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ELEMENTS OF ELECTROMAGNETIC WAVES

*ELEMENTS OF
ELECTROMAGNETIC
WAVES*

BY LAWRENCE A. WARE

*Professor of Electrical Engineering,
State University of Iowa*

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PREFACE

This text has been prepared to meet the need for an elementary introduction to the basic ideas of electromagnetic theory, employing the rationalized MKS units. It is intended as first-course material for either juniors or seniors, and the only prerequisites are calculus and the fundamentals of alternating-current theory. In the treatment of this subject the development has been built up gradually, beginning with the necessary vector algebra. Many illustrative examples and problems have been included to aid in the presentation of the material and to provide sufficient practice in its application. The chapters on wave guides have been included to serve as an application of the theory and are not intended to be, in any sense, complete. One chapter on the basic ideas of antennas is included as an introduction to that field of application. The text is prepared to serve as an introduction to the more advanced treatments of the subject, of which there are many.

In preliminary form the text presented here has been successfully used through two semesters in the Electrical Engineering Department of the State University of Iowa. It is believed that the changes and additions resulting from this trial period have made the book into a practical instrument for teaching electromagnetic theory to electrical engineering students.

The author wishes to express appreciation to Professors G. F. Corcoran and H. R. Reed of the University of Maryland and E. M. Lonsdale of the State University of Iowa for many valuable suggestions and aid given in the course of the preparation of the manuscript for this book.

L. A. WARE

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ELEMENTS OF ELECTROMAGNETIC WAVES

INTRODUCTION TO VECTORS

Summary

In this chapter the basic ideas of vector algebra are introduced. After a short discussion of the nature of vectors and their representation, the subjects of addition and multiplication are treated. Because scalar and vector products are used extensively in electromagnetic theory, examples are given which apply to electricity. The triple scalar and vector products are introduced, and the process of differentiation is defined. This chapter lays a foundation for the treatment in Chapter II of the operator *del* and its applications.

1-1. SCALARS AND VECTORS

Electromagnetic theory, which forms the basis of so much of modern communication engineering, depends upon the methods of vector analysis for its expression in its more useable forms. This chapter is concerned primarily with the development of the basic operations of vector analysis and will, wherever possible, provide illustrations of its application to electricity.

The student is already familiar with the more basic concepts involved. For instance, the distinction between a vector and a scalar is met in every-day life. A simple illustration is provided by the work done on a weight to lift it to a certain position, and the force of gravity acting upon it in that position. Thus, in Fig. 1-1, the work done in lifting the weight W to the position shown is a scalar quantity since it is merely a statement of work accomplished without any reference to any directions or path over which the weight may have been moved. On the other hand, to say that the force of gravity acting on the body is 10 lb is an incomplete state-

ment unless the *direction* of this force is specified. Thus the two statements would have to be somewhat as follows:

Work: $W = 25 \text{ ft-lb}$

Force: $\mathbf{F} = 10 \text{ lb}$ *in a downward direction*

The first is a scalar, and light-face type will be used to represent scalar quantities. The second is a vector, and bold-face type will be used for vectors.

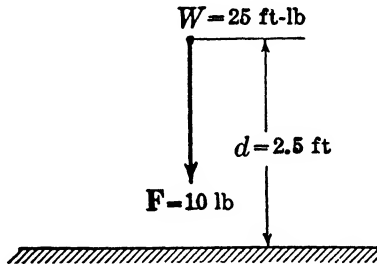


FIG. 1-1. A vector \mathbf{F} and the associated scalar W .

For purposes of further illustration a number of common scalar and vector quantities are listed below:

<i>Scalars</i>	<i>Vectors</i>
Energy	Velocity
Mass	Field intensity
Charge	Force
Volume	Current density
Time	Area
Gravitational and electrical potential	Distance or displacement

1-2. REPRESENTATION OF VECTORS

The previous section presents a fundamental problem. How will it be possible to specify directions for vectors? In Fig. 1-2 is shown a directed quantity or vector, \mathbf{V} , imbedded in a 3-dimensional coordinate system with axes, x , y , and z . The vector \mathbf{V} , lying in the position shown, extends from o to c . The position c , the end of the vector, could just as well have been specified by following the path; $oabc$, which follows along the x , y , and z axes in order. Now oa is a vector in the x direction, ab is a vector in the y direction, and bc is a vector in the z direction. Since both

paths, oc and $oabc$, arrive at the same destination, they are considered as being equivalent, and a vector equation can be written.

$$oc = oa + ab + bc. \quad (1-1)$$

The vector V can now be specified as follows. Give the components oa , ab , and bc with the understanding that they lie along the x , y , and z axes respectively. The system given in equation (1-1), however, is awkward and it is necessary to develop some more convenient representation. By referring to Fig. 1-2, it is

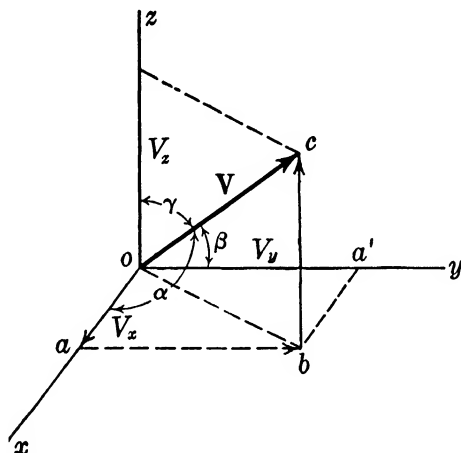


FIG. 1-2. Representation of a vector in rectangular coordinates.

seen that oa can be written as V_x which means the projection of V on the x axis, or the x component of V . This is similarly true of V_y and V_z . Now, if we assume that these components are known, the vector can be written

$$V = V_x (\text{along } x) + V_y (\text{along } y) + V_z (\text{along } z).$$

Now by introducing an abbreviation for the terms (along x), (along y), etc., the expression can be greatly simplified. The standard abbreviations are:

$$\begin{aligned} \mathbf{i} &= (\text{along } x), \\ \mathbf{j} &= (\text{along } y), \\ \mathbf{k} &= (\text{along } z), \end{aligned}$$

which are *unit vectors*, or vectors of unit length and are used merely

to indicate direction. The expression for \mathbf{V} then becomes

$$\mathbf{V} = iV_x + jV_y + kV_z. \quad (1-2)$$

This expression means that \mathbf{V} is the sum of three components whose lengths are V_x , V_y , and V_z and whose respective directions are \mathbf{i} , \mathbf{j} , and \mathbf{k} .

Equation (1-2) leads now to two important ideas. On the right side of the equation are written three terms in the form of products, such as iV_x . Here \mathbf{i} is a vector of unit length and V_x is a scalar. In order for the expression to mean what we have intended, it is necessary for this product to mean merely a vector in the x direction of length V_x . Thus the product of a scalar and a vector is a vector. As an illustration, the product of $\mathbf{V} = \mathbf{i}2 + \mathbf{j}6 - \mathbf{k}5$ by the scalar, $A = 3$, is

$$\begin{aligned} A\mathbf{V} &= 3(\mathbf{i}2 + \mathbf{j}6 - \mathbf{k}5) \\ &= \mathbf{i}6 + \mathbf{j}18 - \mathbf{k}15, \end{aligned}$$

which is a vector in the direction of \mathbf{V} but three times as long. Further, the length of \mathbf{V} , written V , is given by geometry in the following form:

$$\begin{aligned} V &= \sqrt{ob^2 + bc^2} \\ &= \sqrt{oa^2 + ab^2 + bc^2} \\ &= \sqrt{V_x^2 + V_y^2 + V_z^2}. \end{aligned} \quad (1-3)$$

An alternative method of representing a vector is to define *direction cosines* as follows. Let the angle between the vector and the x axis be α , that between \mathbf{V} and the y axis be β , and that between \mathbf{V} and the z axis be γ . Then if the length of \mathbf{V} , V , be stated and the angles, or their cosines, are given, the vector is completely specified. The cosines of the angles are direction cosines and are usually written as follows:

$$\begin{aligned} \cos \alpha &= l, \\ \cos \beta &= m, \\ \cos \gamma &= n. \end{aligned}$$

An identity involving them can be immediately derived from equation (1-3). Let it be noted from Fig. 1-2 that

$$\begin{aligned} V_x &= V \cos \alpha = lV, \\ V_y &= V \cos \beta = mV, \\ V_z &= V \cos \gamma = nV. \end{aligned}$$

Then substituting into equation (1-3) leads to

$$V = \sqrt{l^2V^2 + m^2V^2 + n^2V^2}$$

$$= V \sqrt{l^2 + m^2 + n^2},$$

or

$$\sqrt{l^2 + m^2 + n^2} = 1. \tag{1-4}$$

1-3. ADDITION AND SUBTRACTION OF VECTORS

The concept of addition of vectors has already been made use of in connection with Fig. 1-2. Here it was stated that the vector V

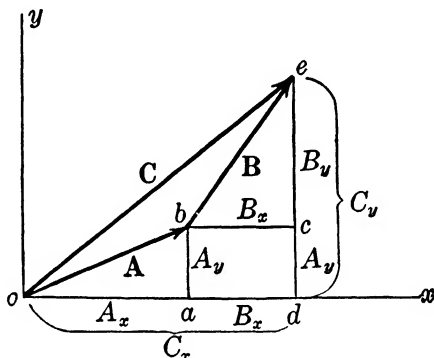


FIG. 1-3. Vector addition.

was the sum of three components, each of which was a vector. Thus there is set up a definition of vector addition which may be stated somewhat as follows: *The sum of a number of vectors is the diagonal resulting when the vectors concerned are placed end to end in their correct directions and the beginning point and end point are connected by a straight line.* In other words, in Fig. 1-2 the vector V is the sum of iV_x , jV_y , and kV_z . Now it is to be shown that if the component vectors themselves are arbitrary vectors a simple rule can be applied to find the sum. Thus if

$$\mathbf{A} = iA_x + jA_y + kA_z,$$

and

$$\mathbf{B} = iB_x + jB_y + kB_z,$$

then

$$\mathbf{C} = \mathbf{A} + \mathbf{B}$$

$$= i(A_x + B_x) + j(A_y + B_y) + k(A_z + B_z).$$

Refer to Fig. 1-3 where two vectors A and B are shown with their components in two dimensions only, in order to simplify the present-

tation. The vectors are drawn end to end, and their components are marked on their respective coordinate systems. It is easily seen that the vector **C** is the sum of vectors **A** and **B**. Inspection of the diagram shows that

$$\begin{aligned} C_x &= oa + ad \\ &= A_x + bc \\ &= A_x + B_x, \\ C_y &= dc + ce \\ &= A_y + B_y. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbf{C} &= iC_x + jC_y \\ &= i(A_x + B_x) + j(A_y + B_y), \end{aligned}$$

or more generally,

$$\begin{aligned} \mathbf{C} &= iC_x + jC_y + kC_z \\ &= i(A_x + B_x) + j(A_y + B_y) + k(A_z + B_z). \end{aligned} \quad (1-5)$$

Now if a vector, or any component of a vector, is negative it merely appears in the equations with a negative sign. Thus the law for addition of vectors also covers the process of subtraction by using the ordinary laws of algebra. Thus if $\mathbf{C} = \mathbf{A} - \mathbf{B}$, then

$$\mathbf{C} = i(A_x - B_x) + j(A_y - B_y) + k(A_z - B_z). \quad (1-6)$$

1-4. ILLUSTRATIVE EXAMPLES: VECTOR ADDITION AND SUBTRACTION

Example 1. Let it be required to find the components and sum of the two following vectors in terms of direction cosines.

Vector **A**: 10 in. long, $l = 0.3$, $m = 0.5$, $n = 0.811$;

Vector **B**: 5 in. long, $l = 0.7$, $m = 0.3$, $n = 0.648$.

The components will be as follows:

$$\begin{aligned} A_x &= 0.3(10) = 3 \text{ in.}, & A_y &= 0.5(10) = 5 \text{ in.}, \\ B_x &= 3.5 \text{ in.}, & B_y &= 1.5 \text{ in.}, \\ A_z &= 0.811(10) = 8.11 \text{ in.}; \\ B_z &= 3.24 \text{ in.} \end{aligned}$$

Thus the two vectors are written as follows:

$$\begin{aligned} \mathbf{A} &= i3 + j5 + k8.11, \\ \mathbf{B} &= i3.5 + j1.5 + k3.24, \end{aligned}$$

and the sum will be

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = i6.5 + j6.5 + k11.35.$$

The length of this vector is given by

$$C = \sqrt{42.3 + 42.3 + 129} = 14.6 \text{ in.}$$

Example 2. Let it be required to find the difference $\mathbf{C}' = \mathbf{A} - \mathbf{B}$. The equation is written

$$\begin{aligned} \mathbf{C}' = \mathbf{A} - \mathbf{B} &= i(3 - 3.5) + j(5 - 1.5) + k(8.11 - 3.24) \\ &= -i0.5 + j3.5 + k4.87, \end{aligned}$$

which has a length of 6.02 in.

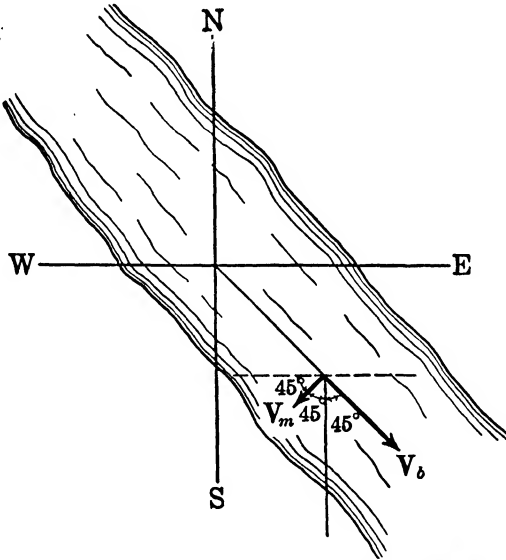


FIG. 1-4. Vector addition of \mathbf{V}_b and \mathbf{V}_m .

Example 3. A river flows SE at 10 mph, and a boat floats upon it with its bow pointed in the direction of travel. A man walks upon the deck in a direction to the right and perpendicular to the direction of the boat's movement. (See Fig. 1-4.) His walking speed is 3 mph. Find his velocity with respect to the earth.

Take the x axis toward the east and the y axis to the north. Then the boat's velocity will be

$$\begin{aligned} \mathbf{V}_b &= i10 \cos 45^\circ - j10 \cos 45^\circ \\ &= i7.07 - j7.07. \end{aligned}$$

The man's velocity with respect to the boat is

$$\begin{aligned} \mathbf{V}_m &= -i3 \cos 45^\circ - j3 \cos 45^\circ \\ &= -i2.12 - j2.12. \end{aligned}$$

Then the absolute velocity is the sum

$$\begin{aligned} \mathbf{V}_{abs} &= i(7.07 - 2.12) - j(7.07 + 2.12) \\ &= i4.95 - j9.19 = 10.44 \angle -61.7^\circ \text{ mph,} \end{aligned}$$

where $\angle -61.7^\circ$ is the angle of \mathbf{V}_{abs} referred to due east as a reference.

1-5. SCALAR MULTIPLICATION

Thus far it has been seen that the operation of addition of vectors is relatively simple. A slight amount of complication enters

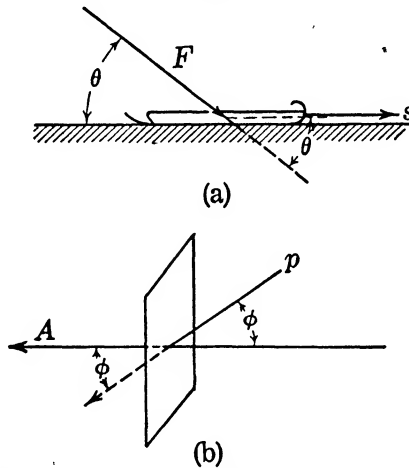


FIG. 1-5. Scalar multiplication of vectors.

the picture, however, when multiplication is introduced. There are two kinds of multiplication, *scalar* and *vector*. Scalar multiplication is so named because the result of such operation is a scalar. The definition is as follows:

$$\mathbf{A} \cdot \mathbf{B} = AB \cos \theta. \tag{1-7}$$

This multiplication is also at times called a *dot product* because of the dot used to indicate the operation. The angle θ is the angle between the vectors \mathbf{A} and \mathbf{B} . This product has many simple applications. Thus in Fig. 1-5a it is seen that if a force is applied to a sled, as shown, at an angle θ , from the horizontal, the work done

is well known to be $W = FS \cos \theta$. Using the above definition of scalar product, this is the same as $W = \mathbf{F} \cdot \mathbf{S}$. In this simple case the advantage of vector notation is merely its brevity.

In Fig. 1-5b let the vector \mathbf{A} represent the area outlined and \mathbf{p} the flow of rain in ounces per second per square inch normal to the direction of fall. Then the amount of water in ounces per second passing through the area will be $pA \cos \phi$, or simply $\mathbf{p} \cdot \mathbf{A}$.

In this connection it is well to point out that areas are represented by vectors drawn perpendicularly to the area and with a length equal to the area. This will be considered in more detail in the next section.

1-6. VECTOR PRODUCTS

Another type of multiplication which is even more useful than the scalar product is *vector multiplication*. This is defined as

$$\mathbf{A} \times \mathbf{B} = AB \sin \theta,$$

where θ is again the angle between the vectors. This product is also called the *cross product*. It is defined to be a vector, and so, to be strictly correct, the equation should be written

$$\mathbf{A} \times \mathbf{B} = nAB \sin \theta. \quad (1-8)$$

Thus both sides of the equation have a vector nature, and \mathbf{n} is a unit vector in the appropriate direction or sense. The *appropriate direction* must be defined. In Fig. 1-6 two vectors are shown, \mathbf{A} and \mathbf{B} . To simplify the argument the coordinate system is oriented so that the vectors lie in the x - y plane. The product $AB \sin \theta$ is equal to the area of the parallelogram $oacb$ because

$$\text{area} = (ob)(ad) = BA \sin \theta.$$

Thus the length of the vector representing the vector product is the area. It seems, therefore, that the area may be represented by a vector. It is now necessary to decide upon its direction. In the previous section a vector was used to represent a certain area and it was drawn perpendicular to the surface, if for no other reason than that the direction perpendicular to the surface is the only definite directional characteristic of the surface. In order to conform to certain conventions, some of which will be met later on, the direction of the vector will be taken in such a manner that if the product is written $\mathbf{A} \times \mathbf{B}$, then the direction of advance of a right-hand screw turned from the first vector to the second through

the smaller angle will be the direction of the vector. Thus in Fig. 1-6 the product $\mathbf{C} = \mathbf{A} \times \mathbf{B}$ is shown along the positive z axis. This rule for finding the direction of the product is known as the "right-hand rule" and is very useful in electricity. Some illustrations of this product will be useful. In Fig. 1-7a is shown a wire carrying a current in a magnetic field. It is already known from

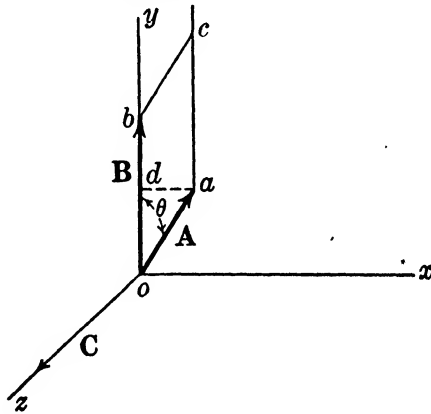


FIG. 1-6. $\mathbf{C} = \mathbf{A} \times \mathbf{B}$.

more elementary considerations that the force per unit length on the wire is given by $F = IB \sin \theta$. This can be written as

$$\mathbf{F} = \mathbf{I} \times \mathbf{B}.$$

Note that in the cross product the \mathbf{I} was written first, thus causing the direction of the force to be along the positive z axis and conforming to the known effects of currents in magnetic fields. It will be observed that the simple statement $\mathbf{F} = \mathbf{I} \times \mathbf{B}$, when properly interpreted, tells the magnitude of the force developed per unit length of wire and the direction of this force. If, for example, in Fig. 1-7a, $B = 0.20$ weber/m², $I = 20$ amp, and $\theta = 60^\circ$, then

$$\mathbf{F} = \mathbf{I} \times \mathbf{B} = \mathbf{k}(0.2)(20)(0.866) = 3.464 \text{ newtons}$$

per meter length of wire directed along the $+z$ direction.

In Fig. 1-7b a wire is moving with velocity \mathbf{v} in the magnetic field \mathbf{B} , which is everywhere directed in the $+x$ direction. The angle between \mathbf{B} and \mathbf{v} is θ , and the voltage generated per unit length of wire is $vB \sin \theta$, or in vector notation, $\mathbf{e} = \mathbf{v} \times \mathbf{B}$, provided the

wire is perpendicular to \mathbf{B} . If the straight conductor is not perpendicular to the \mathbf{B} direction, then the conductor can be resolved into two components, one lying parallel to \mathbf{B} and one perpendicular to \mathbf{B} . From elementary considerations of induced voltages, the component lying in the direction of \mathbf{B} has no voltage induced in it because this component does not cut across the \mathbf{B} vectors.

There remains now the problem of fixing definitely the effect of writing the vectors in equation (1-8) in a certain order. As written

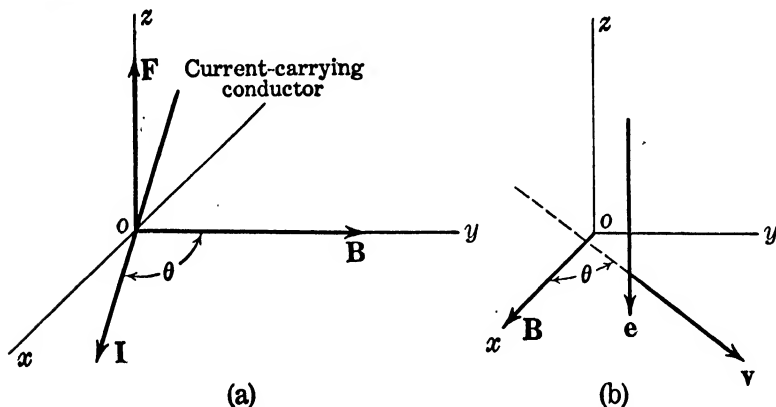


FIG. 1-7. Two cross products, $\mathbf{F} = \mathbf{I} \times \mathbf{B}$, and $\mathbf{e} = \mathbf{v} \times \mathbf{B}$.

there, it was seen that \mathbf{n} is a unit vector along the positive z axis. This vector has previously been taken as \mathbf{k} . Now if \mathbf{A} and \mathbf{B} in Fig. 1-6 are interchanged, with \mathbf{A} along the y axis and \mathbf{B} in the old \mathbf{A} position, then an application of the sign rule (the right-hand rule) indicates that the direction of the product $\mathbf{A} \times \mathbf{B}$ is along the *negative* z axis. Thus it appears that a reversal in position of the vectors will change the sign of the product. This is written as

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}. \quad (1-9)$$

In this connection it is well to point out that for the scalar product the order of the vectors makes no difference, that is,

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}. \quad (1-10)$$

Refer to equation (1-7).

The discussion of multiplication is here interrupted to permit a brief consideration of unit vectors, after which the study of multiplication will be resumed in §1-8.

1-7. UNIT VECTORS

The three standard unit vectors, \mathbf{i} , \mathbf{j} , \mathbf{k} have been mentioned before. It is now necessary to derive a few simple relations among them to facilitate future work. In Fig. 1-8 are shown these vectors in a right-hand coordinate system. A right-hand coordinate system is one where $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ as distinguished from a left-hand system where $\mathbf{i} \times \mathbf{j} = -\mathbf{k}$.

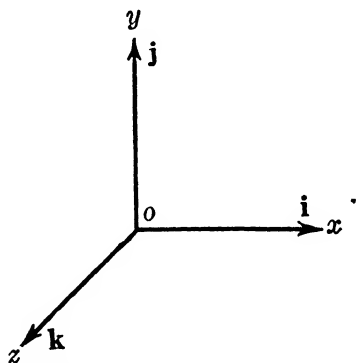


FIG. 1-8. A right-hand system of unit vectors, \mathbf{i} , \mathbf{j} , and \mathbf{k} .

On the basis of what has been presented in §1-5 and §1-6 concerning scalar and vector products, it is now a simple matter to write down the products for pairs of these unit vectors.

$$\begin{aligned} \mathbf{i} \cdot \mathbf{i} &= (1)(1) \cos 0^\circ = 1 = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k}, \\ \mathbf{i} \cdot \mathbf{j} &= (1)(1) \cos 90^\circ = 0 = \mathbf{i} \cdot \mathbf{k} = \mathbf{j} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{k} \\ &= \mathbf{k} \cdot \mathbf{i} = \mathbf{k} \cdot \mathbf{j}. \end{aligned}$$

This information for scalar products can be presented as follows:

	\mathbf{i}	\mathbf{j}	\mathbf{k}	
\mathbf{i}	1	0	0	(1-11)
\mathbf{j}	0	1	0	
\mathbf{k}	0	0	1	

By referring to Fig. 1-8 it is easily seen that

$$\mathbf{i} \times \mathbf{i} = (1)(1) \sin 0^\circ = 0 = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k},$$

and

$$\mathbf{i} \times \mathbf{j} = (1)(1) \sin 90^\circ = \mathbf{k},$$

remembering the right-hand rule. Similarly $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$, etc. The vector products may be conveniently presented as follows:

Second term:	<i>i</i>	<i>j</i>	<i>k</i>	
First term	<i>i</i>	0	<i>k</i>	- <i>j</i>
	<i>j</i>	- <i>k</i>	0	<i>i</i>
	<i>k</i>	<i>j</i>	- <i>i</i>	0

(1-12)

1-8. MULTIPLICATION OF VECTORS (RESUMED)

Equation (1-7) presents the result of scalar multiplication as $\mathbf{A} \cdot \mathbf{B} = AB \cos \theta$. By the use of the unit vectors it is now possible to present another form of this multiplication which, in many cases, is more convenient. Thus, write:

$$\begin{aligned} \mathbf{A} &= iA_x + jA_y + kA_z, \\ \mathbf{B} &= iB_x + jB_y + kB_z. \end{aligned}$$

Then an algebraic multiplication, using dots, produces the result:

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= i \cdot iA_xB_x + i \cdot jA_xB_y + i \cdot kA_xB_z + \\ & \quad j \cdot iA_yB_x + j \cdot jA_yB_y + j \cdot kA_yB_z + \\ & \quad k \cdot iA_zB_x + k \cdot jA_zB_y + k \cdot kA_zB_z. \end{aligned} \tag{1-13}$$

But, on reference to equation (1-11), it is found that all but three of these terms are zero. Thus

$$\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z. \tag{1-14}$$

The same two vectors may be used to obtain another expression for vector product. It is necessary only to rewrite equation (1-13), using crosses instead of dots. One should be especially careful that unit vectors be written always in the correct order. The equation thus obtained is

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= i \times iA_xB_x + i \times jA_xB_y + i \times kA_xB_z + \\ & \quad j \times iA_yB_x + j \times jA_yB_y + j \times kA_yB_z + \\ & \quad k \times iA_zB_x + k \times jA_zB_y + k \times kA_zB_z, \end{aligned}$$

which on reference to equation (1-12) becomes

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= 0 + kA_zB_y - jA_zB_x - \\ & \quad kA_yB_x + 0 + iA_yB_z + \\ & \quad jA_zB_x - iA_zB_y + 0 \\ &= i(A_yB_z - A_zB_y) + j(A_zB_x - A_xB_z) + k(A_xB_y - A_yB_x). \end{aligned} \tag{1-15}$$

This expression may be presented in another form also. It will be found on expansion that

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}. \quad (1-16)$$

It will be found later that the form (1-16) is very useful.

1-9. ILLUSTRATIVE EXAMPLES: PRODUCTS

Example 4. Given two vectors represented by

$$\mathbf{A} = \mathbf{i}2 + \mathbf{j}2 + \mathbf{k}0,$$

$$\mathbf{B} = \mathbf{i}3 + \mathbf{j}4 - \mathbf{k}2,$$

find the dot and cross products and the angle between the vectors.

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= (2)(3) + (4)(2) + 0 = 6 + 8 + 0 = 14 \\ &= AB \cos \theta; \end{aligned}$$

$$A = \sqrt{2^2 + 2^2} = 2.83,$$

$$B = \sqrt{3^2 + 4^2 + 2^2} = 5.39.$$

Therefore,

$$\begin{aligned} \cos \theta &= \frac{14}{2.83(5.39)} = 0.918, \\ \theta &= 23.2^\circ, \end{aligned}$$

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 0 \\ 3 & 4 & -2 \end{vmatrix} \\ &= \mathbf{i}(-4) - \mathbf{j}(-4) + \mathbf{k}(8 - 6) \\ &= -\mathbf{i}4 + \mathbf{j}4 + \mathbf{k}2, \end{aligned}$$

$$\begin{aligned} |\mathbf{A} \times \mathbf{B}| &= \sqrt{16 + 16 + 4} = 6 \\ &= AB \sin \theta, \end{aligned}$$

$$\sin \theta = \frac{6}{2.83(5.39)} = 0.394.$$

Thus, again,

$$\theta = 23.2^\circ$$

Example 5. Show that $\mathbf{A} \times \mathbf{B}$ is at right angles to \mathbf{A} .

If these vectors are at right angles then the scalar product should be zero. Thus

$$\begin{aligned} \mathbf{A} &= \mathbf{i}2 + \mathbf{j}2 + \mathbf{k}0, \\ \mathbf{A} \times \mathbf{B} &= -\mathbf{i}4 + \mathbf{j}4 + \mathbf{k}2, \\ \mathbf{A} \cdot \mathbf{A} \times \mathbf{B} &= -4(2) + 2(4) = 0. \end{aligned}$$

Thus the vectors are perpendicular as they should be in a vector

product. In a similar manner it can be shown that $\mathbf{B} \cdot \mathbf{A} \times \mathbf{B}$ is zero, thus indicating that $\mathbf{A} \times \mathbf{B}$ is also perpendicular to \mathbf{B} .

1-10. TRIPLE SCALAR PRODUCT

The triple scalar product is written $\mathbf{A} \times \mathbf{B} \cdot \mathbf{C}$. In this product the operation $\mathbf{A} \times \mathbf{B}$ must be performed first because if $\mathbf{B} \cdot \mathbf{C}$ were performed first we would be left with a cross product of a vector and a scalar which is meaningless. In order to evaluate the triple scalar product refer to Fig. 1-9. Here $\mathbf{A} \times \mathbf{B}$, based on previous statements, is equal to the base of the parallelepiped. Now $\mathbf{A} \times \mathbf{B} \cdot \mathbf{C}$ is equal to the base multiplied by $C \cos \theta$. This is the

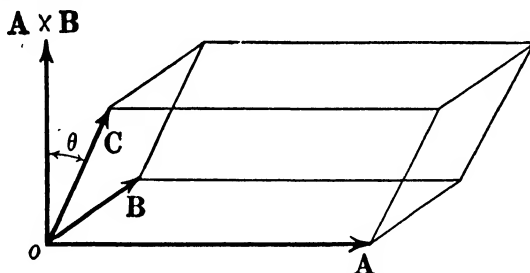


FIG. 1-9. The product $\mathbf{A} \times \mathbf{B} \cdot \mathbf{C}$.

volume of the parallelepiped. Any face of the parallelepiped could be used for the base and it is easily shown that

$$\begin{aligned} \mathbf{A} \times \mathbf{B} \cdot \mathbf{C} &= \mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \mathbf{B} \cdot \mathbf{C} \times \mathbf{A} = -\mathbf{B} \times \mathbf{A} \cdot \mathbf{C} \\ &= \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \end{aligned} \tag{1-17}$$

1-11. TRIPLE VECTOR PRODUCT

This product is defined as

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = \mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = \mathbf{C} \times (\mathbf{B} \times \mathbf{A}). \tag{1-18}$$

The operation in the parenthesis must be performed first. By performing the operations indicated it is easy to show that

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{B} \cdot \mathbf{C})\mathbf{A}. \tag{1-19}$$

See Prob. 1-9 at the close of this chapter.

1-12. DIFFERENTIATION OF A VECTOR

If

$$\mathbf{A} = iA_x + jA_y + kA_z,$$

then

$$\frac{d\mathbf{A}}{dt} = \mathbf{i} \frac{dA_x}{dt} + \mathbf{j} \frac{dA_y}{dt} + \mathbf{k} \frac{dA_z}{dt}, \quad (1-20)$$

since \mathbf{i} , \mathbf{j} , and \mathbf{k} are constant in direction and magnitude and as such are not functions of the independent variable t with respect to which differentiation is being performed.

The differential of a vector is also a vector and is represented in Fig. 1-10.

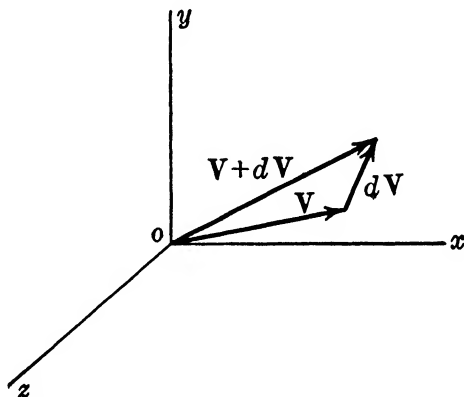


FIG. 1-10. Representing the differential of a vector \mathbf{V} .

The derivatives of dot and cross products are now easily derived.

$$\begin{aligned} \frac{d(\mathbf{A} \cdot \mathbf{B})}{dt} &= \frac{d}{dt} (A_x B_x + A_y B_y + A_z B_z) \\ &= \frac{dA_x}{dt} (B_x) + \frac{dA_y}{dt} (B_y) + \frac{dA_z}{dt} (B_z) + \\ &\quad A_x \frac{dB_x}{dt} + A_y \frac{dB_y}{dt} + A_z \frac{dB_z}{dt} \\ &= \frac{d\mathbf{A}}{dt} \cdot \mathbf{B} + \mathbf{A} \cdot \frac{d\mathbf{B}}{dt}. \end{aligned} \quad (1-21)$$

To show that

$$\frac{d}{dt} (\mathbf{A} \times \mathbf{B}) = \frac{d\mathbf{A}}{dt} \times \mathbf{B} + \mathbf{A} \times \frac{d\mathbf{B}}{dt} \quad (1-22)$$

is straightforward and is left as an exercise for the student. Since the time derivative of a time-varying electric field is of considerable importance in the study of electric waves, a simple numerical example will be given by way of illustration. If the field intensity

vector \mathbf{E} is varying sinusoidally, it might be represented in a particular case as:

$$\mathbf{E} = i400 \sin \omega t + j800 \sin \omega t + \mathbf{k}0 \text{ v/m.}$$

Let it be required to find the current density directed along the $+y$ axis (namely, J_y) which is associated with \mathbf{E} , if the medium (assumed to be isotropic) has a dielectric constant or relative permittivity of 11.31 and ω is $5 \cdot 10^8$ rad/sec.

The electric flux density vector \mathbf{D} is related to the intensity vector \mathbf{E} as $\mathbf{D} = \epsilon_0 \epsilon_r \mathbf{E}$, where ϵ_0 is the absolute permittivity or the permittivity of free space and ϵ_r is the relative permittivity of the medium through which \mathbf{E} is directed. In rationalized mks units (where the electric flux emanating from a coulomb of charge is taken as one coulomb of electric flux) ϵ_0 has a value of $1/(36\pi \cdot 10^9)$. Therefore:

$$\mathbf{D} = \frac{11.31}{36 \pi \cdot 10^9} (i400 \sin \omega t + j800 \sin \omega t + \mathbf{k}0),$$

$$D_y = 10^{-10} [800 \sin (5 \cdot 10^8)t] \text{ coulombs/m}^2,$$

$$J_y = \frac{dD_y}{dt} = 10^{-10} [40 \cdot 10^{10} \cos (5 \cdot 10^8)t] \text{ amp/m}^2,$$

or

$$J_y = 40 \cos \omega t \text{ amp/m}^2 \text{ (directed along } +y\text{).}$$

The current density dD/dt , which is associated with a time-varying electric field, is known as a *displacement* current density to distinguish it from ordinary *conduction* current density. The physical phenomena involved will be discussed in more detail after the subject of "operators" has been considered.

1-13. PROBLEMS

1-1. Two vectors, \mathbf{P} and \mathbf{Q} , are given: $\mathbf{P} = i2 + j3$, $\mathbf{Q} = i3 - j4$. Find the sum, $\mathbf{P} + \mathbf{Q}$, and the difference, $\mathbf{P} - \mathbf{Q}$. Solve graphically and compare results.

1-2. What vector added to $\mathbf{A} = i3 - j5 - \mathbf{k}2$ will produce $\mathbf{C} = i + j + \mathbf{k}$?

1-3. A plot of land has the shape of a parallelogram, one boundary line running east 6 miles and another directly northwest 5 miles. Using methods given in this chapter, find the area.

1-4. Given $\mathbf{P} = i + j + \mathbf{k}$, and $\mathbf{Q} = i3 - j5 + \mathbf{k}A$, find A such that \mathbf{P} and \mathbf{Q} will be perpendicular.

1-5. Given $\mathbf{P} = \mathbf{i} + \mathbf{j}2 + \mathbf{k}3$, $\mathbf{Q} = \mathbf{i} - \mathbf{j}2 + \mathbf{k}2$, find the angle between the vectors.

1-6. Prove that $\mathbf{P} = \mathbf{i} + \mathbf{j}4 + \mathbf{k}3$ and $\mathbf{Q} = \mathbf{i}4 + \mathbf{j}2 - \mathbf{k}4$ are perpendicular.

1-7. Find the cross products of the vectors given in problems 1-4 and 1-6.

1-8. $\mathbf{A} = \mathbf{i}3$, $\mathbf{B} = \mathbf{i}2 + \mathbf{j}4$. Find $\mathbf{A} \times \mathbf{B}$ and sketch the three vectors.

1-9. Prove that equation (1-19) is correct.

1-10. Derive equation (1-22).

1-11. Show that

$$\mathbf{P} \times \mathbf{Q} \cdot \mathbf{R} = \begin{vmatrix} P_x & P_y & P_z \\ Q_x & Q_y & Q_z \\ R_x & R_y & R_z \end{vmatrix}.$$

1-12. Given $\mathbf{A} = \mathbf{i}4$, $\mathbf{B} = \mathbf{i}2 - \mathbf{k}3$, $\mathbf{C} = \mathbf{j}3 + \mathbf{k}3$, find $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ and $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$ and compare the results.

1-13. A section of current-carrying conductor 0.40 m long is lying in the x - z plane and so oriented that the current is properly specified in magnitude and direction by the vector $\mathbf{I} = \mathbf{i}30 + \mathbf{j}0 + \mathbf{k}30$ amp. A uniformly distributed magnetic field which is defined by the magnetic intensity vector, $\mathbf{H} = \mathbf{i}100 + \mathbf{j}100 + \mathbf{k}0$ amp/m is present in the same region as that occupied by the current-carrying conductor. Find the total force, \mathbf{F}_t , in vector form acting on the 0.40 m length of current-carrying conductor assuming a free-space medium where $\mu_r = 1$, and working in the rationalized mks system where $\mu_0 = 4\pi \cdot 10^{-7}$. ($\mathbf{B} = \mu_0 \mu_r \mathbf{H}$.)

1-14. Show that the vector force, $\mathbf{F}_t = \mathbf{i}(-0.00151) + \mathbf{j}(0.00151) + \mathbf{k}(0.00151)$ newton in Prob. 1-13 is directed at right angles to both the \mathbf{H} vector and the \mathbf{I} vector. Draw a vector diagram illustrating the \mathbf{I} , \mathbf{H} , \mathbf{B} , and \mathbf{F}_t vectors including thereon the absolute magnitudes of the four vectors in mks units. ($\mathbf{B} = \mu_0 \mu_r \mathbf{H}$.)

1-15. An electric field specified by the intensity vector

$$\mathbf{E} = \mathbf{i}(12.57 \sin \omega t) + \mathbf{j}(12.57 \sin \omega t) + \mathbf{k}0 \text{ v/m}$$

is assumed to be uniformly distributed in free space. ($\omega = 10^8$ rad/sec.) Find the displacement current in amperes through an area which is represented by the vector

$$\mathbf{A} = 0.30(\mathbf{i} + \mathbf{j}) \text{ m}^2.$$

Solve this problem in the rationalized mks system of units.

Chapter II

APPLICATION OF OPERATORS

Summary

The operator *del*, ∇ , is introduced and through it, the operators known as *grad*, *div*, and *curl* are defined. The Laplacian operator is then defined, and for each operator a number of illustrations are provided.

2-1. OPERATORS

In the preceding material, and for that matter, in all applications of mathematics, use has been made of *operators*. The student is already familiar with operators in many forms. Some of the more common ones are:

\sin , \sinh , \log , \ln , d/dt , \times , $-$, \int , etc.

Each one of these symbols, and many others, merely indicates that a certain operation is assumed to be performed or is to be performed on the quantity specified. For instance $\sin x$ means that the following function of x is understood: $x - x^3/6 + x^5/120 - \dots$, or it may be thought of as the relationship of certain sides of a triangle in which x is one of the angles. The operator \times as used in this book means the application of the procedure given in equation (1-15). The operator d/dt means that the derivative of something is to be obtained. Operators are shorthand symbols in mathematics.

Reciprocal operators are those which effectively cancel each other. As an illustration we have \int and d/dt , or $+$ and $-$, etc. Thus $\int (x + 1) dx = x^2/2 + x + C$, and $(d/dx) (x^2/2 + x + C) = (x + 1)$, the original function, or in general

$$\frac{d}{dx} \int f(x) dx = f(x)$$

However, the operations are not always strictly reciprocal because of integration constants, etc. For instance $\int (d/dx) (x^2) dx$ will not yield unambiguously x^2 , but yields $x^2 + C$ where C would have to be determined.

It must be kept in mind that operators are very common and that there is nothing particularly mysterious about them. They are what they are defined to be. With this brief introduction to operators in general we are ready to introduce a few new ones which are useful in vector analysis as applied to electromagnetic fields. The most important of these new operators is "del" which is written ∇ . This will be taken up in the next section.

2-2. THE DEL OPERATOR

The operator **del** is used to represent the combination of operations presented as follows:

$$\nabla = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \quad (2-1)$$

It means nothing by itself, but operating upon a function of x , y , and z , its character is definite. It is really made up of a combination of two kinds of operators: direction operators, \mathbf{i} , \mathbf{j} , \mathbf{k} , and differentiation operators. Thus del, operating upon a scalar function of x , y , and z , would yield a vector. Let ϕ be a scalar function of x , y , z . Then we define

$$\nabla \phi = \mathbf{i} \frac{\partial \phi}{\partial x} + \mathbf{j} \frac{\partial \phi}{\partial y} + \mathbf{k} \frac{\partial \phi}{\partial z} = \text{grad } \phi. \quad (2-2)$$

The term *grad* is introduced here because it is an alternative way of expressing the application of ∇ to a scalar. They are equivalent. The term *grad* comes from the word *gradient* which is a common term in the treatment of electromagnetic fields. Equation (2-2) can be regarded as defining both ∇ and *grad*. By definition the operator ∇ (equation 2-1) has all the properties of a vector and may be used as such in other operations.

If ϕ in equation (2-2) is a function that at all points of space merely has a definite magnitude but no direction associated with it, it is called a scalar field of which a potential field is a special case. A *scalar field* is represented by the work done in lifting a body to certain points in space against the force of gravity. The value of the work at each point will not have any relationship to any par-

ticular direction. The force of gravity at any point in space, however, is a *vector field*. Since the force of gravity is proportional to the space rate of change of the potential, it is easily seen that the gradient of the potential is proportional to force. Thus, in this case, a vector field may be obtained by taking the gradient of a scalar field.

It can be shown that the gradient $\nabla\phi$ is perpendicular to the surfaces representing equal potentials. Let $(\phi = \text{constant})$ repre-

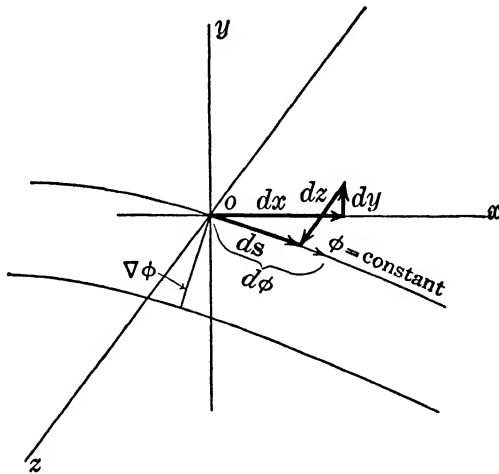


FIG. 2-1. Cross sections of two equipotential surfaces.

sent an equipotential surface, the cross section of which is shown in Fig. 2-1. On the basis of this diagram the following equations can be written:

$$d\phi = \frac{\partial\phi}{\partial x} dx + \frac{\partial\phi}{\partial y} dy + \frac{\partial\phi}{\partial z} dz = 0,$$

because $d\phi$ (shown in Fig. 2-1) is taken along an equipotential surface and hence $d\phi$ represents zero change in ϕ . Also

$$ds = i dx + j dy + k dz.$$

Now write by definition

$$\nabla\phi = i \frac{\partial\phi}{\partial x} + j \frac{\partial\phi}{\partial y} + k \frac{\partial\phi}{\partial z}.$$

The dot, or scalar, product of $\nabla\phi$ and ds becomes

$$\nabla\phi \cdot ds = \frac{\partial\phi}{\partial x} dx + \frac{\partial\phi}{\partial y} dy + \frac{\partial\phi}{\partial z} dz,$$

which is identical with $d\phi$ which is equal to zero. Therefore, $\nabla\phi \cdot ds = 0$, and since ds and $d\phi$ may have been taken in any direction on the surface ($\phi = \text{const.}$), the term $\nabla\phi$ must be perpendicular to the surface.

2-3. ILLUSTRATIVE EXAMPLES: THE DEL OPERATOR

Example 1. As an illustration, consider the electric field between the plates of the condenser shown in Fig. 2-2. Let the distance between plates be d meters and the voltage across the plates, V volts. Then the potential field will be represented by $\phi = ay$ where a is to be determined. When $y = d$, ϕ is given as V volts, and therefore, $a = V/d$ volts/meter. Thus $\phi = Vy/d$ which is a scalar field. Now apply the del operator to ϕ and obtain, from $\mathbf{E} = -\nabla\phi$,

$$\begin{aligned} \mathbf{E} &= -\nabla\phi \\ &= -\left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}\right) \frac{V}{d} y \\ &= -\mathbf{j} \frac{V}{d} \text{ v/m.} \end{aligned}$$

This is a vector field and represents a magnitude and direction at every point. The magnitude is $-V/d$ volts per meter and the direction is along the negative y axis, as one would expect from elementary considerations.

Example 2. A scalar field is given as $\phi = x^2 + z - xy$. Find the gradient $\nabla\phi$ and its magnitude at the point 2,1,3. Applying the del operator we obtain

$$\begin{aligned} \nabla\phi &= \mathbf{i}(2x - y) - \mathbf{j}x + \mathbf{k} \\ &= \mathbf{i}3 - \mathbf{j}2 + \mathbf{k}. \end{aligned}$$

The magnitude of this gradient at 2,1,3 is

$$|\nabla\phi| = \sqrt{(4 - 1)^2 + 2^2 + 1^2} = 3.74.$$

Example 3. In Fig. 2-3 is shown the cross section of two concentric cylindrical conductors which are maintained at a potential difference of 1000 v. The radius of the outer conductor is 0.10 m.

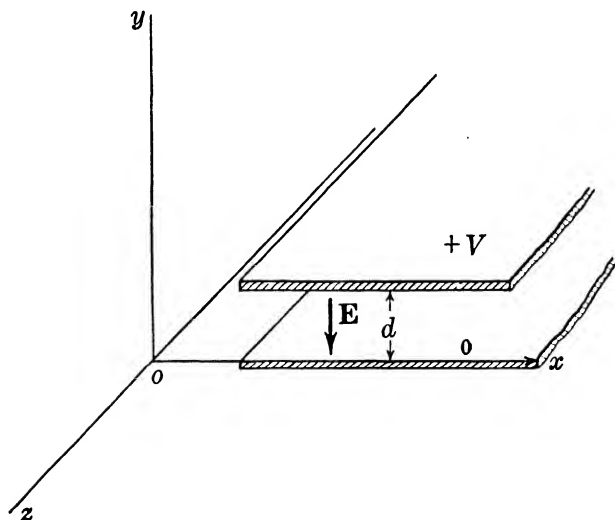


FIG. 2-2. Electric field intensity, \mathbf{E} , between flat plates. $\mathbf{E} = -\nabla\phi$.

