.2a) *and* (11.2c).

If the lemma were false, there would exist constants a and b not both zero, and constants (u), in case $r > 0$, such that

$$(at^1 + b)\dot{\gamma}^i(t^1) - x_h^{i\,1} u^h = 0,$$
$$(at^2 + b)\dot{\gamma}^i(t^2) - x_h^{i\,2} u^h = 0.$$

The matrix (12.0) could not then be of rank $r + 2$. From this contradiction we infer the truth of the lemma.

Characteristic solutions in tensor form

13. Relative to our accessory problem in tensor form we formally define characteristic solutions, characteristic roots, and indices of characteristic roots as in Ch. II, §4. Characteristic solutions are defined by contravariant vectors while characteristic roots and their indices are invariants.

If one refers the neighborhood of g to the normal coordinates of §1 and sets up the corresponding non-parametric problem as in §5, one thereby obtains a special non-parametric accessory problem which we shall term a *normal accessory problem in non-parametric form.* Concerning this accessory problem all the results of the non-parametric theory are available. The principal object of this section will be to relate the general accessory problem in tensor form to this normal accessory problem in non-parametric form.

We represent the neighborhood of g in terms of the normal coordinates of §1. As in §5 we then set

$$(y_1, \cdots, y_n, x) = (x^1, \cdots, x^m)$$

and

$$f(x, y_1, \cdots, y_n, p_1, \cdots, p_n) = F(y_1, \cdots, y_n, x, p_1, \cdots, p_n, 1). \tag{13.1}$$

Corresponding to the x axis as an extremal we set up the form $\Omega(\eta, \eta')$ as in Ch. II except that it will be convenient here to use superscripts on the $n = m - 1$ variables η^μ, instead of subscripts. In terms of the given end conditions

$$x^{is} = x^{is}(\alpha) \qquad (i = 1, \cdots, m),$$

the end conditions in the non-parametric form become

$$\begin{aligned} y_\mu^s &= y_\mu^s(\alpha) = x^{\mu s}(\alpha) \qquad (\mu = 1, \cdots, n), \\ x^s &= x^s(\alpha) = x^{ms}(\alpha). \end{aligned} \tag{13.2}$$

The accessory problem of Ch. II now becomes our *normal accessory problem in non-parametric form.* It is given as follows:

$$\frac{d}{dx}\Omega_{\eta'^\mu} - \Omega_{\eta^\mu} + \lambda\eta^\mu = 0 \qquad (\mu = 1, \cdots, n), \tag{13.3a}$$

$$\eta^{\mu s} - y^s_{\mu h} u^h = 0 \qquad [(\alpha) = (0);\ s = 1, 2], \tag{13.3b}$$

$$y^2_{\mu k} \zeta^{*2}_\mu - y^1_{\mu k} \zeta^{*1}_\mu + b_{hk} u^k = 0 \qquad (h, k = 1, \cdots, r), \tag{13.3c}$$

with

$$b_{hk} = \theta_{hk} + \Big[f x^s_{hk} + f_x x^s_h x^s_k + f_{y_\mu}(x^s_h y^s_{\mu k} + x^s_k y^s_{\mu h}) + f_{p_\mu} y^s_{\mu h k} \Big]_1^2, \tag{13.4}$$

where we have added a star to $\zeta^*_\mu(x)$ to distinguish it from $\zeta_i(t)$ in the parametric form.

The variations $\eta^\mu(x)$ in the non-parametric problem will be distinguished from the variations $\eta^i(t)$ in the parametric problem by the use of the superscripts μ. We understand that $\mu = 1, \cdots, n = m - 1$ and $i = 1, \cdots, m$.

We turn to the accessory problem (11.2) in tensor form. If the coordinates are normal, the components of (η) tangent and orthogonal to g are given by the equations

$$\begin{aligned} \eta^T &= \eta^m, \\ \eta^N_\mu &= \eta^\mu \qquad (\mu = 1, \cdots, n = m - 1), \\ \eta^N_m &= 0, \end{aligned} \tag{13.5}$$

as follows from (11.0) and (11.1). Moreover the last equation in (11.2b) here takes the form

$$L_m(\eta) + \lambda \eta^N_m \equiv L_m(\eta) \equiv 0,$$

and may be discarded. See (13.8)″ and (13.9).

The problem (11.2) then becomes what we term the *normal accessory problem in parametric form.* It is as follows:

$$\ddot{\eta}^m = 0, \tag{13.6a}$$

$$L_\mu(\eta) + \lambda \eta^\mu = 0 \qquad (\mu = 1, \cdots, n), \tag{13.6b}$$

$$\eta^{is} - x^{is}_k u^k = 0 \qquad (i = 1, \cdots, m;\ s = 1, 2), \tag{13.6c}$$

$$x^{i2}_h \zeta^2_i - x^{i1}_h \zeta^1_i + \beta_{hk} u^k = 0 \qquad (h, k = 1, \cdots, r). \tag{13.6d}$$

We shall show that the problem (13.3) is essentially equivalent to the problem (13.6). Before coming to the principal lemma we need to evaluate the partial derivatives of F in terms of those of f.

From the definition of f in (13.1) and the homogeneity of F we have

$$r^m f\left(x, y_1, \cdots, y_n, \frac{r^1}{r^m}, \cdots, \frac{r^n}{r^m}\right) \equiv F(y_1, \cdots, y_n, x, r^1, \cdots, r^m) \qquad (r^m > 0). \tag{13.7}$$

We see then that along the x axis ($r^1 = \cdots = r^n = 0,\ r^m = 1$)

$$F_{r^\mu} = f_{p_\mu}, \qquad F_{r^m} = f, \tag{13.8$'$}$$

$$F_{x^\mu r^\nu} = f_{y_\mu p_\nu}, \qquad F_{x^\mu r^m} = f_{y_\mu} \qquad (\mu, \nu = 1, \cdots, n) \tag{13.8$''$}$$

and

$$\left\| F_{r^i r^j} \right\| \equiv \left\| \begin{matrix} f_{p_\mu p_\nu} & 0 \\ 0 & 0 \end{matrix} \right\| \qquad (i, j = 1, \ldots, m). \tag{13.9}$$

From (13.8) and the fact that the x axis is an extremal, we find that along the x axis

$$F_{r^\mu x^m} = f_{y_\mu}, \qquad F_{r^m x^m} = f_x \qquad (\mu = 1, \cdots, n). \tag{13.10}$$

We shall now prove the following lemma.

LEMMA 13.1. *If $\eta^i(t)$, $i = 1, \cdots, m$, satisfies* (13.6) *with constants λ and (u), the corresponding functions $\eta^\mu(x)$, $\mu = 1, \cdots, n$, satisfy* (13.3) *with the same constants λ and (u).*

To show that the functions $\eta^\mu(x)$ satisfy (13.3a) we observe that any function $\eta^m(t)$ of class C^2 defines a tangential variation $(0, \cdots, 0, \eta^m)$ and satisfies the conditions

$$L_i(0, \cdots, 0, \eta^m) \equiv 0 \qquad (i = 1, \cdots, m).$$

Hence we have identically,

$$L_i(\eta^1, \cdots, \eta^m) \equiv L_i(\eta^1, \cdots, \eta^n, 0).$$

But from (13.8)″ and (13.9) we see that if $t = x$,

$$L_\mu(\eta^1, \cdots, \eta^n, 0) \equiv \frac{d}{dx}\Omega_{\eta'^\mu} - \Omega_{\eta^\mu} \qquad (\mu = 1, \cdots, n).$$

Combining these two identities we find that

$$L_\mu(\eta) + \lambda\eta^\mu \equiv \frac{d}{dx}\Omega_{\eta'^\mu} - \Omega_{\eta^\mu} + \lambda\eta^\mu \qquad (\mu = 1, \cdots, n)$$

provided $t = x$. Thus the lemma is true in so far as the satisfaction of conditions (13.3a) is concerned.

Moreover the conditions (13.3b) are a consequence of conditions (13.6c), in fact are a subset of conditions (13.6c). We continue with the following:

(A). *The conditions* (13.3c) *are satisfied by $\eta^\mu(x)$, the constants (u), and corresponding functions $\overset{*}{\zeta}_\mu$.*

To prove (A) we shall evaluate the various entities entering in (13.6d) in terms of entities entering in (13.3c), substitute our results in (13.6d) and thereby obtain (13.3c).

From the definitions of b_{hk} and β_{hk} we find that

$$\beta_{hk} = b_{hk} - \Big[f_x x_h^s x_k^s + f_{y_\mu}(x_h^s y_{\mu k}^s + x_k y_{\mu h}^s) \Big]_1^2 \tag{13.11}$$

making use thereby of (13.8). Making use of (13.9) we see that

$$\zeta_\mu = \overset{*}{\zeta}_\mu + F_{r^\mu s^m}\eta^m \qquad (\mu = 1, \cdots, n),$$

$$\zeta_m = \qquad F_{r^m s^i}\eta^i \qquad (i = 1, \cdots, m),$$

and then upon using (13.10), (13.8) and (13.6c) we find that (for s not summed)

$$(13.12) \qquad \begin{aligned} \zeta_\mu^s &= \overset{*}{\zeta}{}_\mu^s + f_{y_\mu} x_k^s u^k \qquad (\mu = 1, \cdots, n), \\ \zeta_m^s &= (f_{y_\mu} y_{\mu k}^s + f_x x_k^s) u^k. \end{aligned}$$

Upon substituting the right members of (13.11) and (13.12) in (13.6d), (13.3c) results as stated.

The lemma is thereby proved.

The preceding lemma will be strengthened and completed in the following theorem.

THEOREM 13.1. *If the non-tangency condition holds, then for a given λ there is a one-to-one correspondence between the solutions of the normal accessory problem in parametric form and the solutions of the normal accessory problem in non-parametric form in which a solution*

$$(13.13) \qquad \eta^i = \varphi^i(t) \qquad (i = 1, \cdots, m)$$

of (13.6) *corresponds to the solution*

$$(13.14) \qquad \eta^\mu = \varphi^\mu(x) \qquad (\mu = 1, \cdots, n)$$

of (13.3). *Moreover under this correspondence linearly independent solutions correspond to linearly independent solutions.*

A solution of (13.6) uniquely determines the constants (u) with which it satisfies the terminal conditions (13.6c) since the terminal manifold by hypothesis is regular. According to the preceding lemma a solution $\varphi^i(t)$ of (13.6) with its constant λ and above constants (u) will determine a solution (13.14) which will satisfy (13.3) with the same constants (u) and λ.

On the other hand two solutions (13.13) which determine the same solution (13.14) must be identical. For their difference would be a solution of (13.6) of the form

$$(0, \cdots, 0, \Delta\varphi^m(t))$$

and thus a tangential solution. But we have seen in §12 that if the non-tangency condition holds, tangential solutions of the accessory problem in tensor form must be null. Thus one and only one solution of the form (13.13) gives rise to the solution (13.14).

Finally each solution (13.14) of (13.3) gives rise to a solution (13.13) of (13.6).

For if the solution (13.14) satisfies (13.3) with constants λ and (u), the functions $\varphi^i(t)$, of which $\eta^m = \varphi^m(t)$ satisfies the conditions

$$\ddot{\eta}^m \equiv 0, \qquad \eta^{ms} = x^s_h(0)u^h \qquad (h = 1, \cdots, r),$$

will satisfy (13.6).

The preceding shows that the null solution corresponds to the null solution, and from this it follows that linearly independent solutions correspond to linearly independent solutions.

The theorem is thereby proved.

The general index form

14. We now suppose the Legendre S-condition of §7 holds along g, and that g satisfies the transversality conditions. We are also assuming that the terminal manifold is regular and that the non-tangency condition holds.

Suppose the arc length t on g increases from t^1 to t^2 inclusive. Let

$$a_0, a_1, \cdots, a_p, a_{p+1} \qquad (a_0 = t^1, a_{p+1} = t^2)$$

be a set of increasing values of t so chosen as to divide g into segments on which there are no pairs of conjugate points. We cut across g at the point at which $t = a_q$, $q = 1, \cdots, p$, by a regular n-dimensional manifold M_q of class C^2, of the form

$$x^i = X^i_q(\beta_1, \cdots, \beta_n) \qquad (n = m - 1),$$

intersecting g when $(\beta) = (0)$, but not tangent to g. We term the manifolds M_q *intermediate manifolds*. Let

$$A^1, P^1, \cdots, P^p, A^2 \tag{14.1}$$

be a sequence of points of which A^1 and A^2 are admissible end points determined by parameters (α) in the end conditions, and P^q is on the manifold M_q near g. Points (14.1) sufficiently near g can be successively joined by extremal arcs near g to form a broken extremal E_0. Let (v) be a set of parameters of which the first r are the parameters (α), and the remaining the successive sets of parameters (β) of the points P^q. The value of J along the broken extremal E_0 will be denoted by $J(v)$. The function $J(v)$ will be termed an *index function* belonging to g, to the given functional, and to the end conditions.

Our basic index form is the form

$$P(z) = J_{v_iv_j}(0)z_iz_j \qquad (i, j = 1, \cdots, \delta)$$

where δ is the number of variables (v).

The index form $P(z)$ is an invariant clearly independent of the local representations of R used to evaluate it. If in particular we represent the neighborhood of g by means of a normal system of coordinates (x, y), the index form $P(z)$ may be identified with the index form $Q(z, 0)$ of Ch. II set up for the segment

$$t^1 \leqq x \leqq t^2$$

of the x axis as an extremal. We must of course use the same intermediate manifolds and the same variables (v) in both cases.

By virtue of the correspondence between characteristic solutions of the accessory problem in tensor form and characteristic solutions of the normal accessory problem in non-parametric form as given in Theorem 13.1, and by virtue of the results of Theorems 2.2 and 2.3 of Ch. III concerning $Q(z, 0)$, we have the following fundamental theorem.

THEOREM 14.1. *The nullity of the index form $P(z)$ equals the index of $\lambda = 0$ as a characteristic root of the accessory problem in tensor form, and the index of $P(z)$ equals the number of characteristic roots of this problem which are negative.*

The following is an easy corollary of the theorem and the relation of its conditions to the corresponding conditions in non-parametric form.

COROLLARY. *In order that an extremal g afford a proper, strong, relative minimum to J under our general end conditions, it is sufficient that g satisfy the transversality conditions, that the Weierstrass S-condition hold along g, that $F_1 \neq 0$ along g, that the non-tangency condition hold, and that all characteristic roots of the accessory problem be positive.*

Particular consequences of the hypotheses of the corollary of importance in its proof, are that the Legendre S-condition holds, that $P(z)$ is positive definite, and then from the non-parametric theory, the fact that there will be no pairs of mutual conjugate points on g. The conclusion of Lemma 7.1 also holds and the proof can be completed with a suitable use of $J(v)$ and Mayer fields.

The case of end manifolds

15. We shall now take up the question of the existence of a family of extremals cut transversally by a manifold M. As far as the author knows this has not previously been treated for the case of general dimensionality.

Let M be locally represented in the form

$$x^i = \varphi^i(\alpha^1, \cdots, \alpha^r) \qquad (0 < r < m), \tag{15.1}$$

where the functions φ^i are of class C^3 for (α) near (α_0). We suppose that M is regular, and cuts g transversally when $(\alpha) = (\alpha_0)$, at g's first end point. In the functional J we suppose $\theta(\alpha)$ is of class C^3. We begin by seeking solutions (α) and (r) of the transversality conditions

$$F_{r^i}(\varphi(\alpha), r)\varphi^i_h(\alpha) + \theta_h(\alpha) = 0 \qquad (h = 1, \cdots, r) \tag{15.2$'$}$$

and the side condition

$$r^i r^i = 1. \tag{15.2$''$}$$

Here h indicates differentiation with respect to α^h.

We shall suppose that g is regularly represented by functions $x^i = x^i(t)$ with $t = t_0$ at the initial point of g. We suppose that t is the ordinary arc length of g

in the euclidean space (x). We denote the values of (x) and $(\dot{x})$ on g when $t = t_0$ by (x_0) and (r_0). Our initial solution of (15.2) is then $(\alpha, r) = (\alpha_0, r_0)$.

Let $\sigma_i^k(\alpha)$, $k = 1, \cdots, m - r$, be a set of $m - r$ independent solutions of the homogeneous equations

$$\sigma_i \varphi_h^i(\alpha) = 0 \qquad (h = 1, \cdots, r).$$

These solutions can be so chosen as to be of class C^2 in (α) for (α) near (α_0). Conditions (15.2) can then be written in the form

$$(15.3)' \qquad F_{r^i}(\varphi(\alpha), r) + \rho^k \sigma_i^k(\alpha) = A_i(\alpha),$$

$$(15.3)'' \qquad r^i r^i = 1,$$

where $A_i(\alpha)$ is a particular solution of class C^2 of the equations

$$A_i \varphi_h^i(\alpha) + \theta_h(\alpha) = 0.$$

The variables (ρ) must now be added to our unknowns. Let (ρ_0) represent the set (ρ) which satisfies (15.3) with $(\alpha, r) = (\alpha_0, r_0)$.

The matrix of the partial derivatives of the left members of (15.3) with respect to (r) and (ρ) has the form

$$\left\| \begin{matrix} F_{r^i r^j} & \sigma_i^k \\ 2r^j & 0 \end{matrix} \right\| \qquad (i, j = 1, \cdots, m; k = 1, \cdots, m - r).$$

Now at least one of its $(m + 1)$-square determinants Δ_k obtained by omitting all but the kth of the last $m - r$ columns does not vanish at (α_0, r_0). For we have

$$\Delta_k = -2\, F_1(x, r) r^i \sigma_i^k$$

by virtue of (2.11). Thus $\Delta_k = 0$ for all values of k only if *in the euclidean space* (x) the $(m - r)$ directions (σ^k) are orthogonal to the direction (r_0). But the directions (σ^k) are orthogonal to M at (α_0) and constitute a base for such directions. Any direction orthogonal to all of the directions (σ^k) must be tangent to M at (α_0). Hence if Δ_k were zero for each value of k, the direction (r_0) would be tangent to M at (α_0) contrary to the non-tangency condition. Thus at least one of the determinants Δ_k, say Δ_{m-r}, is not zero.

The equations (15.3) can accordingly be solved for ρ_{m-r} and the variables (r) in terms of the variables (α) and the variables $\rho_1, \cdots, \rho_{m-r-1}$, at least neighboring the initial solution (α_0, r_0, ρ_0). Let us set

$$(15.4)' \qquad v_k = \rho_k, \qquad v_{k0} = \rho_{k0} \qquad (k = 1, \cdots, q = m - r - 1),$$

and write the solution in the form

$$(15.4)'' \qquad r^i = r^i(\alpha, v), \qquad \rho_{m-r} = \rho_{m-r}(\alpha, v),$$

for (α, v) near (α_0, v_0). Following the methods of §4, taking φ as $r^i r^i$, we can now be assured of the existence of an $(m - 1)$-parameter family of extremals of the form

$$x^i = h^i(t, \alpha, v) \tag{15.5}$$

along which t is the arc length in the space (x), and which satisfy the initial conditions

$$\varphi^i(\alpha) \equiv h^i(t_0, \alpha, v), \tag{15.6}$$

$$r^i(\alpha, v) \equiv h^i_t(t_0, \alpha, v), \tag{15.7}$$

where h^i is of class C^2 in its arguments near (t_0, α_0, v_0). The extremal of this family determined by (α, v) will be cut transversally by M when $t = t_0$.

We shall now establish the following theorem.

THEOREM 15.1. *The family of extremals* (15.5) *cutting M transversally are so represented that the jacobian*

$$M(t) = \frac{D(h^1, \cdots, h^m)}{D(t, \alpha_1, \cdots, \alpha_r, v_1, \cdots, v_q)} \qquad [(\alpha, v) = (\alpha_0, v_0); r + q = m - 1],$$

evaluated on g vanishes at $t = t_0$ to the qth order.

Without loss of generality we can suppose a non-singular linear transformation of the variables (x) has been made so that

$$r_0^1 = \cdots = r_0^{m-1} = 0, \tag{15.8}$$

and on g at the initial point of g,

$$\left\| F_{r^i r^j} \right\| = \left\| \begin{matrix} I & 0 \\ 0 & 0 \end{matrix} \right\| \tag{15.9}$$

where I is a unit $(m - 1)$-square matrix.

By virtue of (15.6) we see that the last q columns of $M(t)$ vanish at t_0. We accordingly have

$$M(t) = (t - t_0)^q A(t)$$

where a use of (15.6) and (15.7) discloses the fact that

$$A(t_0) = | r^i_0, \varphi^i_h(\alpha_0), r^i_{v_k}(\alpha_0) |. \tag{15.10}$$

Here $h = 1, \cdots, r$, $k = 1, \cdots, q$, and $A(t)$ is continuous in t for t near t_0. We shall show that $A(t_0) \neq 0$.

To that end we regard (15.3)′ and (15.3)″ as identities in (α, v) subject to (15.4). Upon then differentiating (15.3)″ with respect to v_h and using (15.8) we see that

$$r^m_{v_h}(\alpha_0, v_0) = 0 \qquad (h = 1, \cdots, q).$$

Thus the last q columns of $A(t_0)$ are orthogonal to the first. Upon similarly differentiating (15.3)′ and using (15.9) we find at (α_0, v_0) that

$$r^i_{v_h} + \sigma^h_i + \rho^{m-r}_{v_h}\sigma^{m-r}_i = 0 \qquad (h = 1, \cdots, q). \tag{15.11}$$

From (15.11) we see that the last q columns of $A(t_0)$ are orthogonal to M at (α_0) and are moreover independent directions. In sum the last q columns of $A(t_0)$ represent directions orthogonal to the first $r + 1$ columns of $A(t_0)$. Since the first $r + 1$ columns are likewise independent, $A(t_0) \neq 0$.

The proof of the theorem is now complete.

We can, if we please, change the parameter t in the family (15.5) to the arc length on R. With this understood we can "continue" the family (15.5) as in §4. The resulting jacobians of the form of $M(t)$ will be called the *focal determinants* corresponding to the manifold M. We term their zeros on g the focal points of M. Exactly as in the case of the determinants defining conjugate points in §5, so here, we can introduce normal coordinates and show that the focal determinants vanish at the same points and to the same orders as the focal determinant of M in the non-parametric theory. A first conclusion is that if g affords a weak minimum to J, *it is necessary that there be no focal point of M between the end points of g.*

The basic theorem here is the following:

THEOREM 15.2. *If M cuts g transversally at g's initial point A^1 without being tangent to g at A^1, and if the Legendre S-condition holds along g, the index form $P(z)$ corresponding to the conditions that A^1 lie on M and A^2 be fixed has an index equal to the number of focal points of M on g between A^1 and A^2. The nullity of $P(z)$ equals the index of A^2 as a focal point of M.*

With the aid of this theorem one sees that sufficient conditions that an extremal afford a proper, strong minimum to J, relative to admissible curves which join the manifold M to the second end point of g, are that M cut g transversally without being tangent to g, that $F_1 \neq 0$ along g, that the Weierstrass S-condition hold along g, and that there be no focal point of M on g between M and A^2 including A^2.

The final theorems on the case of two end manifolds as given in Ch. III can be similarly carried over into theorems valid on R. The same is true of the theorems on periodic extremals to which we shall return in Ch. VIII. In general the results of this chapter furnish a mechanism which enables one to pass freely from the parametric to the non-parametric case. The results are freed from the necessity of holding to a single euclidean space or any one coordinate system, and, most important of all, the invariant or tensor forms of the basic elements and hypotheses have been set forth.

CHAPTER VI

THE CRITICAL SETS OF FUNCTIONS

The theory of critical points of functions is concerned with the relations of critical points, classified in the small, with the topological characteristics of the domain on which the functions are defined. The basic relations were first discovered for the case of non-degenerate critical points, that is, for critical points at which the hessian of the function is not zero. To extend the theory one met the difficult and basic problem of characterizing degenerate critical loci so that these loci might be counted as finite sets of non-degenerate critical points. Such an extension led to a radical change in the topological aspects of the theory. Deformations entered more, and combinatorial analysis situs less.

The choice of methods has been largely influenced by the desire to adopt a procedure which might serve as a model for the case of functionals. It has been found that the underlying theory can be given a relatively abstract topological form of great elasticity. This abstract form embraces three different particularized theories, namely, the theory of critical points of the present chapter, the theory of functionals of the following chapter in which the curve replaces the point, and the theory of the space Ω of Ch. VIII in which subgroups of substitutions play so large a part. Each of these three theories remains highly individual in the nature of the deformations peculiar to it.

The present chapter contains a number of applications. It is impossible however to give here an idea of the scope of the theory from this point of view. It will be sufficient to say that such applications are numerous in analysis, geometry, and physics, and the number is constantly increasing (Kiang [1, 2, 3], Birkhoff [7]).

The non-degenerate case

1. Let f be a single-valued function of a point on a circle. Suppose that f is of class C^2 in terms of the arc length on the circle. Suppose also that $f'' \neq 0$ when $f' = 0$. Let M_0 and M_1 be respectively the number of relative minima and maxima of f. We have the relations

$$M_0 \geqq 1,$$

$$M_0 - M_1 = 0.$$

To proceed directly to a general case suppose that f is a single-valued function of the point P on the Riemannian manifold R of Ch. V. We suppose that f is not constant on R. In terms of each set of local coordinates (x) we also suppose that f is a function $\psi(x)$ of class C^2. We term such a function f *admissible.*

A point on R at which each of the first partial derivatives of $\psi(x)$ vanishes will be called a *critical point* of f. Suppose that $(x) = (0)$ defines such a critical point in the system (x). If

$$|\psi_{x^i x^j}(0)| \neq 0 \qquad (i, j = 1, \cdots, m),$$

the critical point $(x) = (0)$ will be termed *non-degenerate*. One sees that the property of non-degeneracy of a critical point is independent of the local coordinate system (x) employed to represent f. If the critical points of f are all non-degenerate, f will be termed *non-degenerate*. In case f is non-degenerate one recognizes that the conditions

$$\psi_{x^i} = 0 \qquad (i = 1, \cdots, m)$$

have at most isolated solutions. The critical points of f on R are then isolated, and hence finite in number.

As is well known, a suitable, non-singular, homogeneous, linear transformation of the variables (x) into a set of variables (z) will effect a reduction

$$\psi_{x^i x^j}(0)x^i x^j = -z_1^2 - \cdots - z_k^2 + z_{k+1}^2 + \cdots + z_r^2,$$

where $0 \leqq k \leqq r$. Here $r = m$ if the critical point is non-degenerate. The number k is called the *index* of the critical point. It is clearly independent of the local coordinate system used to represent f. There are $m + 1$ possible indices for a critical point. A non-degenerate critical point of index zero affords a relative minimum to f, while one of index m affords a relative maximum.

The following theorem comes first in the history of our development of the subject (Morse [1], Morse [20] with van Schaack).

THEOREM 1.1. *The numbers M_i of critical points of index i of a non-degenerate function f defined on R, and the connectivities R_j* (mod 2) *of R, satisfy the following relations*:

$$\begin{aligned}
&M_0 \geqq R_0, \\
&M_0 - M_1 \leqq R_0 - R_1, \\
&M_0 - M_1 + M_2 \geqq R_0 - R_1 + R_2, \qquad (1.1)\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&M_0 - M_1 + \cdots + (-1)^m M_m = R_0 - R_1 + \cdots + (-1)^m R_m.
\end{aligned}$$

A proof of this theorem will be a part of a treatment of the general case which includes the degenerate as well as the non-degenerate case.

Of the relations (1.1) the first is merely a statement of the necessity of the existence of at least R_0 relative minima. The second relation in the form

$$M_1 \geqq M_0 + R_1 - R_0$$

is essentially Birkhoff's minimax principle (Birkhoff [1]) although not stated by Birkhoff in precisely this form. The last relation in the case $m = 2$ was known to Poincaré [1]. The last of these relations for the general m was discovered independently by the author at about the time Lefschetz [3] and Hopf [1] proved the corresponding basic equality concerning the signed index sum of fixed points of a transformation. In this connection we note the following corollary of the theorem.

COROLLARY 1.1. *The numbers M_i and R_i of the theorem satisfy the relations*

$$M_i \geqq R_i \qquad (i = 0, 1, \cdots, m). \tag{1.2}$$

From the set of all relations (1.1) one can thus infer the existence of at least

$$R_0 + R_1 + \cdots + R_m$$

critical points on R.

We regard the above corollary as a statement of the number of critical points which are *topologically necessary*. We term the number

$$Q_i = M_i - R_i$$

the number of critical points *in excess* of those topologically necessary. The relations (1.1) imply much more than the relations (1.2). In fact they imply the relations (1.2) together with the necessary limitations on the numbers Q_i. As a particular example of such a limitation we state the following corollary.

COROLLARY 1.2. *In the non-degenerate case the numbers Q_i satisfy the relations*

$$Q_{i-1} + Q_{i+1} \geqq Q_i \qquad (i = 1, \cdots, m - 1).$$

These relations follow from (1.1) upon comparing each relation with the third following relation.

In particular if R is an m-sphere, $m > 1$, we have the relations

$$M_{i-1} + M_{i+1} \geqq M_i \qquad (i = 2, \cdots, m - 2),$$

together with the special relations

$$M_0 + M_2 \geqq M_1 + 1,$$

$$M_m + M_{m-2} \geqq M_{m-1} + 1.$$

Many other conditions on the numbers Q_i can be derived from the relations (1.1).

We shall now indicate certain extensions of Theorem 1.1 which we shall not use in these Lectures, and shall accordingly not establish. These extensions are of importance in connection with the question of the completeness of the relations (1.1).

Suppose that the domain of definition of f is the interior and boundary of a region Σ of R. Suppose the points on the boundary B of Σ neighboring any one such point satisfy a relation of the form

$$F(x^1, \cdots, x^m) = 0$$

in terms of the local coordinates (x), where $F(x)$ is of class C^3, and

$$F_{x^i}F_{x^i} \neq 0 \qquad (i = 1, \cdots, m).$$

We term Σ a *regular* region. We state the following theorem.

THEOREM 1.2. *If f is non-degenerate on Σ and on the boundary B of Σ possesses a positive directional derivative f_n along the exterior normal, then the numbers M_i of critical points of index i of f on Σ and the connectivities R_i of Σ again satisfy the relations* (1.1).

When the proof of Theorem 1.1 has been completed the reader will be able to construct a proof of Theorem 1.2 upon reading the last section of Morse [1]. We remark that Theorem 1.2 also holds if Σ is a bounded region in euclidean m-space. That the relations between the integers M_i and R_i are the only relations which always hold between these integers alone follows from the following theorem.

THEOREM 1.3. *Corresponding to any prescribed set of integers M_i, R_i, $i = 0, \cdots, m$, positive or zero, satisfying the relations* (1.1) *with M_0 and R_0 positive, there exists a regular region Σ, together with a non-degenerate function f, defined on Σ and assuming an absolute non-critical maximum on the boundary of Σ, such that the integers R_i are the connectivities of Σ and the integers M_i are the numbers of critical points of f of index i.*

While assured of the truth of Theorem 1.3 the author has never published a proof. The theorem is stated in Morse [11]. An independent proof of the theorem has been given by a pupil of Professor Courant, Dr. John. See John [1].

We can extend Theorem 1.2 still further by removing the condition $f_n > 0$ on the boundary B. On B let f equal a function L. As a function of the point on B, L will have its own critical points with their indices. Instead of the assumption $f_n > 0$ on B we now assume merely that f has no critical points on B, and that the function L is non-degenerate as a function of the point on B. These conditions will in general be fulfilled. We term them the *general boundary conditions on f*. The theorem is as follows.

THEOREM 1.4. *Under the general boundary conditions on f the relations* (1.1) *still hold, where R_i is the ith connectivity of Σ and M_i is the number of critical points of index i, not only of f on Σ but also of L at points on B at which $f_n < 0$.*

For a proof of this theorem in euclidean n-space see Morse and Van Schaack (Morse [20]). The proof in general is similar.

W. M. Whyburn [1] has developed certain interesting aspects of the theory of critical points of functions in the case where the critical values are not necessarily isolated.

The equality in the relations (1.1) in the case of a simply connected region in n-space can be derived with the aid of the theory of the Kronecker characteristics, although Kronecker apparently made no such explicit derivation. See Kronecker [1, 2].

The problem of equivalence

2. Before coming to the problem of equivalence we shall enumerate certain conventions concerning singular chains on R. See Lefschetz [1, 2]. We shall vary the form of the basic definitions slightly, in a way that makes the work of the present chapter capable of a natural generalization in later chapters. We wish here to acknowledge the benefit derived from an interchange of views with Dr. A. W. Tucker on the various means of defining singular chains and cycles. See also Alexandroff [1], Alexander [1, 2], Tucker [1].

Let α_k and β_k be two k-simplices in a euclidean space E_n. A non-singular, affine, projective correspondence between α_k and β_k will be termed an *affine correspondence* between α_k and β_k. If α_k lies in a euclidean space E_n, and β_k in a euclidean space E_m with $n \leqq m$, we identify E_n with the linear subspace of E_m determined by the first n coordinate axes of E_m, and define an affine correspondence between α_k and β_k as before.

Indicating closures by adding bars, let φ represent a continuous map of $\bar{\alpha}_k$ on R. The image a_k of α_k under φ will be termed a *k-cell* on R. Let b_k be a second k-cell on R defined with the aid of a map ψ of $\bar{\beta}_k$ on R. Let T be an affine correspondence between α_k and β_k. If points on α_k and β_k which correspond under T possess the same image on R under φ and ψ respectively, the cells a_k and b_k will be regarded as *identical* on R. We shall refer to this statement as the convention of identity.

If α_i is any i-simplex on the boundary of α_k, the image of α_i under φ will be said to be a *boundary i-cell* of a_k on R. The *boundary of* a_k on R is however still to be defined.

We shall deal only with unoriented cells, and with cells mod 2.

By a *closed* i-cell on R we mean an i-cell on R together with its boundary j-cells. By an *i-chain* on R we mean a finite set (possibly null) of closed i-cells on R, no two of which are "identical." By the sum mod 2,

$$z_i + w_i \qquad \text{(mod 2)},$$

of two i-chains z_i and w_i on R, we mean the set of closed i-cells which belong to z_i or w_i but not to both z_i and w_i.

Let k and r be integers with $r < k$. Let a_k and b_r be cells on R given as continuous images of simplices α_k and β_r. Suppose β_r is the affine projective image of α_k under a singular transformation T in which each point of α_k corresponds to a unique point of β_r, and each point of β_r corresponds to at least one point of α_k. If points which correspond on α_k and β_r possess the same images on R on the cells a_k and b_r, then a_k will be termed "degenerate." Cf. Lefschetz [2]. Degenerate k-cells will be counted as if null in any k-chain on R.

The *boundary* z_{i-1} of an i-chain z_i on R is defined as the sum mod 2 of the closed $(i-1)$-cells which are the boundary cells of i-cells of z_i. One then writes

$$z_i \to z_{i-1} \qquad \text{(mod 2)}. \tag{2.1}$$

We observe that the boundary of the sum of a set of i-chains is the sum of the boundaries of the respective chains. It appears that bounding relations such as (2.1) can be added by adding the respective members of the bounding relations, mod 2.

A chain a_j on R whose boundary is null is termed a *j-cycle*. A j-cycle will be said to be *bounding* or *homologous to zero* if a_j is the boundary of some $(j + 1)$-chain a_{j+1} on R. One then writes

$$a_j \sim 0 \qquad \text{(on } R\text{)}.$$

This is understood, mod 2. This phrase will ordinarily be omitted. One sees from the way bounding relations can be added, that homologies

$$a_j \sim 0, \qquad b_j \sim 0 \qquad \text{(on } R\text{)}$$

imply

$$a_j + b_j \sim 0 \qquad \text{(on } R\text{)}.$$

The last relation will also be written in the form

$$a_j \sim b_j.$$

With this understood it appears that valid homologies can be combined into a valid homology by adding the respective members mod 2.

By a *proper* linear combination of a finite set of k-cycles is meant a linear combination of these cycles with coefficients which are not all zero mod 2. By a proper homology between a set of k-cycles is meant an homology

$$\lambda \sim 0$$

in which λ is a proper linear combination of cycles of the set. A set of k-cycles on R will be termed *independent* on a domain A if no proper linear combination of these cycles bounds on A.

Let a class C of k-cycles be distinguished by the possession of certain properties B. By a *maximal* set of cycles of C will be meant a set of cycles of C, every proper linear combination of whose cycles belongs to C and which contains the maximum number of cycles of C of any set with this property. As a convention we admit the possibility that the number of cycles in a maximal set may be infinite.

To *subdivide* a j-chain a_j on R, we subdivide the simplices representing its respective cells, and take the resulting images of the new simplices as the new cells. If two simplices α_i and β_i correspond under an affine collineation T by virtue of which their images a_i and b_i are identical on R, the simplices α_i and β_i shall be subdivided so that the subdivision of β_i may be obtained from that of α_i by applying T. This is clearly possible at least for those modes of subdivision which subdivide cells in the order of dimensionality.

We return to the function f on R. We no longer assume that the critical points are non-degenerate. We shall assume however that the number of critical values of f is finite. This assumption is always fulfilled in the analytic case. We suppose that f is of class C^2 and not identically constant.

By a *critical set* σ will be understood any closed set of critical points on which f is a constant c, and which is at a positive distance from other critical points of f. A critical set may or may not be connected (in the point set sense), or be a finite complex. In the analytic case the critical sets are at most finite in number, with dimensionalities varying from 0 to $m - 1$ inclusive. If σ contains all the critical points at which $f = c$, it will be called a *complete* critical set corresponding to c. In the analytic case a complete critical set is composed of a finite ensemble of connected critical sets.

Since the non-degenerate case occurs in general, and since the relations (1.1) give a complete set of conditions on the existence of non-degenerate critical points, it is natural to seek to assign to each critical set σ an ideal "equivalent" set G of non-degenerate critical points in such a fashion that the relations (1.1) still hold. But the property that the relations (1.1) still hold is only one of the properties that we shall require of this equivalent set G. The problem of equivalence is the problem of specifying the properties which the set G should have in order that it may fairly deserve the name of a set equivalent to σ. This question of equivalence arises in algebraic geometry, for example, when the geometer asks how many double points a multiple point shall be equivalent to, or in the case of fixed points of transformations, when the geometer seeks to count complicated loci of fixed points as equivalent to a finite set of fixed points of simple type.

We shall begin with the case of a complete critical set σ corresponding to a critical value c. Let a and b be any two constants which are not critical values of f, which are such that $a < c < b$, and such that c is the only critical value of f between a and b. If c is the absolute minimum of f, the domain $f < a$ is vacuous.

We shall give a definition of an ideal set of non-degenerate critical points *equivalent* to the complete critical set σ. Later we shall find it possible to extend this definition to critical sets which are not complete.

Relative to the above critical value c and the preceding constants a and b, a *new* k-cycle shall mean a k-cycle which lies on the domain $f < b$ but is independent on $f < b$ of k-cycles on $f < a$. Relative to the critical value c and the constants a and b, a *newly-bounding* k-cycle shall mean a k-cycle on $f < a$ independent on $f < a$, but bounding on $f < b$. It will follow from Lemmas 2.1 and 2.2 that the numbers

$$m_k^+, \qquad m_k^-$$

of cycles in maximal sets of new k-cycles and newly-bounding $(k - 1)$-cycles respectively, are finite and independent of the choice of the numbers a and b among numbers which are not critical values of f, and between which c is the only critical value of f.

We set

$$m_k = m_k^+ + m_k^-,$$

and say that the complete critical set σ is *equivalent* to m_k non-degenerate critical points of index k. We term the integers

$$m_0, m_1, \cdots, m_m$$

the *type numbers* of the critical set σ.

In §8 we shall justify our definition of equivalence by establishing the following four properties of the numbers m_k.

I. *If σ is a set of non-degenerate critical points, the corresponding type number m_k of σ equals the number of non-degenerate critical points of index k in σ.*

II. *The numbers m_k are completely determined by the definition of f in an arbitrarily small neighborhood of the critical set σ.*

III. *If each critical set σ is counted as equivalent to m_k ($k = 0, \cdots, m$) critical points of index k, the relations* (1.1) *still hold.*

IV. *Suppose the function f is analytic and is approximated for parameters*

$$(\mu_1, \cdots, \mu_r)$$

near the set (0) *by a function Φ of the point on R and the parameters (μ) which, in terms of the local coordinates (x) of R and of the parameters (μ), is of the form $F(x, \mu)$, where $F(x, \mu)$ is of class C^2 and non-degenerate for $(\mu) \neq (0)$. If*

$$\Phi \equiv f$$

for $(\mu) = (0)$, then for $(\mu) \neq (0)$ but sufficiently near (0), *Φ will possess at least m_k non-degenerate critical points of index k neighboring the given critical set σ of f.*

Relative to property II we remark that the numbers m_k do not possess property II except by virtue of a deep lying proof. For property II implies the invariance of the numbers m_k with respect to all *functional alterations* of f which leave f invariant neighboring σ, and replace f by a function which is again admissible on R. Moreover examples will show that the numbers m_k^+ and m_k^- do not in general separately possess this property of functional invariance, although their sum m_k does. We here have a distinction between functional and topological invariance. For the numbers m_k^+ and m_k^- are invariant under any homeomorphism of R which preserves the value of f, but are not necessarily invariant with respect to the above functional alterations.

Property III is fundamental in proving the existence of critical points, and property IV interprets this result in terms of non-degenerate functions Φ approximating f. We shall give further point to property IV by showing that when the critical set σ lies in a single coordinate system (x) and f is analytic, an approximating function such as Φ always exists.

That the number of k-cycles in maximal sets of new or newly-bounding k-cycles relative to c is finite follows from the following lemma.

Lemma 2.1. *If a is an ordinary value of f, the connectivities of the domain $f < a$ are finite.*

To establish this lemma we shall make use of the trajectories orthogonal to the manifolds f constant, representing these trajectories in the form

$$\frac{dx^i}{dt} = \frac{g^{ij}f_{x^j}}{g^{ij}f_{x^i}f_{x^j}} \qquad (i, j = 1, \cdots, m).$$

Here g^{ij} is the cofactor of g_{ij} in $| g_{ij} |$ divided by $| g_{ij} |$. Along these trajectories

$$\frac{df}{dt} \equiv \frac{dx^i}{dt} f_{x^i} \equiv 1,$$

so that we can suppose $f = t$ along such trajectories.

Let e be a positive constant so small that no value of f between a and $a - e$ inclusive is a critical value. We can deform the domain $f < a$ onto the domain $f \leqq a - e$, moving each point on the domain

$$a - e \leqq f < a$$

along the orthogonal trajectory through the point so that f decreases at a unit rate with respect to the time τ, stopping the movement when the point reaches the manifold $f = a - e$.

Now let the domain $f \leqq a - e$ be covered by a complex C_m of cells of R, or a subdivision of these cells, taking this subdivision so small that C_m lies on $f < a$. Any cycle on $f < a$ will be homologous, by virtue of the above deformation, to a cycle on $f \leqq a - e$ and hence on C_m. A maximal set of k-cycles on $f < a$, independent on $f < a$, will contain at most the number of k-cycles of cells of C_m which are independent on C_m, and this number is finite. The lemma follows directly.

With Lemma 2.1 we naturally associate the following lemma.

LEMMA 2.2. *If a and b are any two ordinary values of f with no critical values between them, the domains*

$$f < a, \qquad f < b$$

are homeomorphic.

To prove the lemma choose $a - e$ as in the preceding proof. We establish a homeomorphism between the domains $f < a$ and $f < b$ as follows. Let p be a point at which $f = f_0$ where

$$a - e \leqq f_0 < b.$$

Suppose p lies on the orthogonal trajectory λ. We make the point p at which f_0 divides the interval $(a - e, b)$ in a given ratio correspond to the point on λ at which f divides the interval $(a - e, a)$ in the same ratio. The remaining points of $f < b$ shall correspond to themselves. The correspondence between the domains $f < a$ and $f < b$ is now one-to-one and continuous, and the proof of the lemma is complete.

Cycles neighboring σ

3. In this section we suppose that there is just one critical value c between a and b. It will be convenient to say that a point on R at which $f < c$ is *below* c.

Let σ be a critical set of f on which $f = c$. The set σ may or may not be complete, that is, contain all the critical points at which $f = c$. By a neighborhood N of σ we mean an open set of points which includes all points of R within a small positive geodesic distance of σ. We admit only such neighborhoods of σ as lie on the domain

$$a < f < b. \tag{3.1}$$

A neighborhood N of σ will be termed arbitrarily small if its points lie within an arbitrarily small geodesic distance of σ.

We shall state a theorem which affirms the existence of a basic deformation $\theta(t)$. This deformation will be defined for points on a neighborhood N_0 of σ and for a time interval $0 \leqq t < 1$. It will be continuous in that under the deformation each point p of N_0 will be replaced at the time t by a point $q(p, t)$ which coincides with p when $t = 0$, and varies continuously on R with p on N_0 and t on its interval. The theorem is as follows.

THEOREM 3.1. *There exists a deformation $\theta(t)$ defined and continuous for points sufficiently near σ and for t on the interval $0 \leqq t < 1$. The deformation $\theta(t)$ leaves points of σ invariant and deforms a sufficiently small neighborhood N of σ into a neighborhood N_t, the distance of whose points from σ approaches zero uniformly as t approaches 1. It deforms points below c through points below c.*

This theorem is true if f is of class C^2 on R, and satisfies certain other general requirements which do not exclude the possibility of infinitely many distinct critical sets. In this place we shall give its proof for two general cases. In one case f will be assumed non-degenerate. In the other case f will be assumed analytic, but not constant. The next two sections will be occupied with this proof. In the remainder of this section we give certain consequences of the theorem.

Let N^* be a fixed neighborhood of σ whose closure is interior to the domain on which the deformation $\theta(t)$ is defined. We state the following corollary of the theorem.

COROLLARY 3.1. *Corresponding to any neighborhood X of σ on N^*, there exists a neighborhood $M(X)$ of σ so small that $M(X)$ is deformed under $\theta(t)$ only on X. Each k-cycle on $M(X)$ (below c) will then be homologous on X (below c) to a cycle (below c) on an arbitrarily small neighborhood N of σ. If $z_k \sim 0$ on N^* (below c), and z_k is sufficiently near σ, then $z_k \sim 0$ on N (below c).*

In this corollary the phrase (below c) is to be omitted throughout, or read throughout at pleasure.

An ordered pair of neighborhoods VW of σ will be termed admissible if they satisfy the conditions

$$V \subset M(N^*), \qquad W \subset M(V)$$

where $M(X)$ is the neighborhood of Corollary 3.1.

We shall have occasion to use the phrase "corresponding to any admissible pair of neighborhoods VW" many times. For the sake of brevity we shall replace this phrase by the expression corr VW. With this understood we now define two basic types of cycles neighboring σ. We shall refer to these cycles as *belonging* to σ.

By a *spannable k-cycle corr VW*, we shall mean a k-cycle on W, below c, ~ 0 on W, but $\nsim 0$ on V below c.

By a *critical k-cycle corr VW*, we shall mean a k-cycle on W, $\nsim$ on V to a k-cycle on V below c.

Maximal sets of spannable or critical cycles corr VW are of importance in that they depend only on the neighborhood of σ, and that we shall subsequently be able to determine the type numbers m_k of σ with their aid.

The following theorem is an easy consequence of the preceding theorem and corollary.

THEOREM 3.2. *Corresponding respectively to any two choices VW and $V'W'$ of admissible pairs of neighborhoods there exist common maximal sets of spannable or critical k-cycles on any arbitrarily small neighborhood of σ.*

It appears from this theorem that the total number, say μ_k, of cycles in maximal sets of spannable $(k-1)$-cycles and critical k-cycles is independent of the choice of admissible neighborhoods VW. It will turn out that μ_k is finite and that

$$m_k = \mu_k.$$

The neighborhood functions of the next section are of aid in establishing Theorem 3.1 and determining μ_k.

Neighborhood functions

4. Let φ be a function of class C^2 of the point (x) on R neighboring a point p. Suppose p is an ordinary point of both f and φ. The gradient of φ is the vector whose local covariant components are φ_i, where φ_i is the partial derivative of φ with respect to x^i. The contravariant components of this gradient are then $g^{ij}\varphi_j$. See Eisenhart [1]. A regular curve γ orthogonal at each of its points p to the manifold $\varphi =$ const. through p will be called a φ-trajectory. We are restricting ourselves here to ordinary points of φ. The differential equations of the φ-trajectories will be given the form

$$\frac{dx^i}{dt} = \frac{g^{ij}\varphi_j}{g^{ij}\varphi_i\varphi_j} = Y^i(x), \qquad (i, j = 1, \cdots, m). \tag{4.1}$$

The denominator of the middle term is an invariant which is not zero at ordinary points of φ. Along the φ-trajectories we have

$$\frac{d\varphi}{dt} = \frac{dx^i}{dt}\varphi_i \equiv 1, \tag{4.2}$$

so that we can suppose $t = \varphi$ along these trajectories.

The f-trajectories are similarly defined and represented.

We shall now define the (φf)-trajectories. Suppose that the gradients of φ and f at p are not parallel. By the (φf)-vector at the point p we mean a vector which lies in the 2-plane of the gradients of φ and f, which is orthogonal to the gradient of f, and which has a magnitude to be prescribed in (4.5). The contravariant components λ^i of this (φf)-vector will be proportional to

$$g^{ij}(\varphi_j + \sigma f_j) \qquad (i, j = 1, \cdots, m) \tag{4.3}$$

where σ is to be determined so that λ^i is orthogonal to the gradient of f. This gives the condition

$$g^{ij}(f_i\varphi_j + \sigma f_i f_j) = 0, \tag{4.4}$$

from which we see that a particular choice of λ^i is

$$\lambda^i = g^{ij}g^{hk}(f_h f_k \varphi_j - \varphi_h f_k f_j) \qquad (h, k, i, j = 1, \cdots, m). \tag{4.5}$$

We prescribe the magnitude of λ^i by taking it as this vector.

We shall make use of the invariant

$$\lambda^i\varphi_i = g^{ij}g^{hk}[f_h f_k \varphi_i \varphi_j - \varphi_h f_k \varphi_i f_j] = A(x)$$

and shall prove the following lemma.

LEMMA 4.1. *At ordinary points of f and φ at which the gradients of f and φ are not parallel, $A(x) \neq 0$.*

Suppose $A(x)$ were null. Then from (4.3) and the condition $\lambda^i\varphi_i = 0$ we have

$$g^{ij}(\varphi_i\varphi_j + \sigma\varphi_i f_j) = 0 \qquad (i, j = 1, \cdots, m),$$

and combining this condition with (4.4) multiplied by σ we find that

$$g^{ij}[\varphi_i\varphi_j + 2\sigma\varphi_i f_j + \sigma^2 f_i f_j] = g^{ij}[\varphi_i + \sigma f_i][\varphi_j + \sigma f_j] = 0. \tag{4.6}$$

But g^{ij} gives the coefficients of a positive definite quadratic form so that (4.6) holds only if

$$\varphi_i + \sigma f_i = 0 \qquad (i = 1, \cdots, m),$$

contrary to the hypothesis that the gradients of f and φ are not parallel. The lemma is thereby proved.

We note the converse, that $A(x) = 0$ if the gradients of f and φ are parallel.

The (φf)-trajectories will now be defined by the equations

$$\frac{dx^i}{dt} = \frac{\lambda^i(x)}{A(x)} = X^i(x) \qquad (i = 1, \cdots, m). \tag{4.7}$$

We see that along these trajectories

$$\frac{d\varphi}{dt} = \frac{\lambda^i\varphi_i}{A(x)} \equiv 1.$$

We can accordingly take $t = \varphi$ along these trajectories. We also note that

$$\frac{df}{dt} = \frac{\lambda^i f_i}{A(x)} \equiv 0$$

so that f is constant along (φf)-trajectories.

A *neighborhood function* $\varphi(x)$ belonging to the critical set σ of f on which $f = c$, is now defined as a function with the following properties:

(a). It is of class C^2 neighboring σ.

(b). It takes on a proper relative minimum zero on σ.

(c). At points near σ but not on σ, it is ordinary.

(d). At points near σ but not on σ at which $f = c$, the gradients of f and φ are not parallel.

If φ is a neighborhood function, the locus $\varphi = e$ is without singularity for e positive and sufficiently small. The same is true of the intersection of $\varphi = e$ and $f = c$, as follows from (d).

We shall exhibit neighborhood functions φ in certain important cases beginning with the analytic case. We state the following theorem.

THEOREM 4.1. *In the analytic case the invariant function*

$$\varphi = g^{ij}f_if_j \qquad (i, j = 1, \cdots, m)$$

is a neighborhood function corresponding to any critical set σ of f.

That φ satisfies the conditions (a) and (b) upon a neighborhood function is at once evident. We shall finish by proving the following lemma.

LEMMA 4.2. *If f is analytic, any analytic function φ which takes on a proper relative minimum zero on σ is an admissible neighborhood function φ.*

The function φ of the lemma satisfies (a) and (b). It must then satisfy (c). For σ is a set of critical points of φ, and if φ were not ordinary near σ the critical set σ would be a subset of a larger critical set connected to σ. But on all connected critical loci an analytic function is constant. Thus φ would be zero at some points near σ but not on σ, contrary to the nature of a proper minimum. Thus (c) holds.

Now (d) could fail only at points not on σ at which

$$A(x) = 0, \qquad f = c. \tag{4.8}$$

But (4.8) is satisfied on σ. Suppose it were satisfied on a larger analytic locus γ connected with σ. Let h be any regular curve along which (4.8) is satisfied. On $h, f = c$ so that

$$f_i \frac{dx^i}{dt} = 0 \qquad (i = 1, \cdots, m). \tag{4.9}$$

I say that on h,

$$\varphi_i \frac{dx^i}{dt} = 0 \qquad (i = 1, \cdots, m). \tag{4.10}$$

This is certainly true on σ, since $\varphi_i = 0$ on σ. At points not on σ at which $A(x) = 0$ the gradients of f and φ are parallel by virtue of Lemma 4.1, so that (4.10) follows from (4.9). Thus φ is constant on h and hence on γ. It must then be zero on γ. From (b) we see that $\gamma = \sigma$. Thus (d) holds.

The proof of the lemma is now complete and the theorem follows directly.

In the non-analytic case a neighborhood function always exists corresponding to a non-degenerate critical point, as the following theorem states.

THEOREM 4.2. *If in terms of a local coordinate system (x), $(x) = (0)$ is a non-degenerate critical point of f, the function*

$$\varphi = x^i x^i \qquad (i = 1, \cdots, m)$$

is a corresponding neighborhood function.

The function φ clearly satisfies all the requirements upon a neighborhood function except possibly the one involving gradients. But the relevant relations of the gradients of f and φ will be unaltered if we use an orthogonal transformation of the variables (x) to bring f to the form

$$f - c = \frac{a_k x^k x^k}{2} + \eta \qquad (k = 1, \cdots, m) \tag{4.11}$$

where a_k is a constant not zero, and $\eta = o(\rho^2)$, that is, η vanishes to at least the second order with respect to the distance ρ to the origin in the space (x).

At ordinary points of f and φ a condition that the gradients of f and φ be not parallel is the following:

$$(\varphi_i f_k - \varphi_k f_i)(\varphi_i f_k - \varphi_k f_i) = 2(f_k f_k \varphi_i \varphi_i - \varphi_k f_k \varphi_i f_i) \neq 0. \tag{4.12}$$

We have merely to show that (4.12) holds when $f = c$, and $(x) \neq (0)$ neighboring $(x) = (0)$. But the right parenthesis in (4.12) is seen to be of the form

$$8\,[a_k^2 x^k x^k x^i x^i - a_k x^k x^k a_i x^i x^i] + o(\rho^4). \tag{4.13}$$

But on $f = c$, upon using (4.11), we see that the expression (4.13) takes the form

$$8 a_k^2 x^k x^k x^i x^i + o(\rho^4). \tag{4.14}$$

The expression (4.14) however does not vanish for (x) sufficiently near the origin and not (0). Thus φ satisfies condition (d) on a neighborhood function.

The theorem is accordingly proved.

The following theorem will enable us to give a particularly elegant determination of the set of non-degenerate critical points equivalent to an isolated critical point in the analytic case.

THEOREM 4.3. *If f is analytic and $(x) = (0)$ is an isolated critical point, the function $\varphi = x^i x^i$ is an admissible neighborhood function.*

This follows at once from Lemma 4.2.

The determination of spannable and critical cycles

5. We continue with the critical set σ. We suppose that φ is a neighborhood function corresponding to σ. Neighboring σ we shall prove the existence of a basic set of trajectories termed *radial* trajectories. They lead away from σ somewhat after the fashion of rays emanating from a point. The first theorem is the following.

THEOREM 5.1. *If φ is a neighborhood function for σ, then on the domain*

$$H: \qquad 0 < \varphi \leqq r,$$

where r is a sufficiently small positive constant, there exists a "radial" field of trajectories, one through each point of H, satisfying differential equations of the form

$$\frac{dx^i}{dt} = B^i(x) \qquad (B^i B^i \neq 0),$$

where the functions $B^i(x)$ are of class C^1 on H. These trajectories reduce to (φf)-trajectories on $f = c$. On them t may be taken equal to φ.

The φ-trajectories themselves would do except for the fact that they do not in general reduce to (φf)-trajectories on $f = c$. We shall alter the φ-trajectories neighboring $f = c$ so that they will suffice. For the remainder of this proof we shall suppose $c = 0$.

The $(f\varphi)$-trajectories ζ emanating from $f = 0$ on H in general form a field F only for a short distance from $f = 0$, depending upon how near φ is to 0 on the trajectory ζ in question. (Recall that φ is constant on each trajectory ζ.) We shall be precise and say that we can determine a positive function $h(\alpha)$ of class C^1 for $0 < \alpha \leqq r$, such that the field F persists on a trajectory ζ on which $\varphi = \alpha$ where f changes from $-h(\alpha)$ to $h(\alpha)$. We can in fact define $h(\alpha)$ successively on the intervals

$$r \geqq \alpha > \frac{r}{2}, \qquad \frac{r}{2} \geqq \alpha > \frac{r}{4}, \qquad \frac{r}{4} \geqq \alpha > \frac{r}{8}, \cdots,$$

and so define $h(\alpha)$ for $r \geqq \alpha > 0$.

We now let $M(u)$ be a function of u of class C^1, identically one for $u^2 > 1$, and zero for u zero, otherwise positive. Our radial trajectories will be defined as φ-trajectories except for the points on trajectories ζ where f changes from $-h(\varphi)$ to $h(\varphi)$,. At these exceptional points the differential equations of the radial trajectories shall have the form

$$\frac{dx^i}{dt} = X^i(x) + M\left[\frac{f(x)}{h(\varphi(x))}\right](Y^i(x) - X^i(x)) \qquad (i = 1, \cdots, m) \tag{5.1}$$

where X^i and Y^i are the functions appearing in (4.7) and (4.1) respectively.

On $f = 0$ the radial trajectories reduce to the (φf)-trajectories (4.7). For $f = \pm h(\varphi)$ they take the form (4.1). Moreover on them

$$\frac{d\varphi}{dt} = \varphi_i X^i[1 - M] + \varphi_i Y^i M = 1 - M + M = 1 \qquad (i = 1, \cdots, m).$$

This shows that we can take $t = \varphi$ on the radial trajectories.

The theorem follows at once.

By a *radial* deformation we shall hereby mean any continuous deformation neighboring a critical set σ in which each point moves, if at all, on a radial trajectory, and two points for which φ is initially the same are deformed so that at the same time the resulting values of φ are the same. With the aid of suitable radial deformations we can establish the following statements.

(1). For any two positive constants e and η less than r, the domain $\varphi = e$ below c is homeomorphic with the domain $\varphi = \eta$ below c.

(2). If $e < \eta$, the domain $0 < \varphi \leqq \eta$ below c can be radially deformed onto the domain $0 < \varphi \leqq e$ below c, leaving the latter domain fixed, and never increasing φ.

(3). For any closed point set ω on the domain $0 < \varphi \leqq \eta$ below c, there exists a radial deformation on the same domain that leaves the domain $\varphi = \eta$ below c fixed, and deforms the point set ω onto the latter domain.

We note that these radial deformations deform points below c through points below c.

We can satisfy Theorem 3.1 by a particular radial deformation defined as follows.

The radial deformation $R(t)$. Under $R(t)$ the time t varies on the interval $0 \leqq t < 1$. A point on a radial trajectory at which

$$\varphi = r - \theta r \qquad (0 \leqq \theta < 1),$$

shall remain fixed until $t = \theta$, and shall thereafter be replaced by the point on the same radial trajectory at which

$$\varphi = r - tr.$$

The deformation $R(t)$ thereby defined clearly satisfies the conditions of Theorem 3.1.

The following theorem is also established with the aid of radial deformations.

THEOREM 5.2. *Corresponding to admissible neighborhoods VW of σ let e be a positive constant so small that the domain $\varphi \leqq e$ is on W.*

A maximal set of spannable k-cycles corr VW can then be taken as a maximal set of k-cycles on $\varphi = e$ below c, independent on this domain, but bounding on $\varphi \leqq e$.

A maximal set of critical k-cycles corr VW can be taken as a maximal set of k-cycles on $\varphi \leqq e$, independent on this domain of cycles on $\varphi = e$ below c.

The number of cycles in the above sets will be independent of the constant e chosen as above.

The reader has doubtless observed that the above manifolds $\varphi = e$ are without singularity, as are their intersections with $f = c$.

Classification of cycles

6. Having analysed two basic sets of cycles neighboring the critical set σ we are now in a position to determine the change in cycles with respect to bounding as one passes from the domain $f < a$ to the domain $f < b$. We are supposing that $f = c$ on σ, that $a < c < b$, that a and b are not critical values of f, and that c is the only critical value of f between a and b. We also suppose that σ is a complete critical set, that is, the set of *all* critical points at which $f = c$.

We admit the possibility that c is either the absolute minimum or maximum of f. In the former case the domain $f < a$ is vacuous. This case is not excluded in the following. The reader will observe that in this case certain of the chains which appear in the following proofs are null, a case again not excluded. As a convention we understand that a null cycle bounds.

A spannable $(k - 1)$-cycle l_{k-1} corr VW will be called *linkable* if bounding below c. If l_{k-1} is linkable there exists a chain λ_k'' below c such that

$$\lambda_k'' \rightarrow l_{k-1} \qquad \text{(below } c). \tag{6.1}$$

By virtue of the definition of a spannable $(k - 1)$-cycle there also exists a chain λ_k' on W, such that

$$\lambda_k' \rightarrow l_{k-1} \qquad \text{(on } W). \tag{6.2}$$

We set

$$\lambda_k' + \lambda_k'' = \lambda_k, \tag{6.3}$$

and term λ_k a k-cycle *linking* l_{k-1}, *corr* VW. More generally we shall term a k-cycle *linking, corr* VW, if some subdivision links a spannable $(k - 1)$-cycle l_{k-1} in the preceding sense. For the sake of simplicity we shall suppose that a linking k-cycle corr VW is always given with a division into cells by virtue of which it links a spannable $(k - 1)$-cycle corr VW. We shall say that λ_k *belongs* to any critical set to which l_{k-1} belongs.

We shall now establish three lemmas on linking k-cycles. We begin with the following.

LEMMA 6.1. *Let $(l)_{k-1}$ be a set of linkable $(k-1)$-cycles corr VW, and let $(\lambda)_k$ be a set of k-cycles linking the respective $(k-1)$-cycles of the set $(l)_{k-1}$ corr VW. A necessary and sufficient condition that $(l)_{k-1}$ be a maximal set of linkable $(k-1)$-cycles corr VW is that $(\lambda)_k$ be a maximal set of linking k-cycles corr VW.*

We shall first prove the condition sufficient. We assume therefore that $(\lambda)_k$ is maximal and seek to prove $(l)_{k-1}$ maximal.

We shall first show that if u_{k-1} is any proper sum of cycles of $(l)_{k-1}$,

$$u_{k-1} \not\sim 0 \qquad \text{(on } V \text{ below } c). \tag{6.4}$$

To that end let λ_k be the sum of the k-cycles of $(\lambda)_k$ which link the respective cycles of $(l)_{k-1}$ in the sum u_{k-1}. Since $(\lambda)_k$ is a maximal set of linking k-cycles there must exist a spannable $(k-1)$-cycle v_{k-1} linked by λ_k corr VW. By virtue of the definition of a spannable $(k-1)$-cycle corr VW we have

$$v_{k-1} \not\sim 0 \qquad \text{(on } V \text{ below } c).$$

To establish (6.4) it will be sufficient then to show that

$$u_{k-1} \sim v_{k-1} \qquad \text{(on } V \text{ below } c). \tag{6.5}$$

By virtue of the way λ_k is given as a sum of cycles of $(\lambda)_k$ we have

$$\lambda_k = \lambda_k' + \lambda_k''$$

where λ_k' and λ_k'' are chains such that

$$\lambda_k' \to u_{k-1} \qquad \text{(on } W)$$

and

$$\lambda_k'' \to u_{k-1} \qquad \text{(below } c).$$

By virtue of the fact that λ_k links v_{k-1} we have

$$\lambda_k = z_k' + z_k''$$

where z_k' and z_k'' are chains such that

$$z_k' \to v_{k-1} \qquad \text{(on } W),$$

$$z_k'' \to v_{k-1} \qquad \text{(below } c).$$

From our two representations of λ_k we see that

$$\lambda_k' + z_k' \equiv \lambda_k'' + z_k'' \qquad \text{(mod 2)}. \tag{6.6}$$

But since the right member of (6.6) is a chain below c, the left member of (6.6) must reduce mod 2, to a chain below c. Moreover

$$\lambda_k' + z_k' \to u_{k-1} + v_{k-1} \qquad \text{(on } W \text{ below } c),$$

from which (6.5) and (6.4) follow.

Thus $(l)_{k-1}$ is a subset of a maximal set of linkable $(k-1)$-cycles. It remains to prove that $(l)_{k-1}$ is a maximal set of linkable $(k-1)$-cycles corr VW.

To that end suppose $(l)_{k-1}$ contained fewer cycles than a maximal set of linkable $(k-1)$-cycles. There would then exist a set $(u)_{k-1}$ of linkable $(k-1)$-cycles which with the cycles of the set $(l)_{k-1}$ would form a maximal set of linkable $(k-1)$-cycles. Let $(\mu)_k$ be a set of k-cycles linking the respective members of the set $(u)_{k-1}$ corr VW. Any proper sum of cycles of the sets $(\lambda)_k$ and $(\mu)_k$ will be a linking k-cycle contrary to the assumption that $(\lambda)_k$ is maximal. Hence $(l)_{k-1}$ cannot contain fewer cycles than a maximal set of linkable $(k-1)$-cycles, and must therefore be a maximal set of linkable $(k-1)$-cycles corr VW.

To prove the condition necessary we assume that $(l)_{k-1}$ is a maximal set. If $(\lambda)_k$ were not a maximal set, there would exist a larger set of linking k-cycles which would be a maximal set corr VW. By virtue of the sufficiency of the condition already established, there would then exist a maximal set of linkable $(k-1)$-cycles corr VW which would be a larger set than $(l)_{k-1}$ contrary to the hypothesis that $(l)_{k-1}$ is a maximal set.

The condition of the lemma is accordingly necessary, and the lemma is proved.

Our second lemma on linking cycles is the following.

LEMMA 6.2. *If L is the domain below c and $(\lambda)_k$ a maximal set of k-cycles, linking corr VW, any k-cycle which is linking corr VW, is homologous on N^*+L to a combination of k-cycles of $(\lambda)_k$, critical k-cycles corr VW, and k-cycles below c.*

Let $(l)_{k-1}$ be the set of $(k-1)$-cycles linked, corr VW, respectively by the cycles of $(\lambda)_k$. Let z_k be an arbitrary k-cycle linking a $(k-1)$-cycle u_{k-1}, corr VW. By virtue of the preceding lemma we have

$$u_{k-1} \sim l_{k-1} \qquad \text{(on } V \text{ below } c) \tag{6.7}$$

where l_{k-1} is a proper sum of cycles of the set $(l)_{k-1}$. Let λ_k be the sum of the k-cycles linking the respective $(k-1)$-cycles of the sum l_{k-1}. Now λ_k can be represented as in (6.3). Similarly z_k can be represented in the form

$$z_k' + z_k'' = z_k \tag{6.8}$$

where z_k' is on W and z_k'' is below c, and where

$$z_k' \to u_{k-1}, \qquad z_k'' \to u_{k-1}.$$

Upon using (6.3) and (6.8) we see that

$$z_k - \lambda_k = (z_k' - \lambda_k') + (z_k'' - \lambda_k'').$$

Let w_k be the chain on V below c bounded by u_{k-1} and l_{k-1}, by virtue of (6 7). We see that in the congruence

$$(z_k' - \lambda_k' + w_k) + (z_k'' - \lambda_k'' - w_k) \equiv z_k - \lambda_k, \tag{6.9}$$

the first parenthesis is a k-cycle on V, and the second a k-cycle below c

But any k-cycle on V can be deformed on N^* under the deformation $\theta(t)$ of Theorem 3.1 into a k-cycle on W, and hence is homologous on N^* to a combination of critical k-cycles corr VW and cycles below c. From (6.9) we then conclude that $z_k - \lambda_k$ is homologous on $N^* + L$ to a linear combination of critical cycles corr VW and cycles below c. The lemma is thereby proved.

We shall prove the following lemma.

LEMMA 6.3. *No k-cycle λ_k which is a linking k-cycle corr VW is homologous on $N^* + L$ to a combination of critical cycles corr VW and cycles below c.*

Suppose that we had an homology

$$\lambda_k + mc_k + w_k \sim 0 \qquad \text{(on } N^* + L) \tag{6.10}$$

where $m = 0$ or 1, c_k is a critical k-cycle corr VW, and w_k is a cycle on L. Let w_{k+1} be a chain on $N^* + L$ bounded by the left member of (6.10). We can write

$$w_{k+1} = w'_{k+1} + w''_{k+1},$$

where w'_{k+1} is a chain on N^* and w''_{k+1} a chain on L, provided, as we suppose is the case, w_{k+1} is sufficiently finely divided. Thus

$$w'_{k+1} + w''_{k+1} \to \lambda_k + mc_k + w_k.$$

Suppose that

$$w'_{k+1} \to w'_k, \qquad w''_{k+1} \to w''_k.$$

Upon using (6.3), and the preceding bounding relations we see that

$$w'_k + w''_k \equiv \lambda'_k + \lambda''_k + mc_k + w_k. \tag{6.11}$$

From (6.11) it appears that the chain

$$\lambda'_k + mc_k + w'_k \qquad \text{(on } N^*),$$

reduced mod 2, lies on L, since the remaining chains in (6.11) lie on L. But we also see that

$$\lambda'_k + mc_k + w'_k \to l_{k-1}$$

where l_{k-1} is the $(k-1)$-cycle linked by λ_k corr VW.

We arrive at the conclusion that l_{k-1} lies on W, below c, and bounds a chain u_k on N^* below c. Upon applying the deformation $\theta(t)$ of Theorem 3.1 to u_k, we see that u_k will be deformed below c onto V, while l_{k-1} will not be deformed off from V. Hence

$$l_{k-1} \sim 0 \qquad \text{(on } V \text{ below } c).$$

But this is contrary to the fact that l_{k-1} is linked by λ_k corr VW. Thus (6.10) cannot hold, and the lemma is proved.

We now define a deformation $\Lambda(t)$ related to the deformation $\theta(t)$ of Theorem 3.1.

The deformation $\Lambda(t)$, $0 \leqq t < 1$. We extend the definition of $\theta(t)$ so that the resulting deformation $\Lambda(t)$ is continuous over $f < b$ and remains identical with $\theta(t)$ over the neighborhood N^* of §3. To that end let e be a positive constant so small that the set of points not on N^* but at a distance at most e from N^* are within the domain of definition of $\theta(t)$. Under $\Lambda(t)$ each point p at a distance $(1 - \lambda)e$ from N^*, where $0 \leqq \lambda < 1$, shall be deformed as in $\theta(t)$ until $t = \lambda$, and held fast thereafter. Points at a distance e or more from N^* shall be held fast under $\Lambda(t)$.

Our fourth lemma on linking cycles is the following.

LEMMA 6.4. *Let N be an arbitrarily small neighborhood of σ, and L the domain below c. Under $\Lambda(t)$ any cycle z_k which is linking corr VW can be deformed on $V + L$ into a cycle λ_k again linking corr VW, and on the domain $N + L$.*

The cycle z_k lies on $W + L$. From the nature of $\Lambda(t)$ it is clear that z_k can be deformed on $V + L$ into a cycle λ_k on $N + L$. Suppose z_k links a cycle u_{k-1} corr VW. Under $\Lambda(t)$, u_{k-1} will be deformed on V into a cycle v_{k-1}. We must have

$$v_{k-1} \not\sim 0 \qquad \text{(on } V \text{ below } c\text{)},$$

for otherwise

$$u_{k-1} \sim 0 \qquad \text{(on } V \text{ below } c\text{)},$$

contrary to the nature of u_{k-1}. Hence v_{k-1} is a spannable $(k - 1)$-cycle corr VW. Returning to the deformation $\Lambda(t)$ we see that λ_k links v_{k-1}, and the lemma is proved.

We shall make use of the trajectories

$$\frac{dx^i}{dt} = - g^{ij} f_{x^j} \qquad (i, j = 1, \cdots, m) \tag{6.12}$$

orthogonal to the manifolds f constant. We make the convention that there is a trajectory coincident with each critical point at all times t.

The deformation D. Under the deformation D each point on $f < b$ which is at a point p when $t = 0$, shall be replaced at each time t for which $0 \leqq t \leqq 1$ by the point t on the trajectory issuing from p. Under D a point which is not a critical point is so deformed that f is continually decreased. Critical points are held fast under D. With the aid of D we shall establish a deformation lemma in the large.

