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PRINCIPLES OF
REINFORCED CONCRETE CONSTRUCTION

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REINFORCED CONCRETE CONSTRUCTION

BY

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FOURTH EDITION

Seventh Printing

NEW YORK

JOHN WILEY & SONS, Inc.

London: CHAPMAN & HALL, Limited

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FOURTH EDITION
Seventh Printing, June, 1948

PRINTED IN U. S. A.

PREFACE TO THE FOURTH EDITION

THE present edition follows the same general plan as the third edition of this work. The material has been thoroughly revised and due consideration given to the latest experiments and developments in methods of calculation and design. Compared with the third edition, some of the more important changes are the following: Revised formulas and diagrams for the design of beams including circular sections subjected to bending and compression; adaptation of diagrams to the use of any desired value of *n*; discussion of recent tests on beams and columns; amplification of material on flat slabs, including detailed treatment of footings; more adequate development of the analysis of continuous beams; application of the slope-deflection method to building frames, including haunched beams with general results for typical cases; application of method of moment distribution to continuous girders and frames; complete detailed analysis of the arch with methods of arriving at tentative designs; torsional stresses in beams; and effect of shrinkage and plastic flow on stresses in beams, columns, and arches. Much credit is due Professor W. S. Kinne for his work in devising and preparing the new diagrams, and for a large amount of assistance in many other features of this work.

MADISON, WISCONSIN,
May, 1932.

F. E. TURNEAURE
E. R. MAURER.

FOURTH EDITION, THIRD PRINTING

The only important item of revision in this printing is the presentation of a more adequate treatment of the unsymmetrically reinforced beam subjected to both bending and compression. The methods of solution given are general and quite simple and it is believed they will be found useful. Other minor changes have been made and typographical errors corrected.

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CHAPTER I

INTRODUCTORY

1. **Historical Sketch.**—The invention of reinforced concrete is usually credited to Joseph Monier, but his first constructions are antedated by those of Lambot, who in 1850 constructed a small boat of reinforced concrete and in 1855 exhibited the same at the Paris Exposition. In this latter year Lambot took out patents on this form of construction; it was regarded by him as especially well adapted to shipbuilding, reservoir work, etc.

In 1861, Monier, who was a Parisian gardener, constructed tubs and tanks of concrete surrounding a framework or skeleton of wire. In the same year Coignet announced his principles for reinforcing concrete, and proposed construction of beams, arches, pipes, etc. Both he and Monier executed some work in the new material at the Paris Exposition of 1867. In this year Monier took out patents on his reinforcement. It consists of two sets of parallel bars, one set at right angles to and lying upon the other, thus forming a mesh of bars. This system, and slight modifications of it, are extensively used at the present time, particularly for slab reinforcement. Though even the early Monier patents covered principles of wide application, still the early work in reinforced concrete was confined to a comparatively narrow field.

In 1884–5 the German and American rights of the Monier patents fell into the hands of German engineers. One of these, G. A. Wayss, together with J. Bauschinger at once began an experimental investigation of the Monier system, and in 1887 they published their findings.

The investigation proved reinforced concrete a valuable means of construction, and furnished some formulas and methods for design. The practical application of this new method of construction developed first principally in Austria, and for several years the engineers of that country made more use of it than those of any other country. Among these engineers should be mentioned Melan, who in the early 90's originated a system in which I- or T-beams were the principal element of strength, providing compressive as well as tensile strength. In Germany the government regulations hindered the application of reinforced concrete for a time, but later on a large number of " systems " were developed in that country.

In France many systems of reinforcement were invented from time to time, among which should be mentioned that of Hennebique, who was probably the first to use stirrups and "bent-up" bars, an arrangement now universally employed.

In England and America the first use of iron or steel with concrete arose in the effort to fireproof the former by means of the latter. Attempting to utilize also the strength of concrete, Hyatt built beams of concrete reinforced with metal in various ways, and with Kirkaldy of London performed tests on such beams and published the results of the investigation in 1877. The first reinforced-concrete work in the United States was done in 1875 by W. E. Ward, who constructed a building in New York state in which walls, floor-beams, and roof were made of concrete reinforced with metal to provide tensile strength. But the Pacific Coast saw the actual early development of this form of construction. P. H. Jackson, G. W. Percy, and E. L. Ransome were the pioneer workers. Jackson has been credited with reinforced constructions dating as far back as 1877, but Ransome executed the most notable early examples. Among these are a warehouse (1884 or '85), a factory building a few years later, the building of the California Academy of Science (1888 or '89), and the museum building of Leland Stanford Junior University (1892). Percy was the architect of the last two. The museum building contains spans of 45 ft. and is reinforced throughout. This and the Academy building withstood the shocks of the earthquake of 1906 remarkably well—the museum better than its two brick annexes.

Other pioneer constructors in reinforced concrete in this country were F. von Emperger and Edwin Thacher. The former introduced the Melan system (1894) and built the first reinforced arch bridges of considerable span. Thacher's first large reinforced-concrete bridge was built in 1896 and was without precedent here or in Europe.

Reinforced-concrete construction has developed during the past thirty years into a standard form of construction to be used whenever considerations of economy demand it. Uncertainties of behavior and of theory which formerly existed to a considerable extent have been largely eliminated, and design in reinforced concrete is carried out on rational principles in the same manner as in other materials, although in the nature of the case the variation in quality is greater than in such material as structural steel.

2. Use and Advantages of Reinforced Concrete.—A combination of steel and concrete constitutes a form of construction possessing to a large degree the advantages of both materials without their disadvantages. It will be desirable at the outset to consider briefly these advantages in order better to appreciate the field in which this type of construction is likely to be most successful.

Steel is a material especially well suited to resist tensile stresses, and for such purposes the most economical form—the solid compact bar—is well adapted. To resist compressive stresses, steel must be made into more expensive forms, consisting of relatively thin parts widely spread, in order to provide the necessary lateral rigidity. A serious disadvantage in the use of steel in many locations is its lack of durability; and, again, a comparatively low degree of heat destroys its strength, thus rendering it necessary to add a protective covering where a fire-resisting structure is demanded.

Concrete is characterized by low tensile strength, relatively high compressive strength, and great durability. It is a good fireproof material, and therefore serves as a good fireproof covering for steel. It is also found that steel well covered by concrete is thoroughly protected from corrosion.

In the design of structural members these qualities of steel and concrete will lead to the use of the two materials about as follows: For those structural members carrying purely tensile stresses steel must be employed, but it may be surrounded by concrete as a protection against corrosion and fire, or merely for the sake of appearance. For those members sustaining purely compressive stresses concrete is fundamentally the better and cheaper material. With concrete costing 40 cents per cubic foot, for example, and steel 4 cents per pound, or about \$20.00 per cubic foot, and with working stresses of 600 and 16,000 lbs./in.^{2}, respectively, the relative cost of the two materials for carrying a given load is as 40/600 is to 2000/16,000, or as 64 is to 120. For large and compact compressive members plain concrete will therefore naturally be used, especially where durability is a factor. For more slender members, however, such as long columns, plain concrete, being a brittle material, is too much affected by secondary and unknown stresses to be satisfactory; and for such members steel alone, or the two materials in combination, will preferably be used. Steel may be used with concrete in the form of small rods to reinforce the concrete; or it may be used in larger sections and simply surrounded and held rigidly in place by the concrete, most of the load being carried by the steel; or, finally, a steel column may be used and merely fireproofed by the concrete. As the cost of steel in the form of rods is much less than in the form of built members, and as compressive stresses can, in general, be carried more cheaply by concrete than by steel, economical construction will lead to the use of the maximum amount of concrete and the minimum amount of steel consistent with safety, although this principle will be modified by various practical considerations.

For those structural forms in which both tension and compression exist, that is to say, in all forms of beams, the combination of the two materials is particularly advantageous. Here the tensile stresses are carried by steel rods embedded in the concrete near the tension side of the beam. The steel is thus used in its cheapest form, it is thoroughly protected by the concrete, and the compressive stresses are carried by the concrete. Concrete alone cannot be used to any appreciable extent to carry bending stresses on account of its low and uncertain tenacity, but a concrete beam with steel rods embedded in it to carry the tensile stresses is a strong, economical, and very durable form of structure.

From these considerations it follows that reinforced-concrete construction is advantageous to varying degrees in different types of structures. Some of the most important of these types will here be noted, together with the advantages accompanying the use of reinforced concrete in their design.

3. *Buildings.*—This type of construction is especially useful for floor-slabs and to a somewhat less degree for beams, girders, and columns. It is also well adapted for spread footings in foundations.

4. *Culverts and small Girder Bridges.*—Very satisfactory on account of its simplicity and economy as compared to masonry arches, and because of its durability as compared to small steel bridges.

5. *Retaining-walls, Dams, and Abutments.*—Often economical for such structures as compared to ordinary masonry. Plain masonry structures of this kind are designed to resist lateral forces by their weight alone, the resulting compressive stresses, except in extremely large structures, being very small and much below safe values. By the use of reinforced concrete these structures can be designed of a more economical type and so arranged as to utilize the concrete in the form of beams, thus developing more nearly the full compressive strength of the material. The steel reinforcement is fully protected from corrosion, a factor which prevents the use of all-steel frames for structures of this class.

6. *Arch Bridges.*—In this form of structure reinforced concrete possesses less advantage over ordinary masonry than in those forms where the compressive stresses are less important. In an arch the stresses are principally compressive, and these do not require steel reinforcement; it is only to provide for the bending stresses due to moving loads, or as a precaution against undesirable cracks, that steel is serviceable. For short spans no considerable economy can be obtained by its use, but for long spans the reduction in dead load made possible, results in a large economy in many cases. By reason of greater simplicity and the less expensive abutments required, a flat-top culvert or beam bridge, with abutments of reinforced concrete, is more advantageous for short spans than the arch.

7. *Reservoir Walls, Floors, and Roofs.*—Very well adapted as a durable material and lending itself to much lighter design than common masonry.

8. *Conduits and Pipe Lines.*—Reinforced concrete can often be used to great advantage in a water-conduit or large sewer. It is also sometimes used for pipe lines and tanks under pressure, the steel being relied upon to resist the tensile stresses, while the concrete serves as a protection and as a water-tight covering. The amount of steel may

thus be determined by considerations of strength alone, where otherwise a much larger amount of metal would be needed and in a more expensive form.

9. *Elevated Tanks, Bins, etc.*—Advantageous because of its durability and its adaptability in the construction of heavy floors and walls subjected to lateral pressure. Of especial value for coal-bins, either for flooring and lining alone, or for the entire structure. Its use for water tanks has not proved very satisfactory on account of the difficulty of securing imperviousness.

10. *Chimneys and Towers.*—Possesses advantages over brick or stone masonry in the fact that it forms a structure of monolithic character, resulting in greater certainty in the stresses and economy in design.

11. *Piles, Railroad Ties, etc.*—The use of a moderate amount of steel with concrete so as to give to this material a reliable tensile and bending resistance has opened the way for its use in a great variety of forms, not only as complete structures, or important members of structures, but also in many special individual forms. Concrete piles are valuable substitutes for piles of wood where the latter would be subject to deterioration. Reinforced concrete has obvious advantages as a material for railroad ties, but a successful design has not yet been developed. This material is also well adapted to many other special uses, such as fence posts, transmission line poles and all similar purposes where the structure is exposed to the action of the elements.

CHAPTER II

PROPERTIES OF THE MATERIALS

12. In a design where two or more materials are combined in the same member the stresses in the different materials depend upon the elastic properties of the materials as well as upon the superimposed loads. Therefore in making such designs a knowledge of these elastic properties is quite as necessary as a knowledge of the strength of the materials.

CONCRETE

13. **General Requirements.**—The conditions to be met in reinforced-concrete construction require the use, generally, of a concrete of relatively high grade. In this type of construction the strength of the material is of much greater importance than it is in many forms of plain concrete design, as the dimensions of the structures are more directly dependent upon strength and less upon weight. A comparatively strong concrete is therefore found to be economical.

It is especially important, also, that the concrete be of uniform quality and free from voids, as the sections are comparatively small and the stability of the structure, to a much greater extent than is the case with massive concrete, is dependent upon the integrity of every part. Thoroughly sound concrete is also required in order to insure good adhesion to the steel reinforcement and adequate protection of the steel from corrosion and from fire. For exposed structures, density and imperviousness are important qualities. These requirements call for great care in the preparation and placing of the material.

Concrete is subject to great variations in its properties, owing to the great variations in the character and proportions of its ingredients and in its preparation. It is therefore difficult to judge from results of tests made under certain conditions as to what may fairly be expected of a concrete prepared under other conditions. For this reason it is important that special tests be made with the materials actually to be used on the work. Regular and systematic tests should also be made during the progress of construction to serve as a check on the preliminary tests and to prevent any deterioration of quality due to possible changes in materials or in the method of mixing and placing.

14. **Cement.**—Portland cement only should be used; it should meet such standard specifications as those of the American Society for Testing Materials. The rapidity of hardening of different cements varies considerably and may be an element requiring special attention where the structure is to receive its load very early or where such load is to be long deferred.

15. **Fine Aggregate.**—The sand, or fine aggregate (material less than about ¼ in. in size), should be clean and preferably of coarse grain. A fine sand requires more cement paste to completely surround the grains and produce a workable mortar or concrete than a coarse sand, and hence for equal strength requires more cement. Or, if the same amount of cement is used, more water will be needed, resulting in a weaker product. In the case of a very fine sand the difference is very marked, so that unless care is taken and special tests made, the resulting concrete is likely to be porous and deficient in strength and adhesive power. Where the use of fine sand is contemplated, tests of strength may show that a considerable extra cost may be justified in securing a coarser material. The effect of size of sand is shown in Art. 21.

Where practicable the sand should be of such grade that from 40% to 60% will be held on a No. 30 sieve and at least 70% on a No. 50. Not less than 10% should pass the No. 50.

16. **Coarse Aggregate.**—For the coarse aggregate (material exceeding about ¼ in. in size) either broken stone or gravel is satisfactory. As in the case of sand, the coarse aggregate should be graded from fine to coarse. At least 95% should pass the sieve of maximum size desired, 40% to 75% should pass the sieve of half this size, and not more than 10% pass a No. 4 sieve. Pit-run gravel is rarely sufficiently uniform and correct in proportions to be safely used without screening and remixing.

The maximum desirable size of stone or gravel depends upon the size of the structural forms and the size and spacing of the reinforcement, it being desirable to use as large a size of aggregate as will admit of convenient working. Maximum sizes of stone of ¾ in. to 1¼ in. are common, but on heavy work, with rods widely spaced, there is no objection to still larger sizes. Generally speaking, for sizes below 1 in. the density and strength of the concrete increases somewhat with size of coarse aggregate.

Gravel concrete of the same proportions as broken stone concrete is somewhat more fluid and easier to place so that a greater density is likely to be secured. For equal degrees of workability the necessary

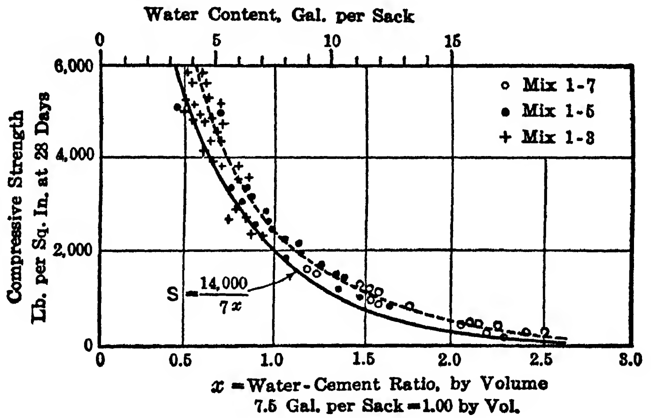


FIG. 1.

proportions of water will be somewhat less, resulting in a greater strength. A difference of 10% to 15% in crushing strength is indicated in a series of tests at the Bureau of Standards.*

17. **General Principles of Proportioning.**—The economical proportioning of concrete requires special attention to two factors: (1) the gradation of the aggregate, and (2) the proportions of water and cement in the cement paste used to fill the voids in the aggregate. To reduce to a minimum the amount of paste needed, the aggregate should be well graded from fine to coarse. With the voids then filled, with some surplus to give workability, the strength of the concrete will be determined closely by the ratio of cement to voids in the aggregate.

* Technologic Paper No. 58.

Inasmuch as the cement paste used in filling the voids is a mixture of water and cement, the proportionate amount of cement in the voids can also be expressed in an inverse manner by the ratio of water to cement, or the "water-cement ratio," a more commonly used basis of analysis.

The relation between compressive strength and water-cement ratio has been found to be quite constant for a great variety of proportions and consistencies within workable limits. This relation is approximately shown by Fig. 1, representing results from a variety of mixes of gravel concrete.* The lower curve is the "Abrams Curve," developed by Abrams in 1918 from his studies on this subject. Its equation is $S = 14{,}000/7^x$, where S = compressive strength in 28 days and x = water-cement ratio by volume. The tests represented in Fig. 1 and other later tests indicate that somewhat higher strengths can usually be obtained by careful proportioning. The Abrams curve may be considered as representing conservative values.

Where no preliminary tests have been made, the specifications of the American Concrete Institute, 1928, provide maximum values of water-cement ratio, corresponding closely to Abrams' curve, as follows:

Compressive Strength at 28 Days, Lbs/in.^{2}	Water-cement Ratio Gallons per Sack
1500	8¼
2000	7½
2500	6¾
3000	6

The recognition of the significance of the proportions of water and cement has led to a general use of the "water-cement ratio" method of proportioning. In proceeding according to this method, the proper proportions of water and cement are determined which will give the strength desired, using Abrams' curve as a basis, or such information as may be available from other tests. Then to a cement paste of these proportions, fine and coarse aggregates are added by trial until the consistency is as dry as desired. The better graded the aggregate, the

* Report Director of Research, Portland Cement Association, 1928.

lower will be the proportion of voids and the greater the amount of aggregate that can be added for a given consistency, leading to greater economy. Actual tests of the product should be made to check the assumptions as to strength, and the proportions then adjusted if necessary. A "lean" mix will be weaker than a "rich" mix for the reason that more water in proportion to cement is required in the former than in the latter, in order to fill the voids and make a workable product. In determining the amount of water used, due allowance must be made for the free water contained in the aggregates.

A great advantage of this method of proportioning is that, with a fixed amount of water per sack of cement (an easily controlled factor), a variation in character of aggregates will be indicated by a change in consistency, and may be corrected by modifying the total amount of aggregate or the proportions of fine and coarse sizes, leaving the water-cement ratio, and hence the strength, undisturbed; whereas the use of an arbitrary proportion of cement to aggregate is likely to result in an increased use of water to correct any deficiency in workability. The important factor is the use of a fixed ratio of water to cement.

18. **Consistency.**—From the period prior to the use of reinforced concrete to the present time there have been striking changes in practice in regard to consistency. The older specifications for mass concrete, which required only sufficient water to be used so that when the concrete was placed in relatively thin layers and thoroughly rammed the water would just flush to the surface, or so that the material would "quake like liver," produced a superior grade of concrete that has proved by years of service to be durable and sound. Then upon the development of reinforced work in which it was impossible to use such dry concrete, the use of water was greatly increased so that the concrete would readily "flow" into place, and tamping and ramming were abandoned even for massive work. Such excessive use of water, together with inadequate mixing, led to the construction of much concrete that has been seriously lacking in durability and strength. Although the weakening effect of excess water has long been known, much more attention has been paid to this matter in recent years; the results of the studies which have been made on the relation between strength and cement-voids ratio, or strength and water-cement ratio, have been utilized to a large extent in modern speci-

fications. The result has been a great improvement in concrete practice and the placing of the subject on a much more definite and rational basis.

Requirements of strength and consistency will vary according to the nature of the structure, and the gradation of available aggregates will also vary. The proportions adopted will therefore be different under different circumstances, but by controlling the water-cement ratio and the consistency it is not difficult to arrive at the most economical proportions in any given case.

Bearing in mind that the consistency should be as dry as practicable, it will be necessary to use a considerable wetter consistency for reinforced work in thin sections than in massive work and in pavements. In no case should the concrete "flow" into place but should require some working with spades, rods, or tampers.

The following range of consistencies, as determined by the slump tests, are suggested in the 1924 specifications of the Joint Committee:

Type of Concrete	Maximum Slump, In.
1. Mass Concrete	3
2. Reinforced Concrete:	
(*a*) Thin vertical sections and columns	6
(*b*) Heavy sections	3
(*c*) Thin confined horizontal sections	8
3. Roads and Pavements:	
(*a*) Hand finished	3
(*b*) Machine finished	1
4. Mortar for Floor Finish	2

19. Proportions of Aggregates.—The report of the Joint Committee contains tables of approximate proportions which will assist in determining upon a proper mix for the particular job. The necessary variations in proportions to provide various strengths and consistencies are illustrated by the following abstract from the tables of the Joint Committee, using values for a fine aggregate of size No. 0 to No. 8, and a coarse aggregate from 3/8 in. to 1½ in.

Consistency Slump, In.	Proportions of Cement, Sand, and Stone by Volume			
	1500-lb Concrete	2000-lb. Concrete	2500-lb. Concrete	3000-lb. Concrete
½ to 1	1 : 3.5 : 5.5	1 : 2.8 : 4.7	1 : 2.2 : 4.0	1 : 1.7 : 3.4
3 to 4	1 : 3.1 : 5.0	1 : 2.4 : 4.1	1 : 1.8 : 3.5	1 : 1.4 : 2.9
6 to 7	1 : 2.4 : 4.2	1 : 1.7 : 3.4	1 : 1.3 : 2.8	1 : 1.0 : 2.2
8 to 10	1 : 1.6 : 3.1	1 : 1.1 : 2.4	1 : 0.8 : 1.8	1 : 0.5 : 1.4

It will be seen that the proportion of cement to maintain a given strength increases very rapidly with increased slump, being about twice as much for the 8 to 10 in. slump as for 3 to 4 in. The economic value of plastic rather than wet consistencies is apparent and should be fully considered in the execution of work. A wet mix is not only uneconomical but promotes segregation of aggregate, water pockets, and bad work.

20. **Mixing of Concrete.**—It is essential that the mixing be thoroughly done. Machine mixing should be required wherever practicable and the time of mixing or number of turns and rate specified. With the ordinary types of rotary mixers at least 1 minute should be required, and still better results can be obtained with longer periods.

21. **Compressive Strength.**—The rating of concrete is indicated chiefly by its compressive strength, and compressive tests are usually the only ones employed in the control of the product. Although shearing and tensile strength are also important, these properties follow a fairly close relation to the compressive strength, and as the latter is usually the determining factor in reinforced-concrete design, the compressive test is the standard for comparison. The shape of the specimen has a considerable effect upon the test results. Formerly the cubical specimen was the standard form, but results of the extensive study made on concrete in recent years have led to the adoption of the cylindrical form, two diameters in height, as the standard, either 6 in. by 12 in., or 8 in. by 16 in., depending upon the maximum size of aggregate. The standard ages for testing are 7 and 28 days.*

On large work it is important that the quality be controlled by

* See American Society for Testing Materials Standards for further details.

field tests of representative samples. Although the strength at 28 days is usually taken as the standard, the effect of further time on the strength is important. In general, long-time tests show that moist cured concrete increases in strength considerably up to 1 year or longer, the strength at 1 year being 50% to 60% greater than at 28 days, and at 3 months, about 35% greater. Long-time tests at the University of Wisconsin of 1 : 2 : 4 concrete stored in water or in the open air showed marked increase in strength up to 4 years but little change beyond that time. Specimens stored in a cellar showed practically no change after 6 months. If adequately supplied with moisture, concrete will thus show increase of strength for a year or longer, but otherwise will reach its full value in a few months. In very dry climates, moisture must be supplied after the forms are removed if satisfactory strength is to be secured.

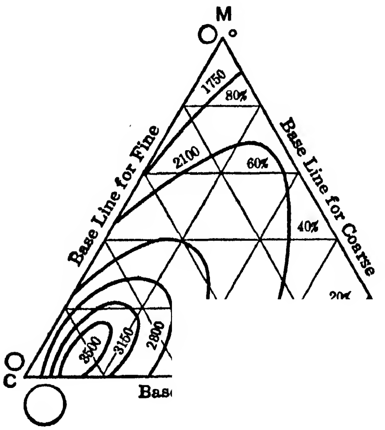


FIG. 2.

The exhaustive studies by Feret on the effect of size of sand are well worth noting even though the more recent studies on the water-cement ratio basis have made these of less practical importance than formerly, although still valid. Fig. 2 shows results obtained by him on 1 : 3 mortar cubes after hardening 1 year in fresh water. The mortars were of plastic consistency. The sand used consisted of mixtures of various proportions of fine (0.c to 0.5 mm.), medium (0.5 to 2 mm.), and coarse (2 to 5 mm.) sand, and in the figure the result from any particular mortar is recorded in the triangle at such distances from the three baselines as will represent the proportions of each size sand used. Lines of equal strength were then drawn in the diagram. Thus the strength of the mortar in which only fine sand was used was only 1400 lbs/in.^{2} The maximum strength of 3500 lbs/in.^{2} was obtained from a mixture containing about 85% of coarse sand and 15% of fine, with a very little sand of medium size. This diagram shows in a striking manner the effect of size of sand.

The explanation of these results on the water-cement ratio basis is the fact that the fine sands require much more water for the same consistency than the coarse, but the very great effect of size is none the less of much significance.

22. Tensile Strength.—The tensile strength of concrete is quite as important as the compressive strength. In fact, the most common type of failure of a reinforced-concrete beam is closely related to the tensile strength of the concrete. The tensile strength is generally from one-eighth to one-twelfth of the compressive strength, but this ratio varies considerably. It decreases somewhat with increase in coarseness of materials and with low water-cement ratios. It is also affected by the nature of aggregate, being less for granite than for hard limestone.

23. Shearing Strength.—In the discussion of beams, the term "shearing strength" is frequently used to designate the strength of a beam against failure from inclined tensile stresses which exist in regions of heavy shear and which are, at the neutral axis, of equal magnitude to the shearing stresses. Such a use of the term is misleading. In this work the authors will use the term to denote the strength of the material against a sliding failure when tested as a rivet or bolt would be tested for shear; or the shearing of metal by punching a hole in a plate. This action in reinforced concrete is sometimes called "punching" shear.

Tests made under the direction of Professor C. M. Spofford on cylinders 5 in. in diameter with ends securely clamped in cylindrical bearings gave results as follows:

Mixture	Shearing Strength, lbs/in.^{2}	Compressive Strength, lbs/in.^{2}	Ratio of Shearing to Compressive Strength
1 : 2 : 4	1480	2350	0.63
1 : 3 : 5	1180	1330	0.89
1 : 3 : 6	1150	1110	1.04

Tests made at the University of Illinois on rectangular specimens tested in a similar manner gave the following average results:

Mixture	Shearing Strength, lbs/in.²	Compressive Strength, lbs/in.²	Ratio of Shearing to Compressive Strength
1 : 2 : 4	1418	3210	0.44
1 : 3 : 6	1250	2290	0.57

Tests made by punching through plates gave shearing strengths varying from 37% to 86% of the compressive, the value depending upon the form of test-piece.*

Tests by Feret on mortar prisms gave results for shearing strength equal to about one-half the crushing strength.

The ordinary crushing failure is really a failure by shearing (combined with compression), and under such conditions the crushing stress is, theoretically, twice the shearing-stress, the angle of shear being 45°. Results of tests give a somewhat greater inclination than 45°, so that the crushing stress is somewhat greater than twice the actual shearing-stress.

We may then conclude, both from theory and from tests, that the shearing strength of concrete, in the sense here used, is about one-half the crushing strength. It is in fact so large that it will need to be considered only in exceptional cases.

24. **Elastic Properties of Concrete.**—*Stress-strain Curve in Compression.*—In the design of combination structures, such as those of steel and concrete, it is necessary to know the relative stresses under like distortions. These will depend upon the moduli of elasticity of the two materials. For purposes of safe design we need to know also the elastic-limit strength.

Fig. 3 represents typical stress-strain curves for concrete in compression obtained from tests on cylinder 6 in. in diameter by 18 in. high. The concrete was 1 : 2 : 4 limestone concrete 30 days old. The ultimate strengths ranged from 1500 to 2300 lbs/in.²

Unlike the elastic line for steel, the line for concrete is slightly curved almost from the beginning, the curvature gradually increasing towards the end. There is no point of sharp curvature as for ductile materials. A release of load at a moderate stress, such as 500 to

* Bulletin No. 8, Univ. of Ill., 1906.

600 lbs/in.^{2}, will usually show a small set indicating imperfect elasticity. A second application of the load will, however, give a straighter

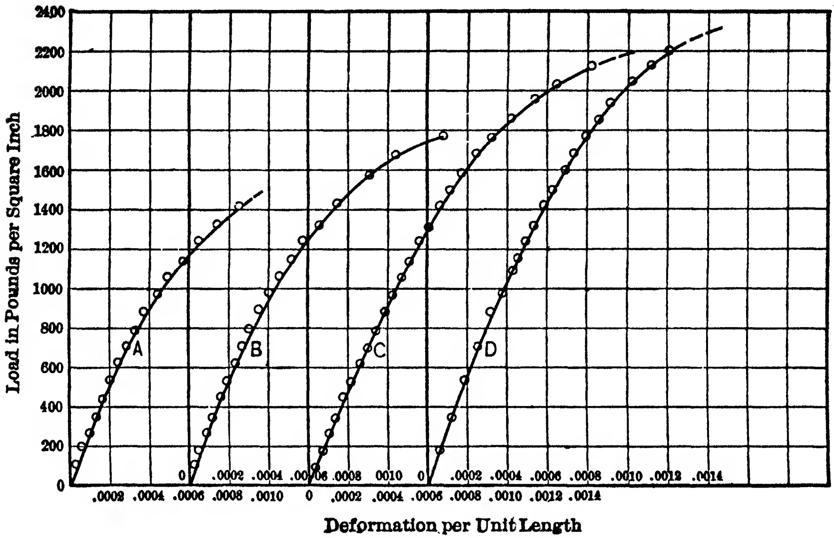


FIG. 3.—Stress-strain Diagrams in Compression.

line than the first and there will be much less permanent set following the release of load. After a few repetitions of load there will be no further set and the stress-strain line will become a straight line up to the load applied. There is a limit of stress, however, beyond which repeated applications of load will continue to add to the permanent deformation and the specimen will ultimately fail. This limit, called the endurance limit, is found to be 50% to 55% of the ultimate strength as determined by the usual test. In a sense this may be taken as the elastic limit of the material.

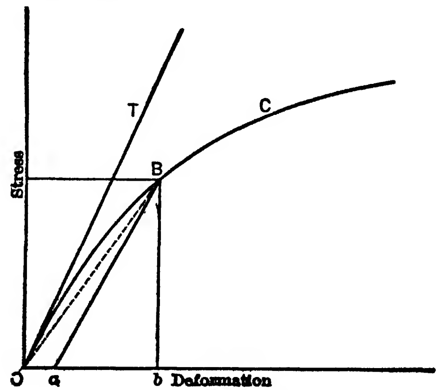


FIG. 4.

25. *Modulus of Elasticity in Compression.*—A typical stress-strain curve for concrete, under progressive loading, is shown in Fig. 4. For low stresses the elastic behavior is fairly well represented by the initial modulus, the slope of the tangent line $O\,T$. For larger stresses, such as represented by bB, a removal of load will result in some permanent set $O\,a$, and if the stress $b\,B$ is not too great, a repetition of the load will produce a new curve following approximately the straight line $a\,B$. The slope of the line $a\,B$ will then represent the elastic behavior of the stressed material for loads below B. But in reinforced-concrete work the principal use of the modulus is to determine the relative stresses in steel and concrete, and these are dependent upon the total deformations from the no-load conditions, or in this case upon the total deformations $O\,b$. Hence for this purpose the slope of the chord $O\,B$ is the ratio desired. This is commonly called the "secant modulus" at load B.

In the case of a beam the stresses in the concrete at any section will vary from zero at the neutral axis to the value bB, for example, at the extreme fibre. At intermediate points the stresses follow approximately the law of the curve $O\,B$. In this case a chord $O\,B$ does not exactly represent the facts, but the error is small, and it is the best line to use if the rectilinear variation of stress be assumed. If a curvilinear law is used, then the modulus should be the slope of the tangent at the origin, the "initial modulus." In neither case is it correct to use the slope of the line $a\,B$.

The value of the modulus for concrete varies greatly as determined by different experimenters and for different kinds of concrete. As a rule the denser and older the concrete the higher the modulus, but this relation is considerably affected by the character of the aggregate and other conditions, so that the relation to strength varies a great deal.

From a study of about 3500 tests of various concrete mixtures Stanton Walker derived the following expression for E:* For the initial modulus, $E = 33{,}000\,S^{5/8}$; and for the tangent modulus at 25% of the ultimate load, $E = 66{,}000\,S^{1/2}$, where S is the ultimate strength. The tangent modulus at 25% of the ultimate is approxi-

* Johnson's Materials of Construction, p. 478.

mately the same as the secant modulus for 50% of the ultimate. The values for various strengths, according to these formulas, would be:

Compressive Strength at 28 Days, lbs/in.^{2}	Initial Modulus, lbs/in.^{2}	Tangent Modulus at 25% of the Compressive Strength, lbs/in.^{2}
1500	3,200,000	2,560,000
2000	3,800,000	2,950,000
2500	4,400,000	3,300,000
3000	4,900,000	3,620,000
3500	5,400,000	3,900,000
4000	5,900,000	4,180,000

The Joint Committee of 1924 adopted values as follows:

Compressive Strength at 28 Days, lbs/in.^{2}	Modulus of Elasticity, lbs/in.^{2}
1500 to 2200	2,000,000
2200 to 2900	2,500,000
Over 2900	3,000,000

The specifications of the American Concrete Institute of 1928 use the value 1000 S, where S is the crushing strength at 28 days. This gives approximately the same values as those of the Joint Committee. A fixed ratio to the ultimate strength has some advantages in calculations, as will be seen later.

26. *Effect of Time on Deformation. Plastic Flow.*—It has been observed in various ways that the deformation of concrete continues to increase for a long time after the load is first applied, this effect being quite distinct from shrinkage due to decrease in moisture content.

Observations by R. E. and H. E. Davis,* on 4-in. by 14-in. cylinders of 1 : 5.05 concrete, water-cement ratio = 1.03, loaded at 28 days and stored in air at 70% humidity, gave the following deformations for the periods noted:

* Proceedings American Concrete Institute, Vol. 27, 1931, p. 837.

Load, lbs/in.^{2}	Instantaneous Deformation, Per Cent	Additional Deformations Due to Flow, Per Cent			
		100 Days	400 Days	800 Days	3½ Years
300	0.01	0.022	0.027	0.030	0.034
600	0.03	0.043	0.054	0.063	0.069
900	0.06	0.072	0.090	0.10	0.111

Plain concrete columns of 1 : 5 mix, water-cement ratio 0.86, when stored in air at 50% humidity gave the following deformations due to plastic flow when loaded at 800 lbs/in.^{2}: 100 days, 0.035%, 1½ years, 0.06%. The instantaneous deformation was about 0.02%, corresponding to E_c = 4,000,000 lbs/in.^{2} Other tests on plastic flow are mentioned in Chapter VII.

The foregoing values indicate the relative magnitude of deformations due to flow under air storage conditions. Under water storage the deformations due to flow are much smaller—about one-third that at 70% humidity. Removal of load after various intervals of time showed instantaneous recoveries somewhat less than the instantaneous deformations, and a further slow plastic recovery of about 0.01% in 2 months, mostly occurring in a few days. The effect of long-sustained load on the modulus of elasticity when later tested in the usual manner was to increase it somewhat and to make the stress-strain curve more nearly a straight line, or, in other words, to stiffen the concrete.

Plastic flow is of much significance in the design of structures; it is a property which results in a more favorable distribution of stress in some cases than the elastic theory calls for. It also greatly modifies the relation between the steel and concrete stresses, as it is equivalent to a large reduction in the value of E. Thus for a concrete with an initial modulus of 3,000,000 and a flow of 0.06% under a load of 600 lbs/in.^{2}, the total deformation will be 0.02 + 0.06 = 0.08%, and the new secant modulus or ratio of stress to total deformation will be 600/0.0008 = 750,000 lbs/in.^{2} In addition to this effect there is the effect of shrinkage for air-exposed concrete which still further modifies the relations. For temporary loads, such as the usual live load, the concrete retains its elasticity and high value of E.

The effect of plastic flow and shrinkage on stresses in beams and columns is discussed in later chapters.

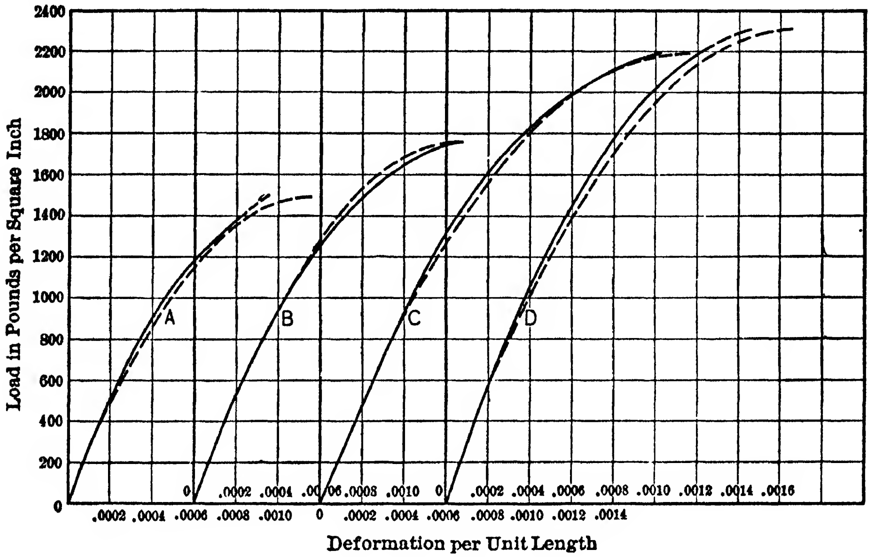


FIG. 5.

27. *Parabola Used for Stress-strain Curve.*—For theoretical calculations involving stresses to the ultimate, the linear law cannot well be used for the stress-strain relation. For this purpose, the ordinary second degree parabola is generally employed and is sufficiently accurate for the purpose. The axis of such parabola is taken as vertical with the vertex at the point of maximum stress. In Fig. 5 the actual curves of Fig. 3 are compared to parabolas (shown in broken lines).

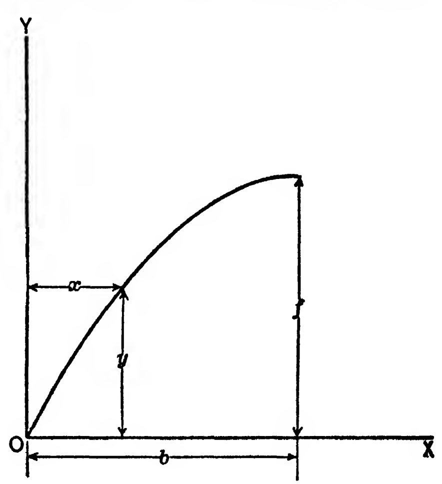


FIG. 6.

If f = stress at rupture and b = ultimate deformation, the equation of the parabola is (Fig. 6):

$$\frac{f-y}{f}=\frac{(b-x)^2}{b^2},$$

or

$$y=f-\left(\frac{b-x}{b}\right)^2 f. \quad \ldots \ldots \quad (1)$$

28. *Poisson's Ratio.*—When a material is subjected to compressive stress a certain amount of lateral expansion takes place. The ratio of such lateral expansion to the longitudinal compression is known as *Poisson's ratio.* In concrete this ratio varies generally from one-sixth to one-twelfth. Talbot found values from 0.1 to 0.16 for working loads for 1 : 2 : 4 concrete at 60 days.* Withey found the following values for 60-day concrete for loads equal to one-fourth the ultimate: for 1 : 3 : 6 mix, 0.08; for 1 : 2 : 4 mix, 0.11; for 1 : 1⅓ mortar, 0.16.†

29. **Coefficient of Expansion.**—Experiments by Professor W. D. Pence ‡ on 1 : 2 : 4 concrete gave an average value of the coefficient of expansion of 0.0000055 per degree Fahrenheit, there being little variation among the several tests. Tests made at Columbia University on 1 : 3 : 6 concrete gave values of about 0.0000065. Other experiments have shown about the same results. A value of 0.000006 may be assumed.

30. **Contraction and Expansion of Concrete Due to Variations in Moisture Content.**—Concrete cured under moist conditions will expand slightly. After hardening it will contract on drying and expand again on wetting. The amount of such change increases with the proportion of water and also with the proportion of cement paste in the mix. From tests of the Portland Cement Association, Lagaard § has prepared a general diagram representing the shrinkage of various mixtures on exposure to air of 50% humidity (a relatively dry air) for 6 months. According to this, a concrete of 1 : 2 : 4 proportions with 6¾ gal. of water per sack of cement will shrink about 0.07%; a 1 : 3 : 5 concrete with 8½ gal. per sack, about 0.06%. Davis found a shrinkage of the columns mentioned in Art. 26 of 0.033% in 100 days and 0.066% in

* Bull. No. 20 Univ. of Ill.

† Bull. No. 466 Univ. of Wis.

‡ Jour. Western Soc. Eng., 1901, p. 549.

§ Johnson's Materials of Construction, 1930, p. 48.

1½ years. The humidity was 50%. Under ordinary air conditions the contraction will be considerably less, but may easily reach 0.04% when the concrete is protected from the weather. This is equivalent to a change of temperature of about 60°.

31. **Weight of Concrete.**—The weight of concrete of the usual proportions will vary from 140 to 150 lbs/ft.^{3}, depending upon the proportions and the specific gravity of the aggregates. For design purposes an average value of 145 lbs/ft.^{3} may be taken. The addition of reinforcing steel in the usual proportions will add from 3 to 5 lbs., so that the weight of reinforced concrete may be taken at 150 lbs/ft.^{3}

32. **Fireproofing Effect of Concrete.**—Laboratory tests and observations of the effects of very hot fires in reinforced-concrete buildings show that concrete is a very good fireproofing material. Although it loses a large part of its strength at temperatures of 1000° F. or above, the heated material is an efficient non-conductor and the heat effect penetrates slowly. The principal question is the thickness of concrete necessary to protect the reinforcing steel from too high temperatures. This depends somewhat upon the character and importance of the member. Such members as main girders and columns, where a failure would involve a considerable portion of the building and where the steel is concentrated in a few rods, should be more thoroughly protected than floor slabs of small span, where a few local failures would be of no importance, and where additional covering would add largely to the expense. Results of fire tests and experience in conflagrations indicate that 2 to 2½ in. will offer practically complete protection, and that a minimum of ¾ in. for floor slabs will usually be sufficient. Large flat surfaces, such as floor slabs, are less exposed than the corners of projecting forms like beams and columns.

33. **Protection of the Steel from Corrosion.**—A continuous coating of Portland cement is an effective protection of steel against corrosion, and the necessary thickness of concrete for such protection is chiefly a matter of the integrity and density of the concrete. Many cases have been cited showing complete protection for more than 20 years when the steel was embedded an inch or more in the concrete. The Joint Committee specifications provide a covering of at least 2 in. where exposed to the weather and 3 in. for footings and for concrete exposed to sea water.

REINFORCING STEEL

34. **General Requirements.**—In general, reinforcing steel must be of such form and size as to be readily incorporated into the concrete

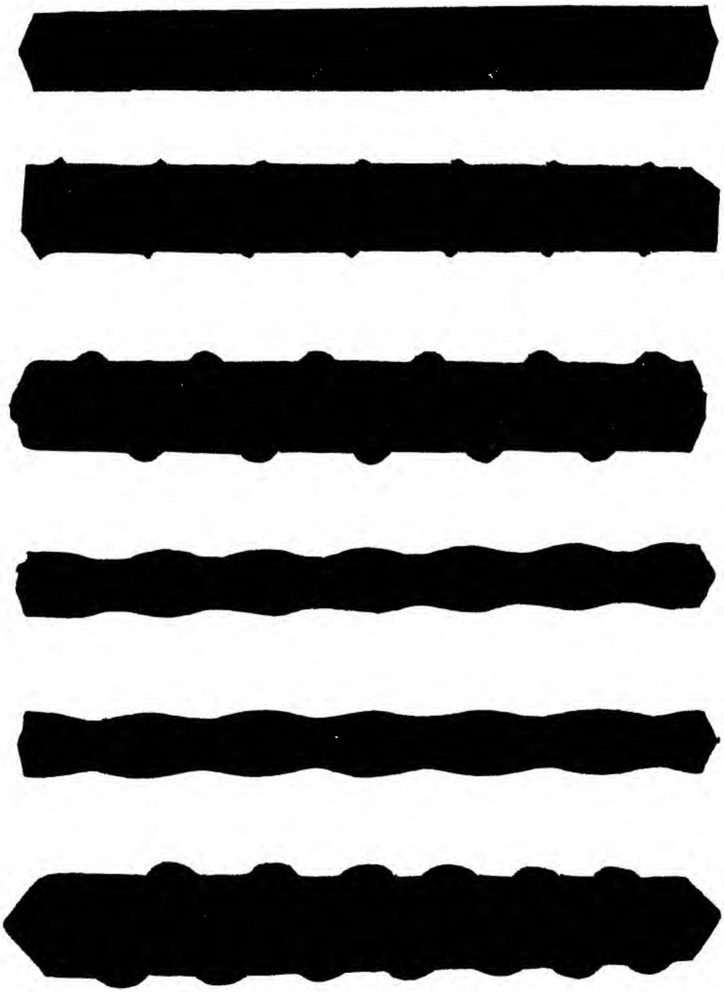


FIG. 7.—Types of Deformed Bars.

so as to make a monolithic structure. To provide the necessary bond strength and to distribute the steel where needed without concentrating the stresses on the concrete too greatly, requires the use of the steel in comparatively small sections. This requirement, as well as that of economy and convenience, leads to the use of the steel in the form of rods or bars. These will vary in size from about 1/4 to 3/8 in. for light floors up to 1 1/2 to 2 in. as maximum sizes for heavy beams or columns. Under certain conditions a riveted skeleton work is preferred for the steel reinforcement, but this is usually where for some reason it is desired to have the steelwork self-supporting or where it is to carry an unusually large proportion of the load.

35. **Forms of Bars.**—Plain round and square bars are largely used, the adhesion of the steel and concrete being depended upon to furnish the necessary bond strength. Plain flat bars are undesirable unless used in connection with riveted reinforcement, as their adhesion to the concrete is much less than that of round or square bars. Many special forms of "deformed" bars have been devised, the principal object of which is to furnish a bond with the concrete independent of adhesion —a mechanical bond, as it is usually called. Some of the most common types of such bars are illustrated in Fig. 7. Wire mesh of cold-drawn steel wire is a convenient form of reinforcement in certain cases.

36. **Quality of Steel.**—Steel bars used in reinforced work are not usually subjected to as severe treatment as those used in ordinary structural work. They must be capable of being bent to the desired form, but this is the only treatment to which the ordinary bars are subjected. In many concrete structures the impact effect is also likely to be less than in all-steel structures; consequently it is considered that a somewhat less ductile material may safely be used. These conditions lead to the use, by many engineers, of high-elastic-limit material; others prefer the use of standard structural steel. Bars made by rerolling rails are included in the high-elastic-limit class. The general requirements of the American Society for Testing Materials specifications for the various grades of steel are as follows:

Grade		Ultimate Strength, lbs/in.^{2}	Yield Point, lbs/in.^{2}
Billet Steel	Structural	55,000 to 70,000	33,000 min.
	Intermediate	70,000 to 90,000	40,000 min.
	Hard	80,000 min.	50,000 min.
Rerolled Rail Steel		80,000 min.	50,000 min.

The tendency of practice is towards the use of a single grade of billet steel—the intermediate—as being satisfactory and promoting economy of stocks and uniformity of practice.

37. **Modulus of Elasticity.**—The modulus of elasticity of all grades of steel is very nearly the same and will be taken at 30,000,000 lbs/in.^{2}

38. **Coefficient of Expansion.**—The coefficient of expansion of steel may be taken at 0.0000065 per 1° F.

PROPERTIES OF CONCRETE AND STEEL IN COMBINATION

39. **Ratio of Moduli of Elasticity,** $E_s/E_c = n$.—So long as the adhesion between steel and concrete is unimpaired the distortion of the two materials will be equal. Their stresses will then be proportional to their moduli of elasticity for the load in question, or as the ratio of $E_s : E_c$. Taking E_s at 30,000,000 and E_c at from 2,000,000 to 4,000,000, the ratio will vary from 15 to 7½. In practice, various values of this ratio are used, depending upon the kind of concrete and the judgment of the designer.

The Joint Committee specifies the following values:

Compressive Strength, lbs/in.^{2}	Value of n
1500 to 2200	15
2200 to 2900	12
2900 and over	10

40. **Tensile Strength and Elongation of Concrete when Reinforced.**—The tensile strength of concrete is ordinarily from 200 to 300 lbs/in.^{2}, and assuming a relatively low value of E_c, the total elongation at rupture would be 0.0001 to 0.00015 part. The stress in the steel corresponding to this elongation is 3000 to 4500 lbs/in.^{2} From these relations it is evident that, where ordinary working stresses are used in the steel, the concrete will crack and be of no assistance in carrying stress.

Some early experiments by Considère on reinforced-concrete beams seemed to indicate that concrete on the tension side would elongate before cracking much more than in an ordinary tension test, and that it could, therefore, be counted upon to carry a certain amount of tension under working conditions. Later experiments failed to confirm these results but showed the formation of cracks at elongations of about 0.0001 to 0.0002, corresponding to a steel stress of 3000 to 6000 lbs/in.[2] However, on account of the presence of the steel, the cracks open up very slowly so that they are at first most difficult to detect.

In some cases, where the stresses in the steel are necessarily very low, it may be proper to consider the tensile resistance of the concrete. This limit may be placed at about 2000 lbs/in.[2], corresponding to an elongation of 0.00006 part and a stress of 150 to 175 lbs/in.[2] in the concrete.

41. Contraction and Expansion of Reinforced Concrete.—The behavior of reinforced concrete as regards contraction and expansion, and the stresses resulting therefrom, may be considered under two conditions: (1) when the structure or part under consideration is not restrained by surrounding structures, and is, therefore, free to contract and expand as a whole; (2) when the part under consideration is restrained so that contraction and expansion are prevented.

1. When the structure is not restrained, then the only stresses will be those resulting from a difference in deformation of the concrete and steel. Temperature changes affect the steel and concrete nearly alike (Arts. 29 and 38), so that the two materials will be but slightly stressed by reason of temperature changes.

The effect of shrinkage in hardening or drying out is more serious. Concrete which is unrestrained either by steel reinforcement or by exterior attachment will shrink or swell proportionally and no stresses will thereby be developed. If restrained by reinforcing material only, a shrinkage will develop tensile stresses in the concrete and compressive stresses in the steel.

If it be assumed that concrete when reinforced tends to shrink the same amount as plain concrete, and that such shrinkage is prevented only so far as the stresses developed in the steel react upon the concrete and cause an opposite movement, then it will be found, using the ordinary values of the modulus of elasticity, that the stresses developed in both the concrete and the steel will be large. These stresses may be estimated as follows:

Let m = shrinkage coefficient of the concrete;
f_c = unit stress in concrete (tensile);
f_s = unit stress in steel (compressive);
p = steel ratio;
$n = E_s/E_c$.

Then the net contraction per unit length as measured by the concrete will be $m - f_c/E_c$, and as measured by the steel will be f_s/E_s. These values are equal. Also, for equilibrium, $f_c = pf_s$. From these equations we get

$$f_c = m E_c \frac{n p}{1 + n p} \quad \quad (1)$$

and

$$f_s = \frac{f_c}{p}. \quad \quad (2)$$

If, for example, $m = 0.0003$, $E_c = 2{,}000{,}000$, $n = 15$, $p = 1\%$, then $f_c = 80$ lbs/in.^{2} tension and $f_s = 8000$ lbs/in.^{2} compression. If $p = 2\%$, $f_c = 140$ and $f_s = 7000$ lbs/in.^{2}

2. When the structure is restrained by outside forces so that it is not free to contract or expand, as in the case of a long wall, then the resulting stresses are likely to be high, and no amount of reinforcement can entirely prevent contraction cracks. The reinforcement can, however, force such cracks to take place, as they do in a beam, at such frequent intervals that the requisite deformation is provided without any one crack becoming large. With a coefficient of expansion of 0.000006, a temperature change of 50° will cause a change of length (if free) of 0.0003 part. The effect of shrinkage may be even greater. A deformation of 0.0003 part corresponds to a stress of about 600 lbs/in.^{2} in the concrete, a stress much beyond the ultimate tensile strength. Hence temperature changes and shrinkage are quite certain to cause cracks; but if the concrete is well reinforced such cracks may be kept very small. For example, a deformation of 0.0003 part on the tension side of a beam corresponds to a stress of about 9000 lbs/in.^{2} in the steel, which is much below the usual working stress and which would not cause cracks easily detected. The prevention of large cracks by means of reinforcement is, then, a matter of using sufficient steel to

force the concrete to crack at small intervals. No crack will open up far until the steel is stressed beyond its elastic limit, hence we may say that the amount of steel should be at least enough to force the concrete to crack at a second point before the steel reaches its elastic limit; that is to say, the elastic limit strength of the steel should be greater than the tensile strength of the concrete. A still larger amount of steel will serve to keep the cracks smaller.

The size and distribution of the cracks will also depend upon the bond strength furnished by the rods. If we assume the cracks to develop successively, the distance between cracks must be sufficient to develop a bond strength equal to the tensile strength of the concrete. Hence, in general, the size and spacing of the cracks will vary inversely with the bond strength of the reinforcing steel per unit of concrete section.

For reinforcement against shrinkage and temperature stresses, a high-elastic-limit steel is desirable, and in order to distribute the deformation as much as possible a mechanical bond is advantageous. The amount of steel necessary for such reinforcement depends upon the thickness and exposure of the structure. For thin walls and exposed locations 0.4% to 0.5% is required, and under very favorable conditions as little as 0.1% has been found to be sufficient. The reinforcement for this purpose should be placed close to the exposed faces of the concrete. In floor slabs longitudinal bars of ¼ in. to ½ in. diameter, spaced about 2 ft. apart, are customary.

CHAPTER III

THEORY OF THE FLEXURE OF BEAMS

42. Kinds of Members.—Structural members are, for convenience, usually divided into *tension members*, *compression members*, and *beams*, according as the forces to be resisted produce in the members simple tension, simple compression, or simple bending. Bending moment is often accompanied by tension or compression, producing what are called *combined stresses* of *bending and tension*, or *bending and compression*. Since reinforced concrete is not used for plain tension members the analysis will be confined to the beam, both under plain bending and under combined stresses, and to the compression member or column. The "flat slab," supported in various ways, will be considered in a separate chapter as a special case of beam. In reinforced-concrete construction the beam is the most important element and is used under a great variety of conditions.

43. Relation of Stress Intensities in Concrete and Steel.—In the following discussion it will be assumed that the concrete and steel adhere perfectly and therefore deform equally. Nearly all reinforced-concrete construction is dependent upon this equal action of the two materials, although simple adhesion is not always entirely depended upon. Several types of deformed bars are used so as to give the steel a grip independent of the adhesion, and in other cases bars are bent or anchored at the ends, but in all cases it is assumed that the materials adhere perfectly and therefore deform equally. Many tests show that under proper design this is for all practical purposes true.

Since the modulus of elasticity of a material is the ratio of stress to deformation, it follows that for *equal* deformations the stresses in different materials will be as their moduli of elasticity. If

f_s = unit stress in steel,
f_c = unit stress in surrounding concrete,
E_s = modulus of elasticity of steel, and
E_c = modulus of elasticity of concrete,

we have the fixed relation

$$f_s/f_c = E_s/E_c.$$

44. **Distribution of Stress in a Homogeneous Beam.**—To assist in forming correct notions of the action of steel reinforcement in a concrete beam, it will be desirable to consider, at the outset, the nature of

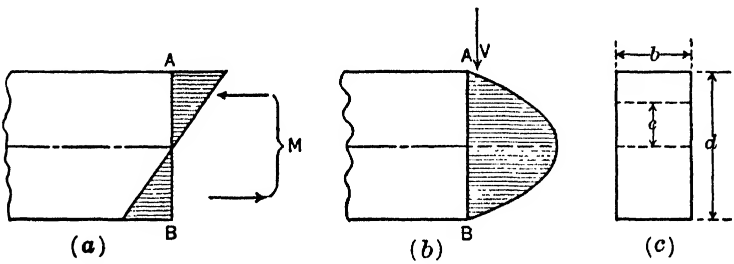


FIG. 1.

the stresses due to bending moment in a plain concrete beam or a beam of any homogeneous material.

Considering a vertical section at any point of the beam, Fig. 1, there will exist in general certain tensile and compressive stresses acting normally to the section and certain vertical or shearing-stresses acting tangentially thereto. In accordance with the common theory of flexure, the normal stress on a vertical section varies in intensity as the distance from the neutral axis, and therefore the variation is represented by the ordinates to a straight line as in Fig. 1 (*a*). The intensity of this stress (fibre stress) at any point is given by the well-known formula

$$f = \frac{Mc}{I}, \quad \ldots \ldots \ldots \quad (1)$$

in which M = bending moment, c = distance of fibre from neutral axis, and I = moment of inertia of the section with respect to the neutral axis.

The shearing-stress intensity is a maximum at the neutral axis and is zero at the outer fibres. At any given point in the section it is given by the equation

$$v = VS/Ib, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

in which V = total vertical shear at the section through the point under consideration, b = breadth of the section at the given point, and S = statical moment of that part of the section above (outside of) the point with respect to the neutral axis. For a rectangular beam the intensity of shear varies as the ordinates to a parabola, as shown in Fig. 1 (*b*), the maximum value being 3/2 times the average, or equal to $\frac{3}{2} \cdot \frac{V}{bd}$. The intensity of the horizontal shearing-stress at any point is equal to that of the vertical shearing-stress.

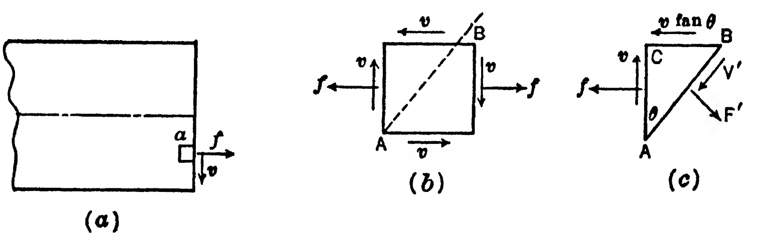


FIG. 2.

A determination of the normal stresses (often called the "direct" or "bending" stresses) and the shearing-stresses above described gives sufficient information for the design of ordinary beams of homogeneous material. For purposes of reinforced-concrete design, however, it is desirable to make a more detailed analysis of the stresses acting at any point in a beam.

Consider an element *a* at any point of a beam, Fig. 2 (*a*), of unit dimensions, and with horizontal and vertical faces. Fig. 2 (*b*) is an enlarged view of this element. The direct and shearing unit stresses may be represented by f and v respectively; their magnitudes will be determined by formulas (1) and (2). Let us examine the stress on an inclined section AB. Fig. 2 (*c*) shows the portion on the left of the section. The stresses on the face AB may be resolved into a tension F' and a shear V'. The shear on CB will be $v \tan \theta$. Taking components normal to $A B$, we find $F' = (f + v \tan \theta) \cos \theta + v \sin \theta$ and $V' = v \cos \theta - (f + v \tan \theta) \sin \theta$.

The stresses per unit of area will be

$$f' = F' \cos \theta = f \cos^2 \theta + 2 v \sin \theta \cos \theta;$$

$$v' = V' \cos \theta = v (\cos^2 \theta - \sin^2 \theta) - f \sin \theta \cos \theta.$$

For a maximum value of f' we find by differentiation,

$$\tan 2 \theta = \frac{2 v}{f}, \qquad (3)$$

and for maximum v'

$$\tan 2 \theta = \frac{f}{2 v}. \qquad (4)$$

Substituting these values of θ in the expressions for f' and v', we find the maximum values to be respectively

$$f'_{\max} = \tfrac{1}{2} f + \sqrt{\tfrac{1}{4} f^2 + v^2}, \qquad (5)$$

$$v'_{\max} = \sqrt{\tfrac{1}{4} f^2 + v^2}. \qquad (6)$$

These equations show that the maximum tensile stress in a beam depends upon both f and v at the point in question.

At all points in a beam where the shear is zero, the direction of the maximum tension is horizontal, as at points of maximum bending moment and along the outer fibres of the beam ($\tan 2 \theta = 0$). Wherever the horizontal fibre stress f is zero (at the neutral surface and at all sections of zero bending moment), the direction of the maximum tension is inclined 45° to the horizontal ($\tan 2 \theta = \infty$), and its intensity is equal to the unit shearing stress at the same place. Above the neutral axis the inclination of the maximum tension is in general greater than 45°, becoming 90° at the upper or compressive fibre. Fig. 3 illustrates the variation in normal stress, shearing-stress, and maximum tensile stress throughout the entire depth of a rectangular beam. The outer normal or fibre stress is

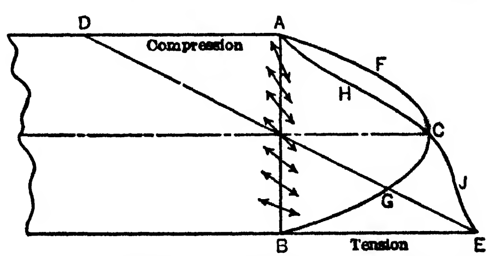


FIG. 3.—Variation in Maximum Tensile Stress.

assumed at 200 lbs/in.^{2}, and the shearing-stress at the neutral axis at 150 lbs/in.^{2} The variation in the fibre stress is shown by the straight line $D E$, and that in the shearing-stress by the parabolic curve $A C B$. By means of eq. (5) the maximum tensile stresses have been computed; these are represented by the line $A H C J E$.

The direction of the maximum tension is shown by the arrows along the line $A B$. The direction of the maximum compression is at right angles to that of maximum tension.

Fig. 4 illustrates the general direction of the maximum tensile stresses in a rectangular beam. The exact direction at any point depends upon the relation between shear and bending moment. Lines

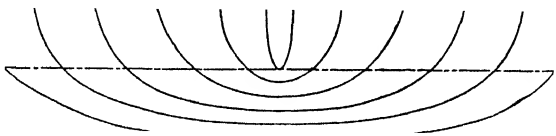


FIG. 4.—Lines of Maximum Tension.

of maximum compression run at right angles to the lines shown and lines of maximum shear at angles of 45° therewith.

45. Purpose and Arrangement of Steel Reinforcement.—The purpose of steel reinforcement is to carry the principal tensile stresses, the concrete being depended upon for the compressive and shearing stresses, its resistance to such stresses being large. If no steel were present the concrete would tend to rupture on lines perpendicular to the direction of maximum tension, as shown in Fig. 4, and hence we may conclude that the ideal tension reinforcement would require the steel to be distributed in the beam along the lines of maximum tension. At the centre of the beam, or section of maximum moment, this direction is horizontal for the entire depth of the beam, and horizontal rods placed near the lower edge of the beam constitute proper and sufficient reinforcement. As we approach the ends of the beam, where the shear is large, the intensity of the inclined tensile stresses becomes of importance, and in many cases these stresses require special attention. Horizontal rods at the bottom are still necessary, but do not entirely reinforce the concrete against tension, so that special consideration must be given to reinforcement in the body of the beam. The arrangement of this reinforcement demands careful consideration.

For purposes of discussion, the subject of beams will first be treated with reference only to the horizontal reinforcement. The inclined tensile stresses will be considered separately under Shear, Chap. IV.

46. The Common Theory of Flexure and Its Limitations for Materials Like Concrete.—The common theory of flexure is based on two main assumptions, namely: (1) a plane cross-section of an unloaded beam will still be plane after bending (Navier's hypothesis); (2) the material of the beam obeys Hooke's law, which is, briefly stated, "stress is proportional to strain." From the first assumption it follows that—*The unit deformations of the fibres at any section of a beam are proportional to their distances from the neutral surface.* In the case of simple bending (all forces at right angles to the beam) the neutral axis lies at the centre of gravity of the section; in the case of bending combined with direct tension or compression, the neutral axis may lie in the section or be merely an imaginary line without the section. From the second assumption it follows that—*The unit stresses in the fibres of any section of a beam also are proportional to the distance of the fibres from the neutral surface.* This may be called the linear law of the distribution of stress.

While these two assumptions are commonly made in the analysis of a beam, they are not in all cases strictly correct, and in some cases Hooke's law cannot be used without large error.

The assumption of plane sections is slightly in error wherever shearing-stresses exist, as these tend to distort a plane section into one slightly S-shaped in form. The resulting error is, however, small and of no practical consequence, especially in concrete, where the effect is much less than in steel.

The assumption that fibre stresses are proportional to deformations (Hooke's law) is practically correct for wrought iron and steel within the elastic limits, and hence the common theory of flexure gives correct results for these materials for working conditions; but for stresses beyond the elastic limit it does not apply. Other materials like timber, stone, cast iron, and concrete do not obey Hooke's law so closely as steel and wrought iron, but for working stresses the variation is not great, and the common theory may be used with closely approximate results. If a more exact analysis is desired for such materials the linear law must be discarded and the actual stress-strain diagrams in compression and tension must be used up to the limit of the actual stresses involved, or some form of mathematical curve approximating the true diagram.

47. Assumptions of Stress Variations Used in Practice.—For use in designing under specified working loads it is the universal practice to use formulas based on the linear law of stress variation, and to neglect any tensile stress in the concrete which may exist below the neutral axis. The ultimate extensibility of concrete is not more than about 0.02%, which corresponds to a steel stress of 6000 lbs/in.^{2} Hence under ordinary working stresses the concrete will be cracked well up towards the neutral axis. Such cracking may also be increased from the effect of shrinkage. Furthermore, whatever tensile stress may exist has little effect upon the resisting moment of the beam. It is therefore entirely neglected and the tension assumed to be taken wholly by the steel. In a study of deflections it is necessary to take into account such tensile stresses, as they are of considerable influence in portions of the beam where the bending moment is small.

To calculate ultimate loads and factors of safety with reference to concrete stresses, the linear law gives results too greatly in error, and for such purposes it is customary to assume a parabolic variation of concrete stress, as explained in Art. 63.

48. Notation.—Fuller explanations of some of these symbols are given in subsequent articles where the formulas are derived; see also Fig. 5.

f_s = unit fibre stress in steel;
f_c = unit fibre stress in concrete at its compressive face;
e_s = unit elongation of the steel due to f_s;
e_c = unit shortening of the concrete due to f_c;
E_s = modulus of elasticity of the steel;
E_c = modulus of the concrete in compression;
n = ratio E_s/E_c;
T = total tension in steel at a section of the beam;
C = total compression in concrete at a section of the beam;
M_s = resisting moment as determined by steel;
M_c = resisting moment as determined by concrete;
M = bending moment or resisting moment in general;
b = breadth of a rectangular beam;

d = distance from the compressive face to the plane of the steel;
k = ratio of the depth of the neutral axis of a section below the top to d;
j = ratio of the arm of the resisting couple to d;
A = area of cross-section of steel;
p = steel ratio, $A/b\,d$.

FLEXURE FORMULAS FOR WORKING LOADS

49. **General Relations.**—Fig. 5 represents a section AB of a beam subjected to bending moment. The neutral axis is at N. The portion

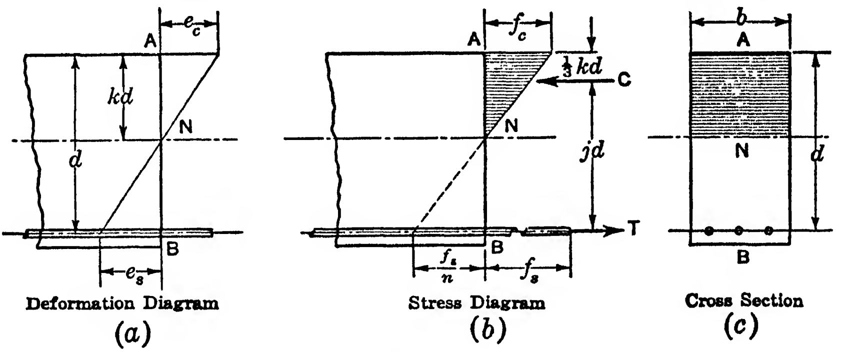


FIG. 5.

of the beam above N is in compression whose maximum value is f_c per unit area, whose average value is $f_c/2$ and whose total value is $\frac{1}{2}f_c\,b\,k\,d = C$. The unit stress in the steel is f_s, and the total steel stress is $f_s A = T$. For the case of simple bending (all loads and reactions transverse to the beam axis), $T = C$, and the resisting moment of the beam is the moment of this couple, $= T\,j\,d = C\,j\,d$. The value of j evidently depends upon the position of the centroid of the compressive stresses which in turn depends upon the position of the neutral axis. It will be found, further, that the position of the neutral axis depends upon the proportion of steel used and the relative moduli of elasticity of steel and concrete. This position having been found, and from this the value of j, the resisting moment is readily calculated for any given values of stress in concrete or steel, or the stress for any given bending moment.

50. Derivation of Formulas.—*Neutral Axis and Arm of Resisting Couple.*—It follows from the assumption of plane sections that the unit deformations of the fibres vary as their distances from the neutral axis; hence, as shown in Fig. 5 (*a*)

$$\frac{e_s}{e_c} = \frac{d - k\,d}{k\,d}.$$

Also, by definition,

$$\frac{e_s}{e_c} = \frac{f_s/E_s}{f_c/E_c} = \frac{f_s}{n\,f_c},$$

whence we have the relation of stresses and distances, as shown in Fig. 5 (*b*),

$$\frac{f_s}{n\,f_c} = \frac{d - k\,d}{k\,d} = \frac{1 - k}{k}. \quad \text{(a)}$$

From $T = C$, we have

$$f_s\,A = \tfrac{1}{2} f_c\,b\,k\,d \quad \text{(b)}$$

or

$$\frac{f_s}{f_c} = \frac{\frac{1}{2}\,b\,k\,d}{A} = \frac{\frac{1}{2}\,b\,k\,d}{p\,b\,d} = \frac{k}{2\,p}. \quad \text{(c)}$$

Eliminating f_s/f_c between (*a*) and (*c*) gives $\dfrac{(1 - k)\,n}{k} = \dfrac{k}{2\,p}$; this, solved for k, gives

$$k = \sqrt{2\,p\,n + (p\,n)^2} - p\,n. \quad \text{(1)}$$

Note that the ratio f_s/f_c, as shown by eqs. (*c*) and (1) is dependent only upon p and n. That is, for a given quality of concrete and amount of steel the ratio of the stresses, f_s/f_c, is fixed; and conversely, with definite working stresses and value of n, the steel ratio p is fixed for all sizes of beams.

Solving (*a*) for k, we have the useful relation

$$k = \frac{1}{1 + f_s/n\,f_c}. \quad \text{(1a)}$$

This is an important relation between the position of the neutral axis and the stresses in steel and concrete. Inasmuch as the value of E_c increases with the strength of the concrete, and likewise the value of the working stress f_c, the quantity $n\,f_c$ is nearly constant, and as the working stress f_s is also nearly constant, it follows that the value of k varies but little.

The arm of the resisting couple is

$$j\,d = d - \tfrac{1}{3}\,k\,d, \quad \text{or} \quad j = 1 - \tfrac{1}{3}\,k. \qquad (2)$$

In Fig. 6 (also Diagram 1 *) are given curves for values of k and j for various values of p and n; also on the left the corresponding values of $f_s/n\,f_c$. Taking $n\,f_c = 12{,}000$, a common value, the values of j are: for $f_s = 18{,}000$, $j = 0.867$; and for $f_s = 20{,}000$, $j = 0.875$. The small range of values of j indicates that a fixed assumed value of 0.86 or 0.87 for all cases could well be adopted. A value of $\tfrac{7}{8}$ is sometimes used.

51. *Resisting Moment for Given Working Stresses f_s and f_c.*—The safe resisting moment of a given beam may depend upon either the strength of the concrete or the strength of the steel, according to the amount of steel actually used. In investigating a given beam we may, therefore, proceed by calculating the safe resisting moment as dependent upon the allowable stress in the concrete and then the resisting moment as dependent upon the allowable stress in the steel and compare results. The smaller of the two values will evidently be the value desired.

The resisting moment in terms of steel stress is

$$M_s = T \cdot j\,d = f_s\,A\,j\,d = f_s\,p\,j\,b\,d^2. \qquad (3)$$

In terms of concrete stress the resisting moment is

$$M_c = C \cdot j\,d = \tfrac{1}{2} f_c\,b\,k\,d\,j\,d = \tfrac{1}{2} f_c\,k\,j\,b\,d^2. \qquad (4)$$

In examining a given design it will generally happen that the values of M_s and M_c will differ, for the reason that the beam was not designed for the precise values of f_s and f_c assumed.

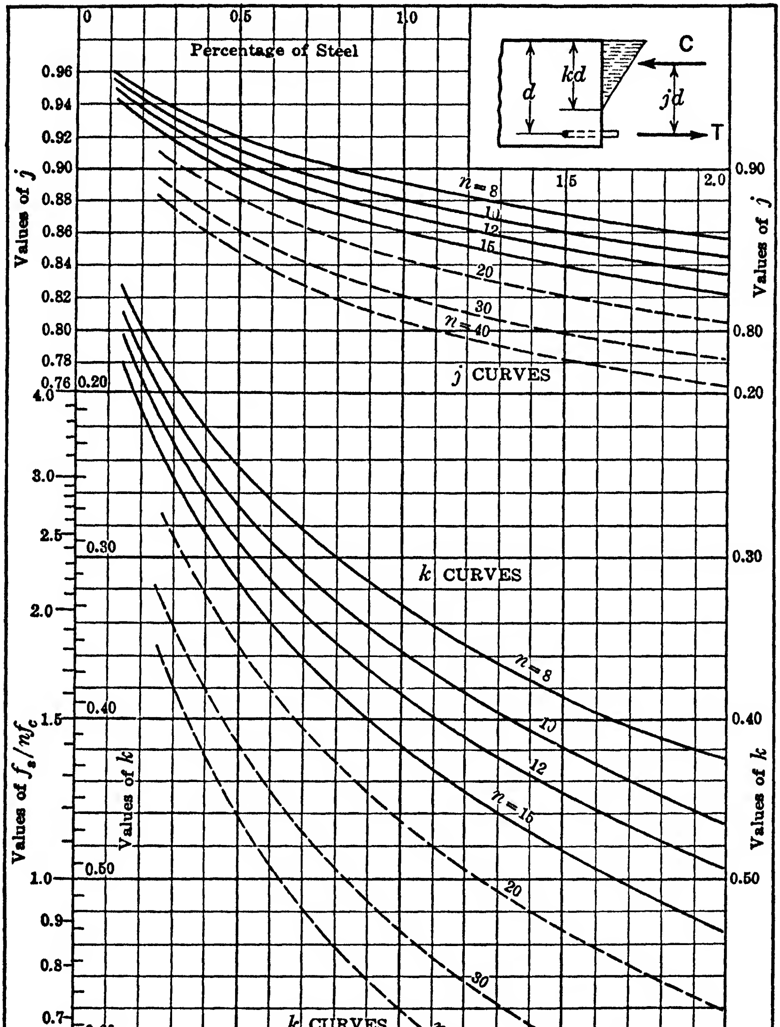
52. *Unit Fibre Stresses for a Given Bending Moment.*—From eqs. (3) and (4) we derive

$$f_s = \frac{M}{A\,j\,d} = \frac{M}{p\,j\,b\,d^2} \qquad (5)$$

$$f_c = \frac{M}{\tfrac{1}{2} j\,k\,b\,d^2} \qquad (6)$$

Dividing (5) by (6) gives $f_s/f_c = k/2\,p$, as in eq. (c).

* The Diagrams are given in Chapter XII.



Percentage of Steel
Values of j
Values of f_s/nf_c
Values of k
j CURVES
k CURVES
n = 8
10
12
15
20
30
n = 40
kd
d
jd
C
T

Fig. 6.

53. *Amount of Steel and Cross-section of Beam for a Given Bending Moment.*—In designing a beam it is desirable to use the correct amount of steel so that the allowable stresses in both steel and concrete will be reached under the given load; that is, such an amount as will give a *balanced* design. The correct steel ratio for any given working stresses is found by substituting the value of k from (a) in (c) and solving for p. Or we may equate eqs. (3) and (4). The result is

$$p = \frac{\frac{1}{2}}{\frac{f_s}{f_c}\left(\frac{f_s}{n\,f_c} + 1\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

This shows that, for a given concrete and ratio of working stresses, p has the same value for all sizes of beams. The value of p for various values of f_s, f_c, and n can be found from Fig. 6, or Diagrams 1, 3, and 4.

If the correct proportion of steel, as given by (7), is used, then the size of the beam, or the value of $b\,d^2$, can be determined by either of eqs. (3) or (4). These give

$$\left.\begin{aligned} b\,d^2 &= \frac{M}{f_s\,p\,j} \\ b\,d^2 &= \frac{M}{\frac{1}{2}f_c\,k\,j} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Equations (8) are convenient to use only when diagrams or tables are available for obtaining the values of p, k, and j. For direct calculation the value of k and j obtained from eqs. (a) and $(1\,a)$ may be substituted in (6), giving

$$b\,d^2 = \frac{6\,M\left(\frac{f_s}{f_c} + n\right)^2}{n\,f_c\left(\frac{3\,f_s}{f_c} + 2\,n\right)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

This gives the required dimensions of the beam in terms of the allowable stresses and the value of n, and assumes an amount of steel used as given in eq. (7).

It is to be noted that the application of eqs. (8) or (9) gives the value of $b\,d^2$ only, as is the case in the design of any rectangular beam. Having determined $b\,d^2$, the values of b and d are to be selected so as to give convenient and economical proportions. The larger the value of d the smaller is the cross-section required and the less the amount of concrete and steel, but the ratio of depth to breadth is limited by practical considerations, such as head room, convenience in placing the steel, and, as shown later, by the shearing-stresses involved. Ordinarily the ratio of $d : b$ will range from about 1½ to 4, the latter for very large beams only. In the case of continuous slabs or floors the width b is, of course, fixed, and the depth only is to be determined. A unit width of 1 ft. is usually assumed in such calculations.

54. Cofficients of Resistance.—From eqs. (3) and (4), Art. 51, it is to be seen that

$$\begin{cases} M_s = f_s\,p\,j \times b\,d^2 \\ M_c = \tfrac{1}{2} f_c\,k\,j \times b\,d^2, \end{cases}$$

in which the quantities $f_s\,p\,j$ and $\tfrac{1}{2} f_c\,k\,j$ are variables dependent upon the unit stress, percentage of steel, and value of n; they are independent of the size of beam. For convenience these products will be called *Coefficients of Resistance* with respect to the steel and the concrete, respectively, and will be denoted by R_s and R_c, that is

$$R_s = f_s\,p\,j; \quad R_c = \tfrac{1}{2} f_c\,k\,j.$$

Then, for given working stresses,

$$M_s = R_s \cdot b\,d^2 \quad \text{and} \quad M_c = R_c \cdot b\,d^2. \quad . \quad . \quad (10)$$

Similarly, for any given beam subjected to a bending moment M,

$$R_s = R_c = M/b\,d^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

For a balanced design R_s must be equal to R_c and, as in eq. (8),

$$b\,d^2 = M/R \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where $R = R_s = R_c$.

Diagrams 3 and 4 give values of R for various values of p and for n = 8, 10, 12, and 15. Values of R_s are read from the f_s curves, and R_c from the f_c curves. For a balanced design the correct value of p is found at the intersection of the given values of f_s and f_c.

55. The Transformed Section.—Instead of considering the concrete and steel as two different materials in the foregoing analysis, we may "transform" the steel into an equivalent amount of concrete and apply the general methods for homogeneous beams. This may be done by replacing the steel by concrete in the same plane and having n times the area of the steel. The resulting imaginary section is called the "transformed section," and its use is quite advantageous in certain cases, particularly in a study of combined bending and compression.

For the case of simple bending, of Art. 50, Fig. 7 shows the transformed section, the steel area A being replaced by the concrete area

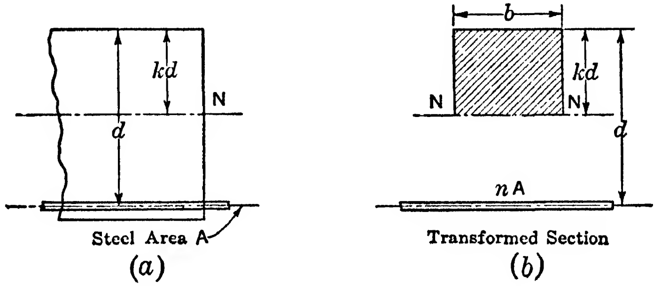


FIG. 7.

$n A$ at the same plane as the steel. Then taking moments about the top edge of the section we have at once

$$k d = \frac{b k d \times \frac{1}{2} k d + n A d}{b k d + n A}.$$

Substituting $p b d$ for A and solving for k gives

$$k = \sqrt{2 p n + (p n)^2} - p n,$$

as in eq. (1).

The moment of inertia of the transformed section may also be found and the fibre stresses determined by the usual formula for beams, $f = M_c/I$, but in this case the methods given in the preceding articles are simpler.

56. Examples.—Problems of practical analysis and design are of three general kinds, namely: (1) Given a beam and the safe working stresses, to find the resisting moment of the beam. (2) Given a beam subjected to a given bending moment, to find the unit stresses in concrete and steel. (3) Given the safe working stresses and the bending moment, to find the dimensions of the beam.

In all the following problems the value of n is taken at 15 and the "depth" is understood to be the depth d from the compression face to the center of gravity of the steel area.

1. Calculate the safe resisting moment of a beam where $b = 10$ in., $d = 14$ in., reinforcement consists of four ¾-in. round bars, allowable stresses are $f_s = 16{,}000$ and $f_c = 800$ lbs./in.^{2}

Solution. The steel area $A = 4 \times 0.442 = 1.77$ sq. in., $p = 1.77/140 = 0.0126$. Then from eq. (1) $k = 0.45$; from (2), $j = 1 - 0.45/3 = 0.85$. (See also Fig. 6 for k and j.) Then, as determined by the steel, the safe moment is (eq. (3))

$$M_s = 16{,}000 \times 1.77 \times 0.85 \times 14 = 337{,}000 \text{ in-lbs.}$$

As determined by the concrete it is (eq. (4))

$$M_c = 400 \times 0.85 \times 0.45 \times 10 \times 14^2 = 300{,}000 \text{ in-lbs.}$$

The safe moment is the latter value.

The stress f_s for $M = 300{,}000$ will be less than the allowable value of 16,000, being equal to $16{,}000 \times 300/337 = 14{,}300$ lbs/in.^{2} Note that the ratio of actual stresses is, from eq. (c), at all times equal to $k/2p = 0.45/0.0252 = 17.9$; and since the ratio of allowable values is $16{,}000/800 = 20$, it is evident that the concrete stress of 800 will control and that the steel stress will be equal to $17.9 \times 800 = 14{,}300$ lbs/in.^{2}

2. Suppose that the beam of the preceding example is 17 in. deep and is subjected to a bending moment of 350,000 in-lbs. Compute the greatest unit stresses in the steel and concrete.

Solution. The steel ratio is $1.768/170 = 0.0104$. Then $k = 0.42$, $j = 0.86$, $jd = 0.86 \times 17 = 14.6$ in., $T = 350{,}000/14.6 = 24{,}000$ lbs., $f_s = T/A = 24{,}000/1.77 = 13{,}600$ lbs/in.^{2} Also $C = T = 24{,}000$ lbs., $f_c = 2 \times C/bkd = 2 \times 24{,}000/10 \times 0.42 \times 17 = 670$ lbs/in.^{2} We may also get from the general relation of eq. (c), $f_c = f_s \times 2p/k = 13{,}600 \times 0.0208/0.42 = 670$ lbs/in.^{2}

(3) Design a beam to withstand a bending moment of 400,000 in-lbs., the working strength of concrete and steel being taken at 700 and 16,000 lbs/in.^{2}, respectively.

Solution. For $n = 15$ and $f_s/f_c = 16{,}000/700 = 22.9$, eq. (7) or Fig. 6 gives $p = 0.0087$. Then from Fig. 6, $k = 0.40$ and $j = 0.87$. Eq. (8) now gives

$$b\,d^2 = \frac{400{,}000}{16{,}000 \times 0.0087 \times 0.87} = 3300,$$

or

$$b\,d^2 = \frac{400{,}000}{350 \times 0.40 \times 0.87} = 3280.$$

Or we may use eq. (9) without getting k and j,

$$b\,d^2 = \frac{6 \times 400{,}000\,(22.9 + 15)^2}{15 \times 700\,(3 \times 22.9 + 30)} = 3340.$$

Many different values of b and d will give the desired value of $b\,d^2$. If b is taken at 10 in., then $d^2 = 3300/10 = 330$, and $d = 18.2$ in. Ordinarily as a matter of economy and practical proportions b ranges from $\frac{1}{4}d$ to $\frac{2}{3}d$. Finally $A = 0.0087 \times 10 \times 18.2 = 1.58$ sq. in.

In practice the dimensions and steel areas as calculated need to be modified somewhat to conform to commercial sizes of bars and convenient dimensions of beams. If the depth is increased beyond the theoretical value, the stress in the steel and hence the required steel area will be decreased in nearly the same proportion. The concrete stress will also be reduced below the allowable value in about the same proportion. If the depth is decreased the amount of steel will need to be increased in much greater proportion in order to lower the neutral axis so as to avoid increasing the concrete stress. This will require a relatively large increase in steel. (For example, with $f_s = 16{,}000$ and $f_c = 700$, a decrease of $2\frac{1}{2}\%$ in depth below the theoretical will increase the amount of steel nearly 20%.) These variations are illustrated in the following article.

57. *Solution of Examples by the Use of Coefficients of Resistance.*—The problems of Art. 56 which have been solved by formulas will now be solved by the use of the diagrams of coefficients of resistance, Diagram 3 or 4.