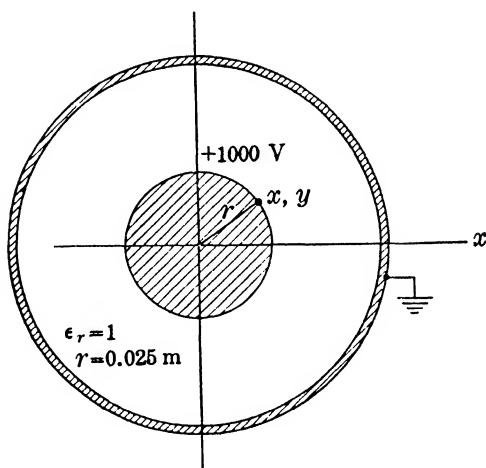


FIG. 2-3. Coaxial cable illustrating Example 3.

Let it be required to find the field intensity, $\mathbf{E} = -\nabla\phi$, at points $(0.025, 0)$, $(0, 0.025)$, $(0.0177, 0.0177)$ on the surface of the inner conductor. It is known that the potential in the space between conductors is given by

$$\phi = -721 \ln 10 \sqrt{x^2 + y^2}.$$

No potential variation exists along the z direction.

$$\mathbf{E} = -\nabla\phi = 721 \left(\mathbf{i} \frac{x}{x^2 + y^2} + \mathbf{j} \frac{y}{x^2 + y^2} \right) \text{ v/m.}$$

$$\text{At } x = 0.025, y = 0;$$

$$\mathbf{E} = \mathbf{i}28,850 \text{ v/m.}$$

$$\text{At } x = 0, y = 0.025;$$

$$\mathbf{E} = \mathbf{j}28,850 \text{ v/m.}$$

$$\text{At } x = 0.0177, y = 0.0177;$$

$$\mathbf{E} = 20,400(\mathbf{i} + \mathbf{j}) \text{ v/m.}$$

It will be observed that in each case $\nabla\phi$ is directed at right angles to the surface of the inner conductor and that the constant magnitude of 28,850 v/m agrees with the value of the \mathbf{E} field vectors at the surface of the inner conductor as found by more elementary methods.

After generalized coordinates have been considered, problems involving cylindrical geometry of this kind will be handled in cylindrical coordinates.

2-4. THE DIVERGENCE OPERATOR

When the operator ∇ is written immediately before a scalar we have seen that a vector is the result of the indicated multiplication and this "product" is defined as the gradient. If ∇ is written before a vector using a dot to indicate a scalar product, a scalar results, as will be shown, and it is *defined* as "divergence." Given a vector \mathbf{V} , the divergence is written

$$\begin{aligned} \text{div } \mathbf{V} &= \nabla \cdot \mathbf{V} = \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) \cdot (\mathbf{i}V_x + \mathbf{j}V_y + \mathbf{k}V_z) \\ &= \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}. \end{aligned} \quad (2-3)$$

This is immediately seen to be a scalar. Its physical significance can be appreciated by considering the following discussion of the equation of continuity.

In Fig. 2-4 an elementary cube is shown in a cartesian system of coordinates. Let it be assumed that fluid is flowing through this volume as indicated by the arrow \mathbf{V} , \mathbf{V} being the mass per unit time per unit cross section and thus equal to the density multiplied by the velocity, or $\mathbf{V} = \delta \mathbf{v}$, where δ may be variable.

The mass of material which flows into the face $AEHD$ is $V_x dy dz$ per unit time. Since at the face opposite (that is, at $BFGC$) the value of V_x may have changed, the material flowing out per unit

time is $[V_x + (\partial V_x / \partial x) dx] dy dz$. Thus, the accumulation of material between the faces of the cube will be the excess of the in-flow over the out-flow, or

$$V_x dy dz - \left[V_x + \frac{\partial V_x}{\partial x} dx \right] dy dz = - \frac{\partial V_x}{\partial x} dx dy dz.$$

Now this procedure may be followed for the flow in the y and z

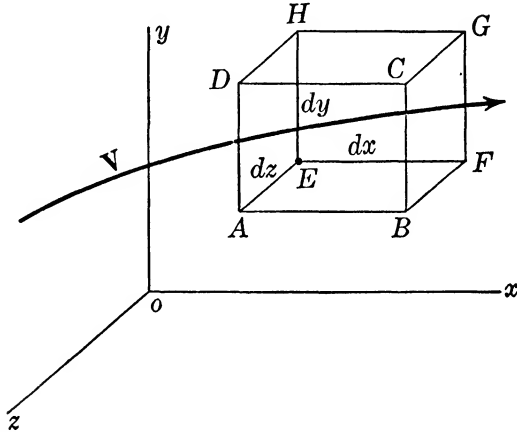


FIG. 2-4. Divergence in a particular case where $\mathbf{V} = \delta \mathbf{v}$, δ being variable.

directions, yielding similar expressions. The total net increase, considering all three components, will then be:

$$- \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \right) dx dy dz.$$

Dividing this expression by the volume of the element there is obtained the increase of material per unit time per unit volume, or

$$- \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \right) = - \nabla \cdot \mathbf{V} \tag{2-4}$$

from equation (2-3).

2-5. ILLUSTRATIVE EXAMPLES: THE DIVERGENCE OPERATOR

Example 4. Given a waterfall as shown in Fig. 2-5, determine the divergence of the velocity at a point y below the reference level O from which all the water may be considered as falling. Since the

water is assumed to be falling in a gravitational field of g acceleration, the velocity is written as

$$\mathbf{v} = i0 + \mathbf{j} \sqrt{2gy} + \mathbf{k}0.$$

Applying the divergence operator, we have

$$\begin{aligned} \nabla \cdot \mathbf{v} &= \frac{\partial v_y}{\partial y} \\ &= \frac{\partial}{\partial y} (2gy)^{\frac{1}{2}} \\ &= \sqrt{\frac{g}{2y}}. \end{aligned}$$

Thus, the divergence decreases as y increases.

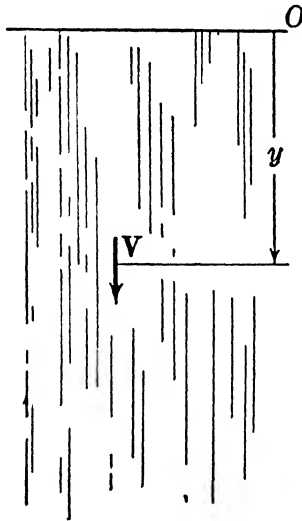


FIG. 2-5. A waterfall where the velocity is $\mathbf{v} \cdot \mathbf{v} = \mathbf{j} \sqrt{2gy}$.

Example 5. Water flows along a river bed in such a way that the horizontal velocity increases from zero as one rises above the bottom. If the water is flowing in the x direction and the y axis is upward, determine whether there is a finite value of $\text{div } \mathbf{v}$. \mathbf{v} may be written

$$\begin{aligned} \mathbf{v} &= i v_x + \mathbf{j}0 + \mathbf{k}0, \quad v_x = my, \\ \mathbf{v} &= i my. \end{aligned}$$

Therefore,

$$\nabla \cdot \mathbf{v} = \frac{\partial}{\partial x} my = 0.$$

Thus, the velocity vector has no divergence in this particular case.

Example 6. An explosion occurs in such a way that at every point in space the velocity of gas is proportional to the distance from the center and directed outward. Find the divergence of the

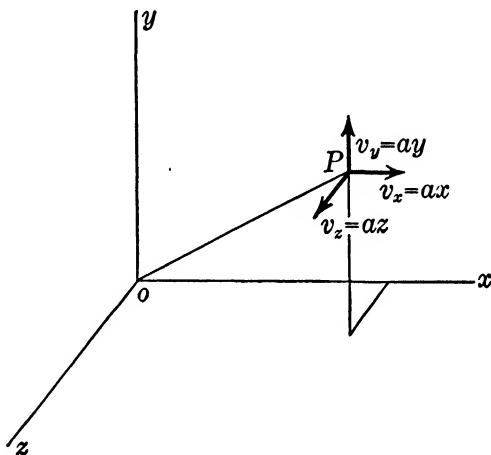


FIG. 2-6. An explosion at point P results in the velocities: $v_x = ax$, $v_y = ay$, $v_z = az$.

gas velocity. Refer to Fig. 2-6. In this figure, $v_x = ax$, $v_y = ay$, and $v_z = az$. Therefore

$$\mathbf{v} = iax + jay + kaz,$$

and

$$\nabla \cdot \mathbf{v} = a + a + a = 3a.$$

Thus, at every point in space the divergence of the velocity is constant and positive.

Example 7. Given an electric field in which free charges are present, determine the divergence of the flux density. Suppose there are ρ positive coulombs per cubic meter; then the excess of lines leaving any cubic meter volume over those entering is ρ mks lines. Thus,

$$\nabla \cdot \mathbf{D} = \rho. \quad (2-5)$$

If the charge per unit volume is zero then this equation reduces

to

$$\nabla \cdot \mathbf{D} = 0. \tag{2-6}$$

Example 8. Given a magnetic field in a free space region, find the divergence of flux density vector \mathbf{B} . Since there are no free sources of magnetic lines in the space, there can be no lines originating in any $dx dy dz$ volume in this space. Thus, there can be no excess or deficiency of lines leaving an elementary volume. Thus

$$\nabla \cdot \mathbf{B} = 0. \tag{2-7}$$

which is an important characteristic of magnetic fields.

2-6. THE LAPLACIAN OPERATOR

It has been seen that the operation $\nabla\phi$ produces a vector. Accordingly, it is possible to operate on the vector by "del-dot" and obtain a scalar, $\nabla \cdot \nabla\phi$. The value of this is determined as follows:

$$\nabla\phi = \mathbf{i} \frac{\partial\phi}{\partial x} + \mathbf{j} \frac{\partial\phi}{\partial y} + \mathbf{k} \frac{\partial\phi}{\partial z},$$

and

$$\nabla \cdot \nabla\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}. \tag{2-8}$$

This operator on ϕ is called the "Laplacian" and is generally written $\nabla^2 (= \nabla \cdot \nabla)$. It is a scalar operator and can be applied to vectors as well as scalars in the same way that the scalar multiplier a can be applied to both scalars and vectors to produce $a\phi$ and $a\mathbf{V}$. Applying the Laplacian to \mathbf{V} we obtain:

$$\nabla^2\mathbf{V} = \frac{\partial^2\mathbf{V}}{\partial x^2} + \frac{\partial^2\mathbf{V}}{\partial y^2} + \frac{\partial^2\mathbf{V}}{\partial z^2}. \tag{2-9}$$

An important expression for the Laplacian is obtained as follows. Write equation (1-19) as shown below, using equation (1-18) and remembering that ordinary algebraic multiplication can be performed in any order.

$$\mathbf{C} \times (\mathbf{B} \times \mathbf{A}) = \mathbf{C}(\mathbf{B} \cdot \mathbf{A}) - (\mathbf{C} \cdot \mathbf{B})\mathbf{A}.$$

Replace \mathbf{C} and \mathbf{B} by the operator ∇ . Then,

$$\begin{aligned} \nabla \times \nabla \times \mathbf{A} &= \nabla(\nabla \cdot \mathbf{A}) - (\nabla \cdot \nabla)\mathbf{A} \\ &= \nabla\nabla \cdot \mathbf{A} - \nabla^2\mathbf{A}, \end{aligned}$$

or

$$\nabla^2\mathbf{A} = \nabla\nabla \cdot \mathbf{A} - \nabla \times \nabla \times \mathbf{A}. \tag{2-9a}$$

The Laplacian enters into the expression of a particularly important equation whose usefulness will appear later. This is Poisson's equation which can be developed as follows: It is known that the electric field intensity is expressed as

$$(1) \quad \mathbf{E} = -\nabla\phi.$$

Also, equation (2-5) expresses the relation between \mathbf{D} and ρ as

$$(2) \quad \nabla \cdot \mathbf{D} = \rho,$$

where ρ is the volume charge density at the point under consideration. From equation (1)

$$-\nabla\phi = \frac{\mathbf{D}}{\epsilon},$$

where ϵ is the dielectric constant, ($\epsilon = \epsilon_0\epsilon_r$). Therefore,

$$(3) \quad -\nabla \cdot \nabla\phi = \frac{\nabla \cdot \mathbf{D}}{\epsilon} = \frac{\rho}{\epsilon}$$

From which

$$\nabla^2\phi = -\frac{\rho}{\epsilon}. \quad (2-9b)$$

This is Poisson's equation. If $\rho = 0$, that is, if the space is free of charges, then

$$\nabla^2\phi = 0. \quad (2-9c)$$

Equation (2-9c) is known as Laplace's equation. Both Laplace's and Poisson's equations will be useful in future work because a large portion of our knowledge of electric fields is contained in the concise statements; $\nabla^2\phi = 0$, and $\nabla^2\phi = -\rho/\epsilon$.

2-7. ILLUSTRATIVE EXAMPLES: THE LAPLACIAN OPERATOR

Example 9. Find $\nabla^2\phi$ where $\phi = x^2 + 3y^2 - z^2$.

$$\frac{\partial\phi}{\partial x} = 2x,$$

$$\frac{\partial\phi}{\partial y} = 6y,$$

$$\frac{\partial\phi}{\partial z} = -2z.$$

$$\frac{\partial^2\phi}{\partial x^2} = 2,$$

$$\frac{\partial^2\phi}{\partial y^2} = 6,$$

$$\frac{\partial^2\phi}{\partial z^2} = -2.$$

Therefore,

$$\nabla^2\phi = 2 + 6 - 2 = 6,$$

from equation (2-8).

Example 10. Find ∇^2V , where $V = i(x^2 + y^2) + j(y^2 + z^2) + k(x^2 + z^2)$.

$$\frac{\partial V}{\partial x} = i2x + 0 + k2x,$$

$$\frac{\partial^2 V}{\partial x^2} = i2 + k2,$$

$$\frac{\partial V}{\partial y} = i2y + j2y,$$

$$\frac{\partial^2 V}{\partial y^2} = i2 + j2,$$

$$\frac{\partial V}{\partial z} = j2z + k2z,$$

$$\frac{\partial^2 V}{\partial z^2} = j2 + k2.$$

Therefore,

$$\nabla^2V = i4 + j4 + k4,$$

by equation (2-9).

2-8. CURL OR ROTATION

The physical significances of gradient and divergence are relatively easy to visualize, at least for certain applications. *Curl*, however, is somewhat more difficult although there are cases where its significance can be seen. *Curl* is defined as follows:

$$\begin{aligned} \text{curl } \mathbf{V} = \nabla \times \mathbf{V} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} \\ &= \mathbf{i} \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \right) + \mathbf{j} \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \right) + \mathbf{k} \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right). \end{aligned} \quad (2-10)$$

It is best to consider this operation from a purely mathematical standpoint and not to attempt to visualize what is meant, at least in complicated cases. Its significance in a few simple cases, however, will be worked out to present a rough picture. In general, the term "rotation" is most descriptive. Anything having simple rotation has a value of curl associated with it. Thus the velocity of points on the end of a rotating shaft has a finite value of curl. On the other hand, fields which appear to have no rotation sometimes have a value of curl, and also sometimes fields which seem to have simple rotation have no curl.

2-9. ILLUSTRATIVE EXAMPLES: CURL OR ROTATION

Example 11. Refer to Fig. 2-7, where the end of a rotating shaft is shown. The velocity v of point P is $r\omega$, where ω is the angular velocity. The x and y components of this velocity are as follows:

$$\begin{aligned}v_x &= -r\omega \sin \theta = -\omega y, \\v_y &= r\omega \cos \theta = \omega x.\end{aligned}$$

The vector expression for velocity is now

$$\begin{aligned}\mathbf{v} &= iv_x + jv_y = -i\omega y + j\omega x, \\ \nabla \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix} \\ &= i0 + j0 + \mathbf{k}(\omega + \omega) \\ &= \mathbf{k}2\omega.\end{aligned}$$

Thus, the curl of velocity of particles in a rotating shaft is simply twice the angular velocity and is represented by a vector parallel

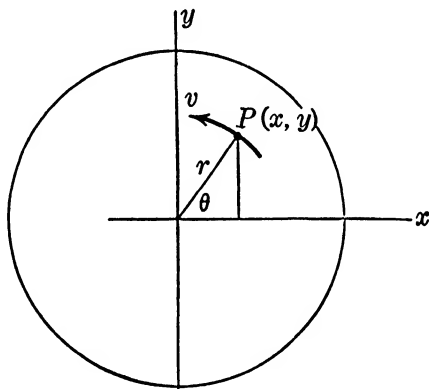


FIG. 2-7. End view of a rotating shaft.

to the axis of rotation. It should be noted here also that the curl is the same for all points in the shaft.

Example 12. Refer to Example 5. Here, horizontal flow of water is the only motion present. However, that there is really a rotation or curl is seen by the fact that if a water wheel were placed somewhere in the water, a rotation of the wheel would result

because water is flowing past it faster on the upper side than on the bottom. In this example, $\mathbf{v} = imy$ and the curl is found as follows:

$$\begin{aligned} \nabla \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ my & 0 & 0 \end{vmatrix} \\ &= \mathbf{i}0 + \mathbf{j}0 - \mathbf{k}m = -\mathbf{k}m. \end{aligned}$$

Accordingly, there is a curl whose direction is horizontal, and perpendicular to the velocity.

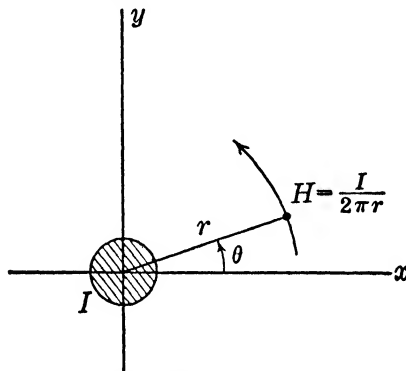


FIG. 2-8. Magnetic field intensity, H , at a distance r from a straight current-carrying conductor.

Example 13. Consider Fig. 2-8 where the magnetic field about a wire is represented. In a manner similar to Example 11, we may write:*

* The equations which follow are based upon the equation

$$H = \frac{I}{2\pi r},$$

which gives the field intensity about a current-carrying conductor in the mks system of units. The student may be more familiar with the equation

$$H = \frac{2I}{r},$$

where H is in gilberts per centimeter, I is in abamperes, and r is in centimeters. This equation in emu may be easily changed to mks units by noting the following relations which are presented in Chapter III.

$$\frac{\text{amperes}}{\text{abamperes}} = 10, \quad \frac{\text{gilberts}}{\text{amperes}} = 0.4\pi, \quad \frac{\text{centimeters}}{\text{meters}} = 100.$$

$$\begin{aligned}
 H_x &= \frac{-I \sin \theta}{2\pi r} \\
 &= \frac{-Iy}{2\pi r^2} \\
 &= \frac{-Iy}{(x^2 + y^2)2\pi} \text{ amp/m,} \\
 H_y &= \frac{I \cos \theta}{2\pi r} \\
 &= \frac{Ix}{2\pi r^2} \\
 &= \frac{Ix}{(x^2 + y^2)2\pi} \text{ amp/m,} \\
 \frac{\partial H_x}{\partial y} &= \frac{-I[(x^2 + y^2) - y(2y)]}{2\pi r^4} \\
 &= \frac{-I(x^2 + y^2 - 2y^2)}{2\pi r^4}, \\
 \frac{\partial H_y}{\partial x} &= \frac{I[(x^2 + y^2) - x(2x)]}{2\pi r^4} \\
 &= \frac{I(x^2 + y^2 - 2x^2)}{2\pi r^4}.
 \end{aligned}$$

Thus,

$$\begin{aligned}
 \text{Curl } \mathbf{H} &= i0 + j0 + \mathbf{k} \left[\frac{I}{2\pi r^4} (-x^2 + y^2 + x^2 - y^2) \right] \\
 &= 0.
 \end{aligned}$$

Accordingly, although there appears to be a rotation involved, there is no curl.

Example 14. If the magnetic field within the wire is considered, however, a different picture results and curl is found to exist as shown below. In Fig. 2-9 the end of a wire of radius a is shown where H at any point is given as

$$\begin{aligned}
 H &= \frac{I}{2\pi r} \cdot \frac{r^2}{a^2} \\
 &= \frac{Ir}{2\pi a^2} \text{ amp/m.}
 \end{aligned}$$

Therefore,

$$H_x = -\frac{Ir \sin \theta}{2\pi a^2}$$

$$\begin{aligned} &= -\frac{Iy}{2\pi a^2}, \\ H_y &= \frac{Ir \cos \theta}{2\pi a^2}, \\ &= \frac{Ix}{2\pi a^2}, \\ \frac{\partial H_x}{\partial y} &= \frac{-I}{2\pi a^2}, \quad \frac{\partial H_y}{\partial x} = \frac{I}{2\pi a^2}. \end{aligned}$$

Thus

$$\nabla \times \mathbf{H} = \mathbf{k} \frac{I}{\pi a^2},$$

and a curl exists.

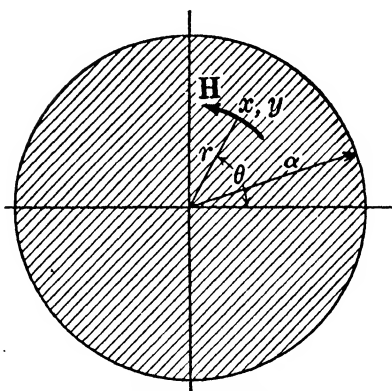


FIG. 2-9. Magnetic field intensity, \mathbf{H} , inside a current-carrying conductor.

Curl may also be looked upon as the limit of the ratio

$$\frac{\text{integral of a quantity around a closed path}}{\text{area of the surface enclosed by the path}}$$

when the area approaches zero. This will be shown to give the appropriate value for one special case. Consider the rotating shaft of Fig. 2-10. The velocity along cd is $r_2\omega$, and that along ba is $r_1\omega$. The velocities along ad and bc are zero. The integral of v around the closed path $abcd$ then is

$$-r_1^2\theta\omega + 0 + r_2^2\theta\omega + 0 = \theta\omega(r_2^2 - r_1^2).$$

The area of the enclosed surface is

$$\frac{\pi r_2^2\theta}{2\pi} - \frac{\pi r_1^2\theta}{2\pi} = \frac{\theta}{2}(r_2^2 - r_1^2).$$

The ratio becomes

$$\frac{\theta\omega (r_2^2 - r_1^2)}{\frac{\theta}{2} (r_2^2 - r_1^2)} = 2\omega.$$

Since this is independent of r and θ , its limit as $r_2 \rightarrow r_1$ and $\theta \rightarrow 0$, is still 2ω . This agrees with the value of curl given previously in Example 11.

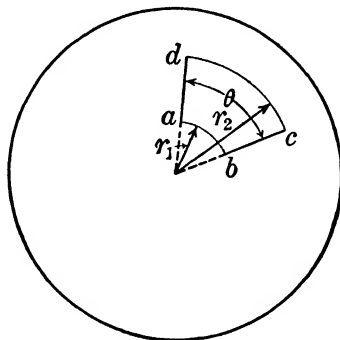


FIG. 2-10. End view of a rotating shaft.

2-10. PROBLEMS

2-1. A scalar field is given by $\phi = x^2 + 2xy + z$. Find the gradient.

2-2. Let $\mathbf{R} = ix + jy + kz$, with $R^2 = x^2 + y^2 + z^2$. (a) Show that $\nabla R^n = nR^{n-2}\mathbf{R}$. (b) Find a function ϕ whose gradient is \mathbf{R} .

2-3. If \mathbf{A} is a constant vector show that $\nabla(\mathbf{A} \cdot \mathbf{R}) = \mathbf{A}$, where \mathbf{R} has the value given in Prob. 2-2.

2-4. Find $\nabla\phi$ where ϕ is given in Prob. 2-16. Find the divergence of this gradient.

2-5. Find $\nabla(\nabla \cdot \mathbf{P})$ where \mathbf{P} is given in Prob. 2-20.

2-6. $V_x = \sin y$, $V_y = V_z = 0$. Sketch the field of \mathbf{V} and find its divergence and curl.

2-7. Two infinite planes are separated by 10 cm. At a point A , taken as the center, an incompressible liquid originates at the rate of $100 \text{ cm}^3/\text{sec}$. The liquid flows outward radially from the center, completely filling the space between the plates. Find the divergence of the velocity of the liquid at a point some distance r , ($\gg 10 \text{ cm}$) from the center.

2-8. $V_x = V_y = 1/\sqrt{x^2 + y^2}$. Sketch this field and find its divergence.

2-9. $V_x = \frac{x}{x^2 + y^2}$, $V_y = \frac{y}{x^2 + y^2}$. Sketch the field of \mathbf{V} and find its divergence and curl.

2-10. Find the curl of the gradient of $P = 1000 - x^2 - y^2$.

2-11. Find the divergence of the curl of the vector field given in Prob. 2-6.

2-12. Let $\mathbf{H} = i2x^2 + j(2xy + y^2) + k3x$. (a) Find the curl of \mathbf{H} . (b) Find the divergence of the curl of \mathbf{H} .

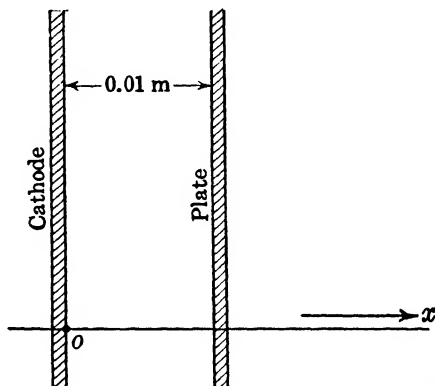


FIG. 2-11. A flat cathode and plate assembly illustrating Prob. 2-23.

2-13. Find $\nabla \times (\nabla \times \mathbf{P})$ where \mathbf{P} is given in Prob. 2-20.

2-14. Using the method given at the end of §2-9, find the curl of the magnetic field \mathbf{H} outside of a wire carrying a current I . Use the mks system of units.

2-15. The gravitational potential field may be represented by $\phi = G + my$, where G and m are constants. Find its Laplacian.

2-16. Find $\nabla^2 \phi$ where $\phi = x^2 + xy^2 + y^3 + 3z$.

2-17. Find the Laplacian of P where $P = -\ln r$. (Use two dimensions.)

2-18. Find the Laplacian of $P = 1000 - x^2 - y^2$.

2-19. Find the Laplacian of P where $P = 1/r$. (Use two dimensions.)

2-20. Find $\nabla^2 \mathbf{P}$ where $\mathbf{P} = i(x^2 + y^2) + j(x + y) + k$.

2-21. Prove that equation (2-9a) is correct by direct expansion.

2-22. Given a sphere of b meters radius containing a negative electric charge which is increasing at such a rate that $\text{div } \mathbf{J} = a$, where \mathbf{J} is the current density, find the time rate of change of charge

density within the sphere, and the total current flowing out of the sphere.

2-23. Given the space between a plane cathode and a plate as shown in Fig. 2-11, assume that the planes are of infinite extent and that the potential of the cathode is 0 and of the plate, +300 v. Let the charge density be ρ given by $\rho = 1/(1 + 100x)$. Applying Poisson's equation, find the field intensity at any point between the cathode and plate. Assume that $\epsilon_r = 1$.

2-24. Repeat Prob. 2-23 for $\rho = 0$, and for $\rho = Q_0$, a constant.

THEOREMS AND LAWS

Summary

The basic operators having been introduced, they are now used to formulate the theorems and laws necessary for a complete description of the electromagnetic field in the time-varying state. A discussion of units is provided, followed by a presentation of the ideas of potential, both scalar and vector. Gauss's and Stokes's theorems are introduced on this basis and they lead directly to a formulation of the four laws of Maxwell.

3-1. BASIC LAWS IN LINE-INTEGRAL FORM

Some of the mathematical tools which are most useful in analyzing scalar and vector fields were considered briefly in Chapters I and II. Emphasis was placed upon the mathematical techniques involved rather than upon the physical significance of the symbols. Attention will now be directed toward the laws governing scalar and vector fields and the fundamental equations of electromagnetic theory.

It is assumed that the reader is familiar with the circuital law of magnetism which in line-integral form reads:

$$\oint \mathbf{H} \cdot d\mathbf{s} = 4\pi I \quad (\text{in unrationalized units}), \quad (3-1)$$

or

$$\oint \mathbf{H} \cdot d\mathbf{s} = I \quad (\text{in rationalized units}), \quad (3-1a)$$

where \mathbf{H} is the magnetic field intensity which, by definition, is the negative of the magnetic potential gradient, $d\mathbf{s}$ is the displacement along the path of integration, and I is the *total* current linked by the closed path of integration.

That integration is to be taken along any closed path in equations (3-1) and (3-1a) is indicated by the closed loop around the integral sign. The matter of rationalized and unrationalized units will be considered in a later section of this chapter.

In a similar manner, a knowledge of Faraday's emf law in line-integral form is assumed.

$$\oint \mathbf{E} \cdot d\mathbf{s} = E = -\frac{d\phi}{dt} \quad (3-2)$$

where E is the total voltage induced around the closed path by the time rate of change of magnetic flux, $(d\phi/dt)$, linking the closed path. \mathbf{E} is the symbol employed here for the electric field intensity which by definition is the negative of the electric potential gradient.

Although scalar potentials will be considered in some detail later, attention will be called at this time to the fact that the essential scalar potentials employed in circuit analysis are:

1. magnetomotive force (usually expressed in amperes or ampere turns).

2. electromotive force (usually expressed in volts).

The analyses of electric and magnetic fields are, however, better carried out in terms of the gradients of the two scalar potentials, namely:

$$\mathbf{H} = -\nabla \text{mmf} \quad (\text{called magnetic field intensity}), \quad (3-3)$$

and

$$\mathbf{E} = -\nabla \phi \quad (\text{called electric field intensity}). \quad (3-4)$$

In order that the basic equations of electromagnetic theory may be generally applicable to any system of units, it is desirable to express the magnetic field density or magnetic flux density (ϕ/\mathbf{A}) as:

$$\mathbf{B} = \mu_0 \mu_r \mathbf{H} = \mu \mathbf{H} = \frac{\phi}{\mathbf{A}}, \quad (3-5)$$

where μ_0 is the permeability of free space which possesses dimensions and as such possesses a numerical value which is dependent upon the particular system of units employed, and

μ_r is the relative permeability of the medium which possesses the same value irrespective of the system of units employed.

In a similar manner, it is desirable to express the electric flux density as:

$$\mathbf{D} = \frac{\psi}{\mathbf{A}} = \epsilon_0 \epsilon_r \mathbf{E} = \epsilon \mathbf{E}, \quad (3-6)$$

where ϵ_0 is the permittivity of free space;

ϵ_r is the relative permittivity or dielectric constant of the medium, and

ψ is the electric flux; and \mathbf{A} the area through which the flux is passing.

The quantities \mathbf{B} and \mathbf{D} are, in one sense, mathematical concepts because they do not lend themselves directly to experimental verification. These concepts are, however, extremely useful in simplifying many calculations and since \mathbf{B} and \mathbf{D} are linearly related, respectively, in isotropic media to the physically realizable vectors \mathbf{H} and \mathbf{E} , the \mathbf{B} and \mathbf{D} vectors are accepted as defined quantities. Their definitions are given by equations (3-5) and (3-6), respectively.

In arriving at the fundamental equations of electromagnetic theory in differential vector form, certain purely mathematical concepts will be invoked. The reader should not expect diagrams or even physical interpretations of the concepts which come from the field of pure mathematics. As the text proceeds, an effort will be made to distinguish between the contributions of physics and those of pure mathematics. Before proceeding, however, the matter of units must be settled upon.

3-2. RATIONALIZED AND UNRATIONALIZED CGS UNITS

The subject of units is a disagreeable one but it is also an essential one from an engineering point of view. It is disagreeable because so many different systems of units are available and each of the commonly used systems has certain distinct advantages within prescribed limits.

The unrationalized ab-cgs system of units, for example, is rather ideally suited to elementary magnetic field calculations, particularly in those cases involving spherical or cylindrical geometry. It will be remembered that, in this system of units, $\mu_0 = 1$. That is,

$$\mathbf{B}' \text{ (in maxwells/cm}^2\text{)} = \mathbf{H}' \text{ (in gilberts/cm)} \quad (3-7)$$

in free space, and

$$\text{mmf (in gilberts)} = 4\pi I_{ab}, \quad (3-8)$$

where I_{ab} is the total current linked by the path around which the mmf is measured expressed in abamperes. [See equation (3-1).]

For example, the magnitude of \mathbf{H} in the region outside and at a distance x from a long straight current-carrying conductor is, according to equation (3-1),

$$H = \frac{4\pi I_{ab}}{2\pi x} = \frac{2I_{ab}}{x} \text{ gilberts/cm.} \quad (3-9)$$

Thus by using the gilbert as the unit of mmf rather than the abampere, certain expressions for \mathbf{H} become more simplified than would otherwise be the case.

There is, however, no basic reason why the abampere might not have been selected as the unit of mmf in the ab-cgs system. From equation (3-8), it is plain that the abampere as a unit of mmf is a unit which is 4π times as large as the gilbert. If \mathbf{H} were to be expressed in abamperes per centimeter in equation (3-5), the same value of \mathbf{B} (in maxwells per square centimeter) could be had if μ_0 were given the value of 4π rather than unity as in the unrationalized ab-cgs system. The rationalized ab-cgs system of units is characterized by the use of \mathbf{H} in abamperes per centimeter and $\mu_0 = 4\pi$.

In the field of electrostatics, the unrationalized stat-cgs system is widely used, and we presume that the reader is reasonably familiar with this system of units which comes into being as a result of making $\epsilon_0 = 1$. It will be remembered that in this system of units, electric flux is defined in such a manner that 4π lines of electric flux emanate from each statcoulomb of charge, the primary unit of charge in the stat-cgs system. That is,

$$\psi = 4\pi Q_s \quad (\text{cgs lines of electric flux}). \quad (3-10)$$

Again there is no basic reason (other than wanting to have ϵ_0 equal to unity) why electric flux might not have been specified in statcoulombs of electric flux, or, what amounts to the same thing, in statcoulombs of *projected* charge. In most of the material that follows, electric flux will be considered as projected charge and will be expressed:

$$\psi = Q \quad (\text{in rationalized units}).$$

The concept of electric flux then takes on a physical significance that it might not otherwise have. The parent charge Q might, for

example, be residing on a conductor like an antenna, but its effect in the free space surrounding the antenna can best be described and manipulated mathematically in terms of the parent charge's *projected* effect, namely, electric flux.

A statcoulomb of electric flux is plainly a unit of flux which is 4π times greater than the stat-cgs line of flux. Thus the statcoulomb per square centimeter ($\mathbf{D}' = Q/A$) is a unit of flux density which is 4π times greater than the stat-cgs line per square centimeter ($\mathbf{D} = 4\pi Q/A$). If now we wish to find \mathbf{E} (still in statvolts per centimeter) in terms of $\mathbf{D} = Q/A$ [for example, by means of equation (3-6)], ϵ_0 will have to be given the value of $1/(4\pi)$. Thus the rationalized stat-cgs system of units is characterized by the facts that the electric flux density is expressed in statcoulombs per square centimeter, and that $\epsilon_0 = 1/4\pi$.

Inasmuch as both electric and magnetic phenomena are to be interrelated in the same equations presently, it is plain that a common system of units must be employed. If a cgs system of units is to be employed, either the stat-cgs or the ab-cgs system might be employed but not both. One fairly common cgs system used in electromagnetic theory is the unrationalized stat-cgs system, sometimes referred to as the Gaussian system of units. In this system all ab-cgs quantities are transformed to their equivalent stat-cgs values on the basis that

$$\frac{\text{statcoulombs}}{\text{ab-coulombs}} \quad (\text{as a ratio}) = 3(10^{10}) = c,$$

where c is the velocity of light expressed in cgs units.

Although the Gaussian system of units is well suited to theoretical derivations, it is not well suited to practical calculations where the unit of electric potential is the volt and the unit of electric current is the ampere. The Gaussian system will be employed in this text in some problems so that the reader may be conversant with this system, but only after a working knowledge of the rationalized mks system is assured. Hereafter, the rationalized mks system will be referred to merely as the *mks system*.

3-3. MKS UNITS DERIVED FROM UNRATIONALIZED CGS UNITS

It appears that more of the literature is being written in mks units now than at any previous time. If we accept the premise

that the present trend will continue, this in itself is some justification for using mks units. In any case, it is the only system of units in common use in which all the following advantages can be obtained:

1. It permits the direct use of volts as a measure of emf; hence the use of \mathbf{E} in volts per meter.
2. It permits the direct use of amperes as a measure of mmf; hence the use of \mathbf{H} in amperes per meter.
3. It permits the direct use of $\mathbf{D} = Q/\mathbf{A}$ in coulombs per square meter; hence the use of $d\mathbf{D}/dt$ in amperes per square meter.
4. It ties together in a common system of units the electric and magnetic field equations.

The price to be paid for the advantages listed above is the rather awkward numerical expressions for μ_0 and ϵ_0 in the general relationships stated in equations (3-5) and (3-6). It will be shown presently that

$$\begin{aligned}\mu_0 &= 4\pi \cdot 10^{-7} \text{ henry/m (in mks units),} \\ \epsilon_0 &= \frac{1}{36\pi \cdot 10^9} \text{ farad/m (in mks units).}\end{aligned}$$

If we assume a working knowledge of the unrationalized ab-cgs system of units (where $\mu_0 = 1$) and decide that we would like to express \mathbf{H} in amperes per meter and still keep \mathbf{B} in webers per square meter, then by the use of equation (3-5) we note that

$$\frac{\mathbf{B} \text{ (in webers/m}^2\text{)}}{\mathbf{B} \text{ (in maxwells/cm}^2\text{)}} = \frac{\mu_0 \mu_r \mathbf{H} \text{ (in amperes/in)}}{1 \mu_r \mathbf{H} \text{ (in gilberts/cm)}}$$

or

$$\begin{aligned}\mu_0 \text{ (in mks)} &= \frac{\text{webers}}{\text{maxwells}} \cdot \frac{\text{gilberts}}{\text{amperes}} \cdot \frac{\text{centimeters}}{\text{meters}} \\ &= 10^{-8} \frac{4\pi}{10} 100 = 4\pi 10^{-7}.\end{aligned}$$

That μ_0 in mks units has the dimensions of henrys per meter (on a one-turn basis) may be seen as follows. From equation (3-5)

$$\frac{\text{webers}}{\text{square meters}} = \mu_0 \frac{\text{amperes}}{\text{meters}},$$

or

$$\mu_0 = \frac{\text{webers}}{\text{amperes meters}} \cdot 1$$

where, on a one-turn basis a weber of flux linking with one ampere of current constitutes the elementary definition of a henry of inductance.

If we assume a working knowledge of the unrationalized stat-cgs units (where $\epsilon_0 = 1$), and decide that we would like to express \mathbf{D} in coulombs per square meter keeping \mathbf{E} in volts per meter, then by the use of equation (3-6) we note that

$$\frac{\mathbf{D} \text{ (in coulombs/m}^2\text{)}}{\mathbf{D} \text{ (in cgs lines/cm}^2\text{)}} = \frac{\epsilon_0 \epsilon_r \mathbf{E} \text{ (in volts/m)}}{1 \epsilon_r \mathbf{E} \text{ (in statvolts/cm)'}}$$

or

$$\begin{aligned} \epsilon_0 \text{ (in mks units)} &= \frac{\text{coulombs}}{4\pi \text{ statcoulombs}} \frac{\text{statvolts}}{\text{volts}} \frac{\text{centimeters}}{\text{meters}} \\ &= \frac{1}{12\pi} \frac{1}{10^9} \frac{1}{300} 100 \\ &= \frac{1}{36\pi 10^9}. \end{aligned}$$

That ϵ_0 in mks units has the dimensions of farads per meter may be seen as follows. From equation (3-6),

$$\frac{\text{coulombs}}{\text{square meters}} = \epsilon_0 \frac{\text{volts}}{\text{meters}},$$

or

$$\epsilon_0 = \frac{\text{coulombs}}{\text{volts}} \frac{1}{\text{meters}},$$

where coulombs per volt constitutes the accepted definition of farads.

A consequence of using any rationalized system of units is that Coulomb's inverse square law must be written as

$$f = \frac{Q_1 Q_2}{4\pi \epsilon_0 \epsilon_r r^2} = \frac{Q_1 Q_2}{4\pi \epsilon_r r^2}, \quad (3-11)$$

where r is the distance of separation between the centers of charge. As applied to mks units, this law becomes

$$f = \frac{9 \cdot 10^9 Q_1 Q_2}{\epsilon_r r^2} \text{ newtons}, \quad (3-11a)$$

where Q_1 and Q_2 are the magnitudes of the charges expressed in coulombs and r is the distance of separation expressed in meters. The newton is the primary unit of force in the mks system of units and is the equivalent of 10^5 dynes or 0.2247 lb.

potential are joined to form a surface, the surface thus formed would be a surface of equal elevation above sea level. The intersections of these surfaces with the earth's surface are the well known contours employed in survey maps.

The absolute electric potential of any point in an electric field is defined as being numerically equal to the work done in bringing a unit positive charge from a region of zero potential to the point whose electric potential is being specified. Thus an electric potential field, ϕ , is a scalar field which, if specified throughout a region, may be represented by $\phi = f(x,y,z)$. Here the scalar ϕ represents electric potential and $\nabla\phi = -\mathbf{E}$ represents the electric potential

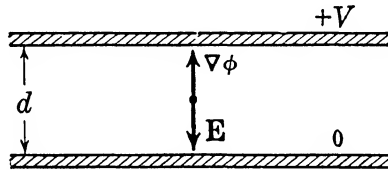


FIG. 3-1. Representation of \mathbf{E} and $\nabla\phi$ between parallel plates.

gradient. Thus $\nabla\phi$ is a vector which is directed normal to the equipotential surfaces and in the direction of increasing ϕ , whereas \mathbf{E} is, by definition, equal in magnitude to the force on the unit test charge and in the same direction as this force.

Every potential field has a gradient, and the force acting upon the test body, unit mass or unit charge, is given by the negative of this gradient. This is, $\mathbf{f} = -\nabla\phi$. That the force on the unit positive charge in an electric field is the negative of the gradient can be seen from a simple example using the condenser plates of Fig. 2-2. If the upper plate is at a potential of $+V$ volts with respect to the lower plate and the separation is d meters, the electric potential may be expressed as $\phi = Vy/d$ volts. The gradient then becomes $\nabla\phi = jV/d$ volts per meter, a vector quantity which is in the positive y direction. From elementary considerations, it is clear that a unit positive charge (a coulomb in this case) would experience a force equal to V/d newtons directed in the negative y direction. Thus,

$$\mathbf{f} = -j \frac{V}{d} = -\nabla\phi = +\mathbf{E}.$$

See Fig. 3-1.

In a corresponding manner it may be seen that whereas the

gradient in a gravitational field is directed upward, the force acting on the unit test mass is directed downward.

Indirectly, the gradient defines the spacing between the equipotential surfaces; the greater the gradient, the closer to one another are the equipotential surfaces, assuming, of course, that the equipotential surfaces in question differ in potential from one another by the same increment. An illustration of this may be seen in any good contour map.

A Note on Scalar Magnetic Potential Difference

If scalar magnetic potential difference, between two points in space, is defined in terms of the work done in moving a unit positive magnetic pole from one point to another, then scalar magnetic potential difference is analogous to scalar electric potential difference and a similar equation applies which may be written as follows.

$$\mathbf{H} = -\nabla F, \quad (3-12)$$

where F is the magnetic potential and \mathbf{H} is related to the current through equation (3-1a).

From equation (3-12) we recognize \mathbf{H} as a vector which equals in magnitude the grad F and is opposite in direction. Let this general discussion be applied to the cylindrical field surrounding the current-carrying conductor of Fig. 3-2. Here F will be assigned a zero value at the point $x = r, y = 0+$. Now proceeding a along circular path of r meters radius where $H = -I_s/2\pi r$ for a distance of $2\pi r$ meters in the positive direction, we have from equation (3-12),

$$\mathbf{H} = -\frac{\partial F}{\partial \mathbf{s}},$$

or

$$\begin{aligned} F &= -\oint \mathbf{H} \cdot d\mathbf{s} \\ &= \frac{I_s}{2\pi r} 2\pi r = I_s. \end{aligned} \quad (3-13)$$

Thus the point $(r, 0-)$ may be considered as being at a magnetic potential of I_s amperes relative to the reference of $F = 0$ at $(r, 0+)$.

Although the procedure given above is open to misinterpretation, it is useful in locating planes of equimagnetic potential. For example, if ∇F is integrated along the circular path for a distance of say $2\pi r/8$ meters, the value of F is $I_s/8$ amperes higher in mag-

netic potential than at the reference point ($x = r, y = 0+$). Since the argument in this case is independent of the magnitude of r , the plane of equimagnetic potential ($I_z/8$ amperes) is readily visualized as a plane surface, the cross section of which is indicated by the $F = I_z/8$ line in Fig. 3-2.

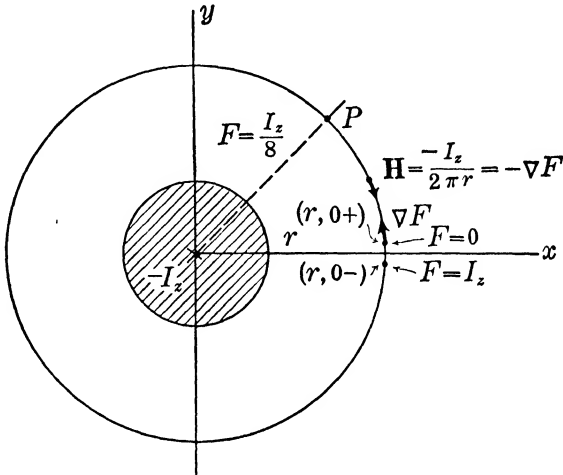


FIG. 3-2. Magnetic potential about a current-carrying conductor.

It will be shown later that the magnetic field can be described better in terms of a *vector* potential field than in terms of a scalar potential field.

3-5. EVALUATION OF ϕ FROM $\nabla\phi$ AND ONE KNOWN VALUE OF ϕ

It has been shown that the gradient of a scalar ϕ is a vector which defines the maximum space rate of increase of the point function ϕ . This vector is known if its magnitude,

$$\sqrt{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2},$$

and its direction cosines are specified, or if $\partial\phi/\partial x$, $\partial\phi/\partial y$, and $\partial\phi/\partial z$ are specified. In either case the gradient may be reduced to the form previously given:

$$\nabla\phi = \text{grad } \phi = \mathbf{i} \frac{\partial\phi}{\partial x} + \mathbf{j} \frac{\partial\phi}{\partial y} + \mathbf{k} \frac{\partial\phi}{\partial z}. \tag{3-14}$$

As in geometry where a line is determined by $df(x)/dx$ and one value of $f(x)$, it is possible to determine the scalar field ϕ , if $\nabla\phi$ and one value of ϕ are known.

Although the general procedure has been discussed previously, it will be illustrated here by means of an electric potential gradient expressed in mks units. Consider the concentric cylindrical con-

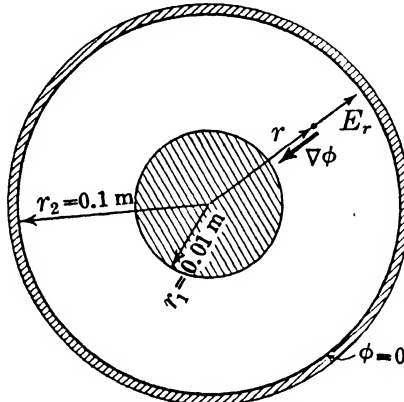


FIG. 3-3. Representation of \mathbf{E} and $\nabla\phi$ between coaxial conductors.

ductors shown in Fig. 3-3 and assume that E_r is known and that the potential ϕ of the outer conductor is zero. If

$$E_r = \frac{180}{r} = \frac{180}{\sqrt{x^2 + y^2}} \text{ v/m,}$$

then

$$\nabla\phi = \mathbf{i} \left(\frac{-180x}{x^2 + y^2} \right) + \mathbf{j} \left(\frac{-180y}{x^2 + y^2} \right) + \mathbf{k}0,$$

or from the x component,

$$\phi = \int \frac{-180x}{x^2 + y^2} dx = -90 \ln(x^2 + y^2) + c_1,$$

and from the y component,

$$\phi = \int \frac{-180y}{x^2 + y^2} dy = -90 \ln(x^2 + y^2) + c_2.$$

It is plain that $c_1 = c_2 = c$ and that at the point $(0.1,0)$, or any other point in the outer conductor, where $\phi = 0$, c may be evaluated

as

$$c = 90 \ln (0.1)^2 = -414 \text{ v,}$$

$$\phi = [-90 \ln (x^2 + y^2) - 414] \text{ v, } r_2 \geq x \geq r_1, r_2 \geq y \geq r_1.$$

If $(x^2 + y^2)$ is constant, then ϕ is constant and the cross sections of the equipotential surfaces which would appear on Fig. 3-3 would be circles, the centers of which are at the origin. This result could, of course, have been anticipated from the symmetry of the conductor surfaces. As a check on the numerical result, let us calculate the potential of the inner conductor as:

$$\phi_{0.01} = -90 \ln (0.01)^2 - 414 = (-180)(-4.6) - 414 = 414 \text{ v,}$$

and

$$\begin{aligned} \phi_{0.01} &= \int_{0.1}^{0.01} -E_x dx \\ &= \int_{0.1}^{0.01} -\frac{180}{x} dx \\ &= -180 \ln \frac{0.01}{0.10} = 414 \text{ v.} \end{aligned}$$

3-6. CONDITION FOR VECTOR \mathbf{V} TO BE $\nabla\phi$

Although every scalar potential field has a gradient associated with it, it is not true that every vector field is a gradient of some scalar field. The problem then arises as to whether or not a specified vector field can be the gradient of some scalar field. In other words, is the vector

$$\mathbf{V} = iV_x + jV_y + kV_z$$

the gradient of some scalar field ϕ ? If it is, then the scalar field ϕ may be found by methods which have been considered previously.