DEFORMATION LEMMA. *Let N be an arbitrary neighborhood of the critical set σ, and let L be the set of points below c. A sufficient number of iterations of the deformation D will provide a deformation Δ of the domain $f < b$ on itself onto the domain $N + L$.*

If a cycle z_k lies on a domain $N_0 + L$ for which N_0 is a sufficiently small neighborhood of σ, and if $z_k \sim 0$ on $f < b$ (below c), then $z_k \sim 0$ on $N + L$ (below c).

By virtue of the continuity of the deformation D there will exist a neighborhood N' of σ so small that D will deform N' only on N.

Let α and β respectively denote the domains $f < a$ and $f < b$. The domain $\beta - \alpha - N'$ has a positive distance from σ, and hence each point p on this domain will be carried by D into a point at which f is at least d less than at p, where d is a positive constant independent of p.

Moreover a sufficiently large number of iterations of D will define a deformation, say D^r, which will carry β into a point set on the domain

$$f < c + d/2.$$

From the choice of d we see then that D^{r+1} will carry all points of β whose rth images are on $\beta - \alpha - N'$ into points on the domain

$$f < (c + d/2) - d = c - d/2,$$

while points whose rth images are on N' will be deformed onto N under D^{r+1}.

The deformation $\Delta = D^{r+1}$ accordingly deforms the domain $f < b$ on itself onto $N + L$.

To establish the final statement of the lemma let N_0 be a neighborhood of σ which is so small that N_0 is deformed under Δ only on N. Suppose the cycle z_k of the lemma bounds a chain z_{k+1} on $f < b$. The deformation Δ will carry z_{k+1} into a chain on $N + L$, deforming z_k on $N + L$. Hence if $z_k \sim 0$ on $f < b$ (below c), it follows that $z_k \sim 0$ on $N + L$ (below c).

The proof of the lemma is now complete.

Before coming to the principal theorem we define a new set of cycles. A k-cycle below c, independent below c of the spannable k-cycles corr VW, is termed an *invariant k-cycle corr VW*. Future theorems justify this term.

From the definition of an invariant k-cycle corr VW it follows that a k-cycle below c is dependent on $f < b$ upon a linear combination of invariant k-cycles. From the definition of critical k-cycles corr VW it follows that any k-cycle on W is dependent on V upon a linear combination of critical k-cycles corr VW and k-cycles below c.

We come to a basic theorem (Morse [11, 12]).

THEOREM 6.1. *A maximal set of k-cycles on $f < b$, independent on $f < b$, is afforded by maximal sets of critical, linking, and invariant k-cycles corresponding to an admissible pair of neighborhoods VW of the critical set σ.*

We shall prove the theorem by proving statements (a) and (b). Statement (a) follows.

(a). *Any k-cycle z_k on $f < b$ is homologous on $f < b$ to a linear combination of the k-cycles of the maximal sets corr VW of the theorem.*

By virtue of the Deformation Lemma we lose no generality if we suppose z_k lies on $W + L$, where L is the domain $f < c$. If sufficiently finely subdivided, z_k can then be represented in the form

$$z_k \equiv z_k' + z_k'' \tag{6.13}$$

where z_k' is a chain on W and z_k'' a chain on L. Suppose z_{k-1} is the common boundary of z_k' and z_k'' so that

$$z_k' \longrightarrow z_{k-1}, \qquad z_k'' \longrightarrow z_{k-1}. \tag{6.14}$$

We admit the possibility that any one of the chains in (6.14) may be null.

The cycle z_{k-1} is necessarily on W below c. It bounds on W and below c. It accordingly satisfies an homology

$$z_{k-1} \sim rl_{k-1} \qquad \text{(on } V \text{ below } c) \tag{6.15}$$

where l_{k-1} is a linkable $(k - 1)$-cycle corr VW, and $r = 1$ or 0.

Let λ_k be a k-cycle linking l_{k-1} on $W + L$. By virtue of (6.15) there exists a chain w_k on V below c such that

$$w_k \longrightarrow z_{k-1} - rl_{k-1}.$$

Upon using (6.3) and (6.13) we obtain the congruence

$$z_k - r\lambda_k \equiv (z_k' - r\lambda_k' + w_k) + (z_k'' - r\lambda_k'' + w_k).$$

The first parenthesis contains a k-cycle on V and the second a k-cycle below c. But k-cycles below c are homologous on $f < b$ to zero or to an invariant k-cycle corr VW, while k-cycles on V are homologous on N^*, and hence on $f < b$, to a linear combination of critical k-cycles corr VW and k-cycles below c. Thus z_k is homologous on $f < b$ to a linear combination of cycles as stated in the theorem.

(b). *The cycles of the maximal sets of critical, linking, and invariant k-cycles corr VW are independent on $f < b$.*

Suppose that there existed an homology of the form

$$m\lambda_k + nc_k + ri_k \sim 0 \qquad (m, n, r = 1 \text{ or } 0) \tag{6.16}$$

where λ_k, c_k, and i_k are respectively linking, critical, and invariant k-cycles corr VW. We shall prove successively that m, n, and r are zero.

Proof that $m = 0$. Suppose $m \neq 0$. By virtue of the Deformation Lemma the homology (6.16) implies a similar homology on $V + L$, provided the cycles in (6.16) lie on $N_0 + L$, where N_0 is a sufficiently small neighborhood of σ. But we have seen that λ_k and c_k can be respectively deformed under $\Lambda(t)$ into cycles λ_k' and c_k' on $N_0 + L$, where λ_k' is a linking cycle corr VW and c_k' is a critical cycle corr VW. We thus have an homology

$$m\lambda_k' + nc_k' + ri_k \sim 0 \qquad (\text{on } f < b).$$

By virtue of the Deformation Lemma this implies a similar homology on $V + L$, contrary to the nature of the linking cycle λ_k' as described in Lemma 6.3.

Proof that $n = 0$. We suppose that $m = 0$ and $n = 1$ in (6.16). We then write (6.16) in the form

$$z_{k+1}' + z_{k+1}'' \longrightarrow nc_k + ri_k \tag{6.17}$$

where z'_{k+1} is a chain on W and z''_{k+1} a chain on L. Let z'_k and z''_k be respectively the boundaries of z'_{k+1} and z''_{k+1}. From (6.17) we see that

$$nc_k + ri_k + z'_k + z''_k \equiv 0. \tag{6.18}$$

Now $z'_k \sim 0$ on W, and we see from (6.18) that

$$nc_k \sim ri_k + z''_k \qquad \text{(on } W\text{)},$$

where the right member is on L. This is contrary to the nature of a critical k-cycle unless $n = 0$. Hence $n = 0$.

Proof that $r = 0$. Returning to (6.18) with $n = 0$, and noting that $z''_k \sim 0$ on L, we have

$$ri_k \sim z'_k \qquad \text{(on } L\text{)}. \tag{6.19}$$

If $z'_k \sim 0$ on L, r must be zero in (6.19), because invariant k-cycles do not bound below c. If $z'_k \not\sim 0$ on L, z'_k is a spannable k-cycle corr VW, and we again infer that $r = 0$, since invariant k-cycles are independent below c of spannable k-cycles corr VW.

Thus in (6.17), $m = n = r = 0$, and the proof of (b) is complete. The theorem follows directly.

The type numbers of a critical set

7. In §2 we associated a set G of m_k ideal non-degenerate critical points of index k ($k = 0, 1, \cdots, m$) with each complete critical set σ, terming this associated set "equivalent" to σ, and terming m_k the kth type number of the set σ. This number m_k was there defined as the sum

$$m_k = m_k^+ + m_k^-. \tag{7.1}$$

Recall that m_k^+ is the number of new k-cycles and m_k^- the number of newly-bounding $(k - 1)$-cycles in maximal sets of such cycles associated with the critical value c.

By a *newly-bounding k-cycle corr VW* we mean a spannable k-cycle corr VW which is not homologous to zero below c. It follows from this definition and from the definition of invariant $(k - 1)$-cycles corr VW, that a maximal set of $(k - 1)$-cycles independent below c, consists of maximal sets of invariant and newly-bounding $(k - 1)$-cycles corr VW. Of these cycles the invariant $(k - 1)$-cycles corr VW remain independent on $f < b$, according to Theorem 6.1. Hence m_k^- equals the number of newly-bounding $(k - 1)$-cycles corr VW in a maximal set of such cycles. It also follows from Theorem 6.1 that m_k^+ is the number of critical and linking k-cycles in maximal sets of such cycles corr VW. Turning to the definitions of these cycles in §3 and §6 we obtain the following theorem.

THEOREM 7.1. *The type number m_k of the critical set σ equals the number of critical k-cycles and spannable $(k - 1)$-cycles in maximal sets of such cycles corresponding to two arbitrarily small admissible neighborhoods VW of σ.*

This theorem is of basic importance in that it shows that the type number m_k of σ depends only upon the nature of f neighboring σ, unlike m_k^+ and m_k^- which in general depend upon f on a larger domain.

The theorem has been proved for the case of complete critical sets. For the case of critical sets σ in general we make the evaluation of m_k given by the theorem serve as the definition of the *type numbers* of σ. If a complete critical set σ is the sum of a finite ensemble of connected critical sets, as is true in the analytic case, we see that the type number m_k of σ is the sum of the corresponding type numbers of the component connected sets.

We shall now further determine the type numbers m_k in the most important cases. The following evaluation of m_k makes use of configurations defined by f and neighborhood functions of σ. It depends upon Theorems 7.1 and 5.2. In it e is an arbitrarily small positive constant.

I. *If σ is a connected critical set possessing a neighborhood function φ, the number m_k is the sum of the numbers of cycles in the following two sets:*

(a). *A maximal set of $(k - 1)$-cycles on $\varphi = e$ below c, independent on this domain, bounding on $\varphi \leqq e$.*

(b). *A maximal set of k-cycles on $\varphi \leqq e$, independent on $\varphi \leqq e$ of the k-cycles on $\varphi = e$ below c.*

We term a critical set on which f takes on a proper relative maximum or minimum, a *maximizing* or *minimizing* set respectively. For a maximizing or minimizing set on which $f = 0$ we note that the functions $-f$ and f are respectively admissible neighborhood functions, and for such sets I holds with $\varphi = -f$ and $\varphi = f$ respectively. In particular we note that for a minimizing set, m_k is the kth connectivity of the domain $f \leqq e$ neighboring σ.

Concerning the numbers m_0 and m_m we have the following theorem.

II. *The type number m_0 is 1 for each connected minimizing set, and null for all other connected critical sets. The type number m_m is 1 for each connected maximizing set and null for all other connected critical sets.*

By virtue of Theorem 7.1, m_0 is the number of critical 0-cycles in a maximal set of such cycles corr VW. If σ is connected, any two of its points can be arcwise connected in any arbitrarily small neighborhood of σ so that there is at most one 0-cycle in a maximal set of critical 0-cycles. If σ is not a minimizing set, there are points arbitrarily near σ at which $f < c$, so that corresponding to N, any point of a sufficiently small neighborhood N_0 of σ can be arcwise connected on N to a point on N_0 below c. Hence m_0 is 0 for connected critical sets which are not minimizing sets; m_0 is 1 for each connected minimizing set.

Before turning to m_m, let it be assumed that R is an m-circuit, that is, possesses no sub-complex of cells which is an m-cycle, and that R satisfies the manifold condition that any chain C_m of m-cells of R which contains a point P of R, either contains all cells incident with P or else possesses a boundary $(m - 1)$-cell incident with P.

We shall show that when $m_m > 0$ the set σ must be maximizing.

Since R is an m-circuit, any m-cycle sufficiently near σ is bounding near σ, so that there are no critical m-cycles. The number m_m must then be the number of spannable $(m - 1)$-cycles in a maximal set of such cycles corr VW.

Suppose that z_{m-1} is such a spannable $(m - 1)$-cycle corr VW. Without loss of generality we can suppose that z_{m-1} is composed of cells of a subdivision of R and bounds a chain z_m of such cells on V, because in any case we could use the Veblen-Alexander deformation to replace z_m by a nearby chain of that nature.

On $z_{m-1}, f < c$, and on z_m there are points at which $f \geqq c$. Let σ_1 be the set of points on z_m at which f takes on its absolute maximum relative to its values on z_m. Each point P of σ_1 will afford a relative or absolute maximum to f on R, since z_m contains an entire neighborhood of P on R. Hence $\sigma_1 \subset \sigma$. The set σ_1 is closed and, since f is constant on σ, contains all points of σ neighboring any point P of σ_1. The set σ_1 must then be identical with σ, since σ is connected. If $m_m > 0$, the set σ must then be maximizing.

It remains to prove that $m_m = 1$ if σ is a connected maximizing set.

Let z_m be the set of all m-cells of R whose closures contain points of σ. If R is sufficiently finely subdivided, z_m will lie on W. Its boundary z_{m-1} will be below c. I say moreover that z_{m-1} will not bound on V below c. For, by virtue of the Veblen-Alexander process, z_{m-1} would then bound a chain z_m' of cells of a subdivision of R below c. We suppose that z_m and z_m' consist of cells of a common subdivision of R. The sum

$$z_m + z_m',$$

reduced mod 2, will then be a non-null, non-singular m-cycle of cells of R covering at most a neighborhood of σ, contrary to the hypothesis that R is an m-circuit. Thus z_{m-1} does not bound on V below c. Hence z_{m-1} is spannable corr VW and $m_m \geqq 1$.

Finally I say that $m_m = 1$. To prove this let w_{m-1} be a second spannable $(m - 1)$-cycle corr VW, consisting of cells of R, and bounding a chain w_m of cells of R on V. We suppose moreover that z_m and w_m belong to a common subdivision, say R', of R. We have

$$w_m - z_m \rightarrow w_{m-1} - z_{m-1}.$$

By virtue of the manifold property of R', as previously assumed, both w_m and z_m must contain each m-cell of R whose closure contains a point of σ, and except for these cells consist of points below c. Hence $w_m - z_m$ reduces, mod 2, to a chain below c. Hence

$$w_{m-1} - z_{m-1} \sim 0 \qquad \text{(on } V \text{ below } c).$$

Thus $m_m = 1$ for a connected maximizing set, and the proof is complete.

The first of the following results was stated by A. B. Brown, but not completely proved by him (Brown [1]).

III. *Suppose $(x) = (0)$ is an isolated critical point in a coordinate system (x) in which f is analytic. If we set $\varphi = x^i x^i$, the jth type number m_j of $(x) = (0)$ is given by the formula*

$$m_j = R_{j-1} - \delta_1^j \qquad (j > 0) \tag{7.2}$$

where R_j is the jth connectivity of the domain $\varphi = e$ below c. In the case of a minimum $m_0 = 1$, otherwise $m_0 = 0$.

The type numbers are similarly evaluated if f is merely of class C^2 and $(x) = (0)$ is a non-degenerate critical point of f.

Relations (7.2) follow from I upon determining the relevant critical and spannable cycles.

First observe that there are no critical k-cycles when $k > 0$, since for $k > 0$ all k-cycles which lie on $\varphi = e$ bound on $\varphi \leqq e$. Turning to spannable $(k - 1)$-cycles we observe that all $(k - 1)$-cycles on $\varphi = e$ are bounding on $\varphi \leqq e$ when $k = 2, \cdots, m$. Hence $m_k = R_{k-1}$ when $k = 2, \cdots, m$. To determine m_1 we observe that there are $R_0 - 1$ spannable 0-cycles independent on $\varphi = e$ below c, each 0-cycle consisting of a pair of points. Hence $m_1 = R_0 - 1$. Finally to determine m_0 we use II. We conclude that $m_0 = 1$ in the case of a minimum. Otherwise $m_0 = 0$.

The second paragraph under III gives a preliminary determination of m_k in case the critical point is non-degenerate. The final result in this case is the following.

THEOREM 7.2. *The jth type number of a non-degenerate critical point of index k equals δ_k^j where δ_k^j is the Kronecker delta $(k, j = 0, 1, \cdots, m)$.*

We suppose that $(x) = (0)$ is the critical point and that $f(0) = 0$. If $k = 0$, the critical point affords a relative minimum to f and the theorem follows from III. If $k > 0$, and $j = 0$, $m_0 = 0$ according to III and the theorem is again true.

If $k > 0$ and $j > 0$, we begin by making a non-singular, linear, homogeneous transformation T of the local coordinates (x) to local coordinates (y) of such a nature that f takes the form

$$f(x) = -y_1^2 - \cdots - y_k^2 + y_{k+1}^2 + \cdots + y_m^2 + \omega(y_1, \cdots, y_m),$$

where ω is of more than the second order with respect to the distance to the origin in the space (y). We now regard f as a function $F(y)$ of the variables (y). Under the transformation T a neighborhood function will be carried into a neighborhood function, and it follows then from I that the type numbers of $(x) = (0)$ as a critical point of $f(x)$ equal those of $(y) = (0)$ as a critical point of $F(y)$.

With F we now consider the 1-parameter family of functions

$$F(y, \mu) = -y_1^2 - \cdots - y_k^2 + y_{k+1}^2 + \cdots + y_m^2 + \mu\omega(y)$$

where μ is a constant on the interval $0 \leqq \mu \leqq 1$. For each value of μ, $F(y, \mu)$ has a non-degenerate critical point of index k at the origin, and the function

$$\varphi = y_i y_i \qquad (i = 1, \cdots, m)$$

is a corresponding neighborhood function, provided

$$y_i y_i \leqq r,$$

where r is a sufficiently small positive constant. Reference to the proof of this fact in §4 shows that this constant r can be chosen independently of the choice of μ on the interval $0 \leqq \mu \leqq 1$. But we have seen under III that the type number m_j of $(y) = (0)$ as a critical point of $F(y, \mu)$ is given by the formula

$$m_j = R_{j-1} - \delta_1^j \qquad (j > 0)$$

where R_j is the jth connectivity of the domain Σ_μ determined by the conditions

$$\varphi = r, \qquad F(y, \mu) < 0.$$

We continue with the following lemma.

LEMMA 7.1. *The domains Σ_μ are homeomorphic for all values of μ on the interval $0 \leqq \mu \leqq 1$, and the type numbers of $(y) = (0)$ as a critical point of $F(y, \mu) = 0$ are accordingly independent of μ.*

To prove this lemma observe first that Σ_μ's boundary B_μ is the domain

$$\varphi = r, \qquad F(y, \mu) = 0,$$

and is without singularity, since φ is a neighborhood function. Let μ_0 be a particular value of μ on the interval (0, 1), and set

$$F(y, \mu_0) = \psi(y).$$

If μ_1 is a second value of μ sufficiently near μ_0, one can use the $(\psi\varphi)$-trajectories of §4 to show that the domains Σ_{μ_0} and Σ_{μ_1} are homeomorphic. To that end one considers a $(\psi\varphi)$-trajectory η through each point of B_{μ_0} and takes ψ as a parameter on this trajectory. If e is a sufficiently small positive constant, points on the trajectory η for which

$$-e \leqq \psi \leqq e$$

will form a field H on $\varphi = r$ neighboring B_{μ_0}. Moreover one shows readily that for μ_1 sufficiently near μ_0 there is one and only one trajectory of H through each point of B_{μ_1}, and that the point of intersection of η with B_{μ_1} varies continuously with its intersection with B_{μ_0}.

Suppose that μ_1 is taken so near μ_0 that each trajectory η meets B_{μ_1} in a point (y) at which ψ equals a value ψ_η such that

$$-e < \psi_\eta < e.$$

We now establish a homeomorphism between Σ_{μ_0} and Σ_{μ_1} by making each point (y) on Σ_{μ_1} and η for which

$$\psi_\eta \geqq \psi(y) \geqq -e \tag{7.3}$$

correspond to that point $(\bar{y})$ on Σ_{μ_0} on the same trajectory at which $\psi(\bar{y})$ divides the interval $(0, -e)$ in the same ratio as that in which $\psi(y)$ divides the interval (7.3).

We make the remaining points of Σ_{μ_0} correspond to themselves. We have thus established a homeomorphism between Σ_{μ_0} and Σ_{μ_1} for μ_1 on a sufficiently small open interval including μ_0. But the whole segment $0 \leqq \mu \leqq 1$ can be covered by a finite set of such intervals, from which it follows that the domains Σ_μ are homeomorphic for $0 \leqq \mu \leqq 1$.

The lemma is thereby proved.

The type numbers of $(x) = (0)$ as a critical point of f are accordingly the same as the type numbers of $(y) = (0)$ regarded as a critical point of the function

$$Q(y) = -y_1^2 - \cdots - y_k^2 + y_{k+1}^2 + \cdots + y_m^2 \qquad (k > 0). \tag{7.4}$$

To determine these type numbers according to III, we have merely to determine the connectivities of the domain

$$y_1^2 + \cdots + y_m^2 = e, \qquad Q(y) < 0 \qquad (k > 0), \tag{7.5}$$

where e is a positive constant. We come then to the following lemma.

LEMMA 7.2. *The connectivities of the domain* (7.5) *are those of the* $(k-1)$*-sphere.*

The connectivities of the domain (7.5) are those of the domain

$$0 < y_1^2 + \cdots + y_m^2 \leqq e, \qquad -y_1^2 - \cdots - y_k^2 + y_{k+1}^2 + \cdots + y_m^2 < 0, \tag{7.6}$$

since every chain or cycle on the domain (7.6) can be radially deformed on (7.6) into a chain or cycle on (7.5). But the domain (7.6) can in turn be deformed on itself into the configuration

$$y_{k+1}^2 + \cdots + y_m^2 = 0, \qquad 0 < y_1^2 + \cdots + y_k^2 \leqq e, \tag{7.7}$$

as follows. Corresponding to each point $(y) = (a)$ on (7.6) we hold

$$(y_1, \cdots, y_k)$$

fast and deform the point (y) in such a manner that the variables

$$|y_{k+1}|, \cdots, |y_m|$$

decrease to zero at rates respectively equal to their initial values

$$|a_{k+1}|, \cdots, |a_m|.$$

As a final simplification we radially deform the configuration (7.7) on itself into the $(k-1)$-sphere

$$y_{k+1}^2 + \cdots + y_m^2 = 0, \qquad y_1^2 + \cdots + y_k^2 = e. \tag{7.8}$$

The connectivities of the configuration (7.5) are then those of the $(k-1)$-sphere (7.8), and the lemma is proved.

In Lemma 7.1 we have seen that the type numbers of $(x) = (0)$ as a critical point of f equal those of $(y) = (0)$ as a critical point of the form $Q(y)$ of (7.4). By virtue of III the latter type numbers are given, for $j > 0$, by the formula

$$m_j = R_{j-1} - \delta_1^j,$$

where R_{j-1} is the $(j-1)$st connectivity of the domain (7.5), or, according to Lemma 7.2, the $(j-1)$st connectivity of the $(k-1)$-sphere. Hence $m_j = \delta_k^j$ where k is the index of the critical point, and Theorem 7.2 is proved.

Theorem 6.1 leads at once to the following theorem. In it the domain $f < a$ (or $f < b$) may be vacuous, and in such a case we understand that its connectivities are null.

THEOREM 7.3. *Let a and b, $a < b$, be any two constants which are not critical values of f. Let ΔR_k denote the kth connectivity of the domain $f < b$ minus that of the domain $f < a$. Let M_k be the sum of the kth type numbers of the critical sets σ on the domain $a < f < b$. Finally let M_k^+ be the number of cycles in the ensemble of maximal sets of new k-cycles relative to the different critical values of f between a and b, and M_k^- be the corresponding sum for newly-bounding $(k-1)$-cycles. Then*

$$\Delta R_k = M_k^+ - M_{k+1}^- \qquad (k = 0, 1, \cdots, m), \tag{7.9}$$
$$M_k = M_k^+ + M_k^-,$$

where $M_0^- = M_{m+1}^- = 0$.

If we eliminate the integers M_k^+ from the relations (7.9), we obtain the following $m + 1$ relations $(i = 0, 1, \cdots, m)$:

$$M_0 - M_1 + \cdots + (-1)^i M_i = \Delta(R_0 - R_1 + \cdots + (-1)^i R_i) + (-1)^i M_{i+1}^-. \tag{7.10}$$

In particular for $i = m$, we have the relation

$$M_0 - M_1 + \cdots + (-1)^m M_m = \Delta(R_0 - R_1 + \cdots + (-1)^m R_m).$$

Theorem 7.3 holds in particular if a is less than the absolute minimum of f on R and b is greater than the absolute maximum of f on R. For this case (7.10) takes the form

$$M_0 - M_1 + \cdots + (-1)^i M_i = R_0 - R_1 + \cdots + (-1)^i R_i + (-1)^i M_{i+1}^- \tag{7.11}$$

where the connectivities R_i are those of the whole manifold R. From (7.11) we then have the following theorem.

THEOREM 7.4. *Between the connectivities R_i of the Riemannian manifold R and the sums M_i of the ith type numbers of critical sets of f, the relations* (1.1) *of Theorem* 1.1 *hold. In particular the validity of Theorem* 1.1 *is established.*

We point out that the two important corollaries of Theorem 1.1 hold with the present interpretation and evaluation of M_i. These corollaries form the basic means of establishing the existence of critical points.

We also state the following corollary of Theorems 7.2 and 7.3.

COROLLARY 7.4. *If c is the only critical value between a and b, and is taken on by just one non-degenerate critical point P of index k, then*

$$M_i = \delta_i^k,$$

and the ith connectivity of the domain $f < b$ minus that of the domain $f < a$ affords a difference ΔR_i which is zero except that either

$$\Delta R_k = 1$$

or

$$\Delta R_{k-1} = -1.$$

We shall say that the critical point P is of *increasing* type if $\Delta R_k = 1$, and of *decreasing* type if $\Delta R_{k-1} = -1$. We see that P will always be of increasing type if $k = 0$. We shall make use of the following remark in subsequent work.

Remark. If $k > 0$, P will be of increasing type if and only if there is a linking k-cycle associated with P.

This follows from the fact brought out in the proof of III of this section, that for $k > 0$ there are no critical cycles associated with an isolated critical point.

The following theorem is useful in later work.

THEOREM 7.5. *Let $\varphi(x^1, \cdots, x^m)$ be a function which is analytic neighboring the origin in the space (x), which vanishes at the origin, and there possesses a non-degenerate critical point of index k, $0 < k \leqq m$. Let Σ be a regular analytic k-dimensional manifold which passes through the origin, and on which $\varphi(x)$ has a proper maximum at the origin. Corresponding to any sufficiently small neighborhood N of the origin there exists a positive constant e so small that the cycle defined by $\varphi = -e$ on Σ is non-bounding on N among points at which $\varphi < 0$.*

Before giving the proof we remark that the theorem can be shown to be false if Σ is not regular.

We begin with the following lemma.

LEMMA 7.3. *There exists a non-singular analytic transformation of the variables (x) neighboring $(x) = (0)$ into variables (y) neighboring $(y) = (0)$, which carries $(x) = (0)$ into $(y) = (0)$, and under which*

$$\varphi(x^1, \cdots, x^m) \equiv -y_1^2 - \cdots - y_k^2 + y_{k+1}^2 + \cdots + y_m^2. \tag{7.12}$$

According to Taylor's Theorem we can write $\varphi(x)$ in the form

$$\varphi(x) = a_{ij}(x)x^i x^j \qquad (i, j = 1, \cdots, m), \tag{7.13}$$

where

$$a_{ij}(x) = \int_0^1 (1 - u) \frac{\partial^2 \varphi}{\partial x^i \partial x^j} (ux^1, \cdots, ux^m)du.$$

See Jordan, *Cours d'Analyse*, vol. I, p. 249. We note that $a_{ij}(x)$ is analytic in (x) for (x) near (0), that it is symmetric in i and j, and that

$$a_{ij}(0) = \frac{1}{2} \frac{\partial^2 \varphi(0)}{\partial x^i \partial x^j}.$$

If $a_{11}(0) \neq 0$, we make the transformation

$$z_1 = \frac{a_{1j}(x)x^j}{|a_{11}(x)|^{1/2}} \qquad (j = 1, \cdots, m),$$

$$z_j = x^j \qquad (j = 2, \cdots, m),$$

as in the Lagrange transformation of quadratic forms. One then verifies the fact that

$$\varphi \equiv \pm z_1^2 + Q(z_2, \cdots, z_m).$$

Here $Q(z)$ is of the same form as $\varphi(x)$ in (7.13), involving merely the variables $z_2, \cdots, z_m$. If the coefficient of z_2^2 in $Q(z)$ is not zero at $(z) = (0)$, we make a similar transformation of the variables $z_2, \cdots, z_m$, and so on until we have reduced $\varphi(x)$ to a form involving squares only of the variables, with coefficients which are ± 1. The transformations involved have all been non-singular and analytic, neighboring the origin.

If $a_{11}(0) = 0$, at least one of the remaining coefficients a_{rs} will not be zero since $|a_{ij}(0)| \neq 0$. If the preliminary transformation

$$x^r = z_r - z_s,$$

$$x^s = z_r + z_s,$$

$$x^i = z_i \qquad (i \neq r, s)$$

be made, the resulting coefficients of z_r^2 and z_s^2 will not be zero at the origin, and upon taking z_r as z_1 we proceed as before.

Finally after a suitable relettering of the variables, φ will be reduced to the form (7.12). The number of minus signs on the right must thereby be exactly k. For if we transform the ordinary quadratic form which gives the terms of second order in φ by using merely the linear terms in the preceding transformations, the quadratic terms in φ will be carried into the form on the right of (7.12). According to the classical law of inertia for quadratic forms the number k is invariant under such transformations.

The proof of the lemma is now complete.

We continue the proof of the theorem by establishing the following statement.

(a). *If the manifold Σ is represented regularly and analytically neighboring the origin by a power series in k parameters (u), in the form*

$$y_i = b_{ij}u_j + \cdots \qquad (i = 1, \cdots, m; j = 1, \cdots, k), \tag{7.14}$$

then

$$|b_{hj}| \neq 0 \qquad (h, j = 1, \cdots, k). \tag{7.15}$$

Since the representation is regular, the complete matrix

$$||b_{ij}|| \qquad (i = 1, \cdots, m; j = 1, \cdots, k)$$

must have the rank k. I say in particular that (7.15) must hold. For otherwise there would exist constants c_j, not all zero, such that

$$b_{hj}c_j = 0 \qquad (h, j = 1, \cdots, k).$$

Upon setting $u_j = tc_j$ in (7.14) and evaluating φ upon the resulting curve γ on Σ, we would find that at $t = 0$

$$\frac{d\varphi}{dt} = 0, \quad \frac{d^2\varphi}{dt^2} = \sum_h 2\,(b_{hj}c_j)^2 > 0 \qquad (h = k+1, \cdots, m; j = 1, \cdots, k),$$

so that φ would have a minimum on γ at the origin, contrary to the hypothesis that φ has a proper maximum on Σ at the origin. Hence (7.15) holds as stated, and statement (a) is proved.

We now introduce a deformation δ with the following property:

(b). *If a is a sufficiently small positive constant, there exists a continuous deformation δ of the domain*

$$y_1^2 + \cdots + y_m^2 \leqq a^2, \qquad \varphi < 0, \tag{7.16}$$

on itself onto its subdomain Σ_a on Σ.

The deformation δ will be defined as the product of two deformations, β and γ.

The deformation β. In defining β we naturally restrict a to values so small that for $y_iy_i \leqq a^2$ the representation (7.12) holds. Under β each point (y) on (7.16) shall be continuously deformed into a point on the domain

$$0 < y_1^2 + \cdots + y_k^2 \leqq a^2, \qquad y_{k+1}^2 + \cdots + y_m^2 = 0, \tag{7.17}$$

by holding $y_1, \cdots, y_k$ fast, and letting the variables

$$|y_{k+1}|, \cdots, |y_m|$$

approach zero at rates equal to their initial absolute values. Under β points at which $\varphi < 0$ will be deformed through such points.

The deformation γ. By virtue of statement (a), points (y) on Σ_a will be determined in a one-to-one continuous manner by their first k coordinates $(y_1, \cdots, y_k)$, provided the constant a in (7.16) is sufficiently small, as we suppose is the case. As applied to Σ_a, the deformation β replaces Σ_a, at each instant of the deformation, by a one-to-one continuous image of Σ_a. The deformation β, as applied to Σ_a, then has a unique inverse β^{-1} which we could apply to (7.17) to deform (7.17) onto Σ_a, except that (7.17) might thereby be deformed outside of (7.16). To avoid this difficulty we first deform (7.17) radially on itself into a similar domain so near the origin that the resulting points of (7.17) are deformed under β^{-1} on (7.16) onto Σ_a. We term the resultant deformation of (7.17) onto Σ_a, the deformation γ.

The deformation $\delta = \beta\gamma$ clearly has the properties ascribed to δ in (b), and (b) is accordingly proved.

We now return to the theorem, and let r be a positive constant so small that the domain excluding the origin and including the points

$$\varphi \geqq -r \qquad \text{(on } \Sigma\text{)}, \tag{7.18}$$

neighboring the origin and connected to the origin, is without singularity or critical point of φ regarded as a function of the point on Σ. Let e be any positive constant less than r. The cycle $\varphi = -e$ on Σ will be non-bounding on (7.18) below 0. For if the cycle $\varphi = -e$ were bounding on (7.18) below 0, a use of the trajectories on Σ orthogonal to the manifolds φ constant on Σ would show that the cycle $\varphi = -e$ on Σ would be bounding on itself, which is impossible.

With r so chosen we choose the constant a as previously, with the additional restriction that the intersection of Σ and the domain

$$y_i y_i \leqq a^2 \qquad (i = 1, \cdots, m) \tag{7.19}$$

be interior to the domain (7.18). Let N then be any neighborhood of the origin on the domain (7.19). Let e be any positive constant less than r, and such that the cycle $\varphi = -e$ on Σ is on N. I say that this cycle will then be non-bounding on N below 0.

For if the above cycle bounded on N below 0, it would bound below 0 on the intersection of (7.19) with Σ by virtue of (b). The above cycle would thus bound on (7.18) below 0, contrary to the nature of the domain (7.18).

The cycle $\varphi = -e$ on Σ is accordingly non-bounding on N below 0, and the theorem is proved.

Justification of the count of equivalent critical points

8. We are counting a critical set with type numbers $m_0, m_1, \cdots, m_m$ as equivalent to a set G of ideal non-degenerate critical points in which the number of points of index k equals m_k ($k = 0, 1, \cdots, m$). In justification of this count we affirmed in §2 that these numbers m_k or this set G had four properties I, II, III, IV. Of these properties, I, II, and III have already been established. It remains to confirm property IV.

To establish property IV we first prove a number of lemmas. In the first lemma we shall be concerned with two ordinary values of f, a and b, with $a < b$. A k-cycle on $f < a$, non-bounding on $f < a$, but bounding on $f < b$, will be called a *newly-bounding* k-cycle relative to the change from a to b. A k-cycle on $f < b$, independent on $f < b$ of k-cycles on $f < a$, will be called a *new* k-cycle relative to the change from a to b. In the following lemma and its proof it will be convenient to abbreviate the phrase "the number of k-cycles in a maximal set of k-cycles" by the phrase *the count of k-cycles.*

LEMMA 8.1. *The sum M_k of the kth type numbers of the critical sets with critical values between a and b will exceed or equal the count u of new k-cycles plus the count v of newly-bounding $(k - 1)$-cycles relative to a change from the domain $f < a$ to the domain $f < b$.*

Let

$$a_1 < a_2 < \cdots < a_r \qquad (a_1 = a;\ a_r = b)$$

be a set of ordinary values of f so chosen as to separate the critical values of f between a and b. Let

$$h_i \qquad (i = 1, \cdots, r - 1)$$

be the count of $(k - 1)$-cycles on $f < a_i$, independent on $f < a_i$, bounding on $f < a_{i+1}$. Let h'_i be the count of such cycles, dependent on $f < a_i$ upon cycles on $f < a$. We have $h'_i \leqq h_i$ and

$$v = \sum_i h'_i \leqq \sum_i h_i \qquad (i = 1, \cdots, r - 1).$$

On the other hand let q_i be the count of k-cycles on $f < a_{i+1}$, independent on $f < a_{i+1}$ of cycles on $f < a_i$. Let q'_i be the count of such cycles independent on $f < b$ of cycles on $f < a_i$. We have $q'_i \leqq q_i$ and

$$u = \sum_i q'_i \leqq \sum_i q_i.$$

Combining these results we find that

$$u + v \leqq \sum_i h_i + \sum_i q_i = M_k$$

and the lemma is proved.

The second lemma concerns the function Φ of the statement IV of §2. By hypothesis,

$$\Phi \equiv f, \qquad (\mu) = (0).$$

LEMMA 8.2. *If a and b are ordinary values of f with $a < b$, then for (μ) sufficiently near (0) the domains $f < b$ and $\Phi < b$ are homeomorphic under a transformation of R by virtue of which the subdomains $f < a$ and $\Phi < a$ are likewise homeomorphic.*

The homeomorphism whose existence is affirmed in the lemma is taken as the identity except for points neighboring $f = a$ and $f = b$. Neighboring these manifolds we utilize the trajectories orthogonal to the manifolds f constant to complete the homeomorphism in the desired manner.

We note that the lemma is also true if either a or b is outside the interval of values which f takes on.

Now let σ be a critical set of f. Suppose $f = c$ on σ. The function f may take on the same value c on other critical sets. This possibility makes the proof of IV of §2 more difficult. We can however meet the difficulty by altering f or Φ slightly neighboring the critical sets in accordance with the following lemma. In this lemma we again use the invariant function

$$\varphi = g^{ij} f_{x^i} f_{x^j} \qquad (i, j = 1, \cdots, m).$$

LEMMA 8.3. *Corresponding to any critical set σ of f and arbitrarily small positive constants e, e_1, and ρ, $e > e_1$, there exists a function Ψ of class C^2 on R, with the following properties:*

(1). *Except when $\varphi < e$ neighboring σ, $\Psi \equiv 0$.*

(2). *When $\varphi < e_1$ neighboring σ, $\Psi \equiv \rho$.*

(3). *For (μ) sufficiently near (0) the function $\Phi + \Psi$ has no other critical points than those of Φ.*

Let r be a positive constant so small that among the points connected to σ for which

$$0 \leqq \varphi \leqq r,$$

$\varphi = 0$ only on σ. Choose $e < r$.

Let $H(z)$ be a function of a single variable z, of class C^2 for $z \geqq 0$, and such that

$$H(z) \equiv 1 \qquad (0 \leqq z \leqq e_1),$$

$$H(z) \equiv 0 \qquad (z \geqq e).$$

Neighboring σ the function Ψ will be defined by setting

$$\Psi \equiv \rho H(\varphi) \qquad (\varphi \leqq e).$$

We then take $\Psi \equiv 0$ elsewhere on R. One sees that Ψ has the properties (1) and (2). Moreover property (3) could fail only when

$$e_1 < \varphi < e. \tag{8.1}$$

But for $\rho = 0$ and $(\mu) = (0)$ we have

$$\Phi + \Psi \equiv f.$$

Moreover $\Phi + \Psi$ is of class C^2 in (x), (μ), and ρ. On the domain (8.1) the gradient of f is not null. Accordingly for ρ and (μ) sufficiently near $\rho = 0$ and $(\mu) = (0)$ respectively, the gradient of $\Phi + \Psi$ is not null.

Thus $\Phi + \Psi$ satisfies (3), and the lemma is proved.

If σ is any critical set of f we shall now justify our definition of the type numbers of σ and the set of ideal non-degenerate critical points equivalent to σ by establishing property IV *of* §2.

By virtue of Lemma 8.3 we lose no generality if we suppose that the critical value c taken on by f on σ is assumed on no other critical sets of f. For the addition to Φ of functions such as Ψ in Lemma 8.3 will not change the position or type numbers of the critical sets of Φ for (μ) sufficiently near (0), but will enable us to make the critical values, belonging to sets other than σ, different from the critical value c.

Suppose c is separated from the other critical values of f and from $\pm\infty$ by constants a and b. We include thereby the special cases where a is less than the absolute minimum of f, or b is greater than the absolute maximum of f. Let σ_μ

be the set of critical points of Φ which lie in the neighborhood of σ, for sets (μ) near (0), but not (0). Let (μ) be taken so near (0) that the critical values of Φ on σ_μ lie between a and b, and so that the homeomorphism of Lemma 8.2 holds.

The type number m_k of σ as a critical set of f will equal the number, say N_k, of newly-bounding $(k-1)$-cycles and new k-cycles in maximal sets of such cycles taken relative to a change from the domain $f < a$ to the domain $f < b$. By virtue of the homeomorphism of Lemma 8.2, the number N_k will be the same if taken relative to a change from the domain $\Phi < a$ to the domain $\Phi < b$. But it follows from Lemma 8.1 that the number M_k of critical points of index k in the set σ_μ will exceed or equal N_k. Thus

$$M_k \geqq N_k = m_k.$$

Statement IV of §2 is thereby proved.

The following theorem gives further content to the preceding theory.

THEOREM 8.1. *Let σ be a critical set of f which lies in a coordinate system (x) in which f is analytic. Corresponding to any arbitrarily small neighborhood N of σ, there exists a function Φ of class C^2 on R which with its first and second partial derivatives approximates f and its first and second partial derivatives arbitrarily closely over R, and which is such that*

$$f \equiv \Phi \qquad (on\ R - N)$$

while Φ has at most non-degenerate critical points on N.

In the coordinate system (x) consider the functions

$$F(x, \mu) = f(x) + \mu_i x^i \qquad (i = 1, \cdots, m)$$

where (μ) is a set of constants not (0). The condition that this function $F(x, \mu)$ have at most non-degenerate critical points in the system (x) is that the equations

$$f_{x^i} + \mu_i = 0$$

have no solution in the system (x) at which the hessian of f vanishes. That a choice of the constants (μ) can be so made arbitrarily near $(\mu) = (0)$ follows from a theorem formulated by Kellogg [1]. Let the constants (μ) then be restricted to such sets of constants.

To define the function Φ of the theorem we make use of the neighborhoods $\varphi < e$ and $\varphi < e_1$ of σ as in the proof of Lemma 8.3, supposing that $e_1 < e$, and that e is so small that points on $\varphi < e$ neighboring σ lie on N and in the coordinate system (x). We then use the function $H(z)$ of the proof of Lemma 8.3 and neighboring σ set

$$\Phi \equiv f + \mu_i x^i H(\varphi) \qquad (\varphi \leqq e),$$

taking Φ as identical with f elsewhere on R. On the neighborhood $\varphi \leqq e_1$ of σ we have

$$\Phi \equiv f + \mu_i x^i \qquad (i = 1, \cdots, m).$$

The remaining properties of Φ are verified as in the proof of Lemma 8.3, provided of course the constants (μ) are sufficiently near (0).

Normals from a point to a manifold

9. Suppose that our manifold R lies in a euclidean space of $m + 1$ dimensions. The problem of determining the straight lines issuing from a prescribed point O and normal to R belongs to the theory of critical points of functions as well as to differential geometry in the large. Since the distance function is involved it is also a special problem in the calculus of variations in the large.

Suppose first that O is not on R. Then the distance from O to R will be a function f of the point on R, analytic if R is locally analytic, of class C^2 if R is represented locally by means of functions of class C^2. One sees at once that a necessary and sufficient condition that a point P on R be a critical point of f is that P be the foot of a normal from O.

Let P be a critical point of f. Let P be taken as the origin in the space $(x_1, \cdots, x_{m+1})$, and the m-plane tangent to R at P as the m-plane $x_{m+1} = 0$. After a suitable rotation of the $x_1, \cdots, x_m$ axes in the m-plane $x_{m+1} = 0$, R can be represented as follows neighboring P:

$$2x_{m+1} = b_i x_i^2 + H(x_1, \cdots, x_m) \qquad (i = 1, \cdots, m). \tag{9.1}$$

Here b_i is a constant, and the first and second partial derivatives of H vanish at the origin.

For such of the constants b_i as are not zero we set

$$r_i = \frac{1}{b_i} \qquad (i = 1, \cdots, m)$$

and call the point P_i on the x_{m+1} axis at which $x_{m+1} = r_i$ a *center of principal normal curvature* or a *focal point* of R corresponding to P. If a constant $b_i = 0$, we say that the corresponding center P_i is at infinity. We define the centers P_i for any axes obtained from our specialized set by a rotation or translation, by imagining each normal to R rigidly fixed to R, and each center P_i rigidly fixed to its normal. The positions of these centers of curvature will be independent of the particular coordinate system into which we have rotated the original system in (9.1).

We shall prove the following lemma.

LEMMA 9.1. *The index of a critical point P of f corresponding to a normal OP equals the number of focal points corresponding to P between O and P exclusive. The point P is a degenerate critical point if and only if O is a focal point corresponding to P.*

If we use the coordinate system in (9.1) it appears that the point O must lie on the x_{m+1} axis, say at a point at which $x_{m+1} = a$. For simplicity suppose

$a > 0$. Neighboring the origin let f be represented by its value $\psi(x_1, \cdots, x_m)$ in terms of the coordinates $(x_1, \cdots, x_m)$. For x_{m+1} given by (9.1) we have

$$\begin{aligned}\psi &= (x_i x_i + (x_{m+1} - a)^2)^{1/2} \\ &= a\left[1 + x_i x_i\left(\frac{1}{a^2} - \frac{b_i}{a}\right) + K(x_1, \cdots, x_m)\right]^{1/2} \\ &= a + \tfrac{1}{2}\, x_i x_i \left(\frac{1 - b_i a}{a}\right) + L(x_1, \cdots, x_m)\end{aligned}$$

where K and L are functions of the same nature as H.

The index of the function f at P is accordingly the number of the coefficients

$$(9.2) \qquad (1 - b_i a) \qquad (i = 1, \cdots, m)$$

which are negative. One sees that the coefficient (9.2) is negative if and only if $b_i \neq 0$ and $0 < r_i < a$, that is, if the corresponding focal point lies between O and P. Moreover the critical point will be degenerate if and only if one of the coefficients (9.2) is zero, that is, if some $r_i = a$ and the corresponding focal point lies at O.

The lemma is thereby established.

We accordingly have the following theorem (Morse [5]).

THEOREM 9.1. *Suppose O is not a focal point of R. Of the straight line segments from O to R which are cut normally by R at their ends P on R, let M_k be the number upon which there are k focal points of P between O and P. Then between the numbers M_k and connectivities R_i of R the relations* (1.1) *of* §1 *hold.*

If O is on R, one must count the point O as a special normal segment upon which there are no focal points of R.

It is thereby understood that in counting the number of focal points of P on OP, a focal point must be counted the number of times that the corresponding coefficient b_i appears in (9.1). Moreover two normal segments OP_1 and OP_2 are to be counted as distinct if $P_1 \neq P_2$ even if OP_1 and OP_2 lie on the same straight line.

We have the following corollary of the theorem.

COROLLARY. *If O is not a focal point of R, there will be at least R_i normals OP from O to R with i focal points of P on OP, and all told at least*

$$R_0 + R_1 + \cdots + R_m$$

normals from O to R.

As an example suppose R is an orientable surface of genus p. Then

$$R_0 = 1, \qquad R_1 = 2p, \qquad R_2 = 1,$$

so that in the non-degenerate case there will be at least one normal OP from O to R with no focal point of P on OP, at least $2p$ normals with 1 such focal point thereon, and at least 1 normal with 2 such focal points thereon. From the relation

$$M_0 - M_1 + M_2 = R_0 - R_1 + R_2 = 2 - 2p,$$

one sees moreover that the number of normals OP is always even.

If O is a focal point of R, degenerate critical sets enter, and these must be counted according to their type numbers.

Symmetric squares of manifolds

10. Let R be a regular analytic m-manifold in a euclidean $(m + 1)$-space $\mathcal{E}$ of coordinates (x). We shall make a study of the *critical chords* of R, that is, chords which are normal to R at both of their ends. The function involved is the length of a variable chord whose end points P', P'' vary independently on R.

We denote the pair of points $(P'P'')$ by (π) and term P' and P'' the *vertices* of (π). We shall regard pairs

$$(P'P''), \qquad (P''P')$$

as identical. With such a convention the ensemble of points (π) will be termed the *symmetric square* of R. We denote it by R^2.

We shall now show how R^2 can be represented by a simplicial complex Π.

Suppose that R is the topological image of a simplicial m-complex K lying in a μ-dimensional euclidean space E. We represent P' and P'' by their images on K and the pairs $(P'P'')$ by the corresponding points on the product $K \times K$. We suppose $K \times K$ represented by a simplicial complex in the euclidean space $E \times E$, the product of E by itself. The complex $K \times K$ represents $R \times R$. To obtain a representation of R^2 we proceed as follows.

Pairs $(P'P'')$ on R^2 have been identified with pairs $(Q'Q'')$ for which

$$Q'' = P',$$
$$Q' = P''.$$

We denote this transformation by T. Points (π) which are images of one another under T will be termed *congruent*. Let $(y_1, \cdots, y_\mu) = (y)$ be the coordinates of a point of E. If (y') and (y'') represent two points on E, the set

$$(y_1', \cdots, y_\mu', y_1'', \cdots, y_\mu'')$$

can be regarded as representing a point on $E \times E$. The transformation T will be represented in the space $E \times E$ by a transformation of the form

$$\begin{aligned} \bar{y}_i'' &= y_i', \\ \bar{y}_i' &= y_i'' \qquad (i = 1, \cdots, \mu). \end{aligned} \tag{10.0}$$

We shall prove the following.

(a). *The complex $K \times K$ in the space $E \times E$ may be sectioned so as to have the following properties. A cell which is not pointwise invariant under T will possess* *no pairs of points which are congruent under T, and will be congruent under T to a second unique cell of the complex.*

We begin by sectioning $K \times K$ in the space $E \times E$ by each of the $(2\mu - 1)$-planes

$$y_i' = y_i'' \qquad (i = 1, \cdots, \mu). \tag{10.1}$$

After such a sectioning a cell which is not pointwise invariant under T will possess no pairs of points which are congruent under T. For a given cell either satisfies each of the conditions (10.1) identically or else for some value of i, say k, satisfies one of the inequalities

$$y_k' < y_k'', \qquad y_k' > y_k'' \tag{10.2}$$

identically. If the cell satisfies the condition $y_k' < y_k''$, for example, it can possess no pairs of points congruent under T, because under T a point satisfying $y_k' < y_k''$ is carried into a point satisfying $y_k'' < y_k'$.

With $K \times K$ so sectioned let w be a finite set of $(2\mu - 1)$-dimensional planes in the space $E \times E$ so chosen that each k-cell of $K \times K$ is on the k-dimensional intersection of a subset of these hyperplanes. Let w' be the set of hyperplanes w together with their images under T. We now further section $K \times K$ by the hyperplanes of w'. The resulting polyhedral complex will have the properties required in (a).

To insure that the cells of $K \times K$ be simplices we further subdivide $K \times K$ in the usual way by introducing a new vertex on each j-cell, $j > 0$, taking the cells in the order of their dimensionality, and adding the "straight" cells determined by the new vertices and the cells on the boundaries of the j-cell. We must take care however to choose these new vertices on congruent cells as congruent points. We thereby obtain a simplicial representation of $K \times K$ satisfying (a).

Finally we identify the pairs of congruent cells of $K \times K$, obtaining thereby a complex which we denote by Π *and which is the one-to-one image of the symmetric square R^2 of R.*

The complex Π could be represented as a simplicial complex on some auxiliary space of sufficiently high dimensions. By the *cells of* R^2 we mean the images on R^2 of the cells of Π.

We shall not be concerned with the ordinary connectivities of R^2, but rather with certain "relative connectivities" of R^2 defined as follows. Cf. Lefschetz [1].

A point (π) on R^2 will be termed *contracted* if its vertices P' and P'' are identical. A cell on R^2 will be termed contracted if composed of contracted points (π). In determining the *relative boundaries* of a k-chain, contracted $(k - 1)$-cells *shall not be counted.* With this understood relative cycles, homologies, and connectivities (mod 2) are defined as are ordinary cycles, homologies, and connectivities.

That these relative connectivities of R^2 are finite can now be proved with the aid of the simplicial representation Π of R^2, applying the Veblen-Alexander deformation to each relative cycle c_k (Lefschetz [1], p. 86). We observe first that if a point on a cell a_i of R^2 is contracted, every point of a_i is contracted. This follows from the fact that $K \times K$ was sectioned by each of the $(2\mu - 1)$-planes (10.1). Recall also that under the Veblen-Alexander process a given point of a_i will be deformed through points all on the same closed cell of R^2. Hence contracted points of a_i will be deformed through contracted points, and contracted cells through contracted cells. Thus each relative cycle of R^2 will be relatively homologous to a cycle of cells of R^2, and the relative connectivities of R^2 will be finite.

We note that any relative cycle of R^2 sufficiently near the set of contracted points on R^2 will be relatively homologous to zero. For any such relative cycle c_k will be relatively homologous under the Veblen-Alexander process to a cycle of contracted cells, and will thus be relatively homologous to zero.

Let (π_0) be any point of R^2 for which $P' \neq P''$. We shall represent the neighborhood of (π_0) as a Riemannian manifold. To that end let $(u^1, \cdots, u^m)$ be coordinates of a point on R in an admissible representation of R neighboring P', and let $(v^1, \cdots, v^m)$ be similar coordinates in an admissible representation of R neighboring P''. Let

$$g'_{ij}du^idu^j, \qquad g''_{hk}dv^hdv^k \qquad (i, j, h, k = 1, \cdots, m)$$

be respectively the corresponding differential forms of R. We now represent the points on R^2 neighboring (π_0) by the *local coordinates* (u) and (v) combined, and assign to these points (π) the differential form

$$ds^2 = g'_{ij}du^idu^j + g''_{hk}dv^hdv^k \qquad (i, j, h, k = 1, \cdots, m). \tag{10.3}$$

We may thus regard R^2 as a Riemannian form with a metric given by (10.3). This statement is to be qualified by the remark that the neighborhoods of contracted points on R^2 do not admit of a convenient parametric representation and corresponding metric.

Critical chords of manifolds

11. We continue with the m-manifold R of the preceding section and its symmetric square R^2. If $(\pi) = (P'P'')$ is a point on R^2, the distance in the euclidean space of R between the points P' and P'' on R will be a function $f(\pi)$ analytic in the local coordinates of R^2 wherever these local coordinates have been defined, that is, neighboring each point of R^2 not a contracted point. Although f takes on its absolute minimum zero at contracted points on R^2, such contracted points cannot properly be included as critical points of f. This corresponds to the use of relative connectivities instead of ordinary connectivities of R^2.

One sees that a necessary and sufficient condition that f have a critical point (π) is that the corresponding chord of R be a critical chord. I say, moreover, that the lengths of critical chords of R are bounded away from zero for all such chords of R. To verify this fact consider a normal to R at a point P. The segment of this normal which consists of points at most a sufficiently small positive distance L from P will have no point other than P in common with R.

Moreover, one choice of the constant L can be made for the whole manifold R, so that all extremal chords must have lengths greater than L. We note the following.

(a). *If L is a positive constant less than the lengths of the critical chords of R, the relative connectivities of the domain $f < L$ on R^2 are all zero.*

For if e is any arbitrarily small positive constant less than L, one can readily show, as in §2, that the domains $f < e$ and $f < L$ are homeomorphic. But if e is sufficiently small, the points on $f < e$ will be arbitrarily near contracted points, and as we have already noted all relative cycles on $f < e$ will then be relatively homologous to zero on $f < e$. Hence statement (a) is true.

We must here ask whether the results of §§1 to 9 still hold when the connectivities there appearing are replaced by the relative connectivities of the preceding section, and the convention is made that zero is not a critical value of our function f. The answer is in the affirmative, but several explanatory remarks are necessary.

In the first place each critical value c is positive, so that the spannable and critical cycles neighboring the corresponding critical sets possess no contracted cells. This part of the theory holds then exactly as before.

In carrying through the rest of the theory it is necessary that all deformations have the property of deforming contracted points through contracted points. The Veblen-Alexander deformation of a singular chain into a chain of cells of R^2 has this property, as we have already remarked. The deformation D which we have defined in §6 in what we have termed the Deformation Lemma, is not in the present case defined for a contracted point. We can avoid the difficulties inherent in this situation by altering D as follows. If L is any positive constant less than the least critical value of f, we can perform D as defined in §6 at least until the point deformed has reached a point at which $f = L$, thereafter holding the point fast. Points for which $f < L$ initially, are to be held fast throughout the altered deformation.

With these changes the theory goes through as before until Theorem 7.3 is reached. In proving this theorem for the case that a is less than the absolute minimum of f, use was made of the properties of the absolute minimizing set. In the present case these properties are replaced by the convention that $f = 0$ is not a critical value, and that the relative connectivities of the domain $f < L$ are all zero.

Theorem 7.4 together with its corollaries then holds as before, relative connectivities replacing ordinary connectivities.

For present purposes it will be convenient to call any set of critical chords which corresponds to a critical set of f, a *critical set of chords*, and to assign to such critical sets of chords the type numbers of the corresponding critical sets. In particular if a critical chord corresponds to a non-degenerate critical point, the chord will be termed *non-degenerate* and assigned the corresponding index. With this understood we restate Theorem 7.4 as follows.

THEOREM 11.1. *Between the sums M_i of the type numbers of the critical sets of chords of R and the relative connectivities R_i of the symmetric square of R the relations* (1.1) *still hold.*

For the sake of a future reference we state the following corollary.

COROLLARY. *If the critical chords of R are all non-degenerate, there exist at least R_i such chords of index i.*

We shall consider the critical chords of any analytic manifold homeomorphic with an m-sphere. To that end we first consider the critical chords of the ellipsoid E_m

$$a_i^2 x_i^2 = 1 \qquad (i = 1, \cdots, m + 1), \tag{11.1}$$

where

$$a_1 > a_2 > \cdots > a_{m+1} > 0.$$

One sees that the only critical chords of E_m are its axes. Concerning these axes we shall prove the following lemma.

LEMMA 11.1. *The axes of the ellipsoid E_m form non-degenerate critical chords, which taken in the order of their lengths have indices given by the respective numbers*

$$m, m + 1, \cdots, 2m.$$

Let the symmetric square of E_m be denoted by R^2. Let the critical chord of E_m on the axis of x_k be denoted by g_k, and the corresponding point (π) on R^2 by (π_k). Let the coordinates (x) of a point near the positive end of g_k $(x_k > 0)$ be denoted by $(u_1, \cdots, u_{m+1})$, and the coordinates (x) of a point near the negative end of g_k $(x_k < 0)$ be denoted by $(v_1, \cdots, v_{m+1})$. We can represent f near the point (π_k) in terms of the parameters

$$u_\alpha, \qquad v_\alpha \qquad (\alpha = 1, \cdots, k - 1, k + 1, \cdots, m + 1).$$

This set of parameters can be regarded as an admissible set of coordinates on the Riemannian manifold R^2 neighboring (π_k).

On E_m near the positive end of g_k we have

$$\begin{aligned} u_k &= \frac{1}{a_k}(1 - a_\alpha^2 u_\alpha^2)^{1/2} \qquad (\alpha = 1, \cdots, k - 1, k + 1, \cdots, m + 1), \\ &= \frac{1}{a_k}(1 - \tfrac{1}{2}a_\alpha^2 u_\alpha^2 + \cdots). \end{aligned} \tag{11.2}$$

Similarly on E_m near the negative end of g_k we have

$$v_k = -\frac{1}{a_k}(1 - \tfrac{1}{2}a_\alpha^2 v_\alpha^2 + \cdots) \quad (\alpha = 1, \cdots, k - 1, k + 1, \cdots, m + 1). \tag{11.3}$$

The length f of the chords determined by the parameters u_α, v_α will be given by the formula

$$f = [(u_\alpha - v_\alpha)(u_\alpha - v_\alpha) + (u_k - v_k)^2]^{1/2},$$

where α is to be summed as previously, but k not summed. Making use of (11.2) and (11.3) we find that

$$f = \left\{(u_\alpha - v_\alpha)(u_\alpha - v_\alpha) + \frac{4}{a_k^2}\left[1 - \frac{a_\alpha^2}{4}(u_\alpha^2 + v_\alpha^2) + \cdots\right]^2\right\}^{1/2}.$$

for variables u_α and v_α sufficiently near zero. Thus

$$f = \frac{2}{a_k}\left\{1 + \frac{a_k^2}{4}(u_\alpha - v_\alpha)(u_\alpha - v_\alpha) - \frac{a_\alpha^2}{2}(u_\alpha^2 + v_\alpha^2) + \cdots\right\}^{1/2}.$$

Whence

$$\text{(11.4)} \qquad \frac{a_k f}{2} = 1 + \tfrac{1}{8}[(a_k^2 - a_\alpha^2)(u_\alpha - v_\alpha)^2 - a_\alpha^2(u_\alpha + v_\alpha)^2] + \cdots .$$

Consider the following quadratic form in u_α and v_α,

$$Q_\alpha \equiv [(a_k^2 - a_\alpha^2)(u_\alpha - v_\alpha)^2 - a_\alpha^2(u_\alpha + v_\alpha)^2] \qquad (\alpha \neq k)$$

with α and k fixed. One sees that Q_α is non-degenerate, and that its index is 1 or 2, according as a_α is less than or greater than a_k. It follows that the bracket in (11.4) has the index

$$m + k - 1.$$

In fact for k fixed, $k - 1$ of the m forms Q_α have the index 2, and all of these forms have an index at least 1. Now k runs from 1 to $m + 1$ so that these indices run from m to $2m$, and the lemma is proved.

We shall say that a critical chord is of *increasing type* if it corresponds to a critical point of increasing type, in the sense of §7. In order to show that each of the critical chords g_k is of increasing type we shall show that there exists a linking μ-cycle Γ_μ belonging to g_k. The integer μ is the index of g_k, that is,

$$\mu = m + k - 1,$$

as we have just seen.

To define Γ_μ we shall subject the space (x) to a deformation in the form of a rotation. In this deformation the time t shall increase from 0 to 1 inclusive. A point whose coordinates $(x) = (z)$ when $t = 0$ shall be replaced at each subsequent moment t by a point (x) such that

$$\text{(11.5)} \qquad \begin{aligned} x_p &= z_q \cos \pi t - z_p \sin \pi t, \\ x_q &= z_p \sin \pi t + z_q \cos \pi t \qquad (p \neq q;\ 0 \leqq t \leqq 1), \\ x_i &= z_i, \end{aligned}$$

where p and q are two distinct, fixed integers on the range $1, \cdots, m+1$, and i takes on integral values from 1 to $m+1$ inclusive, excluding p and q. When $t = 1/2$ we note that

$$x_p = -z_q,$$
$$x_q = z_p.$$

The deformation r_{pq} of points (π). Let (π) be any point on R^2. Let h be the straight line in the space (x) which passes through the vertices of (π). Under the deformation (11.5), h will be replaced at the time t by a straight line which we denote by h_t. If h is sufficiently near the origin, h_t will intersect E_m in two distinct points. We denote this pair of points by (π_t). Under the deformation r_{pq} the point (π) shall be replaced by the point (π_t) at the time t, $0 \leqq t \leqq 1$. The deformation r_{pq} of (π) is defined only for points (π) for which the corresponding straight lines h_t meet E_m in two distinct points.

If w_k is a chain of points (π) on R^2 for which r_{pq} is defined, and if w_{k+1} is the deformation chain derived from w_k, we shall write

$$w_{k+1} = r_{pq}w_k.$$

Any point (π) on w_k and the point (π') which replaces it under r_{pq} when $t = 1$ will be identical on R^2, in accordance with our conventions. The chain w_k and the chain w_k' which replaces w_k under r_{pq} when $t = 1$, will be identical on R^2, and will accordingly disappear from the boundary of w_{k+1}. It is clear that we can then regard w_{k+1} as the product of w_k and a circle whose representative parameter is t.

The chains H_s on R^2. We now consider the set of all chords parallel to the extremal chord g_1. The subset of these chords whose lengths are at least unity will be determined by points (π) to which the deformation r_{pq} will be applicable, provided at least the semi-axes of E_m are sufficiently near unity, and this we suppose to be the case. The points (π) determined by these chords may be regarded as points on a chain H_m on R^2. We suppose the semi-axes of E_m are so near unity that the deformation chain

$$H_{m+k-1} = r_{k-1,\,k} \cdots r_{23}r_{12}H_m \qquad (k = 2, \cdots, m+1)$$

is well defined and possesses a boundary on which f is positive and less than the length of g_1. It is hereby understood that the deformations defining H_{m+k-1} are not to be combined and then applied to H_m, but rather that each deformation is to be applied to the chain which follows it to form a new chain of one higher dimension. We note that

$$H_{m+k-1} = r_{k-1,\,k}H_{m+k-2} \qquad (k > 1).$$

Recalling that the extremal chord g_k is determined by the point (π_k), we note that (π_{k-1}) is replaced by (π_k) under $r_{k-1,\,k}$ when $t = 1/2$. Starting with the fact that (π_1) lies on H_m we then see that (π_k) lies on H_{m+k-1}.

The relative cycles Γ_i *on* R^2. Let B_{m+k-2} be the boundary of H_{m+k-1}. On B_{m+k-2}, f is less than the minimum critical value of f on R^2. Hence there exists a relative bounding relation on R^2 of the form

$$H'_{m+k-1} \to B_{m+k-2} \qquad (k = 1, \cdots, m+1)$$

in which H'_{m+k-1} is a chain below the length of g_1. With the chord g_k we now associate the relative cycle

$$\Gamma_{m+k-1} \equiv H_{m+k-1} + H'_{m+k-1}.$$

We shall presently show that Γ_{m+k-1} is a linking cycle belonging to g_k.

We shall first prove the following lemma.

LEMMA 11.2. *The relative cycle* Γ_m *is a linking cycle on* R^2 *belonging to the critical chord* g_1.

Let (π) be a point on R^2 neighboring the point (π_1) which determines g_1. Let P' and P'' be the respective vertices of (π) neighboring the ends of g_1 at which x_1 is positive and negative. Neighboring (π_1), R^2 can be admissibly represented in terms of the last m coordinates (x) of P' and P'' respectively. Denote these coordinates by

$$(11.6) \qquad x'_\alpha, \quad x''_\alpha \qquad (\alpha = 2, \cdots, m+1),$$

respectively. To represent Γ_m regularly neighboring (π_1) on R^2 it will be sufficient to represent Γ_m regularly in the space of the parameters (11.6). Such a representation can be obtained by setting

$$(11.7) \qquad \begin{aligned} x'_\alpha &= u_\alpha, \\ x''_\alpha &= u_\alpha \end{aligned} \qquad (\alpha = 2, \cdots, m+1),$$

and assigning the variables u_α independent values near 0.

On Γ_m the value of f at the point (π) determined by the variables u_α will be an analytic function

$$f = \varphi(u_2, \cdots, u_{m+1})$$

of the variables u_α for variables u_α near 0. Moreover the function φ has a proper maximum when these variables are null, at the point (π_1). We can accordingly apply Theorem 7.5, and infer that there exists a positive constant e so small that the locus,

$$\varphi(u) = \varphi(0) - e,$$

on R^2, will be a spannable $(m-1)$-cycle C_{m-1} belonging to the function f and its critical point (π_1). But if e is sufficiently small, this cycle bounds on R^2 below $\varphi(0)$, in fact bounds the domain

$$f \leqq \varphi(0) - e.$$

Thus Γ_m is a linking cycle belonging to (π_1) and the extremal chord g_1.

We shall now introduce a principle of use in the representation of deformations.

Let M_r be an r-dimensional manifold in a space of coordinates (x) admitting a representation

$$x_i = h_i(v_1, \cdots, v_r) \qquad (i = 1, \cdots, q)$$

in terms of r parameters (v) neighboring a point (v_0). We suppose that the parameter values (v_0) determine (x_0), and that neighboring (v_0) the functions $h_i(v)$ are of class C^1. Let D be a deformation in the space (x) in which t represents the time, and in which a point whose coordinates x_i assume values z_i when $t = t_0$, is replaced at the time t, for t neighboring t_0, by the point

$$x_i = x_i(z, t) \qquad (i = 1, \cdots, q),$$

where the functions $x_i(z, t)$ are of class C^1 in (z) and t, for sets (z) near (x_0) and t near t_0, and

$$z_i \equiv x_i(z, t_0).$$

For sets (z) near (x_0) and t near t_0, the equations

$$x_i = x_i[h(v), t] \qquad (i = 1, \cdots, q) \tag{11.8}$$

will define an $(r + 1)$-dimensional manifold which will be termed the *deformation manifold* M_{r+1}, corresponding to M_r and D. The representation of M_{r+1} in terms of the parameters (v) and t will be termed the corresponding *product representation*. We now state a principle of use in the sequel.

COMPOSITION PRINCIPLE. *A sufficient condition that the product representation of the deformation manifold M_{r+1} in terms of the parameters (v) and t, be regular when $(v) = (v_0)$ and $t = t_0$, is that the parameters (v) regularly represent the manifold M_r neighboring (v_0), and that the trajectory of the point $(v) = (v_0)$ under D be not tangent to M_r when $t = t_0$.*

This statement follows at once from the representation (11.8) of M_{r+1}.

To show that Γ_{m+k-1} is a linking cycle belonging to g_k we need to know that this cycle admits a regular parametric representation on the Riemannian manifold R^2 neighboring (π_k). Recall that

$$H_{m+k-1} = r_{k-1,k}H_{m+k-2} \qquad (k > 1).$$

Also recall that R^2 admits coordinates (11.6) neighboring (π_1), and that in the space of these coordinates (11.6), H_m admits the regular representation (11.7), and thus a regular representation on R^2. Proceeding inductively we shall assume that H_{m+k-2} admits a regular representation on R^2 neighboring (π_{k-1}) in terms of parameters (v) neighboring (v_0). We then come to the following lemma.

LEMMA 11.3. *The points on H_{m+k-1} neighboring (π_k) result from the deformation of H_{m+k-2} neighboring (π_{k-1}) under $r_{k-1,k}$, for values of t which neighbor $1/2$. If*

H_{m+k-2} admits a regular representation neighboring (π_{k-1}), in terms of parameters (v) neighboring (v_0), the corresponding product representation of H_{m+k-1}, in terms of the parameters (v) and t, will be regular when $(v) = (v_0)$ and $t = 1/2$, provided the semi-axes of E_m are sufficiently near unity.

Let (π) be a point on R^2 neighboring (π_k). Let P' and P'' be the vertices of (π) neighboring the vertices of (π_k) at which x_k is positive and negative respectively. Let

$$x'_\alpha, \quad x''_\alpha \qquad (\alpha = 1, \cdots, k-1, k+1, \cdots, m+1) \tag{11.9}$$

be the x-coordinates of P' and P'' respectively, omitting the kth coordinates. The coordinates (11.9) will serve as Riemannian coordinates of R^2 neighboring (π_k).

Let C_{m+k-2} be the chain on H_{m+k-1} into which H_{m+k-2} is deformed under $r_{k-1,k}$ when $t = 1/2$. The point (π_k) is on C_{m+k-2}, and neighboring (π_k) the chain C_{m+k-2} is *regularly* represented by the parameters (v) which represent the corresponding points on H_{m+k-2}. This is true if the semi-axes of E_m are unity. It is then also true if the semi-axes are sufficiently near unity. We shall now apply the Composition Principle to complete the proof of the lemma.

To that end we note that the chords determined by H_{m+k-2} are parallel to the $(k-1)$-plane of the $x_1, x_2, \cdots, x_{k-1}$ axes in the space (x). Hence the chords determined by C_{m+k-2} are parallel to the $(k-1)$-plane of the

$$x_1, \cdots, x_{k-2}, x_k$$

axes. It follows that on C_{m+k-2},

$$dx'_{k-1} = dx''_{k-1}. \tag{11.10}$$

On the other hand consider the trajectory γ traced by (π_{k-1}) under $r_{k-1,k}$. This trajectory passes through (π_k) when $t = 1/2$. On it the vertices of the points (π) are symmetrically placed relative to the origin in the space (x). In particular on γ at (π_k),

$$\frac{dx'_{k-1}}{dt} = \frac{-dx''_{k-1}}{dt} \neq 0. \tag{11.11}$$

A comparison of (11.10) and (11.11) shows that γ is not tangent to C_{m+k-2} at (π_k). The lemma follows from the Composition Principle.

We can now prove the following theorem.

THEOREM 11.2. *The relative cycle Γ_{m+k-1} is a linking cycle belonging to the critical chord g_k, provided the semi-axes of E_m are sufficiently near unity.*

The proof of this theorem is similar to the proof of Lemma 11.2. It makes use of Theorem 7.5 and depends upon a preliminary verification of three facts.

I. The index of g_k is $m + k - 1$.

II. The cycle Γ_{m+k-1} admits a regular representation neighboring (π_k).

III. On H_{m+k-1} the chord length f assumes an absolute proper maximum at (π_k).

Statements I and II have already been established. We turn therefore to III.

Recall that the chords determined by H_{m+k-1} are parallel to the k-plane λ_k determined by the $x_1, \cdots, x_k$ axes. On λ_k the chord of maximum length is g_k. But all other k-planes parallel to λ_k which are not tangent to E_m either fail to intersect E_m or intersect E_m in an ellipsoid similar to the ellipsoid

$$a_1^2x_1^2 + \cdots + a_k^2x_k^2 = 1,$$

but with semi-axes which are shorter. The chords on these ellipsoids are all shorter than g_k. Statement III is accordingly proved.

As in the proof of Lemma 11.2 we now turn to Theorem 7.5. We infer that the locus

$$f - f(\pi_k) = -e$$

on Γ_{m+k-1}, for a sufficiently small positive constant e, is a spannable $(m + k - 2)$-cycle associated with (π_k), and that this spannable cycle bounds below $f(\pi_k)$. The cycle Γ_{m+k-1} is accordingly a linking cycle associated with g_k, and the theorem is proved.