(1) The percentage of steel is 1.26. Then on Diagram 4 we follow up vertically along the line for $p = 1.26\%$ to the curve for $f_c = 800$ and for $f_s = 16{,}000$. The value of R for $f_c = 800$ is 153, and for $f_s = 16{,}000$ it is 170. Then $M_c = 153 \times b\,d^2 = 300{,}000$, and $M_s = 170 \times b\,d^2 = 333{,}000$ in-lbs. It is obvious from the diagram that the stress f_c determines the safe moment and shows the beam to be over-reinforced.

(2) Here $b\,d^2 = 10 \times 17^2 = 2890$. $R = 350{,}000/b\,d^2 = 121$. $p = 1.04\%$. For these values by interpolation we find $f_c = 670$ and $f_s = 13{,}700$ lbs/in.^{2}

(3) With $f_s = 16{,}000$ and $f_c = 700$, we find at the intersection of these curves, $R = 121$ and $p = 0.87\%$. Hence $b\,d^2 = M/R = 400{,}000/121 = 3300$, as before. With $b = 10$ and $d = 18.2$, $p = 0.087 \times 182 = 1.58$ sq. in. This area will be secured by using two 1-in. round bars. Area $= 1.57$ sq. in.

If it is desired to make $d = 18\frac{1}{2}$ in., then the amount of steel may be slightly reduced but not sufficiently to change the design. For a depth of 19 in. a recalculation should be made. In this case $b\,d^2 = 10 \times 19^2 = 3610$, and $R = 400{,}000/3610 = 111$. For this value of R, and limiting values of $f_c = 700$ and $f_s = 16{,}000$, the diagram shows that the steel stress controls and gives a value of $p = 0.80\%$. $A = 0.0080 \times 190 = 1.52$ sq. in., only slightly less than before. The concrete stress will be about 660 lbs/in.^{2}

It may well be that two large bars will be unsatisfactory on account of bond stress or other reasons and that three or four bars will be desirable.

Assume four ¾-in. round bars giving $A = 1.77$ sq. in., which is more than necessary. With this amount of steel the depth may be somewhat reduced, perhaps to 18 in. With $d = 18$ in., $bd^2 = 3240$, $R = 400{,}000/3240 = 123$, $p = 1.77/180 = 0.98\%$. With $R = 123$ and $p = 0.98\%$, the stresses will be about $f_c = 680$ and $f_s = 14{,}500$ lbs/in.²

58. Effect of Incorrect Assumption of the Value of E_c or n.—As the value of n is subject to considerable uncertainty it will be useful to inquire into the effect of an error in its selection. This can readily be done by the aid of Diagrams 3 and 4. Suppose the value of n is taken at 12 in designing the beam, and the actual value is 10 or 15. The assumed working stresses are 1000 and 18,000 lbs/in.² respectively. From Diagram 4, for $n = 12$, $R = 173.3$, $p = 0.0111$. Then, for $n = 15$, $p = 0.0111$, and $R = 173.3$, we find $f_s = 18{,}600$ and $f_c = 930$; for $n = 10$, $f_s = 17{,}800$ and $f_c = 1120$. Thus if n is taken too low, the steel will be slightly over-stressed and the concrete under-stressed; if taken too high, the reverse will be the case. The relative effect is much greater in the concrete than in the steel. This illustration shows that it is preferable to estimate the value of n too low rather than too high.

59. Effect of Shrinkage and Plastic Flow upon the Stresses in Beams.—The data given in Art. 26 show that, under ordinary dry conditions, the deformation due to shrinkage and plastic flow may easily reach two or three times the elastic deformation due to stress. The result is to modify very considerably the relations of the stresses in concrete and steel. No exact measure of this effect is possible, but it will be useful to attempt a general analysis of this problem. The two sources of non-elastic deformation, shrinkage and flow, may to a certain extent be considered separately.

60. *Shrinkage.*—If unresisted by the steel, the shrinkage will cause a shortening of the beam throughout. The presence of the steel resists the shortening on the tensile side and causes the concrete either to crack or to be under tensile stress, the steel being in compression. If not actually cracked a loading of the beam will cause cracking earlier than if shrinkage were not present, and the final result will be the same as if shrinkage cracks were formed. The effect of all this is to raise the neutral axis in the same manner as would be done if the steel had been elongated by a direct pull and the concrete had been cracked

thereby. Neglecting the tensile strength of the concrete, we would then expect the shrinkage cracks, in the case of an unloaded beam, to extend to the upper or compression surface.

If now a load is applied, the compressive stress will be concentrated on a small area near the top of the section, but this area will increase as the load is increased. The lower limit of this compression area is the actual neutral axis. The position of this neutral axis can be found in the same manner as in Art. 50 by adding the shrinkage of the concrete to the elongation of the steel from stress. Let m = shrinkage per unit length. Then by proportionate strains, as in Art. 50,

$$\frac{f_c/E_c}{k} = \frac{m + f_s/E_s}{1 - k}.$$

Also, as in Art. 50,

$$C = T, \text{ or } f_s A = \tfrac{1}{2} f_c\, k\, b\, d, \text{ or } f_s p = \tfrac{1}{2} f_c k \quad . \quad . \quad (13)$$

From these equations we derive the expression

$$k^2 + k \frac{2 f_s p n}{f_s + m E_s} = \frac{2 f_s p n}{f_s + m E_s},$$

from which

$$k = \sqrt{2 a p n + (a p n)^2} - a p n \quad . \quad . \quad . \quad . \quad (14)$$

in which $a = \dfrac{f_s}{f_s + m E_s}$. Note that this is similar to eq. (1), Art. 50, and the values of k can be found from the same diagrams referred to in that article by using $a p$ for p. From (13) we have, as usual,

$$f_c = \frac{2 f_s p}{k}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Consider, for example, a beam that has been designed for working stresses of 18,000 and 800 lbs/in.^{2} respectively. Then for $n = 15$, $f_s/n f_c = 1.5$, and Fig. 6 gives $p = 0.009$ for a balanced design. If the shrinkage coefficient $m = 0.04\%$ the stresses may be analyzed as follows: For a load such that $f_s = 6000$, $a = 6000/(6000 + 12{,}000) = \tfrac{1}{3}$, $a p = 0.003$. Then from Fig. 6, $k = 0.26$, $j = 0.91$ and from (15) $f_c = 2 \times 6000 \times 0.009/0.26 = 415$ lbs/in.^{2} For a load such that $f_s = 12{,}000$, $a = \tfrac{1}{2}$, $a p = 0.0045$, $k = 0.30$, $j = 0.90$, $f_c = 720$ lbs/in.^{2}; and for $f_s = 18{,}000$, $a = 0.6$, $a p = 0.0054$, $k = 0.33$,

$j = 0.89$, $f_c = 980$ lbs/in.² Without shrinkage the values of f_c corresponding to the three assumed values of f_s would be $\frac{1}{3} \times 800 = 270$, $\frac{2}{3} \times 800 = 530$, and 800 lbs/in.² respectively. However, on account of the higher position of the neutral axis caused by shrinkage, resulting in an increased value of j, the stresses in steel and concrete under full working load will be somewhat less than 18,000 and 980 respectively. Without shrinkage, $k = 0.40$ and $j = 0.867$. The actual stresses will then be reduced in the proportion 0.867/0.89 or to about 17,500 and 950 lbs/in.² The final result is an increase in concrete stress over the design stress of 150 lbs/in.², or nearly 20%. The effect of shrinkage is therefore to decrease the stress in the steel a relatively small amount and to increase the stress in the concrete considerably.

61. Plastic Flow.—As the plastic flow is roughly proportional to the stress, its effects can be studied by using a low value for E_c or a high value for n. If the plastic flow is taken at three times the elastic deformation for a concrete with $E = 3{,}000{,}000$, then the equivalent secant modulus for combined elastic and plastic deformation is 750,000 and $n = 40$. Using $n = 40$ in the foregoing example will give a measure of the combined effect of shrinkage and flow.

For example, for $f_s = 6000$, $a = \frac{1}{3}$, $a\,p = 0.003$. Then for $n = 40$, $k = 0.39$. For $f_s = 12{,}000$, $k = 0.45$; and for $f_s = 18{,}000$, $k = 0.48$. The effect of flow is the reverse of that of shrinkage; it lowers the neutral axis, increases slightly the steel stress, and decreases the concrete stress.

Since plastic flow is a matter of long-sustained loads, the live-load effect will in many cases be little or nothing, and for this part of the load the normal elastic behavior may be expected. The dead-load effect is then shown by using a low value of f_s in the calculations. In the above assumed case, if the dead load is one-third the total design load, then f_s = about 6000, and $k = 0.40$. As this is very nearly the value of k under the elastic conditions ordinarily assumed, it will be changed little by the addition of the live load. For long-continued overloading, approaching the ultimate strength of the beam, the neutral axis will doubtless continue to fall until the steel begins to yield, and then will rise again; but there would seem to be no reason why the ultimate strength will be greatly affected.

The foregoing analysis is intended only to give a general notion of the effect of shrinkage and flow on the stresses in beams. The combined effect is not likely to be large.

62. Approximate Nature of the Calculations.—Since both shrinkage and plastic flow are present in most structures to a greater or less degree, it is obvious that any great refinements in the selection of the value of *n* in beam calculations are quite unnecessary and misleading in their implication of accuracy. A uniform value of 12 or 15 might well be used for all grades of concrete. The lower value is perhaps preferable as being favorable to the concrete while having slight effect on the steel, as indicated in Art. 58. However, it is very desirable to follow adopted standards, and until other specific rules have been formulated it will be necessary to use the commonly specified values

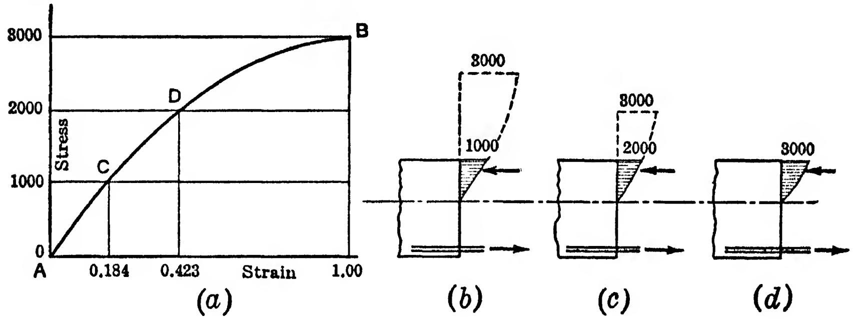


FIG. 8.

of *n*, from 10 to 15, as given in Art. 39. To meet requirements of building codes, designers must arrive at very definite results based on definite assumptions, but the precision of the results as regards the actual stresses which exist in the structure is very much less than indicated by the precision of the usual formulas and calculations. These conditions should be kept in mind and elaborate calculations should be avoided where simpler approximate methods are permitted.

FLEXURE FORMULAS FOR ULTIMATE LOADS

63. Conditions of Stress in the Concrete under Progressively Increasing Loads.—If *AB*, Fig. 8 (*a*), represents the compression stress-strain diagram of a concrete whose ultimate strength is 3000 lbs/in.[2] then in a beam made of this material the stress in the concrete from the neutral plane to the surface will vary according to the ordinates to this curve up to a stress equal to the stress on the extreme fibre. For example, if the load is such that the stress on extreme fibre is 1000 lbs/in.^{2} then the stress variation will be represented by the portion AC of the curve; if loaded to produce a stress of 2000 lbs/in., then the curve AD represents the law of variation; and if loaded to the ultimate, then the entire curve AB represents the variation of stress. The three conditions are represented in Figs. (*b*), (*c*), and (*d*), the scale used for stresses being different in the different sketches. Under working stresses of one-third to two-fifths of the ultimate strength, the variation of stress follows so nearly a straight-line law (Fig. (*b*)) that this is assumed in the analysis. Under higher stresses such an assumption introduces a considerable error, and if correct results are to be obtained a law of stress variation must be assumed which more nearly agrees with the facts.

The case of special importance is where the concrete stress approaches its ultimate value, as in Fig. (*d*), and it is useful to develop formulas for this case, as the application of such formulas helps to give a better notion of the ultimate strength of beams and the factor of safety against failure. As shown in Art. 27, the stress-strain diagram approximates roughly to a parabola, and such a curve will be assumed in the following analysis.

64. General Relations.—In this analysis it is assumed that the amount of reinforcement is sufficient to develop the full compressive strength of the concrete without stressing the steel beyond its yield point. That is to say, the ratio of stress to deformation in the steel remains constant throughout. The stresses and deformations in concrete and steel will then be as represented in Fig. 9. In this figure, $AE = f_c$, $NA = e_c$, and $NB = e_s$.

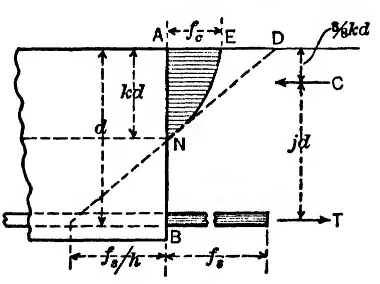


FIG. 9.

In the present connection, the two following properties of a parabola like that of Fig. 9 are useful: (1) The average abscissa of the parabolic arc equals two-thirds the greatest, f_c; (2) the distance from the centroid of the parabolic area to its top equals three-eighths the total height, $k\ d$.

65. Derivation of Formulas.—*Neutral Axis and Arm of Resisting Couple.*—The "initial modulus of elasticity" of the concrete (Art. 25) is denoted by E_c in the present article. It is represented by the slope of the tangent ND with respect to the deformation axis NA, or is equal to AD/NA. For a parabola, vertex at E, $AD = 2AE$, and since $e_c = NA$, we have $f_c = \frac{1}{2} E_c e_c$. Also, $f_s = E_s e_s$, and from the assumption of plane sections it follows that $e_s/e_c = (d - k d)/k d$. Eliminating e_s/e_c from the above equations, and introducing the abbreviation n, gives

$$\frac{f_s}{2 n f_c} = \frac{1 - k}{k}. \quad \ldots \ldots \ldots \quad (a)$$

Then, from $T = C$,

$$f_s p b d = \tfrac{2}{3} f_c b k d. \quad \ldots \ldots \ldots \quad (b)$$

Eliminating f_s/f_c between eqs. (a) and (b) and introducing the abbreviation p, gives $3 p n = k^2/(1 - k)$; this, if solved for k, gives

$$k = \sqrt{3 p n + (3/2 p n)^2} - 3/2 p n. \quad \ldots \quad (1)$$

The distance of the centroid of the compressive stress from the compressive face of the beam is $\frac{3}{8} k d$; therefore, the arm of the resisting couple $T - C$ is given by

$$j d = d - \tfrac{3}{8} k d, \quad \text{or} \quad j = 1 - \tfrac{3}{8} k. \quad \ldots \ldots \quad (2)$$

Fig. 10 gives curves for values of k and j for various values of p and n. Corresponding values of $f_s/n f_c$ are given at the left.

Comparing these curves with those of Fig. 6, it will be noted that the values of k and j are somewhat less, showing that the neutral axis is a little lower under the parabolic assumption of stress variation than under the straight line assumption.

66. *Ultimate Resisting Moment for a Given Ultimate Strength f_c.*—Remembering the assumption made at the outset in regard to the amount of steel (Art. 64), it will be understood that the ultimate resisting moment always depends on the concrete; the value is

$$M_c = C \cdot j d = \tfrac{2}{3} f_c b k d \cdot j d = \tfrac{2}{3} j k f_c b d^2. \quad \ldots \quad (3)$$

If f_s is the steel stress corresponding to f_c we have

$$M_s = T j d = f_s p j b d^2. \quad \ldots \ldots \ldots \quad (4)$$

Placing $M_s = M_c$ of eq. (3) the value of f_s can be found.

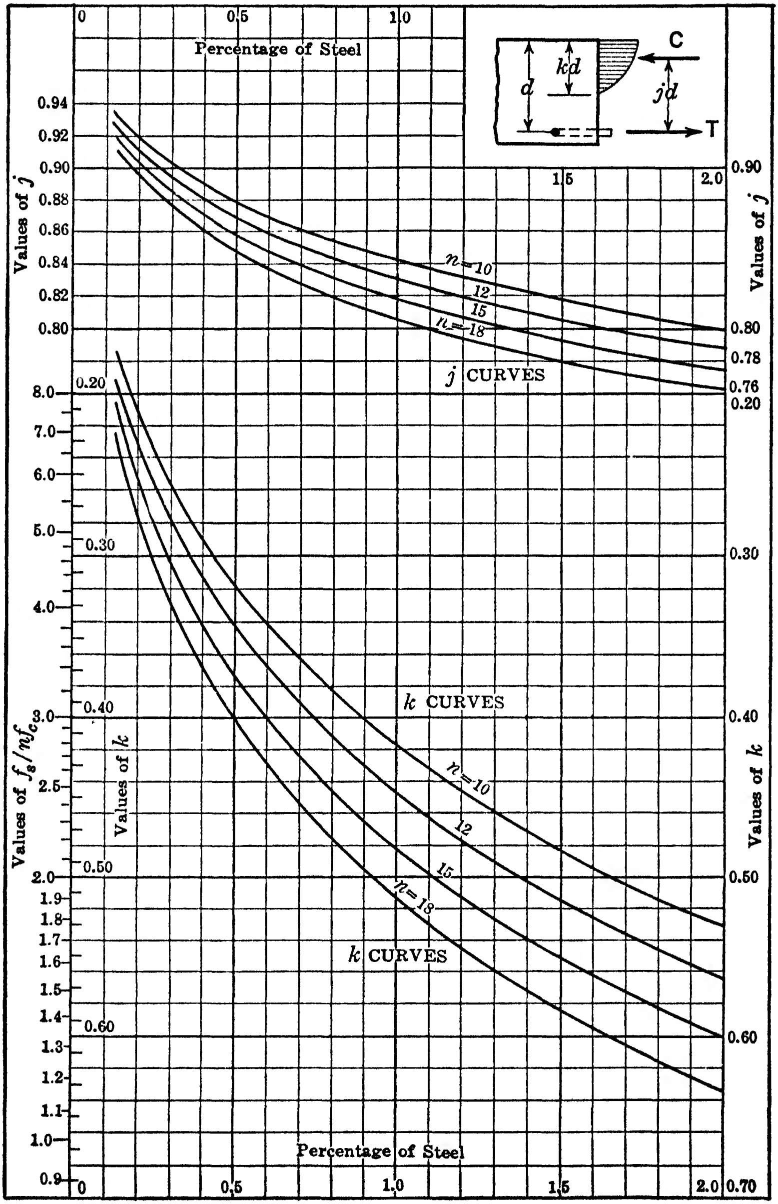

Percentage of Steel
0
0.5
1.0
1.5
2.0
kd
d
jd
C
T
Values of j
0.94
0.92
0.90
0.88
0.86
0.84
0.82
0.80
0.78
0.76
n=10
12
15
n=18
j CURVES
k CURVES
Values of f_s/nf_c
8.0
7.0
6.0
5.0
4.0
3.0
2.5
2.0
1.9
1.8
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1.0
0.9
Values of k
0.20
0.30
0.40
0.50
0.60
0.70

FIG. 10.

If this calculated value of f_s is less than the yield point of the steel, then the ultimate resisting moment is correctly given by (3); but if it exceeds the yield point, then the beam fails by over-stressing of the steel and the formulas of this article do not strictly apply.

67. *Determination of Amount of Steel and Cross-section of Beam for a Given Ultimate Bending Moment.*—To find the proper amount of steel to correspond to given values of f_c and f_s, place M_s of eq. (4) equal to M_c of eq. (3) and substitute the value of k as given by equation (*a*). Solving for p, there results

$$p = \frac{2/3}{\frac{f_s}{f_c}\left(\frac{f_s}{2\,n f_c} + 1\right)}. \qquad (5)$$

If, in any given case, the steel ratio as given by (5), or a higher value, is adopted, then the concrete would crush without straining the steel beyond the yield point, and the ultimate resisting moment of the beam is given by (3), which value equated to the ultimate bending moment, M, to be provided for, gives $2/3 f_c j\,k\,b\,d^2 = M$, or

$$b\,d^2 = \frac{M}{2/3 f_c j\,k}. \qquad (6)$$

From this, d may be computed for any assumed value of b.

If the percentage of steel used is less than that given by (5), with f_s = yield point, then the utlimate resisting moment can be only approximately calculated by eq. (4), but as j varies but little, the results will be sufficiently accurate.

68. Use of the Foregoing Method of Analysis.—In the design and examination of structures the general practice is to make use of certain specified or assumed safe working stresses so selected as to give a safe margin, varying with the circumstances, against failure or undue deformation. In the case of reinforced-concrete beams the assigned working stress in the concrete is such that the straight-line law of stress variation may be assumed and is generally employed. However, if we desire to know the ultimate strength as regards concrete failure, the formulas of this section are useful and are found to give satisfactory results. They are also useful in laboratory studies of the relative strength of concrete in beams and in compression specimens.

69. Example.—1. Find the ultimate strength and factor of safety of the beam of Example 3, Art. 56, if the ultimate strength of the concrete is 2000 lbs/in.^{2} and the steel is assumed to have a sufficiently high yield point so that it will not be over-stressed. Take $n = 15$, $p = 0.0087$. From Fig. 10, $k = 0.46$ and $j = 0.83$. Hence $M_c = \frac{2}{3} \times 2000 \times 0.46 \times 0.83 \times 3300\ (b\,d^2) = 1{,}680{,}000$ in-lbs., giving a factor of safety of 4.2. The steel stress is given by the relation $\frac{f_s}{2\,n f_c} = \frac{1-k}{k}$ or $f_s = 2\,n f_c \left(\frac{1-k}{k}\right)$ $= 30 \times 2000 \times \frac{0.54}{0.46} = 70{,}500$ lbs/in.^{2} It is to be noted that, whereas the ratio of ultimate strength of the concrete to the working stress is only $2000/700 = 2.86$, the factor of safety against failure due to crushing of the concrete is more than 4. This is due to the fact that the extreme fibre stress does not increase proportionately to the increase in load. On the other hand, the steel stress does increase very nearly in proportion to the load, to be exact, a little faster than the load, owing to the slight decrease in the value of j. In this case the ratio of steel stresses is $70{,}500/16{,}000 = 4.4$.

Strictly speaking, the value of the tangent modulus E_c used in the formulas for ultimate load should be somewhat greater than the secant modulus used for working loads, but the difference is unimportant in estimates of this character.

FLEXURE FORMULAS FOR T-BEAMS

70. Use of T-Beams.—Where a concrete floor slab is constructed integrally with the supporting beams so that unity of action is insured,

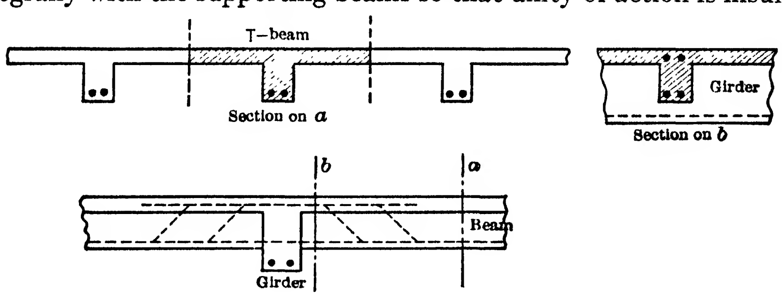


Fig. 11.

then the beam, with a portion of the slab above it, constitutes a so-called T-beam (Fig. 11). In regions of positive bending moment the slab acts as the compression flange, the steel being placed near the bottom. The narrow part of the beam is commonly called the web. In regions of negative bending moment (near the support in the case of continuous beams) the slab is in tension and the lower part of the beam is in compression. If it be assumed that the concrete takes no tension, the beam becomes a rectangular beam with tension side uppermost, and the principal reinforcing steel is placed near the top. Some of the steel is also generally continued straight through near the bottom, thus reinforcing the beam in compression. (See Art. 81.)

The portion of the slab which may be assumed to act in compression, or the effective width of the flange, is usually limited to about eight times its thickness, but, of course, not exceeding the distance centre to centre of beams.

T-beams are also used occasionally where not constructed in connection with a floor slab. Since the concrete in the lower part of a beam takes no tension its only duty is to transmit stress from tension steel to compression concrete (involving mainly shearing-stress) and for this purpose the entire rectangular section is not needed in large beams. Economy can thus be secured by omitting a portion of the concrete, leaving a T-form of section.

71. Assumptions and Notation.—The neutral axis of a T-beam may lie in the *flange* or below the flange in the *web*, depending upon the relative depth of flange and beam, and amount of steel used. If the neutral axis is in the flange then the formulas for rectangular beams

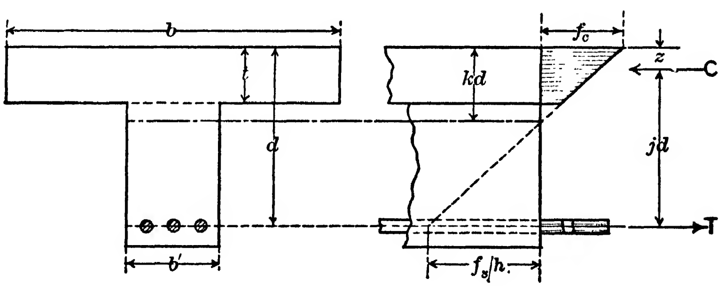


FIG. 12.

apply as the concrete below the neutral axis is of no significance. If it lies in the web the compression area is of different form from that in rectangular beams and different formulas are required. These are developed in the following articles.

Whether or not the neutral axis lies in the flange can readily be determined by means of the formulas and diagrams for rectangular beams, especially Diagram 3 or 4. Having given the amount of reinforcement, or the values of f_c and f_s, the value of k is determined and compared with the slab thickness. The same information can be obtained also by use of the formulas and diagrams for T-beams as explained later.

The following notation is employed in addition to that of Art. 48. (See Fig. 12.)

b = width of flange;
d = effective depth of beam;
b' = width of web;
t = thickness of flange;
z = depth of compression resultant below top of flange;
p = steel ratio = $A/b\,d$.

72. Case I. Compression in Web Neglected.—The amount of the compression in the web is commonly small compared to that in the flange and will be neglected in the analysis of this article. The formulas are thereby greatly simplified and the resulting error is generally very small. To provide for designs in which the web is very large as compared to the flange, formulas which take account of web compression are given in Art. 76.

73. *Neutral Axis and Arm of Resisting Couple.*—Just as in Art. 50, eq. (*a*),

$$\frac{f_s}{n f_c} = \frac{1 - k}{k}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

hence we have, in terms of f_s and f_c,

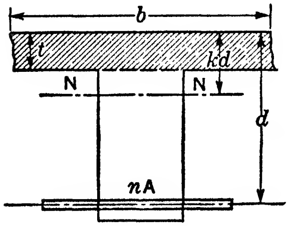


Fig. 13.

$$k = \frac{1}{1 + f_s/n f_c}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Note that in terms of the unit stresses the position of the neutral axis is the same as in rectangular beams.

To derive the value of k in terms of the dimensions of the beam and the steel ratio it will be more convenient in this case to make use of the transformed section than the method of Art. 50. Fig. 13

shows the transformed section. Taking moments about the upper edge we have $n\,A\,d + \frac{1}{2}\,b\,t^2 = (n\,A + b\,t)\,k\,d$, whence

$$k = \frac{n\,A + \frac{1}{2}\,b\,t \cdot \frac{t}{d}}{n\,A + b\,t} \quad . \quad . \quad . \quad . \quad (2)$$

Or in terms of p

$$k = \frac{p\,n + \frac{1}{2}\left(\frac{t}{d}\right)^2}{p\,n + \frac{t}{d}} \quad . \quad . \quad . \quad . \quad (3)$$

The arm of the resisting couple is $d - z$ (see Fig. 12). The distance z is equal to the distance of the centroid of the shaded trapezoid from the top of the beam, that is,

$$z = \frac{3\,k - 2\,\frac{t}{d}}{2\,k - \frac{t}{d}} \cdot \frac{t}{3} \quad . \quad . \quad . \quad . \quad (4)$$

We also have

$$jd = d - z, \quad . \quad . \quad . \quad . \quad (5)$$

and, by substitution from (3) and (4) we have, in terms of t/d and p,

$$j = \frac{6 - 6\,\frac{t}{d} + 2\left(\frac{t}{d}\right)^2 + \left(\frac{t}{d}\right)^3 \Big/ 2\,p\,n}{6 - 3\,\frac{t}{d}} \quad . \quad . \quad . \quad . \quad (6)$$

When $t/d = k$, this reduces to eq. (2), Art. 50 and the neutral axis will be at the junction of web and flange.

On Diagram 5 are plotted curves giving values of k and j for various values of $p\,n$ and of the ratio t/d. This diagram, as well as eq. (6), shows that j is affected very little by changes in the amount of steel. The diagram also gives on the left-hand margin the values of $f_s/n\,f_c$, corresponding to the various values of k as determined from eq. (1). The curves for k and j end at points where $k = t/d$. They become horizontal at these points and the values of k are equal to those for rectangular beams. (See Fig. 6, Art. 50.)

74. *Resisting Moment and Working Stresses.*—The average unit compressive stress in the flange is $\frac{1}{2}\left[f_c + f_c\left(1 - \frac{t}{kd}\right)\right] = f_c\left(1 - \frac{t}{2dk}\right)$, and the whole compression is $f_c\left(1 - \frac{t}{2kd}\right)bt$.

The total tension is $f_s A$; hence, the resisting moment in terms of f_c and f_s is given by the formulas

$$\left.\begin{aligned} M_s &= f_s A \cdot jd \\ M_c &= f_c\left(1 - \frac{t}{2kd}\right) bt \cdot jd \end{aligned}\right\} \quad \ldots \ldots \quad (7)$$

If f_c and f_s are merely the permissible values of stress, then the calculated values of M_c and M_s will not in general be equal; the safe value is evidently the smaller.

The unit stresses, f_s and f_c, produced by a certain bending moment M in a given beam can be computed by solving (7) for f_s and f_c, or from

$$\left.\begin{aligned} C &= T = \frac{M}{jd}; \quad f_s = \frac{T}{A} \\ f_c &= \frac{f_s}{n} \cdot \frac{k}{1-k} = \frac{f_s p}{\left(1 - \frac{t}{2kd}\right)\frac{t}{d}} \end{aligned}\right\} \quad \ldots \ldots \quad (8)$$

After f_s is calculated, f_c can most readily be found from the ratio f_s/nf_c given in Diagram 5.

75. *Approximate Method of Calculating T-beams by Using Full Working Stresses for Concrete.*—A method of calculation applicable to most cases, that is somewhat simpler than the foregoing, is to assume full working stresses for the concrete as well as the steel and then determine the width of flange which will correspond to this assumption. If this is less than the actual width, the design is considered satisfactory. By this method the value of k is dependent only upon the ratio f_s/nf_c. The depth of beam and amount of steel are then determined as for a balanced design. If the calculated width of flange is less than the actual width the resulting value of j will generally be slightly less than the correct value, giving a corresponding excess of steel area but the error is small. (See Art. 80 for examples.)

76. Case II. Compression in Web Not Neglected.—When the web is very large compared to the flange and the flange alone is inadequate to carry the stress, it is desirable to include the web in the calculations. In this case the formulas for the position of neutral axis, arm of resisting couple, and moment of resistance become as follows:

$$kd = \sqrt{\frac{2ndA + (b - b')\,t^2}{b'} + \left(\frac{nA + (b - b')\,t}{b'}\right)^2} - \frac{nA + (b - b')t}{b'} \qquad (9)$$

$$z = kd - \frac{2}{3} \times \frac{b\,(kd)^3 - (b - b')\,(kd - t)^3}{b\,(kd)^2 - (b - b')\,(kd - t)^2}, \quad . \quad . \quad (10)$$

$$jd = d - z; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$\left.\begin{aligned} M_s &= f_s A \cdot jd \\ M_c &= \frac{f_c}{2kd}\,[b\,(kd)^2 - (b - b')\,(kd - t)^2]\,jd \end{aligned}\right\} \quad . \quad . \quad . \quad (12)$$

Equations (12) also give f_s and f_c for given values of M.

77. *Method of Calculation Assuming Full Working Stresses.*—A more expeditious method of arriving at the result in the case here

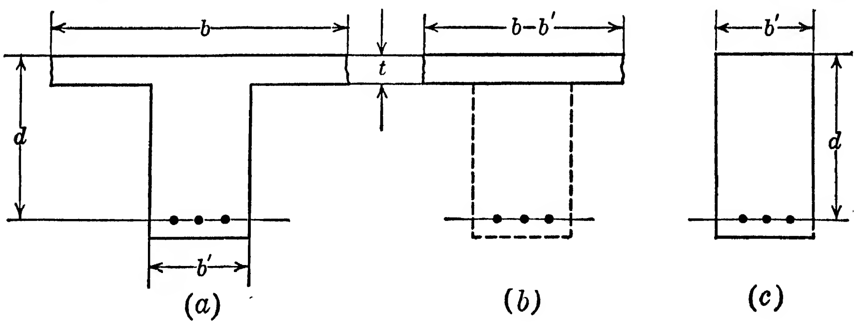


FIG. 14.

considered is by the assumption of full working stresses and proceeding as for a balanced design. The beam may then be considered as made up of two parts as shown in Fig. 14, namely, a T-beam, Fig. (*b*) of width $b - b'$, and without web; and a rectangular beam, Fig. (*c*), of width b' and depth d. The neutral axis will be at the same elevation in both sections and the total resisting moment will be the sum of the separate moments. Where a design is to be made for specified working stresses this method is simple and leads to accurate results; where a given beam is to be reviewed the results will not in general be precise but will be accurate enough for all practical purposes. (See Art. 80 for examples.)

78. Diagrams for Use in Designing.—In addition to Diagram 5, which gives values of k and j in terms of $p\,n$ and t/d, it is convenient to use diagrams to aid in the solution of eq. (7) for moments of resistance. These can be written

$$\left.\begin{aligned} M_s &= (f_s/n)\,p\,n\,j \times b\,d^2 \\ M_c &= f_c\left(1 - \frac{t/d}{2\,k}\right)\frac{t}{d}\,j \times b\,d^2 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \qquad (13)$$

Let $C_s = p\,n\,j$ and $C_c = \left(1 - \frac{t/d}{2\,k}\right)\frac{t}{d}j$, then

$$\left.\begin{aligned} M_s &= C_s \times \frac{f_s}{n}\,b\,d^2 \\ M_c &= C_c \times f_c\,b\,d^2 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (14)$$

The quantities C_s and C_c are functions of $p\,n$ and t/d only (since k and j are also functions of these quantities), hence can be readily shown on diagrams. Diagrams 6 and 7 have been prepared for this purpose. These, together with Diagram 5, facilitate the solution of the various problems of T-beam analysis. Note that $C_s \times f_s/n$ corresponds to R_s, and $C_c f_c$ to R_c for rectangular beams. The use of these diagrams is illustrated in the problems of Art. 80.

Diagram 8 will prove useful when a balanced design is desired and the flange thickness is known. From Fig. 13, Art. 73, the moment of the concrete stress about the steel is

$$M_c = f_c\,b\,t^2\left[\left(\frac{d}{t} - \frac{1}{2}\right) + \frac{1}{k}\left(\frac{t}{3\,d} - \frac{1}{2}\right)\right].$$

Substituting the value of k from eq. (1), Art. 73

$$M_c = f_c\,b\,t^2\left[\left(\frac{d}{t} - \frac{1}{2}\right) + \left(1 + \frac{f_s}{n f_c}\right)\left(\frac{t}{3\,d} - \frac{1}{2}\right)\right] = f_c\,b\,t^2\,C. \quad (15)$$

Values of C are shown on Diagram 8 together with the corresponding values of $p\,n$ and j.

79. Problems of Design.—In practice, various forms of problems will arise: (*a*) The dimensions may be given, to find the safe resisting moment of the beam or the stresses in the steel and concrete under a given load; (*b*) the dimensions of the flange may be given, together with the loading and specified working stresses, to determine suitable web dimensions and steel area; (*c*) the loading and working stresses may be given, to determine suitable proportions for the entire beam.

(*a*) Where all the dimensions are given, the value of k and j are found from eqs. (3) and (6), or from Diagram 5, and thence the values of the moment of resistance from eqs. (7); or with the bending moment given the fibre stresses are found from (7) or (8) or by Diagrams 6 and 7. If the value of k is found to be less than t/d, then the formulas for rectangular beams apply.

(*b*) Generally the flange has been predetermined as it is usually formed by a portion of a floor slab which is already designed. A suitable web must then be selected, together with the necessary amount of steel.

In determining the dimensions of the web, consideration must be given not only to the bending moment but also to the shearing-stresses, space for the necessary bars, and other matters, as fully explained in subsequent articles, but for the present the bending moment only will be considered. For this purpose, the width of the web is immaterial; only the depth is involved. The depth may be made (1) such as to give a balanced design, that is, a beam wherein both steel stress and concrete stress are equal to the assigned working stresses; or, as is frequently the case, (2) the depth may be made greater than this minimum on account of the other considerations mentioned above, in which case the steel is proportioned for the allowed unit stress and the concrete stress will be less than the allowable.

(*c*) When all parts of the beam are to be selected on the basis of given working stresses it is convenient to first select suitable proportions for the web, as in Case (*b*). A flange thickness is then assumed such as to give satisfactory proportions between t and d. The value of t/d is then known and k and j can be determined from (1), (4), and (5) or Diagram 5. The area of steel and the breadth of flange is then found from eq. (7). The smaller the value of t the smaller will be the flange area required, but too slender proportions are to be avoided, as explained in Chap. V.

The procedures followed in solving the various forms of problems are explained in the examples following.

80. Examples.—(1) A T-beam has the following dimensions: $b = 48$ in., $t = 8$ in., $d = 22$ in., $b' = 12$ in., and the steel consists of six 1-in. round bars. If the working stresses are $f_s = 16{,}000$ and $f_c = 700$ lbs./in.^{2}, $n = 15$, what is the safe bending moment for the beam?

Solution. The area of the steel is 4.71 sq. in. and $p = 4.71/48 \times 22 = 0.446\%$. $pn = 0.067$. $t/d = 8/22 = 0.364$. From Diagram 5 (or Fig. 6) it is found that k is less than t/d, hence the neutral axis is in the flange and the beam acts as a rectangular beam and is calculated as such.

(2) Change t in the preceding example to 4 in. and solve.

Solution. Here $t/d = 4/22 = 0.182$. From Diagram 5, $k = 0.33$ and $j = 0.92$. Then $M_s = 16{,}000 \times 4.71 \times 0.92 \times 22 = 1{,}520{,}000$ in-lbs. From eq. (7), Art. 74, $M_c = 700\left(1 - \dfrac{0.182}{2 \times 0.33}\right) 4 \times 48 \times 0.92 \times 22 =$ 1,960,000 in-lbs. The safe moment is therefore 1,520,000 in-lbs. An examination of Diagram 5 shows that the ratio of f_s/nf_c for the given values of t/d and p is about 2.0, hence for $f_s = 16{,}000$, $f_c = 533$, which shows at once that the steel stress determines the strength of the beam. As a check, $152/196 \times 700 = 540$. Diagrams 6 and 7 can also be used for the calculations. With $pn = 0.067$ and $t/d = 0.182$, Diagram 6 gives $C_s = 0.062$ and $M_s = 0.062 \times f_s/n \times bd^2 = 0.062 \times 16{,}000/15 \times 48 \times 22^2 = 1{,}530{,}000$ in-lbs. Diagram 7 gives $C_c = 0.122$ and $M_c = 0.122 \times 700 \times 48 \times 22^2 = 1{,}980{,}000$ in-lbs.

The most useful diagram for such problems is No. 5, as the value of j is accurately given and the ratio of stresses, f_s/nf_c, determines at once the controlling stress.

Suppose in the foregoing case that the allowable stresses were 20,000 and 600 lbs/in.^{2} respectively. The ratio $f_s/nf_c = 20{,}000/9000 = 2.22$, which is larger than the ratio 2.0 given in Diagram 5. Hence the concrete stress determines and the steel stress $f_s = 2 \times 15 \times 600 = 18{,}000$ lbs/in.^{2} The value of $M_s = 18{,}000 \times 4.71 \times 0.92 \times 22 = 1{,}720{,}000$ in-lbs. This will also be obtained from Diagram 7, with $f_c = 600$.

(3) The flange of a T-beam is 40 in. wide and 4 in. thick, and the given bending moment is 1,000,000 in-lbs. With $f_s = 18{,}000$ and $f_c = 800$, determine the depth and amount of steel.

Solutions. (*a*) It will first be assumed that the depth will be selected to give a balanced design—that is, such that both steel and concrete will be fully stressed. A direct solution of this problem can be made from Diagram 8. $f_s/nf_c = \dfrac{18{,}000}{15 \times 800} = 1.5$. From eq. (15), $C = \dfrac{M}{f_c b t^2} = \dfrac{1{,}000{,}000}{800 \times 40 \times 4^2} = 1.96$. For these values of C and f_s/nf_c, we have from Diagram 8, $t/d = 0.288$ and $pn = 0.122$. Then $d = \dfrac{4}{0.288} = 13.92$ in.

Use $d = 14$ in. Hence $A = \dfrac{bdpn}{n} = \dfrac{40 \times 14 \times 0.122}{15} = 4.55$ sq. in. As a check, we have from Diagram 8, $j = 0.882$. Then $A = \dfrac{M}{f_s j d} = \dfrac{1{,}000{,}000}{18{,}000 \times 0.882 \times 14} = 4.50$ sq. in.

(*b*) Quite generally the depth of the beam is determined by other considerations and is made greater than that calculated above. Where the depth is already known the problem is solved more quickly. Thus if the depth is to be 20 in., then $t/d = 0.20$; and from Diagram 5, the value of j is not far from 0.91. Hence $T = 1{,}000{,}000/(20 \times 0.91) = 55{,}000$ and $A = 55{,}000/18{,}000 = 3.06$ sq. in. Then for a check, $p = 3.06/(40 \times 20) = 0.00383$, $pn = 0.057$, and Diagram 5 gives $j = 0.916$; the correct steel area will be 3.04 sq. in.

The best number and size of rods is a matter involving factors which are considered later. Where the depth has been determined in advance, as here assumed, it is necessary to check the concrete stress to make certain it is within safe limits. Here the ratio of stresses from Diagram 5 is seen to be approximately $2.3 \times 15 = 35$, hence $f_c = 510$ lbs/in.^{2}

(4) Design a T-beam to sustain a bending moment of 600,000 in-lbs., all dimensions to be determined. $f_s = 18{,}000$, $f_c = 700$. Assume a ratio $t/d = 0.25$. Then $f_s/nf_c = 1.72$; and from Diagram 5, $pn = 0.104$. From Diagram 6, $C_s = 0.093$ and $b\,d^2 = \dfrac{600{,}000}{0.093 \times 1200} = 5400$. A depth of 16 in. gives a width of 21.2 in.; a depth of 14 in. a width of 27.5 in. In the former case, $t = 0.25 \times 16 = 4$ in.; in the latter, $t = 3.5$ in. If the actual widths are made in whole inches as 22 or 28, then the steel area is best determined directly from the moments. In the first case, $A = \dfrac{600{,}000}{16 \times 0.89 \times 18{,}000} = 2.34$ sq. in. In the second case, $A = \dfrac{600{,}000}{14 \times 0.89 \times 18{,}000} = 2.80$ sq. in. The actual values of p are $\dfrac{2.34}{16 \times 22} = 0.0067$ and $\dfrac{2.80}{14 \times 28} = 0.00715$, and $pn = 0.100$ and 0.107. The actual concrete stress will be slightly below the specified value.

With the same thickness of stem, the total concrete section will be slightly less in the shallower beam but the steel area greater.

(5) Solve example (3 b) by the approximate method explained in Art. 75, namely, by sssuming a width of flange such as to give a balanced design and full working stresses in both concrete and steel.

Solution. The value of $f_s/nf_c = 1.72$ and $t/d = 0.20$. From Diagram 5, $pn = 0.085$ and $p = 0.00567$. From Diagram 6, $C_s = 0.077$, hence $b\,d^2 = \dfrac{1{,}000{,}000}{0.077 \times 1200} = 1080$, and $b = 1080/20^2 = 27$ in. This being less than the actual width of 40 in., the concrete stress is much below the allowable value. The steel area $A = 0.00567 \times 27 \times 20 = 3.06$ sq. in.

The result is practically the same as obtained by taking the full flange width of 40 in. Actually the value of j will be slightly less for a width of 27 in. and the required steel area slightly greater, but the difference is of no practical consequence. The advantage of this method is that the value of j depends only upon the ratio f_s/nf_c and t/d and can readily be tabulated for any given set of working stresses.

(6) Consider a case in which the flange is inadequate and it is desired to take into account the compression area of the web by the method explained in Art. 77. Assume a T-beam, such as a girder in a building, with

$t = 4$ in., $b = 40$ in., $b' = 12$ in., $d = 24$ in., $f_s = 18{,}000$, $f_c = 800$. We have $t/d = 0.167$, $f_s/nf_c = 1.5$. Determine the moment of resistance if fully reinforced, and the amount of the reinforcement.

Solution. The beam is considered as made up of two parts, a rectangular beam 12 in. by 24 in., and a T-beam with flange 28 in. by 4 in.

For the rectangular beam, Diagram 4 gives $R = 140$ and $M = 140 \times 12 \times 24^2 = 970{,}000$ in-lbs. For the T-beam, from Diagrams 5 and 6, $pn = 0.088$ and $C_s = 0.083$. Hence $M = 0.083 \times 1200 \times 28 \times 24^2 = 1{,}650{,}000$ in-lbs. Total $M = 970{,}000 + 1{,}650{,}000 = 2{,}620{,}000$ in-lbs. For the rectangular beam $j = 0.87$ and $A = \dfrac{970{,}000}{0.87 \times 24 \times 18{,}000} = 2.58$ sq. in.

For the T-beam, $j = 0.92$ and $A = \dfrac{1{,}650{,}000}{0.92 \times 24 \times 18{,}000} = 4.10$ sq. in.

If the web be neglected, $C_s = 0.083$ as before, and $M = 0.083 \times 12{,}000 \times 40 \times 24^2 = 2{,}300{,}000$ in-lbs., a value 12% less than the more precise one.

The foregoing represents about the only case for which this exact calculation would be used, namely, where the resisting moment as calculated in the usual way is inadequate and the inclusion of the web may give the desired moment.

If in the above case the given moment is only 2,000,000 in-lbs. and it is desired to calculate the requisite steel, we can proceed in the same manner but the result will not be exact. As before, the retangular beam will carry $M = 970{,}000$ in-lbs. and $A = 2.58$ sq. in. The remaining moment $= 2{,}000{,}000 - 970{,}000 = 1{,}030{,}000$ in-lbs. Then from Diagram 5 we estimate $j = 0.93$ and hence $A = \dfrac{1{,}050{,}000}{0.93 \times 24 \times 18{,}000} = 2.62$ sq. in., $p = \dfrac{2.62}{28 \times 24} = 0.39\%$. The corrected value of j is 0.925, which does not change A appreciably. Total $A = 2.53 + 2.62 = 5.15$ sq. in. There is a slight error in this analysis, as the position of the neutral axis is not the same in the two parts of the beams. For the rectangular beam $k = 0.40$, and for the T-beam $k = 0.32$. Actually it will be between these two values, and j for the rectangular beam will be slightly greater than 0.87 and for the T-beam slightly less than 0.925. If the usual method of calculation is employed by neglecting the compression in the web, we have total $A = \dfrac{2{,}000{,}000}{0.92 \times 24 \times 18{,}000} = 5.02$ sq. in., a slightly less amount than found above, but sufficiently accurate; and, except for the case first mentioned, the use of the precise method is an unnecessary refinement. Where the flange is quite thin and the web heavy this condition can be adequately allowed for by reducing slightly the value of j. A value of $j = 0.90$ would give results accurate within 2 or 3% in all cases.

BEAMS REINFORCED FOR COMPRESSION

81. Use of Compressive Reinforcement.—Generally speaking, it is more economical to carry compressive stresses by concrete than by steel, but, under certain circumstances, it is desirable to reinforce the compressive side of a beam so as to be able to use a smaller beam than would otherwise be required. The most common case is that of the girder continuous over supports and constructed integrally with a floor slab. Such a girder, with slab, acts as a T-beam in carrying the positive moments, as illustrated in the preceding section. At and near the support where the moment is negative, the lower side is in compression, and the beam acts as a rectangular beam. If the beam has been designed mainly with reference to the positive moments at the centre and the shearing-stresses to be carried, the problem at the support will be to use this same section and to reinforce it sufficiently to carry the negative bending moment present, which is generally equal to or greater than the positive moment. The necessary tension steel at the top can readily be provided but it will usually be found that the lower or compression side of the beam will also need to be reinforced.

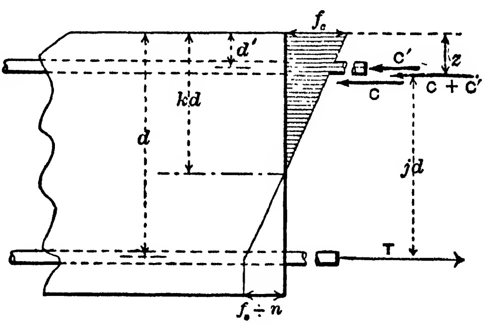


FIG. 15.

82. Assumptions and Notation.—The compression in the concrete is assumed to follow the linear law and the tension in it is neglected; the formulas then apply to working conditions only. In addition to the notation already adopted let

A' = cross-sectional area of the compressive reinforcement;
p' = steel ratio for the compressive reinforcement $= A'/b\,d$;
f_s' = unit stress in the compressive reinforcement;
C' = whole stress in the compressive reinforcement;
d' = distance from the compressive face of the beam to the plane of the compressive reinforcement;
z = distance from the compressive face to the resultant compression, $C + C'$, on the section of the beam.

83. *Neutral Axis and Arm of Resisting Couple.*—From the stress diagram (Fig. 15) it appears that $f_s/n f_c = (d - k\,d)/k\,d$, or

$$f_s = n\frac{1-k}{k}f_c. \quad \quad (1)$$

Similarly, $f_s'/nf_c = (kd - d')/kd$, or

$$f_s' = n\frac{k-d'/d}{k}f_c. \quad \quad (2)$$

For simple flexure, the whole tension T and whole compression $C + C'$ are equal, hence

$$f_s A = \tfrac{1}{2}f_c bkd + f_s'A'. \quad \quad (a)$$

Inserting the values of f_s and f_s' from (1) and (2) in (*a*) gives an equation which may be written thus:

$$k^2 + 2n(p + p')k = 2n(p + p'd'/d), \quad \quad (3)$$

and from this the neutral axis of a given section can be located.

The arm of the resisting couple is the distance between T (see Fig. 15) and the resultant of the compressions C and C'. It follows from the principle of moments and the law of distribution of stress respectively that

$$z = \frac{\tfrac{1}{3}k + d'C'/dC}{1 + C'/C}d, \quad \text{and} \quad \frac{C'}{C} = \frac{2p'n(k - d'/d)}{k^2},$$

from which z can be computed for any given section. Finally the arm $jd = d - z$ or

$$j = (1 - z/d). \quad \quad (4)$$

Diagrams 9, 10, and 11 give values of k and j for various values of pn and $p'n$, and for d'/d of 0.05, 0.10, and 0.15 respectively. The values of f_s/nf_c corresponding to the values of k are also given along the margin, these values being in accordance with eq. (1). This definite relation between k and the ratio f_s/nf_c should be noted; it is the same as for singly reinforced beams, as it depends only upon the fundamental assumptions of plane sections and proportionality of stress and strain.

In the foregoing analysis the slight reduction in the compression area of the concrete caused by the presence of the steel has been neglected as is usually done in this problem. To take this into account the quantity p' or A' in the equations and diagrams should be taken equal to $\frac{n-1}{n}$ times the actual quantity used since $\frac{1}{n}$th part of the steel is needed to make up for the concrete displaced. Conversely, in a design problem where p' or A' is determined this should be multiplied by $\frac{n}{n-1}$ to get the correct amount to be used.

84. *Resisting Moment and Working Stresses.*—In terms of the steel stress f_s, the resisting moment is

$$M_s = f_s A j d = f_s p j b d^2. \quad . \quad . \quad . \quad . \quad . \quad (4a)$$

And making use of the relation between f_s and f_c given by eq. (1), we get in terms of the concrete stress f_c

$$M_c = f_c \frac{p n (1 - k)}{k} j b d^2. \quad . \quad . \quad . \quad . \quad . \quad (4b)$$

The unit stresses in terms of bending moment are

$$\left. \begin{aligned} f_s &= \frac{M}{p j b d^2} \\ f_c &= \frac{k}{p n (1 - k) j} \frac{M}{b d^2} \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

It will be noted that in none of these equations (4) or (5), does the compression reinforcement p' appear. The influence of this factor is contained in the values of k and j.

In Diagrams 9, 10, and 11, the values of $R = \frac{p n (1 - k) j}{k}$ are also plotted for various values of $p n$ and $p' n$. Then $f_c = \frac{M}{R b d^2}$ and $M = R f_c b d^2$. These diagrams are useful in analyzing a given beam for safe moment or finding the stresses due to a given bending moment. The relation between f_s and f_c is given in the diagram for k. The value of f'_s need not be computed as it is always less than $n f_c$.

Example.—Assume a beam with $b = 12$ in., $d = 18$ in., $p = 1.5\%$, $p' = 1.0\%$, $d'/d = 0.1$, $n = 15$. With working stresses of 18,000 and 700 lbs/in.^{2}, what is the safe bending moment?

Solution. Using $p' = 0.01 \times 14/15 = 0.0093$ we get, from Diagram 10, $k = 0.418$, $f_s/n f_c = 1.40$. The value of $f_s/n f_c$ for the given stresses is $18{,}000/10{,}500 = 1.72$, hence it is evident that the allowable concrete stress of 700 lb/in.² will determine the strength of the beam. The stress in the steel will be $700 \times 1.40 \times 15 = 14{,}700$ lb/in.² The value of M_c from the diagram is $f_c R b d^2 = 700 \times 0.275 \times 12 \times 18^2 = 750{,}000$ in-lbs.

The safe moment can also be calculated directly from (4*a*), using 14,700 for f_s, giving $M = 14{,}700 \times 0.015 \times 0.875 \times 12 \times 18^2 = 750{,}000$ in-lbs.

If the steel stress determines the safe moment then this moment is given directly by eq. (4*a*).

85. *Determination of Amount of Reinforcement for a Given Bending Moment.*—This is the usual problem of design, as stated in Art. 81. The size of beam has already been determined from other considerations, and it now remains to calculate the necessary tensile and compressive reinforcement for given working stresses f_s and f_c. Two general methods of doing this will be described.

First Method.—Use Diagrams 9, 10, and 11. Assume an approximate value of j of about 0.87. Calculate the required amount of tensile steel by eq. (4 *a*), $p = \dfrac{M}{f_s j b d^2}$. Then from the known relation $f_s/n f_c$, and the calculated value of p, determine p' from the diagram for k values. The relation $f_s/n f_c$ fixes the neutral axis or value of k, and the value of p' to produce this result is obtained from the diagram. It is also given by solving eq. (3) for p', giving

$$p' = \frac{p(1 - k) - k^2/2n}{k - d'/d}. \qquad (6)$$

Having the value of p', the value of j can be accurately determined and the calculations revised.

Example.—Given $M = 1{,}000{,}000$ in-lbs., $b = 12$ in., $d = 20$ in., $d'/d = 0.1$, $n = 15$, $f_s = 18{,}000$, $f_c = 800$; calculate p and p' or A and A'.

Solution. Assume $j = 0.87$. Then $p = \dfrac{1{,}000{,}000}{18{,}000 \times 0.87 \times 12 \times 20^2} = 0.0133$, $pn = 0.20$. The ratio $f_s/n f_c = 1.5$. Then from Diagram 10 we get $p'n = 0.13$ and $p' = 0.0087$. The corrected value of j is about 0.877, giving $p = 0.0132$ and $p' = 0.0085$. $A = 3.17$ sq. in. $A' = 2.04$ sq. in. The corrected value of A' to allow for reduced concrete area is $2.04 \times 15/14 = 2.10$.

Second Method.—In this method the beam is considered as made up of two parts, an ordinary rectangular beam with the proper amount of tension steel A_1 to make a balanced design, and a beam consisting of the compression steel and the remaining portion of the tension steel. (See Fig. 16.) The moment of resistance of the former is the same as that of an ordinary rectangular beam for the assigned working

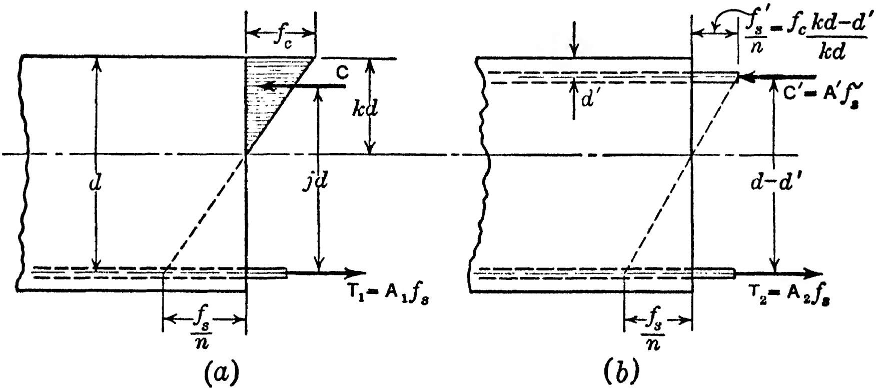


FIG. 16.

stresses and cross-section; the moment of resistance of the latter is equal to the stress in the tensile steel $A_2 f_s$ multiplied by $(d - d')$.

Let M_1 and M_2 denote these two parts of the total moment of resistance. We then have, as in eq. (3) of Art. 51,

$$M_1 = f_s A_1 j d \quad \text{or} \quad \tfrac{1}{2} f_c b k d \cdot j d, \qquad (7)$$

also

$$M_2 = f_s A_2 (d - d') \quad \text{or} \quad f_s' A' (d - d'). \qquad (8)$$

Eq. (7) is identical with eq. (3) for a rectangular beam singly reinforced (Art. 51). Also $M_1 = R b d^2$ as in eq. (10), Art. 54. Hence for this portion of the design the same formulas and diagrams can be used as in Arts. 51 and 54.

Having a beam of given size we therefore first determine M_1. Then $M_2 = M - M_1$, and we proceed to calculate the supplementary tension steel A_2 and the compression steel A' to carry the moment M_2. From eq. (8)

$$A_2 = \frac{M_2}{f_s (d - d')}. \qquad (9)$$

Then from Fig. (*b*), $A'f_s' = A_2 f_s$ and $\frac{f_s}{f_s'} = \frac{d - kd}{kd - d'}$, whence

$$A' = A_2 \frac{1 - k}{k - d'/d}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

To take account of the reduction of the concrete area by the compression steel area A', substitute $A' \frac{n - 1}{n}$ for A' in (10), giving the more exact expression

$$A' = A_2 \frac{1 - k}{k - d'/d} \cdot \frac{n}{n - 1}. \quad . \quad . \quad . \quad . \quad (11)$$

Example.—Solve the preceding example by this method. As an ordinary rectangular beam Diagram 4 gives $R = 140$ and $p = 0.009$, hence $M_1 = 140 \times 12 \times 20^2 = 672{,}000$ in-lbs., and $A_1 = 0.009 \times 240 = 2.16$ sq. in. The moment $M_2 = 1{,}000{,}000 - 672{,}000 = 328{,}000$ in-lbs. Then $A_2 = \frac{328{,}000}{18{,}000 \times 18} = 1.01$ sq. in. Total $A = 2.16 + 1.01 = 3.17$ sq. in. From Diagram 1, $k = 0.40$, hence from eq. (11), $A' = 1.01 \times 0.6/0.3 \times 15/14 = 2.16$ sq. in.

BENDING AND COMPRESSION

86. General Conditions.—In the problems heretofore analyzed, simple bending only was involved, the resultant of external forces being perpendicular to the beam axis. In many cases the forces acting are such as to produce a direct compression as well as bending moment. Common cases are the arch ring, the columns and beams in buildings subjected to lateral forces, and beams in rigid frame bridges. In arches and in building columns the compression is large and the bending relatively small, and the critical stress is the maximum stress in the concrete; compressive stresses exist over all or a large part of the section and the unit stress in the steel will be small. The reinforcement is likely to be symmetrical, and in the analysis of stresses the methods developed for symmetrical sections can be used. In beams of the other type mentioned the compression is relatively small and the bending moment large. The neutral axis will not be greatly shifted by the compression, and the concrete and the tensile steel may be stressed to their permissible values. The reinforcement is likely to be unsymmetrical, and for these cases the method of Art. 96 is well adapted.

Rectangular Sections

87. Notation.—In the analysis of beams under combined stresses, especially arches, it is convenient to consider the external forces on one side of the section combined into a resultant R (Fig. 17) whose

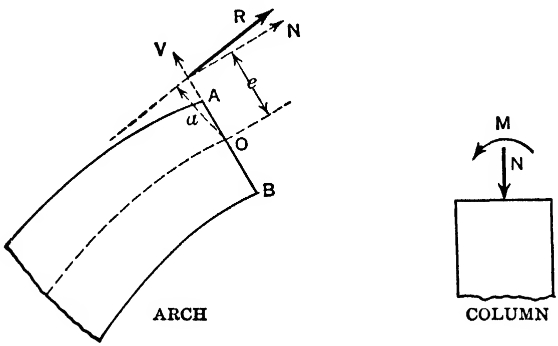


FIG. 17.

direction and line of action are known. If O is the centroid of the section AB, the distance e from O to the point where the resultant R cuts the section is the eccentric distance. Resolving R in directions

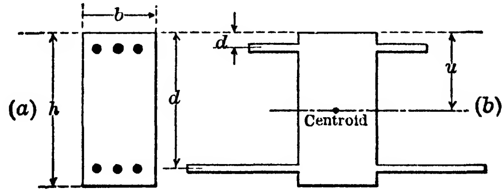


FIG. 18.

normal and parallel to the section gives the component N, which is the direct stress or thrust, and V, which is the shear acting on the section. The bending moment, M, is equal to $R\,a$ or $N\,e$. In the case of columns the direct compression N and usually the bending moment M are known. The eccentricity is then $e = M/N$.

In addition to the notation already adopted, the following will be used (Figs. 18 and 19). For convenience, the face of the beam most highly stressed is called the "compression face" and the opposite face is called the "tension face"; it may be stressed either in tension or compression.

R = resultant force acting on the section;
N = component of R normal to section;
e = eccentric distance of R, e/h = eccentricity;
M = bending moment = Ne;
A' = area of steel near compressive face;
$p' = A'/b\,h$;
A = area of steel near tension face;
$p = A/b\,h$;
d' = distance of compressive steel from face;
u = distance from compressive face to centroid of transformed section;
h = whole height of section;
a = distance from steel to centre of section for symmetrical reinforcement;
A_t = area of transformed section;
I_c = moment of inertia of concrete about centroidal axis of transformed section;
I_s = moment of inertia of steel about centroidal axis of transformed section;
I_t = moment of inertia of transformed section;
f_c = maximum compressive fibre stress in concrete;
f_c' = maximum tensile fibre stress in concrete;
f_s' = stress in steel near compressive face;
f_s = stress in steel near tension face.

88. The Transformed Section.—In the analysis of this problem the use of the transformed section is in some cases very convenient. Where the stresses are wholly compressive or where the tensile stresses are so small as to be permissible such a section can be used for purposes of stress analysis in the same manner as for a beam of homogeneous material.

Thus, if Fig. 18 (*a*) represents an actual section, 18 (*b*) represents the transformed section, the areas of the upper and lower wings being respectively *n* times the areas of the upper and lower reinforcements.

Referring to Fig. 18 it will readily be seen that

$$A_t = bh + n(A + A'), \quad I_t = I_c + nI_s, \quad \ldots \quad (1)$$

$$u = \frac{h/2 + npd + np'd'}{1 + np + np'}, \quad \ldots \quad (2)$$

$$I_c = \tfrac{1}{3}b[u^3 + (h-u)^3] \text{ and } I_s = A(d-u)^2 + A'(u-d')^2. \quad (3a)$$

If the reinforcement is symmetrical, then $u = h/2$ and

$$I_c = 1/12\,bh^3 \text{ and } I_s = 2A(\tfrac{1}{2}h - d')^2. \quad \ldots \quad (3b)$$

To be strictly correct, the compression steel should be multiplied by $\frac{n-1}{n}$; and where there is compression across the entire section, the equations given here can be corrected by using $n - 1$ in place of n. Where one side is in tension and the tension in the concrete is neglected then the tension steel displaces no useful concrete and n is the proper quantity to use for the tension side. For simplicity in the analysis, especially where the same amount of steel is used on each side, the quantity n is used in all the formulas. The resulting error is of no importance.

89. Cases to be Considered.—If the eccentric distance e is within certain limits, then the stress on the section is wholly compressive (Figs. 19 and 20), but if it exceeds this limit, there will be tensile stress on the section (Fig. 21). If it be assumed that the concrete takes no tension then the analysis for these two cases is quite different.

Whether a given problem will fall under case (1) or (2) depends on the eccentricity, the relative amounts of steel and concrete at the section, and on n.

In practice the problem will usually be to find the maximum fibre stress f_c for a given design and a given M. In the arch, for example, a design is selected by the use of empirical formulas or by comparison with previous designs, the moments and thrusts (M and N), determined, and from these the stresses in the concrete. In the column, with N and M known, the total fibre stress is determined and additional reinforcement used if necessary. In this analysis, therefore, we are principally concerned with the development of formulas from which the fibre stress can be determined for a given beam and given values of M and N.

90. Case I. The Fibre Stress is Wholly Compressive.—There are two methods of treatment.

(*a*) The unit fibre stress in the concrete can be computed just as though the beam were homogeneous, but the transformed section

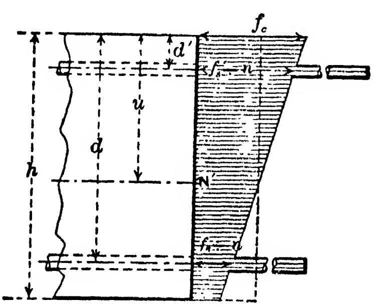


FIG. 19.

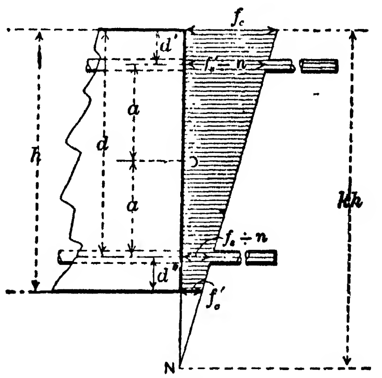


FIG. 20.

must be used. The unit stresses in the steel will be n times those in the concrete in the planes of the reinforcements respectively. Thus, the unit direct stress in the concrete is N/A_t; the unit flexural stress in the concrete on the compression side is Mu/I_t; that in the concrete adjoining the reinforcement on the same side is $M(u - d')/I_t$; and that in the concrete adjoining the other reinforcement is $M(d - u) I_t$. The combined unit stresses are:

$$f_c = \frac{N}{A_t} + \frac{Mu}{I_t}, \quad \ldots \ldots \ldots \quad (4)$$

$$f_s' = n\frac{N}{A_t} + \frac{nM(u - d')}{I_t}, \quad \ldots \ldots \quad (5)$$

$$f_s = n\frac{N}{A_t} - \frac{nM(d - u)}{I_t}. \quad \ldots \ldots \quad (6)$$

These equations—and the stress diagram, Fig. 19—show that f_s is always less than f_s', and f_s' is always less than $n f_c$; hence the unit stresses in both steel reinforcements will always be safe if f_c is a safe value.

(*b*) The method employed for simple flexure, suitably modified, leads to formulas not involving the transformed section, as will now be explained.

From the stress diagram (Fig. 20) it will be seen that

$$f_s' = n f_c (1 - d'/k h), \quad \ldots \quad (7)$$

$$f_s = n f_c (1 - d/k h), \quad \ldots \quad (8)$$

and

$$f_c' = f_c (1 - 1/k). \quad \ldots \quad (9)$$

From the condition that the resultant fibre stress equals N,

$$\tfrac{1}{2} (f_c + f_c') \, b h + f_s' A' + f_s A = N; \quad \ldots \quad (10)$$

and from the condition that the moment of the total fibre stress about the centroidal axis equals M,

$$\tfrac{1}{2}(f_c + f_c') \, b h \frac{h}{6(2k - 1)} + f_s' A' \left(\frac{h}{2} - d'\right) - f_s A \left(\frac{h}{2} - d'\right) = M = N e. \quad (11)$$

From these equations it is possible to compute the unit fibre stresses f_c, f_s, and f_s' in a given case.

The foregoing formulas are general and apply to unsymmetrical as well as to symmetrical sections.

91. *Symmetrical Reinforcement.*—For symmetrical reinforcement the equations simplify greatly, giving, from (10) and (11),

$$k = \frac{1 + 24 n p \left(\frac{a}{h}\right)^2 + 6 (1 + 2 n p) \frac{e}{h}}{12 (1 + 2 n p) \frac{e}{h}} \quad \ldots \quad (12)$$

and by substituting values of A_t and I_t in (4)

$$f_c = C \frac{N}{b h} \quad \ldots \quad (13)$$

in which

$$C = \frac{1}{1 + 2 n p} + \frac{e}{h} \cdot \frac{6}{1 + 24 n p \left(\frac{a}{h}\right)^2}. \quad \ldots \quad (14)$$

The value of the eccentricity e/h for which f_c' is zero is found by placing $k = 1$ in eq. (12). Solving for e/h we get

$$e/h = 1/6 \cdot \frac{1 + 24 n p \left(\frac{a}{h}\right)^2}{1 + 2 n p}. \quad (15)$$

For values of e/h less than this, there is compression on the entire section; for values greater there is some tension.

92. Case II. There is Some Tension at the Section.—(*a*) If the tension in the concrete is so small as to be permissible, and this tension is taken account of in the computations, then the unit fibre stresses in the concrete and steel, if reinforcement is present, may be computed by the method and formulas explained under Case I.* The value of k will be less than unity, f_c will be negative or tension, and f_s' may or may not be negative.

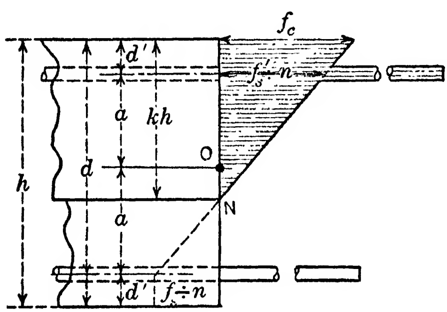


FIG. 21.

(*b*) Generally the tension in the concrete is wholly neglected, in which case a method of analysis similar to that used for simple bending is best used rather than one based upon the transformed section.

In Fig. 21, O denotes a horizontal axis at mid-depth of the beam, M the moment sum of all the external forces on one side of the section with respect to that axis, and N, as before, the algebraic sum of the components of those forces perpendicular to the section. From the stress diagram, it follows that

$$f_s = n f_c \left(\frac{d}{k h} - 1\right) \quad (16)$$

and

$$f_s' = n f_c \left(1 - \frac{d'}{k h}\right) \quad (17)$$

* It is assumed that the linear law of variations of the unit flexural stresses holds for the tension as well as compression.

Since the resultant fibre stress equals N,

$$\tfrac{1}{2} f_c\, b\, k\, h + f_s'\, A' - f_s\, A = N,$$

and since the moment of the fibre stress about the horizontal axis through O equals M,

$$\tfrac{1}{2} f_c\, b\, k\, h \left(\frac{h}{2} - \frac{k\,h}{3}\right) + f_s'\, A' \left(\frac{h}{2} - d'\right) + f_s\, A \left(d - \frac{h}{2}\right) = M.$$

From these four equations k, f_c, f_s, and f_s' can be determined for a given section, reinforcement, M, and N.

93. *Symmetrical Reinforcement.*—In this case the equations simplify. The value of k is given by

$$k^3 - 3\left(\tfrac{1}{2} - \frac{e}{h}\right) k^2 + 12\, n\, p \frac{e}{h} k = 6\, n\, p \left(\frac{e}{h} + 2\frac{a^2}{h^2}\right). \quad . \quad (18)$$

The greatest unit compressive fibre stress is

$$f_c = C \frac{M}{b\, h^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

in which

$$C = \frac{12\, k}{3\, k^2 - 2\, k^3 + 24\, p\, n\, (a/h)^2}. \quad . \quad . \quad . \quad . \quad (20)$$

The steel stresses are given by (16) and (17).

94. Diagrams for Rectangular Sections with Symmetrical Reinforcement.—Diagrams 12 to 22 have been prepared to solve problems pertaining to rectangular sections, symmetrically reinforced. To provide for various values of n, the quantity $p\, n$ is used as an argument instead of p. Four ratios of embedment d'/h have been assumed, namely, 0.05, 0.10, 0.15, and 0.20. Diagram 12 gives the values of e/h from eq. (15), denoting the limit between Cases I and II. For values of e/h less than shown by the curves the case is Case I, no tension on the section; for values greater, the case is Case II, some tension on the section. Diagrams 13 to 16 give the values of C and f_c'/f_c from eqs. (12) and (9) for Case I, and Diagrams 18 to 21 the values of C and k from eqs. (20) and (18) for Case II. Diagrams 17 and 22 give the steel stresses from eqs. (7), (8), (16), and (17). These are not often needed.

To take account of the fact that the steel replaces an equal area of concrete, proper correction can be made for Case I, by using the value $(n - 1)p$ instead of $n\,p$ in reading the diagram. Case II cannot be corrected in this way, as the tension steel replaces no useful concrete. In either case the error involved by using $n\,p$ is small (2% or 3%) compared to other uncertainties in such problems.

95. Examples.—(1) A reinforced-concrete beam of dimensions shown in Fig. 22 is subjected to a direct stress of 100,000 lbs. and a bending moment of 200,000 in-lbs. Find the stresses in concrete and steel. $p = 1\%$. $d'/h = 0.10$. Take $n = 15$.

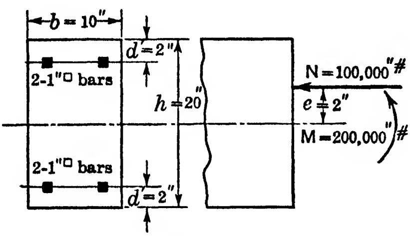


FIG. 22.

Solution. Eccentric distance $= e = 200{,}000/100{,}000 = 2$ in. $e/h = 0.10$. From Diagram 12, we find for $p\,n = 0.15$ and $d'/h = 0.10$, $e/h = 0.20$, which is greater than the given value. Hence Case I governs.

Then from Diagram 14, $C = 1.15$ and $f_c = 1.15 \times \frac{100{,}000}{10 \times 20} = 572$ lbs/in.^{2} Also $f_c' = 0.34 f_c = 195$ lbs/in.^{2} From Diagram 17, $f_s' = 15 \times 575 \times 0.93 = 8000$ lbs/in.^{2}, and $f_s = 15 f_c \times 0.4 = 3450$ lbs/in.^{2}

If a more precise solution is desired, use 14 for n in place of 15, giving slightly greater stresses.

(2) Same beam as in (1) but direct stress = 50,000 lbs. and bending moment = 400,000 in/lbs. Here the value of $e = 8$ in. and $e/h = 0.40$. From Diagram 12, the case is seen to be Case II. Then from Diagram 19, for $h/e = 2.5$, and $n\,p = 0.15$, $C = 6.10$ and $k = 0.67$. Then

$$f_c = 6.10 \times \frac{400{,}000}{10 \times 20^2} = 610 \text{ lbs/in.}^2$$

$$f_s = 15 f_c \left[\frac{0.9}{0.67} - 1\right] = 3150 \text{ lbs./in.}^2$$

(3) A common problem is to determine the amount of reinforcement required to provide for certain bending moments in a column after a preliminary design has been made based on direct stress alone. Suppose that a column has been designed according to the American Concrete Institute specifications for columns with lateral ties for a load of 175,000 lbs. using a 3000-lb. concrete. The column is 15 in. square and is reinforced by four 1-in. square bars, embedded 3 in. $n = 10$, $p = 2/225 = 0.0089$. The allowable load for this column is $P = 675 \times 225\,(1 + 9 \times 0.0178) = 176{,}000$ lbs.

Suppose now the bending moment is 350,000 in-lbs. Then $e = 2$ in., $e/h = 2/15 = 0.133$, and for use in the diagrams $p\,n = (10 - 1) \times 0.0089$

= 0.08, $d'/h = 3/15 = 0.20$. Case I governs. Then from Diagram 16, $C = 1.54$ and $f_c = 1.54 \times 175{,}000/225 = 1200$ lbs/in.² The specifications allow for combined compression and bending 900 lbs/in.² Hence the reinforcement must be increased or a larger column used. The required reinforcement can be determined by writing $900 = C \times 175{,}000/225$, whence $C = 1.16$. Then from the diagram, $np = 0.26$ and $p = 0.26/9 = 0.029$, and the total column reinforcement will be $2 \times 0.029 = 0.058 = 5.8\%$. This is greater than the specifications allow (max. = 4%), hence a larger column will be required. It will be found that a column 16 in. square and with the prescribed stress of 900 lbs/in.² will require a total of 3.8% reinforcement, and a 17-in. column, 2.2%. For further discussion of the design of column, see Chap. VII.

96. Unsymmetrical Reinforcement; Reinforcement on One or Both Sides.—(*a*) *Reinforcement on Tension Side Only.*—Referring to

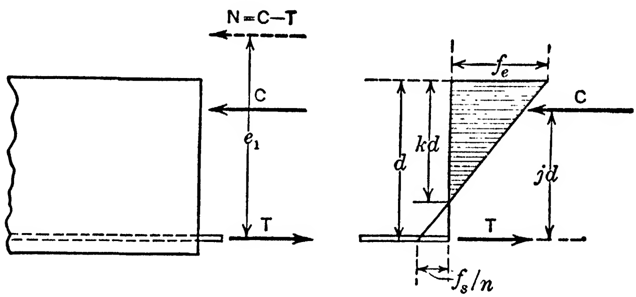


FIG. 23.

Fig. 23 (*a*) and (*b*) the direct compression or thrust is N. For convenience we consider the moment about the tension steel instead of the central axis. This is $M_1 = Ne_1$. Area tension steel = A, $p = A/bd$.

The following analysis of the relation between f_c, f_s, and M_1 will be found useful:

$$M_1 = Ne_1 = \tfrac{1}{2}f_c kj\,bd^2 \quad \ldots\ldots\ldots \quad (1)$$

$$N = \tfrac{1}{2}f_c k\,bd - f_s\,p\,bd. \quad \ldots\ldots\ldots \quad (2)$$

From (1) and (2) we get

$$\frac{d}{e_1} = \frac{k - 2\,pf_s/f_c}{kj} \quad \ldots\ldots\ldots\ldots \quad (3)$$

Substituting the general value of $k = \dfrac{1}{1 + f_s/nf_c}$ in (3)

we derive

$$\frac{d}{e_1} = \frac{\dfrac{f_s}{nf_c} + 1 - 2pn\dfrac{f_s}{nf_c}\left(\dfrac{f_s}{nf_c} + 1\right)^2}{\dfrac{2}{3} + \dfrac{f_s}{nf_c}} \quad . \quad . \quad . \quad . \quad (4)$$

Eq. (4) expresses the relation between d/e_1, f_s/nf_c and pn. This relation is plotted in Diagram 23. Knowing d/e_1, the curves for pn give directly the amount of steel corresponding to any particular value of f_s/nf_c, and conversely.

From Eq. (1)

$$f_c = \frac{M_1}{bd^2} \cdot \frac{2}{kj} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

or in terms of f_s/nf_c

$$f_c = C_c \frac{M_1}{bd^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

in which

$$C_c = \frac{2}{kj} = \frac{6\,(f_s/nf_c + 1)^2}{3\,f_s/nf_c + 2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

We can also write in terms of f_s

$$\frac{f_s}{n} = C_s \frac{M_1}{bd^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

in which

$$C_s = \frac{f_s}{nf_c} C_c = \frac{6\dfrac{f_s}{nf_c}\left(\dfrac{f_s}{nf_c} + 1\right)^2}{3\dfrac{f_s}{nf_c} + 2} \quad . \quad . \quad . \quad (9)$$

Values of C_c and C_s are plotted in Diagram 23 for various values of f_s/nf_c. The use of this diagram will be explained in the examples following. For convenience of reference values of k and j are also plotted in terms of f_s/nf_c.

The problems to be considered are usually similar to the calculation of the compressive reinforcement in a T-beam at points of negative moment; the general dimensions of the beam have already been determined by other conditions and the problem is to calculate the amount of steel needed.

PROBLEM OF DESIGN. In all the problems excepting (6), the following data will be used. Limiting stress values $f_c = 800$ lbs/in.^{2}; $f_s =$ 18,000 lbs/in.^{2}; $n = 15$; $b = 10$ in.; $d = 20$ in.; $e_1 = 30$ in.; $d/e_1 = 0.67$.

It will be instructive first to find the value of N or M_1 which will correspond to the assigned working stresses; that is, which makes a balanced design. The value of $f_s/n f_c = 1.5$, and from Diagram 23 the corresponding value of C_c is 5.8, and $M_1/b\,d^2 = 800/5.8 = 138$ (the same as R_c for simple bending; Art. 54), and $M_1 = 138 \times 4000 =$ 552,000 in-lbs. $N = M_1/e_1 = 18{,}400$ lbs. For smaller values of M_1 the concrete will be understressed, and for higher values the steel will be understressed, or else some compressive steel must be used as explained later. Tension steel only will be here considered.

Examples.—(1) Assume $N = 15{,}000$ lbs.; required, A and f_c. $M_1 =$ $15{,}000 \times 30 = 450{,}000$ in-lbs., and $\frac{M_1}{bd^2} = 112.5$. From eq. (8), $C_s =$ $\frac{18{,}000}{15 \times 112.5} = 10.67$. From Diagram 23, referring to the curve for C_s, we find $f_s/nf_c = 1.73$; and for $d/e_1 = 0.67$, $pn = 0.047$. Then $A = \frac{0.047 \times 200}{15}$ $= 0.63$ sq. in., $f_c = \frac{18{,}000}{15 \times 1.73} = 694$ lbs/in.^{2}

(2) Assume $N = 25{,}000$ lbs.; required, A and f_s. $M_1 = 25{,}000 \times 30 =$ 750,000 in-lbs. $M_1/bd^2 = 187.5$. From eq. (6) $C_c = 800/187.5 = 4.27$, and the diagram gives $f_s/nf_c = 0.7$ and $pn = 0.20$; $A = \frac{0.20}{15} \times 200 =$ 2.67 sq. in.; $f_s = 0.7 \times 15 \times 800 = 8400$ lbs/in.^{2}

A convenient check is to verify the relation $N = C - T$ (Fig. 23(*b*)). In example (2), $k = 0.59$ and $C = 800 \times \frac{1}{2} \times 10 \times 0.59 \times 20 = 47{,}200$ lbs. $T = 2.67 \times 8400 = 22{,}300$ lbs. $C - T = 47{,}200 - 22{,}300 =$ 24,900 lbs. compared to 25,000 lbs. assumed.

PROBLEM OF REVIEW OF GIVEN DESIGN.—This can be directly solved by the use of the diagram.

(3) Assume a tension steel section of 1.2 sq. in.; calculate safe value of N. Value of $pn = \frac{1.2 \times 15}{200} = 0.09$, and Diagram 23 gives $f_s/nf_c = 1.13$. The concrete will determine the limit. $C_c = 5.1$ and $M_1/bd^2 = 800/5.1 =$

157. $M_1 = 628{,}000$ in-lbs. $N = 20{,}900$ lbs. $f_s = 15 \times 800 \times 1.13 = 13{,}560$ lbs/in.[2] For values of pn that will give a value of f_s/nf_c greater than 1.5, the steel will determine and C_s should be used to calculate M_1.

(*b*) *Reinforcement on Both Sides.* PROBLEM OF DESIGN.—It will be assumed that compression reinforcement is to be used sufficient to enable the tension steel to be stressed at some designated safe value. In this case both f_c and f_s are known and the neutral axis is determined at once. The necessary compression steel can best be determined by a process similar to that explained in Art. 85, second method.

(4) Same as Example 2 but compression steel to be used. $N = 25{,}000$ lbs. $M_1 = 750{,}000$ in-lbs. $f_s/nf_c = \dfrac{18{,}000}{15 \times 800} = 1.5$; $k = 0.4$, $j = 0.867$; $kd = 8$ in., $jd = 17.33$ in.

With no compression steel, $M_1/bd^2 = 138$, and $M_1 = 552{,}000$ in-lbs. as calculated previously. For this moment the value of $pn = 0.057$ and steel area $= A_1 = \dfrac{0.057}{15} \times 200 = 0.76$ sq. in. The moment yet to be provided for $= 750{,}000 - 552{,}000 = 198{,}000$ in-lbs., and $N = 6600$ lbs. This is to be resisted by the compression steel and additional tension steel, the arm being 18 in. Total stress in compression steel $= 198{,}000/18 = 11{,}000$ lbs. The unit stress will be $6/8 \times 800 \times 15 = 9000$ lbs/in.[2] Hence $A' = 11{,}000/9000 \times 15/14 = 1.22 \times 15/14 = 1.30$ sq. in. Taking moments about the compression steel we have $6600 \times (30 - 18) = T_2 \times 18$. Whence $T_2 = 4400$ lbs. and $A_2 = \dfrac{4400}{18{,}000} = 0.244$ sq. in. Total $A = 0.76 + 0.244 = 1.004$ sq. in. As a check: $C = 800 \times \frac{1}{2} \times 8 \times 10 = 32{,}000$; $C' = 9000 \times 1.2 = 10{,}900$; $C + C' = 42{,}900$. $T_1 + T_2 = 18{,}000 \times 1.00 = 18{,}000$. $C + C' - (T_1 + T_2) = 24{,}900$ lbs. $= N$.

PROBLEM OF REVIEW OF GIVEN DESIGN.—The direct solution of this problem is cumbersome and requires the use of rather complicated diagrams. The simplest method is solution by trial, only two or three successive approximations being necessary. No diagrams are needed. In this method the process is:

1. Determine by trial calculation the value of N for any selected value of f_c (or f_s) and get the corresponding value of f_s (or f_c).

2. The stresses under the actual value of N, or the safe value of N, can then be determined by proportion, since with a fixed value of e_1, all stresses are proportional to N.

The process can best be explained by the solution of problems. For notation refer to Fig. 23*a*.