This problem will be approached by stating the theorem

$$\text{curl grad } \phi = \nabla \times \nabla\phi = 0. \quad (3-15)$$

The proof of this theorem is straightforward and is left as an exercise for the student. (See Prob. 3-2.) A careful consideration of equation (3-15) will show that *if the curl of a vector is zero, then the vector may be considered as $\nabla\phi$.*

Another way of approaching the problem is to note that if a scalar field exists, then

$$d\phi = \frac{\partial\phi}{\partial x} dx + \frac{\partial\phi}{\partial y} dy + \frac{\partial\phi}{\partial z} dz,$$

or

$$d\phi = V_x dx + V_y dy + V_z dz,$$

must be an exact differential in the sense that

$$\frac{\partial V_x}{\partial y} = \frac{\partial V_y}{\partial x} \quad \text{or} \quad \frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$$

$$\frac{\partial V_y}{\partial z} = \frac{\partial V_z}{\partial y} \quad \text{or} \quad \frac{\partial^2 \phi}{\partial y \partial z} = \frac{\partial^2 \phi}{\partial z \partial y}$$

$$\frac{\partial V_z}{\partial x} = \frac{\partial V_x}{\partial z} \quad \text{or} \quad \frac{\partial^2 \phi}{\partial z \partial x} = \frac{\partial^2 \phi}{\partial x \partial z}$$

This is equivalent to saying that

$$\frac{\partial V_x}{\partial y} - \frac{\partial V_y}{\partial x} = 0,$$

$$\frac{\partial V_y}{\partial z} - \frac{\partial V_z}{\partial y} = 0,$$

$$\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z} = 0.$$

or that $\text{curl } \mathbf{V}$ is zero if $d\phi$ is an exact differential, as it must be if ϕ is to represent a scalar potential field.

Accordingly the problem is solved as follows. If a vector \mathbf{V} is to be tested to determine if it can be the gradient of some scalar field ϕ , merely determine the curl of the vector \mathbf{V} . If the curl turns out to be zero then the vector in question is the gradient of ϕ and, theoretically, ϕ may be determined. In this case ϕ is said to be a lamellar field, or irrotational field. Examples of this type of field will now be given. ;

3-7. ILLUSTRATIVE EXAMPLES: THE LAMELLAR FIELD

Example 1. Given $\mathbf{V} = -ay\mathbf{j} + ax\mathbf{i}$, is this vector the gradient of some scalar field ϕ , and if so what is the form of ϕ ? A sketch of the vector field is shown in Fig. 3-4a.

$$\begin{aligned} \nabla \times \mathbf{V} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -ay & ax & 0 \end{vmatrix} \\ &= \mathbf{k}(a + a) = \mathbf{k}2a \neq 0. \end{aligned}$$

Therefore no scalar field exists of which \mathbf{V} is the gradient.

If, however, the vector \mathbf{V} is changed slightly in form to be as follows: $\mathbf{V} = iay + jax$, then it is easily determined that $\text{curl } \mathbf{V} = 0$

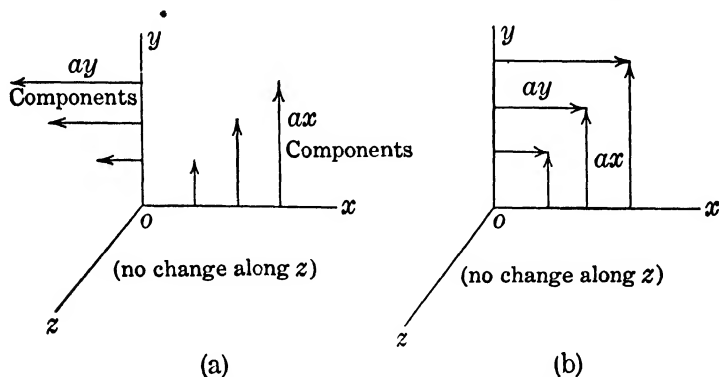


FIG. 3-4. To be used in connection with Example 1. (a) $\text{Curl} \neq 0$. (b) $\text{Curl} = 0$.

and ϕ exists. (See Fig. 3-4b.) For this case ϕ can be determined as follows:

$$\begin{aligned}\nabla\phi &= iay + jax \\ &= i \frac{\partial\phi}{\partial x} + j \frac{\partial\phi}{\partial y} + k \frac{\partial\phi}{\partial z}, \\ \frac{\partial\phi}{\partial x} &= ay, \\ \frac{\partial\phi}{\partial y} &= ax, \\ \frac{\partial\phi}{\partial z} &= 0.\end{aligned}$$

Upon integration of these three equations

$$\begin{aligned}\phi &= ayx + C_1, \\ \phi &= axy + C_2, \\ \phi &= C_3.\end{aligned}$$

Thus it is seen that $\phi = axy + C$ is the required scalar field as can be determined by finding $\nabla\phi$. Notice that C is arbitrary and thus there are innumerable fields which satisfy the conditions.

Example 2. It is known that the force on a body is given by $\mathbf{F} = -j6/y$. Find the potential field, if one exists. First find $\text{curl } \mathbf{F}$. If this is zero, then from what has been said above the gradient of ϕ will be $-j6/y$. Thus

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & \frac{-6}{y} & 0 \end{vmatrix} = 0. \quad .$$

Therefore,

$$\begin{aligned} \nabla \phi &= \mathbf{i} \frac{\partial \phi}{\partial x} + \mathbf{j} \frac{\partial \phi}{\partial y} + \mathbf{k} \frac{\partial \phi}{\partial z} \\ &= \mathbf{i}0 - \mathbf{j} \frac{6}{y} + \mathbf{k}0. \end{aligned}$$

Accordingly,

$$\begin{aligned} \frac{\partial \phi}{\partial y} &= \frac{-6}{y}, \\ \phi &= \int \left(\frac{-6}{y} \right) dy \\ &= -6 \ln y + C. \end{aligned}$$

Example 3. Given

$$\mathbf{E} = \mathbf{i} \frac{180x}{x^2 + y^2} + \mathbf{j} \frac{180y}{x^2 + y^2} + \mathbf{k}0 \text{ v/m.}$$

This is the vector which specified the negative of the potential gradient in Fig. 3-3. Its curl is

$$\begin{aligned} &\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{180x}{x^2 + y^2} & \frac{180y}{x^2 + y^2} & 0 \end{vmatrix} \\ &= 0 + 0 + \mathbf{k} \left[\frac{-180y2x}{(x^2 + y^2)^2} + \frac{180x2y}{(x^2 + y^2)^2} \right] = 0. \end{aligned}$$

Thus the test shows that ϕ exists and as shown in §3-5,

$$\phi = [-90 \ln (x^2 + y^2) - 414] \text{ v.}$$

3-8. VECTOR POTENTIALS

In the case of the electrostatic field, the scalar potential ϕ was given a very definite physical significance but certain ambiguities arose when a corresponding attempt was made for the magnetic scalar potential. Physically, one should not expect to find an exact

correlation between the electric and magnetic fields because they differ in several important respects. For example, magnetic lines of force (\mathbf{H} lines) always form closed paths whereas electric lines of force (\mathbf{E} lines) originate at positive charges and terminate at negative charges. The fact that the magnetic lines always close on themselves implies that no true magnetic charges exist; hence the divergence of the magnetic field is everywhere zero.

The concept of vector potential which is to be employed later in connection with the magnetic field is, unfortunately, not susceptible to easy physical interpretation. At least for the present, vector potential should be accepted as a useful mathematical tool which comes into being when we write

$$\operatorname{div} \operatorname{curl} \mathbf{A} = \nabla \cdot \nabla \times \mathbf{A} = 0. \quad (3-16)$$

The proof of this statement or theorem is simple and is left as an exercise for the reader. (See Prob. 3-7.)

Equation (3-16) is used to define *vector potential* in analogy to the scalar potential which was defined by equation (3-15). Thus if the divergence of a vector, \mathbf{B} for example, is zero, then \mathbf{B} may be considered as the curl of some vector field \mathbf{A} . The field \mathbf{A} is the *vector potential field*. A field of this kind is called *solenoidal*. In many cases the vector potential may be determined in a manner analogous to that used to determine scalar potentials, but it must be recognized that scalar and vector potentials, although they may be derived from the same vector field, are not at all similar. All magnetic fields are solenoidal since $\nabla \cdot \mathbf{B} = 0$. See equation (2-7).

Consider the \mathbf{B} field which is associated with the \mathbf{H} field shown in Fig. 3-2. In this case $H = -I_z/2\pi r$ amp/m, where the minus sign indicates that current is directed along the $-z$ axis.

$$\mathbf{B} = \mathbf{i} \frac{\mu I_z y}{2\pi(x^2 + y^2)} + \mathbf{j} \frac{-\mu I_z x}{2\pi(x^2 + y^2)} + \mathbf{k} 0 \text{ webers/m}^2,$$

$$\nabla \cdot \mathbf{B} = \frac{-2\mu I_z yx}{2\pi(x^2 + y^2)^2} + \frac{2\mu I_z xy}{2\pi(x^2 + y^2)^2} = 0.$$

A vector potential \mathbf{A} exists, therefore, and

$$\nabla \times \mathbf{A} = \mathbf{B}$$

$$= \mathbf{i} \frac{\mu I_z y}{2\pi(x^2 + y^2)} + \mathbf{j} \frac{-\mu I_z x}{2\pi(x^2 + y^2)} + \mathbf{k} 0.$$

The vector \mathbf{A} is defined to the extent that we know now that

$$\begin{aligned}\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial z} &= \frac{\mu I_z y}{2\pi(x^2 + y^2)}, \\ \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} &= \frac{-\mu I_z x}{2\pi(x^2 + y^2)}, \\ \frac{\partial A_y}{\partial x} - \frac{\partial A_z}{\partial y} &= 0.\end{aligned}$$

If now we arbitrarily select $A_x = A_y = 0$ throughout the field, we find that

$$A_z = \frac{\mu I_z}{4\pi} \ln(x^2 + y^2),$$

and that

$$\mathbf{A} = i0 + j0 + \mathbf{k} \frac{\mu I_z}{4\pi} \ln(x^2 + y^2) \text{ webers/m.}$$

Therefore, the loci of the equivector potentials are circles, the centers of which are at the origin. Since the magnetic vector potential is defined indirectly as $\nabla \times \mathbf{A} = \mathbf{B}$ in this case, we know that \mathbf{A} is dimensionally equivalent to magnetic flux per unit length. As a result of the definition $\nabla \times \mathbf{A} = \mathbf{B}$, we find that \mathbf{A} is closely related to the magnetic flux linking the conductor per unit length of conductor, since

$$\begin{aligned}\phi_{ext} \text{ per u.l.} &= \int_{r_1}^r B \, dr \\ &= \int_{r_1}^r \frac{\mu I_z}{2\pi r} \, dr \\ &= \frac{\mu I_z}{2\pi} \ln r - \frac{\mu I_z}{2\pi} \ln r_1,\end{aligned}$$

where r_1 is the radius of the conductor. Thus the magnitude of \mathbf{A} in this case is

$$A = \phi_{ext} \text{ per u.l.} + \text{a constant,}$$

where the constant is $\mu I_z/2\pi \ln r_1$. Considered as a vector, \mathbf{A} is directed along the z axis, that is, opposite in direction to the current source.

If the magnetic vector potential had been defined in such a manner that

$$\nabla \times \mathbf{A}' = \mathbf{H},$$

then

$$\mathbf{A}' = i0 + j0 + \mathbf{k} \frac{Iz}{4\pi} \ln(x^2 + y^2) \text{ amp.}$$

The magnetic vector potential \mathbf{A}' is dimensionally equivalent to current, and it is in this form that magnetic vector potential will be encountered later. It should be noted, however, that the \mathbf{A}' employed here implies that the current source (I) takes the form of a steady or direct current or, at the most, a relatively low-frequency alternating current. As might be anticipated, certain modifications in \mathbf{A}' are necessary when the current source alternates rapidly.

3-9. ILLUSTRATIVE EXAMPLES: VECTOR POTENTIAL

Example 4. Determine the vector potential corresponding to the field $\mathbf{V} = ix + jy - k2z$. First, to test for the existence of a vector potential, find the divergence.

$$\nabla \cdot \mathbf{V} = 1 + 1 - 2 = 0.$$

Thus a vector potential exists and

$$\nabla \times \mathbf{F} = ix + jy - k2z.$$

From equation (2-10):

$$(1) \quad \frac{\partial F'_z}{\partial y} - \frac{\partial F'_y}{\partial z} = x,$$

$$(2) \quad \frac{\partial F'_x}{\partial z} - \frac{\partial F'_z}{\partial x} = y,$$

$$(3) \quad \frac{\partial F'_y}{\partial x} - \frac{\partial F'_x}{\partial y} = -2z.$$

It is seen immediately that \mathbf{F} is arbitrary to a great extent. Thus F'_y , for example, can be set equal to zero. Then

$$\begin{aligned} \frac{\partial F'_z}{\partial y} &= x, \\ -\frac{\partial F'_x}{\partial y} &= -2z. \end{aligned}$$

Therefore,

$$\begin{aligned} F'_z &= xy + f(x,z), \\ F'_x &= 2yz + f'(x,z). \end{aligned}$$

The two equations satisfy (2) if f and f' are the same functions.

Thus,

$$\mathbf{F} = \mathbf{i}[2yz + f(x,z)] + \mathbf{j}0 + \mathbf{k}[xy + f(x,z)].$$

The simplest result is

$$\mathbf{F} = \mathbf{i}2yz + \mathbf{k}xy.$$

Example 5. $\mathbf{V} = \mathbf{j}z + \mathbf{k}y$. Find both a scalar field and vector field associated with \mathbf{V} . The test for scalar field is $\text{curl } \mathbf{V} = 0$, which is easily seen to be true. It is evident also that $\text{div } \mathbf{V} = 0$. Thus both potential fields exist.

For the scalar field:

$$\begin{aligned}\nabla\phi &= \mathbf{i} \frac{\partial\phi}{\partial x} + \mathbf{j} \frac{\partial\phi}{\partial y} + \mathbf{k} \frac{\partial\phi}{\partial z} \\ &= \mathbf{i}0 + \mathbf{j}z + \mathbf{k}y, \\ \frac{\partial\phi}{\partial x} &= 0, \\ \frac{\partial\phi}{\partial y} &= z, \\ \frac{\partial\phi}{\partial z} &= y.\end{aligned}$$

Therefore,

$$\phi = zy + C.$$

For the vector field

$$\text{curl } \mathbf{F} = \mathbf{i}0 + \mathbf{j}z + \mathbf{k}y.$$

From equation (2-10):

$$\begin{aligned}(1) \quad & \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} = 0, \\ (2) \quad & \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} = z, \\ (3) \quad & \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = y.\end{aligned}$$

Again, an arbitrary selection may be made and F_y and F_z will both be taken as constants, C_1 and C_2 . Then,

$$\begin{aligned}\frac{\partial F_x}{\partial z} &= z, \\ -\frac{\partial F_x}{\partial y} &= y.\end{aligned}$$

Therefore,

$$F_z = \frac{z^2}{2} + f(x,y),$$

$$F_z = \frac{-y^2}{2} + f'(x,z).$$

These two values of F_z will be the same if $f(x,y) = -y^2/2$ and $f'(x,z) = z^2/2$. Then,

$$F_z = \frac{z^2}{2} - \frac{y^2}{2} + C_3.$$

Accordingly,

$$\mathbf{F} = \mathbf{i} \left(\frac{z^2 - y^2}{2} + C_3 \right) + \mathbf{j}C_1 + \mathbf{k}C_2.$$

It is to be noted that there is no similarity between these two potential functions and also that a great number of vector potentials can be found. The vector potential is determined uniquely only if six partial derivatives and one value of the potential are given.

3-10. GAUSS'S THEOREM

Gauss's theorem relates the volume integral of the divergence to the surface integral of the vector over the volume. It is a theorem that can be visualized quite easily as follows. It has been seen that the term $\text{div } \mathbf{V}$ indicates the excess of outward flux over inward flux per unit volume. Thus the total excess of outward flux over any volume is $\int \nabla \cdot \mathbf{V} dv$ where dv is the differential of the volume. This excess of outward flux must, of course, flow through the surface of the volume in question and, accordingly, must equal the integral of the normal flux over the surface. This is expressed as $\int_s \mathbf{V} \cdot d\mathbf{a}$ where $d\mathbf{a}$ is the differential of area and the integral with the s indicates integration over the entire surface. Thus it is seen that Gauss's theorem may be expressed

$$\int_s \mathbf{V} \cdot d\mathbf{a} = \int \nabla \cdot \mathbf{V} dv. \quad (3-17)$$

This theorem is also known as the *divergence theorem*.

3-11. ILLUSTRATIVE EXAMPLE: GAUSS'S THEOREM

Example 6. A simple illustration of Gauss's theorem that will aid in visualizing its nature is the following. Assume a sphere about the origin of coordinates as center, having a radius of 0.5 m. Assume that the sphere is filled with an electron gas so that the

electric charge density is -0.0010 coulomb/m³. It is clear, to begin with, that the number of *mk*s lines of electric flux, or coulombs, entering radially into the sphere per square meter will be

$$\frac{\frac{4}{3}\pi r^3(-0.0010)}{4\pi r^2} = -0.000167 \text{ coulomb/m}^2.$$

The negative sign indicates that the electron gas is a sink. The divergence of the vector field is

$$\nabla \cdot \mathbf{V} = -0.001 \text{ coulomb/m}^3.$$

According to equation (3-17), this is to be integrated over the volume. This is simply

$$(-0.001)\frac{4}{3}\pi(0.5)^3 = -0.000523.$$

Then,

$$\int_s \mathbf{V} \cdot d\mathbf{a} = -0.000523.$$

Since the flux is all perpendicular to the surface, this can be written simply

$$\int_s V da = V(4\pi)(0.5)^2 = -0.000523.$$

Accordingly,

$$V = \frac{-0.000523}{\pi} = -0.000167 \text{ coulomb/m}^2,$$

which is the same as given in the first equation in this section.

3-12. STOKES'S THEOREM

It has been explained in Chapter II that the *curl* is a vector quantity associated with a point and that its magnitude can be determined by taking the limit of the ratio of the *line integral of the vector around an area* to the *area itself*. Occasionally there arises the problem of determining the surface integral of the curl of some vector. The above statement suggests a way of getting the answer, not by performing the integration on the curl, but merely by integrating around the boundary of the area. Stokes's theorem states the equality of these two integrals as

$$\int (\nabla \times \mathbf{E}) \cdot d\mathbf{a} = \oint \mathbf{E} \cdot d\mathbf{s}. \quad (3-18)$$

This theorem will be proved in two dimensions by the use of Fig. 3-5. This will hold for one component. The other two could be derived by a repetition of the procedure. In Fig. 3-5 let *C* be

the boundary of the surface under consideration and let $oabco$ be an elementary rectangle on the surface whose sides are dx and dy . First determine the line integral of \mathbf{E} around this rectangle, proceeding in a counterclockwise direction. Along oa we have $E_x dx$. Along ab the field is that at o plus the change due to a transfer to the right through a distance dx . This is $E_y + (\partial E_y / \partial x) dx$. Then the integral along ab is $[E_y + (\partial E_y / \partial x) dx] dy$. We proceed along bc in a negative direction and the integral is, in a similar manner,

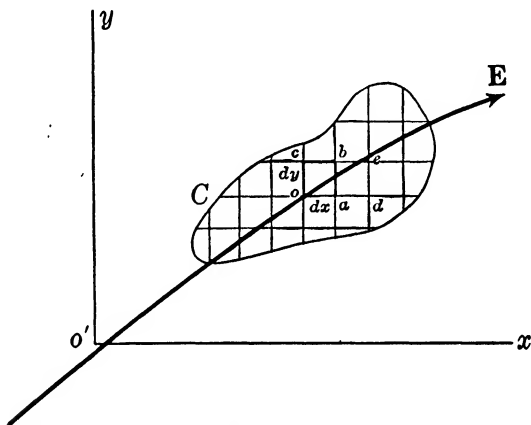


FIG. 3-5. Illustrating $\int \nabla \times \mathbf{E} \cdot d\mathbf{a} = \oint \mathbf{E} \cdot d\mathbf{s}$.

$-[E_x + (\partial E_x / \partial y) dy] dx$. The integral along co is $-E_y dy$. The integral around the entire rectangle becomes:

$$E_x dx + \left(E_y + \frac{\partial E_y}{\partial x} dx \right) dy - \left(E_x + \frac{\partial E_x}{\partial y} dy \right) dx - E_y dy = \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) dx dy, \quad (3-19)$$

where $dx dy$ is the area of the rectangle. Dividing by this area we have the curl (in this case the z component), as we have seen before. Equation (3-19) expresses the curl multiplied by an elementary area. This is equal to the line integral of \mathbf{E} around the area. Now, suppose that we wish to integrate the curl over the surface C in Fig. 3-5. We begin by adding one more rectangle, $adeb$. The new line integral will equal the sum of the line integrals around the two rectangles, but we see immediately that the line integrals along ab will cancel so that the total line integral is only that around $oadebco$. This process of adding rectangles can be continued

indefinitely until the boundary C is reached. At that point it is found that the summation or integral of the curl over the entire surface turns out to be the line integral around the boundary, and the effects along all of the internal common boundaries have canceled out. Therefore, equation (3-18) may be expressed.

3-13. ILLUSTRATIVE EXAMPLES: STOKES'S THEOREM

Example 7. Consider the curl of the velocity of water in the river as described in §2-9, Example 12. It is given as $\nabla \times \mathbf{V} = -\mathbf{k}m$. Assume a section of the river to be 1 m long, reaching from the bottom where the velocity is zero to the surface where the velocity is my . Let $y = 1.4$ meters, and $m = 2$. Then the integral

$$\oint \nabla \times \mathbf{V} \cdot d\mathbf{a} = -2(1)(1.4) = -2.8,$$

because the vector $d\mathbf{a}$ is parallel to the unit vector \mathbf{k} . The integral of \mathbf{V} in the positive direction around this area is as follows:

along the bottom	$V(1) = 0(1) = 0$
upward on downstream side	$0(1.4) = 0$
upstream at the surface	$-2(1.4)(1) = -2.8$
downward on upstream side	$0(1.4) = 0$
	-2.8

The total integral is -2.8 , agreeing with the integral of the curl.

Example 8. Refer to §2-9, Example 14. The curl is constant and equal to $I/\pi a^2$. Its integral over the cross section of the wire is simply the product and is $(I/\pi a^2)(\pi a^2) = I$. Thus $\oint \mathbf{H} \cdot d\mathbf{s}$ (around the surface of the wire) $= H(2\pi a) = I$, by Stokes's theorem. Therefore, $H = I/2\pi a$, agreeing with the known value of the field at this point.

These two illustrations do not demonstrate the usefulness of the theorem but merely serve to aid in a better understanding of its meaning. Better applications will be given later.

3-14. MAXWELL'S FIRST LAW

For the development of Maxwell's equations it is advisable to resort to certain experimentally determined results. For his first law we will make use of equation (3-1a) which is equivalent to Ampere's law. Thus we have at the outset:

$$\oint \mathbf{H} \cdot d\mathbf{s} = \quad (3-20)$$

The current I is the total current flowing through or linking the path of integration and, in general, may be written

$$I = \int_s \mathbf{J} \cdot d\mathbf{a} \quad (3-21)$$

as a surface integral of current density \mathbf{J} . Thus equation (3-20) becomes

$$\oint \mathbf{H} \cdot d\mathbf{s} = \int_s \mathbf{J} \cdot d\mathbf{a}. \quad (3-22)$$

Now from Stokes's theorem, the left part of this equation may be replaced by the integral of the curl with the following result:

$$\int_s \nabla \times \mathbf{H} \cdot d\mathbf{a} = \int_s \mathbf{J} \cdot d\mathbf{a}. \quad (3-23)$$

The two equal integrals are taken over the same area so that, necessarily, the integrands must be equal as follows:

$$\nabla \times \mathbf{H} = \mathbf{J}. \quad \checkmark \quad (3-24)$$

This states that the curl \mathbf{H} at *any point* is equal to the current density *at that point*. If there is no current, the curl is zero and the field is irrotational or lamellar.

At this point an example will be useful in further discussion. Let there be assumed a magnetic field $\mathbf{H} = -iy + jx$. Find the current density at all points.

$$\nabla \times \mathbf{H} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = \mathbf{k}2 = \mathbf{J},$$

from which $\mathbf{J} = \mathbf{k}2$. Thus, for this case the current density is positive and constant over all space.

In this simple example it is noted that, assuming a certain \mathbf{H} , a current in the direction of positive z axis was obtained. A careful consideration of this will show that on account of conventions introduced earlier, the well-known right-hand rule is adhered to. Consider Fig. 3-6. By sketching a few \mathbf{H} vectors, it is seen that the direction of \mathbf{H} is counterclockwise about the z axis. Application of the right-hand rule indicates that the current producing this field should be along the positive z axis, as it is shown to be by the equation. Thus the results of equation (3-24) are in agreement with the usual sign convention.

Equation (3-24) expresses Maxwell's first law. However, Maxwell claimed that this equation is fundamental to general electromagnetic theory only if \mathbf{J} is the *total current density* for either varying or nonvarying fields. In other words, \mathbf{J} must, in general, be made up of the sum of the *conduction* current and the *displacement* current. It is now necessary to determine this total current density so that equation (3-24) can be written in its general form. The total current density \mathbf{J} is defined in such a way that $\text{div } \mathbf{J} = 0$. That $\text{div } \mathbf{J}$ must equal zero is seen from the fact that

$$\text{div } \mathbf{J} = \nabla \cdot \nabla \times \mathbf{H} = 0.$$

The conduction current density \mathbf{i} is, of course, part of this total current density.* That the $\text{div } \mathbf{i} \neq 0$ (and thus \mathbf{i} itself) cannot meet

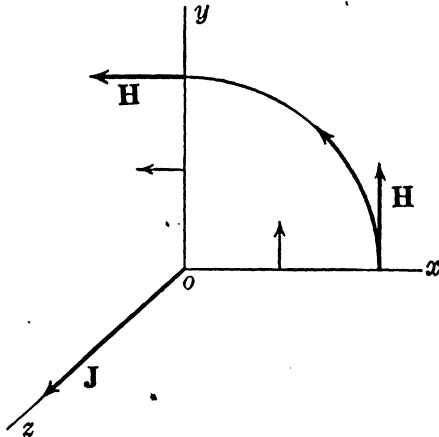


FIG. 3-6. Sign convention in Maxwell's first law.

the requirement for \mathbf{J} , is clear when one considers a simple condenser. If a surface C , Fig. 3-7, is drawn about one plate of a condenser, it is clear that when the condenser is charging, (dE/dt finite) current is flowing onto the plate through the surface C and thus charge is accumulating, but no conduction current flows out through the surface. Accordingly, for this simple case, the divergence is not zero. If \mathbf{i} does not meet the requirement in even one case, it is not suitable for the total current density \mathbf{J} . Thus we can say that the true current density \mathbf{J} is

$$\mathbf{J} = \mathbf{i} + \mathbf{I},$$

* In these sections \mathbf{i} is being used as conduction current density. This will not cause confusion because the unit vector \mathbf{i} does not occur here.

where \mathbf{I} is now some current density which must be added to \mathbf{i} in order to meet the conditions, and which is to be determined.

Accordingly, we may write

$$\nabla \times \mathbf{H} = \mathbf{i} + \mathbf{I}.$$

Taking the divergence of both sides and rearranging, remembering that $\nabla \cdot \nabla \times \mathbf{H} = 0$, we obtain

$$\nabla \cdot \mathbf{i} = -\nabla \cdot \mathbf{I}.$$

Now \mathbf{i} is the conduction current density and by the definition of divergence, the term $\nabla \cdot \mathbf{i}$ means the net current outflow from a unit volume. Current is the time rate of change of charge and,

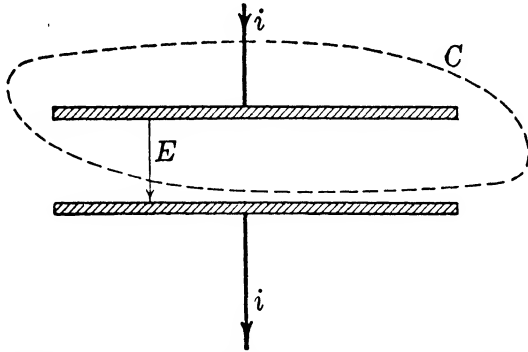


FIG. 3-7. Example of the continuity of current through the surface C .

since charge must be conserved, this net current outflow must equal the rate of decrease of charge in the unit volume. This is expressed by

$$\frac{-\partial \rho}{\partial t}$$

where ρ is the charge density and the negative sign indicates a decrease. Thus

$$\nabla \cdot \mathbf{i} = \frac{-\partial \rho}{\partial t}.$$

Therefore,

$$\nabla \cdot \mathbf{I} = -\nabla \cdot \mathbf{i} = \frac{\partial \rho}{\partial t}.$$

From equation (2-5),

$$\nabla \cdot \mathbf{D} = \rho.$$

From which

$$\frac{d\rho}{dt} = \nabla \cdot \frac{\partial \mathbf{D}}{\partial t} = \nabla \cdot \mathbf{I}.$$

Thus

$$\mathbf{I} = \frac{\partial \mathbf{D}}{\partial t}.$$

From this information we can write

$$\nabla \times \mathbf{H} = \mathbf{i} + \frac{\partial \mathbf{D}}{\partial t}.$$

Two simple changes will now convert this equation into its usual form. The conduction current \mathbf{i} is the ordinary electron flow and is given by $\mathbf{i} = \sigma \mathbf{E}$, where σ is the conductivity of the medium and \mathbf{E} is the electric field intensity. \mathbf{D} is the flux density and is written $\epsilon \mathbf{E}$, where \mathbf{E} has the same meaning as above and ϵ is the dielectric constant. By making these two substitutions, equation (3-24) can be written

$$\nabla \times \mathbf{H} = \sigma \mathbf{E} + \epsilon \frac{\partial \mathbf{E}}{\partial t}. \quad (3-25)$$

This is the usual differential form of Maxwell's first law.

3-15. MAXWELL'S SECOND LAW

The second law of Maxwell has already been touched upon in Chapter II and is really an expression of fundamental definitions. Mathematically, it is expressed as

$$\nabla \cdot \mathbf{D} = \rho. \quad (3-26)$$

The reasoning which lies at the basis of this law is as follows. If ρ is the volume charge density in space, then in each unit volume under consideration there will be ρ units of charge. According to our fundamental ideas each unit charge gives rise to one unit of electric flux, or one *coulomb* of flux. Thus out of the unit volume, there will flow ρ lines in excess of those that may be entering. However, this is the definition of divergence which makes it possible to write an equality between $\nabla \cdot \mathbf{D}$ and ρ . Note that ρ is the net density of positive charge. In the case of electrons, ρ would be written as a negative quantity whereupon the divergence becomes negative. A point where the divergence is negative is called a *sink*, i.e., a point where lines disappear. If the divergence is positive, the point is called a *source*.

Maxwell's second law is really contained in his first law, since the second may be derived from equation (3-24) as follows. By

taking the divergence of (3-24),

$$\nabla \cdot \nabla \times \mathbf{H} = 0 = \nabla \cdot \mathbf{J},$$

where \mathbf{J} is the total current density. Accordingly,

$$\nabla \cdot \mathbf{J} = \nabla \cdot \left(\mathbf{i} + \frac{\partial \mathbf{D}}{\partial t} \right) = 0,$$

or

$$\frac{\partial}{\partial t} \nabla \cdot \mathbf{D} = -\nabla \cdot \mathbf{i} = \frac{\partial \rho}{\partial t}.$$

On integrating and rearranging we obtain

$$\nabla \cdot \mathbf{D} = \rho + C.$$

The constant C is evaluated by taking a special case of one unit charge in a unit spherical volume. The number of lines (coulombs) leaving the small sphere is unity, and by definition the divergence is equal to this quantity. Thus $C = 0$. This development is a repetition of the previous argument in a slightly different form and amounts to the same thing.

3-16. MAXWELL'S THIRD LAW

The third law of Maxwell relates the curl of the electric field intensity to the time rate of change of flux density. This at once brings to mind that basic experiment in which an emf is generated by changing a magnetic field. See equation (3-2). This fundamental experiment will be used to derive the equation which expresses the third law. In Fig. 3-8 let C be a single turn of wire, which may have infinite resistance as far as we are concerned, oriented in any way in space. Let there be an arbitrarily oriented magnetic field which cuts this loop. It is shown in the diagram as being positive in direction. If the magnetic field is changing, the resultant electric field will fill all the space, and the configuration may be illustrated as shown where \mathbf{B} and \mathbf{E} represent the general directions of the fields.

First, the total emf around the loop in the positive direction will be determined. This is clearly the line integral

$$\oint \mathbf{E} \cdot d\mathbf{s}.$$

If this is to be related to the rate of change of the magnetic field, we know from the experiment mentioned above that the same emf

will be given by the expression

$$-\frac{\partial \phi}{\partial t}.$$

It is necessary to inspect this expression carefully to see that it is indeed the same as the line integral. In the first place, $\partial \phi / \partial t$ is the emf generated in a single turn by the changing flux, in *volts*. Second, if ϕ is increasing (a positive derivative) it means that flux lines are cutting through the wire toward the interior of the loop and the generated voltage is around the loop in a clockwise or negative

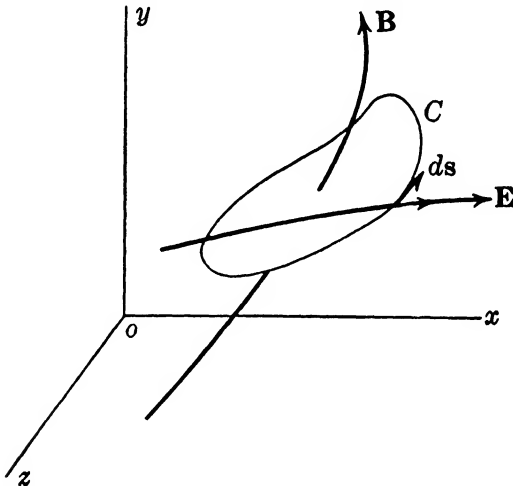


FIG. 3-8. Illustrating $\oint \mathbf{E} \cdot d\mathbf{s} = -\partial \phi / \partial t$.

direction. Therefore, in order to obtain the same emf a negative sign must be used.

With these adjustments, it is then possible to write the equality

$$\oint \mathbf{E} \cdot d\mathbf{s} = -\frac{\partial \phi}{\partial t}. \tag{3-27}$$

The left side of this equation will be changed by using Stokes's theorem and for ϕ will be substituted its equivalent

$$\int_s \mathbf{B} \cdot d\mathbf{a}.$$

Making these changes, we have

$$\int_s \nabla \times \mathbf{E} \cdot d\mathbf{a} = -\int_s \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a}. \tag{3-28}$$

These two integrals are performed over the area of the loop in both cases and accordingly the integrands are equal. Therefore,

$$\nabla \times \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t} = -\mu \frac{\partial \mathbf{H}}{\partial t}. \quad (3-29)$$

This is the expression of Maxwell's third law. It states that wherever there exists a time rate of change of flux density, there will be a consequent electric field intensity produced.

It happens that an effective rate of change of flux may be caused in one or both of two ways: either by a real change in flux density

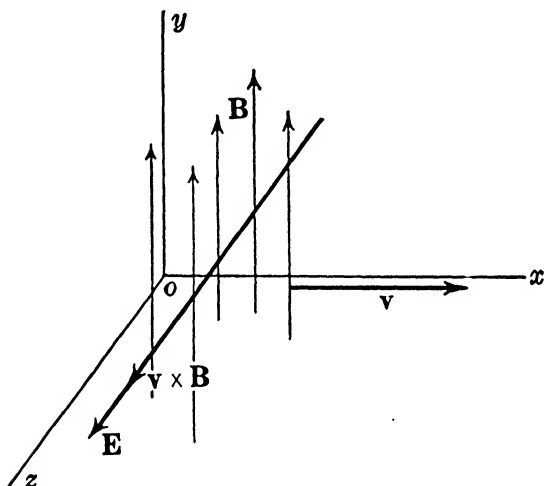


FIG. 3-9. For use in deriving equation (3-31).

with respect to time, or by a movement or velocity of the coordinate system through the magnetic field. In general, then, a term involving this velocity should be added to the right-hand side of equation (3-29). Let a magnetic field and wire be represented as shown in Fig. 3-9 where the coordinate system and wire are to have a velocity in the x direction of \mathbf{v} . For a stationary system

$$\nabla \times \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t}. \quad (3-30)$$

Assume that $\partial \mathbf{B} / \partial t$ is positive and equal to \mathbf{A} , then the electrical field intensity induced in the wire is \mathbf{E} as shown, in the positive z direction. That this is true may be seen by taking the special case already illustrated where

$$\begin{aligned}\nabla \times \mathbf{E} &= -\mathbf{j}A_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & E_z \end{vmatrix} \\ &= \mathbf{i} \frac{\partial E_z}{\partial y} - \mathbf{j} \frac{\partial E_z}{\partial x},\end{aligned}$$

or

$$\frac{\partial E_z}{\partial y} = 0, \quad E_z = C + f(x, z),$$

and

$$\frac{\partial E_z}{\partial x} = A_v, \quad E_z = A_v x + f(y, z),$$

or simply

$$E_z = A_v x + C,$$

which is a positive quantity. A motion of the coordinate system will not affect $\partial \mathbf{B} / \partial t$. However, a velocity \mathbf{v} of the system through the field in the x direction will induce in the wire an emf, $(\mathbf{v} \times \mathbf{B})$, which will be in the same direction as \mathbf{E} . Thus the actual field intensity generated is

$$\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}.$$

Solve this equation for \mathbf{E} and obtain

$$\mathbf{E} = \mathbf{E}' - \mathbf{v} \times \mathbf{B}.$$

By substituting this into equation (3-30) and rearranging, there is obtained the general expression

$$\nabla \times \mathbf{E}' - \nabla \times \mathbf{v} \times \mathbf{B} = -\frac{\partial \mathbf{B}}{\partial t},$$

or

$$\nabla \times \mathbf{E}' = -\left[\frac{\partial \mathbf{B}}{\partial t} - \nabla \times \mathbf{v} \times \mathbf{B} \right]. \quad (3-31)$$

3-17. MAXWELL'S FOURTH LAW

Like the second law, this one has also been touched upon in previous sections. It is the expression

$$\nabla \cdot \mathbf{B} = 0, \quad (3-32)$$

which states merely the fact that flux lines are continuous. In other words the number of flux lines entering a given volume of

space must equal the number leaving. Accordingly, the divergence, which expresses the excess of lines leaving over those entering, is zero. The basis of this law is really the fundamental definition of flux as met with in physics and elementary electrical engineering courses.

Maxwell's fourth law is contained in his third as shown in the next expression. Take the divergence of both sides of equation (3-30) and obtain

$$\nabla \cdot \nabla \times \mathbf{E} = 0 = -\frac{\partial}{\partial t} (\nabla \cdot \mathbf{B}),$$

or

$$\frac{\partial}{\partial t} (\nabla \cdot \mathbf{B}) = 0.$$

From this, on integration, it appears that

$$\nabla \cdot \mathbf{B} = \text{constant} = C_1.$$

However, to evaluate C_1 , it is merely necessary to note that there must have been a time in the past when \mathbf{B} was zero and without divergence. Therefore, the divergence must still have that value (since it is a constant) and $C_1 = 0$. Accordingly,

$$\nabla \cdot \mathbf{B} = 0.$$

3-18. PROBLEMS

3-1. What is the mmf rise once around the C path of Fig. 3-10, expressed in gilberts? in amperes? Assume that 60% of the total

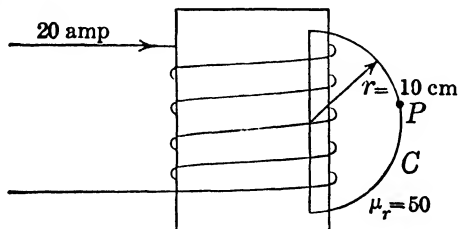


FIG. 3-10. For use in connection with Prob. 3-1.

magnetic potential rise takes place along the straight portion of the C path and that the magnitude of the \mathbf{H} vector is constant over the semicircular part of the path, and find the flux density in webers per square meter at point P in the figure. (It is recognized that the

assumptions made here are not accurately fulfilled in a practical case.)

3-2. Prove that $\text{curl grad } \phi = 0$.

3-3. Find ϕ if $\text{grad } \phi = j6/y$. $\phi = 100$ v when $y = 0.01$ m.

3-4. $\mathbf{V} = ix + jy + kz$. Is this the gradient of a scalar field ϕ and if so, find ϕ .

3-5. $\mathbf{V} = i \frac{x}{\sqrt{x^2 + y^2}} + j \frac{y}{\sqrt{x^2 + y^2}} + k0$. Find a scalar field, if it exists, of which this is the gradient.

3-6. $\mathbf{V} = i10 + j0 + k0$. Find the scalar field of which this is the gradient with the restriction that it must be zero for $x = y = z = 0$.

3-7. Prove that $\text{div curl } \mathbf{F} = 0$.

3-8. Find the magnetic vector potential, based on \mathbf{H} , at a point 0.2 m from the center of a wire carrying a current of 20 amp.

3-9. Find the scalar and vector potential fields for the constant vector, $\mathbf{V} = i5 + j5$.

3-10. Find the vector field \mathbf{F} for $\mathbf{V} = jz + ky$ if it is required that $F_x = F_y = 2$.

3-11. Assume that a sphere of radius a meters is filled with an electron gas in such a manner that the divergence of the flux lines, $\text{div } \mathbf{D} = mr$, where m is a constant and r is the distance from the center. Using Gauss's theorem, find the normal flux per unit area at the surface of the sphere.

3-12. Show that Stokes's theorem is true for a closed path outside of and not including a wire carrying a current I . Let the path be made up of two radial lines and the connecting arcs such that the included angle is about 30° .

3-13. Assume that \mathbf{H} is given at all points of space by the equation $\mathbf{H} = iy$. Find the current density and sketch a few \mathbf{H} and \mathbf{J} vectors.

3-14. Given a section of a conducting cylinder shown in Fig. 3-11. Between the plane faces A and B a constant potential difference V is maintained. Find the distribution of current density in the material.

3-15. If, in Prob. 3-14, the total current is 500 amp, find the current density at a point 1.25 in. from the center of the cylinder.

3-16. Suppose that all space is filled with a magnetic field such that $\mathbf{B} = iat^2$. Find a possible electric field produced, with the restriction that $E_x = E_s = 0$.

3-17. Using the more general equation involving velocity of the coordinate system find a possible electric field produced if $\mathbf{B} = k\mathbf{a}$ and $\mathbf{v} = iv_0$. Solve also for $\mathbf{B} = kat$ and the same velocity. Let $E_x = E_z = 0$.

3-18. Show that in a parallel plate air condenser the current flowing into the condenser under the influence of a voltage $V \sin \omega t$ is equal to the displacement current in the space between the

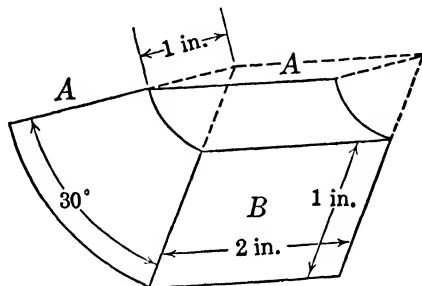


FIG. 3-11. For use in connection with Prob. 3-14.

plates. Let dimensions of the condenser be: area, A square meters, and plate spacing, d meters.

3-19. Given a condenser with two plates separated by a dielectric. The area of the plate is A square meters, and the distance between them is d meters. The dielectric constant is $\epsilon_r = 3$, and $\sigma = 10^{-17}$ mho/m³. A voltage $V = at$ is applied across the plates. Find the total current, the conduction current through the dielectric, and the displacement current.

PLANE WAVES

Summary

Beginning with the Laws of Maxwell, the wave equation for a simple plane wave is derived and solved. Its transmission characteristics are determined, and it is shown how the magnetic component of the field may be found from the electric component, and vice versa. The subjects of energy in the field and power flow are considered, and Poynting's radiation vector is introduced.

4-1. THE WAVE EQUATION

The necessary equations have now been derived to allow an introduction to the study of the simplest of the electromagnetic waves, the plane wave. This will be approached by using Maxwell's equations and other ideas which will be introduced as they are needed.

Maxwell's equations are repeated here for convenience.

$$\nabla \times \mathbf{H} = \sigma \mathbf{E} + \epsilon \frac{\partial \mathbf{E}}{\partial t} \quad (4-1)$$

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \quad (4-2)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (4-3)$$

$$\nabla \cdot \mathbf{D} = \rho \quad (4-4)$$

It is assumed generally that these equations are well established. Most discussions begin by considering them as fundamental and acceptable without argument, such as the relation $2 + 2 = 4$. An inspection of these equations in general leads us to the following initial conclusions.

1. Wherever there exists a changing electric field, there must also exist a magnetic field, and vice versa [from equations (4-1) and (4-2)].

2. The magnetic field is solenoidal, that is, magnetic lines do not end.

3. Electric flux lines terminate on charges and the electric field is solenoidal only where the charge density is zero.

The important point here is that electric and magnetic fields produce each other. From this fact alone one might suspect that there would occur some sort of propagation. Thus far, however, we have no quantitative information on such a propagation, and the details must be obtained from the mathematical development.

These equations really involve six possible dependent variables, \mathbf{E} , \mathbf{H} , ϵ , μ , σ , and ρ . However, for obtaining the first general picture of the wave nature of the electromagnetic field, it will be assumed that $\rho = \sigma = 0$ and that $\mu_r = \epsilon_r = 1$; that is, μ and ϵ have the values of free space. Thus we are, in effect, localizing the possible phenomena to space containing no matter, such as interstellar space, but for all practical purposes space containing air only is essentially equivalent to free space so far as electromagnetic wave propagation is concerned. The two variables \mathbf{E} and \mathbf{H} are too many because, for simple presentations, we would like only one dependent variable. Accordingly, one of these will be eliminated by the following process.

For this simplified case, the equations (4-1) and (4-2) reduce to

$$\nabla \times \mathbf{H} = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \quad (4-5)$$

$$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}. \quad (4-6)$$

First, take the curl of equation (4-6) and then, recognizing that the order of differentiation does not matter, the following is written.

$$\nabla \times (\nabla \times \mathbf{E}) = -\mu_0 \frac{\partial}{\partial t} (\nabla \times \mathbf{H}). \quad (4-7)$$

Equation (4-5) may be substituted into this. Then,

$$\nabla \times (\nabla \times \mathbf{E}) = -\mu_0 \frac{\partial}{\partial t} \left(\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) = -\frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}, \quad (4-8)$$

since $\mu_0 \epsilon_0 = 1/c^2$. The left side of this equation is simplified by using equation (2-9a).