From the fact that the axes of E_m are its only critical chords, and that each of these chords is of increasing type, we deduce the following theorem. See Corollary 7.4 and Remark.

THEOREM 11.3. *The relative connectivities R_i of the symmetric square of the m-sphere are all zero except that*

$$R_m = R_{m+1} = \cdots = R_{2m} = 1.$$

These results were obtained by the author in 1929 (Morse [7]). More recently M. Richardson and P. A. Smith [1] have taken up the abstract topology of involutions including the topology of symmetric products, and have obtained important general results. If suitably modified for the case of relative connectivities the theorems of Smith and Richardson joined with the above theory will greatly enlarge the results on critical chords.

We state the following corollary of the theorem.

COROLLARY. *If R is any regular analytic image of an m-sphere whose extremal chords are non-degenerate, among these extremal chords there must exist $m + 1$ extremal chords with indices varying from m to $2m$ inclusive.*

In the degenerate case the same result holds provided each critical set of chords is counted according to its type numbers.

CHAPTER VII

THE BOUNDARY PROBLEM IN THE LARGE

The problem of extending the theory of critical points of functions to the theory of critical points of functionals presents new analytical and topological difficulties. The point is replaced by a curve whose end points satisfy the given boundary conditions. Three types of curves are used: the ordinary continuous curve, the curve of class D^1, and the broken extremal. The broken extremal is represented by the ensemble (π) of its end points and vertices. The continuous curve is used to give the topological part of the theory a purely topological basis. The broken extremal and points (π) are used to approximate the functional by a function $J(\pi)$, and the curves of class D^1 serve to mediate between the continuous curves and the broken extremals.

The local characterization of the critical sets of the function $J(\pi)$ is made difficult by virtue of the fact that the critical sets are at least p-dimensional, where p is the number of intermediate vertices in a point (π). In general these critical sets are open. These difficulties are surmounted largely with the aid of *J-normal* points (π), that is, points (π) which determine successive extremal arcs on which J has the same value.

The ensemble of continuous curves which satisfy the given boundary conditions form the basic space Ω. In general the space Ω has infinitely many connectivity numbers which are not null. To parallel the work of the preceding chapter and obtain relations between the connectivities of Ω and the type numbers of the critical sets of extremals requires a careful use of deformations. These deformations have two essential characteristics: they are invariant in character, that is, locally independent of the coordinate systems used, and when applied to curves of class D^1 do not increase the value of J. In this connection we find it necessary to introduce a new definition of the *distance* between two curves of Ω of class D^1. This distance possesses two important properties. It is invariant in character, and by virtue of it J may be regarded as a continuous functional.

In the final section of this chapter we apply the preceding results to prove the existence of infinitely many extremals joining any two points on the regular analytic homeomorph of an m-sphere, including thereby a characterization of these extremals.

The calculus of variations in the large was first studied in connection with the absolute minimum. Hilbert was a pioneer in this research. See Bolza [1], p. 428. Tonelli [1] has added many important new conceptions and theorems. Signorini [1] and Birkhoff [1] have effectively used the broken extremal. McShane [1] has extended Tonelli's work. Carathéodory [4] has recently studied the general positive regular problem and obtained novel results. The author is concerned with the minimizimg extremal only as one type of critical extremal and has made little use of the theories of the absolute minimum.

The reader may also refer to a paper by Richmond [2] in which theorems in the large depending upon the existence of a field of extremals are obtained.

The functional domain Ω

1. We are concerned here with the Riemannian space R of Ch. VI. The space R is the homeomorph of an auxiliary simplicial m-circuit K. Locally it possesses an analytic Riemannian metric as described in Ch. VI.

Let A^1 and A^2 be any two *distinct* points on R. The points A^1 and A^2 will be used to designate the initial and final end points, respectively, of an admissible curve on R. Let B_r be an auxiliary connected simplicial r-circuit for which $0 \leqq r < 2m$. Let Z be a set of pairs of distinct points (A^1, A^2) on R, such that the pairs (A^1, A^2) are homeomorphic with the points of B_r. On R the local coordinates (x) of A^1 and A^2 will be respectively denoted by

$$(x^{11}, \cdots, x^{m1}), \qquad (x^{12}, \cdots, x^{m2}).$$

If $r = 0$, the end points A^1 and A^2 are fixed. If $r > 0$, we suppose that the neighborhood of each point of B_r can be represented as the image of a neighborhood of a point (α_0) in an auxiliary euclidean r-space of coordinates (α), and that the corresponding points (A^1, A^2) can be locally represented in the form

$$x^{is} = x^{is}(\alpha) \qquad (i = 1, \cdots, m;\ s = 1, 2),$$

where the functions $x^{is}(\alpha)$ are analytic in (α) for (α) near (α_0), and possess a functional matrix of rank r. We call this set of pairs of points (A^1, A^2), the *terminal manifold* Z.

Let γ be the continuous image on R of a line segment $0 \leqq t \leqq 1$. If the end points $t = 0$ and $t = 1$ of γ determine a pair (A^1, A^2) on Z, γ will be termed *topologically admissible.* When we are dealing with our integral on R we shall suppose that γ is of class D^1 as well as topologically admissible. We then term γ a *restricted* curve. Evaluated on restricted curves the integral defines our basic functional J.

The totality of topologically admissible curves γ will be termed the functional domain Ω determined by R and the manifold Z.

We shall now define chains and cycles on Ω.

Let closures be indicated by adding bars. Let α_i be any i-simplex in an auxiliary euclidean space and p a point on $\bar{\alpha}_i$. Let t be a point on the interval $0 \leqq t \leqq 1$. Denote this interval by t_1. The pairs (p, t) make up a product domain $\bar{\alpha}_i \times t_1$. Let φ represent a continuous map of $\bar{\alpha}_i \times t_1$ on R. The image under φ of the product $p \times t_1$ will be called the *curve determined by* p. We suppose each such curve is topologically admissible. In such a case the image of $\alpha_i \times t_1$ on R will be called an i-cell a_i on Ω. If α_k is any k-simplex on the boundary of α_i, the image of $\alpha_k \times t_1$ under φ will be called a *boundary k-cell* of a_i.

It will be convenient to refer to a product domain such as $\alpha_i \times t_1$ as a *functional i-simplex*.

Let β_k be an auxiliary k-simplex. We do not exclude the case where $\alpha_i = \beta_k$. Suppose that $k \leqq i$ and that T represents an affine projective correspondence which maps α_i onto β_k, covering each point of β_k at least once. Let p be any point on α_i and q its image on β_k. If the points (p, t) and (q, t) on $\alpha_i \times t_1$ and $\beta_k \times t_1$ respectively are now regarded as corresponding, there results an affine projective map of $\alpha_i \times t_1$ onto $\beta_k \times t_1$, covering each point of $\beta_k \times t_1$ at least once. Such a map of $\alpha_i \times t_1$ on $\beta_k \times t_1$ will be termed an *admissible affine map* of $\alpha_i \times t_1$ on $\beta_k \times t$.

Let ψ represent a continuous map of $\bar{\beta}_k \times t_1$ on R. Suppose that for some admissible affine map of $\alpha_i \times t_1$ on $\beta_k \times t_1$, corresponding points have the same images on R under φ and ψ respectively. If $i = k$, the images of $\alpha_i \times t$ and $\beta_i \times t$ on R will then be regarded as *identical* i-cells on Ω. If $i > k$, the image of $\alpha_i \times t_1$ on R will be counted as a null i-cell on Ω.

By a *closed* i-cell on Ω we mean an i-cell on Ω together with its boundary cells on Ω. By an *i-chain* on Ω we mean a finite set (possibly null) of closed i-cells on Ω, no two of which are "identical." By the *sum*, mod 2, of two i-chains z_i and w_i on Ω we mean the set of closed i-cells which belong to z_i or to w_i, but not to both z_i and w_i. The *boundary* z_{i-1} of an i-chain z_i on Ω is defined as the sum, mod 2, of the closed $(i-1)$-cells which are "boundary cells" of i-cells of z_i. We then write

$$z_i \to z_{i-1}. \tag{1.1}$$

As previously a k-chain on Ω whose boundary is null is called a *k-cycle* on Ω. Homologies, independence, maximal sets of k-cycles on Ω, are now formally defined for the case of Ω as for the case of R. We note in particular that z_{i-1} in (1.1) is now an $(i-1)$-cycle on Ω. We then write

$$z_{i-1} \sim 0 \qquad \text{(on } \Omega)$$

as before. If for i-cycles a_i and b_i on Ω, $a_i + b_i \sim 0$, we also write $a_i \sim b_i$. It is then clear that the respective members of valid homologies or bounding relations such as (1.1) can be added mod 2.

Let a_i be an i-cell on Ω given as the image on R of a functional simplex $\alpha_i \times t_1$ under a map φ. To *subdivide* a_i we first subdivide α_i. Let β_i be any one of the resulting simplices. We replace $\bar{a}_i$ by the sum of the images under φ of the closed functional simplices $\bar{\beta}_i \times t_1$. To subdivide a chain on Ω we subdivide its cells in the order of dimensionality. By the *connectivity* P_j of Ω, $j = 0, 1, \cdots$, we mean the maximum number of j-cycles on Ω between which there is no homology, provided such a maximum exists. If no such maximum exists, we say that P_j is infinite. A necessary and sufficient condition that $P_0 = 1$ is that any two admissible curves γ be continuously deformable into each other among admissible curves γ. The number P_0 can be infinite. It will be infinite, for example, if we are dealing with curves joining two fixed points on a torus. In the case of curves on an m-sphere ($m > 1$), with ends fixed, we see that $P_0 = 1$, but we shall find that infinitely many of the remaining connectivities are not null. The same is true for curves on an m-sphere one of whose end points is fixed and the other of which is free to move on a k-manifold with $k < m$.

The connectivities of Ω are invariant under any topological transformation of R which carries admissible end points (A^1, A^2) into admissible end points. For purposes of pure topology the analyticity of R and Z is of course unessential.

Deformations on Ω. The determination of the connectivities of Ω and the relations of these connectivities to our functional J lead us to deformations of curves and chains on Ω. In ordinary topology deformations of chains may not necessarily be point deformations, that is, the deformation of a point may depend upon the cell on which the point is given. So here the broadest class of deformations, namely deformations of chains on Ω, will not in general be curve deformations, that is, will not be uniquely determined when a curve of Ω is given, but only when the curve is given on a cell of some chain of Ω. To define such deformations we proceed as follows.

Let a_i be an i-cell on Ω, the image on R of a functional simplex $\alpha_i \times t_1$. Let p be any point on $\bar{\alpha}_i$, and t and τ be points on the respective intervals

$$0 \leqq t \leqq 1, \qquad 0 \leqq \tau \leqq 1,$$

denoted by t_1 and τ_1 respectively. The sets (p, t, τ) represent points on the product

$$\bar{\alpha}_i \times t_1 \times \tau_1. \tag{1.2}$$

Let $\varphi(p, t, \tau)$ represent a point on R which is the continuous image on R of an arbitrary point (p, t, τ) on (1.2). Suppose moreover that the points

$$\varphi(p, 0, \tau), \qquad \varphi(p, 1, \tau)$$

form an admissible pair of end points (A^1, A^2), and that when $\tau = 0$, $\varphi(p, t, 0)$ is the map which defines a_i. We say then that $\varphi(p, t, \tau)$ defines a *deformation* D of a_i on Ω. Under D a point $\varphi(p, t, 0)$ on a_i is said to be *replaced* at the time τ by the point $\varphi(p, t, \tau)$.

Let β_i be a simplex which is the affine projective image of α_i. The domains

$$\alpha_i \times t_1 \times \tau_1, \qquad \beta_i \times t_1 \times \tau_1 \tag{1.3}$$

then admit an affine projective correspondence in which points (p, t, τ) on $\alpha_i \times t_1 \times \tau_1$ correspond to points (q, t, τ) on $\beta_i \times t_1 \times \tau_1$ whenever p and q correspond on α_i and β_i respectively. This affine correspondence between the domains (1.3) will be termed *admissible*.

Let φ and ψ now represent continuous maps of the respective domains (1.3) on R of such a nature that the maps

$$\varphi(p, t, 0), \qquad \psi(q, t, 0)$$

define the same i-cells a_i of Ω, and φ and ψ define deformations of a_i on Ω. When the domains (1.3) possess an admissible affine correspondence by virtue of which the maps φ and ψ on R of projectively corresponding points on the products (1.3) are identical, the maps φ and ψ will be said to define the *same* deformation of a_i on Ω.

To deform a chain on Ω one deforms its cells, admitting however only such deformations as replace conventionally identical cells by cells which may be regarded as identical. As in ordinary topology one proves that two cycles z_i and w_i which can be deformed into one another on Ω bound a chain on Ω.

The function $J(\pi)$

2. In terms of any set of local coordinates (x) of R and of variables $(r) \neq (0)$ we suppose that the invariant function $F(x, r)$ of Ch. V is here analytic, positive, and positive regular. Moreover $F(x, r)$ shall be homogeneous of order 1 in the variables (r).

As is well known there then exists a positive constant e, small enough to have the following properties. Any extremal arc E on which $J < e$ will give an absolute minimum to J relative to all sensed curves of class D^1 joining E's end points. On E the local coordinates of any point will be analytic functions of the local coordinates of the end points of E and of the distance of P along E from the initial end point Q of E, at least as long as E does not reduce to a point. The set of all extremal segments issuing from Q with $J < e$ will form a field covering a neighborhood of Q in a one-to-one manner, Q alone excepted. We now choose a positive constant ρ less than e, and make the following definition.

Any extremal segment on R for which J is at most ρ will be called an elementary extremal.

An ordered set of $p + 2$ points

$$A^1, P^1, \cdots, P^p, A^2 \tag{2.1}$$

on R, with (A^1, A^2) on the terminal manifold Z, will be denoted by (π). The points (2.1) will be called the *vertices* of (π). It may be possible to join the successive points in (2.1) by elementary extremals. In such a case (π) will be termed *admissible*. The resulting broken extremal will be denoted by $g(\pi)$, and also termed admissible. The value of J taken along $g(\pi)$ will be denoted by $J(\pi)$.

We can regard $J(\pi)$ as a function φ of the parameters (α) locally representing its vertices A^1, A^2 and of the successive sets of coordinates (x) locally representing its vertices P^i. The function φ will be analytic, at least as long as the successive vertices remain distinct. A point (π) whose successive vertices are distinct will be called a *critical point* of $J(\pi)$ if all of the first partial derivatives of the function φ are zero at that point.

If the successive vertices are distinct, the conditions that φ_{α_h} be null are seen to be

$$\Big[F_{r^i}(x, \dot{x})x^{i\,s}_{\alpha_h}\Big]^{s=2}_{s=1} = 0 \qquad (h = 1, \cdots, r). \tag{2.2}$$

Here $(x, \dot{x})$ is to be evaluated on $g(\pi)$ at the final end point of $g(\pi)$ when $s = 2$, and at the initial end point of $g(\pi)$ when $s = 1$. The r conditions (2.2) for $h = 1, \cdots, r$ and $r > 0$ are equivalent to the transversality conditions of Ch. V, §9.

The partial derivative of φ with respect to the ith coordinate of a vertex (x) is seen to be

$$F_{r^i}(x, p) - F_{r^i}(x, q), \tag{2.3}$$

where (p) and (q) are the direction cosines at (x) of the elementary extremals of $g(\pi)$ preceding and following (x) respectively. If the difference (2.3) vanishes for $i = 1, \cdots, m$, I say that $(p) = (q)$. For if these differences all vanish we have

$$p^iF_{r^i}(x, p) - p^iF_{r^i}(x, q) = F(x, p) - p^iF_{r^i}(x, q) = E(x, q, p) = 0, \tag{2.4}$$

where $E(x, p, q)$ is the Weierstrass E-function. But by virtue of the positive regularity of F, (2.4) implies that $(p) = (q)$ as stated. We conclude that a necessary and sufficient condition that an admissible point (π) whose successive vertices are distinct be a critical point, is that $g(\pi)$ be a critical extremal, that is, one satisfying the transversality conditions (2.2).

By a *critical set* σ of $J(\pi)$ we mean any set of critical points on which $J(\pi)$ is constant, and which is at a positive distance from other critical points of $J(\pi)$. A critical set need not be connected. It is not necessarily closed, since it may have limit points (π) whose successive vertices are not all distinct.

To analyse sets of critical points (π) we need to formulate the analytic conditions that an extremal neighboring a given critical extremal g be a critical extremal. To that end let P_0 be a particular point of g and (x) a set of local coordinates neighboring P_0. The extremals neighboring g with directions neighboring those of g at P_0 can be represented in the coordinate system (x) in terms of the arc length t and $2(m - 1)$ parameters (β) neighboring a set (β_0) determining g. These extremals take the form

$$x^i = x^i(t, \beta) \tag{2.5}$$

where the functions $x^i(t, \beta)$ are analytic in their arguments. In any other coordinate system (x) representing the neighborhood of any other point of g, the extremals with points and directions sufficiently near a point and direction of g can again be represented in the form (2.5). We understand that the parameters (β) assigned to an extremal E in this second representation are the same as those which belong to E's continuation, cf. Ch. V, §5, in the first representation, and that the arc length t in the second representation is measured from the same point on the extremal E as in the first representation.

The conditions on the end points of an admissible arc have been locally given in terms of parameters (α), but these parameters (α) can be eliminated and the conditions on end points (x^1), (x^2) neighboring the end points of g given in the form

$$\psi_q(x^{11}, \cdots, x^{m1}, x^{12}, \cdots, x^{m2}) = 0 \qquad (q = 1, \cdots, 2m - r) \tag{2.6}$$

where the functions ψ_q are analytic in their arguments for points (x^1) and (x^2) near the end points of g, and possess a functional matrix of rank $2m - r$. In terms of the parameters t and (β) of (2.5) conditions (2.6) take the form

$$A_q(t^1, t^2, \beta) = 0, \tag{2.7}$$

where t^1 and t^2 are the unknown end values of t, and the functions A_q are analytic in their arguments for (β) near (β_0) and t^1 and t^2 near the values which determine the end points of g. To the conditions (2.7) must be added the transversality conditions. In terms of the parameters (β) and the end values t^1 and t^2 of the arc length t, these conditions take the form

$$B_h(t^1, t^2, \beta) = 0 \qquad (h = 1, \cdots, r), \tag{2.8}$$

where the functions B_h are again analytic for (β) near (β_0) and t^1 and t^2 near the values which determine the end points of g.

The conditions (2.7) and (2.8) are the required conditions that an extremal (β) near g be a critical extremal. These conditions may have no real solution other than the initial solution (t_0^1, t_0^2, β_0) corresponding to the given critical extremal g. If this is not the case the real solutions of (2.7) and (2.8) neighboring the initial solution will be representable by means of functions "in general" analytic on one or more "Gebilde" (Osgood [1], Koopman [1] with Brown) of ρ independent variables, with $\rho > 0$, each G including the initial solution. Moreover any real solution (t^1, t^2, β) neighboring the initial solution can be connected to the initial solution among real solutions of the form

$$\begin{aligned} t^1 &= t^1(\tau), \\ t^2 &= t^2(\tau), \\ \beta_i &= \beta_i(\tau) \qquad [i = 1, \cdots, 2(m-1)], \end{aligned} \tag{2.9}$$

where the functions on the right are analytic in τ for $0 \leqq \tau \leqq 1$, except for at most a finite set of values at which the functions are continuous at least.

We now evaluate J along the extremal E_τ determine by the parameters $\beta_i = \beta_i(\tau)$ in (2.9), taking J between the points determined by $t^1 = t^1(\tau)$ and $t^2 = t^2(\tau)$ respectively. The integral J then becomes a function $J(\tau)$. One can simplify the integral which gives $J(\tau)$ by making a linear transformation from the arc length t to a parameter u which varies between 0 and 1. By virtue of the fact that each extremal E_τ satisfies the Euler and transversality conditions one sees that $J'(\tau) \equiv 0$.

We conclude that J is constant on the critical extremals neighboring g.

We shall regard a family of critical extremals as *connected* if any curve of the family can be continuously deformed into any other curve of the family through curves of the family. By virtue of the preceding analysis in the small we see that J is constant on any connected critical family. We shall continue with a proof of the following statement.

(A). *The critical extremals on which $J < b$, a constant, can be grouped into a finite set of connected families.*

If the contrary were true there would exist an infinite set

$$E_1, E_2, \cdots \tag{2.10}$$

of critical extremals on each of which $J < b$, no two of which could be connected among critical extremals on which $J < b$. Let P_i be a point on E_i. The points P_i will have at least one cluster point P. Let

$$Q_1, Q_2, \cdots \tag{2.11}$$

be a subsequence of the points P_i tending towards P as the index n of Q_n becomes infinite. Let (x) be a local coordinate system neighboring P. In the coordinate system (x) let $(a)_n$ be the set of direction cosines at Q_n of that extremal of the set (2.10) on which Q_n lies. The sets $(a)_n$ will have at least one cluster set (a). The extremal E passing through the point P with direction cosines (a) will be a critical extremal on which $J \leqq b$. But as we have seen in the preceding paragraphs, E will be connected to all critical extremals defined by points Q_n and sets $(a)_n$ sufficiently near P and (a) respectively.

From this contradiction we infer the truth of statement (A).

Let g be a sensed curve of class D^1 and γ a curve segment on g. The value of J taken along γ in the positive sense of g will be termed the *J-length of γ on g.* Let P_0 be the initial point of g and P an arbitrary point of g. The value of J taken along g from P_0 to P will be termed the *J-coordinate* of P on g. If the J-coordinate of P is a differentiable function $h(t)$ of the time t, P will be said to be moving on g at a *J-rate* equal to $| h'(t) |$.

We shall now prove the following lemma.

Lemma 2.1. *Among points (π) for which $J(\pi)$ is less than a constant b, and for which $(p + 1)\rho > b$, there is at most a finite number of distinct connected critical sets.*

Let a point (π) such that the elementary extremals of $g(\pi)$ have equal J-lengths be termed *J-normal.* Let H be any connected family of critical extremals on which $J < b$. We observe that the set of J-normal critical points (π) determined by the extremals of H will form a connected set of points (π). From statement (A) we can infer that all J-normal critical points (π) for which $J(\pi) < b$ can be connected among such critical points to a finite set of such points (π).

We now pass to the case where (π_0) is *any* critical point for which $J(\pi_0) < b$. The point (π_0) will not in general be J-normal. Let (π_1) be the J-normal critical point which determines $g(\pi_0)$. The point (π_0) can be connected to (π_1) among admissible critical points (π). To that end we let each vertex of (π_0) move along $g(\pi_0)$ to the corresponding vertex of (π_1), moving at a J-rate equal to the "J-length on $g(\pi_0)$" of the arc to be traversed. At the end of a unit of time the point (π_0) will coincide with (π_1). The point (π_0) is thereby connected among critical points (π) to the J-normal critical point (π_1).

The lemma follows from the results of the preceding paragraph.

The domain $J(\pi) < b$

3. Let b be an ordinary value of J. Suppose the number $(p + 2)$ of vertices in (π) is fixed and such that

$$(p + 1)\rho > b, \tag{3.1}$$

where ρ is the constant chosen in §2. Understanding that $J(\pi)$ is defined only for admissible points (π) we come to the problem of proving that the connectivities of the domain $J(\pi) < b$ are finite. We shall accomplish this with the aid of certain deformations which we term *J-deformations*. These deformations deform admissible points (π) through admissible points (π). They do not increase $J(\pi)$ beyond its initial value, and they deform chains of points (π) continuously. They are invariantive in their definition.

The deformation D'. Let (π_0) be an admissible point (π). As the time t increases from 0 to 1 let the p intermediate vertices P^i of (π) move along $g(\pi_0)$ from their initial positions on $g(\pi_0)$ to a set of positions on $g(\pi_0)$ which divide $g(\pi_0)$ into $p + 1$ successive arcs of equal J-length, each vertex moving at a J-rate equal to the J-length on g of the arc of $g(\pi_0)$ to be traversed.

The deformation D' thereby defined is a J-deformation. In fact under D', $J(\pi)$ never exceeds its initial value $J(\pi_0)$ by virtue of the minimizing properties of elementary extremals. Moreover, during D' the J-length of each segment of $g(\pi_0)$ between two successive moving vertices varies between its initial value and its final value $J(\pi_0)/(p+1)$. But $J(\pi_0) < b$, and upon using (3.1) we see that

$$\frac{J(\pi_0)}{p+1} < \frac{b}{p+1} < \rho, \tag{3.2}$$

so that the corresponding elementary extremal never exceeds ρ in J-length. Finally chains of admissible points (π) are clearly deformed continuously under D'. Thus D' is a J-deformation.

The deformation D' tends to equalize the J-lengths of the elementary extremals of $g(\pi)$. The deformation D'' now to be defined tends to lessen $J(\pi)$ when $g(\pi)$ has corners, or is an extremal but not a critical extremal.

To define D'' we need certain general facts relating to the possibility of assigning Riemannian metrics to the products of Riemannian spaces or their subspaces. Let $R_1, \cdots, R_q$ be Riemannian m-spaces of which R_k possesses the local coordinates x_k^i, $i = 1, \cdots, m$, and an element of arc ds_k such that

$$ds_k^2 = g_{ij}^k dx_k^i dx_k^j \qquad (i, j = 1, \cdots, m) \tag{3.3}$$

where k is not summed. The combined set of coordinates

$$x_k^i \qquad (i = 1, \cdots, m; k = 1, \cdots, q)$$

will represent a point on the product Σ of the spaces $R_1, \cdots, R_q$. To Σ we can assign a metric defined by the form

$$ds^2 = g^k_{ij} dx^i_k dx^j_k,$$

where k is now to be summed as well as i and j.

On the other hand let A be a Riemannian ν-space with local coordinates (z). Let B be a regular subspace of A, that is, a subset of points of A locally representable in the form

$$z^i = z^i(u^1, \cdots, u^\mu) \qquad (i = 1, \cdots, \nu),$$

where the functions $z^i(u)$ are analytic in the variables (u) neighboring a set (u_0) and possess a functional matrix of rank μ. If A possesses a metric defined by the form

$$ds^2 = g_{ij}(z) dz^i dz^j \qquad (i, j = 1, \cdots, \nu),$$

we understand that the corresponding metric of B is defined by the form

$$ds^2 = g_{ij}[z(u)] \frac{\partial z^i}{\partial u^h} \frac{\partial z^j}{\partial u^k} du^h du^k,$$

or, more concisely,

$$ds^2 = b_{hk}(u) du^h du^k \qquad (h, k = 1, \cdots, \mu).$$

We regard the parameters (u) as the local coordinates of B.

With this understood we consider the $(p + 2)$-fold product A of R by itself, representing a point on A by the local coordinates on R of the points

$$A^1, P^1, \cdots, P^p, A^2$$

previously used to define vertices of a point (π). We can assign a metric to A in the manner just described. To obtain admissible points (π) one must limit the pairs (A^1, A^2) to pairs on our terminal manifold Z. With the vertices A^1, A^2 so limited, the corresponding point (π) defines a point on a regular subspace B of A. Let (u) be a set of $\mu = r + pm$ variables of which the first r are the parameters (α) used in a local representation of the terminal manifold Z, the next m are local coordinates of P^1, the next m are local coordinates of P^2, and so on, the last m being the local coordinates of P^p. The complete set (u) forms a set of local coordinates in a representation of the regular subspace B of A. As in the preceding paragraph we can make use of the metric of A to derive a corresponding metric

$$ds^2 = b_{hk}(u) du^h du^k \qquad (h, k = 1, \cdots, \mu) \tag{3.4}$$

for the space B.

If any two successive vertices of a point (π) on B are at a J-distance at most ρ from each other, the point (π) on B will be admissible. The totality of admissible points (π) on B will be denoted by Π. If (π_0) is an inner point of Π, $J(\pi)$

will be defined for neighboring points (π) on Π. The points on Π neighboring (π_0) can be represented as above in terms of $\mu = r + pm$ parameters (u). In terms of these parameters (u) we then set

$$J(\pi) = \varphi(u), \tag{3.5}$$

obtaining thereby an analytic representation of $J(\pi)$ neighboring (π_0).

The set Σ and constant η. In the forthcoming definition of the deformation D'' we shall refer to the set of all J-normal points (π) on the domain $J \leqq b$ as the set Σ. We shall also refer to a positive constant η defined as follows. The constant η shall be a positive constant so small that any point (π) on B within a geodesic distance η on B of points of Σ will possess successive vertices which are distinct and define elementary extremals of J-length less than ρ. That such a choice is possible follows from the fact that points (π) on Σ determine elementary extremals with lengths which are uniformly bounded from zero and which are at most the constant

$$\frac{b}{p+1} < \rho.$$

The deformation D''. With this choice of η let (π_0) be a point on Π within a distance η of a point of Σ. Neighboring (π_0) we regard Π as a Riemannian manifold with metric defined by (3.4), and with parameters (u) in terms of which $J(\pi)$ equals the function $\varphi(u)$ of (3.5). If (π_0) is an ordinary point of $J(\pi)$, the trajectories on Π orthogonal to the loci on which $J(\pi)$ is constant, can be locally represented by differential equations of the form

$$\frac{du^i}{dt} = -b^{ij}(u)\varphi_{u^j}(u) \qquad (i, j = 1, \cdots, \mu), \tag{3.6}$$

where b^{ij} is the cofactor of the coefficient b_{ij} in (3.4) divided by the determinant $|b_{ij}|$. On these trajectories neighboring an ordinary point (π_0) of $\varphi(u)$ we have

$$\frac{dJ}{dt} = \frac{d\varphi}{dt} = -b^{ij}\varphi_{u^i}\varphi_{u^j} < 0. \tag{3.7}$$

Under the deformation D'' points (π) on Π which are initially at a distance $d \geqq \eta/2$ from the points of Σ shall be held fast. A point (π) which is at a distance $d < \eta/2$ from Σ shall be replaced at the time τ, $0 \leqq \tau \leqq 1$, by the point on the trajectory (3.6) through (π) at which t is larger than at (π) by the amount

$$e(\eta/2 - d)\tau.$$

Here e is a positive constant which we choose so small that the points initially at a distance $d < \eta/2$ from Σ are deformed under D'' through points at most a distance η from Σ. This choice of e is made in order that the points initially at a distance $d < \eta/2$ from Σ may be deformed through points at which the corresponding function $\varphi(u)$ never fails to be analytic through coalescence of some of the vertices of (π).

The deformation D_p. The preceding deformations D' and D'' will now be combined into the product deformation

$$D_p = D''D'. \tag{3.8}$$

It is understood that D' is applied first and D'' then applied to the resulting points. The subscript p indicates that we are dealing with points (π) with p intermediate vertices. Concerning D_p we now prove the following lemma.

LEMMA 3.1. *Under the deformation D_p each ordinary point on the domain*

$$J(\pi) \leqq b$$

is carried into a point (π') at which $J(\pi') < J(\pi)$.

To prove this lemma we divide points (π) on $J(\pi) \leqq b$ into two classes as follows.

Class I shall contain the points (π) which are deformed under D' into points (π_1) at least a distance $\eta/2$ on Π from the J-normal points on $J \leqq b$.

Class II shall contain the remaining points (π) on $J \leqq b$.

If (π) belongs to Class I and (π_1) is its final image under D', there will be at least one elementary extremal of $g(\pi_1)$ with a J-length less than

$$M(\pi_1) - e_1,$$

where $M(\pi_1)$ is the maximum of the J-lengths of the elementary extremals of $g(\pi_1)$ and e_1 is a positive constant independent of the point (π) in Class I. We then have

$$J(\pi_1) \leqq (p + 1)M(\pi_1) - e_1.$$

But from the definition of D',

$$M(\pi_1)(p + 1) \leqq J(\pi),$$

so that

$$J(\pi_1) \leqq J(\pi) - e_1.$$

Now $J(\pi_1)$ will not be increased under D'', and we see that $J(\pi)$ is accordingly decreased by at least e_1 under $D''D'$ if (π) initially belongs to Class I.

If (π) belongs to Class II, but is not a critical point, $J(\pi)$ is decreased under D'' as follows from (3.7).

The lemma is thereby proved.

We need to represent the ensemble of points (π) with $p + 2$ vertices as a complex. With that in view recall that the pairs of end points (A^1, A^2) are the images of points on the terminal manifold Z, while the intermediate vertices lie on R. The totality of points (π) can accordingly be represented by the product complex

$$Z \times R^p.$$

With this understood we can prove the following theorem.

THEOREM 3.1. *If b is an ordinary value of $J(\pi)$, the connectivities of the domain $J(\pi) < b$ are finite.*

First observe that the boundary of the domain $J(\pi) < b$ consists of points at which

$$J(\pi) = b,$$

or at which

$$M(\pi) = \rho,$$

where $M(\pi)$ is the maximum J-length of the elementary extremals of $g(\pi)$. We shall prove that the connectivities of the domain $J(\pi) < b$ are finite by showing that D_p deforms the domain $J(\pi) \leqq b$ on itself onto a complex on its interior.

In particular under D' any point (π_0) will be deformed into a point (π) for which

$$M(\pi) \leqq \frac{b}{p+1} < \rho.$$

Moreover under D'' points at which $M(\pi) < \rho$ are deformed through such points. Hence under D_p all points on the domain $J(\pi) \leqq b$ are carried into points for which $M(\pi) < \rho$.

On the other hand points (π) at which $J(\pi) = b$ are ordinary points by hypothesis, and by virtue of Lemma 3.1 are carried into points (π') at which $J(\pi') < J(\pi) = b$.

In sum D_p deforms the domain $J(\pi) < b$ on itself into a subdomain H at a positive distance from the boundary of the domain $J(\pi) < b$. But if the product complex $Z \times R^p$ is sufficiently finely divided, a subcomplex C of its cells can be chosen so as to include the points of H and to be included on the domain $J(\pi) < b$. By virtue of the deformation D_p any cycle on $J(\pi) < b$ is homologous on $J(\pi) < b$ to a cycle on C. Since all cycles on C are homologous on C to a finite set of such cycles, the same is true of cycles on $J(\pi) < b$ and the theorem is proved.

We continue with the following theorem.

THEOREM 3.2. *If a and b, $a < b$, are two ordinary values of J between which there are no critical values of J, the connectivities of the domains $J(\pi) < a$ and $J(\pi) < b$ will be equal.*

The proof of the preceding lemma makes it clear that under D_p each point (π) on the domain

$$a \leqq J(\pi) \leqq b$$

is carried into a point (π') such that

$$J(\pi') < J(\pi) - d,$$

where d is a positive constant independent of (π). Hence if D_p is repeated a number of times n, such that

$$dn > b - a,$$

the domain $J(\pi) < b$ will be deformed on itself onto the domain $J(\pi) < a$.

Hence any cycle on $J(\pi) < b$ is homologous on $J(\pi) < b$ to a cycle on $J(\pi) < a$, so that the connectivities of $J(\pi) < b$ are at most those of $J(\pi) < a$. But any set H of j-cycles on $J(\pi) < a$ between which there is no proper homology on $J(\pi) < a$, will likewise admit no proper homology on $J(\pi) < b$. For otherwise a use of the product deformation D_p^n would lead to an homology between these same cycles H on $J(\pi) < a$. Thus a maximal set of j-cycles independent on $J(\pi) < a$ is a maximal set of j-cycles independent on $J(\pi) < b$. The number of j-cycles in such a maximal set is the common j-connectivity of the domains $J(\pi) < a$ and $J(\pi) < b$.

Restricted domains on Ω

4. Let z_j be a j-chain on the functional domain Ω. Let a_j be a j-cell of z_j. The closure $\bar{a}_j$ of a_j can be represented as the continuous image on R of the closed functional simplex $\bar{\alpha}_j \times t_1$ of §1. Let P be a point on $\bar{\alpha}_j$. The image on $\bar{a}_j$ of the product $P \times t_1$ on $\bar{\alpha}_j \times t_1$ has been termed the curve on $\bar{a}_j$ determined by P on $\bar{\alpha}_j$. If the curves on $\bar{a}_j$ determined by points P on $\bar{\alpha}_j$ are restricted curves on which the J-length of each curve from its initial point to the image of a point Q on $\bar{\alpha}_j \times t_1$ varies continuously with Q, a_j will be termed a *restricted j-cell.* If z_j is a sum of the closures of restricted j-cells, z_j will be called a *restricted j-chain.* Employing restricted chains and cycles only, one can now formally define the connectivities of Ω as before. We term these connectivities the *restricted connectivities* of Ω. We shall prove the following theorem.

THEOREM 4.1. *The restricted connectivities R_i of the functional domain Ω equal the corresponding unrestricted connectivities P_i of Ω.*

Let k_j be an unrestricted chain on Ω. Let γ be any one of the curves of k_j. Let p be a positive integer and let γ be divided into $p + 1$ segments of equal variation of t. Let (π) denote the point determined by the successive ends of these segments of γ, and let h be any one of these segments. If p is sufficiently large (and we suppose it is), each point of h can be joined to the initial point of h by an elementary extremal μ. Moreover we can suppose that p is chosen independently of the curve γ of k_j under consideration. With p so chosen we shall now define certain deformations.

The deformation δ'. We shall deform the preceding curve γ into $g(\pi)$. Let τ represent the time during this deformation with $0 \leqq \tau \leqq 1$. For each such value of τ we suppose h is divided into two segments λ and λ' in the ratio of τ to $1 - \tau$ with respect to the variation of t on h. For each value of τ we replace the second of these segments of h by itself, while we replace the first by the elementary extremal μ which joins its end points. We make a point on λ which divides λ

in a given ratio with respect to t correspond to the point on μ which divides μ in the same ratio with respect to the variation of J, assigning to this point on μ the same value of t as its correspondent on λ bears. We denote this deformation by δ'.

The deformation δ''. Let g be an arbitrary restricted curve. The deformation δ'' will not change g except in parameterization. To define δ'' we let the point t on g move along g to the point on g which divides g with respect to J-length in the same ratio that t initially divided the interval $(0, 1)$, moving at a constant J-rate along g equal to the "J-length on g" of the arc of g to be traversed. In the resulting parameterization the parameter t again runs from 0 to 1, but is now proportional to the J-length of the arc of g preceding the point t. Such a parameterization will be termed a *J-parameterization.*

The deformation δ_p. We define δ_p as the product

$$\delta_p = \delta''\delta'$$

following δ' by the deformation δ''.

Let a_h be an h-cell of k_j given as the image on R of a functional simplex $\alpha_h \times t_1$. Let γ be the curve on $\bar{a}_h$ which is determined by the point P on $\bar{\alpha}_h$, and let γ_τ be the curve which replaces γ at the time τ under the deformation δ_p. Let τ_1 denote the interval $0 \leqq \tau \leqq 1$. If the point t on γ_τ be regarded as the image of the point (P, t, τ) on the product

$$\alpha_h \times t_1 \times \tau_1,$$

we see that δ_p defines a deformation of a_h on Ω in the sense of §1.

We now come to the proof of the theorem.

We shall denote the restricted j-chain into which an arbitrary j-chain k_j on Ω is deformed under δ_p by $r(k_j)$. Observe that when

$$k_j \rightarrow k_{j-1} \qquad \text{(on } \Omega)$$

we have

$$r(k_j) \rightarrow r(k_{j-1}). \tag{4.1}$$

If k_j is a j-cycle on Ω we see that $r(k_j)$ will be a restricted j-cycle, homologous to k_j on Ω.

It follows that $P_i \leqq R_i$ and that R_i must be infinite with P_i.

To show that $P_i = R_i$, we have merely to show that a restricted j-cycle z_j on Ω which bounds an unrestricted chain z_{j+1} on Ω, necessarily bounds a restricted chain on Ω. We are supposing then that

$$z_{j+1} \rightarrow z_j \qquad \text{(on } \Omega).$$

If the integer p is sufficiently large, the deformation δ_p is applicable to z_{j+1} and we have

$$r(z_{j+1}) \rightarrow r(z_j) \tag{4.2}$$

Let w_{j+1} be the chain through which the cycle z_j is deformed under δ_p. We have (always mod 2),

$$w_{j+1} \to z_j + r(z_j), \tag{4.3}$$

and hence from (4.2) and (4.3),

$$r(z_{j+1}) + w_{j+1} \to z_j. \tag{4.4}$$

Moreover the left member of (4.4) is a restricted chain since z_j is a restricted cycle. Hence z_j bounds a restricted chain if it bounds at all on Ω.

We conclude that $P_i = R_i$ and the theorem is proved.

The restricted domain Ω_b. The set of restricted curves on Ω on which $J < b$ will be denoted by Ω_b. Concerning Ω_b we now prove the following theorem.

THEOREM 4.2. *The restricted connectivities R_i' of Ω_b equal the connectivities R_i'' of the domain $J(\pi) < b$. The connectivities R_i'' are thus independent of the number of vertices $p + 2$ of their points (π), provided only $(p + 1)\rho > b$.*

To prove this theorem we shall begin by showing that any chain c_j of points (π) on $J(\pi) < b$ leads to a chain $\Omega(c_j)$ of restricted curves on Ω_b.

The chain $\Omega(c_j)$. Let b_j be a j-cell, the image on the domain Π of an auxiliary j-simplex α_j. Let (π) be a point on $\bar{b}_j$ and p its image on $\bar{\alpha}_j$. Suppose that $g(\pi)$ has a J-parameterization, with parameter t. We make the point t on $g(\pi)$ correspond to the point (p, t) on $\bar{\alpha}_j \times t_1$. We have thus defined a continuous image of $\bar{\alpha}_j \times t_1$ on R, or if we please, a closed j-cell on Ω *derived* from $\bar{b}_j$. We denote this closed j-cell on Ω by $\Omega(\bar{b}_j)$. With the chain c_j on Π we now associate the sum of the closed j-cells $\Omega(\bar{b}_j)$ on Ω "derived" from the respective j-cells of c_j. We denote this chain on Ω by $\Omega(c_j)$. We shall refer to the integer p giving the number of intermediate vertices of the curves $g(\pi)$ making up $\Omega(c_j)$ as the *index* of $\Omega(c_j)$. We see that $\Omega(c_j)$ will be a cycle on Ω_b if and only if c_j is a cycle on the domain of the points (π).

I say that $R_j' \leq R_j''$. For any restricted j-cycle on Ω_b will be deformed under δ_p into a cycle $\Omega(c_j)$ of index p, derivable from a cycle c_j of points (π). But the cycle c_j will satisfy a relation ($m^i = 1$, or 0)

$$c_{j+1} \to c_j + m^i\gamma^i \qquad (i = 1, \cdots, R_j'')$$

in which γ^i is the ith cycle of a set of R_j'' j-cycles forming a maximal set of j-cycles independent on the domain $J(\pi) < b$, and c_{j+1} is a $(j + 1)$-chain on $J(\pi) < b$. We then have the relation

$$\Omega(c_{j+1}) \to \Omega(c_j) + m^i\Omega(\gamma^i) \qquad \text{(on } \Omega_b\text{)},$$

or

$$\Omega(c_j) \sim m^i\Omega(\gamma^i) \qquad \text{(on } \Omega_b\text{)}.$$

Hence R_j' is at most the number of cycles $\Omega(\gamma^i)$, that is, at most R_j''.

To conclude that $R_j' = R_j''$ one has merely to prove that a cycle $\Omega(c_j)$ of index

p, "derivable" from a cycle c_j of points (π) on the domain $J(\pi) < b$, bounds a restricted chain k_{j+1} on Ω_b only if c_j bounds a chain on $J(\pi) < b$. To that end we shall now show how a chain z_j of restricted curves on Ω_b leads to a chain $\pi(z_j)$ of points (π).

The chain $\pi(z_j)$. Let h be any restricted curve for which $J < b$. The point (π) whose vertices divide h into $p + 1$ segments of equal variation of J will be denoted by $\pi(h)$.

Let b_j be a restricted j-cell on Ω_b, the image on R of a functional simplex $\alpha_j \times t_1$. Let h be a restricted curve of $\bar{b}_j$ determined by an arbitrary point p of $\bar{\alpha}_j$. The point $\pi(h)$ will now be regarded as the image of the point p on $\bar{\alpha}_j$. We thus have a continuous image of $\bar{\alpha}_j$ among points (π), or if we please a closed j-cell $\pi(\bar{b}_j)$ on the domain $J(\pi) < b$, *derived* from the j-cell $\bar{b}_j$ on Ω_b. More generally a chain z_j on Ω_b shall be regarded as *determining* that chain $\pi(z_j)$ on the domain $J(\pi) < b$ which is the sum of the closed j-cells $\pi(\bar{b}_j)$ derived from the respective j-cells $\bar{b}_j$ of z_j.

Suppose that a cycle $\Omega(c_j)$ on Ω_b of index p bounds a restricted chain k_{j+1} on Ω_b. On the domain $J(\pi) < b$ we then have

$$\pi(k_{j+1}) \to \pi(\Omega(c_j)). \tag{4.5}$$

Let (π) be any point on the cycle c_j and $g(\pi)$ the corresponding J-parameterized broken extremal. The point $\pi(g(\pi))$ consists of vertices which lie on $g(\pi)$. The point (π) can be deformed into the point $\pi(g(\pi))$ by moving its vertices along $g(\pi)$ to the corresponding vertices of $\pi(g(\pi))$, moving each vertex at a J-rate equal to the J-length on $g(\pi)$ of the arc of $g(\pi)$ to be traversed. The cycle c_j will thereby be deformed through a chain w_{j+1} on $J(\pi) < b$ into the cycle $\pi(\Omega(c_j))$. We accordingly have the relation

$$w_{j+1} \to c_j + \pi(\Omega(c_j)),$$

and upon using (4.5) we find that

$$\pi(k_{j+1}) + w_{j+1} \to c_j.$$

Thus the cycle c_j of points (π) bounds on $J(\pi) < b$ if $\Omega(c_j)$ bounds on Ω_b. It follows that $R'_j = R''_j$, and the theorem is proved.

The preceding theorem taken with Theorem 3.2 gives us the following.

THEOREM 4.3. *If a and b, $a < b$, are any two ordinary values of J between which there are no critical values of J, the restricted connectivities of the functional domains $J < b$ and $J < a$ are equal.*

The J-distance between restricted curves

5. An unordered pair of points on our Riemannian space R will be said to possess a J-distance equal to the inferior limit of the J-lengths of restricted curves joining the two points. We see that the J-distance between two points of R varies continuously with the points.

We shall now define the J-distance between any two sensed curve segments g_1 and g_2 of class D^1.

To that end regard points on g_1 and g_2 as corresponding if they divide g_1 and g_2 in the same ratio with respect to the J-lengths of their arcs. We now define the J-distance $d(g_1, g_2)$ between g_1 and g_2 as the maximum of the J-distances between corresponding points of g_1 and g_2 plus the absolute value of the difference between the J-lengths of g_1 and g_2. Cf. Fréchet [1]. We see that

$$d(g_1, g_2) = d(g_2, g_1).$$

This definition has the advantage that it is invariant of coordinate systems and that under it

$$| J_{g_1} - J_{g_2} |$$

is arbitrarily small if $d(g_1, g_2)$ is sufficiently small. Moreover it corresponds, as will appear shortly, to vital needs of our developments, particularly in connection with the deformations of restricted chains, where the notion of "uniform J-continuity" is introduced.

If g_3 is a third sensed curve of class D^1 we have the triangle relation

$$d(g_1, g_3) \leqq d(g_1, g_2) + d(g_2, g_3).$$

With the aid of this relation we see that if g_2 is sufficiently near g_3, that is, if $d(g_2, g_3)$ is sufficiently small, $d(g_1, g_3)$ will differ arbitrarily little from $d(g_1, g_2)$.

Let g_1 be any restricted curve. By a *neighborhood* of g_1 on Ω will be meant a set of restricted curves which includes all restricted curves within some small positive J-distance e of g_1. Let A be a set of restricted curves of Ω. The curve g_1 will be called a *limit* curve of curves of A if there is a curve of A in every neighborhood of g_1. The boundary of A is the set of restricted curves which are limit curves of curves of A as well as of $\Omega - A$. Open, closed, and compact sets on Ω are now defined in the usual way. Particular examples of closed and compact sets A are restricted chains and critical sets of extremals.

If A and B are any two sets of restricted curves, $d(A, B)$ will be defined as the inferior limit of the J-distances between curves of A and B. If A and B are compact, $d(A, B)$ will be taken on by at least one pair of curves in A and B respectively. The distance $d(g_1, A)$ varies continuously with g_1, that is, it changes arbitrarily little if g_1 is replaced by a restricted curve sufficiently near g_1.

We shall continue with the following lemma.

LEMMA 5.1. *If a is a curve of class D^1 consisting of an arc a' followed by an arc a'', and b is a similar arc of class D^1 consisting of an arc b' followed by an arc b'', and if*

$$d(a', b') < e, \qquad d(a'', b'') < e,$$

then

$$d(a, b) < 4e.$$

Let us denote the J-lengths of each of the preceding arcs by the letter that designates the arc.

Under the hypotheses of the theorem, $|a - b| < 2e$. Let A' be a point on a' at a J-distance ta' on a' from the initial point on a', $0 \leqq t \leqq 1$. The point A' on a' will "correspond" to the point B' on b' for which the J-distance from the initial point of b' is tb'. Regarded as a point on a, A' will correspond to the point B on b whose J-distance on b from the initial point of b will be

$$\frac{ta'}{a' + a''}(b' + b'').$$

The J-length of the arc of b between B and B' is then seen to be

$$r = \left| ta' \frac{(b' + b'')}{(a' + a'')} - tb' \right|.$$

Upon setting

$$\eta' = b' - a', \qquad \eta'' = b'' - a'',$$

the distance r takes the form

$$\begin{aligned} r &= \left| \frac{t(a'\eta'' - a''\eta')}{a' + a''} \right| \\ &\leqq t\left(\frac{a' \,|\eta''| + a'' \,|\eta'|}{a' + a''}\right), \end{aligned}$$

and this is at most e, since $|\eta''|$ and $|\eta'|$ are at most e. But the J-distance $|A'B|$ is at most the sum of the J-distances $|A'B'|$ and $|B'B|$ and is accordingly at most $2e$. Similarly the J-distance between a point A'' on a, which is given as a point on a'', and the corresponding point on b is less than $2e$. The lemma now follows from the definition of $d(a, b)$.

Let F be any deformation which deforms each restricted curve g_1 in the ordinary sense through a 1-parameter family of continuous curves depending continuously on the curve parameter t and the time τ, $0 \leqq \tau \leqq 1$. We shall say that F is *uniformly J-continuous* over a set B of restricted curves, if corresponding to a positive constant e, there exists a positive constant η, so small that any two restricted curves whatsoever of B within a distance η of each other when $\tau = 0$, remain within a distance e of each other at each subsequent moment τ of the deformation F.

In terms of the previously defined deformations δ' and δ'' we now introduce the deformation

$$\Delta_p' = \delta''\delta'\delta'',$$

and prove the following lemma.

Lemma 5.2. *If $b < (p + 1)\rho$, the deformation Δ_p' is uniformly J-continuous over the set of restricted curves Ω_b on which $J < b$.*

Recall that δ'' alters a restricted curve merely in parameterization, leaving the curve at a zero J-distance from itself. Under δ'' the final image of a curve acquires a "J-parameterization".

Let g be a restricted curve on Ω_b with a J-parameterization. The definition of the deformation δ' involves breaking g into $p + 1$ successive arcs of equal variation of t, here of equal variation of J. At the time τ, $0 \leqq \tau \leqq 1$, the qth one of these arcs is divided into two arcs λ and λ', here in the J-ratio of τ to $1 - \tau$, and the first of these arcs λ is replaced by an elementary extremal μ joining its end points. Let g_1 be a second restricted curve on Ω_b, possessing a J-parameterization, and let λ_1, λ_1', and μ_1 be related to g_1 at the time τ in the definition of δ' as λ, λ', and μ are related to g.

Suppose that

$$d(g, g_1) < \eta \tag{5.1}$$

where η is a positive constant. If the points which divide g and g_1 in the same J-ratio correspond, the respective subsegments λ' and λ_1' of g and g_1 will in particular correspond. It follows that

$$d(\lambda', \lambda_1') < \eta. \tag{5.2}$$

Let e be an arbitrarily small positive constant. If (5.1) holds, the respective end points of μ and μ_1 are within a J-distance η of each other, and hence if η is sufficiently small,

$$d(\mu, \mu_1) < e \tag{5.3}$$

uniformly for all curves g and g_1 on Ω_b.

The arc μ followed by the arc λ' forms a curve which we denote by $\mu + \lambda'$. The arcs μ_1 and λ_1' similarly form a curve $\mu_1 + \lambda_1'$. If we suppose $\eta < e$, as we very well can, it follows from (5.2), (5.3), and Lemma 5.1 that

$$d(\mu + \lambda', \mu_1 + \lambda_1') < 4e. \tag{5.4}$$

That is, the J-distance between the curves which under δ' replace the qth segments of g and g_1 at the time τ is at most $4e$.

Let g^* and g_1^* denote the curves which under δ' replace g and g_1 respectively at the time τ. Upon regarding g^* and g_1^* as the sum of $p + 1$ segments such as appear in (5.4), and applying Lemma 5.1 p times, we see that

$$d(g^*, g_1^*) < 4^p(4e).$$

Thus the J-distance between the deforms of g and g_1 under δ' remains uniformly small if the J-distance between g and g_1 is initially sufficiently small.

Following the deformation $\delta'\delta''$ by the deformation δ'' will further change the restricted curves only in parametrization. The proof of the lemma is now complete.

Cycles on Ω neighboring a critical set ω

6. By a *critical set of extremals* ω we mean a connected set of critical extremals on which J equals a constant c and which are at a positive J-distance, in the sense of the preceding section, from other critical extremals. If ω contains all of the critical extremals on which $J = c$, ω is called *complete*. In the present section ω may or may not be complete.

By a *neighborhood* N of ω will be meant an open set of restricted curves which includes all restricted curves within a small positive J-distance e of ω. We admit only such neighborhoods of ω as consist of curves whose J-distances from other critical sets of extremals is bounded away from zero. We also suppose that the curves of N satisfy the condition

$$a < J < b, \tag{6.1}$$

where a and b are two constants which are not critical values of J and between which c is the sole critical value.

By a J-normal curve of index p we mean a J-parameterized curve $g(\pi)$ determined by a J-normal point (π) of $p + 2$ vertices. With this understood we state an analogue of Theorem 3.1, Ch. VI.

THEOREM 6.1. *There exists a deformation $\theta_p(t)$ of restricted curves which is defined and continuous for restricted curves sufficiently near ω and for t on the interval $0 \leqq t < 5$, and which has the following properties.*

It deforms extremals of ω on themselves, and for $t \geqq 4$ replaces each curve by a J-normal curve of index p. Any sufficiently small neighborhood N of ω is thereby deformed into a neighborhood N_t, the superior limit of the distances of whose curves from ω approaches zero as t approaches 5. Restricted curves below c are deformed through such curves.

The proof of Theorem 6.1 is more difficult than the proof of Theorem 3.1, Ch. VI. The method of proof will be first to deform each restricted curve neighboring ω into a curve $g(\pi)$ for which (π) is J-normal. This deformation will then be followed by a deformation of J-normal points (π), and of the corresponding curves $g(\pi)$. We shall devote the next two sections to the development of these ideas.

Let N^* be a fixed neighborhood of ω whose closure is interior to the domain on which $\theta_p(t)$ is defined.

We state the following corollary of the theorem.

COROLLARY. *Corresponding to any neighborhood X of ω on N^* let $M(X)$ be a neighborhood of ω so small that $M(X)$ is deformed under $\theta_p(t)$ only on X. Each restricted k-cycle on $M(X)$ (below c) will then be homologous on X (below c) to a k-cycle (below c) on an arbitrarily small neighborhood N of ω. If $z_k \sim 0$ on N^* (below c), and is sufficiently near ω, then $z_k \sim 0$ on N (below c).*

The phrase (below c) can be read throughout or omitted at pleasure.

The space Σ of J-normal points

7. We continue with the critical set ω. On extremals of ω, $J = c$. We are supposing that the number $p + 2$ of vertices in points (π) is so large that

$$(p + 1)\, \rho > c,$$

where ρ is the superior limit we have set for the J-lengths of elementary extremals. Let σ^* be the set of J-normal points of $p + 2$ vertices determined by ω. We shall prove the following theorem.

THEOREM 7.1. *The J-normal points (π) of $p + 2$ vertices in a sufficiently small neighborhood of σ^* make up a regular, analytic, Riemannian subspace Σ of the $(p + 2)$-fold product of R by itself.*

Let (π_0) be any point of σ^*. We shall first prove that the J-normal points neighboring (π_0) admit a regular analytic representation of their vertices.

Set $g(\pi_0) = g$. Let s be the arc length along g. Let

$$a^0 < a^1 < \cdots < a^p < a^{p+1}$$

be the values of s on g at the vertices of (π_0). Neighboring the point $s = a^q$ on g let R be referred to local coordinates

$$(x^q, y_1^q, \cdots, y_n^q) \qquad (n = m - 1) \tag{7.1}$$

in such a manner that $y_1^q = \cdots = y_n^q = 0$ and $s = x^q$ along g neighboring $s = a^q$.

Let (π) be any point neighboring (π_0). Let the J-length J_q of the qth elementary extremal of $g(\pi)$ be regarded as a function of the coordinates of the qth and $(q - 1)$th vertices of (π) in (7.1). In the special cases of J_0 and J_{p+1} respectively we understand that the coordinates of A^1 and A^2 are replaced in J_0 and J_{p+1} by the functions of the end parameters (α) which give admissible end points A^1, A^2. The conditions that (π) be a J-normal point are that

$$\begin{aligned} J_0 - J_1 &= 0, \\ &\cdots\cdots\cdots\cdots \\ J_{p-1} - J_p &= 0, \end{aligned} \tag{7.2}$$

where the variables involved are the variables (7.1) for $q = 1, \cdots, p$, and the end parameters α^h.

The parameters (u). We shall show that the equations (7.2) can be solved for the variables

$$x^1, x^2, \cdots, x^p \tag{7.3$'$}$$

as analytic functions of the remaining variables

$$(u) = [y_i^q, \alpha^h] \qquad \left\{ \begin{matrix} q = 1, \cdots, p;\ i = 1, \cdots, n \\ h = 1, \cdots, r \end{matrix} \right\}. \tag{7.3$''$}$$

To continue we consider the jacobian D of the left members of (7.2) with respect to the variables (7.3)′. We evaluate this jacobian at (π_0).

Let $f(s)$ be the value of J taken alone g from $s = a^0$ to the variable end point s. We set

$$f'(a^q) = f'_q \qquad (q = 1, \cdots, p).$$

We make use of the fact that we can set the variables (u) equal to their final values before computing the partial derivatives of the left members of (7.2) with respect to the variables (7.3)′. If we consider the typical case where there are five variables in (7.3)′, we find that at (π_0) the jacobian D is given by the equation

$$D = \begin{vmatrix} 2f'_1 & -f'_2 & 0 & 0 & 0 \\ -f'_1 & 2f'_2 & f'_3 & 0 & 0 \\ 0 & -f'_2 & 2f'_3 & -f'_4 & 0 \\ 0 & 0 & -f'_3 & 2f'_4 & -f'_5 \\ 0 & 0 & 0 & -f'_4 & 2f'_5 \end{vmatrix}$$

Since f' is never zero, D vanishes, if at all, with the determinant

$$\begin{vmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{vmatrix}$$

One can easily show that determinants of this form are never zero.

We can accordingly solve the conditions (7.2) for the coordinates (7.3)′ as analytic functions of the variables (u).

Now the set of coordinates

$$[x^q, y^q] \qquad (q = 0, \cdots, p + 1)$$

make up a local coordinate system for the $(p + 2)$-fold product R^{p+2} of R by itself. One sees that the conditions (7.2) together with the end conditions on A^1, A^2 define a regular analytic subspace Σ of R^{p+2}, at least neighboring (π_0), and that the preceding parameters (u) may be regarded as local coordinates on Σ neighboring (π_0).

Neighboring (π_0) we can assign R^{p+2} a local metric, defining this metric by a differential form ds^2 which is the sum of the forms defining the metrics of R neighboring the respective vertices of (π_0). One can then assign Σ a submetric, making arc length on Σ agree with arc length on R^{p+2}, taking the parameters (u) as local coordinates of Σ.

The theorem is thereby proved.

We now come to the following theorem.

THEOREM 7.2. *The value of $J(\pi)$ on the subspace Σ of J-normal points (π) sufficiently near σ^* is an analytic function of the local coordinates (u) of Σ and possesses no critical points other than points of the set σ^*.*

Let v be any one of the coordinates (u). At a critical point (u) of $J(\pi)$ on Σ we have

$$\frac{\partial J_0}{\partial v} + \frac{\partial J_1}{\partial v} + \cdots + \frac{\partial J_p}{\partial v} = 0$$

Since the equations (7.2) become identities in the variables (u) on Σ, on Σ we have

$$\frac{\partial J_0}{\partial v} - \frac{\partial J_1}{\partial v} = 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{\partial J_{p-1}}{\partial v} - \frac{\partial J_p}{\partial v} = 0.$$

Combining these conditions on a critical point we see that

$$\frac{\partial J_q}{\partial v} = 0 \qquad (q = 0, 1, \cdots, p) \tag{7.4}$$

at each critical point (u).

In terms of the local coordinates $(x^1, y^1) = (x, y)$ of R neighboring the vertex P^1 of (π_0), the conditions (7.4) include in particular the conditions

$$\frac{\partial J_0}{\partial y_i^1} = \frac{\partial J_1}{\partial y_i^1} = 0 \qquad (i = 1, \cdots, n). \tag{7.5}$$

I say that the conditions (7.5) *imply that the broken extremal g corresponding to the critical point (u) has no corner at the vertex P^1 of g.*

To prove this statement observe that on Σ near (π_0) the local coordinate x^1 of the point (x^1, y^1) is an analytic function $X(u)$ of the local coordinates (u) of Σ, and in particular an analytic function of the local coordinates y_i^1. If the integrand of J be put in the non-parametric form $f(x, y, p)$ in the local system $(x, y) = (x^1, y^1)$ we see that, for $h = 1, \cdots, n$,

$$\left[(f - p_h f_{p_h}) \frac{\partial X}{\partial y_i^1} + f_{p_i}\right]^{(\lambda)} = \frac{\partial J_0}{\partial y_i^1} = 0, \tag{7.6)'}$$

$$\left[(f - p_h f_{p_h}) \frac{\partial X}{\partial y_i^1} + f_{p_i}\right]_{(\mu)} = \frac{\partial J_1}{\partial y_i^1} = 0, \tag{7.6)''}$$

where the arguments (x, y, p) corresponding to the upper limit are the coordinates (x^1, y^1) and slopes $p_i = \lambda_i$ at the final end point of the first elementary extremal of g, while those corresponding to the lower limit are the same coordinates (x^1, y^1) with the slopes $p_i = \mu_i$ at the initial end point of the second elementary extremal of g. We set

$$(\lambda_i - \mu_i)\left[(f - p_h f_{p_h})\frac{\partial X}{\partial y_i^1} + f_{p_i}\right]_{(\mu)}^{(\lambda)} = H(\lambda, \mu).$$

We consider $H(\lambda, \mu)$ as a function of the variables (λ) for (λ) near (μ), and expand $H(\lambda, \mu)$ as a power series in the differences $\lambda_i - \mu_i$, holding (x^1, y^1) and (μ) fast. We see that

$$H(\mu, \mu) = 0 \qquad H_{\lambda_i}(\mu, \mu) = 0 \qquad (i = 1, \cdots, n),$$

and that

$$H_{\lambda_h\lambda_k}(\mu, \mu) = 2f_{p_hp_k}(x^1, y^1, \mu) + 2\eta(x^1, y^1, \mu)$$

where $\eta(x^1, y^1, \mu)$ tends to zero as (μ) tends to (0). Accordingly for $h, k = 1, \cdots, n$,

$$H(\lambda, \mu) \equiv [f_{p_hp_k}(x^1, y^1, \mu) + \eta]\,(\lambda_h - \mu_h)\,(\lambda_k - \mu_k) + \cdots, \tag{7.7}$$

where the terms omitted are of higher order than the second in $\lambda_i - \mu_i$.

We now see that (7.6) cannot hold for (λ) and (μ) sufficiently near (0) unless $(\lambda) = (\mu)$. For if (7.6) held for sets (λ) and (μ), $H(\lambda, \mu)$ would vanish for these sets. From (7.7) it would follow that $(\lambda) = (\mu)$ if the broken extremal g lies sufficiently near $g(\pi_0)$. Hence g has no corner at P^1, and the statement in italics is proved.

Similarly the conditions of the form (7.4) corresponding to the remaining vertices P^i of g imply that g has no corners at these vertices. Finally the conditions (7.4) of the form

$$\frac{\partial J_0}{\partial \alpha^h} = \frac{\partial J_p}{\partial \alpha^h} = 0 \qquad (h = 1, \cdots, r),$$

in which the variables (α) are the parameters in the end conditions, imply that g satisfies the transversality conditions. Thus g is a critical extremal, and the theorem is proved.

Theorem 6.1

8. In this section we shall prove Theorem 6.1 and deduce certain consequences therefrom. We begin with the following lemma.

LEMMA 8.1. *There exists a deformation $E_p(t)$, $0 \leqq t \leqq 4$, of the restricted curves neighboring the critical set ω which carries restricted curves into J-normal curves, and leaves J-normal curves invariant. Moreover $E_p(t)$ is uniformly J-continuous over a* *sufficiently small neighborhood of ω. It deforms each extremal of ω on itself into the corresponding J-parameterized extremal.*

To prove this lemma we first apply the deformation Δ_p' of §5 to the restricted curves neighboring ω. Each extremal $g(\pi)$ on ω will thereby be deformed on itself into the corresponding J-normal extremal. Since Δ_p' deforms restricted curves for which $J < b$ in a manner that is uniformly J-continuous, it follows that all restricted curves sufficiently near ω will be deformed into broken extremals $g(\pi)$ for which (π) lies within a prescribed positive J-distance e of the J-normal points σ^* determined by ω. If e is sufficiently small, on each such curve $g(\pi)$ there will exist a unique set of p successive points which together with the end points of $g(\pi)$ are the vertices of a J-normal point (π'). We term (π') the *J-normal image* of (π). If the above constant e is sufficiently small, we see that the J-normal image (π') of (π) will vary continuously with (π).

The deformation Δ_p''. We deform $g(\pi)$ into $g(\pi')$ letting the vertices of a variable point (π_t) move along $g(\pi)$ from the vertices of (π) to the corresponding vertices of (π'), each vertex of (π_t) moving at a J-rate equal to the J-length on $g(\pi)$ of the arc of $g(\pi)$ to be traversed. We assign J-parameterizations to each curve thereby replacing $g(\pi)$, and denote the resulting deformation by Δ_p''. In Δ_p'' the time t varies on the interval $0 \leqq t \leqq 1$.

The deformation Δ_p', defined as the product of deformations δ'', δ', and δ'', in each of which the time runs from 0 to 1 inclusive, may itself be regarded as a deformation in which $0 \leqq t \leqq 3$.

The deformation $E_p(t)$. With this understood we now set

$$E_p(t) = \Delta_p''\Delta_p' \qquad (0 \leqq t \leqq 4),$$

understanding that the deformation Δ_p' is followed by the deformation Δ_p''. Under $E_p(t)$ we can suppose that Δ_p' occupies the time interval $0 \leqq t \leqq 3$, and that Δ_p'' occupies the time interval $3 \leqq t \leqq 4$, so that the time interval for $E_p(t)$ becomes $0 \leqq t \leqq 4$. The symbol $E_p(t)$ stands for the deformation up to the time t. The deformation $E_p(t)$ is applicable to all restricted curves on a sufficiently small neighborhood of ω.

That Δ_p' is uniformly J-continuous over any domain $J < b$ of restricted curves has already been established, provided always that $(p + 1)\rho > b$. A sufficiently small neighborhood of ω will be such a domain. The deformation Δ_p'' is likewise uniformly J-continuous by virtue of Lemma 5.1, if the J-normal images (π') of the points (π) involved depend upon (π) in a uniformly continuous manner. But this dependence of (π') upon (π) will clearly be uniformly continuous if the initial neighborhood of ω is sufficiently small.

The remaining affirmations of the lemma require no further substantiation and the proof is complete.

In the preceding section we have seen that the set of all J-normal points (π) neighboring σ^* make up an analytic Riemannian space Σ and that $J(\pi)$ is an analytic function of the local coordinates (u) of Σ, with the set σ^* of J-normal points as its critical points. With this critical set σ^* on Σ we now associate a neighborhood function φ of the point on Σ neighboring σ^*, exactly as in §4, Ch. VI. Let r be a positive constant so small that the points on Σ which are connected to σ^* and at which

$$\varphi \leqq r \tag{8.1}$$

form a closed domain at each point of which φ enjoys the properties of a neighborhood function.

The radial deformation $R_p(t)$, $0 \leqq t < 1$. With the aid of the preceding function φ we introduce radial trajectories on Σ as in §5, Ch. VI. It will be convenient to say that a J-normal curve $g(\pi)$ lies on a domain $\varphi = k$ (a constant), if the point (π) lies on the domain $\varphi = k$. We shall now define a deformation $R_p(t)$ of J-normal curves on the domain $\varphi \leqq r$. Under $R_p(t)$ the time t shall vary on the interval $0 \leqq t < 1$. We first deform the J-normal points (π) on $\varphi \leqq r$ as follows. Let θ be a constant such that

$$0 \leqq \theta < 1.$$

Each J-normal point (π) at which

$$\varphi = r - \theta r$$

shall remain fixed under $R_p(t)$ until t reaches θ, and shall thereafter be replaced at the time t by the point (π_t) on the radial trajectory through P at which $\varphi = r - tr$. The J-normal curve $g(\pi)$ shall likewise remain fixed until t reaches θ, and shall thereafter be replaced at the time t by $g(\pi_t)$.

The deformation $\theta_p(t)$, $0 \leqq t < 5$. Under the deformation $E_p(t)$, with its time interval $0 \leqq t \leqq 4$, a restricted curve sufficiently near ω will be deformed into a J-parameterized J-normal curve γ on the domain $\varphi \leqq r$. To such a curve γ the deformation $R_p(t)$, $0 \leqq t < 1$, is applicable. It is therefore legitimate to introduce a deformation $\theta_p(t)$ such that

$$\theta_p(t) = E_p(t) \qquad (0 \leqq t \leqq 4)$$

and to continue this deformation so that in so far as the curves which are obtained when $t = 4$ are concerned,

$$\theta_p(t) = R_p(t - 4) \qquad (4 \leqq t < 5),$$

thus defining $\theta_p(t)$ on the time interval $0 \leqq t < 5$. It follows from this definition that $\theta_p(t)$ has the properties ascribed to it in Theorem 6.1.

The proof of Theorem 6.1 is now complete.

Recall that N^* has been chosen as a neighborhood of ω which is so small that its closure is interior to the domain of definition of $\theta_p(t)$. Corresponding to any neighborhood X of ω such that $X \subset N^*$, we let $M(X)$ be a neighborhood of ω so small that $M(X)$ is deformed under $\theta_p(t)$ only on X $(0 \leqq t < 5)$.

An ordered pair of neighborhoods VW of ω will now be termed admissible if they satisfy the conditions

$$V \subset M(N^*), \qquad W \subset M(V).$$

Spannable and critical k-cycles corr VW are now formally defined as in Ch. VI with the present interpretation of the terms involved. These cycles of restricted curves will be spoken of as *belonging* to the critical set ω. The phrase corr VW will be omitted in cases where it is immaterial which pair of admissible neighborhoods VW is used. We continue with the following analogue of Theorem 3.2, Ch. VI.

THEOREM 8.1. *Corresponding respectively to any two choices VW and $V'W'$ of admissible pairs of neighborhoods of ω there exist common maximal sets of spannable or critical k-cycles of restricted curves on any arbitrarily small neighborhood of ω*

Theorem 8.1 follows readily from Theorem 6.1.

Let e be a positive constant less than the constant r of (8.1). A k-cycle of J-normal curves on $\varphi = e$ below c, independent on this domain, but bounding on $\varphi \leqq e$ will be termed a *spannable* k-cycle of J-normal curves corr $\varphi \leqq e$. A k-cycle of J-normal curves on $\varphi \leqq e$ independent on this domain of k-cycles on $\varphi \leqq e$ below c will be called a *critical* k-cycle of J-normal curves corr $\varphi \leqq e$. Maximal sets of spannable or critical k-cycles of J-normal curves corr $\varphi \leqq e$ exist according to the theory of neighborhood functions φ of Ch. VI.

The analogue of Theorem 5.2 of Ch. VI can now be stated as follows.

THEOREM 8.2. *Maximal sets of restricted critical and spannable k-cycles corr VW can be taken respectively as maximal sets of critical and spannable k-cycles of J-normal curves corr $\varphi \leqq e$, provided e is a sufficiently small positive constant.*

Corresponding to the given neighborhood W we choose the constant e so that $e < r$ and so that the domain of J-normal curves on $\varphi \leqq e$ is on W. For this choice of e the theorem holds. We shall give the proof for the case of spannable k-cycles. The proof for critical k-cycles is similar.

Let $(u)_k$ be a maximal set of spannable k-cycles of J-normal curves corr $\varphi \leqq e$. We shall prove that $(u)_k$ is a maximal set of spannable k-cycles corr VW.

To that end let z_k be a spannable k-cycle corr VW. By virtue of the relation of V to W, and Theorem 6.1, we have

$$z_k \sim w_k \qquad \text{(on } V, \text{ below } c)$$

where w_k is a cycle of J-normal curves on the domain $\varphi \leqq e$. The cycle w_k is homologous on $\varphi \leqq e$, below c, to a sum of cycles of $(u)_k$. Hence z_k is homologous on V, below c, to a sum of cycles of $(u)_k$.