(5) Same dimensions as in previous examples. Let $A = 1.5$ sq. in. $A' = 1.0$ sq. in. Find safe value of N. Assume $f_c = 800$ lbs/in.², and for first trial take $f_s = 18{,}000$ lbs/in.²

Then from Diagram 23, $k = 0.4$; $kd = 8.0$ in., $j = 0.867$; $jd = 17.33$ in. $f_s' = 9000$ lbs/in.²

$C = 800 \times \frac{1}{2} \times 10 \times 8 = 32{,}000$ lbs., $C' = 9000$ lbs.; moment of compressive stresses about tension steel $= 32{,}000 \times 17.33 + 9000 \times 18 = 716{,}000$ in-lbs. and $N = 716{,}000/30 = 23{,}900$ lbs. Hence $T = 32{,}000 +$

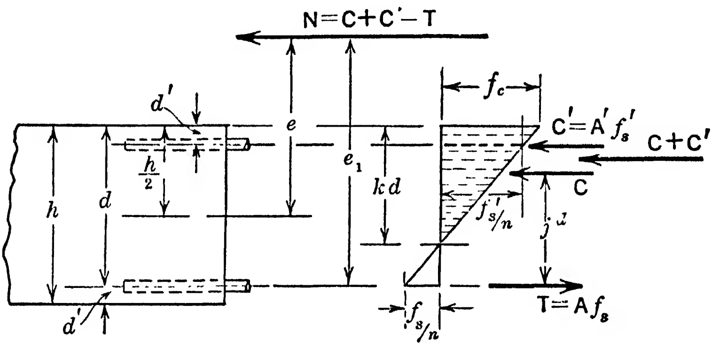


FIG. 23a.

$9000 - 23{,}900 = 17{,}100$ lbs., giving a unit stress of $17{,}100/1.5 = 11{,}400$ lbs/in.², compared to 18,000 lbs/in.² assumed. The correct value of f_s will be between 18,000 and 11,400.

Try $f_s = 14{,}000$ lbs/in.² Then $f_s/nf_c = \dfrac{14{,}000}{15 \times 800} = 1.17$; $k = 0.462$.

$kd = 9.24$; $j = 0.846$, $jd = 16.92$; $f_s' = \dfrac{7.24}{9.24} \times 12{,}000 = 9400$ lbs/in.²
$C = 800 \times \frac{1}{2} \times 10 \times 9.24 = 37{,}000$ lbs.; $C' = 9400$ lbs., $M_1 = 37{,}000 \times 16.92 + 9400 \times 18 = 795{,}000$ in-lbs. $N = 795{,}000/30 = 26{,}500$ lbs. $T = 37{,}000 + 9400 - 26{,}500 = 19{,}900$ lbs. $f_s = 19{,}900/1.5 = 13{,}300$ lbs/in.²

A third trial with $f_s = 13{,}500$ results in a calculated value of 13,500 lbs/in.², and $N = 26{,}800$ lbs., which is the safe value required.

If the amount of tension steel had been only 0.7 sq. in., the same process leads to a value of f_s, corresponding to $f_c = 800$, of 21,800 lbs/in.², and a value of $N = 21{,}800$ lbs. Safe value of $N = 21{,}800 \times 18{,}000/21{,}800 = 18{,}000$ lbs., and $f_c = 800 \times 18/21.8 = 660$ lbs/in.²

(6) Suppose the beam to be an arch ring with $b = 12$ in.; total depth $h = 22$ in. $A = 1.0$ sq. in. $A' = 1.5$ sq. in. Thrust $= N = 50{,}000$ lbs., and eccentricity with respect to central axis $= 11$ in.; depth of embedment $= 2$ in. for each side. Then for this calculation $d = 20$ in.; $e_1 = 11 + 9 = 20$ in.

As the eccentricity is only 11 in., the neutral axis will be relatively low and f_s will be low.

Assume $f_c = 800$, $f_s = 9000$. Then $f_s/nf_c = 0.75$, $k = 0.571$, $j = 0.81$,

$kd = 11.42$ in., $jd = 16.28$ in., $f_s' = 12{,}000 \times \frac{9.42}{11.42} = 9900$ lb/in.^{2}

$$
\begin{aligned}
C &= 400 \times 12 \times 11.42 &&= 55{,}000 \text{ lbs.}\\
C' &= 9900 \times 1.5 &&= 14{,}850 \text{ ''}\\
&\text{Total Compression} &&= 69{,}850 \text{ lbs.}\\
\text{Moment of } C &= 55{,}000 \times 16.2 &&= 891{,}000 \text{ in.-lbs.}\\
\text{'' '' } C' &= 14{,}850 \times 18 &&= 267{,}000 \text{ ''}\\
&\text{Total Moment} &&= 1{,}158{,}000 \text{ in.-lbs.}\\
N &= 1{,}158{,}000/20 &&= 57{,}900 \text{ lbs.}\\
T &= 69{,}850 - 57{,}900 &&= 11{,}950 \text{ lbs.}\\
& & f_s &= 11{,}950 \text{ lbs/in.}^2
\end{aligned}
$$

A second trial with $f_s = 10{,}500$ lbs/in.^{2} results in a calculated value of 10,500 lbs/in.^{2} as assumed, and $N = 55{,}300$ lbs. The actual stresses will then be $f_c = 800 \times 50/55.3 = 723$ lbs/in.^{2} and $f_s = 10{,}500 \times 50/55.3 = 9500$ lbs/in.^{2}

Circular Sections

97. General Conditions.—Where spirally reinforced columns are used the section considered is a circular section with reinforcing bars spaced uniformly around the periphery of the spiral or circular reinforcement. The concrete outside the spiral is not considered in calculating stresses.

The analysis of bending stresses in a circular section can be carried out in the same manner as for rectangular sections, but on account of the fact that the amount of steel in compression or tension for Case II depends upon the location of the neutral axis the resulting equations for this case become rather complex and cannot be solved directly for the unknown. However, a value of k can be assumed, and the values of the coefficients determined for various values of $n\,p$; then another value of k, etc. The results can then be plotted, giving diagrams similar to those for rectangular sections. In the analysis the steel reinforcement is considered as a thin shell of uniform thickness and of a diameter equal to that of the column. The embedment ratio d'/h is therefore zero in all cases.

Additional Notation:

r = radius of section, assumed to be equal to the distance of reinforcing steel from centre;

p = ratio of *total* steel to *total* cross-section.

98. Case I. Compression Over Entire Section.—Fig. 24. The transformed section is used. Neglecting, as before, the reduction of concrete area by the steel, we have

$$A_t = \pi r^2 (1 + n p)$$

$$I_t = \frac{\pi r^4}{4} (1 + 2 n p).$$

From the general equation (4) we have

$$f_c = \frac{N}{A_t} + \frac{M r}{I_t}$$

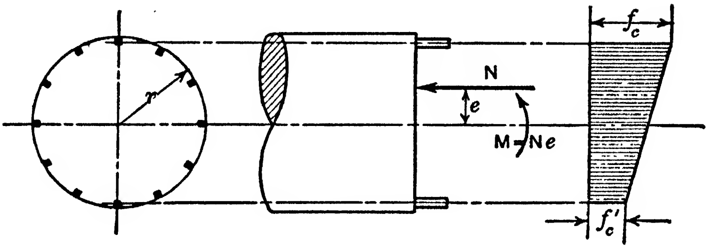


FIG. 24.

$$f_c = \frac{N}{\pi r^2}\left[\frac{1}{1 + n p} + \frac{e}{r} \cdot \frac{4}{1 + 2 n p}\right] \quad . \quad . \quad . \quad . \quad (27)$$

$$f_c = C \frac{N}{\pi r^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

The concrete stress on the opposite side is

$$f_c' = \frac{N}{\pi r^2}\left[\frac{1}{1 + n p} - e/r \frac{4}{1 + 2 n p}\right]. \quad . \quad . \quad . \quad (29)$$

Then

$$f_s' = n f_c; \quad f_s = n f_c'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

For the limiting case, $f_c' = 0$, and from (29) we have

$$e/r = \tfrac{1}{4} \cdot \frac{1 + 2 n p}{1 + n p}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

99. Case II. There is Some Tension on the Section.—The fundamental equations will be given but their detailed derivations will be omitted. Fig. 25 gives sufficient information to indicate the method of integration.

Total normal stress on the section due to both concrete and steel is $N = N_c + N_s$. By integrating over the area above the neutral axis, we have

$$N_c = \frac{f_c r^2}{k}\left\{(2\,k - 1)[\tfrac{1}{2}(\pi - \theta_L) + \tfrac{1}{2}(2\,k - 1)\sin\theta_L] + \tfrac{1}{3}\sin^3\theta_L\right\} \quad (32)$$

in which $\theta_L = \cos^{-1}(2\,k - 1)$. We may then write

$$N_c = f_c\,\pi\,r^2\,K_1 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (33)$$

in which K_1 is a function of k.

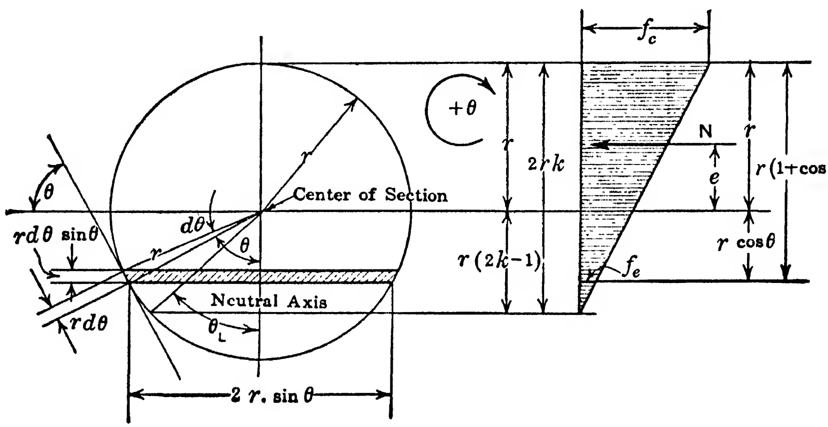


FIG. 25.

Also

$$N_s = n\,p\,\pi\,r^2 f_c\,(1 - \tfrac{1}{2}k) \; . \; . \; . \; . \; . \; . \; . \quad (34)$$

Hence total normal stress

$$N = f_c\,\pi r^2\left[K_1 + n\,p\left(1 - \frac{1}{2\,k}\right)\right]. \quad . \; . \quad (35)$$

The total moment of resistance is

$$M = N\,e = M_c + M_s. \quad . \; . \; . \; . \; . \; . \; . \; . \quad (36)$$

From Fig. 25 we get by integration

$$M_c = f_c\,\pi\,r^3\,K_2, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (37)$$

in which

$$K_2 = \frac{1}{\pi\,k}\,[\tfrac{1}{8}(\pi - \theta_L) + 1/16\cos 2\theta_L \sin 2\theta_L + \tfrac{1}{3}(2k - 1)\sin^3\theta_L]. \quad (38)$$

Also

$$M_s = \frac{1}{4k} n p \pi r^3 f_c, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

and finally

$$M = N e = f_c \pi r^3 \left(K_2 + \frac{n p}{4 k}\right) \quad . \quad . \quad . \quad . \quad . \quad (40)$$

or

$$f_c = \frac{M}{\pi r^3} \times \frac{1}{K_2 + \frac{n p}{4 k}} = C \frac{M}{\pi r^3}. \quad . \quad . \quad . \quad . \quad (41)$$

The tensile steel stress is as usual

$$f_s = n f_c \left(\frac{1}{k} - 1\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

We also get, from eqs. (35) and (40), the reciprocal of the eccentric ratio

$$r/e = \frac{K_1 + n p \left(1 - \frac{1}{2 k}\right)}{K_2 + \frac{n p}{4 k}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

In the process of calculation, we can assume a value of k and calculate the corresponding values of K_1 and K_2. Then for various values of $n p$ calculate the values of r/e from (43) and f_c from (41). Assume another value of k and repeat the process, building up a table from which a diagram can be plotted.

100. Diagrams for Circular Sections.—Diagram 12 gives limiting values of e/r for the two cases; Diagram 24 values of the coefficient C in $f_c = C \frac{N}{\pi r^2}$, eq. (28), also the ratio of f_c' to f_c; and Diagram 25 the coefficient C in $f_c = C \frac{M}{\pi r^3}$, eq. (41), and the values of k. The use of these diagrams is precisely the same as those for rectangular sections.

TESTS ON BENDING STRENGTH OF BEAMS

101. Methods of Failure of a Reinforced-concrete Beam.—A reinforced-concrete beam tested to destruction will usually fail in one of three ways:

(*a*) By the yielding of the steel at or near the section of maximum bending moment.

(*b*) By the crushing of the concrete at the same place.

(*c*) By a diagonal tension failure of the concrete at a place where the shear is large.

Methods (*a*) and (*b*) may be called "moment" failures. Method (*c*) is sometimes called a shear failure, but this term is somewhat misleading, as the concrete in such cases does not fail by shearing, but by diagonal tension.

(*a*) As a beam is progressively loaded and the steel has reached its yield point, any further load will rapidly increase the deformation. The effect of this is to open up large cracks in the tension side and to raise the neutral axis. This causes a rapid increase in the compressive stress in the concrete and ultimate failure soon occurs by the concrete crushing. Such yielding may also result in final failure by diagonal tension if large shear exists near the place of maximum moment. In either case the primary cause of failure is the yielding of the steel and such failure may properly be called a tension failure. Very rarely can the steel be actually broken in a test. The additional load carried after the yield point is reached depends on the excess strength of the concrete, position of loads and other causes, but it is usually not large and cannot be safely considered. The yield point of the steel may therefore be considered its ultimate strength for reinforcing purposes.

(*b*) If the beam is relatively long and the amount of steel is sufficient so that the crushing strength of the concrete is reached before the yield point of the steel, a failure by crushing may result. In this case tension cracks may appear, but will not become large.

(*c*) Diagonal tension failures are likely to occur whenever large shearing-stresses exist together with considerable horizontal or moment stresses, and when no special provision is made for such conditions. This kind of failure is fully discussed in Chap. IV.

Fig. 26 illustrates roughly the appearance of a beam failing in the different ways.

Final failure often results from stresses which are developed after initial failure has occurred, and while the cause of final failure is important from the standpoint of ultimate strength, yet of more importance in design is the initial failure and its cause. Other conditions besides those already mentioned may influence final failure so as often to mislead the observer as to the cause of the initial failure.

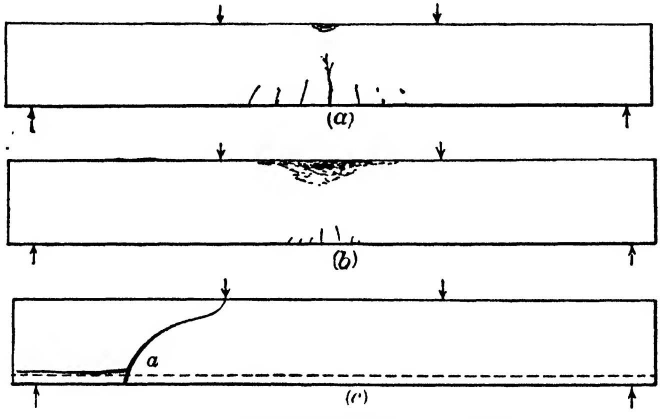


FIG. 26.—Methods of Failure of Beams.

102. *Minor Causes of Failure.*—Slipping of the bars may cause failure, but under usual conditions it will not occur; and as it can readily and economically be obviated by proper construction, it need not be considered as limiting the strength of the beam. Failure by the shearing of the concrete near the support is possible where the load is very close thereto, but as the shearing strength of concrete is about one-half the crushing strength, such failures are exceedingly unlikely and need rarely be considered. The usual so-called "shear" failures are in reality diagonal-tension failures.

103. Action of Beams under Progressive Loading.—When a reinforced-concrete beam is tested to failure the relation of stress and deflection to applied load changes greatly during the progress of the test, for two reasons: (1) the gradual cracking of the concrete on the tension side, and (2) the change in the law of stress variation on the compression side. A clear understanding of the general effect of these changes is necessary to a correct interpretation of laboratory experiments and in the calculations of factors of safety and deflections.

104. *Position of Neutral Axis.*—For low loads under which the concrete has not begun to crack, the beam acts as a homogeneous concrete beam with the steel replaced by concrete at the same level and of a section $(n - 1)$ times the steel section (the transformed section). The neutral axis (for rectangular sections) will then be somewhat below the centre, and the concrete stress can be calculated by the usual beam formula $f_c = M_c/I_t$. As the load increases and the concrete begins to crack, the neutral axis gradually rises and at loads near the ultimate the tensile resistance is of such small influence that the neutral axis will be given very closely by the formulas of Art. 65 for parabolic variation of stress. Fig. 27 represents the situation in detail. A beam is assumed with $n = 9$ (initial modulus of concrete = 3,300,000) and $p = 0.54\%$. The position of the neutral axis at the beginning of the test is indicated by the line *a b* and at the end by *c d*, calculated by eq. (1), Art. 65. If the concrete had no tensile strength, the neutral axis at the beginning would be at *e* as calculated by eq. (1), Art. 50, and the change in position under progressive loading would follow a curve, *e d*. What actually happens is that the neutral axis changes according to some curve connecting *a* and *d*. The curve shown as a full line represents the actual results of a test by Bach with a concrete and steel corresponding to the values assumed above.* For larger values of *p* and of *n* the shifting of the neutral axis will be less as the final position is lower. It is also less in T-beams, as the tension area is of less relative magnitude. These results show that the actual stress in the steel at working loads is considerably less than assumed, but at ultimate loads the analysis is substantially correct.

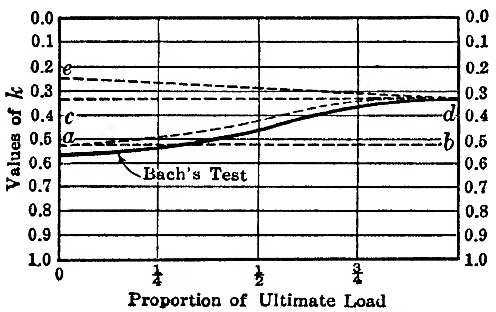
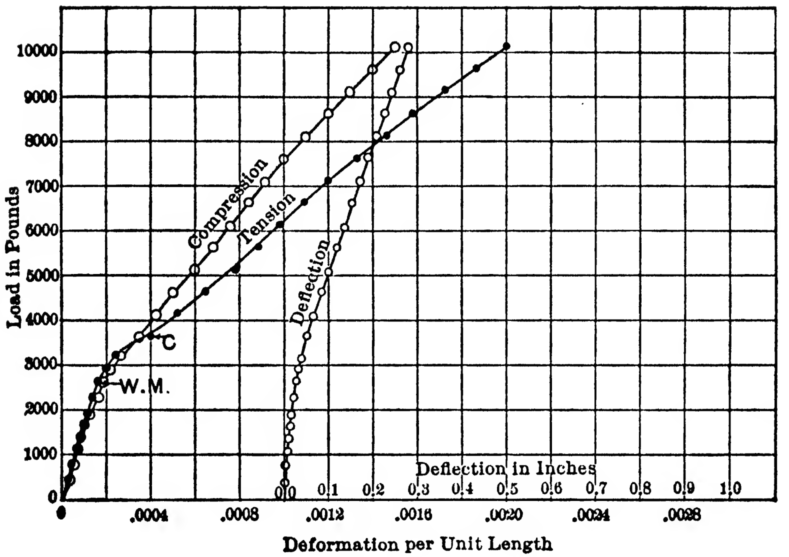


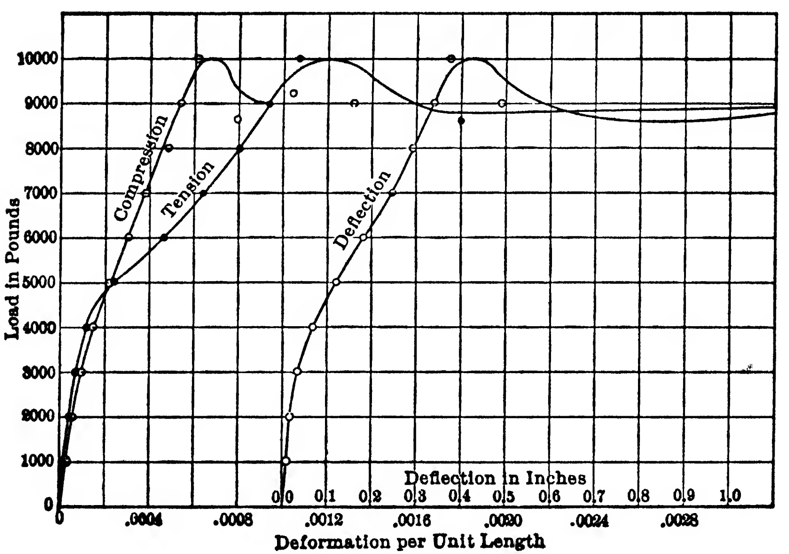
FIG. 27.—Position of Neutral Axis.

* Mit. über Forsch. a. d. Gebiet des Ing., 1907, 45–47.



10000
9000
8000
7000
6000
5000
4000
3000
2000
1000
0
Load in Pounds
Compression
Tension
Deflection
C
W.M.
Deflection in Inches
0.0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0
.0004 .0008 .0012 .0016 .0020 .0024 .0028
Deformation per Unit Length

FIG. 28.



10000
9000
8000
7000
6000
5000
4000
3000
2000
1000
0
Load in Pounds
Compression
Tension
Deflection
Deflection in Inches
0.0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0
0 .0004 .0008 .0012 .0016 .0020 .0024 .0028
Deformation per Unit Length

FIG. 29.

105. *Deflection and Deformation.*—Numerous tests of beams have been made in which extensometers have been used to measure distortions so that the deformation of the steel and of the extreme fibre of the concrete could be calculated. Results of such measurements of deformations and also of centre deflections are shown in Figs. 28 and 29 for two typical beams. In Fig. 28 the proportions were such that the failure occurred by diagonal tension; neither the steel nor the concrete was stressed to the limit of failure. During the first stage of the test, up to a load of about 2500 lbs., the deformations in both steel and concrete are proportional to the loads. Up to this point the tension deformation has not been great enough to begin to rupture the concrete, but with increasing loads and deformations the concrete begins to fail, as shown by the appearance of minute cracks shown by faint "water marks" and indicated on the diagram by the letters *W M*. The deformation at the first "water mark" in this case was about 0.00018, corresponding to a stress of 360 lbs/in.^{2}, assuming a modulus of elasticity of 2,000,000. The first visible crack appeared at the point marked *C*.

In Fig. 29 the amount of steel was small and a tension failure occurred. This is indicated by the great deformations at the end of the test. The curves in the early stages of the test are very similar, in general form, to those in Fig. 28. In the case of the beam of Fig. 28, the steel stress as determined by its deformation at a load of 3000 lbs. was less than one-third of the value which would be obtained by calculations based on the usual formula, neglecting tension in the concrete. The deflection was approximately one-third on the same basis. The error in calculated values becomes less the greater the load.

106. Relation of Ultimate Strength to Yield-point of Steel.—Numerous tests confirm the statement in Art. 101, that the strength of a beam with respect to the steel is closely measured by the yield-point strength of the steel.

Fig. 30 represents some results of tests made by the U. S. Bureau of Standards, which are typical of tests of this class. The beams were 8 by 10 in. in cross-section and 12 ft. long between supports. The concrete was in one case made of a granite aggregate and in the other case of limestone. The yield-point stress of the steel averaged about

41,000 lbs/in.² Failure occurred in all cases by over-stressing of the steel. The line drawn on the diagram is the theoretical strength calculated by eq. (3), Art. 51, using $n = 10$ and $f_s = 41{,}000$. The values of the initial moduli were about 3,800,000 and 4,400,000, respectively, Instead of using $n = 10$, somewhat greater accuracy could be had by using $n = 8$ and the parabolic variation of compressive stress. Calculated values would be slightly less than those indicated by the line. These results indicate that so far as tension failures are concerned the

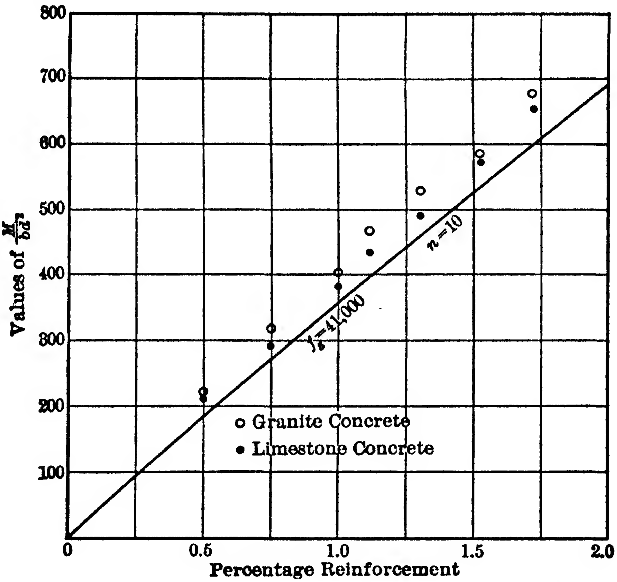


FIG. 30.—Tests of Beams giving Steel-tension Failures.

beam will develop the full yield-point strength of the steel, and also that failure will take place at loads but little greater than those corresponding to this yield-point stress. Numerous other tests show similar results.

107. Relation of Ultimate Strength to Strength of Concrete.—An important question relating to proper working stresses is whether the ultimate compressive strength of concrete in a beam is the same as determined by a direct compression test. The results of various tests indicate that when cured under the same conditions the compressive strength as determined by standard cylinder tests is a good measure of the strength of the concrete in the beam when the latter is calculated on the assumption of parabolic variation of stress. A carefully made series of tests with respect to this question is that by Slater and Lyse.* A series of 36 beams was tested, arranged in 12 groups of 3 beams each with 12 varieties of depths and qualities of concrete. For each group 3 compression cylinders 6 by 12 in. were tested. All specimens were cured in a moist chamber at 70° F. and tested at 28 days. Sufficient reinforcement was used to cause compression failures except in two cases where the failures were by diagonal tension. These values were omitted from the averages. The load was applied at two points 21 in. each side of the centre. All beams were 8 in. wide by 11 ft. long and supported 9 in. from each end. Table No. 1 gives average results of the three tests in each case, with the exception noted above. The calculated stresses in columns 6 and 8 are based upon the parabolic law.

TABLE 1

TESTS OF BEAMS GIVING COMPRESSION FAILURES

(Slater and Lyse)

Group No.	Depth, In.	Steel Ratio, p	Cylinder Strength, Lbs/in.²	Value of n from Cylinder Test	Calculated Stress in Beam Using n from Cylinder Test, Lbs/in.²	Ratio Beam to Cylinder	n From J. C. Specifications	
							Stress in Beam, Lbs/in.²	Ratios
1	2	3	4	5	6	7	8	9
1	10.2	.021	1390	11.1	2340	1.68	2190	1.58
2	10.3	.028	2790	8.6	3130	1.12	2960	1.06
3	10.3	.037	4070	7.9	4250	1.04	4120	1.01
4	10.1	.047	4800	7.1	5050	1.05	4770	.99
5	10.2	.056	5740	6.4	5680	.99	5280	.92
6	14.2	.030	2590	8.1	3670	1.42	3360	1.30
6A	14.1	.039	4130	7.8	4140	1.00	4110	1.00
7	12.2	.028	2950	6.7	3550	1.20	3250	1.10
8	8.0	.031	2760	8.1	3460	1.25	3190	1.16
9	5.9	.032	2900	9.2	3570	1.23	3380	1.17
10	4.1	.030	2820	8.5	3080	1.09	2890	1.03
10A	4.1	.040	3810	7.8	4120	1.08	3980	1.05

* Journal, American Concrete Institute, Jan. 1930, p. 831.

The modulus of elasticity was determined from the cylinder tests and was taken as the tangent modulus at 500 lbs/in.^{2}, which would be slightly smaller than the initial modulus.

Inspection of column 7 of the table shows, with two exceptions, a very close agreement between beam strength and cylinder strength. The large ratio for the weak concrete, Group No. 1, appears to be characteristic of low-strength concretes. Beams of low-strength concrete gave gradual failures, whereas beams of the higher strength broke suddenly. If the beam strength is calculated on the basis of the straight-line law, the ratios would be increased from about 40% to 60% above those given in the table.

As an indication of the difference in results obtained by using values of n varying somewhat from the actual values, the results obtained by using values of n in accordance with the rules of the Joint Committee are given in the last two columns. In general, these calculated values of stress are somewhat lower and hence agree more nearly with the cylinder test values.

The tests here quoted and others that have been made indicate quite conclusively that the full compressive strength can be realized in beams.

CHAPTER IV

SHEAR AND BOND STRESS

THEORY AND GENERAL RELATIONS

108. General Relations.—In Art. 44 it was shown that the direction and intensity of the maximum tensile stress at any point in the body of a beam are dependent upon both the shearing and the bending stress existing at the point in question, and that where the bond stress is small the shearing-stress is the chief factor. The general formula for maximum tensile stress is here repeated. It is

$$t = \tfrac{1}{2}f + \sqrt{\tfrac{1}{4}f^2 + v^2}, \quad \ldots \ldots \quad (1)$$

and its direction is given by the equation

$$\tan 2\theta = 2\,v/f, \quad \ldots \ldots \ldots \quad (2)$$

in which f = horizontal fibre stress (due to bending) at the point, v = vertical or horizontal shearing-stress, t = maximum tensile stress and θ = inclination of the maximum tension to the horizontal. At the neutral plane, for example, the maximum tensile stress is at an inclination of 45° and is equal in intensity to the shearing-stress at that point. Near the end of a simple beam where the bending moment is small, the value of f in eq. (1) is small and the tensile stress t is nearly equal to v at all points in a section.

In order, therefore, to be able to investigate fully the inclined tensile stresses in a beam and to provide proper reinforcement at all points, it is necessary to consider the distribution of the shearing-stresses throughout the depth of the beam.

The stress existing between reinforcing steel and concrete tending to cause slip, or the bond stress, is also dependent upon the shear; it is similar to the action between the flange and the web of a girder caused by the horizontal shear. The bond stress is, therefore, conveniently discussed in connection with shear.

109. Shearing-Stresses in Reinforced Beams.—In Art. 44 the variation in shearing-stress in a homogeneous beam was discussed and the general formula given for the intensity of shear at any point (see eq. (2)). In a reinforced beam the variation in shear differs from that in a homogeneous beam, owing to the concentration of tensile stress in the steel.

110. *Rectangular Beams.*—In Fig. 1 (*a*), is represented a short portion AB of beam length dl, with all shearing and bending stresses

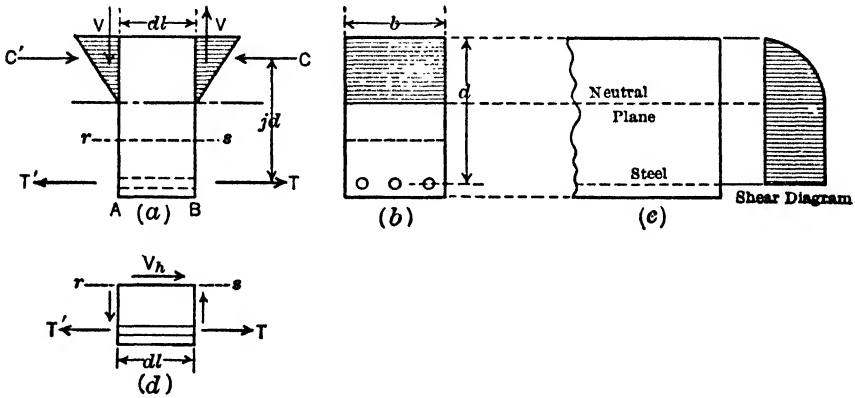


FIG. 1.

indicated. The tension in the concrete is neglected. The total vertical shear on each end is V, the increment of load applied between A and B being neglected. Considering any horizontal section, r–s, below the neutral axis, Fig. (*d*), the total horizontal shear V_h on this section is equal to $T' - T$, and the intensity of the horizontal shearing-stress will be

$$v = \frac{V_h}{b\,d\,l} = \frac{T' - T}{b\,d\,l}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The increment of tensile stress, $T' - T$, may conveniently be replaced by a function of V, from the moment equation, Fig. (*a*), $(T' - T)\,j\,d = V\,d\,l$, whence $T' - T = \dfrac{V\,d\,l}{j\,d}$. Substituting in (3), we have the more convenient expression for intensity of shear,

$$v = \frac{V}{b\,j\,d}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Eq. (4) gives the intensity of the horizontal shearing-stress on any plane between the neutral axis and the steel. It is to be noted, also, from the general principles given in Art. 44, that the vertical shearing-stress per unit area at any point is equal to the horizontal. Above the neutral axis the shear decreases according to the parabolic law as in a homogeneous rectangular beam. Fig. (*c*) represents the law of variation for the case under discussion.

Using ⅞ for an approximate value of j (see Art. 50), we have appromimately

$$v = \frac{8}{7}\frac{V}{b\,d}, \qquad \qquad (5)$$

that is, the shearing-stress at the neutral axis (equal to the maximum) is about one-seventh or 14% more than the average value obtained by dividing the total vertical shear by the sectional area.

111. *T-beams.*—Applying the same method of analysis as in the previous article, it is obvious that the shearing-stress on any section

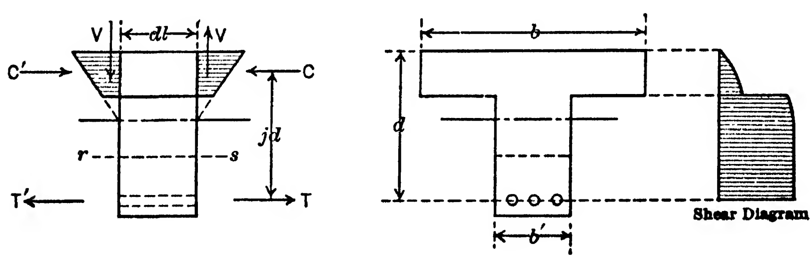


Fig. 2.

r–s (Fig. 2), in the stem below the neutral axis is also given by eq. (4), substituting b' for b. The value of $j\,d$ is the lever arm of the stress couple C–T. Hence, for T-beams,

$$v = \frac{V}{b'\,j\,d}. \qquad \qquad (6)$$

The shearing-stresses in a T-beam below the flange are therefore practically the same as in a rectangular beam having the same depth and the same width as the *stem* of the T. The slab aids in reducing the shear only by its effect in increasing slightly the value of j.

112. *Beams Reinforced for Compression.*—In beams reinforced for compression, eq. (4) will still apply, the value of $j\,d$ being the distance between the tensile steel and the resultant of the compressive stresses as shown in Art. 83.

113. Bond Stress.—The stress on the bond between steel and concrete (Fig. 1, Art. 110) will be equal to $T' - T$ on the length $d\,l$.

If U denotes the bond stress per lineal inch, we then have

$$U = \frac{T' - T}{d\,l},$$

whence we derive

$$U = \frac{V}{j\,d}. \qquad (1)$$

The bond stress per unit area will be equal to U divided by the sum of the perimeters of the steel sections. Or, if o = perimeter of one bar, Σo = sum of perimeters, and u = bond stress per unit area, we have

$$u = \frac{V}{\Sigma o \cdot j\,d}. \qquad (2)$$

The bond stress, therefore, varies directly with the total shear V and inversely with the perimeters of the rods.

114. *Bond Stress for Compressive Reinforcement.*—The question of bond stress for compressive reinforcement will seldom come into consideration. If required it can be calculated most readily by comparing it with the bond stress in the tensile steel. Whatever may be the amounts and positions of the compressive and tensile steels, the total stresses in the two sets of bars will be proportional to their areas and distances from the neutral axis. Using the notation in Art. 82 we have, therefore, $\dfrac{C'}{T} = \dfrac{A'\,(k\,d - d')}{A\,(d - k\,d)}$. This being true at any section, it follows that the *increment* of stress per lineal inch in the reinforcement (the bond stress) will also be proportional to the same quantities. Hence, if U' = bond stress per lineal inch along the compressive reinforcement, we have

$$\frac{U'}{U} = \frac{A'\,(k\,d - d')}{A\,(d - k\,d)}, \qquad (3)$$

hence, from (1),

$$U' = \frac{V}{j\,d} \times \frac{A'(k - d'/d)}{A(1 - k)}. \quad . \quad . \quad . \quad . \quad (4)$$

Since the compressive steel will generally be nearer the neutral axis than the tensile steel it follows that if the compression bars are no larger in diameter than the tension bars, the bond stress in the former will be no greater than in the latter.

115. Diagonal Tension and Shear.—It has been shown in Art. 44 (Fig. 4) that the direction of the maximum tensile stress at any point in the interior of a homogeneous beam is, in general, not horizontal but is inclined at some angle to the horizontal. The direction and intensity of this tensile stress are functions of the horizontal fibre stress and of the shearing-stress at the point in question. At the bottom fibre the maximum tension is horizontal; at the neutral axis it is at 45°

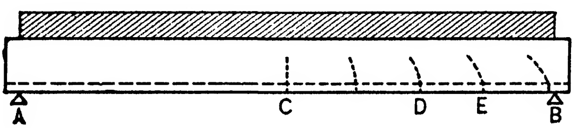


FIG. 3.

inclination, and equal in value to the shearing-stress. At sections of zero shear it is horizontal at all points.

In the reinforced beam the same relation exists between shear, direct stress and inclined tension at any given point as given by eq. (5), Art. 44, as this equation is perfectly general in scope. In the reinforced beam, however, the direct stress, f, does not vary in the same manner as in the homogeneous beam on account of the concentration of the tension in the steel; so that, as a result, the direction of maximum tension at various depths is somewhat different from that shown in Fig. 4. In this case large shearing-stresses exist immediately above the steel, hence the maximum tensile stresses become considerably inclined just above the steel, the exact direction depending upon the relation between the shear and the horizontal tension.

It will be of assistance in gaining a knowledge of this question to consider in some detail the development of stresses in a reinforced-concrete beam as it is progressively loaded, and in particular the tendency for the concrete to crack and the direction of such cracks. Consider a uniformly loaded beam, Fig. 3, and assume the load to be progressively applied. Assume, further, that the reinforcement is uniform throughout, consisting of horizontal bars only. Fig. 4 shows the form of moment and shear curves at all times, the actual values depending, of course, upon the load. As the load increases, the tensile stress in the steel rods will increase, and when this stress has reached a certain value (about 4000–6000 lbs/in.^{2}), the concrete will begin to crack. This will occur first at or near the centre. As the load increases, these first cracks will gradually extend towards the neutral axis and cracking will begin at points *D*, *E*, etc., farther and farther from the

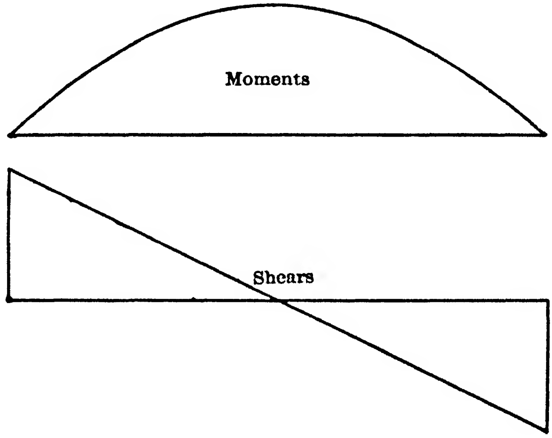
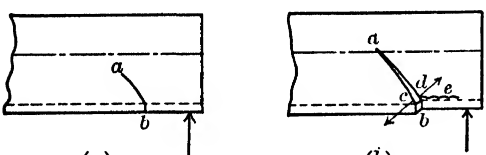


Fig. 4.

centre as the stresses in steel and concrete increase. At and near the centre where the shear is zero, or very small, the cracks will be vertical, as the direction of maximum tension is horizontal. As we pass towards the support the shearing-stresses become larger, so that the direction of maximum tension just above the rods becomes more and more inclined to the horizontal, and the cracks will not be vertical but will take an inclined direction, the inclination being greater as the end is approached. At point *E* the crack will be more inclined than at *D*, as the relative shearing-stress will be greater; near the support where the moment stresses are nearly zero, the theoretical direction of maximum tension will be about 45° at all depths. The form of the cracks is generally somewhat curved, as shown, corresponding roughly to the change in direction of maximum tension. The general direction of cracks under progressive loading is well shown in Fig. 16, Art. 138.

116. Failure from Diagonal Tension.—Consider a beam with horizontal reinforcement only. Under ordinary working loads the reinforcement at the centre is calculated for a unit stress of about 18,000 lbs/in.[2] It must happen, therefore, that under such loads the concrete is actually cracked in the vicinity of maximum moment. As shown in the preceding article, these cracks will be vertical, or nearly so, and hence at right angles to the reinforcement. So long as this condition exists no danger is involved, as the opening of the cracks is strictly limited by the deformation of the steel. Where the cracks are inclined, however, the reinforcement is not at right angles to the cracks and a different action is involved. Suppose, for example, that near the end of the beam, Fig. 5, a crack $a\ b$ starts at about 45° inclination. As the load is increased it tends to open up as shown in Fig. (*b*). The movement of the point c with reference to d is at 45° inclination, that is to say, the point c tends to move downward as well as towards the left with respect to d. The horizontal rod is effective reinforcement against excessive horizontal movement, but it offers very little resistance against vertical motion. A very small opening of the inclined crack thus brings a heavy *vertical* load on the bars at c, which is transferred across the crack by the bars and causes heavy tensile stress along the line $d\ e$. A horizontal crack then starts at d and the concrete very quickly strips or tears off along the line $d\ e$ and the beam fails. Or the crack may extend further upwards, always at right angles to the maximum tensile stress, until the concrete crushes at the top or fails by shear.



(*a*) (*b*)

Fig. 5.

It is thus seen that horizontal rods only are not adequate reinforcement against diagonal tension, and innumerable tests show that these diagonal tension cracks will develop and lead to failure regardless of the horizontal reinforcement, although the amount of the latter affects to some extent the maximum load carried.

117. The Shear as a Measure of Diagonal Tension.—Before a beam begins to crack, the diagonal tensile stress at the neutral axis is a maximum at 45° inclination, and is equal in intensity to the shearing stress $v = V/bjd$. Below the neutral axis it will be greater than v on account of the effect of the direct stress f, and above it will be less. For practical purposes of design it has been found satisfactory to consider the diagonal tension as being a maximum along a 45° line, and having an average value of $v = V/bjd$. Tests have shown that diagonal tension cracks in the central part of the beam will begin to form when the diagonal tension thus calculated is approximately equal to the tensile strength of the concrete, and if the beam is not reinforced for these stresses it will fail.

118. Methods of Reinforcement against Diagonal Tension.— There are in use many methods of placing steel in a beam so as to reinforce it against diagonal tension failure. Theoretically, the most effective way to reinforce against tension failure in any direction is to place reinforcement across the probable lines of rupture, or in the direction of the maximum tensile stresses. From these considerations the ideal web reinforcement would be a system of rods arranged somewhat as shown in Fig. 6 attached at their lower ends to the horizontal rods, or consisting of numerous horizontal rods bent up as indicated. The figure also indicates roughly the manner in which the inclination of diagonal cracks near the bottom tends to vary from nearly vertical at the centre to a large inclination at the end. The exact conditions depend upon the nature of the loading, concentrated loads tending to extend the region of large shear to greater distances from the support. It is, however, not practicable nor necessary to have the inclination of the reinforcing rods exactly the same as the lines of maximum tension, and various arrangements will serve to accomplish the purpose.

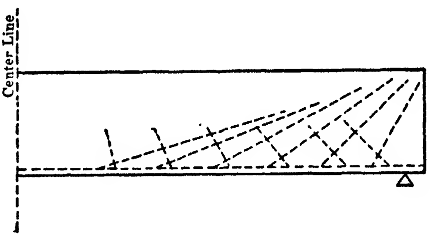


FIG. 6.

The most commonly used methods of arranging shear reinforcement are: (1) bent-up bars, (2) vertical secondary members called stirrups, and (3) inclined secondary members. Fig. 7 illustrates these various arrangements.

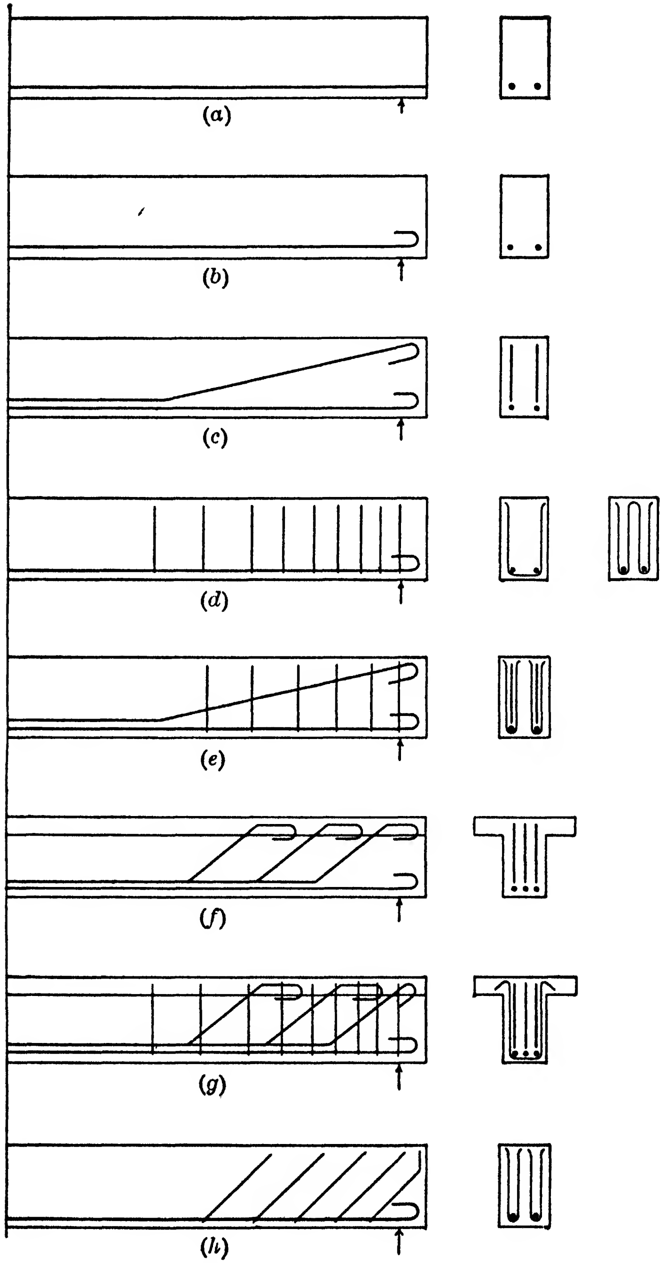


FIG. 7.—General Methods of Arranging Beam Reinforcement.

Fig. (*c*) illustrates a method adapted to relatively small shearing-stresses. A part of the horizontal rods are bent up at a small angle. For heavier stresses several rods may be bent up as in Fig. (*f*). Fig. (*d*) shows the use of the vertical stirrup. This device has long been successfully used and its design has been well standardized. Figs. (*e*) and (*g*) show a combination of bent rods and stirrups which is a very effective arrangement. Fig. (*h*) shows inclined stirrups or secondary members connected to the horizontal bars.

In all cases the reinforcement needs to be sufficiently bonded or anchored at its upper end to develop its full strength in the upper part of the beam. At the bottom, or on the tension side of the beam, it will have its maximum stress; it must be thoroughly anchored at this point, which is generally accomplished by looping the stirrup around the horizontal bars.

119. Action of Diagonal Tension Reinforcement.—To aid in understanding the action of web reinforcement placed in various ways, consider the deformations which occur in the body of a beam under increasing loads, and in a region of large shearing stress. In Fig. 8, *c d* is a vertical bar or stirrup and *f e* is a diagonal one. Under low loads and before any cracking has occurred there will be a tensile stress and deformation along the line, *f e*, and a compressive stress and deformation along the line *a b*, these stresses being approximately equal to the vertical shearing-stress $v = V/bjd$. In a vertical direction *c d* there will be no compression or tension and no distortion (except in the vicinity of load concentrations). Hence a bar *c d* will not be stressed, and a bar *f e* will receive a stress proportional to the stress in the concrete and the value of n, or approximately nv per square inch. At a stress of perhaps 200 lbs/in.^{2} the concrete will begin to crack along a diagonal line such as *a b*, and the stress in *f e* will be perhaps $15 \times 200 = 3000$ lbs/in.^{2}, a small value. From this point on, as the crack widens there will be a vertical or downward component to the movement due to the separation of the concrete along the line *a b*, and the bar *c d* will come into action but its stress will be considerably less than that in the diagonal bar *f e*.

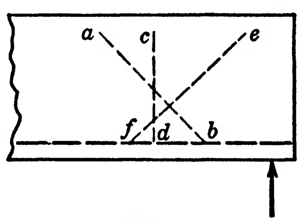


Fig. 8.

It may therefore be concluded from theoretical grounds that initial diagonal cracking is not delayed by vertical reinforcement and that such reinforcement receives no appreciable stress until cracking begins; that diagonal reinforcement is stressed with the concrete and hence acts to delay somewhat the formation of cracks, but as the unit stress in the steel when cracking begins is small as compared to its working stress, such aid is of little importance; that after cracking begins both vertical and diagonal bars will be stressed, but the former less than the latter where both are present.

Tests fully confirm these conclusions. They show also that either vertical or diagonal bars are effective reinforcement against diagonal tension failures when properly placed and proportioned. When both types are present the diagonal bars will be stressed higher than the vertical ones but the stresses become quickly equalized when the yield point is reached in the former.

120. Calculations of Stresses in Diagonal Tension Reinforcement.—It has been shown by tests that beams without diagonal tension reinforcement show an ultimate strength in diagonal tension, as measured by the shearing-stress, generally of from 6 to 10% of the crushing strength of the concrete, depending somewhat on the proportions of the beam. A safe working value is about 2% of the compressive strength. Where a higher stress must be carried, some form of reinforcement is required, and a method of estimating the stresses in such reinforcement must be applied. As already described, the usual reinforcement consists of: (1) vertical stirrups, (2) inclined stirrups, and (3) bent-up bars. Owing to the complex nature of the forces acting, it is not possible to calculate the stresses in such reinforcement with any high degree of accuracy, but a good estimate of the requirements can be determined on rational grounds, checked by the results of tests.

121. *Relative Proportion of Diagonal Tension Carried by Concrete and Steel.*—In calculating the stresses in the reinforcement an important question arises as to the mutual action of concrete and steel, and whether the concrete can still be counted upon to carry a portion of the stress when stressed beyond the safe value for unreinforced beams, or whether the steel must be proportioned to carry the entire load. Many tests to failure, with strain gage measurements, show conclusively that the concrete does continue to carry a part of the load up to rupture. The amount thus carried by the concrete is variously estimated, appearing to be a fairly constant amount for low loads but increasing at high loads, approximately in proportion to the load or stress in the reinforcing steel. A formula deduced from tests and expressing this relation approximately is (see Art. 139)

$$v = 0.005 f_v + r f_v \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where v = total shearing-stress per square inch, $= V/b\,j\,d$; f_v = stress in steel; and r = steel ratio (ratio of volume of steel to volume of concrete) for the diagonal tension reinforcement at the section in question. The term $0.005 f_v$ may be considered to represent the part carried by the concrete, and $r f_v$ that by the steel, as calculated in the usual way. This empirical formula applies to either vertical or inclined reinforcement at 45°, and for concrete of 2100–5400 lbs./in.^{2} compressive strength. By this formula the amount carried by the concrete for a fixed working stress f_v is a fixed amount $= 0.005 f_v$. With $f_v = 16{,}000$ lbs/in.^{2} it becomes 80 lbs/in.^{2} The Joint Committee of 1924 specifies an amount equal to 2% of the compressive strength of the concrete, which is the allowed value for beams without diagonal tension reinforcement and without special anchorage to the horizontal bars. The American Concrete Institute, 1928, increased this to 3% where special anchorage is used.

122. *Stress in Reinforcement.*—Fig. 9 shows three arrangements of reinforcement: vertical, inclined at 45°, and inclined at any angle θ. The shearing-stress is $v = V/b\,j\,d$, and this is taken to be the diagonal tensile stress along the line $a\,c$. The concrete takes a certain part of this, as may be assumed, the reinforcement the remainder. Let v' represent the part taken by the steel.

(*a*) *Vertical Stirrups.*—Fig. 9 (*a*). The vertical stirrups, spaced a distance s apart, are assumed to carry the vertical component of the diagonal tension v' along the line $a\,c$, and the horizontal bars the horizontal component. The diagonal tension over the distance centre to centre of stirrups on line $a\,c$ is $\dfrac{v'\,b\,s}{\cos 45°}$ (width of beam $= b$) and its vertical component $= v'\,b\,s$. Hence if P = stress on each stirrup then

$$P = v'\,b\,s. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

(*b*) *Inclined Stirrups at* 45°.—Fig. 9 (*b*). The distance along $a\,c$, centre to centre of stirrups, is $s \cos 45°$, and the diagonal tension on this area is $v'\,b\,s \cos 45°$. Its vertical component is $v'\,b\,s \cos^2 45 = \tfrac{1}{2} v'\,b\,s$, which is the vertical component of P. Hence

$$P = \frac{\tfrac{1}{2} v'\,b\,s}{\cos 45°} = 0.7\, v'\,b\,s. \qquad (3)$$

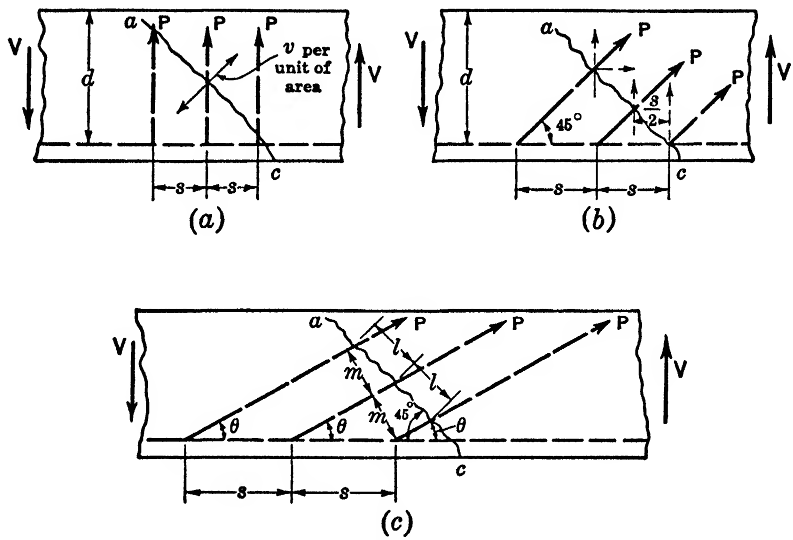


FIG. 9.

(*c*) *Inclined Bars or Stirrups at Angle θ with the Horizontal.*—Fig. 9 (*c*). The distance l along the diagonal $a\,c$ is $l = \dfrac{m}{\cos(45° - \theta)}$ and $m = s \sin \theta$. Hence $l = \dfrac{s \sin \theta}{\cos(45° - \theta)} = \dfrac{s \sin \theta}{\cos 45\,(\cos \theta + \sin \theta)}$. The vertical component of stress on length l is $v'\,b\,l \cos 45° = v'\,b \dfrac{s \sin \theta}{\cos \theta + \sin \theta}$. This is also the vertical component of the stress P, or $P \sin \theta$. Hence

$$P = \frac{v'\,b\,s}{\cos \theta + \sin \theta}. \qquad (4)$$

Eq. (4) is general and reduces to (2) and (3) for $\theta = 90°$ and $45°$

respectively. For angles between 45° and 90° the simpler formula $P = v' b s \sin\theta$ is commonly used and is sufficiently exact.

123. Steel Ratio for Diagonal Tension Reinforcement.—Assuming the stirrups extending the full depth of the beam, the total amount of steel for a length s of beam, in case (a) is $\frac{P}{f_v} d = \frac{v' b s}{f_v} \times d$, and the ratio r of volume of steel to volume of beam $= \frac{v' b s d}{f_v} \Big/ b d s = \frac{v'}{f_v}$. In case ($b$) the length of each bar $= 1.41\, d$, and the volume of steel in length $s = \frac{P}{f_v} \times 1.41\, d = 0.7 \frac{v' b s}{f_v} \times 1.41\, d = \frac{v' b s d}{f_v}$, the same as for vertical stirrups, and $r = \frac{v'}{f_{v'}}$. For case (c) the length of bar is $\frac{d}{\sin\theta}$, and the steel ratio becomes $r = \frac{v' b s}{f_v (\cos\theta + \sin\theta)} \times \frac{d}{\sin\theta} \Big/ b d s$ $= \frac{v'}{f_v \sin\theta (\cos\theta + \sin\theta)}$. For angles less than 45° this is larger than the values for the other cases. The minimum value of r is for $\theta = 67\frac{1}{2}°$, giving $r = 0.825 \frac{v'}{f_v}$. This value of θ is the theoretical angle for maximum efficiency of the steel reinforcement for equal spacing of bars. This is of little practical importance, as the reinforcement consists generally of vertical stirrups or bent rods or both. But the foregoing analysis is an indication that very flat angles should not be used for heavy stresses. The steel ratio $r = \frac{v'}{f_v}$ gives $v' = r f_v$, which shows more clearly the meaning of eq. (1), $f_v = 0.005 f_c + r f_v$. The part $r f_v$ is seen here to be the portion v' carried by the steel.

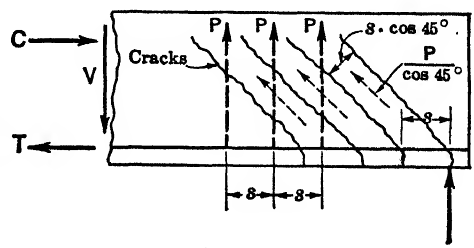


FIG. 10.

124. Diagonal Compression.—It will clarify the problem and be of interest to consider what happens to the web stresses carried by the steel. The web of a beam may be considered analogous to a lattice truss, the steel carrying tension and the concrete compression. Assuming cracks at 45°, the compressive stress in the concrete will have to be carried from top to bottom along a 45° line. With vertical bars, Fig. 10, each carrying a stress P and spaced a distance s apart, the vertical component of the compressive stress in each diagonal block of concrete will be P, and the stress itself is $\frac{P}{\cos 45^\circ}$. The cross-section is $b\,s\cos 45^\circ$. Hence the unit compressive stress $= \frac{P}{b\,s\cos^2 45^\circ} = \frac{2\,P}{b\,s}$. Then from (2) $P = v'\,b\,s$. Hence the unit compressive stress in the concrete $= 2\,v'$, or twice the diagonal tension v'.

For inclined rods at 45°, Fig. 11, the stress in each concrete block will equal the tension in a bar $= P$, and the stress itself $= \frac{P}{\cos 45^\circ}$. From (3) $P = 0.7\,v'\,b\,s$, hence in this case the compressive stress in the concrete $= \frac{v'b\,s}{b\,s} = v'$, or one-half its value for vertical stirrups. This shows that diagonal reinforcement results in less stress and strain than vertical reinforcement; it is another way of showing that diagonal reinforcement is more in harmony with the nature of the distortions of a solid beam. With the limiting values of v' allowed in practice the diagonal compression is generally of no practical importance, though it may be in special cases.

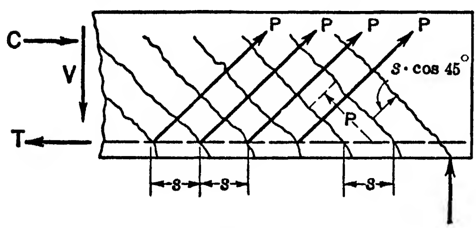


FIG. 11.

125. Limiting Value of Shearing or Diagonal Tensile Stresses.— With well-designed and carefully placed reinforcement it is possible to develop a total shearing-strength (diagonal tension) of about 30 to 35% of the compressive strength of the material. To accomplish this in practice with certainty, considering the nature of the details involved (closely spaced rods, careful anchorage, etc.) requires an amount of inspecton and skill of workmanship quite beyond the ordinary, and is impracticable. It is also unnecessary, as the proportions of beams determined from bending moment and other considerations are usually such as to bring the shearing-stresses to values much below those above mentioned. To promote safe construction in this respect the maximum allowable values of shearing-stresses are specified. The 1916 Joint Committee specified a maximum shearing-stress of 5% of the compressive strength. Improvements in practice in web reinforcement led the Joint Committee of 1924 to allow a maximum of 12%. This high value is not generally followed. The American Concrete Institute specifications, 1928, permit 12% only under very special susupervision, otherwise 9%. All these values are based on well-designed reinforcement.

126. Spacing and Other Details.—In arranging the details of diagonal tension reinforcement of large beams, it is generally convenient to select a certain size of stirrup and then calculate the necessary

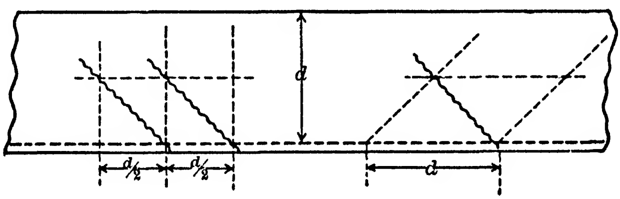


FIG. 12.

spacing at various points along the beam. Frequently, the maximum allowable spacing of bent bars must also be determined. Both of these problems are simple, and require merely the use of eqs. (2), (3), or (4), solved for s.

To be reasonably effective the reinforcement should be so spaced that at least one rod will intersect any 45° line of rupture below the centre of the beam. As shown by the sketch, Fig. 12, this requires a spacing of vertical reinforcement not greater than $d/2$, and for diagonal rods, a horizontal spacing not greater than d. Considerable gain in strength is obtained by rods spaced somewhat further apart, but tests show little value from vertical rods spaced a distance apart equal to d. The Joint Committee recommends a spacing not exceeding $s = \dfrac{45\,d}{\theta + 10}$. This gives, for $\theta = 90°$, $s = 0.45\,d$; and for $\theta = 45°$, $s = 0.82\,d$. For shearing-stresses exceeding 6% of the compressive strength the maximum spacing is made two-thirds of the above values. The American Concrete Institute specifies, for a shearing-stress not exceeding 6% of the compressive strength, a spacing measured at right angles to the direction of the stirrup of $\frac{3}{4}d$, and for greater shearing-stresses, of $\frac{3}{8}d$. For $\theta = 45°$ this gives a horizontal spacing of 1.06 d and 0.53 d respectively.

The bond strength of the reinforcement must be carefully guarded, especially in the case of large bent-up bars. This strength should be provided in the upper portion of the beam above the neutral axis, (within 0.3 d of the compression face is required by the Joint Committee). Plain and bent-up bars often lack sufficient bond strength to render them fully effective, in which case the ends of the bars should be bent into hooks.

127. Calculation of Spacing or Vertical Stirrups along a Beam.— For any given size of stirrup the spacing at any point is given by the formula $s = P/v'b$, where v' is the shearing-stress to be carried by the reinforcement at the point in question. The spacing will vary from point to point, but in practice it is convenient to use but a few different values. No great accuracy is required or has any justification, as the calculations are at best only approximate.

The most common problem of stirrup spacing is for a beam supporting a uniform dead load and a uniform live load. In this case it is customary, and sufficiently exact, to assume the shear to vary uniformly from end to centre, the centre shear being equal to one-fourth the end live load shear. The conditions are illustrated in Fig. 13. Ordinates to the line df represent the shearing-stress intensity along the beam. The concrete is assumed to carry a portion of the shear represented by the ordinates to the line gh, and the remainder of the shear, from gh to dh, must be carried by the steel. The area gdh, multiplied by the width of beam b, will be the total shear in pounds carried by all the web steel.

The spacing of vertical stirrups at any point will be $s = P/yb$, where P = strength of stirrup. The total number required will be $\dfrac{\text{area } dgh \times b}{P}$. If a considerable number are required, the proper spacing may be calculated at several points, a curve sketched in as shown in the figure, and convenient values laid off as indicated. Or a general table may be made up for determining the relative spacing for any triangular shear area as follows:

The problem is to divide up the shear triangle $d\ g\ h$, Fig. 13 (c), into

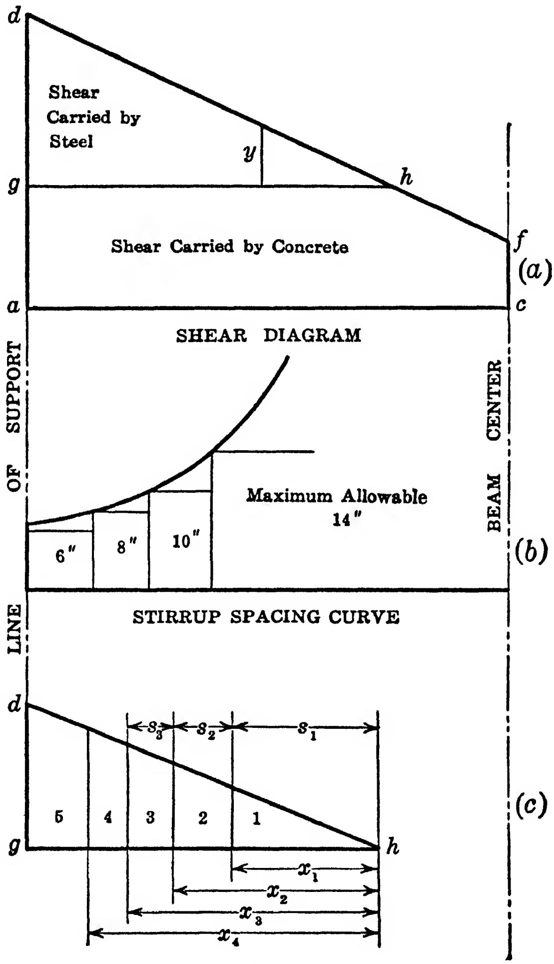


FIG. 13.

a given number of equal areas and then place the stirrups approximately at the centres of gravity of the several strips, adjusting the actual spaces to convenient dimensions. Take the length $g\ h$ as 100 units, and assume 10 stirrups required. The value of x_1 is determined from the proportion $\frac{x_1^2}{100^2} = 1/10$, whence $x_1 = 100\sqrt{0.1} = 31.8$; likewise $\frac{x_2^2}{100^2} = 2/10$ and $x_2 = 100\sqrt{0.2}$; etc. The several values of x for the 10 divisions are given below; then by subtraction we get the widths of the several strips for equal areas, beginning at the right end. These values may be taken as the stirrup spacing required.

No. of Strip....	1	2	3	4	5	6	7	8	9	10
x..............	31.8	44.8	54.8	63.4	70.7	77 5	83.8	89.6	95.0	100
Width of Strip..	31.8	13.0	10.0	8.6	7.3	6.8	6.3	5.8	5.4	5.0

For any other number less than 10, the widths may be determined by proportion, using the proper value of x for the basis. Thus if 7 stirrups are required over a length of 60 in., then the widths will be found by multiplying the first 7 values above given by the ratio 60/83.8. To provide for a larger number of stirrups a table for 20 divisions can be made up and used in the same manner.

Example.—Suppose shearing-stresses at end and centre are respectively 160 and 30 lbs/in.², and that the concrete is good for 75 lbs/in.² Half span length = 180 in.; b = 12 in. The length $g\,h$ in Fig. 13 (a), in which the shear is greater than 75, is found by the proportion $\frac{g\,h}{180} = \frac{160 - 75}{160 - 30}$, whence $g\,h = 118$ in. Total shear over this length = $\frac{(160 - 75) \times 118}{2} \times 12 = 60{,}000$ lbs. Use ½-in. square bars. Area of double bar = ½ sq. in. Allowable stress = ½ × 16,000 = 8000 lbs. Number required = 60/8 = 8.

For 8 stirrups $x_8 = 89.6$. Hence the required widths are obtained by multiplying the first 8 values of the table by 118/89.6. They are as follows:

42, 17, 13, 11.4, 9.6, 9.0, 8.3, 7.6. Total 117.9 in.

These values are then rounded off, using a maximum spacing in accordance with specifications.

TESTS ON BOND STRENGTH

128. Nature of Bond Resistance.—When slipping takes place between a plain, smooth bar and the surrounding concrete under a gradually increasing force, the progressive action appears to be as follows: Until the load reaches a value sufficient to produce a bond stress of 200 to 300 lbs/in.^{2} (depending upon conditions) there is no measureable movement; the adhesion between cement and steel appears to hold the two materials firmly together. As the load increases, slipping begins, bringing into action the frictional resistance. This resistance increases as the slip increases and reaches a maximum for a slip of about 0.01 in. After this point the resistance gradually falls off. The resistance to slip depends upon the smoothness of the rods, and the character and age of concrete. In the case of bars having corrugations or ribs, the initial action is about the same as for plain bars, but the resistance continues to increase with increase of slip until failure occurs by the splitting of the concrete or the shearing through of the rods at a relatively high value.

129. **Methods of Testing.**—Tests of bond are generally made by embedding the rod in a block of concrete and pulling it therefrom, the rod being stressed in tension and the concrete in compression. Tests have also been made by *pushing* the rods through the block. Neither of these methods of testing is entirely satisfactory for use in beam analysis, as they do not altogether simulate the action in a beam where both the rod and the concrete are in tension. Various experimenters have, accordingly, made efforts to determine the bond strength by tests on beams.

130. **Results of Tests in Direct Tension.**—Table 2 contains in condensed form the results of some of the most important tests made by direct tension.

The variation of bond resistance during a test is illustrated in Fig. 14, the solid line being the average load-slip curve for the third group of tests of Table 2. It is seen that the maximum resistance occurs at a slip of about 0.01 in., after which the resistance gradually falls off. Generally the slip begins at a load of from 60 to 80% of the maximum. After a slip of 0.10 in. occurs, the frictional resistance is still about two-thirds of the maximum resistance. In these tests the slip was measured at the free end.

In general, the stronger the concrete the greater the bond strength, whether the difference is due to age or richness of mixture, the bond strength being approximately proportional to the strength of the con-

TABLE 2

BOND TESTS BY DIRECT TENSION, PLAIN BARS

All Concrete 1 : 2 : 4; 60–90 Days Old

Authority	Steel Bars		Depth Embedded, Inches	Bond Resistance		
	Kind	Size, Inches		At End Slip of 0.0005 in.	At End Slip of 0.001 in.	Maximum
(1)	Round	3/16 to 3/4	6	....	....	400
		3/16 to 3/4	8	....	....	310
(2)	Round	1/2 to 1 1/4	25 diam.	....	....	410
		1/2 to 1 1/4	40 diam.	....	....	390
(3)	Round	1/2	8	323	339	381
		5/8	8	266	295	405
		3/4	8	275	303	387
		1	8	247	281	385
		1 1/4	8	269	296	397
(3)	Flat	1/2 × 1	6	359	395	459
		1/4 × 2	4	239	263	293
(3)	Polished Round	1	5	149	....	152
		3/4	5	137	146	160
		3/4	6	170	192	255

(1) Withey; *Bull., Univ. of Wis.*, No. 175, 1907.
(2) Van Ornum; *Eng. News*, Vol. LIX, 1908, p. 142.
(3) Abrams; *Bull.* No. 71, *Univ. of Ill. Eng. Exp. Sta.*, 1913.

crete. The results of Abrams' tests show a value for the maximum bond resistance of about one-fourth the compressive strength of 6-in. cubes, and the resistance at a slip of 0.0005 in. about one-sixth of this compressive strength. Tests indicate little difference due to size of rod. The bond strength of flat bars is considerably less than round bars, and that of polished or smooth bars very much less than ordinary bars. Rusted bars show considerably higher bond resistance than bars with the ordinary mill scale, such as usually tested.

131. Bond Strength of Deformed Bars.—The initial action of deformed bars is very similar to that of plain bars. When the adhesion is broken a small slip takes place under increasing load at about the same rate at first as for plain bars. After reaching a movement of about 0.01 in., however, the resistance continues to increase as the projections begin to bear hard against the concrete. If splitting of the concrete does not take place, the resistance continues to rise to a high value, but conditions are not usually such as to make available this

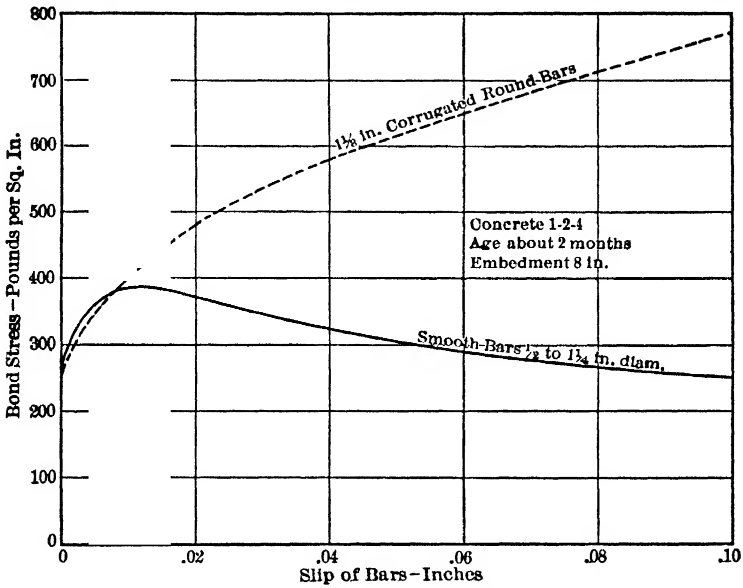


FIG. 14.—Load-slip Curves for Plain and Corrugated Bars.

higher value of the bond resistance. Either the concrete splits or the slip is so great as to cause failure of the beam in other ways.

Important tests on deformed bars are reported in the bulletin of Abrams already referred to. Fig. 14 shows typical curves of load-slip relations of a corrugated round bar as compared to plain round. It is to be noted that the action is practically the same up to a slip of 0.01 in., beyond which the resistance of the deformed bar continues to increase, but at a decreasing rate.

132. Results of Bond Tests on Beams.—In a series of tests by Mr. Withey * at the University of Wisconsin, test beams were arranged

* Engineering Record, 1908, LVII, p. 798.

TABLE 3

BOND TESTS ON BEAMS

UNIVERSITY OF WISCONSIN, 1907

Concrete 1 : 2 : 4; Age 60 Days

No. of Tests	Diam. of Rod, Inches	Bond Strength, Lbs/in.^{2}	Average of Group (*A*), Lbs/in.^{2}	Average Bond Strength by Direct Tension (*B*) Lbs/in.^{2}	Ratio *B* : *A*
38		345			
39	3/8	298	278	394	1.42
40		190			
41		361			
42	1/2	312	286	455	1.54
43		186			
7		362			
8		264			
9	5/8	201	272	467	1.75
36		254			
37		278			
44		207			
45	3/4	289	264	502	1.90
46		295			
47		136			
48	1	174	163	487	2.99
49		180			

as shown in Fig. 15. The stresses in the exposed rods were determined by means of extensometers. The conditions were similar in many respects to those obtaining in an ordinary beam, but the beam was

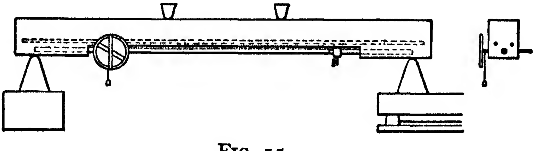


FIG. 15.

prevented from failing by the upper auxiliary rods. Table 3 gives the principal results of these tests. The table also contains results of comparative tests made at the same time by the usual direct tension method. The last column gives the ratio of the two results. The rods were of ordinary mild steel and were free from rust. The beams were 5 by 5 in. in section and 5 ft. 6 in. long.

These tests indicate that the bond was not affected by size of rod except in the case of the 1-in. size. This difference is undoubtedly due to other factors not explained. Excepting the tests on this size the results are quite uniform, the average of all being 275 lb/in.^{2}, with maximum variations of 32% below and 32% above the average. The results obtained by direct tension are much higher, averaging about 475 lbs./in.^{2}, or 75% greater.

It has been observed in beam tests that the maximum resistance is reached for a very small end slip, about 0.001 in., whereas in the tension tests the corresponding slip is about 0.01 in. This appears to be due to the progressive slipping action in the beam from centre or load points towards end, so that the average slip was a considerable amount when the end slip began. Some slip of bar necessarily occurs even under safe loads on account of the necessary stretch of bar to which the concrete adjusts itself by slight cracks at intervals, accompanied by more or less slipping. A very small end slip, therefore, indicates that the maximum bond resistance has been nearly reached.

133. Bond Stress in Beams with Bent Rods.—Table 4 gives results of tests by Bach on beams showing the effect of bending up some of the rods for shear reinforcement. The results with straight rods in the rectangular beams are about the same as previously given. In the T-beams with straight rods only they are rather low. The important results are those where four of the rods are bent leaving only one straight. Calculating the bond stress at the end of the beam by the usual formula, and taking into account only the one rod at the bottom of the beam, gives the values shown in the table, the average being 493 lb/in.^{2} This is about 2½ times the value for straight rods and stirrups. The same general result is shown in Table 8, Art. 137, where with some of the rods bent, a bond strength calculated for the straight rods, of over 600 lbs/in.^{2} was obtained. Inasmuch as the actual bond strength in these cases must have been practically the same as in other tests, these results show that where some of the bars are bent up so as to reinforce the web of the beam the bond stress on the remaining straight bars near the end of the beam is much less

TABLE 4

BOND TESTS ON BEAMS WITH BENT BARS

(Bach)

Concrete 1 : 4; Age 6 Months; Beams Loaded at Quarter Points

No.	Kind of Beam	Reinforcement	Calculated Bond Stress at beginning of Slip, Lbs/in.[2]	Average for Group, Lbs/in.[2]
2	Rectangular Beams	Straight rods only	312	291
3			300	
4			271	
5			281	
161		Straight rods and stirrups	330*	330
162	T-Beams	Straight rods only	158	158
163		Straight rods and stirrups	182	195
164			208	
165		1 straight, 4 bent	408	493
166		1 straight, 4 bent, with stirrups	498	
167		1 straight, 4 bent, with stirrups	545	
168		1 straight, 4 bent	522	

* Average of three.

than the theoretical values obtained by the usual formula. Where such an arrangement of rods is used it is evident that the allowable bond stress on the straight rods may be very considerably increased.

The Joint Committee permits bent-up bars to be included in the calculations that are within a distance of $\frac{1}{3}d$ from the horizontal reinforcing.

134. Hooked and Anchored Ends.—It is a common practice, where increased bond resistance is required beyond that readily secured by increasing the straight length of the bar, to bend the ends of the rods into short hooks. Tests show that initial slip is not delayed by bending the bar, but that the ultimate strength is much increased. Circular bends of full 180° are the most effective form, and should be of a diameter not less than 4 bar diameters, preferably 6 or 8 diameters.

Tests by Mylrae, using ½-in. round bars, with bends of 180° of various diameters, embedded in blocks 9 in. square, gave the following relations between slip and bond stress:

TABLE 5

TESTS ON HOOKS

(Mylrae *)

Diam. of Hook in Bar Diameters	Strength of Concrete, Lbs/in.²	Bond Stress at Slip of 0.01 in., Lbs/in.²	Bond Stress at Slip of 0.1 in., Lbs/in.²	Bond Stress at Failure, Lbs/in.²	Slip at Failure, In.
3 *d*	1970	280	710	928	0.157
6 *d*	1970	190	....	705	0.098
6 *d*	2700	470	670	692	0.139
8 *d*	2130	260	760	804	0.110
8 *d*	4000	460	740	809	0.196
12 *d*	1620	320	....	540	0.143

* Proc. Am. Concr. Inst. 1928, p. 240.

The length of embedment was the full bend of 180° plus 1½ in. of straight bar at the end of the hook and 3 in. at the other. Ultimate failure occurred by splitting of the concrete, the steel stress being from 51,000 to 64,000 lbs/in.² The yield point was about 70,000 lbs/in.² In other tests where the concrete was prevented from splitting by the use of small spiral reinforcement around the hooks, the yield-point strength of the steel was developed in a 2200-lb. concrete for bends of 6- and 8-bar diameters. In these tests the slip as recorded was measured at the point of maximum stress instead of at the free end, as in some of the other tests. This accounts for the relatively low values of bond stress at a slip of 0.01 in.

From these tests it appears that the behavior of hooked bars is quite similar to that of deformed bars; the initial slip occurs for about the same stress as for straight bars but the resistance continues to increase with the slip.

The chief value of hooks is to secure greater bond strength in a limited space and to increase the margin of safety against ultimate failure. They are of special value in web reinforcement and at the ends of simply supported beams where they provide added bond strength at a point of maximum bond stress.

The Joint Committee allows the same value for bond stress around a circular hook as for the same length of straight rods. A diameter not less than 4 d is required.

The use of bolted anchor plates is convenient in special cases where the full strength must be developed in a very short distance. Such anchorage acts like other types; it does not delay formation of first cracks but increases ultimate strength.

TESTS RELATING TO DIAGONAL TENSILE STRENGTH OF BEAMS

135. Importance of Tests.—A large number of laboratory tests have been made to determine the strength of beams in diagonal tension having various arrangements of web reinforcement. In the early period of development of reinforced concrete the problem was to determine the necessary amount of web reinforcement that would be sufficient to cause tension or moment failures in beams of usual proportions, so that a large proportion of the tests do not show the actual strength of the web reinforcement. Inadequate bond strength also, in many cases, led to failures in diagonal tension where the web reinforcement would otherwise have been adequate. The important features of design which needed to be determined by tests were (1) the strength of beams in diagonal tension having no web reinforcement (reinforced with straight bars only); and (2) the amount and arrangement of web reinforcement necessary to develop the full bending strength of beams of ordinary proportions, and, conversely, the limiting shearing-stresses permissible with the designs in common use. In the later tests attention has been given to a more exact study of stresses in web reinforcement, the development of cracks in the concrete, and correct methods of design, in an effort to place the design of web reinforcement on a basis comparable to that used in the design of reinforcement for bending stresses.

As the mathematical analysis of this problem is at best very approximate, it is necessary to rely to a considerable extent on the results of tests, especially in fixing the limitations of practical design.

136. Tests on Beams without Web Reinforcement.—In Table 6 are given the results of a large number of tests on rectangular beams

TABLE 6

TESTS OF BEAMS REINFORCED WITH STRAIGHT BARS ONLY

Diagonal Tension Failures

Authority and Reference	Kind of Concrete	Crushing Strength, Lbs/in.²	Cross-section, $b \times d$, Inches	Span Length, Feet	Per Cent Reinforcement	No. of Tests	Shearing-stress at Failure $v = V/bjd$: Min. and Max. of Groups, lbs in.²	Shearing-stress at Failure $v = V/bjd$: Average of Group, lbs in.²	Ratio of Shearing-stress to Crushing Strength
Marburg, A.S.T.M., IV, 1904, p. 508.	1 : 2 : 4	1700	7 × 8	5 and 8	0.96–1.36	13	84–124	104	0.061
Carson, Boston Transit Com., 1904.	1 : 2 : 3	4100	6 × 7¼	5	0.82–1.64	6	203–265	246	0.060
	approx.		6 × 7¼	5	0.9 –2.3	4	175–262	220	0.054
Talbot, 1908 Series, *Bull. Univ. of Ill.*, No. 29, 1909.	1 : 1 : 2	4150	8 × 10	6 and 12	0.98–1 47	4	134–229	184	0.044
	1 : 1½ : 3	3315	8 × 10	6	0 98	2	142–181	161	0.049
	1 : 2 : 4	2475	8 × 10	6 and 12	0.98–2.21	24	128–214	164	0.066
	1 : 3 : 6	1403	8 × 10	6	0.98	2	104–170	137	0.098
	1 : 4 : 8	1296	8 × 10	6 and 12	0 98	6	65–113	90	0.069
	1 : 5 : 10	940	8 × 10	6 and 12	0.98	7	37– 82	63	0.067
Richart, *Bull. Univ. of Ill.*, No. 166, 1927.	1 : 2 : 4	3312	8 × 10	6	1.23–1.92	3	145–182	164	0.050
	1 : 2 : 4	2910	8 × 15	10	1.47	2	126–130	128	0.044
	1 : 2 : 4	2700*	8 × 10	9½	2.74–3.7	6	213–259	239	0.089
	1 : 2 : 4	4160*	8 × 21†	9	2.33	4	459–599	535	0.129
	1 : 2 : 4	3590*	8 × 21	9	0.93	2	565–603	584	0.163
Richart and Larson, *Bull. Univ. of Ill.*, No. 175, 1928.	1 : 2 : 4	1630	8 × 15	2 ft. 8 in. cantilever	0.74	1	142	142	0.087
	1 : 2 : 4	3362*	8 × 15	12	0.93	2	229–231	230	0.068

* Tests on cylinders; all others on cubes.

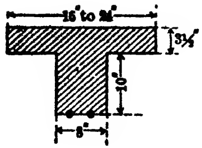
† Web of T-beam.

which were reinforced with straight rods only and which failed by diagonal tension failures. The shearing-stress given in the table is the stress at failure calculated by the formula $v = V/bjd$.

Taking into consideration the difference in strength between the cube and standard cylinder it is seen that the ratio of shearing-stress at failure to the compressive strength as determined by cylinder tests has a minimum value of about 0.06, most values being above 0.07. In general, the greater the ratio of depth to span length the greater the relative shearing-stress at failure.

137. Tests on Beams with Web Reinforcement of Ordinary Proportions.—Tables 7 and 8 give results of tests on T-beams which illustrate several features pertaining to the strength in diagonal tension of web reinforcement. Referring to Table 7 the yield-point strength of the corrugated bars was about 48,000 lbs/in.^{2}; that of the ¾-in. round bars 41,000 lbs/in.^{2}; and of the ¼-in. round material, 47,000 lbs/in.^{2} These limits correspond closely to the stresses in the horizontal steel at failure, excepting in the case of beams G_1 and G_2 which failed by diagonal tension. The table contains results of value with respect to shearing-stresses and the use of stirrups and bent rods for shear reinforcement. In the progress of the tests the occurrence of the first diagonal crack was carefully noted, and the maximum shearing-stress at this load is calculated and given in the table. It will be noted that there is a fairly close agreement between this value and the tensile strength of the concrete as given in the next column. The average value for the maximum shearing-stress is 179 lbs/in.^{2} whereas the average tensile strength is 187 lbs/in.^{2} This would indicate that in spite of stirrups the concrete will crack at a diagonal tensile stress about equal to its tensile strength. The table also gives the maximum shearing-stress at ultimate load, the amount of web reinforcement including the bent-up bars, and the calculated stress in web steel assuming all shear to be carried by it. Inasmuch as the yield point of this material, with the exception of the wire mesh of beams F_1 and F_2, was about 47,000 lbs/in.^{2}, it is evident that a considerable amount of the shear was carried by the concrete. In beams G_1 and G_2, which failed by breaking of the stirrups, if we assume a yield-point strength of 47,000 lbs/in.^{2}, the amount carried by the concrete would be 150 lbs/in.^{2}, a value about equal to its tensile strength. In all other

TABLE 7



T-BEAM TESTS, UNIVERSITY OF WISCONSIN*

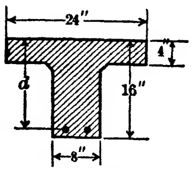
Concrete 1 : 2 : 4; age, 28 days; compressive strength = 1940 lbs/in.^{2}
Beams D had 24-in. flanges, all others 16-in. Span length = 10 ft.; loaded at third points.
Stirrups uniformly spaced between loads and supports.
All beams failed in tension except G_1 and G_2, which failed by breaking of stirrups.