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}. \quad (4-9)$$

Now, since $\rho = 0$, and $\mathbf{D} = \epsilon_0 \mathbf{E}$, we find from equation (4-4) that $\nabla \cdot \mathbf{E} = 0$. Therefore,

$$\nabla^2 \mathbf{E} = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}. \quad (4-10)$$

This is the *wave equation* for free space and involves all the components of \mathbf{E} . It can be written in any one of the various coordinate systems and in each system is subject to simplification by suitable orientation. For the present, cartesian coordinates will be employed. (See Chapter V for a discussion of general coordinates.)

4-2. THE WAVE AND ITS VELOCITY

Equation (4-10), written out in terms of its components, is as follows:

$$\left. \begin{aligned} \frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} &= \frac{1}{c^2} \frac{\partial^2 E_x}{\partial t^2}, \\ \frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2} &= \frac{1}{c^2} \frac{\partial^2 E_y}{\partial t^2}, \\ \frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} &= \frac{1}{c^2} \frac{\partial^2 E_z}{\partial t^2}. \end{aligned} \right\} \quad (4-11)$$

These equations provide for an electric field with the electric intensity vector in any direction, and for variations with respect to x , y , z , and t . A simple plane wave will now result if the following restrictions are applied. Allow only a y component of \mathbf{E} and assume no variations in either \mathbf{E} or \mathbf{H} with respect to y and z . Accordingly, equations (4-11) will reduce to

$$\frac{\partial^2 E_y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 E_y}{\partial t^2}. \quad (4-12)$$

This represents a considerable simplification but does not limit the argument. The conclusions derived from this equation are applicable to a more general solution of equation (4-10) and the only limitations are as before stated;

$$\rho = \sigma = 0, \quad \text{and} \quad \epsilon_r = \mu_r = 1.$$

The solution of equation (4-12) is, in general,

$$E_y = f_1(x + ct) + f_2(x - ct), \quad (4-13)$$

which can be shown easily to be the case by direct substitution. We shall be interested principally in sinusoidal functions, so for present purposes E_y will be taken to be simply

$$E_y = A \cos \beta(x - ct) = A \cos (\omega t - \beta x). \quad (4-14)$$

Equation (4-14) represents a wave traveling in the positive x direction.* A plot of the equation for a given instant of time, $t = 0$, is shown in Fig. 4-1. At a slightly later time the curve will be displaced somewhat toward the right of its original position as shown. In order to determine the velocity of the wave, note that if attention is centered on a point such as at $x = 0$, ($t = 0$), where

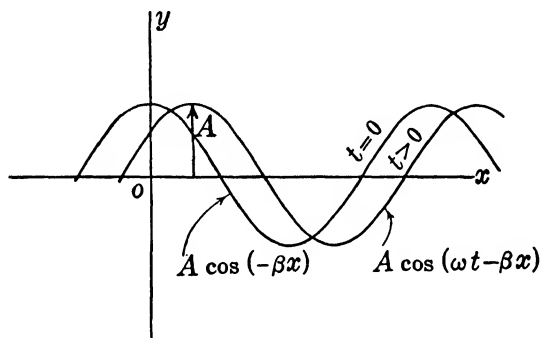


FIG. 4-1. Representations of $E_y = A \cos (\omega t - \beta x)$ for $t = 0$, and $t > 0$.

the magnitude of the wave has its positive crest value, it will be necessary to increase x as time elapses in order to maintain this crest value. Thus, in order to move along the x axis with the wave always at its positive crest value, $(\omega t - \beta x)$ must be kept at the value zero or some multiple of π , that is, $n\pi$ where n is any integer. Thus,

$$\omega t - \beta x = \beta ct - \beta x = 0, \pi, 2\pi, \text{ etc.}$$

Thus,

$$ct - x = 0, \frac{\pi}{\beta}, \frac{2\pi}{\beta}, \text{ etc.}$$

or,

$$x = ct + \text{constant.}$$

The velocity is given by

* It will be shown in Chapter VI that this wave, upon striking a conducting surface, will result in a reflected wave which travels in the negative x direction, that is, a wave defined by $\cos (\omega t + \beta x)$.

$$v = \frac{dx}{dt} = \frac{d(ct)}{dt} = c, \quad (4-15)$$

and it appears that the velocity of the wave is equal to the velocity of light ($3 \cdot 10^8$ m/sec). That this velocity will be the same in the more general case where all components of the coordinate system are considered is seen by the fact that by turning the coordinate system one can always cause the wave to travel along the positive x direction, and since velocity cannot be a function of the type of coordinate system used it must be the same and equal to c in any direction.

It was seen above that only one part of the solution (4-13) was used. The remaining part represents a wave which travels in the negative x direction. Otherwise the argument remains the same.

It now appears that equation (4-12) represents a wave filling all space and traveling in the positive x direction at the velocity of light. This wave has a component in the y direction which varies as time passes. Thus the "vibration" is perpendicular to the direction of propagation and the wave is known as a transverse wave. It is a plane wave because a surface passing through all points of equal phase would be a plane. This plane, for the case considered here, is perpendicular to the x axis.

4-3. THE ASSOCIATED MAGNETIC FIELD

It was stated previously that a changing electric field produces a magnetic field. If such is true then the above development should lead to a magnetic field intensity associated with E_y . The electric field intensity given by equation (4-14) can be substituted into equation (4-5) leading to

$$\begin{aligned} \nabla \times \mathbf{H} &= \epsilon_0 \frac{\partial}{\partial t} jA \cos(\omega t - \beta x) \\ &= -j\epsilon_0 A \omega \sin(\omega t - \beta x), \end{aligned} \quad (4-16)$$

which must equal the j component of curl \mathbf{H} . Curl \mathbf{H} is given by

$$\begin{aligned} \nabla \times \mathbf{H} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} \\ &= \mathbf{i} \dots + \mathbf{j} \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right) + \mathbf{k} \dots \\ &= -j\epsilon_0 A \omega \sin(\omega t - \beta x). \end{aligned} \quad (4-17)$$

It has already been specified that all changes with respect to z are zero. Therefore

$$\frac{\partial H_z}{\partial x} = A \epsilon_0 \omega \sin(\omega t - \beta x),$$

which, upon integration, yields

$$\begin{aligned} H_z &= \frac{A \epsilon_0 \omega}{\beta} \cos(\omega t - \beta x) \\ &= A \sqrt{\frac{\epsilon_0}{\mu_0}} \cos(\omega t - \beta x). \end{aligned} \tag{4-18}$$

Three important observations are made from this equation.

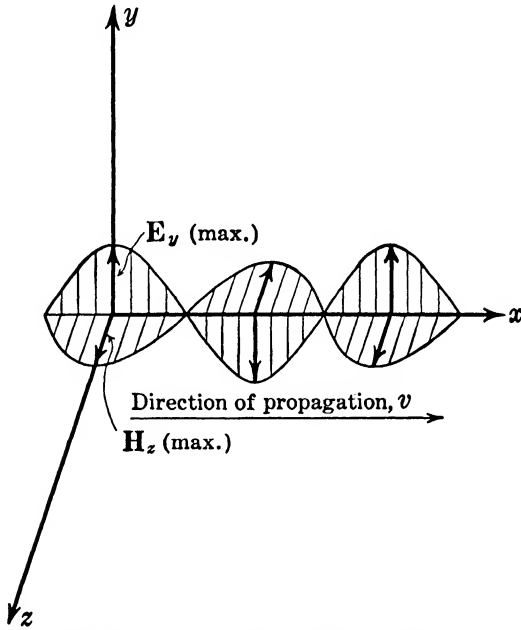


FIG. 4-2. Illustrating $E_y = A \cos(\omega t - \beta x)$ and $H_z = A \cos(\omega t - \beta x)$.

1. E_y and H_z are in time phase with one another.
2. The electric and magnetic field intensities are in space quadrature, that is, perpendicular to one another.
3. The ratio of E_y to H_z is a simple constant in free space and equals 377 ohms.

Another observation can be made which is useful later. The electric and magnetic components are in such direction that the

product $\mathbf{E} \times \mathbf{H}$ yields a vector in the direction of propagation. The waves are represented in Fig. 4-2.

4-4. THE EQUATIONS FOR ARBITRARY μ_r AND ϵ_r , ($\sigma = 0$)

In the preceding development it was assumed that $\epsilon_r = \mu_r = 1$. To obtain the equations for arbitrary values of ϵ_r and μ_r , it is necessary only to substitute into the above equation ϵ and μ for ϵ_0 and μ_0 , and thus we have

$$\frac{\partial^2 E_y}{\partial x^2} = \epsilon\mu \frac{\partial^2 E_y}{\partial t^2}. \quad (4-19)$$

Note that the velocity of the wave turned out to be the square root of the reciprocal of the coefficient of the second derivative with respect to t . Accordingly, the velocity of the wave in the new medium is

$$v = \frac{1}{\sqrt{\epsilon\mu}} = \frac{c}{\sqrt{\epsilon_r\mu_r}}. \quad (4-20)$$

This indicates that the velocity will be less than the velocity of light in any medium where either ϵ_r or μ_r or both are greater than unity.

The relationship between the magnitudes of E_y and H_z is also affected. The solution of equation (4-19) by analogy with the previous solution is

$$E_y = A \cos(\omega t - \beta x), \quad (4-21)$$

where

$$\beta = \frac{\omega}{v} = \frac{\omega}{c} \sqrt{\mu_r\epsilon_r} = \omega \sqrt{\mu\epsilon}. \quad (4-22)$$

From equation (4-21); using equation (4-1) with $\sigma = 0$,

$$\begin{aligned} \epsilon \frac{\partial \mathbf{E}}{\partial t} &= -j\epsilon A \omega \sin(\omega t - \beta x) \\ &= -j \frac{\partial H_z}{\partial x}, \end{aligned}$$

and on integration,

$$\begin{aligned} H_z &= \frac{A\epsilon\omega}{\beta} \cos(\omega t - \beta x) \\ &= A \sqrt{\frac{\epsilon}{\mu}} \cos(\omega t - \beta x). \end{aligned} \quad (4-23)$$

It is interesting to note the order of magnitude of the electric and magnetic intensities. With $\mu_r = \epsilon_r = 1$, these are such that if E_y is given as 377 v/m, then the magnetic intensity is 1 amp/m.

Whereas 377 v/m is an appreciable voltage intensity, 1 amp/m is a very weak magnetic field in practical cases. The ratio of intensities is as follows:

$$\frac{E_y}{H_x} = \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{\mu_r \mu_0}{\epsilon_r \epsilon_0}}$$

Thus for arbitrary values of μ_r and ϵ_r , this ratio may deviate quite widely from the value

$$\sqrt{\frac{\mu_0}{\epsilon_0}} = 377 \text{ ohms.}$$

4-5. ENERGY IN THE ELECTROMAGNETIC FIELD

It is well known from previous work in electricity that electric and magnetic fields represent stored energy. For instance, it is known that the stored energy in a condenser is

$$W = \frac{CV^2}{2} = \frac{QV}{2}, \quad (4-24)$$

where C is the capacitance, V is the potential difference across the plates, and Q is the charge on one plate. This equation may be used to obtain an expression for the energy stored per unit volume in an electric field. We shall take as a condenser two small plates of area da , each coinciding with a portion of two equipotential surfaces separated by a distance ds . See Fig. 4-3. Let these plates be so placed that all the lines of flux leaving one plate (no more and no less) pass through the volume in question to the other plate. Let the flux density in the space be \mathbf{D} , and the dielectric constant, ϵ . The charge on each plate is then $\mathbf{D} \cdot da$ because $Q = \psi = \mathbf{D} \cdot \mathbf{A}$. The differential energy for this elementary condenser then becomes, according to equation (4-24),

$$dW = \frac{1}{2} \mathbf{D} \cdot da \mathbf{E} \cdot ds = \frac{1}{2} DE da ds,$$

since the angle between both \mathbf{D} and da , and \mathbf{E} and ds is zero as shown in Fig. 4-3. It will be remembered that $\mathbf{D} = \epsilon \mathbf{E}$. Therefore,

$$dW = \frac{1}{2} \epsilon E^2 da ds.$$

The energy per unit volume then becomes $(\epsilon/2)E^2$ on dividing by dv ($= da ds$), or in any volume, the energy is

$$W = \frac{\epsilon}{2} \int E^2 dv. \quad (4-25)$$

In order to calculate the energy stored in a particular magnetic field, use will be made of a toroidal coil as shown in Fig. 4-4. The student is already familiar with this coil and with the equation relating the flux to the current. Let the coil be made of a great

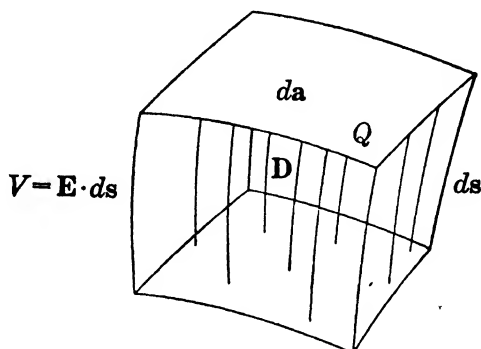


FIG. 4-3. An elementary condenser formed of portions of two equipotential surfaces.

many turns of fine wire wound very closely together so that effectively the thickness of the conducting wall is infinitesimal. Let the current per turn be I amperes, and the turns, N . The cross section of the toroidal core is da and the mean length around the coil is l .

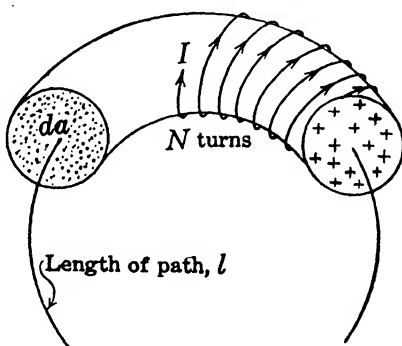


FIG. 4-4. A special magnetic field for obtaining equation (4-27).

The relative permeability of the core material is μ_r . The total flux through the coil is then

$$\phi = B da = \mu H da = \mu \frac{NI}{l} da,$$

from which

$$NI = \frac{\phi l}{\mu da}$$

The total energy in a magnetic field is known to be

$$W = \frac{LI^2}{2},$$

and from the ordinary definition of self inductance

$$L = \frac{N\phi}{I} = \frac{N\mu H da}{I}.$$

Accordingly, the differential of energy in the field for this elementary volume is

$$\begin{aligned} dW &= \frac{1}{2} \frac{N\mu H da}{I} I^2 \\ &= \frac{\mu}{2} H^2 da(l) \\ &= \frac{\mu}{2} H^2 dv. \end{aligned}$$

This is the energy in the element of volume $l da = dv$. The energy per unit volume is

$$\frac{\mu}{2} H^2, \quad (4-26)$$

and the energy in any volume is

$$W_m = \frac{\mu}{2} \int H^2 dv. \quad (4-27)$$

By means of equations (4-25) and (4-27) it is now possible to derive an equation for the power conveyed by the electromagnetic wave. Consider a small closed surface in space and let the vector \mathbf{P} represent the power flow out through this surface per unit area. The total power flow (in *joules per second* or *watts*) out through the surface is given by the integral over the entire surface,

$$\int_s \mathbf{P} \cdot d\mathbf{a}. \quad (4-28)$$

Now, the total energy in this elementary volume is, from equations (4-25) and (4-27)

$$\frac{1}{2} \int (\mu H^2 + \epsilon E^2) dv. \quad (4-29)$$

The rate of decrease of this stored energy is

$$-\frac{\partial}{\partial t} \int (\mu H^2 + \epsilon E^2) dv = \int_s \mathbf{P} \cdot d\mathbf{a}. \quad (4-30)$$

The equality must be true because the rate of decrease must equal the out-flow given by expression (4-28). Now perform the indicated differentiation and obtain

$$-\int \left(\mu H \frac{\partial H}{\partial t} + \epsilon E \frac{\partial E}{\partial t} \right) dv. \quad (4-31)$$

Now since $\mu H (\partial H/\partial t)$ and $\epsilon E (\partial E/\partial t)$ represent merely the scalar values of rates of change, it is possible to express them as the scalar products of the vectors. Then expression (4-31) becomes

$$-\int \left(\mu \mathbf{H} \cdot \frac{\partial \mathbf{H}}{\partial t} + \epsilon \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} \right) dv. \quad (4-32)$$

In free space we may make substitutions from Maxwell's equation for the derivatives. Therefore, expression (4-32) becomes

$$\int [\mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H})] dv. \quad (4-33)$$

It can be shown that

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H}). \quad (4-34)$$

Applying this to expression (4-33) and referring to equation (4-30), we have

$$\int_s \mathbf{P} \cdot d\mathbf{a} = \int \nabla \cdot (\mathbf{E} \times \mathbf{H}) dv,$$

which equals

$$\int_s (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{a} \quad (4-35)$$

by Gauss's theorem, and since the integrals are over the same surface, we obtain equation (4-36) which expresses the power flow per unit area in the electromagnetic field.

$$\mathbf{P} = (\mathbf{E} \times \mathbf{H}) \text{ joules/sec/m}^2, \text{ or watts/m}^2. \quad (4-36)$$

The vector \mathbf{P} is known as the Poynting vector. It is well named because not only does it give the value of the power but also its direction is the direction of the flow of power. Note that its direction is the same as the direction of travel of the wave as mentioned at the end of §4-3.

4-6. ILLUSTRATIVE EXAMPLES: ENERGY IN THE ELECTROMAGNETIC FIELD

Example 1. In a space in which $\mu_r = 1$, $\epsilon_r = 4.6$, an electric field is given by

$$E_y = 1.3 \cos(\omega t - \beta x),$$

where $f = 10^6$ cps. Find the velocity of the wave, the value of β , the wavelength, the expression for H_z , and the value of E_y/H_z .

This velocity will be less than the velocity of light as indicated by equation (4-20). From this equation

$$v = \frac{3 \cdot 10^8}{\sqrt{4.6}} = 1.4 \cdot 10^8 \text{ m/sec.}$$

From equation (4-22) β may be found.

$$\begin{aligned} \beta &= \frac{2\pi f}{v} = \frac{2\pi \cdot 10^6}{1.4 \cdot 10^8} \\ &= 0.0449 \text{ rad/m.} \end{aligned}$$

From this result the wavelength is

$$\begin{aligned} \lambda &= \frac{2\pi}{\beta} = \frac{2\pi}{0.0449} \\ &= 140 \text{ m.} \end{aligned}$$

Equation (4-23) may be used to obtain H_z . In this equation $A = 1.3$ v/m.

$$\begin{aligned} H_z &= 1.3 \sqrt{\frac{\epsilon}{\mu}} \cos(\omega t - \beta x) \\ &= 1.3 \frac{\sqrt{4.6}}{377} \cos(\omega t - \beta x) \\ &= 0.0074 \cos(\omega t - \beta x) \text{ amp/m.} \end{aligned}$$

Accordingly,

$$\frac{E_y}{H_z} = \frac{1.3}{0.0074} = 176 \text{ ohms.}$$

Example 2. In the preceding example, determine the power flow through a cross-sectional area of 10 cm^2 . Poynting's vector, equation (4-36), may be used for this calculation. Since this equation gives power in watts per square meter, the required result will be

$$\mathbf{P} = 0.001(\mathbf{E} \times \mathbf{H}) = i0.001E_yH_z,$$

because E_y and H_z are mutually perpendicular and the right-hand rule for direction is used to obtain the unit vector i . Thus, using peak values for E_y and H_z ,

$$\begin{aligned} P &= 0.001(1.3)(0.0074) \\ &= 9.62 \text{ microwatts in the positive } x \text{ direction.} \end{aligned}$$

It should be recognized here that when peak values are used for E_y and H_z , the peak power is obtained. If the entire expressions for

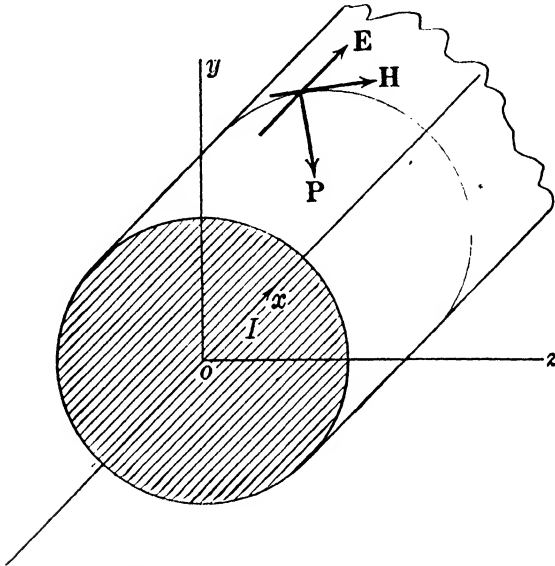


FIG. 4-5. A wire carrying a current I , to be used in Example 3.

E_y and H_z were used, then power as a function of x and t would result. Both treatments are used in subsequent material.

Example 3. A conductor is 0.05 m in radius and is carrying a current of 2000 amp in the positive x direction. The conductivity of the material is $5.8(10^7)$ mhos/m². Find the power flow through the surface of the conductor per meter length of wire and its direction.

At the surface of the wire the magnetic field will be given by

$$H = \frac{2000}{2\pi(0.05)} = 6360 \text{ amp/m.}$$

This field is in the direction shown in Fig. 4-5. The current density in the wire is

$$J = \frac{2000}{(0.05)^2 \pi} = 254,500 \text{ amp/m}^2.$$

The field intensity in the plus x direction is, from $\mathbf{J} = \sigma \mathbf{E}$,

$$E = \frac{J}{\sigma} = \frac{254,500}{5.8 \cdot 10^7} = 0.004395 \text{ v/m}.$$

By the expression for Poynting's vector

$$P = EH \sin \theta,$$

where θ is the angle between \mathbf{E} and \mathbf{H} and is 90° in this case. Thus

$$\begin{aligned} P &= (0.004395)(6360) \\ &= 27.95 \text{ watts/m}^2. \end{aligned}$$

Let this be multiplied by the area of 1 m length of wire which is $A = 2\pi(0.05) = 0.314 \text{ m}^2$. Then the total power is

$$P_t = (0.314)(27.95) = 8.78 \text{ watts}.$$

This power crosses the boundary of the wire and, by the right-hand rule, is directed along the radius into the wire.

This result can be easily checked by the usual I^2R expression for 1 m of wire. The resistivity is

$$\rho = \frac{1}{\sigma} = \frac{10^{-7}}{5.8}.$$

The resistance of 1 m of wire then becomes

$$R = \frac{10^{-7}}{5.8\pi(0.05)^2} = 2.195 \cdot 10^{-6} \text{ ohm}.$$

The power is

$$\begin{aligned} P_t &= I^2R = 2.195 \cdot 10^{-6} (4)10^6 \\ &= 8.78 \text{ watts}. \end{aligned}$$

This example shows that it is possible to look upon power transmission as a field phenomenon since the power dissipated in 1 m of the wire on account of its resistance is the same as the power flowing into the wire from the field.

4-7. PROBLEMS

4-1. Show that equation (4-13) is a solution of equation (4-12).

4-2. Assuming a given electric field intensity of

$$E_y = 10 \cos(\omega t - \beta x),$$

where the intensity is in volts per meter and the frequency is 10^8 , find the associated magnetic field and β and ν for the two cases: (a) free space, (b) $\epsilon_r = 3.5$, $\mu_r = 1$.

4-3. Beginning with equations (4-1) and (4-2), show that equation (4-19) is correct.

4-4. Show that equation (4-34) is correct.

4-5. Find the Poynting vector \mathbf{P} for the two cases of Prop. 4-2.

4-6. If b is equal to unity in $E_y = A \sin b(x - ct)$, find the wavelength and the frequency.

4-7. Prove that \mathbf{E} and \mathbf{H} are perpendicular no matter what function of $(x - ct)$ is assumed as the solution.

4-8. Determine the fraction of the total energy conveyed by the electric field if $\epsilon_r = 4$ and $\mu_r = 1$.

4-9. Show that, if μH ($\partial H/\partial t$) is the scalar value of the rate of change of H , it is just as well represented by $\mu \mathbf{H} \cdot (\partial \mathbf{H}/\partial t)$.

4-10. Determine, in any manner, the electric and magnetic fields between coaxial conductors of radii a and b . ($a < b$). Using Poynting's vector, show that the power conveyed by the field between the conductors is given by VI , where V is the potential difference between the conductors, and I is the current per conductor.

4-11. An electric field component is given as

$$\mathbf{E} = (jA_1 + \mathbf{k}A_2) \cos(\omega t - \beta x)$$

where $A_1 = 3.0$, and $A_2 = 4.0$ v/m. In this space $\epsilon_r = 3$, $\mu_r = 1$, and $f = 10^8$ cps. Find: (a) the value of β ; (b) direction and velocity of propagation; (c) expression for \mathbf{H} .

CYLINDRICAL AND SPHERICAL COORDINATES

Summary

This chapter is devoted to the development of expressions for *grad*, *div*, *curl*, and the *Laplacian* in a general curvilinear coordinate system, because, in many applications, fields in cylindrical and spherical space must be considered. It is shown how, from this system, the expressions for cartesian, cylindrical, and spherical systems may be easily derived.

5-1. GENERAL CURVILINEAR COORDINATES

Thus far the treatment of the subject has intentionally been restricted to the familiar cartesian system of coordinates. There are many applications, however, in which the problem is greatly simplified by treatment in some other system, principally cylindrical or spherical. This is true even though the equations in these systems sometimes appear more complicated. As an illustration, the solution of the problem of transmission of electromagnetic waves through cylindrical tubes naturally fits into the cylindrical system of coordinates.

There are three ways in which expressions for gradient, divergence, etc., may be developed in the curvilinear systems.

- ✓ 1. A transformation may be made from the cartesian system.
2. Derivations may be made in each system separately.
- ✓ 3. Derivatives may be made in a general curvilinear system from which the expressions for the different systems may be obtained by a simple substitution.

The third of the methods will be used here. It will be found that by its use it will be equally easy to obtain expressions for cartesian, cylindrical, spherical, and other coordinate systems.

A general curvilinear orthogonal system is characterized by the fact that, in general, the edges of the elementary volume are curves, and the lengths of the edges are the products of a factor and the differential of the coordinate unit. Also, the edges are mutually perpendicular. Reference to Fig. 5-1 will help to clarify the procedure. The edges of the elementary volume are the *distances* λdp , μdq , and νds , arranged in this order in a right-handed system. Note especially that, in general, the differential of the coordinate is not a length in the same way that $d\theta$ is not a length

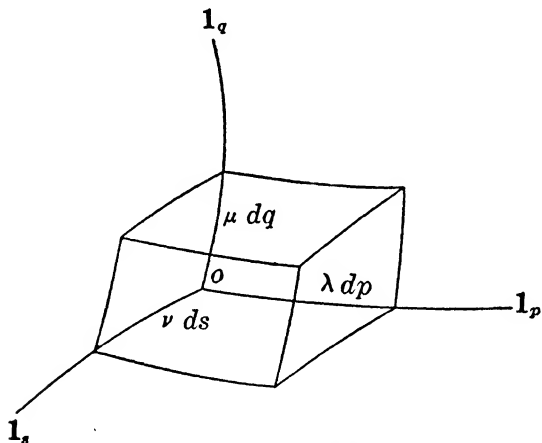


FIG. 5-1. Elemental volume in general coordinates.

and must be multiplied by some factor in order to produce a length, just as $d\theta$ is transformed into a differential of length by multiplication by r . Thus λdp is a length which can be identified with any particular coordinate in a particular system by assigning appropriate definitions to λ and p . As an illustration, in the cartesian system $\lambda = 1$, $p = x$, producing dx ; in the cylindrical system $\lambda = 1$, $p = r$, producing dr , etc. At this point a list of these identifications will be given. Refer to Fig. 5-2.

	λ	p	μ	q	ν	s	}	(5-1)
Cartesian	1	x	1	y	1	z		
Cylindrical	1	r	r	θ	1	z		
Spherical	1	r	r	θ	$(r \sin \theta)$	φ		

If the reader is unfamiliar with the scheme of general coordinates as proposed here, he will find that a careful study of Fig. 5-2 will

reveal the reasons for the various identifications listed in (5-1). In Fig. 5-2b, for example, the three differential displacements which are mutually orthogonal and which characterize the familiar cylindrical coordinate system are:

$$\begin{aligned} dr &= \lambda dp \\ r d\theta &= \mu dq \\ dz &= \nu ds \end{aligned}$$

In a like manner, in Fig. 5-2c, the three differential displacements

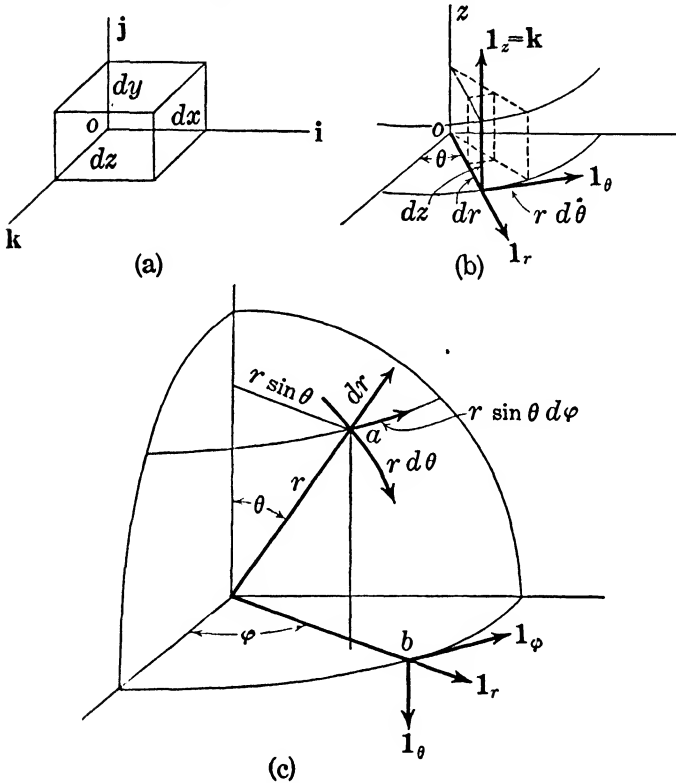


FIG. 5-2. Rectangular, cylindrical, and spherical systems.

which are mutually orthogonal (say at point a) and which characterize the spherical coordinate system in general coordinates are:

$$\begin{aligned} dr &= \lambda dp \\ r d\theta &= \mu dq \\ r \sin \theta d\phi &= \nu ds \end{aligned}$$

It is now necessary to assign unit vectors to this system in order to specify directions. These will be as follows:

$$\left. \begin{array}{l} \text{For the } \lambda \, dp \text{ direction, the unit vector is } \mathbf{1}_p. \\ \text{For the } \mu \, dq \text{ direction, the unit vector is } \mathbf{1}_q. \\ \text{For the } \nu \, ds \text{ direction, the unit vector is } \mathbf{1}_s. \end{array} \right\} \quad (5-2)$$

Now we are prepared to make use of this system of coordinates for deriving the various operator expressions.

5-2. THE GRADIENT

Gradient has been defined as the rate of change of a function with respect to the distance in the required direction. Accordingly, in general coordinates, it becomes

$$\nabla \phi = \mathbf{1}_p \frac{1}{\lambda} \frac{\partial \phi}{\partial p} + \mathbf{1}_q \frac{1}{\mu} \frac{\partial \phi}{\partial q} + \mathbf{1}_s \frac{1}{\nu} \frac{\partial \phi}{\partial s} \quad (5-3)$$

Note here that, for each component of the gradient, the change in the scalar ϕ along each of the orthogonal directions ($\mathbf{1}_p$, $\mathbf{1}_q$, and $\mathbf{1}_s$) is divided by the change in the *distance* along the appropriate axis. Equation (5-3) will now be written over into the three fundamental systems by the use of the list of identifications (5-1) and (5-2).

Cartesian System

$$\nabla \phi = \mathbf{1}_x \frac{\partial \phi}{\partial x} + \mathbf{1}_y \frac{\partial \phi}{\partial y} + \mathbf{1}_z \frac{\partial \phi}{\partial z}$$

However, it has been usual to replace the 1's, in this system, as follows:

$$\begin{array}{l} \mathbf{i} \text{ is written for } \mathbf{1}_x \\ \mathbf{j} \text{ is written for } \mathbf{1}_y \\ \mathbf{k} \text{ is written for } \mathbf{1}_z \end{array}$$

Therefore,

$$\nabla \phi = \mathbf{i} \frac{\partial \phi}{\partial x} + \mathbf{j} \frac{\partial \phi}{\partial y} + \mathbf{k} \frac{\partial \phi}{\partial z}, \quad (5-4)$$

which is identical with equation (2-2).

Cylindrical System

$$\nabla \phi = \mathbf{1}_r \frac{\partial \phi}{\partial r} + \mathbf{1}_\theta \frac{1}{r} \frac{\partial \phi}{\partial \theta} + \mathbf{k} \frac{\partial \phi}{\partial z} \quad (5-5)$$

Note again that here \mathbf{k} has been written for $\mathbf{1}_z$.

Spherical System

$$\nabla\phi = \mathbf{1}_r \frac{\partial\phi}{\partial r} + \mathbf{1}_\theta \frac{1}{r} \frac{\partial\phi}{\partial\theta} + \mathbf{1}_\varphi \frac{1}{r \sin\theta} \frac{\partial\phi}{\partial\varphi}. \tag{5-6}$$

5-3. THE DIVERGENCE

In the derivation of the general expression for divergence, only one component will be worked out. On the basis of this one, the other components can be readily written down. In Fig. 5-3 are shown two opposite sides of an elementary volume. The field \mathbf{V} exists in the space occupied by the volume and the component of

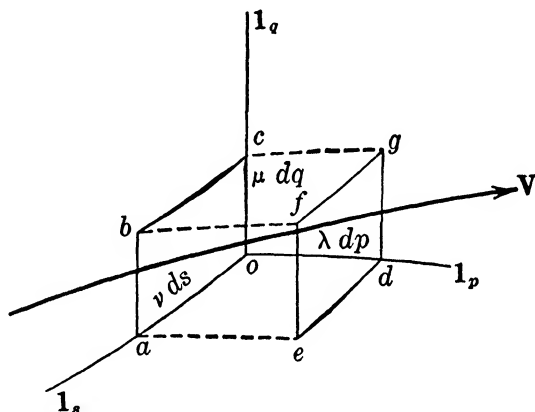


FIG. 5-3. To be used in the development of §5-3.

\mathbf{V} , with which we will be immediately concerned, is V_p . The vector component V_p enters the face $oabc$ and emerges, after some possible change in value, from face $defg$. The influx is

$$\text{influx} = V_{p\mu\nu} dq ds. \tag{5-7}$$

The outflux at the opposite face is, in general, the influx plus any differential change caused by movement to a new position along the p axis. Thus the outflux becomes

$$V_{p\mu\nu} dq ds + \frac{\partial V_{p\mu\nu} dq ds}{\lambda \partial p} \lambda dp,$$

but since p , q , and s are independent, and λ , μ , and ν may be functions of the coordinates, we may write the outflux as

$$V_{p\mu\nu} dq ds + \frac{\partial V_{p\mu\nu}}{\partial p} dp dq ds. \tag{5-8}$$

By subtracting expression (5-7) from (5-8),

$$\text{excess}_{(p \text{ component})} = \frac{\partial(V_{p\mu\nu})}{\partial p} dp dq ds. \quad (5-9)$$

Now, because of the symmetry of the terms, the other two components can be written down immediately by a suitable cyclic change in the symbols. Thus,

$$\text{excess}_{(q \text{ component})} = \frac{\partial(V_{q\nu\lambda})}{\partial q} dq ds dp, \quad (5-10)$$

and

$$\text{excess}_{(s \text{ component})} = \frac{\partial(V_{s\lambda\mu})}{\partial s} ds dp dq. \quad (5-11)$$

The total excess will be the sum of these three components and this sum represents the change in \mathbf{V} which results from the passage of the field \mathbf{V} through the elementary volume $\lambda\mu\nu dp dq ds$. If this sum is then divided by the volume of the element, $\lambda\mu\nu dp dq ds$, the divergence is, by definition:

$$\nabla \cdot \mathbf{V} = \frac{1}{\lambda\mu\nu} \left[\frac{\partial(V_{p\mu\nu})}{\partial p} + \frac{\partial(V_{q\nu\lambda})}{\partial q} + \frac{\partial(V_{s\lambda\mu})}{\partial s} \right]. \quad (5-12)$$

This can now be written in the three systems by the use of the lists (5-1), (5-2).

Cartesian System

Here λ , μ , and ν have constant magnitudes equal to unity. Hence,

$$\nabla \cdot \mathbf{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}. \quad (5-13)$$

Cylindrical System

Here $\mu = r$ and as such μ is a function of r . It follows that

$$\begin{aligned} \nabla \cdot \mathbf{V} &= \frac{1}{r} \left[\frac{\partial(V_r r)}{\partial r} + \frac{\partial(V_\theta)}{\partial \theta} + \frac{\partial(V_z r)}{\partial z} \right] \\ &= \frac{1}{r} \left[r \frac{\partial V_r}{\partial r} + V_r + \frac{\partial V_\theta}{\partial \theta} + r \frac{\partial V_z}{\partial z} \right] \\ &= \frac{\partial V_r}{\partial r} + \frac{V_r}{r} + \frac{1}{r} \frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z}. \end{aligned} \quad (5-14)$$

Spherical System

It will be recalled that in the spherical coordinate system, $\lambda = 1$, $\mu = r$, $\nu = r \sin \theta$, $p = r$, $q = \theta$, $s = \varphi$. These substituted into equation (5-12) produce

$$\nabla \cdot \mathbf{V} = \frac{1}{r^2 \sin \theta} \left[\frac{\partial(V_r r^2 \sin \theta)}{\partial r} + \frac{\partial(V_\theta r \sin \theta)}{\partial \theta} + \frac{\partial(V_\varphi r)}{\partial \varphi} \right].$$

The expansion of this expression is entirely straightforward and is not repeated here. The result is

$$\nabla \cdot \mathbf{V} = \frac{1}{r^2} \frac{\partial(V_r r^2)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial(V_\theta \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial V_\varphi}{\partial \varphi}. \quad (5-15)$$

5-4. THE CURL

The derivation of the expression for curl in the general coordinate system follows a procedure which has been used before. It is

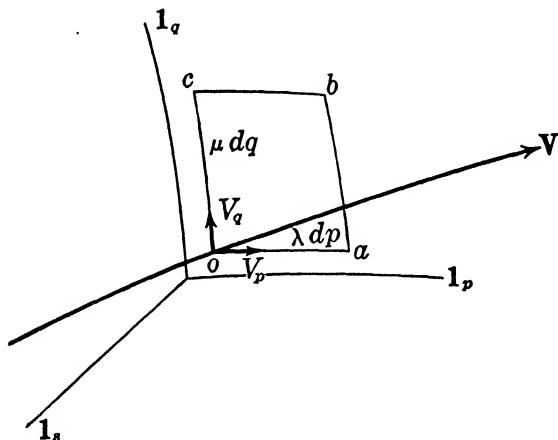


FIG. 5-4. For derivation of curl in general coordinates.

based on the fact that curl is the limit of the ratio of the line integral of a vector around the periphery of a surface to the area of the surface. As in the case of the divergence, only one component will be derived from which the others can be written out by using the cyclic property of the symbols as previously mentioned.

The line integral of \mathbf{V} in a positive direction around the elementary rectangle of Fig. 5-4 is determined as follows:

$$\begin{aligned} \text{along } oa, \text{ (line integral)}_{oa} &= V_p \lambda dp; \\ \text{along } co, \text{ (line integral)}_{co} &= -V_q \mu dq \end{aligned}$$

Now along ab the line integral is that along oc plus the increment added owing to the transfer to the right through a distance λdp . Thus,

$$\text{along } ab, (\text{line integral})_{ab} = V_{q\mu} dq + \frac{\partial V_{q\mu}}{\partial p} dq dp,$$

but p and q are independent so,

$$(\text{line integral})_{ab} = V_{q\mu} dq + \frac{\partial V_{q\mu}}{\partial p} dq dp.$$

Similarly along bc ,

$$(\text{line integral})_{bc} = -V_{p\lambda} dp - \frac{\partial V_{p\lambda}}{\partial q} dq dp.$$

The line integral around the area is then:

$$\begin{aligned} (\text{line integral})_t &= V_{p\lambda} dp - V_{q\mu} dq + V_{q\mu} dq + \frac{\partial V_{q\mu}}{\partial p} dp dq \\ &\quad - V_{p\lambda} dp - \frac{\partial V_{p\lambda}}{\partial q} dp dq \\ &= \left[\frac{\partial V_{q\mu}}{\partial p} - \frac{\partial V_{p\lambda}}{\partial q} \right] dp dq. \end{aligned}$$

Dividing by the area, $\lambda\mu dp dq$, we have the s component of the curl.

$$\text{curl}_s = \left[\frac{\partial V_{q\mu}}{\partial p} - \frac{\partial V_{p\lambda}}{\partial q} \right] \frac{1}{\lambda\mu}.$$

By a cyclic change of the symbols the other components may now be written.

$$\text{curl}_p = \left[\frac{\partial V_{s\nu}}{\partial q} - \frac{\partial V_{q\mu}}{\partial s} \right] \frac{1}{\mu\nu}$$

$$\text{curl}_q = \left[\frac{\partial V_{p\lambda}}{\partial s} - \frac{\partial V_{s\nu}}{\partial p} \right] \frac{1}{\nu\lambda}$$

Accordingly,

$$\begin{aligned} \text{curl } \mathbf{V} &= \mathbf{1}_p \frac{1}{\mu\nu} \left[\frac{\partial V_{s\nu}}{\partial q} - \frac{\partial V_{q\mu}}{\partial s} \right] + \mathbf{1}_q \frac{1}{\nu\lambda} \left[\frac{\partial V_{p\lambda}}{\partial s} - \frac{\partial V_{s\nu}}{\partial p} \right] + \\ &\quad \mathbf{1}_s \frac{1}{\lambda\mu} \left[\frac{\partial V_{q\mu}}{\partial p} - \frac{\partial V_{p\lambda}}{\partial q} \right]. \end{aligned} \quad (5-16)$$

Again the substitution of the appropriate letters for λ , μ , ν , p , q , and s will produce the expressions for the curl in the three principal coordinate systems.

Cartesian

$$\lambda = 1, \mu = 1, \nu = 1, p = x, q = y, s = z.$$

$$\text{curl } \mathbf{V} = \mathbf{i} \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \right) + \mathbf{j} \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \right) + \mathbf{k} \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right). \quad (5-17)$$

Cylindrical

The procedure for writing out the various expressions is now well enough understood so that the following relations will be written down without detailed derivation. The equation for curl in cylindrical coordinates is

$$\nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{1}_r & \mathbf{1}_\theta & \mathbf{k} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ V_r & rV_\theta & V_z \end{vmatrix}. \quad (5-18)$$

Spherical

$$\nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{1}_r & \mathbf{1}_\theta & \mathbf{1}_\varphi \\ r^2 \sin \theta & r \sin \theta & r \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ V_r & rV_\theta & r \sin \theta V_\varphi \end{vmatrix}. \quad (5-19)$$

5-5. THE LAPLACIAN

In writing the Laplacian in general coordinates it is sufficient to notice that it is given by the expression

$$\nabla^2 \phi = \nabla \cdot \nabla \phi = \text{div grad } \phi.$$

Before going farther it is well to point out here that this operator is defined also when applied to a vector even though $\nabla \mathbf{V}$ has no meaning by itself. Thus we may have the Laplacian of a vector written as

$$\nabla^2 \mathbf{A},$$

which is important in subsequent material. The general form of the Laplacian may be obtained by combining expressions (5-3) and (5-12), the $\nabla \phi$ of equation (5-3) being substituted for \mathbf{V} of equation (5-12). When this is done the following is easily obtained.

$$\nabla^2\phi = \frac{1}{\lambda\mu\nu} \left[\frac{\partial}{\partial p} \left(\frac{\mu\nu}{\lambda} \frac{\partial\phi}{\partial p} \right) + \frac{\partial}{\partial q} \left(\frac{\nu\lambda}{\mu} \frac{\partial\phi}{\partial q} \right) + \frac{\partial}{\partial s} \left(\frac{\lambda\mu}{\nu} \frac{\partial\phi}{\partial s} \right) \right]. \quad (5-20)$$

This equation written out in the three systems is as follows:

Cartesian

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}. \quad (5-21)$$

Cylindrical

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r} \frac{\partial\phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2}. \quad (5-22)$$

Spherical

$$\nabla^2\phi = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial\phi}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial\phi}{\partial\theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2\phi}{\partial\varphi^2}. \quad (5-23)$$

When the Laplacian operator is applied to a vector, however, a somewhat unexpected difficulty arises. If \mathbf{A} is substituted for ϕ in equation (5-20), there will appear such terms as $\partial\mathbf{A}/\partial r$, etc., which will involve derivatives of unit vectors. This difficulty does not arise in the case of cartesian coordinates, for the unit vectors, \mathbf{i} , \mathbf{j} , and \mathbf{k} are fixed in magnitude and direction at all points of space. However, this is not true in general in the other systems. For example, note that in Fig. 5-2c, at points a and b , the unit vectors are not in the same direction, indicating a change in the unit vectors with a change in the coordinates. Accordingly, derivatives of unit vectors with respect to the coordinates must be defined. These will be considered under the appropriate systems below.

Cartesian

Here \mathbf{i} , \mathbf{j} , and \mathbf{k} are constants and the Laplacian becomes simply, from equation (5-21),

$$\begin{aligned} \nabla^2\mathbf{A} &= \frac{\partial^2\mathbf{A}}{\partial x^2} + \frac{\partial^2\mathbf{A}}{\partial y^2} + \frac{\partial^2\mathbf{A}}{\partial z^2} \\ &= \mathbf{i} \frac{\partial^2 A_x}{\partial x^2} + \mathbf{j} \frac{\partial^2 A_y}{\partial x^2} + \mathbf{k} \frac{\partial^2 A_z}{\partial x^2} + \\ &\quad \mathbf{i} \frac{\partial^2 A_x}{\partial y^2} + \mathbf{j} \frac{\partial^2 A_y}{\partial y^2} + \mathbf{k} \frac{\partial^2 A_z}{\partial y^2} + \\ &\quad \mathbf{i} \frac{\partial^2 A_x}{\partial z^2} + \mathbf{j} \frac{\partial^2 A_y}{\partial z^2} + \mathbf{k} \frac{\partial^2 A_z}{\partial z^2} \\ &= \mathbf{i}\nabla^2 A_x + \mathbf{j}\nabla^2 A_y + \mathbf{k}\nabla^2 A_z. \end{aligned} \quad (5-24)$$

Cylindrical

If **A** is substituted for ϕ in equation (5-20), the derivatives of $\mathbf{1}_r$, $\mathbf{1}_\theta$, and \mathbf{k} with respect to r , θ , and z will appear. Let these derivatives be evaluated first. Reference to Fig. 5-2b will lead to the following results immediately.

$$\left. \begin{aligned} \frac{\partial \mathbf{1}_r}{\partial r} = 0, \quad \frac{\partial \mathbf{1}_r}{\partial z} = 0, \\ \frac{\partial \mathbf{1}_\theta}{\partial r} = 0, \quad \frac{\partial \mathbf{1}_\theta}{\partial z} = 0, \\ \frac{\partial \mathbf{k}}{\partial r} = 0, \quad \frac{\partial \mathbf{k}}{\partial \theta} = 0, \quad \frac{\partial \mathbf{k}}{\partial z} = 0. \end{aligned} \right\} \quad (5-25)$$

The partial derivatives $\partial \mathbf{1}_r / \partial \theta$, and $\partial \mathbf{1}_\theta / \partial \theta$ may be found by the use of the diagram in Fig. 5-5. From Fig. 5-5a we have the

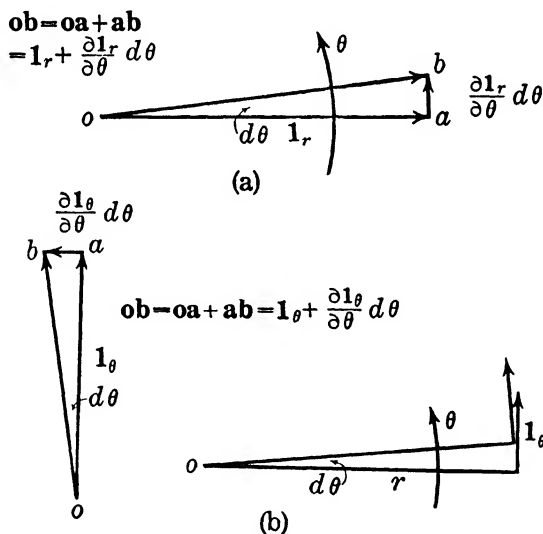


FIG. 5-5. Representation of derivatives of unit vectors in the cylindrical system.

vectors oa and ob such that the difference between them is

$$\frac{\partial \mathbf{1}_r}{\partial \theta} d\theta,$$

which is a vector in the $\mathbf{1}_\theta$ direction and has a magnitude of

$|\mathbf{1}_r| d\theta = d\theta$. Thus,

$$\frac{\partial \mathbf{1}_r}{\partial \theta} d\theta = \mathbf{1}_\theta d\theta,$$

or

$$\frac{\partial \mathbf{1}_r}{\partial \theta} = \mathbf{1}_\theta. \quad (5-26)$$

In a similar manner in Fig. 5-5b it is seen that the difference between the two unit vectors is

$$\frac{\partial \mathbf{1}_\theta}{\partial \theta} d\theta,$$

which is in the direction $-\mathbf{1}_r$, and which has a magnitude of $d\theta$ as in the previous example. Thus, again

$$\frac{\partial \mathbf{1}_\theta}{\partial \theta} d\theta = -\mathbf{1}_r d\theta,$$

or

$$\frac{\partial \mathbf{1}_\theta}{\partial \theta} = -\mathbf{1}_r. \quad (5-27)$$

Equations (5-25), (5-26), (5-27) may now be used to work out the expansion of (5-20) for a vector \mathbf{A} . The following result is obtained after considerable mathematical manipulation.

$$\nabla^2 \mathbf{A} = \mathbf{1}_r \left(\nabla^2 A_r - \frac{2}{r^2} \frac{\partial A_\theta}{\partial \theta} - \frac{A_r}{r^2} \right) + \mathbf{1}_\theta \left(\nabla^2 A_\theta + \frac{2}{r^2} \frac{\partial A_r}{\partial \theta} - \frac{A_\theta}{r^2} \right) + \mathbf{k} \nabla^2 A_z. \quad (5-28)$$

Spherical

In the spherical coordinate system the derivatives of the unit vectors are found in a manner analogous with that used above. The nonzero derivatives are as follows:

$$\left. \begin{aligned} \frac{\partial \mathbf{1}_r}{\partial \theta} &= \mathbf{1}_\theta, & \frac{\partial \mathbf{1}_r}{\partial \varphi} &= \mathbf{1}_\varphi \sin \theta, \\ \frac{\partial \mathbf{1}_\theta}{\partial \theta} &= -\mathbf{1}_r, & \frac{\partial \mathbf{1}_\theta}{\partial \varphi} &= \mathbf{1}_\varphi \cos \theta, \\ \frac{\partial \mathbf{1}_\varphi}{\partial \varphi} &= -(\mathbf{1}_r \sin \theta + \mathbf{1}_\theta \cos \theta). \end{aligned} \right\} \quad (5-29)$$

The application of these derivatives to the expansion of expression (5-20) for a vector is long and involved and the result, not being of immediate importance, is not given here.