Now let u_k be any proper linear combination of the cycles of the set $(u)_k$. I say that $u_k \not\sim 0$ on V below c. For if u_k bounded a chain w_{k+1} on V below c, an application of $\theta_p(t)$ up to a time t sufficiently near $t = 5$ would deform w_{k+1} below c into a chain v_{k+1} of J-normal curves on $\varphi \leqq e$. The cycle u_k would at the same time be deformed below c through a chain u_{k+1} of J-normal curves on $\varphi \leqq e$ so that

$$u_{k+1} + v_{k+1} \rightarrow u_k$$

on the domain $\varphi \leqq e$ below c. Since this is contrary to the nature of the set $(u)_k$ we infer that $u_k \not\sim 0$ on V below c.

The critical cycles can be similarly treated. The proof of the theorem is now complete.

Cycles on the domains $J < b$ and $J < a$

9. Having analysed the restricted cycles neighboring the critical set ω we shall now examine the changes in cycles of restricted curves with respect to bounding, as one passes from the domain $J < a$ to the domain $J < b$. We are here supposing that ω is the complete set of critical extremals on which J equals a critical value c. The constants a and b are any two constants which are not critical values of J and between which c is the only critical value.

We shall use the deformation $\theta_p(t)$ of Theorem 6.1 to define a basic deformation $\Lambda_p(t)$, $0 \leqq t < 5$.

The deformation $\Lambda_p(t)$. We shall extend the definition of $\theta_p(t)$ so that the resulting deformation $\Lambda_p(t)$ is J-continuous over the restricted domain $J < b$ and remains identical with $\theta_p(t)$ over the neighborhood N^* of §6. To that end let e be a positive constant so small that the set of restricted curves not on N^*, but at a J-distance at most e from N^*, are within the domain of definition of $\theta_p(t)$. Under $\Lambda(t)$ each restricted curve at a distance say $(1 - \lambda)e$ from N^*, where $0 \leqq \lambda \leqq 1$, shall be deformed as in $\theta_p(t)$ until $t = \lambda 5$ and held fast thereafter. Restricted curves at a distance e or more from N^* shall be held fast under $\Lambda_p(t)$.

We shall make use of admissible pairs of neighborhoods VW of ω as previously defined. Corr VW *linkable* and *linking* cycles are formally defined as in §6, Ch. VI. Lemmas 6.1–6.4 of Ch. VI then hold with the interpretations of the present chapter, the proofs remaining formally the same. One naturally replaces points by restricted curves, and the deformations $\theta(t)$ and $\Lambda(t)$ of Ch. VI by the deformations $\theta_p(t)$ and $\Lambda_p(t)$ respectively.

The Deformation Lemma of §6, Ch. VI, is here replaced by the following lemma.

DEFORMATION LEMMA. *Let N be an arbitrary neighborhood of ω and L the set of restricted curves below c. There exists a J-deformation Δ_p of the restricted curves on $J < b$ which is uniformly J-continuous on $J < b$, which deforms extremals of ω on themselves, and which deforms restricted curves on $J < b$ into rectricted curves on $N + L$.*

If a cycle z_k lies on a restricted domain $N_0 + L$ for which N_0 is a sufficiently small neighborhood of ω and if $z_k \sim 0$ on $J < b$ (below c), then $z_k \sim 0$ on $N + L$ (below c).

To establish this lemma we begin by applying the deformation Δ_p' of §5 to the restricted curves on the domain $J < b$. The resulting curves will be J-parameterized broken extremals determined by points (π) with p intermediate vertices. To these points (π) we now apply the deformation D_p of §3, and let D_p' denote the deformation of the curves $g(\pi)$ thereby generated. In D_p' we understand that a J-parameterization has been given to each curve which replaces $g(\pi)$.

The curves $g(\pi)$ to which D_p' is applied are such that $J(\pi) < b$. Such of these curves as are not extremals of ω and for which $a < J(\pi)$ will be lessened in J-length under D_p', as follows from Lemma 3.1. One can then prove exactly as under the Deformation Lemma of §6, Ch. VI, that a product deformation $D_p'^n$ for which n is a sufficiently large positive integer will deform these curves $g(\pi)$ below c onto the domain $N + L$. Hence the product deformation

$$\Delta_p = D_p'^n \Delta_p'$$

will J-deform the restricted curves of $J < b$ on $J < b$ onto the domain $N + L$.

Moreover Δ_p' is uniformly J-continuous over the restricted curves of $J < b$, as we have seen. It follows in similar fashion that D_p' is uniformly J-continuous over the domain of curves $g(\pi)$ for which $J(\pi) < b$, so that Δ_p is uniformly J-continuous over the restricted domain $J < b$.

To establish the final statement of the lemma, observe that Δ_p deforms a restricted curve representing an extremal of ω into the same extremal of ω. From the J-continuity of Δ_p it then follows that there exists a neighborhood N_0 of ω which is so small that restricted curves on N_0 are deformed under Δ_p only on N. Suppose then that the cycle z_k of the lemma bounds a chain z_{k+1} on $J < b$. The deformation Δ_p will carry z_{k+1} into a chain on $N + L$, deforming z_k on $N + L$. Hence if $z_k \sim 0$ on $J < b$ (below c), it follows that $z_k \sim 0$ on $N + L$ (below c).

The proof of the lemma is now complete.

As in Ch. VI an *invariant* k-cycle corr VW is here defined as a k-cycle below c, independent below c of spannable k-cycles corr VW. Using the preceding Deformation Lemma where we formerly used the Deformation Lemma of §6, Ch. VI, we see that Theorem 6.1 of Ch. VI and its proof hold here in essentially the same form as in Ch. VI, J replacing f. For the sake of reference we repeat the theorem in this place.

THEOREM 9.1. *A maximal set of restricted k-cycles independent on $J < b$ is afforded by maximal sets of critical, linking, and invariant k-cycles of restricted curves corresponding to an admissible pair of neighborhoods VW of the critical set ω.*

The existence of critical extremals

10. Let c be a critical value of J and e a positive constant so small that c is the only critical value on the closed interval $(c - e, c + e)$. Let ω be the complete set of critical extremals on which $J = c$. Relative to the critical value c,

restricted k-cycles which are independent on $J < c + e$ of k-cycles below c will be termed *new* k-cycles. By virtue of Theorem 9.1 a maximal set of such cycles can be obtained by combining maximal sets of critical and linking k-cycles corresponding to an admissible pair of neighborhoods VW of ω.

Relative to the critical value c we shall also consider maximal sets of restricted $(k - 1)$-cycles independent below c but bounding on $J < c + e$. We term such cycles *newly-bounding* $(k - 1)$-cycles. It follows from Theorem 9.1 that a maximal set of such cycles will be afforded by a maximal set of spannable $(k - 1)$-cycles corr VW which are independent below c.

The number of k-cycles in a maximal set of new k-cycles depends upon more than the neighborhood of ω, as examples would show. The same is true of the number of cycles in a maximal set of newly-bounding $(k - 1)$-cycles relative to c.

It is a remarkable fact, however, that the sum m_k of the number of new k-cycles and newly-bounding $(k - 1)$-cycles in maximal sets relative to c depends only upon the nature of J neighboring ω.

In fact if α, β, and γ respectively denote the numbers of cycles in maximal sets of critical k-cycles, linking k-cycles, and newly-bounding $(k - 1)$-cycles, we see that

$$m_k = \alpha + \beta + \gamma.$$

Of these numbers α depends only on the nature of J neighboring ω, while $\beta + \gamma$ equals the number of spannable $(k - 1)$-cycles in a maximal set and likewise depends only on the nature of J neighboring ω. Thus the statement in italics is true.

But critical k-cycles and spannable $(k - 1)$-cycles are well defined not only for *complete* critical sets but for critical sets in general. The definition of the type number m_k can then be consistently extended as follows.

DEFINITION. *The kth type number m_k of a critical set ω shall be the number of critical k-cycles and spannable $(k - 1)$-cycles of restricted curves in maximal sets of such cycles corresponding to neighborhoods VW of the critical set ω.*

A critical set ω which is the sum of a finite number of component critical sets with the same critical value, possesses a type number m_k which is the sum of the kth type numbers of the component critical sets.

Let a and b be any two constants which are not critical values of J, with $a < b$. Let M_k^+ be the number of linking and critical k-cycles in maximal sets associated with the different complete sets of critical points with critical values between a b. Let M_k^- be the number of newly-bounding $(k - 1)$-cycles in maximal sets associated with these same complete critical sets. Let ΔR_k be the kth connectivity of the restricted domain $J < b$ minus the kth connectivity of the restricted domain $J < a$. Let M_k be the sum of the kth type numbers of the critical sets with critical values between a and b. By virtue of Theorem 9.1 we have

$$\begin{aligned} \Delta R_k &= M_k^+ + M_{k+1}^- \qquad && (k = 0, 1, \cdots), \\ M_k &= M_k^+ + M_k^-, && M_0^- = 0. \end{aligned} \tag{10.0}$$

If a is less than the absolute minimum of J, the connectivities of the domain $J < a$ are null by convention and ΔR_k in (10.0) equals the kth connectivity R_k of the domain $J < b$. Upon eliminating the numbers M_k^+ from the relations (10.0) one then finds that

$$(10.1) \quad M_0 - M_1 + \cdots + (-1)^i M_i = R_0 - R_1 + \cdots + (-1)^i R_i + (-1)^i M_{i+1}^-.$$

If r is a sufficiently large positive integer, then for $k > r$ the numbers M_k and R_k are null. In particular M_{r+1}^- will be null. From (10.1) we then obtain the following theorem.

THEOREM 10.1. *The connectivities R_k of the restricted domain $J < b$ and the sums M_k of the kth type numbers of the critical sets of extremals with critical values less than b satisfy the relations*

$$(10.2) \quad \begin{aligned} M_0 &\geqq R_0, \\ M_0 - M_1 &\leqq R_0 - R_1, \\ &\cdots\cdots \\ M_0 - M_1 + \cdots + (-1)^r M_r &= R_0 - R_1 + \cdots + (-1)^r R_r, \end{aligned}$$

for any sufficiently large integer r.

Another particular consequence of (10.0) is that

$$(10.3) \quad M_k \geqq \Delta R_k.$$

From (10.3) we can deduce a basic theorem on the existence of extremals which are *topologically necessary*. The theorem is as follows.

THEOREM 10.2. *If the connectivities of the functional domain Ω are*

$$P_0, P_1, P_2, \cdots,$$

the sum N_k of the kth type numbers of all critical sets of extremals satisfies the relation

$$(10.4) \quad N_k \geqq P_k \qquad (k = 0, 1, \cdots).$$

In particular if P_k is infinite, N_k must be infinite.

If P_k is finite, let $Q_k = P_k$. If P_k is infinite, let Q_k be any positive integer. In either case there will exist Q_k independent k-cycles on Ω. As we have seen in §4 these k-cycles are homologous on Ω to restricted k-cycles independent among restricted k-cycles. These Q_k k-cycles will lie on some restricted domain $J < b$, for which b is sufficiently large. If R_k is the restricted kth connectivity of $J < b$, we will have

$$(10.5) \quad R_k \geqq Q_k.$$

On the other hand if M_k is the sum of the kth type numbers of the critical extremals for which $J < b$, we also have

$$(10.6) \quad N_k \geqq M_k.$$

From (10.3), (10.5), and (10.6) it follows that

$$N_k \geqq Q_k,$$

from which (10.4) and the theorem follow at once.

The number

$$E_i = N_i - P_i \qquad (i = 0, 1, \cdots) \tag{10.7}$$

will be called the count of critical extremals of index i in excess of those topologically necessary.

We shall investigate the limitations on these numbers E_i. We begin with the following lemma.

LEMMA 10.1. *If the connectivities $P_0, P_1, \cdots, P_r$ of Ω are finite, there will exist a restricted domain $J < b$ whose connectivities $R_0, \cdots, R_r$ are at least $P_0, \cdots, P_r$ respectively.*

If then β is a sufficiently large positive constant greater than b, if M_k is the kth type number sum of critical sets on $J < b$, and M'_k the corresponding sum for critical sets on $J < \beta$, there will exist an integer μ_k between M_k and M'_k inclusive such that

$$\begin{aligned} \mu_0 &\geqq P_0, \\ \mu_0 - \mu_1 &\leqq P_0 - P_1, \\ &\cdots\cdots\cdots\cdots \\ \mu_0 - \mu_1 + \cdots + (-1)^r\mu_r &\gtreqless P_0 - P_1 + \cdots + (-1)^r P_r, \end{aligned} \tag{10.8}$$

where the sign $>$ or $\lessdot$ is understood in the last relation according as r is even or odd.

The first statement in the lemma is a consequence of previous remarks.

To prove the second statement of the lemma we first note that

$$R_k = P_k + q_k \qquad (k = 0, \cdots, r) \tag{10.9}$$

where $q_k \geqq 0$. For a sufficiently large constant β there must then exist a set (h) of q_k $(k = 0, \cdots, r)$ restricted k-cycles, independent on $J < b$, but bounding on $J < \beta$. I say that the lemma holds for this choice of β. We begin by proving statement (a).

(a). *The sum of the numbers of newly-bounding k-cycles in maximal sets of such cycles relative to the critical values between b and β must be at least q_k $(k = 0, \cdots, r)$.*

Suppose $c_1 < \cdots < c_\nu$ are the critical values of J between b and β. Of the k-cycles which are newly bounding relative to c_i suppose there are p_i in a maximal set. If $q_k > p_1$, there will be a set of at least $q_k - p_1$ k-cycles on $J < c_1$ dependent on $J < c_1$ upon cycles of (h) and independent on $J < c_2$. If $q_k > p_1 + p_2$, there will similarly be a set of at least $q_k - p_1 - p_2$ k-cycles on $J < c_2$ dependent upon $J < c_2$ upon cycles of (h) and independent on $J < c_3$. Proceeding in this fashion we see that if

$$t = [q_k - p_1 - p_2 - \cdots - p_\nu] > 0,$$

there will be at least t k-cycles on $J < c_\nu$, dependent on $J < c_\nu$ upon cycles of (h) and independent on $J < \beta$. But all cycles dependent on (h) are bounding on $J < \beta$ so that $t \leqq 0$.

Statement (a) is thereby proved.

From statement (a) as applied to $(k-1)$-cycles we see that

$$M_k' \geqq M_k + q_{k-1} \qquad (k = 0, \cdots, r).$$

We set

$$\mu_k = M_k + q_{k-1} \qquad (k = 0, \cdots, r),$$

and observe that

$$M_k \leqq \mu_k \leqq M_k'.$$

We now substitute the right members of the relations

$$M_k = \mu_k - q_{k-1},$$
$$R_k = P_k + q_k,$$

for M_k and R_k in (10.1). Relations (10.8) are thereby obtained, and the lemma is proved.

The following theorem is a generalization of Theorem 1.1 of Ch. VI.

THEOREM 10.3 *Let $N_0, N_1, \cdots$ be the type number sums for all critical sets of extremals, and let $P_0, P_1, \cdots$ be the connectivities of the function space Ω. If the numbers N_i are finite, they satisfy the infinite set of inqualities*

$$\begin{aligned} N_0 &\geqq P_0, \\ N_0 - N_1 &\leqq P_0 - P_1, \\ N_0 - N_1 + N_2 &\geqq P_0 - P_1 + P_2, \\ &\cdots\cdots\cdots\cdots\cdots\cdots . \end{aligned} \tag{10.10}$$

If all of the integers N_i are finite for $i < r + 1$, the first $r + 1$ relations in (10.10) *still hold.*

The first statement in the theorem is a consequence of the last. We shall prove the last.

Suppose that the integers

$$N_0, \cdots, N_r$$

are finite. It follows from (10.4) that the connectivities

$$P_0, \cdots, P_r$$

of Ω must be finite. We can then apply the preceding lemma. In applying this lemma we can take b so large that the type numbers

$$m_0, \cdots, m_r$$

of the critical sets for which $J > b$ are all null, for otherwise some of the numbers $N_0, \cdots, N_r$ would be infinite. With this choice of b the numbers

$$\mu_0, \cdots, \mu_r$$

in the lemma must be the numbers $N_0, \cdots, N_r$ respectively. The first $r + 1$ relations in (10.10) then follow from (10.8).

We shall now prove the following theorem.

THEOREM 10.4. *If the connectivities $P_0, P_1, \cdots$ of Ω are finite, and if E_i is the count of extremals of index i in excess of those topologically necessary, we have the relations*

$$E_{i+1} + E_{i-1} \geqq E_i \qquad (i = 0, 1, \cdots).$$

In particular if E_i is infinite, at least one of the two numbers E_{i+1} and E_{i-1} must be infinite.

To prove the theorem we refer to relations (10.8). Upon combining each inequality in (10.8) with the third preceding inequality we find that

$$(\mu_{i+1} - P_{i+1}) + (\mu_{i-1} - P_{i-1}) \geqq (\mu_i - P_i) \qquad (i = 0, 1, \cdots). \tag{10.11}$$

For $i = 0$ and $i = 1$ relations (10.11) reduce respectively to the second and third relations in (10.8). If N_{i+1}, N_i, and N_{i-1} are finite, and we take the constant b in Lemma 10.1 large enough we must have

$$N_{i+1} = \mu_{i+1}, \qquad N_i = \mu_i, \qquad N_{i-1} = \mu_{i-1},$$

and the theorem follows from (10.11).

If N_i is infinite, (10.11) still holds. In this case there must be infinitely many critical sets whose ith type numbers are not null. The corresponding set of critical values cannot be bounded above. If we take the constant b of Lemma 10.1 successively as the constants of a sequence becoming positively infinite, the number μ_i in Lemma 10.1 will become infinite with b and from (10.11) we see that either μ_{i+1} or else μ_{i-1} will become infinite with b. Thus the theorem is true if N_i is infinite. If either N_{i+1} or N_{i-1} is infinite, E_{i+1} or E_{i-1} is infinite respectively, and the theorem is again true.

The non-degenerate critical extremal

11. A critical extremal g is termed *non-degenerate* if the associated "index form" of Ch. V, §14, is non-degenerate. We have seen in Ch. V that a necessary and sufficient condition that g be degenerate in the case of non-tangency, is that the associated boundary value problem in tensor form possess a characteristic root which is zero. The property of degeneracy or non-degeneracy is an invariant property, that is, one independent of the local coordinate systems employed.

Since no characteristic root of a given critical extremal will in general be zero, it appears that the general case is the non-degenerate case. It is therefore fair to say that any adequate theory of the calculus of variations in the large must admit a definitive specialization in the non-degenerate case. We shall accordingly examine the non-degenerate case in the light of the general theory.

A first theorem of importance is the following.

THEOREM 11.1. *If g is a non-degenerate critical extremal, there is no connected family of critical extremals which contains both g and critical extremals other than g.*

To prove this theorem we refer the neighborhood of g to normal coordinates (x, y) as in Ch. V, §1, with x the arc length along g and $y_i = 0$ on g. We then cut across g with intermediate n-planes, x constant, so placed as to divide g into $p + 1$ segments of equal variation of x. We take p so large that each of these segments is less than ρ in J-length. We refer these n-planes to their coordinates (y) as parameters (β), and set up the index function $J(v, 0)$ of §1, Ch. III, giving the value of J along the broken extremal determined by (v).

If g belonged to a connected family of critical extremals which contained critical extremals other than g, there would be critical extremals determined by sets $(v) \neq (0)$ arbitrarily near (0). But for each of these sets (v) the first partial derivatives of $J(v, 0)$ are zero. This is impossible, for when g is non-degenerate the point $(v) = (0)$ is an isolated critical point of the function $J(v, 0)$. We accordingly infer the truth of the theorem.

To determine the type numbers of g we return to the space of points (π). Let c be the J-length of g. We are supposing that the number $p + 2$ of vertices in points (π) is such that $(p + 1)\rho > c$. Let σ denote the set of critical points of $J(\pi)$ with vertices on g, and (π^*) the J-normal point of σ. The set of J-normal points neighboring (π^*) make up an analytic Riemannian manifold Σ, as we have seen in §7. On Σ, $J(\pi)$ has a critical point in (π^*). Let φ_Σ be a neighborhood function belonging to $J(\pi)$ on Σ, and to the critical point (π^*). From Theorem 8.2 and the definition of the type numbers of g we have the following lemma.

LEMMA 11.1. *The kth type number of g is the number of critical k-cycles and spannable $(k - 1)$-cycles of J-normal points (π) on the domain Σ of J-normal points (π) belonging to the function f defined by $J(\pi)$ on Σ, and to the critical point (π^*) of f determined by g.*

As previously, we denote the space of admissible points (π) with $p + 2$ vertices by Π. We are supposing that $(p + 1)\rho > c$. We now define a class of submanifolds of Π with vertices neighboring g.

Proper sections of Π *belonging to* g. Let t be the arc length measured along g. Let t^1 and t^2 be the values of t at the end points of g, and let

$$t_1 < \cdots < t_p$$

be a set of values of t between t^1 and t^2 which divide g into a set of segments in J-length less than our basic constant ρ. Let

$$M^q \qquad (q = 1, \cdots, p)$$

be a regular $(m - 1)$-manifold cutting g at the point $t = t_q$ without being tangent to g. Points (π) whose intermediate vertices lie on the manifolds M^q, and whose terminal vertices satisfy the terminal conditions form a regular submanifold S of the space Π, at least neighboring the point (π) on S which determines g. We term S a *proper section* of Π belonging to g.

Let (π_0) be an admissible point (π) which determines g. Suppose none of the elementary extremals of $g(\pi_0)$ are null. If (π) is a point which is sufficiently near (π_0), there will be a unique point (π') on S, and a unique point (π'') on Σ whose vertices lie respectively on $g(\pi)$. The points (π') and (π'') will be respectively termed the *extremal projections* of (π) on S and Σ.

We shall now prove the following lemma.

LEMMA 11.2. *Let S be a proper section of the space Π belonging to g, and Σ the manifold of J-normal points (π) belonging to g.*

(a). *The points (π) on S on any sufficiently small neighborhood of σ can be J-deformed on the corresponding broken extremals $g(\pi)$ into their extremal projections on Σ.*

(b). *Among points (π) on S sufficiently near σ, a k-cycle on S will bound on S (below c), if and only if its extremal projection on Σ bounds (below c).*

(c) *The lemma also holds if S and Σ are interchanged.*

To deform a point (π) on S into its extremal projection (π'') on Σ we deform each vertex of (π) along $g(\pi)$ to the corresponding vertex of (π''), moving the given vertex at a J-rate equal to the J-length on $g(\pi)$ to be traversed. Statement (a) follows readily.

Now let z_k be a k-cycle on S. If z_k bounds a chain z_{k+1} on S (below c), its extremal projection on Σ will bound the extremal projection of z_{k+1} on Σ (below c), if z_{k+1} is on a sufficiently small neighborhood of σ.

It remains to prove that among points sufficiently near σ, z_k bounds on S (below c) if its extremal projection u_k on Σ bounds on Σ (below c). Suppose then that u_k bounds a chain u_{k+1} on Σ (below c). Let w_{k+1} be the deformation chain generated by z_k in its deformation into u_k. If z_k and u_{k+1} are sufficiently near σ, z_k will bound the extremal projection on Σ of

$$w_{k+1} + u_{k+1}.$$

One can interchange S and Σ in the preceding proof. Thus the lemma holds as stated.

Let (u) be a set of parameters neighboring $(u) = (u_0)$ regularly representing S neighboring its intersection with σ. Let $F(u)$ denote the value of $J(\pi)$ at the point (π) determined by (u) The function $F(u)$ is an index function corresponding to g, in the sense of §1, Ch. III. It follows that $(u) = (u_0)$ is a non-degenerate critical point of $F(u)$ of index k. Let $\varphi_s(u)$ be a neighborhood function belonging to F and the critical point (u_0). If e is a sufficiently small positive constant, there will exist maximal sets of spannable and critical k-cycles on the domain $\varphi_s < e$ of S, belonging to F and its critical point (u_0).

It follows from the preceding lemma that maximal sets of spannable and critical k-cycles on the domain $\varphi_\Sigma < e'$ on Σ have extremal projections on S which are maximal sets of spannable and critical k-cycles on the domain $\varphi_S < e$ of S if e' is a sufficiently small positive constant. With this understood the following theorem appears as a consequence of the two preceding lemmas.

THEOREM 11.2. *Maximal sets of spannable and critical k-cycles of restricted curves, belonging to a non-degenerate critical extremal g, can be chosen among the cycles of broken extremals $g(\pi)$ determined by points (π) on a proper section S of Π belonging to the extremal g.*

We state the following corollary.

COROLLARY. *The kth type number of a non-degenerate critical extremal g equals the number of critical k-cycles and spannable $(k-1)$-cycles in maximal sets of such cycles belonging to the index function F defined by $J(\pi)$ on a proper section S of Π, and to the non-degenerate critical point of F determined by g.*

Theorem 7.2 of Ch. VI taken with the preceding corollary gives us the following theorem.

THEOREM 11.3. *The jth type number of a non-degenerate critical extremal of index k is δ_k^j $(j = 0, 1, \cdots)$.*

Theorem 9.1 then yields the following corollary.

COROLLARY. *If a and b are two ordinary values of J between which there lies just one critical value c taken on by just one non-degenerate critical extremal of index k, the only changes in the restricted connectivities as one passes from the domain $J < a$ to the domain $J < b$ are that either*

Case 1: $$\Delta R_k = 1,$$

or

Case 2: $$\Delta R_{k-1} = -1.$$

Case 1 *always occurs if $k = 0$. If $k > 0$, Case* 1 *or Case* 2 *occurs according as a spannable $(k-1)$-cycle γ_{k-1} associated with g is or is not bounding below c.*

In verifying the corollary in case $k > 0$ one notes that a linking k-cycle is associated with g according as γ_{k-1} is or is not bounding below c. In case γ_{k-1} is not bounding below c, γ_{k-1} is what we have called a newly-bounding $(k-1)$-cycle associated with g, and in this case $\Delta R_{k-1} = -1$. If $k = 0$, there are no linking or newly-bounding cycles, and just one critical k-cycle in maximal sets of such cycles. In this case $\Delta R_0 = 1$.

We term the extremal g of increasing or decreasing type according as $\Delta R_k = 1$ or $\Delta R_{k-1} = -1$.

From Theorem 10.2 we now obtain the following important consequence.

THEOREM 11.4. *If all the critical extremals are non-degenerate, the number N_k of distinct extremals of index k is such that*

$$N_k \geqq P_k,$$

where P_k is the kth connectivity of the unrestricted functional domain Ω.

A problem in which all the critical extremals are non-degenerate will be called *non-degenerate*. In the next section we shall prove that the non-degenerate problem is the general problem, at least in two important classes of problems.

The connectivities P_k are topological invariants of the Riemannian manifold R and the manifold of end points Z. One can frequently determine these connectivities by a study of a particular extremal problem defined on R. We formulate this result in the following corollary.

COROLLARY. *If corresponding to a given Riemannian space R and terminal manifold Z, there exists an integral J defined on R such that all the critical extremals are non-degenerate and of increasing type, then the connectivity P_k of the functional domain Ω equals the number of distinct extremals of index k.*

The non-degenerate problem

12. It follows from Theorem 11.1 that a non-degenerate critical extremal g is isolated from other critical extremals in the sense that there exists no other critical extremal with an arbitrarily small J-distance from g. It then follows that the number of critical extremals with J-lengths less than a constant b is finite in a non-degenerate problem. For otherwise the initial points and directions of such critical extremals would have a cluster point and direction (P, λ) on R, and the extremal through P with direction λ, taken with a suitable limiting length, would be a critical extremal which was not isolated, and hence would be degenerate.

Accordingly in a non-degenerate problem the critical extremals are either finite in number, or else form a countably infinite sequence of extremals

$$g_1, g_2, \cdots$$

whose J-lengths become infinite with their subscripts.

We have seen in Ch. VI that in the case of an analytic function $f(x_1, \cdots, x_m)$, defined over a region in euclidean space, the non-degenerate case may be regarded as the general case, in that f can always be approximated arbitrarily closely by an analytic function whose critical points are non-degenerate. We shall establish the corresponding fact for functional problems, at least in case the problem has at most one variable end point. The proofs are necessarily more difficult than in the case of a function f, because in the case of a functional problem there are in general infinitely many critical extremals, and in approximating such a problem these critical extremals must all be replaced by non-degenerate critical extremals.

We begin with the fixed end point problem. Let P be a point on R. Let g be an extremal issuing from P. Let the extremals through P with initial directions neighboring that of g be represented neighboring P as in Ch. V, §5, in the form

$$x^i = x^i(t, u), \tag{12.1}$$

where t is the arc length along the extremals measured from P, and (u) is a set of $n = m - 1$ parameters chosen as in Ch. V, with $(u) = (u_0)$ defining g. Let the extremal determined by (u) be denoted by $g(u)$.

Let $t = t_0$ represent a conjugate point Q of P on $g(u_0)$ and let (z) be a set of local coordinates on R neighboring Q. Let the continuation of the extremal $g(u)$ be represented, neighboring Q, in the form

$$z^i = h^i(t, u), \tag{12.2}$$

where t is the arc length along $g(u)$ measured from P. In (12.2) t is confined to values near t_0. As in (12.1) the parameters (u) neighbor (u_0). We shall confine the sets (t, u) to a closed convex neighborhood N of the sets (t_0, u_0) on which the functions (12.2) are analytic. The conjugate points of P on the extremals $g(u)$ for (t, u) on N are given by the zeros of the jacobian

$$D(t, u) = \frac{D(z^1, \cdots, z^m)}{D(t, u^1, \cdots, u^n)} \qquad (n = m - 1) \tag{12.3}$$

provided N is sufficiently small.

We shall prove the following lemma.

LEMMA 12.1. *In the space (z) the conjugate points of P on the extremals $g(u)$, at points (t, u) on N, form a set whose measure is null.*

In proving this lemma it will be convenient to set

$$(t, u^1, \cdots, u^n) = (v^1, \cdots, v^m) = (v)$$

and $(t_0, u_0) = (v_0)$. We also write (12.2) in the form

$$z^i = h^i(t, u) = z^i(v)$$

and set

$$D(t, u) = \Delta(v).$$

Let (a) be any set (v) such that $\Delta(a) = 0$. The rank r of the determinant $\Delta(a)$ will lie between 0 and m exclusive. In the space (z) let the r-plane

$$z^i = z^i(a) + \frac{\partial z^i(a)}{\partial v^h}(v^h - a^h) \qquad (i, h = 1, \cdots, m) \tag{12.4}$$

be denoted by ω_a. The distance $d(v, a)$ of the point $z^i(v)$ from the r-plane ω_a will be at most the square root of the quantity

$$\sum_i \left[z^i(v) - z^i(a) - \frac{\partial z^i(a)}{\partial v^h}(v^h - a^h) \right]^2 = \sum_i [b^i_{hk}(v, a)(v^h - a^h)(v^k - a^k)]^2. \tag{12.5}$$

The bracket on the right is obtained by applying Taylor's formula with the remainder as a term of the second order to each of the m differences appearing in the bracket on the left. The coefficients $b^i_{hk}(v, a)$ are accordingly less in absolute value than some constant independent of the choice of (a) and (v) on N. If we set

$$\rho^2 = (v^h - a^h)(v^h - a^h) \qquad (h = 1, \cdots, m),$$

taking ρ itself as positive, we see from (12.5) that

$$d(v, a) \leqq k\rho^2$$

where k is a positive constant independent of the choice of (v) and (a) on N.

Let $D(v, a)$ be the distance in the space (z) from the point $z^i(v)$ to the point $z^i(a)$. One sees that

$$D(v, a) \leqq \lambda\rho$$

where λ is a constant independent of (v) and (a) on N.

Let s^σ_a be a region in the space (v) consisting of the points (v) interior to and on an $(m - 1)$-sphere of radius σ with center at (a). Suppose $\Delta(a) = 0$. Restricting points (v) to N, the points $z^i(v)$ corresponding to points (v) on s^σ_a will be included in a region v^σ_a in the space (z) consisting of the points (z) on ω_a at most a distance $\lambda\sigma$ from the point $[z(a)]$, together with the points on perpendiculars to ω_a at most a distance $k\sigma^2$ from the points (z) already chosen on ω_a. This follows from the choice of λ and k.

To determine a useful upper bound for the volume v^σ_a of v^σ_a let the space (z) neighboring $[z(a)]$ be referred to rectangular coordinates (x), with the origin at $(z) = (a)$, and with the coordinate axes of $x^1, \cdots, x^r$ in the r-plane ω_a. It appears then that v^σ_a will lie in the rectangular region

$$\begin{aligned} &| x^h | \leqq \lambda\sigma && (h = 1, \cdots, r), \\ &| x^k | \leqq k\sigma^2 && (k = r + 1, \cdots, m). \end{aligned} \tag{12.6}$$

The volume v^σ_a will be less than the volume of the region (12.6), that is,

$$v^\sigma_a < \lambda^r\sigma^r k^{m-r}\sigma^{2(m-r)} = \lambda^r k^{m-r}\sigma^m\sigma^{m-r}. \tag{12.7}$$

Let e be an arbitrarily small positive constant. Let s^σ_a represent the volume of the region s^σ_a in the space (v). From (12.7) we infer that there exists a positive constant ρ_e independent of (a) on N, and so small that, when $\sigma \leqq \rho_e$,

$$v^\sigma_a < es^\sigma_a \tag{12.8}$$

for all points $(v) = (a)$, for which $\Delta(a) = 0$.

Let us now cover N in the space (v) by a set of non-overlapping congruent m-cubes K with diameters at most ρ_e. We prefer those m-cubes which contain points $(v) = (a)$ of N at which $\Delta(a) = 0$. Let each preferred cube K be included in a spherical region s^σ_a such that (a) lies on K, $\Delta(a) = 0$, and σ is as small as is consistent with s_a^σ containing K. It appears that σ will then be at most the diameter of K, so that if K denotes the volume of the m-cube K we have

$$s_a^\sigma < \mu K \tag{12.9}$$

where μ is a numerical constant depending only on the dimension m. Combining (12.9) and (12.8) we have the result

$$v_a^\sigma < e\mu K. \tag{12.10}$$

With each preferred m-cube K in the space (v) we have then associated a region v_a^σ in the space (z) containing all of the points $[z(v)]$ for which (v) is on the intersection of K and N. A set of regions v_a^σ which includes a region v_a^σ for each preferred m-cube K will contain all conjugate points $[z(v)]$ of P for which (v) lies on N. The total volume V of this set of regions v_a^σ will be such that

$$V < e\mu\Sigma K,$$

where the sum Σ extends over the preferred m-cubes. Now ΣK is bounded regardless of the diameters of the cubes K, μ is fixed, and e is arbitrarily small. Hence V is arbitrarily small.

The lemma follows directly.

We shall now prove the following theorem.

THEOREM 12.1. *The set of points on R which are the conjugate points of a fixed point P has the measure sero on R.*

Recall that the volume of an elementary region on R is given by the invariant integral

$$\int \cdots \int | g_{ij}(x) | \, dx^1 \cdots dx^m.$$

From the preceding lemma it then appears that the set of conjugate points $[z(v)]$ defined by the vanishing of $\Delta(v)$ in the neighborhood N in the space (v) has a measure zero on R. This is a result in the small. To proceed we need a representation of the extremals through P as a whole.

Let (x) be a coordinate system containing P. Let (α) be the set of direction cosines in the space (x) of a ray issuing from P. Let the extremal issuing from P with the direction (α) be denoted by $\gamma(\alpha)$. Let the point on $\gamma(\alpha)$ at a distance $t > 0$ from P along g on R be represented by (t, α). The sets (t, α) can be represented as points on a domain Σ which is the product of the interval $0 < t < \infty$ and the unit $(m - 1)$-sphere

$$\alpha_1^2 + \cdots + \alpha_m^2 = 1.$$

If the extremal $\gamma(\alpha)$ is identical with the extremal determined by the parameters (u) in (12.3), we say that the set (t, u) in (12.3) represents the point (t, α) on Σ. The sets $(t, u) = (v)$ which lie on the neighborhood N in the space (v) thus represent the neighborhood of a point on Σ. Moreover the neighborhood of each point on Σ can be similarly represented in such a fashion that for each neighborhood the points $[z(v)]$ which are conjugate to P have a measure zero on R. The points (t, α) on Σ for which t lies between two finite positive constants can be included on a finite set of such neighborhoods. All the points on Σ can accordingly be included on a countably infinite set of such neighborhoods. The set of conjugate points of P on R can thus be included in a countably infinite set of regions of the type of v_a^σ in such a fashion that the sum of the volumes on R of the regions v_a^σ is arbitrarily small.

The proof of the theorem is now complete.

A study of the geodesics on a torus would disclose the fact that there are points P on the torus, the set of whose conjugate points on the torus is everywhere dense. In spite of this fact the preceding theorem gives us the following corollary.

COROLLARY. *The set of points Q on R which are conjugate to a fixed point P on no extremal through P is everywhere dense on R.*

A pair of points P and Q of which P and Q are mutually conjugate on no extremal joining P to Q will be termed *non-degenerate*. All other pairs will be termed degenerate. It follows from the preceding corollary that if P and Q are a degenerate pair, an arbitrarily small and suitably chosen displacement of Q or P will replace P and Q by a non-degenerate pair. We see that the non-degenerate pair and corresponding non-degenerate problem may properly be considered as representing the general case.

In §14 we shall prove similar theorems concerning focal points of a manifold.

The fixed end point problem

13. This section presents a study of the most important special problem in the large. In it we not only obtain precise results in the non-degenerate case, but also show how the degenerate case may be treated as a limiting case of the non-degenerate case.

As a matter of notation it will be convenient to denote the functional domain Ω corresponding to two fixed points A^1, A^2 by

$$\Omega(A^1, A^2). \tag{13.1}$$

We have seen in Theorem 11.3 that the type numbers of a non-degenerate critical extremal g of index k are all null except that $M_k = 1$. In the case of the fixed end point problem we have also seen in Ch. III that the index of g is the number of conjugate points of A^1 on g between A^1 and A^2. The theorems of §10 can now be re-interpreted for a non-degenerate fixed end point problem. In particular Theorem 10.2 gives us the following.

THEOREM 13.1. *If A^1, A^2 is a non-degenerate pair of points on R, the number of extremals g joining A^1 to A^2 with k conjugate points of A^1 on g between A^1 and A^2 must be at least as great as the kth connectivity P_k of the functional domain $\Omega(A^1, A^2)$.*

We shall show how this theorem leads by a limiting process to a theorem applicable both to the degenerate and to the non-degenerate case. To that end we turn to the extremals represented in (12.1). We denote the extremal in (12.1) determined by (u) by $g(u)$. We shall use the following lemma.

LEMMA 13.1. *If on the extremals* (12.1) *through the point P, the value $t = t_0$ determines the kth conjugate point of P on $g(u_0)$, the kth conjugate point of P on the extremal $g(u)$ exists for (u) sufficiently near (u_0) and lies at a distance, $t(u)$, from P along $g(u)$, which varies continuously with (u). If on the other hand $t = t_0$ is not the kth conjugate point of P on $g(u_0)$, then the kth conjugate point of P on $g(u)$, if it exists, will not be determined by values of t near t_0, provided (u) is sufficiently near (u_0).*

It is understood that conjugate points are counted with their indices.

To prove the first statement of the lemma, suppose $t = t_0$ is the kth conjugate point of P on $g(u_0)$. Let t' and t'' be two values of t such that

$$t'' < t_0 < t',$$

and such that t' and t'' separate t_0 from the other values of t which define conjugate points on $g(u_0)$. On $g(u)$ we take A^1 and A^2 respectively as the point P and the point $Q(u)$ on $g(u)$ at which $t = t'$. For these end points we then set up the index form corresponding to $g(u)$ essentially as in §14, Ch. V.

More explicitly we cut across the extremal $g(u_0)$ with the intermediate manifolds M_q of Ch. V, and choose the parameters (v) of Ch. V, as the ensemble of the successive sets of parameters (β) determining points on the intermediate manifolds M_q. We evaluate J along the broken extremal whose end points are P and $Q(u)$ and whose intermediate vertices are on the respective manifolds M_q at the points determined by (v), restricting (v) to sets near (0). We denote the resulting function by $J(v, u)$. We see that $g(u)$ will meet the manifold M_q in a point (β) whose parameters β_h will be functions $\beta_h(u)$ of class C^2 in the variables (u) for (u) near (u_0). We denote the ensemble of these successive sets $[\beta(u)]$ by $[v(u)]$, and define the index form corresponding to $g(u)$ as the form

$$P(z, u) = \frac{\partial^2 J}{\partial v^i \partial v^j}(v(u), u) z^i z^j \qquad (i, j = 1, \cdots, np).$$

The form $P(z, u)$ is non-singular for $(u) = (u_0)$, since $Q(u_0)$ is not a conjugate point of P on $g(u_0)$ for $t = t'$. For a sufficiently small variation of (u) from (u_0) the form $P(z, u)$ will remain non-singular and unchanged in index. But its index is the number of conjugate points of P on $g(u)$ for which $0 < t < t'$. Thus the kth conjugate point of P on $g(u)$ must exist and lie at a distance $t(u)$ on $g(u)$ from P such that $t(u) < t'$.

We now prove in a similar fashion that $t(u) > t''$. Since t' and t'' can be taken arbitrarily close to $t = t_0$ it follows that $t(u)$ must be continuous at (u_0). But $t(u)$ is similarly shown to be continuous at other nearby values of (u), and the proof of the first statement of the lemma is complete.

The second statement of the lemma follows from a similar use of the index form.

We shall show that the connectivities of $\Omega(P, Q)$ are independent of the choice of P, Q on R. We begin with a definition.

The extension of a curve g by curves p and q. Let p, g, and q be continuous images on R of the line segment $0 \leqq t \leqq 1$. Suppose p, g, and q can be joined in the order written to form a single continuous curve g^*. We assign a parameter τ to the points of g^* such that $0 \leqq \tau \leqq 1$, and such that the variation of τ corresponding to any segment γ of g^* is proportional to the sum of the variations of the parameters t on the segments of γ on p, g, and q. We term g^* so parameterized the extension of g by p and q.

The extension of a j-cycle on Ω by curves p and q. Let a_i be an i-cell on $\Omega(P, Q)$, the image on R of a functional simplex $c_i \times t_1$. Let p and q respectively be sensed curves on R which join a point P' to P and Q to a point Q'. Let g be any curve of a_i represented by a product $\omega \times t_1$ of a point ω on c_i and the line segment $0 \leqq t \leqq 1$. We "extend" g by p and q as in the preceding paragraph, forming thereby a curve g^* with parameter τ. We represent g^* by the product $\omega \times \tau_1$. We thereby obtain a new i-cell a_i^* on $\Omega(P', Q')$, an image on R of the functional simplex $c_i \times \tau_1$. We term a_i^* an extension of a_i by p and q.

The *extension* of an i-chain w_i on Ω by p and q will now be defined as the chain obtained by extending each of the cells of w_i by p and q. It will be observed that the extensions of cells on Ω which are conventionally identical will again be conventionally identical.

Let g be the continuous image on R of the line segment $0 \leqq t \leqq 1$. Let U represent any point on R. We suppose that we have given a deformation T of q of the form

$$U = U(t, \mu) \tag{13.2}$$

where U is a continuous point function of the curve parameter t and the time μ for

$$\begin{aligned} 0 &\leqq t \leqq 1, \\ 0 &\leqq \mu \leqq 1, \end{aligned} \tag{13.3}$$

where (13.2) defines g when $\mu = 0$. Under T we understand that the point $U(t, 0)$ on g is *replaced* at the time μ by the point $U(t, \mu)$. Under T the curve g is deformed into a curve g' on which

$$U = U(t, 1) \qquad (0 \leqq t \leqq 1).$$

Let p be the curve traced by the initial end point of g under T, taking the time μ as the parameter. Let q be the curve traced by the final end point of g' under the inverse of T, taking $\mu' = 1 - \mu$ as the parameter. The curve g' "extended" by p and q affords a sensed curve g^* joining the end points of g on R. On g^* the parameter is τ, with $0 \leqq \tau \leqq 1$.

It is clear that g can be deformed into g^*, holding its end points fast. We need however to make such a deformation more explicit, and in particular to show that it can be so defined as to be determined completely by the preceding deformation T.

We accordingly introduce a deformation T' of g into g^*, deriving T' from T as follows.

In the (t, μ)-plane consider the square (13.3). In this square the side

$$0 \leqq t \leqq 1, \qquad \mu = 0, \tag{13.4}$$

represents g, while the three remaining sides may be regarded as a curve γ representing g^*. To define T' we join each point t of the segment $0 \leqq t \leqq 1$ of the t axis by a straight line to that point of γ which represents the point on g^* at which $\tau = t$. We let the points (t, μ) on (13.4) move along the resulting straight line segments at rates which equal the lengths of these segments. The corresponding points $U(t, \mu)$ on R will move on R so as to define a deformation T' of g into g^*. Under T' the end points of g remain fixed. During T' we assign the same parameter t to the moving point as it initially possessed on g.

We continue with the following lemma.

LEMMA 13.2. *Let w_j be a j-cycle on $\Omega(P, Q)$ which is non-bounding on $\Omega(P, Q)$. Suppose that there exists a deformation T which deforms P and Q in a unique manner into points P' and Q' on R, and deforms w_j into a j-cycle z_j on $\Omega(P', Q')$. The cycle z_j will be non-bounding on $\Omega(P', Q')$.*

Let p be the curve traced by P under T and q the curve traced by Q' under the inverse of T. Let $E(z_j)$ represent the extension of z_j by p and q. The cycle $E(z_j)$ will lie on $\Omega(P, Q)$. We now apply the deformation T' "derived" from T to each curve of w_j. We thereby obtain a deformation of w_j into $E(z_j)$, holding P and Q fast. We thus find that

$$w_j \sim E(z_j) \qquad [\text{on } \Omega(P, Q)]. \tag{13.5}$$

If the lemma were false, z_j would bound a chain z_{j+1} on $\Omega(P', Q')$. Let $E(z_{j+1})$ be the extension of z_{j+1} by p and q. We have

$$E(z_{j+1}) \rightarrow E(z_j) \qquad [\text{on } \Omega(P, Q)], \tag{13.6}$$

so that $E(z_j) \sim 0$ on $\Omega(P, Q)$. Hence

$$w_j \sim 0 \qquad [\text{on } \Omega(P, Q)],$$

contrary to hypothesis.

We infer the truth of the lemma.

We shall now prove the following lemma.

LEMMA 13.3. *Suppose a j-cycle w_j on $\Omega(P, Q)$ has been extended by curves p and q to form a j-cycle z_j on $\Omega(P', Q')$. If w_j is non-bounding on $\Omega(P, Q)$, z_j will be non-bounding on $\Omega(P', Q')$.*

Let g be any curve of w_j. Let g^* denote the curve obtained by extending g by p and q. We shall now define a deformation of w_j into z_j. In it we let each point U, initially on g, move along g^* to the point on g^* which possesses a parameter τ equal to the parameter t initially possessed by U on g. We let U thereby move so that its image on the τ axis representing g^* proceeds at a rate equal to the length of the segment of the τ axis to be traversed. One thereby deforms w_j into z_j. By virtue of the preceding lemma z_j is non-bounding on $\Omega(P', Q')$.

We now come to a theorem of purely topological content.

THEOREM 13.2. *The connectivities of the functional domain $\Omega(P, Q)$ are independent of the choice of the points P, Q.*

Let P, Q and P', Q' be two pairs of points on R. Let p and q be respectively two sensed curves which joint P' to P and Q to Q'. Let w_j be any j-cycle on $\Omega(P, Q)$. Let w_j be "extended" by p and q to form a cycle z_j on $\Omega(P', Q')$. By virtue of the preceding lemma z_j will be non-bounding on $\Omega(P', Q')$ if w_j is non-bounding on $\Omega(P, Q)$. It follows that the jth connectivity of $\Omega(P', Q')$ must be at least that of $\Omega(P, Q)$. Upon interchanging the rôles of P, Q and P' Q' one sees that the connectivities of $\Omega(P, Q)$ and $\Omega(P', Q')$ must be equal.

The two point connectivities of R. Since the connectivities P_k of $\Omega(P, Q)$ are independent of the choice of points P, Q on R, we can properly omit reference to P and Q and term P_k the kth two point connectivity of R.

The function $J_k(P, Q)$. Suppose the kth two point connectivity of R is not zero. Corresponding to any two points P and Q on R we let

$$J_k(P, Q) \qquad (k = 0, 1, \cdots)$$

be the inferior limit of constants c such that there is at least one restricted k-cycle which is non-bounding on $\Omega(P, Q)$ below c. We continue with the following lemma.

LEMMA 13.4. *The function $J_k(P, Q)$ is continuous in P and Q for arbitrary choices of P and Q on R.*

Let P, Q and P', Q' be two pairs of points on R. Let $| XY |$ denote the J-distance between points X and Y on R. Suppose that

$$| PP' | < e, \qquad | QQ' | < e, \tag{13.7}$$

where e is an arbitrarily small positive constant. By virtue of the definition of $J_k(P, Q)$ there exists a restricted k-cycle w_k on $\Omega(P, Q)$ which is non-bounding on $\Omega(P, Q)$ and on which

$$J < J_k(P, Q) + e.$$

Let p and q be two sensed elementary extremals joining P' to P and Q to Q' respectively. Let w_k' denote the k-cycle on $\Omega(P', Q')$ obtained by extending w_k

by p and q. On w_k' we see that the maximum of J exceeds the maximum of J on w_k by at most $2e$. Hence on w_k',

$$J < J_k(P, Q) + 3e,$$

so that

$$J_k(P', Q') < J_k(P, Q) + 3e.$$

But we can reverse the rôles of P, Q and P', Q' and hence conclude that

$$| J_k(P', Q') - J_k(P, Q) | < 3e,$$

subject to (13.7)

The lemma is thereby proved.

We shall now prove a general theorem.

THEOREM 13.3. *If the kth two-point connectivity of R is not zero, there exists an extremal g_k joining an arbitrary pair of points P, Q on R, with the following properties. The J-length of g_k is $J_k(P, Q)$ and varies continuously with P and Q on R. If P, Q is a non-degenerate pair of points, there are exactly k conjugate points of P on g_k. If P, Q is a degenerate pair of points, there are at least k and at most $k + m - 1$ conjugate points of P on g_k including Q.*

Suppose first that the pair P, Q is non-degenerate. Set $J_k(P, Q) = c$. The number c must then be a critical value of J, and among the extremals of J-length c there must be at least one extremal g for which the number $M_k = 1$. Otherwise if c' and c'' were two constants not critical values of J separating c from other critical values of J, with $c' < c < c''$, every restricted k-cycle of $\Omega(P, Q)$ for which $J < c''$ would be homologous on $\Omega(P, Q)$ to a k-cycle on $J < c'$, so that c could not equal $J_k(P, Q)$.

In case P, Q is a degenerate pair, let

$$Q_i \qquad (i = 1, 2, \cdots)$$

be a sequence of points tending towards Q as i becomes infinite, and such that the pairs P, Q_i are non-degenerate and distinct. Let γ_i be an extremal satisfying the theorem for the pair P, Q_i and for the given k. Let λ_i represent the initial direction of γ_i at P. The directions λ_i will have a cluster direction λ at P and the extremal g issuing from P with the direction λ will reach Q after traversing a J-length $J_k(P, Q)$.

Suppose there are exactly h conjugate points of P on g including Q. It follows from Lemma 13.1 that $h \geqq k$. But there will be at least $h - (m - 1)$ conjugate points of P on g excluding Q. By virtue of Lemma 13.1 there will then be at least $h - (m - 1)$ conjugate points of P on each extremal γ_i with initial directions sufficiently near λ. But the number of conjugate points of P on γ_i is exactly k so that

$$k \geqq h - (m - 1).$$

It follows that

$$h \leqq k + m - 1,$$

and the proof of the theorem is complete.

It would be a mistake to believe that the extremals g_k affirmed to exist in the preceding theorem could always be chosen so as to vary continuously with their end points P, Q. Simple examples show that the contrary is true.

The case $P = Q$ presents no special difficulty. If however $P = Q$ and $k = 0$, the extremal g_k of the theorem reduces to a point. For $k > 0$ the extremals g_k exist and possess the same properties as when $P \neq Q$.

The one variable end point problem

14. We here suppose that the second end point A^2 is fixed at a point Q and that the first end point rests on a non-singular analytic $(m - 1)$-manifold M, the image on R of an auxiliary simplicial circuit B.

We suppose that M is *orientable.* To define this term let us understand that a covariant (or contravariant) vector at a point P on R varies continuously with P if its components in each local coordinate system vary continuously with P. Starting with a point P_0 on M and a unit covariant vector λ^0 normal to M at P_0, let P vary continuously along a path on M which starts and ends at P_0, and let a unit covariant vector λ normal to M at P vary continuously with P starting from the vector λ^0. If λ returns to λ^0 no matter what the path, M is termed *orientable.* If M were non-orientable, we could replace M by an orientable covering manifold and proceed in essentially the same way.

We begin with a study of the transversality conditions in the large. Let (x) be a local coordinate system and $(x) = (a)$ a particular point (x). Let r^i be the components of a unit contravariant vector defining a direction at $(x) = (a)$. Let λ_i be the components of a unit covariant vector orthogonal to M at the point $(x) = (a)$. A necessary and sufficient condition that the extremal tangent to (r) at $(x) = (a)$ cut M transversally at $(x) = (a)$ is that there exist a constant μ such that

$$F_{r^i}(a, r) - \mu\lambda_i = 0 \qquad (i = 1, \cdots, m), \tag{14.1$'$}$$

$$g_{ij}(a)r^ir^j = 1. \tag{14.1$''$}$$

In the search for all directions (r) which cut M transversally at $(x) = (a)$ we lose no generality if we impose the condition,

$$\mu > 0. \tag{14.1$'''$}$$

For if (14.1)$'$ and (14.1)$''$ are satisfied we have

$$\mu\lambda_ir^i = r^iF_{r^i}(a, r) = F(a, r) > 0. \tag{14.2}$$

We will then either have

$$\lambda_i r^i > 0 \tag{14.3}$$

and hence $\mu > 0$, or else upon replacing λ_i by $\lambda'_i = -\lambda_i$, and μ by $\mu' = -\mu$, we will have

$$F_{r^i}(a, r) - \mu' \lambda'_i = 0$$

with (14.1)″ satisfied, and $\mu' > 0$ as before.

We shall now prove the following lemma.

LEMMA 14.1. *The conditions* (14.1) *define a one-to-one analytic correspondence between the totality of unit contravariant and covariant vectors* (r) *and* (λ) *at the point* $(x) = (a)$.

If a unit contravariant vector (r) is given, conditions (14.1) uniquely determine the components of a unit covariant vector (λ) as analytic functions of the components of (r).

On the other hand suppose sets (r), μ, (λ) are initially given satisfying (14.1), with (λ) a unit covariant vector. One can then vary the vector (λ) independently among unit covariant vectors, and solve the system (14.1) for the variables (r) and μ as analytic functions of the components of (λ). For the jacobian of the system (14.1) with respect to the variables (r) and μ is seen to be

$$\begin{vmatrix} F_{r^i r^j} & -\lambda_i \\ 2g_{ij}r^i & 0 \end{vmatrix} = 2F_1 r^i \lambda_i, \tag{14.4}$$

and this is not zero by virtue of (14.2), and the hypothesis that $F_1 \neq 0$. Thus the relation between the vectors (r) and (λ) defined by (14.1) is locally analytic and one-to-one. But one can continue this correspondence by varying (λ) subject to the condition

$$g^{ij}\lambda_i\lambda_j = 1. \tag{14.5}$$

Regarding (λ) as a point, condition (14.5) requires (λ) to rest on an $(m - 1)$-ellipsoid. It follows from the topological properties of an ellipsoid (or $(m - 1)$-sphere) that the above correspondence is one-to-one in the large.

Let (λ) and $(-\lambda)$ be the two unit covariant vectors normal to M at $(x) = (a)$. According to Lemma 14.1 there are two unique contravariant vectors (r) and $(\bar{r})$ which satisfy (14.1) with (λ) and $(-\lambda)$ respectively. Upon using (14.2) we see that the sums

$$\lambda_i r^i, \qquad \lambda_i \bar{r}^i$$

have opposite signs, a property which makes it possible to distinguish between r^i and $\bar{r}^i$.

The unit contravariant vector $r(P)$. Corresponding to each point P on M let $\lambda(P)$ be a unit covariant vector normal to M at P and so chosen as to vary continuously with P on M. Such a choice of $\lambda(P)$ is possible since M is orientable. Corresponding to $\lambda(P)$ let $r(P)$ be the unique unit contravariant vector $r(P)$ which cuts M transversally at P and which is chosen from the two vectors which cut M transversally at P by requiring that

$$\lambda_i r^i > 0, \tag{14.6}$$

where λ_i and r^i are the local components of $\lambda(P)$ and $r(P)$ respectively.

Let M be regularly and analytically represented in the coordinate system (x), neighboring a point $(x) = (a)$ on M, by functions

$$x^i = x^i(u^1, \cdots, u^n) \qquad (n = m - 1), \tag{14.7}$$

where $x^i(u_0) = a^i$, and the parameters (u) neighbor (u_0). We shall now prove the following lemma.

LEMMA 14.2. *In the system (x) the components r^i of the contravariant vector $r(P)$ are analytic functions $r^i(u)$ of the parameters (u) locally representing M.*

First recall that the components λ_i of the preceding covariant vector $\lambda(P)$ normal to M at P are analytic functions $\lambda_i(u)$ of the parameters (u) of P, for (u) neighboring (u_0). We now write (14.1) in the form

$$\begin{aligned} &F_{r^i}(x(u), r) - \mu\lambda_i(u) = 0 \qquad (\mu > 0), \\ &g_{ij}(x(u))r^i r^j = 1. \end{aligned} \tag{14.8}$$

As previously we see that we can solve for the variables r^i as analytic functions

$$r^i = R^i(u)$$

of the variables (u) for (u) near (u_0). The solution $R^i(u)$ so obtained satisfies the condition

$$R^i(u)\lambda_i(u) > 0$$

as follows from (14.8). But the vector $r(P)$ satisfies the same conditions at the point (u) as the vector $R^i(u)$, and is thereby uniquely determined. Thus

$$r^i(u) \equiv R^i(u)$$

neighboring (u_0).

The proof of the lemma is now complete.

The unit contravariant vector $\bar{r}(P)$. Corresponding to each point P on M let $\bar{r}(P)$ denote the unit contravariant vector which cuts M transversally at P and in terms of the components λ_i of the preceding vector $\lambda(P)$ satisfies the condition

$$\lambda_i \bar{r}^i < 0.$$

One readily proves that the contravariant components of $\bar{r}(P)$ are again analytic functions $\bar{r}^i(u)$ of the local parameters (u) of M.

We now parallel the results in §12 on the measure of the conjugate points of a fixed point.

We represent M locally as in (14.7). Let $g(u)$ be the extremal issuing from the point $x^i(u)$ on M with the direction $r^i(u)$ of Lemma 14.2. Let t be the distance along $g(u)$ from M. Let Q be a focal point of M on $g(u_0)$ with $t = t_0$. Let (z) be a set of local coordinates of M neighboring Q. Neighboring Q the extremal $g(u)$ can be represented in the form

$$z^i = h^i(t, u).$$

We can prove as under Lemma 12.1 that the focal points of M on the extremals $g(u)$ for t near t_0 have a measure in the space (z) which is null. We are then led to a similar result concerning the extremals $\bar{g}(u)$ issuing from the point (u) on M with the directions $\bar{r}^i(u)$.

Theorem 12.1 is here replaced by the following theorem.

THEOREM 14.1. *The set of focal points of M on R has a measure on R which is null.*

The proof of this theorem can be given essentially as was the proof of Theorem 12.1. The representation in the large of the extremals cutting M transversally is necessarily somewhat different.

To obtain such a representation let $g(P)$ and $\bar{g}(P)$ be respectively the extremals issuing from the point P on M with directions $r(P)$ and $\bar{r}(P)$. The point on $g(P)$ at a distance t from M on $g(P)$ will be represented by the pair (P, t). The sets (P, t) form a domain Σ representable as the product, $M \times t^*$, of M and the interval

$$t^*: \qquad -\infty < t < \infty.$$

With the aid of Σ one can prove as in §12 that the focal points of M belonging to the extremals $g(P)$ have a measure zero on R. One can then establish a similar result for the extremals $\bar{g}(P)$, and thereby complete the proof of the theorem.

We state the following corollary.

COROLLARY. *The set of points on R which are not focal points of M is everywhere dense on R.*

Focal points of M belonging to the extremals $g(P)$ or $\bar{g}(P)$ at points at which $t > 0$ will be called *positive* focal points of M, while focal points at which $t < 0$ will be called *negative* focal points of M. Positive focal points are relevant in the problem in which the first end point rests on M and the second is fixed, while negative focal points are relevant when the first end point is fixed and the second end point rests on M.

Let $\Omega(M, Q)$ denote the functional domain Ω in the problem in which the first end point rests on M and the second end point is fixed at Q. Theorem 13.1 is then replaced by the following.

THEOREM 14.2. *If Q is not a positive focal point of M nor on M, the number of extremals g which join M to Q, which cut M transversally at their initial points and* *possess k positive focal points of M thereon between M and Q, is at least as great as the kth connectivity of the functional domain $\Omega(M, Q)$.*

We can prove that the connectivities of $\Omega(M, Q)$ are independent of the choice of Q among points Q on R, following the analogous proof in §13. One uses the extension of a curve g by curves p and q as defined in §13. In the present case we need q only, and take p as null. We record the theorem as follows.

THEOREM 14.3. *The connectivities of the functional domain $\Omega(M, Q)$ are independent of the choice of Q on R.*

The function $J_k(M, Q)$. Suppose the kth connectivity of $\Omega(M, Q)$ is not zero. Let

$$J_k(M, Q)$$

be the inferior limit of constants c such that there is at least one restricted k-cycle which is non-bounding on $\Omega(M, Q)$ below c. We can prove, essentially as in §13, that the function $J_k(M, Q)$ is continuous with respect to a variation of Q on R.

We come finally to the analogue of Theorem 13.3.

THEOREM 14.4. *If the kth connectivity of $\Omega(M, Q)$ is not zero and Q is not on M, there exists an extremal g_k which joins M to Q with the following properties.*

The extremal g_k cuts M transversally at its initial point on M. The J-length of g_k is $J_k(M, Q)$ and varies continuously with Q. If Q is not a positive focal point of M, there are exactly k positive focal points of M and g_k on g_k between M and Q exclusive. If Q is a positive focal point of M and g_k, there are at least k and at most $k + m - 1$ positive focal points of M and g_k on g_k including Q.

The proof of Theorem 14.4 is practically identical with that of Theorem 13.3 and can be omitted.

One can admit that Q lies on M if one understands that g_0 reduces to a point in that case.

The two point functional connectivities of an m-sphere

15. We shall now suppose that the Riemannian space R is the topological image of a unit m-sphere S_m in a euclidean $(m + 1)$-space (w) and that the terminal manifold is represented by two distinct points A^1 and A^2 on S_m. We shall determine the connectivities P_k of the corresponding functional domain $\Omega(A^1, A^2)$.

We have seen in §13 that the connectivities P_k of $\Omega(A^1, A^2)$ are independent of the choice of the points A^1, A^2 on R. Without loss of generality we can take A^1 and A^2 at the interesections of S_m with the positive w_1 and w_2 axes. Moreover the connectivities P_k are independent of the particular image R of S_m which is chosen. We can accordingly take R as S_m itself. We shall suppose moreover that J is the integral of arc length on S_m and shall make use of the Corollary of Theorem 11.4 to determine the connectivities P_k.

Let A be the point on S_m diametrically opposite to A^1. The extremals joining A^1 to A^2 on S_m are arcs of great circles simply or multiply covered. We naturally regard two extremals as distinct if they overlap each other, but are not identical in length. Taken in the order of their lengths, our critical extremals are then an infinite sequence of geodesics

$$g_0, g_1, \cdots,$$

of which g_0 is the arc of the great circle of length less than π joining A^1 to A^2, and g_1 is the residue of the same great circle. The arcs g_0 and g_1 have opposite senses relative to their common great circle γ. The geodesic g_{2r} joins A^1 to A^2 on γ, agreeing in sense with g_0 and passing A r times. The geodesic g_{2r+1} joins A^1 to A^2 on γ, agreeing in sense with g_1, passing A $r + 1$ times.

To apply the theory of the non-degenerate extremal we need to know the index of g_p. Observe first that g_p is non-degenerate since its end points are not conjugate on g_p. Its index equals the number of conjugate points of A^1 on g_p at distances from A^1 on g_p less than the length of g_p. Each conjugate point P of A^1 is thereby counted a number of times equal to its index. The index of a conjugate point P on g_p is the number of linearly independent solutions of the Jacobi equations corresponding to g which vanish at the points t^1 and t^2 on the t axis corresponding to A^1 and P. Now the conjugate points on S_m are diametrically opposite points so that every extremal through A^1 passes through P. Consequently every secondary extremal vanishing at t^1 must also vanish at t^2. The index of each conjugate point of A^1 is accordingly $m - 1$, and the index k of g_p is thus seen to be

$$k = p(m - 1). \tag{15.1}$$

For, starting from A^1, g_p passes A and A^1 in turn, until p passages have been made all told.

We state the following lemma.

LEMMA 15.1. *The geodesics* $g_0, g_1, \cdots$ *are of increasing type in the sense of* §11.

This is true of g_0 by virtue of the Corollary of Theorem 11.3, since g_0 is non-degenerate and has the index zero.

To prove the lemma for $g_1, g_2, \cdots$ we shall associate a restricted cycle λ_k with g_p, and show that λ_k is a linking cycle corresponding to g_p. Here k is the index of g_p and equals $p(m - 1)$.

Corresponding to g_p, $p > 0$, we introduce a set of p constants

$$e_q \qquad (q = 1, \cdots, p),$$

such that

$$0 < e_1 < e_2 < \cdots < e_p < 1. \tag{15.2}$$

In the space (w) let

$$M^q \qquad (q = 1, \cdots, p)$$

be an $(m - 1)$-sphere formed by the intersection with S_m of an m-sphere S_m^q with radius e_q, and center at A if q is odd, and at A^1 if q is even. Let P^q be an arbitrary point on M^q. No point on any of the spheres M^q is diametrically opposite to a point on any other such sphere, or to A^1 or A^2, as follows from the choice of the radii e_q. Hence the points

$$A^1, P^1, \cdots, P^p, A^2 \tag{15.3}$$

can be successively joined by unique minimizing arcs of great circles. The points (15.3) will be regarded as the vertices of a point (π). If the constant ρ of §2 limiting the lengths of elementary extremals is taken near enough to the number π, these points (π) as well as the broken geodesics $g(\pi)$ will be admissible. We suppose $g(\pi)$ represented in terms of a parameter t proportional to the arc length measured from the initial point of $g(\pi)$ and running from 0 to 1. The totality of the above points (π) defines a cycle c_k representable by the product

$$\Gamma_k = A^1 \times M^1 \times M^2 \times \cdots \times M^p \times A^2.$$

We note that the dimension k of c_k is given by the equation

$$k = p(m - 1),$$

and equals the index of g_p.

The totality of the curves $g(\pi)$ determined by points (π) on Γ_k forms a restricted k-cycle λ_k on Ω as we shall now see.

Suppose Γ_k has been subdivided into cells so that it may be regarded as the sum of a set of closed k-cells $\bar{a}_k$ which are the images of closed simplices $\bar{\alpha}_k$. Let (π) be a point on a_k and P the corresponding point on $\bar{\alpha}_k$. We represent the point t on $g(\pi)$ by the pair (P, t), thus representing $g(\pi)$ by the product $P \times t_1$ of P and the line segment $t_1 : 0 \leqq t \leqq 1$. The ensemble of curves $g(\pi)$ determined by points (π) on $\bar{a}_k$ will thus be represented by the closed functional simplex $\bar{\alpha}_k \times t_1$. In this way we see that the totality of curves $g(\pi)$ determined by points (π) on Γ_k can be represented as the curves of a restricted k-cycle λ_k on Ω.

We continue with a proof of the following statement.

(A). *The integral J assumes an absolute, proper maximum over $\lambda_{p(m-1)}$ on the curve g_p.*

Set $k = p(m - 1)$. The vertices of g_p as a curve of λ_k are respectively A^1, the p successive first intersections of g_p with the m-spheres M^q, and finally the point A^2. Let $h_1, \cdots, h_{p+1}$ be the corresponding set of elementary extremals making up g_p. Denote the length of h_q by h_q. We see that

$$\begin{aligned}
h_1 &= \pi - e_1, \\
h_2 &= \pi - (e_2 - e_1), \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
h_p &= \pi - (e_p - e_{p-1}), \\
h_{p+1} &= \pi/2 + e_p.
\end{aligned} \tag{15.4}$$

That g_p has the maximum length among curves of λ_k follows from the fact that on any broken extremal of λ_k the respective elementary extremals have at most the lengths of the corresponding elementary extremals in (15.4).

Let γ be any broken extremal of λ_k whose length equals that of g_p. Upon considering the elementary extremals of γ in inverse order one sees that the requirement that their respective lengths be the lengths (15.4) uniquely determines these elementary extremals. Thus $\gamma = g_p$, and statement (A) is proved.

We shall now prove statement (B).

(B). *The set of curves on λ_k for which*

$$J = c - e^2,$$

where c is the length of g_p, forms a spannable $(k - 1)$-cycle s_{k-1} belonging to g_p, provided e is a sufficiently small positive constant.

To prove (B) we regard the portions of the $(m - 1)$-spheres M^q neighboring the first points of intersection of g_p with M as a set of manifolds defining a "proper section" S belonging to g_p in the space (π) of $p + 2$ vertices. Let (π_0) be the point on S which determines g_p. Let (v) be a set of parameters regularly representing S neighboring (π_0). The number of parameters (v) equals the index k of g_p. Suppose that the set $(v) = (v_0)$ corresponds to (π_0). Let $F(v)$ be the value of $J(\pi)$ at the point (π) on S determined by (v). According to the result of Theorem 11.2 a spannable $(k - 1)$-cycle associated with g can be chosen among the cycles of broken extremals $g(\pi)$ determined by points (π) on S. Inasmuch as $F(v)$ takes on a proper maximum on S at $(v) = (v_0)$ the locus

$$F = J = c - e^2$$

will be such a spannable $(k - 1)$-cycle if e is a sufficiently small positive constant.

Statement (B) is accordingly proved.

We can now complete the proof of Lemma 15.1. We see that the cycle s_{k-1} of (B) bounds below c, in fact bounds a chain of restricted curves on λ_k for which

$$J \leqq c - e^2.$$

Thus λ_k is a linking k-cycle associated with g_p, and the lemma is proved.

The Corollary of Theorem 11.4 leads to the following theorem.

THEOREM 15.1. *The two point connectivities of the m-sphere are all zero except the connectivities $P_{p(m-1)}$, $p = 0, 1, \cdots$, and these connectivities are* 1.

This follows from the fact that the index of the extremal g_p is $p(m - 1)$.

Let R be a regular, analytic homeomorph of an m-sphere, and let A^1 and A^2 be any two distinct points on R. With A^1 and A^2 we associate a sequence of numbers

$$n_0, n_1, n_2, \cdots,$$

which we call the *conjugate number sequence* for A^1, A^2. In this sequence n_k denotes the number of extremals g (possibly infinite) on which there are k conjugate points of A^1 on g, including A^2. From Theorem 13.3 and Theorem 15.1 we derive the following corollary.

COROLLARY. *In the conjugate number sequence corresponding to any two distinct points on a regular, analytic homeomorph of an m-sphere there are at most $m - 2$ consecutive zeros if the pair of end points A^1, A^2 is non-degenerate, and at most $2m - 3$ consecutive zeros if the pair A^1, A^2 is degenerate.*

When $k = p(m - 1)$, the extremal affirmed by Theorem 13.3 to exist and have the length $J_k(A^1, A^2)$, will have at least

$$k = p(m - 1)$$

conjugate points on it including A^2, while for $k = (p + 1)(m - 1)$, the corresponding extremal will have at most

$$(p + 1)(m - 1) + (m - 1)$$

conjugate points on it. The number of integers between these two integers is seen to be $2m - 3$.

For example the conjugate number sequence for two points on a 3-sphere not diametrically opposed is

$$1, 0, 1, 0, 1, 0, \cdots .$$

We are supposing the integral is the arc length. If the two points on the 3-sphere are diametrically opposed, the conjugate number sequence is

$$0, 0, \infty, 0, 0, 0, \infty, 0, 0, 0, \infty, \cdots ,$$

so that there are $2m - 3 = 3$ consecutive zeros periodically recurring.

The preceding results lead also to the following corollary.

COROLLARY. *On any regular, analytic homeomorph of an m-sphere there exist infinitely many geodesics*

$$g_1, g_2, \cdots$$

joining any two fixed points A^1 and A^2. The length of g_n and the number of conjugate points of A^1 on g_n become infinite with n.

CHAPTER VIII

CLOSED EXTREMALS

We continue with the Riemannian space R of the last two chapters. We shall be concerned in this chapter with closed extremals, that is, with extremals which return to the same point with the same direction. We shall treat the *reversible* case, that is, the case in which an extremal reversed in sense is again an extremal.

The absence of end conditions of the nature of those of Ch. VII, and the condition of reversibility necessitate a new approach to the topological aspects of the problem. The fundamental entity here will not be the continuous closed curve, but rather the closed broken extremal determined by a point (π). It might seem at first glance that no purely topological basis could be obtained thereby, but the contrary is the case. By considering points (π) with arbitrarily many vertices, and by introducing an abstract semi-topological definition of elementary extremals, we free the basic topology from the metric employed.