No. of Beam	Kind of Reinforcement	Shearing Stress at First Crack, Lbs./in.^{2}	Tensile Strength of Concrete, Lbs./in.^{2}	At Maximum Load		Web Reinforcement, Per Cent	Calculated Stress in Web Steel at Maximum Load, Lbs./in.^{2}
				Loads, Lbs.	Shearing Stress, Lbs./in.^{2}		
A_1	Four ¾" cor. bars (2 bent).....	128	256	66,600	361	0.62	58,000
A_2	Fourteen ¼" cor. stirrups......	171	167	66,200	357		58,000
B_1	Four ¾" cor. bars (2 bent).....	226	260	65,600	357	0.59	60,000
B_2	Sixteen ¼" round stirrups......	150	157	62,400	354		60,000
C_1	Five ¾" round rods (3 bent)....	181	197	60,000	322	0.66	49,000
C_2	Sixteen ¼" round stirrups......	185	216	57,400	316		48,000
D_1	Six ¾" cor. bars (3 bent).......	247	182	96,200	526	1.14	46,000
D_2	Fourteen ⅜" cor. stirrups......	306	...	101,400	544		48,000
E_1	Six ¾" cor. bars (3 bent).......	244	148	92,800	505	1.58	32,000
E_2	Twenty-four ⅜" cor. stirrups...	215	184	88,000	485		31,000
F_1	Four ¾" cor. bars (2 bent).....	158	142	67,600	368	0.63	58,000
F_2	No. 11 wire mesh, 1 inch.......	150	174	65,400	352		56,000
G_1	Four ¾" cor. bars (straight)....	126	184	48,000	259	0.24	106,000
G_2	Sixteen ¼" round stirrups......	149	164	48,200	260		106,000

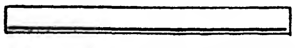
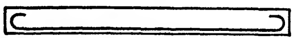
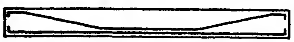
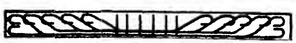
* Bulletin No. 2, Vol. 4, 1908.

TABLE 8

T-BEAM TESTS BY BACH *



Compressive strength of concrete in cubes = 3320 lbs/in.[2]
Yield point of steel = 46,000 lbs/in.[2]
Reinforcement at centre = approximately 1.2%.
Stirrups made of round rods 0.28 in. in diam., excepting No. 6, in which flat strips were used 0.08" × 0.8" in section.
Span length, 13'–2"; loaded at 8 points uniformly spaced.
Each result is the average of three tests.

Type of Reinforcement	No.	Web Reinforcement at End, Per Cent, $= 100r$	Calculated Stresses at Failure, Lbs/in.[2]				Kind of Failure
			Bond Stress in Horizontal Bars	Shearing-stress	Stress in Web Reinforcement		
					$f_s = \frac{v}{r}$	$f_s = \frac{v - 250}{r}$	
	1	....	203	252	....	....	Diagonal tension
	2	0.50	290	360	70,000	20,000	Diagonal tension and bond
	3	....	230	273	....	....	Diagonal tension
	4	0.50	394	495	97,000	48,000	Diagonal tension
	5	....	465	405	....	....	Diagonal tension
	6	....	640	550	....	....	Tension
	7	0.60	600	470	80,000	37,000	Tension
	8	0.60	660	520	85,000	45,000	Tension
	9	0.60	610	568	59,000	33,000	Tension
	10	0.66	800	552	85,000	46,000	Diagonal tension

* Deutscher Ausschuss für Eisenbeton. Heft 20, 1912.

cases the web reinforcement was ample and hence no conclusions can be drawn as to relative value of the different kinds of reinforcement.

Table 8 gives results of tests by Bach in which a variety of forms of reinforcement was used. Comparing beams Nos. 1 and 3 it appears that the bond strength was sufficient without the hooks and the results are about equal. In No. 2 the bond strength was insufficient but was made adequate by the use of hooks as in No. 4. The stresses in the web reinforcement were calculated on two assumptions: (1) assuming the steel to carry all the shear, and (2) assuming the concrete to carry the amount of 250 lb/in.^{2}, this being the strength shown in group No. 1. By the former method the steel is greatly over-stressed; by the latter method the steel is stressed to its yield point in three groups, two of which failed in diagonal tension. These results indicate the substantial correctness of the second method of calculation.

Considering straight bars only, the calculated bond stress, where several bars are bent up, is greatly in excess of the probable bond strength, showing the value of bent bars in reducing the actual bond stress.

138. *Development of Cracks during Tests.*—An important question pertaining to the use of various kinds of shear reinforcement is the manner of development of cracks when tested to destruction. According to the analysis of Art. 119 vertical stirrups are not in a position to be stressed in tension until the concrete begins to deform more in tension than in compression, and they cannot take much stress until cracks begin to form. Inclined web members, bent bars or stirrups, can, on the other hand, take some stress before the concrete begins to crack, but the amount of such stress is not large. On this subject the tests of Bach are very instructive, as the development of cracks in each beam is shown at several stages of the tests. In Fig. 16 is shown the behavior of four of the beams tested. The kind of reinforcement is indicated and the total load given in kilograms.

Comparing *A* and *B* the results are seen to be about the same. More important are the results shown in *C* and *D*. In *C* there are vertical stirrups only; in *D* the beam is very thoroughly reinforced by both stirrups and bent rods. Notwithstandng this difference the development of cracks is nearly the same in both cases, but *D* is stronger than *C*. At a load of 12,000 kg. all beams show considerable cracking and about as much in those fully reinforced as in *A* and *B*. This load corresponds to a stress of about 12,000 lbs/in.^{2} in the steel.

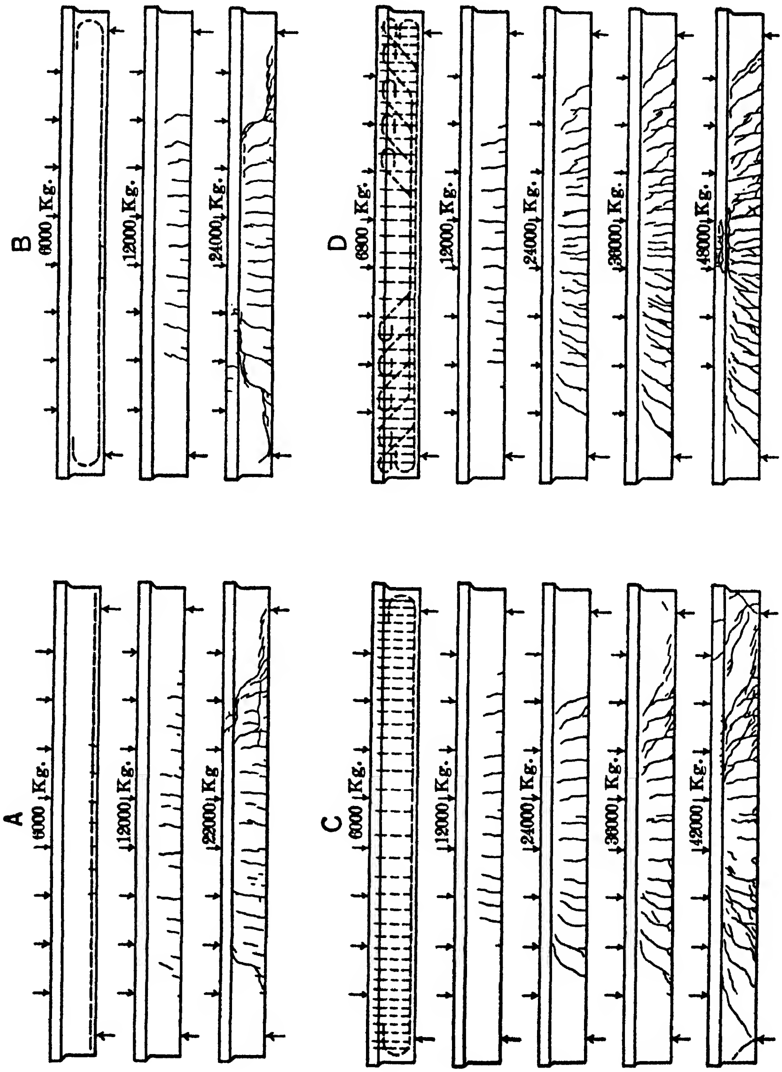


FIG. 16.—Development of Cracks in T-beams.

It may be said, therefore, that no kind of web reinforcement will prevent incipient cracking. What web reinforcement accomplishes is to prevent diagonal cracks from opening up to a dangerous extent and greatly to increase the ultimate strength of the beam. In this respect the vertical stirrup appears from these tests to be about as effective as bent rods.

139. Tests on Beams with Heavily Reinforced Webs.—Tests by Slater, Lord, and Zipprodt showed the possibility of developing very high shearing-stresses by the use of sufficient web steel. The beams tested were mostly of plate-girder form, with thin webs and wide flanges, both top and bottom, heavily reinforced in tension and most of them in compression. Flanges were from 12 in. to 17½ in. wide, and webs 2 in. to 12 in. thick. The concrete was of very high strength. Table 9 gives significant data from those tests having large amounts of web reinforcement and for web thickness from 3 to 8½ in. The span length was 9 ft. 6 in., and depth 36 in. The yield point of the web steel was about 60,000 lbs/in.^{2} All beams listed in the table failed by diagonal tension.

Group 1 had no web reinforcement. These failed at shearing values of 7% to 12% of the compressive strength, about the same values as shown by other tests. Group 2 had vertical web reinforcement spaced 4 in. apart, well hooked around the horizontal bars at top and bottom. Shearing-stresses were developed of 1000 to 1800 lbs/in.^{2}, the lower values being determined by the yield point of the web steel. For those where the yield point was not reached, the shearing-stresses were from 24% to 32% of the compressive strength. Shearing-stresses corresponding to 0.01-in. cracks were from 400 to 600 lbs/in.^{2}, or about the same as the stress at failure for unreinforced beams. In Group (3) diagonal bars at 45°, also spaced 4 in. apart, were used, with results markedly better than those indicated for vertical bars. Only one beam was sufficiently reinforced to prevent failure by over-stressing of the steel, and this gave a shearing strength of 45% of the compressive strength. Others gave values of about one-third of the compressive strength. The effect of diagonal bars in reducing the size of cracks is indicated by the high shearing-stresses of 1200 to 1600 lbs/in.^{2} prevailing for an average crack width of 0.01 in.

The tests indicated that a shearing-stress of 25% of the compressive strength can readily be developed by well-placed vertical bars. With diagonal bars this proportion can be increased to one-third or more.

The authors deduced from the tests the formula $v = (0.005 + r) f_v$

TABLE 9

TESTS ON BEAMS OF I-BEAM FORM

(Slater, Lord, and Zipprodt *)

Group	No.	Flange Width, In.	Web Thick-ness, In.	Web Reinforce-ment, Per Cent	Shearing-stress at Failure, Lbs/in.^{2}	Measured Stress in Web Steel, Lbs/in.^{2}	Shearing-stress for 0.01-in. Cracks, Av. of 5 Largest	Com-pressive Strength of Cylinders, Lbs/in.^{2}
1	109	17.5	8.5	0	460	......	......	6050
	110	17.5	8.5	0	390	......	......	5500
	111	17.5	8.5	0	470	......	......	4060
	107	15.0	6.0	0	630	......	......	6180
	108	15.0	6.0	0	590	......	......	3650
				Vertical Bars				
2	13	17.5	8.5	1.3	1030	y. p.	420	5970
	14	17.5	8.5	1.33	1050	......	710	5780
	18	17.5	8.5	1.3	970	y. p.	450	3510
	12	15.0	6.0	1.88	1250	y. p.	540	5340
	17	15.0	6.0	1.86	950	43,000	410	4000
	11	12.0	4.0	2.48	1340	46,000	870	5400
	5	12.0	3.0	3.35	1750	42,500	520	5940
	22	12.0	3.0	2.51	1800	59,500	670	5700
	34	12.0	3.0	1.55	1550	y. p.	620	5550
				Diagonal Bars				
3	67	12.0	3.0	3.54	2320	64,000	......	5120
	68	12.0	3.0	3.5	2150	y. p. (68–74)	......	5530
	69	12.0	3.0	2.38	1860		1550	5500
	70	12.0	3.0	2.52	1840		1630	5310
	72	12.0	3.0	1.58	1790		1200	5400
	73	12.0	3.0	1.49	1730		1360	5190
	74	12.0	4.0	1.16	1220		1000	4380

Web bars spaced 4 in. apart and hooked about the horizontal bars. All failures by diagonal tension.

* Technologic Paper, Bureau of Standards, No. 314, 1926.

as representing approximately the relation between the total shearing-stress v per square inch, the ratio of web reinforcement r, and the actual stress f_v in the web steel (See Art. 121.)

140. Conclusions Regarding Strength in Diagonal Tension.—From the foregoing data it would appear that the ultimate diagonal tensile strength, as measured by shearing-stress, of beams with no web reinforcement is from 6% to 10% of the compressive strength of the concrete; that for beams of ordinary strength concrete a diagonal tensile strength of 20% of the compressive strength can readily be developed, using from 1% to 1.25% of web steel; that a maximum of about 30% of the compressive strength can be developed by careful design, which for high-strength concrete requires an unusually large amount of web steel. Such high values are required only where thin webs are necessary. Diagonal bars are especially advantageous where high stresses are to be carried, but for ordinary cases vertical bars are about as effective.

CHAPTER V

DESIGN OF BEAMS

141. Working Stresses and Factors of Safety.—In the design of steel structures it has come to be the practice to make use of definite working stresses rather than factors of safety. These working stresses are based, for the most part, on the elastic-limit strength of the material, although the margin of safety between the elastic-limit and the ultimate strength (indicated by strength and ductility) receives consideration. The working stresses are made sufficiently below the elastic limit to provide for:

(*a*) Variations and imperfections in material, design, and workmanship.
(*b*) Uncalculated deformation stresses, such as secondary stresses, stresses due to unequal settlement, and, usually, those due to temperature changes.
(*c*) Dynamic effect of live load if not provided for by an allowance for impact.
(*d*) Possible increase in live load over that assumed, or rare applications of excessive loads.
(*e*) Deterioration of the structure.

The more accurately the various elements are determined in any case the closer may the working stress approach the elastic limit. Where the dynamic effect of the live load does not enter, or is otherwise fully provided for, and where items (*d*) and (*e*) are of small moment, working stresses for steel structures will vary from about one-half to two-thirds the elastic-limit strength of the material. Were it certain that the elastic limit of the material would never be exceeded, the working stresses could be placed still higher and the margin of strength between the elastic limit and the ultimate strength would be of no importance. This is, however, not the case, and the fact that the ultimate strength is, in most materials, much higher than the elastic limit constitutes an important element of safety. The *ductility* of the material, or its possible deformation beyond the elastic limit, is also of great importance, especially by reason of its effect on deformation stresses such as mentioned in item (*b*).

In reinforced-concrete design the problem is complicated by the use of two unlike materials whose stress-deformation relations are quite unlike, as pointed out in Chap. II. The time effect is also of importance as having a large influence on the deformation of the concrete.

142. Relative Effect of Dead and Live Loads.—The tendency of practice in the treatment of live-load stresses is to reduce them to equivalent dead-load stresses by the application of some sort of impact formula or by other means of estimation. The resulting stresses are then considered on the same basis as the usual dead-load stresses and a single set of working stresses applied. The question of impact coefficients, or the relation between live- and dead-load working stresses, requires little special attention in connection with reinforced-concrete structures, but is of great importance in the case of railway bridges and in floor members of highway bridges. The proper coefficient to use, or the relation between live- and dead-load working stresses, varies much under different conditions and must be left to the judgment of the designer, or to formulas or rules prepared especially for the purpose. Further discussion of this question will not be undertaken here.

In case of buildings the question of impact is of minor importance. When a building is fully loaded, the portion of the load which is in motion and capable of producing a dynamic effect is generally but a very small percentage of the total live load. In most cases, therefore, in building construction it is not necessary to treat the live-load stresses differently from the dead-load stresses, and the design is based on a single set of working stresses. Special cases will arise, however, where the dynamic effect of the live load requires consideration, as, for example, in the case of floors supporting moving machinery.

Whatever the effect of live load may be, it can more readily be taken account of by adding to the resulting live-load stresses a percentage which, in the judgment of the engineer, will reduce them to their dead-load equivalent, and then apply a single set of working stresses, or factor of safety, to the sum of the stresses. The discussion of working stresses in the following articles will relate to the proper basal working stress for dead load, or for live load suitably increased for impact.

Allowance is commonly made in the design of large girders and columns which receive their load from large areas for the fact that such large areas, especially if on two or more floors, are seldom or never loaded to the extent assumed for smaller areas. This allowance varies with different conditions, but relates solely to the selection of the amount of live load rather than to its effect.

143. Working Stresses in Tension and Compression.—The strength of a beam is limited usually by:

(*a*) The compressive strength of the concrete,
(*b*) The elastic-limit strength of the horizontal steel, or
(*c*) The strength of the beam in diagonal tension.

In this article the first two elements only will be considered.

As shown in Chap. II, the permanent elastic limit of concrete is about 60% of its ultimate strength. Plastic flow will occur under lower stresses, but such flow does not appear to impair the elasticity of the concrete under live-load variations or reduce its ultimate strength. A working stress of 40% of the compressive strength at 28 days is commonly used. As regards the ultimate strength of beams with respect to the concrete stress it has been shown (Art. 69) that, because of the curved form of the stress-deformation diagram of concrete, the stress in the concrete does not increase in proportion to the load, and that the factor of safety is larger than the ratio of the compressive strength to the working stress. Thus for a working stress of one-third of the ultimate, the factor of safety is about 4.3 and a working stress of 40% gives a factor of safety of about 3.6.

With respect to the steel the ultimate beam strength is closely determined by the yield point, which ranges from 33,000 to 40,000 lbs/in.^{2} for structural grade to about 50,000 lbs/in.^{2} for rail steel.

For the structural grade the Joint Committee in 1924 specified a working stress of 16,000 lb/in.^{2}, but 18,000 lb/in.^{2} is used in the American Concrete Institute Specifications. The latter value is in general use in structural-steel design and is quite as applicable to reinforced concrete. It gives a factor of safety of about 2. For the harder grades of steel the Joint Committee uses 18,000 lbs/in.^{2}, and the American Concrete Institute Specifications 20,000 lbs/in.^{2} Compared to the elastic limit of the harder grades of steel the value of 20,000 gives a larger factor of safety than 2, but higher stresses than this are objectionable on account of the lesser ductility of the harder steels and the increased deformations involved, resulting in a greater degree of cracking of the concrete.

Comparing the steel and concrete stresses, it is seen that, with a concrete working stress of 40% of its compressive strength, the ultimate strength of the beam will depend upon the steel, although its elastic limit will probably be determined by the concrete. The Joint Committee in 1916 specified a concrete stress of 32½% of the strength of the concrete; but on the basis of further experience and consideration of the question the Committee in 1924 raised this to 40%. This value is used in the American Concrete Institute Specifications.

144. Working Stresses in Shear (Diagonal Tension).—

JOINT COMMITTEE, 1924

(*a*) Beams without web reinforcement	$0.02 f_c'$
(*b*) Beams without web reinforcement but with special anchorage for longitudinal bars	$0.03 f_c'$
(*c*) Beams with web reinforcement	$0.06 f_c'$
(*d*) Beams with web reinforcement and special anchorage for longitudinal bars	$0.12 f_c'$

f_c' = compressive strength at 28 days.

From the results given in Arts. 136 and 137, the ultimate strength in diagonal tension has a minimum value of about $0.06 f_c'$. A working stress of $0.02 f_c'$ therefore gives a minimum factor of safety of about 3. Inasmuch as failures due to high shearing-stresses are apt to come without warning, a relatively high safety factor is desirable. The value of $0.03 f_c'$ for case (*b*) is based upon the greater certainty of bond strength here provided for. In view of the results referred to, it does not appear that such a value would insure a factor of more than 2 in all cases, especially for high-strength concrete and low ratio of depth to span, as would occur for concentrated loads. For beams of the usual proportions and for uniform loads the value of $0.03 f_c'$ is fairly conservative.

For beams with web reinforcement, the value of $0.12 f_c'$ presupposes a high degree of excellence in web steel design and placement and appears to be too high for the usual quality of workmanship encountered in building construction. The later specifications of the American Concrete Institute of 1928 use a value of $0.09 f_c'$, for ordinary practice, but permit $0.12 f_c'$ under special provision for design and inspection. Under such circumstances a factor of safety of at least 2 can readily be secured.

145. Working Bond Stresses and Anchorage.—Joint Committee, 1924: also American Concrete Institute, 1928.

(*a*) Plain bars, $0.04 f_c'$.
(*b*) Deformed bars, $0.05 f_c'$.
(*c*) Bent-up bars within a distance of $\frac{1}{3} d$ from the tension reinforcement may be included in computing $\Sigma\, o$.

Double the above values are allowed where special anchorage is provided.

From the data of Arts. 130–133 the bond strength of plain bars is about $0.10 f_c'$ to $0.12 f_c'$, which indicates a factor of safety of about $2\frac{1}{2}$ with respect to bond.

For deformed bars the working stress may be somewhat greater than for plain bars, but as the initial slip occurs at about the same load, the increase in working stress should not be large.

Since adequate bond strength can be provided for at small expense it is obviously uneconomical to limit the strength of a beam by its bond strength, and the factor of safety should be ample.

The "special anchorage" referred to here and also in Art. 144 is provided by extending the bars beyond the critical section sufficiently to take care of the deficiency of bond strength. In the case of a simply supported beam the critical section is at the face of the support, and in a continuous beam, the point of inflection. This extension of bars in effect distributes the bond stress somewhat differently from the results of the theoretical formula $u = v\, b/\Sigma\, o$, but gives adequate total strength. Hooks may be used of a diameter not less than four bar diameters with a bond stress the same as for straight bars.

146. Size, Length, and Spacing of Horizontal Bars.—*Size of Bars.*—The total required sectional area is determined by the moment. The number and size are determined by convenience in placing, necessary space between bars, the use of part of the bars for web reinforcement, and the bond strength required. For equal total sectional areas, the larger the bars the greater the bond stress. The size may therefore be limited by the bond stress. If a portion of the bars are bent up, then the size and number of the straight bars must still meet the bond stress requirements.

A convenient expression for the minimum sum total of bar perimeters, in terms of shear and bond stress, can be derived as follows: The shearing-stress is $v = \frac{V}{v\,b\,j\,d}$, and the bond stress is

$$u = \frac{V}{j\,d\,\Sigma o}$$

from which

$$u = \frac{v\,b}{\Sigma o} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1a)$$

and

$$\Sigma o = \frac{v\,b}{u} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1b)$$

In (*a*), u is the bond stress for a given value of Σo, and in (*b*), Σo is the necessary total bar perimeter for a given bond stress u. For instance, if $v = 160$ and $u = 80$, then Σo must be at least $2 \times b$. If this is not readily secured then an alternative is to provide some of the bond strength by hooks or straight extension of the bars termed "special anchorage" in the specifications.

147. *Lengths of Bent-up Bars.*—The minimum lengths of the several horizontal bars are determined by the bending moments to be carried. In determining these lengths the same methods, either analytical or graphical, may be employed as in the design of plate girder flanges. If the moment is due to a uniform load, the parabolic formula may be used. The lengths of the several rods are given by the equation

$$x_n = \frac{l}{\sqrt{A}} \sqrt{a_1 + a_2 + \ldots + a_n}, \quad . \quad . \quad . \quad . \quad (1)$$

in which x_n = length of the nth rod in order of length counting the shortest one as number one;

l = length of span;

A = total steel area at centre;

$a_1 + a_2 + \ldots + a_n$ = sum of areas of all rods up to the one in question (the nth rod).

If the amount of steel actually used is considerably greater than required, then the formula may be modified by making A = required area and deducting the excess from the area of the first rods. Thus,

$$x_n = \frac{l}{\sqrt{A_r}} \sqrt{a_1 - (A_n - A_r) + a_2 + \ldots + a_n}, \quad . \quad . \quad (2)$$

in which A_r = area required;

A_n = area used.

For unsymmetrical or concentrated loading the actual moment curve must be determined and the length of bars determined therefrom as in plate girder design.

The required theoretical lengths having been found the rods may be bent up at these points or at any desired place between these points and the supports. Or, if not bent up, they may be discontinued a few inches beyond the theoretical points and the ends bent into hooks for better bond.

148. *Spacing of Bars.*—The requirement in general as to spacing is that the bars must be spaced sufficiently far apart to readily admit the concrete between and beneath them and to give sufficient section along the plane of the rods to prevent failure by tension or shear. A common requirement is a clear spacing of 1½ bar diameters (diagonal width for square bars) but not less than 1 in. nor less than 1⅓ times the maximum size of the coarse aggregate.

149. Proportioning of Rectangular Beams.—*General Design.*—The size of a rectangular beam must meet two conditions. To resist the bending moment, we have from Art. 54, eq. (12),

$$b\,d^2 = \frac{M}{R}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and from Art. 110 the size necessary to resist the shear is

$$b\,d = \frac{V}{vj}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where v is the allowable maximum shearing-stress and V = maximum total vertical (end) shear. With the value of $b\,d^2$ determined from (1), corresponding values of b and d are selected to give convenient and economical proportions. For any given value of $b\,d^2$, the value of $b\,d$ will decrease as d increases. Hence the deeper the beam the less the required cross-section $b\,d$. Therefore, until the value of $b\,d$ becomes as small as determined by (2), the deeper the beam the less the cross-section and the less the amount of concrete.

In general, then, we may say that for rectangular beams the ratio of depth to breadth should be made as great as consistent with convenience of proportions, space for bars, head room, cost of forms, and similar practical requirements. Generally speaking the convenient ratio of depth to breadth will vary from a minimum of about 1½ to 1 for small beams up to 3 or 4 to 1 for very large beams. Ample width for placing of bars is an important factor.

150. *Ratio of Length to Depth for Equal Strength in Shear and Moment.*—It is instructive to consider the relation between the proportions of a beam (length-depth ratio) and the maximum moment and shearing-stresses therein.

For a uniform load we have $M = \frac{1}{8} w\,l^2$ and $V = w\,l/2$. Also from (8), Art. 53, $b\,d^2 = M/f_s\,p\,j$; and from (2) Art. 149, $b\,d = V/v\,j$. Substituting the values of M and V above, and dividing, we find that

$$\frac{l}{d} = \frac{4 f_s\,p}{v} \quad \text{. . . . .} \quad (3)$$

For a single concentrated load a similar analysis gives

$$\frac{l}{d} = \frac{2 f_s\,p}{v} \quad \text{. . . . .} \quad (4)$$

For $f_s = 18{,}000$ and $f_c = 1000$ ($f'_c = 2500$), $p = 0.0111$. We then have

For $v = 50$, no web reinforcement.

For uniform load . . . . $l/d = 16.0$
For a concentrated load $l/d = 8.0$

For $v = 225$, full web reinforcement.

For uniform load . . . . $l/d = 3.55$
For a concentrated load $l/d = 1.77$

For larger ratios of l/d than the above values the strength in shear will be greater than that in bending. From these ratios it will be easy to judge quickly whether or not the shear is likely to be the determining factor in any case. It is evident that only in relatively deep beams or for special cases of concentrated load will the shearing strength come into question or will it be necessary to use the higher values of allowable stress. It is only in T-beams that shearing-stresses become troublesome.

151. **Design of a Rectangular Beam.**—(Fig. 1.) Design a rectangular beam 16 ft. long to support a live load of 4000 lb. per ft. Use a concrete of 2500-lb. strength and the 1924 Joint Committee specifications with certain exceptions as noted. The following are the working stresses and certain constants:

$f_s = 18{,}000$ lbs/in.^{2}
$f_c = 1000$ lbs/in.^{2}
f_v for web reinforcement $= 16{,}000$ lbs/in.^{2}
$v = 0.03 f'_c = 75$ lbs/in.^{2} without web reinforcement, bars to have special anchorage.
$v = 0.06 f'_c = 150$ lbs/in.^{2} with web reinforcement.
$v = 0.09 f'_c = 225$ lbs/in.^{2} with web reinforcement and special anchorage.
$n = 12$; $k = 0.4$; $j = 0.87$; $p = 0.00111$; $R = 173$.
$u = 100$ lbs/in.^{2}; or 200 lbs/in.^{2} with special anchorage.
Assume weight of beam = 400 lbs. per ft.

Bending Moment.

$$M = \tfrac{1}{8} \times 4400 \times 16^2 \times 12 = 1{,}690{,}000 \text{ in-lbs.}$$

$$b\,d^2 = \frac{1{,}690{,}000}{173} = 9750.$$ This value of $b\,d^2$ can be obtained as follows:

30 in. $\times$ 11 in., $b\,d^2 = 9900$, $b\,d = 330$ sq. in.
28 in. $\times$ $12\frac{1}{2}$ in., $b\,d^2 = 9800$, $b\,d = 350$ sq. in.
26 in. $\times$ $14\frac{1}{2}$ in., $b\,d^2 = 9800$, $b\,d = 377$ sq. in.

The deeper the beam the smaller the cross-section required, and with no limitation of head room the 30-in. beam might be used if the shearing-stresses are not exceeded; but these proportions are rather extreme and the space for bars quite small. We will therefore adopt 28 in. × 12½ in. Steel area = 0.00111 × 350 = 3.88 sq. in.

Use five ⅞-in. round bars = 3.00 sq. in.
two ¾-in. round bars = 0.88 sq. in.
3.88 sq. in.

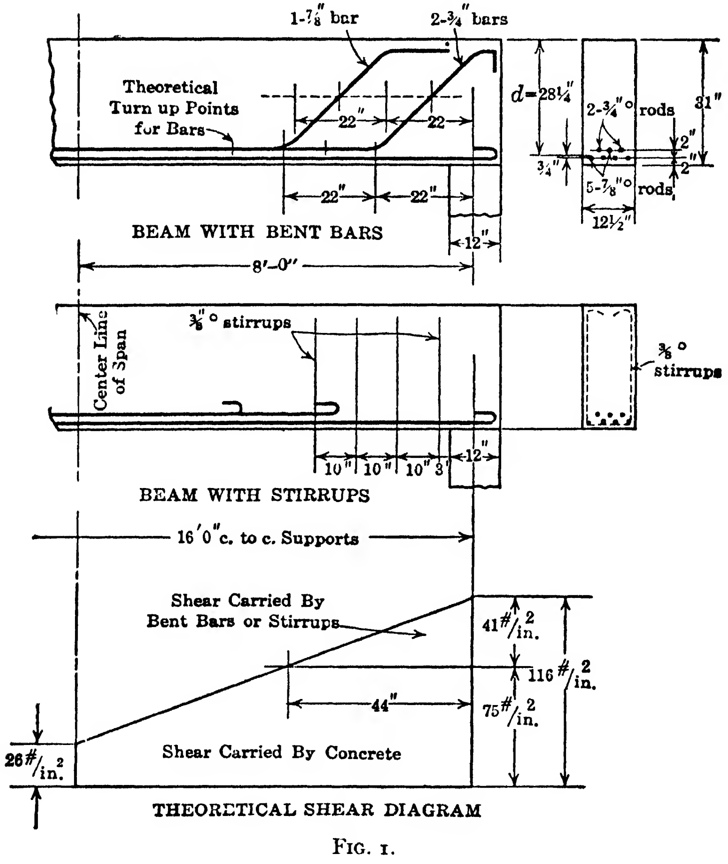


FIG. 1.

Place in two layers as shown in Fig. 1. Total depth of beam = 31 in., making the depth to centroid of tension area slightly over 28 in. Check for weight. Weight per foot $= \frac{12\frac{1}{2} \times 31 \times 150}{144} = 402$ lbs/ft.

Shearing-stresses.

At end, $V = 4400 \times 8 = 35{,}200$ lbs. $jd = 0.87 \times 28 = 24.4$ in.

$$v = \frac{35{,}200}{24.4 \times 12.5} = 116 \text{ lbs/in.}^2$$

At centre, live load shear $= \dfrac{4000 \times 8}{4} = 8000$ lbs.

$$v = \frac{8000}{24.4 \times 12.5} = 26 \text{ lbs/in.}^2$$

Assume maximum shear varies uniformly from centre to end.
Distance from centre for $v = 75$ is found by the proportion

$$\frac{x}{8} = \frac{75 - 26}{116 - 26} = \frac{49}{90}. \quad x = 4.33 \text{ ft.} = 4 \text{ ft. } 4 \text{ in.}$$

From this point to the end, a distance of 44 in., web reinforcement must be used. The shear diagram is shown in Fig. 1. At the end the shearing-stress to be carried by the steel is 41 lbs/in.^{2}

Bond Stresses.

For special anchorage at least one-half of the reinforcement must extend to the end. This requires four bars. For four ⅞-in. bars, $\Sigma\, o = 4 \times 2.75 = 11.0$ sq. in., $u = \dfrac{v\,b}{\Sigma\, o}$. At end $u = \dfrac{116 \times 12.5}{11.0} = 132$ lbs/in.^{2} Special anchorage requires the four bars to extend beyond the face of the support a sufficient distance to develop one-third the working stress at a bond stress of 100 lbs/in.^{2}

The length of embedment for *full* working stress is found by the relation $18{,}000 \times \frac{1}{4}\,\pi\, D^2 = 100 \times \pi\, D\, l$, whence $l = 45\, D$, where l = length of embedment and D = diameter of bar. For one-third working stress the embedment is 15 diameters $= 15 \times \frac{7}{8} = 13$ in. This will be supplied by hooks as shown. Diameter of hook to be not less than 4 bar diameters.

Web Reinforcement.

Try bent-up bars at 45°. Three bars can be bent up. Maximum spacing by Joint Committee specifications $= \dfrac{45}{10 + 45}\, d = 0.82\, d = 23$ in. Length of beam needing reinforcement = 44 in. It will then be sufficient to bend up the bars at two points only. The arrangement shown in Fig. 1 will be adopted. A spacing of 22 in. is used, and it is assumed that each diagonal group is effective over a horizontal distance of 22 in. spaced symmetrically with respect to the points where the reinforcement crosses the beam centre.

The stresses in the bent-up bars will now be checked. Taking the theoretical end shearing-stress at $v' = 41$ lbs/in.^{2}, the total stress in the end two bars will be $P = 0.7 \times 41 \times 12.5 \times 22 = 7900$ lbs., a stress of only 9000 lbs/in.^{2} In the ⅞-in. bar it will be less.

The length of embedment in the upper half of the beam must be sufficient to develop the stress in the bars at a bond stress of 100 lbs/in.^{2} For the ¾-in. bars this length $= 3950/100 \times 2.35 = 17$ in.

Necessary Length of Bars for Moment.

$$\begin{aligned}\text{Total sectional area at centre} &= 3.88 \text{ sq. in.}\\ \text{Area } 7/8\text{-in. bar} &= 0.60 \text{ sq. in.}\\ \text{Area } 3/4\text{-in. bar} &= 0.44 \text{ sq. in.}\end{aligned}$$

Hence if the 7/8-in. bar is the first to be bent up its necessary half length is given by the proportion

$$\frac{x^2}{8^2} = \frac{0.60}{3.88};\ x = 3.15 \text{ ft., or } 4.85 \text{ ft. from end.}$$

For the two 3/4-in. bars, $\frac{x^2}{8^2} = \frac{1.48}{3.88}$; $x = 5.0$ ft., or 3 ft. from end. These requirements are met in the design.

Use of Stirrups for Web Reinforcement.

Suppose vertical stirrups are used instead of bent-up bars. Use a stirrup made of 3/8-in. round material. Working stress per stirrup = $16{,}000 \times 2 \times 0.11 = 3520$ lbs. Spacing at end for $v' = 41$ is $s = \frac{3520}{41 \times 12.5} = 6.9$ in. At 1 ft. from end $v' = 30$ and $s = \frac{3520}{30 \times 12.5} = 9.4$ in. At 2 ft. from end $v' = 19$ and $s = \frac{3520}{19 \times 12.5} = 14.8$ in. Maximum allowable spacing $= 0.45\,d = 12.6$ in. To meet the requirements of maximum spacing four stirrups will be needed. They may be spaced as shown in Fig. 1.

If the reaction be considered as wholly applied at the centre of bearing and the theoretical end shear of 41 lbs/in.^{2} be used, the first stirrup should be placed about 4½ in. from the centre of bearing, which would bring it within the area of the support. This is unnecessary as the actual shearing-stresses fall off rapidly from the edge of support towards the centre. If the first stirrup is placed about ⅓ s from the face of the support, the diagonal tensile stresses at this point will be adequately taken care of. In the case of continuous beams the face of the support is generally considered as the critical section for shear and bond and the first stirrup placed ½ s from that point.

152. Proportioning of T-Beams.—*General Design.*—T-beams occur in practice generally where a floor-slab and beam are built as a monolithic structure, as in floor construction. Occasionally, also, where heavy girders are required it is expedient to design the beam in the form of a T. Inasmuch as the only purpose of the concrete below the neutral axis is to bind together the tension and compression flanges, its section is determined by the shearing-stresses involved and space required for bars, and a considerable saving can thus often be effected over the rectangular form. Where the flange is a part of a slab its thickness is determined in the design of the floor, but the width of slab which can be taken as effective flange width must be estimated. A common rule of practice is to count a width of slab not greater than one-fourth of the span length, but this should in fact depend also upon thickness of slab and of the stem of the T. Tests on T-beams with very wide flanges show that the compressive stresses are quite uniformly distributed over the entire width. A still more even distribution of stress is to be expected in a series of T-beams in a continuous floor structure. If, however, the slab or flange is made too wide and thin the shearing-stresses along the line aa' and cc', Fig. 2, will be excessive. The Joint Committee specifies a maximum width of overhang x of 8 times the thickness. For isolated T-beams the total width b to be not greater than $4t$.

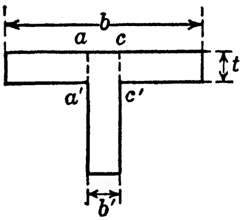


FIG. 2.

Where a T-beam is not connected with a floor system, the size of flange may be selected to meet the conditions at hand. In this case the stem of the beam should first be determined approximately, on the basis of thesh earing-stresses to be carried. A suitable flange can then be selected by a few trials, as explained in Art. 79. The deeper the beam the less the amount of steel required for constant cross-section. But T-beams should not be made too deep in proportion to width, as such forms are relatively weak at the junction of stem and flange. All re-entrant angles in rigid material such as concrete are points of weakness and such angles should therefore be modified by curved lines or by a beveled fillet. A width of beam sufficient to carry the shear and to give plenty of space for the bars is usually ample. The maximum desirable ratio of depth to width may be taken at about two for small beams up to three or four for very large and massive work. Depths are often determined by available head room. Beams of excessive depths are objectionable as being more difficult and troublesome to reinforce properly; the cost of web reinforcement also becomes relatively greater. The flanges should be thoroughly bonded to the web by means of web reinforcement running well up into the flange and, where the flange is wide, by additional cross-reinforcement in the plane of the flange.

153. Economical Proportions.—Where a floor-slab forms the flange of a T-beam, the economical proportions of the stem may be considered. Here the slab forms practically all the compressive area and its cost is not affected by the proportions of the stem. The required steel area is given by the formula $M/f_s j d$. Then let C = cost of beam per unit length, c = cost of concrete per unit volume, and r = ratio of cost of steel to cost of concrete per unit volume. The total cost of the stem per unit length will then be (cost of stirrups and variation in cost of forms neglected),

$$C = c\left[(d + a - t)\,b' + \frac{r\,M}{f_s j\,d}\right] \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In this expression a is the distance from center of steel to bottom face of beam, assumed as constant.

From this expression it is evident that with a fixed value of $b'\,d$ to meet shear requirements the cost decreases with decreased width b' and increased depth d. The width b' must, however, be made sufficient to accommodate the bars conveniently. If b' is assumed constant we find, by differentiating (1), the value of d for minimum cost will be

$$d = \sqrt{\frac{rM}{f_s b' j}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If this value is less than required for shear then the shear governs, but in case b' is taken relatively large, the economical value of d may be greater than required for shear.

The depth of stem is also frequently influenced by the question of head room required and other architectural considerations.

154. Design of a T-Beam.—(Fig. 3.)—A beam and slab floor (beams one way only), spans an opening of 24 ft. centre to centre. The slab is 5 in. thick, the beams are 10 ft. on centres, and the live load is 250 lbs/ft.^{2} The beams are simply supported and are to be reinforced against diagonal tension by means of bent-up bars and stirrups. Assume a 2500-lb. concrete and the same specifications as in the problem of Art. 151. Live load = 2500 lbs. per lineal foot. The 5-in. slab 10 ft. wide weighs 625 lbs/ft.

Assume weight of stem of beam	=	275 lbs/ft.
Slab	=	625 lbs/ft.
Live load	=	2500 lbs/ft.
Total	=	3400 lbs/ft.

Dimensions of Stem and Steel Area.

$$M = \tfrac{1}{8} \times 3400 \times 24^2 \times 12 = 2{,}950{,}000 \text{ in-lbs.}$$

The relation of depth to width will be investigated by the use of eq. (2), Art. 153. The width should probably be sufficient to give space for 4 or 5

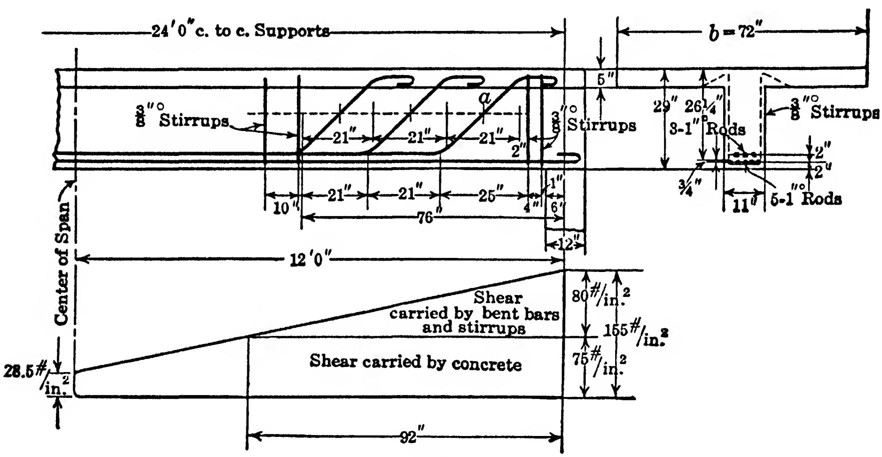


FIG. 3.

large bars, say 12 in. Assume cost of steel = 4¢ per lb. and concrete $10 per cu. yd. Then $r = \dfrac{0.04 \times 490}{10/27} = 53$. Assume $j = 0.9$. Then by eq. (2), Art. 153,

$$d = \sqrt{\frac{53 \times 2{,}950{,}000}{18{,}000 \times 12 \times 0.90}} = 28.3 \text{ in.}$$

With bars in two layers the total depth would be about 3 in. greater or about $31\frac{1}{2}$ in., and the stem below the slab $31\frac{1}{2} - 5 = 26\frac{1}{2}$ in. If b' were made 11 in., then $d = 28.3 \times \sqrt{12/11} = 29.6$ in.

The true economical depth will be somewhat less than the calculated value, and in this case the depth of stem below the slab will be assumed at 24 in. With 3 in. from lower face to center of steel area the value of d will be $5 + 24 - 3 = 26$ in.

Then $t/d = 0.19$, and from Diagram 5, j may be taken at 0.92, $j\,d = 0.92 \times 26 = 24.0$ in.

$$\text{Steel area} = \frac{2{,}950{,}000}{24 \times 18{,}000} = 6.85 \text{ sq. in.}$$

Use three 1-in. square bars = 3.00 sq. in.
five 1-in. round bars = 3.92 sq. in.

Total = 6.92 sq. in.

Placing the five round bars in one layer will require a minimum width of 11 in., which will be used.

$$\text{Check weight of stem, } \frac{11 \times 24 \times 150}{144} = 275 \text{ lbs/ft.}$$

Check value of j. Assume width of flange = ¼ of span = 72 in.; $b\,d = 72 \times 26 = 1870$; $p = 6.92/1870 = 0.0037$. From Diagram 5 with $t/d = 0.19$ and $p\,n = 12 \times 0.0037 = 0.0444$, $j = 0.92 +$ as assumed. Hence the calculated stress in steel is correct. The stress in the concrete will be small. Diagram 5 gives $f_s/n\,f_c = 2.75$, whence $f_c = 540$ lbs/in.^{2}

(If the calculations are made by the method given in Art. 77, assuming full concrete stress of 1000 lbs/in.^{2}, we have $f_s/n\,f_c = \dfrac{18{,}000}{12{,}000} = 1.5$, and Diagram 5 gives $p\,n = 0.095$ and a value of j slightly less than before, but the difference is immaterial. The width of flange which corresponds to this assumption is found from $A_s = p\,b\,d$, or $6.85 = 0.093 \times b \times 26/12$; when $b = 34$ in.)

Shearing-stresses and Web Reinforcement.

At end, $V = 3400 \times 12 = 40{,}800$ lbs.
At centre, $V = 2500 \times 12 \times \frac{1}{4} = 7500$ lbs.

$$v_{\text{end}} = \frac{40{,}800}{11 \times 24} = 155 \text{ lbs/in.}^2$$

$$v_{\text{centre}} = \frac{7500}{11 \times 24} = 28.5 \text{ lbs/in.}^2$$

For point of 75-lb. shearing-stress

$$\frac{x}{12} = \frac{75 - 29}{155 - 29} = 0.365;\ x = 4.38 \text{ ft. from centre.}$$

Distance from end $= 12 - 4.38 = 7.62$ ft. $= 92$ in.

$$\text{Area of shear triangle} = \frac{(155 - 75) \times 92}{2} \times 11 = 40{,}500 \text{ lbs.}$$

Bending of Bars.

Not more than one-half the tension reinforcement may be bent up. For this purpose the three square bars will be used.

Points of bend. Total area = 6.92. Each bar = 1 sq. in. Then distance from centre must be at least, for

First bar: $\frac{x_1^2}{12^2} = \frac{1}{6.92}$, $x_1 = 4.45$ ft.

Second bar: $\frac{x_2^2}{12^2} = \frac{2}{6.92}$, $x_2 = 5.9$ ft.

Third bar: $\frac{x_3^2}{12^2} = \frac{3}{6.92}$, $x_3 = 7.9$ ft.

The end shear to be carried by web steel = 155 − 75 = 80 lbs/in.² Since total shearing-stress is greater than 0.06 f_c', the spacing of web reinforcement must be two-thirds the value given by the formula $s = \frac{45}{10 + \theta} d$. For bent bars at 45°, $s = 30/55\, d$ = 14.2 in. As it is difficult to secure this spacing for bent bars at the end of a beam of this depth, one or two stirrups will be used in this region.

For stirrups, try ½-in. round steel. Allowable stress = P = 2 × 0.196 × 16,000 = 6300 lbs.

$$s = \frac{6300}{80 \times 11} = 7.1 \text{ in.}$$

For a ⅜-in. stirrup, P = 2 × 0.11 × 16,000 = 3520 lbs.

$$s = \frac{3520}{80 \times 11} = 4.1 \text{ in.}$$

Use two ⅜-in. stirrups at 4-in. spacing, placing the first one 1 in. from support.

Using a bent bar next, the total shear is now less than the limiting value of 150 lbs/in.², so that a 21-in. spacing may be used. The centre of this space is about 24 in. from centre of support. Shear = v = 70/92 × 80 = 59. Stress in bar = 0.7 × 59 × 11 × 21 = 9800 lbs., a low stress for a 1-in. square bar. Using two more 21-in. spaces brings us 77 in. from the centre of support, or 15 in. from the point where no reinforcement is required. It is unnecessary to calculate the shear in this region. Use two stirrups spaced 10 in. apart as shown.

The arrangement of bent bars adopted meets the requirements of length for moment.

Bond.

At the end there are five 1-in. round bars, Σo = 13.35 in.

$$\text{Shear} = 155. \quad \text{Bond stress} = \frac{155 \times 11}{13.35} = 128 \text{ lbs/in.}^2,$$

requiring special anchorage. The bars must extend beyond the face sufficiently to develop a stress of 6000 lbs/in.^{2} This will require 15 diameters or 15 in. Use a hook of 4-in. diameter as shown.

In the case of bar a the stress is 9800 lbs. $\Sigma o = 4$ in. Length of embedment above centre of beam $= \dfrac{9800}{4 \times 100} = 24.5$ in., requiring a horizontal extension of at least 10 in. including the hook. Stresses in the others are less but it is well to extend them in the same manner. The ⅜-in. stirrups require an embedment of 45 diameters for full strength $= 45 \times \frac{3}{8} = 17$ in. above the centre of the beam. The necessary length is secured by using hooks as shown.

End Bearing.

This may be calculated on the basis of an allowable bearing pressure of $0.25 f_c' = 625$ lbs/in.^{2} if the wall or pedestal is of the same strength as the beam. This will require a bearing area of 40,800/625 = 65 sq. in., which is more than provided for in the assumed length of bearing of 12 in.

Effect of Using Theoretical Depth.

It will be interesting to determine the effect it would have in cost, under the conditions previously assumed, if the theoretical depth of 29.6 in. be used with the width b' the same as before.

$$\text{Depth of stem} = 27.6 \text{ in.}$$

$$\text{Required steel area} = \frac{2{,}950{,}000}{18{,}000 \times 0.91 \times 29.6} = 6.08 \text{ sq. in.}$$

$$\text{Cost of steel} = 6.08 \times 10/3 \times 4 = 81.1¢ \text{ per ft.}$$

$$\text{Concrete in stem} = \frac{27.6 \times 11}{144} = 2.11 \text{ cu. ft. per ft.}$$

$$\text{Cost of concrete} = \frac{2.11 \times 1000}{27} = 78.1¢ \text{ per ft.}$$

$$\text{Total cost} = 81.1 + 78.1 = 159.2¢ \text{ per ft.}$$

For the design as made we have

$$\text{Cost of steel} = 6.92 \times 10/3 \times 4 = 92.3¢ \text{ per ft.}$$

$$\text{Cost of concrete} = \frac{24 \times 11 \times 1000}{144 \times 27} = 67.9¢ \text{ per ft.}$$

$$\text{Total cost} = 92.3 + 67.9 = 160.2¢ \text{ per ft.}$$

The difference is trifling and is about balanced by the extra length of web reinforcement required in the deeper beam. This calculation shows, however, that, as is always the case, a considerable variation of dimensions on either side of the theoretical values for a minimum will affect the result but little.

155. Design of Continuous Beams.—In steel work, when several successive beam spans are constructed, connected to a series of columns, each is usually designed as a simple beam, the end details being unsuited to carry the negative bending moments which would result from continuity of action. In reinforced concrete, on the other hand, it is desirable to construct the beams as a continuous or monolithic piece of work rather than as disconnected, simply supported beams. This method of design has been a gradual development. At first the structure was merely tied together by overlapping some of the lower reinforcing bars at the support; then to prevent cracks on the upper surface over or near the support, some reinforcement was placed in the upper part of the beam. These conditions led to the design and construction of such beams as true continuous girders, proportioning the reinforcement over the supports to carry the negative moments involved.

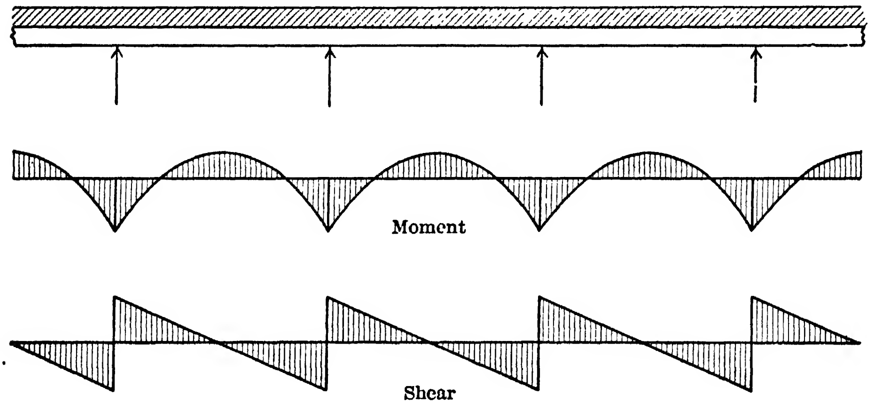
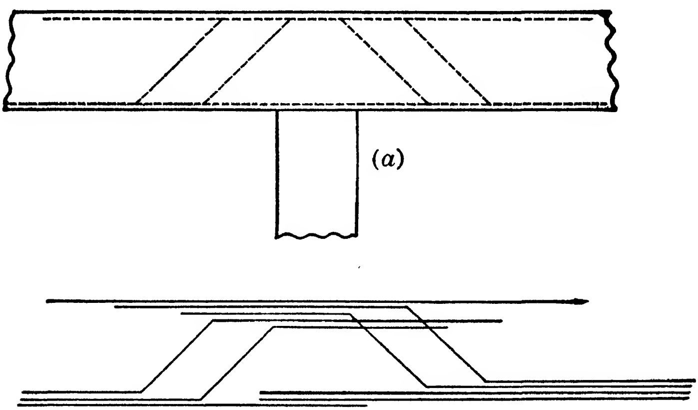


Fig. 4.

Fig. 4 represents the general variation in moment and shear in a continuous beam uniformly loaded. The negative moment is a maximum over the support, and is larger than the positive moment between supports. It decreases rapidly from the support. The point of inflecton is about 0.2 l from the support. The shear is about the same as in a simple beam; it changes sign suddenly at the supports.

Owing to action of live load the moments and shears will vary considerably from those represented in the diagram, and the points of inflection will shift more or less. Each span must be designed for its maximum positive moment at or near centre, and maximum negative moment at the support. The necessary length of positive and negative reinforcement must be determined by locating the points of inflection. This information also enables the points for bending up of the several rods to be determined. Calculation of these quantities is a problem involving the theory of continuous girders and rigid frames and, as it most commonly arises in connection with building construction, its discussion is reserved mainly to that chapter. Some general features of design only will be mentioned here.



(b)

FIG. 5.

156. *Arrangement of Reinforcement.*—Various arrangements of reinforcement may be used. Generally, some of the lower bars are bent up and extended over the support to furnish part of the upper reinforcement. Some of the lower reinforcement should extend well into the column or over the support and preferably overlap so as to provide good anchorage and bind the structure thoroughly together against contraction cracks. Fig. 5 illustrates a common arrangement of reinforcement for a continuous beam. A portion of the lower bars are extended straight through, and a portion are bent up and extended across the top to furnish part of the negative reinforcement. Fig. (*b*) shows a scheme of arrangement in detail. Additional short top bars are often used to make up the necessary reinforcement for negative

moment, and may be bent down at their ends to furnish additional shear reinforcement. In this type of beam the maximum moment occurs at the same point as the maximum shear which gives conditions more unfavorable as regards diagonal cracks than in the case of the simple beam. Special attention must therefore be given to the shear reinforcement.

Diagonal tension cracks tend to form first near the support, the shear being a maximum at this place; and they tend to open up at the *top* of the beam, this being the tension face. Fig. 7 illustrates the general form of such cracks in this case. It is important to take due account of these conditions in the arrangement of web reinforcement. Stirrups must be thoroughly anchored at their upper ends preferably by hooking them around some of the upper bars. Inadequate shear reinforcement in continuous beams is not uncommon, due mainly to the overlooking of the necessary relation of such reinforcement to the tension side of the beam.

157. *Bond Stress.*—As the bond stress is a linear function of the shear, it follows that it changes sign suddenly at the support. This condition gives rise to a sudden change in the *direction* of bond stress. Thus, in Fig. 6, on the left of the support, the concrete pulls towards the left on the upper rod, and on the right it pulls towards the right, as shown by the small arrows. Any slipping increases the deformation of the concrete at once, and hence increases the tension in the concrete at the centre. Likewise, at the bottom, any slip tends to increase the compressive stress in the concrete. It follows, therefore, that where rods continue over the support in continuous beams, the bond stress should be well provided for each side of the centre, otherwise the deformations and stresses in the concrete will be excessive. As a matter of fact, the exact theoretical conditions can hardly be realized, and the variation in bond stress must follow more nearly a rounded curve such as shown by the dotted line in Fig. 6.

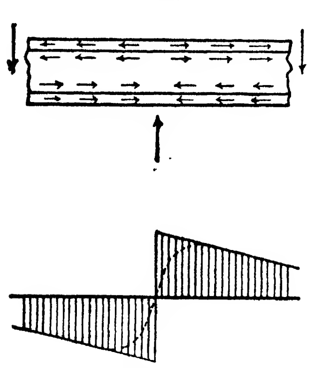


Fig. 6.

158. *T-beams as Continuous Beams.*—In floor construction a

T-beam designed for positive moments becomes a rectangular beam over the supports where the moments are negative (see Fig. 7). The tension side is uppermost, and, neglecting the tension in the concrete, the flange of the T is of no value in resisting this negative moment. From the moment diagram of Fig. 4 it is seen that the negative moments are larger than the positive moments, and hence it will generally be found that a continuous T-beam designed for positive moments will furnish inadequate compressive area at the support for the nega-

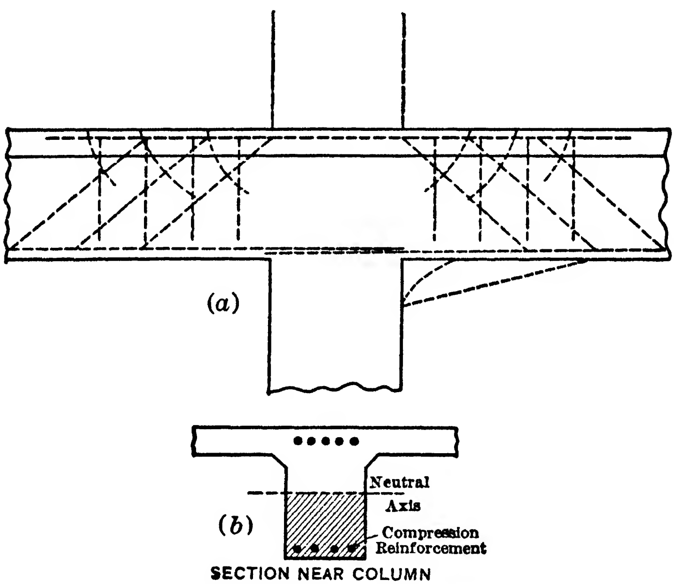


FIG. 7.

tive moment. This condition is met by the use of compressive reinforcement, and, generally, to some extent, by allowing somewhat greater unit stress at this point than elsewhere. Fig. 4 shows that the negative moment falls off very rapidly from the theoretical maximum over the support so that within a few inches from the support the moment will be much below the maximum. In view of this condition, the allowable unit stress may reasonably be increased slightly. Increasing the depth of the beam near the support as indicated in Fig. 7 (*a*) or increasing the width of the beam are also common methods of providing for the increased negative moment.

CHAPTER VI

DEFLECTIONS OF BEAMS

159. General Theory; Live Loads Only.—Deflection formulas for homogeneous beams can be interpreted semi-rationally to make them applicable to reinforced-concrete beams. So interpreted, they yield results in fair agreement with measured deflections due to live load. For deflection due to long-continued dead load, see Art. 164.

The common deflection formulas pertain to a beam, the cross-sections of which have equal moments of inertia. This condition is not fulfilled in most reinforced-concrete beams, because of the presence of bent-up bars and stirrups. Still, the amounts of steel in, and hence the moments of inertia of, sections in the middle third or middle half are commonly constant. And since the middle half contributes nearly 85% of the maximum deflection in the case of a simple beam constant in section and uniformly loaded, and 82% when the beam is loaded at the two outer quarter points, it must be that a small change in the moments of inertia of end sections of a simple beam produces a much smaller change in the maximum deflection.

More definitely, consider for example the deflections of two simple beams of equal span with equal uniform loads W. Suppose that in one beam all cross-sections have a moment of inertia I_1, and that in the other beam, cross-sections in the middle half and in the outer quarters have moments of inertia equal to I_1 and I_2 respectively. Suppose also that $I_2 = 0.75 I_1$; then the deflections of these beams respectively are as 1 to 1.06. That is to say, a reduction of 25% in the moment of inertia of the cross-sections of the outer quarters of the first beam changes the deflection only 6%.

The common deflection formulas imply that the material of the beam obeys Hooke's law ("stress is proportional to strain"), up to working stresses at least, and that the moduli of elasticity of the material for tension and compression are equal. Although it is true that concrete does not obey the law strictly, still its stress-strain relation for compression is nearly linear up to working stresses. But the stress-strain relation for tension is far from linear, and the assumption that it is, herein made for simplicity in formulas, must be regarded as a rough approximation. It is true that the "initial moduli" of concrete for compression and tension are nearly equal, but the deflection of a beam depends on the elongations and shortenings of all the fibres, and hence not upon initial modulus but on some sort of a mean value. This is not the modulus corresponding to the mean unit fibre stress, but certainly the average or secant modulus is more nearly correct than the initial modulus, or that at the maximum unit stress.

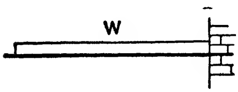
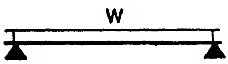
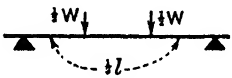
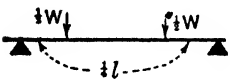
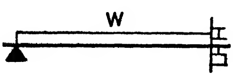
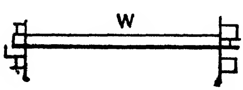
In view of the foregoing and indications of test data, we have chosen the following basal deflection formula for a concrete beam,

$$D = C_1 \frac{Wl^3}{E_c I_0}, \qquad \qquad (1)$$

wherein W = total load·
l = span;
E_c = secant modulus of elasticity of concrete for working stress in compression;
I_0 = moment of inertia of middle cross-sections more fully described below, and
C_1 = a coefficient depending on manner of support and loading (see Table 10).

Before using eq. (1), one must decide on how to calculate I_0. Thus the question is raised whether to neglect the tensile fibre stress in the concrete or not. As already fully explained (Art. 47), a reinforced-concrete beam under full working load contains one or more cracks at or near the section of maximum bending moment or else the condition there is near the cracking stage; and to compute the maximum unit fibre stresses at such section, engineers rightly assume the presence of a tension crack, and that it has extended, in effect, to the neutral surface. But deflection depends on stress at *all* sections, and cracking does increase deflection. Ordinarily cracks are not numerous, and besides they do not affect deflection nearly so much as fibre stress. These effects are entirely different in order of magnitude. The first

TABLE 10

Case	C_1	C_2	C
	$1/3 = 0.333$	1	$1/3 = 0.333$
	$1/8 = 0.125$	$1/2 = 0.50$	$1/4 = 0.250$
	$1/48 = 0.0208$	$1/4 = 0.25$	$1/12 = 0.083$
	$5/384 = 0.0130$	$1/8 = 0.125$	$5/48 = 0.104$
	$23/1296 = 0.0177$	$1/6 = 0.167$	$23/216 = 0.106$
	$11/766 = 0.0144$	$1/8 = 0.125$	$44/383 = 0.115$
	$\sqrt{5}/240 = 0.0093$	$3/16 = 0.187$	$\sqrt{5}/45 = 0.050$
	$1/185 = 0.0054$	$1/8 = 0.125$	$8/185 = 0.043$
	$1/192 = 0.0052$	$1/8 = 0.125$	$1/24 = 0.0416$
	$1/384 = 0.0026$	$1/12 = 0.083$	$1/32 = 0.0312$

is not noticeable at all in careful measurements on deflections due to increasing loads, whereas the latter certainly would be if fair measurements of fibre stress *at* a section of a beam were possible. For these reasons and indications of certain test data, we shall, for the calculation of I_0 in eq. (1), regard the concrete from the tensile steel to the compression face as intact, the concrete taking its share of tension; the concrete from the centre of the tensile steel to the tension face is neglected.

It is possible to express the deflections—as in the case of homogeneous beams—in terms of maximum bending moment M or fibre stresses f_c and f_s. Thus, since

$$M = C_2 W l$$

wherein C_2 is a coefficient like C_1 depending on kind of load and manner of support of beam (see Table 10), therefore

$$D = \frac{C_1}{C_2} \frac{M l^2}{E_c I_0} = C \frac{M l^2}{E_c I_0}, \qquad \qquad (2)$$

C being an abbreviation for C_1/C_2 (see Table 10).

Now, as explained in Art. 47, maximum fibre stresses in concrete and steel, at the cross-section of maximum bending moment, are universally calculated on the basis of a cracked section, or negligible tension in concrete. So that if I' denotes the moment of inertia of cross-section (cracked or tensile value of concrete neglected), then

$$M = f_c I'/k d = (f_s/n) I'/(1 - k)d.$$

Combining these two equations with (2) appropriately (eliminating M and k) gives

$$D = C \frac{l^2}{d} \frac{I'}{I_0} \frac{f_c + f_s/n}{E_c}$$

$$= C \frac{l^2}{d} \frac{I'}{I_0} \frac{n f_c + f_s}{E_s}. \qquad \qquad (3)$$

Eqs. (1), (2), and (3) will be applied and illustrated in articles following.

160. Rectangular Cross-Section Beam; Tension Bars Only.—The notation used here is like that of Art. 48. It is represented in Fig. 1; the cross hatching suggests that "full" tension is ascribed to concrete in calculation for I_0. As a first step in this calculation, it is necessary to locate the neutral axis of the cross-hatched (concrete-steel) section. This is done in the manner employed in Art. 50 or 55; but there any tension in concrete was neglected. Using the transformed section as in Art. 55, taking moments about the upper edge, we get

$$b\,d \times \tfrac{1}{2} d + n\,p\,b\,d \times d = (b\,d + n\,p\,b\,d)\,k\,d;$$

hence
$$k = \frac{1 + 2\,p\,n}{2\,(1 + p\,n)}. \quad \text{. . . . .} \quad (1)$$

The moment of inertia of the equivalent steel area about the neutral axis is practically

$$n\,p\,b\,d\,(1 - k)^2\,d^2 = p\,n\,(1 - k)^2\,b\,d^2.$$

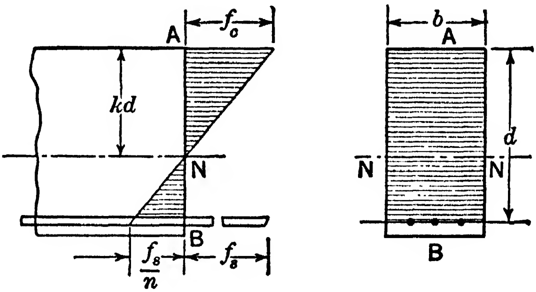


FIG. 1.

The moments of inertia of the tensile and compressive concrete areas respectively are $\tfrac{1}{3}\,b(1 - k)^3\,d^3$ (nearly) and $\tfrac{1}{3}\,b\,k^3\,d^3$; hence,

$$I_0 = \tfrac{1}{3}\,[k^3 + (1 - k)^3 + 3\,p\,n\,(1 - k)^2]\,b\,d^3.$$

Expanding the binomials in this equation, inserting the value of k from eq. (1), and then reducing, gives

$$I_0 = \frac{1 + 4\,p\,n}{12\,(1 + p\,n)}\,b\,d^3. \quad \text{. . . . .} \quad (2)$$

Substituting this in eq. (1) of Art. 159, we get

$$D = C_1\,\frac{12\,(1 + p\,n)}{1 + 4\,p\,n}\,\frac{W\,l^3}{E_c\,b\,d^3} = C_1\,\alpha\,\frac{W\,l^3}{E_c\,b\,d^3}, \quad \text{. . .} \quad (3)$$

where α stands for $12(1 + p\,n) \div (1 + 4\,p\,n)$. Value of α for any particular case (of $p\,n$) may be taken from the graph in Fig. 2.

Probable deflection due to a proposed or design live load may be computed more readily by means of eq. (5), below, which may be deduced as follows: From eq. (2) of Art. 159 and eq. (3) above

$$D = \frac{C\,\alpha\,M\,l^2}{E_c\,b\,d^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

From Art. 54, $M = R\,b\,d^2$; hence,

$$D = C\,\alpha\frac{l^2}{d}\frac{R}{E_c}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

See diagrams 3 and 4 for values of R.

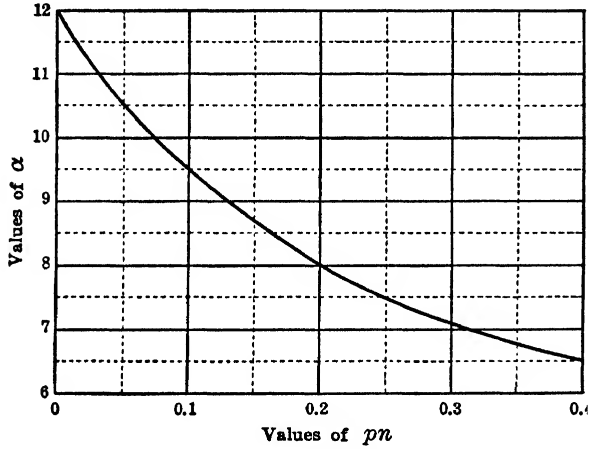


FIG. 2.

Example.—1. A concrete beam rests on end supports 16 ft. centre to centre; the breadth of its section is 10 in., the depth (to the steel) is 15 in., the reinforcement consists of four ¾-in. rods extending along the whole length (and stirrups). What is its probable deflection when sustaining a uniform load of 10,000 lbs., including its own weight?

The value of C_1 is 5/384 (see Table 10). The amount of steel is 1.767 in.^{2}; hence $p = 1.767 \div 150 = 0.012$. If $n = 8$, $n\,p = 0.096$ and $\alpha = 9.5$ (see Fig. 2). Therefore, eq. (3) becomes

$$D = \frac{5}{384}\,9.5\,\frac{10{,}000\,(16 \times 12)^3}{(30{,}000{,}000 \div 8)\,10 \times 15^3} = 0.069 \text{ in.}$$

Example.—2. The deflection of the beam described in the preceding example is desired, (*i*) when it is loaded so that the working compressive fibre stress is 500 lbs/in.^{2}, and (*ii*) when the working stress in the steel is 14,000 lbs/in.^{2}

(*i*) The value of C is 5/48 (see Table 10). For $n = 8$, $p = 0.012$ (as in example 1), and $f_c = 800$, diagram 3 gives $R = 123$. Therefore, eq. (5) becomes

$$D = \frac{5}{48} 9.5 \frac{(16 \times 12)^2 \, 123}{15\,(30{,}000\,000 \div 8)} = 0.083 \text{ in.}$$

(*ii*) For $n = 8$, $p = 0.012$, and $f_s = 14{,}000$, diagram 3 gives $R = 148$. Therefore, eq. (5) becomes

$$D = \frac{5}{48} 9.5 \frac{(16 \times 12)^2 \, 148}{15\,(30{,}000{,}000 \div 8)} = 0.10 \text{ in.}$$

161. Rectangular Cross-Section Beam with Compression Bars.— The notation used here is like that of Art. 82. It is represented in Fig. 3. The tensile and compressive steel areas respectively are $p\,b\,d$ and $p'\,b\,d$; the corresponding steel unit stresses are f_s and f_s'. It should be plain from Fig. 3 that

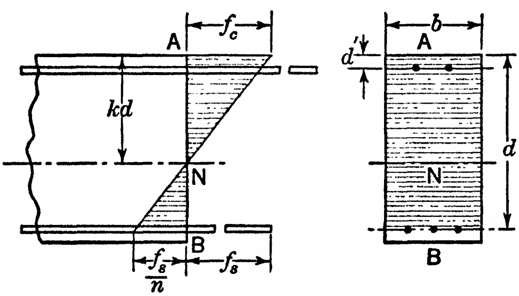


FIG. 3.

$$f_s = n f_c \frac{1 - k}{k} \quad \text{and} \quad f_s' = n f_c \frac{1 - d'/d}{k}.$$

Proceeding as in the preceding article, allowing full value of tensile concrete, we get

$$k = \frac{1 + 2\,p\,n + 2\,p'\,n\,d'/d}{2\,(1 + p\,n + p'\,n)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The moments of inertia of the "weighted" tensile and compressive steel areas respectively are practically,

$$n\,p\,b\,d\,(1 - k)^2\,d^2 \quad \text{and} \quad n\,p'\,b\,d\,(k - d'/d)^2\,d^2$$

The moments of inertia of the tensile and compressive concrete areas respectively are nearly

$$\tfrac{1}{3}\,b\,(1 - k)^3\,d^3 \quad \text{and} \quad \tfrac{1}{3}\,b\,(k\,d)^3;$$

hence,

$$I_0 = [\tfrac{1}{3}\,k^3 + \tfrac{1}{3}\,(1 - k)^3 + p\,n\,(1 - k)^2 + p'\,n\,(k - d'/d)^2]\,b\,d^3. \; (2)$$

Substitution for k in eq. (2) from eq. (1), as in Art. 160, is not practical; but we may write as in Art. 159

$$D = C_1 \beta \frac{W l^3}{E_c b d^3}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where β stands for the reciprocal of the (bracketed) coefficient of $b d^3$ in eq. (2). The numerical value of β for any ordinary case (value of $p n$, $p' n$, and d'/d) may be taken from the curves in Fig. 4.

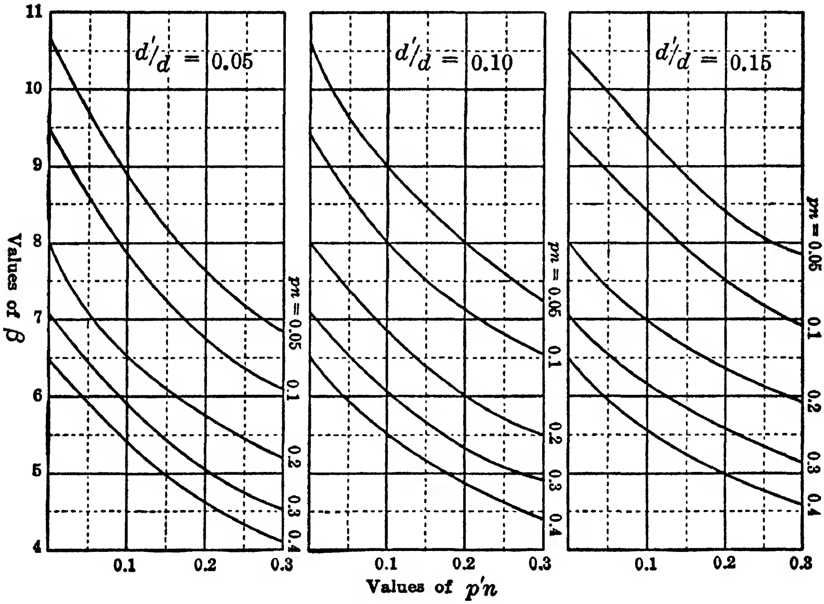


FIG. 4.

For a proposed or design live load the probable deflection may be computed from eq. (4) below, which may be deduced from eq. (2) of Art. 159, and $M = f_c R b d^2$ from Art. 83. Combining these two equations so as to eliminate M, we find

$$D = C \beta \frac{l^2}{d} \frac{f_c R}{E_c}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The value of R for any given case may be taken from diagrams 8, 9, or 10.

Example.—1. A 12 by 20 in. beam rests on end supports 16 ft. centre to centre. It has 1% compressive steel 2 in. from the top, and 2½% tensile steel 2 in. from the bottom. How much deflection would be caused by a live load of 2250 lbs/ft.?

Solution: From Table 10, $C_1 = 5/384$. If $n = 12$, $p\,n = 0.025 \times 12 = 0.3$, and $p'\,n = 0.01 \times 12 = 0.12$; $d'/d = 2/18 = 0.111$. From Fig. 4,

$$\text{with } d'/d = 0.10, \quad \beta = 5.9;$$
$$\text{with } d'/d = 0.15, \quad \beta = 6.1;$$

and so for $d'/d = 0.11$, we take $\beta = 5.9$. Hence (see eq. (3))

$$D = \frac{5}{384}\,5.9\,\frac{(2250 \times 16) \times (16 \times 12)^3}{(30{,}000{,}000 \div 12)\,12 \times 18^3} = 0.11 \text{ in.}$$

Example.—2. Suppose that the foregoing beam is to support a centre live load so as to produce stresses $f_c = 750$ and $f_s = 18{,}000$ lbs/in.² The probable deflection due to this load is desired.

Solution: With $n = 12, f_s/nf_c = 2$; $d'/d = 2/18 = 0.11$. Entering diagram 9 at $f_s/nf_c = 2$, we may pick out various values of p and p'. Choosing $p = 1\%$ say, then $n\,p = 0.12$; note intersection of horizontal line at $f_s/nf_c = 2$ and curve $n\,p = 0.12$. This intersection corresponds to $p' = 0.85\,p$ (or $p' = 0.85\%$). From this intersection trace down to R curve $n\,p = 0.12$, in lower part of diagram, and read $R = 0.21$ (about). From Fig. 4, with $d'/d = 0.1$, $p\,n = 0.12$ and $p'\,n = 0.85$, note that $\beta = 8$ (about). From Table 10, $C = 1/12$. Hence eq. (4) becomes

$$D = \frac{1}{12} \times 8\,\frac{(16 \times 12)^2}{18}\,\frac{750 \times 0.21}{2{,}500{,}000} = 0.09 \text{ in.}$$

162. Deflection of T-Beams.—The notation used here is like that of Art. 71. It is represented in Fig. 5.

Allowing for full value of the tensile concrete, we get, as in Art. 73

$$k = \frac{\left[\frac{b'}{b} - \frac{b'}{b}\left(\frac{t}{d}\right)^2 + \left(\frac{t}{d}\right)^2\right] + 2\,p\,n}{2\left[\frac{b'}{b} - \frac{b'}{b}\frac{t}{d} + \frac{t}{d} + p\,n\right]}. \quad . \quad . \quad . \quad . \quad (1)$$

The moment of inertia of the transformed concrete section, the steel "weighted" n-fold, is given by

$$I_0 = \tfrac{1}{3}\left[k^3 - \left(1 - \frac{b'}{b}\right)\left(k - \frac{t}{d}\right)^3 + \frac{b'}{b}(1 - k)^3 + 3\,p\,n\,(1 - k)^2\right] b\,d^3. \quad . \quad . \quad . \quad (2)$$

The deflection formula (1) of Art. 159 becomes

$$D = C_1 \gamma \frac{W l^3}{E_c b d^3}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where γ stands for the reciprocal of the coefficient of $b d^3$ in eq. (2). Because there are so many variables ($n p$, b'/b and t/d) in eqs. (1) and (2), graphical representation of γ is not practical.

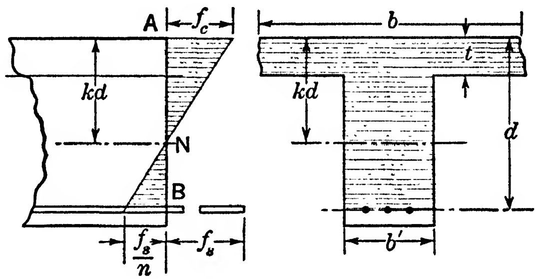


FIG. 5.

Because the flanges of T-beams are generally parts of a floor slab, the concrete in the flanges is generally subjected not only to longitudinal but also to transverse fibre stress. These latter influence deflection but are not taken into account in eq. (3), which is not so reliable as formulas for beams rectangular in cross-section.

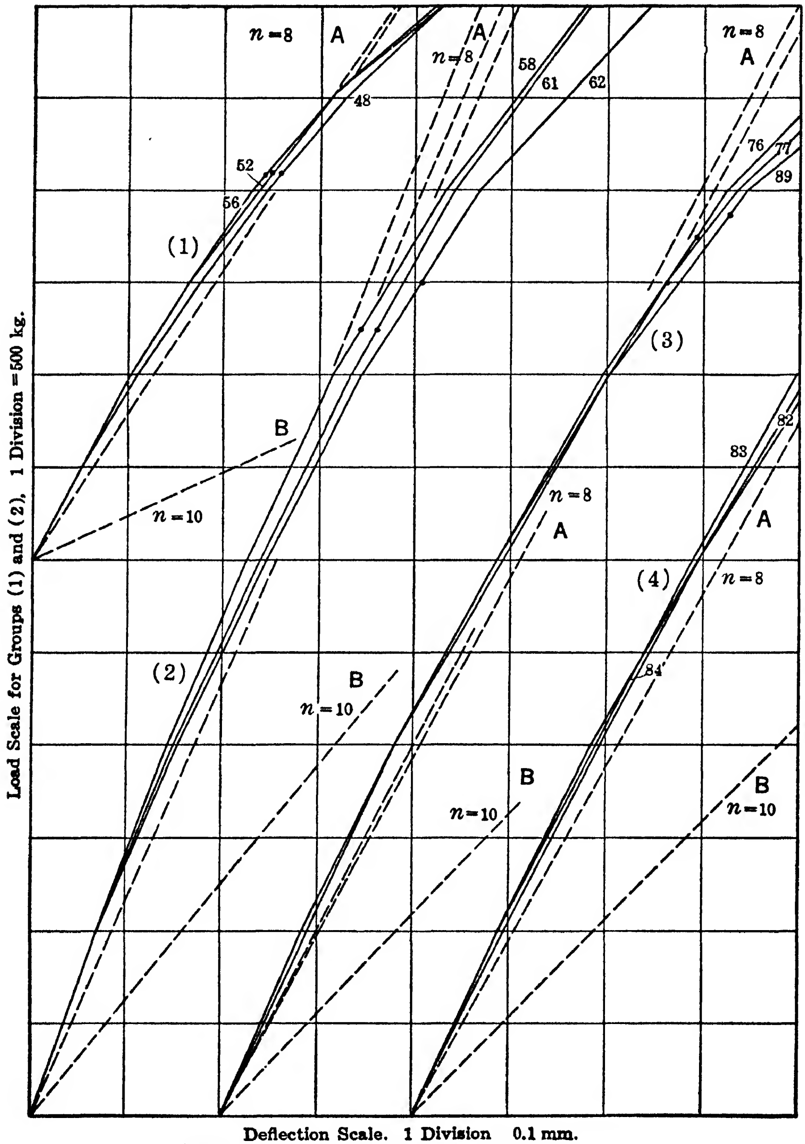
Example.—A T-beam rests on end supports 10 ft. apart, and sustains loads of 5000 lbs. at its third points. The dimensions of the section are $b = 16$ in., $b' = 8$ in., $d = 10$ in., and $t = 3\frac{1}{4}$ in.; and the reinforcement consists of three $\frac{3}{4}$-in. square bars. What is the probable deflection due to the load?

Solution. The steel ratio is 0.011; and with $n = 8$, eq. (1) gives $k = 0.485$; hence $\gamma = 11.3$. Now for loads at third points, $C_1 = 23/1296$ (see Table 10); hence,

$$D = \frac{23}{1296} \frac{11.3 \times 10{,}000 \times 120^3}{3{,}750{,}000 \times 16 \times 10^3} = 0.06 \text{ in.}$$

163. Experiments on Deflection of Beams; Comparisons with Formulas.—(*i*) The first selection of test data is from an extensive investigation by Bach * on about 50 rectangular and 20 T-beams. The *rectangular beams* were 2 m. long, 30 cm. deep, and 15, 20, or 30 cm. wide. They were reinforced with a single straight bar, several straight bars, straight bars with stirrups, or several bars, some bent up; the percentages varied from about 0.4 to 1.35. The *T-beams* were 3 m.

* Mitteilungen über Forschungsarbeiten auf dem Gebiete des Ingenieurwesens, Heft 39, 45, 46, 47 (1907).



Load-deflection Graphs for Four Groups of Beams.

FIG. 6.

long, 45 cm. wide, 48 cm. deep, flange 10 and web 20 cm. thick. They were reinforced with straight bars, with or without stirrups, or bars, some bent up, with or without stirrups; the percentage of steel was about 0.8 in all of them.

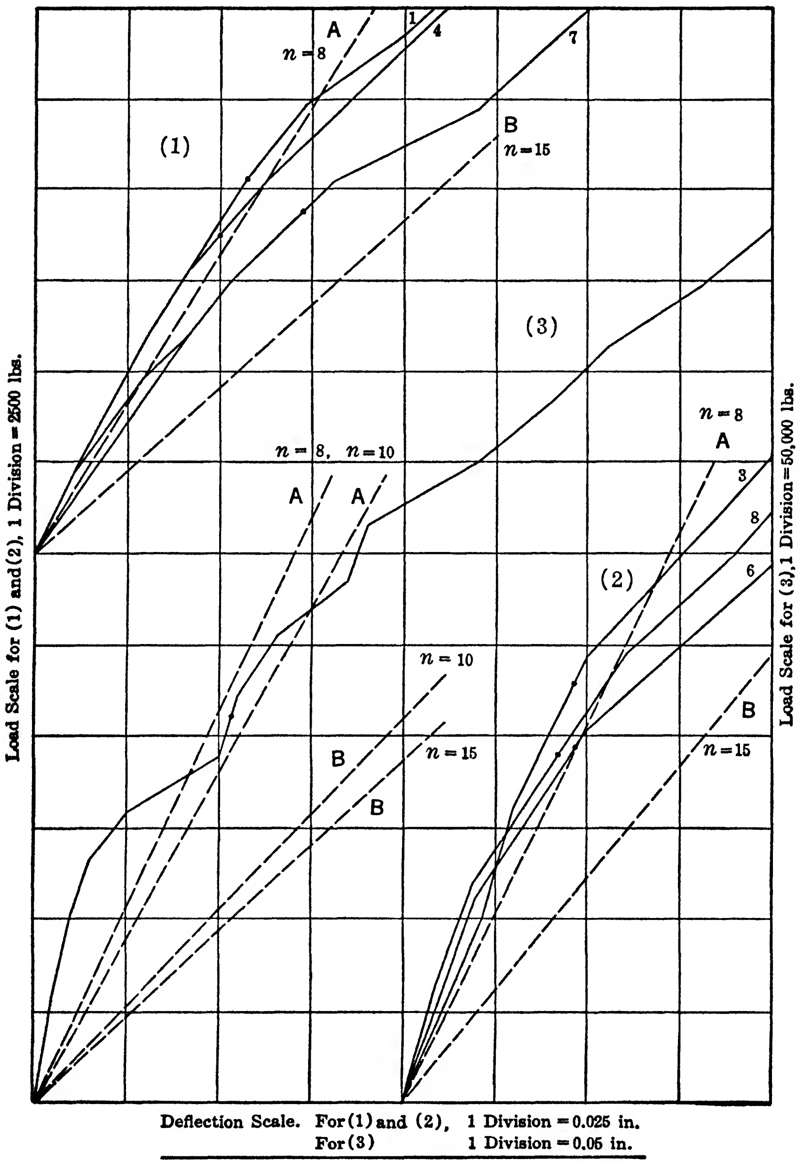
Beams were made in sets of three as nearly alike as possible. They were tested on end supports, and loaded at third or quarter points. Deflections were measured at five or seven points along the beam, to the nearest 0.004 mm.