5-6. PROBLEMS

5-1. Given two concentric cylinders with radii 3 and 5 cm. The outer cylinder is at a potential of 1000 v and the inner one is at zero. If $\epsilon_r = 1$, find the potential and electric field intensity at any point between the cylinders, using Laplace's equation.

5-2. Given an isolated sphere of radius a with a charge $+Q$ on it. If the surrounding space has a relative permittivity, ϵ_r , find the electric field intensity at all points, using Laplace's equation.

5-3. Two parallel plates of a condenser are separated by d meters, and have an area of A square meters each. If the charge density per unit area on each plate is $+m$ and $-m$ respectively, use Laplace's equation to obtain the electric field intensity at any point between the plates and the capacitance per unit area.

5-4. Using cylindrical coordinates find the curl of the magnetic field between concentric conductors if the radii are a and b . ($a < b$) Assume a current I flowing in the inner conductor.

5-5. Find the curl of the magnetic field within a wire carrying a current I .

5-6. Starting with equation (5-12), show that

$$\nabla \cdot \mathbf{V} = \frac{\partial V_r}{\partial r} + \frac{2V_r}{r} + \frac{1}{r} \frac{\partial V_\theta}{\partial \theta} + \frac{V_\theta}{r \tan \theta} + \frac{1}{r \sin \theta} \frac{\partial V_\varphi}{\partial \varphi}$$

5-7. Show by use of equations (5-3) and (5-12) that both equations (5-20) and (5-22) are correct.

5-8. Show by means of reasoning similar to that employed in §5-3, that the s component of the "excess" is $\frac{\partial(V_s \lambda \mu)}{\partial s} ds dp dq$. See equation (5-11).

5-9. Show by means of a diagram similar to that given in Fig. 5-4, that the q component of curl as given in equation (5-16) is correct.

5-10. Show that equation (5-23) is correct.

5-11. Show that equation (5-28) is correct.

5-12. Show that the expressions for derivatives of unit vectors given under equations (5-29) are correct.

5-13. Two charged plates are set at an angle θ , as shown in Fig. 5-6. The charges on the plates are $+Q$ and $-Q$ and the areas are A units each. Find: (a) the electric field intensity between the

plates; (b) the distribution of charge density on the plates. Assume no fringing effects.

5-14. Given a coaxial cable whose radii are a and b , ($a < b$). Assume an electron cloud in the space between the conductors

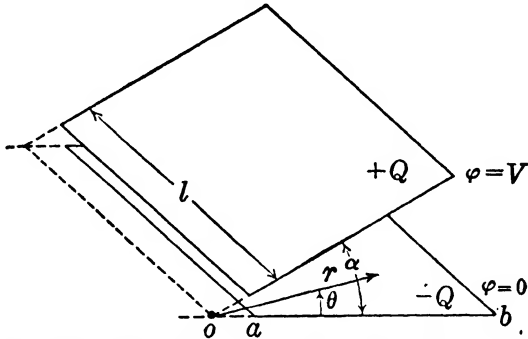


FIG. 5-6. For use in connection with Prob. 5-13.

whose density in coulombs per cubic meter is given by $-mr$, where $a < r < b$. Find the electric field if the potentials at $r = a$, and $r = b$ are 0 and 100 v, respectively. Assume that $\epsilon_r = 1$.

REFLECTION PHENOMENA IN PLANE WAVES

Summary

The main thread of the argument is again taken up in this chapter, and the boundary conditions leading to reflection phenomena are treated. The simpler case of normal reflection against a perfect conductor is first considered, followed by a discussion of the resulting standing wave, and power-flow phenomena. Next, normal reflection against a general boundary is discussed and also the special case of reflection against a dielectric for which the conductivity is zero. This is followed by a treatment of the waves in a conductor resulting from an incident wave, and the depth of penetration is defined. The chapter ends with a short discussion of the effect on the standing wave produced by an imperfect conductor, and with an extensive example illustrative of a number of the points brought out in the chapter.

6-1. BOUNDARY CONDITIONS

In preparation for an introduction to surface effects and reflections it is necessary to consider briefly the effect on electric fields produced by passage through a boundary. If the electric fields on either side of a boundary are known, the magnetic fields can be found by methods previously considered.

First consider an electric field perpendicular to the surface of a material having specified values of ϵ_r , μ_r , and σ . Refer to Fig. 6-1 in which a dielectric plate lies between condenser plates so that it is perpendicular to the lines of flux. Then, $\nabla \cdot \mathbf{D} = 0$ where there are no charges such as at the boundary at a . Accordingly, if a small volume be considered which contains the point a , it is seen that as many lines of flux leave it as enter. Therefore the normal \mathbf{D} must be continuous through the boundary. This leads immediately to

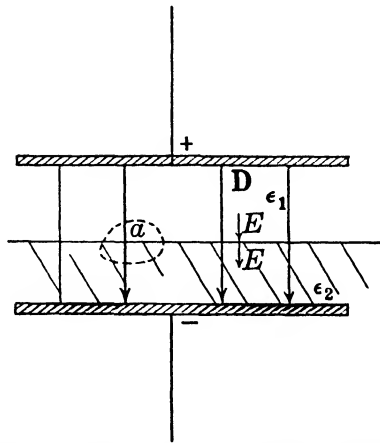


FIG. 6-1. Continuity of normal component of \mathbf{D} at a dielectric discontinuity.

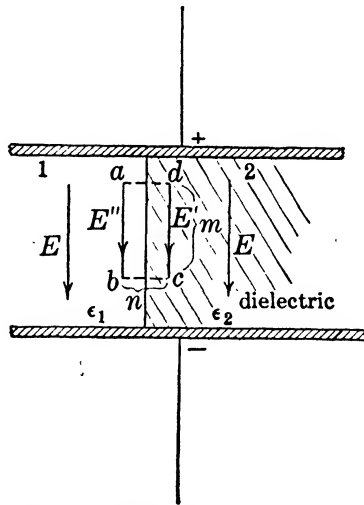


FIG. 6-2. Continuity of tangential component of \mathbf{E} at a dielectric discontinuity.

the effect on \mathbf{E} because in the space above a , $\mathbf{E} = \mathbf{D}/\epsilon_1$, whereas below the surface in the dielectric $\mathbf{E} = \mathbf{D}/\epsilon_2$. In other words, the normal electric field intensity decreases on passing into a medium of higher dielectric constant, and vice versa.

For the purpose of investigating the tangential component of \mathbf{E} , refer to Fig. 6-2. Consider a small section of the boundary at

a. To the left of the boundary the field intensity is E'' and to the right it is E' . The length and width of the rectangular section are respectively m and n . The line integral of \mathbf{E} around the rectangular path must be given by the value $\partial\phi/\partial t$ where ϕ is the flux threading the area. The line integral is the sum of the following components:

$$\begin{array}{ll} \text{Along } ab, & (E'')m. \\ \text{Along } bc, & (0)n. \\ \text{Along } cd, & (-E')m. \\ \text{Along } da, & (0)n. \end{array}$$

Thus

$$(E'' - E')m = \frac{\partial\phi}{\partial t}.$$

But $\phi = Bmn$, therefore

$$(E'' - E') = n \frac{\partial B}{\partial t}.$$

Now, in the limit when we are right at the boundary, $n = 0$. Thus, by substituting 0 for n above, we have

$$(E'' - E') = 0,$$

or

$$E'' = E'.$$

This indicates that the tangential components of \mathbf{E} at a boundary must be continuous, that is, the same on both sides. However, the tangential component of \mathbf{D} changes at a boundary, because in the case under consideration, for example, to the left of the boundary where the dielectric constant may be taken as ϵ_1 , $\mathbf{D} = \epsilon_1\mathbf{E}$, while to the right where ϵ is ϵ_2 , $\mathbf{D} = \epsilon_2\mathbf{E}$. Thus the tangential component of \mathbf{D} increases on passing from a medium of low ϵ_r to one of higher ϵ_r .

A similar argument will show that the tangential components of \mathbf{H} are continuous across a boundary and thus that tangential \mathbf{B} will increase on passing from a material of low relative permeability to one of higher permeability.

Since $\nabla \cdot \mathbf{B} = 0$, it is also clear that at a boundary the normal component is continuous. The argument is the same as that applied to \mathbf{D} above.

6-2. EXPONENTIAL NOTATION: GENERAL EQUATIONS

In Chapter IV a simple treatment of plane waves in non-conducting space was presented. It is now advisable, on the basis of

the introduction given, to change the method of treatment slightly in order to treat more carefully the problem of reflection and transmission in general. This discussion will begin with equations (4-1) and (4-2) and will repeat in a slightly different manner some of the procedure followed in Chapter IV. Rewrite equations (4-1) and (4-2) as follows:

$$\nabla \times \mathbf{H} = \sigma \mathbf{E} + \epsilon \frac{\partial \mathbf{E}}{\partial t}, \quad (6-1)$$

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}. \quad (6-2)$$

In the text material to be covered later only sinusoidal variations with time will be considered. Thus it is helpful to eliminate from the above equations the differentiations with respect to time. As a matter of convenience this will be done by substituting for \mathbf{E} and \mathbf{H} , above, the terms

$$\mathbf{E} = \bar{\mathcal{E}} e^{j\omega t},$$

and

$$\mathbf{H} = \bar{\mathcal{H}} e^{j\omega t},$$

where the exponential terms contain all the reference to time, and the script $\bar{\mathcal{E}}$'s and $\bar{\mathcal{H}}$'s now are only functions of x , y , and z .* The exponential can be used as the time function because

$$e^{j\omega t} = \cos \omega t + j \sin \omega t.$$

It is immediately seen that the substitution of $e^{j\omega t}$ is equivalent to the substitution of *sine* and *cosine* functions. That $e^{j\omega t}$ includes an imaginary term $j \sin \omega t$ is not a source of difficulty because when all work has been completed and a practical answer is desired, the imaginary terms may be discarded. The reason for the use of $e^{j\omega t}$ is the simplicity of its differentiation and the ease with which it is handled in the subsequent material.

If this substitution be made into equations (6-1) and (6-2) we obtain

$$\nabla \times \bar{\mathcal{H}} e^{j\omega t} = \sigma \bar{\mathcal{E}} e^{j\omega t} + j\omega \epsilon \bar{\mathcal{E}} e^{j\omega t}, \quad (6-3)$$

where it must be remembered now that $\bar{\mathcal{E}}$ and $\bar{\mathcal{H}}$ are no longer functions of t . The last term is obtained from

* See: *Communication Circuits*, L. A. Ware and H. R. Reed, John Wiley and Sons, New York, 1944, p. 205.

$$\begin{aligned}\frac{\partial}{\partial t} \epsilon \bar{\mathcal{E}} e^{j\omega t} &= j\omega \epsilon \bar{\mathcal{E}} e^{j\omega t} + \epsilon e^{j\omega t} \frac{\partial \bar{\mathcal{E}}}{\partial t}, \\ &= j\omega \epsilon \bar{\mathcal{E}} e^{j\omega t} + 0.\end{aligned}\quad (6-4)$$

Equation (6-2) now becomes

$$\nabla \times \bar{\mathcal{E}} e^{j\omega t} = -j\mu\omega \bar{\mathcal{H}} e^{j\omega t}.\quad (6-5)$$

Now divide out the exponential and rearrange equations (6-3) and (6-5) to give

$$\nabla \times \bar{\mathcal{H}} = (\sigma + j\omega\epsilon)\bar{\mathcal{E}},\quad (6-6)$$

and

$$\nabla \times \bar{\mathcal{E}} = -j\mu\omega \bar{\mathcal{H}}.\quad (6-7)$$

In these equations it must be carefully noted also that the j used is merely the usual $\sqrt{-1}$, namely a 90° time operator, and is not even remotely related to the \mathbf{j} unit vector considered in earlier matter. The letters will be distinguished by writing the above j in light face type.

The solution of equations (6-6) and (6-7) can now be found as outlined in Chapter IV. Write them as

$$\nabla \times \bar{\mathcal{H}} = A\bar{\mathcal{E}},\quad (6-8)$$

and

$$\nabla \times \bar{\mathcal{E}} = B\bar{\mathcal{H}},\quad (6-9)$$

where

$$A = \sigma + j\omega\epsilon,\quad (6-10)$$

and

$$B = -j\omega\mu.\quad (6-11)$$

By taking the curl of equation (6-9) and using equation (6-8),

$$\begin{aligned}\nabla \times (\nabla \times \bar{\mathcal{E}}) &= B\nabla \times \bar{\mathcal{H}} \\ &= AB\bar{\mathcal{E}}.\end{aligned}$$

Now in a manner similar to that used on equation (4-8)

$$\begin{aligned}\nabla^2 \bar{\mathcal{E}} &= -AB\bar{\mathcal{E}}, \\ &= +j\omega\mu(\sigma + j\omega\epsilon)\bar{\mathcal{E}} \\ &= -(\omega^2\mu\epsilon - j\sigma\omega\mu)\bar{\mathcal{E}} \\ &= -b^2\bar{\mathcal{E}},\end{aligned}\quad (6-12)$$

where

$$b = \sqrt{\omega^2\mu\epsilon - j\sigma\omega\mu}.\quad (6-13)$$

In an exactly similar manner an equation involving $\overline{\mathcal{C}}$ can be derived. It is found to be the same expression as equation (6-12).

$$\nabla^2 \overline{\mathcal{C}} = -b^2 \overline{\mathcal{C}}. \quad (6-12a)$$

This is the general *wave equation* and involves all the parameters which may occur in the field (except t which has been eliminated). The equation, however, is true for expressions involving t also, that is, the equation could just as well be written in block letters.

6-3. ILLUSTRATIVE EXAMPLES: EXPONENTIAL NOTATION

The material of the preceding section is somewhat clarified by presenting a few simple illustrations of the operations involved.

Example 1. First, let it be required to show that the real part of $(\partial/\partial t)e^{j\omega t}$ is the same as the derivative of the real part of $e^{j\omega t}$, that is, the derivative of $\cos \omega t$.

$$\begin{aligned} \frac{\partial}{\partial t} e^{j\omega t} &= j\omega e^{j\omega t} = j\omega(\cos \omega t + j \sin \omega t) \\ &= j\omega \cos \omega t - \omega \sin \omega t. \end{aligned}$$

and the real part is $-\omega \sin \omega t$. Now the derivative of the real part of $e^{j\omega t}$ is

$$\frac{\partial}{\partial t} \cos \omega t = -\omega \sin \omega t,$$

which is the same as above.

Example 2. As a second illustration assume that Ae^{at} is the solution to the following equation, with the understanding that only the real part is to be retained.

$$\frac{d^2 y}{dt^2} + b^2 y = 0.$$

Substitution of $y = Ae^{at}$ into the equation results in

$$Aa^2 e^{at} + b^2 A e^{at} = 0,$$

from which

$$a = \pm jb,$$

and

$$y = A_1 e^{jbt} + A_2 e^{-jbt},$$

or

$$y_{(real)} = A_1 \cos bt + A_2 \cos bt = A' \cos bt.$$

Now assume directly that the solution to the equation is

$$y = B \cos at.$$

Substitution into the original equation results in

$$-Ba^2 \cos at + b^2B \cos at = 0,$$

or

$$a = \pm b.$$

Thus

$$\begin{aligned} y &= B_1 \cos bt + B_2 \cos (-bt) \\ &= B' \cos bt. \end{aligned}$$

which is the same as arrived at above.

The foregoing examples illustrate the use of the exponential in simple cases, but they do not show the simplification which may result from its use in more complicated problems. As the student becomes more familiar with the use of the exponential, its value will become more apparent.

6-4. REFLECTION FROM PERFECT CONDUCTORS

To serve partly as a review, equation (6-12) will be applied to the previously treated plane wave in non-conducting space. By analogy with the previous treatment we shall pick ϵ_y as the only component of electric field intensity present and shall assume no variations with respect to y and z so that partials with respect to y and z are zero. Referring to equation (4-11) and using the right-hand side of equation (6-12), we can now write

$$\frac{\partial^2 \epsilon_y}{\partial x^2} = -b^2 \epsilon_y. \quad (6-14)$$

All the other terms are zero. It will be noted that this is a considerable simplification and occurs only because we arbitrarily fix the position of the $\bar{\epsilon}$ vector and effectively specify the direction of propagation.

The solution to this equation is known to be

$$\epsilon_y = \epsilon_0 e^{ax}.$$

From this and equation (6-14),

$$\epsilon_0 a^2 e^{ax} = -b^2 \epsilon_0 e^{ax},$$

or

$$\begin{aligned} a^2 &= -b^2, \\ a &= \pm jb \\ &= \pm j \sqrt{\omega^2 \mu \epsilon - j \sigma \omega \mu}. \end{aligned} \quad (6-15)$$

The solution including the time term is then

$$E_y = (E_2 e^{bx} + E_1 e^{-bx}) e^{j\omega t}, \quad (6-16)$$

where E_1 and E_2 are constants of integration.

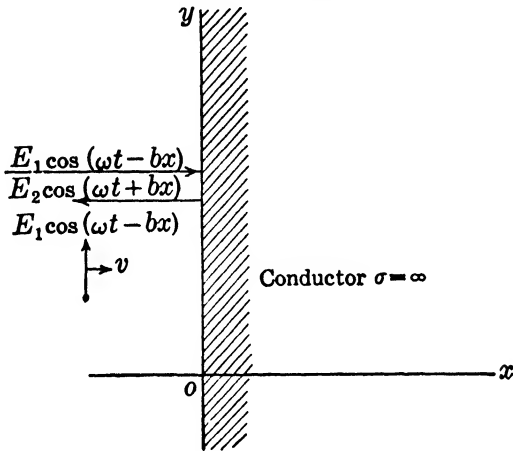


FIG. 6-3. Reflection of a wave against a perfect conductor.

In this section we are concerned with a medium to the left of the boundary in Fig. 6-3 where σ is zero. Therefore $b = \omega \sqrt{\mu \epsilon}$, which is the same as the β which appeared in Chapter IV, and

$$E_y = E_1 e^{j(\omega t - bx)} + E_2 e^{j(\omega t + bx)} \quad (6-17)$$

$$\begin{aligned} &= E_1 [\cos(\omega t - bx) + j \sin(\omega t - bx)] + \\ &\quad E_2 [\cos(\omega t + bx) + j \sin(\omega t + bx)], \end{aligned} \quad (6-18)$$

from which, as stated above, the imaginary terms are to be removed. Then

$$E_y = E_1 \cos(\omega t - bx) + E_2 \cos(\omega t + bx). \quad (6-19)$$

The two terms on the right represent two waves, one traveling in the direction of positive x and the other in the direction of negative x . For reflection against a perfect conductor, the first term will represent the incident wave and the second, the reflected wave as in Fig. 6-3. At the boundary the tangential electric field

intensity E_y must be zero because just inside the conductor the conductivity is infinite and the material cannot support an electric intensity of any amount. Thus at $x = 0$,

$$E_y = E_1 \cos \omega t + E_2 \cos \omega t = 0,$$

and

$$E_1 = -E_2. \quad (6-20)$$

It is seen immediately that the electric field is reflected with a reversal of phase.

This general situation is shown in Fig. 6-4 which represents the two waves, the incident wave E_1 , and the reflected wave E_2 , after the initial transient period has been passed. The waves shown at a , as well as in the subsequent diagrams, should be considered as extending indefinitely to the left, in the negative x direction. At a is shown the condition at a time which may be taken as $t = 0$. It is seen that at all points to the left of the origin, the resultant amplitude is zero. At b where $\omega t = \pi/8$, the wave E_1 has progressed $\pi/8$ radians to the right, and the reflected wave E_2 has moved $\pi/8$ radians to the left, creating a condition where the resultant amplitude at $x = -\lambda/4$ is beginning to build up in a negative direction. Note that conditions are always such that at $x = 0$, the amplitude is zero, thus meeting the boundary condition imposed by the perfect conductor. At $x = -\lambda/2$ the amplitude is still zero, and at $x = -3\lambda/4$ the amplitude is positive. At c this progress has continued. An inspection of the waves for values of ωt up to π will indicate clearly that the resultant amplitude at $x = -\lambda/4$ oscillates between values of $-2A$ and $+2A$ where A is the amplitude of each wave. At points $x = -\lambda/2, -\lambda, \text{etc.}$, the resultant amplitude is at all times zero. Thus a *standing wave* is produced, so called because effectively it does not move in space since the nodes, or points of zero resultant amplitude, are fixed. It is thus seen that, whereas a wire of unit length placed parallel to the electric vector at any point where an incident wave alone is progressing will have induced in it an effective emf equal to $A/\sqrt{2}$, a wire placed in the space to the left of $x = 0$ in Fig. 6-4 will have induced in it an effective emf which will depend on the *position* of the wire. Thus at $x = -\lambda/2$, zero emf will be induced, whereas at $x = -\lambda/4$ the effective emf will be $2A/\sqrt{2} = \sqrt{2}A$. Accordingly, a simple method exists of detecting standing waves and of determining the wavelength, because if a wire were moved from the position $x = 0$

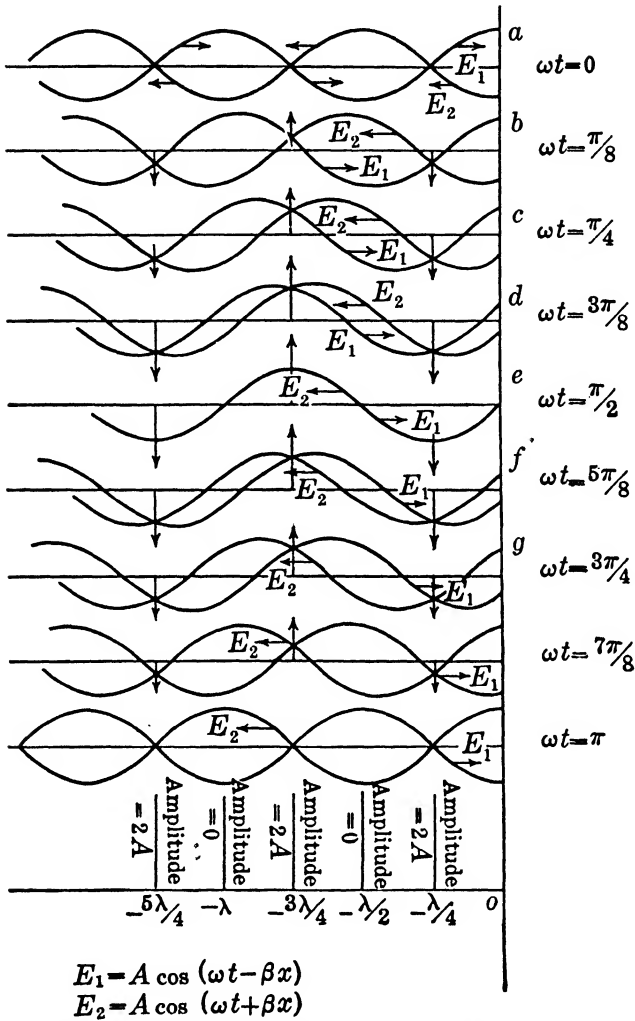


FIG. 6-4. Representation of a standing wave caused by a perfect reflector.

to the left, reading continuously the induced emf, the first node would be found at $x = 0$ and the second at $x = -\lambda/2$. A measurement of the distance traversed then is known to give $\lambda/2$, and λ can be calculated. This is, of course, true of movement from $x = -\lambda/2$ to $x = -\lambda$, or, in general, movement from one position of zero amplitude (or maximum amplitude) to another of zero

amplitude (or maximum) will be over an integral number of half-wavelengths.

6-5. THE MAGNETIC COMPONENT

The total electric field intensity in the space outside the conductor of Fig. 6-3 is given by

$$E_y = E_1[\cos(\omega t - bx) - \cos(\omega t + bx)]. \quad (6-21)$$

Now the magnetic wave can be found from this by using equation (4-2).

$$\begin{aligned} \frac{\partial \mathbf{H}}{\partial t} &= -\frac{1}{\mu} \nabla \times \mathbf{E}, \\ \frac{\partial \mathbf{H}}{\partial t} &= -\frac{1}{\mu} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E_y & 0 \end{vmatrix} \\ &= -\mathbf{k} \frac{1}{\mu} \frac{\partial E_y}{\partial x}, \\ &= -\mathbf{k} \frac{1}{\mu} E_1 [b \sin(\omega t - bx) + b \sin(\omega t + bx)] \\ &= -\mathbf{k} \frac{b}{\mu} E_1 [\sin(\omega t - bx) + \sin(\omega t + bx)], \end{aligned}$$

or

$$\mathbf{H} = -\mathbf{k} \frac{b}{\omega \mu} E_1 [-\cos(\omega t - bx) - \cos(\omega t + bx)], \quad (6-22)$$

or

$$H_x = \sqrt{\frac{\epsilon}{\mu}} E_1 [\cos(\omega t - bx) + \cos(\omega t + bx)], \quad (6-23)$$

because

$$b = \omega \sqrt{\epsilon \mu}.$$

Now in free space,

$$\epsilon = \frac{1}{36\pi \cdot 10^9},$$

and

$$\mu = 4\pi \cdot 10^{-7},$$

so H_x turns out to be

$$H_x = \frac{E_1}{120\pi} [\cos(\omega t - bx) + \cos(\omega t + bx)] \text{ amp/m}, \quad (6-23a)$$

if E_1 is expressed in volts per meter. Equation (6-23) indicates that the magnetic wave is reflected *without* a change of phase because of the similarity in signs of the *cosine* terms.

It is essential that a clear picture be formed of this wave. A trigonometric transformation applied to equation (6-21) gives

$$E_y = 2E_1 \sin bx \sin \omega t. \quad (6-24)$$

This is known as a standing wave for it will be noted that there are values of x for which E_y never has any value but zero (except in the initial transient period during which the wave is being set up). (See Fig. 6-4). Thus the wave does not move in space. Such points are known as nodes and occur at

$$x = -\frac{n\pi}{b} = -\frac{n\pi}{\omega \sqrt{\mu\epsilon}}. \quad (6-25)$$

Now $1/\sqrt{\mu\epsilon} = v$, the velocity of the wave, and

$$\frac{v}{\omega} = \frac{v}{2\pi f} = \frac{\lambda}{2\pi}.$$

Thus

$$x = -\frac{n\pi\lambda}{2\pi} = -n\frac{\lambda}{2}. \quad (6-26)$$

From this it is seen that nodes occur at multiples of $\lambda/2$, and, of course, a node occurs at $x = 0$, where the reflection takes place, as stated above.

A trigonometric transformation will also simplify equation (6-23) and the result is

$$H_z = 2\sqrt{\frac{\epsilon}{\mu}} E_1 \cos bx \cos \omega t.$$

Immediately it is seen that there is a 90° phase difference between the resultant magnetic and electric waves. However, the magnetic wave is, of course, also a standing wave.

Additional insight into this wave is obtained by calculating the Poynting vector for the field. The power is

$$\mathbf{P} = (\mathbf{E} \times \mathbf{H}).$$

In this simple case the product, $\mathbf{E} \times \mathbf{H}$, merely becomes

$$E_y H_z \sin 90^\circ = E_y H_z,$$

and

$$\begin{aligned} P &= 4\sqrt{\frac{\epsilon}{\mu}} E_1^2 (\cos bx \sin bx)(\cos \omega t \sin \omega t) \\ &= \sqrt{\frac{\epsilon}{\mu}} E_1^2 \sin 2bx \sin 2\omega t. \end{aligned} \quad (6-27)$$

This equation indicates that there are planes across which no power is transmitted. These planes occur at

$$2bx = 0, -\pi, -2\pi, \text{ etc.},$$

or

$$x = -\frac{n\pi}{2b} = -n\frac{\lambda}{4} \quad (6-28)$$

where n is an integer. When $n = 0$, the node at the reflecting surface is obtained. Between these planes, however, there is found to be an oscillation of energy back and forth along the x axis. Thus consider the point at $2bx = \pi/2$. Here the sine is 1.00, and

$$P = \sqrt{\frac{\epsilon}{\mu}} E_1^2 \sin 2\omega t \text{ watts/m}^2. \quad (6-29)$$

Thus, as time passes, P is alternately positive and negative. When E_y is a maximum, H_z is zero and all the energy is in the electric field and in a position in space corresponding to the position of the electric wave. Some time later the electric field becomes zero and H_z is maximum when all the energy is stored in the magnetic field and in a different region because of the phase difference between E_y and H_z .

6-6. ILLUSTRATIVE EXAMPLE: THE MAGNETIC COMPONENT

Example 3. Let it be assumed that an incident wave, $e_1 = 0.202 \cos(\omega t - bx)$ volts/meter, is propagated through a space where $\epsilon_r = 3.5$ and $\mu_r = 1$, and strikes perpendicularly a perfect reflector. Let it be required to find the reflected magnetic wave, its velocity of propagation, and the position of the power node nearest the reflector (not $x = 0$), if $f = 10^8$ cps. Equation (6-23) may be used to obtain the reflected magnetic component. It will be

$$\begin{aligned} H_{s(\text{reflected})} &= \sqrt{\frac{\epsilon}{\mu}} E_1 \cos(\omega t + bx), \\ &= \frac{\sqrt{3.5}}{377} (0.202) \cos(\omega t + bx), \\ &= 0.001 \cos(\omega t + bx) \text{ amp/m.} \end{aligned}$$

In order to find the velocity of propagation it is necessary to determine b . This is given by equation (6-15) where $\sigma = 0$. Thus

$$b = \frac{\omega}{c} \sqrt{\epsilon_r},$$

but this is ω/v , so

$$v = \frac{c}{\sqrt{\epsilon_r}} = \frac{c}{\sqrt{3.5}} = 1.605 \cdot 10^8 \text{ m/sec.}$$

This is, of course, the same as the expression for velocity of the wave as previously given in Chapter IV.

The position of the nearest power node is obtained from equation (6-28) by setting $n = 1$, giving $x = -\lambda/4$. The wavelength is given by the standard equation, $v = f\lambda$, or

$$\lambda = \frac{v}{f} = \frac{1.605 \cdot 10^8}{10^8} = 1.605 \text{ m.}$$

Thus

$$x = -\frac{1.605}{4} = -0.401 \text{ m.}$$

6-7. REFLECTION FROM A GENERAL BOUNDARY

In treating the more general case where the reflecting boundary may have any value of ϵ_3 , μ_3 , and σ , use will be made of the general equation (6-16). The two parts of the equation will be written as follows:

$$\text{incident wave, } e_1 = E_1 e^{j(\omega t - b_1 x)}, \quad (6-30)$$

$$\text{reflected wave, } e_2 = E_2 e^{j(\omega t + b_1 x)}, \quad (6-31)$$

where E_1 and E_2 are the amplitudes of the waves. Let the magnitudes of the incident and reflected magnetic components be represented by H_1 and H_2 respectively, and it has already been seen from §6-5 that for medium (1),

$$H_1 = \sqrt{\frac{\epsilon_1}{\mu_1}} E_1, \quad (6-32)$$

$$H_2 = -\sqrt{\frac{\epsilon_1}{\mu_1}} E_2. \quad (6-33)$$

In these equations subscripts 1 and 2 refer to the transmitted and reflected components respectively, in medium (1), while subscript 3 refers to the medium to the right of the boundary (medium 3).

Now, for future use, by analogy with transmission line work, we *define* an impedance

$$\frac{E_1}{H_1} = \sqrt{\frac{\mu_1}{\epsilon_1}} = Z_1. \quad (6-34)$$

$$\frac{E_2}{H_2} = -\sqrt{\frac{\mu_1}{\epsilon_1}} = Z_2 = -Z_1. \quad (6-35)$$

The term Z may be considered as the impedance of the medium through which the wave is traveling. Its unit is

$$\frac{\text{volts/meter}}{\text{amperes/meter}} = \text{ohms}.$$

In free space the value of Z is 377 ohms.

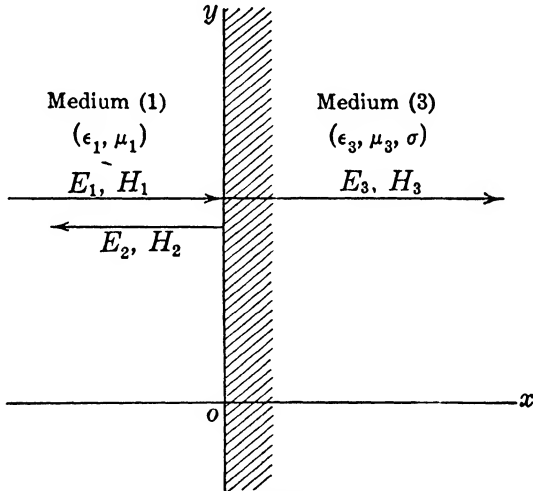


FIG. 6-5. Reflection of a wave against an imperfect conductor.

Consider Fig. 6-5 where an incident wave E_1 is meeting a conductor at $x = 0$. Because the medium to the right of $x = 0$, medium (3), can now support an electric field intensity on account of its finite conductivity, it is possible that E_1 will produce not only a reflected wave E_2 but also a transmitted wave E_3 . According to §6-1, the tangential components of E and H at the surface must be continuous. Therefore,

$$\text{and} \quad \left. \begin{aligned} E_1 + E_2 &= E_3, \\ H_1 + H_2 &= H_3. \end{aligned} \right\} \quad (6-36)$$

The three E waves may now be written as follows:

$$\left. \begin{array}{l} \text{incident,} \quad E_1 e^{j(\omega t - b_1 x)}, \\ \text{reflected,} \quad E_2 e^{j(\omega t + b_1 x)}, \\ \text{transmitted,} \quad E_3 e^{j(\omega t - b_3 x)}, \end{array} \right\} \quad (6-37)$$

where b_1 has been shown to be $\omega \sqrt{\mu_1 \epsilon_1}$ in a region where $\sigma = 0$. See paragraph just above equation (6-17). It is now necessary to write the H components in terms of the E components. This has already been done for H_1 and H_2 . See equations (6-32) and (6-33). It is seen from equation (6-22) that the magnitude of the transmitted H wave is given by

$$H_3 = \frac{b_3}{\omega \mu_3} E_3, \quad (6-38)$$

where b_3 is the more complete expression given by equation (6-15) involving σ .

From equations (6-15) and (6-38) the impedance of the conducting material is defined as

$$\begin{aligned} \frac{E_3}{H_3} &= \frac{\omega \mu_3}{\sqrt{\omega^2 \mu_3 \epsilon_3 - j \sigma \omega \mu_3}}, \\ &= \frac{1}{\sqrt{\frac{\epsilon_3}{\mu_3} - j \frac{\sigma}{\omega \mu_3}}} \equiv Z_3. \end{aligned} \quad (6-39)$$

From this equation it is seen that Z_3 is complex and involves a phase shift between E_3 and H_3 . This will be referred to later.

It is now possible to derive the reflection equation which will be valid for medium (1), with $\sigma = 0$, and a conductor as medium (3). Substituting the values of H from equations (6-34), (6-35), and (6-39) into equation (6-36), there results

$$E_1 + E_2 = E_3,$$

and

$$\frac{E_1}{Z_1} - \frac{E_2}{Z_1} = \frac{E_3}{Z_3},$$

from which

$$\frac{E_2}{E_1} = \frac{\frac{Z_3}{Z_1} - 1}{\frac{Z_3}{Z_1} + 1}.$$

Let

$$\frac{Z_3}{Z_1} = K,$$

and this becomes

$$\frac{E_2}{E_1} = \frac{K - 1}{K + 1}. \quad (6-40)$$

In the above, it must be kept in mind that the E 's, H 's, Z 's, and K may be complex quantities.

Before going farther it is well to briefly consider the meaning of this equation as it ties in very well with previous material. In the first place, if $K = 1$, that is, if $Z_3 = Z_1$, there is no reflected wave,

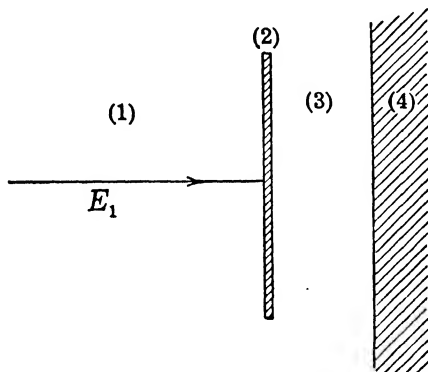


FIG. 6-6. Use of "377-ohm" cloth to reduce standing waves.

as one would expect, because then the boundary becomes non-existent.* On the other hand, if $K = 0$, that is, if $Z_3 = 0$, for a perfect conductor where $\sigma = \infty$, E_2 becomes $-E_1$ which we found previously to be the case. Equation (6-40), however, is very

* An application of this principle is the use of the so-called 377-ohm cloth for the purpose of matching a conducting wall to free space for the elimination of reflections. See Fig. 6-6. Here the wall, (4), is a perfect conductor and as such would set up standing waves to the left of it in the absence of the screen (2). However, if the screen is in place and set at a distance from the wall of one quarter-wavelength, where the intensity of the electric field is a maximum, the standing wave in medium (1) can be eliminated provided the screen has the proper intrinsic, or characteristic, impedance. It is found that this impedance must be that of the medium (1), here considered as free space. Thus the Z of the cloth is 377 ohms per unit area. See the following reference: *Fields and Waves in Modern Radio*, S. Ramo, and J. R. Whinnery, John Wiley and Sons, New York, 1944, p. 277.

general and takes care of cases where both Z_1 and Z_3 may be complex although here only Z_3 is allowed to be so.

Now it is necessary to determine E_3 . This is obtained easily from the same two equations and is

$$\frac{E_3}{E_1} = \frac{2K}{K+1}. \quad (6-41)$$

This is clearly consistent with equation (6-40) because if $K = 0$, then the transmitted wave must be zero. Also if there is no discontinuity at the boundary, $Z_3 = Z_1$, $K = 1$, $E_3 = E_1$, which means that the original wave is transmitted unchanged.

We now have a means of crossing the boundary and can consider in some detail the transmitted wave.

6-8. REFLECTION OF WAVES AGAINST A DIELECTRIC

In the event that both mediums (1) and (3) are dielectrics with $\sigma = 0$, rather simple relations are obtained for reflected and transmitted components. In this case

$$K = \frac{Z_3}{Z_1} = \sqrt{\frac{\mu_3}{\epsilon_3}} \sqrt{\frac{\epsilon_1}{\mu_1}} = \sqrt{\frac{\mu_3 \epsilon_1}{\mu_1 \epsilon_3}}. \quad (6-42)$$

which is a real number and may be greater or less than unity. The reflected electric wave is given by equation (6-40)

$$E_2 = \frac{K-1}{K+1} E_1. \quad (6-43)$$

From this it is seen that if K is less than unity a reversal of phase results, whereas if $K > 1$, there is no reversal. The limiting case is thus given as follows:

$$\begin{aligned} \text{for reversal, } & \mu_3 \epsilon_1 < \mu_1 \epsilon_3; \\ \text{for no reversal, } & \mu_3 \epsilon_1 > \mu_1 \epsilon_3. \end{aligned}$$

The transmitted wave amplitude is given by equation (6-41) which indicates that the transmitted component remains at all times in phase with the incident wave at the boundary.

As an illustration, consider an incident wave $e_1 = \cos(\omega t - b_1 x)$ in a medium where $\epsilon_{r1} = 3.5$ and $\mu_{r1} = 1$, striking a dielectric where $\epsilon_{r3} = 3.0$ and $\mu_{r3} = 1$. From these data

$$K = \sqrt{\frac{3.5}{3.0}} = 1.08.$$

From equation (6-43)

$$E_2 = \frac{1.08 - 1}{1.08 + 1} = 0.0384,$$

and from equation (6-41)

$$E_3 = \frac{2(1.08)}{2.08} = 1.0384.$$

The velocity in the medium (1) will be

$$v_1 = \frac{c}{\sqrt{3.5}} = 1.605 \cdot 10^8 \text{ m/sec},$$

and in medium (3),

$$v_3 = \frac{c}{\sqrt{3.0}} = 1.73 \cdot 10^8 \text{ m/sec}.$$

6-9. WAVES IN A CONDUCTOR

If a wave meets a conductor whose conductivity is not infinite, we have seen that E_3 (and H_3) will have finite values which, from equation (6-41), can be expressed in terms of the incident wave amplitude as

$$E_3 = \frac{2KE_1}{K+1}. \quad (6-44)$$

From equation (6-37) the transmitted wave is

$$\left. \begin{array}{l} \text{electric component, } E_3 e^{j(\omega t - b_3 x)}; \\ \text{magnetic component, } \frac{E_3}{Z_3} e^{j(\omega t - b_3 x)}. \end{array} \right\} \quad (6-45)$$

To begin with, a great simplification will result by noting that for a reasonably good conductor such as Cu, Ag, and Al,

$$b_3 \doteq \sqrt{-j\sigma\omega\mu_3} \quad (6-46)$$

on account of the very large value of the σ term relative to the ϵ_3 term.

Consider, for example, a conductor which has a conductivity of 0.001 times the conductivity of copper. (For Cu, $\sigma = 57.8 \cdot 10^6$ mhos/m³). At a frequency such that $\omega = 10^9$ rad/sec we have, from equation (6-15),

$$\begin{aligned}
 b_3 &= \sqrt{(10^{18})4\pi 10^{-7} \frac{1}{36\pi 10^9} - j57.8 (10^3)(10^9)4\pi 10^{-7}} \\
 &= \sqrt{11.1 - j72.6 (10^7)}.
 \end{aligned}$$

Thus it is obvious that, even in this material of relatively low conductivity, it is permissible to use the approximate expression for b given by equation (6-46).

From equation (6-46),

$$b_3 = \sqrt{-j} \sqrt{\sigma\omega\mu_3} \quad (6-47)$$

$$\begin{aligned}
 &= \sqrt{\frac{\sigma\omega\mu_3}{2}} (1 - j) \\
 &= s(1 - j)
 \end{aligned} \quad (6-48)$$

where

$$s = \sqrt{\frac{\sigma\omega\mu_3}{2}}; \quad (6-49)$$

and from equation (6-39), neglecting the ϵ_3 term,

$$Z_3 = \frac{1}{\sqrt{-j} \sqrt{\frac{\sigma}{\omega\mu_3}}}, \quad (6-50)$$

$$\begin{aligned}
 &= \sqrt{\frac{\omega\mu_3}{\sigma}} \frac{1}{\sqrt{-j}}, \\
 &= \sqrt{\frac{\omega\mu_3}{2\sigma}} (1 + j).
 \end{aligned} \quad (6-51)^*$$

By using equation (6-48), the third expression (6-37) becomes

$$E_3 e^{j(\omega t - sx + j\sigma x)} = E_3 e^{-sx} e^{j(\omega t - sx)}, \quad (6-52)$$

or the real component is

$$E_3 e^{-sx} \cos(\omega t - sx). \quad (6-53)$$

This is an interesting equation because it states that the transmitted wave will be damped on account of the exponential term, and the

* The question may arise in this connection as to why the imaginary part of this result is not discarded since it has been done in certain other cases. The answer is that the j appearing in equation (6-51), and in certain other equations, did not enter through the assumption of $e^{j\omega t}$ as the law of time variation. It is only the imaginary part arising from the time-variation assumption which is thrown away. In this connection it is of interest to note the following. In $\epsilon^{-\gamma} e^{j\omega t}$, the j entering through the

velocity will be a function of the frequency and conductivity. The numerical value of s for copper at $\omega = 10^6$ rad/sec, for example, is 1905 m^{-1} , which shows that $sx = 1$ at a distance of 0.525 mm into the conductor.

The *propagation constant*, by analogy with transmission lines, can now be defined by using expression (6-52) and written

$$E_3 e^{-(s+j\beta)x} e^{j\omega t}.$$

The term $(s + j\beta)$ in this expression is known as the propagation constant, and is usually written as

$$\gamma = \alpha + j\beta,$$

where α is the *attenuation* constant and β is the *phase* constant. It may be noted that for transmission of the wave in metals, $\alpha = \beta$, and furthermore, that the attenuation is extraordinarily high as compared with transmission line attenuation.

The magnetic component of the transmitted wave may be found by dividing expression (6-52) by Z_3 , obtaining

$$\frac{E_3 e^{-sx} e^{j(\omega t - sx)}}{1 + j} \sqrt{\frac{2\sigma}{\omega\mu_3}} = \sqrt{\frac{\sigma}{\omega\mu_3}} E_3 e^{-sx} e^{j[\omega t - (\pi/4) - sx]}, \quad (6-54)$$

because

$$\frac{1}{1 + j} = \frac{\varepsilon^{-j\pi/4}}{\sqrt{2}}.$$

γ ($= \alpha + j\beta$) should remain but the imaginary part arising from the $e^{j\omega t}$ is to be discarded. It will be shown that the real part of $e^{-\gamma x} e^{j\omega t}$ is the same as $\varepsilon^{-\gamma x} \cos \omega t$.

$$\begin{aligned} e^{-\gamma x} e^{j\omega t} &= \varepsilon^{-\alpha x - j\beta x} e^{j\omega t}, \\ &= \varepsilon^{-\alpha x} e^{j(\omega t - \beta x)}; \end{aligned}$$

or the real part is

$$\varepsilon^{-\alpha x} \cos(\omega t - \beta x).$$

Now

$$\begin{aligned} \varepsilon^{-\gamma x} \cos \omega t &= \varepsilon^{-\alpha x} \varepsilon^{-j\beta x} \cos \omega t \\ &= \varepsilon^{-\alpha x} (\cos \beta x - j \sin \beta x) \cos \omega t \\ &= \varepsilon^{-\alpha x} \cos \omega t \angle -\beta x \\ &= \varepsilon^{-\alpha x} \cos(\omega t - \beta x). \end{aligned}$$

In the first development the j term was discarded, and in the second, it was not discarded because the assumption did not require it; however, the result was the same. In future material it is instructive to note in all cases what happens to the j operator. Such attention to this detail will gradually increase one's familiarity with the peculiarities of this operator.