We shall defer this part of the theory until §12 is reached. In §12 we shall define a *metric in the small*, assigning to such a metric the properties ordinarily assigned to a metric, together with certain additional properties peculiar to the needs of a theory in the large. Infinitely many metrics of the nature prescribed turn out to be possible. The existence of such metrics as a class is a topologically invariant fact, and the connectivities P_i, which are central in our theory, are proved independent of the particular metric used to define them.

Our basic domain Ω is here the set of all spaces representing admissible points (π). There are infinitely many spaces (π) since there are points (π) with arbitrarily many vertices. The central topological problem is the proper definition of an homology on Ω. It is made difficult by the fact that we must regard a circular permutation of the vertices of a point (π) as giving an *equivalent* point (π). Another difficulty arises from the fact that the vertices of a point (π) may coalesce. By far the greatest difficulty, however, arises from the fact that homologies of some sort must exist between cycles on spaces (π) with different numbers of vertices. To meet this difficulty we introduce *special homologies* not defined in terms of bounding.

To be useful, our various conventions must lead to definitions of connectivities which are topological invariants, and which are related to the analytic characteristics of closed extremals in the same way that the connectivities of Ω in the preceding chapter are related to the type numbers of the critical sets of extremals. In this chapter we present a solution of these problems. It is essentially a theory of the function space attached to closed curves on a Riemannian space R.

The complexes K, K^p, and Π^p

1. Recall that the basic simplicial circuit K used to define the Riemannian space R lies in a euclidean space E of μ dimensions. As previously we are concerned with sets of points

$$P^1, \cdots, P^p \tag{1.1}$$

on K. We denote such sets of points by (π), and represent (π) as a point on the p-fold product K^p of K by itself. We suppose K^p represented by a simplicial complex in the euclidean space E^p, the p-fold product of E by itself. We shall be concerned in this chapter with the domain of points (π) on K^p in which two points (π) whose vertices have the same circular order, either direct or inverse, shall be regarded as identical. To obtain this domain one must identify the points of K^p under transformations of a *group* G^p. This group we now define.

The transformations T_r and S_r. We are concerned with transformations T_r of the vertices (1.1) into vertices

$$Q^1, \cdots, Q^p. \tag{1.2}$$

Under T_r the point P^i is replaced by the point

$$Q^i = P^{i+r} \qquad (r = 0, 1, \cdots, p-1) \tag{1.3}$$

where we understand that i is any integer, positive, negative, or zero, and that the superscripts in (1.3) are to be reduced, mod p, to a residue $1, \cdots, p$. We are also concerned with transformations U_r under which the point P^i is replaced by the point

$$Q^i = P^{-i+r} \tag{1.4}$$

with the same reduction, mod p, of the superscripts. The transformation U_r is its own inverse. We note that $U_r = U_0T_r$, where we understand that the transformation T_r is followed by the transformation U_0. The transformations T_r, U_0, and their products form a group G^p whose elements we denote by G_i.

Let the coordinates y_h, $h = 1, \cdots, \mu$, of the point P^i in the euclidean space E be denoted by

$$y_h^i \qquad (i = 1, \cdots, p). \tag{1.5}$$

Let us regard the set (1.5), taken for all the above values of h and i, as the coordinates of a point (π) on E^p. By an *element* in the space E^p we shall mean a set of points in E^p satisfying a finite number of homogeneous linear equations and inequalities between the coordinates of a point (π) of E^p. We can regard a transformation of the group G^p as a point transformation of the space E^p. Concerning G^p and E^p we shall now prove the following lemma.

Lemma 1.1. *The space E^p can be divided by hyperplanes into a finite number of elements M, with the property that under G^p an element is carried into an element,* *and that an element M which is carried into itself under a transformation G_i is pointwise invariant under G_i.*

Let i, j be integers of the set $1, \cdots, p$. Let h be one of the integers $1, \cdots, \mu$. Our elements M will be defined by subsets of the conditions

$$(1.6)' \qquad y_h^i < y_h^j \qquad (i \neq j),$$

$$(1.6)'' \qquad y_h^i = y_h^j \qquad (i < j).$$

For each pair of integers α and β on the range $1, \cdots, p$, with $\alpha < \beta$, the conditions (1.6) include just three conditions involving both α and β as superscripts, namely

$$y_h^\alpha < y_h^\beta, \qquad y_h^\beta < y_h^\alpha, \qquad y_h^\alpha = y_h^\beta.$$

Suppose the vertices (1.1) determine a point (π) with coordinates (1.5). Suppose that the vertices (1.1) are replaced by a new set (1.2) under a transformation T_r of the group G^p. We denote this new set of vertices by (π'). The point (π') will possess a set of coordinates (1.5) in general numerically different from the coordinates (1.5) of (π). If the coordinates of (π') satisfy one of the conditions A of (1.6), the coordinates of (π) will satisfy the condition B obtained from A by replacing y_h^i and y_h^j respectively by

$$y_h^{i+r}, \qquad y_h^{j+r}.$$

We say that condition A corresponds to B under T_r. We see that T_r permutes the conditions (1.6) in a one-to-one manner preserving the equality when the equality holds. If (π') is the image of (π) under U_r we obtain the conditions on (π) by replacing y_h^i and y_h^j in the conditions satisfied by (π') by

$$y_h^{-i+r}, \qquad y_h^{-j+r},$$

respectively.

Let (π) be a point of E^p and H the set of all conditions (1.6) satisfied by (π). The set of all points on E^p which can be connected to (π) among points which satisfy H will be termed an *element* H. We shall prove that the elements H can serve as the elements M of the lemma.

Observe that two elements H are either identical or possess no points in common. Consequently when one point of an element H is congruent to a point of the same element, under a transformation G_i, H must be carried into itself under G_i. It remains to prove that H is then pointwise invariant under G_i.

(a). Suppose first that an element H is self-congruent under T_r, where $r \not\equiv 0$, mod p. We shall prove that H is pointwise invariant under T_r.

To that end consider the set of conditions

$$(1.7) \qquad y_h^i < y_h^{i+r}, \qquad y_h^{i+r} < y_h^{i+2r}, \cdots,$$

reducing the superscripts, mod p, to the range $1, \cdots, p$, and holding i, h, and r

fast. I say that these conditions are incompatible if $r \not\equiv 0$, mod p. For if q is the smallest positive integer such that

$$qr \equiv 0 \pmod{p},$$

the qth condition in (1.7) reduces to the form

$$y_h^{i+(q-1)r} < y_h^i,$$

and is incompatible with the ensemble of conditions preceding it in (1.7).

Moreover no point (π) on the element H can satisfy the first condition in (1.7). For in such a case the image (π') of (π) under T_{-r} would satisfy the second condition in (1.7) by virtue of the form of T_{-r}. But under the hypothesis of (a), (π') and (π) both belong to H, so that (π) also satisfies the second condition in (1.7). Similarly (π) satisfies the remaining conditions in (1.7). But the conditions (1.7) are incompatible, so that no point of H can satisfy the first condition in (1.7).

One can prove similarly that no point of H can satisfy the condition

$$y_h^i > y_h^{i+r},$$

so that H must satisfy the equation

$$y_h^i = y_h^{i+r}.$$

Hence H must be pointwise invariant under T_r. The conclusion of (a) is accordingly established.

(b). We now consider the case in which H is self-congruent under U_r, and prove that H is then pointwise invariant under U_r.

Under U_r the regions

$$y_h^i < y_h^{-i+r}, \qquad y_h^{-i+r} < y_h^i$$

are interchanged. Hence H can be self-congruent under U_r only if it satisfies the conditions

$$y_h^i = y^{-i+r}$$

and hence is pointwise invariant under U_r.

If we take the hyperplanes (1.6)″ as the hyperplanes of the lemma, the lemma follows from the preceding analysis.

Suppose K^p is subdivided into a simplicial complex in the space E^p. Let A be a finite set of hyperplanes of dimension one less than that of E^p, so chosen that each i-cell of K^p is on the i-dimensional intersection of a subset of these hyperplanes. Let B be the set of hyperplanes congruent under G^p to the hyperplanes of A. Let the cells of K^p be sectioned (Lefschetz [1], p. 67) by the respective hyperplanes of B. Let the resulting cells be further sectioned by the set of all hyperplanes (1.6)″ The resulting complex will be divided into cells, each of which lies on some one element of M, and is carried by transformations of

G^p into cells of the complex. So divided our complex will be again denoted by K^p.

Let Π_0^p *be the complex obtained from* K^p *by identifying cells of* K^p *which are congruent under* G^p*, and subdividing the resulting complex in such a manner that it is the image of a simplicial complex.*

A point (π) whose successive vertices represent points on R which can be joined by elementary extremals on R, will be termed admissible.

The subdomain of Π_0^p *which consists of admissible points* (π) *will be denoted by* Π^p.

The infinite space Ω

2. In this section we combine the different domains Π^p into a domain Ω, and introduce new conceptions of homologies necessary if the basic relations of Ch. VII are to be preserved.

By the *domain* Ω we mean the set of all points (π) on the respective domains Π^p. Any infinite set of k-chains

$$z^p \qquad (p = 3, 4, \cdots)$$

on the respective domains Π^p, all null except at most a finite number, will be termed a *k-chain* z on Ω. The chain z^p will be termed the pth *component* of z. The *sum*, mod 2, of a finite number of k-chains z on Ω shall be defined as the chain on Ω whose pth component is the sum, mod 2, of the pth components of the given chains z.

A point (π) on Π^p will be termed *contracted* if its vertices are coincident. A cell on Π^p will be termed contracted if composed of contracted points (π). In determining boundaries, cycles, and homologies on Π^p, contracted cells shall not be counted. With this understood the *boundary* of a k-chain z on Ω shall be the $(k - 1)$-chain whose pth component is the boundary on Π^p of the pth component of z. A k-chain on Ω without boundary will be termed a k-cycle. The qualification mod 2 is to be understood throughout.

The relations between homologies and bounding on Ω will not be the ordinary ones by virtue of the conventions we now introduce. It is by virtue of these conventions that we shall have homologies between cycles whose components lie on different domains Π^p.

By the *r-fold partition*, $r > 0$, of a point (π) on Π^p we mean the point (π) on Π^{rp} obtained by inserting $r - 1$ vertices on $g(\pi)$ between each pair of successive vertices P' and P'' of (π) so as to divide the elementary extremal joining P' to P'' into r segments of equal J-length. If $P' = P''$, the vertices added are identical with P' and P''. Let $\bar{a}^p$ be a closed k-cell on Π^p given as the continuous image on Π^p of a closed k-simplex $\bar{\alpha}$. By the *r-fold partition of* $\bar{a}^p$ on Π^{rp}, we mean the closed cell $\bar{c}^p$ on Π^{rp}, obtained by replacing the image (π) on Π^p of each point of $\bar{\alpha}$ by its r-fold partition on Π^{rp}. The cell $\bar{c}^p$ is thus the continuous image on Π^{rp} of $\bar{\alpha}$. Let z^p be a k-chain on Π^p. By the *r-fold partition* of z^p on Π^{rp} we mean the sum on Π^{rp} of the r-fold partitions of the respective closed k-cells of z^p.

Let z and w be two n-cycles on Ω with pth components z^p and w^p respectively. If for each p we have

$$z^p \sim w^p \qquad (\text{on } \Pi^p), \tag{2.1}$$

we shall say that we have a *simple homology*

$$z * w \qquad (\text{on } \Omega). \tag{2.2}$$

On the other hand let z be a k-cycle on Ω with at most one non-null component z^p. We shall refer to z as the cycle z^p on Ω. In the same sense let w^q be a second k-cycle w on Ω. If w^q is the r-fold partition of z^p we shall say that we have a *special homology*

$$z * w \qquad (\text{on } \Omega). \tag{2.3}$$

We shall also write

$$z^p * w^q \qquad (\text{on } \Omega). \tag{2.4}$$

We now define an Ω-homology as one formally *generated* by the addition, mod 2, of the respective right and left members of a finite set of simple and special homologies between k-cycles on Ω. If the resulting sums are cycles u and v respectively, we write

$$u * v \qquad (\text{on } \Omega). \tag{2.5}$$

We also write (2.5) in the form $u + v * 0$. Our Ω-homologies thus admit the usual formal linear operations.

We note that a set of generating homologies sum to a homology of the form (2.5) in which u and v are unique k-cycles on Ω. On the other hand an Ω-homology of the form (2.5) can be generated by the addition of simple and special homologies in infinitely many ways. Unlike an ordinary homology an Ω-homology (2.5) does not in general imply that its members bound a chain on Ω. A simple homology in a set of generating homologies does however imply that its members bound a chain Γ on Ω. We term such a chain Γ a *bounded chain* implied by the corresponding homology. An Ω-homology will be said to *hold* on a subdomain Ω_0 of Ω if the members of a set of generating homologies are on Ω_0 and if the members of the respective simple homologies bound chains on Ω_0.

Suppose we have an Ω-homology of the form

$$z * 0. \tag{2.6}$$

Let H be a set of simple and special homologies generating (2.6). The simple homologies of H can be combined into homologies of the form

$$u^p \sim 0 \qquad (\text{on } \Pi^p), \tag{2.7}$$

the homologies (2.7) including just one homology for each integer p, with all but a finite set of the cycles u^p null. Suppose there are μ special homologies in the set H. The ith of these special homologies will then take the form

$$v^{p_i} * w^{q_i} \qquad (i = 1, \cdots, \mu) \tag{2.8}$$

where the two members of (2.8) are k-cycles on Π^{p_i} and Π^{q_i} respectively, and one of these cycles is the partition of the other with $p_i \neq q_i$. The pth component of z in (2.6) will then be a k-cycle z^p on Π^p of the form

$$z^p \equiv u^p + \delta_p^{p_i} v^{p_i} + \delta_p^{q_i} w^{q_i} \qquad (i = 1, \cdots, \mu), \tag{2.9}$$

where δ_j^i is the Kronecker delta, and the terms involving i are to be summed with respect to i, holding p fast.

A chain on Π^p will be said to have an *index* p. A chain z on Ω will be said to possess an index equal to the least common multiple of the indices of its non-null components. A set of homologies generating an Ω-homology will be said to possess an index equal to any multiple of the members of the respective generating homologies. If an Ω-homology $z * 0$ can be generated by a set of homologies with index p, we shall say that $z * 0$ *with index* p.

Let z be a chain on Ω with index p. Let q be any positive integral multiple of p. If each non-null component of z is replaced by its partition on Π^q, and the resulting chains added mod 2, on Π^q, one obtains a chain w^q on Π^q which will be termed the partition of z on Π^q. With this understood we state the following.

Any k-cycle z on Ω such that $z * 0$, possesses a partition $w^q \sim 0$ on Π^q, provided q is a suitably chosen positive integer.

Let q be an index of a set of homologies generating the homology $z * 0$. Each of these generating homologies implies a homology on Π^q between the partitions on Π^q of the members of the given generating homology. In particular a special homology thereby implies an homology on Π^q between identical cycles. If w^q is the partition of z on Π^q, it appears from the definition of an Ω-homology, that the homologies which we have obtained on Π^q sum to an homology reducible to the form

$$w^q \sim 0 \qquad (\text{on } \Pi^q).$$

The statement in italics is accordingly proved.

A set of k-cycles on Ω will be termed *Ω-dependent* if a proper linear combination of cycles of the set is Ω-homologous to zero. By the *connectivity*

$$P_k \qquad (k = 0, 1, 2, \cdots)$$

of Ω we mean the number of k-cycles on Ω in a maximal set of Ω-independent k-cycles. We admit that P_k may be infinite. The following statement covers a particular case of interest.

A necessary and sufficient condition that the connectivity P_0 of Ω be null, is that every closed curve on R be deformable into a point on R.

We shall prove the condition sufficient. To that end let (π) be any point on Ω. We regard (π) as a 0-cycle on Ω, and wish to show that $(\pi) * 0$ on Ω. The curve $g(\pi)$ is deformable into a point on R by hypothesis. One sees then that a sufficiently high partition of (π) on Π^p can be deformed on Π^p into a contracted point (π_0) on Π^p. Hence

$$(\pi) * (\pi_0) \qquad \text{(on } \Omega).$$

But according to our conventions the contracted point (π_0) can be omitted in the count of boundaries so that $(\pi) * 0$ on Ω.

Conversely it will follow from our latter work that the condition is necessary. Inasmuch as we shall not use this fact we omit further details.

Critical sets of extremals

3. In this section we shall make a study of the existence of closed extremals from the point of view of the theory of analytic functions. To that end we shall say that a continuous family of closed curves is *connected* if any closed curve of the family can be continuously deformed into any other closed curve of the family through the mediation of curves of the family. A connected family of closed extremals which is a proper subset of no connected family of closed extremals will be called a *maximal connected* set of closed extremals. We shall prove the following theorem.

THEOREM 3.1. *The number of maximal connected sets of closed extremals on which J is less than a constant b is finite. On each such set J is constant.*

Let g be a closed extremal of J-length ω. Let g be given a positive sense. Let Q be a point on g. The neighborhood of Q can be represented regularly and analytically in terms of coordinates

$$(x, y_1, \cdots, y_n)$$

such that along g neighboring Q

$$y_1 = \cdots = y_n = 0 \qquad (n = m - 1)$$

and x is the arc length on g measured from Q. The extremals neighboring g can be represented near Q by giving the coordinates y_i of their points as functions $\varphi_i(x, \alpha)$ of x and $2n$ parameters (α) which give the initial values of (y) and (y') when $x = 0$. The functions $\varphi_i(x, \alpha)$ will be analytic in their arguments for x near 0 and (α) near (0).

For sets (α) sufficiently near zero the extremal g_α, determined by (α) when $x = 0$, will return to the neighborhood of Q after traversing a J-length ω, and will then be representable in the form

$$y_i = \psi_i(x, \alpha),$$

where $\psi_i(x, \alpha)$ is analytic in x and (α), for (x, α) near the set $(0, 0)$. In order that g_α be periodic with respect to its J-length, and possess a period near ω, it is necessary and sufficient that

$$
\begin{aligned}
\varphi_i(0, \alpha) &= \psi_i(0, \alpha), \\
\varphi_{ix}(0, \alpha) &= \psi_{ix}(0, \alpha),
\end{aligned}
\tag{3.1}
$$

for (α) sufficiently near (0).

The equations (3.1) may be satisfied for real sets (α) only when $(\alpha) = (0)$, or they may be satisfied identically. Apart from these special cases the real solutions (α) of (3.1) will be representable as functions "in general" analytic on one or more locally connected "Gebilde" G (Osgood [1]) of r independent variables with $0 < r < 2n$. Each G includes the point $(\alpha) = (0)$. To each set (α) on G corresponds a closed extremal. Corresponding to any regular curve Γ on one of the above Gebilde G, one obtains a 1-parameter family of closed extremals. Upon differentiating the J-lengths J of these closed extremals with respect to the arc length along Γ we see that $J' \equiv 0$. It follows that J is constant on G, in fact takes on the value ω.

To come to the theorem let us suppose the theorem is false in that there exist infinitely many maximal connected sets of closed extremals on which $J < b$. In each of these sets we choose an extremal g^*, and on g^* a point p. Let λ be a unit contravariant vector tangent to g^* at p. Let L be the J-length of g^*. The sets (p, λ, L) are infinite in number. They have at least one cluster set (p_0, λ_0, L_0). We see that $0 < L_0 \leqq b$.

The extremal g_0 passing through the point p_0 with the direction λ_0 will be closed, and with respect to its J-length possess a period L_0. But as we have seen in an earlier paragraph, closed extremals sufficiently near g_0, with periods sufficiently near L_0, will be connected to g_0 among closed extremals of the same class, contrary to the choice of (p_0, λ_0, L_0) as a cluster set of the sets (p, λ, L).

From this contradiction we infer the truth of the theorem.

Corresponding to any point (π) on Π^p let the value of J taken along $g(\pi)$ be denoted by $J(\pi)$. Neighboring any point (π_0) we can regard $J(\pi)$ as a function ψ of the coordinates (x) in the sets locally representing the vertices of (π) on R. The function ψ will be analytic provided consecutive vertices of (π) remain distinct. Of the points (π) whose successive vertices are distinct, a point (π) at which ψ has a critical point will be called a *critical point* of $J(\pi)$. As in Ch. VII, so here, it follows that a necessary and sufficient condition that a point on Π^p with consecutive vertices distinct be a critical point of $J(\pi)$, is that $g(\pi)$ be a closed extremal. We define a *critical set* of $J(\pi)$ on Π^p as a set of critical points of $J(\pi)$ on which $J(\pi)$ is constant, and which is at a positive distance from other critical points of $J(\pi)$.

By a *critical set of closed extremals* we mean a set of closed extremals on which J is constant, and which contains the whole of each maximal connected set of closed extremals of which it contains a single extremal. A critical set of closed extremals will be termed *complete* if it contains all of the closed extremals on which J equals a given constant c.

Let A be a critical set of closed extremals of J-length c. In terms of the constant ρ of Ch. VII §2, let p be an integer so large that $p\rho > c$. Of points (π) on Π^p whose consecutive vertices are distinct let σ^p be the set which determines extremals $g(\pi)$ of A. We see that σ^p is a critical set of $J(\pi)$ on Π^p.

The ensemble of the sets

$$\sigma^3, \sigma^4, \cdots$$

will be termed the critical set σ on Ω determined by A. The set σ^p will be termed the component of σ on Π^p.

The domain Π^p

4. In this section we shall begin an analysis of the domain Π^p, and the critical sets on this domain. This analysis is preliminary to a similar analysis of the domain Ω. In the present section p is a fixed integer greater than 2.

A first difference between the developments in the present section and those of Ch. VII arises from the convention that contracted $(k - 1)$-cells are omitted from the boundaries of k-chains on Π^p. A necessary and sufficient condition that a point (π) be contracted is that as a point of K^p it be invariant under the transformation T_1 of §1. Now the cells of K^p have been so chosen as to be pointwise invariant under T_1 whenever they possess a pair of points congruent under T_1. Hence if one point of a cell of Π^p is contracted so are all points of that cell.

We shall now prove the following lemma.

LEMMA 4.1. *Corresponding to an arbitrary positive constant b there exists an arbitrarily small positive constant $\delta(b)$, such that any k-cycle on Π^p below $\delta(b)$ is homologous to zero below b on* Π^p.

A cycle z^p on Π^p below a sufficiently small positive constant will be arbitrarily near the subcomplex of contracted cells of Π^p, and will be homologous to a cycle u^p of contracted cells. Moreover there exists a chain on Π^p bounded by z^p and u^p arbitrarily near the contracted cells of Π^p, if z^p itself is sufficiently near these contracted cells. This follows readily from the Veblen-Alexander deformation. By virtue of our conventions the cycle u^p of contracted cells can be dropped from the homology $z^p \sim u^p$, so that $z^p \sim 0$.

The statement of the lemma involving b and $\delta(b)$ follows from the fact that $J(\pi)$ is continuous on Π^p, and equals zero on the contracted cells.

A second departure from Ch. VII comes in new demands which we must put on "J-deformations" of points (π) on Π^p. As in Ch. VII such deformations should carry admissible points (π) into admissible points (π) and not increase the value of $J(\pi)$ beyond its initial value. But in the present chapter points (π) on K^p which are obtained one from the other under transformations of the group G^p must be deformed through points with the same property. Moreover our deformations should vary contracted points through contracted points.

In §3, Ch. VII, we made use of a deformation $D''D'$. We shall now define deformations D^* and D^{**}, analogous to D' and D'' respectively. The deformation D^* tends to distribute the vertices of a point (π) more evenly on $g(\pi)$, while D^{**} tends to decrease the J-length of $J(\pi)$ if the vertices of $g(\pi)$ are already fairly evenly distributed on $g(\pi)$.

The deformation D^*. Let (π) be a point on Π^p. Let g^* be an unending curve "covering" $g(\pi)$. On g^* let s represent the J-length measured in a prescribed sense from a prescribed point on g^*. The vertices of (π) will be represented by infinitely many copies on g^*. Let

$$P^1, \cdots, P^p \tag{4.1}$$

be a set of copies of consecutive vertices of (π) which appear consecutively on g^* in the order (4.1). If c is the J-length of $g(\pi)$ let

$$Q^1, \cdots, Q^p \tag{4.2}$$

be a set of consecutive points on g^* which delimit successive segments of g^* of J-length c/p, and which are so placed that the average value of s for the points (4.2) is the same as for the points (4.1). The deformation D^* is now defined as one in which the vertices (4.1) move along g^* to the corresponding vertices in (4.2), moving at J-rates equal to the J-lengths to be traversed on g^*.

One sees that the point (π') on Π^p which is determined by the vertices (4.2) will be independent of the particular set (4.1) chosen as above to represent the point (π). In particular one might replace P^1 in (4.1) by a point on g^* for which s is c greater. The average s for the points (4.1) is now c/p greater than previously. The corresponding new set (4.2) will now be obtained from the old set (4.2) by replacing Q^1 by a point on g^* for which s is c greater. But this new set (4.2) determines the same point (π') on Π^p. One also sees that the point (π') determined by (π) is independent of the sense assigned to g^* and of the point from which s is measured. The same is true of the points through which (π) is deformed under D^*. Thus D^* has the properties required of a J-deformation.

The deformation D^{**}. We begin by assigning a metric to Π^p neighboring points (π) whose consecutive vertices are distinct. Let (π_0) be such a point and

$$P_0^1, \cdots, P_0^p$$

its successive vertices. Let

$$x_q^1, \cdots, x_q^m \qquad (q = 1, \cdots, p) \tag{4.3}$$

be local coordinates on R neighboring P_0^q. Suppose that the form

$$ds_q^2 = g_{ij}^q dx_q^i dx_q^j \qquad (q \text{ not summed})$$

defines the metric on R neighboring the vertex P_0^q. We then assign the metric (q summed)

$$ds^2 = g_{ij}^q dx_q^i dx_q^j \qquad (i, j = 1, \cdots, m; q = 1, \cdots, p) \tag{4.4}$$

to Π^p neighboring (π_0). We note that the form (4.4) is *invariant* under the transformations of points (π) on K^p which are defined by members of the group G^p, so that the form (4.4) is *uniquely* defined on Π^p.

In conformity with Ch. VII we denote the set of all parameters (4.3) by (u), and write (4.4) in the form

$$ds^2 = b_{hk}(u)du^h du^k \qquad (h, k = 1, \cdots, mp). \tag{4.5}$$

We denote the value of $J(\pi)$ at the point determined by (u) by $\varphi(u)$.

Let b be any ordinary value of J. Suppose moreover that $p\rho > b > \delta(b)$ where $\delta(b)$ is the constant of Lemma 4.1. Suppose further that $\delta(b)$ is less than any critical value of J. Let Σ denote the set of J-normal points of Π^p for which

$$\delta(b) \leqq J(\pi) \leqq b. \tag{4.6}$$

Let η be a positive constant so small that any point on Π^p within a distance η on Π^p of points of Σ will define elementary extremals of positive J-lengths less than ρ. Recall that it is only for the case of distinct consecutive vertices that $\varphi(u)$ is assuredly analytic.

We now define D^{**} in the same manner as D'' was defined in Ch. VII, making use of the preceding metric (4.5) of Σ, and of the preceding constant η. The deformation D^{**} is *uniquely* defined for each point of Π^p, by virtue of the invariance of the form (4.4) under transformations of the group G^p, and in particular by virtue of the corresponding invariance of the trajectories (3.6) of Ch. VII.

Let R_i denote the connectivities of the domain $J(\pi) < b$ on Π^p. We take these connectivities in the ordinary sense, modified only by our conventions concerning the omission of contracted cells. We shall prove the following theorem.

THEOREM 4.1. *Let a and b, $a < b$, be ordinary values of J between which there are no critical values of J. The connectivities of the domains $J(\pi) < b$ and $J(\pi) < a$ on Π^p are finite and equal. If there are no critical values less than b, the connectivities of the domain $J(\pi) < b$ are null.*

We begin with a definition.

The deformation D_p^*. We replace the deformation $D_p = D''D'$ of §3, Ch. VII, by the deformation

$$D_p^* = D^{**}D^*.$$

With its aid we shall now prove statement (a).

(a). *The connectivities of the domain*

$$J(\pi) < b \qquad (on\ \Pi^p) \tag{4.7}$$

are finite.

As in Ch. VII so here it follows that D_p^* will deform the domain (4.7) into a subdomain H whose boundary points are inner points of the domain (4.7). If Π^p is sufficiently finely subdivided, a subcomplex Γ of Π^p can then be chosen so as to contain all the points of H, and to consist of points of (4.7). Any k-cycle on Π^p is thus deformable on Π^p under D_p^* into a k-cycle *on* Γ. Moreover, D_p^* deforms a cycle through cycles, since D_p^* deforms contracted cells through contracted cells.

To show, as in Ch. VII, that any k-cycle z on (4.7) is homologous (not counting contracted cells on boundaries) to a k-cycle of cells *of* Γ we use the Veblen-Alexander process. In this deformation a point of z never leaves the closed cell of Γ on which it is originally found, and this has the consequence that contracted cells are deformed through contracted cells. Hence z will be deformed through k-cycles into a k-cycle. The connectivities of (4.7) will then be at most the connectivities of Γ and will thus be finite. Statement (a) is accordingly proved.

To prove that the connectivities of the domain (4.7) are null if there are no critical values less than b we note that a sufficient number of iterations of D_p^* will deform a k-cycle on (4.7) into a k-cycle on the domain

$$J(\pi) \leqq \delta(b), \tag{4.8}$$

where $\delta(b)$ is the constant of (4.6) and Lemma 4.1. But by virtue of Lemma 4.1 all cycles on (4.8) are homologous to zero on the domain $J(\pi) < b$. Hence the connectivities of the domain (4.7) are null if there are no critical values less than b.

Finally the connectivities of the domains

$$J(\pi) < a, \qquad J(\pi) < b \tag{4.9}$$

are equal if there are no critical values of J between a and b. This follows from the result of the preceding paragraph if both a and b are less than the least critical value of $J(\pi)$. In case a and b are both greater than the least critical value of $J(\pi)$ the connectivities of the domains (4.9) are again equal as can be proved formally after the manner of proof of Theorem 3.2, Ch. VII.

The proof of the theorem is now complete.

Critical sets on Π^p

5. Let ζ be a critical set of closed extremals on which $J = c$. Suppose the integer p so chosen that $p\rho > c$. Let σ^p denote the set of critical points on Π^p which determine curves $g(\pi)$ of ζ. We state the following theorem.

THEOREM 5.1. *The set Σ of all J-normal points on Π^p neighboring σ^p forms a regular analytic Riemannian manifold.*

The proof of this theorem can be brought under the proof of Theorem 7.1, Ch. VII, as follows.

Let (π_0) be any J-normal point of σ^p. Let the vertices of (π_0) be denoted by

$$P_0^1, \cdots, P_0^p,$$

taking these vertices in one of their circular orders. Let (π) be a J-normal point neighboring (π_0). Let the vertices of (π) neighboring the respective vertices of (π_0) be denoted by

$$P^1, \cdots, P^p.$$

To make use of the proof in Ch. VII we cut $g(\pi)$ at P^p, forming thereby a broken extremal whose end points A^1 and A^2 both cover P^p, but are regarded as distinct. In the notation of Ch. VII this broken extremal possesses the vertices

$$A^1, P^1, \cdots, P^{p-1}, A^2,$$

where A^1 and A^2 are subject to end conditions which make A^1 and A^2 cover the same point on R. The end parameters of Ch. VII may be taken as any set of admissible coordinates of R representing A^1 or A^2 neighboring P_0^p. One now completes the proof as in Ch. VII.

We continue with the following theorem.

THEOREM 5.2. *The value of $J(\pi)$ on the subspace Σ of J-normal points (π) sufficiently near σ^p is an analytic function of the local coordinates of Σ, and possesses no critical points other than J-normal points of the set σ^p.*

The proof of this theorem does not differ essentially from the proof of Theorem 7.2, Ch. VII, and will be omitted.

The radial deformation $R_p(t)$, $0 \leqq t < 1$. Let φ be a neighborhood function belonging to the function $J(\pi)$ on Σ and to the critical set of that function. Let r be a positive constant so small, that the points on Σ which are connected to σ^p, and for which

$$\varphi \leqq r,$$

form a closed domain at each point of which φ enjoys the properties of a neighborhood function. With the aid of φ we introduce "radial trajectories" on Σ neighboring σ^p as in Ch. VI. Under the deformation $R_p(t)$ a J-normal point P at which

$$\varphi = r - \theta r \qquad (0 \leqq \theta < 1)$$

shall remain fixed until t reaches the constant θ, and shall thereafter be replaced at the time t by the point on the radial trajectory through P at which $\varphi = r - tr$. The deformation $R_p(t)$ is thereby defined.

We continue with the following lemma, the analogue of Lemma 8.1 of Ch. VII.

LEMMA 5.1. *There exists a J-deformation $E_p(t)$, with time interval $0 \leqq t \leqq 2$, which deforms points (π) on Π^p neighboring σ^p into J-normal points (π), and leaves J-normal points invariant. Moreover $E_p(t)$ deforms the vertices of each point (π) on σ^p through points (π) on the extremal $g(\pi)$ into a J-normal point of σ^p.*

We begin with the deformation D^* of §4, thereby deforming points (π) on Π^p sufficiently near σ^p into points (π) arbitrarily near J-normal points of σ^p. We then continue with a deformation Δ^* defined as follows.

The deformation Δ^*, $0 \leqq t \leqq 1$. This deformation is a deformation of points (π) on Π^p formally defined as was D^*, except that the points (4.2) shall here be a set of consecutive points on g^* in §4 which define a J-normal point (π') with vertices on $g(\pi)$, and, as in §4, are so placed that the average value of s for the points (4.2) is the same as for the points (4.1). If the given point (π) is sufficiently near a J-normal point of σ^p, we see that (π') is uniquely determined, lies on Π^p, and varies continuously with (π).

The deformation $E_p(t)$, $0 \leqq t \leqq 2$. We now set

$$E_p(t) = \Delta^* D^*,$$

understanding that D^* occupies the first unit interval of time in $E_p(t)$, and Δ^* the second unit interval of time. We see that $E_p(t)$ so defined satisfies the requirements of the lemma.

The deformation $\theta_p(t)$, $0 \leqq t < 3$. Proceeding formally as in Ch. VII, we now combine the deformations $E_p(t)$ and $R_p(t)$ of this section into the deformation $\theta_p(t)$. We understand that in $\theta_p(t)$, $E_p(t)$ occupies the time interval $0 \leqq t \leqq 2$ and $R_p(t)$ the time interval $2 \leqq t < 3$. The deformation $\theta_p(t)$ is applicable to points (π) sufficiently near σ^p. Its characteristic properties are enumerated as follows.

THEOREM 5.3. *Under $\theta_p(t)$, $0 \leqq t < 3$, σ^p is deformed on itself. For $t \geqq 2$ each point of Π^p which is sufficiently near σ^p is replaced by a J-normal point (π). Any sufficiently small neighborhood of σ^p is deformed at the time t into a neighborhood N_t^p, the superior limit of the distances of whose points from σ^p approaches zero as t approaches 3. Under $\theta_p(t)$ points below c are deformed through points below c.*

Let a and b be two constants which are not critical values of J, and between which c is the only critical value of J. Let the integer p be so chosen that $\rho p > b$. Denote the complete set of critical points on Π^p corresponding to the critical value c by σ^p. In terms of the deformation D_p^* of §4, we state the following lemma.

LEMMA 5.2. *Let N^p be an arbitrary neighborhood of σ^p on Π^p, and let L^p be the set of points on Π^p below c. A sufficient number of iterations of the deformation D_p^* will afford a deformation Δ_p which will deform the domain $J(\pi) < b$ on itself onto $N^p + L^p$.*

If a k-cycle z lies on a domain $N_0^p + L^p$ for which N_0^p is a sufficiently small neighborhood of σ^p, and if $z \sim 0$ on $J(\pi) < b$ (below c), then $z \sim 0$ on the domain $N_0^p + L^p$ (below c).

This lemma is the analogue of the Deformation Lemma of §6, Ch. VI. Its proof is the counterpart of the corresponding proof in Ch. VI, J replacing f, and D_p^* replacing D.

Let N^{*p} be a fixed neighborhood of σ^p whose closure is interior to the domain on which the preceding deformation $\theta_p(t)$ is defined.

Certain lemmas and theorems of Ch. VI now hold here in the same form as in Ch. VI, except at most for the substitution of $J(\pi)$ for f, $\theta_p(t)$ for $\theta(t)$, and σ^p for σ, and the addition of the superscript p to the neighborhoods. The basic domain is Π^p. We shall add a star to a theorem of Ch. VI to indicate that it shall be taken with the present interpretations.

We first take over Corollary 3.1 of Ch. VI, denoting its counterpart here by Corollary 3.1*. With the aid of the neighborhoods N^{*p} and $M^p(X)$ appearing in Corollary 3.1*, admissible pairs of neighborhoods V^pW^p of σ^p are formally defined as are the neighborhoods VW of Ch. VI. We then add the definition of spannable and critical k-cycles corr V^pW^p, belonging to σ^p, as before. We next have Theorem 5.2* where the domains $\varphi \leqq e$ and $\varphi = e$ are to be interpreted as the domains of J-normal points (π) of the present section. From Theorem 5.2* we infer that the number of cycles in maximal sets of spannable and critical k-cycles corr V^pW^p is finite.

Linking and invariant k-cycles on Π^p corr V^pW^p, are now formally defined as in Ch. VI, §6. We then obtain Lemmas 6.1*, 6.2*, 6.3* and 6.4* from the corresponding lemmas in Ch. VI. The proofs are unchanged except in notation and connotation. In proving Lemma 6.4* one replaces the deformation $\Lambda(t)$ of Ch. VI by a deformation $\Lambda_p(t)$ defined as follows.

The deformation $\Lambda_p(t)$, $0 \leqq t < 3$. The deformation $\Lambda_p(t)$ is defined in terms of $\theta_p(t)$, as $\Lambda(t)$ was defined in terms of $\theta(t)$ in Ch. VI. It is defined for all points (π) on the domain $J(\pi) < b$, and is continuous on this domain. It is identical with the deformation $\theta_p(t)$ of Theorem 5.3 on the neighborhood N^{*p} of that theorem.

Lemma 5.2 of the present section is used in place of the Deformation Lemma of Ch. VI. With its aid one proves Theorem 6.1*. We restate Theorem 6.1* as follows.

THEOREM 5.4. *A maximal set of k-cycles on the domain $J < b$ of Π^p, independent on $J < b$, is afforded by maximal sets of critical, linking, and invariant k-cycles corresponding to an admissible pair of neighborhoods V^pW^p of the critical set σ^p.*

Theorem 5.3, characterizing $\theta_p(t)$, and Theorem 5.4, together with the deformation $\Lambda_p(t)$, will be frequently used in the sequel.

Critical sets on Ω

6. Let σ be a critical set on Ω on which $J = c$. Let σ^p be the component of σ on Π^p. Corresponding to those integers p for which $\rho p > c$ let N^p be an arbitrary neighborhood of σ^p on Π^p. When $\rho p \leqq c$ we shall understand that N^p is null. The set of neighborhoods

$$N^3, N^4, \cdots \tag{6.1}$$

will be termed a *neighborhood* N of σ on Ω. The neighborhood N^p in (6.1) will be termed the component of N of *index p*. A neighborhood N of σ will be termed arbitrarily small if the components N^p of N can be taken as arbitrarily small neighborhoods of their respective sets σ^p. If X and N are two neighborhoods of σ such that

$$X^p \subset N^p \qquad (p = 3, 4, \cdots), \tag{6.2}$$

we shall write $X \subset N$.

Let N^* designate the neighborhood of σ on Ω whose component on Π^p for $p\rho > c$ is the neighborhood N^{*p} of σ^p, defined in §5. Let X be an arbitrary neighborhood of σ on Ω such that

$$X \subset N^*. \tag{6.3}$$

Corresponding to X, a neighborhood $M(X)$ of σ on Ω will be chosen with the following property.

(A). *Let p be any positive integer, and q any integral multiple of p. The component $M^q(X)$ shall be so small that any point (π) which lies on $M^p(X)$, and possesses a partition on $M^q(X)$, will be deformed under $\theta_p(t)$ through points which possess partitions on X^q.*

We do not exclude the case where $p = q$. When $p = q$ the preceding condition means that $M^p(X)$ shall be so small that any point (π) on $M^p(X)$ will be deformed under $\theta_p(t)$ on X^p. This part of the condition on $M^p(X)$ is similar to the condition on $M^p(X)$ of §5.

It is clear that the neighborhoods $M^q(X)$ can be successively chosen in the order of the integers q so as to satisfy the preceding conditions. With this understood we state an analogue of Corollary 3.1 of Ch. VI. Entirely new considerations enter into its proof.

THEOREM 6.1. *If X is an arbitrary neighborhood of σ on N^*, any k-cycle z on $M(X)$ (below c) is Ω-homologous on X (below c) to a k-cycle (below c) on an arbitrarily small neighborhood N of σ.*

Corresponding to an arbitrarily small neighborhood N of σ there exists a neighborhood N_0 of σ whose components are so small that any k-cycle z on N_0, such that $z * 0$ on $M(N^*)$ (below c) "with index" q, will have the property that $z * 0$ on N (below c) with index q.

To see that the k-cycle z on $M(X)$ is Ω-homologous on X to a cycle on N we have merely to apply the deformations $\theta_p(t)$ to the corresponding components z^p of z, continuing $\theta_p(t)$ up to a suitable time t, $0 \leqq t < 3$, dependent on N^p. The cycle z^p will thereby be homologous (below c) to a cycle on N^p (below c).

We shall now prove the second statement of the theorem.

To that end let N_0 be a neighborhood of σ for which N_0^p is so small that it is deformed only on N^p under $\theta_p(t)$. For this choice of N_0 we shall prove that the second statement of the theorem is true.

We suppose then that z is a k-cycle on N_0 such that

$$z * 0 \qquad [\text{on } M(N^*)]. \tag{6.4}$$

As we have seen in §2 the Ω-homology (6.4) can be obtained by the formal addition of a set of ordinary homologies

$$u^p \sim 0 \qquad [\text{on } M(N^*),\ p = 3, 4, \cdots] \tag{6.5}$$

and a set of μ special homologies of the form

$$v^{p_i} * w^{q_i} \qquad [\text{on } M(N^*),\ i = 1, \cdots, \mu]. \tag{6.6}$$

As we have noted, all but a finite set of the cycles u^p are null. Moreover, as in §2, we have

$$z^p = u^p + \delta_p^{p_i} v^{p_i} + \delta_p^{q_i} w^{q_i} \qquad (i = 1, \cdots, \mu), \tag{6.7}$$

summing with respect to i.

Suppose that $p_i > q_i$ in (6.6). The cycle v^{p_i} is then a partition of the cycle w^{q_i}. By virtue of the conditions (A) on $M(N^*)$, the k-cycle w^{q_i} can be deformed under $\theta_{q_i}(t)$ on N^{*q_i} into a k-cycle arbitrarily near σ^{q_i} in such a fashion that the partition of w^{q_i} remains on N^{*p_i}. We accordingly infer the existence of homologies

$$\begin{aligned} w^{q_i} &\sim w_0^{q_i} && (\text{on } N^{*q_i}), \\ v^{p_i} &\sim v_0^{p_i} && (\text{on } N^{*p_i}), \end{aligned} \tag{6.8}$$

in which the k-cycle $v_0^{p_i}$ is a partition of $w_0^{q_i}$, and the cycles $v_0^{p_i}$ and $w_0^{q_i}$ lie on N_0. We record the special homologies

$$v_0^{p_i} * w_0^{q_i} \qquad (\text{on } N_0,\ i = 1, \cdots, \mu). \tag{6.9}$$

If we set

$$u_0^p = u^p + \delta_p^{p_i}(v^{p_i} - v_0^{p_i}) + \delta_p^{q_i}(w^{q_i} - w_0^{q_i}) \qquad (i = 1, \cdots, \mu), \tag{6.10}$$

summing with respect to i, we see that (6.7) can be given the form

$$z^p \equiv u_0^p + \delta_p^{p_i} v_0^{p_i} + \delta_p^{q_i} w_0^{q_i}. \tag{6.11}$$

From (6.11) it appears that u_0^p lies on N_0^p, since the remaining cycles in (6.11) lie on N_0^p. From (6.5), (6.8) and (6.10) we see that

$$u_0^p \sim 0 \qquad (\text{on } N^{*p}). \tag{6.12}$$

But from (6.11) it follows that the homology (6.4) may be regarded as generated by the homologies (6.12) and the special homologies (6.9). Observe that the cycles in these generating homologies all lie on N_0, and that an index of the original homologies (6.5) and (6.6) is also an index of the homologies (6.9) and (6.12).

It remains to show that the homologies (6.12) imply homologies

$$u_0^p \sim 0 \qquad (\text{on } N^p). \tag{6.12$'$}$$

To that end let v^p denote the chain on N^{*p} bounded by u_0^p. The deformation $\theta_p(t)$ applied to v^p up to a suitable time t_0, will deform v^p into a chain x^p as near σ^p as we please, in particular into a chain on N^p. But this same deformation $\theta_p(t)$ will deform u_0^p, according to the choice of N_0, through a cycle w^p on N^p. Hence

$$x^p + w^p \to u_0^p \qquad \text{(on } N^p\text{)},$$

and (6.12)′ is established. The proof of the theorem is now complete.

By an *admissible pair* of neighborhoods VW of σ we mean neighborhoods such that

$$V \subset M(N^*), \qquad W \subset M(V).$$

We shall understand that a given pair of neighborhoods VW is admissible unless otherwise stated.

By a spannable k-cycle, corr VW, we shall mean a k-cycle on W, below c, *bounding on W but not Ω-homologous to zero on V below c.*

By a critical k-cycle, corr VW, we shall mean a k-cycle on W, not Ω-homologous on V to a k-cycle on V below c.

We call attention to the fact that the distinction between bounding relations and Ω-homologies makes a real difference in the above definition of spannable k-cycles. We shall prove in §9 that the number of k-cycles in maximal sets of critical and spannable k-cycles, corr VW, is finite. We shall use this fact in the remainder of this section.

Theorem 6.1 leads readily to the following theorem.

THEOREM 6.2. *Corresponding to two admissible pairs of neighborhoods VW and $V'W'$ of σ, there exist common maximal sets of spannable and critical k-cycles on an arbitrarily small neighborhood of σ.*

By a *linkable* k-cycle corr VW we mean a spannable k-cycle, corr VW, which bounds a chain on Ω below c. The present theory here departs from the earlier theory in that in §5 a linkable k-cycle corr V^pW^p could be defined either as above, or as a spannable k-cycle, corr V^pW^p, homologous to zero below c. In §5 these two definitions would have been equivalent. In the present theory however we cannot replace the condition of bounding below c, by the condition of being Ω-homologous to zero below c.

Let l be a linkable $(k-1)$-cycle corr VW. We now formally define a k-cycle λ *linking* l, as in Ch. VI. The components of λ and l on Π^p are denoted by λ^p and l^p respectively. I say that there is at least one integer p for which λ^p links l^p on Π^p corr V^pW^p, in the sense of §5. For λ^p could fail to be linking in this sense only if l^p bounded on V^p below c. If, for each integer p, l^p bounded on V^p below c, the cycle l would bound on V below c, contrary to hypothesis. Thus a linking cycle λ has at least one component λ^p which is linking corr V^pW^p on Π^p in the sense of §5.

On the other hand a cycle l^p on Π^p which is linkable corr V^pW^p, in the sense of

§5, need not be linkable on Ω corr VW, since l^p may be Ω-homologous to zero on V below c, and hence not be spannable corr VW.

An *invariant* k-cycle corr VW will be defined as a k-cycle on Ω below c, Ω-independent below c of spannable k-cycles corr VW. Invariant linking, and critical k-cycles corr VW, on Ω, will be *distinguished* from cycles which are invariant, linking, and critical k-cycles on Π^p, corr V^pW^p, in the sense of §5, by the use of the qualifying phrase corr VW, instead of the phrase corr V^pW^p, used in §5.

With this understood we now establish a basic lemma by means of which the invariant, linking, and critical cycles on Π^p, corr V^pW^p, can be expressed in terms of maximal sets of cycles, corr VW. In this lemma we shall refer to the domain on Ω which consists of points (π) below c as the domain L. The lemma follows.

LEMMA 6.1. (a). *A k-cycle on Π^p which is an invariant k-cycle corr V^pW^p, is Ω-homologous on $V + L$ to a linear combination of invariant k-cycles corr VW.*

(b). *A k-cycle on Π^p which is a critical k-cycle corr V^pW^p is Ω-homologous on $V + L$ to a linear combination of invariant and critical k-cycles corr VW.*

(c). *A k-cycle on Π^p which is a linking k-cycle corr V^pW^p, is Ω-homologous on $V + L$ to a linear combination of linking, critical and invariant k-cycles corr VW.*

Statement (a) is true of an invariant k-cycle corr V^pW^p, because it is true more generally of any k-cycle on Ω below c. This follows at once from the definition of an invariant k-cycle corr VW.

Statement (b) is true of a critical k-cycle corr V^pW^p, because it is true more generally of any k-cycle on W. This follows from the definition of a critical k-cycle corr VW.

We come therefore to the proof of statement (c). We suppose z^p is a k-cycle on Π^p which is a linking cycle corr V^pW^p. The k-cycle z^p links a $(k - 1)$-cycle u^p, corr V^pW^p, by hypothesis. If u^p is not Ω-homologous to zero on V below c, z^p is a linking k-cycle corr VW, by virtue of the definition of such cycles, and statement (c) is true.

It remains to prove that statement (c) is true when

$$u^p * 0 \qquad \text{(on } V\text{, below } c). \tag{6.13}$$

Suppose q is an index of the homology (6.13), that is a multiple of the indices involved in a set of homologies generating (6.13). Let N be a neighborhood of σ which is so small that when r is a divisor of q, points (π) on N^r have partitions on W^q. Corresponding to N let N_0 be a neighborhood of σ chosen as in Theorem 6.1.

Let the cycle z^p be deformed under $\Lambda_p(t)$ into a cycle z_0^p on $N_0 + L$. Suppose that u^p is thereby deformed into a $(k - 1)$-cycle u_0^p. Since z^p lies on $W^p + L^p$, and u^p lies on W^p below c, these deformations imply the respective homologies

$$z^p \sim z_0^p \qquad \text{(on } V + L), \tag{6.14}$$

$$u^p \sim u_0^p \qquad \text{(on } V\text{, below } c). \tag{6.15}$$

From (6.13) and (6.15) we see that

$$u_0^p * 0 \qquad \text{(on } V\text{, below } c\text{).} \tag{6.16}$$

According to our choice of N_0, (6.16) implies an Ω-homology

$$u_0^p * 0 \qquad \text{(on } N\text{, below } c\text{).} \tag{6.17}$$

By virtue of the choice of N, z_0^p and u_0^p possess partitions z_1^q and u_1^q, on $W^q + L^q$, as do the chains involved in the homologies generating (6.17). Hence (6.17) implies that

$$u_1^q \sim 0 \qquad \text{(on } W^q\text{, below } c\text{).} \tag{6.18}$$

But z_1^q is the sum of a k-chain on W^q, and a chain below c, with u_1^q as the common boundary. It follows from (6.18), that for suitable integers m, n,

$$z_1^q \sim mc^q + ne^q \qquad \text{(on } V^q + L^q\text{)} \tag{6.19}$$

where c^q is a critical k-cycle corr V^qW^q, and e^q is a k-cycle on Π^q below c. The special homology

$$z_0^p * z_1^q \qquad \text{(on } V + L\text{)}$$

and the homologies (6.19) and (6.14) combine into the Ω-homology

$$z^p * mc^q + ne^q \qquad \text{(on } V + L\text{).}$$

The proof of the lemma is now complete.

Let maximal sets of linking, critical, and invariant k-cycles corr VW be represented by

$$(\lambda)_k, \qquad (c)_k, \qquad (i)_k, \tag{6.20}$$

respectively. It will follow from the results of the next two sections that the number of cycles in these sets is finite. Let q be a multiple of the indices of these cycles. Let $N \subset W$ be a neighborhood of σ which is so small that for integers p which are divisors of q the components N^p of N possess partitions on W^q. Let each non-null component of the cycles (6.20) be deformed under the deformations $\Lambda_p(t)$ of the same index, into a cycle on $N + L$. The sets (6.20) will thereby be replaced by sets

$$(\lambda)_k', \qquad (c)_k', \qquad (i)_k', \tag{6.21}$$

which remain maximal sets of linking, critical, and invariant k-cycles corr VW. Finally let each cycle in the sets (6.21) be replaced by its partition on Π^q, and the resulting maximal sets be denoted by

$$(\lambda)_k^*, \qquad (c)_k^*, \qquad (i)_k^*. \tag{6.22}$$

The resulting cycles will be Ω-homologous to the cycles which they replace and will again constitute maximal sets of linking, critical, and invariant k-cycles corr VW. These final cycles will each possess but one component which is not null, namely a component on Π^q.

Our second lemma is the following.

LEMMA 6.2. *There exist maximal sets of linking, critical, and invariant k-cycles corr VW, all of whose components are null, save their respective components on a domain Π^q with suitable integer q. These sets form subsets respectively of maximal sets of linking, critical, and invariant k-cycles, corr V^qW^q, on Π^q.*

The first statement of the lemma has already been proved, the corresponding sets being represented by (6.22).

To prove the second statement of the lemma let λ^q be a sum of the qth components of a subset of the linking cycles in (6.22). Let u^q be the sum of the $(k - 1)$-cycles on W^q linked by the respective k-cycles in the sums λ^q. The cycle u^q cannot bound below c on V^q, because it would then not be spannable corr VW. Hence λ^q is a linking k-cycle corr V^qW^q, and the lemma is established for linking cycles.

Let c^q be a sum of the qth components of any subset of the given critical k-cycles in (6.22). The cycle c^q cannot be dependent on V^q upon k-cycles below c, because c^q would then be Ω-dependent on V on k-cycles below c, and fail to be a critical k-cycle corr VW. Hence c^q is a critical k-cycle corr V^qW^q, and the lemma is established for critical cycles.

Let u^q be a sum of the qth components of any subset of the given invariant k-cycles of (6.22). If u^q were not an invariant k-cycle corr V^qW^q, we would have an homology of the form

$$u^q \sim v^q \qquad \text{(on } \Pi^q \text{ below } c) \tag{6.23}$$

in which v^q would be a spannable k-cycle corr V^qW^q or null. If

$$v^q * 0 \qquad \text{(on } V \text{ below } c), \tag{6.24}$$

we would have $u^q * 0$ below c, contrary to the nature of u^q as an invariant k-cycle corr VW. If on the other hand (6.24) does not hold, v^q is a spannable k-cycle corr VW, and (6.23) is contrary to the nature of u^q as an invariant k-cycle corr VW. Hence (6.23) cannot hold, and u^q is an invariant k-cycle corr V^qW^q. The lemma is accordingly established for invariant k-cycles.

Suppose that σ is the complete set of critical points on Ω on which $J = c$. Let b be an ordinary value of J such that b is greater than c, and separates c from greater critical values of J. The following theorem can now be established. It depends upon the two preceding lemmas.

THEOREM 6.3. *A maximal set of k-cycles on the domain $J < b$ of Ω, Ω-independent on $J < b$, is afforded by maximal sets of critical, linking, and invariant k-cycles, corresponding to an admissible pair of neighborhoods VW of the critical set σ.*

Let the sets (6.20) respectively represent the maximal sets of linking, critical, and invariant k-cycles of the theorem. We shall first prove that any k-cycle z

on the domain $J < b$ of Ω, is Ω-homologous on $J < b$ to a linear combination of cycles of the sets (6.20).

Let q be the index of z, and let u^q be the partition of z on Π^q. According to Theorem 5.4, u^q will be homologous on the domain $J(\pi) < b$ of Π^q to a linear combination of linking, critical, and invariant k-cycles corr V^qW^q. But according to Lemma 6.1, linking, critical, and invariant k-cycles corr V^qW^q are Ω-homologous on $J < b$ to linear combinations of the cycles in the maximal sets (6.20) of the theorem.

It remains to prove that the cycles in (6.20) are Ω-independent on $J < b$.

In the contrary case there would exist a sum w of k-cycles of (6.20) such that

$$w * 0 \qquad (\text{on } J < b). \tag{6.25}$$

Let q be a multiple of the indices of the cycles (6.20) and of the chains "involved" in (6.25) For this q let the respective cycles in (6.20) be replaced by the cycles (6.22), that is, by Ω-homologous linking, critical, and invariant k-cycles corr VW, each with all components null save one on Π^q. Let w_0 be the sum of the cycles in (6.22) which correspond respectively to the cycles of (6.20) in the sum w. We have

$$w_0 \sim 0 \qquad (\text{on } \Pi^q,\ J < b). \tag{6.26}$$

By virtue of Lemma 6.2, the components on Π^q of the cycles (6.22) form subsets of the maximal sets of linking, critical, and invariant k-cycles on Π^q corr V^qW^q. It follows from Theorem 5.4 that an homology such as (6.26) is impossible. Hence (6.25) is impossible, and the cycles of the sets (6.20) are Ω-independent on $J < b$.

The proof of the theorem is now complete.

For reasons which we have given at length in Ch. VI and Ch. VII, we now define the kth *type number* m_k of a critical set σ as the number of critical k-cycles and spannable $(k - 1)$-cycles in maximal sets of such cycles corresponding to neighborhoods VW of the critical set σ. That a critical set with type numbers m_k, $k = 0, 1, \cdots$, can be considered equivalent to a set of non-degenerate closed extremals of J-length c, containing m_k, $k = 0, 1, \cdots$, closed extremals of index k, will be seen in §11.

From Theorem 6.3 and the definition of the type numbers of a critical set σ, we obtain the following theorem.

THEOREM 6.4. *Between the connectivities P_k of Ω and the sums N_k of the kth type numbers of all critical sets of extremals we have the relations*

$$N_k \geqq P_k \qquad (k = 0, 1, \cdots).$$

In particular if P_k is infinite, N_k is infinite.

This theorem parallels Theorem 10.2 of Ch. VII.

Theorems 10.3 and 10.4 of Ch. VII likewise hold here with the interpretations of the present chapter.

The proofs of these theorems with the interpretations of the present chapter, depend upon Theorem 6.3 and are similar to the corresponding proofs in Ch. VII.

Extremals determined by sets of cycles. In the preceding theory the critical set of extremals has come first, and has served to determine various sets of cycles on Ω. We here reverse the process and see how a set of Ω-independent k-cycles determines a minimal set of closed extremals.

We begin with several definitions.

Let σ be a critical set on Ω and VW an admissible pair of neighborhoods of σ on Ω. Let c be the value of J on σ. Let w be the sum of a k-cycle below c (possibly null) and a proper linear combination of the k-cycles of maximal sets of critical and linking k-cycles belonging to σ, corr VW. We term w a *new* cycle belonging to σ or to the set of closed extremals determined by σ. If w is Ω-homologous to no cycle below c or to no new cycle "belonging" to σ, w will be termed a *reduced* new cycle and σ the corresponding *reduced critical* set.

Let u be a k-cycle on Ω, not Ω-homologous to zero. There will exist a positive constant c such that u is Ω-homologous to no cycle below c, but is Ω-homologous to a k-cycle below $c + e$, where e is an arbitrarily small positive constant. There will be a critical set of closed extremals with J-lengths c. We term c the *minimum critical value* belonging to u. We understand that a cycle Ω-homologous to zero has no minimum critical value.

Corresponding to u there will be one or more reduced new k-cycles Ω-homologous to u, belonging to reduced critical sets with the minimum critical value c. The ensemble σ_u of the reduced critical sets with critical value c corresponding to all reduced new k-cycles Ω-homologous to u will be termed the *minimal* critical set determined by u. Let (u) now be a set of Ω-independent k-cycles. The ensemble of the minimal sets σ_u determined by all proper linear combinations u of cycles of (u) will be termed the minimal set H of critical points determined by (u).

The set K of closed extremals determined by points (π) on H will be termed the minimal set of closed extremals determined by (u).

Two sets of cycles (u) and (v) on Ω will be termed Ω-*equivalent* if every cycle Ω-dependent on cycles of (u) is Ω-dependent on cycles of (v) and conversely. It is clear from the preceding definitions that Ω-equivalent sets of k-cycles determine the same minimal sets of closed extremals.

We continue with the following theorem.

THEOREM 6.5. *The sum M_k of the kth type numbers of the critical sets in the minimal set of closed extremals determined by a finite set (u) of Ω-independent k cycles is at least the number p_k of cycles in the set (u).*

Let H be the minimal set of critical points (π) on Ω determined by (u). Let

$$c_1 < c_2 < \cdots < c_p \tag{6.27}$$

be the critical values assumed by J on H. Let σ_i be the subset of H on which

$J = c_i$ and let $(a)_i$ be a maximal set of new k-cycles belonging to σ_i. The ensemble of cycles in the sets

$$(6.28) \qquad (a)_1, \cdots, (a)_p$$

will be at most M_k in number. But by virtue of the definition of H each cycle of (u) is Ω-dependent on the cycles of (6.28), at least if the set (6.28) be suitably chosen. Hence

$$M_k \geqq p_k,$$

and the theorem is proved.

The extension of a chain on Π^p

7. The proof that the connectivities of the domain $J < b$ on Ω are finite depends upon certain novel consequences of our "special" homologies. We shall now develop this aspect of the theory. We begin with a number of definitions.

Deformation chains. The loci introduced by deformations of chains may be divided into simplicial cells in many ways. It is essential for our purposes that this division be made in a particular way which we shall now describe.

Let α_k be an auxiliary k-simplex, and t_1 the line segment $0 \leqq t \leqq 1$. We represent the product $\bar{\alpha}_k \times t_1$ by a right prism ζ in an auxiliary euclidean space. We suppose $\bar{\alpha}_k$ is the base of the prism, and that a point $Q = (p, t)$ on ζ is determined by giving the point p on $\bar{\alpha}_k$ into which Q projects, and the distance t of Q from $\bar{\alpha}_k$. To subdivide ζ into simplices, we first divide it into two prisms ζ' and ζ'' by the locus $t = 1/2$. We then divide the prisms, ζ' and ζ'', into simplices, first dividing their lateral prismatic faces in the order of dimensionality as follows. Let ζ^* be a prism whose lateral faces have already been divided into simplices. Let Z be the center of gravity of ζ^*. We divide ζ^* into the simplices which are determined by Z and the simplices on the boundary of ζ^*. In this way we arrive at a canonical subdivision of ζ.

Let a_k be a k-cell on a basic complex C, and F a continuous deformation of $\bar{a}_k$ on C. We can suppose that the deformation F is defined by giving a continous point function $F(p, t)$ of the point (p, t) on ζ. We understand thereby that $F(p, t)$ is a point on C, that $F(p, 0)$ defines $\bar{a}_k$, and that the point $F(p, 0)$ on $\bar{a}_k$ is replaced under the deformation F at the time t by the point $F(p, t)$. The point function $F(p, t)$ defines a map on C of each of the closed k-simplices of ζ. The sum of the resulting closed k-cells on C will be termed the *deformation chain a_{k+1} derived from a_k under the deformation F.*

The image under F of the prism ζ, unreduced on C mod 2, will be termed the *unreduced deformation chain* derived from a_k. In the sequel we shall apply various deformations D to deformation chains H. Inasmuch as these deformations D depend upon the unreduced deformation chains H for their definition, it is hereby understood that the operation of reduction mod 2 is deferred until after the deformations D are made.

The deformation chain a_{k+1} derived from a_k, as we have defined it, possesses the following basic property of symmetry. If the prism ζ be reflected in the hyperplane $t = 1/2$, and then mapped on C by means of the point function $F(p, t)$, the sum of the resulting images of the closed k-simplices of ζ, viewed as a k-chain on C, will be "identical" with a_{k+1}.

The deformation φ on K^p. We understand that a point (π) on K^p is given with a definite ordering of its vertices

$$P^1, \cdots, P^p. \tag{7.0}$$

The point (π) on K^p and a point (π) on Π^p will be said to be *K-images* of one another, if the vertices (7.0) taken in their circular order agree with the vertices of (π) on Π^p, taken in one of their two circular orders. We shall restrict ourselves to points (π) on K^p which define admissible elementary extremals. Let (π) be such a point. As the time increases from 0 to 1 let each vertex of (π) move along the elementary extremal which follows it on $g(\pi)$ at J-rate equal to the J-length of that elementary extremal. Denote the resulting deformation of K^p by φ.

The extension of a chain on K^p. We understand that a variable point

$$Q^1, \cdots, Q^p$$

on K^p has the point (7.0) as a limit point on K^p, only if for each integer i on the range $1, \cdots, p$, Q^i tends to P^i as a limit point. Cells and chains on K^p are defined on K^p with this notion of continuity. By the *extension* of a k-cell a on K^p we now mean the deformation chain on K^p derived from a under the deformation φ. We denote this chain by Ea. By the extension Ez of a k-chain z on K^p we mean the sum, mod 2, of the deformation chains derived from the k-cells of z. If we indicate the boundary of a chain by prefixing the letter B, we see that

$$BEa \equiv a + T_1 a + EBa \qquad (\text{on } K^p), \tag{7.1}$$

where $T_1 a$ denotes the image of a on K^p under the transformation T_1. For a chain z on K^p we then have

$$BEz \equiv z + T_1 z + EBz \qquad (\text{on } K^p). \tag{7.2}$$

The extension of a k-chain of cells of Π^p. Let e be any point or k-cell of Π^p and e' a point or k-cell of K^p which is a "K-image" of e. The cell e' will be unique if and only if it is pointwise invariant under each transformation of G^p. By the extension $\mathcal{E}e$ of e on Π^p we mean the K-image on Π^p of Ee'. One sees that $\mathcal{E}e$ is independent of the K-image e' on K^p used to define Ee'. This fact depends in part upon the symmetry of the deformation chain as we have defined it.

Let w be a k-chain of cells of Π^p. By the extension $\mathcal{E}w$ of w we mean the sum, mod 2, of the extensions of the k-cells of w on Π^p. From (7.1) we see that

$$B\mathcal{E}w = \mathcal{E}Bw \qquad (\text{on } \Pi^p) \tag{7.3}$$

since a and $T_1 a$ in (7.1) have identical K-images on Π^p.

In proving the theorem of this section we shall make use of a deformation η. In defining η it will be convenient to denote an elementary extremal with end points Q', Q'' by $(Q' Q'')$.

The deformation η. Let there be given a sequence of $r + 1$ points

$$Q^1, \cdots, Q^{r+1} \qquad (r > 1) \tag{7.4}$$

on R such that the sum of the J-lengths of the elementary extremals

$$(Q^1Q^2), \cdots, (Q^rQ^{r+1}) \tag{7.5}$$

is at most the constant ρ of §2, Ch. VII. The deformation η shall be so defined as to hold the points Q^1 and Q^{r+1} fast, and deform the points

$$Q^2, \cdots, Q^r \tag{7.6}$$

into that sequence of $r - 1$ points on the extremal

$$(Q^1Q^{r+1}) \tag{7.7}$$

which divide this extremal into r segments of equal J-length. The deformation η shall also be such that the points (7.6) will be deformed in the same manner if the points (7.4) are relettered in the inverse order.

The deformation η can be defined as follows.

Let h denote the broken extremal formed by the sequence of elementary extremals in (7.5). We begin by defining a *deformation* η_0 of the curve h into the extremal (7.7). Let P be the point which divides h equally with respect to J-length. As the time t increases from 0 to 1 let two points P_1 and P_2 move away from Q on h towards Q^1 and Q^{r+1} respectively, at J-rates equal to half the J-length of h. At the time t let the point on the segment of h between P_1 and P_2 which divides that segment in a given J-ratio, be replaced by that point on (P_1P_2) which divides (P_1P_2) in the same J-ratio. Under η_0, h will be deformed into the extremal (7.7).

We now use the deformation η_0 to define the deformation η. To that end let h_t be the curve which replaces h at the time t in the deformation η_0, $0 \leqq t \leqq 1$. To define η we replace the sequence of points (7.6) at the time t by the sequence of points on h_t which divide h_t in the same J-ratios as the points (7.6) divide h at the time $t = 0$. We thereby deform the points (7.6) into a sequence of points on the extremal (7.7). This last sequence of points is finally deformed along the extremal (7.7) into a sequence of points which divide the extremal (7.7) into segments of equal J-length, each point moving at a J-rate equal to the J-length to be traversed. The deformation η is thereby defined.

We now come to an important consequence of the introduction of our special homologies.

THEOREM 7.1. *If the extension* $\mathcal{E}z$ *of a chain of* n*-cells of* Π^p *is an* $(n + 1)$*-cycle, it satisfies the* Ω*-homology*

$$\mathcal{E}z * 0. \tag{7.8}$$

If z *is below* c, *the homology* (7.8) *holds below* c.

We shall establish this theorem with the aid of a deformation ψ on K^{2p}, and the K-image ψ_1 of ψ on Π^{2p}. The r-fold partition of a chain w on K^p or Π^p, respectively, will be denoted by

$$p^r w.$$

The deformation ψ *on* K^{2p}. Let (π) be an admissible point on K^p. The extension $E\pi$ of (π) on K^p may be regarded as a curve on K^p. Consider the curve

$$p^2 E\pi \qquad \text{(on } K^{2p}\text{)}$$

as the continuous image of $E\pi$. The deformation ψ is a deformation of $p^2E\pi$ on K^{2p}. The time in the deformation ψ will be denoted by τ (not t).

The curve $E\pi$ is the trajectory traced by (π) under the deformation φ of this section. Let (π_t) be the point thereby replacing (π) at the time t, $0 \leqq t \leqq 1$. The odd vertices of the partition $p^2\pi_t$ coincide with the successive vertices of (π_t), and thus lie on $g(\pi)$. The even vertices of $p^2\pi_t$ do not in general lie on $g(\pi)$. The object of the deformation ψ is to deform $p^2\pi_t$ on K^{2p}, so as to hold its odd vertices fast, and deform its even vertices onto $g(\pi)$, carrying $p^2\pi_t$ into a final image (π_t^g) which we will now describe.

The final image (π_t^g) *of* $p^2\pi_t$. Let h' and h'' be two successive elementary extremals of $g(\pi)$. Let h' and h'' be bisected in J-lengths by points P' and P'' on R. When $t = 0$ one of the even vertices P_t of $p^2\pi_t$ will coincide with P'. As t increases from 0 to 1, this vertex P_t will move from P' to P'', but not in general on $g(\pi)$. Let P^* be the common end point of h' and h''. The final image Q_t of P_t under ψ will now be defined as follows. For t fixed on the interval

$$0 \leqq t \leqq \tfrac{1}{2}, \tag{7.9}$$

Q_t shall be the point on the elementary extremal $(P'P^*)$ which divides $(P'P^*)$ in the same J-ratio as the ratio in which t divides the interval (7.9). For t fixed on the interval

$$\tfrac{1}{2} \leqq t \leqq 1, \tag{7.10}$$

Q_t shall be the point on the elementary extremal (P^*P'') which divides (P^*P'') in the same J-ratio as the ratio in which t divides the interval (7.10). The final image (π_t^g) of $p^2\pi_t$ is thereby defined.

We can deform P_t into the corresponding final vertex Q_t, and thus deform $p^2\pi_t$ into (π_t^g). To that end let M' and M'' be the odd vertices common to $p^2\pi_t$ and (π_t^g) between which P_t and Q_t lie on $p^2\pi_t$ and (π_t^g) respectively. The vertex P_t lies on the elementary extremal $(M'M'')$. We use the inverse of the deformation η to deform P_t into Q_t, holding M' and M'' fast. The point $p^2\pi_t$ is thereby deformed into the point (π_t^g), and the curve $p^2E\pi$ into a curve $E^g\pi$, all of whose vertices lie on $g(\pi)$.

The definition of the deformation ψ is now complete. We observe that it deforms points on K^{2p} so as not to increase the value of J on the corresponding broken extremal.

The preceding deformation ψ on K^{2p} has a deformation ψ_1 as its K-image on Π^{2p}.

The deformation ψ_1 *on* Π^{2p}. Suppose the preceding point (π) on K^p is the K-image of a point (π_1) on Π^p. The curve $E\pi$ on K^p will have $\mathcal{E}\pi_1$ as its K-image on Π^p. The K-image of the curve $p^2E\pi$ on K^{2p} will be the curve $p^2\mathcal{E}\pi_1$ on Π^{2p}. The deformation ψ will have as its K-image on Π^{2p} a deformation ψ_1 of $p^2\mathcal{E}\pi_1$ into the K-image on Π^{2p} of $E^g\pi$. Denote this K-image of $E^g\pi$ on Π^{2p} by $\mathcal{E}^g\pi$. The deformation ψ_1 thereby defined will be independent of the particular K-image of (π_1) used to define ψ on K^{2p}.

The curve $p^2\mathcal{E}\pi_1$ and the curves replacing $p^2\mathcal{E}\pi_1$ under ψ_1 are closed curves on Π^{2p}. Moreover the final curve $\mathcal{E}^g\pi_1$, regarded as a 1-chain on Π^{2p}, *reduces to zero*, mod 2. This follows from the fact that on K^{2p} the application of T_1 to the 1-cell traced by (π^g_t) as t increases from 0 to $\frac{1}{2}$, yields the 1-cell traced by (π^g_t) as t increases from $\frac{1}{2}$ to 1.

With the deformation ψ_1 defined as above, the main body of the proof of Theorem 7.1 can be incorporated in the following lemma.

LEMMA 7.1. *The 2-fold partition $p^2\mathcal{E}z$ of the cycle $\mathcal{E}z$ of the theorem, can be J-deformed on Π^{2p} into a set of n-cells whose sum reduces to zero*, mod 2.