Fig. 6 shows the load-deflection curves for four sets of beams. Groups (1) and (2) relate to rectangular beams; in (1) the beams were 15 cm. wide, reinforced with 0.5% of steel in 3 rods, 2 bent up at each end; in (2) the beams were 20 cm. wide and reinforced with 1.35% of steel in three rods, two bent up at each end. Groups (3) and (4) relate to T-beams; in the beams of (3) there were three straight rods (0.8%) and 24 stirrups; and in those of (4) there were five rods, four bent up at each end (0.77%), and 24 stirrups. Only a part of each curve is given. The dot on each corresponds to one-quarter ultimate load; dots on extensions of group (4) would be a trifle higher than in group (3).

The dashed lines marked *A* are graphs of the deflection formulas corresponding to the various beams. The deflection formula agrees as well with other sets in Bach's tests except in a few cases in which the reinforcement consisted of a single straight rod and stirrups.

(*ii*) The second selection pertains to some tests made by Talbot * (see Fig. 7). Groups (1) and (2) relate to two sets of T-beams, 12 in. deep (over all), flange 3¼ and web 8 in. thick; the span was 10 ft., and loads at third points. The three in group (1) were 16 in. wide, reinforced with straight bars (about 1%) and stirrups; the three in group (2) were 24 in. wide, reinforced as others except that some rods were bent up. Graph 3 is for a very large rectangular beam; its breadth was 25 in., depth to steel 30.5 in., span 23.5 ft., and percentage of steel 1.25. Only a portion of each graph is shown; the dot on each corresponds to one-fourth the ultimate applied load. The dashed lines marked *A* are the graphs of formulas for the corresponding beams.

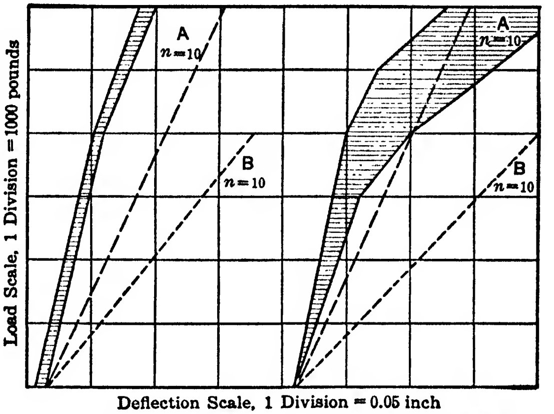
* Bulletin Univ. of Ill. Eng. Exp. Station, No. 12 (1907); Eng. News, Vol. LX, p. 145 (1908).



Load-deflection Graphs for Three Groups of Beams.

FIG. 7.

(*iii*) Fig. 8 affords still other comparisons of test deflections * and formulas. The hatched area at the left is the field of four load-deflection graphs for four beams with 1.47% steel; that at the right is the field for graphs for five beams with 0.74% steel. All beams were 8 by 11 in. in cross-section, 12 ft. long; they were tested on end supports, loads at third points. Lines marked *A* are graphs of the deflection formulas for the beams respectively.



Load-deflection Graphs for Two Groups of Beams

FIG. 8.

NOTE: Professor G. A. Maney has proposed † the following formula for deflection of beams:

$$D = C\frac{l^2}{d}(e_c + e_s). \quad (1)$$

where e_c and e_s denote unit deformation in extreme fibres of concrete and of (tensile) steel at the place of maximum bending moment. In the paper referred to, he shows very close agreements between deflections measured in tests of 11 beams, and deflections computed from the formula, he having substituted *measured* (test) values of e_c and e_s.

For design, such values of e_c and e_s being unavailable, Maney rewrites the formula, in other publications, thus:

$$D = C\frac{l^2}{d}\frac{nf_c + f_s}{E_s}. \quad (2)$$

In the case of a design load, f_c and f_s are working stresses, it being assumed that the designed cross-section will be consistent with both f_c and f_s. In the case of a constructed beam under load, f_c and f_s denote stresses due to that load. In either case, the calculations (for cross-section or for stresses) are to be made by standard methods or formulas.

It may be noted that eq. (3) of Art. 159 is like eq. (2) above except for the factor I'/I_0, where I' and I_0 respectively denote moments of inertia of cross-section, including allowance for steel, without and with tensile con-

* From U. S. Bureau of Standards Technologic Paper No. 2 (1911), by R. L. Humphrey and L. H. Losse.

† Proc. A.S.T.M., Vol. XIV (1914), p. 310.

crete. Eq. (2) gives larger deflections than eq. (3) in the ratio I_0/I'. Thus for rectangular cross-sections,

when	pn = 0.1	0.2	0.3	0.4,
then	ratio = 1.85	1.40	1.23	1.12.

For further comparison, Maney's formula is shown and marked B in Figs. 6, 7, and 8.

Stirrups and Bent-up Rods do not affect the stiffness of beams materially for working loads; but they do increase the ultimate deflection as well as the strength. Bach's tests clearly show this to be true, for example:

(1) Column a of the following table gives the average deflections for three beams (numbers 7, 13, and 14) corresponding to the loads tabulated; the beams were reinforced with a single straight rod (p about 0.9%). Column b gives the average deflections for another set of 3 (29, 32, and 37); these were reinforced like the first set but with sixteen stirrups added. The fourth column gives the percentage differences between the deflections of the two sets of beams up to 4000 kg. The average ultimate deflections of the two sets were 1.78 and 2.3 mm., and the ultimate loads 18,900 and 23,250 kg. respectively.

DEFLECTIONS OF RECTANGULAR BEAMS

Loads (Kilos)	Deflection (Millimeters)					
	a	b	Diff.	A	B	Diff.
500	0.052	0.052	0%	0.048	0.050	+ 4.0%
1000	.110	.107	−2.7	.107	.110	+ 2.8
1500	.175	.165	−5.7	.167	.173	+ 3.6
2000	.245	.232	−4.9	.232	.248	+ 6.8
2500	.322	.307	−4.6	.308	.330	+ 6.9
3000	.428	.417	−2.6	.403	.403	+ 6.7
3500	.608	.580	−4.6	.538	.585	+ 8.7
4000	.793	.767	−3.3	.775	.902	+14.0

(2) Column A of the same table gives the average deflections for a set of beams (40, 43, and 45) which were reinforced with three straight rods (p = 0.55%); and B the average deflections for a set (49, 51, and 53) reinforced like the first, but two of the rods were bent

up at each end. The last column gives the percentage differences between the average deflections of the two sets of beams. The average ultimate deflection of sets *A* and *B* were 3.38 and 3.45 mm. and their average ultimate loads 8250 and 8600 kg. respectively.

(3) The numbered columns in the following table give the average deflections of six sets of T-beams, three in each set, for the loads tabulated. The beams were alike except as to reinforcement. Beams of set 1 were reinforced with three straight rods (p = 0.8%); set 2 like 1 and 24 stirrups; set 3 like 1 and 48 stirrups; set 4 with five rods (p = 0.87%), four bent up at each end; set 5 like 4 and 24 stirrups; and set 6 like 5 except that a hook was formed at each end of the fifth rod. The horizontal lines in the table are drawn to correspond to one-quarter ultimate loads. The last column of the table gives the greatest percentage difference for the various working loads. The average ultimate deflections were 2.4, 3.2, 3.8, 6.0, 5.8, and 9.4 mm.; the average ultimate loads 23,000, 30,500, 37,800, 33,300, 41,000, 46,000 kg. respectively. All average ultimate deflections are not reliable.

DEFLECTIONS OF T-BEAMS

Loads (Kilos)	Deflections (Millimeters)						
	1	2	3	4	5	6	Diff.
2,000	0.090	0.087	0.088	0.093	0.092	0.095	9.2%
4,000	.193	.183	.180	.190	.190	.192	7.2
6,000	.307	.290	.287	.303	.293	.303	10.5
8,000	0.435	0.400	.398	.422	.407	.420	6.0
10,000	0.577	0.535	0.568	0.553	0.553	0.563	1.8

164. Deflection Due to Shrinkage and Plastic Flow.—*Shrinkage.*—For a beam singly reinforced, shrinkage of the concrete will force the beam into a curved form, the radius of curvature depending upon the amount of shrinkage, amount of steel, and depth of beam. The solution of the problem may be arrived at as follows:

It will be asssumed first that the concrete is not cracked in the process, but it will be stressed in tension on the side containing the steel, and the steel will be stressed in compression. On the unreinforced side the concrete will be in compression. There will be an axis of zero stress in the concrete (neutral axis) the position of which will first be determined. To simplify the problem the concrete below the steel will be ignored. The stresses in concrete and steel are in equilibrium.

Let f_s = stress in steel after shrinkage has taken place, and f'_c and f_c = extreme fibre stresses in the concrete on the tension and compressive faces respectively. Let P = total compression in steel. This acts as an eccentric force on the concrete of the beam, producing a direct tensile stress of $P/b\,d$ and a bending moment of $P\dfrac{d}{2}$. The total fibre stress in the concrete on the tension side is

$$f'_c = \frac{P}{b\,d} + \frac{P\dfrac{d}{2} \times \dfrac{d}{2}}{\dfrac{1}{12}\,b\,d^3} = \frac{P}{b\,d} + \frac{3\,P}{b\,d} = 4\,\frac{P}{b\,d}.$$

And on the compression side it is

$$f_c = + \frac{P}{b\,d} - \frac{3\,P}{b\,d} = -\,2\,\frac{P}{b\,d}.$$

Hence, $f_c = f'_c/2$, which places the neutral axis $\frac{1}{3}\,d$ from the top, or

$$k = \tfrac{1}{3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Let m = coefficient of shrinkage. This will represent the actual shrinkage at the neutral plane. At the bottom fibre it will be $m - f'_c/E_c$, and the stress f_s will be equal to $(m - f'_c/E_c)\,E_s = m\,E_s - f'_c\,n$. As the total compression must equal total tension, we have, noting that $k = \frac{1}{3}$ and $f_c = \frac{1}{2}f'_c$,

$$f_s\,p\,b\,d + \tfrac{1}{4}f'_c \times \tfrac{1}{3}\,b\,d = \tfrac{1}{2}f'_c \times \tfrac{2}{3}\,b\,d$$

from which

$$f'_c = 4\,p\,f_s. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hence, $f_s = m\,E_s - 4\,p\,n\,f_s$ or

$$f_s = \frac{m}{1 + 4\,p\,n}\,E_s. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Also, as above,

$$f_c = \frac{f'_c}{2} = 2\,p\,f_s. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The deflection will now be considered. Assuming a circular curve the centre deflection for a simply supported span is

$$D = \frac{1}{8}\,\delta\,\frac{l^2}{y}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where δ = unit deformation of a fibre distant y from the neutral plane and l = span length. Consider the bottom fibre for the calculation. The value of δ is

$$\frac{f'_c}{E_c} = \frac{4\,p\,f_s}{E_c} = \frac{4\,p\,m\,E_s}{(1 + 4\,p\,n)\,E_c} = \frac{4\,p\,n}{1 + 4\,p\,n} \times m; \quad y = \frac{2}{3}\,d.$$

Hence

$$D = \frac{3}{16} \cdot \frac{4\,p\,n}{1 + 4\,p\,n} \cdot m \cdot \frac{l^2}{d}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Example.—Assume $m = 0.0002$, $p = 0.008$, $n = 15$; $l = 16$ ft., $d = 16$ in.

From eq. (3) $f_s = \dfrac{0.0002}{1 + 0.48} \times 30{,}000{,}000 = 4050$ lbs/in.^{2} compression; $f'_c = 4 \times 0.008 \times 4050 = 130$ lbs/in.^{2} tension; $f_c = 65$ lbs/in.^{2} compression.

$$D = \frac{3}{16} \cdot \frac{0.48}{1.48} \times 0.0002 \times \frac{l^2}{d} = 0.0000122\,\frac{l^2}{d} = 0.028 \text{ in.}$$

Applying the same shrinkage coefficient to the beam of example 1 of Art. 160 gives a deflection of 0.025 in., which is about one-third that due to load.

For continuous beams, if we neglect the effect of any reinforcement on the compression side, the deflection due to shrinkage will be about one-half that of a simply supported beam of the same span.

The effect of compressive reinforcement is to decrease the effect of shrinkage as it reduces the ratio δ/y in eq. (5). In fact, the neutral plane may be entirely above the beam if the compressive reinforcement is large.

If the concrete be assumed as cracked throughout, due to shrinkage and loading, the deflection due to shrinkage will be greater than calculated above. In this case there is no stress in the concrete or steel and the neutral plane may be considered to be at the top. The unit deformation δ will now be equal to m, and $y = d$.

Hence

$$D = \frac{1}{8} m \frac{l^2}{d}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

For $m = 0.0002$, $D = 0.000025 \frac{l^2}{d}$.

For the particular value of $pn = 0.12$ of the foregoing example, this deflection is twice the value on the "no cracked" assumption.

As the deflection depends upon the behavior of the concrete throughout its entire volume, the true value will probably lie closer to the result given by eq. (6) than eq. (7).

The foregoing gives a means of making a fair estimate of the effect of shrinkage on deflection. With larger shrinkage coefficients such as 0.0006, which is quite likely to occur, the shrinkage deflection is likely to be as large as that due to load. The effect of cracking will also be of more influence.

Plastic Flow.—As indicated elsewhere this effect may be taken account of by using a low value of E_c or high value of n, such as 40 or 45. The formulas and diagrams of the preceding articles may be used. It will be found that the deflection is greatly increased but not in proportion to the increase in n. For example, comparing the deflection for $E_c = 2{,}000{,}000$ with that for $E_c = 750{,}000$, corresponding to $n = 40$, the ratios are 4.35 : 7.3.

Considering the effects of both shrinkage and plastic flow, it is apparent from the foregoing analysis that the effect of long-sustained loads applied to beams under relatively dry conditions will cause deflections of as much as 3 or 4 times the elastic deflection at the beginning. Several tests that have been reported show values corresponding to these estimates.*

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* See summary of published data in Proc. Am. Concr. Inst., Vol. 27, 1931, p. 886.

CHAPTER VII

COLUMNS

165. The Relative Length of Concrete Columns.—Short columns or piers of a length up to 4 or 5 diameters may be built without reinforcement but for greater lengths some reinforcement should be used to provide for bending stresses that are likely to occur. In ordinary construction the ratio of unsupported length to least width will seldom exceed 12 or 15, and results of tests indicate little or no effect of length up to ratios as high as 20 or 25. Hence the reinforced column can usually be designed as a " short column." For columns with a slenderness ratio greater than about 40 (ratio of length to diameter about 12) a long-column formula should be used.

166. Kinds of Reinforced Columns.—There are three general types of reinforced columns in use:

(1) Columns reinforced by longitudinal rods.

(2) Columns reinforced by longitudinal rods and circumferential or hoop reinforcement, consisting usually of spirally wound wire closely spaced.

(3) Columns composed of structural-steel or cast-iron cores surrounded by concrete with hoop reinforcement.

Longitudinal reinforcement aids the concrete by carrying a part of the load directly, the stresses in the two materials being proportional to their moduli of elasticity. Hoops and bands support the concrete laterally, preventing lateral expansion to a greater or less degree, and thus strengthening the concrete. Although the use of circumferential reinforcement alone increases the ultimate strength of a column and renders it capable of sustaining increased deformations, it is not convenient to use such reinforcement without some longitudinal bars to hold it in place, resulting in a column of type 2.

THEORY AND GENERAL RELATIONS

167. Columns with Longitudinal Reinforcement.—So long as the steel and coucrete adhere, the relative intensities of stress in the two materials will be proportional to their moduli of elasticity, using the secant modulus as explained in Art. 24.

Let A = total cross-section of column;
A_c = cross-section of concrete;
A_s = cross-section of steel;
p = ratio of steel area to total area = A_s/A;
f_c = unit stress in concrete;
f_s = unit stress in steel;
f = average unit stress in entire section;
n = ratio of moduli of steel and concrete at the given stress f_c, = E_s/E_c;
P = total strength of a reinforced column for the stress f_c.

Then

$$P = f_c A_c + f_s A_s = f_c (A - pA) + f_c n p A,$$

whence

$$P = f_c A [1 + (n - 1) p]. \qquad (1)$$

The average unit stress will be

$$f = P/A = f_c [1 + (n - 1) p] \qquad (2)$$

The relative increase in strength caused by the steel is shown by the term $(n - 1)p$. Thus if $p = 2\%$ and $n = 15$, the relative increase in strength due to the steel is $14 \times 0.02 = 28\%$.

The unit stress f_s in the steel is always equal to $n f_c$ and for the usual working values allowed in the concrete will always be relatively low. The stronger the concrete and the higher the value of f_c, the smaller is n and the product $n f_c$ $(= f_s)$ remains about constant, about 7000 to 10,000 lbs/in.² This is an uneconomical stress for steeel, but it is unavoidable. In fact, at ordinary prices it is cheaper to carry compressive stresses by concrete than by steel, and the use of steel is justified only because of other factors of the problem, such as bending stress, increased reliability, increase in allowable working stress in the concrete, and reduction in space required. Steel costs in place from 40 to 50 times the same volume of concrete, exclusive of form work, while the ratio of unit stresses (value of n) is from 10 to 15, thus indicating that, neglecting indirect advantages, steel is four or five times as expensive as concrete. The general conclusion which may be stated is that the use of a larger amount of steel than is necessary to secure the indirect advantages heretofore listed is uneconomical.

168. Columns with Hoop Reinforcement.—Whenever a material which is subjected to compression in one direction is restrained laterally, then lateral compressive stresses are developed which tend to neutralize the effect of the principal compressive stresses and thus to increase the resistance to rupture. Were the compressive stresses equal in all directions there would be no rupture, and the tendency to rupture may be measured by the amount by which the longitudinal stress exceeds the lateral, that is, by the amount of the *unbalanced* longitudinal stress.

Within limits of elasticity of both the concrete and the surrounding steel, it is possible to deduce a theoretical relation between the lateral and the longitudinal stresses, and thence the portion of the longitudinal stress remaining unbalanced. This relation depends upon the relation between the lateral and longitudinal deformations of concrete under a compressive force acting in one direction only. This is Poisson's ratio, and its value for concrete has been determined by various experimenters to be from 1/6 to 1/12 for ordinary working stresses.

Let μ = Poisson's ratio, f_c = unbalanced or excess of longitudinal over lateral compressive unit stress, f = total longitudinal unit stress, f_s = unit tensile stress in steel, p = steel ratio = ratio of volume of steel to volume of concrete.

We find approximately

$$f = f_c\left(1 + \frac{\mu n p}{2}\right), \quad \ldots \ldots \quad (3)$$

and

$$f_s = \mu n f_c.* \quad \ldots \ldots \quad (4)$$

* *Demonstration.* Let μ = Poisson's ratio; p = steel ratio considered as a thin cylinder of equivalent area surrounding the concrete; A_s = cross-section of this steel cylinder; r = radius. Then

$$A_s = p\pi r^2 \quad \text{and} \quad \text{thickness of cylinder} = \frac{p\pi r^2}{2\pi r} = p\frac{r}{2}.$$

With no steel banding the stress f would cause a proportionate lateral swelling of $\frac{f}{E_c}\mu$. If the actual stress in the steel is f_s then the compression per sq. in. developed in the

Taking Poisson's ratio at ⅛, eq. (3) becomes $f = f_c\,(1 + n\,p/16)$, and eq. (4), $f_s = 1/9\,n\,f_c$. As the term $n\,p/16$ is not more than 2 per cent it is apparent that within the limits of elasticity the hoop reinforcement is of little aid in strengthening the column, much less than longitudinal steel, as shown in eq. (2), Art. 167. The steel stress of ⅛ $n\,f_c$ is very small.

Although it is thus evident that hoop reinforcement can have little effect upon the concrete stress within the elastic limit, such reinforcement is quite effective in increasing the ultimate strength of a column and its ability to withstand deformations without rupture. (See results of tests described in Art. 173.)

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concrete by the steel reinforcement $= f_s\,p\,\frac{r}{2} \div r = \frac{f_s\,p}{2}$. This compression caused by the banding is equal in all horizontal directions, and has the same effect on distortion as two pairs of equal compressive forces acting on two sets of faces of a cube. The resultant lateral compression due to these horizontal forces is equal to $\frac{f_s\,p}{2\,E_c}\,(1 - \mu)$. Combining this compression with the lateral swelling caused by f we have the net lateral deformation equal to $\frac{f}{E_c}\mu - \frac{f_s\,p}{2\,E_s}\,(1 - \mu)$. This net deformation must equal the actual deformation in the steel under the stress f_s, which is $\frac{f_s}{E_s}$ or $\frac{f_s}{n\,E_c}$. Hence we have

$$\frac{f}{E_c}\,\mu - \frac{f_s\,p}{2\,E_c}\,(1 - \mu) = \frac{f_s}{n\,E_c}.$$

A part of f may be considered to be balanced by the lateral compression of $\frac{f_s\,p}{2}$; it is the unbalanced portion only which is significant. Call this unbalanced portion f_c; then $f = f_c + \frac{f_s\,p}{2}$. Then eliminating f_s from these two equations we find for f_c' the value

$$f = f_c\left(1 + \frac{n\,p\,\mu}{n\,p\,(1 - 2\,\mu) + 2}\right). \quad \ldots \ldots \quad (a)$$

We also have

$$f_s = \frac{2\,\mu\,n}{n\,p\,(1 - 2\,\mu) + 2}\,f_c. \quad \ldots \ldots \quad (b)$$

For ordinary values of p eqs. (a) and (b) are reduced approximately to

$$f = f_c\left(1 + \frac{\mu\,n\,p}{2}\right) \quad \ldots \ldots \quad (1)$$

and

$$f_s = \mu\,n\,f_c. \quad \ldots \ldots \quad (2)$$

169. Columns with Both Longitudinal and Hoop Reinforcement.—From the theoretical considerations of the preceding article it would appear that the addition of bands or hoops to columns having longitudinal reinforcement would not have much effect upon the deformation of such columns until the ordinary elastic-limit strength of the concrete has been passed. The effect of such hooping would be to maintain the integrity of the concrete beyond the usual limit of deformation and so enable the longitudinal steel to be stressed to a higher value. As an effective and reliable structural unit this type of column is superior to either of the others.

170. Concrete Columns as Long Columns.—Columns whose slenderness ratio exceeds about 40 should have their working stress reduced by application of a long-column formula. Such a formula can be derived from the theoretical form of the Rankine formula, which is, for pivoted ends

$$P' = \frac{P}{1 + \frac{f}{\pi^2 E}\left(\frac{l}{r}\right)^2}, \qquad (5)$$

in which P is the strength of a short column, f and E are the ultimate strength and the modulus of elasticity of the concrete, l = length, and r = least radius of gyration. This formula gives results materially too low when applied to steel columns, but it is believed that it is not too conservative for material like concrete. The value of f/E may be taken at 1/1000, giving finally the formula

$$P' = \frac{P}{1 + \frac{1}{10,000}\left(\frac{l}{r}\right)^2}. \qquad (6)$$

For the usual fixed-end conditions the constant in the denominator may be made 1/20,000 giving

$$P' = \frac{P}{1 + \frac{1}{20,000}\left(\frac{l}{r}\right)^2}. \qquad (7)$$

It may be observed that if formulas be derived from the Rankine-Gordon formulas for steel columns, by taking account of the difference in ultimate strength and modulus of elasticity of the materials, the resulting formulas would contain constants of very nearly the same value as for steel, namely, 1/18,000 and 1/36,000. The higher values above given represent a larger degree of safety, which is to be desired. For a value of l/r of 100, or a length of about 30 diameters, the formula for fixed ends gives an ultimate strength of two-thirds that of the short column. Because of the fact that it is difficult to secure thoroughly homogeneous concrete, and that variations in quality will affect the strength of long columns more seriously than any other structural form, long columns should generally be avoided.

Formula (7) is practically the same as that suggested by Bach, as the result of tests on columns up to values of l/r of about 100.*

The Joint Committee, 1924, specifies the straight-line formula

$$P' = P\left(1.33 - \frac{1}{120} \cdot \frac{l}{r}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

for columns having a value of l/r greater than 40. This gives somewhat higher results than formula (7) for values of l/r from 40 to 55, and lower results for higher values.

The American Concrete Institute Specifications of 1928 use formula (8) for columns reinforced by longitudinal steel only ("tied columns"), but for spirally reinforced or composite columns allow

$$P' = P\left(1.50 - \frac{1}{100}\frac{l}{r}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

for value of l/r greater than 50. The results are the same for $l/r = 100$.

The unsupported length l is to be taken as the clear length between built-in slabs or beams which support the column in all directions.

TESTS OF COLUMNS

171. Tests of Plain Concrete Columns.—Although unreinforced concrete is not used for columns beyond a length of about 4 diameters it will be useful to consider some results of tests on plain concrete columns, especially with reference to the compressive strength as determined on short specimens. These results show in general con-

* Zeit. Ver. Deutsch. Ing., 1913, p. 1969.

siderably less strength per square inch than the same concrete in the short cylindrical specimen. This is due not so much to the effect of flexibility, as such a column is very rigid, but rather to the increased effect of the non-homogeneous character of the material and to various unavoidable imperfections and irregularities giving rise to small bending moments.

Table 11 gives results of tests made at the University of Illinois.

TABLE 11

UNIVERSITY OF ILLINOIS, 1907 *

All Columns were 12 in. in Diameter by 10 ft. Long

Group	No. of Columns	Kind of Concrete	Crushing Strength, Lbs. per Sq. In.			Age in Days
			Average	Minimum	Maximum	
1	2	1 : 1½ : 3	2300	2120	2480	62–66
2	7	1 : 2 : 4	1740	1165	2210	58–72
3	2	1 : 3 : 6	1030	955	1110	61–62
4	2	1 : 4 : 8	575	575	575	63
5	6	1 : 2 : 4	2025	1770	2680	181–203
6	2	1 : 2 : 3¾	2710	2650	2770	12 mo. 16 mo.

* From Bulletin No. 20, Eng. Exp. Sta., University of Illinois.

In general, it was found that the richer mixtures tended to fail by true shear failures, whereas the poorer mixtures generally failed by gradual crushing. The very superior results obtained on the 1 : 1½ : 3 mixture as compared with the 1 : 2 : 4 mixture, or poorer, should be noted. It shows the value of the use of rich mixtures for columns, the increase in strength over the 1 : 2 : 4 concrete being about 32% while the increase in cost would not be over 10 or 15%. The great variation in individual tests in Table 10 should be noted, the results for group 2 varying from 33% below to 27% above the average. Results of comparative tests on short cylinders of 1 : 2 : 4 concrete, stored in damp sand for 9 to 11 months, gave an average crushing strength of 2650 lbs/in.^{2}; tests on 12-in. cubes stored in air at age of 60 days gave an average value of about 1950 lbs/in.^{2}, and at age of about 200 days, of 2350 lbs/in.^{2}

Tests made at the University of Wisconsin in 1908 gave an average value of about 2000 lbs/in.^{2} in 60 days on 1 : 2 : 4 concrete, the results there obtained being very uniform. This was about 85% of the cylinder strength. Other tests of plain concrete columns noted in Art. 173 showed values of from 80 to 85% of the strength of cylinder specimen.

172. Tests of Reinforced Columns.—*General Behavior of Columns.*—The phenomenon of "flow" of concrete under sustained loads has been noted in a general way for many years and has been confirmed by various tests on beams and observations on beams and columns in service. This action of concrete, together with the considerable amount of shrinkage which takes place under ordinary inside conditions, should be kept in mind in studying the results of column tests that have heretofore been made in the usual short-time operation. The subjects of shrinkage and flow are discussed in detail in Art. 174.

Columns with longitudinal reinforcement only exhibit about the same characteristics in a test as columns of plain concrete. When the stress on the concrete has reached its ordinary ultimate strength, the concrete fails on shearing planes, the rods bending or buckling at the same time.

It is important to note that in this type of column the steel gives very little aid in preventing the concrete shearing out; it merely serves to carry part of the load until the concrete is over-loaded. With ordinary materials the deformation reached at the ultimate strength of the concrete brings the stress in the steel up to about its elastic limit and sometimes beyond. For example, if the ultimate strength of the concrete is 3000 lbs/in.^{2}, and E_c at rupture = 2,000,000, the stress in the steel corresponding to the ultimate strength of the concrete is $3000 \times 15 = 45{,}000$ lbs/in.^{2}, which may exceed its elastic limit. Inasmuch as the steel rapidly deforms as soon as the yield point is reached, this limit may be taken as the maximum possible stress in the steel.

The ultimate strength of such a column is thus approximately equal to the strength of the concrete plus the yield-point strength of the steel. For high-elastic-limit steel the ultimate strength of the concrete is likely to be reached first, and in that case would determine the strength of the column. Here the influence of shrinkage and flow would act to increase the ratio of steel to concrete stress and so to utilize the higher elastic limit of the steel.

In Fig. 1 curves *C* and *D* are typical stress-strain diagrams of a plain concrete column and of a column reinforced with longitudinal

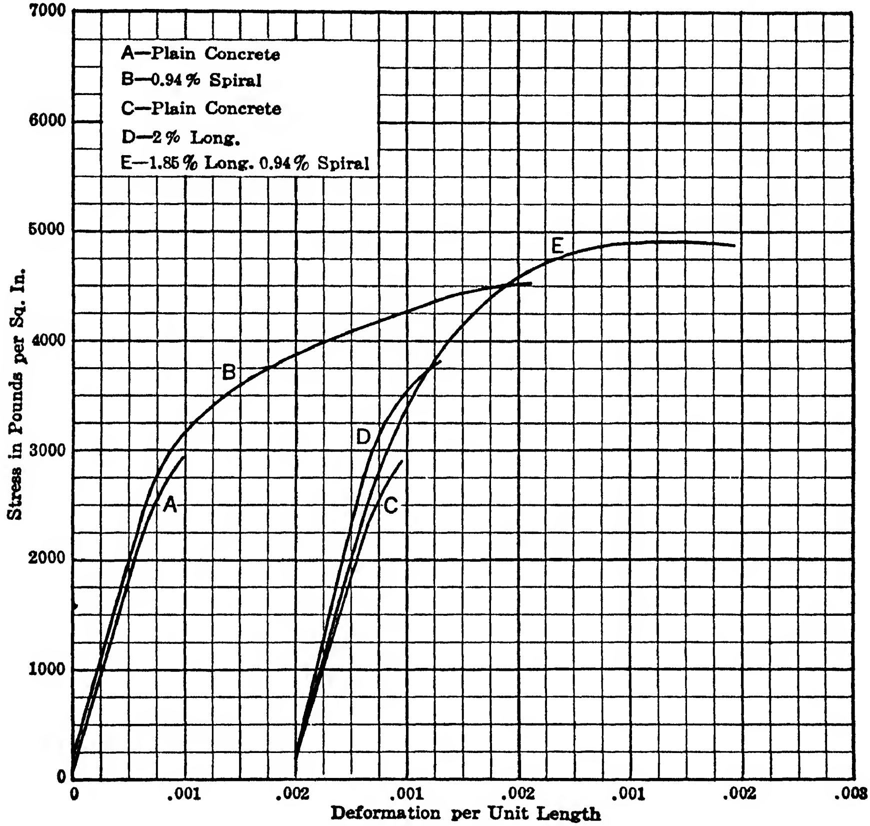

FIG. 1.

rods. The ordinates represent the average stress per square inch on the entire column. The effect of the steel in *D* is to carry part of the load, giving less strain for the same total load. The ultimate deformations are not greatly different, that of the reinforced column generally being a little the greater. In both cases the failure is sudden and the total deformation small. The curves are similar in form to the ordinary compression curve for concrete.

Columns with hoop reinforcement, with or without longitudinal steel, are likely to show a much greater deformation before rupture than those without hoops, although this effect appears to be dependent upon the strength of the concrete and amount of reinforcement. High-strength concrete shows less effect than low-strength material. As shown in Art. 168, the steel cannot receive any appreciable effect until the load approaches the ultimate strength of the concrete.

Curves *A* and *B* show a comparison between a plain concrete column and a hooped column. The deformation is practically the same up to the crushing point of the plain concrete. Beyond this the hooped column undergoes a greatly increased deformation before rupture takes place. Curves *C*, *D*, and *E* represent a plain concrete column, a column with 2% longitudinal reinforcement only, and a column with 1.85% longitudinal and 0.94% spiral or hoop reinforcement. Note the effect of the spiral reinforcement in increasing the deformation and the ultimate strength but not the behavior of the column within the elastic strength.

In these tests the elastic limit of the longitudinal steel was about 36,000 lbs/in.^{2}, which stress would be reached at a deformation of 0.0012. From this diagram it is evident that in all the hooped columns the longitudinal steel passed far beyond its elastic limit.

In the following articles the results of some of the more important tests are given. These serve to illustrate the effects of various conditions and give reliable data as a basis for establishing suitable working stresses. The tests are grouped as follows:

A. Tests of columns with longitudinal reinforcement only.
B. Tests of columns with hoop reinforcement only.
C. Tests of columns, University of Wisconsin Series.
D. Tests of columns of the American Concrete Institute.

173. *Results of Tests.—A. Tests of Columns with Longitudinal Reinforcement Only.*—The following is a summary of results of tests made by Professor A. N. Talbot on columns of 1 : 2 : 3¾ concrete reinforced with longitudinal steel of an elastic limit of about 40,000 lbs. per sq. in. Age of columns 59 to 71 days. Length of columns 6 ft. to 12 ft.

TABLE 12

TESTS ON COLUMNS WITH LONGITUDINAL REINFORCEMENT

University of Illinois, 1906 *

No. of Columns	Cross-section, Inches	Amount of Longitudinal Reinforcement, Per Cent	Crushing Strength, Lbs. Per Sq. In.		
			Average	Minimum	Maximum
4	12 × 12	1.2	1809	1587	1936
7	9 × 9	1.5	1710	1280	2335
6	{ 9 × 9, 12 × 12 }	0	1550	1079	2206

* Bulletin No. 10, Eng. Exp. Sta. 1907.

Comparing the reinforced with the plain concrete, the average strength of the 12 by 12 in. columns with 1.2% reinforcement is about 1.17 times as great, and the 9 by 9 in. columns with 1.5% reinforcement is about 1.10 times as great. These tests indicate less effect of reinforcement than theory would indicate. Strain measurements indicated that the steel stress was close to the elastic limit and that the low values for ultimate strength of the reinforced columns appeared to be due to a lower actual crushing strength of the concrete in these columns than in the plain columns. This condition has been observed in other tests and is probably due to the interference of the reinforcing bars in the placing of the concrete.

Tests made by the Bureau of Standards for the American Concrete Institute gave the following average results for 1 : 1½ : 3 concrete.*

Amount of Reinforcement, Per Cent	Strength, Lbs/in.^{2}	Increase of Strength per 1% of Reinforcement
0	2745	
1.00	3750	1050
2.03	3690	470
4.07	4535	450

The columns were 9 ft. 4 in. long by 20 in. in diameter. The elastic limit of the steel was about 36,000 lbs/in.^{2} In this case the steel

* Proc. American Concrete Institute, 1915, vol. 3, p. 424.

evidently was fully effective up to its elastic limit as such limit would supply a strength of but 360 lbs/in.^{2} per 1% steel.

B. Columns with Hoop Reinforcement.—The effect of hoop reinforcement alone is well shown by the results given in Table 13. Two forms of hooping were used, electrically welded bands 1 in. wide and of various gage thickness, and spirally wound wire at a pitch of 1 in. The columns were 10 ft. long by 12 in. in diameter. A thin film of mortar covered the hooping. Taking the elastic-limit strength as a basis, the effect of the hooping is, on the average, somewhat more than would be expected of longitudinal steel.

TABLE 13

TESTS OF HOOPED COLUMNS

UNIVERSITY OF ILLINOIS, 1907

Concrete, 1 : 2 : 4; Age, 56–69 Days; Length, 10 ft.; Diam., 12 in.

No. of Columns	Reinforcement		Average Strength, Lbs/in.^{2}	Excess Over Plain Concrete per 1% of Steel	Elastic Limit of Steel, Lbs/in.^{2}
	Kind	Per Cent			
7	....	0	1740		
3	Electric Welded Bands	1.07	2239	560	48,000
3		2.09	2877	540	
3		3.21	3202	458	
1		1.39	2735	710	
2		1.02	2226	480	
2	High Carbon Wire	0.85	2505	920	60,000
3		1.69	3437	1000	115,000
3	Mild Steel Wire	0.84	2168	510	38,500
3		1.65	2736	600	54,000

C. Tests of Columns, University of Wisconsin Series.—The results of an extensive series of tests of reinforced columns made by Mr. M. O. Withey at the University of Wisconsin are given in Table 13 and in Figs. 2 and 3. Columns *W* were unreinforced; *E* were reinforced by longitudinal bars only with wire ties widely spaced; *B* were rein-

* Bul. No. 466, Univ. of Wis., 1911.

forced by latticed angles forming a square 8 by 8 in. in section; the remaining columns by spiral wire and varying proportions of longitudinal reinforcement wired to the spirals on the inner side. The diameter of the columns was made equal to the outside diameter of the spirals, but the calculations have been made with reference to the inside diameter of 10 in. The yield point of the steel reinforcement was 38,000 to 45,000 lbs/in.^{2} for the longitudinal steel and 80,000 to

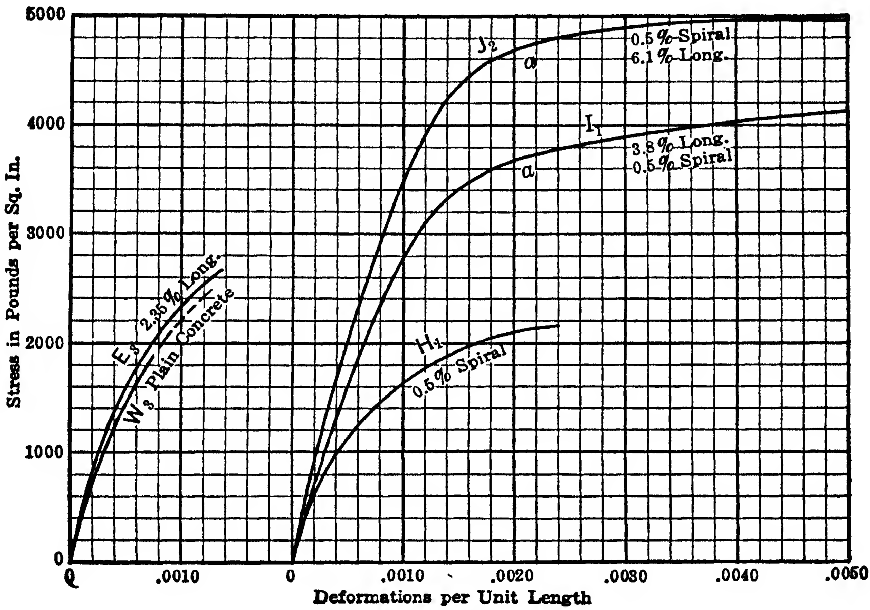

FIG. 2.

105,000 lbs/in.^{2} for the spiral steel. The columns were sprinkled once a week.

The characteristics of the various columns are well brought out in Fig. 2. Column W_3 is of plain concrete and E_3 is reinforced with longitudinal reinforcement only. The behavior of the two columns is about the same. Comparing H_1, I_1, and J_2, the "toughening" effect of the hooping is well shown, also the effect of longitudinal reinforcement in giving added strength.

Columns reinforced both longitudinally and spirally showed a fairly sharp change in curvature of the stress-strain curve, as at *a* in

TABLE 14

TESTS OF REINFORCED COLUMNS

UNIVERSITY OF WISCONSIN

Length of Columns, *A* . . . *C* = 120 in.; *H* . . . *U* = 100 in.; Age, 2 Months

No.	Reinforcement			Concrete		Cross-section, Sq. in.	Average Strength of Columns, Lbs/in.²		
	Kind	Per Cent Vertical	Per Cent Lateral	Mix	Compressive Strength of Cyl., Lbs/in.²		At Yield Point	At Max. Load	No. of Tests
W	None....................	0	0	1 : 2 : 4	2420	86.6	....	2600	3
E	9 5/8" rods with 1/4" ties, 1 ft. c. to c..............	2.35	0.11	1 : 2 : 4	2300	118	....	2470	3
B	4 2"×2"×3/16" latticed angles.................	4.5	....	1 : 2 : 4	2200	64	....	3740	2
C	1" spiral, 1" pitch.........	0	2.0	1 : 2 : 4	2180	78.5	2380	4030	4
D	9 5/8" rods and 1/4" spiral, 1" pitch...............	3.50	2.00	1 : 2 : 4	2250	78.5	3580	4750	4
H	No. 7 wire spiral, 2" pitch	0	0.50	1 : 2 : 3½	1750	78.5	1850	2230	2
G	8 1/2" rods and No. 7 wire spiral, 2" pitch.........	2.0	0.50	1 : 2 : 3½	1760	78 5	2710	3300	2
I	8 11/16" rods and No. 7 wire spiral, 2" pitch.....	3.78	0.50	1 : 2 : 3½	2180	78.5	3470	4160	2
J	8 7/8" rods and No. 7 wire spiral, 2" pitch.........	6.11	0.50	1 : 2 : 3½	2050	78.5	4240	5120	2
L	No. 7 wire spiral, 1" pitch	0	1.00	1 : 2 : 3½	1770	78.5	1370	2640	2
K	8 1/2" rods and No. 7 wire spiral, 1" pitch.........	2.0	1.00	1 : 2 : 3½	2000	78.5	2610	3900	2
N	8 11/16" rods and No. 7 wire spiral, 1" pitch......	3.78	1.00	1 : 2 : 3½	1800	78.5	3370	4190	2
M	8 7/8" rods and No. 7 wire spiral, 1" pitch.........	6.11	1.00	1 : 2 : 3½	1680	78.5	3760	4680	2
P	8 1" rods and No. 7 wire spiral, 1" pitch.........	8.0	1.00	1 : 2 : 4	2360	78.5	5666	6920	2
O	8 7/8" rods and 1/4" spiral, 1" pitch...............	6.11	1.96	1 : 2 : 4	2480	78.5	4430	6580	2
R	8 1" rods and 1/4" spiral, 1" pitch..............	8.0	1.96	1 : 2 : 4	2380	78.5	5190	6960	2
Q	8 1 1/8" rods and 1/4" spiral, 1" pitch...............	10.12	1.96	1 : 2 : 4	2300	78.5	5760	7090	2
S	No. 7 wire spiral, 1" pitch	0	1.00	1 : 1 : 2	4070	78.5	4050	5850	2
T	8 7/8" rods and No. 7 wire spiral, 1" pitch..........	6.11	1.00	1 : 1 : 2	4400	78.5	5760	7290	2
V	No. 7 wire spiral, 1" pitch	0	1.00	1 : 1½	4870	78.5	3570	5340	2
U	8 7/8" rods and No. 7 wire spiral, 1" pitch.........	6.11	1.00	1 : 1½	4550	78.5	5950	8150	2

Fig. 2 for columns I_1 and J_2. This point, which may be called the yield point of the column, was found to be at a deformation corresponding to the yield point of the longitudinal steel, and the load then carried was closely equal to the compressive strength of the concrete plus the yield-point strength of the steel, or

$$P_1 = f_c' A (1 - p) + f_s A p$$

where P_1 = yield point, f_c' = ultimate strength of a plain concrete column, f_s = yield-point strength of steel, and p = steel ratio of longi-

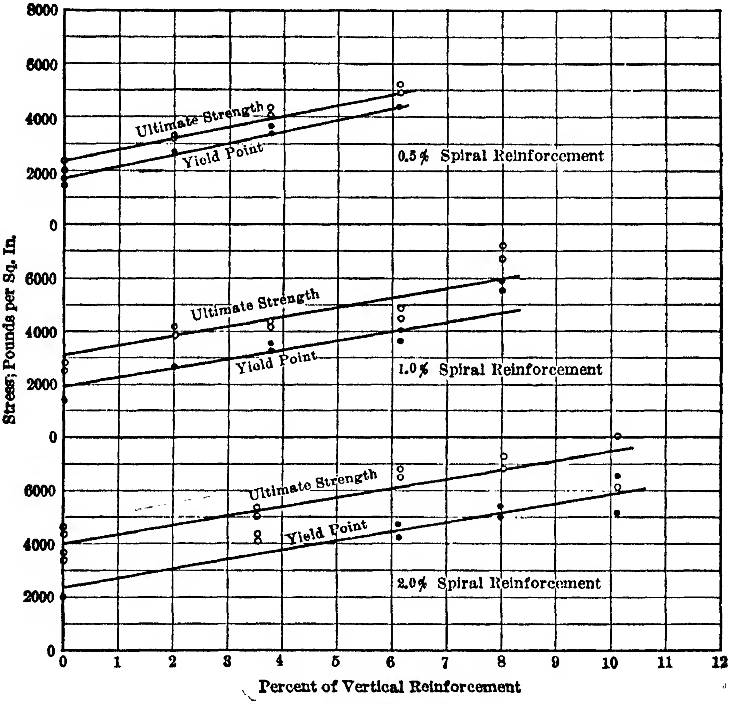

Fig. 3.

tudinal steel. The ultimate strength varied with the amount of spiral reinforcement, the excess over the yield point increasing with the amount of spiral reinforcement and equal to from 500 to 1000 lbs/in.^{2} for each 1% spiral reinforcement. In every case the total deformation was from double to several times the deformation corresponding to the yield point of the longitudinal steel, showing that the steel was stressed beyond that point. Removal of load after stressing far beyond the yield point and reloading showed a higher yield point as in the testing of steel.

Fig. 3 shows individual tests plotted with reference to amount of reinforcement, with lines drawn representing fair average values of ultimate strength and yield point. These diagrams show the small effect of spiral steel upon yield point and the relatively large effect on ultimate. The effect of longitudinal steel on both yield point and ultimate is closely proportional to amount of steel used.

D. Tests of Columns of the American Concrete Institute.—An important series of column tests is that being conducted (1931–32) at Lehigh University and the University of Illinois under the supervision of a committee of the American Concrete Institute.* Some interesting results already published will be noted.

The columns were 8 in. in diameter and 60 in. long. Reinforcement was about 1% spiral and from 1.5% to 6% longitudinal, the latter steel being of three grades, structural, intermediate, and rail steel, with yield points from 42,000 to 68,000 lbs/in.^{2} Plain concrete columns were also tested. The cylinder strength of the concrete ranged from about 2200 to 7300 lbs/in.^{2} at 56 days.

Effect of End Conditions.—Three types of end arrangements were investigated: (*a*) reinforcing bars bearing directly upon the plates of the testing machine; (*b*) columns provided with enlarged heads or capitals 18 in. long and 14 in. diameter, with rods stopped 3 in. short of the end; (*c*) rods stopped 3 in. short and spliced by dowel rods lapped 20 and 30 diameters and bearing against the plates. The highest results were obtained from type (*a*). Results from type (*b*) were from 2% to 4% lower. Results from type (*c*) were from 5% to 15% lower than (*a*). The longitudinal reinforcement in these columns was 4%.

Plain Concrete Columns.—The ratio of strength of columns to cylinders was about 85% at Lehigh and 80% at Illinois.

Reinforced Columns.—The results of these tests showed that in general the longitudinal steel was stressed to its yield point before failure for all grades of steel, the deformations at rupture being equal

* Proc. American Concrete Institute, Vol. 27–1931, pp. 677 to 835.

to and in many cases much greater than the deformation corresponding to the yield point of the steel; and by comparing the results for different steel ratios and yield points they indicate that the effect of the longitudinal reinforcement was to contribute a strength equal to the yield-point strength of the steel. Assuming the strength of the concrete to be the same as that of a plain concrete column, the effect of the spiral steel could then be estimated. A group of the tests at Lehigh indicated that the increase of strength due to 1.2% of spiral steel was from 400 to 570 lbs/in.^{2} for various amounts of longitudinal steel. In the Illinois tests the effect of the spiral steel appeared to be greater.

Tests made by slow loading during the latter part of the run, allowing 4 hours between each increment of load, showed little effect on ultimate strength, but the deformations were in general greatly increased for the last one or two increments in the case of the reinforced columns. For plain concrete the effect was small.

Columns of the higher grade of concrete, tested at the usual speed of loading, failed suddenly, but for concrete of 2000- to 3500-lb. strength, the failure was gradual. Generally speaking, the deformations of these columns were less than in most of the tests on hooped columns heretofore made, owing in part to the relatively high strength concrete used and, possibly, to the small amount of spiral reinforcement compared to the strength of the concrete.

174. Effect of Shrinkage and Flow on Reinforced Columns.—Observations on the columns of buildings have shown deformations over a period of four to six years, due to shrinkage and flow, of 0.03% to 0.15%, corresponding to steel stresses of 9000 and 45,000 lbs/in.^{2}, assuming the yield point not exceeded. The larger the amount of reinforcement, the smaller the effect.* Lateral shrinkage has also been observed corresponding to a steel stress of 11,000 to 15,000 lbs/in.^{2}

Tests by R. E. and H. E. Davis † on plain and reinforced short prisms, 10 in. by 20 in., subjected to sustained load for 1½ years, gave results as follows:

Plain Concrete.—Load of 800 lbs/in.^{2} Initial deformation 0.02%, giving a value of $E = 4{,}000{,}000$ lbs/in.^{2} The deformations due to shrinkage and flow were:

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* Proc., American Concrete Institute, 1921, p 150.

† Proc., American Concrete Institute, 1931, p. 837.

	100 Days	1½ Years
Shrinkage, per cent.........	0.033	0.066
Flow, per cent.............	0.035	0.060

Reinforced Concrete.—Longitudinal reinforcement 1.91%, hoop reinforcement 1.3%. Load of 880 lbs/in.^{2} Initial deformation 0.019%.

	100 Days	1½ Years
Shrinkage, per cent.........	0.023	0.044
Flow, per cent.............	0.022	0.038

The amount of flow under different loads was roughly proportional to the load applied. The rate of change decreased greatly with age (compare 100 days with 1½ years). At 1½ years the steel stress might be calculated as follows:

Initial stress........................	5,700	lbs/in.^{2}
Stress due to shrinkage...............	13,200	lbs/in.^{2}
Stress due to flow....................	11,400	lbs/in.^{2}
Total........................	30,300	lbs/in.^{2}

The stress in the concrete changed from 775 initially to a final stress of 300 lbs/in.^{2} The lateral shrinkage corresponded to a compressive stress in the spiral steel of 13,800 lbs/in.^{2}

The concrete used in these tests was a 1 : 2.64 : 2.32 mix with water-cement ratio of 0.86 by volume. The specimens were stored in air at 50% humidity.

The column tests of the American Concrete Institute Committee mentioned in Art. 173 included also tests on the effect of sustained loads, and results of observations for 20 weeks have been given. After curing in the moist room for 56 days, the columns were removed and loaded with loads calculated by the design formula of the American Concrete Institute, except in a few cases, where the New York Building Code was used. The deformations due to shrinkage and flow (not including initial deformation) amounted generally to from 0.035% to

0.07%, corresponding to a steel stress of 10,000 to 20,000 lbs/in.^{2} Under moist storage the deformations were very much less, generally from 0.01% to 0.02%.

The general effect of shrinkage and flow is to reduce the stress intensity in the concrete and increase that in the steel as compared to the initial or design values. Under increasing loads, in a long-time test, the steel will reach its yield point at an earlier load than otherwise, but if the ultimate strength of the column is measured by the ultimate strength of the concrete plus the yield-point strength of the steel it will not be changed. As compared to the usual short-time test the deformations will be greatly increased.

The relations can best be understood by a rough general analysis of the problem.

Let m = shrinkage coefficient of plain concrete;

E_f = ratio of stress to flow, which increases with age but which, with certain limits, is approximately constant for various stresses;

E_c = usual modulus of elasticity of concrete under short-time loading.

Then if f_c = actual stress in the concrete, the total unit deformation in concrete is

$$\Delta_c = m + \frac{f_c}{E_f} + \frac{f_c}{E_c}.$$

The unit deformation of the longitudinal steel is

$$\Delta_s = \frac{f_s}{E_s}.$$

Placing these equal and solving, we have as the ratio of stresses

$$\frac{f_s}{f_c} = \frac{m\,E_s}{f_c} + \frac{E_s}{E_f} + \frac{E_s}{E_c}. \qquad (1)$$

Assume for illustration values as follows:

m = 0.06%;

E_f = 1,000,000 for usual working loads = 0.06% for a load of 600 lbs/in.^{2};

E_c = 3,000,000 for a 3000-lb. concrete;

E_s = 30,000,000.

The tests and observations previously referred to indicate that these values are not unreasonable. Inserting the numerical quantities in (1), we have:

$$\frac{f_s}{f_c} = \frac{18,000}{f_c} + 30 + 10 = \frac{18,000}{f_c} + 40. \quad . \quad . \quad . \quad (2)$$

This expression replaces the value of n in the usual relation $f_s/f_c = n$. The term $18,000/f_c$ represents shrinkage, and 30 represents flow, and these, taken together, are several times as large as the term 10, the value of n for elastic stresses.

The total load on the column is

$$P = A\,(1 - p)f_c + A\,p\,f_s$$
$$= A\,[f_c(1 - p) + 18,000\,p + 40\,p\,f_c].$$

The average stress is (neglecting $p\,f_c$)

$$f = \frac{P}{A} = f_c\,(1 + 40\,p) + 18,000\,p\,. \quad . \quad . \quad . \quad (3)$$

and

$$f_c = \frac{f - 18,000\,p}{1 + 40\,p} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$f_s = 18,000 + 40 f_c. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

It will be instructive to estimate from these relations the actual stresses in concrete and steel in columns designed by a working formula such as that of the American Concrete Institute. This is

$$f = \frac{P}{A} = f_c\,[1 + (n - 1)\,p]$$

where $f_c = 300 + (0.10 + 4\,p)f_c'$.

Assuming $f_c' = 3000$ lbs/in.^{2}, we have the various design values of f_c, f_s, and f given in cols. 2, 3, and 4 of Table 15. If we then assume the entire load to be permanent and to cause flow we get from eqs. (4) and (5) the probable actual values of f_c and f_s shown in the second group, cols. 5 and 6.

In building columns only a portion of the load is sufficiently permanent to cause flow, mainly the dead load. If the live load is twice the dead load, a not uncommon ratio, and only 50% of the live load is considered in column design, then one-half of the column load will be dead load and one-half live load. The dead load will cause flow, the live load little or none, and the live-load stresses may therefore be distributed according to the usual value of n (n = 10). Values calculated on this assumption are given in the last group, cols. 7 and 8.

TABLE 15

EFFECT OF SHRINKAGE AND PLASTIC FLOW ON STRESSES

Steel Ratio	Stresses by Design Formula (n = 10)			Stresses after Shrinkage and Flow under Full Load		Stresses under One-half Load as Permanent Dead Load and One-half Load as Temporary Live Load not Producing Flow					
						Dead Load		Live Load		Total	
p	f_c	f_s	f	f_c	f_s	f_c	f_s	f_c	f_s	f_c	f_s
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
0.01	720	7,200	780	430	35,200	150	24,000	360	3600	510	27,600
0.02	840	8,400	990	350	32,000	80	21,200	420	4200	500	25,400
0.03	960	9,600	1220	310	30,400	30	19,200	480	4800	510	24,000
0.04	1080	10,800	1470	290	29,600	10	18,400	540	5400	550	23,800
0.06	1320	13,200	2030	280	29,300	−20	17,200	660	6600	640	23,800

The figures in the table show the great effect of shrinkage and flow in reducing the concrete stresses and increasing those in the steel. The smaller the percentage of steel the higher the stresses with the design formula employed here. With half live load the stresses are more nearly uniform in both steel and concrete.

A comparison of the values in cols. 5 and 6 with the results of the tests of the American Concrete Institute referred to in Art. 173, shows that values approximating these were obtained in 5 months.

175. Factor of Safety.—The relative factor of safety with respect to the ultimate strength of a column cannot readily be judged from the values of stresses such as given in the foregoing analysis. This is because of the effect of the amount of steel and the fact that its strength is limited by its yield point. Assuming the column strength to be equal to the compressive strength of the concrete (85% of 3000 lbs. = 2550 lbs/in.^{2}) plus the yield point of the steel (taken at 45,000 lbs/in.^{2}), we have the following relations:

Steel Ratio p	Working Value of f	Assumed Ultimate Strength $2550 (1 - p) + 45{,}000\, p$	Factor of Safety
0.01	780	2975	3.8
0.02	990	3400	3.4
0.03	1220	3825	3.1
0.04	1470	4250	2.9
0.06	2030	5100	2.5

It will be seen that whereas the actual unit stresses shown in Table 15 are highest for the lowest percentage of reinforcement, the factor of safety is the highest for such columns. Considering that the concrete is a more variable material than the steel, and that a larger proportion of the strength is contributed by the concrete for the lower steel ratios, it would seem proper that the factor of safety be somewhat higher for such columns. The difference is, however, greater than would appear reasonable, and the allowable value of f_c in the formula increases rather too rapidly with increased value of p.

176. Tests of Composite Columns.—Tests of this type of column are not numerous. Results show the same general effect as for the other type of column with longitudinal reinforcement. With properly designed hooping the strength of such columns is approximately equal to the combined strength of reinforcement and concrete. As the modulus of elasticity of cast iron used for such purposes is only about 12,000,000 lbs/in.^{2} the ratio of stress in the iron to that in the concrete will be only about 40% as much as for steel, but the effect of shrinkage and flow will generally bring the cast iron into full action.

177. Conclusions Regarding Strength of Columns.—Summarizing the results of tests discussed in the foregoing paragraphs, the following general conclusions may be drawn:

The strength of plain concrete in columns is equal to from 80% to 85% of the strength in cylinder specimens.

The strength of columns reinforced with longitudinal bars only ("tied" columns) is greater than that of plain concrete, the excess being in some cases equal to the theoretical, in other cases less. The results appear to be more uncertain than those of the hooped type.

Columns with longitudinal and hoop reinforcement have a strength somewhat in excess of the sum of the crushing strength of the concrete and the yield point of the steel, the excess depending upon the amount of spiral reinforcement. Deformations increase rapidly when the yield point is reached.

The effect of shrinkage and plastic flow under sustained loads is greatly to increase the stress in the longitudinal steel and to reduce the stress in the concrete. This effect probably is such as to utilize the yield point of the steel in the tied column in the same manner as in the hooped columns. Its effect on the influence of the spiral steel on ultimate strength is problematical and awaits determination.

178. Working Stresses for Columns.—The working stresses usually specified distinguish between the two general types of columns. The hooped column is considered to be a more reliable structural unit than the tied column, and a higher working stress is allowed. The unit stresses allowed on the concrete by the American Concrete Institute Specifications are:

For the tied column $f_c = 0.225 f_c'$
For the hooped column $f_c = 300 + (0.10 + 4p) f_c'$

where f_c' is the cylinder strength of the concrete. The Joint Committee allows only $0.20 f_c'$ for the tied column.

In the hooped or spirally reinforced column the area considered effective is that within the outer surface of the spiral reinforcement, and the fireproofing layer of 1½ in. or more is neglected. Because of the closely spaced spiral steel this outer layer is not well bonded to the interior, and for loads approaching the ultimate it breaks away and is ineffective. In the tied column the entire section is taken into account, as there is full continuity of the concrete throughout. This difference in treatment leads to the somewhat anomalous result that in certain cases the total cross-section of the spiral column will need to be greater than the tied column. This is difficult to avoid as the characteristics mentioned are well established. In both types of columns after the value of f_c has been determined the column section is calculated by the same general formula

$$P/A = f_c[1 + (n - 1) p].$$

For composite columns the core and concrete are assumed to act practically independently and each allowed its own working stress, the concrete, as specified by the American Concrete Institute at 25% of the crushing strength.

179. Column Details.—Details of design and construction involve the hoop or spiral reinforcement, securing of bars in place, splicing of bars, and fireproofing of steel by a sufficient thickness of concrete. These details are described in the specifications. Splices are made by lapping the bars, the specifications requiring a lap of 30 diameters for plain bars and 24 diameters for deformed bars. Some of the tests mentioned in Art. 173 indicate that this is not quite adequate for full strength. Splices are made by extending the bars of the lower column into the base of the upper. At the foundation the column bars must rest on a foundation plate or dowel bars be constructed in the footing and extended up into the column to provide a proper splice.

Spiral reinforcement is made of cold drawn wire of ¼ to ½ in. in diameter. Vertical spacing, or pitch, must not exceed one-sixth the core diameter nor exceed 3 in. As the spiral steel is not generally directly allowed for in the unit stresses, it is used only to the extent sufficient to secure the general effect of hooping. From the tests quoted it would appear that this should be from ½ to 1 per cent. The American Concrete Institute Specifications require an amount equal to one-fourth the longitudinal steel. Some building codes allow directly for the spiral steel. For example, the New York code specifies

$$P/A = f_c\left[1 + (n - 1) p + \frac{2 f_s p'}{f_c}\right],$$

in which

p' = ratio of spiral steel;
f_s = working stress in spiral steel.

The value of f_c is taken as a constant, depending upon the concrete strength.

In view of the uncertain effect of spiral steel on ultimate strength beyond the load determined by the action of the concrete and longitudinal steel, it would seem undesirable to take it directly into account in this way.

180. Bending Stresses in Columns.—The bending moments in columns caused by the action of floor and wind loads are often of considerable amount. The resulting bending stresses, when combined with the axial stresses, are allowed to exceed the usual working axial stresses by a certain amount. The American Concrete Institute Specifications allow a fixed excess stress of $0.15 f_c'$ for hooped columns and $0.075 f_c'$ for tied columns. The latter is one-third the working stress.

CHAPTER VIII

ANALYSIS OF FLAT SLABS

181. General Conditions.—In the preceding chapters, we have dealt with beams on the assumption that they were individual units carrying definite loads, and that the stresses on any transverse section were uniformly distributed in a lateral direction,—that is, were of equal intensity at all points equidistant from the neutral axis. This treatment holds good, of course, for beams or slabs of indefinite width, supported along lines transverse to the axis, provided the loading is uniformly distributed over the entire width of the beam. In such a case it is convenient and satisfactory to analyze a strip of beam 1 ft. wide. But there are many cases, both in building and in bridge construction, of broad beams or slabs in which conditions of support or loading are such that they cannot be analyzed by this method. The important cases of this kind are the following:

A. Broad beams or slabs, supported along two sides only, but loaded with concentrated loads, such as bridge floors supporting concentrated loads.

B. Rectangular slabs supported on four sides.

C. Footings, in which a flat slab supports one or more columns.

D. Floor slabs supported directly on columns, this arrangement constituting the so-called "flat-slab" construction.

In all these cases the structure is a statically indeterminate one, that is to say, the stresses cannot be determined in detail from the principle of statics alone; relative deflections or deformations must be considered. Various investigators have quite fully analyzed some of these cases, but the process is too complex to be applied in practice to specific problems. The results obtained, however, give valuable information regarding the distribution of stresses which can be used to supplement the information obtained by the application of the principles of statics and so enable safe and economical designs to be made. The several cases mentioned above will be considered in the order given.

A. SLAB BEAMS SUPPORTED ALONG TWO SIDES

182. The Problem of Analyses.—Where concentrated loads are applied to beams of great width, it is necessary to determine approximately the manner in which such loads are distributed laterally over the beam. A common example is a bridge floor consisting of a concrete slab resting on steel or concrete beams, the beams running either parallel or transverse to the axis of the roadway. In either case the stresses caused by concentrated loads, such as the wheels of a truck, involve a determination of the extent to which such a concentration is distributed laterally over the beam. The problem may be considered in two parts, (1) the lateral distribution of load over a beam of given width, and (2) the determination of the "effective" width of a beam of indefinite or very great width. Only a rough theoretical treatment of this problem can be given, but the results of analysis may serve as a basis for reasonable rules of practice, and for a rational interpretation of experimental data.

183. Lateral Distribution of Concentrated Loads on Slab Beams.—Fig. 1 represents a plan view of a slab beam of length l and width b, simply supported at $A\,A'$ and $B\,B'$. The principal reinforcement is longitudinal, but the beam is also generally reinforced laterally to some extent. A concentrated load W is applied at the centre. The problem is to determine the relative proportion of this load carried by a central element $a\,a$ and by any other element $d\,d$ or $c\,c$.

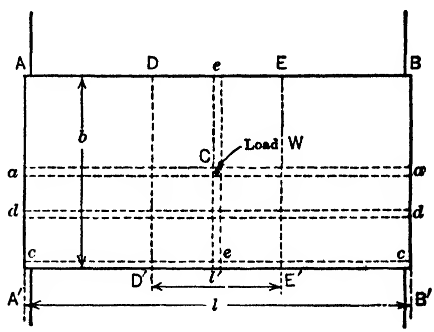


Fig. 1.

In general, the relative loads carried by the various longitudinal elements will be proportional to their deflections, and it is evident that the point e on $c\,c$ will not deflect as far as the point C under the load. The difference in these deflections will be measured by the upward deflection of point e with respect to C, due to the lateral transfer of load. This upward deflection will now be estimated and compared to the downward deflection of the entire beam along the line $e\ e$.

It is obvious that the lateral transfer of load from C toward the edges of the beam is not accomplished entirely by a narrow strip $e\ e$ under the load, but to a variable extent throughout the entire length of the beam. For a certain length l', it may be assumed that the lateral transfer of load will be practically uniform, and it will be sufficiently exact for our purpose to assume the transfer of load to be concentrated over this length l'. This length may safely be taken at $\frac{1}{4}$ or $\frac{1}{3}$ of the length l. It will then be assumed that $\frac{1}{2}$ the load W is transferred laterally in each direction over a length of $l' = \frac{1}{4}\ l$. It will also be assumed that the transverse strips $e\ e$, etc., are so rigid that they transfer the load equally to each longitudinal strip $c\ c$, $d\ d$, etc. Each half of the beam $D\ E\ D'\ E'$ acts, then, as a cantilever beam loaded with a load W at the centre and a uniform upward load equal to $W/2$ on each half, causing a small upward deflection of the edges $D\ E$ and $D'\ E'$ with reference to the centre C. The downward deflection of the entire beam on the line $e\ e$ will be a maximum at C and a minimum at e; its average value will be about the same as if the load W were uniformly distributed along the line $e\ e$. It will be so calculated.

Let Δ = downward deflection of beam centre, assuming the load W uniformly distributed along the line $e\ e$;

Δ' = upward deflection of point e relative to C;

I = moment of inertia of a longitudinal strip of beam, $a\ a$, one unit wide.

It will be sufficiently accurate for present purposes to assume that the moment of inertia of a transverse unit strip $e\ e$ is the same as that of a longitudinal strip. There will be differences in the amount of steel, but the concrete will be the same, and, under working conditions, the differences in steel will have little effect on deflections. Hence, the total moment of inertia of the longitudinal beam will be $b\ I$ and of the transverse beam $D\ E\ D'\ E'$ will be $l'\ I$.

Under the above assumptions, the deflections will be as follows:

$$\Delta = \frac{W l^3}{48 E I b}, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\Delta' = \frac{\frac{W}{2}\left(\frac{b}{2}\right)^3}{8 E I l'} = \frac{W b^3}{32 E I l'}. \quad . \quad . \quad . \quad (2)$$

Comparing Δ' with Δ, we have

$$\frac{\Delta'}{\Delta} = 1.5 \left(\frac{b}{l}\right)^4. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

That is, the ratio of deflections is approximately proportional to the fourth power of the ratio b/l.

If, for example, $b/l = \frac{1}{3}$, then $\Delta'/\Delta = 1.5/81 = 0.02$; that is to say, the edges of such a beam will be deflected upwards with respect to the centre only about 2% as much as the *average* downward deflection, or about 3% as much as the *maximum*. The load W is therefore distributed transversely practically uniformly on a beam of such proportions.

184. Effective Width of Beams of Indefinite Width.—To solve this problem we will first estimate the width of a beam over which the distribution may be considered practically uniform. For such a beam, it would be reasonable to place a limit of variation of deflection at, say, 20% from the average. Then placing $\Delta'/\Delta = 0.2$, we have,

$$\left(\frac{b}{l}\right)^4 = \frac{1}{7.5},$$

from which

$$b = 0.6\, l. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If the length l' be assumed at $l/3$, instead of $\frac{1}{4}$, as would appear reasonable from the results obtained above, the result would be

$$\Delta'/\Delta = 1.12 \left(\frac{b}{l}\right)^4,$$

and for $\Delta'/\Delta = 0.2$,

$$b = 0.65\, l.$$

Applying this information to a beam of indefinite width it should be noted that in the latter case the edges of the strip of width b are not free to bend up as cantilever beams, but are restrained by the portion outside the width considered, hence the ratio of Δ' to Δ for the width b is considerably less than here assumed.

From this analysis, it would appear that it is reasonable to assume an even distribution of load over a width of two-thirds the length of the beam. This may be called the " effective width " of the beam for concentrated loads. Where the concentrated load itself is distributed over a considerable width, such width may be added to the width b as above determined.

185. Transverse Bending Moments in Slab Beams.—The average bending moment in a longitudinal strip one unit wide will be

$$M = \frac{W\,l}{4\,b}. \qquad (1)$$

The bending moment M' in a transverse strip one unit wide, assuming $l' = l/3$, and acting as assumed in Art. 183, will be

$$M' = \frac{W}{2\,l'} \times \frac{b}{4} = \frac{3}{8}\frac{W\,b}{l}. \qquad (2)$$

Making $b = 0.6\, l$ gives the following values:

$$M = 0.42\, W, \qquad (3)$$

$$M' = 0.22\, W. \qquad (4)$$

Equations (3) and (4) indicate that the transverse bending moment, under the conditions assumed, is approximately one-half the longitudinal moment. Tests of slabs appear to show that under working loads the transverse moment is considerably less than this, good results being obtained on beams with little or no transverse reinforcement. An amount of transverse reinforcement equal to ¼ to ⅓ the longitudinal is ample in any case. For loads approaching the ultimate, the transverse moments will become relatively greater, as the deflection will depend more and more on the steel, and less on the concrete. Final failure from concentrated loads generally occurs by the load punching through the slab, a combination of punching shear and diagonal tension.

186. Tests of Slab Beams.—Tests have been made by Professor C. T. Morris * on slabs 4 in. and 7 in. thick, spans from 3½ to 7 ft. and widths from 1 ft. to 7 ft. The longitudinal reinforcement amounted to 1.04% and the transverse reinforcement 0.2% to 0.78%. His general conclusions were as follows:

1. The effective width is affected very little by the percentage of transverse reinforcement.

2. The effective width decreases somewhat as the load increases.

3. The effective width in percentage of the span decreases as the span increases.

4. The following formula will give a safe value of effective width where the total width of slab is greater than $1.35\,l + 4$ ft.

$$b = 0.6\,l + 1.7,$$

where b = effective width in feet, and
l = span-length in feet.

The Office of Public Roads, U. S. Department of Agriculture, conducted a series of tests on a slab beam 12 in. thick of 16-ft. span length, and a width of 32 ft. For a single concentrated load the effective width was found to be about 11 ft., for loads producing concrete stresses about equal to ordinary working stresses. The slab had 0.75% longitudinal reinforcement, but no transverse reinforcement. This value of the effective width is about $0.7\,l$.† The specifications for Highway Bridges of the American Association of Highway Officials specify an effective width of $0.7\,l + W$, where W = width of area over

* Bulletin No. 28, Highway Department, Ohio, 1915.
† *Engineering Record*, Vol. 71, 1915, p. 26.

which the load is applied; but the effective width is not to exceed 7 ft.

187. Distribution of Load from Continuous Slabs to Supporting Joists.—Where a continuous slab is supported by several beams or joists, it becomes necessary to estimate the maximum load on any one joist caused by a concentrated load on the slab. It is a problem depending upon the relative flexibility of the slab and the supporting beams. It will be advantageous to derive certain theoretical formulas covering two simple cases. These will indicate the probable range of values and the effects of the various elements of the problem.

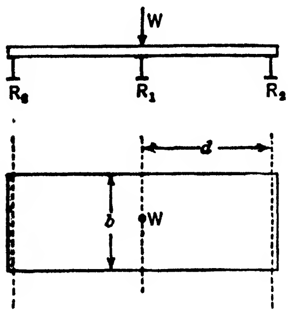


FIG. 2.

We will assume, first, a slab of width b resting upon three longitudinal supporting beams, Fig. 2, and supporting a concentrated load W placed at the centre. The problem is to determine the proportion of the load W carried by the centre beam, or the values of the reactions R_1 and R_2.

Let E = modulus of elasticity of slab;
I = moment of inertia of slab = $1/12\, b h^3$;
K = coefficient of flexibility of supporting beam = deflection in inches for a load of 1 lb. applied where the slab is placed. This can readily be computed for any given length and section of beam;
d = spacing of beams in inches;
a = a constant = $K E I/d^3$.

Then, by placing the deflection of the slab at the centre, equal to the difference in deflection of the centre and outside beams, we can solve for the value of R_1, obtaining

$$R_1 = \frac{1 + 3a}{1 + 9a} \times W. \qquad (1)$$

Then

$$R_2 = \frac{W - R_1}{2}. \qquad (2)$$

If five supports are assumed instead of three, a similar analysis gives the following results (Fig. 3):

$$R_1 = \frac{7 + 108\,a + 36\,a^2}{7 + 204\,a + 180\,a^2} \times W, \quad . \quad (3)$$

$$R_2 = \frac{11 + 6\,a}{16 + 24\,a}(W - R_1), \quad . \quad . \quad . \quad (4)$$

$$R_3 = \frac{W - R_1 - 2\,R_2}{2}. \quad . \quad . \quad . \quad (5)$$

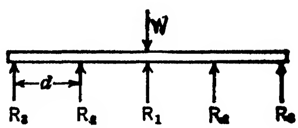


FIG. 3.

To show how the values of R_1 vary with values of the constant a, the curves of this function for values of a from 0.05 to 3.0 are plotted in Fig. 4. It will be seen that for values of a less than 1.0, the value of R_1 is about the same, whether three supports or five supports are used. Small values of a correspond to rigid or widely spaced supports

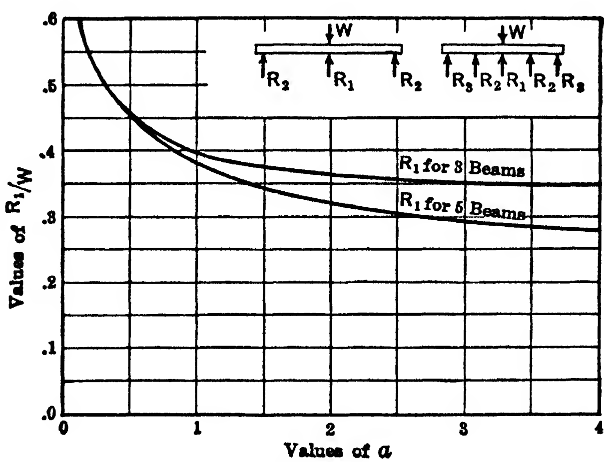


FIG. 4.

and thin or flexible slabs; large values of a, to stiff slabs and flexible supports.

Ordinary values of the constant a may be estimated as follows:

For the usual case of a wide slab resting upon joists, the effective width b may be taken at least as great as d. The value of I is then equal to $1/12\, d\, h^3$. Assume $E = 3{,}000{,}000$. The quantity K is the deflection of the supporting beam for a 1-lb. load distributed over the length b. For a centre load, the deflection is $\frac{1}{48}\frac{W\,l^3}{E_b\,I_b}$, and for a uniformly distributed load it is $\frac{5}{384}\frac{W\,l^3}{E_b\,I_b}$, where W = total load on beam, and l, E_b, and I_b refer to the beam. As the load is only partially distributed, the value of K or the deflection for a load of 1 lb. may be taken at about $\frac{1}{60}\frac{l^3}{E_b\,I_b}$.

Assuming a beam of steel, E_b = 30,000,000. Then if $b = d$, we have

$$a = \frac{K\,E\,I}{d^3} = \frac{l^3}{60\,E_b\,I_b} \times \frac{E \cdot 1/12\,d\,h^3}{d^3}$$

$$= \frac{l^3 d}{7200\,I_b} \times \frac{h^3}{d^3}. \qquad (6)$$

The value of $\frac{l^3\,d}{7200\,I_b}$ will ordinarily range from 150 to 300, hence, approximately

$$a = (150 \text{ to } 300)\,\frac{h^3}{d^3}. \qquad (7)$$

We have, then, for a very thin slab such that $h/d = 1/10$, $a = 0.15$ to 0.30, and from Fig. 4, R_1 = from $0.5\,W$ to $0.55\,W$. For $h/d = 1/7$, $a = 0.45$ to 0.9, and $R_1 = 0.45\,W$ to $0.50\,W$, and for $h/d = 1/5$, $a = 1.2$ to 2.4, and $R_1 = 0.32\,W$ to $0.38\,W$.

As the ratio of thickness to stringer spacing will generally range from 1/5 to 1/7, the value of R_1 may, ordinarily, be taken at about $0.40\,W$.

The tests by Professor C. T. Morris for the Ohio State Highway Department already mentioned included tests on the distribution of load to several beams. Slabs, 6, 7, and 8 in. thick, supported on three 10-in., 25-lb. I-beams, gave a load on the centre beam from 32% to 48% of the total load. The value of a was in this case about $130\,h^3/d^3$ and h/d ranged from 1/5 to 1/7. The corresponding values of R_1 from Fig. 4 are about 0.4 to 0.5. The specifications for Highway Bridges of the American Association of Highway Officials use the formula $R_1/W = d/6$ for concrete floors, where d is stringer spacing in feet.

188. Bending Moment in Slabs Supported on Several Beams.—The bending moment in a slab resting on several supports will depend upon the relative values of the several reactions and the spacing of the beams. For three supports and $R_1 = 0.4\,W$, $M = 0.3\,W\,d$. For five supports and $R_1 = 0.4\,W$, $M = 0.36\,W\,d$, and for $R_1 = 0.35\,W$, $M = 0.40\,W\,d$. For four supports, load placed at the centre, the value of M ranges from $0.35\,W\,d$ for $a = 0.5$ to $0.44\,W\,d$ for $a = 2$.

The above values relate to the effect of a load concentrated along a narrow line along the centre parallel to the beams. Considering the fact that any heavy load is distributed over a considerable width, it would appear that a value of $M = 0.35\,W\,d$ would be ample, as representing the total moment in the portion of the slab considered. For heavy loads, distributed laterally over a considerable distance, such as loads from road rollers, etc., the moment as above found may be divided by the number of stringers over which the load extends. Or, if c = width of load, then $M = 0.35\,Wd\Big/\frac{c+d}{d}$. If $c = d$, then $M = 0.35\,Wd/2$, etc.

The bending moment per foot of width of beam will be equal to the total moment as above determined, divided by the effective width. Taking this as equal to d, we find the moment per foot of width to be equal to $0.35\,W$ or $0.35\,W\Big/\frac{c+d}{d}$, as the case may be. Thus, the moment per foot, in the case of a narrow, concentrated load, is independent of the stringer spacing. It would follow that for such a case, the thickness of slab (neglecting effect of dead load) is also independent of stringer spacing. This appears to be an unreasonable result but, considering the greater distribution of load in a longitudinal direction which results from a wide stringer spacing, it can be seen that within reasonable limits the stringer spacing will have little effect on the bending moment per foot of slab for a concentrated load.

B. RECTANGULAR SLABS SUPPORTED ON FOUR SIDES

189. General Conditions.—Assuming the slab to be a plate of homogeneous material and the supports rigid, various investigators have analyzed the stresses in square and rectangular slabs for both free and restrained edges, and also when continuous over supports. From the results of these investigations, suggested coefficients for a great variety of conditions are contained in a paper by Westergaard,* to which the student is referred for more detailed information. No attempt can be made here to consider this subject at length, but it will be instructive to discuss briefly certain elements of the problem, in order to assist in the selection of coefficients or in the use of approximate solutions by the simple methods of statics.

190. The Square Slab (Fig. 5).—It will be assumed to be simply supported at the edges. Consider the bending moment along the central line $a\,b$. From considerations of possible deflections, this moment is obviously not uniform, being a maximum at the centre

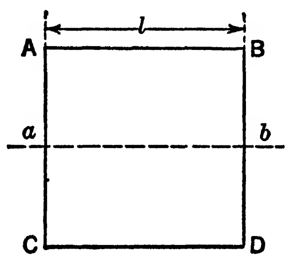


FIG. 5.

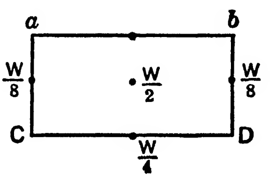


FIG. 6.

and zero at the edges. The pressure along the edges is likewise not uniform, being a maximum at the centres and very small or nothing at the corners. If we then consider the half slab, Fig. 6, cut along $a\,b$ and supported along the margin a—C—D—b, we will attempt to determine the total moment on $a\,b$ by a moment equation about this axis. The external forces are the half load $W/2$, and the reactions $W/4$ and $W/8$. The only uncertainty is the locations of the centroids of pressure along the sides $a\,C$ and $b\,D$. Assuming the pressure to vary as a parabola, the points of application of the reactions $W/8$ are $3/16\,l$ from $a\,b$. Then we derive at once:

$$M_{ab} = \frac{W}{4} \times \frac{l}{2} + 2\left(\frac{W}{8} \times \frac{3}{16}l\right) - \frac{W}{2} \times \frac{l}{4}$$

* Proc. Am. Concr. Inst., 1926, Vol. 22, p. 26. Contains bibliography.

whence

$$M_{ab} = 3/64\,W\,l = 0.047\,W\,l. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This is ¾ the value of 1/16 $W\,l$ found by assuming uniform pressure all around, and may be considered as a reasonably close approximate value. The average moment per unit width = 3/64 W.

Comparison with the more exact analyses given by Westergaard indicates that this value is still too high. He gives theoretical values of about 0.042 W at centre and 0.032 W at quarter point, assuming Poisson's ratio at ⅛. Hencky suggests 0.6 × 1/16 W = 0.037 W. The A.C.I. specifications provide for ¾ of the full half, giving 3/64 W as found in eq. (1). Westergaard suggests a working coefficient of 1/30 or 0.033 W, basing this on a reduction of the theoretical values of about 28%, similar to the reduction made in the coefficients for " flat slab " floor design as noted in Art. 216.

191. Oblong Slabs.—For oblong slabs the proportion of the load carried by the longitudinal system decreases rapidly with increasing ratio of length to breadth. A good notion of the relative distribution of load in the two directions can be obtained from a consideration of relative deflections.

Fig. 7 represents an oblong slab of length l and breadth b. Consider a central strip 1 ft. wide along the line $a\,a'$ and also along the line $m\,m'$. Assume equal moments of inertia which is sufficiently accurate for deflection calculations. Let w_1 = load per foot on the strip $m\,m'$ and w_2 = load per foot on $a\,a'$. The deflection of a beam uniformly loaded is proportional to $w\,l^4$; hence, since the deflections of the two beams are equal, we have $w_1\,l^4 = w_2\,b^4$ or $w_1 : w_2 = b^4 : l^4$. That is the amount of the load carried (per square foot) by the two systems is inversely proportional to the fourth power of the respective dimensions. For points near the ends of the slab the proportion carried by the longitudinal system will be greater, and for points near the sides it will be smaller.

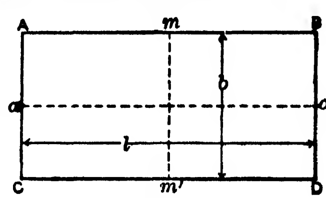


FIG. 7.

In accordance with this theory the proportions of the total load carried by the two systems for various ratios of $l : b$ are as follows:

Ratio $l : b$	1.0	1.1	1.2	1.3	1.4	1.5
Proportion of load carried by transverse system	0.50	0.59	0.67	0.75	0.80	0.83
Proportion of load carried by longitudinal system	0.50	0.41	0.33	0.25	0.20	0.17

More exact theoretical analyses indicate that these values are about right for the *relative proportions* of load carried by the two systems, but that for calculating the actual moments the coefficients should be considerably reduced, as in the case of the square panel, the reduction being greatest for the square panel and decreasing as the ratio of length to width becomes larger. Obviously, for a very long panel the slab would need to be designed for practically 100% to be carried by the transverse system.

For panels of a length of 1.5 or more times the width, the longitudinal reinforcement, besides being small in amount, is also very inefficient, as its unit stresses are limited by the limited deflections. For equal deflections and different span lengths the unit fibre stresses are inversely proportional to l^2; hence, for a ratio of $l/b = 1.5$, the unit stresses in the transverse steel in the central portion will be $1.5^2 = 2.25$, times those in the longitudinal steel. Therefore for ratios of 1.5 or higher the load in the central portion of the beam is carried almost wholly transversely, but there will always be a considerable part of the load in the areas near the ends that will be carried to the adjacent beams by the longitudinal reinforcement.

192. Working Coefficients.—The specifications of the A.C.I. provide that the proportion of load to be carried by the transverse system shall be determined by the formula $R = l/b - \frac{1}{2}$, where R = proportion of total load, l = length, and b = width of panel. The remainder is assigned to the longitudinal system. Furthermore, in the outer quarters of the slab the number of bars may be reduced 50% below that called for by this rule. This in effect gives a total bending moment across the slab of $\frac{3}{4}$ of that called for by the load prescribed. For $l/b = 1.5$, the entire load is to be assigned to the transverse system. In Table 16 are given the resulting coefficients to be applied to the total load to get the total moment across the slab by the usual formula $\frac{1}{8} W l$, where l = span length in the direction considered. There are also given similar coefficients suggested by Westergaard, obtained by a reduction of about 28% from the theoretical values, and a third proposed set of values deduced as described below.