The real magnetic component now becomes

$$H_{(real)} = \sqrt{\frac{\sigma}{\omega\mu_3}} E_3 e^{-sx} \cos \left[\omega t - \left(\frac{\pi}{4} + sx \right) \right]. \quad (6-55)$$

It is seen that the magnetic component is damped in the same way and at the same rate as the electric component. Also there is now a 45° displacement between the two waves. The magnetic component lags the electric component.

6-10. TRANSMISSION IN METAL

The *depth of penetration* of the wave into the metal is defined as that distance x ($=\delta$) which causes the exponent of the exponential in equation (6-53) to become unity. That is, the depth of penetration is defined as the distance traveled before the magnitude of the wave has dropped to $1/\epsilon$ of its initial value. Thus

$$\begin{aligned} s\delta &= 1, \quad \text{or} \quad \delta = \frac{1}{s} \\ \delta &= \sqrt{\frac{2}{\omega\sigma\mu_3}}. \end{aligned} \quad (6-56)$$

For all reasonably good conductors this distance is very small at high frequencies. In order to obtain an idea of the magnitude of this depth, δ is plotted against ω for copper in Fig. 6-7. In this example we have

$$\begin{aligned} \sigma &= 57.8 (10^6) \text{ mhos/m}^3, \\ \mu_3 &= 4\pi 10^{-7}, \\ c &= 3 (10^8) \text{ m/sec.} \end{aligned}$$

The velocity in the metal is very low and is given by reference to equation (6-52) as

$$v = \frac{\omega}{s} = \sqrt{\frac{2\omega}{\sigma\mu_3}}, \quad (6-57)$$

where s is the coefficient of x in the imaginary exponent. In Fig. 6-7 the velocity in copper is also plotted against ω . It must be kept in mind that equations (6-56) and (6-57) are valid only where the ω term in equation (6-13) is at least 10 times as large as the ω^2 term.

In connection with transmission in metals it is also of interest to find the magnitude of Z_3 . One should expect it to be very low,

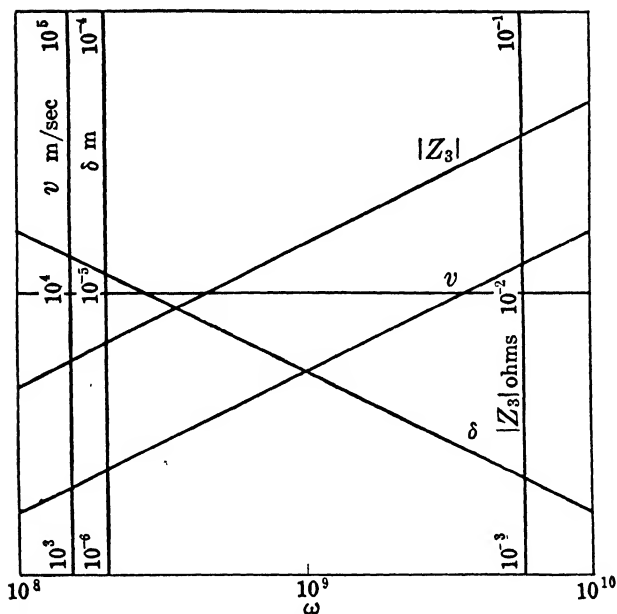


FIG. 6-7. Variation of δ , v , and Z_3 for copper as a function of ω .

on the basis of known characteristics of copper. From equation (6-51)

$$\begin{aligned} |Z_3| &= \sqrt{\frac{\omega\mu_3}{2\sigma}} |(1+j)| \\ &= \sqrt{\frac{\omega\mu_3}{\sigma}} \text{ ohms.} \end{aligned} \quad (6-58)$$

This impedance is also plotted against ω in Fig. 6-7.

6-11. THE REFLECTED WAVE

The incident wave has already been given as

$$E_1 e^{j(\omega t - b_1 x)}$$

of which the real part is

$$E_1 \cos(\omega t - b_1 x). \quad (6-59)$$

Now by equation (6-40)

$$E_2 = E_1 \frac{K - 1}{K + 1},$$

and

$$K = \frac{Z_3}{Z_1} = \sqrt{\frac{\omega\epsilon_1\mu_3}{2\sigma\mu_1}} (1 + j) = M + jM,$$

where

$$M = \sqrt{\frac{\omega\epsilon_1\mu_3}{2\sigma\mu_1}}, \quad (6-60)$$

and ϵ_1 and μ_1 refer to medium (1).

By substituting $M + jM$ for K , we have

$$\begin{aligned} \frac{K - 1}{K + 1} &= \frac{-1}{2M + 1} + \frac{j2M}{2M + 1} \\ &\doteq -1 + j2M, \end{aligned} \quad (6-61)$$

because M is extremely small, permitting one to disregard terms in M^2 . For example in a copper conductor at a frequency of 10^{10} cps,

$$M = 6.93 (10^{-6}).$$

Now we obtain

$$E_2 = E_1(-1 + j2M), \quad (6-62)$$

which indicates that the reflection does not take place with an exact reversal of phase as it did for a perfect conductor, but with a change of phase of slightly less than 180° . It is of interest to determine the angle between an exact reversal and the approximate true position of E_2 . This angle is given by

$$\begin{aligned} \theta &= \tan^{-1} 2M \doteq 2M = \sqrt{\frac{2\omega\epsilon_1\mu_3}{\sigma\mu_1}} \\ &= \sqrt{\frac{2\omega\epsilon_1}{\sigma}} \text{ rad} \end{aligned} \quad (6-63)$$

because of the equality of μ_1 and μ_3 in this case. This angle becomes of the order 0.008° for copper at 10^{10} cps. This is certainly negligible and, accordingly, any good conductor may be considered as a perfect reflector as far as the conditions in medium (1) are practically concerned.

6-12. ILLUSTRATIVE EXAMPLE: REFLECTED WAVE

Example 4. Assume that an incident wave is given by

$$e_1 = E_1 \cos(\omega t - b_1 x).$$

This wave strikes normally upon a surface of aluminum. The parameters are as follows: $E_1 = 1$ v/m, $\epsilon_{r1} = 3.0$, $\mu_{r3} = 1$, $\sigma = 35.3$ (10^6) mhos/m³, $f = 3$ (10^9) cps. Find the following: (a) the transmitted electric amplitude, E_3 ; (b) the transmitted magnetic amplitude, H_3 ; (c) the depth of penetration, δ ; (d) the reflected magnetic amplitude, H_2 ; (e) the incident power; (f) the transmitted power; (g) the velocity of the wave in the metal.

First, calculate s , Z_1 , Z_3 , and K .

$$s = \sqrt{\frac{\omega\sigma\mu_3}{2}} = 6.47 (10^5) \text{ m}^{-1}.$$

$$Z_1 = \sqrt{\frac{\mu_1}{\epsilon_1}} = \frac{377}{\sqrt{3}} = 218 \text{ ohms.}$$

$$Z_3 = (1 + j) \sqrt{\frac{\omega\mu_3}{2\sigma}} = 0.0183(1 + j) \text{ ohm}$$

$$= 0.0259 \angle 45^\circ \text{ ohm.}$$

$$K = \frac{Z_3}{Z_1} = \frac{0.0259 \angle 45^\circ}{218} = 0.000119 \angle 45^\circ$$

$$= 0.000084 + j0.000084.$$

(a) From equation (6-41),

$$E_3 = \frac{2K(1)}{K + 1},$$

$$\doteq 23.8 (10^{-5}) \angle 45^\circ \text{ v/m,}$$

which is very small in magnitude relative to E_1 and 45° ahead of E_1 . (Note that $K \ll 1$.)

(b) Equation (6-45) gives the magnitude of the magnetic component as follows:

$$H_3 = \frac{E_3}{Z_3} = \frac{23.8 (10^{-5}) \angle 45^\circ}{0.0259 \angle 45^\circ}$$

$$= 9.2 (10^{-3}) \text{ amp/m.}$$

(c) The depth of penetration, δ , is given by equation (6-56):

$$\delta = \frac{1}{s} = \frac{1}{6.47 \cdot 10^5}$$

$$= 1.543 (10^{-6}) \text{ m.}$$

(d) The reflected magnetic component is given by

$$\begin{aligned} H_2 &= \frac{E_2}{-Z_1} = \frac{E_1(-1 + j2M)}{-Z_1} = \frac{E_1}{Z_1} \\ &= \frac{1}{218} = 0.00459 \text{ amp/m,} \end{aligned}$$

which is *nearly* in phase with H_1 .

(e) The incident power is obtained from Poynting's vector and the values of E_1 and H_1 . H_1 is given by equation (6-34).

$$\begin{aligned} H_1 &= \frac{E_1}{Z_1} = \frac{1}{218} = 0.00459 \text{ amp/m;} \\ P &= |\mathbf{E} \times \mathbf{H}| = E_1 H_1 = 1(0.00459), \\ &= 0.00459 \text{ watt/m}^2, \end{aligned}$$

as a peak value, or

$$P_{av} = 0.00229 \text{ watt/m}^2$$

(f) The transmitted power is

$$\mathbf{P} = \mathbf{E} \times \mathbf{H}.$$

However, it must be noted here that E_3 and H_3 are out of time phase by 45° so, by analogy with a-c theory, it is necessary to multiply the product of $|E_3|$ and $|H_3|$ by the *cosine* of the angle between them. Thus

$$\begin{aligned} P &= |E_3||H_3| \cos 45^\circ \\ &= 23.8 (10^{-5}) 9.2 (10^{-3}) (0.707) \\ &= 155 (10^{-8}) \text{ watt/m}^2 \end{aligned}$$

or

$$P_{av} = 77.5 (10^{-8}) \text{ watt/m}^2.$$

(g) The velocity in the metal is given by equation (6-57).

$$\begin{aligned} v &= \sqrt{\frac{2\omega}{\sigma\mu_3}} = \sqrt{\frac{2(2\pi)(3)10^9}{35.3 (10^6) 4\pi (10^{-7})}} = 10^4 \sqrt{\frac{300}{35.3}} \\ &= 29,100 \text{ m/sec.} \end{aligned}$$

6-13. NOTE ON THE REFRACTIVE INDEX

The student who has had some contact with optics will no doubt ask how the constants μ and ϵ are related to the refractive index which, in optics, describes the action of a wave crossing a boundary. This relation is worked out as follows.

The refractive index is defined as the ratio of the velocity of the wave in a vacuum to the velocity in the medium under consideration. The velocity in a vacuum is

$$v_1 = \frac{1}{\sqrt{\mu_0 \epsilon_0}}.$$

The velocity v_2 in the medium can be determined in general by the use of the complete expression for b_3 applied to the third expression (6-37).

$$\begin{aligned} b_3 &= \sqrt{\omega^2 \mu \epsilon - j \omega \sigma \mu} \\ &= \sqrt{\omega \mu \sqrt{\omega^2 \epsilon^2 + \sigma^2}} \angle -\frac{1}{2} \tan^{-1} \frac{\sigma}{\omega \epsilon} \end{aligned} \quad (6-64)$$

$$= A + jB. \quad (6-65)$$

Now, by using expression (6-37), the electric wave is represented by

$$E_3 e^{j(\omega t - Ax - jBx)} = E_3 e^{j(\omega t - Ax)} e^{Bx}, \quad (6-66)$$

from which it is seen that the velocity will be given by

$$v_2 = \frac{\omega}{A}. \quad (6-67)$$

From equation (6-64),

$$\begin{aligned} A &= \sqrt{\omega \mu \sqrt{\omega^2 \epsilon^2 + \sigma^2}} \cos \left[-\frac{1}{2} \tan^{-1} \frac{\sigma}{\omega \epsilon} \right] \\ &= \sqrt{\frac{\omega \mu}{2} (\omega \epsilon + \sqrt{\omega^2 \epsilon^2 + \sigma^2})}, \end{aligned} \quad (6-68)$$

and

$$v_2 = \frac{\omega}{A} = \frac{1}{\sqrt{\frac{\mu \epsilon}{2} + \frac{\mu}{2\omega} \sqrt{\omega^2 \epsilon^2 + \sigma^2}}}. \quad (6-69)$$

Now the refractive index becomes

$$\begin{aligned} n &= \frac{1}{\sqrt{\mu_0 \epsilon_0}} \sqrt{\frac{\mu \epsilon}{2} + \frac{\mu}{2\omega} \sqrt{\omega^2 \epsilon^2 + \sigma^2}} \\ &= \sqrt{\frac{\mu_r \epsilon_r}{2} + \frac{\mu_r}{2\omega \epsilon_0} \sqrt{\omega^2 \epsilon^2 + \sigma^2}}. \end{aligned} \quad (6-70)$$

For the special case of $\sigma = 0$,

$$n = \sqrt{\mu_r \epsilon_r}.$$

6-14. PROBLEMS

Table of conductivities

Unit: mho/m³

$$\text{Ag } \sigma = 61.2 \cdot 10^6$$

$$\text{Cu } \sigma = 57.8 \cdot 10^6$$

$$\text{Al } \sigma = 35.3 \cdot 10^6$$

6-1. An electric field intensity vector passes from one dielectric having an $\epsilon_r = 3$ into another where $\epsilon_r = 5.5$. The electric field is at an angle of 35° to the surface of the dielectric in the first medium. What is the angle between the surface and the field in the second dielectric? Is it the same for both **D** and **E**?

6-2. A plane sinusoidal wave is traveling in a space where $\mu_r = 1$, and $\epsilon_r = 3.5$. (a) Find the velocity of the wave. (b) How much of the total power, expressed as a percentage, is conveyed by the electric component?

6-3. A plane sinusoidal electric wave of unit amplitude is traveling in a medium where $\mu_r = 1$ and $\epsilon_r = 3$. This wave reflects from a perfect conductor. Find the amplitude of the reflected magnetic component.

6-4. A plane sinusoidal wave for which $E_1 = 1$ v/m is traveling in a medium in which $\mu_r = \epsilon_r = 1$. The frequency is 10^8 cps. The wave reflects from a perfect conductor placed normal to the direction of travel. Find the positions of the three planes nearest the reflector across which no resultant energy is transmitted in the steady state.

6-5. Show that the two terms under the radical in equation (6-13), namely $\omega^2\mu\epsilon$ and $\sigma\omega\mu$, are dimensionally equivalent.

6-6. An incident wave is expressed as

$$e_1 = \cos(\omega t - bx)$$

where $f = 10^8$ cps. This wave meets a boundary of silver and is partially reflected and transmitted. In the silver, $\epsilon_r = \mu_r = 1$. Find: (a) amplitude of the transmitted electric wave at the surface; (b) velocity of this wave.

6-7. For Prob. 6-6, find E_2 from equation (6-40).

6-8. For Prob. 6-6, find the complete expression H_z for the magnetic component in the metal at the surface.

6-9. For Prob. 6-6, find the depth of penetration into the metal.

6-10. Determine the depth of penetration at $\omega = 10^5$ rad/sec in a sheet of iron, the resistivity of which is 10 times that of copper and the relative permeability of which is 100.

6-11. Determine the depth of penetration into a medium the conductivity of which is 10^{-8} that of copper if $\mu_r = \epsilon_r = 1$, and $\omega = 10^{10}$. (*Note:* The complete expression for b_3 must be used here.)

6-12. Show that s as defined in equation (6-49) is dimensionally equivalent to $(\text{length})^{-1}$.

6-13. Using Poynting's vector, find the maximum and average power flow into the metal per unit cross section for Prob. 6-6.

6-14. In Prob. 6-6, calculate the power flow in the incident wave and compare with results of Prob. 6-13.

6-15. Discuss the nature of the composite wave in medium (1) of Prob. 6-6. Is it a standing wave? Are there any planes across which energy does not pass in the steady state?

WAVES GUIDED BY PARALLEL PLATES

Summary

To provide a simple transfer to the treatment of rectangular wave guides, the subject of waves guided by parallel plates is introduced. The restricting effect on the electromagnetic field caused by the presence of the plates is brought in through the boundary conditions which apply. After the components of the TEM mode are written down there follows a discussion of the power transmitted. Next, the TE mode of transmission is defined and treated at some length to determine the components of the field, the properties of transmission, and the configuration of the field. Power relations are treated briefly. Finally, the TM_1 mode of transmission is touched upon.

7-1. GENERAL CONSIDERATIONS

In the preceding chapters, plane waves in space with no boundaries except perhaps a reflecting wall have been treated in a very elementary manner. Now, as a simple introduction to guided waves, the transmission of a wave between two infinite parallel conducting plates will be considered. This will serve as a simple introduction to the subject of wave-guide transmission. More complete analyses may be found elsewhere for those who wish to pursue the subject farther.*

* A. B. Bronwell and R. E. Beam, *Theory and Application of Microwaves*, New York, McGraw-Hill Book Co., 1947.

Marchand, Nathan, *Ultrahigh Frequency Transmission and Radiation*, New York, John Wiley and Sons, 1947.

S. Ramo and J. R. Whinnery, *Fields and Waves in Modern Radio*, New

The conducting planes shown in Fig. 7-1 lie in the xz plane of a cartesian coordinate system and the separation of the planes, one from the other, is symbolized as d . Since the space between the parallel plates is the region under consideration and since cartesian coordinates are employed, Maxwell's equations already derived for cartesian coordinates are suitable for use here. For the present, we shall assume an absence of free charge carriers in the region between the plates ($\sigma = 0$), in which case the general wave equation takes the form

$$\frac{\partial^2 \mathbf{E}}{\partial x^2} + \frac{\partial^2 \mathbf{E}}{\partial y^2} + \frac{\partial^2 \mathbf{E}}{\partial z^2} = \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}. \quad (7-1)$$

In this form of the wave equation, \mathbf{E} is to be interpreted as a function of x , y , z , and t . The manner in which the electromagnetic field is introduced into the free-space region between the parallel plates will define the mode of propagation and the particular components of \mathbf{E} which are present in any specified mode of propagation.

7-2. TRANSVERSE ELECTRIC-MAGNETIC OR TEM MODE OF PROPAGATION

In the absence of the E_x and H_x components of the electromagnetic field (in Fig. 7-1), both the electric and magnetic fields will be transverse to the direction of propagation, assuming that the direction of wave propagation actually takes place along the x direction. Under these conditions, the mode of propagation is called the TEM mode.

Probably the simplest way in which to introduce an electromagnetic field into the region in question is to connect the terminals

York, John Wiley and Sons, 1944.

R. I. Sarbacher, and W. A. Edson, *Hyper and Ultra-High Frequency Engineering*, New York, John Wiley and Sons, 1943.

J. A. Stratton, *Electromagnetic Theory*, New York, McGraw-Hill Book Co., 1941.

H. H. Skilling, *Fundamentals of Electric Waves*, New York, John Wiley and Sons, 1948.

J. C. Slater, *Microwave Transmission*, New York, McGraw-Hill Book Co., 1942.

S. A. Schelkunoff, *Electromagnetic Waves*, New York, D. Van Nostrand Co., 1943.

L. A. Ware and H. R. Reed, *Communication Circuits*, New York, John Wiley and Sons, 1944.

of an oscillator (or other a-c generator) to the conducting plates. This potential difference will in turn establish a y component of the electric field, E_y . In the absence of E_x , E_z , and H_x , the TEM mode of propagation can be shown to exist. In thinking about the

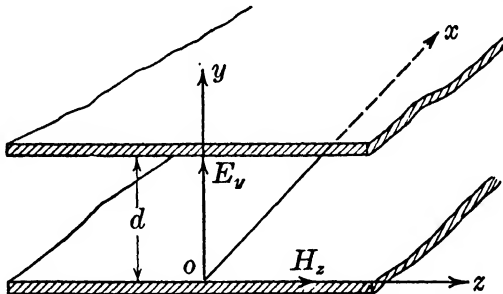


FIG. 7-1. Infinitely extended parallel plates.

restriction $E_x = E_z = H_x = 0$, it may prove helpful to consider the arrangement shown in Fig. 7-2, where the conductors of a coaxial cable which are separated by a distance d are energized in the usual manner, namely, by the application of a time-varying voltage between the two conductors.

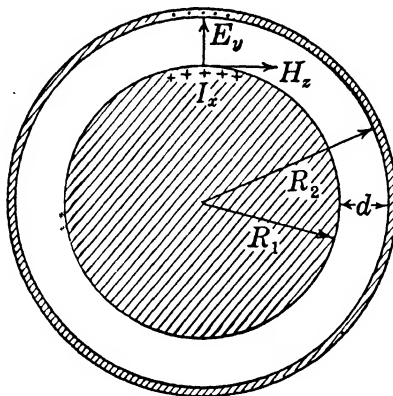


FIG. 7-2. Cross section of coaxial cable in which $R_2 \rightarrow \infty$.

With the restriction $E_x = E_z = H_x = 0$, and with the further restriction that $\partial E_y / \partial y = \partial E_y / \partial z = 0$, equation (7-1) becomes

$$\frac{\partial^2 E_y}{\partial x^2} = \mu\epsilon \frac{\partial^2 E_y}{\partial t^2}, \tag{7-2}$$

where E_y is a function of x and t only.

The solution of equation (7-2) of primary interest to us is that which yields propagation in the $+x$ direction, namely,

$$E_y = A \cos(\omega t - \beta x), \quad (7-3)$$

where A is an arbitrary constant which depends for its value upon the magnitude of the potential difference impressed upon the conducting plates, and $\beta = \omega \sqrt{\mu\epsilon}$.

The fact that equation (7-3) is a solution to equation (7-2) may be verified by direct substitution of E_y , as given by equation (7-3), into equation (7-2). During this process β^2 is evaluated as $\omega^2\mu\epsilon$. We learn from equation (7-3) that E_y can exist as a traveling wave, the phase velocity of which is

$$v = \frac{x}{t} = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\epsilon}}. \quad (7-4)$$

In case $\mu_r = \epsilon_r = 1$, the phase velocity is equal to the velocity of light, $3(10^8)$ m/sec.

Although equation (7-3) represents a solution to the wave equation, a more informative discussion of the TEM mode of propagation will result from a detailed examination of Maxwell's equations subject only to the restriction

$$E_x = E_z = H_x = 0. \quad (7-5)$$

Then

$$\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial z} = -\mu \frac{\partial H_x}{\partial t}$$

yields

$$\frac{\partial E_y}{\partial z} = 0. \quad (7-6)$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\mu \frac{\partial H_y}{\partial t}$$

yields

$$\begin{aligned} H_y &= 0. & (7-7)^* \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} &= -\mu \frac{\partial H_x}{\partial t} \end{aligned}$$

* In this connection we disregard any magnetic or electric field component which does not vary with time.

yields

$$\frac{\partial E_y}{\partial x} = -\mu \frac{\partial H_x}{\partial t},$$

or

$$\frac{\partial^2 E_y}{\partial x^2} = -\mu \frac{\partial^2 H_x}{\partial t \partial x} \tag{7-8}$$

$$\frac{\partial H_x}{\partial y} - \frac{\partial H_y}{\partial z} = \epsilon \frac{\partial E_x}{\partial t}$$

yields

$$H_y = 0,$$

since

$$\frac{\partial H_x}{\partial y} = 0. \tag{7-9}$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = \epsilon \frac{\partial E_y}{\partial t}$$

yields

$$-\frac{\partial H_x}{\partial x} = \epsilon \frac{\partial E_y}{\partial t},$$

or

$$-\frac{\partial^2 H_x}{\partial x \partial t} = \epsilon \frac{\partial^2 E_y}{\partial t^2} \tag{7-10}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial t} = \epsilon \frac{\partial E_x}{\partial t}$$

yields

$$0 = 0. \tag{7-11}$$

From equations (7-8) and (7-10), the form of the wave equation given in (7-2) follows directly. That is

$$\frac{\partial^2 E_y}{\partial x^2} = \mu \epsilon \frac{\partial^2 E_y}{\partial t^2} \tag{7-2}$$

This equation may be solved by letting $E_y = \mathcal{E}_{yz} e^{i\omega t}$, where \mathcal{E}_{yz} is a function of x only. Equation (7-2) then reduces to

$$\frac{\partial^2 \mathcal{E}_{yz}}{\partial x^2} = -\omega^2 \mu \epsilon \mathcal{E}_{yz}, \tag{7-12}$$

where it is reasonable to assume that \mathcal{E}_{yz} will take the form $A e^{-\gamma x}$. In this case, γ would play the same role as in ordinary transmission line theory. Substituting $\mathcal{E}_{yz} = A e^{-\gamma x}$ into equation (7-12) results in

$$\gamma^2 A e^{-\gamma x} = -\omega^2 \mu \epsilon A e^{-\gamma x}, \tag{7-13}$$

and

$$\gamma = j\omega \sqrt{\mu\epsilon} = j\beta. \quad (7-14)$$

By taking the real part of

$$E_y = A e^{-i\beta x} e^{j\omega t},$$

we obtain

$$E_y = A \cos(\omega t - \beta x), \quad (7-3)$$

which is the solution in which we are particularly interested.

If the region between the plates is free space or air

$$E_y = A \cos \omega \left(t - \frac{x}{c} \right), \quad (7-15)$$

where c is the velocity of light.

From equation (7-8) it follows that in free space

$$\frac{\partial H_z}{\partial t} = -\frac{1}{\mu_0} \frac{\partial E_y}{\partial x} = -\frac{A\omega}{\mu_0 c} \sin \omega \left(t - \frac{x}{c} \right),$$

or

$$H_z = A \sqrt{\frac{\epsilon_0}{\mu_0}} \cos \omega \left(t - \frac{x}{c} \right) + k_1, \quad (7-16)$$

where k_1 is zero for the reason referred to in the footnote for equation (7-7).

It will be observed that H_z is in time phase with E_y . Furthermore, H_z represents the entire magnetic field in this case since H_x was originally restricted to zero to obtain the TEM mode of transmission and H_y was found to be equal to zero from equation (7-7). Since E_x and E_z have been assumed to be equal to zero throughout this discussion, it follows that E_y represents the entire \mathbf{E} field. The Poynting vector ($\mathbf{E} \times \mathbf{H}$) is directed along the $+x$ axis and has a magnitude equal to

$$P = A^2 \sqrt{\frac{\epsilon_0}{\mu_0}} \cos^2 \omega \left(t - \frac{x}{c} \right) \text{ watts/m}^2. \quad (7-17)$$

The time-averaged value of power transmitted per meter length of the z -axis is

$$P_{av} = \frac{A^2 d}{2} \sqrt{\frac{\epsilon_0}{\mu_0}} \text{ watts.} \quad (7-18)$$

The same result as that obtained above might be had in terms of the potential difference between the conductors and the conduction

current in the conductors. These are the quantities normally employed in calculating transmitted power in conventional circuit arrangements. From equation (7-15), it is plain that the maximum voltage between the conductors is

$$V_{max} = E_{y(max)}d = Ad \text{ volts,} \quad (7-19)$$

since E_y does not vary along the y direction.

The conductor current associated with H_x may be found by applying the circuital law of magnetism to a rectangular loop which

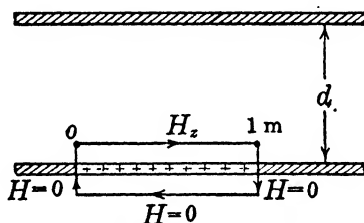


FIG. 7-3. Rectangular loop used to evaluate I_x .

will encompass the conductor current I_x along a unit length in the z direction. (See Fig. 7-3)

$$\begin{aligned} \oint \mathbf{H} \cdot d\mathbf{l} &= I_x \\ &= \int_0^1 H_x dz \\ &= A \sqrt{\frac{\epsilon_0}{\mu_0}} \text{ amp, max.} \end{aligned} \quad (7-20)$$

In thinking about this evaluation of I_x , it may prove helpful to consider the coaxial arrangement shown in Fig. 7-2.

The time-averaged power associated with the voltage between conductors, equation (7-19), and the conductor current, equation (7-20), is

$$\begin{aligned} P_{av} &= \frac{1}{2} V_{max} I_{max} \\ &= \frac{A^2 d}{2} \sqrt{\frac{\epsilon_0}{\mu_0}} \text{ watts/m of } z \text{ axis.} \end{aligned} \quad (7-21)$$

If the impedance of the parallel-plate transmission system is defined on a per-unit-length basis as V_{max}/I_{max} , then the impedance per meter length of z axis is

$$\begin{aligned} z/\text{u.l.} &= \frac{V_{max}}{I_{max}} \\ &= \sqrt{\frac{\mu_0}{\epsilon_0}} d \\ &= 120\pi d \text{ ohms.} \end{aligned} \tag{7-22}$$

The characteristic impedance of the coaxial cable shown in Fig. 7-2 is

$$Z_0 = \sqrt{\frac{L}{C}} = 60 \ln \frac{R_2}{R_1} \text{ ohms,} \tag{7-23}$$

if a free-space region between the conductors is assumed and if the ohmic resistance of the conductors is neglected. The impedance per unit length of z axis is

$$Z/\text{u.l.} = 2\pi R_2 \left[60 \ln \frac{R_2}{R_1} \right] \text{ ohms.} \tag{7-24}$$

In writing the above equation, the fact is observed that the per-unit-length impedances combine in parallel to form the characteristic impedance of the cable, namely, $60 \ln (R_2/R_1)$.

Since*

$$\begin{aligned} \ln x &= \frac{x-1}{x} + \frac{1}{2} \left(\frac{x-1}{x} \right)^2 + \frac{1}{3} \left(\frac{x-1}{x} \right)^3 + \dots \quad \left(x > \frac{1}{2} \right), \\ R_2 \ln \frac{R_2}{R_1} &= R_2 \left[\frac{\frac{R_2}{R_1} - 1}{\frac{R_2}{R_1}} + \frac{1}{2} \left(\frac{\frac{R_2}{R_1} - 1}{\frac{R_2}{R_1}} \right)^2 + \dots \right] \doteq R_2 - R_1 = d, \end{aligned}$$

as R_2 approaches an infinitely large value. Therefore

$$Z/\text{u.l.} = 120\pi(R_2 - R_1) = 120\pi d \text{ ohms,} \tag{7-25}$$

* In order to obtain this type of expansion for $\ln x$, one may express

$$\frac{1}{1-z} = 1 + z + z^2 + \dots,$$

and

$$-\ln(1-z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \dots$$

Letting $z = \frac{(x-1)}{x}$,

$$-\ln \left(1 - \frac{x-1}{x} \right) = \ln x = \frac{x-1}{x} + \frac{1}{2} \left(\frac{x-1}{x} \right)^2 + \frac{1}{3} \left(\frac{x-1}{x} \right)^3 + \dots$$

the same result as that obtained for the parallel-plate transmission system. [See equation (7-22).]

7-3. THE TRANSVERSE ELECTRIC OR TE MODE OF PROPAGATION

The wave known as the TE wave or mode is characterized by the fact that it possesses only one component of the electric vector and that in a transverse direction relative to the direction of propagation. If we wish to distinguish between the TEM mode shown in Fig. 7-1 and the TE mode, then the \mathbf{E} field of the latter must be directed along the z axis in Fig. 7-4. $E_x = E_y = 0$ in the TE

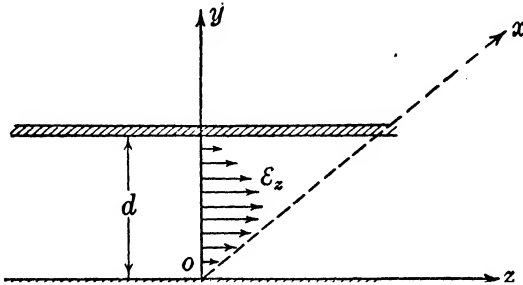


FIG. 7-4. The wave $\mathcal{E}_z = A\epsilon^{-\gamma z} \sin(\pi y/d)$ between parallel plates.

mode. This is equivalent to postulating a source (or generator) at $x = -\infty$ and a sink (or load) at $x = +\infty$. It is recognized, of course, that no practical generator can establish precisely the \mathbf{E} which is here postulated. One might, however, consider that a sinusoidally varying potential difference is applied to a pair of widely separated condenser plates at $x = -\infty$ in such a manner that the \mathbf{E} field in the vicinity of the origin (in Fig. 7-4) is directed only along the z axis.

Since only cosinusoidal variations with respect to time are to be considered, the derivatives with respect to t will be written as follows:

$$\frac{\partial}{\partial t} = j\omega,$$

$$\frac{\partial^2}{\partial t^2} = -\omega^2.$$

Assuming that the space between plates shown in Fig. 7-4 is an insulator ($\sigma = 0$), the wave equation for the TE mode is

$$\frac{\partial^2 \mathcal{E}_z}{\partial x^2} + \frac{\partial^2 \mathcal{E}_z}{\partial y^2} = -\omega^2 \mu \epsilon \mathcal{E}_z. \quad (7-26)$$

The direction of motion of the wave will be taken as along the positive x axis, and, since the planes of equal phase are parallel to the yz plane, there is no change with respect to z . Accordingly, the derivative of E_z with respect to z will be zero.

From equation (6-7),

$$\nabla \times \bar{\mathcal{E}} = -j\omega\mu \bar{\mathcal{H}},$$

or

$$\bar{\mathcal{H}} = j \frac{1}{\omega\mu} (\nabla \times \bar{\mathcal{E}}), \quad (7-27)$$

from which $\bar{\mathcal{H}}$ (or \mathbf{H}) may be found once $\bar{\mathcal{E}}$ is determined.

A solution of equation (7-26) will be obtained by assuming $\mathcal{E}_z = A\epsilon^{-\gamma x}\epsilon^{a y}$ with the boundary conditions fixed by the fact that both at $y = 0$, and $y = d$, \mathcal{E}_z must be zero because of the proximity to the perfect conducting planes. Also γ must be positive to prevent the possibility of infinite amplitude at $x = \infty$. Thus

$$\begin{aligned} \frac{\partial \mathcal{E}_z}{\partial x} &= -A\gamma\epsilon^{-\gamma x}\epsilon^{a y}, \\ \frac{\partial^2 \mathcal{E}_z}{\partial x^2} &= A\gamma^2\epsilon^{-\gamma x}\epsilon^{a y}, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \mathcal{E}_z}{\partial y} &= Aa\epsilon^{-\gamma x}\epsilon^{a y}, \\ \frac{\partial^2 \mathcal{E}_z}{\partial y^2} &= Aa^2\epsilon^{-\gamma x}\epsilon^{a y}. \end{aligned}$$

Then

$$A\gamma^2\epsilon^{-\gamma x}\epsilon^{a y} + Aa^2\epsilon^{-\gamma x}\epsilon^{a y} = -\omega^2\mu\epsilon A\epsilon^{-\gamma x}\epsilon^{a y},$$

or

$$\gamma^2 + a^2 = -\omega^2\mu\epsilon$$

or

$$a^2 = -\gamma^2 - \omega^2\mu\epsilon.$$

Thus

$$a = \pm j \sqrt{\gamma^2 + \omega^2\mu\epsilon} = \pm j b. \quad (7-28)$$

The solution then becomes

$$\mathcal{E}_z = \epsilon^{-\gamma x} [A_1\epsilon^{j b y} + A_2\epsilon^{-j b y}]. \quad (7-29)$$

Applying the boundary conditions, we obtain two equations from which A_1 and A_2 may be determined.

$$0 = A_1 + A_2, \quad (7-30)$$

$$0 = A_1 \varepsilon^{jbd} + A_2 \varepsilon^{-jbd}. \quad (7-31)$$

From equation (7-30),

$$A_1 = -A_2 (= A'),$$

and equation (7-31) becomes

$$\begin{aligned} 0 &= A'(\varepsilon^{jbd} - \varepsilon^{-jbd}), \\ &= A \sin bd, \end{aligned}$$

from which $bd = n\pi$, or

$$b = \frac{n\pi}{d}, \quad (7-32)$$

where n is any integer or zero. The value of n is defined by the mode of excitation which is assumed. The solution (7-29) now becomes

$$\varepsilon_z = A \varepsilon^{-\gamma x} \sin \frac{n\pi y}{d}, \quad (7-33)$$

where A is a constant depending on the magnitude of the excitation voltage. The space variation of ε_z along the y direction is shown in Fig. 7-4 for the case of $n = 1$. The boundary conditions require that $\varepsilon_z = 0$ at $y = 0$ and at $y = d$.

From equation (7-28) the term γ may be found.

$$\gamma^2 + \omega^2 \mu \varepsilon = b^2 = \frac{n^2 \pi^2}{d^2},$$

or

$$\gamma = \pm \sqrt{\frac{n^2 \pi^2}{d^2} - \omega^2 \mu \varepsilon} = \pm j\beta, \quad (7-34)$$

where

$$\beta = \sqrt{\omega^2 \mu \varepsilon - \frac{n^2 \pi^2}{d^2}}. \quad (7-34a)$$

By analogy with conventional long-line theory, γ is the *propagation* constant and β is the phase constant.

The complete solution now becomes, using equations (7-33) and (7-34),

$$E_z = A \varepsilon^{-\gamma x} \varepsilon^{j\omega t} \sin \frac{n\pi y}{d}, \quad (7-35)$$

where γ has the positive value in equation (7-34). The positive

value is used, otherwise when γ is real, the wave becomes infinite at $x = \infty$. If γ is real, a rapid damping of the wave occurs. On the other hand, if γ is imaginary we have a wave propagated in the positive x direction as noted in a previous section. If γ is imaginary, the real part of the solution can be written as follows:

$$E_z = A \cos (\omega t - \beta x) \sin \frac{n\pi y}{d}, \quad (7-36)$$

where β has the value given in equation (7-34a).

7-4. PROPERTIES OF THE TE WAVES

From equations (7-4) and (7-34a) it is easily seen that the phase velocity of the wave is given by

$$v = \frac{\omega}{\beta} = \frac{\omega}{\sqrt{\omega^2 \mu \epsilon - \frac{n^2 \pi^2}{d^2}}}. \quad (7-37)$$

In free space where $\mu \epsilon = 1/c^2$,

$$v = \frac{1}{\sqrt{\frac{1}{c^2} - \frac{n^2 \pi^2}{\omega^2 d^2}}}. \quad (7-37a)$$

Immediately, one peculiarity of these waves in free space is noted. The phase velocity is higher than the velocity of light except for one case, that of $n = 0$, when the velocity will be equal to c . In the limiting case where $1/c = n\pi/\omega d$, the velocity becomes infinite. It should be noted that we are speaking of the *phase* velocity, that is, the velocity of a crest of the wave in steady state. It can be shown that wave groups are not transmitted at this velocity but at a rate lower than c (in the case of waveguides).*

* An expression for the group velocity mentioned here is easily derived for a rather simple combination of two waves. This simple treatment is sufficiently general because any combination of waves can be built up by combining waves two at a time and using the principle of superposition. Let the two waves be given as

$$i_1 = A \sin (\omega_1 t - \beta_1 x),$$

and

$$i_2 = A \sin (\omega_2 t - \beta_2 x),$$

where ω_2 differs very slightly from ω_1 , and β_2 differs very slightly from β_1 . Thus we can set $\omega_2 - \omega_1 = d\omega$, and $\beta_2 - \beta_1 = d\beta$. Let these two waves

The condition $c = \omega d/n\pi$ is the boundary condition between free transmission and attenuation, for suppose that d , or ω , is decreased just sufficiently to cause expression (7-37) to become imaginary, then γ in equation (7-35) immediately becomes real and attenuation rapidly decreases the amplitude of the wave. Thus this condition is used to define a critical or *cutoff* d or f which marks

be superposed so that

$$i = i_1 + i_2 = A [\sin (\omega_1 t - \beta_1 x) + \sin (\omega_2 t - \beta_2 x)].$$

This equation is readily simplified by a trigonometrical transformation into

$$\begin{aligned} i &= 2A \sin \frac{(\omega_1 + \omega_2)t - (\beta_1 + \beta_2)x}{2} \cos \frac{(\omega_2 - \omega_1)t - (\beta_2 - \beta_1)x}{2} \\ &\doteq 2A \sin (\omega t - \beta x) \cos \frac{t(d\omega) - x(d\beta)}{2} \end{aligned} \tag{a}$$

where $\omega = (\omega_1 + \omega_2)/2$ and $\beta = (\beta_1 + \beta_2)/2$.

Now equation (a) represents a wave whose amplitude is the variable function

$$2A \cos \frac{t(d\omega) - x(d\beta)}{2}, \tag{b}$$

and whose frequency and wavelength are the means of those of the two original waves. The expression for amplitude, expression (b), also represents a wave moving in the positive x direction, whose radial velocity is $\omega' = d\omega/2$, and whose phase constant is $\beta' = d\beta/2$. From these two expressions the velocity of the crest of the composite wave or wave group is

$$v_g = \frac{\omega'}{\beta'} = \frac{d\omega}{d\beta}. \tag{c}$$

This is known as the group velocity, and, as an illustration, the group velocity will be determined corresponding to equation (7-37).

$$\frac{d\beta}{d\omega} = \frac{1}{2} \left[\omega^2 \mu \epsilon - \frac{n^2 \pi^2}{d^2} \right]^{-1/2} 2\omega \mu \epsilon,$$

or

$$v_g = \frac{d\omega}{d\beta} = \frac{1}{\omega \mu \epsilon} \sqrt{\omega^2 \mu \epsilon - \frac{n^2 \pi^2}{d^2}} = \frac{1}{v \mu \epsilon} = \frac{c^2}{v}. \tag{7-37b}$$

It is thus seen that the velocity of light is the geometric mean between the phase velocity and the group velocity for the wave guide.

the dividing line between transmission and no transmission. For free space these cutoff values are

$$d_0 = \frac{n\pi c}{\omega} = \frac{n\pi c}{2\pi f} = \frac{nc}{2f} \tag{7-38}$$

or

$$f_0 = \frac{nc}{2d} \tag{7-39}$$

For a given frequency, d must be larger than the critical value, equation (7-38), for transmission, or, for a given separation, f must be larger than the critical value, equation (7-39).

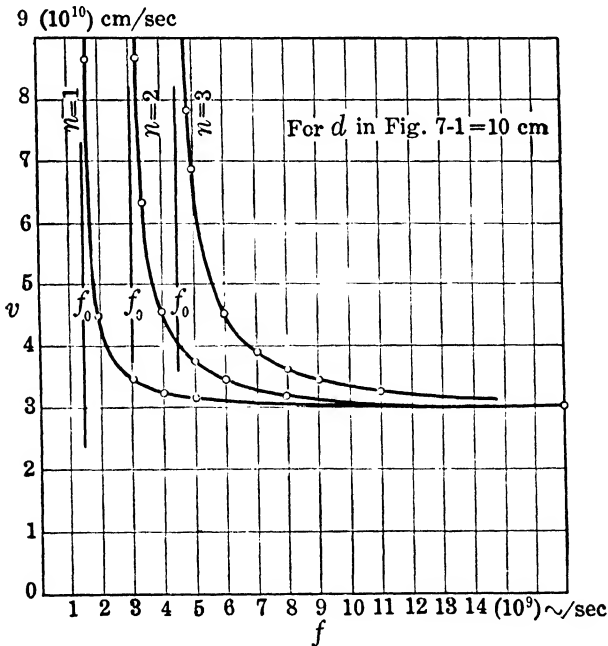


FIG. 7-5. Phase velocity vs frequency for three TE modes of excitation in Fig. 7-1.

The velocity of the wave as a function of f is shown in Fig. 7-5 for the parameters $d = 10$ cm, $n = 1, 2, 3$.

In Fig. 7-6, f_0 is plotted vs d for $n = 1, 2, 3$. In this diagram transmission results only for combinations of f and d which lie to the right of the curve.

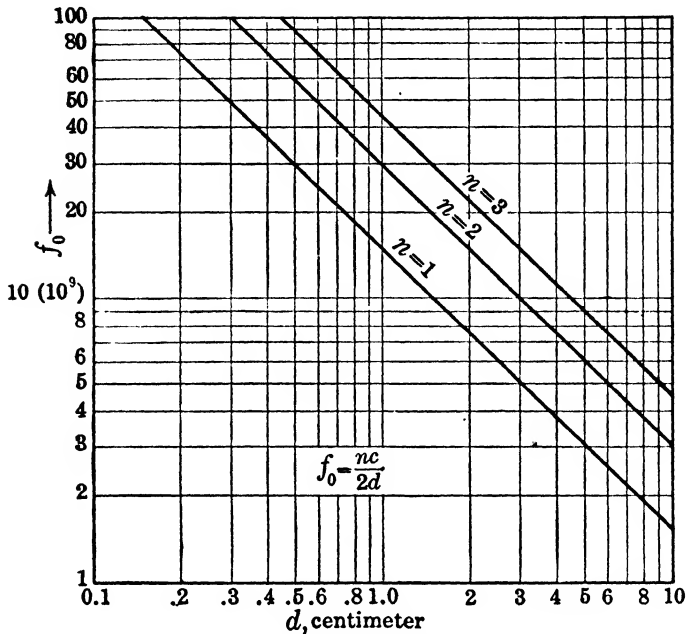


Fig. 7-6. Cutoff frequency vs separation of plates for three TE modes.

7-5. THE MAGNETIC FIELD COMPONENTS OF THE TE WAVE

If $\bar{\mathcal{E}}$, from equation (7-33), is substituted into equation (7-27), the $\bar{\mathcal{H}}$ field may be obtained. Thus

$$\bar{\mathcal{H}} = j \frac{1}{\omega\mu} \left(\nabla \times \mathbf{k} A e^{-\gamma z} \sin \frac{n\pi y}{d} \right) \tag{7-40}$$

$$= j \frac{1}{\omega\mu} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & A e^{-\gamma z} \sin \frac{n\pi y}{d} \end{vmatrix}$$

$$= j \frac{1}{\omega\mu} \left[\mathbf{i} A e^{-\gamma z} \frac{n\pi}{d} \cos \frac{n\pi y}{d} + \mathbf{j} \gamma A e^{-\gamma z} \sin \frac{n\pi y}{d} \right]. \tag{7-41}$$

It has been seen that in order for transmission to take place γ must be imaginary. Since interest lies only in that case, γ will be written as $j\beta$. Then the complete magnetic components become

$$H_x = j \frac{An\pi}{\mu\omega d} e^{j(\omega t - \beta x)} \cos \frac{n\pi y}{d}, \quad (7-42)$$

or for the real part,

$$H_{x(real)} = \frac{-An\pi}{\omega\mu d} \sin(\omega t - \beta x) \cos \frac{n\pi y}{d}. \quad (7-43)$$

Also

$$H_y = \frac{-\beta A}{\omega\mu} e^{j(\omega t - \beta x)} \sin \frac{n\pi y}{d}, \quad (7-44)$$

or for the real part.

$$H_{y(real)} = \frac{-\beta A}{\omega\mu} \cos(\omega t - \beta x) \sin \frac{n\pi y}{d}. \quad (7-45)$$

In similar terms, equation (7-36) is repeated here

$$E_{z(real)} = A \cos(\omega t - \beta x) \sin \frac{n\pi y}{d}. \quad (7-46)$$

7-6. CONFIGURATION OF THE FIELD

It is seen immediately that there are three components of the field present, E_z , H_x , and H_y . It thus appears that there exists a component of \mathbf{H} in the direction of propagation. This provides a basis for calling this type of wave an H wave, a name used interchangeably with TE in a number of texts.

A very great number of possible field configurations exist depending upon the value of n . However, in this elementary introduction, the case for $n = 1$ is the only one to be considered. If $n = 1$ is substituted into equations (7-46), (7-45), and (7-43), we obtain

$$E_z = A \cos(\omega t - \beta x) \sin \frac{\pi y}{d}, \quad (7-47)$$

$$H_y = \frac{-\beta A}{\omega\mu} \cos(\omega t - \beta x) \sin \frac{\pi y}{d}, \quad (7-48)$$

and

$$H_x = \frac{-A\pi}{\omega\mu d} \sin(\omega t - \beta x) \cos \frac{\pi y}{d}. \quad (7-49)$$

where

$$\beta = \sqrt{\omega^2 \mu \epsilon - \frac{\pi^2}{d^2}}, \quad (7-50)$$

or in free space,

$$\beta = \sqrt{\frac{\omega^2}{c^2} - \frac{\pi^2}{d^2}}, \quad (7-50a)$$

from equation (7-34). It is clear that the time-varying field, E_z , which has been postulated will give rise to a displacement current directed along the z axis and that a small enclosing path located in the x - y plane will have a finite value of mmf acting around it. The shape of the H field in the x - y plane may be determined as

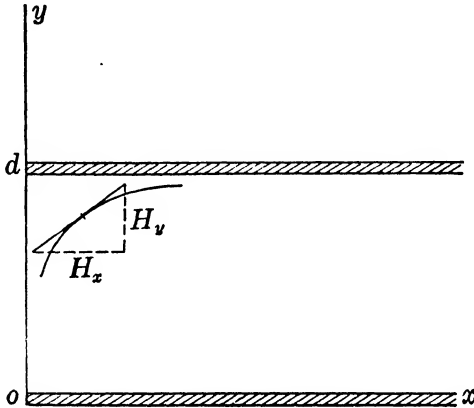


FIG. 7-7. A portion of an H loop in the TE_1 mode.

follows. See Fig. 7-7. The slope of the magnetic lines at any point is

$$\begin{aligned} \frac{dy}{dx} &= \frac{H_y}{H_x} \\ &= \frac{\beta d}{\pi} \cot(\omega t - \beta x) \tan \frac{\pi y}{d}, \end{aligned}$$

or

$$\cot \frac{\pi y}{d} dy = \frac{\beta d}{\pi} \cot(\omega t - \beta x) dx. \tag{7-51}$$

When equation (7-51) is integrated, the following result is obtained.

$$\sin \frac{\pi y}{d} \sin(\omega t - \beta x) = C, \tag{7-52}$$

in which C is an integration constant. This equation gives the shape of the magnetic loops in the field. Its actual use is left as an exercise.