To prove this lemma we regard each point of z as typified by the point (π_1) on Π^p used in the definition of ψ_1. We then regard $p^2\mathcal{E}z$ "unreduced" as a locus of the curves $p^2\mathcal{E}\pi_1$, and the cells of $p^2\mathcal{E}z$ as the images of the cells of the deformation chain $\mathcal{E}z$, each point on $\mathcal{E}z$ corresponding to its 2-fold partition on $p^2\mathcal{E}z$. But under ψ_1 each curve $p^2\mathcal{E}\pi_1$ is deformed on Π^{2p} into a set of 1-cells which sum, mod 2, to zero. By virtue of our canonical division of a deformation chain it follows that the set of $(n + 1)$-cells into which $p^2\mathcal{E}z$ is thereby deformed on Π^{2p} likewise sum, mod 2, to zero. The proof of the lemma is now complete.

To turn to the theorem, recall that

$$\mathcal{E}z * p^2\mathcal{E}z \tag{7.11}$$

by virtue of our special homologies. But according to the preceding lemma,

$$p^2\mathcal{E}z \sim 0 \qquad (\text{on } \Pi^{2p}). \tag{7.12}$$

From (7.11) and (7.12) we see that (7.8) holds as stated.

The r-fold join of a cycle

8. The r-fold partition of a point (π) on Π^p always exists. It is a point on Π^{rp}. On the other hand an arbitrary point on Π^{rp} is not in general the r-fold partition of any point on Π^p. The process of taking the partition of a point on Π^p does not then admit an inverse applicable to all points on Π^{rp}. Nevertheless there is another process applicable to a limited class of chains on Π^{rp} which for our purposes takes the place of an inverse of a partition. The chain on Π^p which is thereby made to correspond to the given chain z^{rp} on Π^{rp} is termed the *join*

of z^{rp} on Π^p. In this section we shall define and analyse the join of a chain on Π^{rp}. The results obtained are fundamental in our final theory of Ω-homologies.

The complex Θ. Let r be an integer greater than 1. Let G_r^q denote the subgroup of G^q generated by the transformations T_r and U_0. The complex Π^q was formed from K^q by identifying the cells of K^q which were the images of one another under transformations of the group G^q. Let Θ be the complex similarly formed from K^q by identifying the cells of K^q under the transformations of the group G_r^q.

A point (π) on Θ represents a class of points (π') on K^q obtainable from any member of the class by means of the transformations of G_r^q. Points of the class (π') on K^q and the corresponding point (π) on Θ will be termed K-images of one another. Points (π) and (π'') on Θ and Π^q respectively which possess a common K-image on K^q will likewise be termed K-images of one another.

The r-fold join on Π^p of a k-cycle z^{rp} on Π^{rp} will be defined only for those k-cycles which satisfy the following two conditions.

A. *The k-cycle z^{rp} on Π^{rp} shall be the K-image, reduced* mod 2, *of a k-cycle w^{rp} on Θ.*

B. *The points (π) on z^{rp} shall determine curves $g(\pi)$ on which the J-lengths of r successive elementary extremals is at most ρ.*

To define the r-fold join of z^{rp} let (π) be a point on w^{rp}, and

$$P^1, P^2, \cdots, P^q \qquad (q = rp) \tag{8.1}$$

the vertices of a point on K^{rp} which is the K-image of (π). By arbitrarily preferring the vertices

$$P^r, P^{2r}, \cdots, P^{rp} \tag{8.2}$$

we obtain a point on K^p. Let $\varphi(\pi)$ denote the K-image on Π^p of the point (8.2) on K^p. The point $\varphi(\pi)$ is uniquely determined by (π). That is, it does not depend upon the particular K-image (8.1) which is selected to represent (π) on K^{rp}. For any other K-image on K^{rp} of (π) would be obtainable from the point (8.1) by applying a transformation of the form T_{mr} or $T_{mr}U_0$ to the point (8.1), and would accordingly lead to the same point $\varphi(\pi)$ on Π^p. It follows that the points $\varphi(\pi)$ form a continuous image on Π^p of w^{rp}. Reduced, mod 2, this image is a k-cycle on Π^p which we term an *r-fold join of z^{rp} determined by w^{rp}.*

We shall now prove the following theorem.

THEOREM 8.1. *A k-cycle z^{rp} which possesses a join z^p on Π^p satisfies the Ω-homology*

$$z^{rp} * z^p. \tag{8.3}$$

This Ω-homology can in particular be realized by using the deformation η to deform z^{rp} into the r-fold partition $p^r z^p$ of z^p on Π^{rp}.

Let (π) be a point on w^{rp} and (π') the corresponding point $\varphi(\pi)$ on a join z^p of z^{rp}. Let P' and P'' be two successive vertices of (π'). Holding P' and P'' fast we can use the deformation η of §7 to deform the vertices of (π) between P' and P'' into the correspondingly ordered vertices of $p^r\pi'$ between P' and P''. We thus have

$$z^{rp} \sim p^r z^p \qquad (\text{on } \Pi^{rp})$$

and

$$z^p * p^r z^p$$

from which (8.3) follows as stated.

We shall say that an n-cycle z^q is *simple* if it possesses the following properties. Each $(n-1)$-cell is incident with just two n-cells. The n and $(n-1)$-cells incident with an $(n-2)$-cell form a circular sequence in which n and $(n-1)$-cells alternate, and in which each n-cell is incident with the preceding and following $(n-1)$-cells and no other $(n-1)$-cells of the sequence.

We have given conventions under which the cells of a sum shall be regarded as identical. If these conventions are made optional so that cells previously regarded as identical may or may not be regarded as identical at pleasure, the resulting chain will be termed an *unreduced* chain. With this understood it is clear that any n-cycle can be replaced by an unreduced n-cycle which consists of the same n-cells, but for which the conventions of identity relating to the cells of lower dimensionality have been so altered that the new cycle is simple.

We now state an important lemma.

LEMMA 8.1. *Let z^q be a "simple" k-cycle of cells of Π^q, $q = rp$, no cell of which has a K-image on K^q invariant under a transformation of G^q other than a power of T_{2r}. There then exists an unreduced $(k-1)$-chain y^q of cells of z^q such that the chain*

$$z^q + \mathfrak{L} y^q \qquad (on\ \Pi^q), \tag{8.4}$$

unreduced mod 2, *is the K-image of a cycle w^q on Θ.*

We shall first define the k-chain w^q on Θ, and after deriving certain properties of w^q obtain a chain y^q with the required properties.

Definition of w^q. Corresponding to each closed k-cell $\bar{a}_k$ of z^q let an arbitrary K-image $\bar{b}_k$ be chosen among the closed k-cells of K^q. We form the sum

$$u_k = \Sigma \bar{b}_k \qquad (\text{on } K^q) \tag{8.5}$$

of these cells. Let us denote the K-image on Θ of a chain z on K^q by Θz. The chain w^q will be defined as a sum

$$w^q = \Theta(u_k + Ev_{k-1}) \qquad (\text{on } \Theta) \tag{8.6}$$

where v_{k-1} is a $(k-1)$-chain on K^q still to be defined, and Ev_{k-1} is the extension of v_{k-1} on K^q.

The chain v_{k-1} shall consist of a sum of $(k-1)$-chains a^{v}_{k-1} on K^q, one corresponding to each $(k-1)$-cell a_{k-1} of z^q.

Definition of a_{k-1}^v. On the closed k-cells $\bar{b}_k$ in the sum (8.5) let b_k' and b_k'' denote the K-images of the two k-cells incident with a_{k-1} on z^q. Let b_{k-1}' and b_{k-1}'' be the K-images of a_{k-1} on the boundaries of b_k' and b_k'' respectively. There will then exist an integer m between 0 and $q-1$ inclusive, such that one of the two following relations holds:

(8.7) $$b_{k-1}'' = T_m b_{k-1}', \qquad \text{Case I;}$$

(8.8) $$b_{k-1}'' = U_0 T_m b_{k-1}', \qquad \text{Case II.}$$

We then define a_{k-1}^v by the congruence

(8.9) $$a_{k-1}^v \equiv T_0\bar{b}_{k-1}' + T_1\bar{b}_{k-1}' + \cdots + T_{m-1}\bar{b}_{k-1}' \qquad (\text{on } K^q)$$

understanding that the right member is null if $m = 0$.

For the sake of brevity we write (8.9) symbolically in the form

$$a_{k-1}^v \equiv (T_0 + \cdots + T_{m-1})\bar{b}_{k-1}'.$$

We introduce the symbol

$$T_i^j = T_i + T_{i+1} + \cdots + T_j,$$

understanding that T_i^j is null if $j < i$. We then write (8.9) in the form

(8.10) $$a_{k-1}^v \equiv T_0^{m-1}\bar{b}_{k-1}'.$$

The chain $\Theta E v_{k-1}$ in (8.6) shall consist of a sum of k-chains, each of the form

(8.11) $$\Theta E T_0^{m-1}\bar{b}_{k-1}' \qquad (\text{on } \Theta),$$

one corresponding to each $(k-1)$-cell a_{k-1} of z^q.

We shall now prove the following statement.

(a). *If one interchanges the rôles of* b_{k-1}' *and* b_{k-1}'' *in the definition of* a_{k-1}^v, *the chain* (8.11) *is unaltered,* mod 2.

We first suppose that Case I holds. Let us interchange the rôles of b_{k-1}' and b_{k-1}'' and put (8.7) in the form

$$b_{k-1}' = T_\beta b_{k-1}''.$$

Here $\beta = 0$ if $m = 0$. If $m = 0$, (a) is clearly true. If $m > 0$, $\beta = q - m$ and we proceed as follows.

The new chain (8.11) will be the chain

$$\Theta E T_0^{q-m-1}\bar{b}_{k-1}'',$$

which may be written in the form

(8.12) $$\Theta E T_m^{q-1}\bar{b}_{k-1}'$$

upon using (8.7). The chains (8.12) and (8.11) accordingly have the sum

(8.13) $$\Theta E T_0^{q-1}\bar{b}_{k-1}'.$$

Now two chains such as

$$\Theta E T_i \bar{b}'_{k-1}, \qquad \Theta E T_{i+r} \bar{b}'_{k-1} \tag{8.14}$$

are equal, mod 2, on Θ. But q is an even multiple of r by hypothesis, and the chain (8.13) accordingly involves $q/2$ pairs of chains such as the pair (8.14). Thus (8.13) reduces to zero, mod 2, and statement (a) is proved in Case I.

In Case II, we first rewrite (8.8) in the form

$$b'_{k-1} = U_0 T_m b''_{k-1} \tag{8.15}$$

as is possible since $U_0 T_m$ is its own inverse. Having thus interchanged the rôles of b'_{k-1} and b''_{k-1}, the chain (8.11) is replaced by the chain

$$\Theta E T_0^{m-1} \bar{b}''_{k-1}. \tag{8.16}$$

Upon using (8.8), chain (8.16) takes the form

$$\Theta E T_0^{m-1} U_0 T_m \bar{b}'_{k-1}. \tag{8.17}$$

To reduce (8.17) to the form (8.11), observe that

$$T_i U_0 = U_0 T_{-i}$$

so that (8.17) becomes

$$\Theta E U_0 T_1^m \bar{b}'_{k-1}. \tag{8.18}$$

Reference to the definition of E shows that if w is any chain on K^q,

$$E U_0 w \equiv U_0 E T_{-1} w, \tag{8.19}$$

so that (8.18) takes the form

$$\Theta U_0 E T_0^{m-1} \bar{b}'_{k-1}. \tag{8.20}$$

Finally for any chain w on K^q

$$\Theta U_0 w = \Theta w,$$

so that (8.20) reduces to the chain

$$\Theta E T_0^{m-1} \bar{b}'_{k-1}, \tag{8.21}$$

and is thus equal to the chain (8.11) as stated.

The proof of (a) is now complete.

To replace a cell w on K^q by $U_0 w$ will be termed changing the *sense* of w. We shall now prove the following.

(b). *The senses of the cells b'_k and b''_k can be separately or jointly changed at pleasure, without changing the chain* (8.11) *corresponding to a_{k-1}.*

We shall first establish (b) for the case in which the cells b'_k, b''_k are replaced by the cells

$$\beta'_k = b'_k, \qquad \beta''_k = U_0 b''_k.$$

The cells b'_{k-1} and b''_{k-1} will then be replaced by the cells

$$\beta'_{k-1} = b'_{k-1}, \qquad \beta''_{k-1} = U_0 b''_{k-1}. \tag{8.22}$$

Suppose now that Case I holds for b'_{k-1} and b''_{k-1}. From (8.7) and (8.22) we see that

$$\beta''_{k-1} = U_0 T_m \beta'_{k-1},$$

so that Case II holds for β'_{k-1} and β''_{k-1}. One sees that the chain (8.11) remains unchanged.

Suppose now that Case II holds for b'_{k-1} and b''_{k-1}. From (8.8) and (8.22) we find that

$$\beta''_{k-1} = T_m \beta'_{k-1}$$

and we see that the chain (8.11) is again unchanged.

Finally I say that all other changes of sense of b'_k and b''_k reduce to the cases just considered. For by virtue of (a) the rôles of b'_k and b''_k can be interchanged without changing (8.11). Moreover to change the senses of b'_k and b''_k jointly it is sufficient to change their senses separately in succession, thereby producing no change, mod 2, in (8.11).

Statement (b) is thus established.

Let z be a chain on a given domain. The boundary of z on the same domain will be denoted by Bz. With this understood we shall prove the following statement.

(c). *The boundary of the chain w^q on Θ is the cycle*

$$Bw^q \equiv \Sigma^* \Theta E T_0^{m-1} B \bar{b}'_{k-1} \tag{8.23}$$

where the sum Σ^* contains one term corresponding to each $(k-1)$-cell a_{k-1} of z^q and where b'_{k-1} and m are determined with the aid of a_{k-1} as previously.

From (8.5), (8.6), and (8.11) we see that

$$w^q \equiv \Sigma \Theta \bar{b}_k + \Sigma^* \Theta E T_0^{m-1} \bar{b}'_{k-1}, \tag{8.24}$$

where the terms in the sums Σ and Σ^* correspond respectively to the k- and $(k-1)$-cells of z^q, and are summed for all these cells. We note the relation

$$B\Theta u \equiv \Theta B u \qquad \text{(on } \Theta\text{)} \tag{8.25}$$

where u is any k-chain on K^q. From (8.24) we then find that

$$Bw^q \equiv \Sigma \Theta B \bar{b}_k + \Sigma^* \Theta B E T_0^{m-1} \bar{b}'_{k-1}. \tag{8.26}$$

To evaluate the sum Σ^*, we refer to (7.2) and see that

$$\begin{aligned} BET_0^{m-1}\bar{b}'_{k-1} &\equiv T_0^{m-1}\bar{b}'_{k-1} + T_1^m \bar{b}'_{k-1} + EBT_0^{m-1}\bar{b}'_{k-1} \\ &\equiv \bar{b}'_{k-1} + T_m \bar{b}'_{k-1} + ET_0^{m-1} B\bar{b}'_{k-1}. \end{aligned}$$

Upon using (8.7) or (8.8), according to the case in hand, we find that

$$\Theta BET_0^{m-1}\bar{b}'_{k-1} \equiv \Theta\bar{b}'_{k-1} + \Theta\bar{b}''_{k-1} + \Theta ET_0^{m-1}B\bar{b}'_{k-1}. \tag{8.27}$$

Independently of the preceding, we note that

$$\Sigma\Theta B\bar{b}_k \equiv \Sigma^*[\Theta\bar{b}'_{k-1} + \Theta\bar{b}''_{k-1}], \tag{8.28}$$

summing as in (8.24). With the aid of (8.26), (8.27), and (8.28) we obtain (8.23) as written.

Statement (c) is thereby proved.

We continue with a proof of the following.

(d). *Let a_{k-2} be an arbitrary $(k-2)$-cell of z^q. The subset of $(k-1)$-cells on the boundary of w^q obtained from (8.23) by omitting all $(k-2)$-cells of $B\bar{b}'_{k-1}$ save those which have a_{k-2} as a K-image on Π^q sum to zero,* mod 2. *Hence $Bw^q \equiv 0$ on Θ and w^q is a cycle.*

Recall that z^q is a simple k-cycle. The k- and $(k-1)$-cells of z^q incident with a_{k-2} taken in their circular order about a_{k-2} will be denoted by

$$a_k^{(1)}\, a_{k-1}^{(1)}\, a_k^{(2)}\, a_{k-1}^{(2)} \cdots a_k^{(s)}\, a_{k-1}^{(s)} \qquad (\text{on } \Pi^q). \tag{8.29}$$

In forming the sum (8.5) we have selected K-images on K^q of the respective k-cells of z^q. Using these same K-images on K^q, let the respective K-images of the k-cells in (8.29) be denoted by

$$b_k^{(1)} b_k^{(2)} \cdots b_k^{(s)} \qquad (\text{on } K^q). \tag{8.30}$$

Let

$$b_{k-1}^{(i)\prime}, \qquad b_{k-1}^{(i)\prime\prime} \qquad (i = 1, 2, \cdots, s)$$

be the K-images of $a_{k-1}^{(i)}$ on the boundaries of $b_k^{(i)}$ and $b_k^{(i+1)}$ respectively, understanding that $b_k^{(s+1)} = b_k^{(1)}$. By virtue of (c) we will lose no generality if we suppose the senses of $b_k^{(2)}, \cdots, b_k^{(s)}$ have been successively changed so that

$$b_{k-1}^{(i)\prime\prime} = T_{m^{(i)}} b_{k-1}^{(i)\prime} \qquad (i = 1, \cdots, s-1) \tag{8.31}$$

where $m^{(i)}$ is an integer between 0 and $q-1$ inclusive.

Case II thus does not then occur corresponding to $a_{k-1}^{(i)}$, for $i = 1, \cdots, s-1$. Nor will Case II then occur corresponding to $a_{k-1}^{(s)}$ as we shall now prove.

To that end let $\beta^{(i)}$ be the K-image of a_{k-2} on $\bar{b}_k^{(i)}$, $i = 1, \cdots, s$. Observe that $\beta^{(i+1)}$ and $\beta^{(i)}$ lie respectively on the boundaries of $b_{k-1}^{(i)\prime\prime}$ and $b_{k-1}^{(i)\prime}$. From (8.31) it then follows that

$$\beta^{(i+1)} = T_{m^{(i)}}\beta^{(i)} \qquad (i = 1, \cdots, s-1). \tag{8.32}$$

If $a_{k-1}^{(s)}$ came under Case II, we would have a relation of the form

$$\beta^{(1)} = U_0 T_{m^{(s)}}\beta^{(s)} \qquad (0 \leqq m^{(s)} \leqq q-1), \tag{8.33}$$

and we could infer from (8.32) and (8.33) that

$$\beta^{(1)} = U_0 T_\mu \beta^{(1)} \tag{8.34}$$

where

$$\mu = m^{(1)} + \cdots + m^{(s)}. \tag{8.35}$$

But (8.34) is contrary to an hypothesis of the lemma. We infer that $a_{k-1}^{(s)}$ comes under Case I as stated.

The relations (8.32) may now be completed by the relation

$$\beta^{(1)} = T_{m^{(s)}}\beta^{(s)}. \tag{8.36}$$

Relations (8.32) and (8.35) combine into a relation

$$\beta^{(1)} = T_{\mu}\beta^{(1)} \tag{8.37}$$

where μ is given by (8.35). By virtue of the principal hypothesis of the lemma we see that μ has the form

$$\mu = 2\nu r \tag{8.38}$$

where ν is an integer, positive, or zero.

To return to statement (d) we observe that the terms in the sum Σ^* of (8.23) corresponding to the $(k-1)$-cells of z^q incident with a_{k-2}, take the form

$$\sum_{i=1}^{s} \Theta E T_0^{m^{(i)}-1} B b_{k-1}^{(i)\prime} \qquad (\text{on } \Theta). \tag{8.39}$$

To establish statement (d) we omit all of the $(k-2)$-cells of $Bb_{k-1}^{(i)\prime}$ save $\beta^{(i)}$. With this omission (8.39) reduces to the chain

$$\sum_{i=1}^{s} \Theta E T_0^{m^{(i)}-1} \beta^{(i)}. \tag{8.40}$$

Upon using (8.32) the chain (8.40) takes the form

$$\Theta E T_0^{\mu-1}\beta^{(1)} \equiv \Theta E(T_0 + T_1 + \cdots + T_{\mu-1})\beta^{(1)} \tag{8.41}$$

where μ is given by (8.35). But, on Θ, pairs of chains of the form

$$\Theta E T_\alpha \beta^{(1)}, \qquad \Theta E T_{\alpha+r}\beta^{(1)}$$

are equal, mod 2. Since $\mu = 2\nu r$, there are ν such pairs in the sum (8.41). The chain (8.41) accordingly reduces to zero, mod 2.

Hence the boundary of w^q on Θ reduces to zero, mod 2, and (d) is established.

We now return to the proof of the lemma. On K^q we set

$$u_k = \Sigma \bar{b}_k,$$

$$v_{k-1} = \Sigma^* T_0^{m-1} \bar{b}'_{k-1}.$$

We see from (8.24) that the cycle w^q on Θ takes the form

$$w^q = \Theta(u_k + E v_{k-1}).$$

Now z^q is the K-image on Π^q of u_k. We let y^q be the K-image on Π^q of v_{k-1} unreduced mod 2. We observe that y^q is a sum of $(k-1)$-cells of z^q. The cycle w^q on Θ is then the K-image on Π^q of the cycle

$$z^q + \mathcal{E}y^q \qquad (\text{on } \Pi^q),$$

unreduced mod 2.

The lemma is thereby proved.

We can now prove the following theorem.

THEOREM 8.2. *Let z^q $(q = rp)$ be a k-cycle of cells of Π^q (below c), no cell of which has a K-image on K^q invariant under a transformation of G^q other than a power of T_{2r}. Suppose also that points (π) on z^q determine curves $g(\pi)$ on which the J-lengths of r successive elementary extremals are at most ρ. Then z^q is Ω-homologous (below c) to a k-cycle on Π^p.*

By virtue of the preceding lemma there exists a $(k-1)$-chain y^q of $(k-1)$-cells of z^q such that

$$z^q + \mathcal{E}y^q \qquad (\text{on } \Pi^q) \tag{8.42}$$

unreduced mod 2, is the K-image of a k-cycle on Θ. The k-cycle (8.42) accordingly possesses a join x^p on Π^p. According to Theorem 8.1 we have

$$z^q + \mathcal{E}y^q * x^p \tag{8.43}$$

below c, if z^q is below c. But $\mathcal{E}y^q$ is a cycle on Π^q since $z^q + \mathcal{E}y^q$ is a cycle, and, by virtue of Theorem 7.1,

$$\mathcal{E}y^q * 0. \tag{8.44}$$

Hence

$$z^q * x^p \tag{8.45}$$

below c if y^q is below c. The proof is accordingly complete.

Note. For the sake of reference it is important that we have more intimate knowledge of how the homologies (8.43) and (8.44) are generated. Reference to the proof of Theorem 7.1 shows that (8.44) is the result of applying the deformation ψ_1 to the 2-fold partition of $\mathcal{E}y^q$. Reference to Theorem 8.1 shows that (8.43) is effected by using the deformation η to deform the cycle (8.42) into the r-fold partition of x^p on Π^q.

Finiteness of the basic maximal sets

9. Let σ be a critical set on Ω on which $J = c$. Some of the closed extremals determined by σ may be multiply covered. In case a closed extremal γ possesses at most a finite number of multiple points, and is covered ν times by a closed extremal g, we shall say that g possesses the multiplicity ν. A given critical set may possess closed extremals with several different multiplicities.

We shall now prove a lemma which has immediate bearing on the finiteness of maximal sets of spannable and critical k-cycles corresponding to an admissible pair of neighborhoods VW of σ.

LEMMA 9.1. *Corresponding to the critical set σ there exists a positive integer p with the following property. If VW is an admissible pair of neighborhoods of σ, any k-cycle on W (below c) is Ω-homologous on V (below c) to a J-normal k-cycle which is arbitrarily near σ, and whose components are null except at most its components on W^p.*

We shall prove that the lemma is satisfied by any integer p such that $p\rho > c$, and such that p is an even multiple of the multiplicities of the closed extremals of the given critical set.

Let p be such an integer. Let u be any k-cycle on W. Let q be a multiple of p and the number of vertices in the non-null components of u, say $q = rp$. Without loss of generality we can suppose that the components of u are so near σ that u possesses a partition z^q on W^q. Without loss of generality we can also suppose that z^q is composed of J-normal points on W^q, because in any case such a cycle would be obtained from z^q by an application of the deformation $\theta_q(t)$.

We shall now investigate the applicability of Theorem 8.2 to z^q. We shall first verify the fact that J-normal points on W^q sufficiently near σ possess no K-images on W^q invariant under transformations of G^q, other than powers of T_{2r}, where $q = rp$.

Let (π) be any J-normal point of σ^q. Suppose that $g(\pi)$ has the multiplicity ν. There will then be $q = s\nu$ vertices in (π), where s is a positive integer. It is clear that a K-image of (π) on K^q will be invariant under no transformations of G^q other than powers of T_s.

From the fact that

$$q = rp = s\nu$$

and that p is an even multiple of ν we infer that s is an even multiple of r. Thus K-images of (π) on K^p are invariant at most under powers of T_{2r}. Finally it is clear that J-normal points sufficiently near σ will have this same property. Without loss of generality we can then assume that z^q is so near σ that the K-images of its points are invariant at most under powers of T_{2r}.

In order to apply Theorem 8.2 to z^q we must know that its points (π) define curves $g(\pi)$, the J-lengths of r of whose consecutive elementary extremals is at most ρ. If (π) is a J-normal critical point of σ^q, the J-length of r of its elementary extremals is

$$r\frac{c}{q} = \frac{c}{p} < \rho.$$

Moreover if z^q is a J-normal cycle sufficiently near σ, the J-length of r of its elementary extremals will still be less than ρ.

Theorem 8.2 is thus applicable to z^q provided z^q is a k-cycle of cells of Π^q. But if z^q is not a k-cycle of cells of Π^q, upon subdividing Π^q and z^q sufficiently finely, one can obtain a cycle u^q of cells of Π^q homologous to z^q below c if z is below c, and so near z^q that Theorem 8.2 is applicable to u^q. We conclude that z^q is Ω-homologous to a k-cycle on Π^p. Reference to the note following Theorem 8.2 makes it further clear that if z^q is sufficiently near σ^q, as we suppose it is, u^q and hence z^q is Ω-homologous on V to a k-cycle on W^p, the homology holding below c, if z^q is below c.

Finally any k-cycle on W^p is homologous on V^p (below c) to a J-normal k-cycle on W^p arbitrarily near σ^p. The proof of the lemma is now complete.

We now establish an important consequence of the preceding lemma.

THEOREM 9.1. *There exists at most a finite number of spannable or critical k-cycles corr VW in maximal sets of such cycles.*

We shall give the proof of the theorem for the case of spannable k-cycles.

Let z be any spannable k-cycle corr VW. According to the preceding lemma z is Ω-homologous on V below c to a k-cycle on W^p where p is a positive integer dependent only on σ. But there are at most a finite number of k-cycles on W^p below c, independent on V^p below c. The theorem is accordingly true for the case of spannable k-cycles.

The proof for the case of critical k-cycles is similar.

The following theorem is an easy consequence of the final statement in Theorem 4.1.

THEOREM 9.2. *If b is a positive number less than the least critical value of J, the Ω-connectivities of the domain $J < b$ are null.*

For if z is a k-cycle on $J < b$, any non-null component of z on Π^p is homologous to zero on Π^p, according to Theorem 4.1, so that z is Ω-homologous to zero on $J < b$ as stated.

We conclude this section with the following theorem.

THEOREM 9.3. *If b is any ordinary value of J, the Ω-connectivities of the domain $J < b$ are finite.*

We have already seen that the number of k-cycles in maximal sets of spannable or critical k-cycles corresponding to a critical set σ is finite. The number of linking k-cycles in a maximal set corresponding to any complete critical set σ is then finite, for it is at most the number of spannable $(k - 1)$-cycles in a maximal set corresponding to σ.

To establish the theorem we let

$$c_1 < c_2 < \cdots < c_m$$

be the critical values of J less than b, and let

$$\sigma_1, \sigma_2, \cdots, \sigma_m$$

be the corresponding complete critical sets. There are no invariant k-cycles corresponding to c_1 and σ_1, since there are no k-cycles below c_1 except those Ω-homologous to zero. Let β_1 be a constant such that $c_1 < \beta_1 < c_2$. According to Theorem 6.3 a maximal set of k-cycles on the domain $J < \beta_1$, Ω-independent on this domain, will be afforded by maximal sets of critical, linking, and invariant k-cycles corresponding to the critical set σ_1. Since these maximal sets are finite, the Ω-connectivities of $J < \beta_1$ are finite.

We now assume the theorem is true for any domain $J < \beta_r$ for which $c_r < \beta_r < c_{r+1}$, and prove it is true for the domain $J < \beta_{r+1}$, reasoning as in the preceding paragraphs. It follows by mathematical induction that the theorem is true as stated.

Numerical invariants of a closed extremal g

10. In this section we shall define the index and nullity of a closed extremal g in a way that will be independent of the coordinate systems used to cover the neighborhood of g. We first introduce two important conceptions.

Proper sections S of R^p belonging to g. Let c be the J-length of g, and p a positive integer such that $p\rho > c$. Let (π_0) be an inner point of Π^p such that $g(\pi_0) = g$. Suppose that none of the elementary extremals of $g(\pi_0)$ reduce to points. Let

$$P_0^q \qquad (q = 1, \cdots, p)$$

be the qth vertex of (π_0), taking these vertices in one of their two circular orders. Let M^q be a regular analytic $(m - 1)$-manifold intersecting g at P_0^q, but not tangent to g. A manifold of points (π) whose qth vertex P^q is subject to no other restriction than to lie on M^q neighboring P_0^q will be called a *proper section* S of R^p belonging to g.

The boundary problem associated with S. With g and S we shall now associate a boundary problem of the type studied in Ch. V. To define such a problem we cut g at P_0^1, forming thereby an extremal segment γ of J-length c, with end points A^1 and A^2 which are copies of P_0^1. Let (x) be an arbitrary coordinate system on R neighboring P_0^1. We shall regard the points A^1 and A^2 on γ as distinct, and shall provide them with neighborhoods which we shall also regard as distinct. These neighborhoods will be represented by copies of the coordinate system x^i denoted by x^{i1} and x^{i2} respectively. We suppose that M^1 is regularly represented in the form

$$x^i = x^i(\alpha_1, \cdots, \alpha_n) \qquad (n = m - 1) \tag{10.1}$$

and that P_0^1 corresponds to the parameter values $(\alpha) = (0)$. With g and S we now associate a boundary problem B in which the end conditions refer to points neighboring A^1 and A^2 respectively, and have the form

$$x^{is} = x^i(\alpha) \qquad (s = 1, 2; i = 1, \cdots, m) \tag{10.2}$$

where the functions $x^i(\alpha)$ are those defining M^1. We see that γ will be a critical extremal in the boundary problem B.

Let (v) be a set of parameters in a regular analytic representation of S neighboring (π_0). Suppose that $(v) = (0)$ corresponds to (π_0). On S the value of

$J(\pi)$ at the point (π) determined by (v) will be a function $f(v)$, analytic in (v) at $(v) = (0)$. The point $(v) = (0)$ will be a critical point of $f(v)$. The form

$$Q(v) = f_{v_i v_j}(0) v_i v_j \qquad (i, j = 1, \cdots, pn) \tag{10.3}$$

will be an index form corresponding to γ as a critical extremal in the preceding boundary problem B.

We shall now prove the following theorem.

THEOREM 10.1. *The index and nullity of the form* (10.3) *are independent of the proper section S of R^p on which the function $J = f(v)$ is defined.*

Suppose g has the length ω. We shall combine the conditions (11.2a) and (11.2b) of Ch. V into a system

$$\ddot{\eta}^T = 0, \tag{10.4a}$$

$$L_i(\eta) + \lambda \eta_i^N = 0 \qquad (i = 1, \cdots, m). \tag{10.4b}$$

We admit solutions of the system (10.4) in the form of contravariant tensors $\eta^i(t)$, locally of class C^2 in terms of the arc length t along g. Recall that η^T is an invariant, and that the left member of (10.4b) is a tensor which is covariant with respect to admissible changes of coordinates (x) along g. In the system (10.4) we are free from the necessity of having a single coordinate system along g, and in particular free from the difficulties which arise in connection with such coordinate systems when R is non-orientable.

The theorem is a consequence of the following lemma.

LEMMA 10.1. *The index of the form* (10.3) *equals the number of solutions of the system* (10.4) *of period ω which are independent of tangential solutions of* (10.4), *and correspond to negative values of λ. The nullity of the form* (10.3) *equals the number of solutions of* (10.4) *of period ω which are independent of tangential solutions of* (10.4), *and correspond to a null value of λ.*

Corresponding to the extremal segment γ and the end conditions (10.2), the accessory boundary problem (11.2) of Ch. V here takes the form

$$\ddot{\eta}^T = 0 \qquad (s = 1, 2;\ i = 1, \cdots, m), \tag{10.5a}$$

$$L_i(\eta) + \lambda \eta_i^N = 0, \tag{10.5b}$$

$$\eta^{is} - x_k^i u^k = 0, \tag{10.5c}$$

$$x_h^i(\zeta_i^1 - \zeta_i^2) = 0 \qquad (h, k = 1, \cdots, n = m - 1), \tag{10.5d}$$

where x_h^i is the partial derivative of $x^i(\alpha)$ with respect to α_h, evaluated for $(\alpha) = (0)$. According to Theorem 14.1 of Ch. V the nullity of $Q(v)$ will be the index of $\lambda = 0$ as a characteristic root of the system (10.5), and the index of $Q(v)$ the number of characteristic roots of (10.5) which are negative, counting these roots with their indices.

To compare the system (10.5) with the system (10.4), first recall that the manifold (10.1) is not tangent to g. The end conditions (10.2) are then seen to satisfy the non-tangency condition of §12, Ch. V. It follows from Lemma 12.1 of Ch. V that the system (10.5) admits no non-null tangential solutions, so that no proper linear combination of independent characteristic solutions of (10.5) is a tangential solution.

We shall now show that each characteristic solution (η) *of* (10.5) *has the period* ω *in* t.

First we note that for such a solution

$$\eta^{i2} = \eta^{i1} \qquad (i = 1, \cdots, m) \tag{10.6}$$

as follows from (10.5c). It remains to prove that

$$\zeta_i^2 = \zeta_i^1. \tag{10.7}$$

To that end let $x^i = \gamma^i(t)$ be a representation of g neighboring P_0^1, measuring the arc length t from P_0^1. Upon using the definition of ζ_i as Ω_{η^i}, together with (10.6), we see that

$$\dot\gamma^i(0)(\zeta_i^2 - \zeta_i^1) = 0. \tag{10.8}$$

We combine (10.8) with (10.5d) to form the system

$$\begin{aligned} \dot\gamma^i(0)\,(\zeta_i^2 - \zeta_i^1) &= 0, \\ x_h^i(0)\,(\zeta_i^2 - \zeta_i^1) &= 0. \end{aligned} \tag{10.9}$$

But the m-square determinant

$$\begin{vmatrix} \dot\gamma^i(0) \\ x_h^i(0) \end{vmatrix} \qquad (h = 1, \cdots, n; i = 1, \cdots, m)$$

is not zero since the manifold $x^i = x^i(\alpha)$ is not tangent to g when $(\alpha) = (0)$. From (10.9) we conclude that (10.7) holds, and hence that each characteristic solution of (10.5) has the period ω in t.

The lemma and the theorem follow directly.

We now define the index and nullity of g as the index and nullity of the index form (10.3) *determined by any proper section S belonging to g of the space R^ν. If the nullity of g is zero, we term g non-degenerate.*

We add the following theorem.

THEOREM 10.2. *If g is a non-degenerate closed extremal, there is no connected family of closed extremals which contains both g and closed extremals other than g.*

If the theorem were false, there would be a connected family of critical extremals corresponding to the boundary problem (10.2) which would contain g and closed extremals other than g. We would then have a contradiction to Theorem 11.1 of Ch. VII. We accordingly infer the truth of Theorem 10.2.

The non-degenerate closed extremal

11. In this section we shall prove that the ith type number of a non-degenerate closed extremal g of index k is δ_i^k, $i = 0, 1, \cdots$. We shall accomplish this by showing that the type numbers of the critical set σ defined by g on Ω are the same as those of g considered as a critical extremal, in the sense of Ch. VII, in a boundary problem B with end conditions of the form (10.2).

Let (π) be an arbitrary point on Π^p. Let (π') be any point on Π^p whose vertices lie on $g(\pi)$ and possess a circular order in agreement with their order on $g(\pi)$. With (π) and (π') we now associate a third point (π'') on the extension $\mathcal{E}\pi$ of (π).

The mean correspondent of (π') *on* $\mathcal{E}\pi$. Let g^* be an unending curve covering $g(\pi)$. On g^* let s represent the J-length measured in a prescribed sense from a prescribed point on g^*. The vertices of (π') will be represented by infinitely many copies on g^*. Let

$$P^1, \cdots, P^p \tag{11.0}$$

be a set of copies of the p vertices of (π') which appear consecutively on g^* in the order (11.0). Let

$$Q^1, \cdots, Q^p \tag{11.1}$$

be a set of consecutive points on g^* which define a point (π'') on $\mathcal{E}\pi$ such that the average value of s for the points (11.1) is the same as for the points (11.0). We term (π'') the *mean correspondent of* (π') *on* $\mathcal{E}\pi$.

We observe that the mean correspondent of (π') on $\mathcal{E}\pi$ will be independent of the particular set of vertices (11.0) chosen to represent the point (π'), and of the sense assigned to $g(\pi)$, as well as of the point on $g(\pi)$ from which s is measured.

The deformation F_p. We now deform (π') into its mean correspondent (π'') on $\mathcal{E}\pi$ moving each vertex P^i along g^* to the corresponding vertex Q^i, moving P^i at a J-rate along g^* equal to the J-distance to be traversed. We term this the deformation F_p.

Let c be the J-length of g and p a positive integer such that $p\rho > c$. Let σ^p be the critical set of $J(\pi)$ determined by g on Π^p, and σ the corresponding critical set on Ω. Let S be a proper section of R^p belonging to g, as defined in §10. Let q be a second integer such that $q\rho > c$. Let (π_0) be a point of σ^q none of whose elementary extremals are null. If (π) is a point on Π^q sufficiently near (π_0), there will be a unique point $b^p(\pi)$ on S with vertices on $g(\pi)$. We term $b^p(\pi)$ the *extremal projection* of (π) on S, and state the following lemma.

LEMMA 11.1. *Let S be a proper section of R^p belonging to g. Corresponding to S and an arbitrary neighborhood N of σ on Ω, any J-normal k-cycle z^p on Π^p (below c) sufficiently near σ^p will possess an extremal projection $b^p(z^p)$ on S such that*

$$z^p * b^p(z^p) \qquad [\text{on } N \text{ (below } c)]. \tag{11.2}$$

If z^p lies on a sufficiently small neighborhood of σ^p, it will possess the properties enumerated in the following paragraph.

The extension $\mathcal{E}w^p$ of any cycle w^p on z^p will satisfy the relation

$$\mathcal{E}w^p * 0 \qquad \text{[on } N \text{ (below } c)] \tag{11.3}$$

as follows from Theorem 7.1. The extremal projection $b^p(z^p)$ of z^p on S will exist, and be the continuous image of z^p, points on z^p corresponding to their extremal projections on S. The extremal projection $b^p(\pi)$ of any arbitrary point (π) on z^p will possess a mean correspondent $\beta^p(\pi)$ on $\mathcal{E}\pi$ which varies continuously with (π) on z^p. As (π) ranges over z^p, its image $\beta^p(\pi)$ will define a cycle $\beta^p(z^p)$, the continuous image of z^p. Moreover the deformation F_p of $b^p(\pi)$ into $\beta^p(\pi)$ will deform $b^p(z^p)$ continuously on N^p (below c) into $\beta^p(z^p)$. Accordingly

$$\beta^p(z^p) \sim b^p(z^p) \qquad \text{[on } N^p \text{ (below } c)].$$

We regard the deformation chain $\mathcal{E}z^p$, unreduced mod 2, as the product of z^p and a circle. We see that $\beta^p(z^p)$ is a singular k-cycle on $\mathcal{E}z^p$. According to the theory of product chains, of which $\mathcal{E}z^p$ is an instance, there will exist a $(k-1)$-cycle w^p on z^p, such that

$$\beta^p(z^p) \sim z^p + \mathcal{E}w^p \qquad (\text{on } \mathcal{E}z^p). \tag{11.4}$$

The lemma now follows from the three preceding homologies.

Note. The homology (11.4). For the sake of an application in Ch. IX we shall here exhibit a $(k+1)$-chain on $\mathcal{E}z^p$ bounded by the members of (11.4).

According to the definition of the extension of a chain, $\mathcal{E}z^p$, unreduced mod 2, can be regarded as the product of z^p and a circle whose parameter t represents the time in the deformation defining $\mathcal{E}z^p$, with $0 \leqq t < 1$. Any point (π') on $\mathcal{E}z^p$ is thereby determined by a pair (π, t) in which (π) is a point on z^p and t a value of t on the interval $0 \leqq t < 1$. It will be convenient to regard the point (π') as represented not only by the pair (π, t), but also by all pairs of the form $(\pi, t + n)$ where n is any integer, positive, negative, or zero. We have thereby covered $\mathcal{E}z^p$ by an unending succession of copies of $\mathcal{E}z^p$ in the form of a product W of z^p and the unlimited t axis. On W, z^p is represented by chains corresponding to integral values of t.

Let a_i be any i-cell of z^p. Let b_i be the corresponding i-cell of $\beta^p(z^p)$. If z^p is sufficiently finely divided, as we suppose it is, the cell b_i will be represented on W by at least one and at most two continuous image cells b_i^1 and b_i^2, whose closures lie on W at points at which

$$0 < t \leqq 2.$$

If there are two such cells b_i^1 and b_i^2, the values of t on one such cell, say b_i^2, will exceed the values of t at the corresponding points on the other b_i^1, by unity.

Let (π) be any point on $\bar{a}_i$ and (π, t^1) the corresponding point on b_i^1. With the point (π) on $\bar{a}_i$ we now associate the closed 1-cell on W of points (π, t) for which

$$0 \leqq t \leqq t^1.$$

The ensemble of these 1-cells on W as (π) ranges over $\bar{a}_i$ will be denoted by w^1_{i+1}, and termed the first *associate* of a_i. If b^2_i exists, we similarly define a second associate w^2_{i+1} of a_i, replacing t^1 by $t^1 + 1$.

We shall now describe a division of the point sets w^1_{i+1} and w^2_{i+1} into cells.

First suppose that on w^1_{i+1},

$$0 \leqq t \leqq 1.$$

If a_j is any boundary cell of $\bar{a}_i$ and w^1_{j+1} its first associate, w^1_{j+1} will lie on the (geometric) boundary of w^1_{i+1}. We proceed inductively, supposing that w^1_{j+1} has already received a division into cells. In case $j = 0$, w^1_{j+1} shall consist of two consecutive 1-cells, the point of division being arbitrary. For any j the geometric boundary of w^1_{j+1} will include an image of $\bar{a}_j$ on w^1_{j+1} consisting of points at which $t = 0$, and an image of the corresponding cell $\bar{b}^1_j$, the locus of the other extremities of the 1-cells making up w^1_{j+1}. We suppose that the cells of these images are included among the cells of w^1_{j+1}. We now choose an arbitrary inner point P on w^1_{i+1}, and join P by suitable "straight" cells to the boundary cells of w^1_{i+1}, thus completing the division of w^1_{i+1} into cells in case $0 \leqq t \leqq 1$ on w^1_{i+1}.

A second associate of a_i will exist if and only if $0 \leqq t \leqq 1$ on w^1_{i+1}. In such a case we let $w^{1\,\prime}_{i+1}$ denote the copy of w^1_{i+1} on W obtained by adding 1 to the parameter t in the pair (π, t) which represents an arbitrary point on w^1_{i+1}. Let $\mathcal{E}_1\bar{a}_i$ represent the closure of an image of $\mathcal{E}\bar{a}_i$ on W on which $0 \leqq t \leqq 1$. We suppose $\mathcal{E}_1\bar{a}_i$ has the division into cells which we have accorded a deformation chain. We now give w^2_{i+1} a division into cells such that

$$w^2_{i+1} \equiv \mathcal{E}_1\bar{a}_i + w^{1\,\prime}_{i+1} \tag{11.5}$$

becomes a valid congruence.

In case there are points on a set w^1_{i+1} at which $t > 1$, there may be a cell a_j on the boundary of a_i such that w^2_{j+1} is on the geometric boundary of w^1_{i+1}. We then divide such sets w^2_{j+1} into cells in the order of their dimensionality in accordance with (11.5) and finally divide the sets w^1_{i+1} in the order of their dimensionality as before.

Let $u^1_{i+1}(a_i)$ denote the image on $\mathcal{E}\bar{a}_i$ of w^1_{i+1}. In case w^2_{i+1} exists let $u^2_{i+1}(a_i)$ denote the image on $\mathcal{E}\bar{a}_i$ of w^2_{i+1}, $i = 0, \cdots, k$.

Let a_k be a k-cell of z^p, and a_{k-1} any one of its boundary cells. We see that on $\mathcal{E}z^p$,

$$u^1_{k+1}(a_k) \to \bar{a}_k + \bar{b}_k + \sum_{a_{k-1}} u^s_k\,(a_{k-1}) \qquad (s = 1, \text{or } 2) \tag{11.6}$$

where there is one chain $u^s_k(a_{k-1})$ in the sum Σ corresponding to each $(k-1)$-cell a_{k-1} on the boundary of a_k. Summing over all k-cells of z^p we find that

$$\sum_{a_k} u^1_{k+1}(a_k) \to z^p + \beta^p(z^p) + \sum_{a_k} \sum_{a_{k-1}} u^s_k(a_{k-1}). \tag{11.7}$$

To evaluate the double sum we invert the order of summation, and first consider the sum

$$\sum_{a_k} u_k^s(a_{k-1}) \tag{11.8}$$

in which a_{k-1} is fixed, and there is one term corresponding to each cell a_k of z^p incident with a_{k-1}. A pair of cells (a_k, a_{k-1}) corresponding to which $s = 2$ in (11.8) will be said to be of even type, otherwise of odd type. For pairs (a_k, a_{k-1}) of even type,

$$u_k^2(a_{k-1}) \equiv \mathcal{E}\bar{a}_{k-1} + u_k^1(a_{k-1}) \qquad \text{(on } \Pi^p)$$

by virtue of (11.5). Hence (11.8) reduces to a sum

$$\Sigma\mathcal{E}\bar{a}_{k-1}$$

taken for all pairs (a_k, a_{k-1}) of even type with a_{k-1} fixed and a_k incident with a_{k-1}. More generally we have

$$\sum_{a_k} \sum_{a_{k-1}} u_k^s(a_{k-1}) \equiv \Sigma^*\mathcal{E}\bar{a}_{k-1} \tag{11.9}$$

where the sum Σ^* contains a term $\mathcal{E}\bar{a}_{k-1}$ corresponding to each mutually incident pair (a_k, a_{k-1}) of k- and $(k-1)$-cells on z^p of even type.

I say finally that the sum

$$\Sigma^*\bar{a}_{k-1} = w^p, \tag{11.10}$$

in which the terms are derived from those on the right of (11.9), defines a $(k-1)$-cycle w^p on Π^p. To see this we replace z^p by a "simple" k-cycle consisting of the same k-cells. The sums (11.9) and (11.10) will not thereby be altered. Upon considering the circular sequence of k- and $(k-1)$-cells then incident with a given $(k-2)$-cell a_{k-2}, $k > 1$, one sees that the number of pairs (a_k, a_{k-1}) of successive cells in this sequence which are of even type must itself be even. This statement presupposes a sufficiently fine subdivision of z^p. There are then an even number of the cells $\bar{a}_{k-1}$ in (11.10) incident with a_{k-2} so that (11.10) defines a $(k-1)$-cycle. The cases $k = 0$ and 1 require no comment.

The homology (11.4) can accordingly be written in the form

$$\sum_{a_k} u_{k+1}^1 \to z^p + \beta^p(z^p) + \mathcal{E}w^p \tag{11.11}$$

where w^p is a cycle of $(k-1)$-cells of z^p. The analysis of the homology (11.4) is now complete.

We continue with the following theorem.

THEOREM 11.1. *Let S be a proper section of R^p belonging to g. Corresponding to S and an arbitrarily small neighborhood N of σ, there exists an arbitrarily small* *neighborhood N_0 of σ with the following property. Any k-cycle u on N_0 (below c) will be Ω-homologous on N (below c) to a cycle on S.*

Let r be any positive integer, and p the integer of the theorem. Let S^{rp} be a proper section of R^{rp} belonging to g. Let

$$M^1, \cdots, M^{rp}$$

be the successive manifolds on R which cut across g to define S^{rp}. We suppose that the manifolds

$$M^r, M^{2r}, \cdots, M^{rp} \tag{11.12}$$

are the manifolds which define the section S of the theorem.

If the respective components of N_0 are sufficiently small, the statements of the following paragraph will be true.

The components u^r of the k-cycle u of the theorem will satisfy the homologies

$$u^r \sim v^r \qquad [\text{on } N \text{ (below } c)]$$

where v^r is a J-normal k-cycle. The cycle v^r will possess a partition w^{rp} on N (below c). Hence

$$v^r * w^{rp} \qquad [\text{on } N \text{ (below } c)].$$

The cycle w^{rp} will possess an extremal projection x^{rp} on S^{rp} such that

$$w^{rp} * x^{rp} \qquad [\text{on } N \text{ (below } c)],$$

by virtue of the preceding lemma. Let y^p be the extremal projection of x^{rp} on S, and z^{rp} the r-fold partition of y^p. Let (π) be an arbitrary point on x^{rp}, (π') the extremal projection of (π) on S, and (π'') the r-fold partition of (π'). The point (π'') lies on z^{rp}. Its vertices on the manifolds (11.12) of S are also vertices of (π). We can accordingly use the deformation η of §7 to deform each point (π) on x^{rp} into the corresponding point (π'') on z^{rp}, holding the vertices on S fast. We will then have the homology

$$x^{rp} \sim z^{rp} \qquad [\text{on } N \text{ (below } c)].$$

But since z^{rp} is the partition of y^p we have

$$z^{rp} * y^p \qquad [\text{on } N \text{ (below } c)].$$

Combining the preceding homologies we see that

$$u^r * y^p \qquad [\text{on } N \text{ (below } c)].$$

The theorem follows readily.

We continue with the following theorem.

THEOREM 11.2. *Let S be a proper section of R^r belonging to g. Corresponding to S and to an arbitrary neighborhood N of σ, there exists an arbitrarily small neighborhood N_0 of σ with the following property. If z^p is any cycle on S such that*

$$z^p * 0 \qquad [\textit{on } N_0 \text{ (below } c)], \tag{11.13}$$

then

$$z^p \sim 0 \qquad [\textit{on } S \textit{ and } N^p \textit{ (below } c)]. \tag{11.14}$$

If the neighborhood N_0 of the theorem is sufficiently small, the following statements are true.

The deformation $\theta_p(t)$ of §5 will deform the k-cycle z^p of the theorem into a J-normal cycle w^p such that

$$w^p * 0 \qquad [\text{on } N \text{ (below } c)] \tag{11.15}$$

where the chains involved in (11.15) possess continuous extremal projections on S which lie on N^p (below c). Suppose the extremal projections on S of the $(k + 1)$-chains involved in (11.15) sum to a $(k + 1)$-chain β^p. Under $\theta_p(t)$, z^p will be deformed through a $(k + 1)$-chain which will possess a continuous extremal projection b^p on S. Moreover b^p will lie on N^p (below c). Hence

$$b^p + \beta^p \to z^p$$

and the theorem is proved.

The two preceding theorems lead at once to the following theorem.

THEOREM 11.3. *Let S be a proper section of R^p belonging to a non-degenerate closed extremal g. Maximal sets of spannable and critical k-cycles on Ω belonging to the critical set σ determined by g, can be taken as maximal sets of spannable and critical cycles on S belonging to the function $J(\pi)$ on S and to the critical point (π_0) on S determined by g.*

The type number of g or σ is accordingly the type number of (π_0) as a critical point of the function f defined by $J(\pi)$ on S. Since f is non-degenerate, the ith type number of f is then δ_i^k where k is the index of (π_0) as a critical point of f. Inasmuch as the index of g is by definition (see §10) the index of (π_0) as a critical point of f, we have the following corollary of the theorem.

COROLLARY. *The ith type number of a non-degenerate critical extremal g of index k is δ_i^k, $i = 0, 1, \cdots$.*

From Theorem 6.3 and the preceding corollary we deduce the following.

THEOREM 11.4. *If a and b are two ordinary values of J between which there is just one critical value c taken on by just one non-degenerate closed extremal g, the Ω-connectivities of the domain $J < b$ minus those of the domain $J < a$ afford differences which are all null except that*

Case I: $$\Delta P_k = 1,$$

or

Case II: $$\Delta P_{k-1} = -1,$$

where k is the index of g.

If Case I occurs, g is said to be of *increasing type*. Case I will occur if a spannable $(k - 1)$-cycle is associated with g and this cycle bounds below c.

Metrics with elementary arcs

12. The space Ω and its Ω-homologies depend upon the elementary extremals used. Nevertheless we shall prove that the connectivities of Ω are topological invariants of R, or, if one pleases, of the m-circuit K of which R is the homeomorph. To accomplish this we shall postulate the essential properties of J-distances and elementary extremals in an abstract form. We shall thereby define a *metric with elementary arcs*. Corresponding to each such metric a space Ω will be defined as previously. We shall then show that the connectivities of Ω are independent of the defining metric among metrics which are topologically equivalent.

The basic elements form a set of "points" P, Q, R, etc. termed a "space" S. To each ordered pair of points P and Q of the set S there is assigned a real number PQ, satisfying the following postulates.

POSTULATE 1. *$PQ = 0$ if and only if $P = Q$.*

POSTULATE 2.

$$PR \leqq PQ + RQ. \tag{12.1}$$

The number PQ is termed the *distance* from P to Q. See Lindenbaum [1], Fréchet [1].

These postulates are of the well known type used to define a "metric space." They are not sufficient for our purposes, but before proceeding further it will be convenient to indicate several of their consequences. The first is as follows.

I. *The distance PQ is never negative and $PQ = QP$.*

Upon setting $R = P$ in (12.1) we see that $PQ \geqq 0$. Upon setting $P = Q$ in (12.1) we then see that $QR = RQ$.

Before stating the second proposition let it be understood that a function f of a finite number of points $P^1, \cdots, P^n$ of S is continuous at a particular set $Q^1, \cdots, Q^n$ of such points of S, if corresponding to an arbitrarily small positive constant e, there exists a positive constant d so small that

$$|f(P^1, \cdots, P^n) - f(Q^1, \cdots, Q^n)| < e$$

whenever

$$P^iQ^i < d \qquad (i = 1, \cdots, n).$$

We now state the second proposition.

II. *The distance PR is a uniformly continuous function of P and R as P and R range over S.*

This proposition follows readily from Postulate 2.

Let Σ be a second space possessing a metric satisfying Postulates 1 and 2. Let $P(Q)$ be a point on S uniquely determined by an arbitrary point Q on Σ.

The map $P(Q)$ of Σ on S will be said to be continuous at a point Q_0 on Σ if the distance

$$P(Q)P(Q_0) \qquad \text{(on } S\text{)}$$

is arbitrarily small when the distance QQ_0 on Σ is sufficiently small. A homeomorphism between S and Σ can now be defined in the usual way. In particular a *simple arc* γ on S will be defined as the homeomorph on S of the line segment $0 \leqq t \leqq 1$. If P, Q, R are three distinct points on γ, Q will be said to be "between" P and R on γ if the image of Q on the t axis lies between the images of P and R respectively.

We state two additional postulates concerning our space S.

Corresponding to S there shall exist a positive constant ρ, and corresponding to each ordered pair of points P, R on S for which

$$0 < PR \leqq \rho, \tag{12.2}$$

there shall exist at least one simple arc $[PR]$ on S, with end points at P and R respectively, of such a nature that the following two postulates hold.

POSTULATE 3. *If Q is any point on $[PR]$ between P and R, then*

$$PR = PQ + QR. \tag{12.3}$$

POSTULATE 4. *If Q is any point not on $[PR]$, then*

$$PR < PQ + QR. \tag{12.4}$$

A simple arc satisfying Postulates 3 and 4 will be termed *elementary*. A space S for which distances and elementary arcs can be defined so as to satisfy Postulates 1 to 4 will be said to possess a *metric with elementary arcs*. We shall denote such a metric by M_ρ.

We continue with a number of propositions depending on our four postulates.

III. *Corresponding to any pair of points P, R of S for which* (12.2) *holds, there is but one elementary arc $[PR]$ on S. Moreover as a set of points on S, $[PR] = [RP]$.*

Let $[PR]$ be one elementary arc joining P to R. If there is any other such elementary arc, let Q be any point on that arc between P and R. The point Q must satisfy (12.3), and by virtue of (12.4) must then lie on $[PR]$.

One proves similarly that $[PR] = [RP]$.

IV. *Any two distinct points on an elementary arc $[PR]$ bound an elementary arc on $[PR]$.*

We shall first prove IV for the case in which one of the two points on $[PR]$ is P, and the other point a point Q between P and R.

Let W be any intermediate point on $[PQ]$. From (12.3) we have

$$PW + WQ = PQ. \tag{12.5}$$

Adding QR to both sides of (12.5), and applying Postulate 3 to the resulting right member we see that

$$PW + WQ + QR = PR. \tag{12.6}$$

Upon applying Postulate 2 to points W, Q, and R, and then to P, W, and R, (12.4) takes the form

$$PW + WR = PR.$$

Hence W must lie on $[PR]$.

Regard $[PQ]$ as the homeomorph of a line segment $0 \leqq t \leqq 1$, and let $W(t)$ be the point on $[PQ]$ corresponding to t. Regard $[PR]$ as the homeomorph of a line segment $0 \leqq \tau \leqq 1$, and let $\varphi(t)$ be the value of τ at the point $W(t)$. We see that $\varphi(t)$ is a continuous function of t. But the relation between the values of t and $\varphi(t)$ must be one-to-one. It follows that $\varphi(t)$ is an increasing function of t. Hence the point W of the preceding paragraph must lie on the segment of $[PR]$ bounded by P and Q.

Proposition IV is accordingly true if the given points are P and a point Q interior to $[PR]$. Upon selecting a point U on $[PQ]$, not P or Q, one proves similarly that $[UQ]$ is an arc of $[PQ]$ and hence of $[PR]$.

Proposition IV is established.

V. *On an elementary arc $[PR]$, regarded as the homeomorph of a line segment $0 \leqq t \leqq 1$, the distance $d(t)$ of the point t from the point $t = 0$ is a continuously increasing function of t.*

The continuity of $d(t)$ follows from II. That $d(t)$ increases with t now follows from IV and Postulate 3.

The metric geometry which we have developed up to this point is closely connected with the interesting and extensive geometry developed by Menger [2, 3, 4]. Menger however starts with a different notion of "betweenness," and is not concerned with our problem in the large and function space Ω.

The space S is said to be *compact* if each infinite set of points on S has at least one limit point on S.

VI. *If S is compact, the point Q on an elementary arc $[PR]$, at a distance s from P, is a continuous point function $Q(P, R, s)$ of P, R, and s, provided*

$$0 \leqq s \leqq PR \leqq \rho. \tag{12.7}$$

To prove this proposition let

$$P_n,\ R_n,\ s_n \qquad (n = 1, 2, \cdots)$$

be an infinite sequence of points P_n and R_n, and numbers s_n, such that

$$0 \leqq s_n \leqq P_nR_n \leqq \rho,$$

and such that

$$\lim_{n\to\infty} P_n = P, \qquad \lim_{n\to\infty} R_n = R, \qquad \lim_{n\to\infty} s_n = s.$$

The points

$$Q(P_n, R_n, s_n) = Q_n$$

will have at least one limit point on S, say Q^*. Without loss of generality we can suppose that the sequence Q_n has no limit point other than Q^*, since such a condition could be obtained by selecting a suitable subsequence of the points Q_n.

From Postulate 3 we see that

$$P_nQ_n + Q_nR_n = P_nR_n. \tag{12.8}$$

Letting n become infinite in (12.8), and using II we find that

$$PQ^* + Q^*R = PR. \tag{12.9}$$

It now follows from (12.9) and Postulate 4 that Q^* lies on $[PR]$. But since

$$PQ^* = \lim_{n\to\infty} P_nQ_n = \lim_{n\to\infty} s_n = s,$$

it follows from V that Q^* is uniquely determined as the point $Q(P, R, s)$. Hence

$$\lim_{n\to\infty} Q(P_n, R_n, s_n) = Q(P, R, s).$$

Proposition VI is thereby proved.

Let S' and S'' be two compact metric spaces provided respectively with metrics $M'_{\rho'}$ and $M''_{\rho''}$. We suppose S' and S'' homeomorphic, and represent corresponding points on S' and S'' by the same letters P, Q, R, etc. We shall avoid ambiguity by denoting the distance between points P and Q, on S' and S'', by the symbols

$$d'(P, Q), \qquad d''(P, Q),$$

respectively. In terms of these distances we state the following lemma. Its form is convenient for future use.

Lemma 12.1. *There exists a positive constant $r' < \rho'$, such that two points P, Q of S' and S'' which satisfy the condition*

$$d'(P, Q) < r' \tag{12.10}$$

also satisfy the condition

$$d''(P, Q) < \frac{\rho''}{8}. \tag{12.11}$$

Similarly there exists a positive constant $r'' < \rho''$, such that two points P, Q of S' and S'' which satisfy the condition

$$d''(P, Q) < r'' \tag{12.12}$$

also satisfy the condition

$$d'(P, Q) < \frac{\rho'}{8}. \tag{12.13}$$

To establish the existence of the constant r' we have merely to note that $d''(P, Q)$ is a uniformly continuous function of P and Q on S', and that $d''(P, Q) = 0$ when $d'(P, Q) = 0$. The existence of the constant r'' is similarly established.

The space Ω determined by S. Let S be a compact metric space with metric M_ρ. The distance between any two points P and Q on an elementary arc $[PQ]$ will be termed the M-length of $[PQ]$ or the M-distance between P and Q. A set of p points on S, given in a circular order, two successive points of which have an M-distance at most ρ, will be termed an admissible point (π). We suppose that $p > 2$. The space Π^p corresponding to S will now be defined as the totality of all admissible points (π) with p vertices, regarding two points (π) which can be obtained one from the other by a transformation of the group G^p of §1 as identical. Contracted points on Π^p are defined as previously. With this understood the space Ω will be taken as the ensemble of the spaces Π^p, $p = 3, 4 \cdots$.

Chains and cycles on Ω are now defined as in §2. Replacing J-length and J-distance by M-length and M-distance respectively, partitions of chains on Π^p are defined as in §2. Ordinary and special homologies on Ω are then introduced as previously, leading finally to the definition of the connectivities of Ω. It will be convenient to term these connectivities of Ω the *circular connectivities* of S.

We now return to the spaces S' and S'' and denote the corresponding spaces Ω by Ω' and Ω'' respectively. We regard corresponding points on S' and S'' as identical. We note that points (π) which are admissible on Ω' or Ω'' may not be admissible on Ω'' or Ω' respectively. Points (π) or chains of points (π) which are admissible both on Ω' and Ω'' will ordinarily be denoted by the same symbol.

Let r_0 be a positive constant such that

$$r_0 < r', \qquad r_0 < r'',$$

where r' and r'' are the constants of Lemma 12.1. Two points P, Q on S' and S'' will be said to be *admissible rel* r_0 if

$$d'(P, Q) < \frac{r_0}{8}, \qquad d''(P, Q) < \frac{r_0}{8}. \tag{12.14}$$

A point (π) will be said to be *admissible rel* r_0 if each pair of successive vertices of (π) satisfy (12.14). We now state the following lemma.

LEMMA 12.2. *Let z be a k-cycle on Π'^p and Π''^p which is admissible rel r_0. If z' and z'' are respectively r-fold partitions of z on Π'^{rp} and Π''^{rp}, then*

$$z' \sim z'' \tag{12.15}$$

on both Π'^{rp} and Π''^{rp}.

We shall show that (12.15) holds on Π'^{rp}. The proof that (12.15) holds on Π''^{rp} is similar.

Let (π) be a point on z, and P, Q a pair of successive vertices of (π). Let h' and h'' be the elementary arcs determined by P, Q on S' and S'' respectively.

Let Q' and Q'' be points which divide h' and h'' in the ratio of μ to ν on S' and S'' respectively. We suppose that μ and ν are positive integers such that

$$\mu + \nu = r.$$

The points Q' and Q'' will be vertices on the r-fold partitions of (π) relative to S' and S'' respectively.

We shall now prove that

$$d'(Q', Q'') < \frac{\rho'}{4}.$$

We start with the relation

$$d'(Q', Q'') \leqq d'(Q', P) + d'(P, Q''). \tag{12.16}$$

With the aid of (12.14), and the fact that $r_0 < \rho'$, we see that

$$d'(Q', P) \leqq d'(Q, P) < \frac{r_0}{8} < \frac{\rho'}{8}$$

Similarly

$$d''(P, Q'') \leqq d''(P, Q) < \frac{r_0}{8},$$

and from Lemma 12.1 we then infer that

$$d'(P, Q'') < \frac{\rho'}{8}.$$

From (12.16) we thus find that

$$d'(Q', Q'') < \frac{\rho'}{8} + \frac{\rho'}{8} = \frac{\rho'}{4}$$

as stated.

The points Q' and Q'' can then be joined by an elementary arc k on S'. We deform Q' along k to Q'', moving Q' so that its distance on S' from its initial position increases at a rate equal to the length of k on S'. The cycle z' will thereby be deformed into the cycle z''.

Let (π') be the r-fold partition of (π) relative to S', and let (π_t) be the point through which (π') is deformed. It remains to show that (π_t) is admissible on Π''^r, that is, that successive vertices of (π_t) possess a distance at most ρ' on S'. To that end let Q_t be the point into which the vertex Q' is deformed at the time t. We note that

$$\begin{aligned} d'(Q_t, P) &\leqq d'(Q_t, Q') + d'(Q', P) \\ &< \frac{\rho'}{4} + \frac{r_0}{8} \\ &< \frac{\rho'}{2}. \end{aligned}$$

Thus the distance on S' between Q_t and its adjacent vertices in (π_t) will be at most ρ'. The point (π_t) is accordingly admissible relative to S'.

The proof of the lemma is now complete.

The principal theorem in this section is the following.

THEOREM 12.1. *The circular connectivities of two homeomorphic, compact, metric spaces possessing elementary arcs are the same*

As previously we denote the two spaces by S' and S'' and represent corresponding chains on Ω' and Ω'' by the same symbol.

To prove the theorem let H be any set of k-cycles on Ω' satisfying no proper Ω'-homology. Without loss of generality we can suppose that the cycles of H are admissible rel r_0, and consist of points (π) with a fixed number of vertices. For if this were not the case, suitable partitions of the cycles of H would satisfy these requirements and would again form a maximal set of k-cycles on Ω' subject to no proper Ω'-homology.

So chosen, the cycles of H are cycles of Ω'' as well We shall now prove that no proper combination z of k-cycles of H satisfies an Ω''-homology

$$z * 0. \tag{12.17}$$

If the contrary were true there would exist a subdomain Π''^q of Ω'' upon which the chains "involved" in the Ω''-homology (12.17) would possess partitions. Without loss of generality we can suppose q so large a positive integer that these partitions lie on Π'^q as well as on Π''^q. The relation (12.17) thus implies an homology

$$z'' \sim 0 \tag{12.18}$$

on both Π''^q and Π'^q, where z'' is a partition of z relative to S''.

Let z' be a partition on Π'^q of z, relative to S'. It follows from Lemma 12.1 that

$$z' \sim z'' \tag{12.19}$$

on both Π''^q and Π'^q. Moreover we have the special homology

$$z * z' \qquad (\text{on } \Omega'). \tag{12.20}$$

From (12.18), (12.19), and (12.20) we infer that

$$z * 0 \qquad (\text{on } \Omega'),$$

contrary to the nature of the set H.

We conclude that (12.17) is impossible, and that the kth connectivity of Ω'' is at least as great as that of Ω'. Interchanging the rôles of Ω' and Ω'' we infer that the kth connectivities of Ω' and Ω'' are equal.

The proof of the theorem is now complete.

Returning now to our Riemannian manifold R we see that our elementary extremals and J-lengths serve to define a particular metric on R with elementary arcs. We can thus regard R as a compact metric space S' with metric $M'_{\rho'}$. If the functional J is now replaced by another functional of the same general character, R will give rise to a metric space S'' with a different metric $M''_{\rho''}$. If points of S' and S'' which represent the same point of R are regarded as corresponding, S' and S'' are seen to be homeomorphic. We are thus led to the following theorem.

THEOREM 12.2. *The circular connectivities of R are independent of a change in the functional J with whose aid Ω is defined, provided J is replaced by a functional of the same general character.*

We add the following theorem.

THEOREM 12.3. *The circular connectivities of homeomorphic Riemannian manifolds are equal. For such manifolds the circular connectivities never fail to exist.*

By virtue of the preceding theorem any admissible function J can be used to define the circular connectivities of R. In particular one can always take J as the integral of arc length.

Moreover by virtue of the preceding abstract theory it is *immaterial* whether the homeomorphism between two given Riemannian manifolds R' and R'' can be effected by analytic transformations or not, and this point is highly important. It is sufficient that the compact metric spaces S' and S'' respectively defined by R' and R'' and their geodesics be homeomorphic. This condition of homeomorphism between S' and S'' is always fulfilled if R' and R'' are homeomorphic.

CHAPTER IX

SOLUTION OF THE POINCARÉ CONTINUATION PROBLEM

The problem of the existence of closed geodesics on a convex surface was considered by Poincaré in connection with his studies in celestial mechanics (Poincaré [2], Birkhoff [1, 3], Morse [7, 17], Schnirrelmann and Lusternik [1]). Poincaré supposed that the given surface was a member of a family of convex surfaces S_α depending analytically on a parameter α ranging over a finite interval $0 \leqq \alpha \leqq 1$. He supposed that one member of the family, say S_0, was an ellipsoid with unequal axes. On the ellipsoid there are three principal ellipses. According to Poincaré, upon varying the parameter α, closed geodesics appear and disappear in pairs and the analytic continuation of the three principal ellipses will lead to an odd number of closed geodesics. One should recall that there exist closed geodesics on S_0 other than the principal ellipses, but if the semi-axes of the ellipsoid are unequal and sufficiently near unity, the remaining closed geodesics have lengths which are arbitrarily large. Certainly then for values of α sufficiently near zero, an odd number of closed geodesics on S_α can be obtained from the principal ellipses on S_0 following the method of analytic continuation.

Just how far this method can be carried is not clear. It is unquestionably useful in limited cases. Relative to its general use the writer wishes to point out certain difficulties and limitations.

Among the difficulties are the following. (1). The principle that closed geodesics appear and disappear in pairs needs to take account of the fact that infinite families of closed geodesics of the same length can appear on S_α for isolated values of α when, for example, the surface becomes a sphere or a spheroid. (2). As one varies α on the whole interval $0 \leqq \alpha \leqq 1$, it is conceivable that the continuation of the principal ellipses may coalesce with the continuation of some of the closed geodesics on S_0 whose lengths were initially very large. In fact one is really dealing with the continuation of an infinite class of closed geodesics of infinitely many lengths, and not merely with a finite odd number of such geodesics.

Limitations on the method are the following. (1). If one passes from the 2-dimensional to the m-dimensional ellipsoid, it appears that the number of principal ellipses is sometimes odd and sometimes even, depending on m, and the Poincaré principle fails when the number is even. (2). The Poincaré method when valid affirms the existence of at least one closed geodesic, whereas in the m-dimensional case we shall see that the existence of

$$\frac{m(m+1)}{2}$$

closed geodesics with lengths commensurate in size with the lengths of the principal ellipses on an m-ellipsoid can in general be affirmed to exist. (3). The characterization of classes of closed geodesics according to the oddness or evenness of the number of closed geodesics therein is obviously inadequate in view of the possibility of classification by means of type numbers, and offers no characterization for families of closed geodesics. (4). We shall see that it is possible to give an adequate and general theorem on the *continuation of the type numbers* of a critical set of closed geodesics, backed by existence theorems applicable to *each* member of the family S_α. (5). The Poincaré method does not *distinguish* adequately between the continuation of the principal ellipses and the continuation of the remaining geodesics on the ellipsoid. We shall show that it is possible to *relate* the principal ellipses on the m-ellipsoid to a topologically defined class of closed geodesics on any regular, analytic homeomorph of the m-sphere. The geodesics so related to the principal ellipses on the m-ellipsoid stand in remarkable metric, as well as topological, relations to these ellipses.

The existence of three closed geodesics on any closed convex surface subject to certain limitations was first established by Birkhoff ([2], p. 180). Birkhoff ([3], p. 139) also established the existence of at least one closed geodesic on any analytic homeomorph of an m-sphere.

The Poincaré problem of the continuation of a closed geodesic on a convex surface with respect to a parameter α will be solved as a part of a more general problem which we formulate as follows.

GENERAL PROBLEM. I. *To define numerical characteristics of sets of closed geodesics on R, the possession of which by a particular set of closed geodesics is sufficient to guarantee the existence of a corresponding set H of closed geodesics on any other admissible Riemannian manifold homeomorphic with R.*

II. *To show that the preceding set H varies analytically (in a manner to be made precise) with any parameter α with which R varies analytically.*

We shall solve this general problem, and apply our results to any Riemannian manifold which is the homeomorph of an m-sphere. To that end we shall first determine the circular connectivities of the m-sphere as defined in Ch. VIII. We shall show in what sense the $m(m+1)/2$ principal ellipses on an m-ellipsoid with unequal axes can be continued analytically, as the m-ellipsoid is varied analytically. In the final section we shall state and prove a basic continuation theorem.