TABLE 16

COEFFICIENTS C FOR RECTANGULAR SLABS SIMPLY SUPPORTED ON ALL SIDES

Total Moment = $C \times \frac{1}{8} W l$

Ratio l/b		1	1.1	1.2	1.3	1.4	1.5
A. C. I.	Transverse	0.375	0.45	0.525	0.60	0.675	0.75
	Longitudinal	.375	.30	.225	.15	.075	.00
Westergaard	Transverse	.267	.33	.38	.44	.50	.54
	Longitudinal	.267	.21	.17	.14	.12	.10
Proposed	Transverse	.30	.39	.48	.57	.66	.75
	Longitudinal	.30	.24	.18	.12	.045	.00
Totals	A. C. I.	.75	.75	.75	.75	.75	.75
	Westergaard	.525	.54	.55	.58	.62	.64
	Proposed	.60	.63	.66	.69	.705	.75

It would seem that in view of current practice in regard to flat-slab floors the coefficients specified in the A. C. I. Specifications are high, but that the general relation between the load on transverse and longitudinal systems is satisfactory. As a compromise proposition it is suggested that the coefficients for the square slab be reduced 20% below those values and that this reduction be gradually diminished to nothing for a ratio $l/b = 1.5$. The formula for the proportions carried by the two systems would then be as follows:

$$\left.\begin{array}{l}\text{For the transverse system } R = 1.2\, l/b - 0.8 \\ \text{For the longitudinal system } R = 1.2 - 0.8\, l/b\end{array}\right\} \quad . \quad . \quad . \quad (2)$$

Then as in the A. C. I. Specifications, the number of bars in the outer quarters are to be reduced 50%, giving a total bending moment equal to ¾ that found by application of the loads of eq. (2). Thus for a slab 12 ft. × 10 ft., Fig. 8, supported on all sides and carrying a load of 150 lbs/ft.², $W = 150 \times 120 = 18{,}000$ lbs., $l/b = 1.2$. The proportions of load used for moment calculations would be, for the transverse system $m\,m'$, $R = 64\%$, and for the longitudinal system $a\,a'$, $R = 24\%$. Then for the transverse system the moment per foot of width for the central half $= \frac{1}{8} \times 18{,}000 \times 10/12 \times 0.64 = 1200$ ft-lbs., and for the outer quarters one-half this value. For the longitudinal system, for the central half, $M = \frac{1}{8} \times 18{,}000 \times 12/10 \times 0.24 = 648$ ft-lbs. per ft. of width, and for the outer quarters one-half as much. For slabs longer than 1.5 b it is suggested that the number of bars in the end portions for a length of ⅓ b be reduced 50%.

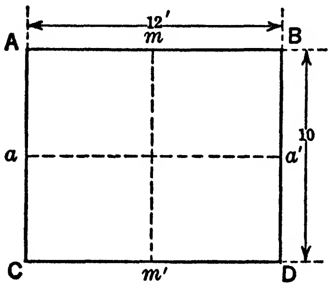


FIG. 8.

193. Coefficients for Continuous Panels.—The A. C. I. Specifications provide that the same coefficients are to be used for rectangular slabs of several panels as for other forms of continuous beams, varying from ⅛ to 1/16, according to the conditions. A study of the coefficients suggested by Westergaard leads to the conclusion that the coefficients of the A. C. I. Specifications for negative moments are satisfactory but that for positive moments they are relatively high, as in the case of the single panel. A fairly consistent set of values can therefore be obtained by applying the proposed set of coefficients of Table 16 for all positive moments, using ⅛, 1/12, or 1/16, according to circumstances of continuity, and the A. C. I. coefficients for the negative moments.

194. Distribution of Slab Loads to Supporting Beams.—Where the floor-slab is reinforced in one direction only the load will practically all be transmitted to the corresponding beams, but at the ends of the panels a small part will be transferred directly to the girder. This may be neglected in the calculations. In the case of reinforcement in two directions, for square or oblong panels, the loads on the beams will not be uniformly distributed but will be considerably greater at the centre than at the ends. The moments in the supporting beams will therefore be increased somewhat as compared to the results for a uniform load. The difference will, however, not be as great as if the load were concentrated at the centre, and it is shown in Art. 235 that for such a case the moment coefficients for girders of two or more spans are not increased more than 8 or 10%. In general, therefore, the usual coefficients may be used for the supporting beams.

If desired, the correct value in any case can be determined by considering that the beams, together with the slab, must sustain the entire bending moment across the several panels, the value of which can be found by considering the structure as a whole. The moment in the beams will then equal the total moment, less the amount taken by the slabs. This will be illustrated by the case of the single panel supported on four beams.

Consider the problem of Art. 192. In accordance with the slab analysis there given, the total slab moment on section *a b*, Fig. 8, is $\frac{1}{8} \times 18{,}000 \times 10 \times 0.48 \times \frac{3}{4} = 8100$ ft-lbs. The total bending moment on *a b* must be equal to $\frac{1}{8} \times 18{,}000 \times 10 = 22{,}500$ ft-lbs. Hence the moment in each beam *A C* and *B D* $= \frac{22{,}500 - 8100}{2} = 7200$ ft-lbs. In a similar manner the moment in each of the beams *A B* and *C D* is found to be $23{,}350/2 = 11{,}675$ ft-lbs.

C. CIRCULAR SLABS SUPPORTED AT THE CENTRE; FLAT-SLAB FOOTINGS

195. In this type of structure it is possible to determine certain facts regarding bending moments by statics alone, but to arrive at a satisfactory estimate of the variation in moments along a section it is desirable to consider the results of a theoretical analysis under certain assumed conditions. These theoretical results will also be helpful in a study of the flat slab floor later on.

196. **Circumferential and Radial Bending Moments.**—Let *A B*, Fig. 9, represent a circular plate loaded on the upper surface and supported over a certain area *C* at the centre by means of a column or pier. The plate will be subjected to bending stresses which will deflect it into a sort of umbrella-shaped figure. Any element *a* will be subjected to moment stresses in both directions *F G* and *D E*, each producing tension in the upper part and compression in the

lower part. The bending moment producing the stresses in the direction FG at a is called a *circumferential* moment, and that in the direction DE is called a *radial* moment. By assuming certain definite conditions regarding the distribution of the pressure over the area C, it is possible to determine the values of these moments at any point in the plate. Such calculations cannot represent the facts very exactly, as the assumed conditions cannot be realized in practice but they are useful in determining the probable distribution of moments along a section.

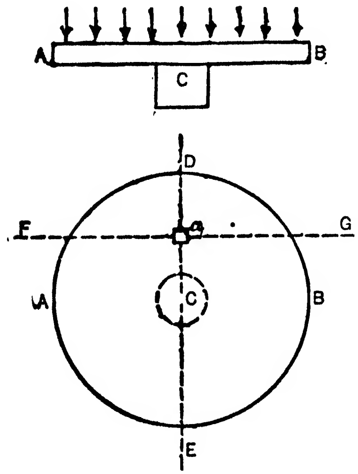


Fig. 9.

The results of such an analysis are given in Figs. 10 and 11 for two assumed conditions of support.* In Fig. 10, it is assumed that the plate and support are rigidly joined so that the plate is fixed at the edge of support, similar to a fixed end beam; in Fig. 11, the reaction of the support is assumed to be uniformly distributed over the area. The full lines in each case give the coefficients for radial moments and the dotted lines for circumferential moments. The values of the moments per unit width are then

$$\left. \begin{aligned} M_r &= C_r W \\ M_c &= C_c W \end{aligned} \right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

in which M_r and M_c are, respectively, the radial and circumferential bending moments in foot-pounds per foot of width, or inch-pounds per inch, and C_r and C_c are the corresponding coefficients given by the full and dotted lines respectively. Ratios of radius R of the plate to radius r of the support, have been taken at values from 3 to 7.

A noteworthy characteristic in Fig. 10 is the very high radial moments at the support and the very great reduction a short distance therefrom. The conditions here assumed cannot exist in practice

* Dr. H. T. Eddy, *Year Book, Univ. of Min.*, 1899, or Morley's *Strength of Materials*, Chapter XIII. Poisson's ratio assumed at 1/10.

as the support cannot be perfectly rigid as assumed, and its compressibility would have a large effect in reducing the high radial moments at the edge. Furthermore, the deformations of reinforcing

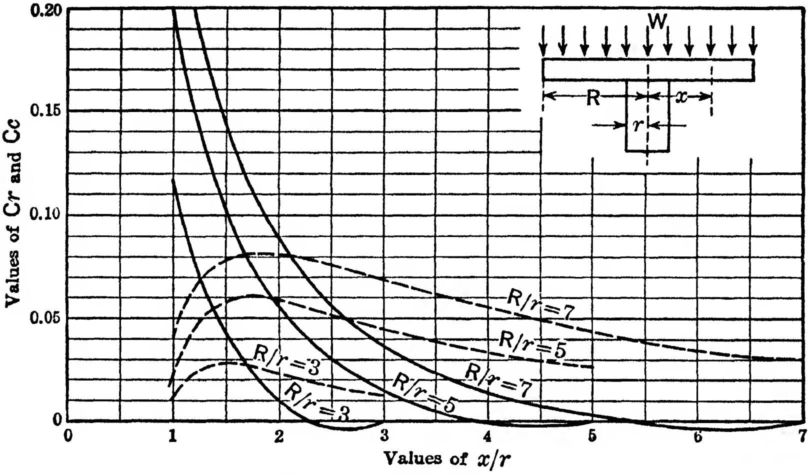


FIG. 10.—Moment Coefficients for Circular Slabs.

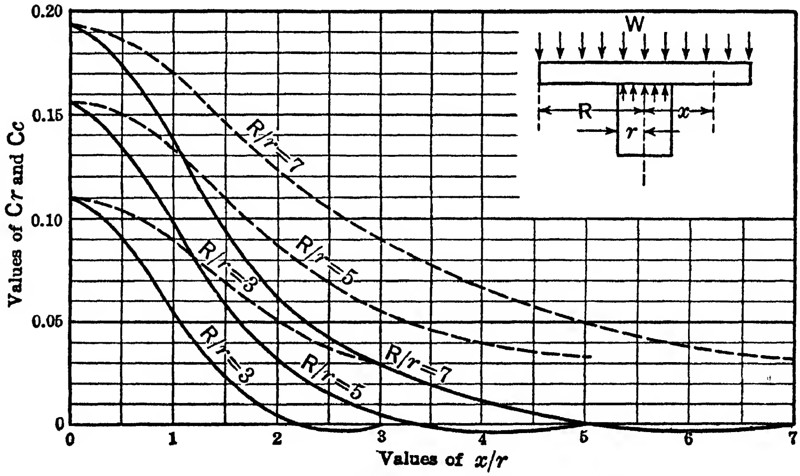


FIG. 11.—Moment Coefficients for Circular Slabs.

bars in the area over the support would result in a very considerable angular moment at and near the edge of support, seriously vitiating the assumption of fixity and reducing the radial moments at this point. For relatively small supports the moment distribution will follow more closely the curves of Fig. 11; for large supports, or small ratios R/r, more nearly those of Fig. 10, but with the radial moments near the support decreased and the circumferential moments increased.

197. Total Bending Moment on a Central Section and its Distribution.—Consider the two cases represented in Fig. 12. Fig. (*a*) represents a plate fixed over the area *B C D*, as in Fig. 10; Fig. (*b*), one with uniformly distributed upward reaction, as in Fig. 11. The critical section for bending moments in (*a*) is the section *A B C D E*, and for (*b*) the section *A B*. The total bending moments on these

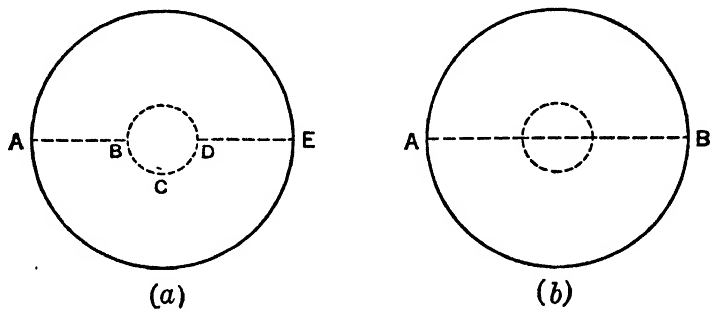


FIG. 12.

sections can be determined by statics. In (*a*) the moment of the load on the half disk *A B C D E F* about axis *A E* is $\frac{W'}{2}\left(\frac{4}{3\pi}\frac{R^3-r^3}{R^2-r^2}\right)$, and the moment of the shears along *B C D*, about the same axis, is $W'/2 \times 2\,r/\pi$. Hence total moment is

$$M_1 = \frac{W'}{2}\left(\frac{4}{3\pi}\frac{R^3 - r^3}{R^2 - r^2} - \frac{2}{\pi}r\right) \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in which W' is the total load on the plate outside of the area directly above the support. If W is the total load then $\frac{W'}{W} = \frac{R^2 - r^2}{R^2}$.

In (*b*) the moment of the load on the half plate about axis $A\,B$ = $\frac{W}{2} \times \frac{4}{3\pi} \times R$ and the moment of the upward forces = $\frac{W}{2} \times \frac{4}{3\pi} \times r$. The total bending moment =

$$M_2 = \frac{W}{2} \times \frac{4}{3\pi}(R - r). \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Example.—Determine the total bending moment and its distribution by the two different assumptions as to support, for $R = 6$ ft., $r = 2$ ft., $w = 500$ lbs/ft.[2]

Total Moment. Case I.—Refer to Fig. 12. The half load on the entire plate $= \frac{1}{2}\pi R^2 \times 500 = 28{,}300$ lbs. The load on the half ring $ABCDEF$ $= 28{,}300 \times \frac{36-4}{36} = 25{,}200$ lbs. $M_1 = 25{,}200\left(\frac{4}{3\pi}\frac{216-8}{36-4} - \frac{2}{\pi}\times 2\right) =$ $25{,}200 \times 1.48 = 37{,}500$ ft-lbs. This is an average moment of $37{,}500/12 =$ 3220 ft-lbs. per ft.

Case II.—$M_2 = 28{,}300 \times 4/3\,\pi \times 4 = 28{,}300 \times 1.70 = 48{,}000$ ft-lbs. $= 4000$ ft-lbs. per ft. on the average.

Distribution of Moment.—Case I.— From the diagram of Fig. 10, the coefficient for radial moment for $R/r = 3$ is about 0.115. The radial

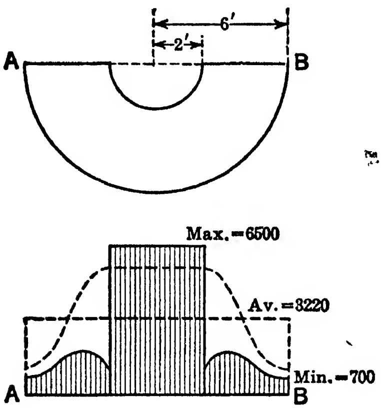


FIG. 13.

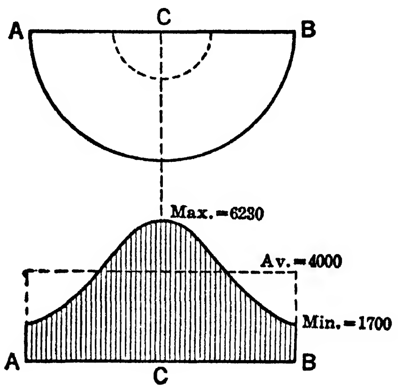


FIG. 14.

moment around the half circle BCD will then be $0.115 \times 56{,}600 = 6500$ ft-lbs. per ft.

If the radial moments along BCD be resolved into components normal and parallel to AE, the normal components will be equivalent to a uniform bending moment along the projection BD of an intensity equal to that at C, or 6500 ft-lbs. per ft.

The circumferential moments from D to E will vary in accordance with the dotted curve for $R/r = 3$ in Fig. 10. The theoretical distribution along the entire section is shown in Fig. 13. Owing to the factors already mentioned there is no such sudden change at the margin of the support as indicated in the figure, and the actual curve will be some such curve as shown by the dotted line; but the total and average moment to be resisted on the given section will be unchanged.

Case II.—Here the moment distribution along the axis AB is shown by the dotted curve of Fig. 11 for $R/r = 3$. The moment per foot at $C = 0.11 \times 56{,}600 = 6230$ ft-lbs. per ft., and Fig. 14 shows the distribution.

Remarks.—From the foregoing study it is seen that the total and average moment across the central section is somewhat greater where the supporting forces are considered uniformly distributed over the area, Case II, than where the support is assumed as rigid, Case I. Also, from Figs. 10 and 11, it is to be noted that, the larger the plate in proportion to the support (R/r), the greater the variation in bending moment across the section.

Column Footings

198. General Conditions.—The circular plate analysis here given may be applied in general to column footings. Single footings are commonly made square in form, and the column load is applied either directly, or through an intermediate pier or pedestal of larger section than the column.

To transmit the load from the longitudinal reinforcing bars into the pedestal or footing, either dowel bars are used, extending upward into the column and downward into the pedestal or footing a sufficient length to transmit the stresses involved through bond resistance, or a base plate is used upon which the column reinforcing bars bear directly. The former arrangement is more economical and is commonly employed. Where a composite column with structural steel or cast-iron core is used, a base plate is necessary. Where the footing is relatively large as compared to the column, an intermediate pedestal of a width of about ¼ that of the footing is commonly employed, as it reduces the necessary thickness of the footing and results in a much more uniform distribution of bending moment than would otherwise be the case, as pointed out in Art. 197.

The area of the footing is determined by the bearing pressure allowed on the soil, or, in case of a pile foundation, on the necessary number and spacing of piles.

199. Bending Stresses.—Assume a square footing supporting a square column or pedestal, Fig. 15. The position of the critical section for bending moment depends upon the rigidity of connection between column base and slab and also upon the shape of column base. Where a base plate is used, it is customary to assume the load on the footing to be uniformly distributed over the area of the plate, giving conditions corresponding to Case II, Art. 196; but where concrete is built on concrete, the connection is assumed rigid, as in Case I. In the first case, the bending moment will be a maximum on section $A\,D$; in the second case, a maximum on a section along the face of the column $F\,G$. Furthermore, with a relatively large area of contact, the square form and the effect of the torsional resisting moments along the sides $B\,F$ and $G\,C$ are likely to promote failure along the straight line $E\,H$ rather than along the irregular line $A\,B\,F\,G\,C\,D$ as might be assumed. Results of tests show failures occurring some times along one section and sometimes along the other.

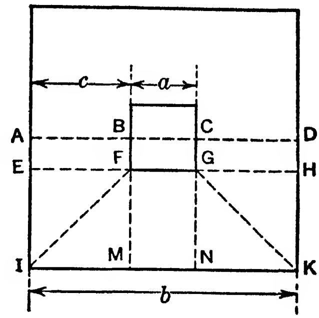


FIG. 15.

Assuming the critical section to be along $E\,H$, the total bending moment is readily found. If w = net upward pressure per square foot = soil pressure less weight of concrete, the total moment on section $EH = w\left(b \times \frac{b-a}{2} \times \frac{b-a}{4}\right) = \frac{1}{8} w b (b-a)^2$. The distribution of this moment will follow roughly the curve of Fig. 13, about one-half the moment being concentrated in the centre 35% to 40% of the width. In practice, such footings are usually reinforced by two sets of bars placed parallel to the sides of the square and spaced uniformly. With this arrangement the stresses will not be uniform, but for footings of ordinary proportions the variation will not be serious.

An important series of tests by Professor Talbot * gave valuable information on the various elements of this problem. Tests on footings 5 ft. square with the column section 1 ft. square showed that the stresses in the bars were fairly uniform over a width somewhat greater than the width of the column, and that the stresses in these central bars were about 1⅓ times the average of all. These results were obtained on the assumption that the yield point of the steel measured the strength in bending where tension failures occurred. From the results of these tests Talbot proposed (1) that the critical section be

* Univ. of Ill. Eng. Exp. Sta. Bul. No. 67, 1913.

taken on a line at the face of the column, as EH, Fig. 15; (2) that in calculating the bending moment the area of pressure be taken as the trapezoid $IFGK$, and that the centres of pressure on the triangles IFM and NGK be taken at 6/10 of the width of the projection FM from the face; and (3) that, for footings having two-way reinforcement spaced uniformly over the footing, the width of footing over which the bars are effective may be taken equal to the width of pier plus twice the depth of footing plus one-half the remaining distance to the edge of footing. The moment thus calculated is expressed by the formula

$$M = \frac{w}{2}(a + 1.2\,c)\,c^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where c = width of projection from column to edge of footing, $= (b - a)/2$. For $b/a = 4$, the result is 30% less than the theoretical total value previously given. For smaller ratios the difference is less and for larger ratios it is greater. Considering the limitation proposed regarding the effective width of reinforcement, these rules appear to be reasonable and have been adopted by the Joint Committee. Although the resulting stresses in the centrally located bars may be somewhat high, any over-stressing of these bars results in a more even distribution. In the specifications of the A. C. I. the same rules are used without limitation of width of effective reinforcement, the entire width being assumed as effective. For large ratios of b/a this results in relatively high stresses in the central bars, but other limitations regarding shearing-stresses preclude the use of very large ratios of b/a.

200. Shearing-Stresses.—As a measure of diagonal tensile stress the tests already mentioned indicated that the critical section may be taken at a distance from the face of the pier about equal to the thickness of the footing. As diagonal tension cracks are approximately at a slope of 45°, it may be expected that true diagonal tensile stresses are not fully effective on sections much nearer the face of the pier than the depth of the footing. The critical section for shear is therefore taken along the four sides of a square having a width equal to the width of pier or column, plus twice the thickness of the footing. The total shearing force is equal to the net upward pressure upon the area outside this square, equal to the soil pressure less weight of concrete.

201. Thickness of Square Footing as Determined by Shearing-Stress.—Assuming the critical section as above explained, a definite relation exists between the ratio of depth of footing to breadth, the soil pressure, and the allowable shearing-stress; and as the shearing-stress generally determines the thickness of footing, a general formula for thickness will be useful.

Let k = ratio of width of pier or column to width of footing = a/b;
w = net upward pressure per square foot;
d = depth of footing in feet.

Then the total shear area in the footing is that on the four sides of a square of width $= a + 2\,d = k\,b + 2\,d$. The total shearing force $= w\,[b^2 - (k\,b + 2\,d)^2]$, and the shearing force per foot of width $= V = \dfrac{w\,[b^2 - (k\,b + 2\,d)^2]}{4\,(k\,b + 2\,d)}$. If v_c = allowable shearing unit stress, then $v_c = \dfrac{V}{12 \times j\,d \times 12}$. Taking $j = 7/8$, we get $d = \dfrac{V}{126\,v_c}$. Substituting the value of V, we have

$$d = \frac{w}{504\,v_c} \cdot \frac{b^2 - (k\,b + 2\,d)^2}{k\,b + 2\,d}$$

or, more conveniently, and with sufficient accuracy,

$$d/b = \frac{w}{500\,v_c} \cdot \frac{1 - (k + 2\,d/b)^2}{k + 2\,d/b} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The quantity $\dfrac{w}{500\,v_c}$ for any given problem is a constant. Let this be represented by C. Then solving (5) we get finally

$$d/b = \frac{\sqrt{2\,C + 4\,C^2 + \frac{1}{4}\,k^2} - \frac{1}{2}\,k\,(1 + 4\,C)}{2 + 4\,C} \quad . \quad . \quad (6)$$

in which $C = \dfrac{w}{500\,v_c}$ and $k = a/b$. Note that w is in lbs/ft.^{2} and v_c is in lbs/in.^{2}

For sloped footings the requirements for thickness are the same with respect to the section at a distance from the pier equal to the depth of the footing at the pier. Stirrups in footings are difficult to place and impracticable, and hence the allowable shearing-stresses are those for beams without web reinforcement, usually 3% of the concrete strength, and special anchorage is required.

202. Bond Stress.—Bond stress in footings is very important as it is likely to limit the size of bars that may be used. The critical section for bond stress is considered to be at the face of the pier, and the shearing-stresses will be much greater than those previously considered. Total upward pressure, or shearing force, $V, = w\,c\,\frac{b+a}{2} = \frac{w\,(b^2 - a^2)}{4}$. If $\Sigma\,o$ = sum of perimeters of all bars effective for moment, then

$$\Sigma\,o = \frac{V}{u\,j\,d} = \frac{w\,(b^2 - a^2)}{4\,u\,j\,d}. \qquad (7)$$

A definite value for bar diameter can be obtained from the relation between total steel area required for moment and the total perimeter sum required for bond. From Art. 199, $M = \frac{w}{2}\,(a + 1.2\,c)\,c^2$ and steel area $= A = \frac{M}{f_s\,j\,d} = \frac{w\,(a + 1.2\,c)\,c^2}{2\,f_s\,j\,d}$. The diameter of a round or square bar $= D = \frac{4 \times \text{area}}{\text{perimeter}} = \frac{4\,A}{\Sigma\,o}$. Substituting the foregoing values of A and $\Sigma\,o$ we get

$$D = \frac{(1.2\,b + 0.8\,a)(b - a)}{f_s\,(b + a)} \times u. \qquad (8)$$

For $a = 0.25\,b$, a common value, and $f_s = 18{,}000$, eq. (8) becomes $D = 0.000047\,b\,u$, in which b and D are in like units, and u is in lbs/in.^{2} For a 2000-lb. concrete, for example, and $u = 120$, $D = 0.0056\,b$. A footing 90 in. wide would thus permit a maximum diameter of bars of ½ in.

It will be noted by reference to the specifications that the allowable bond stress in two-way footings is smaller than in ordinary beams and one-way footings. This is due to the influence which the tensile stresses and strains at right angles to the bars in question may have in causing incipient cracks and reducing bond resistance.

All bars should be hooked at the ends to secure adequate bond strength.

203. Pile Footings.—When the footing is placed on piles, the concentration of the load on the individual piles should be used in determining the actual pressures upon the portion of the footing under consideration.

204. The Pier or Pedestal.—To reduce the stresses in the footing it is usually economical to enlarge the area of pressure upon the footing by supporting the column upon a low pier or pedestal of a width of about one-quarter that of the footing. The pier, if unreinforced, can carry a load proportioned for plain concrete, and the allowable unit pressure on its top is dependent upon the ratio of its area to the area of the column. According to the A. C. I. Specifications this allowable pressure increases with increased area by the relation $p = f_c \sqrt[3]{A/A'}$, where A = area of footing or pedestal, A' = area of columm, f_c = allowable compressive stress for plain concrete, and p = allowable pressure. The pedestal itself may be spirally reinforced, in which case the allowable stress f_c can be correspondingly increased.

205. Examples.—1. Design a flat top footing to support a column load of 1,000,000 lbs., the column being 36 in. in diameter or 1020 sq. in. in cross-section. Allowable soil pressure = 4000 lbs/ft.^{2} Use a 3000-lb. concrete; f_s = 18,000 lbs/in.^{2}, v_c = 90 lbs/in.^{2}, u = 180 lbs/in.^{2} Use a pedestal of width one-quarter that of the footing. A.C.I. Specifications.

Solution.—Estimating the depth of footing to be about 2 ft., its weight will be 300 lbs/ft.^{2} and the upward net pressure producing bending and shearing-stresses in the concrete will be 4000 − 300 = 3700 lbs/ft.^{2} Required area = 1,000,000/3700 = 270 sq. ft., requiring a footing 16 ft. 6 in. square. Then in eq. (6) for d/b, C = 3700/45,000 = 0.082; k = 0.25; and $\dfrac{d}{b} = \dfrac{\sqrt{0.164 + 0.027 + 0.015} - 0.125 \times 1.33}{2.33} = 0.124$. Hence depth = 0.124 × 16.5 = 2.05 ft. = 25 in. Allowing 4 in. from centre of steel to lower surface, the total depth will be 29 in. and weight per square foot = 360 lbs. Correcting the values, it will be found that the width should be made 16 ft. 8 in. The depth will not be changed. Width of pedestal = 4 ft. 2 in., width of projection c = 6 ft. 3 in.

Total bending moment = $\frac{1}{2} \times 3640\,(4.18 + 1.2 \times 6.25)(6.25)^2 = 835{,}000$ ft-lbs. Total steel area, using $j = \frac{7}{8}$, $= \dfrac{835{,}000 \times 12}{18{,}000 \times \frac{7}{8} \times 25} = 25.4$ sq. in. The bar diameter D must not exceed 0.0000467 bu., = 0.0000467 × 200 ×

180 = 1.68 in. We may use twenty-six 1-in. square bars or twenty $1\frac{1}{8}$-in. square bars. The latter will be preferable. The steel ratio is $\frac{25.3}{25 \times 200}$ = 0.0051, which indicates that the compressive stress in the concrete is low, about 600 lbs/in.^{2} It will be observed that the depth was determined by the shearing stresses.

If the pedestal is 4 ft. 2 in. square, its area = 17.5 sq. ft. The allowable bearing pressure on the pedestal = $\sqrt[3]{17.5/8.5}$ = 1.25 times the usual working stress, which for plain concrete is one-quarter the ultimate strength. The actual pressure is 1,000,000/1020 = 980 lbs/in.^{2} and 980/1.25 = 790 lbs/in.^{2} If plain concrete is used it must therefore have a strength of 4 × 790 = 3160 lbs/in.^{2}, which can readily be supplied; otherwise the pedestal would need to be spirally reinforced. The pedestal will project a distance equal to $\frac{50 - 36}{2} = 7$ in. from the column, and according to the specification its height must be at least twice this projection = 14 in.

2. Change the width of pedestal to one-third the width of footing and determine the effect.

Solution.—Assuming again a weight of footing of 300 lbs/ft.^{2}, the value of C in eq. (4) = 0.082 as before, $k = \frac{1}{3}$, b = 16 ft. 6 in. Then $\frac{d}{b} = \frac{\sqrt{0.164 + 0.027 + 0.028} - 0.167 \times 1.33}{2.33} = 0.116$. Hence depth = 0.116 × 16.5 = 1.91 ft. = 22 in. Total depth = 26 in. Weight per square foot = 325 lbs. Recalculation shows that the dimension of 16 ft. 6 in. will be correct. The width of pedestal will be 5 ft. 6 in. Width of projection c = 5 ft. 6 in.

Total bending moment = $3675/2\,(5.5 + 1.2 \times 5.5)(5.5)^2$ = 670,000 ft-lbs. Steel area = $\frac{670{,}000 \times 12}{18{,}000 \times \frac{7}{8} \times 22}$ = 23.3 sq. in., requiring nineteen $1\frac{1}{8}$-in. bars.

The pedestal area = 30.2 sq. in. and the allowable pressure = $\sqrt[3]{30.2/8.5}$ = 1.52 times normal value. Actual pressure = 980 lbs/in.^{2} and 980/1.52 = 650. A plain concrete of ultimate strength of 4 × 650 = 2600 lbs/in.^{2} is required. The pedestal will project $\frac{66 - 36}{2} = 15$ in. from the column, and according to the specifications its height must be at least twice this value or 30 in.

Comparing the two designs, we have the following quantities:

No. 1. Pedestal = 20.3 cu. ft.
Footing = 670
Total concrete = 690 cu. ft.

Using twenty $1\frac{1}{8}$-in. bars the weight of steel = 3140 lbs.

No. 2. Pedestal = 75.5 cu. ft.
Footing = 590.0
Total = 665.5 cu. ft.

Using nineteen $1\frac{1}{8}$-in. bars the weight of steel = 2940 lbs.

The saving in material for No. 2 may or may not be overcome by the extra excavation required, depending upon circumstances.

206. Combined Footings.—Where it is not possible to place the column over the centre of the footing, as is frequently the case with exterior columns, it is necessary to combine two or more footings in order to maintain a reasonably uniform earth pressure. The object to be attained is to bring the centre of gravity of the bearing area of the footing to coincide closely with the centre of gravity of the load. In accomplishing this the dead load should be given principal consideration, as settlement of foundation is dependent mainly upon the steady load and but little upon ordinary live loads. This is of considerable importance where exterior walls and tower loads are to be combined with loads from interior columns. The stresses in the footings themselves must be determined on the basis of the maximum loads assumed for the columns resting thereon.

Combined footings may be considered as of two types: (1) a continuous broad slab footing supporting the columns; (2) essentially separate footings connected by a concrete beam to resist the moment due to eccentricity. These types will be analyzed by an example treated in different ways.

Examples.—Design a combined footing for an exterior and an interior column with column loads of 300,000 and 450,000 lbs. respectively. The columns are 22 in. and 28 in. square in section and spaced 20 ft. apart. Use a 2500-lb. concrete with f_c = 1000 lbs/in.^{2}, f_s = 18,000 lbs/in.^{2}, shearing-stress without diagonal tension reinforcement $v_c = 75$, and with such reinforcement, 150 lbs/in.^{2} Bond stress without special anchorage = 90, and with special anchorage = 180 lbs/in.^{2}, $n = 12$. Allowable earth pressure = 6000 lbs/ft.^{2}

1. *Rectangular Slab Design* (Fig. 16).—The centre of gravity of the loads is at a distance from A equal to $\frac{450 \times 20}{750} = 12$ ft. This should be made the center of the footing area. The length of the slab will then be $2 \times 12 + 2 \times 11/12 = 25$ ft. 10 in. Assume the slab weight to be 500 lbs/ft.^{2} The net value of bearing pressure to support the columns = 6000 − 500 = 5500 lbs/ft.^{2} Required area = 750,000/5500 = 137 sq. ft. Required width = 137/25.83 = 5.3 ft. or 5 ft. 4 in.

Bending Moments.—In a footing of this kind the bending moment will usually determine the section and will be calculated first. The line of zero shear, *D–D*, is at a distance from the left end determined by the equation $300{,}000 = 5500 \times 5.3 \times x$, whence $x = 10.3$ ft. $M = 300{,}000 \times 9.4 -$

$5500 \times 5.3 \times (10.3)^2/2 = 1{,}270{,}000$ ft-lbs. $= 240{,}000$ ft-lbs. per ft. of slab. For the given unit stresses the coefficient of resistance $R = 174$, hence $d^2 = 240{,}000/174 = 1380$ and $d = 37$ in. The total depth should be 4 in. more or 41 in. The assumed weight is sufficiently exact.

For the units assumed the steel ratio $p = 0.0111$ and the steel area per foot of width $= 0.0111 \times 12 \times 37 = 4.93$ sq. in. requiring $1\frac{1}{8}$-in. bars spaced about 3 in. apart. In the vicinity of Column *B* there will be a line of zero moment, *E–E*, which is found to be 1.4 ft. from the column centre. The reinforcing bars may then be stopped off between *D–D* and *E–E* in accordance with the moment requirement, extending a part of the bars beyond the line of the column as in a continuous beam. At the other end the bars should be treated as the positive reinforcement of a simply supported beam.

Between *E–E* and the right end the moment is positive and the bottom bars are determined by the moment at the right face of the column as in an

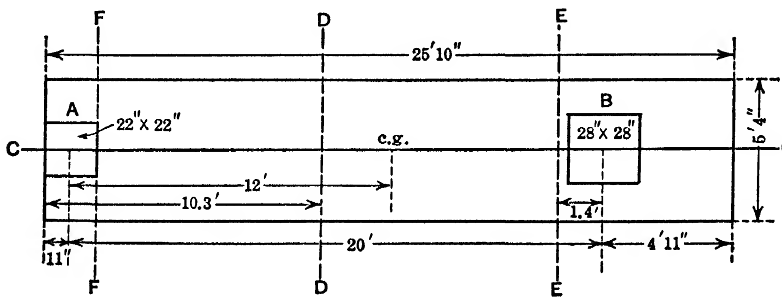


FIG. 16.

ordinary footing. Transverse bars are calculated in a similar manner. With the depth of slab used the transverse bars are hardly needed as the projection beyond the column is only about one-half the depth of the slab. The theoretical steel area along the section *C–C*, for Column *B*, may be figured for a width of slab of 28 in. + 2×3 ft. = 8.3 ft. Upward pressure = $5500 \times 8.3 \times 1.6 = 64{,}000$ lb. Moment $= 64{,}000 \times 1.6/2 = 51{,}200$ ft.-lbs. Steel area $= \dfrac{51{,}200 \times 12}{18{,}000 \times \frac{7}{8} \times 37} = 1.0$ sq. in. Use $\frac{1}{2}$-in. round bars spaced 12 in. apart for a width of 8 ft.; also the same for a width of about 5 ft. opposite Column *A*. They may be omitted in the intermediate section.

Shear and Bond Stress.—The maximum shearing-stress will occur on section *F–F* adjacent to Column *A*. Inasmuch as the top of the beam is in tension and a diagonal crack will tend to start at the top, the critical section is adjacent to the column instead of at a distance therefrom equal to the depth of footing, as in the ordinary footing problem. The shear on section *F–F* is $300{,}000 - 5500 \times 5.3 \times 1.83 = 247{,}000$ lbs. Shearing-stress $v = \dfrac{247{,}000}{12 \times 5.3 \times \frac{7}{8} \times 37} = 120$ lbs/in.^{2} This value will require the use of stirrups calculated as in an ordinary simply supported beam (inverted). Such stirrups must be looped about or hooked over the bars at the top.

The required bond area for top bars at section *F–F* is equal to $\Sigma o = \frac{120 \times 12}{90} = 16$ sq. in. per ft. of slab, or with special anchorage 8 sq. in. Using the latter value, about one-half the bars will need to be extended to the end and provided with long hooks. The bars so extended should be concentrated mainly in the central portion of the slab.

The shearing-stress near Column *B* may be calculated on a section to the left of the column a distance equal to the depth of 37 in. This stress will be $450{,}000 - 5500 \times 5.3 \times 10.33 = 150{,}000$ lbs. Shearing-stress $= \frac{150{,}000}{12 \times 5.3 \times 7/8 \times 37} = 73$ lbs/in.^{2}, which requires no stirrups.

2. *Separate Footings with Beam Connection.*—Since the foregoing calculations show that the bending moment is the determining factor in the dimensions of the combined footing it may be concluded that a more economical design can be made by using a narrow and deep beam along the centre and spread out the footings under the columns. This will have the effect of concentrating the upward pressure near the ends of the beam and so reducing the bending moment.

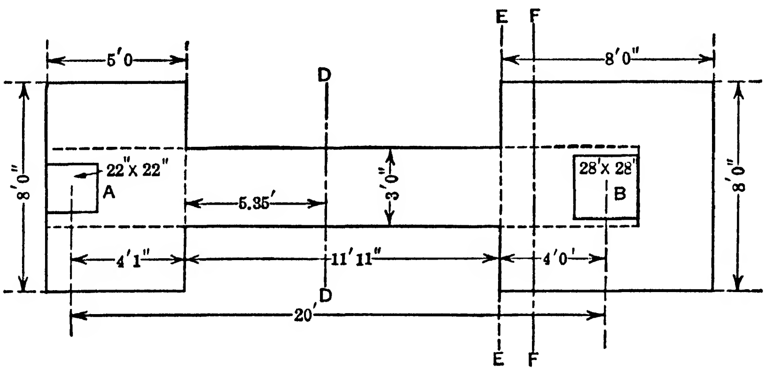


FIG. 17.

The same total area of footing will be required as before, namely, 135 sq. ft. Also, as before, the centre of gravity of the footing area should coincide with the centre of gravity of the column loads, namely, about 12 ft. from Column *A*. Assuming a connecting beam 3 ft. wide, trial calculations lead to the dimensions shown in Fig. 17. This gives a total area of 140 sq. ft. and a centre of gravity 12.0 ft. from *A*. The average earth pressure, exclusive of weight of footings, $= 750{,}000/140 = 5350$ lbs/ft.^{2}

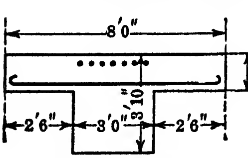
Bending Moment.—The point of zero shear, section *D–D*, is 5.35 ft. from the face of the *A* footing. The bending moment $= 300{,}000 \times 9.45 - 40 \times 5350 \times (2.5 + 5.35) - 5350 \times 5.35 \times 3 \times 5.35/2 = 926{,}000$ ft-lbs. $= 309{,}000$ ft-lbs. per ft. of beam width. Then $d^2 = 309{,}000/173.3 = 1780$

and d = 42 in. Total depth = 46 in. Steel area = 0.0111 × 12 × 42 = 5.6 sq. in. per ft. of width.

As this thickness is greater than required for type 1, it is obvious that no material will be saved if the entire footing is made of uniform thickness. Two methods of reducing the amount of concrete may be considered: (*a*) design the slabs under *A* and *B* in the same manner as ordinary footings and run the connecting beam into the slabs only so far as necessary to make a proper connection; (*b*) extend the beam through the footing or nearly so and design the footings as projections from the beam in the form of flanges of a T-beam. Both methods will be considered.

(*a*) As Column *B* is centrally placed, the required thickness of its footing will be less than for Column *A*. The shear in the footing slab is that due to the net earth pressure of 5350 lbs/ft.² By eq. (6), Art. 201, with v = 75 lbs/in.², the ratio d/b = 0.155. Hence d = 0.155 × 8.0 = 1.24 ft. = 1 ft. 3 in. Make total depth 1 ft. 7 in. It is then necessary to investigate the bending moment on section *E–E* to determine the necessary depth of beam at that point. Upward pressure on footing = 5350 lbs/ft.² Hence M = 450,000 × 4 − 5350 × 64 × 4 = 428,000 ft-lbs. = 142,700 ft-lbs. per ft. of beam. Then d^2 = 142,700/174 = 820, d = 28.7 in., which is thicker than the slab. Sufficient depth will be secured by making the beam the full depth of 46 in. for a distance of 4 or 5 ft. and then sloping it in a straight line to the slab depth at the right face of the column. The remaining part of the footing under this column may then be designed for moment as an ordinary square footing with upward pressure of 5350 lbs/ft.² and thickness of 1 ft. 7 in.

At Column *A* the footing may also be reduced in thickness as for Column *B*. The same depth of 1 ft. 7 in. may be used and the beam tapered in the same manner. The slab here should be reinforced by bars at right angles to the beam as in (*b*) below.



Section F-F

Fig. 18.

Shearing-Stress in Beam.—The maximum will be on section *E–E* and will be found to be about 94 lbs/in.², requiring a few stirrups.

(*b*) In this arrangement the beam is extended nearly through the *B* footing. The section through *F–F* will be as shown in Fig. 18. The beam will be 3 ft. × 3 ft. 10 in. as before. The flanges are subjected to an upward pressure of 5350 lbs/ft.²

The section for shear calculation may be taken a distance d from the beam. This will result in a depth of about 0.9 ft. Use d = 12 in. and a total thickness of 16 in. For d = 12 in. the shearing-stress at 12 in. from the beam will be 65 lbs/in². For bending moment A_s = 1.27 sq. in. per ft. of width, and the bond stress, calculated for the edge of the beam, will require a total $\Sigma\, o$ of 7.07 sq. in. Use three ¾-in. round bars per ft., giving A = 1.33 sq. in. and $\Sigma\, o$ = 7.1 in. The compressive stress in the concrete will be about 870 lbs/in.² The same dimensions are required for the *A* footing. The thickness required for moment will be less. The main beam may be made of uniform depth, or tapered somewhat as in (*a*). It may also be stopped somewhat short of the extreme edge of the *B* footing and the overhanging portion reinforced by bars parallel to the beam. This type of footing is substantially the same as (*a*) but the reinforcement is somewhat simpler and the calculations more direct and definite.

Comparing the rectangular flat-slab type I and type II (*b*) with the reduced thickness of footings, the volume of concrete is as follows:

Type I. Flat slab. 472 cu. ft.
Type II. Beam 23 ft. long of uniform depth, with flanges, 360 cu. ft.

For light loads or small areas the plain slab of type I is likely to be the more economical.

207. Combined Footings on Piles.—Where the footings rest on piles the connecting beam is not subjected to upward pressure between the separate pile foundations. This condition brings the line of maximum moment in the beam to the edge of the pile foundation for the exterior column, and makes the arrangement described under (*a*) more advantageous.

208. Trapezoidal Footings.—Where both columns must be placed at the edge of the footing and carry unequal loads, a trapezoidal form must be used in order to have the centre of area in the proper place. In Fig. 19, if d_1 = distance from one side to the centre of gravity and

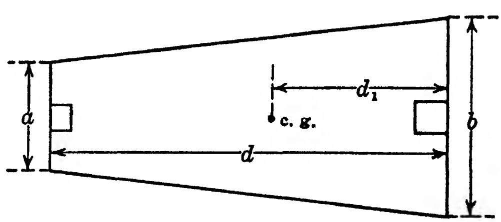


FIG. 19.

A = required area, then the dimensions of the footing are given by the equations

$$b = A\,\frac{4\,d - 6\,d_1}{d^2} \quad \ldots\ldots\ldots \quad (1)$$

$$a = \frac{2\,A}{d} - b \quad \ldots\ldots\ldots\ldots \quad (2)$$

D. FLAT-SLAB FLOORS

209. General Description.—The "flat-slab floor" is the name given to a type of floor and column design in which the floor is built in the form of a continuous flat slab of uniform, or nearly uniform, thickness, and supported directly upon the columns, the floor and columns

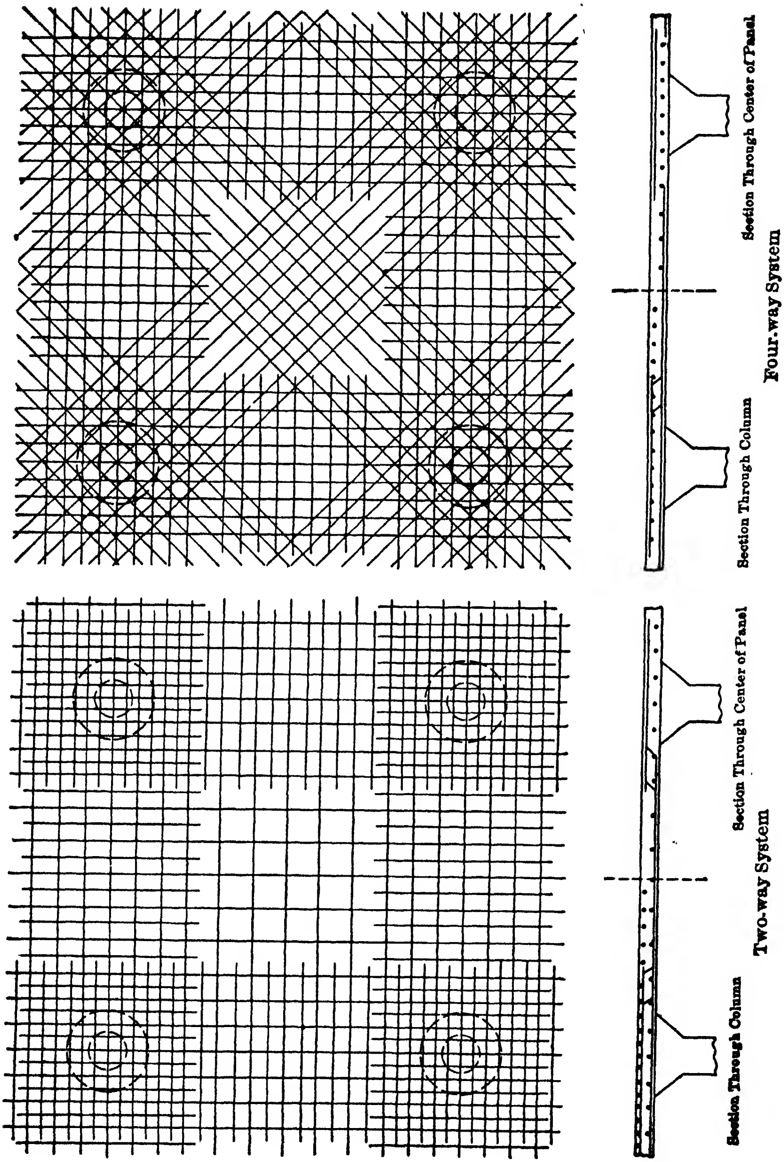


FIG. 20.—Flat Slab Reinforcement.

being built monolithically. In supporting the load, the floor acts as a continuous plate rather than a combination of beams and slabs, and the reinforcement must be arranged accordingly. Fig. 20 illustrates two common arrangements of columns, slab, and reinforcement: the two-way system and the four-way system. Various combinations of straight and bent-up bars are used to secure the necessary steel area. A three-way system is sometimes employed in which the columns are placed at the apices of a system of equilateral triangles. Another system employs a combination of circular and straight bar reinforcement, concentric rings being used above the columns and also at panel centres, and radial bars over the columns.

In order to extend the area of support and reduce the stresses in the slab, the columns are usually enlarged at their top, forming column capitals. The stresses in the slab may also be further reduced by thickening a portion of the slab over and near the column, forming what is commonly called a "dropped panel." (See Fig. 27.)

The flat-slab type of floor is especially advantageous for relatively heavy loads, large areas with few openings, and a uniform column spacing.

210. Nature of Stresses Involved. — Let Fig. 21 represent a portion of a flat-slab floor, including four column supports, assumed as equally spaced. Assume a uniformly distributed load. The column heads underneath the slab are represented by full circles. Considering the nature of the bending moments and deformations at different points, it is plain that if we draw radial lines, *A B*, *A C*, *A D*, from a column centre *A*, the curvature of the slab along such lines will be convex upwards for a certain distance from the centre, then concave upwards,

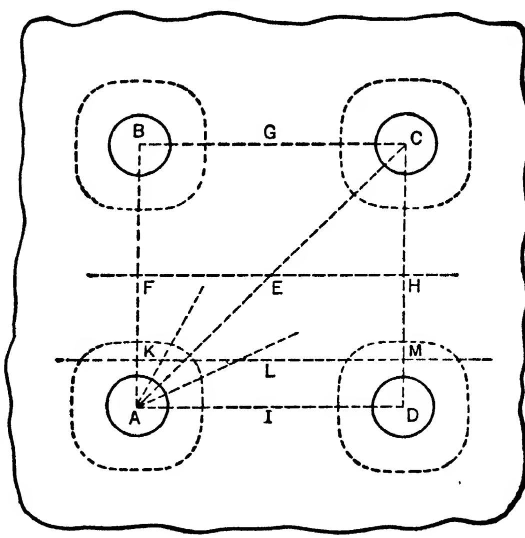


Fig. 21.

then again convex upwards in the vicinity of the other support. That is to say, there will be points of inflection along these lines where the radial moment changes sign. If we connect these points of inflection around the column heads, we will get "lines" of inflection, as shown by the dotted lines, in Fig. 21, which may be considered roughly as circles. The areas of the slab within these lines of inflection are stressed in a manner similar to a circular slab supported at the centre as described in Art. 196. The bending moments along all radial lines are negative, and are a maximum at the edge of the column head.

Considering the nature of the bending moments in other than radial directions, we will be aided in our conception if we note that the lowest point of the deflected panel is at the centre *E*, and that the intermediate points, *F*, *G*, etc., are higher than *E*, but lower than the supports. Therefore, if we consider the bending moments along the line *F H*, we will find negative moments at *F* and *H*, and a positive moment at *E*. Likewise along the line *K M* there will be negative moments at *K* and *M* and a positive moment at *L*. Expressed in another way, there will exist ridges along the lines *A B*, *B C*, etc., with low points, or "saddles" at the centre points. The moments transverse to these ridges are negative at all points. The negative moments transverse to a radial line, as *A K*, correspond to the "circumferential" moments discussed in Art. 196.

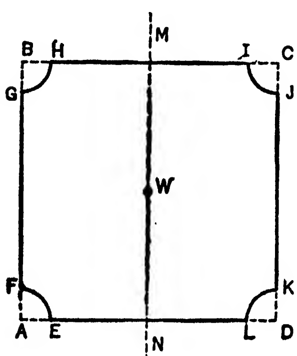


FIG. 22.

Consider now a single panel, Fig. 22, separated from its supports and the surrounding panels. The total load = *W*. The reactions around the panel consist of shears along the edges of the column heads, *E F*, *G H*, etc., and negative bending moments around the entire periphery. Assuming that this panel is surrounded by exactly equal panels, similarly stressed, there will be no shear along the lines *F G*, *H I*, etc., and the total shear at each column head will be $W/4$, which may be assumed as uniformly distributed over the quarter perimeter. The negative bending moments around the column heads act in a radial direction, and may be assumed as uniformly distributed.

The negative moments along the lines *F G*, *H I*, etc., cannot be assumed as uniform; they are evidently a maximum near the columns, and a minimum at the centre. The bending moments transverse to a centre line *M N* are all positive and somewhat greater at *M* and *N* than at the centre.

A theoretical analysis of the moments at various sections can be made under certain assumptions, as in the case of the circular slab, but such analysis is useful mainly in estimating the distribution of certain total moments that can be readily determined by the principles of the statics.

211. The Total Bending Moment.—Let Fig. 23 represent one-half of a panel. In addition to the forces already noted, there will be positive bending moments along the line *I J*, but no shear (by reason of symmetry). The half panel load will act at its centre of gravity *O*, and the shear reactions *R*, along the column head, may be considered as acting at their centroids, a distance *b* from the centre line *A B*. Assuming a uniform shear, it can be shown that $b = c/\pi$, where $c =$ diameter of column capital.

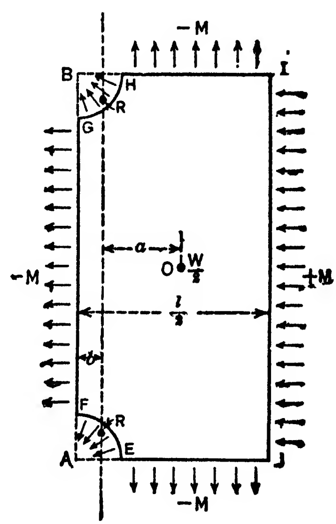


FIG. 23.

Considering the moments transverse to the axis *A B*, the external moment is $Wa/2$. This is resisted by the positive moments along *I J* and the negative moments along *H G F E*. The negative moments along *H I* and *E J* have no components transverse to *A B*. The radial moments along *H G* and *F E* may be resolved into components transverse and parallel to *A B*. Assuming the radial moments as uniform, it will be found that the transverse components will also be uniform in amount per foot of projection on the axis *A B*. We may therefore say that the sum of the positive moments along *I J* and the negative moments along *A B* (including the resolved radial moments around the column capital) is equal to the external moment $W\,a/2$. Taking account of the portion of the load supported directly by the column, and considering the usual size of column capital, the value of the total moment $W\, a/2$ is given very closely by the equation:

$$M = \tfrac{1}{8} W l \left(1 - \frac{2}{3}\frac{c}{l}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where M = numerical sum of negative and positive moments;
W = total load on panel = $w\, l^2$;
w = load per unit area, assumed as uniform;
l = distance centre to centre of columns;
c = diameter of column capital.

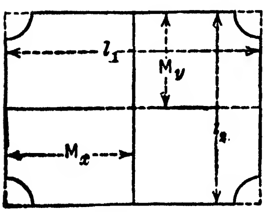


FIG. 24.

212. *Oblong Panels.*—The above analysis holds true for oblong panels as well as square panels, providing there are a number of consecutive panels of the same size, so that the shears along the centre lines may be taken at zero. Referring to Fig. 24, the equations for the sums of the negative and positive moments in the two directions are:

$$\left.\begin{aligned} M_x &= \tfrac{1}{8} W l_1 \left(1 - \frac{2}{3}\frac{c}{l_1}\right)^2 \\ M_y &= \tfrac{1}{8} W l_2 \left(1 - \frac{2}{3}\frac{c}{l_2}\right)^2 \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where l_1 and l_2 are the dimensions of the panel centre to centre and M_x and M_y the sums of the moments on sections parallel to l_2 and l_1 respectively.

213. **Distribution of the Bending Moment.**—The analysis, up to this point, has been based on statics alone, and gives correct values for total bending moments under the conditions assumed. The distribution of these moments cannot be so readily determined; it depends upon the relative rigidity of the different parts of the structure.

Considering first the relative amounts of bending moment carried along the line $E\,F\,G\,H$ (negative), Fig. 23, and the line $I\,J$ (positive) the slab is similar to a beam fixed at the ends, although on account of the short length of the fixed support it will be less rigidly fixed than such a beam. We may then expect the total negative moment to be less and the total positive moment greater than the proportions of 2 to 1 for a fixed-end beam, but the negative moment will still be somewhat greater than the positive.

Again consider the distribution of the negative and positive moments across the panel. For this purpose, it will be helpful to consider the slab in two parts (Fig. 25): a strip *a*, to include the column head, and a central strip *b*. The strip *a* is evidently more rigid than *b*, and the proportion of the moments carried by *a* will

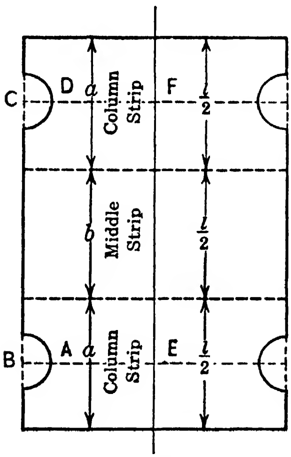


FIG. 25.

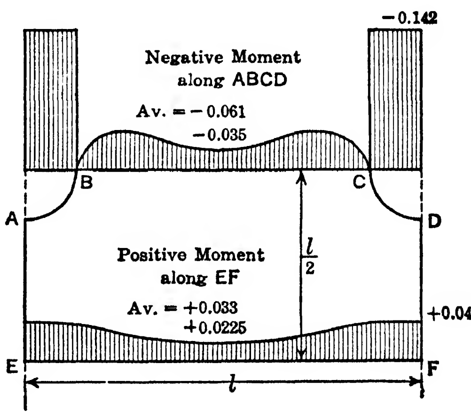


FIG. 26.

therefore be greater than that carried by *b*. Furthermore the variation in negative moment along the side *A B* will be greater than that in the positive moment along the centre line *I J*.

Results of detailed analysis by Westergaard and Slater* give valuable information regarding moment distribution. Fig. 26 gives their results for a value of $c = 0.25\,l$. The sum of the shaded areas is closely equal to the total moment as given by the equation $M = \frac{1}{8} w\, l\, (l - \frac{2}{3} c)^2$. The rectangular area above the column represents the component of the radial stresses taken perpendicular to the edge *B C*. In this analysis the slab at the edge of the column capital is

* Proc. A. C. I., 1921, Vol. 17, p. 415.

assumed to be rigidly fixed in a horizontal position, corresponding to the assumption in Fig. 10 of Art. 196. This results in very high radial stresses at the edge of the column, as indicated in the figure, with zero circumferential stresses at that point. In the actual structure these radial stresses will be much less and the negative moments much more uniform, but not as uniform as the positive.

For various ratios of c/l, Table 17, from the paper by Westergaard and Slater, gives the percentages of the total moment carried as negative or positive moment in the two halves of the slab.

TABLE 17

CALCULATED PERCENTAGES OF TOTAL MOMENT RESISTED AT EACH SECTION OF A FLAT SLAB

		c/l				
		0.15	0.20	0.25	0.30	Average
Negative moments	Column section	48.3	48.4	48.3	48.4	48
	Mid-section	17.0	16.7	16.6	16.3	17
	Total negative	65.3	65.1	64.9	64.7	65
Positive moments	Column section	20.9	20.9	20.8	20.7	21
	Mid-section	13.8	14.0	14.3	14.6	14
	Total positive	34.7	34.9	35.1	35.3	35

214. Effect of a Drop Panel (Fig. 27).—The effect of the drop panel is to stiffen considerably the column-head section of the slab and therefore to cause a somewhat greater proportion of the bending moment to be carried by this part and less by the central section. Furthermore, the negative moments will be somewhat increased and the positive moments decreased.

215. Tests of Flat-Slab Floors.—On account of the difficulty of making theoretical analyses, experimental results have been more relied upon in the design of flat slabs than in most types of construction. A considerable number of such tests have been made where the test load has been from 2½ to 4 times the design load and the stresses in the reinforcement determined by strain gage measurement. In a few cases the loading was carried practically to the ultimate. The results of a number of such tests are summarized by Westergaard and Slater * and factors of safety estimated in each case from the behavior of the slab under its maximum load. Assuming the design load to be based on a steel stress of 16,000 lbs/in.^{2} for a total moment given by the formula $M = \frac{1}{8} w l (l - \frac{2}{3} c)^2$, and the ultimate load to be determined on the basis of a yield point of 40,000 lbs/in.^{2}, the factor of safety was found to range from 2.67 to 5.57, averaging 3.8. In those tested to failure or nearly so, the factor was 4 or more. The theoretical factor is 40,000/16,000 = 2.5. With a test load of 2 to 2½ times the design live load the measured stresses in the reinforcement were much below the theoretical values.

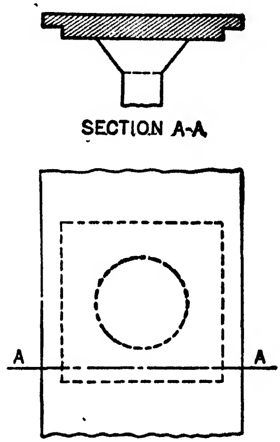


FIG. 27.

In addition to the high safety factors shown in such tests, consideration should also be given to the fact that this type of structure is exceedingly reliable and capable of heavy over-load without danger of sudden failure. The shearing-stresses are generally low, and the danger of shear failures which exist in ordinary beam construction hardly exists. From all these considerations it seems proper and admissible that higher unit stresses should be used in this type of design, or that some reduction be made in the theoretical moment coefficients. The latter is the usual method of modification.

The foregoing discussion relates primarily to the stresses in the steel; the compressive stresses in the concrete must correspond to the full theoretical bending moment.

216. Working Coefficients for Moments.—The results of such tests as noted in the preceding paragraph and experience for many years with existing structures led the Joint Committee of 1924 to adopt a value for total moment for an interior panel, for calculating stresses in the steel given by the formula

$$M_0 = 0.09\ Wl\,(1 - \tfrac{2}{3} c/l)^2 \quad . \quad . \quad . \quad . \quad . \quad (3)$$

* Proc. A. C. I., 1921, p. 509.

thus using a coefficient of 0.09 instead of ⅛, the theoretical value, a reduction of 28%. The corresponding factor of safety as determined by the tests referred to averages about 2.7 with a few tests at about 2.0. Although this is somewhat less than is customary in rectangular beams for the same working stresses, the nature of the structure fully warrants some distinction in this respect.

The proportions of total moment assumed as carried by the various sections as specified by the Joint Committee are as follows:

PERCENTAGES OF TOTAL MOMENT M_0 OF EQ. (1) AT EACH SECTION OF A FLAT SLAB PANEL (JOINT COMMITTEE)

(For all values of c/l)

Strip	Slabs without Dropped Panels		Slabs with Dropped Panels	
	Negative	Positive	Negative	Positive
Slabs with 2-way Reinforcement				
Column.........	46	22	50	20
Middle..........	16	16	15	15
Total.........	62	38	65	35
Slabs with 4-Way Reinforcement				
Column.........	50	20	54	19
Middle..........	10	20	8	19
Total.........	60	40	62	38

For exterior or wall panels the positive moments will be greater and the negative moments at the wall or marginal beam will be less than those for an interior panel, the actual values depending upon the relative rigidity of the marginal support. Empirical rules for these moments are given in general specifications. See also Art. 264 on Torsion.

The column strip width is taken as one-half the width of the panel, occupying the two quarter panel areas outside the middle strip.

217. Thickness of Slab.—The thickness of the slab is usually fixed by a definite formula in the specifications in terms of load, dimensions of the panel, size of column capital, and strength of concrete. Such a formula may be arrived at as follows:

The full bending is, from eq. (1), $M = 0.125\, w\, l^3\, (1 - \frac{2}{3}\, c/l)^2$. If the amount carried as negative moment in the column strip be taken at 50% of the total, then the negative moment in foot-pounds per foot of width is $0.125\, w\, l^2\, (1 - \frac{2}{3}\, c)^2$. For a 2000-lb. concrete we may take $f_c = 0.45 \times 2000 = 900$ lbs/in.^{2}; also let $f_s = 18{,}000$ lbs/in.^{2} If the steel is calculated by the use of a coefficient of 0.09 instead of 0.125 then the theoretical stress for full moment will be $18{,}000 \times 0.125/0.09 = 25{,}000$ lbs/in.^{2} The thickness should then be such as to make a balanced design for $f_c = 900$ and $f_s = 25{,}000$ lbs/in.^{2} For these values, from Diagram 4, $R = 140$ and $p = 0.64\%$. Then from the relation $M = R\, b\, d^2$ we have

$$d^2 = \frac{0.125\, w\, l^2\, (1 - \frac{2}{3}\, c/l)^2}{R}$$

whence

$$d = 0.03\, l\, (1 - \tfrac{2}{3}\, c/l)\, \sqrt{w}. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

To this value is added 1½ in. to get total thickness.

The formula of the Joint Committee for a 2000-lb. concrete is

$$t = 0.038\, (1 - 1.44\, c/l)\, l\sqrt{w} + 1\tfrac{1}{2}. \quad . \quad . \quad . \quad . \quad (5)$$

This gives the same value as (4) for $c/l = 0.225$; for smaller values of c/l, somewhat greater, and for larger values somewhat less, which is a desirable provision.

218. Calculation of Reinforcement.—In calculating stresses in the reinforcement, all bars crossing the sections in question may be counted, provided adequate provision is made for bond strength. Rods crossing diagonally may be considered to have a value determined by multiplying their area by the sine of the angle which they make with the side of the panel considered. Thus, a diagonal bar on line *A C* (at 45°), Fig. 25, would be counted for negative moment on line *A D* and positive moment on line *M N* at 0.707 of its full area.

219. Shear and Diagonal Tension.—The conditions regarding diagonal tensile stresses are similar to those in the case of footings discussed in Art. 200. The line of critical shear may be taken as a circle around the column capital at a distance therefrom equal to the effective depth of the slab, or slab and dropped panel, usually assumed at 1½ in. less than the total thickness. Where a dropped panel is used a similar critical section surrounding the dropped panel should be investigated. The shearing-stresses are usually limited to values not requiring diagonal tension reinforcement. The bending moments generally determine the thickness.

CHAPTER IX

BUILDING CONSTRUCTION

220. The foregoing chapters have dealt with the analysis of various elements of construction without much reference to the structure as a whole. The use of these elements in many forms of construction is simple and direct and requires no further discussion; but in other cases, when these elements are combined into whole structures, certain problems of stress determination arise which differ somewhat from those involved in similar structures of steel or timber. In the present chapter the use of reinforced concrete will be considered with especial reference to building construction.

221. The Building Frame.—Based on the type of framework used, buildings may consist of:

1. A steel framework of columns, girders, and beams with reinforced concrete used for floor slabs only. In this case the framework is designed as a steel structure, and it is only the concrete floor that concerns us here.

2. A complete framework of reinforced concrete consisting of columns and floor members designed and built to act as an integral structure. The columns in this case may consist of one of the usual types of reinforced-concrete columns, or may consist of structural steel surrounded by concrete to which the concrete floor is adequately connected. In either case the monolithic character of the construction involves the calculation of stresses in continuous girders, and of open frames composed of floor members and columns; and as such calculations are inseparably connected with reinforced-concrete building design, suitable methods of analysis will be explained in this chapter. For buildings of only two or three stories in height, bearing walls may be used in part to support the floors, but for higher buildings, a complete skeleton of columns and floor members will be more economical,

the external walls being made relatively thin and resting on floor members at each story.

222. Arrangement of Columns.—The spacing and arrangement of columns is a matter largely of convenience and architectural considerations. Where not otherwise controlled, the economical spacing is not far from 20 ft., with somewhat less for exterior panels. A considerable variation in span will affect total cost but little. Exterior columns are for convenience usually made rectangular or square in form; interior columns square for beam and girder floors, and round or octagonal for the flat-slab type. Except where governed by considerations of space, high percentages of reinforcement are not econom-

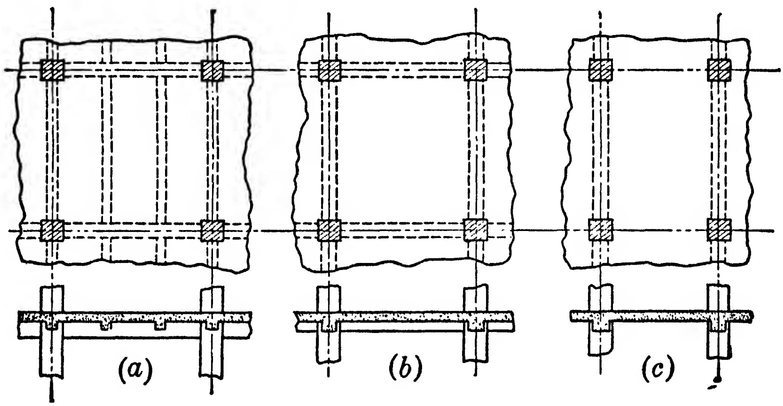


FIG. 1.

ical, as the load can be carried more cheaply by concrete than by steel. The most economical type of column will depend largely upon the requirements of the building code for the locality. As in all cases of numerous units, uniformity of size is a factor of importance in securing maximum economy and to a certain extent will justify the use of excess material for some of the units.

223. Types of Concrete Floors.—Concrete floors may be divided into four types:

1. Floors of reinforced concrete placed on a steel frame of girders and beams.

2. The beam and girder type in which the slabs are built integrally with a framework of beams and girders, the girders being built into

the columns. This type is especially advantageous where the span lengths vary considerably and where there are many openings. Fig. 1 shows various arrangements: in (*a*) girders and beams with oblong slabs reinforced transversely; (*b*) square panels with two-way reinforcement; (*c*) beams in one direction only, suitable for small span lengths.

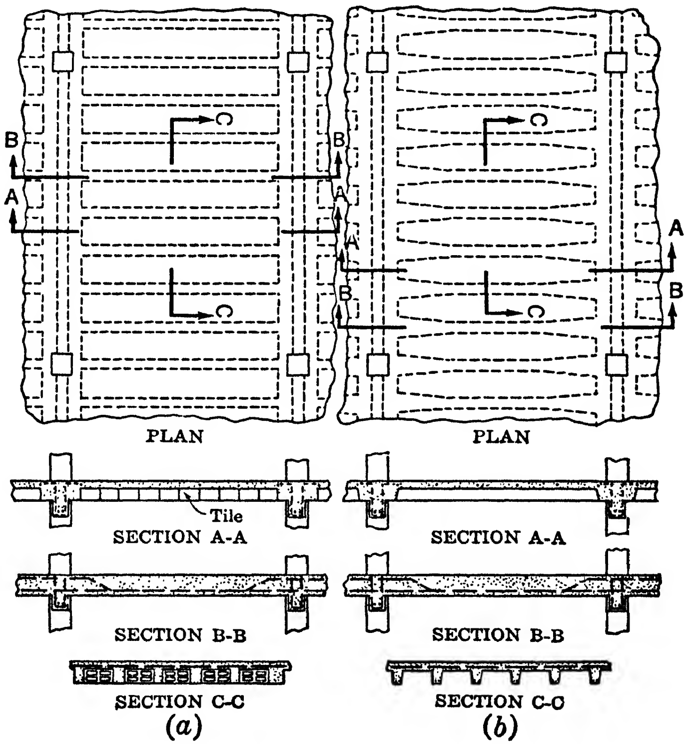


FIG. 2.

3. The girderless floor, or flat-slab type. Especially suited to large continuous areas with uniform spacing of columns.

4. The Joist Type of construction for relatively light loads. This is essentially a beam and girder type in which the beams are small and closely spaced and the slab is relatively thin. Fig. 2 illustrates two standard methods of construction. In (*a*), hollow clay tile are used between the beams. In (*b*), a metal form or tile is used which

may be made quite thin and left in place, or relatively thick and removed and used repeatedly.

224. Loads on Buildings.—The dead load is the weight of all parts of the structure and is estimated in detail from unit weights of the various materials used. These are readily found in various structural handbooks. The weight of ordinary reinforced concrete is taken at 150 lbs/ft.^{3} The live load is the extraneous load placed upon the structure. It varies greatly in amount and character, depending upon the use to which the building is put. Usually the live load is specified and treated as a uniformly distributed load over a portion or all the area of a floor, depending upon the effect produced. Occasionally, as in the case of floors supporting heavy machinery or loaded trucks or cars, the live load may be specified as concentrated loads. Where the live load includes heavy moving parts, as in the case of some machinery, elevators, etc., some account must be taken of the dynamic effect. Whatever this may be, it is usually added as a percentage to the actual static live load. The wind load is the horizontal pressure due to the maximum wind velocity anticipated and is specified at from 20 to 30 lbs/ft.^{2} for the exposed surface of the building. This is of much importance in high and narrow buildings.

The live load on columns in the lower stories of a many-storied building is reduced somewhat below the value found by assuming all floors to be fully loaded. The amount of this reduction varies in different building codes, but for general purpose buildings it is about 5% per story to a maximum of 50%. For warehouses a smaller reduction is made

Some of the commonly specified live loads are as follows:

	Lbs/ft.^{2}
Assembly halls	100
Office buildings	75
Residences	50
Class rooms	50
Corridors and stairs	100
Garages	100
Storage purposes	100 to 250
Roofs	30 to 40

For exact values the building code for the city or state in question must be consulted.

225. Problems of Analysis.—In the design of the beams, girders, and columns of a building there must be determined with a satisfactory degree of accuracy the maximum bending moments, both positive and negative, in the beams and girders, the shearing-stresses therein, and the bending moments and compressive stresses in the columns. This determination includes the analysis of continuous girders and of open quadrangular frames. Exact analysis of such structures can in most cases be made only on the basis of an assumed design, as the relative rigidity of the various parts must be known. Then from the results of such analysis the design must be corrected.

In practice, a design closely meeting requirements can usually be made on the basis of assumed or standard coefficients determined from experience or by a generalized treatment so that the necessary corrections are not difficult to make. This analytical discussion will be divided into two parts: (1) the analysis of continuous beams with especial reference to the determination of working coefficients for ordinary cases; (2) general methods of analysis of quadrangular frames and beams of differing moments of inertia and span lengths.

ANALYSIS OF CONTINUOUS BEAMS

226. General Conditions.—Where continuous beams rest upon walls or are framed into other beams, as in the beam and girder arrangement, the restraint offered at the supports is usually so small that in calculating bending moments such beams are assumed as acting as freely supported continuous girders and calculated as such. In the design, provision is then made for small negative moments at the end supports.*

The exact determination of the maximum bending moments at all sections of a beam continuous over several spans is a tedious and time-consuming problem. For several reasons such a complete solution is generally unnecessary and of little value. In the ordinary case the beams in question (especially floor slabs), are continuous over several spans, and the loading required to produce the theoretical maximum stresses involves unreasonable assumptions as to position of live load. It would be necessary, in general, not only to load alternate panels completely, leaving intermediate panels entirely unloaded, but it

* See Art. 264 for discussion of torsional resistance of girders.

would be necessary also to have such load conditions extend over a very considerable width transversely of the beams in question, as the monolithic character of a concrete floor produces a wide lateral distribution of concentrated loads. Where the span lengths are equal a sufficiently exact analysis may be arrived at by considering certain simple cases of continuous beams. Where span lengths are unequal, and in other special cases, a more exact analysis should be made.

227. The Theorem of Three Moments.—The calculation of moments, shears, and reactions for continuous beams is based on the theorem of three moments. which expresses the relation between the

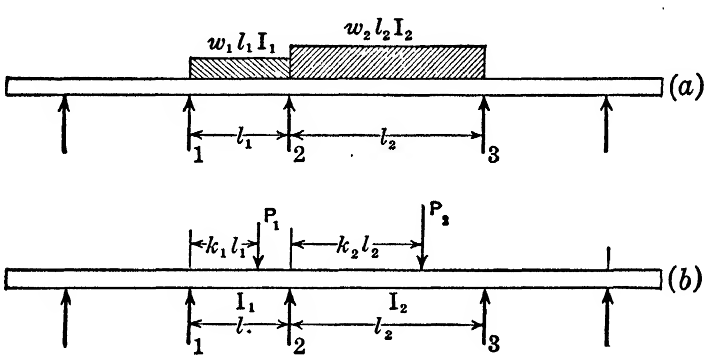


FIG. 3.

bending moments at any three consecutive supports and the loads on the two included spans. Referring to Fig. 3, let supports 1, 2, and 3 be any three consecutive supports of a continuous girder of any number of spans; l_1 and l_2 the included span lengths, and I_1 and I_2 the respective moments of inertia, and M_1, M_2, and M_3 the bending moments at the respective supports. In (*a*) the load consists of uniform loads of w_1 and w_2 per unit length on the two spans, and in (*b*) the load consists of any number of concentrated loads represented by P_1 and P_2, respectively, and distant $k_1 l_1$ and $k_2 l_2$, respectively, from the supports on the left

Let $I/l = K$, in general. Then assuming uniform modulus of elasticity:

For uniform loads *

$$\frac{M_1}{K_1} + 2\,M_2\left(\frac{1}{K_1} + \frac{1}{K_2}\right) + \frac{M_3}{K_2} = -\frac{1}{4}\,\frac{w_1\,l_1^2}{K_1} - \frac{1}{4}\,\frac{w_2\,l_2^2}{K_2}. \quad . \quad (1a)$$

For concentrated loads *

$$\frac{M_1}{K_1} + 2\,M_2\left(\frac{1}{K_1} + \frac{1}{K_2}\right) + \frac{M_3}{K_2} = -\frac{\Sigma\,[P_1\,l_1\,(k_1 - k_1^3)]}{K_1} - \frac{\Sigma\,[P_2\,l_2\,(2\,k_2 - 3\,k_2^2 + k_2^3)]}{K_2}. \quad (1b)$$

By the use of these equations the bending moments at all supports can be determined, and thence the shears and reactions. For example, a two-span girder supported at the ends is solved at once by applying the equation to the moments at the three supports, the values of M_1 and M_3 being zero. In a three-span girder the theorem is applied to the first and second spans and to the second and third spans, placing M_1 and M_4 = zero. Two equations are thus formed with two unknown moments M_2 and M_3, and these moments determined. In a similar manner three equations are written for a four-span girder, etc. If the ends are fixed so that the end moments are not zero, two additional equations are obtained by assuming two additional end spans, each of zero length, and then considering the structure as supported at the ends, making the end moments zero. Such an assumption is equivalent to fixing the direction of the end tangent to the curved beam axis.

Having determined the moments at supports, the moments and shears may be determined at any desired point.

228. The Moment of Inertia of a Beam.—The moment of inertia of a reinforced-concrete beam is somewhat uncertain. Where this property is used in problems relating to strength, as in the application of the transformed section, the moment of inertia should be calculated for the compression area of the concrete and the total area of the steel. The area of concrete in tension should be omitted. It is a question of strength.

In the present series of problems we are attempting to determine the relative proportion of load carried by the several supports; and

* See text books on mechanics for derivation of these equations.

in the case of the open frames discussed later on, the relative moments carried by the various members meeting at a joint. In this class of problems, it is a question of the relative rigidities of the various beams concerned, a measure of which is the ratio of the moment of inertia divided by the length of the member. Now the deflection of a reinforced-concrete beam is due to the strains in every part of the beam, and, for a considerable portion of the entire volume below the neutral axis, tensile stresses in the concrete exist and exert a large influence on the deflection. No very exact method can be devised, but in Chap. VI it is shown that satisfactory results for deflection are obtained by taking account of the concrete to the centre of the steel and using the amount of steel at mid-section. Ordinarily about the same result will be obtained by using the gross section of concrete and omitting the steel; this method of calculation is recommended and will be used in the examples following.

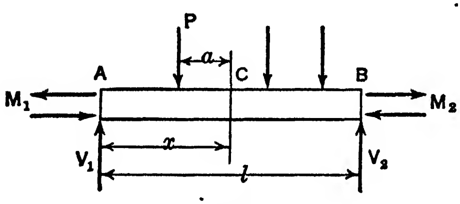


FIG. 4.

For columns, the calculation of moment of inertia should include the steel, as there is usually compression over the entire section, and both steel and concrete are fully effective. Three or four per cent of steel will add largely to the moment of inertia of a column.

229. Unit for Values of I/l or K.—Since the quantity K appears to the first power in all terms it is immaterial what unit is used in its calculation provided it is the same for all. Any convenient unit may therefore be chosen. It is only the *relative* values of K that are significant.

230. Shears and Moments in Any Span.—In Fig. 4 the end moments M_1 and M_2 are known. They are represented as negative moments, the usual case. The beam is also loaded in any manner. It is required to determine the shears V_1 and V_2 and the moment M_x at any point C. Let V'_1, V'_2, and M'_x represent the shears and moment due to the vertical loads alone, considering the beam as simply supported. The effect of M_1 and M_2 can then be calculated separately and added. We thus find readily, taking moments about B,

$$V_1 = V'_1 + \frac{M_1 - M_2}{l}; \quad \ldots\ldots\ldots \quad (2)$$

$$V_2 = V'_2 + \frac{M_2 - M_1}{l}; \quad \ldots\ldots\ldots \quad (3)$$

$$M_x = V_1 x - P a - M_1$$

$$= V'_1 x - P a + (M_1 - M_2)\frac{x}{l} - M_1$$

$$= M'_x - \left[M_1 + (M_2 - M_1)\frac{x}{l} \right]. \quad \ldots \quad (4)$$

For the centre point

$$M_c = M'_c - \frac{M_1 + M_2}{2}. \quad \ldots\ldots\ldots \quad (5)$$

That is, *the centre moment is equal to the centre moment in a simply supported beam minus the average of the two end moments.**

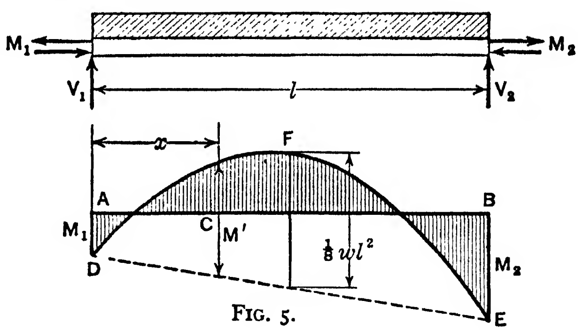


FIG. 5.

Fig. 5 illustrates the case for a uniform load. M_1 and M_2 are the end moments and the curve DEF is a parabola plotted from the axis $D\,E$, with centre ordinate = ⅛ $w\,l^2$. The ordinates from the axis $D\,E$ to the curve represent values of M', the moment in a simple beam. The resultant moment, represented by the ordinates from $A\,B$ to the curve, is equal to M' minus the ordinate from $A\,B$ to $D\,E$, which is equal to $M_1 + (M_2 - M_1)\,x/l$, as given in eq. (4). At the centre this is $\frac{M_1 + M_2}{2}$ as in eq. (5). For concentrated loads the moment diagram for M' will be plotted from the axis $D\,E$ in the same manner, the total or resultant moment being represented by the ordinates from $A\,B$.

* The numerical values of the moments are here of most significance, as the negative end moments and the positive centre moment act together to resist the external forces.

Fig. 6 shows the shear diagram corresponding to the moment diagram. The value of the shear at A is $V'_1 + \frac{M_1 - M_2}{l}$ or $\frac{1}{2} w l - \frac{M_2 - M_1}{l}$, and the shear throughout the beam is equal to the shear in a simple beam reduced by the quantity $\frac{M_2 - M_1}{l}$ as shown. For concentrated loads the shear diagram can be constructed in the same manner.

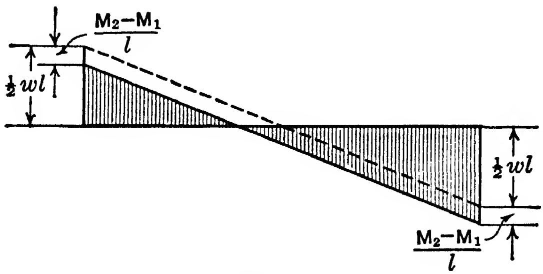


FIG. 6.

Example.—Calculate the moments and shears in the beam shown in Fig. 7 (a). It is fixed at the left and simply supported at the right.

Solution.—In Fig. 7 (b) there is shown an imaginary span 1–2 of zero length, and the spans are numbered to include this. Then applying eq. (1a) to spans 1 and 2, we have, M_1 being zero, and $K_1 = \infty$,

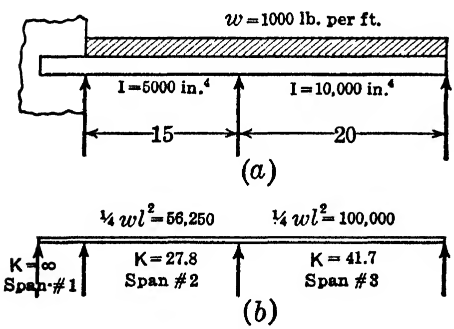


FIG. 7.

$$\frac{2\,M_2}{27.8} + \frac{M_3}{27.8} = -\frac{56{,}250}{27.8}. \qquad (a)$$

Applying eq. (1a) to spaces 2 and 3, M_4 being zero,

$$\frac{M_2}{27.8} + 2\,M_3\left(\frac{1}{27.8} + \frac{1}{41.7}\right) = -\frac{56{,}250}{27.8} - \frac{100{,}000}{41.7}. \quad . \quad . \quad (b)$$

Simplifying

$$2\,M_2 + M_3 = -\,56{,}250$$

$$M_2 + 3.33\,M_3 = -\,122{,}950$$

Solving, we get finally

$$M_2 = -\,11{,}500 \text{ ft-lbs.}$$

$$M_3 = -\,33{,}300 \text{ ft-lbs.}$$

The shear at 2 is equal to $\frac{1}{2} w l_2 - \frac{33,300 - 11,500}{15} = 7500 - 1450 =$ 6050 lbs. On the left of 3 it is $7500 + 1450 = 8950$ lbs. On the right of 3 it is $= \frac{1}{2} w l_3 - \frac{0 - 33,300}{20} = 10,000 + 1660 = 11,660$ lbs. At 4 it is $10,000 - 1660 = 8340$ lbs.

Fig. 8 shows the moment and shear diagrams. The centre moment for span 2–3 is $\frac{1}{8} w l_2^2 - \frac{33,300 + 15,500}{2} = 28,120 - 22,400 = 5720$ ft-lbs., and for 3–4 is $\frac{1}{8} w l_3^2 - 33,300/2 = 50,000 - 16,600 = 33,400$ ft-lbs.