7-7. THE POWER TRANSMITTED BY THE FIELD

It is easy now to apply Poynting's vector to the expressions (7-47) and (7-48) to obtain the power transmitted along the tube.

The E_x and H_y components are the only ones contributing to power transmission in the x direction. Components E_x and H_x give only power transmission perpendicular to the x axis, and for the values of E_x and H_x given by equations (7-47) and (7-49) respectively it is a simple matter to show that the time-averaged value of the product $E_x H_x$ is zero. Hence no average power is transmitted along the y direction.

Since E_x and H_y are perpendicular, Poynting's vector becomes

$$\mathbf{P} = \mathbf{i} \frac{\beta A^2}{\omega \mu} \cos^2 (\omega t - \beta x) \sin^2 \frac{\pi y}{d},$$

because $(\mathbf{k} \times \mathbf{j}) = -\mathbf{i}$. If this expression is multiplied by dy and integrated from $y = 0$ to $y = d$, we obtain the power transmitted per unit width of the guide. The result of this integration is

$$P = \frac{\beta A^2 d}{2\omega \mu} \cos^2 (\omega t - \beta x), \quad (7-53)$$

or the average power transmitted is obtained by averaging over one cycle.

$$P_{av} = \frac{\beta A^2 d}{4\omega \mu} \text{ watts.} \quad (7-54)$$

7-8. DIFFERENTIAL EQUATION FOR THE TM_1 WAVE

The mode of transmission considered above is known as the TE_1 mode, the subscript referring to the value of n . In the TM_1 mode, by analogy, we would expect to set $n = 1$ in the solution for a wave which has the following general configuration. In the first place we expect that the only component of \mathbf{H} present will be H_x , taking the place of the previous E_x component. It is known from previous work that the electric and magnetic fields are perpendicular, so it appears that the only \mathbf{E} components present will be E_x and E_y . Thus the general configuration of the TM_1 field is known. Again a sinusoidal variation with respect to t will be assumed.

Since there exists only one component of \mathbf{H} , it is convenient to find, first, a solution for it in general terms. Then from it the electric components E_x and E_y will be determined. The boundary conditions, $E_x = 0$ at $y = 0$ and $y = d$, will then be applied, whereupon all the field components will be known.

In order to solve for H_x it is necessary to solve equation (6-12a) which will reduce to the following for the conditions outlined above:

$$\frac{\partial^2 \mathcal{H}_z}{\partial x^2} + \frac{\partial^2 \mathcal{H}_z}{\partial y^2} = -\omega^2 \mu \epsilon \mathcal{H}_z. \tag{7-55}$$

After an expression is obtained for \mathcal{H}_z , $\bar{\mathbf{E}}$ can be found from equation (4-1).

$$\bar{\mathbf{E}} = -\frac{j}{\omega \epsilon} (\nabla \times \bar{\mathcal{H}}). \tag{7-56}$$

By analogy with equation (7-29) we can immediately write down the solution to equation (7-55) as follows:

$$\mathcal{H}_z = \epsilon^{-\gamma z} (B_1 e^{j\beta y} + B_2 e^{-j\beta y}), \tag{7-57}$$

where γ will be replaced by $j\beta$ and β is

$$\beta = \sqrt{\frac{\omega^2}{c^2} - \frac{\pi^2}{d^2}}, \tag{7-58}$$

from equation (7-34). We have already decided to treat only the case for $n = 1$, so n has been omitted in writing equation (7-58).

The field $\bar{\mathbf{E}}$ is determined now from equation (7-56) using expression (7-57) for \mathcal{H}_z .

$$\begin{aligned} \bar{\mathbf{E}} &= -\frac{j}{\omega \epsilon} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & \epsilon^{-j\beta z} (B_1 e^{j\beta y} + B_2 e^{-j\beta y}) \end{vmatrix} \\ &= -\frac{j}{\omega \epsilon} [i \epsilon^{-j\beta z} (B_1 j \beta e^{j\beta y} - B_2 j \beta e^{-j\beta y}) + \\ &\qquad\qquad\qquad j j \beta \epsilon^{-j\beta z} (B_1 e^{j\beta y} + B_2 e^{-j\beta y})]. \end{aligned} \tag{7-59}$$

Thus

$$\mathcal{E}_x = \frac{b}{\omega \epsilon} \epsilon^{-j\beta z} (B_1 e^{j\beta y} - B_2 e^{-j\beta y}), \tag{7-60}$$

and

$$\mathcal{E}_y = \frac{\beta}{\omega \epsilon} \epsilon^{-j\beta z} (B_1 e^{j\beta y} + B_2 e^{-j\beta y}). \tag{7-61}$$

The boundary condition must now be applied to equation (7-60):

$$\begin{aligned} 0 &= B_1 - B_2 \\ 0 &= B_1 e^{j\beta d} - B_2 e^{-j\beta d}. \end{aligned}$$

Therefore,

$$B_1 = B_2 \left(\equiv \frac{B}{2} \right),$$

and

$$\sin bd = 0,$$

from which

$$bd = n\pi$$

or, setting $n = 1$,

$$b = \frac{\pi}{d} \tag{7-62}$$

as in the case of the TE_1 wave.

The components now become

$$E_x = \frac{Bb}{2\omega\epsilon} e^{j(\omega t - \beta x)} (e^{j(\pi y)/d} - e^{-j(\pi y)/d}),$$

$$E_y = \frac{B\beta}{2\omega\epsilon} e^{j(\omega t - \beta x)} (e^{j(\pi y)/d} + e^{-j(\pi y)/d}),$$

$$H_z = \frac{B}{2} e^{j(\omega t - \beta x)} (e^{j(\pi y)/d} + e^{-j(\pi y)/d}),$$

or the real parts are

$$E_{x(\text{real})} = -\frac{B\pi}{\epsilon\omega d} \sin(\omega t - \beta x) \sin \frac{\pi y}{d}, \tag{7-63}$$

$$E_{y(\text{real})} = \frac{B\beta}{\omega\epsilon} \cos(\omega t - \beta x) \cos \frac{\pi y}{d}, \tag{7-64}$$

$$H_{z(\text{real})} = B \cos(\omega t - \beta x) \cos \frac{\pi y}{d}. \tag{7-65}$$

7-9. PROPERTIES OF THE TM_1 WAVE

Since b and β for this wave are the same as for the TE_1 wave, the transmission properties, velocity, and critical or cutoff values of d and f are given by the equations of §7-4.

7-10. CONFIGURATION OF THE FIELD

By a method similar to that used before, the shape of the electric field "loops" may be determined. The slope of the electric lines at any point is given by

$$\frac{dy}{dx} = \frac{E_y}{E_x} = \frac{-\beta d}{\pi} \cot(\omega t - \beta x) \cot \frac{\pi y}{d}. \tag{7-66}$$

The equation of the curve is given by solving this equation for y .

$$\tan \frac{\pi y}{d} dy = \frac{-\beta d}{\pi} \cot(\omega t - \beta x) dx,$$

or

$$\cos \frac{\pi y}{d} \sin (\omega t - \beta x) = C, \quad (7-67)$$

where C is a constant of integration and depends upon the location of the loop.

7-11. POWER TRANSMISSION IN THE TM_1 MODE

From equations (7-64) and (7-65) the power is obtained by using Poynting's vector.

$$\mathbf{P} = \mathbf{E} \times \mathbf{H}.$$

The components contributing to transmission in the x direction are E_y and H_z . Since they are perpendicular, P becomes, very simply,

$$P = \frac{\beta B^2}{\omega \epsilon} \cos^2 (\omega t - \beta x) \cos^2 \frac{\pi y}{d}. \quad (7-68)$$

This equation, integrated from $y = 0$ to $y = d$, and averaged over one cycle, gives the following expression for the average power transmitted per unit width of the guide.

$$P_{av} = \frac{\beta B^2 d}{4\omega \epsilon} \text{ watts.} \quad (7-69)$$

7-12. THE TEM MODE OF TRANSMISSION

In writing equation (7-62) for the TM_1 mode, n was set equal to unity. However, there is another very simple possibility, that of setting $n = 0$. If this is done, it amounts to replacing π/d ($=b$) by zero in equations (7-63), (7-64), and (7-65). By doing this, there results

$$\left. \begin{aligned} E_x &= 0, \\ E_y &= \frac{B\beta}{\omega \epsilon} \cos (\omega t - \beta x), \\ H_z &= B \cos (\omega t - \beta x). \end{aligned} \right\} \quad (7-70)$$

However, β has a very simple expression for $n = 0$, obtained from equation (7-34). It is

$$\beta = \omega \sqrt{\mu \epsilon}. \quad (7-71)$$

The above equations then become:

$$E_x = 0, \quad (7-72)$$

$$E_y = B \sqrt{\frac{\mu}{\epsilon}} \cos (\omega t - \beta x), \quad (7-73)$$

$$H_z = B \cos (\omega t - \beta x). \quad (7-74)$$

It is immediately noted that the x component disappears, and we obtain the same result as in §7-2. The velocity becomes the velocity of light, and the guide no longer acts as a high-pass filter. Note that $B \sqrt{\mu/\epsilon} = A$.

7-13. PROBLEMS

7-1. If $\bar{\mathcal{E}} = \mathbf{k}Ae^{-\gamma z}\epsilon^{ay}$, find $\bar{\mathcal{H}}$ by using equation

$$\bar{\mathcal{H}} = j \frac{1}{\omega\mu} (\nabla \times \bar{\mathcal{E}}).$$

7-2. Given the following parameters for a TE wave between parallel plates. $d = 6$ cm, $f = 2f_0$, $A = 1$, $n = 1$. Find: (a) f_0 , (b) v , the phase velocity.

7-3. In Prob. 7-2, plot E_x vs y and vs x , assuming that $t = 0$. Repeat for $\omega t = \pi/2$. (When plotting against y let $x = 0$, and when plotting against x let $y = d/2$.)

7-4. Find H_x and H_y for Prob. 7-2, and plot each vs x and y . Observe the same restrictions as in Prob. 7-3.

7-5. Plot an H loop for Prob. 7-2 at $t = 0$, with the requirement that it pass through the point $x = 0.2$ cm, and $y = 3$ cm.

7-6. Find the maximum power flow per unit area in the field of the guide of Prob. 7-2.

7-7. Show by direct substitution that

$$H_x = \frac{-A\pi}{\omega d\mu} \sin(\omega t - \beta x) \cos \frac{\pi y}{d}$$

is a solution of

$$\nabla^2 \mathbf{H} = \mu\epsilon \frac{\partial^2 \mathbf{H}}{\partial t^2}.$$

7-8. Show by direct substitution that

$$E_y = \frac{\beta A}{\omega\epsilon} \cos(\omega t - \beta x) \cos \frac{\pi y}{d}$$

is a solution of

$$\nabla^2 \mathbf{E} = \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

7-9. Given the following parameters for a TM_1 wave between parallel plates. $d = 10$ cm, $f = 2f_0$, $B = 1$. Plot an electric field loop passing through point $x = 0.2$ cm and $y = 7.5$ cm at $t = 0$.

7-10. Find the phase velocity of the wave in Prob. 7-9.

7-11. In Prob. 7-9, plot E_x and E_y vs x and y assuming $t = 0$. (When plotting vs x , let $y = d/2$, and when plotting vs y , let $x = 0$.)

7-12. The TEM mode is being transmitted between plates 10 cm apart. $B = 1$, and $f = 10^8$. Find the power transmitted per unit width of the guide.

7-13. A Z_0 is defined as follows: $Z_0 = E_z/H_y$. Calculate Z_0 in ohms from equations (7-47) and (7-48) for $f = 1.5f_0$, $d = 10$ cm.

7-14. Find the maximum power flow per square meter in the x direction based on equations (7-47) and (7-48) for conditions as given in Prob. 7-13.

7-15. According to equations (7-47) and (7-49) there is a power flow in the $-y$ direction. Find its average value per square meter for the conditions of Prob. 7-13.

RECTANGULAR WAVE GUIDES

Summary

This chapter continues the discussion of Chapter VII by adding two more conducting walls, thus forming a rectangular guide. The properties of the TE modes of transmission are derived. The $TE_{0,1}$ mode is treated specifically, and the field configuration and expressions for power and for characteristic impedance are determined. On the basis of the equations for the $TE_{0,1}$ mode relations for currents in the guide walls are derived. The $TE_{1,1}$ mode is considered briefly.

8-1. THE DIFFERENTIAL EQUATIONS FOR THE TE MODES

In Chapter VII it was shown that an electromagnetic wave can be transmitted between two parallel plates and that the variation with x and t , the direction of propagation, and time, respectively, can be expressed by the exponential product

$$e^{-\gamma x} e^{j\omega t} \quad (8-1)$$

where γ is known as the propagation constant and is expressed in general as

$$\gamma = \alpha + j\beta, \quad (8-2)$$

where α is the attenuation constant and β , the phase constant. In the preceding chapter, it was found that with perfect conductors, α was zero if the frequency was above some critical value known as the cutoff frequency. It might be expected that transmission of some kind will still result even if walls are added so that we have a rectangular tube instead of an infinite plane. That the addition of walls to form a rectangular guide may be considered possible is seen by reference to Fig. 8-1 where the end of such a guide is shown,

with an electric line drawn for the TE mode. The addition of boundaries at ab and cd would not seem to do violence to any *fundamental principles, since the electric lines could terminate on charges on these boundaries and are perpendicular to the surfaces, as required.* It is also now obvious that if walls are added as shown the electric line could be drawn in either the y or z direction equally well because the tube presents the same aspect to either field. If this mode seems to be a possibility, it seems that equations could

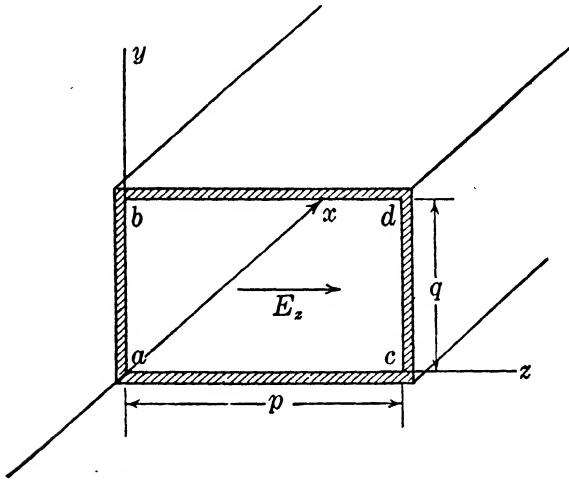


FIG. 8-1. Rectangular wave guide with inside dimensions p and q .

be set up which will predict them. It will be found that such equations in general will indicate an infinite number of possible modes of transmission. The specific one treated in some detail in the following pages will be the one in which the electric lines are drawn parallel to the y axis, a mode known as the $TE_{0,1}$.

We begin with equations (4-11), boldly making the assumption, which by this time may seem somewhat reasonable, that the variations with respect to x and t are given by expression (8-1). With this substitution, equations (4-11) become

$$\gamma^2 \mathcal{E}_z + \frac{\partial^2 \mathcal{E}_z}{\partial y^2} + \frac{\partial^2 \mathcal{E}_z}{\partial z^2} = -\omega^2 \mu \epsilon \mathcal{E}_z, \quad (8-3)$$

$$\gamma^2 \mathcal{E}_y + \frac{\partial^2 \mathcal{E}_y}{\partial y^2} + \frac{\partial^2 \mathcal{E}_y}{\partial z^2} = -\omega^2 \mu \epsilon \mathcal{E}_y, \quad (8-4)$$

$$\gamma^2 \mathcal{E}_x + \frac{\partial^2 \mathcal{E}_x}{\partial y^2} + \frac{\partial^2 \mathcal{E}_x}{\partial z^2} = -\omega^2 \mu \epsilon \mathcal{E}_x. \quad (8-5)$$

These equations provide for the existence of all three components of $\bar{\mathbf{E}}$, (or \mathbf{E}). The solutions will divide themselves into two main groups, those in which $\varepsilon_x = 0$, and those in which $\mathcal{K}_x = 0$. The groups are known respectively as the TE and TM groups, for the same reason as given in Chapter VII. The set to be considered here is the one containing the TE modes. In this set the $\text{TE}_{0,1}$ mode is the most important and will be worked out in some detail, whereas the $\text{TE}_{1,1}$ mode will be considered briefly. The TM modes, having little practical importance, are omitted but may be handled in an exactly similar method.

When $\varepsilon_x = 0$, we are left with only equations (8-4) and (8-5). These two equations are exactly the same in form so that when a solution is obtained for one it can also be applied to the other. Let these two equations be represented for the present by the following:

$$\begin{aligned} \frac{\partial^2 Q}{\partial y^2} + \frac{\partial^2 Q}{\partial z^2} &= -(\omega^2 \mu \epsilon + \gamma^2) Q \\ &= -h^2 Q \end{aligned} \quad (8-6)$$

and

$$h^2 = (\omega^2 \mu \epsilon + \gamma^2)$$

where Q may represent either ε_y or ε_z . This is the equation which will lead to all the TE mode solutions.

8-2. SOLUTION OF THE TE MODE EQUATION

Equation (8-6) involves two variables, y and z . It can easily be solved by assuming a solution as follows:

$$Q = YZ, \quad (8-7)$$

where Y is a function of y only, and Z a function of z only. Substituting (8-7) into equation (8-6) there is obtained

$$Z \frac{\partial^2 Y}{\partial y^2} + Y \frac{\partial^2 Z}{\partial z^2} = -h^2 YZ,$$

or

$$\frac{1}{Y} \frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z} \frac{\partial^2 Z}{\partial z^2} = -h^2. \quad (8-8)$$

Now the two terms on the left are entirely independent of each other, and their sum must be $(-h^2)$. Thus each term is a constant, and we have the following, where M and N are constants:

$$\frac{1}{Y} \frac{\partial^2 Y}{\partial y^2} = -M^2,$$

$$\frac{1}{Z} \frac{\partial^2 Z}{\partial z^2} = -N^2,$$

and

$$M^2 + N^2 = h^2. \quad (8-9)$$

From the above equations we have

$$\frac{d^2 Y}{dy^2} = -M^2 Y, \quad (8-10)$$

$$\frac{d^2 Z}{dz^2} = -N^2 Z, \quad (8-11)$$

of which the solutions are known to be circular functions. Thus

$$Y = A' \cos My + B' \sin My, \quad (8-12)$$

and

$$Z = C' \cos Nz + D' \sin Nz, \quad (8-13)$$

or

$$Q = (A' \cos My + B' \sin My)(C' \cos Nz + D' \sin Nz) \\ = \mathcal{E}_y. \quad (8-14)$$

The boundary conditions are $\mathcal{E}_y = 0$, when $z = 0, p$. To meet this boundary condition, equation (8-14) indicates that

$$C' \cos Nz + D' \sin Nz = 0, \quad z = 0, p.$$

Therefore

$$C' = 0,$$

and

$$Np = n\pi, \quad (n = 0, 1, 2, 3 \dots),$$

or

$$N = \frac{n\pi}{p}. \quad (8-15)$$

Therefore

$$\mathcal{E}_y = (A \cos My + B \sin My) \sin \frac{n\pi z}{p}, \quad (8-16)$$

where $A = A'D'$, $B = B'D'$, and in which A , B , and M are to be determined.

Now \mathcal{E}_z could be determined in the same manner, but if the expression were obtained it would contain undetermined constants and the work of correlating the two solutions would be somewhat

Then using equations (8-16), and (8-17),

$$\begin{aligned}\frac{\partial \varepsilon_x}{\partial z} &= -\frac{\partial \varepsilon_y}{\partial y} \\ &= -\frac{\partial}{\partial y} (A \cos My + B \sin My) \sin \frac{n\pi z}{p} \\ &= (AM \sin My - BM \cos My) \sin \frac{n\pi z}{p},\end{aligned}$$

and on integration,

$$\varepsilon_x = -\frac{Mp}{n\pi} (A \sin My - B \cos My) \cos \frac{n\pi z}{p} = 0, \quad (8-25)$$

when $y = 0, q$. Therefore,

$$B = 0,$$

and

$$Mq = m\pi, \quad (m = 0, 1, 2, 3, \dots),$$

or

$$M = \frac{m\pi}{q}. \quad (8-26)$$

Therefore

$$\varepsilon_x = -\frac{mp}{nq} A \sin \frac{m\pi y}{q} \cos \frac{n\pi z}{p}. \quad (8-27)$$

From equation (8-16),

$$\varepsilon_y = A \cos \frac{m\pi y}{q} \sin \frac{n\pi z}{p}, \quad (8-28)$$

and, of course,

$$\varepsilon_z = 0. \quad (8-29)$$

Thus we have determined all components of $\bar{\varepsilon}$. The propagation constant can now be calculated from equation (8-9).

$$\begin{aligned}h^2 &= M^2 + N^2 = \left(\frac{m\pi}{q}\right)^2 + \left(\frac{n\pi}{p}\right)^2 \\ &= \omega^2\mu\epsilon + \gamma^2,\end{aligned}$$

or

$$\gamma_{m,n} = (\alpha + j\beta)_{m,n} = \sqrt{\left(\frac{m\pi}{q}\right)^2 + \left(\frac{n\pi}{p}\right)^2 - \omega^2\mu\epsilon}. \quad (8-30)$$

We are primarily interested in conditions for good transmission, that is, with $\alpha = 0$. Accordingly, we are restricted to the imaginary value of the radical in equation (8-30). Therefore we may

write

$$(\alpha + j\beta)_{m,n} = j \sqrt{\omega^2 \mu \epsilon - \left(\frac{m\pi}{q}\right)^2 - \left(\frac{n\pi}{p}\right)^2},$$

or

$$\beta_{m,n} = \sqrt{\omega^2 \mu \epsilon - \left(\frac{m\pi}{q}\right)^2 - \left(\frac{n\pi}{p}\right)^2}, \quad (8-31)$$

which is the phase constant for any TE mode, $TE_{m,n}$.

8-3. THE MAGNETIC COMPONENTS OF THE TE MODES

The magnetic components may be obtained by the use of equations (8-27), (8-28), and (8-29) in connection with equations (8-20), (8-21), and (8-22). First from equation (8-20),

$$\begin{aligned} \mathcal{H}_x &= \frac{j}{\omega\mu} \left(\frac{\partial \mathcal{E}_z}{\partial y} - \frac{\partial \mathcal{E}_y}{\partial z} \right) \\ &= \frac{-jA}{\omega\mu} \left(\frac{m^2 \pi^2 p}{nq^2 \pi} + \frac{n\pi}{p} \right) \cos My \cos Nz \\ &= -j \frac{Ap}{\omega\mu n\pi} (\omega^2 \mu \epsilon + \gamma^2) \cos My \cos Nz, \end{aligned} \quad (8-32)$$

from equation (8-9). From equations (8-21) and (8-27) we have

$$\begin{aligned} \mathcal{H}_y &= \frac{j\gamma}{\omega\mu} \mathcal{E}_z \\ &= -j \frac{\gamma}{\omega\mu} \frac{mp}{nq} A \sin My \cos Nz. \end{aligned} \quad (8-33)$$

From equations (8-22), (8-28),

$$\begin{aligned} \mathcal{H}_z &= \frac{-j\gamma}{\omega\mu} \mathcal{E}_y \\ &= -j \frac{A\gamma}{\omega\mu} \cos My \sin Nz. \end{aligned} \quad (8-34)$$

Now it is possible to collect, all in one place, the six components of the field in their complete form, that is, with the variables x and t restored and with the imaginary parts discarded. First

$$E_x = 0. \quad (8-35)$$

From equation (8-28), remembering that $\gamma = j\beta$,

$$E_y = A e^{j(\omega t - \beta x)} \cos My \sin Nz, \quad (8-36)$$

or

$$E_{y(\text{real})} = A \cos(\omega t - \beta x) \cos My \sin Nz. \quad (8-37)$$

From equation (8-27), in a like manner,

$$E_{z(\text{real})} = -A \frac{mp}{nq} \cos(\omega t - \beta x) \sin My \cos Nz. \quad (8-38)$$

From equation (8-32),

$$H_x = -j \frac{Ap}{\omega\mu n\pi} (\omega^2\mu\epsilon + \gamma^2) \epsilon^{j(\omega t - \beta x)} \cos My \cos Nz, \quad (8-39)$$

or

$$H_{z(\text{real})} = \frac{Ap}{\omega\mu n\pi} (\omega^2\mu\epsilon - \beta^2) \sin(\omega t - \beta x) \cos My \cos Nz, \quad (8-40)$$

or using equation (8-31),

$$H_{z(\text{real})} = \frac{Ap}{\omega\mu n\pi} (M^2 + N^2) \sin(\omega t - \beta x) \cos My \cos Nz. \quad (8-41)$$

From equation (8-33),

$$H_{y(\text{real})} = \frac{A\beta mp}{\omega\mu nq} \cos(\omega t - \beta x) \sin My \cos Nz, \quad (8-42)$$

and from equation (8-34), similarly,

$$H_x = -j \frac{A\gamma}{\omega\mu} \epsilon^{j(\omega t - \beta x)} \cos My \sin Nz, \quad (8-43)$$

or

$$H_{z(\text{real})} = \frac{A\beta}{\omega\mu} \cos(\omega t - \beta x) \cos My \sin Nz. \quad (8-44)$$

8-4. CHARACTERISTICS OF THE TE TRANSMISSION

Equations (8-31), (8-35), (8-37), (8-38), (8-40), (8-42), and (8-44) give us all the information concerning the modes of transmission under the TE classification. The modes are specified by stating the value of m and n . Thus there may exist modes $TE_{0,1}$, $TE_{1,0}$, $TE_{1,1}$, $TE_{1,2}$, etc. The $TE_{0,1}$ mode is the first to be considered here. In the first place, equation (8-31) gives us much information about the actual transmission, the other equations having to do only with the field configuration. From equation (8-31) may be obtained information concerning the cutoff frequency, phase velocity, and wavelength in the guide.

The cutoff frequency is that frequency below which β becomes imaginary making γ real and thus resulting in a case of high attenuation. The limiting case is expressed by setting the quantity under the radical in equation (8-31) equal to zero. Thus

$$\omega_0 \sqrt{\mu\epsilon} = \sqrt{M^2 + N^2},$$

or

$$\begin{aligned} f_0 &= \frac{1}{2 \sqrt{\mu\epsilon}} \sqrt{\left(\frac{m}{q}\right)^2 + \left(\frac{n}{p}\right)^2} \\ &= \frac{v}{2} \sqrt{\left(\frac{m}{q}\right)^2 + \left(\frac{n}{p}\right)^2}, \end{aligned} \quad (8-45)$$

where v is the phase velocity in this space and equals the velocity of light, c , if $\epsilon_r = \mu_r = 1$. The cutoff frequency f_0 is thus a function of the mode through m and n and a function of the dimensions through p and q . On account of the reciprocal relationship between f_0 and the dimensions, only small tubes are suitable for high frequencies. The tube acts like a high-pass filter, passing only frequencies above f_0 .

The phase velocity in the guide is given by

$$v_p = \frac{\omega}{\beta} = \frac{\omega}{\sqrt{\omega^2 \mu\epsilon - \left(\frac{m\pi}{q}\right)^2 - \left(\frac{n\pi}{p}\right)^2}}. \quad (8-46)$$

From this equation it is seen that the phase velocity is always greater than the corresponding phase velocity in free space having the same values of μ and ϵ . The lowest value which v_p can have would be obtained by setting $m = n = 0$, or $p = q = \infty$, in which case

$$v_p = \frac{\omega}{\omega \sqrt{\mu\epsilon}} = \frac{1}{\sqrt{\mu\epsilon}} = v. \quad (8-47)$$

This v is equal to or less than the velocity of light, depending upon the values of μ_r and ϵ_r .

The wavelength in the guide becomes

$$\lambda_g = \frac{2\pi}{\beta} = \frac{2\pi}{\sqrt{\omega^2 \mu\epsilon - M^2 - N^2}}. \quad (8-48)$$

8-5. ILLUSTRATIVE EXAMPLE: CHARACTERISTICS OF THE TE TRANSMISSION

Example 1. Let it be assumed that a tube is given with $p = 0.03$ m, and $q = 0.02$ m. Required to find the cutoff frequency f_0 and the phase velocity and wavelength for a frequency of $1.5f_0$ for the $TE_{0,1}$ mode. The cutoff frequency is given by equation (8-45), and for the $TE_{0,1}$ mode, $m = 0$, $n = 1$. In the space within the guide, $\mu_r = \epsilon_r = 1$, and $v = c$. Thus

$$f_0 = \frac{3(10^8)}{2} \frac{1}{0.03} = 5(10^9) \text{ cps.}$$

Now β will be calculated for use in equation (8-46) to find the phase velocity and in equation (8-48) for the determination of the wavelength. ($\mu\epsilon = 1/c^2$ for this case, and $f = 7.5 \cdot 10^9$)

$$\begin{aligned} \beta &= \sqrt{\frac{4\pi^2(7.5)^2(10^{18})}{9(10^{16})} - \left(\frac{\pi}{0.03}\right)^2} \\ &= 117.1 \text{ rad/m.} \end{aligned}$$

From this, the velocity becomes

$$v_p = \frac{2\pi(7.5)10^9}{117.1} = 4.02 (10^8) \text{ m/sec,}$$

and the wavelength is

$$\lambda_g = \frac{2\pi}{117.1} = 0.0536 \text{ m.}$$

The corresponding wavelength in free space is

$$\lambda = \frac{3(10^8)}{7.5(10^9)} = 0.04 \text{ m.}$$

Comparing these values it is seen that in the same way as with waves guided between parallel plates, the velocity is considerably above the velocity of light and the wavelength is longer than in free space.

8-6. THE $TE_{0,1}$ MODE

The equations given above are applicable to the general $TE_{m,n}$ mode. In this section the field components and field configuration will be considered for the $TE_{0,1}$ mode. In this mode, as we have seen, $m = 0$, and $n = 1$. Accordingly we have for equations (8-45), (8-46), and (8-48) the following special forms:

$$f_{0(0,1)} = \frac{c}{2p} \quad (8-49)$$

Also

$$v_{p(0,1)} = \frac{c}{\sqrt{1 - \left(\frac{c}{2pf}\right)^2}} \quad (8-50)$$

The wavelength in the guide is

$$\lambda_{g(0,1)} = \frac{1}{\sqrt{\left(\frac{f}{c}\right)^2 - \left(\frac{1}{2p}\right)^2}} \quad (8-51)$$

The components of the field will be obtained from equations (8-35), (8-37), (8-38), (8-41), (8-42), and (8-44) by substituting $m = 0$, $n = 1$, $M = 0$, and $N = \pi/p$. First, of course,

$$E_x = 0. \quad (8-52)$$

Then

$$E_y = A \cos(\omega t - \beta x) \sin \frac{\pi z}{p}, \quad (8-53)$$

$$E_z = 0,$$

$$H_x = \frac{A\pi}{\omega\mu p} \sin(\omega t - \beta x) \cos \frac{\pi z}{p}, \quad (8-54)$$

$$H_y = 0, \quad (8-55)$$

$$H_z = \frac{A\beta}{\omega\mu} \cos(\omega t - \beta x) \sin \frac{\pi z}{p}, \quad (8-56)$$

where

$$\beta = \sqrt{\omega^2\mu\epsilon - \left(\frac{\pi}{p}\right)^2}$$

from equation (8-31). It is seen that there are only three components in the $TE_{0,1}$ mode, one longitudinal component and two transverse. The electric field is entirely transverse, having no longitudinal component. Thus the symbol TE is used meaning "transverse electric." In Fig. 8-2 is presented a diagram of these lines for one half-wavelength in the guide. At the point a where $z = p/2$, $x = 0$, $y = q/2$, and taking $t = 0$, the fields are

$$E_y = A, \quad (8-57)$$

$$H_x = 0, \quad (8-58)$$

$$H_z = \frac{A\beta}{\omega\mu}. \quad (8-59)$$

These components are shown by arrows and it is to be noted that E_y and H_z are in the appropriate directions to produce power flow in the $+x$ direction, according to Poynting's vector. At this point there is no longitudinal component of \mathbf{H} . A half-wavelength

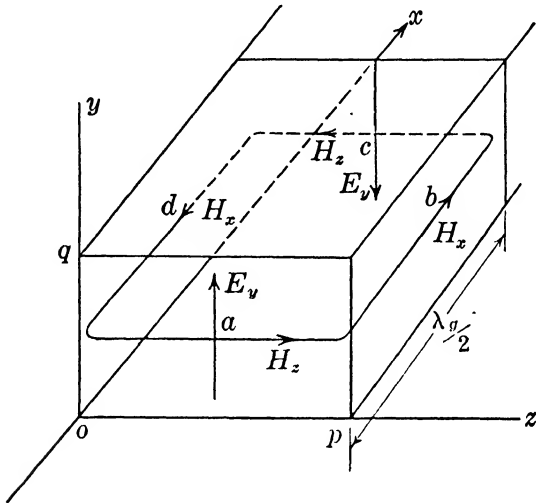


FIG. 8-2. General configuration over one half-wavelength of the $TE_{0,1}$ mode.

further down the guide where $\beta x = \pi$, and still at the center, as at the point c , it is found that the fields have reversed.

$$E_y = -A, \tag{8-57a}$$

$$H_z = 0, \tag{8-58a}$$

$$H_z = -\frac{A\beta}{\omega\mu}. \tag{8-59a}$$

Here the Poynting vector is still in the $+x$ direction. At the point where $\beta x = \pi/2$, represented by b and d , the fields are

$$E_y = 0, \tag{8-57b}$$

$$H_z = -\frac{A\pi}{\omega\mu p} \cos \frac{\pi z}{p}, \tag{8-58b}$$

$$H_z = 0. \tag{8-59b}$$

At the center of the guide where $z = p/2$, $H_z = 0$, but at the sides it has a value such that at $z = p$,

$$H_z = \frac{A\pi}{\omega\mu p} \quad (\text{a positive quantity}),$$

and at $z = 0$,

$$H_x = -\frac{A\pi}{\omega\mu p} \quad (\text{a negative quantity}).$$

These vectors are shown also, and the diagram thus becomes consistent with the idea that the magnetic lines are continuous loops.

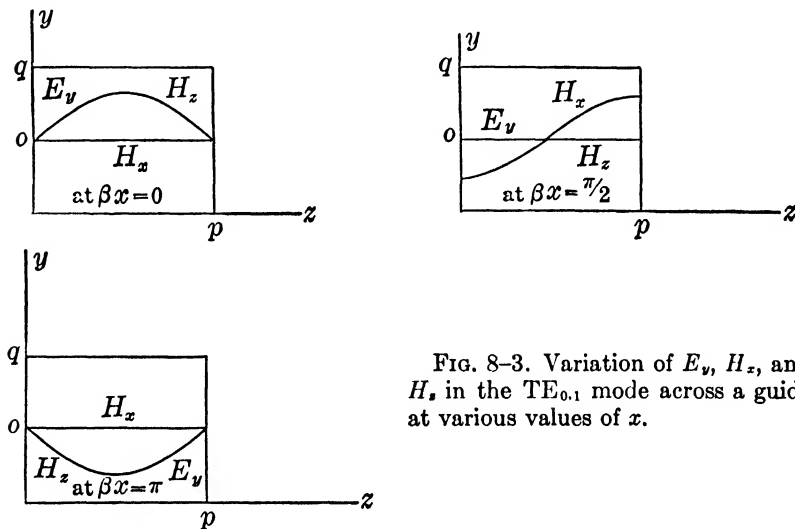


FIG. 8-3. Variation of E_y , H_x , and H_z in the $TE_{0,1}$ mode across a guide at various values of x .

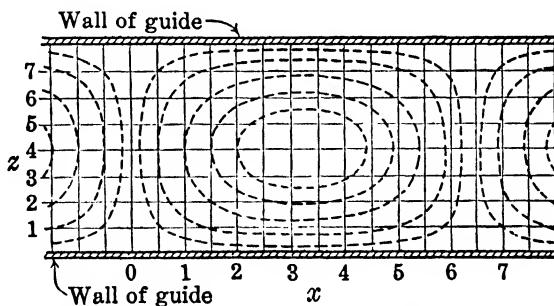


FIG. 8-4. Magnetic lines of force in a rectangular guide. $TE_{0,1}$ mode. (Modified by permission from "*Communication Circuits*" by L. A. Ware and H. R. Reed, published by John Wiley and Sons, Inc.)

In Fig. 8-3 these three components are plotted in various ways in order to present a better idea of the field configuration. Refer to Chapter VII where a similar discussion occurs concerning loops in the parallel plate guide. The form of the magnetic loops for the $TE_{0,1}$ mode is shown in Fig. 8-4.

8-7. POWER TRANSMITTED BY THE TE_{0,1} MODE

The power transmitted by the TE_{0,1} mode per unit cross section of the guide is given by Poynting's vector

$$\mathbf{P} = \mathbf{E} \times \mathbf{H} \text{ watts/m}^2.$$

The only field components which will contribute to longitudinal transmission are E_y and H_z . Accordingly, using equations (8-53) and (8-56) we obtain

$$\mathbf{P} = \mathbf{i} \frac{A^2\beta}{\omega\mu} \cos^2(\omega t - \beta x) \sin^2 \frac{\pi z}{p}. \quad (8-60)$$

This is the same as obtained for the parallel-plate guide of Chapter VII. The time-averaged value of power density is

$$P = \frac{A^2\beta}{2\omega\mu} \sin^2 \frac{\pi z}{p}. \quad (8-61)$$

The total power is obtained by integrating over the cross section of the tube

$$\begin{aligned} P_t &= \int_0^p \int_0^q \frac{A^2\beta}{2\omega\mu} \sin^2 \frac{\pi z}{p} dy dz \\ &= \frac{A^2\beta pq}{4\omega\mu} \text{ watts.} \end{aligned} \quad (8-62)$$

This equation should be compared with equation (7-54). The voltage at which the guide operates may be considered as the rms value of the potential difference between the lower and upper wall. The peak value of field intensity is A . Thus the potential difference becomes

$$V_{max} = Aq,$$

or the rms value is

$$V = \frac{Aq}{\sqrt{2}} \text{ v.} \quad (8-63)$$

Equations (8-62) and (8-63) may now be used to find an impedance which will be *defined* as the characteristic impedance and represented by Z_0 . It will be a real quantity and is such that

$$P_t = \frac{V^2}{Z_0}. \quad (8-64)$$

Thus

$$Z_0 = \frac{V^2}{P_t} = \frac{2q\omega\mu}{\beta p} \text{ ohms.} \quad (8-65)$$

This characteristic impedance is seen to depend upon the dimensions and upon the frequency. It can be varied in value by changing q which does not change the other transmission properties, except attenuation. Characteristic impedance can also be defined as the ratio of $E_{y(max)}$ to $H_{z(max)}$. This produces a simpler form but, in general, a different value for the characteristic impedance from that given by equation (8-65).

$$Z_0' = \frac{E_{y(max)}}{H_{z(max)}} = \frac{\omega\mu}{\beta} \text{ ohms.} \quad (8-66)$$

This equation produces the same value for Z_0 as given above only in case the guide is half as high as it is wide, or $p = 2q$. Equation (8-65), however, comes nearer to conforming to our usual ideas of impedance of transmission lines.

Power will be transmitted also in the z direction but such power merely reflects back and forth, and since the walls are assumed to be perfect conductors, it does not represent any power loss. This can be seen by noting that the average value of $E_y H_x$ is zero because the average value of $\cos x \sin x$ is zero.

8-8. ILLUSTRATIVE EXAMPLE: $TE_{0,1}$ MODE

Example 2. Let it be assumed in Example 1 that the guide is transmitting 10 kw at the frequency of $7.5(10^9)$ cps. Required to find the value of Z_0 by the use of both equations (8-65) and (8-66) and the maximum voltage across the guide. From Example 1, $\beta = 117.1$ rad/m. For the space in the guide, $\mu = \mu_0\mu_r = 4\pi(10^{-7})$. From equation (8-65)

$$\begin{aligned} Z_0 &= \frac{2(0.02)4\pi(10^{-7})2\pi(7.5) 10^9}{117.1 (0.03)} \\ &= 674 \text{ ohms.} \end{aligned}$$

From equation (8-66)

$$\begin{aligned} Z_0' &= \frac{2\pi(7.5)(10^9)4\pi 10^{-7}}{117.1} \\ &= 505 \text{ ohms.} \end{aligned}$$

currents may thus be calculated, all those indicated by i_1 , i_2 , i_3 , and i_4 flowing into the lower plate of the half-wave section of guide. Along the edges oa , and bc , the magnetic field is given by

$$H_x = \frac{A\beta}{\omega\mu} \cos(\omega t - \beta x) \sin \frac{\pi z}{p},$$

where x is to be set at the value zero for the path oa , and at $\lambda_g/2$ for the path bc . The time t will be taken as zero, that is, the value giving maximum H_x . Thus at $x = 0$,

$$H_x = \frac{A\beta}{\omega\mu} \sin \frac{\pi z}{p}, \quad (8-67)$$

and at $x = \lambda_g/2$,

$$H_x = -\frac{A\beta}{\omega\mu} \sin \frac{\pi z}{p}. \quad (8-68)$$

The current, i_1 , is now determined by integrating equation (8-67) over the range $z = 0$ to p . This is

$$\begin{aligned} i_1 &= \int_0^p \frac{A\beta}{\omega\mu} \sin \frac{\pi z}{p} dz \\ &= \frac{2A\beta p}{\omega\mu\pi}. \end{aligned} \quad (8-69)$$

According to equation (8-68), the current i_2 will be the same except for sign and the total longitudinal current *entering* the half-wave-length plate is

$$I_x = \frac{4A\beta p}{\omega\mu\pi} \text{ amp.} \quad (8-70)$$

The current flows into the plate from the wave-guide sections before and after the section shown. Simultaneously there is an opposite and equal flow of current in the upper plate at the ends marked A and B . It should be recognized that the current distribution from $z = 0$ to p is not uniform but, of course, follows the same law as the magnetic field, because at all points the current and magnetic field are proportional.

The currents i_3 and i_4 flowing into the plate from the sides of the guide are determined from the expression for H_x ,

$$H_x = \frac{A\pi}{\omega\mu p} \sin(\omega t - \beta x) \cos \frac{\pi z}{p}. \quad (8-54)$$

For i_3 , $z = 0$, and the integration is from $x = \lambda_g/2$ to zero, with

$t = 0$, along co . Therefore,

$$\begin{aligned} i_3 &= \frac{-A\pi}{\omega\mu p} \int_{\lambda_0/2}^0 \sin \beta x \, dx \\ &= \frac{2A\pi}{\beta\omega\mu p}. \end{aligned} \quad (8-71)$$

For i_4 , equation (8-54) becomes, along ab , (at $z = p$),

$$H_x = \frac{A\pi}{\omega\mu p} \sin \beta x, \quad (8-72)$$

and accordingly will result in an equal current flowing into the plate from the right. Thus the total lateral current is

$$I_z = \frac{4A\pi}{\beta\omega\mu p}. \quad (8-73)$$

8-10. THE CHARACTERISTIC IMPEDANCE

Using equation (8-69), it is now possible to define a third characteristic impedance based upon the fact that in transmission lines it is the impedance, $Z_0 = V/I$, at any point of an infinite line operating under steady state conditions. The rms potential difference between the upper and lower plates at $z = p/2$, is given by equation (8-63) as $V_{rms} = Aq/\sqrt{2}$. The current expression (8-69) gives the maximum value on account of the selection of $t = 0$, so the rms current is

$$I_{rms} = \frac{\sqrt{2} A\beta p}{\omega\mu\pi}. \quad (8-74)$$

Thus a characteristic impedance can be defined as

$$\begin{aligned} Z_0 &= \frac{V_{rms}}{I_{rms}} = \frac{Aq}{\sqrt{2}} \frac{\omega\mu\pi}{\sqrt{2} \sqrt{2} A\beta p} \\ &= \frac{\omega\mu\pi q}{2\beta p} \text{ ohms.} \end{aligned} \quad (8-75)$$

It will be noted that this is still different from the two previous expressions for Z_0 .

8-11. CURRENT CONTINUITY

On referring again to Fig. 8-5, it is seen that current is flowing from all sides into the lower plate in the same way as into the plate of a condenser. If the condenser analogy holds here, then the

total current

$$\begin{aligned} I_t' &= I_x + I_z = \frac{4A\beta p}{\omega\mu\pi} + \frac{4A\pi}{\beta\omega\mu p} \\ &= \frac{4A}{\omega\mu} \left(\frac{\beta p}{\pi} + \frac{\pi}{\beta p} \right) = \frac{4A\omega\epsilon p}{\beta\pi} \end{aligned} \quad (8-76)$$

must equal the maximum displacement current in the dielectric between the plates. To test this, equation (8-53) will be used for the electric field between the plates.

$$E_y = A \cos(\omega t - \beta x) \sin \frac{\pi z}{p}. \quad (8-53)$$

The flux density at any point is

$$D_y = A\epsilon \cos(\omega t - \beta x) \sin \frac{\pi z}{p}, \quad (8-77)$$

and the displacement current density becomes

$$I' = \frac{dD_y}{dt} = -A\epsilon\omega \sin(\omega t - \beta x) \sin \frac{\pi z}{p}. \quad (8-78)$$

This must be integrated over the surface of the plate, as follows, to give I_t .

$$\begin{aligned} I_t &= -A\epsilon\omega \int_{x=0}^{a/2} \int_{z=0}^p \sin(\omega t - \beta x) \sin \frac{\pi z}{p} dz dx \\ &= \frac{4A\epsilon\omega p}{\beta\pi} \cos \omega t. \end{aligned} \quad (8-79)$$

The maximum value occurs at $t = 0$, and is

$$I_t = \frac{4A\epsilon\omega p}{\beta\pi}, \quad (8-80)$$

which agrees with equation (8-76) in both magnitude and time (phase).

8-12. THE $TE_{1,1}$ MODE

The $TE_{0,1}$ mode which has been treated in some detail above is the most commonly used because of its simplicity and the fact that, at the same frequency, no other mode may exist in the guide. However, in order to present a better picture of the general TE modes, the $TE_{1,1}$ mode will be briefly considered. The TM modes are omitted because of the fact that they are not extensively used

in practice. Also the procedure for deriving the TM modes is almost a repetition of that for the TE modes.

For the TE_{1,1} mode we may first go back to the equations (8-45), (8-46) and (8-48) and obtain the following for an air dielectric in the guide ($m = 1, n = 1$):

$$f_{o(1,1)} = \frac{c}{2} \sqrt{\left(\frac{1}{q}\right)^2 + \left(\frac{1}{p}\right)^2} \quad (8-81)$$

$$\begin{aligned} v_{p(1,1)} &= \frac{\omega}{\sqrt{\omega^2 \mu \epsilon - \left(\frac{\pi}{p}\right)^2 - \left(\frac{\pi}{q}\right)^2}} \\ &= \frac{f}{\sqrt{\left(\frac{f}{c}\right)^2 - \left(\frac{1}{2p}\right)^2 - \left(\frac{1}{2q}\right)^2}}, \end{aligned} \quad (8-82)$$

and

$$\lambda_{o(1,1)} = \frac{1}{\sqrt{\left(\frac{f}{c}\right)^2 - \left(\frac{1}{2p}\right)^2 - \left(\frac{1}{2q}\right)^2}}. \quad (8-83)$$

It is immediately seen that these quantities are dependent upon both dimensions of the guide.