Although we confine ourselves to the reversible case, the methods and results hold with obvious changes in the irreversible case with positive integrand.

It seems appropriate in this place to point out the fundamental difference between the methods employed by Birkhoff and those employed by the author in the study of periodic orbits. These methods are complementary. The method most frequently employed by Birkhoff is based on the theory of fixed points of transformations. Once a periodic orbit is given, the transformation theory of Birkhoff is decidedly effective in discovering the infinitely many other closed orbits which may exist nearby. The Birkhoff theory also affords a deep characterization of local stability together with a development of the basic conceptions of recurrence and transitivity in the large. The reader is referred to Birkhoff's numerous papers on this subject. The methods of the author are generalizations for functionals of the theory of critical points of functions. The theory of the author as developed so far has been successful in obtaining a priori existence theorems on closed extremals, in classifying such extremals in the large, and in solving the general continuation problem.

It is not implied that either theory is inapplicable to the domain of the other. In fact for dynamical systems with two degrees of freedom the transformation theory is highly successful both in the small and in the large. This is doubtless due to the fact that the theory of the distribution of vectors in a 2-space is essentially equivalent to the theory of critical points of a function of two variables. In fact such a system of vectors, if suitably altered in length, will in general become the gradients of a function. This relation between the fixed point theory and the critical point theory does not persist however in higher spaces.

On the other hand investigations now under way by the author as to the manner in which the "index" of a closed extremal g is related to the index of multiples of g indicate a closer connection between the formal aspects of the two theories than has yet been disclosed. A preliminary paper by Hedlund [1] in the case $n = 2$ bears this out. In all events the further study of the inter-relations between the two theories seems likely to be one of the most fascinating subjects for research in the future.

The reader may also refer to a paper by Schnirrelmann and Lusternik [1]. In addition to refining the results of Birkhoff concerning the closed geodesics on the homeomorph of a 2-sphere these authors introduce a deformation principle which leads to certain types of critical points of functions and functionals.

Regular submanifolds of R^p

1. We continue the theory of non-degenerate closed extremals g of §11, Ch. VIII. As previously let c be the J-length of g, and p any positive integer such that $p\rho > c$. It (π_0) be a point on R^p which determines g. Suppose that the length of none of the elementary extremals of $g(\pi_0)$ is 0 or ρ. Let the arc length on g be measured in an arbitrary sense from an arbitrary point. Starting with an arbitrary vertex of (π_0) let the values of t at successive vertices of (π_0) be

$$t_1 < t_2 < \cdots < t_p.$$

Let the local coordinates (x) on R neighboring the qth vertex of (π_0) be denoted by (x^q), and neighboring this vertex suppose g has the representation

$$x_i^q = x_i^q(t) \qquad (q = 1, \cdots, p; i = 1, \cdots, m).$$

Let $u_1, \cdots, u_p$ be parameters which assume values near $t_1, \cdots, t_p$ respectively. The p-manifold

$$
\begin{aligned}
x_i^1 &= x_i^1(u_1) \qquad\qquad (i = 1, \cdots, m), \\
&\cdots\cdots\cdots \\
x_i^p &= x_i^p(u_p)
\end{aligned} \tag{1.0}
$$

will be a regular submanifold of R^p neighboring (π_0). It will be termed the *extremal manifold* on R^p neighboring (π_0).

Let Z be an arbitrary regular analytic submanifold of R^p passing through (π_0). We here *admit* only those regular submanifolds of R^p which consist of points (π) none of whose elementary extremals have the length 0 or ρ. If the manifold Z has no tangent line in common with the extremal manifold (1.0) at the point (π_0), Z will be termed a *non-tangential submanifold* of R^p belonging to g. If Z is regularly represented in terms of parameters (v) in the form

$$x_i^q = \varphi_i^q(v) \qquad\qquad (q = 1, \cdots, p; i = 1, \cdots, m)$$

with $(v) = (0)$ corresponding to (π_0), a necessary and sufficient condition that Z be non-tangential is that when $(v) = (0)$, the columns of the functional matrix of the functions $\varphi_i^q(v)$ be independent of the columns of the functional matrix of the functions $x_i^q(u)$ when $(u) = (t_1, \cdots, t_p)$.

We shall prove the following theorem.

THEOREM 1.1. *Corresponding to any non-tangential submanifold Z of R^p belonging to g, there exists a proper section S of R^p belonging to g, possessing the following properties. If (π_0) is the point on Z which determines g, the correspondence between points on Z sufficiently near (π_0) and their extremal projections on S is one which is analytic and non-singular, and in which points (π) on Z can be J-deformed into their extremal projections on S by suitably moving their vertices along the corresponding curves $g(\pi)$.*

We shall confine the proof of the theorem to the case of orientable manifolds R. On such a manifold the neighborhood of g can be referred to an analytic coordinate system

$$(x, y_1, \cdots, y_n) \qquad\qquad (n = m - 1)$$

such that g corresponds to a segment of the x axis of length ω, and two points (x, y) whose coordinates y_i are the same but whose coordinates x differ by an integral multiple of ω, represent the same point on R. Such a coordinate system will be explicitly exhibited in the case of the closed geodesics to which we shall apply the theorem, so that any general proof of the existence of such a coordinate system can be omitted.

Let the qth vertex of (π_0) be represented by the point $x = a^q$ on the x axis, with

$$a^1 < a^2 < \cdots < a^p < a^1 + \omega. \tag{1.1}$$

In terms of the parameters (v) representing Z the qth vertex of (π) will have an image in the space (x, y) such that

$$x = x^q(v) \qquad (q = 1, \cdots, p) \tag{1.2}$$

where

$$x^q(0) = a^q,$$

and where the functions $x^q(v)$ are analytic in the variables (v) for (v) neighboring (0). It will now be convenient to set

$$x^{p+1}(v) \equiv x^1(v) + \omega.$$

The elementary extremal of $g(\pi)$ which begins at the qth vertex of (π) will have an image in the space (x, y) of the form

$$y_i = y_i^q(x, v) \qquad (i = 1, \cdots, n) \tag{1.3}$$

for x on the interval

$$x^q(v) \leqq x \leqq x^{q+1}(v) \qquad (q = 1, \cdots, p) \tag{1.4}$$

where the functions $y_i^q(x, v)$ are analytic in x and (v) for x on the interval (1.4) and (v) neighboring (0).

We shall now determine the analytic consequences of the fact that Z is a non-tangential submanifold of R^p. The qth vertex of a point (π) on Z has a representation in the space (x, y) of the form

$$y_i = y_i^q(x^q(v), v),$$
$$x = x^q(v).$$

Since the x axis is an extremal, we see that

$$y_{ix}^q(x, 0) \equiv 0. \tag{1.5}$$

Let us indicate partial differentiation with respect to v_h by adding the subscript h. A necessary and sufficient condition that Z have no tangent line in common with the extremal manifold (1.0) at the point (π_0) is that the matrix

$$\| y_{ih}^q(a^q, 0) \| \qquad (h = 1, \cdots, r;\ i = 1, \cdots, n;\ q = 1, \cdots, p) \tag{1.6}$$

of qn rows and r columns have the rank r.

We are now in a position to choose the proper section S of R^p whose existence is affirmed in the theorem. We take the manifold on R on which the qth vertex of the point (π) on S rests as the image on R of the n-plane

$$x = a^q + e, \tag{1.7}$$

where e is a positive constant yet to be determined. The extremal projection on S of the point (v) on Z will be a point with qth vertex

$$
\begin{aligned}
y_i &= y_i^q(a^q + e, v), \\
x &= a^q + e.
\end{aligned}
\tag{1.8}
$$

We choose the constant e so small that the functional matrix

$$
\| y_{ih}^q(a^q + e, 0) \|
\tag{1.9}
$$

has the rank r. This is possible since the matrix (1.6) has the rank r.

From the fact that the matrix (1.9) has the rank r it follows that the relation (1.8) between the point (v) on Z and the corresponding point on S is one-to-one, non-singular, and analytic, provided (v) is sufficiently near (0). Finally any point (π) on Z (below c), sufficiently near (π_0) can be deformed (below c) into its extremal projection (π') on S by moving the qth vertex of (π) along the qth elementary extremal of $g(\pi)$ to the qth vertex of (π'), moving each vertex at a J-rate equal to the J-length to be traversed.

The proof of the theorem is now complete.

We continue with the following lemma.

LEMMA 1.1. *Let $f(v_1, \cdots, v_k)$ be an analytic function of the variables (v) neighboring $(v) = (0)$, assuming a proper, relative maximum c when $(v) = (0)$. If e is a sufficiently small positive constant, the closure of the domain*

$$
f(v) \geqq c - e
\tag{1.10}
$$

neighboring $(v) = (0)$ will contain no critical points of f other than $(v) = (0)$, and will have the locus

$$
f(v) = c - e
\tag{1.11}
$$

for its boundary neighboring $(v) = (0)$. Moreover any k-cycle on (1.10), below c, which is not homologous to zero on (1.10), below c, will be homologous on (1.10), below c, to the $(k - 1)$-cycle (1.11).

The point $(v) = (0)$ affords a proper maximum to $f(v)$, and must accordingly be an isolated critical point. The lemma then follows readily except at most for the concluding statement. This final statement of the lemma is a consequence of the results on maximizing critical sets in §7, Ch. VI.

We first note that the function $\varphi = c - f$ is a neighborhood function corresponding to f and the critical point $(v) = (0)$, at least on the domain (1.10). According to the results on maximizing critical sets in Ch. VI the type number m_k of $(v) = (0)$ equals unity, so that there is a single spannable $(k - 1)$-cycle in a maximal set of such cycles corr $\varphi \leqq e$. But the cycle (1.11) is such a spannable $(k - 1)$-cycle, and any cycle on (1.10), below c, which is not homologous to zero, below c, is likewise a spannable $(k - 1)$-cycle, so that the concluding statement of the lemma is true.

The principal theorem of this section is the following.

THEOREM 1.2. *Let Z be a non-tangential submanifold of R^p belonging to a non-degenerate closed extremal g. Suppose the index k of g is positive and equals the dimension of Z. If $J(\pi)$ assumes a proper maximum c on Z at the point (π_0) which determines g, the locus*

$$J(\pi) = c - e \qquad (on\ Z) \tag{1.12}$$

will be a spannable $(k - 1)$-cycle on Ω belonging to g, provided e is a sufficiently small positive constant.

Let (v) be a set of parameters in a regular representation of Z neighboring (π_0). Suppose that $(v) = (0)$ corresponds to (π_0). Let $f(v)$ be the value of $J(\pi)$ at the point (π) determined by (v). The function $f(v)$ has a proper maximum c at the origin $(v) = (0)$. We now identify this function $f(v)$ with the function $f(v)$ of the preceding lemma, and choose e so that the lemma is satisfied.

Corresponding to Z let S be the proper section of R^p belonging to g whose existence is affirmed in the preceding theorem. Let T denote the correspondence between a point (π) on Z and its extremal projection on S. We suppose that the constant e in (1.10) is so small that the correspondence T between the domain (1.10) on Z, and its extremal projection ζ on S will be analytic and non-singular. If the extremal projection on ζ of a point (v) on (1.10) be assigned the parameters (v), ζ appears as a regular, analytic, k-dimensional submanifold of S.

Let $F(v)$ denote the value of $J(\pi)$ at the point on ζ determined by (v). We shall continue with a proof of the following statement.

(A). *If η is a sufficiently small positive constant, the locus*

$$F(v) = c - \eta \qquad (on\ S) \tag{1.13}$$

neighboring $(v) = (0)$ will be a spannable $(k - 1)$-cycle s_{k-1} belonging to the critical set σ determined by g on Ω.

To prove (A) first observe that

$$F(v) \leqq f(v), \qquad F(0) = f(0) = c.$$

It thus appears that $F(v)$ takes on a proper maximum c when $(v) = (0)$. On the other hand let (u) be a set of parameters in a regular representation of S neighboring the point (π_1) which determines g. Suppose that $(u) = (0)$ corresponds to (π_1). Let $\psi(u)$ be the value of $J(\pi)$ at the point (π) on S determined by (u). By virtue of the definition of the index and nullity of g in §10, Ch. VIII, $\psi(u)$ will have in $(u) = (0)$ a non-degenerate critical point of index k. Now ζ is a regular, analytic submanifold of S, and as such will be represented in the space (u) by a regular, analytic sub-k-manifold on which ψ will assume a proper maximum at the origin. It follows from Theorem 7.5 of Ch. VI that if η is a sufficiently small positive constant, the locus (1.13) will be a spannable $(k - 1)$-cycle on S belonging to the function $\psi(u)$ and the critical point $(u) = (0)$.

But according to Theorem 11.3 of Ch. VIII such a spannable $(k - 1)$-cycle, if sufficiently near (π_1), will be a spannable $(k - 1)$-cycle on Ω belonging to the critical set σ determined by g.

Statement (A) is accordingly proved.

The cycle s_{k-1} of (A) lies on S. Suppose that it is the extremal projection on S of the cycle z_{k-1} on Z. As stated in the preceding theorem z_{k-1} can be J-deformed on Π^p, into s_{k-1} on S, arbitrarily near σ^p, if s_{k-1} is sufficiently near σ^p. If then η is sufficiently small, z_{k-1} will share with s_{k-1} the property of being a spannable cycle on Ω belonging to σ.

But z_{k-1} lies on the domain of Z defined by (1.10), and being spannable cannot bound on this domain below c. It follows from the preceding lemma that z_{k-1} is homologous below c on the domain (1.10) of Z to the $(k - 1)$-cycle (1.11) of Z. The $(k - 1)$-cycle (1.11) must then share with z_{k-1} and s_{k-1} the property of being a spannable $(k - 1)$-cycle on Ω belonging to the critical set σ.

The proof of the theorem is now complete.

Geodesics on m-ellipsoids

2. We shall reduce the determination of the circular connectivities of the m-sphere to an analysis of the closed geodesics on ellipsoids. We begin with an m-ellipsoid $E_m(a_1, \cdots, a_{m+1})$ given by the condition

$$a_1^2w_1^2 + \cdots + a_{m+1}^2w_{m+1}^2 = 1, \tag{2.1}$$

where

$$a_i > 0 \qquad (i = 1, \cdots, m + 1). \tag{2.2}$$

By the principal ellipse g_{ij} of $E_m(a)$, $i \neq j$, we mean the ellipse in which the 2-plane of the w_i, w_j axes meets $E_m(a)$. The number of principal ellipses is $(m + 1)m/2$. These principal ellipses are closed geodesics. The determination of the indices of the principal ellipses on $E_m(a)$ can be reduced to a determination of the indices of the principal ellipses of an ordinary ellipsoid $E_2(a)$. We now investigate the principal ellipses on E_2.

The ordinary ellipsoid $E_2(a)$. Let S represent a regular, analytic, orientable surface S, and g a simple closed geodesic on S. Since S is orientable, g has two sides, one of which may be termed positive and the other negative. Neighboring g let S be referred to coordinates (x, y) of which y represents the geodesic distance from a point P to g, taken as positive on the positive side of g and negative on the negative side of g. Let x represent the distance along g in a prescribed sense from a prescribed point on g to the foot of a geodesic through P and orthogonal to g. If ω is the length of g, each point P will possess infinitely many x-coordinates, $x + \mu\omega$, where μ is an integer, positive, negative, or zero. It will simplify matters if we think of the problem as given in the (x, y)-plane with the x axis an extremal. For our purposes the integral of arc length can be taken in non-parametric form with x as the independent variable and y the dependent variable. The integrand then has the period ω in x, and the problem is of the same nature as the problem in §11, Ch. III.

As shown in Bolza [1], p. 231, the Jacobi equation corresponding to g or the x axis takes the form

$$\frac{d^2y}{dx^2} + K(x)y = 0, \tag{2.3}$$

where $K(x)$ is the total curvature of S at the point x on g. The nullity of g as a closed extremal, determined in accordance with Theorem 10.1 of Ch. VIII, will be the number of linearly independent solutions of (2.3) of period ω. The nullity of g is thus 0, 1, or 2. We continue with the following lemma.

LEMMA 2.1. *If the nullity of g is 1 and $u(x)$ is a non-null solution of* (2.3) *with period ω, the only points x for which x is conjugate to $x + \omega$ are the points at which $u(x) = 0$.*

Let $x = a$ be a point conjugate to $x = a + \omega$. Let $v(x)$ be a solution of (2.3) such that

$$v(a) = 0, \qquad v'(a) \neq 0.$$

We make use of Abel's integral, by virtue of which

$$u(x)v'(x) - u'(x)v(x) \equiv \text{const.} \tag{2.4}$$

By hypothesis

$$v(a + \omega) = 0, \qquad v'(a + \omega) \neq 0.$$

Upon setting $x = a$ and then $a + \omega$ in (2.4), we find that

$$u(a)[v'(a) - v'(a + \omega)] = 0. \tag{2.5}$$

Condition (2.5) leads to two cases.

Case I. $u(a) = 0$.

Case II. $u(a) \neq 0$.

If Case II holds, it follows from (2.5) that

$$v'(a) = v'(a + \omega),$$

and since

$$v(a) = v(a + \omega) = 0,$$

we conclude that $v(x)$ has the period ω. But $v(x)$ is independent of $u(x)$ since in Case II

$$v(a) = 0, \qquad u(a) \neq 0,$$

from which it follows that the nullity of g is 2. From this contradiction we infer that Case II is impossible.

Thus Case I holds and the lemma is proved.

Let s denote the arc length along the ellipse g_{ij}. Let the length of g_{ij} be denoted by g_{ij}. We shall prove the following lemma.

LEMMA 2.2. *Corresponding to constants $a_1 > a_2 > a_3$ sufficiently near unity, the principal ellipses of $E_2(a_1, a_2, a_3)$ have the following properties.*

(a). *To each point s_0 on g_{12} there corresponds just one conjugate point s for which $s_0 < s < s_0 + g_{12}$ while s_0 is never conjugate to $s_0 + g_{12}$.*

(b). *To each point s_0 on g_{23} there correspond just two conjugate points x for which $s_0 < s < s_0 + g_{23}$ while s_0 is never conjugate to $s_0 + g_{23}$.*

(c). *On g_{13} opposite umbilical points are conjugate to each other and to no other points. The geodesic g_{13} is non-degenerate.*

We need the fact that the total curvature $K(s)$ of $E_2(a_1, a_2, a_3)$ at the point s on g_{12} will increase with a_3. To see this one notes that $K(s)$ is the product of the curvature k_1 of g_{12} at the point s, and the curvature k_2 of the ellipse in which a plane orthogonal to g_{12} at the point s cuts $E_2(a_1, a_2, a_3)$. An increase of a_3 will not alter k_1, but will diminish the axis of the ellipse orthogonal to g_{12}. It will accordingly increase k_2, and hence $K(s)$.

We shall prove the following statement. In this statement we set the length $g_{12} = \omega$.

(A). *On the ellipse g_{12} of $E_2(a_1, a_2, a_2)$ the distance Δs from a point s to its first following conjugate point, measured in the sense of increasing s, exceeds $\omega/2$ for all points s except the intersections of g_{12} with the w_1 axis, for which points $\Delta s = \omega/2$.*

Let s be measured from the point of intersection of g_{12} with the positive w_1 axis. All geodesics through the point $s = 0$ on g_{12} on $E_2(a_1, a_2, a_2)$ go through the opposite point on g_{12}, and form a field otherwise. These geodesics are ellipses. It follows that the point $s = 0$ is conjugate to the points $s = \omega/2$ and $s = \omega$ on g_{12}, and that any solution of (2.3) which vanishes when $x = s = 0$ has the period ω.

On the other hand on the ellipse g_{12} of $E_2(a_1, a_2, a_2)$, $K(s)$ is less than the total curvature at the same point on $E_2(a_1, a_2, a_1)$. But on $E_2(a_1, a_2, a_1)$ the point $s = \omega/4$ on g_{12} is conjugate to the opposite point on g_{12} and to no other points on g_{12}, as one proves by considering the geodesics through $s = \omega/4$ on g_{12}. An application of the Sturm Comparison Theorem to (2.3) now shows that the distance from the point $s = \omega/4$ on g_{12} to its first conjugate point on g_{12} exceeds $\omega/2$, on $E_2(a_1, a_2, a_2)$. It follows from Lemma 2.1 that the same is true for all points of g_{12} on $E_2(a_1, a_2, a_2)$ except the points conjugate to $s = 0$. To see this one varies the initial point s from $\omega/4$ to 0 or to $\omega/2$. If during this variation the distance Δs from s to its first conjugate point should reduce to $\omega/2$, the symmetry of the ellipsoid shows that the point s would be conjugate to the point $s + \omega$, contrary to Lemma 2.1, unless $s = 0$ or $\omega/2$.

Thus statement (A) is proved.

We can now prove (a) of Lemma 2.2.

The curvature $K(s)$ on the ellipse g_{12} of $E_2(a_1, a_2, a_3)$ is less than that at the same point on $E_2(a_1, a_2, a_2)$. It follows from (A) and the Sturm Comparison Theorem, that to each point s on the ellipse g_{12} of $E_2(a_1, a_2, a_3)$ there corresponds at most one conjugate point prior to or including $s + \omega$. If we compare $E_2(a_1, a_2, a_3)$ with the unit sphere, we see that if the constants a_i are sufficiently near unity, there will be exactly one conjugate point of the point s prior to the point $s + \omega$ on g_{12}, and the point $s + \omega$ will not be conjugate to s. The proof of (a) is now complete.

To prove (b) one first proves the following statement, setting the length $g_{23} = \omega$.

(B). *On the ellipse g_{23} of $E_2(a_2, a_2, a_3)$ the distance Δs from a point s to the first following conjugate point is less than $\omega/2$ for all points except the intersections of g_{23} with the w_3-axis, for which points $\Delta s = \omega/2$ ($g_{23} = \omega$).*

To prove (B) we compare g_{23} on $E_2(a_2, a_2, a_3)$ with g_{23} on $E_2(a_3, a_2, a_3)$, and use Lemma 2.1 as in the proof of (A). To prove (b) we make use of (B), comparing g_{23} on $E_2(a_1, a_2, a_3)$ with g_{23} on $E_2(a_2, a_2, a_3)$.

To prove (c) we recall that the geodesics through an umbilical point pass through the opposite umbilical point, but form a field otherwise. Hence each umbilical point on g_{12} is conjugate to the opposite umbilical point, and to no other points on g_{12}.

We shall now prove that the nullity of g_{13} is not 2. If the nullity of g_{13} were 2, each point s on g_{13} would be conjugate to the corresponding point $s + \omega$. After a slight decrease of a_2 on $E_2(a_1, a_2, a_3)$, no point $s + \omega$ would be conjugate to the corresponding point s, contrary to the fact that there would still be umbilical points on g_{13} if $a_1 > a_2 > a_3$. Thus the nullity of g_{13} cannot be 2.

Finally the nullity of g_{13} cannot be 1. If the nullity of g_{13} were 1, set $s = x$ and let $u(x)$ be the non-null periodic solution of (2.3). According to Lemma 2.1, $u(x)$ would vanish at each umbilical point, since each umbilical point s_0 is conjugate to the point $s_0 + \omega$. Thus $u(x)$ would vanish at each of the four umbilical points, contrary to the fact that these four points are not mutually conjugate.

Thus the nullity of g_{13} must be zero, and statement (c) is proved.

We shall conclude this section with a proof of the following theorem.

THEOREM 2.1. *If the constants $a_1 > a_2 > a_3$ of the ellipsoid*

$$a_1^2 w_1^2 + a_2^2 w_2^2 + a_3^2 w_3^2 = 1 \tag{2.6}$$

are sufficiently near 1, *the principal ellipses g_{12}, g_{13}, and g_{23} are non-degenerate and possess the indices* 1, 2 *and* 3 *respectively.*

Let distances s on g_{13} be measured from an umbilical point. By virtue of (c) in Lemma 2.2, there is just one point on g_{13} conjugate to the point $s = 0$ on the interval $0 < s < \omega$, where $\omega = g_{13}$. Moreover the point $s = 0$ is conjugate to the point $s = \omega$. It follows from (A) in §11, Ch. III, that the index of g_{13} is 2.

It follows similarly from (a) and (c) in Lemma 2.2, and from (B) in §11, Ch. III, that the indices of g_{12} and g_{13} are 1 and 3 respectively if the semi-axes of (2.6) are sufficiently near unity.

In Lemma 2.2 (c) we have expressly affirmed that g_{13} is non-degenerate. That g_{12} and g_{23} are non-degenerate if the semi-axes of $E_2(a)$ are sufficiently near unity follows respectively from the statement in Lemma 2.2 (a) that on g_{12} a point s is never conjugate to $s + g_{12}$, and the statement in (b) that on g_{23} a point s is never conjugate to $s + g_{23}$.

The proof of the theorem is now complete.

The indices of the ellipses g_{ij}

3. We shall determine the indices of the principal ellipses g_{ij} of the m-ellipsoid $E_m(a)$ of (2.1). By the *principal ellipsoids* of $E_m(a)$ we mean those 2-dimensional ellipsoids which are obtained from $E_m(a)$ by setting all of the coordinates (w) equal to zero save three.

The equations of the geodesics on $E_m(a)$ can be put in the form

$$(3.1)' \qquad w_j'' + \lambda w_j a_j^2 = 0 \qquad (j \text{ not summed}),$$

$$(3.1)'' \qquad a_i^2 w_i'^2 = 1 \qquad (i, j = 1, \cdots, m+1),$$

where the independent variable is the arc length s, and (w) and λ are dependent variables to be determined as analytic functions of s by the conditions (3.1). It follows from the equations (3.1) that the geodesics on any principal ellipsoid of $E_m(a)$ are geodesics on $E_m(a)$.

We shall prove the following theorem.

THEOREM 3.1. *The index and nullity of the principal ellipse g_{ij} on the ellipsoid $E_m(a)$ is the sum of the indices and nullities of g_{ij} regarded as an ellipse on each of the $m - 1$ principal 2-dimensional ellipsoids on which it lies.*

For simplicity we shall give the proof of this theorem for the ellipse $g_{m,\, m+1}$. We denote this ellipse by g.

We shall need a parametric representation of $E_m(a)$ neighboring g. Such a representation can be given in terms of parameters

$$(x, y_1, \cdots, y_n) \qquad (n = m - 1)$$

with the points $(y) = (0)$ corresponding to g. In fact if we set

$$r^2(y) = 1 - a_h^2 y_h^2 \qquad (h = 1, \cdots, n),$$

we can represent $E_m(a)$ near g parametrically in the form

$$w_h = y_h \qquad (h = 1, \cdots, n),$$

$$(3.2) \qquad w_m = \frac{r(y) \cos x}{a_m} \qquad (r > 0),$$

$$w_{m+1} = \frac{r(y) \sin x}{a_{m+1}}.$$

The variables (y) are limited to sets near (0). If the representation is to be one-to-one, it will be necessary to limit x to some such interval as the interval $0 < x \leqq 2\pi$.

As in §10, Ch. VIII, we associate g with a boundary problem in the space (x, y) in which the integral is the arc length on $E_m(a)$, and the end conditions require that $x = 0$ at the first end point, $x = \omega$ at the second end point, and that at these end points the coordinates y_i be the same. As we have seen in §10, Ch. VIII, the index of g will be that of an index form corresponding to the critical extremal

$$(y) = (0) \qquad\qquad (0 \leqq x \leqq 2\pi)$$

in this boundary problem. We proceed to set up such an index form.

We cut across the x axis by the four n-planes

$$x = 0, \qquad x = \frac{\pi}{2}, \qquad x = \pi, \qquad x = \frac{3\pi}{2}. \tag{3.3}$$

Let

$$P^1, \qquad P^2, \qquad P^3, \qquad P^4 \tag{3.4}$$

be points on the respective n-planes (3.3) neighboring the x axis, and let

$$(y_1^q, \cdots, y_n^q) \qquad\qquad (q = 1, \cdots, 4) \tag{3.5}$$

be the y-coordinates of the point P^q. Let

$$(v_1, \cdots, v_\delta) \qquad\qquad (\delta = 4n) \tag{3.6}$$

denote the ensemble of the coordinates (3.5). Let the images on $E_m(a)$ of the points (3.4) be joined in circular order by minimizing geodesics, and let the sum of the lengths of these geodesics be denoted by $J(v)$. The required index form will be the form

$$Q(v) = J_{v_\alpha v_\beta}(0)v_\alpha v_\beta \qquad\qquad (\alpha, \beta = 1, \cdots, \delta). \tag{3.7}$$

It follows from the symmetry of the ellipsoid $E_m(a)$ with respect to the m-plane $w_k = 0$, that the function $J(v)$ and the form $Q(v)$ will be unchanged in value if for a fixed k we replace

$$y_k^1, \quad y_k^2, \quad y_k^3, \quad y_k^4 \qquad\qquad (k = 1, \cdots, n) \tag{3.8}$$

by

$$-y_k^1, \ -y_k^2, \ -y_k^3, \ -y_k^4.$$

It follows that $Q(v)$ can contain terms which are constant multiples of the product

$$y_h^i y_k^j \qquad\qquad (i, j = 1, \cdots, 4; h, k = 1, \cdots, n)$$

only if $h = k$. We can accordingly write $Q(v)$ as a sum

$$(3.8)' \qquad Q(v) \equiv Q_1(y_1^1, y_1^2, y_1^3, y_1^4) + \cdots + Q_n(y_n^1, y_n^2, y_n^3, y_n^4)$$

where Q_k is a quadratic form in the arguments (3.8).

Hence the index of $Q(v)$ will be the sum of the indices of the separate forms Q_k. But if we set all the variables (v) equal to zero save those in Q_k, we see that Q_k is an index form which would be associated with g, were g regarded as a closed geodesic on the principal ellipsoid which lies in the space of the

$$w_k, w_m, w_{m+1} \qquad (k = 1, \cdots, m-1)$$

axes.

Similar results hold relative to the nullity of $Q(v)$, and the proof of the theorem is complete for the case of $g_{m,\,m+1}$. The analysis is not essentially different for the case of g_{ij} in general, and will be omitted.

We can use the preceding theorem and Theorem 2.1 to establish the following theorem.

THEOREM 3.2. *Corresponding to constants $a_1 > \cdots > a_{m+1}$ sufficiently near unity, the principal ellipses g_{ij} of the m-ellipsoid $E_m(a)$ will be non-degenerate and possess indices k given by the formula*

$$k = m + i + j - 4.$$

We suppose that $i < j$. Let p, q, and h be three distinct integers on the range $1, \cdots, m+1$. Let

$$E_{pqh}$$

be the principal ellipsoid which lies in the 3-space of the w_p, w_q, w_h axes. In particular consider the principal ellipsoids E_{ijh} which contain g_{ij}. There will be $(i-1)$ such ellipsoids for which $h < i$. On each such ellipsoid, g_{ij} will be of the type of g_{23} in Theorem 2.1, and have the index 3, provided always the constants $a_1 > \cdots > a_{m+1}$ are sufficiently near unity. There will be $(j-i-1)$ ellipsoids E_{ijh} for which h lies between i and j. On such ellipsoids, g_{ij} will be of the type of g_{13} in Theorem 2.1, and will have the index 2. There will be $(m-j+1)$ ellipsoids E_{ijh} for which $h > j$. On such ellipsoids, g_{ij} will be of the type of g_{12} in Theorem 2.1, and will have the index 1. By virtue of the preceding theorem the index of g_{ij} on $E_m(a)$ will be the sum of these indices. Thus

$$k = 3(i-1) + 2(j-i-1) + (m-j+1) = m + i + j - 4$$

as stated.

That g_{ij} will be non-degenerate if the constants $a_1 > \cdots > a_{m+1}$ are sufficiently near unity is similarly proved.

The geodesics $g_{ij}^r(a)$. Let the closed geodesic which covers g_{ij} r times on $E_m(a)$ be denoted by $g_{ij}^r(a)$. The preceding theorem will now be extended to include an evaluation of the indices of the geodesics $g_{ij}^r(a)$.

THEOREM 3.3. *Corresponding to constants $a_1 > \cdots > a_{m+1}$ sufficiently near unity, the geodesics $g^r_{ij}(a)$ for which r is less than a fixed integer s will be non-degenerate and possess indices k^r_{ij} given by the formula*

$$k^r_{ij} = m + i + j - 4 + 2(r - 1)(m - 1). \tag{3.9}$$

To prove this theorem we review the proof of Theorem 2.1 and verify the truth of the following extension. If corresponding to the integer r the constants $a_1 > a_2 > a_3$ of the ellipsoid $E_2(a)$ of Theorem 2.1 are sufficiently near unity, the geodesics

$$g^r_{12}(a), \qquad g^r_{13}(a), \qquad g^r_{23}(a)$$

will be non-degenerate and possess indices given respectively by the formulae

$$2(r - 1) + 1, \qquad 2(r - 1) + 2, \qquad 2(r - 1) + 3.$$

The proof of this statement depends upon an obvious restatement of Lemma 2.2.

We next review Theorem 3.1, and verify the fact that the index and nullity of $g^r_{ij}(a)$ are the respective sums of the indices and nullities of $g^r_{ij}(a)$ on each of the $(m - 1)$ principal 2-dimensional ellipsoids on which $g^r_{ij}(a)$ lies. We then reason as in the proof of Theorem 3.2, and conclude that $g^r_{ij}(a)$ is non-degenerate, and that its index k^r_{ij} is given by the formula

$$\begin{aligned} k^r_{ij} = {} & [2(r - 1) + 3]\,(i - 1) + [2(r - 1) + 2]\,(j - i - 1) \\ & + [2(r - 1) + 1]\,(m - j + 1), \end{aligned}$$

provided the constants $a_1 > \cdots > a_{m+1}$ are sufficiently near unity.

This evaluation of k^r_{ij} reduces to that given in (3.9), and the proof of the theorem is complete.

We verify the important property that for arbitrary integers $i < j$ on the range $1, \cdots, m + 1$,

$$k^r_{ij} \leqq k^{r+1}_{12},$$

and that in particular for any positive integer r,

$$k^r_{m,\,m+1} = k^{r+1}_{12}.$$

The exclusiveness of the closed geodesics g^r_{ij}

4. In this section we are concerned again with the m-ellipsoid

$$a^2_i w^2_i = 1 \qquad (i = 1, \cdots, m + 1) \tag{4.1}$$

where $a_i > 0$. We shall make use of the equations of the geodesics on (4.1) in the form (3.1). If $w_i = w_i(s)$ represents a geodesic on (4.1), the function $\lambda(s)$ in (3.1) can be determined as follows. Two differentiations of (4.1) with respect to s yield the identity

$$a^2_i w'_i w'_i + a^2_i w_i w''_i = 0 \qquad (i = 1, \cdots, m + 1)$$

Upon substituting $-\lambda a_i^2 v_i$ for w_i'' in accordance with (3.1) with i not summed, we find that

$$\lambda = \frac{a_i^2 w_i' w_i'}{a_i^4 w_i w_i}. \tag{4.2}$$

But since s is the arc length, we have

$$w_i' w_i' \equiv 1. \tag{4.3}$$

When the constants a_i in (4.1) are all unity, we see from (4.2) that $\lambda(s) \equiv 1$. Accordingly, for constants a_i sufficiently near unity, λ will be uniformly near 1 for any point (w) on the ellipsoid (4.1) and set (w') satisfying (4.3).

Before coming to the theorem it will be convenient to state a lemma.

LEMMA 4.1. *Let $\varphi(s)$ be a function which is continuous in s and has the period ω. If $u(s)$ is a solution ($\not\equiv 0$) of the differential equation*

$$w'' + \varphi(s)w = 0, \tag{4.4}$$

of period ω, such that all solutions of (4.4) *of period ω are dependent on $u(s)$, the only solutions of* (4.1) *whose zeros s have the period ω are dependent on $u(s)$.*

The proof of this lemma is similar to that of Lemma 2.1 and will be omitted.

The principal theorem of this section is the following.

THEOREM 4.1. *Let N be an arbitrarily large positive constant. Corresponding to constants $a_1 > \cdots > a_{m+1}$ sufficiently near unity, there are no closed geodesics on $E_m(a)$ with lengths less than N other than the geodesics $g_{ij}^r(a)$.*

We begin by stating limitations on the constants a_i under which we can prove the theorem is true.

Corresponding to any set (a) of $m + 1$ positive constants, let g be a geodesic on the m-ellipsoid $E_m(a)$, and $\lambda_g(s)$ the value of λ which with g satisfies the system (3.1). Let c be any number, and $w_j(s, c)$ a solution of the differential equation

$$w'' + \lambda_g(s)a_j^2 w = 0 \tag{4.5$'$}$$

such that

$$w_j(c, c) = 0, \qquad w_j'(c, c) = 1, \tag{4.5$''$}$$

where j is one of the integers $1, \cdots, m + 1$.

Upon referring to (4.2) we see that $\lambda_g(s) \equiv 1$ when $(a) = (1)$, independently of the choice of the geodesic g on (4.1). For $(a) = (1)$ the function $w_j(s, c)$ will then vanish at the points $s = c + n\pi$, where n is an integer, positive, negative, or zero. For $(a) = (1)$ the qth zero of $w_j(s, c)$ following $s = c$ will thus lie on the interval

$$c + (2q - 1)\frac{\pi}{2} < s < c + (2q + 1)\frac{\pi}{2}. \tag{4.6}$$

Let μ be a positive integer such that the constant N of the theorem satisfies the condition

$$N < (2\mu + 1)\frac{\pi}{2}.$$

Let q be a positive integer at most μ. We restrict the constants (a) to so small a neighborhood K of the set $(a) = (1)$ that corresponding to any geodesic g on $E_m(a)$, the qth zero $s > c$ of the function $w_j(s, c)$ lies on the interval (4.6) and is the only zero of $w_j(s, c)$ on the closure of that interval. Such a neighborhood K can be chosen independently of the choice of c, or of the geodesic g on $E_m(a)$, or of the integer j in (4.5)′.

We shall now prove the following statement.

(α). *Corresponding to any set* (a) *which lies on the preceding neighborhood* K *of the set* $(a) = (1)$ *and satisfies the conditions*

$$a_1 > \cdots > a_{m+1} > 0, \tag{4.7}$$

any closed geodesic g *on* $E_m(a)$ *with a length* $\omega < N$ *will be a geodesic* $g_{ij}^r(a)$.

Let g be represented in the form

$$w_i = u_i(s) \qquad (i = 1, \cdots, m + 1). \tag{4.8}$$

Of the functions $u_i(s)$ at least two functions, say

$$u_h(s), \qquad u_k(s) \qquad (h < k),$$

are not identically zero. Let $\lambda(s)$ be the function with which g satisfies (3.1). The functions $u_h(s)$ and $u_k(s)$ are solutions of the differential equations

$$w'' + \lambda(s)a_h^2 w = 0, \tag{4.9}$$

$$w'' + \lambda(s)a_k^2 w = 0, \tag{4.10}$$

respectively. They have the period ω in s. We continue with a proof of the following statement.

(β). *All solutions of* (4.10) *of period* ω *are linearly dependent on* $u_k(s)$.

Suppose (β) false. Then all solutions of (4.10) have the period ω.

The solution $u_h(s)$ of (4.9) vanishes at least once, since $\lambda(s) > 0$. Suppose that $u_h(c) = 0$. According to our choice of the neighborhood K, the qth zero, $0 < q \leqq \mu$, of $u_h(s)$ following $s = c$ is on the interval (4.6), and is the only zero of $u_h(s)$ on the closure of that interval. Since $u_h(c + \omega) = 0$, and $\omega < N$, the point $s = c + \omega$ must lie on one of the intervals (4.6), say the pth.

Let $w_k(s)$ be a solution of (4.10) such that

$$w_k(c) = 0, \qquad w_k'(c) = 1.$$

As previously noted $w_k(s)$ has the period ω if (β) is false. Hence $s = c + \omega$ is a zero of $w_k(s)$. Inasmuch as $w_k(s)$ has just one zero on each of the intervals (4.6), the point $s = c + \omega$ must be the pth zero following $s = c$, not only of $u_h(s)$ but also of $w_k(s)$.

Upon applying the well known Sturm Comparison Theorem to the differential equations (4.9) and (4.10) and their solutions $u_h(s)$ and $w_k(s)$ respectively, we see that the pth zero of $u_h(s)$ and $w_k(s)$ following $s = c$ cannot be common to $u_h(s)$ and $w_k(s)$. From this contradiction we infer the truth of statement (β).

Now let $w_j(s, a)$, $j = 1, \cdots, m + 1$, be a solution of the differential equation

$$w'' + \lambda(s)a_j^2 w = 0,$$

such that

$$w_j(a, a) = 0, \qquad w_j'(a, a) = 1.$$

Let $D_j(a)$ be the distance along the s axis from $s = a$ to the pth zero of $w_j(s, a)$, choosing p as previously so that

$$(2p - 1)\frac{\pi}{2} < \omega < (2p + 1)\frac{\pi}{2}.$$

We shall now prove statement (γ).

(γ). *The function $D_k(s) - \omega$ is positive except at the zeros of $u_k(s)$.*

It is clear that $D_k(s) = \omega$ at the zeros of $u_k(s)$ by virtue of our choice of p. According to (β), all solutions of (4.10) of period ω are dependent on $u_k(s)$, and according to Lemma 4.1, the zeros of $u_k(s)$ are then the only values of s at which $D_k(s) = \omega$. Between each pair of successive zeros of $u_k(s)$, $D_k(s) - \omega$ must have one sign. To determine that sign we compare $u_h(s)$ with $u_k(s)$, recalling that a_k in (4.10) is less than a_h in (4.9). From the Sturm Theorems we can then infer that there is at least one zero of $u_h(s)$ between each two consecutive zeros of $u_k(s)$. Suppose that $u_h(a) = 0$. I say that

$$D_k(a) > \omega. \tag{4.11}$$

To establish (4.11) we compare $u_h(a)$ with $w_k(s, a)$, recalling that

$$u_h(a) = w_k(a, a) = 0.$$

According to the Sturm Comparison Theorem the pth zero of $w_k(s, a)$ following $s = a$, follows the corresponding zero of $u_h(a)$, namely $s = a + \omega$, so that (4.11) holds as stated, and (γ) is proved.

One can similarly prove the following.

(δ). *The function $D_h(s) - \omega$ is negative except at the zeros of $u_h(s)$.*

We now complete the proof of (α).

Let $u_i(s)$ be any one of the functions $u_i(s)$ in (4.8), other than $u_h(s)$ and $u_k(s)$. I say that $u_i(s) \equiv 0$. For if $h < k < i$ and $u_i(s) \not\equiv 0$, we could treat k and i as we have just treated h and k, and infer that $D_k(s) < \omega$ except at the zeros of $u_k(s)$, contrary to (γ). If the integers h, k, i are in some order other than the order $h < k < i$, we can interchange their rôles and arrive at a similar contradiction. Hence $u_i(s) \equiv 0$ for i not h or k. Thus g must be one of the geodesics g_{hk}^r.

The proof of (α) and of the theorem is now complete.

The linking cycles $\Lambda_{12}^r(a)$

5. We shall eventually determine the circular connectivities of the m-sphere as defined at the end of Ch. VIII. To that end we shall prove that the geodesics $g_{ij}^r(a)$ for which r is at most a prescribed positive integer s will be non-degenerate and possess linking cycles, provided the constants a_i are sufficiently near unity and satisfy the conditions

$$a_1 > \cdots > a_{m+1}. \tag{5.1}$$

We shall explicitly exhibit these linking cycles. We begin with the geodesics $g_{12}^r(a)$.

The cycles $\Lambda_{12}^r(a)$. Recall that $E_m(a)$ reduces to the unit m-sphere

$$w_i w_i = 1 \qquad (i = 1, \cdots, m+1), \tag{5.2}$$

when the constants $a_i = 1$. We denote $E_m(1)$ by S_m.

Let A^1 and A^2 respectively denote the points of intersection of S_m with the positive w_1 and w_2 axes, and let A be the point diametrically opposite to A^1 on S. Corresponding to $g_{12}^r(1)$ we introduce a set of $2r - 1$ constants e_q, such that

$$0 < e_1 < \cdots < e_{2r-1} < 1. \tag{5.3}$$

In the space (w) let

$$M^q \qquad (q = 1, \cdots, 2r-1) \tag{5.4}$$

be an $(m-1)$-sphere formed by the intersection of S_m with an m-sphere S_m^q of radius e_q, with center at A^1 if q is even, and with center at A if q is odd. Let

$$P^1, \cdots, P^p \qquad (p = 2r+1) \tag{5.5}$$

be a circular sequence of $2r + 1$ points on S_m of which

$$P^1 = A^1, \qquad P^p = A^2, \tag{5.6}$$

and of which the points

$$P^2, \cdots, P^{2r} \tag{5.7}$$

lie on the respective $(m-1)$-spheres

$$M^1, \cdots, M^{2r-1}. \tag{5.8}$$

Our integral on S_m is the arc length. The constant limiting the lengths of the corresponding elementary extremals can be taken as any positive constant less than π. If we choose ρ less than π but differing from π sufficiently little, any two successive points in the circular sequence (5.5) can be joined by an elementary extremal on S_m, and the points (5.5) will be the vertices of a point (π) interior to the corresponding space Π^p.

The ensemble of the points (π) as the vertices (5.7) range over their respective $(m - 1)$-spheres (5.8) is a k-cycle of dimension

$$k = (2r - 1)(m - 1). \tag{5.9}$$

We take this k-cycle as our definition of $\Lambda_{12}^r(1)$.

We shall take the constant ρ limiting elementary geodesics on $E_m(a)$ as the constant ρ chosen above for $S_m = E_m(1)$. Such a choice is permissible for constants a_i sufficiently near unity. We shall denote the spaces Π^q, R^q, and Ω then determined by the integral of arc length on $E_m(a)$ by

$$\Pi^q(a), \qquad R^q(a), \qquad \Omega(a),$$

respectively.

Let (π) be an inner point of $\Pi^q(1)$. The vertices of (π) lie on the m-sphere $E_m(1)$. If the constants a_i are sufficiently near unity, the central projections of the vertices of (π) on $E_m(a)$ will define a point (π') on $\Pi^q(a)$. We term (π') the *central projection* of (π) on $\Pi^q(a)$. If the constants a_i are sufficiently near unity, the cycle $\Lambda_{12}^r(1)$ will have a central projection on $\Pi^{2r+1}(a)$, and we take this central projection of $\Lambda_{12}^r(1)$ as our definition of $\Lambda_{12}^r(a)$.

We shall prove the following lemma.

LEMMA 5.1. *Corresponding to a prescribed positive integer r, constants $a_1 > \cdots > a_{m+1}$ can be chosen so near unity that the cycle $\Lambda_{12}^r(a)$ will be a linking cycle on $\Omega(a)$ belonging to the critical set σ determined by $g_{12}^r(a)$.*

The lemma will follow after we have verified the truth of statements (α), (β), and (γ) below.

(α). *The dimensionality k of the cycle $\Lambda_{12}^r(a)$, as given by* (5.9), *equals the index k_{12}^r, as given by* (3.9).

We have merely to set $i = 1$ and $j = 2$ in the index formula

$$k_{ij}^r = m + i + j - 4 + 2(r - 1)(m - 1),$$

to obtain (5.9) as stated.

(β). *On $\Lambda_{12}^r(1)$, $J(\pi)$ assumes a proper absolute maximum at the point (π_0) which determines $g_{12}^r(1)$. In terms of k_{12}^r parameters (v) regularly representing $\Lambda_{12}^r(1)$ neighboring (π_0), $J(\pi)$ has a non-degenerate critical point of index k_{12}^r at the point (v_0) corresponding to (π_0).*

The first statement under (β) is a consequence of statement (A) in the proof of Lemma 15.1 of Ch. VII. We refer to the case where the number of vertices, $p + 2$, in the cycle $\lambda_{p(m-1)}$ of (A) equals the number of vertices, $2r + 1$, in the present cycle $\Lambda_{12}^r(1)$, that is the case where

$$p = 2r - 1.$$

Upon adding the elementary geodesic A^1A^2 to the geodesic g_{2r-1} in (A) of §15, Ch. VII, one obtains the closed geodesic $g_{12}^r(1)$. More generally the addition of the elementary geodesic A^1A^2 to the broken geodesics of $\lambda_{p(m-1)}$ of (A) yields the respective broken geodesics $g(\pi)$ determined by points (π) on $\Lambda_{1\,2}^{r}(1)$. The first statement in (β) is thus a consequence of (A) in §15, Ch. VII.

Let $f(v)$ be the value of $J(\pi)$ at the point (π) on $\Lambda_{1\,2}^{r}(1)$ determined by the parameters (v). The function $f(v)$ forms an index function belonging to g_{2r-1}, regarding g_{2r-1} as a critical extremal of the boundary problem in which the functional is the arc length on S_m and the end points are fixed at A^1 and A^2. Since A^1 and A^2 are not conjugate on g_{2r-1}, the point $(v) = (v_0)$ is a non-degenerate critical point of $f(v)$. Since $f(v)$ assumes a maximum at (v_0), the critical point (v_0) must have an index equal to the number of parameters (v), namely $k_{1\,2}^{r}$.

Thus (β) is proved.

(γ). *On $\Lambda_{1\,2}^{r}(a)$, $J(\pi)$ assumes a proper, absolute maximum at the point (π_1) which determines $g_{1\,2}^{r}(a)$, provided the constants a_i are sufficiently near unity.*

To represent $\Lambda_{1\,2}^{r}(a)$ neighboring (π_1), we make use of the preceding representation of $\Lambda_{1\,2}^{r}(1)$ neighboring (π_0) in terms of the parameters (v). We represent points on $\Lambda_{1\,2}^{r}(a)$ and $\Lambda_{1\,2}^{r}(1)$ which are central projections of one another by the same parameters (v). Let $f(v, a)$ then denote the value of $J(\pi)$ at the point (π) on $\Lambda_{1\,2}^{r}(a)$ determined by (v). According to (β) the function $f(v, 1)$ has a non-degenerate critical point of index $k_{1\,2}^{r}$ when $(v) = (v_0)$. Hence for constants a_i sufficiently near unity, $f(v, a)$ will have a *unique*, non-degenerate critical point of index $k_{1\,2}^{r}$ neighboring (v_0). The coordinates (v) of this critical point will be analytic functions $[v(a)]$ of the variables a_i. But $g_{1\,2}^{r}(1)$ projects centrally into $g_{1\,2}^{r}(a)$. We infer that

$$[v(a)] \equiv (v_0).$$

Let e be a positive constant so small that the domain of points (v) which satisfy the condition

$$f(v, 1) \geqq f(v_0, 1) - e \tag{5.10}$$

and are connected to (v_0) in the space (v) contains no critical points of $f(v, 1)$, other than (v_0). We now place two restrictions on the constants a_i.

Let H be the closure of the set of points (π) on $\Lambda_{1\,2}^{r}(a)$ which are not represented by points (v) on the domain (5.10). The first restriction on the constants a_i is that they be so near 1 that the value of $J(\pi)$ at points (π) on H is everywhere less than $f(v_0, a)$. The second restriction on the constants a_i is that they be so near 1 that $f(v, a)$ has no critical points on the domain (5.10) other than the point (v_0). By virtue of these two restrictions on the constants a_i, $J(\pi)$ assumes its absolute maximum on $\Lambda_{1\,2}^{r}(a)$ at the point (π_1) corresponding to (v_0), that is, at the point (π_1) which determines $g_{1\,2}^{r}(a)$.

Statement (γ) is accordingly proved.

We can now prove the lemma.

Let the constants a_i be chosen so near unity that on $\Lambda_{1\,2}^{r}(a)$ $J(\pi)$ assumes a proper absolute maximum at the point (π_1) which determines $g_{1\,2}^{r}(a)$. Neighboring (π_1) the points on $\Lambda_{1\,2}^{r}(a)$ form what has been termed in Theorem 1.2 a "non-tangential" submanifold Z of $R^{2r+1}(a)$ belonging to $g_{1\,2}^{r}(a)$. It follows from Theorem 1.2 that if c is the length of $g_{12}^r(a)$, and if e is a sufficiently small positive constant, the locus

$$J(\pi) = c - e \qquad [\text{on } \Lambda_{12}^r(a)]$$

s a spannable $(k - 1)$-cycle u_{k-1} belonging to the critical set σ determined by $g_{12}^r(a)$. Moreover the cycle u_{k-1} is bounding below c, in fact bounds the chain of points on $\Lambda_{12}^r(a)$ at which $J(\pi)$ is at most $c - e$.

Thus $\Lambda_{12}^r(a)$ is a linking k-cycle on $\Pi^{2r+1}(a)$ belonging to the critical set σ on $\Omega(a)$ determined by $g_{12}^r(a)$, and the lemma is proved.

Symmetric chains and cycles

6. The proof of the existence of linking cycles belonging to the geodesics $g_{ij}^r(a)$ is most conveniently made with the aid of the symmetry properties of $E_m(a)$. Before turning to the main theorems of this section we recall certain facts about quadratic forms.

Let $Q(v)$ be a quadratic form in μ variables (v). The constants λ and sets $(v) \neq (0)$ which satisfy the conditions

$$Q_{v_i} - 2\lambda v_i = 0 \qquad (i = 1, \cdots, \mu)$$

are respectively the so-called characteristic roots and characteristic solutions belonging to Q. If Q has the index k, there will be k mutually orthogonal characteristic solutions

$$(v^h) \qquad (h = 1, \cdots, k)$$

belonging respectively to characteristic roots λ_h which are negative. The k-plane L_k consisting of points (v) linearly dependent on the sets (v^h) will be termed the *index hyperplane belonging to Q.* This index hyperplane is uniquely determined by Q. Moreover on L_k for normalized sets (v^h),

$$Q(v) = \lambda_h(v_i^h v_i)^2 \qquad (h = 1, \cdots, k; i = 1, \cdots, \mu) \tag{6.1}$$

as is well known. We thus see that $Q(v)$ is negative definite on L_k.

Any orthogonal transformation V of the variables (v) leaves the characteristic roots of Q invariant. If V leaves Q invariant as well, it transforms characteristic solutions of Q into such solutions. We embody these facts in the following lemma.

LEMMA 6.1. *An orthogonal transformation of the variables (v) which leaves the form $Q(v)$ invariant leaves the corresponding index hyperplane L_k invariant. On L_k, $Q(v)$ is negative definite.*

We turn to a function $f(v)$ analytic in the preceding variables (v) at $(v) = (0)$, possessing in $(v) = (0)$ a critical point of index k. We suppose that $0 < k < \mu$, and that $f(0) = 0$. We set

$$Q(v) = f_{v_i v_j}(0) v_i v_j \qquad (i, j = 1, \cdots, \mu) \tag{6.2}$$

and let L_k be the index hyperplane corresponding to Q. We term L_k the *index hyperplane* belonging to f and the point $(v) = (0)$.

Suppose that $(v) = (0)$ is a non-degenerate critical point of f. We shall investigate the sign of f on normals to L_k near the origin. To that end let an orthogonal transformation from the variables (v) to the variables (x) be used, of such a nature that $Q(v)$ takes the form

$$Q(v) \equiv \lambda_i x_i^2 \qquad (i = 1, \cdots, \mu).$$

We suppose moreover that L_k has been carried into the k-plane of the first k axes in the space (x). The roots $\lambda_1, \cdots, \lambda_k$ are accordingly negative, and the roots $\lambda_{k+1}, \cdots, \lambda_\mu$ positive $(0 < k < \mu)$.

Let $a_1, \cdots, a_k$ and $c_{k+1}, \cdots, c_\mu$ be parameters such that

$$\lambda_1 a_1^2 + \cdots + \lambda_k a_k^2 = -1, \qquad \lambda_{k+1} c_{k+1}^2 + \cdots + \lambda_\mu c_\mu^2 = 1. \tag{6.3}$$

Set

$$\begin{aligned} x_i &= \sigma a_i & (i &= 1, \cdots, k), \\ x_j &= \rho c_j & (j &= k+1, \cdots, \mu). \end{aligned} \tag{6.4}$$

If the parameters σ, a_i and c_j in (6.4) are held fast while ρ is varied, (6.4) defines a straight line normal to L_k at the point at which $\rho = 0$ in (6.4). Moreover any normal to L_k can be expressed in the form (6.4). On such a normal f takes the form

$$f = \rho^2 - \sigma^2 + F(\rho, \sigma),$$

where $F(\rho, \sigma)$ is a power series in ρ and σ involving terms of at least the third degree, with coefficients which are polynomials in the parameters a_i and c_j, subject to (6.3).

The equation

$$0 = f = \rho^2 - \sigma^2 + F(\rho, \sigma)$$

possesses solutions of the form

$$\begin{aligned} \rho &= \sigma + \cdots, \\ \rho &= -\sigma + \cdots, \end{aligned} \tag{6.5}$$

where the terms omitted are power series in σ of degree higher than the first with coefficients which are analytic functions of the parameters a_i and c_j satisfying (6.3). From (6.5) we see that on each normal to L_k sufficiently near the origin but not through the origin, f vanishes just twice. On each such normal the segment on which $f < 0$ includes the point $\rho = 0$ on L_k. On the normals to L_k through the origin, f is never negative sufficiently near the origin.

These results lead to the following lemma.

LEMMA 6.2. *Let $f(v_1, \cdots, v_\mu)$ be a function which is analytic in the variables (v) when $(v) = (0)$, and for which $(v) = (0)$ is a non-degenerate critical point of index k, with $0 < k < \mu$. Suppose moreover that $f(0) = 0$.*

The domain $f < 0$ neighboring $(v) = (0)$ can then be continuously deformed among points at which $f < 0$ onto the index hyperplane L_k belonging to f and $(v) = (0)$, by moving an arbitrary point P at which $f < 0$ along the normal to L_k through P to the foot of the normal on L_k, moving P at a rate equal to the distance to be traversed.

The transformations V_k. Corresponding to a fixed integer k between 1 and $m + 1$ inclusive, let V_k denote the transformation

$$
\begin{aligned}
w_i' &= w_i \qquad (i = 1, \cdots, k-1, k+1, \cdots, m+1),\\
w_k' &= -w_k.
\end{aligned}
\tag{6.6}
$$

The m-ellipsoid $E_m(a)$ is invariant under each transformation V_k. It will be convenient to suppose that the polyhedral complex K of §1, Ch. VIII, to which $E_m(a)$ is supposed homeomorphic, has been so divided into cells that its cells are carried into cells of K under transformations V_k.

By the transformation V_k as applied to a point (π) we mean the transformation effected by applying V_k to each of the vertices of (π).

We modify the division of K^p and Π^p into cells as follows. Let the ith coordinate w_i of the qth vertex of a point (π) on K^p be denoted by

$$w_i^q \qquad (q = 1, \cdots, p; i = 1, \cdots, m).$$

We begin as in §1 of Ch. VIII, sectioning and subdividing that portion H^p of K^p for which

$$w_i^q \geqq 0.$$

We add the hyperplanes $w_i^q = 0$ to the sectioning hyperplanes. We then apply the transformations V_k to the resulting cells of H^p, thereby obtaining a cellular division of the whole of K^p. In making any further subdivision of K^p we first subdivide H^p, and then apply the transformations V_k to the resulting cells, thereby obtaining a subdivision of the whole of K^p. With K^p so divided we define Π^p as in Ch. VIII.

A chain z^p on Π^p will be called *symmetric* if an arbitrary i-cell of z^p is transformed into an i-cell of z^p under each transformation V_k. An homology will be termed *symmetric*, if the cycles involved are symmetric and the chain bounded can be chosen so as to be symmetric. A deformation will be termed symmetric if points (π) which are images of one another under a transformation V_k are replaced at the time t by points (π) which are likewise images of one another under V_k. The Veblen-Alexander process of reduction of a k-cycle w^p on Π^p to a k-cycle of cells of Π^p will lead to a symmetric homology if w^p is symmetric, provided 0-cells of w^p which are images one of the other under transformations V are assigned to 0-cells of Π^p with the same property.

We shall now prove the following lemma.

LEMMA 6.3. *Corresponding to a geodesic $g = g^r_{ij}(a)$ and constants a_i sufficiently near unity, there exists a symmetric proper section S of R^p belonging to g. The integer p can be taken as any multiple of $4r$, and the parameters (v) regularly representing S neighboring the point $(v) = (0)$ determining g, can be chosen so that the transformations V_k correspond to orthogonal linear transformations of the variables (v).*

We shall give the proof of the lemma for the case of the geodesic $g = g^r_{m,\,m+1}(a)$, and refer the neighborhood of g to coordinates

$$(x, y_1, \cdots, y_n) \qquad (n = m - 1)$$

as in §3. In the space (x, y), g can be represented by a segment of the x axis on which

$$0 < x \leqq 2r\pi.$$

We take the qth manifold of S as the locus on which

$$x = \frac{2qr\pi}{p} \qquad (q = 1, \cdots, p) \tag{6.7}$$

neighboring g. We see that S so defined is a symmetric proper section of R^p belonging to g, provided the constants a_i are sufficiently near unity and p is a multiple of $4r$.

Let the ith coordinate y_i of the point on the qth manifold (6.7) be denoted by

$$y^q_i \qquad (q = 1, \cdots, p;\, i = 1, \cdots, n). \tag{6.8}$$

We denote the ensemble of the variables (6.8) by (v). The variables (v) parameterize S. On S, in terms of the variables (6.8), and for values of k on the range $1, \cdots, m - 1$, the transformation V_k takes the form

$$y'^q_k = -y^q_k \qquad (q = 1, \cdots, p). \tag{6.9$'$}$$

The remaining transformations, V_m and V_{m+1}, interchange the manifolds (6.7) without changing the coordinates (y) of points thereon. The transformations V_m and V_{m+1} when applied to S thus define substitutions of the form

$$y'^{q'}_i = y^q_i \qquad (i = 1, \cdots, n) \tag{6.9$''$}$$

in which q and q' are integers independent of i.

The lemma follows from the nature of the transformations (6.9).

The number μ of variables (6.8) is at least $4r(m - 1)$, since p is assumed to be a multiple of $4r$. On the other hand the index k^r_{ij} of $g^r_{ij}(a)$ is at most

$$k^r_{m,\,m+1} = (2r + 1)(m - 1) < 4r(m - 1).$$

Hence

$$0 < k^r_{ij} < \mu,$$

a fact of value in the application of Lemma 6.2.

We continue with the following lemma.

Lemma 6.4. *Suppose the constants $a_1 > \cdots > a_{m+1}$ are so near unity that the geodesic $g = g^r_{ij}(a)$ is non-degenerate and possesses a symmetric proper section S on R^p. Let c be the length of g, k its index, and σ the critical set determined by g.*

(a). *Any symmetric, non-spannable j-cycle z^p on Π^p, below c, sufficiently near σ^p, for which $j \geqq k - 1$, will possess a 2-fold partition symmetrically homologous to zero, below c, on Π^{2p} neighboring σ^{2p}.*

(b). *Any symmetric j-cycle z^p on Π^p sufficiently near σ^p, for which $j > 0$, will possess a 2-fold partition symmetrically homologous to zero on Π^{2p} neighboring σ^{2p}.*

We shall first prove (a). Our previous analysis would make it a relatively simple matter to show that (a) holds, were the condition of symmetry removed. Our problem is then to review the homologies by virtue of which we know that (a) holds disregarding symmetry, and then to show that the homologies involved may be taken as symmetric.

It will be sufficient to prove the lemma for the case that z^p is a J-normal cycle. For in any case an application of the deformation $\theta_p(t)$ of §5, Ch. VIII, would deform the given cycle, below c, neighboring σ^p into a J-normal cycle. Moreover an examination of the definition of the deformation $\theta_p(t)$ shows that this deformation is symmetric.

Suppose then that z^p in (a) is J-normal. By virtue of (11.4) in Ch. VIII we have an homology

$$z^p \sim b^p(z^p) + \mathcal{E}w^p \qquad \text{(on } N^p, \text{ below } c) \tag{6.10}$$

where $b^p(z^p)$ is the extremal projection of z^p on S, and w^p is a $(k-1)$-cycle on z^p. The analysis of the homology (11.4) in Ch. VIII shows that this homology is symmetric if z^p and S are symmetric, and that w^p is symmetric. That the 2-fold partition of $\mathcal{E}w^p$ is symmetrically homologous to zero on Π^{2p}, below c, neighboring σ^{2p}, follows from the symmetry of the deformation used in Theorem 7.1 of Ch. VIII to prove the 2-fold partition of $\mathcal{E}w^p$ homologous to zero.

The lemma will follow from (6.10) after we have proved that

$$b^p(z^p) \sim 0 \qquad \text{(on } S, \text{ below } c) \tag{6.11}$$

neighboring c, and that this homology is symmetric.

To that end let (v) be a set of parameters representing S as in Lemma 6.3, and let $f(v)$ be the value of $J(\pi)$ at the point (π) determined by (v). Let L_k be the index hyperplane belonging to f and to the critical point $(v) = (0)$. It follows from Lemma 6.3 that L_k is symmetric. It follows from Lemma 6.2 that the cycle $b^p(z^p)$ can be symmetrically deformed along the normals to L_k, below c, into a cycle u^p on L_k.

On L_k, $f(v)$ assumes a proper, non-degenerate maximum at the origin. Let $\varphi(\alpha)$ represent the value of $f(v)$ on L_k in terms of parameters (α) regularly representing L_k. If e is a sufficiently small positive constant, the locus

$$\varphi = c - e$$

on L_k neighboring the point $(v) = (0)$ will be a spannable $(k - 1)$-cycle belonging to g, as follows from Theorem 7.5 of Ch. VI. If u^p is sufficiently near the point $(v) = (0)$ on L_k, and the constant e is sufficiently small, u^p can be symmetrically deformed on L_k into a j-cycle v^p on $\varphi = c - e$, below c, with the aid of trajectories on L_k orthogonal to the manifolds φ constant. We now give the locus $\varphi = c - e$ a symmetric division into cells, and infer from the Veblen-Alexander process that v^p is symmetrically homologous on $\varphi = c - e$ to a cycle x^p of cells of $\varphi = c - e$. But since v^p and $\varphi = c - e$ have dimensionalities j and $k - 1$ respectively, and $j \geqq k - 1$, the cycle x^p will either be null, mod 2, or identical with the cycle $\varphi = c - e$. Since $\varphi = c - e$ defines a spannable $(k - 1)$-cycle belonging to g, and x^p is not such a cycle in accordance with our hypothesis in (a), we infer that x^p must be null.

Thus (6.11) holds as stated, and (a) follows from (6.10).

To prove (b) we again make use of (6.10) omitting the condition, below c, again noting that (6.10) is a symmetric homology, and that $\mathcal{E}w^p$ possesses a 2-fold partition symmetrically homologous to zero on Π^{2p} neighboring σ^{2p}. We shall conclude the proof of (b) by showing that

$$b^p(z^p) \sim 0 \qquad \text{(on } \Pi^p) \tag{6.12}$$

neighboring σ^p, and that this homology is symmetric.

To that end we symmetrically deform the cycle $b^p(z^p)$ into a j-cycle u^p on L_k as before. We then use the trajectories orthogonal to the manifolds φ constant to deform u^p symmetrically on L_k into the point $(v) = (0)$. The homology (6.12) thus holds as stated, and (b) is proved.

The proof of the lemma is now complete.

The following hypothesis will be made in several of the following theorems. It will be validated in §7.

Inductive Hypothesis. *If corresponding to an arbitrary positive integer s, constants $a_1, \cdots, a_{m+1}$ are chosen sufficiently near unity, each geodesic $g^r_{ij}(a)$ for which $r < s$ will possess a linking cycle on Ω.*

We let

$$c^r_{ij}(a)$$

denote the length of $g^r_{ij}(a)$, and state the following theorem.

Theorem 6.1. *If corresponding to the positive integer s, constants $a_1, \cdots, a_{m+1}$ are chosen sufficiently near unity, and each geodesic $g^r_{ij}(a)$ for which $r < s$ possesses a linking cycle on Ω, any symmetric j-cycle w^p of dimension*

$$j \geqq k^s_{12} - 1,$$

on which $J(\pi)$ is at most a constant M, and which is Ω-homologous to zero below $c^s_{12}(a)$, will possess a partition which is symmetrically homologous to zero below M.

We begin by replacing w^p by a t-fold partition w^q where

$$t = 4(s - 1)!,$$

and

$$q = tp.$$

This is done in order that q may be a multiple of the integer $4r$ corresponding to each geodesic g^r_{ij} for which $r < s$. We then take the constants $a_1 > \cdots > a_{m+1}$ so near unity that the geodesics $g^r_{ij}(a)$ for which $r < s$ are non-degenerate, possess the respective indices k^r_{ij}, have lengths less than $c^s_{12}(a)$, admit symmetric proper sections S of R^q of the type of Lemma 6.3, and in accordance with Theorem 4.1 are the only closed geodesics with lengths less than $c^s_{12}(a)$.

Of the geodesics whose lengths are less than $c^s_{12}(a)$, let G_1 be the set of geodesics of greatest length c_1. Let σ^q be the critical set on $\Pi^q(a)$ determined by G_1.

We shall prove the following statement.

(α) *The 4-fold partition of the symmetric cycle w^q is symmetrically homologous below M to a cycle w^{4q} below c_1.*

Let L be the domain of points (π) on $\Pi^q(a)$ below c_1. Let N^q be an arbitrarily small neighborhood of the critical set σ^q determined by G_1. We make use of the deformation D^*_q of §4, Ch. VIII, and then of the deformation $\Lambda_q(t)$ of §5, Ch. VIII, to deform w^q into a cycle z^q on $N^q + L^q$, such that the points of z^q which lie on N^q are J-normal. An examination of the definitions of D^*_q and $\Lambda_q(t)$ shows that these deformations are symmetric.

We symmetrically subdivide z^q so finely that it can be written in the form

$$z^q \equiv u^q + v^q, \tag{6.13}$$

where u^q is a symmetric J-normal chain on N^q, and v^q is a symmetric chain below c_1. Let x^q be the common boundary of u^q and v^q. The cycle x^q will be a symmetric J-normal $(j - 1)$-cycle on N^q below c_1. It will not be a spannable cycle belonging to σ^q, because z^q would then be a linking j-cycle belonging to σ^q, and could not be Ω-homologous to zero below $c^s_{12}(a)$.

If the neighborhood N^q is sufficiently small, the statements in the following paragraph are true.

Corresponding to each distinct critical set σ^q_0 determined by G_1, the neighborhood N^q will contain a neighborhood of σ^q_0 at a positive distance from the residue of N^q. The subcycle of x^q in none of these separate neighborhoods is spannable, for otherwise x^q would be spannable. It follows from Lemma 6.4 (a), that the 2-fold partition of each of these subcycles bounds a symmetric j-chain on $\Pi^{2q}(a)$ neighboring σ^{2q} below c_1. Hence (6.13) can be replaced by a congruence

$$\alpha^{2q} \equiv \beta^{2q} + \gamma^{2q}$$

in which α^{2q} is the 2-fold partition of z^q, β^{2q} is a symmetric j-cycle on Π^{2q} neighboring σ^{2q}, and γ^{2q} is a symmetric j-cycle on Π^{2q} below c_1. It follows from Lemma 6.4 (b) that the 2-fold partition of α^{2q} is symmetrically homologous to zero neighboring σ^{4q}. We conclude that the 4-fold partition of w^q is symmetrically homologous below M to the 2-fold partition of γ^{2q}.

Statement (α) is thereby proved.

We now repeat the reasoning used in the proof of (α), replacing $c_{12}^s(a)$ by the length c_1 of the geodesics of the set G_1, and w^q by the cycle w^{4q} of (α). By virtue of the hypothesis that the geodesics $g_{ij}^r(a)$ for which $r < s$ each possess linking cycles, we see that there are no cycles below c_1 Ω-homologous to zero below $c_{12}^s(a)$ which are not Ω-homologous to zero below c_1. The cycle w^{4q} must in particular be Ω-homologous to zero below c_1, since it is Ω-homologous to zero below $c_{12}^s(a)$. Let G_2 be the set of geodesics $g_{ij}^r(a)$ whose lengths equal the maximum c_2 of the lengths $g_{ij}^r(a)$ less than c_1. Proceeding as in the proof of (α) we can show that the 4-fold partition of w^{4q} is symmetrically homologous below c_1 to a cycle below c_2.

Continuing this process we find that a suitable partition of w^q is symmetrically homologous below $c_{12}^s(a)$ to a cycle $w^{q'}$ on which J is less than an arbitrarily small positive constant. The Veblen-Alexander process will then suffice to show that $w^{q'}$ is symmetrically homologous to a cycle of contracted cells on $\Pi^{q'}$, and thus symmetrically homologous to zero.

The proof of the theorem is complete.

Note. Let ν be the number of geodesics $g_{ij}^r(a)$ for which $r < s$. Let μ be the integer

$$\mu = 4^{\nu+1}(s-1)!.$$

The preceding proof shows that a μ-fold partition of w^p will be symmetrically homologous to zero below $c_{12}^s(a)$.

The linking cycles $\lambda_{ij}^r(a)$

7. In this section we shall establish the following. Corresponding to a fixed positive integer s, constants

$$a_1 > \cdots > a_{m+1}$$

can be chosen so near unity, that the geodesics $g_{ij}^r(a)$ for which $r \leqq s$, possess linking cycles $\lambda_{ij}^r(a)$. We begin with the following lemma.