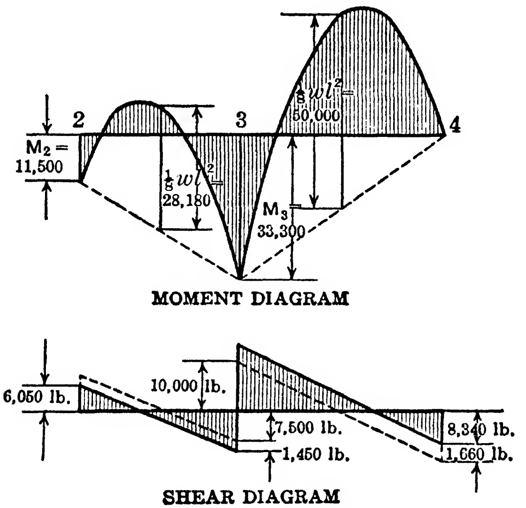


FIG. 8.

Parabolas drawn as shown, with centre ordinates of $\frac{1}{8} w l^2$ for each span, will give the complete moment diagram.

Beams of Equal Span Lengths and Equal Moments of Inertia

231. Where the lengths and moments of inertia of the various spans of a continuous girder are equal or nearly so, specifications usually permit the moments to be calculated by the use of certain general coefficients, which, although not very exact, are sufficiently accurate for most purposes. It will therefore be desirable to consider in some detail the analysis of the girder of equal spans and uniform moment of inertia with especial reference to the determination of suitable coefficients for general use. The following cases will be discussed separately:

1. Beams of two and three equal spans.
2. Beams of numerous spans.

As a general rule, it is necessary to calculate the maximum moments at centres of spans and at supports only.

232. Beams of Two and Three Equal Spans.—Inasmuch as the

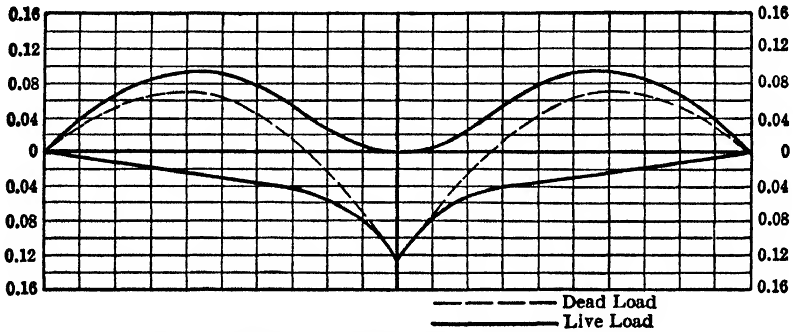


FIG. 9.—Moment Coefficients for a Two-span Beam.

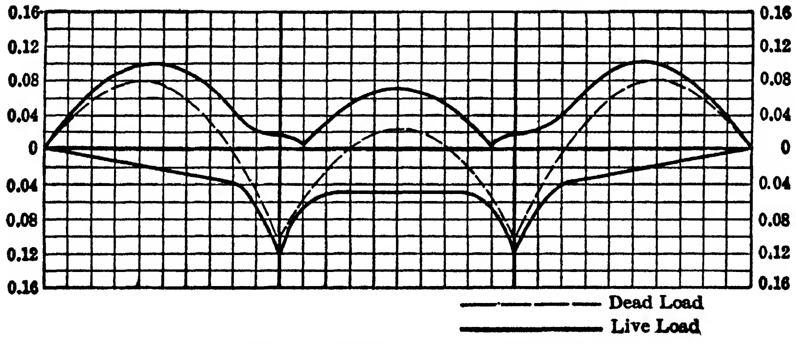


FIG. 10.—Moment Coefficients for a Three-span Beam.

conditions for a theoretical maximum are more likely to occur in beams of two or three spans than where the number of spans is large, an exact analysis will be made of maximum moments at all points for the beam of two spans and for the beam of three spans. The spans will be assumed equal and the beam considered as continuous but freely supported at all points. Assume the dead load to be a uniformly distributed load, = w per lineal foot, and the live load to be also a uniform load, $= p$ per lineal foot, but distributed over such portions of the beam as to cause a maximum moment at the given section. The live-load moments have been calculated for such position of the load as to cause the maximum moment at all points. The results are shown in Figs. 9 and 10. Dead-load moments are given by the dotted lines, live load by full lines. The ordinates as plotted are the coefficients of the quantities $w\,l^2$ and $p\,l^2$.

The coefficients of $w\,l^2$ and $p\,l^2$ for the maximum positive and negative moments for the two beams are as follows:

	Maximum Near Centre of Span (+)	Maximum at Support (−)
Beams of two spans (Fig. 9):		
Dead load	0.070	0.125
Live load	.095	.125
Beams of three spans (Fig. 10):		
Dead load — 1st span	.080	.100
Dead load — 2d span	.025	
Live load — 1st span	.100	.117
Live load — 2d span	.075	

If the dead and live loads are combined into a single unit for the purposes of calculation, the proper coefficient for $(w + p)$ will depend on the relation of dead and live load. If, for example, the dead load is one-third the live load, then there results:

Beam of two spans:

Maximum positive moment $= 0.089\,(w + p)\,l^2$,
Maximum negative moment $= 0.125\,(w + p)\,l^2$.

Beam of three spans:

Maximum positive moment $\begin{cases} \text{end span} = 0.095\,(w + p)\,l^2. \\ \text{centre span} = 0.062\,(w + p)\,l^2. \end{cases}$
Maximum negative moment $= 0.113\,(w + p)\,l^2$.

In Figs. 11 and 12 the curves show the maximum and minimum moments throughout the beam for the case where $p = 3\,w$, expressed as coefficient of the sum of dead and live load $(w + p)$. These curves are particularly useful in showing the relative distances from the supports over which positive and negative moments may occur. (See Art. 236.)

In the two-span beam, with $p = 3w$, Figs. 11 and 12 show negative moments occurring for about 55% of the length of the span; and for the three-span beam, they occur throughout the middle span. This is of considerable significance as three-span girders are very common in certain types of buildings, and in many cases the central span is

FIG. 11.

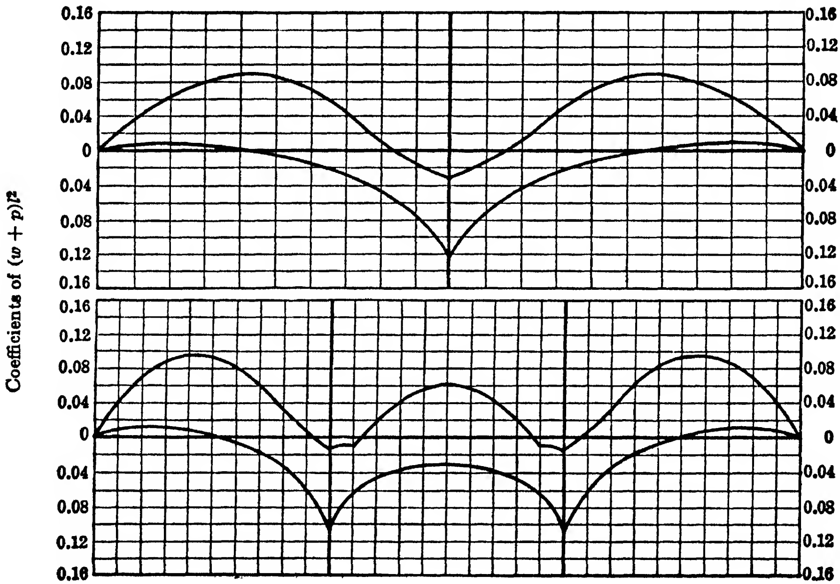


FIG. 12.

relatively short, resulting in heavy negative moments throughout the span.

233. Beams of Numerous Spans.—*Position of Loads for Maximum Moments and Shears.*—To assist in getting a clear understanding of the effect of loads in various spans upon moments and shears at particular points, several influence lines are shown in Fig. 13 for a six-span girder.* Figs. (*b*) and (*c*) are the influence lines for

* Influence lines may be drawn by calculating the value of the moment or shear in question due to a load of unity placed at various points along the beam.

moment at (*a*) and (*b*), the centres of the first and third spans. Figs. (*d*) and (*e*) for moment at supports 2 and 4 and Figs. (*f*) and (*g*) for shears at (*a*) and (*b*). A maximum positive moment at or near the centre of a span requires each alternate span to be loaded,

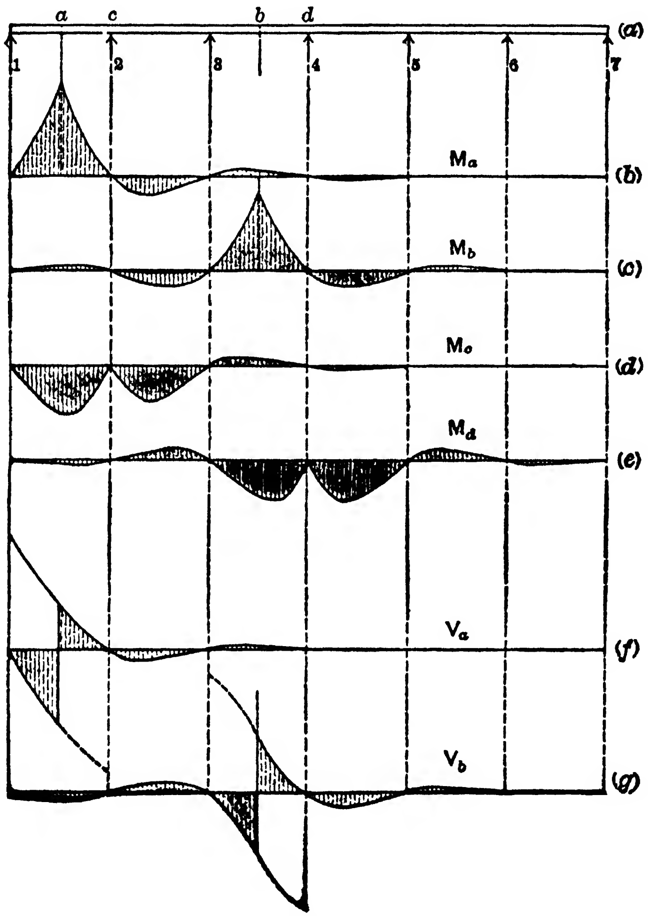


FIG. 13.

and a maximum negative moment at the support requires the two adjacent spans to be loaded and then each alternate span. The small effect of loads on remote spans is to be noted. For maximum shears the general rule of loads in alternate spans is seen to hold true as for moments, but the effect of remote loads upon the maximum shears is relatively less than in the case of moments.

If the load is a uniform one on any span, the shaded area below the influence line represents to scale the actual moment or shear due to this load. In special cases of heavy concentrated loads influence lines may be drawn, the actual moments calculated therefrom, and the maximum value determined by trial.

234. Moment Coefficients for Uniform Loads.—In ordinary construction of equal or nearly equal spans, exact bending moments are seldom calculated but are determined by applying specified coefficients which have been standardized by various engineering and code committees. A study of the moments in several cases will show the basis for these coefficients.

In the case of several spans it will be practically correct in calculating maximum positive moments to consider that the maximum moment at the centre of the span is the maximum desired. (Strictly the maximum is generally not quite at the centre.) The loading required for maximum live-load moments is illustrated in Fig. 14, which shows in (*a*) the loading for maximum positive moment in spans 2–3, 4–5, and 6–7, and in (*b*) the loading for maximum negative moment at support No. 3. For the former case each alternate span is loaded, and for the latter, the two adjoining spans are loaded and then each alternate span. Calculations of maximum positive and negative moments have been made for each span and each support for girders of four, five, six, and seven spans, and the significant results are given in Table 17. Included also are the results for two and three spans previously determined. It is found in general that, for all spans and supports, except the end span and support adjacent thereto, the maximum positive and negative moments do not vary greatly for the different spans, but for these end spans and supports they are considerably larger than for intermediate spans. The results are, accordingly, arranged in two groups in the table. For the intermediate spans the greatest value for the several spans is given.

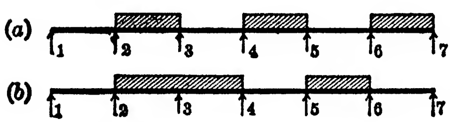


FIG. 14.

TABLE 17

COEFFICIENTS OF wl^2 OR pl^2 FOR MAXIMUM MOMENTS IN CONTINUOUS BEAMS

Number of Spans	Intermediate Spans and Supports				End Span and 2d Support			
	At Centre (+)		At Support (−)		At Centre (+)		At Support (−)	
	Dead	Live	Dead	Live	Dead	Live	Dead	Live
Two......	....	....	....	....	0.070	0.095	0.125	0.125
Three.....	0.025	0.075	....	....	.080	.100	.100	.117
Four......	.036	.081	.071	0.107	.071	.098	.107	.120 (.115)
Five......	.046	.086	.079	.111 (.106)	.072	.099	.105	.120 (.116)
Six.......	.043	.084	.086	.116 (.106)	.072	.099	.106	.120 (.116)
Seven.....	.044	.084	.085	.114 (.106)	.072	.099	.106	.120 (.116)

The quantities in parentheses are the coefficients for live-load moments over supports where the two adjoining spans only are loaded. The effect of loading each alternate span in addition to these two spans is seen to be small; and considering that such a loading would be extremely improbable, and also the fact that a comparatively small amount of load on the other spans would neutralize this effect, it is apparent that the quantities in parentheses may be taken as reasonable maximum values. The two-span beam should preferably be treated as a special case.

Finally, leaving out of account the two-span beam, the following values may be taken as reasonable maximum values of the coefficients for beams of any number of spans:

	Intermediate Spans		End Spans	
	At Centre	At Support	At Centre	At Support
Dead-load moments..................	0.045	0.085	0.075	0.105
Live-load moments..................	.085	.105	.100	.115

Further simplification is made by combining the dead and live loads and applying a single coefficient to the sum. Taking the above values as a basis, the combined coefficients for various ratios of live to dead load are shown below. Thus for a ratio of 3 to 1, the combined coefficient would be, for centre moment in intermediate spans,

$$\frac{3 \times 0.085 + 1 \times 0.045}{4} = 0.075.$$

Ratio of Live to Dead Load	Intermediate Spans		End Spans	
	At Centre	At Support	At Centre	At Support
2 : 1	0.072	0.098	0.092	0.112
3 : 1	.075	.100	.094	.112
4 : 1	.077	.101	.095	.113
5 : 1	.078	.102	.096	.113

It will be seen from this table that for ordinary proportions a single coefficient may well be used for both dead and live loads.

In adopting final values consideration should be given to certain modifying influences. The beams and slabs are not freely supported as assumed, but are, to a considerable extent, fixed at the supports. This tends to reduce the maximum moments, especially in the end span. Furthermore, the negative moments decrease very rapidly from the support towards the centre, so that any slight excess of stress would extend but a few inches at most. In the case of slabs it is also convenient to use the same amount of steel for positive as for negative reinforcement. The supports, also, are of considerable width, so that if the span lengths be taken centre to centre, the negative moment at the edge of the support is considerably less than the calculated maximum.

The coefficients adopted by the Joint Committee of 1924 and by the American Concrete Institute, 1928, applicable to both dead- and live-loads, are as follows:

	Joint Committee, 1924	A. C. I., 1928
Two-span beams:		
Positive moment	1/10	1/10
Negative moment at centre support	1/8	1/8
Beams of more than two spans:		
Intermediate spans:		
Positive moment	1/12	1/12
Negative moment	1/12	1/12
End spans:		
Positive moment	1/10	1/10
Negative moment	1/10	1/10
Negative moment at end support	1/24	1/16

A negative moment is assumed at the end, owing to some restraint from walls or longitudinal beams or girders.

235. Effect of Applying Loads at Panel Points.—Where beams frame into girders at one or more panel points, the load on the girder consists of equal concentrated loads, an arrangement which modifies somewhat the moments in the girder. The effect is not great, but in general the moment is made more nearly equal at centre and support and in the three-panel arrangement is reduced about 10%.

236. Position of Point of Inflection.—In determining the length of rods required for reinforcement, and in calculating bond stress, it is necessary to ascertain approximately the variation of bending moment along the beam under the assumed load conditions. For positive moment the span in question is fully loaded and the adjacent spans not loaded.

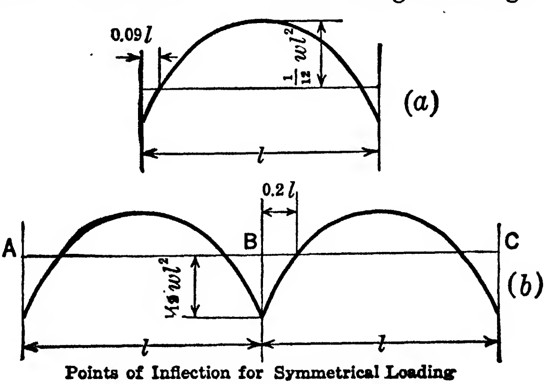


FIG. 15.

If the centre moment is taken at $1/12\ w\,l^2$ the point of zero moment will be 0.09 l from the end, as shown in Fig. 15 (a). For an end span the points of zero moment will be at the free end and about 0.1 l from the first interior support. Between points of inflection the beam acts precisely as a simply supported beam.

For maximum negative moments, the point of zero moments will be about 0.2 l from the support, as in Fig. 15 (*b*).

Negative moments may also exist throughout the entire length of a span (when adjacent spans are loaded), as shown in Fig. 12. Where the spans are of equal length these negative moments through the central part of the beam are small and may generally be neglected, but where the spans are of unequal length they may be important and must be provided for. (See Art. 241 for illustration.)

237. Shears in Continuous Beams.—As shown by eq. (2), Art. 230, and as illustrated by the numerical example given, the end shear in a continuous beam is equal to the end shear of a simply supported beam plus the difference between the end moments divided by span length. Such shear is therefore a maximum when the span is fully loaded and other loads so placed as to cause a maximum moment difference at the two ends. As shown by the influence lines of Art. 233 this is the same loading as for maximum negative moment.

The span having the maximum shear will be the end span, since the moment at one end of this span is zero. And the shear will be $\frac{1}{2} w l^2 + M_2/l$, where M_2 is the moment at the second support. From Table 17 this shear can be readily computed for beams of various numbers of spans. For the two-span-girder the moment at the second support is $0.125 \, w l^2$, hence shear at this support $= (0.5 + 0.125) \, w l = 0.625 \, w l$. For a three-span beam, the dead-load shear $= (0.5 + 0.1) \, w l = 0.6 \, w l$, and the live-load shear $= 0.617 \, w l$, etc.

For intermediate spans the dead-load shears are practically $0.5 \, w l$, actually $0.5036 \, w l$ as a maximum in the four-span beam. The live-load shears are a little greater, being practically $0.60 \, w l$ for the second span and less for the others. It is common practice to assume the shears to be the same as for simple beams, but for end spans this value should be increased by 20% or to $0.6 \, w l$.

Beams of Unequal Span Lengths and Moments of Inertia

238. Where span lengths or moments of inertia differ considerably, it is not permissible to use the general coefficients for moments, but the maximum values must be determined by more exact methods. The most expeditious way of accomplishing this is to calculate first the moments at all supports for each span loaded in turn. Then from these results the maximum combinations can be found.

239. General Formulas for Moments for Load in a Single Span.— General formulas will first be developed for a girder of several spans for a uniform load in a single span. These formulas will then be simplified for the special but common cases of the two-span and the three-span girder.

Fig. 16 represents a girder of several spans. In general, let

$K = I/l$ for any span.
$A = -\frac{1}{4} w l^2$ for the loaded span. (See eq. (1*a*), Art. 227.)

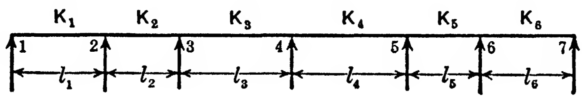


Fig. 16.

We will first determine the relation between moments at successive supports along the unloaded portions of the beam. Suppose span 5 or 6 to be loaded and consider the moments on the left. Assuming the moment M_1 to be zero (or any definite value), it will be found possible to express the moment at any support up to the loaded span in terms of the moment at the next support towards the load, thus $M_2 = -M_3 a_2$, $M_3 = -M_4 a_3$, etc., where a_2, a_3, etc., are coefficients to be determined and depend only upon the quantities K.

Beginning with spans 1 and 2 and applying the theorem of three moments we have (assuming $M_1 = 0$)

$$2 M_2 \left(\frac{1}{K_1} + \frac{1}{K_2}\right) + \frac{M_3}{K_2} = 0$$

whence

$$M_2 = -M_3 \frac{K_1}{2(K_1 + K_2)} = -M_3 a_2 \quad \text{or} \quad a_2 = \frac{K_1}{2(K_1 + K_2)}.$$

Then for spans 2 and 3

$$\frac{M_2}{K_2} + 2 M_3\left(\frac{1}{K_2} + \frac{1}{K_3}\right) + \frac{M_4}{K_3} = 0.$$

Substituting the value $-M_3\,a_2$ for M_2, and solving for M_3 gives

$$M_3 = -M_4 \frac{K_2}{2\,(K_2+K_3) - a_2\,K_3} = -M_4\,a_3$$

or

$$a_3 = \frac{K_2}{2\,(K_2 + K_3) - a_2\,K_3}.$$

Proceeding further it will be found that the general expression for the coefficient a_n for any support on the left of the load is

$$a_n = \frac{K_{n-1}}{2\,(K_{n-1} + K_n) - a_{n-1}\,K_n} = \frac{K_{n-1}/K_n}{2\,(1 + K_{n-1}/K_n) - a_{n-1}}. \qquad (1)$$

in which n = number of support or span from the left end.

It is sufficiently exact in most cases to assume $a_{n-1} = 1/4$ except for a_1 which will be zero when $M_1 = 0$.

For

$$a_{n-1} = 1/4,\ a_n = \frac{1}{2 + 1.75\,K_n/K_{n-1}}. \quad . \quad . \quad . \quad . \qquad (1a)$$

Beginning at the end, it is therefore a simple process to calculate the numerical values of a for successive supports, up to the last support but two on the right and these values will hold good for all supports on the left of any loaded span.

If the beam is assumed as fixed at 1 then $M_1 = 1/2\,M_2$, or $a_1 = 0.5$, and a_2 is given by the general formula, eq. (1). If partially fixed, M_1 may be taken at $1/4\,M_2$ and $a_1 = 0.25$.

Likewise for moments on the *right* of the loaded span similar coefficients may be calculated, beginning at the right end. These coefficients are indicated as b, and the general formula is

$$b_n = \frac{K_n}{2\,(K_n + K_{n-1}) - b_{n+1}\,K_{n-1}} = \frac{K_n/K_{n-1}}{2\,(1 + K_n/K_{n-1}) - b_{n+1}}. \quad . \qquad (2)$$

The approximate expression is

$$b_n = \frac{1}{2 + 1.75\,K_{n-1}/K_n}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2a)$$

These coefficients a and b having been calculated, it is possible to derive workable formulas for the moments at the left and right supports of any loaded span, all other spans being unloaded. Having these values, then the moments at other supports for this loading are obtained by using the coefficients already calculated. It will be simplest to take a particular span in deriving these formulas.

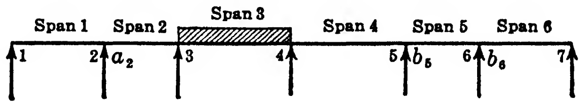


FIG. 17.

Assume span 3–4, Fig. 17, to be loaded with a uniform load and that the coefficients a_2 and b_5 are already calculated. Then applying the 3-moment equation to spans 2 and 3 and to 3 and 4, and bearing in mind that $M_2 = - a_2 M_3$, $M_5 = - b_5 M_4$, and $-\frac{1}{4} w l^2 = A$;

$$- \frac{a_2 M_3}{K_2} + 2 M_3 \left(\frac{1}{K_2} + \frac{1}{K_3}\right) + \frac{M_4}{K_3} = \frac{A}{K_3}$$

$$\frac{M_3}{K_3} + 2 M_4 \left(\frac{1}{K_3} + \frac{1}{K_4}\right) - \frac{b_5 M_4}{K_4} = \frac{A}{K_3}.$$

Solving these equations for M_3 and M_4 we have

$$M_3 = \frac{K_2 [2 (K_4 + K_3) - b_5 K_3] - K_2 K_4}{[2 (K_4 + K_3) - b_5 K_3] [2 (K_2 + K_3) - a_2 K_3] - K_2 K_4} \times A \quad (3)$$

$$M_4 = \frac{K_4 [2 (K_2 + K_3) - a_2 K_3] - K_2 K_4}{[2 (K_4 + K_3) - b_5 K_3] [2 (K_2 + K_3) - a_2 K_3] - K_2 K_4} \times A \quad (4)$$

Sufficiently accurate results will be obtained by assuming all values of a and $b = 0.25$. They will vary somewhat from this, but as the terms containing these quantities are combined with terms about sixteen times as large, any error arising from this assumption will be very small. However, as the coefficients, or their approximate equivalent, are required in any event, it will be found that the exact process is about as expeditious as an approximate one.

240. The Two-Span Beam.—Application of the general formulas (3) and (4) of Art. 239 gives the following abbreviated expressions:

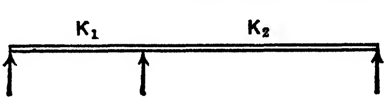


FIG. 18.

For the first span loaded

$$M_2 = \frac{K_2}{2 (K_1 + K_2)} A \quad . \; . \; . \; . \; . \; . \; . \quad (5)$$

and for the second span loaded

$$M_2 = \frac{K_1}{2\,(K_1 + K_2)} \cdot A \quad . \; . \; . \; . \; . \; . \; . \quad (6)$$

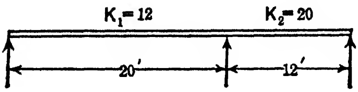


Fig. 19.

Example.—Spans 20 and 12 ft. Uniform moment of inertia. The values of K will be in the proportion 12 to 20 as indicated. Live load = 2000 lbs/ft. Dead load = 1000 lbs/ft. Calculate maximum positive and negative moments throughout.

Live Load.

$$A_1 = -\tfrac{1}{4}\, w\, l^2 = -\tfrac{1}{4} \times 2000 \times 20^2 = -\ 200{,}000 \text{ ft-lbs.}$$
$$A_2 = - \quad - \tfrac{1}{4} \times 2000 \times 12^2 = 72{,}000 \text{ ft-lbs.}$$

First span loaded

$$M_2 = -\ 200{,}000 \times 20/64 = -\ 62{,}500 \text{ ft-lbs.}$$

Second span loaded

$$M_2 = -\ 72{,}000 \times 12/64 = -\ 13{,}500 \text{ ft-lbs.}$$

Both spans loaded

$$M_2 = -\ 62{,}500 - 13{,}500 = -\ 76{,}000 \text{ ft-lbs.}$$

Dead Load.

$$M_2 = -\ \tfrac{1}{2} \times 76{,}000 = -\ 38{,}000 \text{ ft-lbs.}$$

Figs. 20 (*a*), (*b*), and (*c*) show live-load moment diagrams for the three conditions of loading, and (*d*) the dead-load diagram. The moments are given in thousands of foot-pounds. The curved portions are parabolas with centre ordinates = $\frac{1}{8} w l^2$. Adding the dead-load diagram to (*a*) and (*b*) gives maximum positive moments in spans 1 and 2 and minimum positive or maximum negative moments in spans 2 and 1, except near the centre support. These curves are shown in Fig. (*e*). The curves *a b c* and *d e f* give maximum positive, and *a b′ d* and *c e′ f* the minimum. For both spans loaded the moment at 2 = − 76 − 38 = − 114, shown at *g*, and, for a short distance from 2, partial loading of both spans will give negative moments somewhat greater than shown by the curves *a b′ d* and *c e′ f*. These values can be found sufficiently well by sketching curves from *g* tangent to the others at about the 2/10 point, shown by dotted lines. The least moment at 2 is the dead-load moment of −38.0, and the diagram can be completed for maximum and minimum by the dotted lines from *h* to the upper curves, although this is of no particular value. The shaded area shows the range of values at all points.

The maximum positive moments at the centres of the spans are

First span	68,750 + 31,000 = 99,750 ft-lbs.
Second span	29,250 − 1000 = 28,250 ft-lbs.

Minimum centre moments are

First span	$-6750 + 31{,}000 = +\ 24{,}250$ ft-lbs.
Second span	$-31{,}250 - 1000 = -\ 32{,}250$ ft-lbs.

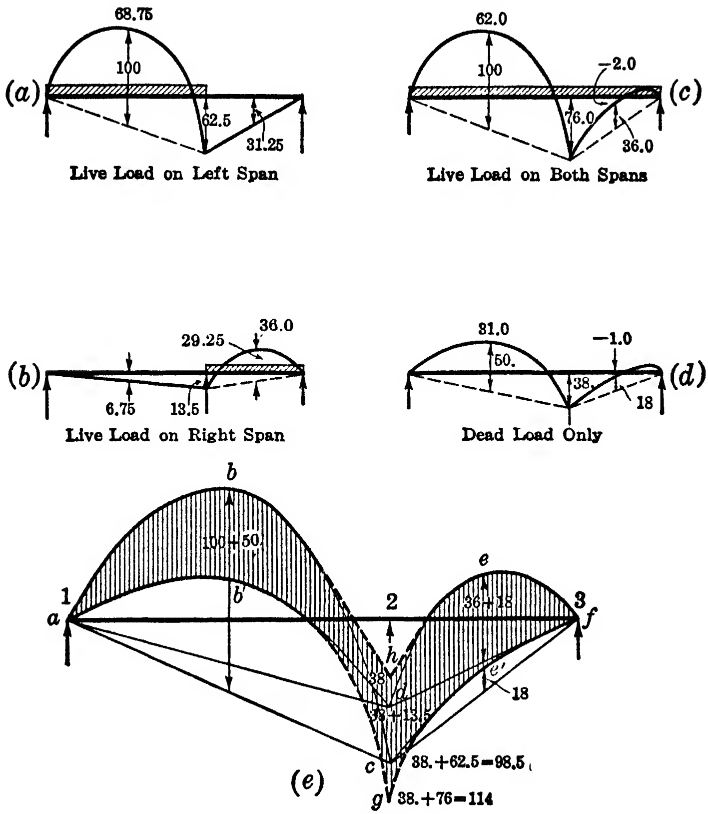


FIG. 20.

Maximum moment at centre support $= -76{,}000 - 38{,}000 = -114{,}000$ ft-lbs.

241. The Three-Span Beam.—The coefficients a and b are

$$a_2 = \frac{K_1}{2\,(K_1 + K_2)}$$

$$b_3 = \frac{K_3}{2\,(K_3 + K_2)}$$

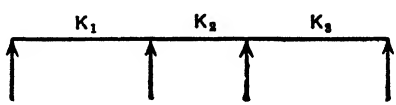


FIG. 21.

First Span Loaded.—The value of M_2 is obtained from eq. (4)

for M_4 by reducing subscripts by 2. $K_0 = 0$, $a_1 = 0$. This gives

$$M_2 = \frac{K_2\,(2\,K_1)}{[2\,(K_2 + K_1) - b_3\,K_1]\,2\,K_1} \times A.$$

Substituting the value of b_3, we get

$$M_2 = \frac{2\,K_2\,(K_2 + K_3)}{4\,(K_2 + K_3)\,(K_2 + K_1) - K_1\,K_3} A. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Then

$$M_3 = -\,M_2 \times b_3 = -\,\frac{K_2\,K_3^3}{4\,(K_2 + K_3)\,(K_2 + K_1) - K_1\,K_3} A. \quad (8)$$

Centre Span Loaded.—Applying the general formula, we get, after reducing,

$$M_2 = \frac{2\,K_1\,(K_2 + K_3) - K_1\,K_3}{4\,(K_2 + K_8)\,(K_2 + K_1) - K_1\,K_3} A, \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$M_3 = \frac{2\,K_3\,(K_2 + K_1) - K_1\,K_3}{4\,(K_2 + K_3)\,(K_2 + K_1) - K_1\,K_3} A. \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Note that all equations for M_2 and M_3 have the same denominator. For the third span loaded, use eqs. (7) and (8) with change of subscripts.

Example.—Live load = 2000 lbs/ft., dead load = 1000 lbs/ft. Spans and relative values of K as shown. All calculations will be made in kips and kip-feet.

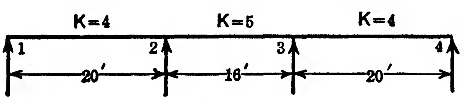


FIG. 22.

First Span Loaded.—$A = -\frac{1}{4} \times 2 \times 20^2 = -\,200$. From eq. (7),

$$M_2 = -\,\frac{2 \times 5 \times 9}{4 \times 9 \times 9 - 16} \times 200 = -\,\frac{90}{308} \times 200 = -\,58.4.$$ From eq. (7),

$$M_3 = 58.4 \times \frac{4}{2 \times 9} = +\,13.0.$$

Second Span Loaded.—$A = -\frac{1}{4} \times 2 \times 16^2 = -\,128$. Then from (9),

$$M_2 = -\,\frac{2 \times 4 \times 9 - 16}{308} \times 128 = -\,\frac{56}{308} \times 128 = -\,23.2. \quad M_3 = M_2.$$

Third Span Loaded.—$M_2 = + 13.0$. $M_3 = - 58.4$.

For all spans loaded, $M_2 = M_3 = - 58.4 + 13.0 - 23.2 = - 68.6$.

For dead load, $M_2 = M_3 = - 68.6/2 = - 34.3$.

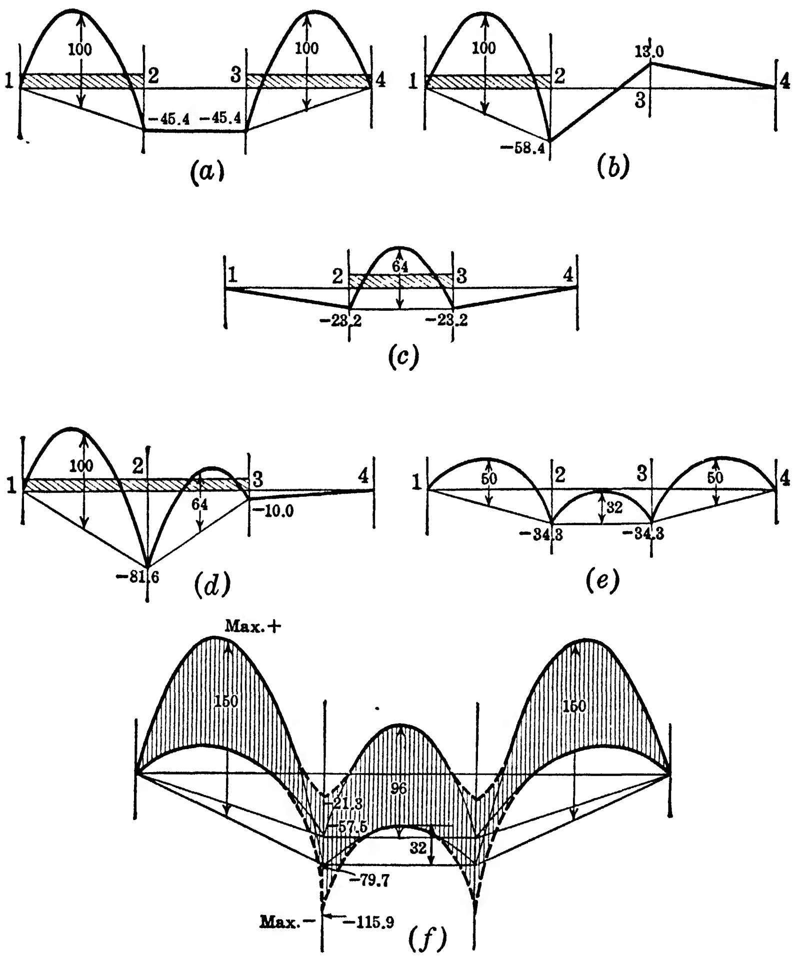


Fig. 23.

Fig. 23 (*a*) to (*d*) shows moment diagrams for various combinations of live loads. Fig. (*a*) for maximum positive moments in end spans, and maximum negative in centre span. Fig. (*c*) for maximum positive moment in centre span and maximum negative moments in end spans. For maximum moments at and near supports, Fig. (*d*) for maximum negative at 2, and Fig. (*b*) for maximum positive at 3. Combining with dead load gives Fig. (*f*), with shaded area showing range of moments. Note the large negative moments throughout the centre span.

242. Beam of Several Spans.—This case will be worked out by the direct application of the general formulas of Art. 239 to a numerical problem. Span lengths and relative values of K as shown in Fig. 24. Values of K shown in circles. Live load = 3000 lbs/ft.; dead load =

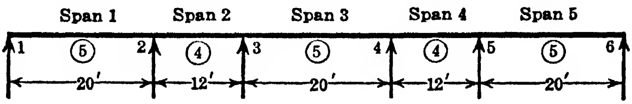


FIG. 24.

1000 lbs/ft. For the long spans, for live load, $A = -300$; for the short spans, $A = -108$.

Coefficients a and b: From eq. (1),

$$a_2 = 5/18 = 0.278$$

$$a_3 = \frac{4}{18 - 0.28 \times 5} = 0.241$$

$$a_4 = \frac{5}{18 - 0.24 \times 4} = 0.293$$

$$b_5 = a_2 = 0.278; \quad b_4 = a_3 = 0.241; \quad b_3 = a_4 = 0.293.$$

Span 1 *Loaded:* From eq. (4),

$$M_2 = -\frac{4 \times 2 \times 5}{(2 \times 9 - 0.293 \times 5) \times 2 \times 5} \times 300 = -72.6.$$

Then

$$M_3 = +72.6 \times 0.293 = +21.3$$

$$M_4 = -21.2 \times 0.241 = -5.1$$

$$M_5 = +5.1 \times 0.278 = +1.4.$$

Span 2 *Loaded:* From eq. (3),

$$M_2 = -\frac{5\,(2 \times 9 - 0.241 \times 4) - 5 \times 5}{(2 \times 9 - 241 \times 4)\,(2 \times 9) - 5 \times 5} \times 108 - \frac{85.2 - 25}{306.6 - 25} \times 108 = -23.1.$$

Then from (4),

$$M_3 = -\frac{5\,(2 \times 9 - 0) - 25}{306.6 - 25} \times 108 = -\,24.9$$

$$M_4 = +\,24.9 \times 241 = +\,6.0$$

$$M_5 = -\,6.0 \times 0.278 = -\,1.7.$$

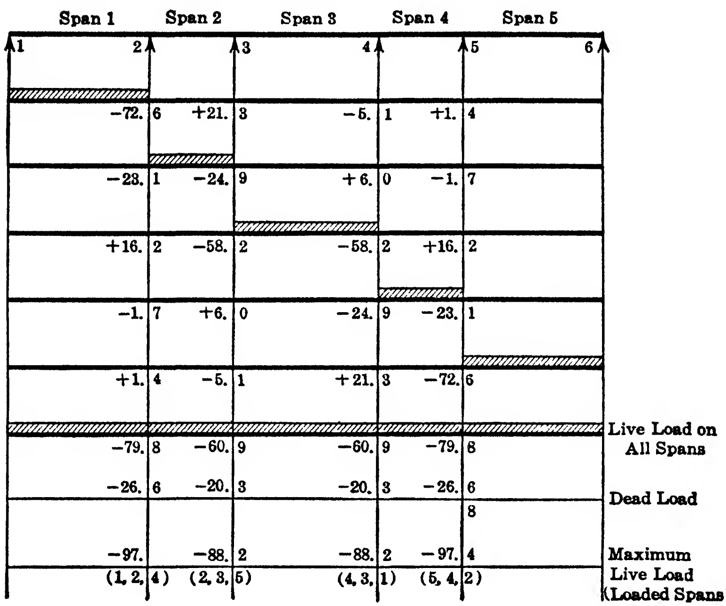


FIG. 25.

Span 3 *Loaded:* From eq. (3),

$$M_3 = \frac{4\,(2 \times 9 - 0.278 \times 5) - 4 \times 4}{(2 \times 9 - 0.278 \times 5)\,(2 \times 9 - 0.278 \times 5) - 4 \times 4} \times 300$$

$$= -\frac{66.4 - 16}{276 - 16} \times 300 = -\,58.2$$

$$M_4 = M_3 = -\,58.2$$

$$M_2 = +\,58.2 \times 0.278 = +\,16.2$$

$$M_5 = M_2 = +\,16.2.$$

The results are assembled in Fig. 25, together with the moments for all spans loaded, the dead load moments of one-third these values, and the maximum live-load moments.

Fig. 26 gives a summary of maximum values at span centres and at supports. Complete diagrams as illustrated in Art. 241 may readily

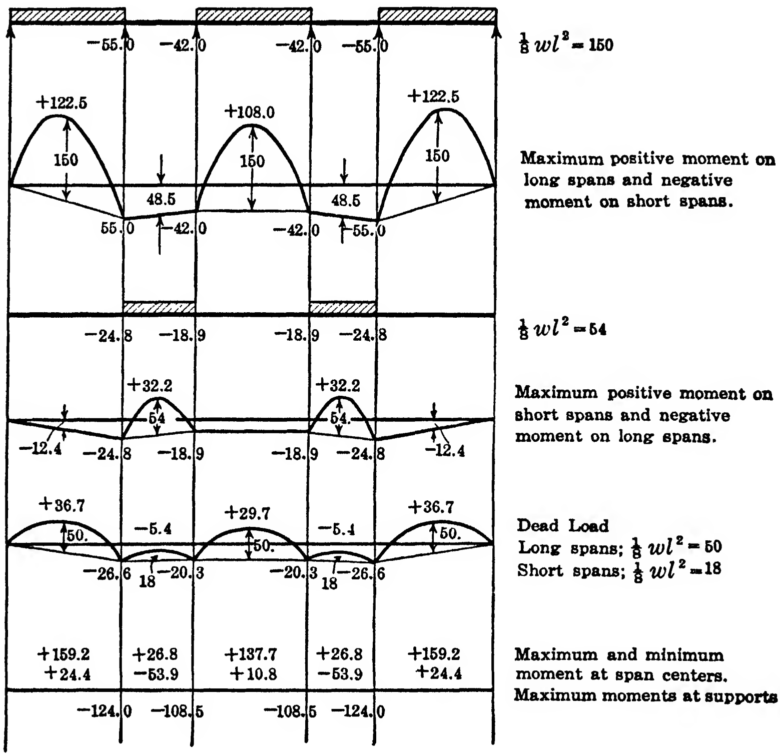


FIG. 26.

be prepared from the data given. Note the large negative moments throughout the short spans.

It will be instructive to compare the foregoing values with those obtained by the use of coefficients, such as specified for equal spans. Combining dead and live load we have for the long spans $1/16\, w\, l^2 = 100$; $1/12\, w\, l^2 = 133.3$; $1/10\, w\, l^2 = 160$. For the short spans $1/16\, w\, l^2 = 36$; $1/12\, w\, l^2 = 48$; $1/10\, w\, l^2 = 57.6$. It would be difficult to select from these coefficients values that could be relied upon to give satisfactory results in the foregoing problem.

243. Formulas for Concentrated Loads.—In special cases of heavy concentrated live loads it will not be satisfactory to use an approximate equivalent uniform load. In this case, referring to eq. (1*b*), Art. 227, let

$$B_1 = -\Sigma P_3 l_3 (k - k^3)$$

$$B_2 = -\Sigma P_3 l_3 (2k - 3k^2 + k^3).$$

Then equations of Art. 239, for the relations between M_3 and M_4, become

$$\frac{-a_2 M_3}{K_2} + 2 M_3 \left(\frac{1}{K_2} + \frac{1}{K_3}\right) + \frac{M_4}{K_2} = \frac{B_2}{K_3}, \quad . \quad . \quad . \quad (11)$$

$$\frac{M_3}{K_3} + 2 M_4 \left(\frac{1}{K_3} + \frac{1}{K_4}\right) - \frac{b_5 M_4}{K_4} = \frac{B_1}{K_3}. \quad . \quad . \quad . \quad . \quad (12)$$

These are best solved after substituting numerical values for K and B. The values of a and b are the same as in Art. 239.

GENERAL METHODS OF ANALYSIS OF FRAMES AND BEAMS

244. Stress Calculation in Frames.—In the usual skeleton type of reinforced-concrete construction, the girders together with the columns constitute a structural frame rigidly connected at the joints. In this case the beams cannot be calculated as simply supported continuous girders, but the effect of the columns must be considered. For a complete general treatment of such frames reference is made to other works,* but the fundamental equations will be explained and some of the simpler cases analyzed.

A load placed in any panel of a rigid frame will cause moments and deflections in all members, and a rigid analysis requires the inclusion of all members in the equations. But as in the case of the continuous girder of several spans, the effect of any particular load rapidly diminishes with distance, so that it is necessary to consider only a few spans or panels adjacent to the one in question. Fig. 27 illustrates the character of bending that takes place in a frame due

* See "Modern Framed Structures," Part II.

to a load in a single span. Each beam and column is bent more or less and each joint is twisted through a small angle. The analysis of this problem is based on the relation between the bending moments at the ends of the members, the angles of twist at the joints, and the external loads on the structure. If there is lateral pressure, such as wind load, the joints are also displaced laterally, and this introduces another factor in the problem. It is also true that unsymmetrical vertical loads will cause a slight lateral displacement, but this is very small compared to other movements and will be neglected.

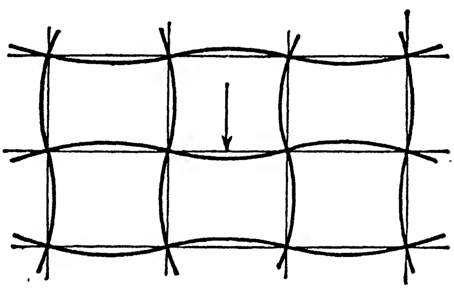


FIG. 27.

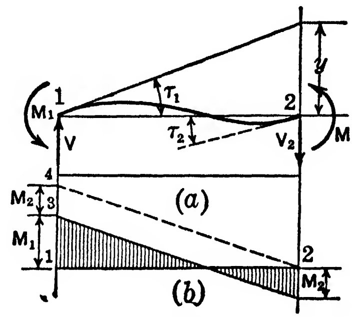


FIG. 28.

245. General Slope-Deflection Equations.—A beam 1–2, Fig. 28, is subjected to the end moments M_1 and M_2, bending it as shown. Fig. (*b*) represents the moment diagram. By the principle of area-moments, the deflection y, measured from the tangent at 1, is equal to $1/EI$ times the moment of the moment-area about point 2. This is equal to the moment of the area 1–4–2, minus the moment of the area 3–4–2–5. Or

$$y = \frac{1}{EI}\left[\frac{M_1 + M_2}{2} \cdot l \cdot \frac{2l}{3} - M_2 l \cdot \frac{l}{2}\right]$$

$$= \frac{l^2}{6EI}(2M_1 - M_2).$$

The angle $\tau_1 = \frac{y}{l} = \frac{l}{6EI}(2M_1 - M_2)$

Likewise $\tau_2 = \frac{l}{6EI}(2M_2 - M_1)$ . . . . . (1)

These equations may be solved for M_1 and M_2, giving

$$\left.\begin{aligned} M_1 &= \frac{2\,E\,I}{l}(2\,\tau_1 + \tau_2) \\ M_2 &= \frac{2\,E\,I}{l}(2\,\tau_2 + \tau_1) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If the beam is also rotated through an angle α, due to displacement of the joints, as shown in Fig. 29, the deflection angles τ_1 and τ_2 are now equal to $\theta_1 - \alpha$ and $\theta_2 - \alpha$, respectively, where θ_1 and θ_2 are the angles through which the tangents at 1 and 2 have turned, that is, the twist angles. Substituting, we have

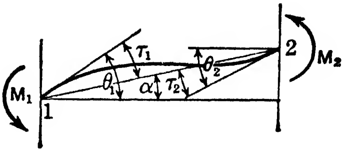


FIG. 29.

$$\left.\begin{aligned} M_1 &= \frac{2\,E\,I}{l}(2\,\theta_1 + \theta_2 - 3\,\alpha) \\ M_2 &= \frac{2\,E\,I}{l}(2\,\theta_2 + \theta_1 - 3\,\alpha) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (3)$$

Eqs. (3) are the fundamental slope-deflection equations, expressing the values of the end moments in a restrained beam in terms of the angles of end slope, or twist, θ, and the deflection angle, α.

If the beam supports intermediate loads, the equations become

$$\left.\begin{aligned} M_1 &= \frac{2\,E\,I}{l}(2\,\theta_1 + \theta_2 - 3\,\alpha) + C_1 \\ M_2 &= \frac{2\,E\,I}{l}(2\,\theta_2 + \theta_1 - 3\,\alpha) - C_2 \end{aligned}\right\} \quad . \quad . \quad . \quad (4)$$

where C_1 and C_2 are the numerical values of end bending moments at 1 and 2 due to the applied loads, considering the beam as fixed at the ends.

If the beam is hinged at 2, then $M_2 = 0$, and

If hinged at 1,
$$\left.\begin{aligned} M_1 &= \frac{3EI}{l}(\theta_1 - \alpha) + D_1 \\ M_2 &= \frac{3EI}{l}(\theta_2 - \alpha) - D_2 \end{aligned}\right\} \quad \ldots\ldots \quad (5)$$

in which $D_1 = C_1 + C_2/2$ and $D_2 = C_2 + C_1/2$.

246. Signs of the Moments.—In dealing with general problems involving several members meeting at a point at various angles, it is convenient to adopt a rule for sign of moment related to the *direction*

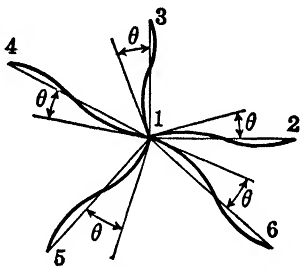


FIG. 30.

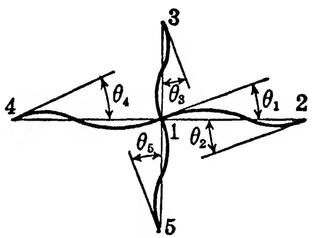


FIG. 31.

of rotation of the bending forces with respect to the joint centre, and not, as in horizontal beams, a sign depending upon the nature of the fibre stress on one side of the beam. The direction of bending which will be taken as *positive* is shown in Fig. 30, and the angle θ through which the joint has turned is also positive. Thus positive *bending* is for right-handed bending of the beam with respect to the joint, and positive *twist* is for left-handed turning of the joint. The angle α is also positive when denoting a left-handed rotation about the joint.

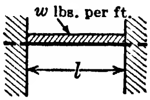
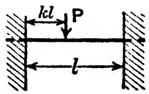
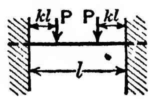
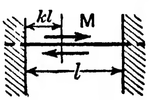
247. Equilibrium of Moments at a Joint.—In Fig. 31 are represented four members radiating from the joint 1. Each joint is twisted through an angle θ but no displacement of joints 2, 3, etc., is here assumed. Let $K = I/l$ for any member.

For equilibrium of moments at joint 1 we must have

$$M_{1-2} + M_{1-3} + M_{1-4} + M_{1-5} = 0,$$

TABLE 18

VALUES OF C_1 AND C_2 IN EQ. (4), ART. 245.

Loading Condition	C_1	C_2
	$\frac{1}{12} w l^2$	$\frac{1}{12} w l^2$
	$P k (1 - k)^2 l$	$P k^2 (1 - k) l$
	$P k (1 - k) l$	$P k (1 - k) l$
	$M (1 - k)(1 - 3 k)$	$M k (2 - 3 k)$

or in detail from eq. (4), α being zero,

$$2 E K_{1-2} (2 \theta_1 + \theta_2) + 2 E K_{1-3} (2 \theta_1 + \theta_3) + 2 E K_{1-4} (2 \theta_1 + \theta_4) + 2 E K_{1-5} (2 \theta_1 + \theta_5) = 0.$$

Collecting terms,

$$4 E (K_{1-2} + K_{1-3} + K_{1-4} + K_{1-5}) \theta_1 + 2 E K_{1-2} \theta_2 + 2 E K_{1-3} \theta_3 + 2 E K_{1-4} \theta_4 + 2 E K_{1-5} \theta_5 = 0.$$

And if any member supports a load, a term C_1 or C_2 must be added, depending on the circumstances.

Expressed in general terms, the equation may be written as follows:

$$4 E \Sigma K \theta_1 + 2 E \Sigma K \theta_{2 \ldots 5} + \Sigma C = 0 \quad . \quad . \quad (6)$$

248. Use of Eq. (6) in the Solution of Problems.—In the solution of problems in which the answer depends upon the relative rigidity of the various members, a preliminary design must first be made, or some assumption regarding the relative values of I for the various members. The values of $K = I/l$ for each member can then be computed. As noted in Art. 228, the value of I for beams may be calculated by taking the gross section of the concrete and omitting the steel. For columns the steel should be included at $n - 1$ times its section. No great accuracy is possible or necessary in such problems. (The effect of distortion due to direct stress may be neglected in dealing with unbraced frames, as this is relatively small compared to the effect of bending.)

Having the values of K, eq. (6) can be written out for each joint giving as many equations as values of θ. Where displacement of joints takes place, as for wind pressures, the value of α enters for the columns, and additional equations are needed. An exact solution of this problem is too complex to be discussed here, but approximate methods will be mentioned later.*

If the far end of any member in Fig. 31 is hinged, as at joint 2, then the term for that member in eq. (6) is $3\ E\ K\ \theta$, in place of $4\ E\ K\ \theta$, and θ_{2-1} drops out of the equation [see eq. (5)], and the value of C becomes $C_1 + C_2/2$. If the far end is fixed then $\theta_{2-1} = 0$. Having the values of θ (and α if present), then the moments are calculated from eq. (4).

249. Use of Quantities E and I/l.—Where the value of E is constant throughout, it may, in most problems, be omitted from the equations. Likewise the values of $I/l = K$ may be expressed in relative terms only, in any convenient unit. The result of these changes will be that the calculated values of θ and α will be in terms of units involving the value of E and the units used for I/l. But on substituting the values of θ and α back in the equations for moments, the results will be correct and in the same units as used in calculating C_1 and C_2. If the values of θ and α are desired in terms of radians, or if one or more of these quantities are used in terms of radians, then the actual values of E and I/l must be used. This will not be the case in the common problems of moment determination, and the omission of E and the use of I/l as above suggested result in magnitudes for θ and α much more convenient to handle.

* See "Modern Framed Structures," Part II, for detailed treatment.

250. Example.—Fig. 32 shows a frame of columns and beams hinged at 3, 4, 5, and 6, and fixed at 7 and 8. Spans loaded as shown. Required the bending moments at 2 in beams and columns. Suppose the value of K for the columns is twice that of the beams. Then these quantities can be represented as 2 and 1, respectively, as shown in circles. For beam 1–2, the numerical values of C_1 and $C_2 = 1/12\, w l^2 = 100 \times 400 = 40{,}000$ ft-lbs. For beam 2–6, hinged at the end, the value of $D_1 = C_1 + C_2/2 = 40{,}000 + 20{,}000 = 60{,}000$. It is convenient to write these moments along the members as shown, with signs as they will appear in eq. (6). For the fixed ends 7 and 8, $\theta = 0$. Equations like (6), omitting the quantity E, will now be written out for joints 1 and 2.

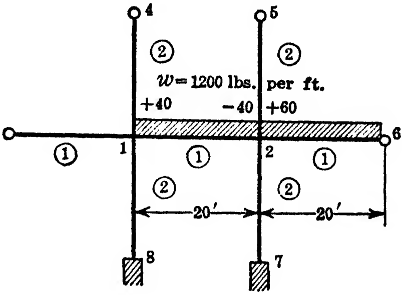


FIG. 32.

Joint 1

$$[4 \times (1 + 2) + 3 \times (1 + 2)]\, \theta_1 + 2 \times 1 \times \theta_2 + 40 = 0.$$

Joint 2

$$[4 \times (1 + 2) + 3 \times (1 + 2)]\, \theta_2 + 2 \times 1 \times \theta_1 + 60 - 40 = 0,$$

or

$$21\, \theta_1 + 2\, \theta_2 = -40$$

and

$$21\, \theta_2 + 2\, \theta_1 = -20.$$

Solving, we get

$$\theta_1 = -1.83;\ \theta_2 = -0.78.$$

Then

$$M_{2-6} = 3 \times 1 \times \theta_2 + 60 = +57.7 = +57{,}700 \text{ ft-lbs.}$$

$$M_{2-5} = 3 \times 2 \times \theta_2 = -4.7 = -4700 \text{ ft-lbs.}$$

$$M_{2-1} = 4 \times 1 \times \theta_2 + 2 \times 1 \times \theta_1 - 40 = -46{,}800 \text{ ft-lbs.}$$

$$M_{2-7} = 4 \times 2 \times \theta_2 = -6.2 = -6200 \text{ ft-lbs.}$$

The sum of these moments is zero. The direction of bending with reference to the joint is indicated by the sign, plus indicating right-handed rotation, and minus, left-handed rotation. The beam 2–6 therefore has a moment of 57,700 ft-lbs. at 2, bending to the *right*, or producing tension in the top fibres. Beam 2–1 has a moment of 46,800 ft-lbs. bending to the *left*, also producing tension in the top fibres. The columns 2–5 and 2–7 have relatively small bending moments as shown.

Note that in this problem, with hinges at 3, 4, 5, and 6 and fixed ends at 7 and 8, there are but two unknowns to be determined.

Approximate Calculation of Maximum Moments in Beams and Columns for Uniform Live Loads

251. The exact theoretical calculation of maximum moments in a building frame consisting of several stories and panels is a long, tedious operation, as the number of equations to be solved in each problem will be very large. However, as already pointed out in Art. 244, the accuracy of such a solution is more apparent than real, and equally satisfactory results can be obtained by making certain assumptions that will greatly simplify the process. The stresses in any particular beam or column are but little affected by loads and conditions of structure in remote panels, and hence closely approximate results may be reached by limiting the structure under consideration to one or two panels in all directions from the member in question, assuming the beams and columns at these limits to be definitely fixed or hinged. This method of analysis will be illustrated by the various problems which follow. In these calculations, a certain degree of uniformity as to values of K is assumed for beams and columns, as indicated. Where the actual values of K differ considerably from the assumed relations, the more general method of analysis must be used as indicated in Art. 247.

The following cases will be discussed and approximate formulas and coefficients deduced:

A. Positive Moment in an Interior Beam
B. Negative Moment in an Interior Beam
C. Moment in an Interior Column
D. Moment in an Exterior Beam
E. Moment in an Exterior Column

252. **A. Positive Moment in an Interior Beam.**—*Load on One Panel Only.*—In Fig. 33, panel 1–2 is loaded. Required the moment at the centre. It will be assumed that the beams are all alike as to values of K and that the columns in each story are all alike, and that the building extends in each direction a sufficient distance to give substantially symmetrical conditions about the centre. To solve this problem approximately, various assumptions may be made as to the extent of the surrounding structure to be considered.

(*a*) The structure extending for two panels each way from joints 1 and 2 may be taken, and the members assumed as fixed at the limits, *a*, *b*, *d*, etc. This will give rise to four equations for the four joints at 1, *A*, *B*, and *C*. Results on this assumption will be quite accurate, as the effect of the structure beyond these assumed limits will be very small.

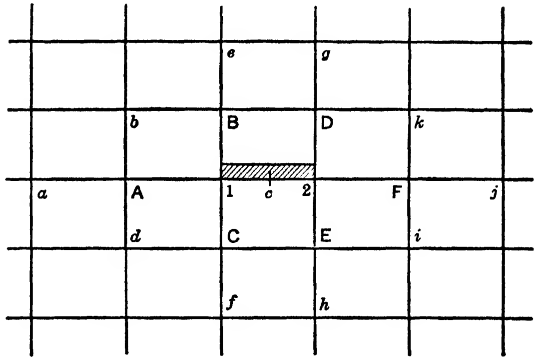


FIG. 33.

(*b*) The structure considered may be limited to one-panel length from joints 1 and 2, and the members assumed as fixed at *A*, *B*, *C*, etc., giving only one equation, that for joint 1.

(*c*) The same extent of structure as in (*b*) but ends assumed as hinged at *A*, *B*, *C*, etc.

Comparing first the assumptions (*a*) and (*b*), it has been found* that the difference in results amounts to less than 1.5% for as wide range in ratios of *K* for columns to *K* for beams as from ½ to 2, and hence only (*b*) and (*c*) will be considered here.

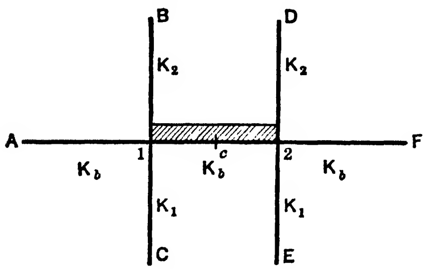


FIG. 34.

(*b*) *Members Assumed as Fixed at Far Ends, A, B, C, etc.,* Fig. 34. Values of *K* as shown. $C_1 = 1/12\, w\, l.^2$ By symmetry, $\theta_2 = -\theta_1$. The moment equation for joint 1 is then

$$6\,K_b\,\theta_1 + 4\,(K_1 + K_2)\theta_1 + C_1 = 0$$

whence

$$\theta_1 = -\frac{C_1}{6\,K_b + 4\,K_1 + 4\,K_2}.$$

* "Modern Framed Structures," Part II, p. 535.

Then

$$M_{1-2} = 2\,K_b\,\theta_1 + C_1 = C_1\left(1 - \frac{K_b}{3\,K_b + 2\,K_1 + 2\,K_2}\right).$$

Centre moment

$$= M_c = \tfrac{1}{8}\,w\,l^2 - M_{1-2} = C_1\left[0.5 + \frac{K_b}{3\,K_b + 2\,K_1 + 2\,K_2}\right]. \quad (1)$$

(*c*) *Members Assumed as Hinged at Far Ends, A, B, C, etc.*—For hinged ends, the restraint coefficient for these members will be 3 instead of 4, giving for joint 1,

$$5\,K_b\,\theta_1 + 3\,(K_1 + K_2)\,\theta_1 + C_1 = 0.$$

$$\theta_1 = -\frac{C_1}{5\,K_b + 3\,K_1 + 3\,K_2}$$

and

$$M_c = C_1\left[0.5 + \frac{2\,K_b}{5\,K_b + 3\,K_1 + 3\,K_2}\right]. \quad . \quad . \quad . \quad (2)$$

Comparison of Values of M_c for Various Proportions of Beams and Columns.—Calculated values of the coefficients of C_1 from eqs. (1) and (2) for various ratios of values of K for beams and columns are given below. K_c = the average of K_1 and K_2. The values of M_c are also given in terms of coefficients of $w\,l^2$ in order to compare with the coefficients prescribed in specifications.

CENTRE MOMENTS IN BEAM M_c

Ratio K_b/K_c	Coefficient of $C_1 = \frac{1}{12}\,w\,l^2$		Coefficients of $w\,l^2$	
	Ends Fixed Case (*b*)	Ends Hinged Case (*c*)	Ends Fixed Case (*b*)	Ends Hinged Case (*c*)
0	0.50	0.50	0.0417	0.0417
1/10	.523	.531	.0436	.0442
1/4	.553	.569	.0461	.0474
1/2	.591	.618	.0492	.0501
1	.643	.682	.0536	.0568
2	.700	.750	.0583	.0625
3	.730	.785	.0608	.0654
4	.750	.807	.0625	.0672
6	.773	.833	.0644	.0694
10	.795	.858	.0662	.0715
α	.833	.900	.0694	.0750

The maximum difference in the two sets of values is for the last ratio, namely, for columns of no stiffness, or for a continuous girder freely supported, and is about 8%. For usual proportions the difference is not more than 7%. Compare these results with the coefficient of $1/16\,w\,l^2$ or $0.0625\,w\,l^2$ commonly specified.

The corresponding moments in the columns at joint 1 are as follows:

MOMENTS IN COLUMNS

Ratio K_b/K_c	Coefficients of $C_1 = 1/12\,w\,l^2$	
	Ends Fixed Case (*b*)	Ends Hinged Case (*c*)
¼	0.461	0.420
½	.364	.324
1	.285	.274
2	.200	.187
3	.154	.143
4	.125	.116

253. Effect of Loads in Other Panels.—In the foregoing analysis a single span only was loaded. For a theoretical maximum, other spans should be loaded as shown in Fig. 35. Such a loading will never occur, but its effect can be studied to advantage. With such conditions we have, from symmetry, referring to Fig. 34, $\theta_A = -\theta_1$, and can also assume $\theta_B = \theta_D = -\theta_1$, hence the equation for joint 1 becomes

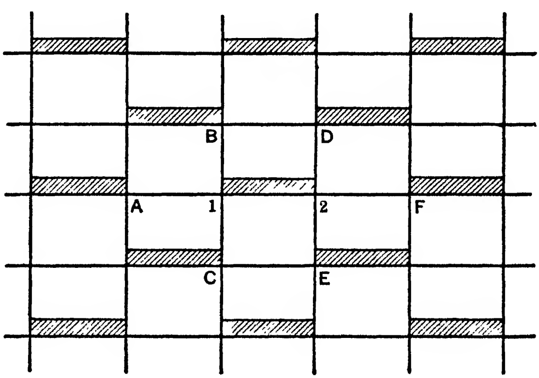


FIG. 35.

$$(4\,K_b + 2\,K_1 + 2\,K_2)\,\theta_1 + C_1 = 0$$

or

$$\theta_1 = -\frac{C_1}{4K_b + 2K_1 + 2K_2}$$

and

$$M_c = C_1\left(0.5 + \frac{K_b}{2K_b + K_1 + K_2}\right). \quad . \quad . \quad . \quad . \quad (3)$$

Assuming K_c = average of K_1 and K_2 we have for various ratios K_b to K_c

Ratio K_b/K_c	Centre Moments in Beam M_c
½	$0.667\ C_1 = 0.056\ w\ l^2$
1	$.750\ C_1 = .062\ w\ l^2$
2	$.833\ C_1 = .069\ w\ l^2$
4	$.900\ C_1 = .075\ w\ l^2$

These values are about 10% higher than those for Case (*c*) of Art. 252. Considering the extreme assumptions here made it would

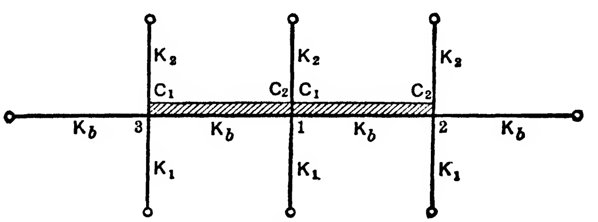


FIG. 36.

appear that it is sufficiently accurate to calculate positive moments by the assumptions of Case (*c*), namely: *that one span only be loaded and that the joints at the far ends of adjacent members be considered as hinged.*

The value of the moment is given by eq. (2).

254. B. Maximum Negative Moment in an Interior Beam.—For maximum negative moment at the left end of beam 1–2, span 3–1 and 1–2 are loaded as shown in Fig. 36. Assume hinged ends as shown. As in Art. 252 the results will be larger than from the assumption of fixed ends. Due to symmetry $\theta_1 = 0$, $C_2 = C_1 = 1/12\, w\, l^2$. Hence for joint 2

$$[(4 + 3)\, K_b + 3\,(K_1 + K_2)]\, \theta_2 - C_1 = 0$$

$$\theta_2 = \frac{C_1}{7\, K_b + 3\, K_1 + 3\, K_2}$$

$$M_{1-2} = C_1 \left(1 + \frac{2\, K_b}{7\, K_b + 3\, K_1 + 3\, K_2}\right). \quad . \quad . \quad . \quad (4)$$

Assuming K_c = average of K_1 and K_2, as before, we have, for various ratios K_b/K_c

Ratio K_b/K_c	End Moments in Beam M_{1-2}
¼	$1.07\ C_1 = 0.089\ w\, l^2$
½	$1.105\ C_1 = .092\ w\, l^2$
1	$1.154\ C_1 = .096\ w\, l^2$
2	$1.200\ C_1 = .100\ w\, l^2$
4	$1.235\ C_1 = .103\ w\, l^2$

The coefficient of $w\, l^2$ usually specified is $1/12 = 0.0833$, a rather low value.

Fig. 37 shows the theoretical loading in several surrounding panels for maximum moment at 1. Analysis of the effect of the loads outside of the panels A and B shows that their effect is to add about 5 to 8% to the moment due to the loads in A and B as calculated in the preceding analysis. Considering the unlikelihood of such loading and its small relative effect, the foregoing method of analysis appears to be sufficiently precise.

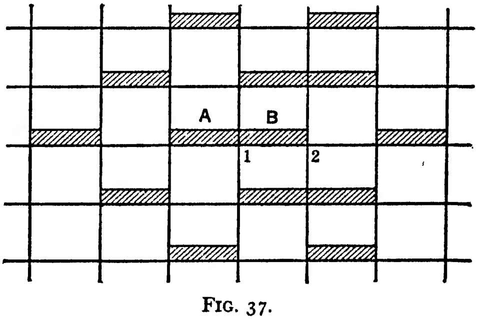


FIG. 37.

255. C. Moment in an Interior Column.—The critical loading for columns is that which will give a large bending stress simultaneously with a large direct compression. Fig. 38 shows a loading giving a large bending moment in 1–3 with the load on the column equal to the full vertical load less ½ panel load. Fig. 39 shows a loading giving full vertical load on column 1–3 but a somewhat smaller moment at 1. Which condition gives the greater total stress depends upon the relative amounts of moment and direct stress.

Referring to Fig. 38, with alternate spans loaded as shown, we may assume $\theta_a = -\theta_1$, $\theta_2 = -\theta_1$, and $\theta_3 = \theta_1$. Assume also fixed

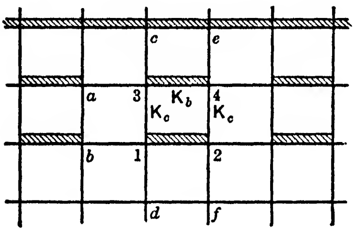


FIG. 38.

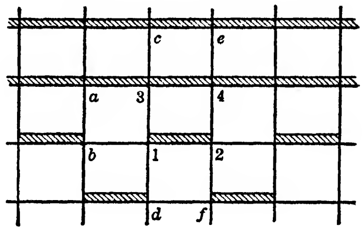


FIG. 39.

ends at c and d, and all beams alike and all columns alike. Then for joint 1

$$(4\,K_b + 10\,K_c)\,\theta_1 = -C_1;\ \theta_1 = \frac{-C_1}{4\,K_b + 10\,K_c}$$

$$M_{1-3} = 6\,K_c\,\theta_1 = -C_1\frac{6\,K_c}{4\,K_b + 10\,K_c}.$$

Omitting the alternate loading to the left of a and b as being very unlikely, and assuming hinged ends at a and b, gives for joint 1

$$(5\,K_b + 10\,K_c)\,\theta_1 = -C_1$$

and

$$M_{1-3} = -C_1\frac{6\,K_c}{5\,K_b + 10\,K_c} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

which may be taken as a fair value for the moment.

Then in Fig. 39, alternate loads as shown, fixed end at 3, $\theta_b = -\theta_1$, $\theta_2 = -\theta_1$, $\theta_d = -\theta_1$.

For joint 1

$$(4\,K_b + 6\,K_c)\,\theta_1 = -C_1$$

and

$$M_{1-3} = - C_1 \frac{4 K_c}{4 K_b + 6 K_c}.$$

Omitting the alternate loading on the lower floor and assuming a hinge at d gives

$$M_{1-3} = - C_1 \frac{4 K_c}{4 K_b + 7 K_c} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

a more likely value.

Comparing (5) and (6) it will be seen that (5) gives somewhat the larger value but the direct stress will be ½ panel load less.

For various ratios of K_b/K_c, eq. (5) gives the following coefficients:

Ratio K_b/K_c	Moments in Column M_{1-3}
¼	$0.53\,C_1 = 0.044\,w\,l^2$
½	$.48\,C_1 = .040\,w\,l^2$
1	$.40\,C_1 = .033\,w\,l^2$
2	$.30\,C_1 = .025\,w\,l^2$
4	$.20\,C_1 = .017\,w\,l^2$

It will be noted that the moment decreases rapidly with increased ratio of K_b/K_c.

256. D. Moments in an Exterior Beam (Fig. 40).—Here the restraint at the end of the beam is less than at an interior support and hence the end moment at 2 will be somewhat less and the end moment at 1 and the centre moment somewhat greater than for an interior beam.

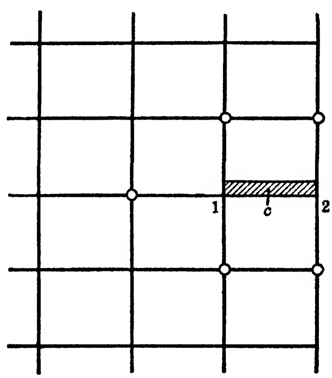


FIG. 40.

Positive Moment at the Centre.—Hinged joints as shown in Fig. 40. There will be two unknowns, θ_1 and θ_2, and the two equations are:

$$\text{Joint 1: } (7\,K_b + 6\,K_c)\,\theta_1 + 2\,K_b\,\theta_2 + C_1 = 0$$

$$\text{Joint 2: } (4\,K_b + 6\,K_c)\,\theta_2 + 2\,K_b\,\theta_1 - C_1 = 0.$$

A solution of these equations gives the following results for various ratios K_b/K_c.

Ratio K_b/K_c	Center Moments M_c in Beam
¼	$0.57\ C_1 = 0.048\ w\ l^2$
½	$.63\ C_1 = .052\ w\ l^2$
1	$.71\ C_1 = .059\ w\ l^2$
2	$.82\ C_1 = .068\ w\ l^2$
3	$.88\ C_1 = .073\ w\ l^2$
4	$.92\ C_1 = .077\ w\ l^2$

These values are not greatly different from those for the interior panel. If the outer column is considerably smaller than the interior ones, the average value of K_c may be used in the equations.

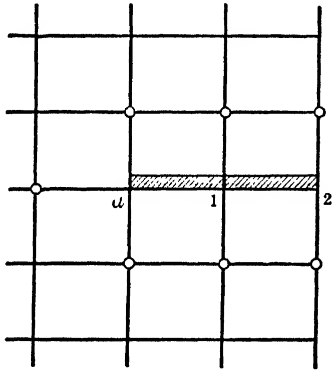


FIG. 41.

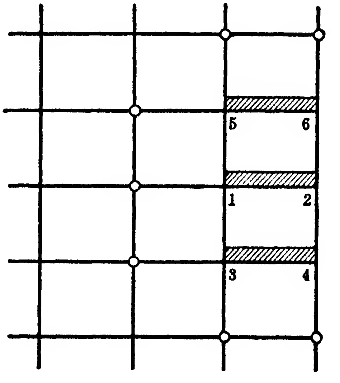


FIG. 42.

Negative Moment at Interior Support.—Hinged ends as in Fig. 41. These moments will be about 7 to 8% larger than for an interior panel as given in Art. 254. They are as follows:

Ratio K_b/K_c	End Moments in Beam at Interior Support, M_{1-2}
¼	$1.07\ C_1 = 0.089\ w\ l^2$
½	$1.12\ C_1 = .093\ w\ l^2$
1	$1.19\ C_1 = .100\ w\ l^2$
2	$1.26\ C_1 = .105\ w\ l^2$
3	$1.29\ C_1 = .107\ w\ l^2$
4	$1.31\ C_1 = .110\ w\ l^2$

Note the specified coefficients in Art. 234 of $1/10$ to $1/12\, w\, l^2$ for positive moment and $1/10\, w\, l^2$ for negative moment at 1.

Negative Moment at Exterior Support. Fig. 42.—Loading as shown. Assume $\theta_1 = \theta_5 = \theta_3$; $\theta_2 = \theta_6 = \theta_4$.

Equations:

$$\text{Joint 1: } (7\, K_b + 12\, K_c)\, \theta_1 + 2\, K_b\, \theta_2 + C_1 = 0$$
$$\text{Joint 2: } (4\, K_b + 12\, K_c)\, \theta_2 + 2\, K_b\, \theta_1 - C_1 = 0$$

The results are:

Ratio K_b/K_c	End Moments in Beam at Exterior Support, M_{2-1}
½	$0.92\, C_1 = 0.077\, w\, l^2$
1	$.84\, C_1 = .070\, w\, l^2$
2	$.71\, C_1 = .059\, w\, l^2$
3	$.62\, C_1 = .052\, w\, l^2$
4	$.55\, C_1 = .046\, w\, l^2$

257. E. Moment in Exterior Column.—Loading and structure as shown in Fig. 43. The moment in question is M_{2-4}. Assume $\theta_1 = \theta_3$; $\theta_2 = \theta_4$.

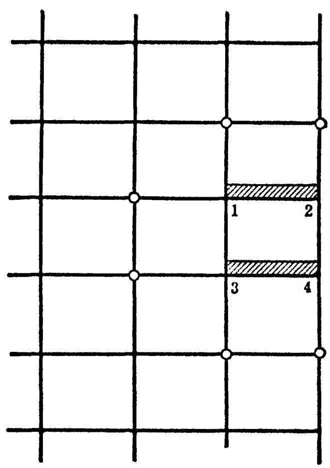


FIG. 43.

Equations:

Joint 1:

$$(7\, K_b + 9\, K_c)\, \theta_1 + 2\, K_b\, \theta_2 + C_1 = 0$$

Joint 2:

$$(4\, K_b + 9\, K_c)\, \theta_2 + 2\, K_b\, \theta_1 - C_1 = 0$$

The results are:

Ratio K_b/K_c	Moment in Column, M_{2-4}
½	$0.61\, C_1 = 0.051\, w\, l^2$
1	$.53\, C_1 = .044\, w\, l^2$
2	$.43\, C_1 = .036\, w\, l^2$
3	$.36\, C_1 = .030\, w\, l^2$
4	$.31\, C_1 = .026\, w\, l^2$

Specification requirements for an end panel are $1/12\, w\, l^2$, $= 0.083\, w\, l^2$, for the beam, and $1/24\, w\, l^2$, $=0.041\, w\, l^2$, for the column

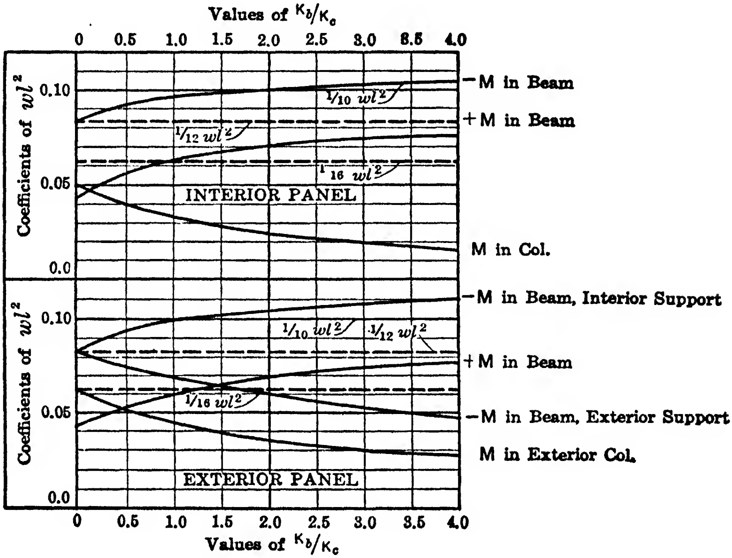


FIG. 44.

(assuming equal columns above and below) for values of K_b/K_c less than 4, and $1/16\, w\, l^2$, $=0.0625\, w\, l^2$, for the beam, and $0.0312\, w\, l^2$ for the column for larger ratios.

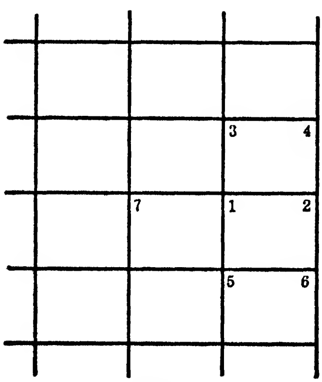


FIG. 45.

258. Recapitulation. — Following in Table 19 is a summary of recommended equations for maximum live-load moments in beams and columns. Value of I/l for all beams $= K_b$. Value of I/l for columns $= K_c$. Average of K for the columns of two stories may be used for K_c.

In Fig. 44 are given the values of the coefficient of $w\, l^2$ calculated from these equations for various ratios K_b/K_c. For the interior column the curve shown is for condition No. 3, Table 19.

259. Dead-Load Moments.—The dead load is usually so nearly uniform that for interior panels the moments may be taken as in a fixed end beam, namely: $1/12\, w\, l^2$ for negative end moments and $1/24\, w\, l^2$ for positive centre moments. For an end panel the end moment at inside support is somewhat greater, at end support somewhat less, and the centre moment is somewhat greater than these values.

The following analysis shows the approximate values for beam and column moments in end panels. Consider beam 1–2 and column 2–6 in Fig. 45. Assume $\theta_3 = \theta_5 = \theta_1 = \theta_4 = \theta_6 = \theta_2$, beam fixed at 7.

Equations:

$$\text{Joint 1: } (8\, K_b + 12\, K_c)\, \theta_1 + 2\, K_b\, \theta_2 = 0$$

$$\text{Joint 2: } (4\, K_b + 12\, K_c)\, \theta_2 + 24\, K_b\, \theta_1 - C_1 = 0$$

The resulting moments are as follows:

Ratio K_b/K_c	End Moment M_{1-2}	End Moment M_{2-1} Coefficients of $w\, l^2$	Centre Moment in Beam, M_c	Column Moment M_{2-6}
½	0.089	0.072	0.044	0.036
1	.092	.063	.047	.032
2	.096	.051	.051	.026
3	.098	.043	.055	.022
4	.100	.036	.057	.018

260. Shears in Beams.—The end shearing-stress is a maximum for a fully loaded beam and under conditions where the difference between end moments is a maximum. For an interior panel loaded as in Fig. 36, Art. 254, the shear at the left end of 1–2 is from 0.53 to $0.56\, w\, l$.

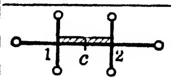
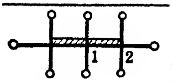
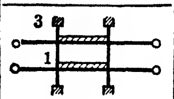
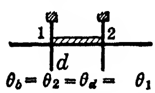
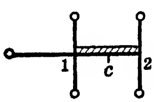
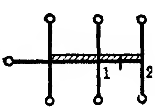
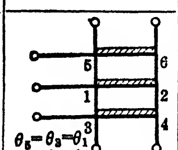
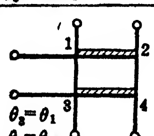
For an exterior panel, as in Fig. 41, the shear ranges from 0.56 to $0.58\, w\, l$. In general, the shear may be taken at about 10% in excess of the end shear for a simple beam.

261. Beams and Columns Having Unequal Values of *K*.—For the general case of unequal values of *K*, the foregoing assumptions may be made as to loading and limitation of structure by assuming hinged or fixed ends at the boundaries. A slope-deflection equation is then written out for each joint within the structure so limited, generally

TABLE 19

FORMULAS FOR APPROXIMATE MAXIMUM LIVE-LOAD BENDING MOMENTS IN BEAMS AND COLUMNS

$K_b = I/l$ for beams; $K_c = I/l$ for columns; $C_1 = \frac{1}{12} w l^2$

Ref. No.	Assumed Structure and Loading	Moment Required	Recommended Formula
1		Positive moment in interior beam.	$M_c = \left[0.5 + \frac{2 K_b}{5 K_b + 6 K_c}\right] C_1$
2		Negative moment in interior beam.	$M_{1-2} = \left[1 + \frac{2 K_b}{7 K_b + 6 K_c}\right] C_1$
3		Maximum moment in interior column.	$M_{1-3} = \frac{6 K_c}{5 K_b + 10 K_c} C_1$
4		Moment in interior column, max. vert. loading.	$M_{1-3} = \frac{4 K_c}{4 K_b + 7 K_c} C_1$
5		Positive moment in exterior beam.	*For* M_c Equations for θ_1 and θ_2 1. $(7 K_b + 6 K_c)\,\theta_1 + 2 K_b \theta_2 = -C_1$ 2. $(4 K_b + 6 K_c)\,\theta_2 + 2 K_b \theta_1 = C_1$ $M_c = 0.5\, C_1 + K_b (\theta_2 - \theta_1)$
6		Negative moment in exterior beam. Inside end.	*For* M_{1-2} Equations for θ_3, θ_1, and θ_2 1. $(8 K_b + 6 K_c)\,\theta_1 + 2 K_b \theta_3 + 2 K_b \theta_2 = 0$ 2. $(4 K_b + 6 K_c)\,\theta_2 + 2 K_b \theta_1 = C_1$ 3. $(7 K_b + 6 K_c)\,\theta_3 + 2 K_b \theta_1 = -C_1$ $M_{1-2} = (4\,\theta_1 + 2\,\theta_2) K_b - C_1$
7		Negative moment in exterior beam. Outside end.	*For* M_{2-1} Equations for θ_1 and θ_2 1. $(7 K_b + 12 K_c)\,\theta_1 + 2 K_b \theta_2 = -C_1$ 2. $(4 K_b + 12 K_c)\,\theta_2 + 2 K_b \theta_1 = C_1$ $M_{2-1} = (4\,\theta_2 + 2\,\theta_1) K_b - C_1$
8		Moment in exterior column	*For* M_{3-4} Equation for θ_1 and θ_2 1. $(7 K_b + 9 K_c)\,\theta_1 + 2 K_b \theta_2 = -C_1$ 2. $(4 K_b + 9 K_c)\,\theta_2 + 2 K_b \theta_1 = C_1$ $M_{3-4} = (4\,\theta_4 + 2\,\theta_3) K_c$

but two or three such equations being required. On substitution of numerical quantities for the several values of K the equations are readily solved for the values of θ and the desired moments then found. The method of moment distribution explained later is also a useful method for such a case.

262. Beams with Concentrated Loads.—In the case of heavy concentrated loads the use of an assumed equivalent uniform load is likely to be unsatisfactory. Calculations should be made by using the correct value of C_1 and C_2, Art. 245, for the actual load in each beam. For such loading no generalizations are possible, and each case will need to be worked out independently.