The components of the electric field may again be found from equations (8-35), (8-37) and (8-38).

$$\begin{aligned} E_x &= 0, \\ E_y &= A \cos(\omega t - \beta x) \cos \frac{\pi y}{q} \sin \frac{\pi z}{p}, \end{aligned} \quad (8-84)$$

$$E_z = \frac{-Ap}{q} \cos(\omega t - \beta x) \sin \frac{\pi y}{q} \cos \frac{\pi z}{p}. \quad (8-85)$$

There are thus two components of the electric field instead of one as before. Both of these, however, are transverse, as required. The magnetic components are obtained from equations (8-41), (8-42), and (8-44).

$$H_x = \frac{Ap\pi}{\omega\mu} \left[\left(\frac{1}{q}\right)^2 + \left(\frac{1}{p}\right)^2 \right] \sin(\omega t - \beta x) \cos \frac{\pi y}{q} \cos \frac{\pi z}{p}, \quad (8-86)$$

$$H_y = \frac{A\beta p}{\omega\mu q} \cos(\omega t - \beta x) \sin \frac{\pi y}{q} \cos \frac{\pi z}{p}, \quad (8-87)$$

$$H_z = \frac{A\beta}{\omega\mu} \cos(\omega t - \beta x) \cos \frac{\pi y}{q} \sin \frac{\pi z}{p}, \quad (8-88)$$

and all three components are present.

8-13. CONFIGURATION OF THE $TE_{1,1}$ ELECTRIC FIELD

The equations of the preceding section indicate that the electric field is a 2-dimensional field perpendicular to the direction of propagation as the name TE requires. The magnetic field is, in

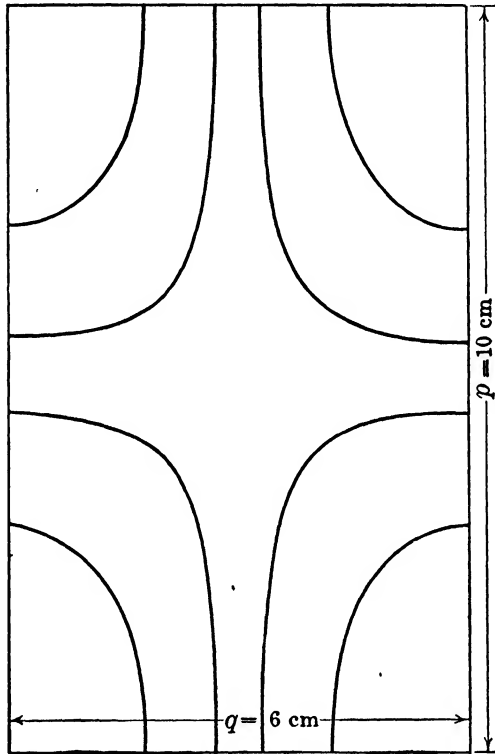


FIG. 8-6. Configuration of electric lines in the yz plane in a guide transmitting the $TE_{1,1}$ mode.

general, 3-dimensional. The shape of the electric lines can be found as follows, in a manner similar to that used previously.

$$\begin{aligned} \frac{dy}{dz} &= \frac{E_y}{E_z} \\ &= -\frac{q}{p} \cot \frac{\pi y}{q} \tan \frac{\pi z}{p}, \end{aligned}$$

or

$$\tan \frac{\pi y}{q} dy = -\frac{q}{p} \tan \frac{\pi z}{p} dz. \quad (8-89)$$

On integration of this equation, there is obtained

$$\cos \frac{\pi y}{q} \cos \frac{\pi z}{p} = C, \quad (8-90)$$

where C is a constant depending upon the location of the electric line under consideration. In Fig. 8-6 a few of these electric lines have been plotted for a guide with dimensions $p = 10$ cm, $q = 6$ cm. It should be kept in mind that the magnetic lines must always lie perpendicular to the electric lines at every point.

8-14. POWER TRANSMITTED BY THE $TE_{1,1}$ MODE

Poynting's vector will give the power transmitted per unit area of the guide. If all components listed above are included in this calculation, it will be found that the component H_x contributes nothing to the power transmission and could therefore be omitted. However, the complete expression will be written out here.

$$\mathbf{P} = \mathbf{E} \times \mathbf{H} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ E_x & E_y & E_z \\ H_x & H_y & H_z \end{vmatrix}, \quad (8-91)$$

or

$$\begin{aligned} P_x &= E_y H_z - E_z H_y \\ &= \frac{A^2 \beta}{\omega \mu} \cos^2 (\omega t - \beta x) \cos^2 \frac{\pi y}{q} \sin^2 \frac{\pi z}{p} + \\ &\quad \frac{A^2 \beta p^2}{\omega \mu q^2} \cos^2 (\omega t - \beta x) \sin^2 \frac{\pi y}{q} \cos^2 \frac{\pi z}{p} \\ &= \frac{A^2 \beta}{\omega \mu} \cos^2 (\omega t - \beta x) \left[\cos^2 \frac{\pi y}{q} \sin^2 \frac{\pi z}{p} + \frac{p^2}{q^2} \sin^2 \frac{\pi y}{q} \cos^2 \frac{\pi z}{p} \right], \end{aligned}$$

which, on integration over the cross section of the guide and averaging over one cycle, becomes

$$P_{(x)} = \frac{A^2 \beta p}{8 \omega \mu q} (p^2 + q^2) \text{ watts.} \quad (8-92)$$

This is the power flow along the axis of the tube, that is, in the x direction. The power flow in the y and z directions can be determined as follows, from equation (8-91),

$$\begin{aligned}
 P_y &= E_x H_z - E_z H_x \\
 &= \frac{-A^2 p^2 \pi}{\omega \mu q} \left[\left(\frac{1}{q} \right)^2 + \left(\frac{1}{p} \right)^2 \right] \cos(\omega t - \beta x) \\
 &\quad \sin(\omega t - \beta x) \sin \frac{\pi y}{q} \cos \frac{\pi y}{q} \cos^2 \frac{\pi z}{p}. \quad (8-93)
 \end{aligned}$$

On account of the $\cos(\omega t - \beta x) \sin(\omega t - \beta x)$ product, this becomes zero when averaged over one cycle. Thus no net power is transmitted in the y direction. Similarly,

$$\begin{aligned}
 P_x &= E_z H_y - E_y H_z \\
 &= \frac{-A^2 \pi p}{\omega \mu} \left[\left(\frac{1}{q} \right)^2 + \left(\frac{1}{p} \right)^2 \right] \cos(\omega t - \beta x) \sin(\omega t - \beta x) \\
 &\quad \cos^2 \frac{\pi y}{q} \sin \frac{\pi z}{p} \cos \frac{\pi z}{p}. \quad (8-94)
 \end{aligned}$$

The average of this is also zero for the same reason as above. Thus the only power transmitted is given by equation (8-92). This should be compared with equation (8-62) remembering that the two β 's involved are not the same.

8-15. PROBLEMS

8-1. A rectangular wave guide is given with the following dimensions: $p = 3$ cm, $q = 1$ cm. The mode to be transmitted is the $TE_{0,1}$ and the frequency is $1.5f_0$. Let A in equation (8-53) be 10^5 v/m. Find the components of the electromagnetic field at $z = 1.5$ cm, $y = 0.5$ cm, for $x = 0.0$ and $x = 2.0$ cm. Let $t = 0$.

8-2. For the conditions of Prob. 8-1, find Z_0 (by three methods) for $f = 1.1f_0$, $1.5f_0$, and $2.0f_0$.

8-3. For the conditions of Prob. 8-1, find v_p for $f = 1.1f_0$, $1.5f_0$, and $2.0f_0$.

8-4. For the conditions of Prob. 8-1, find λ_g for $f = 1.1f_0$, $1.5f_0$, and $2.0f_0$.

8-5. For Prob. 8-1, calculate the average power in watts being transmitted at frequencies of $1.5f_0$ and $2.0f_0$.

8-6. Minimum attenuation for a rectangular guide occurs when $q/p = 1.18$. Assume that this change is made in the guide of Prob. 8-1, leaving $p = 3$ cm, and find the new value of Z_0 by equations (8-65) and (8-75).

8-7. What is the disadvantage of operating as mentioned in Prob. 8-6? Find the absolute minimum frequency which can be transmitted in this tube.

8-8. Assume that the tube of Prob. 8-6 is transmitting 5 kw. Find the total rms voltage across the tube and the equivalent current flow based on the equation $P_t = VI$. Compare this current with that given by the equations of §8-9.

8-9. A guide with dimensions $p = 6$ cm and $q = 3$ cm is transmitting power in the $TE_{0,1}$ mode at a frequency of $2f_0$. (a) Find the rms value of longitudinal current if the value of A is 5000 v/m. (b) Find the rms total current flow into any half-wave plate of the guide. (See §8-9.)

8-10. Given a tube with $p = 5$ cm and $q = 2$ cm. (a) Find the cutoff frequencies for both the $TE_{0,1}$ and the $TE_{1,1}$ modes. (b) Using a frequency of one and a half times the higher of these cutoff frequencies, determine the ratio of powers transmitted in the two modes, that is, $P_{0,1}/P_{1,1}$.

8-11. For the guide of Prob. 8-10, plot a set of four E curves as shown in Fig. 8-6, one of which is to pass through the point $y = 0$, $z = 2.4$ cm. Use the $TE_{1,1}$ mode.

8-12. Equations (8-87) and (8-88) may be used to obtain the equations of the yz projection of the magnetic lines. Derive this equation and plot a curve for the guide of Prob. 8-10 passing through the point $y = 0.5$ cm, $z = 2.5$ cm.

8-13. Using the equations of §8-11 and the fact that the maximum voltage across the center of the guide is Aq volts, find the effective capacitance of a half-wave section of the guide. Compare this with the ordinary capacitance of the two plates if the side walls were not present.

Summary

The subject of radiation begins by a discussion of potentials and the presentation of a suitable expression for retarded potential. On the basis of the retarded potential, which depends upon the current flowing in the antenna, the action of an antenna of very short length is studied, including a brief treatment of radiation resistance. Using the equation for the very short antenna, the properties of the half-wave antenna are derived and an expression for radiation resistance written down. A short treatment of the receiving loop antenna is presented in the last part of the chapter, and the directional effects are investigated.

9-1. DYNAMIC POTENTIALS

Previously obtained results indicate that an electromagnetic wave can be propagated through space of any kind. It seems that this propagation is the result of the mutual action of varying electric and magnetic fields. However, thus far, there has been no discussion of the means of establishing these waves. This subject is to be considered under the title of "Radiation" which concerns itself with the establishment of electromagnetic waves as a result of currents flowing in circuits, and as an introduction, it is convenient first to consider more carefully the general subject of potentials.

Specifically, unique potentials will first be found from which \mathbf{E} and \mathbf{H} may be determined. The discussion begins by first listing Maxwell's equations again, for convenience, and by calling attention to the two relations for potentials previously presented. Maxwell's equations are

$$\nabla \times \mathbf{H} = \sigma \mathbf{E} + \epsilon \frac{\partial \mathbf{E}}{\partial t}, \quad (9-1)$$

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}, \quad (9-2)$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon}, \quad (9-3)$$

$$\nabla \cdot \mathbf{H} = 0. \quad (9-4)$$

The two potentials already mentioned are, first, the scalar potential ϕ from which \mathbf{E} is found,

$$\mathbf{E} = -\nabla\phi. \quad (9-5)$$

and the vector potential \mathbf{A} from which \mathbf{H} is determined,

$$\mathbf{H} = \nabla \times \mathbf{A}. \quad (9-6)$$

The potential ϕ is not suitable, however, for the general case of time-varying fields represented by equations (9-1) to (9-4). That this is true may be seen by taking the curl of equation (9-5).

$$\nabla \times \mathbf{E} = -\nabla \times \nabla\phi = 0. \quad (9-7)$$

This is identically zero. However, according to equation (9-2), it must be equal to

$$-\mu \frac{\partial \mathbf{H}}{\partial t},$$

a finite quantity if the field is changing. Thus ϕ cannot be used. Some other potential function must be found and if one should think, by analogy with equation (9-6), that some vector potential could be used he is soon disillusioned because, if

$$\mathbf{E} = \nabla \times \mathbf{G},$$

then

$$\nabla \cdot \mathbf{E} = \nabla \cdot \nabla \times \mathbf{G} = 0.$$

But by equation (9-3) this should be proportional to charge density. Thus a pure unadulterated vector potential cannot be used.

In the meantime, however, we note that equation (9-6) is entirely satisfactory because from it

$$\nabla \cdot \mathbf{H} = \nabla \cdot \nabla \times \mathbf{A} = 0,$$

agreeing with equation (9-4). Equation (9-6) does not uniquely define the vector potential because there are many vector fields whose curls are equal to \mathbf{H} . Nevertheless, we begin by accepting

\mathbf{A} as the vector potential and will substitute equation (9-6) into equation (9-2)

$$\nabla \times \mathbf{E} = -\mu \left(\nabla \times \frac{\partial \mathbf{A}}{\partial t} \right), \quad (9-8)$$

or

$$\nabla \times \left(\mathbf{E} + \mu \frac{\partial \mathbf{A}}{\partial t} \right) = 0. \quad (9-9)$$

Now, since the curl is zero, there exists a scalar potential F such that

$$\mathbf{E} + \mu \frac{\partial \mathbf{A}}{\partial t} = -\nabla F, \quad (9-10)$$

or

$$\mathbf{E} = -\nabla F - \mu \frac{\partial \mathbf{A}}{\partial t}. \quad (9-11)$$

This expression for \mathbf{E} in terms of both a scalar field F and the vector field \mathbf{A} is satisfactory because it satisfies equation (9-2), and is made to satisfy equation (9-3) as follows. Substitute expression (9-11) into equation (9-3).

$$-\nabla \cdot \nabla F - \mu \frac{\partial}{\partial t} \nabla \cdot \mathbf{A} = \frac{\rho}{\epsilon},$$

or

$$\nabla^2 F + \mu \frac{\partial}{\partial t} \nabla \cdot \mathbf{A} = -\frac{\rho}{\epsilon}. \quad (9-12)$$

The substitution of expressions (9-11) and (9-6) into equation (9-1) gives

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mathbf{j}_c - \epsilon \nabla \frac{\partial F}{\partial t} - \epsilon \mu \frac{\partial^2 \mathbf{A}}{\partial t^2}, \quad (9-13)$$

where $\mathbf{j}_c = \sigma \mathbf{E}$, the conduction current density. Equations (9-12) and (9-13) relate ρ , \mathbf{j}_c , F , and \mathbf{A} but are somewhat complicated. Furthermore, the potentials are not yet definitely defined since many \mathbf{A} 's would satisfy the equations, and F 's may differ among themselves by a constant. Thus one more relation is needed to connect F and \mathbf{A} and, arbitrarily but wisely, we pick the following:

$$\nabla \cdot \mathbf{A} = -\epsilon \frac{\partial F}{\partial t}. \quad (9-14)$$

By using this expression, equations (9-12) and (9-13) may be rewritten as follows:

$$\nabla^2 F = \mu\epsilon \frac{\partial^2 F}{\partial t^2} - \frac{\rho}{\epsilon} \quad (9-15)$$

$$\nabla^2 \mathbf{A} = \mu\epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} - \mathbf{j}_c \quad (9-16)$$

The potentials F and \mathbf{A} given by equations (9-15) and (9-16) are called the *dynamic potentials* because they apply to fields which are changing in time.

9-2. SPECIAL CASES

Certain special cases of equations (9-15) and (9-16) are of interest.

Case 1. $\rho = 0$. Equation (9-15) reduces to the wave equation. In this case the potential F is propagated with the velocity $1/\sqrt{\mu\epsilon}$.

Case 2. $\mathbf{j}_c = 0$. Equation (9-16) reduces to the wave equation and the potential \mathbf{A} is propagated with the velocity $1/\sqrt{\mu\epsilon}$.

Case 3. Variation with respect to t very low or zero. Equations (9-15) and (9-16) reduce to

$$\nabla^2 F = -\frac{\rho}{\epsilon}, \quad (9-17)$$

which is Poisson's equation, and

$$\nabla^2 \mathbf{A} = -\mathbf{j}_c \quad (9-18)$$

Case 4. Time variation as $e^{j\omega t}$ ($= \cos \omega t + j \sin \omega t$). If we assume steady-state conditions with this specific variation with respect to time, which is very common, it is possible to write both \mathbf{E} and \mathbf{H} in terms of \mathbf{A} alone in a rather simple manner. Equation (9-14) becomes

$$\begin{aligned} \nabla \cdot \bar{\mathbf{A}} e^{j\omega t} &= -\epsilon \frac{\partial}{\partial t} \mathcal{F} e^{j\omega t} \\ &= -j\epsilon\omega \mathcal{F} e^{j\omega t}, \end{aligned}$$

from which

$$\mathcal{F} = j \frac{1}{\omega\epsilon} \nabla \cdot \bar{\mathbf{A}}, \quad (9-19)$$

and

$$\nabla \mathcal{F} = j \frac{1}{\omega\epsilon} \nabla \nabla \cdot \bar{\mathbf{A}},$$

and equation (9-11) becomes

$$\bar{\mathbf{E}} = -j \frac{1}{\epsilon\omega} \nabla \nabla \cdot \bar{\mathbf{A}} - j\mu\omega \bar{\mathbf{A}},$$

or

$$\mathbf{E} = -j\epsilon^{j\omega t} \left(\frac{1}{\epsilon\omega} \nabla \nabla \cdot \bar{\mathbf{A}} - \mu\omega \bar{\mathbf{A}} \right), \quad (9-20)$$

and of course,

$$\mathbf{H} = \epsilon^{j\omega t} (\nabla \times \bar{\mathbf{A}}). \quad (9-21)$$

Thus both \mathbf{E} and \mathbf{H} are given in terms of a single potential function \mathbf{A} ($= \bar{\mathbf{A}}\epsilon^{j\omega t}$).

9-3. THE SOLUTION OF THE EQUATION

In any practical case the values of σ and \mathbf{j}_c may be known and it may be required to obtain \mathbf{E} and \mathbf{H} . Thus, solutions to equations (9-15) and (9-16) must be had from which we may determine \mathbf{E} and \mathbf{H} . An inspection of these equations with reference to §9-2 indicates that the disturbance travels at a velocity of v , ($= 1/\sqrt{\mu\epsilon}$). If r , which is the distance in a spherical coordinate system from the source of disturbance, is very small, the time effects are negligible and the potentials can be written as follows:

$$F = \int \frac{\rho(t)}{4\pi r} d\tau, \quad (9-22)$$

and

$$\mathbf{A} = \int \frac{\mathbf{j}_c(t)}{4\pi r} d\tau, \quad (9-23)$$

where the integration is to be performed throughout a volume enclosing the source of disturbance whether it be a time-varying charge or a current variation. The student is already familiar with the simple expression for potential of a fixed charge. It often appears in basic electrical engineering courses in the cgs system of units as follows:

$$\phi = \frac{Q}{\epsilon r},$$

where ϕ is in statvolts, Q is in statcoulombs, and r is in centimeters. If this equation is transformed to the mks system, using ϵ for free space, the following is obtained.

$$\phi = \frac{Q}{4\pi r},$$

where ϕ is now in volts, Q is in coulombs, and r is in meters. On changing the Q to charge density, $\rho(t)$, and allowing it to be a function of time, as indicated, we obtain

$$\phi = \frac{\rho(t)}{4\pi r},$$

which can be integrated over the volume enclosing the charge to obtain equation (9-22),

$$F = \int \frac{\rho(t)}{4\pi r} d\tau, \quad (9-22)$$

where F is now the total potential in volts produced by the summation of all the elementary charges under consideration. Equation (9-23) can be written down by analogy.

Because of the finite velocity of propagation, it would be expected that the effect at any distant point will be expressed in similar form but proportional to the state of affairs at the source *at an earlier time*, $[t - (r/v)]$, allowing for the time required for transmission. Thus it might be expected that the new potentials are given by

$$F = \int \frac{\rho\left(t - \frac{r}{v}\right)}{4\pi r} d\tau, \quad (9-24)$$

and

$$\mathbf{A} = \int \frac{\mathbf{j}_c\left(t - \frac{r}{v}\right)}{4\pi r} d\tau, \quad (9-25)$$

where $\rho[t - (r/v)]$ and $\mathbf{j}_c[t - (r/v)]$ mean functions of $[t - (r/v)]$ and are not products.

As a further discussion of the \mathbf{A} potential, consider the case of the thin straight wire oriented along the x axis of Fig. 9-1. From equation (9-25) we may write down

$$A_x = \int_{-a/2}^{a/2} \frac{I\left(t - \frac{r}{v}\right)}{4\pi r} dx, \quad (9-25a)$$

where now the volume integral is, in effect, an integration along x because of the infinitesimal diameter of the wire, and I is the current carried by the wire. It is observed that the vector potential at a point r from the origin is considered to be a function of $[t - (r/v)]$ in order to account for the finite length of time $[t' = (r/v)]$ required for the effect of the change of antenna current to reach the point in question, at a distance r from the antenna. Since only linear current densities are to be used here, the form (9-25a) of

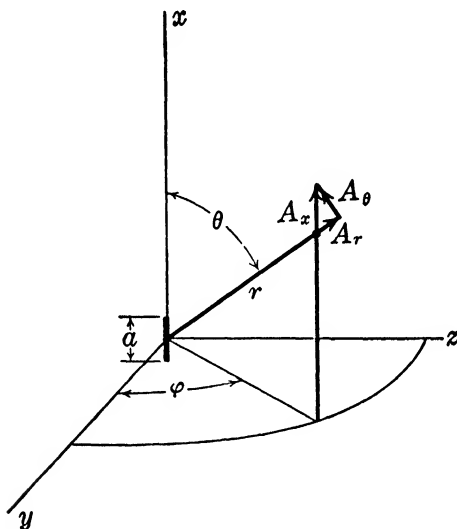


FIG. 9-1. Magnetic potential produced by a short wire.

the solution for A_x is applicable for our needs. The only justification which need be made for equations (9-25) and (25a) is that they represent solutions to equation (9-16) which agree with experimentally determined results.

It will be remembered that a similar situation is encountered even in the steady-state d-c evaluation of H in the vicinity of a short segment of current-carrying conductor. From Ampere's law

$$H = \int_{-a/2}^{a/2} \frac{I_{dc} \sin \theta}{4\pi r^2} dx.$$

The justification usually given for the use of this equation is that it serves to give results which agree with experimentally determined results.

9-4. RADIATION FROM A SHORT WIRE

In Fig. 9-1 is represented a wire of length a carrying a current $I \sin \omega t$. The current function which must now be used in equation (9-25a) is

$$I \sin \omega \left(t - \frac{r}{v} \right) = I \sin (\omega t - \beta r),$$

so that we have

$$A_x = \int \frac{I \sin (\omega t - \beta r)}{4\pi r} dx, \tag{9-26}$$

where $\beta = \omega/v$. Now let a be very small compared to the distance r and to the wavelength, λ . Then the integrand is practically constant over the range of integration and equation (9-26) becomes simply

$$A_x = \frac{Ia}{4\pi r} \sin (\omega t - \beta r), \tag{9-27}$$

from which, referring to Fig. 9-1,

$$A_r = \frac{Ia}{4\pi r} \sin (\omega t - \beta r) \cos \theta. \tag{9-28}$$

$$A_\theta = - \frac{Ia}{4\pi r} \sin (\omega t - \beta r) \sin \theta. \tag{9-29}$$

$$A_\phi = 0. \tag{9-30}$$

Now, according to equation (9-6), the magnetic field is given by

$$H_\phi = \frac{Ia}{4\pi r} \sin \theta \left[\beta \cos (\omega t - \beta r) + \frac{1}{r} \sin (\omega t - \beta r) \right]. \tag{9-31}$$

The electric field can be found either by using Maxwell's equation or from equation (9-11). In order to use equation (9-11), it is necessary to find F which can be done through equation (9-14), as follows;

$$\frac{\partial F}{\partial t} = - \frac{1}{\epsilon} \nabla \cdot \mathbf{A}, \tag{9-32}$$

or

$$F = - \frac{1}{\epsilon} \nabla \cdot \int \mathbf{A} dt, \tag{9-33}$$

and from (9-11),

$$\mathbf{E} = \frac{1}{\epsilon} \nabla \nabla \cdot \int \mathbf{A} dt - \mu \frac{\partial \mathbf{A}}{\partial t}. \tag{9-34}$$

Using the expression for the components of \mathbf{A} above, this finally results in the following:

$$E_r = \frac{Ia}{2\pi r} \cos \theta \left[\frac{1}{r} \sqrt{\frac{\mu}{\epsilon}} \sin(\omega t - \beta r) - \frac{1}{\omega \epsilon r^2} \cos(\omega t - \beta r) \right]. \quad (9-35)$$

$$E_\theta = \frac{Ia}{4\pi r} \sin \theta \left[\omega \mu \cos(\omega t - \beta r) - \frac{1}{r} \sqrt{\frac{\mu}{\epsilon}} \sin(\omega t - \beta r) - \frac{1}{\omega \epsilon r^2} \cos(\omega t - \beta r) \right]. \quad (9-36)$$

$$E_\phi = 0.$$

In order to write equations (9-31), (9-35), (9-36) in terms of the wavelength, the following substitution will be made:

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega \sqrt{\mu \epsilon}}.$$

Now we have:

$$H_\phi = \frac{\beta I a \sin \theta}{4\pi r} \left[\cos(\omega t - \beta r) + \frac{\lambda}{2\pi r} \sin(\omega t - \beta r) \right]. \quad (9-37)$$

$$E_r = \frac{\mu I a \omega \cos \theta}{4\pi^2 r} \left[\frac{\lambda}{r} \sin(\omega t - \beta r) - \frac{\lambda^2}{2\pi r^2} \cos(\omega t - \beta r) \right]. \quad (9-38)$$

$$E_\theta = \frac{\mu I a \omega \sin \theta}{4\pi r} \left[\cos(\omega t - \beta r) - \frac{\lambda}{2\pi r} \sin(\omega t - \beta r) - \frac{\lambda^2}{4\pi^2 r^2} \cos(\omega t - \beta r) \right]. \quad (9-39)$$

$$E_\phi = 0.$$

These equations give the electromagnetic field at any point a distance r from the wire, the only restriction being that r be great enough so that the approximate integration of equation (9-26) holds. In radio work we are usually concerned with the field at a considerable distance from the antenna. The next section will treat this specific case.

9-5. THE RADIO FIELD

At a considerable distance from the current-carrying wire, r becomes sufficiently large that λ/r and its powers are negligible. This means that the electric component E_r drops out rather rapidly leaving only H_ϕ and E_θ which become

$$H_{\phi} = \frac{\beta I a \sin \theta}{4\pi r} \cos(\omega t - \beta r). \quad (9-40)$$

$$E_{\theta} = \frac{\omega I a \mu \sin \theta}{4\pi r} \cos(\omega t - \beta r). \quad (9-41)$$

These equations state that the electric and magnetic components are such that the direction of propagation, given by Poynting's vector, is in the positive r direction.

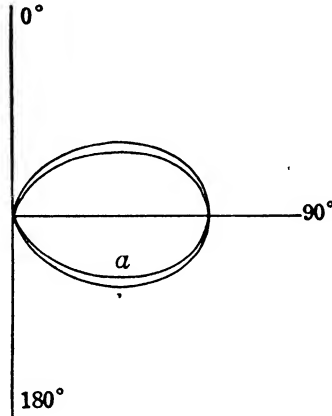


FIG. 9-2. Field patterns on a power basis: (a) from equation (9-63), for a half-wave dipole; (b) from equation (9-42), for a short wire.

The power, from Poynting's vector, becomes

$$P_r = \frac{\omega\beta\mu I^2 a^2 \sin^2 \theta}{16\pi^2 r^2} \cos^2(\omega t - \beta r). \quad (9-42)$$

At a great distance from the source, the wave is practically a plane wave traveling outward at the velocity of light (for free space). The power at any distance is inversely proportional to the square of the distance, indicating that there is no decrease in the total flow of power since the area is proportional to the square of the distance.

Equation (9-42) also throws some light on the shape of the radiation pattern. As far as the directional coordinates are concerned, E_{θ} and H_{ϕ} depend only on θ , and in the same way. Thus, in equation (9-42) the pattern is given by plotting

$$p = \sin^2 \theta,$$

as shown in Fig. 9-2. Propagation will be equal for all values of ϕ so that the pattern is really the figure formed by rotating the

curve $p = \sin^2 \theta$ about its vertical axis, giving a doughnut-shaped volume.

The field quite near to the antenna, but not too near, is given by the last terms of the brackets in equations (9-37), (9-38), (9-39). Thus, at close range,

$$H_\phi = \frac{Ia \sin \theta}{4\pi r^2} \sin(\omega t - \beta r). \quad (9-43)$$

$$E_r = \frac{-Ia\mu\omega\lambda^2 \cos \theta}{8\pi^2 r^3} \cos(\omega t - \beta r). \quad (9-44)$$

$$E_\theta = \frac{-\lambda^2\mu\omega Ia \sin \theta}{16\pi^2 r^3} \cos(\omega t - \beta r). \quad (9-45)$$

It is noted here that E_θ and H_ϕ are 90° out of phase so that the radially transmitted power is alternating in character. These components make up the *induction* field, and since they decrease as the square or cube of the radius, they soon disappear in comparison with the radiation components of equations (9-40) and (9-41) which only decrease as the first power of r .

9-6. RADIATION RESISTANCE

The radiation resistance of an antenna can be defined as that value of R which multiplied by the square of the current flowing in the antenna gives the *total* power radiated. In order to calculate R it is necessary to evaluate

$$\mathbf{P} \cdot d\mathbf{a}$$

over a sphere whose center is at the antenna and whose radius is large enough to make valid the equation (9-42). The integration is easily accomplished as follows:

$$\begin{aligned} \int \mathbf{P} \cdot d\mathbf{a} &= \int \mathbf{E} \times \mathbf{H} \cdot d\mathbf{a} \\ &= \int_0^\pi \frac{I^2 a^2 \mu \omega \beta \sin^2 \theta}{16\pi^2 r^2} \cos^2(\omega t - \beta r) (2\pi r^2 \sin \theta) d\theta \\ &= \frac{I^2 a^2 \mu \omega \beta \cos^2(\omega t - \beta r)}{8\pi} \int_0^\pi \sin^3 \theta d\theta \\ &= \frac{I^2 a^2 \mu \omega \beta}{6\pi} \cos^2(\omega t - \beta r). \end{aligned} \quad (9-46)$$

The elemental area, $d\mathbf{a} = (2\pi r \sin \theta) r d\theta$, employed in the integration is shown in Fig. 9-3.

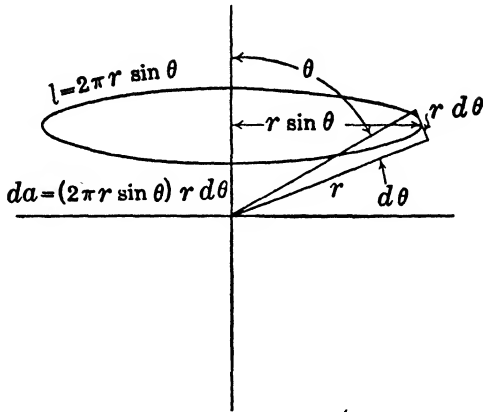


FIG. 9-3. Representing elemental area on a spherical surface for use in obtaining equation (9-46).

The average power becomes

$$P_{av} = \frac{I^2 a^2 \mu \omega \beta}{12\pi} \text{ watts.} \tag{9-47}$$

However, this average power must be given by

$$P_{av} = I^2_{eff} R,$$

or

$$R = \frac{P_{av}}{I^2_{eff}} = \frac{a^2 \mu \omega \beta}{6\pi} \text{ ohms,} \tag{9-48}$$

which, in free space, is

$$R = 80\pi^2 \left(\frac{a}{\lambda}\right)^2 \text{ ohms.} \tag{9-49}$$

9-7. THE HALF-WAVE DIPOLE

In microwave work the half-wave dipole is a very common and comparatively simple form of antenna. Such a dipole is represented in Fig. 9-4, where it is seen that it is made up of two quarter-wave antennas placed end to end and fed in the middle in such a manner that the current flow at any point in the antenna at any instant is given by

$$\begin{aligned} i &= I \cos \frac{2\pi y}{\lambda} \sin \omega t \\ &= I \cos \beta y \sin \omega t. \end{aligned} \tag{9-50}$$

This current distribution is represented by the curve in Fig. 9-4. It must be kept in mind that this is an assumed current distribution and does not necessarily conform exactly to facts. The error in the final result, however, is not significant.

A half-wave antenna is no longer small compared to a wavelength and, therefore, the procedure here is to calculate the contribution provided by each elementary length dy of the antenna according to previous equations and then to integrate the results over the actual length of the antenna. Since we are interested now only

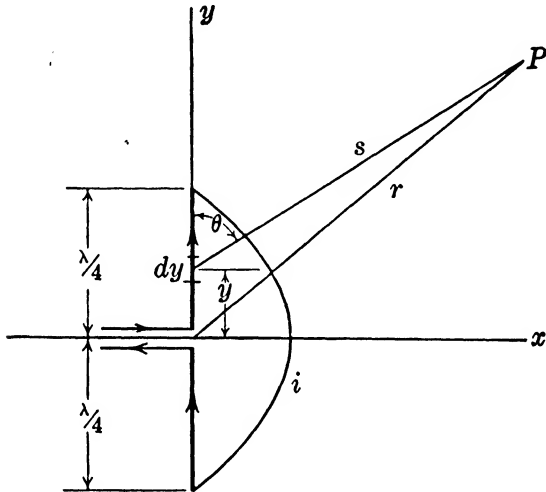


FIG. 9-4. Half-wave dipole.

in the field at a considerable distance from the antenna, the basic equation to be used is equation (9-41).

$$dE_{\theta} = \frac{\mu\omega I' dy \sin \theta}{4\pi s} \cos(\omega t - \beta s), \tag{9-51}$$

where the length a has been replaced by dy , and r replaced by s . The term I' is the coefficient of $\sin \omega t$ in equation (9-50). From equations (9-50) and (9-51), there is obtained

$$E_{\theta} = \int_{-\lambda/4}^{\lambda/4} \frac{\omega\mu I \cos \beta y \sin \theta}{4\pi s} \cos(\omega t - \beta s) dy. \tag{9-52}$$

If r (and accordingly, s) is very large, s may be replaced by r in the denominator. However, s must be retained in the *cosine* term

because here its difference from r can greatly affect the value of the cosine. The distance s can be written in terms of r (refer to Fig. 9-4) as follows:

$$s = r - y \cos \theta. \quad (9-53)$$

Thus equation (9-52) becomes

$$E_{\theta} = \frac{\mu\omega I \sin \theta}{4\pi r} \int_{-\lambda/4}^{\lambda/4} \cos \beta y \cos (\omega t - \beta r + \beta y \cos \theta) dy. \quad (9-54)$$

The integral can be written as

$$\int_{-\lambda/4}^{\lambda/4} \cos \beta y \cos (gy + h) dy, \quad (9-55)$$

where $g = \beta \cos \theta$, and $h = \omega t - \beta r$. Through a trigonometrical transformation, equation (9-55) can be integrated easily and will result in the following expression for E_{θ} .

$$E_{\theta} = \frac{I\mu\omega}{2\pi r\beta} \cos (\omega t - \beta r) \frac{\cos \left(\frac{\pi}{2} \cos \theta \right)}{\sin \theta}. \quad (9-56)$$

Now, in free space, the ratio of E_{θ} to H_{ϕ} is a constant and equal to $\sqrt{\mu/\epsilon}$ as seen from previous material. [See equations (9-40), (9-41)]. Thus H_{ϕ} can be written as follows:

$$\begin{aligned} H_{\phi} &= \frac{I\mu\omega}{2\pi r\beta} \sqrt{\frac{\epsilon}{\mu}} \cos (\omega t - \beta r) \frac{\cos \left(\frac{\pi}{2} \cos \theta \right)}{\sin \theta} \\ &= \frac{I}{2\pi r} \cos (\omega t - \beta r) \frac{\cos \left(\frac{\pi}{2} \cos \theta \right)}{\sin \theta}. \end{aligned} \quad (9-56a)$$

9-8. RADIATION RESISTANCE FOR A $\lambda/2$ DIPOLE

The average power radiated by the dipole can be calculated from Poynting's vector as follows:

$$P_r = \frac{I^2}{4\pi r^2} \sqrt{\frac{\mu}{\epsilon}} \frac{\cos^2 \left(\frac{\pi}{2} \cos \theta \right)}{\sin^2 \theta} \cos^2 (\omega t - \beta r), \quad (9-57)$$

or the total power is given by

$$P_t = \int \mathbf{P} \cdot d\mathbf{a} = \oint P_r da, \quad (9-58)$$

where the integral is taken over a sphere whose radius is r , ($\gg \lambda$).

$$P_t = \frac{I^2}{4\pi^2 r^2} \sqrt{\frac{\mu}{\epsilon}} \cos^2(\omega t - \beta r) \int_0^\pi \frac{\cos^2\left(\frac{\pi}{2} \cos \theta\right)}{\sin^2 \theta} 2\pi r^2 \sin \theta d\theta. \quad (9-59)$$

$$P_t = \frac{I^2}{2\pi} \sqrt{\frac{\mu}{\epsilon}} \cos^2(\omega t - \beta r) \int_0^\pi \frac{\cos^2\left(\frac{\pi}{2} \cos \theta\right)}{\sin \theta} d\theta. \quad (9-60)$$

The value of the definite integral is 1.219. See Prob. 9-7. Therefore,

$$\begin{aligned} P_{t(average)} &= \frac{1.219 I^2}{4\pi} \sqrt{\frac{\mu}{\epsilon}} \\ &= R I_{eff}^2 \\ &= \frac{R I^2}{2}. \end{aligned} \quad (9-61)$$

Accordingly, since $\sqrt{\mu/\epsilon} = 377$,

$$R = \frac{1.219(2)377}{4\pi} = 73.1 \text{ ohms.} \quad (9-62)$$

This is the usual value taken for the input resistance to a half-wave dipole, and is to be matched by the transmission line feeding the antenna. It should be noticed that this quantity is a constant for a *half-wave dipole* and, accordingly, is the resistance for different sizes of antennas depending on frequency. As an example:

f	Length of antenna	R
10^6	150 meters	73.1 ohms
10^7	15	73.1
10^8	1.5	73.1
10^9	0.15	73.1
10^{10}	0.015	73.1
10^{11}	0.0015	73.1

The field pattern on a power basis is given by plotting

$$p = \frac{\cos^2\left(\frac{\pi}{2} \cos \theta\right)}{\sin^2 \theta} \quad (9-63)$$

vs θ . This field pattern, α , is presented on Fig. 9-2 so that a com-

parison can be made easily with that for an antenna of elementary length.

9-9. LOOP ANTENNAS

The use of a loop as a receiving antenna provides a relatively simple example of some of the ideas which have been treated in the preceding section. There are two ways in which the process of reception by a loop may be handled. One is to find the emf induced in the loop by the electric component of the radio field and the other

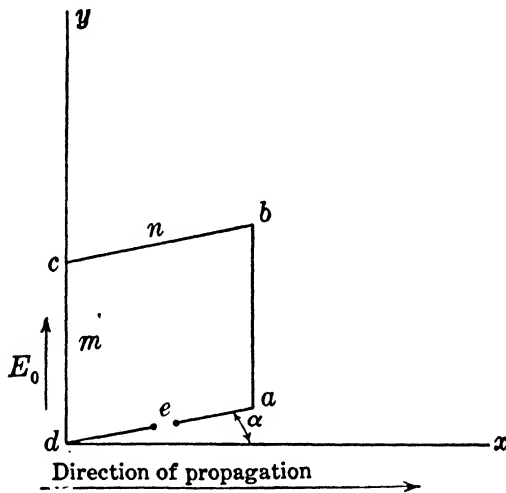


FIG. 9-5. Reception by using a loop antenna.

is to calculate the induced emf from the changing magnetic component of the field.

It is assumed that the loop is so placed that two of its sides are parallel to the electric component of the field. In these two sides will be induced an electric intensity equal to the field intensity, and the total emf available can be determined by integrating around the loop.

Let a loop be as shown in Fig. 9-5, placed so that its sides dc and ab are parallel to the electric field components, and its plane at an angle of α with the direction of propagation. Let it be assumed that the maximum field intensity is E_0 and that the frequency is f . The induced emf in side dc is $mE_0 \cos(\omega t - \beta r)$. The emf along the side ab will lag behind that at dc by an angle given by $n\beta \cos \alpha$. The emf's at da and cb are, of course, equal and

opposite, and thus cancel. The total instantaneous emf induced around the loop will then be

$$\begin{aligned} e &= mE_0[\cos(\omega t - \beta r - n\beta \cos \alpha) - \cos(\omega t - \beta r)] \\ &= mE_0(2) \sin\left(\omega t - \beta r - \frac{n\pi}{\lambda} \cos \alpha\right) \sin\left(\frac{n\pi}{\lambda} \cos \alpha\right) \\ &= 2mE_0 \sin\left(\frac{n\pi}{\lambda} \cos \alpha\right) \sin\left(\omega t - \beta r - \frac{n\pi}{\lambda} \cos \alpha\right), \end{aligned} \quad (9-64)$$

or

$$E_{eff} = \sqrt{2} mE_0 \sin\left(\frac{n\pi}{\lambda} \cos \alpha\right). \quad (9-65a)$$

$$E_{eff} = 2mE_{0(eff)} \sin\left(\frac{n\pi}{\lambda} \cos \alpha\right). \quad (9-65b)$$

This is the one turn. For N turns, equation (9-65) is to be multiplied by N .

In order to obtain the maximum induced voltage at the antenna, it is necessary to set

$$\frac{n\pi}{\lambda} \cos \alpha = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \text{ etc.} \quad (9-66)$$

or

$$n = \frac{\lambda}{2 \cos \alpha}, \frac{3\lambda}{2 \cos \alpha}, \text{ etc.} \quad (9-67)$$

This result seems obvious enough since in Fig. 9-5, if maximum induced emf in the loop is to be desired, the field along dc must be in the direction $d-c$, and the field along ab must be in the direction $b-a$, or just reversed. This condition is, of course, met by placing dc and ab one half-wavelength apart in a radial direction.

In order to treat the problem by the use of the magnetic component, refer again to Fig. 9-5 and consider that the position of the point d is r_0 units from the origin. The B field, which is given by multiplying expression (9-56a) by μ , is perpendicular to the paper. It is now necessary to integrate this expression over the area of the loop to find ϕ and then to determine the emf through the equation

$$e = - \frac{d\phi}{dt}. \quad (9-68)$$

A simplification may be made immediately in equation (9-56a) by noting that, in integrating over the small area at a great distance

from the origin, the r appearing in the denominator may be considered a constant, r_0 . The integration then will appear as follows;

$$\phi = B_0 \int_{r_0}^{r_0+n \cos \alpha} \int_{(d)}^{(a)} \cos (\omega t - \beta r) dy dr, \quad (9-69)$$

where B_0 is the maximum magnetic field density at r_0 . Now the integration in the y direction from d to a is very easily performed by merely multiplying by m . Then we have

$$\begin{aligned} \phi &= mB_0 \int_{r_0}^{r_0+n \cos \alpha} \cos (\omega t - \beta r) dr \\ &= \frac{-mB_0\lambda}{2\pi} [\sin (\omega t - \beta r_0 - n\beta \cos \alpha) - \sin (\omega t - \beta r_0)], \end{aligned} \quad (9-70)$$

and from equation (9-68), the instantaneous voltage becomes

$$e = \frac{\omega m B_0}{\beta} \cos (\omega t - \beta r_0 - n\beta \cos \alpha) - \cos (\omega t - \beta r_0),$$

which through the same transformation applied to obtain equation (9-64), becomes

$$e = \frac{2\omega m B_0}{\beta} \sin \left(\frac{n\pi}{\lambda} \cos \alpha \right) \sin \left(\omega t - \beta r_0 - \frac{n\pi}{\lambda} \cos \alpha \right).$$

Now,

$$B_0 = \mu H_0,$$

and

$$H_0 = \sqrt{\frac{\epsilon}{\mu}} E_0,$$

so

$$e = 2mE_0 \sin \left(\frac{n\pi}{\lambda} \cos \alpha \right) \sin \left(\omega t - \beta r_0 - \frac{n\pi}{\lambda} \cos \alpha \right), \quad (9-71)$$

which is the same as equation (9-64), since $r = r_0$, and will reduce likewise to equation (9-65b).*

* In relation to the problem of antennas, the following question may arise concerning the calculation of induced voltages by the electric and magnetic fields separately: "If both fields may be used to calculate the induced emf and if both give the same result, why is it that the actual induced voltage is not twice that produced by a single field?"

That such is not true may be seen in the following discussion. Refer to Fig. 9-6. Here the wire W is in an electric field of a certain intensity E volts/meter. We have seen in previous chapters that the tangential component of \mathbf{E} just outside a conductor must be equal to the field inside

The directional effect of the antenna is easily derivable from equation (9-65). For instance, if $\alpha = 90^\circ$, E_{eff} is zero for all values of n , thus providing a means of determining direction to the transmitter. However, for a large antenna, other zeros may occur

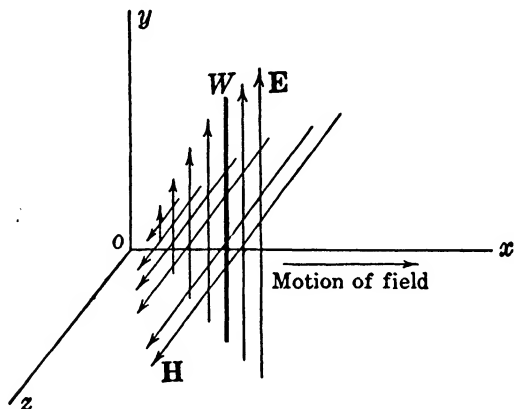


FIG. 9-6. The induced voltage in the wire may be calculated from either \mathbf{E} or \mathbf{H} , (B).

so that this is not a safe criterion. The small loop antenna, to be treated in the next section, gives a definite indication of direction.

9-10. THE SMALL LOOP ANTENNA

Except for very high frequencies, a loop antenna with n equal to one-half a wavelength is not practical. Thus there is some interest in loop antennas whose dimensions are very small compared to a wavelength. If, in equation (9-65b), n is made small, the equation reduces to

the surface of the conductor. Thus the field intensity inside the wire cannot be different from the actual field itself. Therefore, the induced emf may be calculated on the basis of the \mathbf{E} field alone.

Further insight into this may be obtained. It is known that the power transmitted by the electric field is equal to the power transmitted by the magnetic field; also that the induced voltages are the same. These two fields may be thought of as two generators in parallel supplying power to the antenna as a load. Like generators in parallel supply a voltage equal to that of each generator. The power taken will then depend on the impedance of the antenna or load, as is the case with parallel generators. Thus, again, the induced emf is that produced by each field alone.

$$\begin{aligned}
 E_{\theta} &= 2mE_{0(\theta)} \frac{n\pi}{\lambda} \cos \alpha \\
 &= \frac{2A\pi}{\lambda} E_{0(\theta)} \cos \alpha \text{ volts for one turn,} \quad (9-72)
 \end{aligned}$$

where A is the area of the loop. This equation will apply to any shape of loop as long as the largest dimension is small compared to a wavelength. Here it is quite easy to see that for $\alpha = 90^\circ$ zero signal will be received and, further, that if $\alpha = 0^\circ$, maximum signal is received. As mentioned above, these conditions are not necessarily met in the case of a large loop antenna.

9-11. PROBLEMS

9-1. Integrate equation (9-54).

9-2. A half-wave dipole is absorbing 1 kw from the supply line. Find the field intensity in microvolts per meter at a position where $r = 5$ miles, and $\theta = 45^\circ$. Repeat for $\theta = 90^\circ$. ($f = 800$ mc/sec.)

9-3. In Problem 9-2 find the power radiated per square meter at $r = 5$ miles and $\theta = 90^\circ$ and compare with the value which would be obtained if the power were radiated equally in all directions.

9-4. A certain radio field has a frequency of 10^9 cps and a field intensity of $100 \mu\text{v}/\text{m}$. A rectangular loop of a single turn, 5 in. by 6 in., is used to pick up this field and is oriented in such a way that the 6-in. sides are parallel to the electric component and the plane of the coil is parallel to the direction of propagation. Find the rms voltage supplied by the loop.

9-5. A certain short antenna has a length of $\lambda/10$ and is radiating at 10^7 cps with a maximum current in the antenna of 5 amp. Find the distance r at which the induction field E_θ has the same magnitude as the radio field E_θ . Let $\theta = 90^\circ$.

9-6. At a point in space where the electric field intensity is $500 \mu\text{v}/\text{m}$, rms, there is placed a small loop antenna of one turn. The area of the antenna is 4 in.^2 and it is placed so that its plane is parallel to both the direction of propagation and the electric vector. The frequency is $3(10^8)$ cps. (a) Find the magnetic flux threading the loop at any instant. (b) Find the rate of change of this flux and thus the induced voltage as a function of time.

9-7. Show by graphical means that the definite integral in equation (9-60) has the value 1.219.

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