LEMMA 7.1. *The index of the geodesic $g_{12}^s(1)$ is $(2s-1)(m-1) = k_{12}^s$.*

Let S be a proper section of $R^{4s}(a)$ belonging to $g_{12}^s(a)$, set up as in Lemma 6.3. The parameters (v) representing S are explicitly given by (6.8). The value of $J(\pi)$ at the point (π) determined by (v) will be a function $f(v, a)$ which is analytic neighboring $(v) = (0)$ and $(a) = (1)$. The form

$$Q(v, a) = f_{v_i v_j}(0, a)v_i v_j \tag{7.1}$$

will be an index form corresponding to the geodesic $g_{12}^s(a)$. If the constants $a_1 > \cdots > a_{m+1}$ are sufficiently near unity, we have seen that the index k_{12}^s of

$g_{12}^s(a)$ is $(2s - 1)(m - 1)$. Upon letting the variables a_i tend to unity, we see that the index k of $g_{12}^s(1)$ must satisfy the condition

$$k \leqq (2s - 1)(m - 1). \tag{7.2}$$

If on the other hand we set the last n parameters (v) equal to zero, $Q(v, 1)$ reduces to the index form in a fixed end point problem corresponding to a segment λ of $g_{12}^s(1)$ of length $2\pi s$. Inasmuch as there are $(2s - 1)(m - 1)$ conjugate points of either end point of λ between the end points of λ we must have

$$k \geqq (2s - 1)(m - 1). \tag{7.3}$$

The lemma follows from (7.2) and (7.3).

For constants $a_1 > \cdots > a_{m+1}$ sufficiently near unity we shall now define a symmetric chain $\Gamma_{12}^s(a)$ whose boundary $\gamma_{12}^s(a)$ is a symmetric spannable cycle belonging to $g_{12}^s(a)$.

The chain $\Gamma_{12}^s(a)$ *and cycle* $\gamma_{12}^s(a)$. Set

$$t = 4^{\nu+2}(s!). \tag{7.4}$$

The reason for this choice of t will appear in the proof of Lemma 7.3. Let S be a symmetric proper section of $R^t(1)$ belonging to $g_{12}^s(1)$, set up as in Lemma 6.3 with parameters (v). Let $f(v)$ be the function defined by $J(\pi)$ on S, and L_k the index hyperplane corresponding to $f(v)$ and the critical point $(v) = (0)$. Let (u) be a set of k parameters regularly representing L_k neighboring the point, say $(u) = (0)$, which determines $g_{12}^s(1)$, and let $\varphi(u)$ be the function defined by $J(\pi)$ on L_k.

The function $\varphi(u)$ assumes a non-degenerate maximum $2\pi s$ when $(u) = (0)$. Moreover L_k and the function φ are symmetric. If e is a sufficiently small positive constant, the domain

$$\varphi \geqq 2\pi s - e, \tag{7.5}$$

on L_k neighboring $(u) = (0)$, will be free from critical points of φ, save the point $(u) = (0)$. We divide the domain (7.5) into cells in symmetric fashion, and denote the resulting k-chain by $\Gamma_{12}^s(1)$, and its boundary by $\gamma_{12}^s(1)$. On $\gamma_{12}^s(1)$ we have

$$\varphi = 2\pi s - e. \tag{7.6}$$

We record the fact that

$$\Gamma_{12}^s(1) \to \gamma_{12}^s(1) \qquad [\text{on } \Pi^t(1)].$$

For constants a_i sufficiently near unity the chain $\Gamma_{12}^s(1)$ and its boundary $\gamma_{12}^s(1)$ will possess central projections on $\Pi^t(a)$. We denote these projections by $\Gamma_{12}^s(a)$ and $\gamma_{12}^s(a)$ respectively. We have

$$\Gamma_{12}^s(a) \to \gamma_{12}^s(a) \qquad [\text{on } \Pi^t(a)]. \tag{7.7}$$

We shall establish the following lemma.

LEMMA 7.2. *For constants $a_1 > \cdots > a_{m+1}$ sufficiently near unity and a sufficiently small positive constant η, the locus*

$$(7.7)' \qquad J(\pi) = c^s_{12}(a) - \eta$$

on $\Gamma^s_{12}(a)$ is a spannable cycle belonging to $g^s_{12}(a)$, homologous on $\Gamma^s_{12}(a)$, below $c^s_{12}(a)$, to $\gamma^s_{12}(a)$.

According to Theorem 3.3, the index k^s_{12} of $g^s_{12}(a)$, for constants $a_1 > \cdots > a_{m+1}$ sufficiently near unity, is given by the formula

$$k^s_{12} = (2s - 1)(m - 1).$$

By virtue of Lemma 7.1 this is also the index of $g^s_{12}(1)$, and hence equals the dimension k of the index hyperplane L_k on which the preceding function $\varphi(u)$ was defined.

Now let the section S of $R'(1)$ used in defining $\Gamma^s_{12}(1)$ be projected centrally onto $R'(a)$. The resulting manifold $S(a)$ will be a proper section of $R'(a)$ belonging to $g^s_{12}(a)$ if the constants a_i are sufficiently near unity. The index hyperplane L_k belonging to the function defined by $J(\pi)$ on $S(1)$ will project centrally into a regular k-manifold $L_k(a)$ on $S(a)$, on which $J(\pi)$ will assume a proper, nondegenerate maximum at the point (π) which determines $g^s_{12}(a)$, provided the constants $a_1 > \cdots > a_{m+1}$ are sufficiently near unity. For such constants, and for a sufficiently small positive constant η, the cycle defined by $(7.7)'$ on $\Gamma^s_{12}(a)$ will be a spannable $(k - 1)$-cycle belonging to the function defined by $J(\pi)$ on $S(a)$ and to the critical point of this function determined by $g^s_{12}(a)$.

That the cycle $(7.7)'$ will be a spannable $(k - 1)$-cycle on Ω belonging to $g^s_{12}(a)$ follows now from Theorem 11.3 of Ch. VIII.

If the constants a_i are sufficiently near unity, one can use the trajectories orthogonal to the loci, $J(\pi)$ constant on $\Gamma^s_{12}(a)$, to deform the cycle $(7.7)'$ on $\Gamma^s_{12}(a)$ below $c^s_{12}(a)$ into $\gamma^s_{12}(a)$.

The proof of the lemma is complete.

We continue with the following lemma.

LEMMA 7.3. *If corresponding to a positive integer s, constants $a_1 > \cdots > a_{m+1}$ are chosen sufficiently near unity, and if the geodesics $g^r_{ij}(a)$ for which $r < s$ then possess linking cycles, the cycle $\gamma^s_{12}(a)$ will be symmetrically homologous to zero on the domain $J(\pi) \leq M$, where M is the maximum of $J(\pi)$ on $\gamma^s_{12}(a)$.*

Let $p = 4s$. The integer t of (7.4) and p satisfy the relation

$$t = \mu p$$

where

$$\mu = 4^{\nu+1}(s - 1)!.$$

The cycle $\gamma^s_{12}(a)$ has been defined with a K-ordering of its points (π), that is, an ordering with a definite first, second, $\cdots$, and tth vertex. There will accordingly exist a μ-fold join $\bar{\gamma}_{12}(a)$ of $\gamma_{12}^s(a)$ on $\Pi^{4s}(a)$ in the sense of §8, Ch. VIII, at least if the constants a_i are so near unity that a succession of μ elementary extremals determined by an arbitrary point (π) of $\gamma_{12}^s(a)$ has a J-length at most ρ. The cycle $\bar{\gamma}_{12}(a)$ is obtained by preferring the qth vertices of the points (π) of $\gamma_{12}^s(a)$ for which q is a multiple of μ. Let $\gamma_{12}^*(a)$ be the μ-fold partition of $\bar{\gamma}_{12}(a)$. The cycle $\gamma_{12}^*(a)$ will lie on the domain $\Pi^t(a)$ on which $\gamma_{12}^s(a)$ lies, and can be symmetrically deformed into $\gamma_{12}^s(a)$ by using the deformation η of §7, Ch. VIII, holding the common vertices fast. We will thus have the symmetric homology

$$(7.8)' \qquad \gamma_{12}^*(a) \sim \gamma_{12}^s(a)$$

on the domain $J(\pi) \leqq M$ of $\Pi^t(a)$. Since $\gamma_{12}^*(a)$ is a partition of $\bar{\gamma}_{12}(a)$, we also have

$$(7.8)'' \qquad \gamma_{12}^*(a) * \bar{\gamma}_{12}(a).$$

For constants $a_1 > \cdots > a_{m+1}$ sufficiently near unity $g_{12}^s(a)$ possesses a linking cycle, as we have seen in §5. Hence any spannable cycle belonging to $g_{12}^s(a)$, or cycle Ω-homologous to such a cycle below $c_{12}^s(a)$, will be Ω-homologous to zero below $c_{12}^s(a)$. But $\gamma_{12}^s(a)$ is such a cycle, according to Lemma 7.2, and $\bar{\gamma}_{12}(a)$ is another such cycle by virtue of (7.8).

We can apply Theorem 6.1 and the appended note to $\bar{\gamma}_{12}(a)$, and infer that its μ-fold partition $\gamma_{12}^*(a)$ is symmetrically homologous to zero on the domain $J(\pi) \leqq M$. According to (7.8)$'$ the cycle $\gamma_{12}^s(a)$ must also be symmetrically homologous to zero on the domain $J(\pi) \leqq M$, and the lemma is proved.

We are led to the following theorem.

THEOREM 7.1. *If corresponding to a prescribed positive integer s, the geodesics $g_{ij}^r(a)$ possess linking cycles when $r < s$, and if the constants $a_1 > \cdots > a_{m+1}$ are sufficiently near unity, the cycle $\gamma_{12}^s(1)$ will be symmetrically homologous to zero on the domain $J(\pi) \leqq 2\pi s - e$, where $2\pi s - e$ is the value of $J(\pi)$ on $\gamma_{12}^s(1)$.*

It follows from the preceding lemma that there exists a symmetric chain such that

$$w(a) \to \gamma_{12}^s(a) \qquad [\text{on } \Pi^t(a)]$$

where $w(a)$ is a chain on which $J(\pi)$ is at most the maximum M of $J(\pi)$ on $\gamma_{12}^s(a)$, provided the constants $a_1 > \cdots > a_{m+1}$ are sufficiently near unity. But if the constants a_i are sufficiently near unity M will be so near $2\pi s - e$ that the central projection of $w(a)$ on $\Pi^t(1)$ will be a chain z on which $J(\pi) < 2\pi s$. We thus have

$$z \to \gamma_{12}^s(1) \qquad [\text{on } \Pi^t(1), \text{ below } 2\pi s].$$

We now use the symmetric deformation D_p^* of §4, Ch. VIII, with $p = t$ to deform z on $\Pi^t(1)$ into a chain u below $2\pi s - e$. The cycle $\gamma_{12}^s(1)$ will thereby generate a deformation chain v, and we have

$$u + v \to \gamma_{12}^s(1) \qquad [\text{on } \Pi^t(1)]$$

where $u + v$ is a symmetric chain on which $J(\pi) \leqq 2\pi s - e$.

The proof of the theorem is now complete.

The linking cycles $\lambda^s_{12}(a)$. Under the hypotheses of the preceding theorem there exists a symmetric chain $M^s_{12}(1)$ on $\Pi^t(1)$ on which $J(\pi) \leqq 2\pi s - e$, and which is such that

$$(7.9)' \qquad M^s_{12}(1) \to \gamma^s_{12}(1).$$

For constants a_i sufficiently near unity the central projection of $M^s_{12}(1)$ on $\Pi^t(a)$ is well defined, and will be denoted by $M^s_{12}(a)$. Moreover on $M^s_{12}(a)$,

$$J(\pi) < c^s_{12}(a)$$

if the constants a_i are sufficiently near unity. We will then have

$$(7.9)'' \qquad M^s_{12}(a) \to \gamma^s_{12}(a) \qquad [\text{below } c^s_{12}(a)].$$

On the other hand we have seen in (7.7) that

$$(7.10) \qquad \Gamma^s_{12}(a) \to \gamma^s_{12}(a)$$

where $\Gamma^s_{12}(a)$ is a symmetric chain on which $J(\pi)$ assumes a proper absolute maximum $c^s_{12}(a)$ at the point (π) determined by $g^s_{12}(a)$; and the locus

$$J(\pi) = c^s_{12}(a) - \eta \qquad (\eta > 0)$$

on $\Gamma^s_{12}(a)$ is a spannable cycle belonging to $g^s_{12}(a)$ if the constant η is sufficiently small, and the constants $a_1 > \cdots > a_{m+1}$ are sufficiently near unity.

We set

$$(7.11) \qquad \lambda^s_{12}(a) = M^s_{12}(a) + \Gamma^s_{12}(a).$$

Except in the case $s = 1$, the definition of $\lambda^s_{12}(a)$ has been made to depend upon the inductive hypothesis that the geodesics $g^r_{ij}(a)$ for which r is less than a prescribed integer s possess linking cycles if the constants $a_1 > \cdots > a_{m+1}$ are sufficiently near unity. For constants $a_1 > \cdots > a_{m+1}$ sufficiently near unity the cycle $\lambda^s_{12}(a)$ is then a *symmetric linking cycle belonging to* $g^s_{12}(a)$.

In order to define the cycles $\lambda^r_{ij}(a)$ in general, we introduce a deformation R_{pq} of points on $E_m(a)$.

The deformation R_{pq}. We begin by defining a deformation of the space (w) in the form of a rotation. In this deformation the time t shall increase from 0 to 1 inclusive. A point whose coordinates (w) afford a set (z) when $t = 0$, shall be replaced at the time t by a point (w) such that

$$(7.12) \qquad \begin{aligned} w_p &= z_p \cos \pi t - z_q \sin \pi t && (p \neq q), \\ w_q &= z_p \sin \pi t + z_q \cos \pi t && (0 \leqq t \leqq 1), \\ w_i &= z_i, \end{aligned}$$

where p and q are two distinct integers on the range $1, \cdots, m + 1$, and i takes on all integral values from 1 to $m + 1$, excluding p and q.

The deformation R_{pq} of $E_m(a)$ is now defined as a deformation in which each point (w) on $E_m(a)$ moves so that its central projection on $E_m(1)$ is subjected to the deformation (7.12). Under R_{pq}

$$w_p = -z_q, \qquad w_q = z_p, \tag{7.13)$'$}$$

when $t = 1/2$, while when $t = 1$

$$w_p = -z_p, \qquad w_q = -z_q. \tag{7.13)$''$}$$

By the deformation R_{pq} of points (π) on $\Pi^s(a)$ we mean a deformation in which the vertices of (π) are deformed on $E_m(a)$ under R_{pq}. A chain on $\Pi^s(a)$ whose central projection on $\Pi^s(1)$ consists of inner points of $\Pi^s(1)$ will thereby be deformed under R_{pq} so as to remain on $\Pi^s(a)$, provided the constants a_i are sufficiently near unity.

If w^s is a symmetric k-chain on $\Pi^s(a)$, the initial and final images of w^s under R_{pq} are identical. We let

$$R_{pq}w^s \tag{7.14}$$

denote the deformation $(k + 1)$-chain derived from w^s under R_{pq}. If w^s is a k-cycle, the chain (7.14) reduces to a $(k + 1)$-cycle, mod 2.

The cycles $\lambda^s_{\mu\nu}$. For positive integers $\mu < \nu \leqq m + 1$ we now set

$$\lambda^s_{\mu\nu}(a) = R_{\mu,\mu-1} \cdots R_{32}R_{21}R_{\nu,\nu-1} \cdots R_{43}R_{32}\lambda^s_{12}(a). \tag{7.15}$$

If $\mu = 1$, the first symbol on the right is $R_{\nu,\nu-1}$. The successive operations of forming deformation cycles in (7.15) are to be performed in the order R_{32}, R_{43} etc., each operation producing a cycle of one higher dimension.

Referring to (7.11) we set

$$M^s_{\mu\nu}(a) = R_{\mu,\mu-1} \cdots R_{21}R_{\nu,\nu-1} \cdots R_{32}M^s_{12}(a) \tag{7.16)$'$}$$

and

$$\Gamma^s_{\mu\nu}(a) = R_{\mu,\mu-1} \cdots R_{21}R_{\nu,\nu-1} \cdots R_{32}\Gamma^s_{12}(a). \tag{7.16)$''$}$$

We see that

$$\lambda^s_{\mu\nu}(a) = M^s_{\mu\nu}(a) + \Gamma^s_{\mu\nu}(a). \tag{7.16)$'''$}$$

In each of the preceding deformations the time t runs from 0 to 1 inclusive. Denote the point into which a point (π) is deformed at the time t under R_{pq} by

$$R^t_{pq}\pi.$$

Let $\pi^s_{12}(a)$ be the point on $\lambda^s_{12}(a)$ which determines $g^s_{12}(a)$. We now set

$$\pi^s_{\mu\nu}(a) = R^{1/2}_{\mu,\mu-1} \cdots R^{1/2}_{21} R^{1/2}_{\nu,\nu-1} \cdots R^{1/2}_{32} \pi^s_{12}(a).$$

We observe that the point $\pi^s_{\mu\nu}(a)$ lies on $\lambda^s_{\mu\nu}(a)$, and determines $g^s_{\mu\nu}(a)$.

We shall prove a lemma concerning $\Gamma^s_{\mu\nu}(a)$. In this connection we point out that the definition of $\Gamma^s_{\mu\nu}(a)$ is independent of any inductive hypothesis.

LEMMA 7.4. *On each chain $\Gamma^s_{\mu\nu}(a)$ for which s is a prescribed positive integer $J(\pi)$ will assume a proper, absolute maximum equal to the length $c^s_{\mu\nu}(a)$ of $g^s_{\mu\nu}(a)$, at the point $\pi^s_{\mu\nu}(a)$, provided the constants $a_1 > \cdots > a_{m+1}$ are sufficiently near unity.*

In proving this lemma we shall make use of the following property of ellipses.

Corresponding to any positive constant a, there exists a positive constant e with the following property. Let E' and E'' be two ellipses in the xy plane with centers at the origin but with arbitrary orientations. Suppose no points of E'' are exterior to E'. Let b' and b'' be segments of E' and E'', respectively, which subtend a common angle at the origin, in magnitude at least a. If the ellipses E' and E'' possess semi-axes which differ from unity by at most e, the lengths β' and β'' of the segments b' and b'' respectively satisfy the condition

$$\beta'' \leqq \beta'. \tag{7.17}$$

The proof of these statements can be given by elementary methods, and will be left to the reader.

To establish the lemma we shall make use of a function $H(\pi)$ defined as follows. Let b be any elementary extremal on $E_m(a)$ of positive length and with end points P and Q. Let λ be the 2-plane determined by P, Q and the origin. The 2-plane λ will intersect $E_m(a)$ in an ellipse. Of the arcs of this ellipse bounded by P and Q, let b' be the shorter. We term b' the *elliptical arc corresponding to b*. To define $H(\pi)$ we replace each non-null elementary arc of $g(\pi)$ by the corresponding elliptical arc and leave null arcs unchanged. We denote the value of the arc length J taken along the resulting curve by $H(\pi)$. We observe that

$$J(\pi) \leqq H(\pi).$$

We shall now establish Lemma 7.4 with $H(\pi)$ replacing $J(\pi)$.

Observe first that $H(\pi) \equiv J(\pi)$ when $(a) = (1)$. Hence $H(\pi)$ has a non-degenerate, absolute maximum on $\Gamma^s_{12}(1)$ when $(\pi) = \pi^s_{12}(1)$. Hence for constants a_i such that $|a_i - 1| < \eta$, where η is a sufficiently small positive constant, the function $H(\pi)$ will have a non-degenerate, absolute maximum on $\Gamma^s_{12}(a)$ when $(\pi) = \pi^s_{12}(a)$. There is no limitation in this statement on the relative sizes of the constants a_i.

Let the constants (a) be chosen so that

$$a_1 > \cdots > a_{m+1}$$

and

$$|a_i - 1| < \eta. \tag{7.18}$$

For these constants $(a_1, a_2, \cdots, a_{m+1})$ we consider the chain

$$\Gamma_{1\nu}(a_\mu, a_\nu, \cdots, a_\nu, a_{\nu+1}, \cdots, a_{m+1}). \tag{7.19}$$

The chain (7.19) can be obtained from the chain Γ^s_{12} with the same arguments, by using the formula

$$\Gamma^s_{1\nu} = R_{\nu\,\nu-1} \cdots R_{32}\Gamma^s_{12}.$$

But for the arguments in (7.19) the deformations here involved become rotations of the vertices of points (π). For these arguments $H(\pi)$ assumes a proper, absolute maximum on Γ_{12}^s equal to the length of an ellipse with semi-axes $1/a_\mu$ and $1/a_\nu$. This length equals the number

$$c_{\mu\nu}^s(a_1, a_2, \cdots, a_{m+1}) = c_{\mu\nu}^s(a),$$

and is the length of

$$g_{12}^s(a_\mu, a_\nu, \cdots, a_\nu, a_{\nu+1}, \cdots, a_{m+1}). \tag{7.20}$$

We see then that $H(\pi)$ assumes a proper, absolute maximum relative to its values on (7.19) at each point (π) on the cycle

$$R_{\nu,\nu-1} \cdots R_{32}\pi_{12}^s(a_\mu, a_\nu, \cdots, a_\nu, a_{\nu+1}, \cdots, a_{m+1}). \tag{7.21}$$

This maximum equals $c_{\mu\nu}^s(a)$.

We shall next prove the following statement.

(α). *For points (π') on the chain*

$$\Gamma_{1\nu}(a_\mu, \cdots, a_\mu, a_{\mu+1}, \cdots, a_{m+1}) \qquad (\mu < \nu) \tag{7.22}$$

the function $H(\pi')$ assumes a proper, absolute maximum $c_{\mu\nu}^s(a)$ at the point (π') which determines the geodesic

$$g_{1\nu}^s(a_\mu, \cdots, a_\mu, a_{\mu+1}, \cdots, a_{m+1}),$$

provided the constants $a_1 > \cdots > a_{m+1}$ are sufficiently near unity.

We shall prove (α) for the case $s = 1$. The proof for a general s is not essentially different.

In proving (α) we shall compare each point (π') on the chain (7.22) with its central projection (π) on the chain (7.19), taking $s = 1$. Points (π) on (7.19) will be divided into two classes. The first class shall consist of the points (π) on the cycle (7.21), while the second class shall consist of the remaining points on (7.19).

Points (π) on (7.21), $s = 1$. For such points (π), $g(\pi)$ is an ellipse obtainable from the ellipse

$$g_{1\nu}^1(a_\mu, a_\nu, \cdots, a_\nu, a_{\nu+1}, \cdots, a_{m+1}) \tag{7.23}$$

by a rotation in which the intersection of the ellipse with the w_1 axis is fixed. For such points (π), $H(\pi)$ equals the length of this ellipse, namely

$$c_{\mu\nu}^1(a_1, \cdots, a_{m+1}) = c_{\mu\nu}^1(a). \tag{7.24}$$

Let $\gamma(\pi)$ be the central projection of the preceding ellipse $g(\pi)$ on the m-ellipsoid

$$E_m(a_\mu, \cdots, a_\mu, a_{\mu+1}, \cdots, a_{m+1}). \tag{7.25}$$

Let (π') be the central projection of (π) on the chain (7.22). The value of $H(\pi')$ is the length of $\gamma(\pi)$. The ellipse $g(\pi)$ lies on the m-ellipsoid

$$E_m(a_\mu, a_\nu, \cdots, a_\nu, a_{\nu+1}, \cdots, a_{m+1}). \tag{7.26}$$

Its center is at the origin, it intersects the w_1 axis, and it lies in the space of the $w_1, \cdots, w_\nu$ axes. Now $\mu < \nu$ and $a_\mu > a_\nu$. We see that $\gamma(\pi)$ and $g(\pi)$ have their intersections with the w_1 axis in common, but that $\gamma(\pi)$ is otherwise interior to $g(\pi)$, except in the special case where $g(\pi)$ and $\gamma(\pi)$ are the ellipse (7.23). But the length of the ellipse $g(\pi)$ is $c^1_{\mu\nu}(a)$ and the length of the ellipse $\gamma(\pi)$ is $H(\pi')$. Hence

$$H(\pi') < c^1_{\mu\nu}(a) \tag{7.27}$$

for points (π') which project centrally into points (π) on (7.21), except in the case where (π') determines the ellipse (7.23).

Points (π) *on* (7.19) *but not on* (7.21). For such points (π),

$$H(\pi) < c^s_{\mu\nu}(a) \tag{7.28}$$

as stated in connection with (7.21). But if each elliptical arc β' determined by such points (π) be compared with its central projection β'' on the m-ellipsoid (7.25), it follows from (7.17) that if η in (7.18) is sufficiently small

$$H(\pi') \leqq H(\pi). \tag{7.29}$$

From (7.28) and (7.29) we see that

$$H(\pi') < c^s_{\mu\nu}(a) \tag{7.30}$$

for points (π') on the chain (7.22) which do not project centrally into points (π) on (7.21).

Statement (α) follows from (7.27) and (7.30).

We shall now prove statement (β).

(β). *The function* $H(\pi)$ *assumes a proper, absolute maximum* $c^s_{\mu\nu}(a)$ *on the chain*

$$\Gamma^s_{\mu\nu}(a_1, \cdots, a_{m+1}) = \Gamma^s_{\mu\nu}(a) \qquad (\mu < \nu) \tag{7.31}$$

at the point $\pi^s_{\mu\nu}(a)$ *thereon which determines* $g^s_{\mu\nu}(a)$.

We shall compare the chain (7.31) with the chain

$$\Gamma^s_{\mu\nu}(a_\mu, \cdots, a_\mu, a_{\mu+1}, \cdots, a_{m+1}). \tag{7.32}$$

The latter chain is given by the formula

$$\Gamma^s_{\mu\nu} = R_{\mu,\mu-1} \cdots R_{21}\Gamma^s_{1\nu}$$

where the arguments in $\Gamma^s_{1\nu}$ are the same as in (7.32). It follows from (α) that $H(\pi)$ assumes a proper, absolute maximum $c^s_{\mu\nu}(a)$ on the chain (7.32) at each point of the cycle

$$R_{\mu,\mu-1} \cdots R_{21}\pi^s_{1\nu}(a_\mu, \cdots, a_\mu, a_{\mu+1}, \cdots, a_{m+1}). \tag{7.33}$$

For $s = 1$ the curves $g(\pi)$ determined by points (π) on (7.33) consist of ellipses obtainable from the ellipse

$$g^1_{1\nu}(a_\mu, \cdots, a_\mu, a_{\mu+1}, \cdots, a_{m+1})$$

by a rotation in which the points of intersection of this ellipse with the w_ν axis are fixed. The central projections of these ellipses on the ellipsoid $E_m(a)$ will be ellipses of lesser length, except for the ellipse

$$g^1_{\mu\nu}(a_\mu, \cdots, a_\mu, a_{\mu+1}, \cdots, a_{m+1}).$$

The last ellipse is identical with the ellipse $g^1_{\mu\nu}(a)$, and has the length $c^1_{\mu\nu}(a)$.

For points (π) on (7.32) which are not on (7.33),

$$H(\pi) < c^s_{\mu\nu}(a) \tag{7.34}$$

as follows from (α). But if (π) is any point on the chain (7.32), and (π') its central projection on (7.31), we have

$$H(\pi') \leqq H(\pi)$$

in accordance with (7.17), provided the constant η in (7.18) is sufficiently small. Hence for points (π') on (7.31) whose central projections (π) do not lie on (7.33), we have

$$H(\pi') < c^s_{\mu\nu}(a).$$

Statement (β) follows from this result and the result of the preceding paragraph.

To return to the lemma we observe that

$$J(\pi) = H(\pi) = c^s_{\mu\nu}(a)$$

at the point $\pi^s_{\mu\nu}(a)$ on $\Gamma^s_{\mu\nu}(a)$ which determines $g^s_{\mu\nu}(a)$. By virtue of statement (β),

$$H(\pi) < c^s_{\mu\nu}(a) \tag{7.35}$$

at all other points (π) on $\Gamma^s_{\mu\nu}(a)$ if the constants $a_1 > \cdots > a_{m+1}$ are sufficiently near unity. But $J(\pi) \leqq H(\pi)$, so that (7.35) gives us the relation

$$J(\pi) < c^s_{\mu\nu}(a) \qquad [(\pi) \neq \pi^s_{\mu\nu}(a)]$$

for (π) on $\Gamma^s_{\mu\nu}(a)$.

The proof of the lemma is now complete.

We continue with the following lemma.

LEMMA 7.5. *Points (π) on the chain $\Gamma^s_{\mu\nu}(1)$ neighboring $\pi^s_{\mu\nu}(1)$ make up a non-tangential manifold Σ, belonging to $g^s_{\mu\nu}(1)$ in the sense of §1. The dimension of Σ equals $k^s_{\mu\nu}$.*

The lemma is true of $\Gamma^s_{12}(1)$ by virtue of the definition of $\Gamma^s_{12}(1)$ as the domain (7.5).

Let (v) be a set of $k^s_{1\,2}$ parameters regularly representing $\Gamma^s_{1\,2}(1)$ neighboring $\pi^s_{1\,2}(1)$, with $(v) = (0)$ corresponding to $\pi^s_{1\,2}(1)$ Recall that

$$\Gamma^s_{\mu\,\nu}(1) = R_{\mu,\,\mu-1} \cdots R_{21}R_{\nu,\,\nu-1} \cdots R_{32}\Gamma^s_{1\,2}(1). \tag{7.36}$$

Let the time t in the respective deformations employed in (7.36), taken in the order written, be denoted by

$$(\tau_{\mu-1}, \cdots, \tau_1, t_{\nu-2}, \cdots, t_1).$$

Recall that $0 \leqq t \leqq 1$ for each such parameter. The general point (π) on $\Gamma^s_{\mu\,\nu}(1)$ is obtained from an arbitrary point (v) on $\Gamma^s_{1\,2}(1)$ by subjecting that point to R_{32} up to a time t_1, subjecting the resulting point to R_{43} up to a time t_2, and so on until all the deformations in (7.36) have been employed, the final deformation $R_{\mu,\,\mu-1}$ continuing up to a time $\tau_{\mu-1}$. The general point (π) is thus representable by means of the parameters

$$(\tau) = (\tau_1, \cdots, \tau_{\mu-1}),$$
$$(t) = (t_1, \cdots, t_{\nu-2}),$$

and the parameters (v) of the initial point. In particular the point $\pi^s_{\mu\,\nu}(1)$ on $\Gamma^s_{\mu\,\nu}(1)$ is determined as above by parameters (τ) and (t) each of which equals $1/2$, and parameters $(v) = (0)$. We shall prove the following.

(α). *In terms of the parameters (t), (τ), and (v), $\Gamma^s_{\mu\,\nu}(1)$ is regular at the point $\pi^s_{\mu\,\nu}(1)$ on $\Gamma^s_{\mu\,\nu}(1)$ which corresponds to the parameter values*

$$(v) = (0), \qquad (t) = (\tfrac{1}{2}), \qquad (\tau) = (\tfrac{1}{2}). \tag{7.37}$$

We shall establish (α) by showing that in the space of the points (π) the directions tangent to the parametric curves on $\Gamma^s_{\mu\,\nu}(1)$ through the point $\pi^s_{\mu\,\nu}(1)$ are independent. Of these directions, those involving the variables (v) alone are independent among themselves, since the same is true of $\Gamma^s_{1\,2}(1)$, and since the deformations in (7.36) subject the vertices of points (π) to a rigid motion.

We consider the curves on which the parameters (t) and (τ) vary. In terms of the parameters (t), (τ) and (v) let M^q be the manifold on the unit m-sphere on which the qth vertex of the point (π) on $\Gamma^1_{\mu\,\nu}(1)$ varies for parameter values (t), (τ), and (v) near the values (7.37). Let A' and A'' be the intersections of the positive w_μ and w_ν axes with the unit m-sphere. Of the manifolds M^q let L' and L'' be two particular manifolds which pass through A' and A'' for parameter values (7.37). Let the parametric curves on L'' through A'' on which one only of the parameters

$$t_1, \cdots, t_{\nu-2}, \tau_1, \cdots, \tau_{\mu-1}$$

vary and on which the remaining parameters have the values (7.37), be respectively denoted by

$$h_1, \cdots, h_{\nu-2}, k_1, \cdots, \kappa_{\mu-1}. \tag{7.38}$$

On taking account of the deformations used in (7.36) one sees that in the neighborhood of A'' the curves $h_1, \cdots, h_{\nu-2}$ consist respectively of segments of the circles

(7.39) $$g^1_{1\nu}, \cdots, g^1_{\mu-1,\nu}, g^1_{\mu+1,\nu}, \cdots, g_{\nu-1,\nu}, \qquad [(a) = (1)],$$

while the curves $k_1, \cdots, k_{\mu-1}$ reduce to the point A''. If $\mu = 1$, the curves (7.39) become the set

$$g^1_{2\nu}, \cdots, g^1_{\nu-1,\nu}.$$

To determine the curves (7.39) let Q_i denote the intersection of the unit m-sphere with the positive w_i axis. Under R_{32}, Q_2 is rotated into the point Q_3, reaching Q_3 when $t_1 = 1/2$. The path of Q_2 is thus a segment of g^1_{23}. Under R_{43}, Q_2 is fixed, while Q_3 is rotated into Q_4 when $t_2 = 1/2$. Thus the path g^1_{23} is rotated into the path g^1_{24} when $t_2 = 1/2$. The successive application of the deformations $R_{54}, \cdots, R_{\nu,\nu-1}$ up to times $t_3 = \cdots = t_{\nu-2} = 1/2$ respectively will rotate g^1_{24} into the path $g_{2\nu}$. The deformations $R_{21}, \cdots, R_{\mu,\mu-1}$ are now to be successively applied to $g^1_{2\nu}$ up to the times $\tau_1 = \cdots = \tau_{\mu-1} = 1/2$ respectively. Of these deformations R_{21} alone affects $g^1_{2\nu}$, rotating $g^1_{2\nu}$ into $g^1_{1\nu}$. Thus the parametric curve h_1 is a segment of $g^1_{2\nu}$ as stated. Similar reasoning will establish that the remainder of the parametric curves (7.38) lie on the corresponding circles in (7.39).

Let

$$\Sigma_t, \qquad \Sigma_\tau, \qquad \Sigma_v,$$

respectively, denote the submanifolds of Σ through the point $\pi^s_{\mu\nu}(1)$ on which the parameters (t), (τ), and (v) vary neighboring the sets (7.37). We have seen that the manifold Σ_v is regular. That the manifold Σ_t is regular follows from the mutual orthogonality of the circles (7.39).

To show that Σ_τ is regular at $\pi^s_{\mu\nu}(1)$ we consider the parametric curves on L', which pass through A' corresponding to the values (7.37), and on which one only of the parameters,

$$\tau_1, \cdots, \tau_{\mu-1},$$

varies. Following the trajectory of the point Q_1 under the successive deformations in (7.36) one finds that these parametric curves consist respectively of segments of the mutually orthogonal circles

(7.40) $$g_{1\mu}, \cdots, g_{\mu-1,\mu} \qquad [(a) = (1)].$$

That the manifolds Σ_t and Σ_τ have no tangents in common at the point $\pi^s_{\mu\nu}(1)$ follows from the fact that the curves $k_1, \cdots, k_{\mu-1}$ in (7.38) which result from a variation of the respective parameters (τ) of Σ_τ reduce to points, while the remaining curves in (7.38) which result from a variation of the respective parameters (t) on Σ_t have mutually orthogonal directions. Thus the submanifold Σ^* of Σ through $\pi^s_{\mu\nu}(1)$, on which (t) and (τ) alone vary, is regular in terms of the parameters (t) and (τ).

That the manifolds Σ^* and Σ_v have no tangent line in common at $\pi^s_{\mu\nu}(1)$ follows from the fact that on analytic curves tangent to Σ_v at $(v) = (0)$, $J(\pi)$ assumes a non-degenerate maximum when $(v) = (0)$, while on Σ^* $J(\pi)$ is constant. Thus the directions of the tangents to the respective parametric curves of Σ through $\pi^s_{\mu\nu}(1)$ are independent.

Statement (α) now follows.

We continue with a proof of statement (β).

(β). *The manifold Σ is a non-tangential manifold belonging to $g^s_{\mu\nu}(1)$.*

Let the manifolds on R on which lie the successive vertices of a point (π) on Σ_t, Σ_τ, or Σ_v be termed *vertex manifolds*. For points (π) neighboring $\pi^s_{\mu\nu}(1)$ these vertex manifolds are readily seen to be orthogonal to $g^s_{\mu\nu}(1)$. It follows that the corresponding vertex manifolds of Σ are orthogonal to $g^s_{\mu\nu}(1)$. Upon recalling the definition of non-tangential manifolds belonging to $g^s_{\mu\nu}(1)$ one sees that Σ must belong in that category, and (β) is proved.

We complete the proof of the lemma by proving statement (γ).

(γ). *The dimension j of $\Gamma^s_{\mu\nu}(1)$ equals $k^s_{\mu\nu}$.*

First recall that the dimension of $\Gamma^s_{12}(1)$ is k^s_{12}. The number of parameters (v) thus equals k^s_{12}. The number of parameters (t) and (τ) is $\mu + \nu - 3$, so that

$$j = k^s_{12} + \mu + \nu - 3.$$

But

$$k^s_{12} = (2s - 1)(m - 1).$$

Hence

$$\begin{aligned} j &= (2s - 1)(m - 1) + \mu + \nu - 3 \\ &= m + \mu + \nu - 4 + 2(s - 1)(m - 1). \end{aligned}$$

This is the value of $k^s_{\mu\nu}$ as stated.

The preceding lemma leads to the following:

LEMMA 7.6. *Let s be a prescribed positive integer, and $c^s_{\mu\nu}(a)$ the length of $g^s_{\mu\nu}(a)$. If the constants $a_1 > \cdots > a_{m+1}$ are sufficiently near unity and e is a sufficiently small positive constant the locus,*

$$J(\pi) = c^s_{\mu\nu}(a) - e$$

on $\Gamma^s_{\mu\nu}(a)$, will be a spannable cycle belonging to $g^s_{\mu\nu}(a)$.

We shall prove this lemma with the aid of Theorem 1.2.

To that end first observe that the points (π) on $\Gamma^s_{\mu\nu}(a)$ neighboring $\pi^s_{\mu\nu}(a)$ form an analytic manifold $\Sigma(a)$. According to Lemma 7.5 this manifold is a "non-tangential" manifold, belonging to $g^s_{\mu\nu}(1)$ when $(a) = (1)$. By virtue of the definition of such non-tangential manifolds one sees that $\Gamma^s_{\mu\nu}(a)$ will remain a non-tangential manifold belonging to $g^s_{\mu\nu}(a)$, if the constants a_i are sufficiently near unity.

According to Lemma 7.5 the dimension of $\Sigma(a)$ will equal $k_{\mu\nu}^{s}$, and thus equal the index of $g_{\mu\nu}^{s}(a)$. According to Lemma 7.4, $J(\pi)$ will assume a proper, absolute maximum $c_{\mu\nu}^{s}(a)$ on $\Gamma_{\mu\nu}^{s}(a)$ at the point $\pi_{\mu\nu}^{s}(a)$ if the constants $a_1 > \cdots > a_{m+1}$ are sufficiently near unity.

The lemma follows from Theorem 1.2.

We come to a basic theorem.

THEOREM 7.2. *Let s be a prescribed positive integer. If the constants $a_1 > \cdots > a_{m+1}$ are sufficiently near unity, the cycles $\lambda_{\mu\nu}^{r}(a)$ exist and are linking cycles belonging to the geodesics $g_{\mu\nu}^{r}(a)$, for all integers $r \leqq s$.*

We first consider the case $s = 1$.

If the constants $a_1 > \cdots > a_{m+1}$ are sufficiently near unity, the following statements are true. The cycle $\lambda_{12}^{1}(a)$ will be a linking cycle belonging to $g_{12}^{1}(a)$, the definition of $\lambda_{12}^{1}(a)$ depending upon no inductive hypothesis. If e is a sufficiently small positive constant, the locus

$$J(\pi) = c_{\mu\nu}^{1}(a) - e \tag{7.41}$$

on $\Gamma_{\mu\nu}^{1}(a)$ will be a spannable cycle belonging to $g_{\mu\nu}^{1}(a)$, according to Lemma 7.6. On $\Gamma_{\mu\nu}^{1}(a)$, $J(\pi)$ will assume a proper, absolute maximum $c_{\mu\nu}^{1}(a)$ at the point $\pi_{\mu\nu}^{1}(a)$. The chain $M_{12}^{1}(1)$ of (7.9)′ exists, by virtue of Theorem 7.1, and the chain $M_{\mu\nu}^{1}(a)$ is then defined by (7.16)′ for $\nu > 1$, and will lie below $c_{\mu\nu}^{1}(a)$. The cycle $\lambda_{\mu\nu}^{1}(a)$ can now be defined by the congruence

$$\lambda_{\mu\nu}^{1}(a) \equiv M_{\mu\nu}^{1}(a) + \Gamma_{\mu\nu}^{1}(a),$$

as in (7.16)‴ We see that the cycle (7.41) on $\Gamma_{\mu\nu}^{1}(a)$ will bound below $c_{\mu\nu}^{1}(a)$ on $\lambda_{\mu\nu}^{1}(a)$. Hence $\lambda_{\mu\nu}^{1}(a)$ will be a linking cycle belonging to $g_{\mu\nu}^{1}(a)$. The theorem is thus true when $s = 1$.

Proceeding inductively we assume that the theorem is true when s is replaced by $s - 1$. This inductive hypothesis enables us to apply Theorem 7.1 and infer that $M_{12}^{s}(a)$ exists as in (7.9)″. The cycle $\lambda_{12}^{s}(a)$ can then be defined as in (7.11), and the cycle $\lambda_{\mu\nu}^{s}(a)$ as in (7.15). We prove that $\lambda_{\mu\nu}^{s}(a)$ is a linking cycle belonging to $g_{\mu\nu}^{s}(a)$ as in the preceding paragraph.

The proof of the theorem is complete.

The circular connectivities of the m-sphere

8. Before coming to the problem of the existence of closed geodesics we shall solve the basic topological problem of the determination of the circular connectivities of the m-sphere. Recall that these connectivities are the Ω-connectivities of the space Ω determined by an admissible metric on the m-sphere. An admissible metric is any metric with elementary arcs of the nature defined in §12, Ch. VIII. These circular connectivities will be independent of the metric used and, as we have seen, are topological invariants. On Riemannian manifolds the metric can be defined, if one pleases, by the integral of arc length, and the elementary arcs defined by means of geodesics.

We shall make use of the notation of the preceding chapter, referring to the m-ellipsoid $E_m(a)$ and to $\Omega(a)$. We shall take the constant ρ which limits the lengths of elementary extremals as a fixed number such that

$$\frac{\pi}{2} < \rho < \pi.$$

The circular connectivities of the m-sphere will be found by determining the Ω-connectivities of $\Omega(1)$.

We begin with the following theorem.

THEOREM 8.1. *Corresponding to any positive integer k, a maximal set of k-cycles, Ω-independent on $\Omega(1)$, consists of the cycles of the set*

$$\lambda^r_{ij}(1) \qquad (r = 1, 2, \cdots; i, j = 1, \cdots, m+1; i < j)$$

of dimension k.

Statement (α) will now be proved.

(α). *Any k-cycle z on $\Omega(1)$ is Ω-homologous to a linear combination of the k-cycles of the theorem.*

On z suppose that $J(\pi)$ is less than $2\pi s$, where s is some positive integer. Without loss of generality we can assume that the elementary arcs determined by z have lengths at most $\rho/2$, because in any case a 2-fold partition of z would be Ω-homologous to z and have this property.

Let

$$1 > \alpha_1 > \cdots > \alpha_{m+1} > 0$$

be a fixed set of constants, and let

$$a_1 > \cdots > a_{m+1}$$

be a set of constants on the range

$$a_i = \alpha_i + t(1 - \alpha_i) \qquad (0 \leqq t < 1). \tag{8.0}$$

The constants a_i will satisfy the condition $a_i < 1$ so that the m-sphere $E_m(1)$ will be interior to $E_m(a)$. It follows that the central projection on $\Omega(1)$ of any point (π) on $\Omega(a)$ will be admissible; for an elementary geodesic λ on $E_m(a)$ will project centrally into a shorter curve μ on $E_m(1)$. The elementary geodesic joining the end points of μ on $E_m(1)$ will then be shorter than λ. Thus the central projection on $\Omega(1)$ of an arbitrary point (π) on $\Omega(a)$ will be admissible. Moreover if the above constants α_i are chosen sufficiently near unity, the statements of the following paragraph are true.

The central projection $z(a)$ of z on $\Omega(a)$ is admissible for constants (a) given by (8.0). On $E_m(\alpha)$ there are no closed geodesics with lengths less than $2\pi s$ other than the geodesics $g^r_{ij}(\alpha)$ for which $r < s$. The geodesics $g^r_{ij}(\alpha)$ for which $r < s$ are non-degenerate, and possess the cycles $\lambda^r_{ij}(\alpha)$ as linking cycles. The value of $J(\pi)$ on $z(\alpha)$ is less than $2\pi s$.

It follows from the theory developed in Ch. VIII that $z(\alpha)$ is Ω-homologous to a linear combination, say $v(\alpha)$, of the cycles $\lambda_{ij}^r(\alpha)$ for which $r < s$. There will then exist a partition, $v'(\alpha)$ of $v(\alpha)$, and a partition $z'(\alpha)$ of $z(\alpha)$, both on a domain $\Pi^q(\pi)$, together with a $(k + 1)$-chain $w'(\alpha)$, also on $\Pi^q(\alpha)$, such that

$$(8.1) \qquad w'(\alpha) \to v'(\alpha) + z'(\alpha).$$

If we denote the central projection, on $\Omega(1)$, of a cycle x of $\Omega(a)$, by $c[x]$, it follows from (8.1) that

$$c[w'(\alpha)] \to c[v'(\alpha)] + c[z'(\alpha)]$$

so that

$$(8.1a) \qquad c[v'(\alpha)] \sim c[z'(\alpha)] \qquad [\text{on } \Pi^q(1)].$$

Let $v(a)$ denote the central projection of $v(\alpha)$ on $\Omega(a)$, and $v'(a)$ the partition of $v(a)$ on $\Pi^q(a)$. If we let t in (8.0) range from 0 to 1 inclusive, the cycle $c[v'(a)]$ will generate a $(k + 1)$-chain v^* on $\Pi^q(1)$ such that

$$v^* \to c[v'(\alpha)] + v'(1).$$

Thus

$$(8.1b) \qquad v'(1) \sim c[v'(\alpha)] \qquad [\text{on } \Pi^q(1)].$$

It follows similarly that

$$(8.1c) \qquad z'(1) \sim c[z'(\alpha)] \qquad [\text{on } \Pi^q(1)].$$

But $z'(1)$ and $v'(1)$ are respectively partitions of $z(1)$ and $v(1)$, so that

$$(8.1d) \qquad z'(1) * z(1), \qquad v'(1) * v(1).$$

From the homologies (8.1a) to (8.1d) we see that

$$(8.2) \qquad z(1) * v(1) \qquad [\text{on } \Omega(1)].$$

But $z(1)$ is the given cycle z, and $v(1)$ is a linear combination of the k-cycles of the set $\lambda_{ij}^r(1)$.

Statement (α) follows then from (8.2).

We continue with a proof of statement (β).

(β). *The k-cycles of the theorem are Ω-independent.*

Suppose (β) is false, and that u is a proper linear combination of k-cycles of the set $\lambda_{ij}^r(1)$, Ω-homologous to zero on $\Omega(1)$. There will then exist a partition of u, say w, on a domain $\Pi^q(1)$, together with a $(k + 1)$-chain z on $\Pi^q(1)$, such that

$$z \to w \qquad [\text{on } \Pi^q(1)].$$

On z suppose that $J(\pi)$ is less than $2\pi s$. If q is taken sufficiently large, the elementary extremals determined by z and w will be at most $\rho/2$ in length.

Let $w(a)$ and $z(a)$ be central projections of w and z respectively on $\Pi^q(a)$. For constants a_i sufficiently near unity, we will have

$$z(a) \to w(a) \qquad [\text{on } \Pi^q(a)].$$

But if these constants a_i are sufficiently near unity, the k-cycles of the set $\lambda_{ij}^r(a)$ for which $r \leqq s$ are Ω-independent. It is impossible therefore that $z(a) \to w(a)$. We conclude that u is not Ω-homologous to zero on $\Omega(1)$, and that (β) is true.

The proof of the theorem is complete.

The kth circular connectivity of the m-sphere is then the number of k-cycles in the set $\lambda_{ij}^r(1)$. We thus have the following corollary of the theorem.

COROLLARY. *The kth circular connectivity P_k of the m-sphere is the number of distinct integral solutions i, j, r of the diophantine equation*

$$k = m + i + j - 4 + 2(r - 1)(m - 1) \tag{8.3}$$

in which $m + 1 \geqq i > j > 0, r > 0$, and k and m are fixed.

The sequence of circular connectivities

$$P_0 P_1 P_2 \cdots \tag{8.4}$$

can readily be determined from (8.3) for a given m. We give the determination for the cases $m = 2, 3, 4$, and 5:

$(m = 2)$ $\quad 01\underline{12}12 \cdots;$

$(m = 3)$ $\quad 001\underline{1212}1212 \cdots;$

$(m = 4)$ $\quad 0001\underline{122212}122212 \cdots;$

$(m = 5)$ $\quad 00001\underline{122322212}122322212 \cdots.$

The numbers underlined represent a group which thereafter repeats periodically.

The first general existence theorem is the following.

THEOREM 8.2. *Corresponding to any admissible functional J, defined on any Riemannian manifold R which is the topological image of the m-sphere, there exist critical sets of closed extremals on R whose kth type number sum is at least the kth circular connectivity P_k of the m-sphere.*

There also exists a number L_k which depends only on R, J, and k, and which is such that if all closed extremals on R with J-lengths less than L_k are non-degenerate, there will be at least P_k non-degenerate, closed extremals on R of index k, with J-lengths at most L_k.

By virtue of the topological invariance of the circular connectivities there will exist a set $(\lambda)_k$ of P_k cycles on the space Ω determined by R and the functional J. Of the minimum critical values, "determined" in the sense of §6, Ch. VIII, by linear combinations of cycles $(\lambda)_k$, let L_k be the maximum. It follows from Theorem 6.5 in Ch. VIII that the minimal set K of closed extremals determined by the set $(\lambda)_k$ will lie on the domain $J \leqq L_k$, and will consist of critical sets of extremals whose kth type number sum M_k is at least P_k. In case all closed extremals with J-lengths at most L_k are non-degenerate, the number M_k is the number of closed extremals of index k in the set K.

The theorem thus holds as stated.

Topologically related closed extremals

9. Let R' and R'' be Riemannian manifolds of the nature of the preceding manifold R. Suppose that R' and R'' admit a homeomorphism T. Let Ω' and Ω'' be the respective spaces Ω defined by functionals J' and J'' on R' and R''. Identifying R' and R'' and their metrics with the spaces S' and S'' respectively of §12, Ch. VIII, we introduce the conception of points (π) which are admissible rel r_0 as defined in (12.14) of Ch. VIII. Points (π') and (π'') on Ω' and Ω'' respectively which are admissible rel r_0 will be said to *correspond* under T if their vertices taken in some one of their circular orders, direct or inverse, correspond under T.

To avoid ambiguity a partition on Ω' of a cycle z on Ω' will be called an *Ω'-partition*. An Ω''-partition of a cycle on Ω'' is similarly defined. Cycles z' and z'' on Ω' and Ω'', respectively, will be said to *correspond after partition* if suitable Ω'- and Ω''-partitions of z' and z'', respectively, are admissible rel r_0 and correspond under T.

If z is a cycle on Ω', two cycles on Ω'' which correspond to z after two Ω'-partitions of z are mutually Ω''-homologous. It will be sufficient to prove this statement for the case where z is a cycle on Π'^p. Let u and v be r- and s-fold Ω'-partitions of z which admit correspondents on Ω''. Let the correspondents of u and v on Ω'' also be denoted by u and v. We wish to prove that $u * v$ on Ω''. To that end let w be the correspondent on Ω'' of the rs-fold Ω'-partition of z. Observe that w is also the correspondent on Ω'' of the s-fold Ω'-partition of u. By virtue of Lemma 12.2 of Ch. VIII, w is homologous to the s-fold Ω''-partition $\bar{u}$ of u, so that we have

$$w \sim \bar{u}, \qquad \bar{u} * u \qquad \text{(on } \Omega''\text{)}.$$

Hence

$$w * u \qquad \text{(on } \Omega''\text{)}.$$

Similarly

$$w * v \qquad \text{(on } \Omega''\text{)}.$$

Hence

$$u * v \qquad \text{(on } \Omega''\text{)}$$

as stated.

As seen in the proof of Theorem 12.1 of Ch. VIII, a set of cycles on Ω' which satisfy no Ω-homology on Ω' will correspond after partition to a set of cycles on Ω'' which satisfy no Ω-homology on Ω''.

Let (u') and (u'') be finite sets of Ω-independent cycles on Ω' and Ω'' respectively, with members which correspond respectively after partition. The sets (u') and (u'') will determine minimal sets K' and K'' of closed extremals on R' and R'' respectively, in the sense of §6, Ch. VIII. We shall then say that K' and K'' are *topologically related* under the homeomorphism T.

To illustrate this conception we return to the m-ellipsoid $E_m(a)$ and the space $\Omega(a)$ determined by the integral of arc length on $E_m(a)$. We state the following theorem.

THEOREM 9.1. *If the constants $a_1 > \cdots > a_{m+1}$ are sufficiently near unity, the minimal set of closed geodesics on $E_m(a)$ determined by the linking cycle $\lambda^1_{h\,k}(a)$ is the ellipse $g^1_{h\,k}(a)$.*

We choose the constants $a_1 > \cdots > a_{m+1}$ so near unity that the principal ellipses $g^1_{i\,j}(a)$ are non-degenerate, possess the cycles $\lambda^1_{i\,j}(a)$ as Ω-independent linking cycles respectively, and are the only geodesics on $E_m(a)$ with lengths less than 3π. Let λ be one of the cycles $\lambda^1_{i\,j}(a)$. Let K be the minimal set of closed geodesics determined by λ. Suppose K includes a geodesic $g^1_{h\,k}(a)$. In such a case λ must be the cycle $\lambda^1_{h\,k}(a)$ as we shall now prove.

Recall that $g^1_{h\,k}(a)$ determines the point $\pi^1_{h\,k}(a)$ on $\Omega(a)$. By virtue of the definition of a minimal set of closed geodesics belonging to λ there will exist, among the "reduced new cycles" which are Ω-homologous to λ, at least one, say μ, for which the corresponding reduced critical set σ will include the point $\pi^1_{h\,k}(a)$. The cycle μ will be Ω-homologous among points (π) neighboring σ and below $J(\pi^1_{h\,k}) + e^2$ to a linear combination L of the cycles $\lambda^1_{i\,j}(a)$. By virtue of the definition of a reduced new cycle L must include the cycle $\lambda^1_{h\,k}(a)$. We have $\lambda * L$. Since the cycles $\lambda^1_{i\,j}(a)$ are Ω-independent this is possible only if

$$\lambda = \lambda^1_{h\,k}(a).$$

Hence K consists of the single ellipse $g^1_{h\,k}(a)$.

The proof of the theorem is now complete.

Let R be a Riemannian manifold homeomorphic with the ellipsoid $E_m(a)$ of the theorem. There exists a well defined minimal set of closed extremals on R topologically related to each principal ellipse $g^1_{h\,k}(a)$ on $E_m(a)$. For $\lambda^1_{h\,k}(a)$ will correspond after partition to a well defined cycle u on the space Ω determined by the integral J on R. The minimal set of closed extremals on R determined by u will be topologically related to $g^1_{h\,k}(a)$ in accordance with our definitions.

We consider the complete set of cycles $\lambda^1_{i\,j}(a)$. It is clear that this set of cycles determines the principal ellipses on $E_m(a)$ as a minimal set of closed geodesics. A set of cycles on Ω which correspond to the cycles $\lambda^1_{i\,j}(a)$ after partition will determine a minimal set G of closed extremals on R. If we combine this result with Theorem 6.5 of Ch. VIII, we obtain the following.

THEOREM 9.2. *Let R be a Riemannian manifold homeomorphic with an m-ellipsoid $E_m(a)$ for which $a_1 > \cdots > a_{m+1}$. If the constants (a) are sufficiently near unity, there exists a set G of closed extremals which is topologically related on R to the principal ellipses on $E_m(a)$, and which has a kth type number sum at least as great as the number of principal ellipses on $E_m(a)$ of index k.*

We also note the following. On the space $\Omega(0)$ determined by the m-sphere $E_m(0)$ the cycles $\lambda^1_{i\,j}(0)$ form a set which is Ω-independent. The corresponding minimal set of closed geodesics on $E_m(0)$ is the set of great circles on $E_m(0)$. For these great circles form a connected set of closed geodesics for which the corresponding critical set on $\Omega(0)$ includes all points (π) on the cycles $\lambda^1_{i\,j}(0)$, while the cycles $\lambda^1_{i\,j}(0)$ are not Ω-homologous to cycles below 2π, since there are no critical values below 2π.

Metric relations between topologically related closed geodesics. Let T represent a homeomorphism between two Riemannian manifolds R' and R''. Let the functionals J' and J'' be the integrals of arc length on R' and R'' respectively. If the homeomorphism T can be locally effected by a non-singular analytic transformation of coordinates, minimal sets of closed geodesics which are topologically related on R' and R'' respectively stand in noteworthy metric relations.

Let P' and P'' be points which correspond on R' and R'' respectively. Let (x) be local coordinates on R' neighboring P'. By virtue of the transformation T we can take the coordinates (x) as local coordinates on R'' neighboring P''. Let ds' and ds'' be the differentials of arc on R' and R'' respectively, expressed in terms of the coordinates (x) and their differentials (dx). The ratio

$$\frac{ds''}{ds'} = M(x, dx) \qquad [(dx) \neq (0)]$$

will be a positive continuous function of the variables (x) and (dx) for points (x) neighboring P' and sets $(dx) \neq (0)$. It will be homogeneous of order zero in the variables (dx), and thus depend only upon (x) and the direction Γ defined at the point (x) by the differentials (dx). We can regard the preceding ratio as locally defining a function

$$\frac{ds''}{ds'} = \mu(P, \Gamma)$$

of the point P on R' and an arbitrary direction Γ on R' at P.

Let μ_1 and μ_2 be respectively the absolute minimum and maximum of $\mu(P, \Gamma)$, for points P on R' and directions Γ on R' at P. We have

$$\mu_1 \leqq \mu(P, \Gamma) \leqq \mu_2.$$

If L' is the length of any regular curve on R', and L'' the length of the corresponding curve on R'', we see that

$$\mu_1 L' \leqq L'' \leqq \mu_2 L'.$$

We shall prove the following lemma.

LEMMA 9.1. *Let Ω' and Ω'' be the spaces Ω respectively determined by the integrals of arc length on R' and R''. On Ω' and Ω'' let z' and z'' be corresponding k-cycles not Ω-homologous to zero. If c' and c'' are respectively the minimum critical values "determined" by z' and z'', we have the relations*

$$\mu_1 c' \leqq c'' \leqq \mu_2 c'.$$

By hypothesis the cycles z' is Ω-homologous on Ω' to a cycle w' below $c' + e$, where e is an arbitrarily small positive constant. If we take sufficiently high Ω'-partitions of the chains involved in this homology, the resulting chains possess images on Ω'' under T, and we see that z'' is Ω-homologous on Ω'' to the image w'' on Ω'' of a partition of w'. But w'' will lie below $\mu_2 c' + \mu_2 e$, from which it follows that

$$c'' \leqq \mu_2 c'.$$

Upon interchanging the rôles of R' and R'', z' and z'', and c' and c'', replacing μ_2 by $1/\mu_1$, we see that

$$c' \leqq \frac{1}{\mu_1} c''.$$

The lemma follows from the preceding inequalities.

We state the following theorem.

THEOREM 9.3. *Let R be a Riemannian manifold which is the non-singular, analytic homeomorph of the unit m-sphere $E_m(0)$, and which is such that the ratio of the differential of arc length on R to the corresponding differential on $E_m(0)$ has an absolute maximum μ_2 and absolute minimum μ_1. There exists a set G of closed geodesics which is "topologically related" on R to the great circles on $E_m(0)$ and which has the following properties.*

(1). *The geodesics of the set G have lengths between $2\pi\mu_1$ and $2\pi\mu_2$ inclusive.*

(2). *The kth type number sum of the geodesics of G is at least the number of principal ellipses of index k on any ellipsoid $E_m(a)$ for which the constants $a_1 > \cdots > a_{m+1}$ are sufficiently near unity.*

(3). *If the closed geodesics on R with lengths between $2\pi\mu_1$ and $2\pi\mu_2$ inclusive are non-degenerate, there exists a subset of non-degenerate geodesics of G which correspond in a one-to-one manner to the principal ellipses on $E_m(a)$ in such a fashion that corresponding geodesics have the same index.*

To establish (1) we identify $E_m(0)$ with R' and R with R''. The set of cycles $\lambda_{i\,j}^1(0)$ on Ω' will have the great circles on $E_m(0)$ as a minimal set of closed geodesics. The set of cycles on Ω'' corresponding to the cycles $\lambda_{i\,j}^1(0)$ after partition will determine a minimal set G of closed geodesics on R. By virtue of the preceding lemma the geodesics of G will have lengths between $2\pi\mu_1$ and $2\pi\mu_2$ inclusive.

To establish (2) recall that the number of cycles $\lambda_{i\,j}^1(0)$ of dimension k equals the number of principal ellipses of index k on $E_m(a)$, provided the constants

$a_1 > \cdots > a_{m+1}$ are sufficiently near unity. Statement (2) follows from Theorem 6.5 of Ch. VIII.

To verify statement (3) recall that the kth type number sum M_k of a set of non-degenerate geodesics is the number of these geodesics with index k. Under the hypotheses of (3) the geodesics of the set G are non-degenerate. It follows from statement (2) that the number M_k of geodesics with index k in G is at least as great as the number of principal ellipses on $E_m(a)$ with index k. The one-to-one correspondence affirmed to exist in (3) can accordingly be set up as stated.

We also note the following. When $\mu_2 < 2\mu_1$, none of the geodesics whose existence is affirmed in the theorem can cover any other such geodesics an integral number of times.

Continuation theorems

10. We shall conclude with two theorems on the analytic continuation of closed geodesics. With Poincaré the theory of the continuation of closed geodesics was used to establish the existence of the basic geodesics. For us the existence of the basic geodesics has been established by other means. The theory of their continuation serves to describe their variation and the variation of their type numbers with variation of the manifold.

We start with an analytic Riemannian m-manifold R, given in the large as previously. We suppose that R is the initial member R_0 of a 1-parameter family R_α of homeomorphic Riemannian manifolds depending on a parameter α which varies on the interval

$$0 \leqq \alpha \leqq 1. \tag{10.1}$$

Let P_0 be any point on R. Let (x) be any admissible coordinate system representing R neighboring P_0, with $(x) = (a)$ corresponding to P_0. We represent the point on R_α which corresponds to the point (x) on R by these same coordinates (x), and suppose that the differential of arc on R_α takes the form

$$ds^2 = g_{ij}(x, \alpha)dx^i dx^j,$$

where the coefficients $g_{ij}(x, \alpha)$ are analytic in the variables (x) and α for (x) near (a) and α any number on the interval (10.1).

Let g' and g'' be two closed curves on R. Let k represent a homeomorphism between g' and g''. Let D_k be the minimum of the geodesic distances between points of g' and g'' which correspond under k. Let $d(g', g'')$ be the greatest lower bound of the numbers D_k for all homeomorphisms k between g' and g''. The number $d(g', g'')$ will be called the distance between g' and g'' on R. Cf. Fréchet [1].

Suppose now that g' and g'' lie on $R_{\alpha'}$ and $R_{\alpha''}$ respectively. Let γ' be the homeomorph of g'' on $R_{\alpha'}$ and γ'' the homeomorph of g' on $R_{\alpha''}$. Of the two numbers

$$d(g', \gamma'), \qquad d(g'', \gamma'')$$

on $R_{\alpha'}$ and $R_{\alpha''}$, let δ be the minimum. We define the *distance* between g' and g'' as the number

$$d(g', g'') = [(\alpha' - \alpha'')^2 + \delta^2]^{1/2}.$$

Let H_α be a set of closed curves on R_α defined for all values of $\alpha \neq \alpha_0$ sufficiently near α_0. Let H be a set of closed curves on R_{α_0}. The set H_α will be said to *tend* to H as a limiting set as α tends to α_0 if for $| \alpha - \alpha_0 |$ sufficiently small each closed curve of H_α (or H) is within an arbitrarily small distance of some closed curve of H (or H_α) respectively.

Let c be the length of a closed geodesic on R_{α_0}. Let J_0 be the integral of arc length on R_{α_0}. Let a and b be ordinary values of J_0 which separate c from other critical values of J_0; $a < c < b$. Let H_α be the set of all closed geodesics on R_α with lengths between a and b. We state the following:

(A). *As α tends to α_0, the set H_α of closed geodesics on R_α tends to a subset of H_{α_0} on R_{α_0} as a limiting set.*

We observe that the set H_α may be vacuous when $\alpha \neq \alpha_0$. The subset of H_{α_0} is then the null set. The proof of statement (A) is contained in the analysis of critical sets of closed extremals in §3 of Ch. VIII. For our present purposes the parameter α must be added to the variables employed in Ch. VIII.

Statement (A) contains no affirmation concerning the continued existence of critical sets of closed geodesics on R_α as α is varied. The following theorem makes such an affirmation and describes the variation of critical sets of geodesics with reference to their type numbers. In this theorem a finite ensemble of critical sets of closed extremals will be termed a *composite* set of closed extremals.

First Continuation Theorem. *Let K be a critical set of closed geodesics on R_{α_0}. For α sufficiently near α_0 and not α_0, there exists a composite set K_α of closed geodesics on R_α which tends to a subset of K as α tends to α_0, and which possesses a kth type number sum at least as great as that of K ($k = 0, 1, \cdots$).*

The set K_α is null at most when the type numbers of K are null.

To prove this theorem we regard the integral

$$J_\alpha = \int_{t^1}^{t^2} \left(g_{ij}(x, \alpha) \frac{dx^i}{dt} \frac{dx^j}{dt} \right)^{1/2} dt$$

as a functional on R. The length of a curve g on R_α is given by the value of J_α along the homeomorph of g on R. The geodesics on R_α will be represented by the extremals of J_α on R. The preceding theorem is equivalent to the following lemma concerning the functional J_α on R.

Lemma. *Let G be a critical set of closed extremals belonging to the functional J_{α_0} on R. For $\alpha \neq \alpha_0$ and sufficiently near α_0 there exists a composite set G_α of closed extremals belonging to J_α which tends to a subset of G as α tends to α_0 and which possesses a kth type number sum at least as great as that of G.*

The lemma will be made to depend upon the corresponding statement in IV of

§2, Ch. VI concerning a function Φ of a parameter μ and of a point P on a Riemanniann manifold R.

Let ρ be a positive constant uniformly limiting the lengths of admissible elementary extremals determined by J_α on R for values of α near α_0. Let c be the value of J_{α_0} on the extremals of G, and p be an integer so large that $p\rho > c$.

Let R^p be the Riemannian manifold of points (π) with p vertices on R. As previously, we assign R^p an element of arc ds whose square is the sum of the squares of the elements of arc of the respective vertices of a point (π) on R^p.

Let σ be the set of J-normal points (π) on R^p which belong to J_{α_0} and are determined by the extremals of G. Neighboring σ let Σ_α be the set of all J-normal points (π) on R^p which belong to J_α. The analysis of J-normal points (π) in §7, Ch. VII, shows that for α sufficiently near α_0, and among points sufficiently near σ, Σ_α forms a regular, analytic, Riemannian submanifold of R^p. More precisely, let (π_0) be any point of σ, and (x) a set of pm coordinates locally representing a neighborhood of (π_0) on R^p. There then exists a set of parameters (u) such that the points on Σ_α neighboring (π_0) can be represented in the form

$$x^i = \varphi^i(u, \alpha) \qquad (i = 1, \cdots, mp). \tag{10.2}$$

The values of α in (10.2) are restricted to values near α_0 and the sets (u) to sets near the set (u_0) which determines (π_0) on Σ_{α_0}. Let r be the number of variables (u). The functions $\varphi^i(u, \alpha)$ are analytic in their arguments and possess a matrix of first partial derivatives with respect to the variables (u) which has a rank r when $(u) = (u_0)$ and $\alpha = \alpha_0$.

The projection of Σ_α *on* Σ_{α_0}. We here make a digression in which we show that Σ_α can be projected onto Σ_{α_0} neighboring σ by means of geodesics on R^p orthogonal to Σ_{α_0}. For α sufficiently near α_0 this will lead to an analytic homeomorphism between Σ_α and Σ_{α_0} at least if Σ_α be restricted to points which project into points on Σ_{α_0} sufficiently near σ. We shall obtain a representation of this homeomorphism.

By using the power series representation of geodesics common in Riemannian geometry, one can set up a non-singular analytic transformation of local coordinates (x) on R^p, into coordinates (y) of such a nature that Σ_{α_0} is represented in the space (y) by the coordinate r-plane ζ of the first r axes y^i, and such that the geodesics orthogonal to ζ are represented near (π_0) by the set of straight lines orthogonal to ζ in the space (y).

In terms of the coordinates (y) of R^p and the parameters (u) in (10.2) Σ_α will have a regular, analytic representation near (π_0) of the form

$$y^i = \psi^i(u, \alpha) \qquad (i = 1, \cdots, mp). \tag{10.3}$$

Since this representation reduces to the coordinate r-plane ζ when $\alpha = \alpha_0$, we have

$$0 \equiv \psi^i(u, \alpha_0) \qquad (i = r + 1, \cdots, mp).$$

The point (y) on Σ_α which is given by (10.3) will project orthogonally into the point

$$(y) = [\psi^1(u, \alpha), \cdots, \psi^r(u, \alpha), 0, \cdots, 0]$$

on ζ. We regard $(y^1, \cdots, y^r)$ as a set of parameters representing the point

$$[y^1, \cdots, y^r, 0, \cdots, 0]$$

on Σ_{α_0}. The relations between the parameters (u) which represent a point Q on Σ_α and the parameters (y) which represent the projection of Q on Σ_{α_0} will take the form

$$y^i = \psi^i(u, \alpha) \qquad (i = 1, \cdots, r).$$

These relations have the property that for $\alpha = \alpha_0$,

$$\frac{D(\psi^1, \cdots, \psi^r)}{D(u^1, \cdots, u^r)} \neq 0,$$

as follows from the regularity of the representation of Σ_{α_0}.

Thus the relation between Σ_α and its orthogonal projection on Σ_{α_0} is non-singular, analytic, and one-to-one provided α be sufficiently small and Σ_α be restricted to points whose projections lie sufficiently near σ.

We now return to a proof of the theorem.

Let P be any point on Σ_{α_0} near σ and (π) the point on Σ_α which projects orthogonally into P. Let

$$\Phi(P, \alpha)$$

be the value of J_α along the broken extremal $g(\pi)$ which belongs to J_α. In terms of coordinates (x) locally representing the point P on Σ_{α_0} neighboring an arbitrary point P_0 of Σ_{α_0}, the function Φ will become a function $F(x, \alpha)$ which is analytic in its arguments (x) and α of the nature of the function $F(x, \mu)$ of IV, §2, Ch. VI.

When $\alpha = \alpha_0$, the function Φ possesses the critical set σ. For values of $\alpha \neq \alpha_0$ sufficiently near α_0 the results of Ch. VI show that Φ will possess a set σ_α of critical points neighboring σ with a kth type number sum at least as great as that of σ. This statement is the basis of the proof of the theorem.

We now return to Σ_α and the functional J_α on R. Let σ_α' be the set of points on Σ_α which projects on R^p into σ_α on Σ_{α_0}. The set σ_α' is composed of critical sets of J-normal points (π) on Σ_α belonging to the functional J_α on Σ_α. The kth type number sum of σ_α' will equal that of σ_α, and hence be at least as great as that of σ. Let G_α be the set of closed extremals on R, belonging to J_α and determined by points (π) on σ_α'. By virtue of the relations between critical sets of J-normal points and the corresponding sets of closed extremals, the kth type number sum of G_α will equal that of σ_α', and hence be at least as great as that of σ, or at least as great as that of G.

The set G_α is the set of closed extremals whose existence is affirmed in the lemma.

The proof of the First Continuation Theorem is now complete.

We return to the homeomorphic manifolds R_α, $0 \leqq \alpha \leqq 1$. Let Ω_α be the space Ω defined by the integral of arc length on R_α. Let λ be a k-cycle on Ω_0 which is not Ω-homologous to zero on Ω_0. As we have seen there will exist a cycle λ_α on Ω_α which "corresponds to λ after partition." The cycle λ_α will not be Ω-homologous to zero on Ω_α.

We now state a second theorem:

SECOND CONTINUATION THEOREM. *For each value of α on the interval $0 \leqq \alpha \leqq 1$ there exists a minimal set M_α of closed geodesics on R_α belonging to the k-cycle λ_α. As α tends to a particular value α_0, M_α tends to M_{α_0}. The kth type number of M_α is at least one.*

A proper discussion of this theorem and its implications is beyond the scope of these Lectures. A suitable treatment will be given elsewhere.

As an example we suppose that R_α is a family of analytic m-manifolds which for $\alpha = 0$ reduces to the m-ellipsoid $E_m(a)$ of Theorem 9.3. We take λ as one of the cycles $\lambda^1_{i\,j}(a)$. The corresponding ellipse $g^1_{i\,j}(a)$ on $E_m(a)$ will thus "continue" into the set M_α in the sense of the preceding theorem. The set M_α never fails to exist and is uniquely defined.

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