263. Beams with Haunches.*—Economy is often secured by increasing the depth or breadth of a beam near the ends, thus increas-

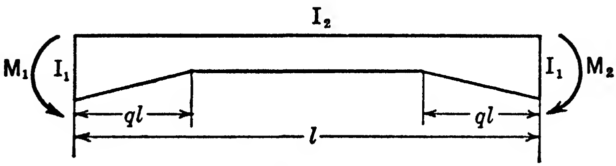


FIG. 46.

ing the stiffness in that region. This serves to reduce the central moment and increase the end moments. Where a beam has haunches (Fig. 46), the general slope deflection equations for uniform loads become

$$\left.\begin{aligned} M_1 &= A\,E\frac{I_2}{l}[B\,\theta_1 + C\,\theta_2 - 3\,\alpha] + C\,C_1 \\ M_2 &= A\,E\frac{I_2}{l}[B\,\theta_2 + C\,\theta_1 - 3\,\alpha] - C\,C_1 \end{aligned}\right\} \quad . \; . \; . \quad (1)$$

in which A, B, and C are coefficients depending upon the proportions of the haunch and $C_1 = 1/12\,w\,l^2$, as usual. Table 20 gives values of the coefficients A, B, and C for various proportions as represented by the ratio q of length of haunch to length of beam, and the ratio n of minimum to maximum moment of inertia, I_2/I_1. In the beam without haunches, $A = 2$, $B = 2$, and $C = 1$, as in the usual slope-deflection equation.

* For design methods for Beams with Haunches, see Example 1, Art. 336.

For the right end hinged

$$M_1 = A\,E\frac{I_2}{l}\left[\frac{(B+C)\,(B-C)}{B}\,\theta_1-\left(1-\frac{C}{B}\right)3\,\alpha\right]+C\left(1+\frac{C}{B}\right)C_1 \quad (2)$$

For the left end hinged

$$M_2 = A\,E\frac{I_2}{l}\left[\frac{(B+C)\,(B-C)}{B}\,\theta_2-\left(1-\frac{C}{B}\right)3\,\alpha\right]-C\left(1+\frac{C}{B}\right)C_1$$

For concentrated loads the term C_1 in eqs. (1) and (2) is replaced by $\Sigma\, D_1\, P\, l$ and $\Sigma\, D_2\, P\, l$ for the formulas for M_1 and M_2, respectively, D_1 and D_2 being coefficients depending upon the position of the concentrated load P. These coefficients are given in Table 21.

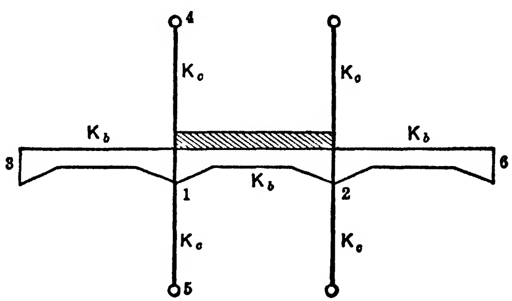


FIG. 47.

Example.—Find the centre moment in beam 1–2, Fig. 47. Assume $q = 0.25$, $I_2/I_1 = n = 0.5$; also $K_b = I_2/l = K_c = 1$; $\theta_1 = -\theta_2$; beams hinged at 3 and 6.

From Table 21, $A = 2.84$, $B = 1.91$, $C = 1.09$. The moments at joint 1 are: from eqs. (1) and (2), omitting E,

$$M_{1-2} = A\,(B-C)\,\theta_1 + C\,C_1 = 2.33\,\theta_1 + 1.09\,C_1$$

$$M_{1-3} = A\left[\frac{(B+C)\,(B-C)}{B}\right]\theta_1 = 3.65\,\theta_1$$

$$M_{1-4} = 3\,\theta_1;\ M_{1-5} = 3\,\theta_1.$$

Adding and placing equal to zero we find $\theta_1 = 0.091\,C_1$.

Then

$$M_{1-2} = [(2.33 \times 0.091) + 1.09]\,C_1 = 0.88\,C_1$$

$$M_c = (1.5 - 0.88)\,C_1 = 0.62\,C_1.$$

For a beam without haunches the results are

$$M_{1-2} = 0.82\,C_1;\ M_c = 0.68\,C_1.$$

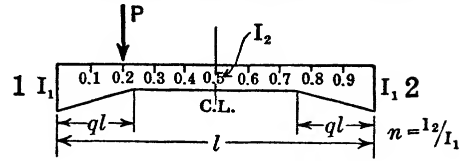
TABLE 20

VALUES OF A, B, AND C IN SLOPE-DEFLECTION EQUATIONS FOR HAUNCHED BEAMS

q		Values of $n = I_2/I_1$								q
		0.03	0.05	0.10	0.20	0.30	0.40	0.50		
0.35	A	15.293	11.857	8.152	5.460	4.274	3.579	3.112	A	0.35
	B	1.648	1.677	1.728	1.793	1.839	1.874	1.903	B	
	C	1.352	1.323	1.272	1.207	1.161	1.126	1.097	C	
0.30	A	11.195	9.229	6.847	4.894	3.958	3.385	2.989	A	0.30
	B	1.671	1.697	1.742	1.801	1.844	1.877	1.905	B	
	C	1.329	1.303	1.258	1.199	1.156	1.123	1.096	C	
0.25	A	8.137	7.071	5.638	4.312	3.618	3.169	2.848	A	0.25
	B	1.704	1.726	1.765	1.817	1.855	1.885	1.910	B	
	C	1.296	1.274	1.235	1.183	1.145	1.115	1.090	C	
0.20	A	5.953	5.400	4.589	3.750	3.267	2.938	2.692	A	0.20
	B	1.747	1.764	1.796	1.840	1.872	1.897	1.920	B	
	C	1.254	1.236	1.204	1.160	1.128	1.102	1.080	C	
0.15	A	4.416	4.145	3.716	3.226	2.920	2.698	2.525	A	0.15
	B	1.798	1.811	1.836	1.869	1.895	1.916	1.933	B	
	C	1.202	1.189	1.164	1.131	1.105	1.084	1.067	C	

TABLE 21

VALUES OF D_1 AND D_2 IN SLOPE-DEFLECTION EQUATIONS FOR HAUNCHED BEAMS, CONCENTRATED LOADING



q	n	Values of D_1 Read Down — Load Points 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	n	q
0.35	0.03	0.0957	0.1789	0.2385	0.2459	0.1871	0.1057	0.0467	0.0166	0.0035	0.03	0.35
	0.05	0.0943	0.1735	0.2271	0.2314	0.1812	0.1099	0.0523	0.0199	0.0044	0.05	
	0.10	0.0922	0.1649	0.2096	0.2115	0.1714	0.1127	0.0595	0.0244	0.0056	0.10	
	0.20	0.0896	0.1559	0.1913	0.1913	0.1595	0.1116	0.0640	0.0281	0.0068	0.20	
	0.30	0.0878	0.1489	0.1803	0.1794	0.1516	0.1092	0.0652	0.0297	0.0074	0.30	
	0.40	0.0864	0.1442	0.1724	0.1709	0.1456	0.1069	0.0654	0.0306	0.0079	0.40	
	0.50	0.0851	0.1404	0.1662	0.1644	0.1408	0.1046	0.0653	0.0312	0.0082	0.50	
0.30	0.03	0.0966	0.1825	0.2425	0.2376	0.1800	0.1032	0.0409	0.0132	0.0026	0.03	0.30
	0.05	0.0954	0.1771	0.2303	0.2257	0.1752	0.1066	0.0475	0.0167	0.0033	0.05	
	0.10	0.0932	0.1681	0.2123	0.2083	0.1671	0.1093	0.0556	0.0216	0.0048	0.10	
	0.20	0.0904	0.1578	0.1933	0.1898	0.1568	0.1091	0.0613	0.0261	9.0062	0.20	
	0.30	0.0881	0.1501	0.1810	0.1777	0.1500	0.1067	0.0628	0.0279	0.0068	0.30	
	0.40	0.0869	0.1456	0.1735	0.1704	0.1444	0.1056	0.0642	0.0297	0.0075	0.40	
	0.50	0.0856	0.1415	0.1675	0.1640	0.1399	0.1037	0.0644	0.0305	0.0079	0.50	

q	n	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	n	q
0.25	0.03	0.0973	0.1846	0.2355	0.2242	0.1719	0.1030	0.0418	0.0110	0.0020	0.03	0.25
	0.05	0.0962	0.1791	0.2253	0.2153	0.1682	0.1050	0.0471	0.0145	0.0027	0.05	
	0.10	0.0941	0.1703	0.2095	0.2015	0.1616	0.1068	0.0540	0.0193	0.0040	0.10	
	0.20	0.0912	0.1593	0.1920	0.1858	0.1531	0.1067	0.0596	0.0243	0.0056	0.20	
	0.30	0.0891	0.1522	0.1810	0.1759	0.1471	0.1055	0.0619	0.0270	0.0065	0.30	
	0.40	0.0874	0.1467	0.1731	0.1686	0.1424	0.1040	0.0630	0.0286	0.0071	0.40	
	0.50	0.0860	0.1423	0.1668	0.1628	0.1385	0.1026	0.0635	0.0298	0.0077	0.50	
0.20	0.03	0.0978	0.1837	0.2213	0.2085	0.1632	0.1032	0.0464	0.0107	0.0015	0.03	0.20
	0.05	0.0968	0.1786	0.2138	0.2023	0.1603	0.1041	0.0498	0.0136	0.0022	0.05	
	0.10	0.0947	0.1701	0.2017	0.1922	0.1553	0.1048	0.0546	0.0182	0.0034	0.10	
	0.20	0.0918	0.1594	0.1873	0.1799	0.1486	0.1045	0.0590	0.0232	0.0049	0.20	
	0.30	0.0897	0.1526	0.1779	0.1718	0.1438	0.1036	0.0610	0.0260	0.0060	0.30	
	0.40	0.0879	0.1469	0.1710	0.1657	0.1399	0.1025	0.0621	0.0278	0.0067	0.40	
	0.50	0.0864	0.1425	0.1653	0.1606	0.1366	0.1013	0.0628	0.0291	0.0073	0.50	
0.15	0.03	0.0980	0.1756	0.2036	0.1920	0.1541	0.1029	0.0520	0.0143	0.0013	0.03	0.15
	0.05	0.0970	0.1716	0.1987	0.1880	0.1520	0.1031	0.0538	0.0164	0.0019	0.05	
	0.10	0.0950	0.1647	0.1903	0.1811	0.1484	0.1031	0.0565	0.0198	0.0030	0.10	
	0.20	0.0922	0.1559	0.1798	0.1724	0.1434	0.1025	0.0594	0.0237	0.0045	0.20	
	0.30	0.0900	0.1498	0.1726	0.1663	0.1398	0.1017	0.0609	0.0260	0.0056	0.30	
	0.40	0.0882	0.1451	0.1670	0.1616	0.1368	0.1009	0.0618	0.0277	0.0064	0.40	
	0.50	0.0866	0.1412	0.1625	0.1576	0.1343	0.1000	0.0623	0.0289	0.0070	0.50	
q	n	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	n	q

Load Points

VALUES OF D_2 READ UP

Torsional Stresses in Marginal Beams

264. General Formulas.—Torsional stresses in marginal beams due to the deflection of adjacent loaded panels is a matter deserving of consideration, although suitable methods of analysis of such stresses or of design of members have not as yet been provided for in general specifications or rules of practice. Here again any exact analysis is difficult, but certain simplified cases can be worked out which will serve to indicate the conditions under which such stresses will be large and also their probable magnitude. Fig. 48 represents a beam and girder design in which B is the marginal girder or beam and A is the cross-beam framed into B at some point between the supporting columns. A load on A produces a deflection and an angular twist on beam B with resulting torsional stresses. The resisting moment at the point of connection depends upon the relative bending flexibility of beam A, the torsional flexibility of beam B, and the angular movement of the columns to which beam B is attached. This problem will be analyzed on two different assumptions as represented in Fig. 49 (a) and (b). In (a) the beam A will be assumed as simply supported at 2 and fixed at 3; in (b) it will be assumed as fixed at 2. The actual conditions will be somewhere between these two extremes.

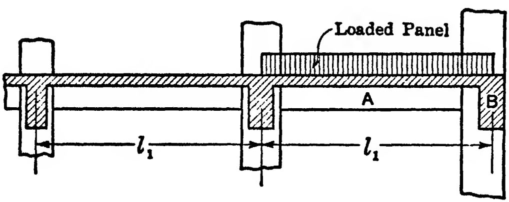


FIG. 48.

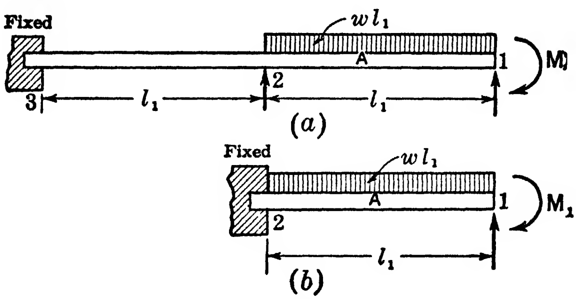


FIG. 49.

Notation:

l_1 = length of beam A;
l_2 = length of beam B;
b and h = dimensions of beam B;
I_1 = moment of inertia of beam A;
E_s = shearing modulus of elasticity of concrete;
T = torsional torque or moment;
$K = I_1/l_1$;
θ_1 = angular movement at 1, the point of connection;
θ_c = angular movement of columns to which B is attached.

The angle of twist θ_1 at joint 1 will be determined first in terms of the moment M_1 and dimension of beam A, and then in terms of the torsional rigidity of beam B and the column movement θ_c. Equating these values will enable the value M_1 to be determined in terms of load, the dimensions of beams A and B and of the column twist θ_c.

Referring to Fig. 49 (a) we have, from the slope deflection equations, joint 2,

$$8\,E\,K\,\theta_2 + 2\,E\,K\,\theta_1 + C_1 = 0$$

in which $C_1 = 1/12\,w\,l_1^2$, w being the load per lineal foot on beam A. Also

$$-M_1 = 4\,E\,K\,\theta_1 + 2\,E\,K\,\theta_2 - C_1$$

(M_1 is shown as a negative moment.) From these we get

$$\theta_1 = \frac{5\,C_1 - 4\,M_1}{14\,E\,K} \quad \quad (1)$$

From Fig. 49 (b) we have at once $-M_1 = 4\,E\,K\,\theta_1 - C_1$, whence

$$\theta_1 = \frac{C_1 - M_1}{4\,E\,K} \quad \quad (2)$$

The angle of torsional twist for rectangular sections is given by the formula

$$\theta_t = \frac{3.33^{*}\,T\,l}{E_s} \cdot \frac{b^2 + h^2}{b^3\,h^3} \quad \quad (3)$$

where l = length of beam subjected to a uniform torque T.

* See Maurer and Withey's "Strength of Materials." Some authorities give somewhat higher values. Professor Young in his investigations uses the factor 3.57.

The relation between T and M_1, and the value of l in eq. (3), depends upon the number of beams in a panel. For a single beam at the centre of B, $T = \frac{1}{2} M_1$ and $l = \frac{1}{2} l_2$; for two intermediate beams, $T = M_1$ and $l = \frac{1}{3} l_2$; for a larger number of beams; T may be taken equal to M_1 and l equal to the spacing of cross-beams, although this does not exactly represent the conditions.

The value of E_s is theoretically equal to $\dfrac{E}{2(1+\lambda)}$ where E = usual compression modulus and λ = Poisson's ratio, equal to about 1/10. It will be sufficiently accurate for our purposes to take $E_s = 0.42\,E$ to $0.45\,E$ * and to write eq. (3)

$$\theta_t = \frac{8\,T\,l}{E} \cdot \frac{b^2 + h^2}{b^3\,h^3}. \quad \quad (4)$$

The total angular movement of beam B will be $\theta_t + \theta_c$, which is equal to θ_1 of eq. (1) or (2). Using eq. (1) and solving for M_1, we get

For a single beam at centre of B,

$$M_1 = \frac{5\,C_1 - 14\,E\,K\,\theta_c}{4 + 28\,\dfrac{l_2}{l_1}\,I_1 \cdot \dfrac{b^2 + h^2}{b^3\,h^3}}. \quad \quad (5)$$

For two beams (three panels),

$$M_1 = \frac{5\,C_1 - 14\,E\,K\,\theta_c}{4 + 37\,\dfrac{l_2}{l_1}\,I_1 \cdot \dfrac{b^2 + h^2}{b^3\,h^3}}. \quad \quad (6)$$

From eq. (2), for a fixed condition at the interior end of beam A,

For a single beam at centre of B,

$$M_1 = \frac{C_1 - 4\,E\,K\,\theta_c}{1 + 8\,\dfrac{l_2}{l_1}\,I_1 \cdot \dfrac{b^2 + h^2}{b^3\,h^3}}. \quad \quad (7)$$

For two beams (3 panels),

$$M_1 = \frac{C_1 - 4\,E\,K\,\theta_c}{1 + 10.7\,\dfrac{l^2}{l_1}\,I_1 \cdot \dfrac{b^2 + h^2}{b^3\,h^3}}. \quad \quad (8)$$

---

* Professor Young determined E_s for 1 : 2 : 4 concrete to be about 1,500,000 lbs/in.^{2}

For a flat-slab construction the problem is less definite. Some useful estimate may be made by considering the central portion of the slab, say one-third of the width, and treating this as a single beam. The condition as to fixity represented by eq. (1) will be more accurate than that represented by (2). Using eq. (1), and assuming $l = l_2/3$, we derive for the value of M_1

$$M_1 = \frac{5\,C_1 - 14\,E\,K\,\theta_c}{4 + 18.7\,\frac{l_2}{l_1}\,I_1\,\frac{b^2 + h^2}{b^3\,h^3}}. \quad . \quad . \quad . \quad . \quad . \quad (9)$$

In this case the value of w in the formula $C_1 = 1/12\,w\,l^2$ is the load per foot assumed as carried by the portion of the flat slab considered. This will be not more than about one-half the average load on the panel.

The analysis here given is obviously far from exact. In particular it ignores the horizontal rigidity of the floor system which prevents the beam B from deflecting horizontally at the top, which is assumed in calculating torsional twist. Its deflection must be wholly at the bottom. There is, however, little resistance to this action, so that the final results are probably not greatly affected. It is useful to note that in eqs. (5) to (9) the second term in the numerator represents the effect of column twist, and the second term in the denominator represents the effect of the torsional flexibility of beam B. Omitting both these terms gives the value of M_1 for a fixed-end condition.

265. Value of Column Deflection θ_c.—In Art. 256 an analysis is made of end moments in a beam connected to an exterior column. The value of angular twist (θ_2 in Art. 256) is obtained by the elimination of θ_1 in the two joint equations. We get from this

$$\theta_c = \frac{3 + 4\,K_c/K_b}{E\,K_b\,[8 + 44\,K_c/K_b + 48\,(K_c/K_b)^2]}\,C_1. \quad . \quad . \quad (10)$$

(For present purposes the quantity E, which had been omitted in the analysis of Art. 256, must be restored.) In eq. (10), K_b refers to the beam and K_c to the column. $C_1 = 1/12\,w^2$, where w = load per foot on the beam connected to the column. It may or may not be the same value as used in eqs. (5) to (9).

The influence of the relative rigidities of beam and column on θ_c is large. For example,

$$\text{For } K_b = \tfrac{1}{2} K_c, \quad \theta_c = 0.038 \frac{C_1}{E K_b}$$

$$K_b = K_c, \quad \theta_c = 0.07 \frac{C_1}{E K_b}$$

$$K_b = 2 K_c, \quad \theta_c = 0.119 \frac{C_1}{E K_b}.$$

In calculating θ_c, in a beam and girder type of construction, it is evident that the loads on the intermediate beams will have some effect on the column deflection, but much less than the load on the beam connected directly to the column. A fair estimate can be made by assuming one-half of the panel as rigidly attached to the column and the load per lineal foot and the value of K_b taken accordingly.

For the flat-slab type the column strip is relatively rigid and a width of two-thirds of the panel width may be taken as controlling the column deflection.

266. Stress Due to Torsional Moments.—The maximum shearing stress v in a rectangular section, due to torsion, is

$$v = T \frac{1.8\, b + 3\, h}{b^2 h^2} \qquad (11)$$

The value of T is equal to M, or $\tfrac{1}{2} M$ as already indicated. The maximum stress occurs at the centre of the long side of a rectangular section.

267. Examples. 1. Beam and girder design; three panels. Spans $l_1 = l_2 = 18$ ft. Floor slab = 4 in., beam A below slab = 10 in. by 14 in., beam B below slab = 12 in. by 20 in., $w = 300$ lbs/ft.^{2} = 1800 lbs/ft. on beam A. Columns are 20 in. by 20 in. with 1.5% reinforcement. Use eq. (6).

Solution. Calculate first θ_c, eq. (10). The value of I for one beam and 6 ft. of floor slab = 10,700 in.^{4} For one-half the panel it is $1\tfrac{1}{2} \times$ 10,700 = 16,000 in.^{4} The value of I for the column is also about 16,000 in.^{4} If the length of column = 12 ft., then $K_c = 1.5\, K_b$ and eq. (10) gives $\theta_c = 0.05\, C_1/E K_b$.

In this equation C_1 and K_b pertain to a half panel, and their ratio will be the same as the ratio C_1/K_b for a single beam so that the value of θ_c can be inserted directly in eq. (6).

In eq. (6) the quantity $\frac{b^2 + h^2}{b^3 h^3} = \frac{1}{33{,}000}$.

Then we have

$$M_1 = \frac{(5 - 0.7)\,C_1}{4 + 37 \times \frac{10{,}700}{33{,}000}} = \frac{4.3\,C_1}{4 + 12} = 0.27\,C_1$$

$$C_1 = 1/12 \times 300 \times 6 \times 18^2 \times 12 = 584{,}000 \text{ in-lbs.}$$

Hence

$$M_1 = 157{,}000 \text{ in-lbs.}$$

Then from eq. (11)

$$T = M_1$$

and

$$v = 157{,}000 \times \frac{72 + 21.6}{12^2 \times 24^2} = 177 \text{ lbs/in.}^2$$

Note the effect of column deflection is to reduce the moment 7/50 or about 14%. The flexibility of beam B has a large effect as indicated by the term 12 in the denominator.

If eq. (8) be used, $M_1 = 0.18\,C_1$ and $v = 117$ lbs/in.²

2. Flat slab. 9 in. thick; same load as in (1). Also assume same column as in example 1.

For calculating θ_c use ⅔ of panel. $I_b = 8800$ in.⁴ and $K_c/K_b = 2.7$. $\theta_c = 0.029\,C_1/E\,K_b$, a very small value. In this case the load per square foot assumed in calculating C_1 is twice that in C_1 of eq. (9), hence in applying eq. (9) we get

$$M_1 = \frac{(5 - 14 \times 2 \times 0.029)\,C_1}{4 + 18.7 \times \frac{4400}{33{,}000}} = \frac{(5 - 0.81)\,C_1}{4 + 2.5} = 0.64\,C_1.$$

The value of $C_1 = \frac{150 \times 6 \times 18^2 \times 12}{12} = 292{,}000$ in-lbs. and $M_1 =$ 187,000 in-lbs. Shearing-stress $v = 210$ lbs/in.² Note the effect of the flexibility of the flat slab in increasing the value of M_1 in terms of the load on the slab.

268. Effect of Form and Size of Marginal Beam.—Suppose in example 1 that $h = 12$ in., $b = 12$ in. The value of the term $\frac{b^2 + h^2}{b^3 h^3}$ becomes 1/10,400 and $M_1 = \frac{4.3}{4 + 37}\,C_1 = 0.105\,C_1 = 61{,}000$ in-lbs.

$v = 166$ lbs/in.^{2} The decrease in depth decreases the torque greatly, but the actual shearing stress only about 7%. If the area $b\,h$ remain constant and $b = h = 17$ in., the value of $M_1 = \dfrac{4.3}{4 + 37 \times \dfrac{10{,}700}{42{,}000}} C_1 =$

$\dfrac{4.3}{4 + 9.4} C_1 = 0.32\,C_1 = 187{,}000$ in-lbs. $v = 187{,}000 \times \dfrac{4.8}{17^3} = 183$ lbs/in^2. Changing proportions, but retaining the same total section, changes maximum stress comparatively little.

269. Character of Shearing-Stresses Due to Torsion.—The shearing-stresses due to torsion act in a manner similar to those due to vertical load. At the centres of the sides, where these stresses are a maximum, the accompanying diagonal tensile and compressive stresses are a maximum at 45° and are of equal intensity to the shearing-stresses. Tension failures would occur in a spiral direction around the member, and reinforcement for these stresses would naturally consist of spirally arranged wire wrapped around the member and placed near the surfaces. On the inner surface of the marginal beam, these diagonal stresses act in the same direction as those due to vertical shear, and the reinforcement for the vertical shear is also of aid against torsional stresses. On the outside surface the two sets of diagonal stresses act in opposite directions and tend to neutralize each other.

270. Tests of Torsional Strength of Concrete.—Tests conducted by Messrs. Young, Sagar, and Hughes * gave an average shearing strength of rectangular beams 5 in. by 5 in., 5 in. by 7½ in., and 5 in. by 10 in., all 5 ft. long, 1 : 2 : 4 concrete (crushing strength 1700 lbs/in.^{2}) as follows:

	Maximum Shearing-Stress Lbs/in.^{2}, Average
No reinforcement	536
Longitudinal reinforcement only	541
Longitudinal and 4 spirals in a 5-ft. length	645
Longitudinal and 8 spirals in a 5-ft. length	789

* Bul. No. 3, 1922, Univ. of Toronto, School of Engineering Research.

Tests made for the Deutscher Ausschuss für Eisenbeton quoted in the same bulletin gave values of about 500 lbs/in.^{2} for rectangular beams with no spiral reinforcement and 1160 lbs/in.^{2} for rectangular beams with 0.5% spiralling. This was 1 : 2 : 3 concrete of a compressive strength of 3530 lbs/in.^{2}

From these tests it would appear that the torsional strength of unreinforced concrete, as measured by the maximum shearing-stress, is at least double the strength shown by beams under transverse shear.

271. **Conclusions.**—In drawing conclusions from the foregoing analysis it should be borne in mind that the calculated stresses are based upon the assumption of elastic behavior, and the resulting stresses are primarily a function of distortion. They are " deformation " stresses, similar to the secondary stresses in steel trusses, and are strictly limited by the deformations of the surrounding structure. Under these circumstances a stress of 40% of the ultimate is not to be feared. The chief objection to such stresses is their effect upon the safety of the beam against diagonal tension failures due to vertical load. On this account it would appear advisable to use relatively low shearing unit stresses in marginal beams and to place ample reinforcement near the inside surface of such beams. Such reinforcement is most effective if placed at 45° to the horizontal axis. Special cases where long beams frame into girders a short distance from columns should be given special consideration, and the portion of girder between beam and column should be well reinforced, preferably by diagonally placed reinforcement.

Analysis of Frames by Method of Moment Distribution

272. **The Principle of Moment Distribution.**—A method of reaching results by successive approximations, called the method of moment distribution, developed largely by Professor Hardy Cross,* has certain advantages over the method of algebraic elimination previously described. It is easy to remember, requires no writing out or manipulation of formulas, and deals with numerical values of moments throughout. Its development and use will be explained.

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* Proc. Am. So. C. E., May, 1930.

Fig. 50 represents four (any other number may be taken) members connected together rigidly at 1 and fixed at their far ends. Imagine an external moment M applied at the joint 1. This will cause certain bending moments in all the members at each end, which can be determined by use of the equations developed in Art. 245. In this case, for equilibrium, the sum of the bending moments in all the members at joint 1 must balance the moment M. From this condition the twist angle θ_1 at 1 can be determined. Since all members are assumed as fixed at the far ends the twist angles θ are zero at these points. Hence for joint 1 we have from eq. (6), Art. 245:

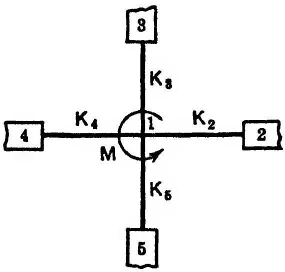


FIG. 50.

$$4\,(K_2 + K_3 + K_4 + K_5)\,\theta_1 + M = 0$$

whence

$$\theta_1 = -\frac{M}{4\,(K_2 + K_3 + K_4 + K_5)}. \qquad (1)$$

Then

$$\left.\begin{aligned} M_{1\text{-}2} &= 4\,K_2\,\theta_1 = -M\frac{K_2}{K_2 + K_3 + K_4 + K_5} \\ M_{1\text{-}3} &= -M\frac{K_3}{K_2 + K_3 + K_4 + K_5}, \\ &\text{etc.} \qquad\qquad \text{etc.} \end{aligned}\right\} \qquad (2)$$

The sum of the four moments is equal to $-M$.

From an inspection of these several values it is seen that they are proportional to the values of K for the several members. Hence the general relation that with the far ends fixed *the applied moment M is distributed among the several members in proportion to the values of K, or in proportion to the relative rigidities of the members.*

If one of the members is hinged at the far end, as at 2, then, from Art. 243, it is seen that the multiplier of K_2 in eq. (6) will be 3 and not 4. This may be taken into account most readily by using three-fourths of the value of K for any hinged-end member in applying the foregoing rule of distribution.

273. Unbalanced Moment Due to Loads.—Fig. 51 shows one member loaded. From Art. 247 the equation for joint 1 will be $4(K_2 + K_3 + K_4 + K_5)\,\theta_1 + C_1 = 0$, where C_1 is the moment in 1–2 at 1, considering the beam as fixed at the ends.

Then

$$\theta_1 = -\frac{C_1}{4(K_2 + K_3 + K_4 + K_5)} \quad \cdots \cdots \quad (3)$$

and

$$\left.\begin{aligned} M_{1-2} &= 4\,\theta_1 K_2 + C_1 = C_1 - C_1 \frac{K_2}{K_2 + K_3 + K_4 + K_5} \\ M_{1-3} &= -C_1 \frac{K_3}{K_2 + K_3 + K_4 + K_5} \\ M_{1-4} &= -C_1 \frac{K_4}{K_2 + K_3 + K_4 + K_5} \\ M_{1-5} &= -C_1 \frac{K_5}{K_2 + K_3 + K_4 + K_5}. \end{aligned}\right\} \quad (4)$$

Comparing these results with those obtained for the external moment M, it is seen that the moment C_1 can be treated exactly as an external moment. For the loaded member the total moment is found by adding the value so obtained to the moment C_1.

If other members are loaded then the algebraic sum of all the moments C_1 or C_2 at 1 is the quantity to be distributed.

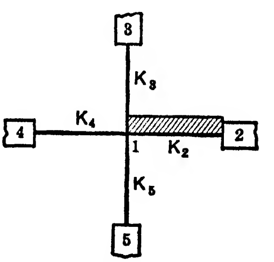


FIG. 51.

Example.—Fig. 52. Beams 1–2 and 1–4 are loaded as shown, hinge at 2. Relative values of K shown in circles.

For 1–2, $C_1 = 1/12\,w\,l^2 = 80{,}000$ ft-lbs., and, the end being hinged, the moment to be considered is $1\tfrac{1}{2}C_1 = 120{,}000$ ft-lbs. For 1–4, $C_2 = 40{,}000$ ft-lbs. Fig. (*b*) shows the moments in thousands of foot-pounds, with the proper signs for convenient use in balancing moments.

The sum of the moments at 1 is +80, which is the moment to be balanced and distributed. Using $\tfrac{3}{4}K$ for 1–2, the value of $\Sigma K = 11$, hence the distributed moments are as follows:

$$1\text{–}2 = -80 \times 3/11 = -21.8$$
$$1\text{–}4 = -80 \times 4/11 = -29.1$$
$$1\text{–}3 = 1\text{–}5 = -80 \times 2/11 = -14.5.$$

Writing these values along the members and adding, the final moments are

$$
\begin{aligned}
M_{1-2} &= +120 - 21.8 = +98.2 \\
M_{1-4} &= -40 - 29.1 = -69.1 \\
M_{1-3} &= \qquad\qquad -14.5 \\
M_{1-5} &= \qquad\qquad -14.5 \\
\text{Total} &= \qquad\quad +0.1
\end{aligned}
$$

274. Moments at the Far Ends of the Members.—When the moment M of Art. 272 or the moment of $+80$ of the foregoing example is distributed among the members at joint 1, there will result certain bending moments at the far ends of the members, except where hinged. Where fixed, these far end moments will be one-half the distributed moments at the central joint, as readily shown by the moment equations for any member. Thus for 1–3, Fig. 52, $\theta_3 = 0$, and hence

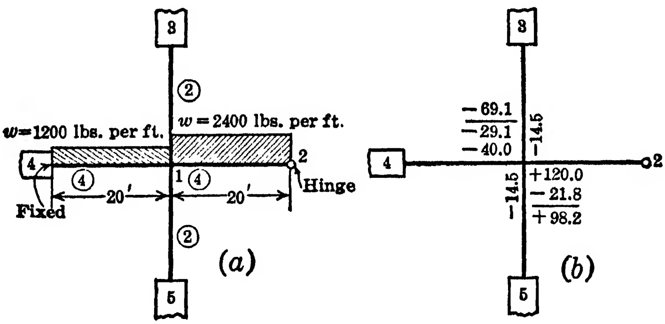


FIG. 52.

$$M_{1-3} = 4\,E\,K\,\theta_1 \quad \text{and} \quad M_{3-1} = 2\,E\,K\,\theta_1$$

hence

$$M_{3-1} = \tfrac{1}{2}\,M_{1-3}$$

A useful conception of the process above described is to imagine the joint 1 held fast at first, making the beams 1–2 and 1–4 truly fixed at 1 and hence carrying the moments of $+120$ and -40 as shown. Then imagine this joint released and allowed to twist until the bending moments are equalized. This action results in "distributing" the unbalanced moment of 80 among the members in proportion to the values of K (or $\frac{3}{4}K$ when hinged at the far end). To the moments so distributed are added the original moments of +120 and −40 in the members 1–2 and 1–4. The release of joint 1 at the same time causes moments at the far ends, where fixed, of one-half the distributed moments at the joint 1.

275. Structure of Several Joints.—Fig. 53 shows a part of a building frame loaded to give approximately the maximum moment at 1 in beam 1–2. The structure is limited by assuming hinged joints as shown. Assume relative values of K as shown in Fig. 54 by the figures in circles. Load on spans 1–2 and 1–3 = 2000 lbs/ft. The moments at joint 1 will be calculated by both the slope-deflection and the moment-distribution methods.

Slope-Deflection Method.—There are three interior joints, 1, 2,

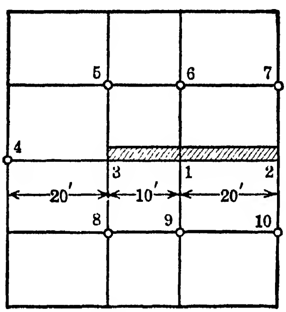


FIG. 53.

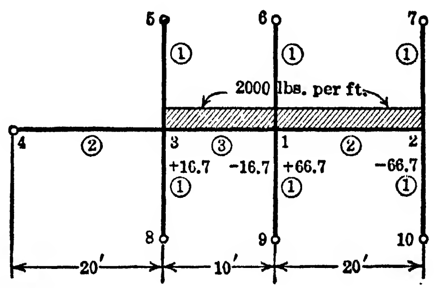


FIG. 54.

and 3, for which the twist angles θ are to be determined. For span 1–2, $C_1 = C_2 = 1/12 \times 2000 \times 20^2 = 66{,}700$ ft-lbs.; for span 3–1, $C_1 = C_2 = 16{,}700$ ft-lbs. Use the thousand foot-pound units 66.7 and 16.7. The equations are:

$$\text{Joint 1: } (8 + 3 + 12 + 3)\,\theta_1 + 4\,\theta_2 + 4\,\theta_3 + 66.7 - 16.7 = 0$$

or

$$26\,\theta_1 + 4\,\theta_2 + 4\,\theta_3 = -\,50 \quad \text{. . . . . . . } (a)$$

$$\text{Joint 2: } (3 + 3 + 8)\,\theta_2 + 4\,\theta_1 - 66.7 = 0$$

or

$$14\,\theta_2 + 4\,\theta_1 = 66.7 \quad \text{. . . . . . . } (b)$$

$$\text{Joint 3: } (12 + 3 + 6 + 3)\,\theta_3 + 6\,\theta_1 + 16.7 = 0$$

or

$$24\,\theta_3 + 6\,\theta_1 = -\,16.7 \quad \text{. . . . . . } (c)$$

Although these three equations are readily solved directly for the unknowns, it will be instructive to use the method of successive approximations. Then for the first approximation we have from (*a*), assuming θ_2 and $\theta_3 = 0$ (fixed ends):

$$\theta_1 = -\frac{50}{26} = -1.92.$$

Likewise from (*b*) and (*c*)

$$\theta_2 = \frac{66.7}{14} = +4.75$$

$$\theta_3 = \frac{-16.7}{24} = -0.69.$$

These are the first approximate values.

In the second approximation substitute these values of θ_2 and θ_3 in (*a*) getting

$$\theta_1 = \frac{-50 - 4 \times 4.75 + 4 \times 0.69}{26} = -2.54.$$

Likewise

$$\theta_2 = \frac{66.7 + 4 \times 1.92}{14} = +5.32$$

$$\theta_3 = \frac{-16.7 + 6 \times 1.92}{24} = -0.21.$$

Repeating the process we get for the third approximations

$$\theta_1 = -2.70; \; \theta_2 = +5.49; \; \theta_3 = -0.07$$

and for the fourth

$$\theta_1 = -2.75; \; \theta_2 = +5.53; \; \theta_3 = 0$$

(Values found by direct elimination are -2.78; $+5.55$; and 0.)

The third approximations are quite accurate enough, and using these values for the moments in 1–2 we have

$$M_{1-2} = 8\,\theta_1 + 4\,\theta_2 + 66.7 = -21.6 + 21.9 + 66.7 = +66.4$$
$$= +66{,}400 \text{ ft-lbs.}$$

$$M_{1-3} = 12\,\theta_1 + 6\,\theta_3 - 16.7 = -32.4 - 0.4 - 16.7 = -49.5$$
$$= -49{,}500 \text{ ft-lbs.}$$

The bending moments in columns 1–6 and 1–9 are

$$M_{1-6} = M_{1-9} = -\ 8100 \text{ ft-lbs.}$$

Checking totals at 1, $+66.4 - 49.5 - 8.1 - 8.1 = +\ 0.7$. As this total should be zero, the degree of approximation is indicated.

Moment-Distribution Method.—Fig. 55 shows the frame with the values of C_1 and C_2 written adjacent to the members 3–1 and 1–2, using signs in accordance with the direction of bending. These would be the actual moments if all joints were rigidly held from turning. Let joint 1 be now released so it will turn until the moments are balanced. The unbalanced moment is $66.7 - 16.7 = +\ 50.0$, and

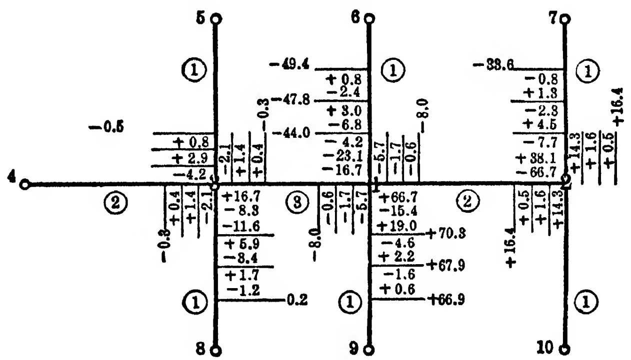


FIG. 55.

the moment of -50 will be distributed to the four members in proportion to their rigidities, measured by the value of K for 1–2 and 1–3 and by ¾ K for the columns. Distributing this moment in the proportions 2, ¾, 3 and ¾, we get the values -15.4, -5.75, -23.1, and -5.75, which are conveniently written below or above the respective member for horizontal members, and on the right and left for vertical members.

Then release joint 2, distributing the unbalanced moment of -66.7, giving the values $+38.1$, $+14.3$, and $+14.3$ as shown; also then joint 3. Finally there must be written down the moments at the far ends of the members arising from the release of each joint, the "carry over" moment. These moments are one-half of those produced at the joint in question. Thus for the members radiating from joint 1, the moment of -15.4 in 1–2 will result in a moment of -7.7 at 2; the moment of -23.1 in 1–3 will result in a moment of -11.6 in 3–1; the moment of $+38.1$ in 2–1 results in a moment of $+19.0$ in 1–2, and the moment of -8.3 in 3–1 results in a moment of -4.2 in 1–3. All these far end moments are written down in Fig. 55 to the nearest tenth. This completes one approximation, and for clearness a line is drawn below or above the last figures thus obtained.

The total values of moments now shown are the same as would be found by using the first approximations of θ obtained in Art. 275. Thus $M_{1-2} = 66.7 + 8\,\theta_1 + 4\,\theta_2 = 66.7 - 15.4 + 19.0 = 70.3$; $M_{1-6} = 3\,\theta_1 = -\ 5.76$; $M_{1-3} = -\ 16.7 + 12\,\theta_1 + 6\,\theta_3 = -\ 16.7 - 23.1 - 4.2 = -\ 44.0$. The several quantities are identical.

An inspection of Fig. 55 shows that the total moments at the joints are now unbalanced. The unbalanced moment at joint 1 is

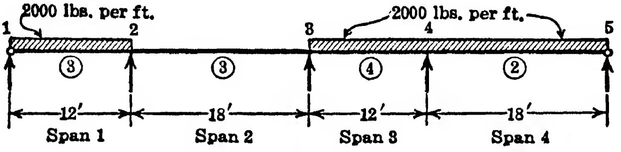


FIG. 56.

equal to $+19.0 - 4.20 = +\ 14.8$; at 2 it is -7.7; and at 3 it is -11.6, these being the half moments or "carry over" moments caused by releasing joints surrounding the one in question.

The second approximation is now obtained by distributing these unbalanced moments in the same manner as in the first process. The total moment of M_{1-2} is now $70.3 - 4.6 + 2.2 = -\ 67.9$. The second approximations by the slope-deflection method give $M_{1-2} = -\ 66.7 - 8 \times 2.54 + 4 \times 5.32 = -\ 67.7$. For the third approximation the moment $M_{1-2} = +\ 66.9$ and $+66.4$ by the two methods. Further distribution results in successive values of M_{1-2} of -66.8 and -66.7. The number of successive distributions required for the desired accuracy depends upon the proportion of the structure and load distribution. This can easily be determined by noting the changes as the work progresses.

276. The Continuous Girder.—This is but a special case of the general one already treated. For purposes of comparison an example will be solved by three methods: (*a*) by the three-moment equations, (*b*) by the slope-deflection equations, and (*c*) by moment distribution.

Fig. 56 shows a beam of four spans loaded for maximum moment at 4. Hinged at 1 and 5. Values of K as shown.

(*a*) *Method of Three-Moment Equation*, eq. (1), Art. 227.—For spans 1 and 3, $\frac{1}{4}\,w\,l^2 = 72{,}000$ ft-lbs.; for span 4, $\frac{1}{4}\,w\,l^2 = 162{,}000$ ft/lbs. Applying the equation to spans 1 and 2,

$$2\,M_2\,(\tfrac{1}{3} + \tfrac{1}{3}) + \tfrac{1}{3}\,M_3 = -\,72/3.$$

To spans 2 and 3,

$$\tfrac{1}{3}\,M_2 + 2\,M_3\,(\tfrac{1}{3} + \tfrac{1}{4}) + \tfrac{1}{4}\,M_4 = -\,72/4.$$

To spans 3 and 4,

$$\tfrac{1}{4}\,M_3 + 2\,M_4\,(\tfrac{1}{4} + \tfrac{1}{2}) = -\,72/4 - 162/2.$$

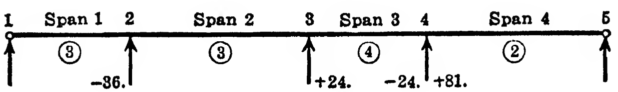


FIG. 57.

Simplifying, these equations become

$$4\,M_2 + M_3 = -\,72$$

$$4\,M_2 + 14\,M_3 + 3\,M_4 = -\,216$$

$$M_3 + 6\,M_4 = -\,396.$$

Solving, we get $M_4 = -\,66{,}700$ ft-lbs. Also $M_3 = +\,4300$ ft-lbs., $M_2 = -\,19{,}100$ ft-lbs. In this analysis a minus sign indicates a negative bending moment or tension in top fibres.

(*b*) *Slope-Deflection Method.*—For spans 1 and 3, $C_1 = C_2 = 1/12 \times 2000 \times 12^2 = 24{,}000$; for span 4, $C_1 = 54{,}000$. Fig. 57 shows the proper values written on the members, taking account of the hinged ends at 1 and 5.

The equations are:

Joint 2: $(12 + 9)\,\theta_2 + 6\,\theta_3 - 36 = 0$

Joint 3: $(12 + 16)\,\theta_3 + 6\,\theta_2 + 8\,\theta_4 + 24 = 0$

Joint 4: $(16 + 6)\,\theta_4 + 8\,\theta_3 + 81 - 24 = 0$

or

$$21\,\theta_2 + 6\,\theta_3 = 36$$
$$6\,\theta_2 + 28\,\theta_3 + 8\,\theta_4 = -\,24$$
$$8\theta_3 + 22\,\theta_4 = -\,57.$$

The solution gives:

$$\theta_4 = -\,2.37$$
$$\theta_3 = -\,0.59$$
$$\theta_2 = +\,1.89.$$

Then $M_{4-5} = 6\,\theta_4 + 81 = -\,14.22 + 81 = +\,66.8$ or 66,800 ft-lbs.

As a check:

$$M_{4-3} = 16\,\theta_4 + 8\,\theta_3 - 24 = -\,37.9 - 4.8 - 24 = -\,66.7.$$

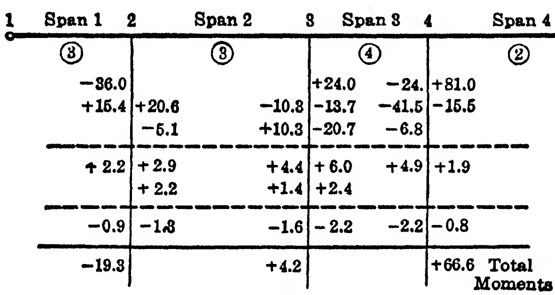


FIG. 58.

Also $M_{2-3} = +\,19{,}100$ ft-lbs., $M_{3-4} = -\,4400$ ft-lbs. Here a plus sign indicates right-handed bending around the joint and hence for a moment on the right of the joint it indicates tension on top fibres.

(*c*) *Moment-Distribution Method.*—The work is fully given in Fig. 58 up to the third distribution. Total moments are also shown which are seen to be very nearly correct. In other cases the same degree of precision might require a further distribution.

277. The Moment Distribution Method Applied to Beams with Haunches.—Referring to eq. (1), Art. 263, and assuming the right end fixed, we have, in general, omitting *E*,

$$M_{1-2} = A\;K\,B\,\theta_1 + C\,C_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where *A*, *B*, and *C* are coefficients given in Table 20. If several members meet at joint 1 and are fixed at the far ends, there would be an equation like (5) for each member. The sum of the moments at 1 would be zero, hence we may write

$$\Sigma A\ K B \times \theta_1 + \Sigma C C_1 = 0$$

or

$$\theta_1 = -\frac{\Sigma C C_1}{\Sigma A\ K B}$$

and any moment M_{1-2} is

$$M_{1-2} = -A\ K B \frac{\Sigma C C_1}{\Sigma A\ K B} + C C_1 \quad . \quad . \quad . \quad . \quad (6)$$

Considering $\Sigma C C_1$ as the total applied moment, it is seen that the amount taken by member 1–2 is in proportion to $A\ K B/\Sigma A\ K B$, and that the total moment on 1–2 is equal to this distributed moment plus the end moment $C C_1$ on this beam. (Compare with the equations in Art. 272.) Hence for beams with haunches, each value of $K = I_2/l$ is to be first multiplied by $A B$ for that member, then the applied moments $\Sigma C C_1$ distributed in proportion to this modified value of K. For members without haunches the multiplier is 4 for fixed-end and 3 for hinged-end members. As can be seen from eq. (6), the quantity $C C_1$ is the end moment considering the beam fixed at both ends.

Having distributed the moments at joint 1, the resulting moments at the far ends of the members are found by multiplying the distributed moments for the several members by the quantity C/B for that member in place of the usual ½. (This relation is obtained from eq. (1), Art. 263, by placing θ_2 and $C_1 = 0$.)

If the far end of any member is hinged, multiply K for that member by $A\ (B + C)(B - C)/B$ in place of $A\ B$.

278. Exact Method of Moment Distribution for Continuous Girders. —By applying suitable coefficients to the values of K it is possible to obtain correct results from the first distribution. This method is especially advantageous in solving the general problem of the continuous girder of several spans where it is desired to determine the moments at each support for a load in each span separately. This method will now be explained.

Fig. 59 shows a beam of 6 spans with an external moment M applied at 3. Calculate first the coefficients a and b for each support as described in Art. 239. Their values are given by the following general formulas:

$$a_n = \frac{K_{n-1}/K_n}{2\,(1 + K_{n-1}/K_n) - a_{n-1}},$$

$$b_n = \frac{K_n/K_{n-1}}{2\,(1 + K_n/K_{n-1}) - b_{n-1}}.$$

They represent the ratios of two successive bending moments to the left and right respectively of the loaded portion of the beam. Following the sign convention used in the slope-deflection equations we thus have $M_{4-3} = + b_4 M_{3-4}$, and $M_{2-3} = + a_2 M_{3-2}$.

From the slope-deflection equations for span 3–4, we have, neglecting E,

$$M_{3-4} = 4\,K_3\,\theta_3 + 2\,K_3\,\theta_4 \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

$$M_{4-3} = 4\,K_3\,\theta_4 + 2\,K_3\,\theta_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

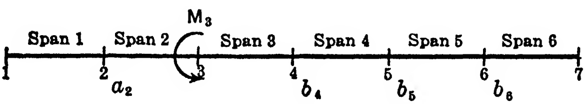


FIG. 59.

From (b) we have $2\,K_3\,\theta_4 = \dfrac{M_{4-3}}{2} - K_3\,\theta_3$, and substituting in (a)

$$M_{3-4} = 3\,K_3\,\theta_3 + \frac{M_{4-3}}{2} = 3\,K_3\,\theta_3 + \frac{b_4}{2}\,M_{3-4}$$

from which

$$M_{3-4} = \frac{3\,K_3\,\theta_3}{1 - \dfrac{b_4}{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

In a similar manner we get

$$M_{3-2} = \frac{3\,K_2\,\theta_3}{1 - \dfrac{a_2}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (d)$$

whence the ratio

$$\frac{M_{3-4}}{M_{3-2}} = \frac{K_3 \Big/ \left(1 - \dfrac{b_4}{2}\right)}{K_2 \Big/ \left(1 - \dfrac{a_2}{2}\right)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

We also have

$$M_{3-4} + M_{3-2} = - M. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

From the relations shown in (7) and (8) it is observed that the applied moment M is distributed between the members 3–4 and 3–2 in proportion to the quantities $K_3/(1 - b_4/2)$ and $K_2/(1 - a_2/2)$, that is, the moments on the right and left of joint 3 are proportional to the values of K for these spans when modified by dividing them by the quantities $1 - b_4/2$ and $1 - a_2/2$, respectively. As these divisors are independent of what particular span is loaded, the adjusted or modified values of K can be calculated once for all and used in the distribution of the moment applied at any joint.

The general formulas for the adjusted K values are

$$\left.\begin{array}{ll} \text{From left to right} & K'_n = \dfrac{K_n}{1 - \dfrac{a_n}{2}} \\[2em] \text{From right to left} & K'_n = \dfrac{K_n}{1 - \dfrac{b_{n+1}}{2}} \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where n = number of span or support from the left end. Having the moments 3–4 and 3–2 calculated, the moments at other supports are determined by applying the coefficients a and b as already illustrated in Art. 239.

279. Example.—The problem of Art. 242 will be solved by this method. Fig. 60. Live load per foot = 3000 lbs. Values of a, b, and adjusted values K' are shown in Fig. (b).

Span 1 Loaded.—$C_1 = 1/12 \times w\,l^2 = 100$. The applied moment at 2 is -150. This is to be distributed on the left and right in the ratio of 5/4.68 giving $+ 77.5$ and $+ 72.5$ as shown. Then $M_{3-4} = - 72.5 \times b_3 = - 21.3$; $M_{4-5} = 21.3 \times 0.242 = + 5.2$ and $M_{5-6} = - 5.2 \times 0.278 = - 1.4$ as shown in Fig. (c).

Span 2 Loaded.—Fig. (d). $C_1 = 36$. Two operations are required here, as the applied moments at 2 and 3 are distributed separately until the total moments M_{2-1} and M_{3-4} are determined. At 2 the moment 36 is distributed left and right in the ratio 5.0/4.68, giving -18.6 and -17.4 as shown. At 3 the ratio is 4.64/5.70, giving $+ 16.2$ and $+ 19.8$. Then the moment of -17.4 at 2 gives rise to a moment of $+ 17.4 \times b_3 = + 17.4 \times 0.293 = + 5.1$ at 3, which is to be added to $+ 19.8$, giving a total of $+ 24.9$. Then $M_{4-5} = - 24.9 \times 0.242 = - 6.0$ and $M_{5-6} = + 6.0 \times 0.278 = + 1.7$.

Similarly on the left, the moment of + 16.2 in 3–2 produces a moment of −16.2 × 0.278 = − 4.5 in 2–1, making a total of − 23.1.

Note that the signs used here follow the convention used in the slope-deflection analysis, hence a plus sign for the moment on the right of a support indicates right-hand rotation and hence a curvature convex upwards

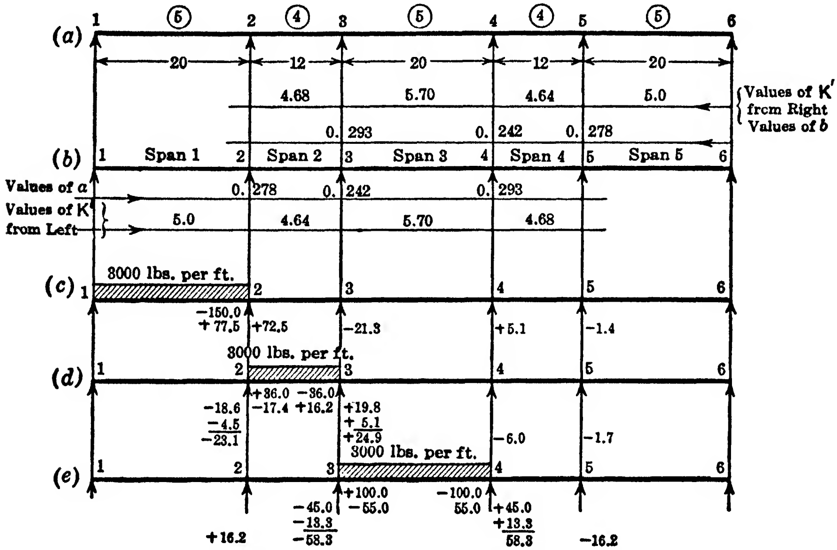


FIG. 60.

or a *negative* bending moment at the support in the ordinary sense. To show the character of the moments in this sense, therefore, change signs when on the right of the support. Signs on the left are correct.

Span 3 Loaded.—C_1 = 100. All the work is indicated in Fig. 60 (*e*).

280. Example of Beam and Girder Design.—

Data:

Interior panel 20 ft. square c. to c. of columns.
Columns 24 in. square.
Live load 300 lbs/ft.²
Use 2500-lb. concrete. $n = 12$.

Unit Stresses:

f_s = 18,000 lbs/in.²; for stirrups 16,000 lbs/in.²

$f_c = \begin{cases} 1000 \text{ lbs/in.}^2 \text{ in general.} \\ 1125 \text{ lbs/in.}^2 \text{ adjacent to supports.} \end{cases}$

$$v = \begin{cases} 50 \text{ lbs/in.}^2 \text{ without web reinforcement or special anchorage.} \\ 75 \text{ lbs/in.}^2 \text{ without web reinforcement but with special anchorage.} \\ 150 \text{ lbs/in.}^2 \text{ with web reinforcement but without special anchorage.} \\ 225 \text{ lbs/in.}^2 \text{ with web reinforcement and with special anchorage.} \end{cases}$$

$$u = \begin{cases} 100 \text{ lbs/in.}^2 \text{ without special anchorage.} \\ 200 \text{ lbs/in.}^2 \text{ with special anchorage.} \end{cases}$$

Constants:

For $f_c = 1000$ lbs/in.²	For $f_c = 1125$ lbs/in.²
$k = 0.4.$	$k = 0.429.$
$j = 0.867.$	$j = 0.857.$
$p = 0.0111.$	$p = 0.0134.$
$R = 173.$	$R = 207.$

Design of Floor Slab.—

The panel will be divided into three bays as shown in Fig. 61. A preliminary design indicates that the beams will be 10 in. wide. Clear span of slab = 5 ft. 10 in.

Load per foot of width

Live load	= 300 lbs.
Assume dead load	= 50 lbs.
Total	350 lbs.

Center moment = $\frac{1}{16} w l^2$ = 8930 in-lbs.
End moment = $\frac{1}{12} w l^2$ = 11,900 in-lbs.
From the relation $R = M/b\,d^2$ we have

$$\text{for center } d = \left(\frac{8{,}930}{12 \times 173}\right)^{\frac{1}{2}} = 2.09 \text{ in.}$$

$$\text{for end } d^* = \left(\frac{11.900}{12 \times 207}\right)^{\frac{1}{2}} = 2.19 \text{ in.}$$

$$\text{or} \quad d = \left(\frac{11{,}900}{12 \times 173}\right)^{\frac{1}{2}} = 2.40 \text{ in.}$$

* The allowable value of f_c = 1125 lbs/in.² for compressive stress adjacent to supports is intended primarily for beams, but there is no valid reason for not using it for slabs if advantageous to do so.

Make total slab thickness 3.5 in. with $d = 2.5$ in., giving a covering of about ¾ in. (Building codes may require a minimum slab thickness of 4 in.)

Check weight. $w = 44$ lbs/ft.²

Steel Area.

For negative moment $A_s = \dfrac{11{,}900}{0.86 \times 2.50 \times 18{,}000} = 0.308$ sq. in/ft.

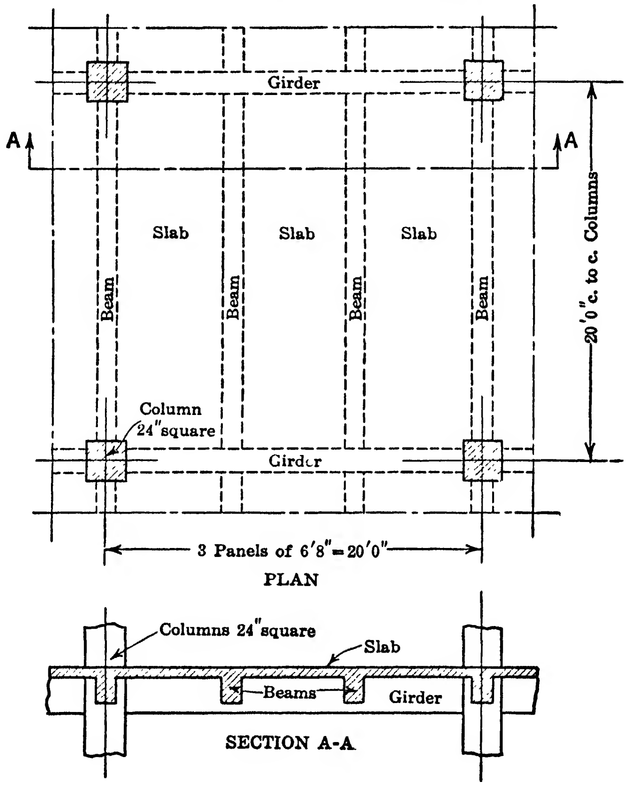


FIG. 61.

For positive moment $A_s = \dfrac{8{,}930}{0.87 \times 2.50 \times 18{,}000} = 0.228$ sq. in/ft.

Use ⅜-in. bars spaced 6 in. apart for positive reinforcement, giving 0.22 sq. in/ft. For negative reinforcement use ⅜-in. bars spaced 4 in. apart giving 0.33 sq. in/ft.

(The use of the values of p in calculating steel area will give larger areas than necessary, as the slab is thicker than theoretically required.) Note that the negative reinforcement is provided by additional bars placed near the top of the slab. For slabs of greater thickness than about 5 in., negative reinforcement is generally provided by bars bent up from the positive reinforcement.

Shearing Stress:

At end $V = \frac{1}{2} \times 344 \times 5.83 = 1000$ lbs.

$$v = \frac{1000}{12 \times 0.86 \times 2.5} = 39 \text{ lbs/in.}^2$$, a safe value without reinforcement.

Bond Stress:

At face of support, bars spaced 4.0 in.;

$$\Sigma o = 1.18 \times \frac{12}{4.0} = 3.54 \text{ in.} \quad u = \frac{1000}{3.54 \times 0.86 \times 2.5} = 132 \text{ lbs/in.}^2$$

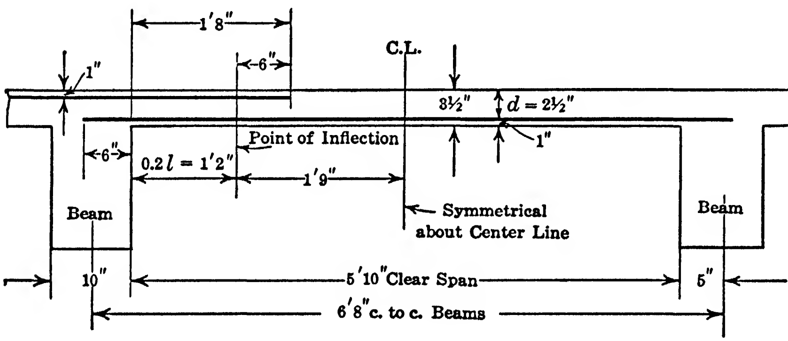


FIG. 62.

This is allowable with special anchorage, which requires at least one-third of the tension reinforcement to extend a sufficient distance beyond point of inflection to develop one-third of the working stress at a bond stress of 100 lbs/in.² For a working stress of 18,000, this requires an embedment of 15 diameters, or 6 in. for a ⅜-in. bar. The point of inflection is taken 2/10 l from the end. Fig. 62 shows the arrangement.

For positive moment the critical point for bond is the point of inflection. $V = 344 \times 0.3 \times 5.83 = 600$ lbs.

$$\Sigma o = 2.36 \quad \text{and} \quad u = \frac{600}{2.36 \times 0.87 \times 2.5} = 118 \text{ lbs/in.}^2$$

This also requires special anchorage which is secured by extending one-third of the bars 6 in. beyond the face of the support. See Fig. 62.

Design of Beam:

Assume width of girder at 15 in. Clear span of beam = 18 ft. 9 in. Fig. 66.

Load.

Live load = 300 × 6.67	=	2000 lbs/ft.
Slab weight = 44 × 6.67	=	294 lbs/ft.
Assume beam to be 10 in. by 16 in., weight	=	166 lbs/ft.
Total	=	2460 lbs/ft.

End Shear = $2460 \times \frac{18.75}{2}$ = 23,100 lbs.

Using $v = 150$, the necessary value of $b\,j\,d$ = 23,100/150 = 154.
At end of beam $j = 0.857$, hence $b\,d = 154/0.857 = 180$ sq. in.
This area may be secured by using 10 in. by 18 in. or 12 in. by 15 in. The former will be the more economical of steel and will be used, assuming head room not in question.

Centre moment = $\frac{1}{12}\,w\,l^2 = \frac{1}{12} \times 2460 \times 18.75^2 \times 12$ = 861,000 in-lbs.*

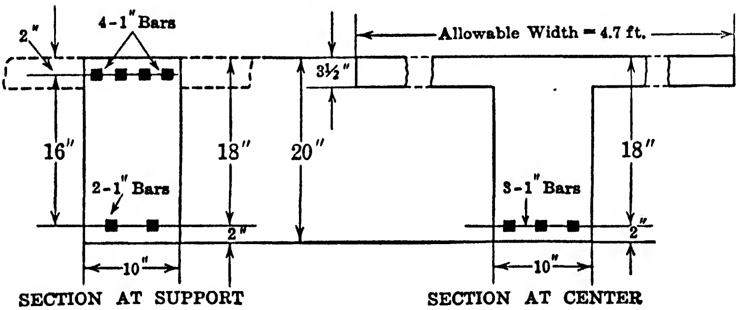


FIG. 63.

The beam is a T-beam and $t/d = 3.5/18 = 0.194$. From Diagram 5 the value of j may be taken at about 0.91. Hence $A_s = \frac{861{,}000}{0.91 \times 18 \times 18{,}000}$ = 2.92 sq. in. Use three 1-in. square bars. $A = 3$ sq. in. Check concrete stress by determining necessary width of flange to make $f_c = 1000$. For $t/d = 0.194$ and $f_s/n\,f_c = 18{,}000/12{,}000 = 1.5$, Diagram 5 gives $p\,n = 0.095$ or $p = 0.008$. Then for a balanced design $b\,d = 3.00/0.008$ = 375, and $b = 375/18 = 20.8$ in. This requires a width of flange of only 5.4 in. each side of stem, hence the concrete is much under-stressed. The usable width of flange is determined either by spacing c. to c. of beams, = 6.67 ft., or by one-fourth the span length, = $\frac{18.75}{4}$ = 4.7 ft., or by the imitation of overhanging width of 8 times the thickness, = $2 \times 8 \times 3.5 + 10$ = 66 in. The allowable width is therefore 4.7 ft. Then the actual value of p is 0.0030 and from Diagram 5 we find $f_s/n\,f_c = 3.2$ and $f_c = 470$ lbs/in.²

End Moment = $\frac{1}{12}\,w\,l^2$ = 861,000 in-lbs.

* Assumed to come under paragraph 107 of the specifications. At the columns the beam will be well restrained; at the girders it would be partially restrained.

Here the beam is a rectangular beam and will need to be reinforced for compression.

Moment carried by rectangular beam without compressive reinforcement is

$$M_1 = R\,b\,d^2 = 207 \times 10 \times 18^2 = 670{,}000 \text{ in-lbs.}$$

$$A_1 = p\,b\,d = 0.0134 \times 10 \times 18 = 2.41 \text{ sq. in.}$$

Moment to be carried by additional steel top and bottom is

$$M_2 = 861{,}000 - 670{,}000 = 191{,}000 \text{ in-lb.}$$

Lever arm for steel = 16 in. as shown in Fig. 63. Hence

$$A_2 = \frac{191{,}000}{16 \times 18{,}000} = 0.66 \text{ sq. in.}$$

Total tension steel = 2.41 + 0.66 = 3.07 sq. in.

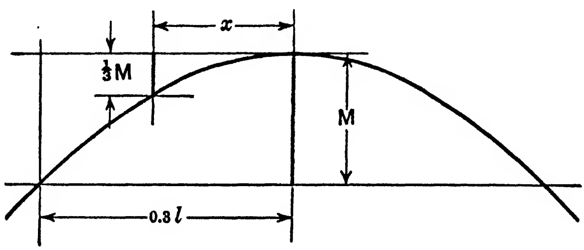


FIG. 64

The amount of compressive steel A' is found from A_2 by proportion (see eq. (11), Art. 85) and is

$$A' = 0.66 \times \frac{1-k}{k - 2/18} \times \frac{n}{n-1} = 0.66 \times \frac{0.57}{0.32} \times \frac{12}{11} = 1.28 \text{ sq. in.}$$

The required steel area is furnished by four 1-in. bars at top, giving $A = 4$ sq. in., and by two bars at bottom, giving $A' = 2$ sq. in. This requires bending up two of the three bars and lapping all bars at the end both at top and bottom, as shown in Fig. 66.

Bond Stress and Length of Bars:

End shear = 23,100 lbs. $v = \dfrac{23{,}100}{0.86 \times 18 \times 10} = 150 \text{ lbs/in.}^2$ Maximum center shear, assumed equal to one-fourth end live load shear = 4700 lbs. $v = \dfrac{4700}{0.86 \times 18 \times 10} = 30 \text{ lbs/in.}^2$

Shear at point of inflection, two-tenths point, $= 30 + (150 - 30) \times 0.6 = 102 \text{ lbs/in.}^2$

Bond stress at end, top bars. Four bars give

$$\Sigma o = 4 \times 4.0 = 16.0 \text{ in.}$$

$$u = \frac{150 \times 10}{16.0} = 93.8 \text{ lbs/in.}^2$$

Special anchorage is not required.

Bond stress at point of inflection, bottom bars. Stress per lineal inch = 102 × 10 = 1020 lbs., requiring a value of Σo of 1020/100 = 10.2 in., without special anchorage, or with special anchorage, 5.1 in. Three bars are sufficient at the lower value of u, and two bars are sufficient at the higher value. Hence one bar may be bent up where not needed for moment and the second bar bent up at the point of inflection, or the two-tenths point, extending one bar to the end for at least 15 diameters or 15 in. beyond the face of the support. Here these bars serve also as compressive reinforcement and hence must be extended through into the opposite beam sufficiently

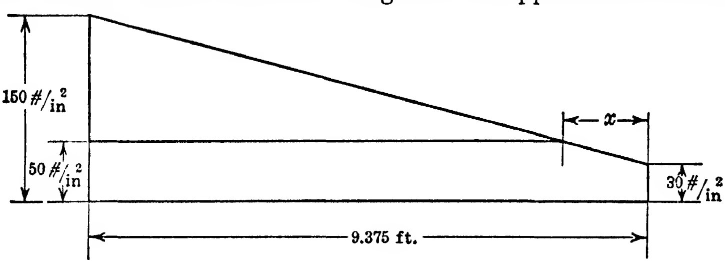


FIG. 65.

to take care of bond stress for the compressive stress therein. The unit stress in the steel $= 1125 \times \frac{k - 2/18}{k} \times n = 1125 \times \frac{0.32}{0.43} \times 12 = 10{,}000$ lbs/in.² At the value of $u = 100$, the required length for bond is

$$\frac{10{,}000 \times 1.0}{100 \times 4.0} = 25.0 \text{ in.}$$

The negative bending moment falls off very rapidly from the support, and assuming it to become zero at the two-tenths point, it will be found that the moment drops from 861,000 to 670,000 (the amount the beam can resist without compressive reinforcement) in a distance of about 10 in. Hence an extension of 2 ft. 0 in. beyond the support is sufficient.

The bars for negative moment are four in number. Two of these must extend 15 in. beyond the two-tenths point to provide the special anchorage required.

The point of bending of the first lower bar is found from the assumption of parabolic moment curve crossing the axis at the point of inflection. From Fig. 64, we have the proportion $\frac{x^2}{(0.3\, l)^2} = \frac{\frac{1}{3} M}{1\, M}$ or $x = 0.173\, l = 3.25$

ft. from center. The second bar may be bent up at the point of inflection or 5.625 ft. from the center. In the design it is found convenient to bend up the first bar 4 ft. 6 in. from the center, as it is more effective for web reinforcement.

Diagonal Tension Reinforcement. The point at which the shearing stress becomes 50 is found from Fig. 65 to be $\frac{x}{50-30} = \frac{9.375}{150-30}$. $x = 1.56$ ft. Web reinforcement will be required from this point to the end. Using stirrups of 3/8-in. round steel, the allowable stress per stirrup

$$= 16{,}000 \times 2 \times 0.11 = 3520 \text{ lbs.}$$

For vertical stirrups, spacing $= s = \frac{P}{v' b} = \frac{3520}{10 \times v'} = \frac{352}{v'}$, where v' is the shear to be carried by the reinforcement, the concrete carrying 50 lbs.

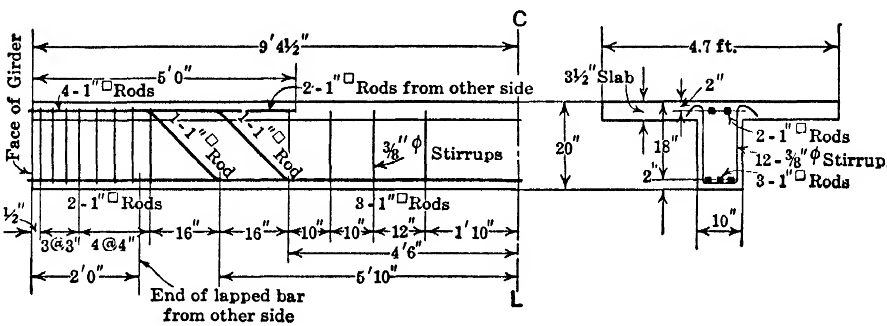


FIG. 66.

The requisite spacing is shown in Fig. 66. Maximum allowable spacing $= \frac{3}{4} \times 18 = 13.5$ in. Adopt spacing as shown, omitting stirrups where the bars are bent. Stress in the bar is $P = 0.7\, v' \times 10 \times 16$. For the two bars this is respectively equal to 5100 and 5700 lbs. The allowable stress is $16{,}000 \times 1.0 = 16{,}000$ lbs. Hence no stirrups are required to supplement the bars. The complete details are shown in Fig. 66.

The beams connecting to the columns will be only 18 ft. clear span and the steel can be reduced slightly if found practicable.

Design of Girder.—Clear span = 18 ft. See Fig. 68.

Load from beams at each point $= 2 \times 23{,}100 = 46{,}200$ lbs.

Load directly on girder of assumed width of 15 in. is

Live load $= 300 \times 1\frac{1}{4}$	= 375 lbs/ft.
Assume stem of 15 in. × 28 in., total depth 31 in., weight	= 430 lbs/ft.
Slab on girder $= 44 \times 1\frac{1}{4}$	= 55 lbs/ft.
Total	= 860 lbs/ft.

End shear = 46,200 + ½ × 860 × 18 = 53,940 lbs.

Required value of $b'jd$ = 53,940/150 = 360 sq. in. Assuming j = 0.86, $b'd$ = 360/0.86 = 418 sq. in. Provide area = 15 × 28 = 420 sq. in.

Use 15 in. by 28 in. beam. Then t/d = 0.125; j = 0.93.

Moments. As shown in Art. 235, with a 3-panel arrangement the bending moments will be about 10% less than for continuous loading, and the moments might logically be reduced by this amount from the values specified for continuous loading. However, the table of coefficients of Art. 234 indicate that the values of $\frac{1}{16}$ and $\frac{1}{12}$ are small enough. They will therefore be used and the entire load treated as uniformly distributed.

Load from the two long beams = 46,200 × 2	=	92,400 lbs.
Load directly on girder = 860 × 18	=	15,500 lbs.
Load on short beam joined to column	=	44,000 lbs.
Total panel load	=	151,900 lbs.

Center moment = $\frac{1}{16}$ × 151,900 × 18 × 12 = 2,050,000 in-lbs.
End moment = $\frac{1}{12}$ × 151,900 × 18 × 12 = 2,730,000 in-lbs.

$$\text{For center moment } A = \frac{2{,}050{,}000}{0.93 \times 28 \times 18{,}000} = 4.37 \text{ sq. in.}$$

A check of concrete stress shows that the necessary width of flange is but 26 in.

At support the moment carried by the rectangular beam is

M_1 = 207 × 15 × 28² = 2,440,000 in-lb.
A_1 = 0.0134 × 15 × 28 = 5.63 sq. in.
M_2 = 2,730,000 − 2,440,000 = 290,000 in-lb.

$$\text{Additional top steel} = A_2 = \frac{290{,}000}{18{,}000 \times 25} = 0.65 \text{ sq. in.}$$

Total top steel = A = 5.63 + 0.65 = 6.28 sq. in.

$$\text{Compression steel required} = A' = 0.65 \times \frac{0.57}{0.32} \times \frac{12}{11} = 1.3 \text{ sq. in.}$$

The following bars will be used.

Six 1-in. round bars for positive reinforcement

$$A = 6 \times 0.785 = 4.71 \text{ sq. in.}$$

Bend up four each way giving eight bars for top at end:

$$A = 8 \times 0.785 = 6.28 \text{ sq. in.}$$

Extend two straight through giving two bars for compression reinforcement which is sufficient.

Bond Stress. Fig. 67 represents approximately the moment diagram for positive moments. The point of inflection is about 2.83 ft. from the end.

The shear at this point is $V = 53{,}940 - 860 \times 2.83 = 51{,}510$ lbs. For six bars $\Sigma o = 6 \times 3.14 = 18.84$ in. Bond stress $= \frac{51{,}510}{18.84 \times 0.93 \times 28}$ $= 105$ lbs/in.^{2} Four bars will be sufficient with special anchorage.

At support $V = 53{,}940$. Σo for 8 bars $= 25.12$ in. Hence

$$u = \frac{53{,}940}{0.86 \times 28 \times 25.12} = 89 \text{ lbs/in.}^2$$

Special anchorage not necessary.

Length of bars. For positive moment, the point of inflection is 2.83 ft. from end. Four bars will be needed up to this point. Two bars can be bent up here extending two bars straight through. If two bars are bent

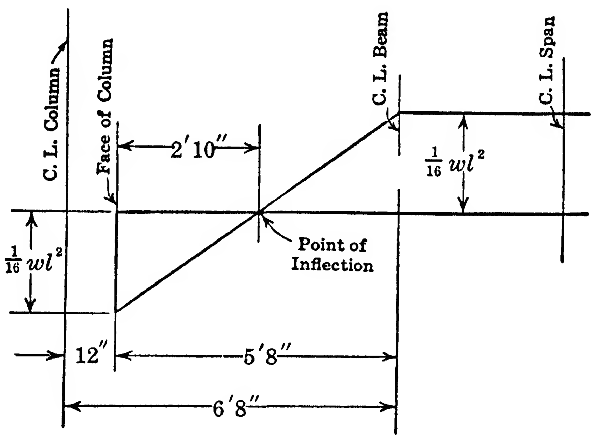


FIG. 67.

up 2 ft. 10 in. from support at 45° they will come within 9 in. of the face of the column at the top and will not serve as tension steel on the left side of the column. Hence to take care of diagonal tension next to the column, bend up two bars 2 ft. 10 in. from column as shown. These will be extended through to serve as top steel on the other side. From Fig. 67, two bars may be bent up 12 in. from the beam center giving a spacing of 22 in. between bars, which is allowable. These will also be extended through the column. There will then be needed two straight bars at the top to make up the eight bars required. The bond stress at support is 89 lbs/in.^{2}, which requires no special anchorage. Hence one-third of the top bars must extend at least to point of inflection for negative moment, which may be taken at the two-tenths point, or 3.6 ft. from the column. The length of bars must also be such as to develop their full strength at a bond stress of 100 lbs/in.^{2} This requires 45 diameters or 45 in., which determines the length.

The bond stress for compression steel will not require a greater embedment than the 15 diameters required for special anchorage.

Stirrups. These will be needed from the beam to the first bent bar and also from the column to the second bent bar. Shear 1 ft. from beam $=49{,}800$ lbs. $v = \dfrac{49{,}800}{15 \times 0.93 \times 28} = 127$ lbs/in.² Concrete carries 50 lbs.; stirrups 77 lbs. Use ½-in. round material giving allowable stress $P = 2 \times 0.196 \times 16{,}000 = 6260$ lbs. Spacing $= P/vb = \dfrac{6260}{77 \times 15} = 5.40$ in. Use 5-in. spacing. At the column, shear $= 53{,}940$ lbs.

$$v = \frac{53{,}940}{15 \times 0.86 \times 28} = 149 \text{ lbs/in.}^2 \quad \text{Spacing} = \frac{6260}{99 \times 15} = 4.20 \text{ in.}$$

The stirrups must have a length in the upper or lower half of the beam for proper anchorage at a bond stress of 100 lbs/in.² For a ½-in. stirrup

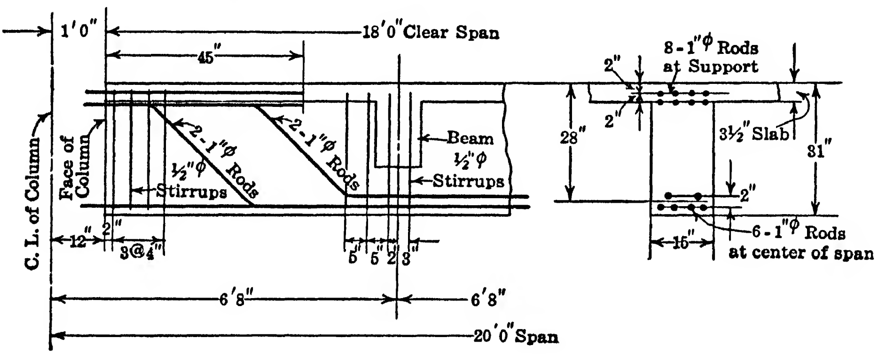


FIG. 68.

$\Sigma o = 3.14$ in. Stress $= 6260$ lbs. Required length $= \dfrac{6260}{3.14 \times 100} = 20$ in. Use a hook of the necessary length.

Fig. 68 shows the arrangement adopted.

Between the column and the beam the average shear is 51,435 lbs. $v = \dfrac{51{,}435}{15 \times 0.93 \times 28} = 132$ lbs/in.² The bent bars, spaced at 22 in., are stressed at $P = 0.7\, b\, v'\, s = 0.7 \times (132 - 50) \times 15 \times 22 = 18{,}900$ lbs. Allowable stress $= 2 \times 0.785 \times 16{,}000 = 25{,}000$ lbs.

Longitudinal Reinforcement in Floor Slab. Not less than 0.25% required. Use ½-in. bars spaced at 18 in. This also meets the requirements for transverse reinforcement in the girder flange.

Moment in Columns. Assume equal size of columns above and below and a story height of 12 ft.

From Art. 255 the bending moment at the top of a column may be taken at $M = \dfrac{4K_c}{4K_b + 7K_c} \times C$; where $K_b = I/l$ for beam, and

$K_c = I/l$ for columns below and above the floor. For uniform loads, $C = \frac{1}{12} W l$.

$$M = \frac{4K_c}{6\,K_b + 8\,K_c}\,C.$$

Moment of inertia of floor. Inasmuch as the use of the value of moment of inertia is to express the relative rigidities of floor and column it will be proper to include a considerable width of slab as part of the floor member. Just how much is very difficult to say, but the specified limit of one-fourth the span length for T-beam design is undoubtedly small enough and on the conservative side as increased stiffness of beam reduces column moment. A width of one-third the span would probably represent the rigidity of the floor better than one-fourth. A width of one-fourth will, however, be assumed. On this basis the value of moment of inertia is found to be $59{,}200$ in.^{4} and $K_b = I/l = \frac{59{,}200}{240} = 247$.

Moment of Inertia of Column. Assume the column to be made up as shown (Fig. 69), reinforced with eight $1\frac{1}{4}$-in. square bars. The sectional area of 3 bars is 4.69 sq. in. Then

$$I = \tfrac{1}{12} \times 24 \times 24^3 + 4.69 \times 2 \times 10^2 \times (n - 1) = 27{,}600 + 9400 = 37{,}000 \text{ in.}^4$$

and $K_c = I/l = 37{,}000/144 = 256$. In this case the values of K_b and K_c are so nearly alike that they may be assumed as equal.

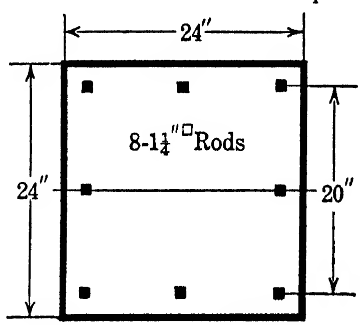


FIG. 69.

The column moment is therefore

$$M = \tfrac{4}{11} \times \tfrac{1}{12}\, W l = \tfrac{1}{33}\, W l = 0.03\, W l$$

For live load alone, $W = 300 \times 20 \times 20 = 120{,}000$ lbs., and $M = 0.03 \times 120{,}000 \times 18 \times 12 = 778{,}000$ in-lbs.

The bending stress $= f = \dfrac{778{,}000 \times 12}{37{,}000} = 252$ lbs/in.^{2} Provision must be made for this stress in the design of the column so that the total stress shall not exceed the value allowed for combined compression and bending.

If the entire floor width had been considered in calculating the value of I for the beam the value of $I_b = 93{,}000$, $K_b = 390$, $M = 0.025\, W l$ and fiber stress $= 210$ lbs/in.^{2}

For an exterior column the specifications give a value of $\frac{1}{12}\, w\, l^2$ for the end of the girder, this moment being resisted by the columns above and below. One-half of this is $\frac{1}{24}\, w\, l^2 = 0.04\, w\, l^2$ as compared to $0.03\, W l$ above calculated.

281. Example of Flat Slab Design.

Typical Interior Panel

Data

Four-way flat slab with drop panels and capitals.
Specifications: 1924 Joint Committee Report, except as noted.
Live Load: 250 lbs /ft.^{2}
Panel: 20 ft. square center to center of columns.
Unit Stresses: f_s = 18,000 lbs/in.^{2}; f_c' = 2500 lbs/in.^{2};
f_c = 1000 lbs/in.^{2};
v = 75 lbs/in.^{2} or as given in Art. 131, J.C.R.;
$n = 12$.

Loads

Live load = 250 lbs/ft.^{2}
Dead load = 100 lbs/ft.^{2} (assumed).

Total load = 350 lbs/ft.^{2}

Thickness of Main Slab

$t_2 = 0.02l\sqrt{w} + 1$. Eq. (38), J.C.R.
$t_2 = 0.02 \times 20 \times \sqrt{350} + 1 = 8.48$ in.

Minimum allowable $t_2 = \dfrac{l}{32} = \dfrac{20 \times 12}{32} = 7\frac{1}{2}$ in.

(See Art. 145, J.C.R.)
Use $8\frac{1}{2}$-in. slab.

Check Weight of Slab

$$\text{Weight} = \frac{8\frac{1}{2} \times 150}{12} = 106 \text{ lbs/ft.}^2$$

Size of Column Capital

Minimum $c = 0.225l = 0.225 \times 20 = 4.5$ ft.
Use a 4.5-ft. capital.

Size of Drop Panel

Minimum $b = l/3 = 20/3 = 6.67$ ft.
Maximum $b = l/2 = 20/2 = 10.0$ ft.
Use a 7.5-ft. drop.

Thickness of Drop Panel

Maximum $t_1 = 1.5 \times t_2 = 1.5 \times 8.5 = 12.75$ in.

Minimum $t_1 = 0.038\ (1-1.44\ c/l)l\sqrt{Rw'l_1/b_1} + 1\frac{1}{2}$. Eq. (37), J.C.R.

$t_1 = 0.038(1-1.44 \times 4.5/20)20 \times \sqrt{0.54 \times 356 \times 20/7.5}$
$+1\frac{1}{2} = 12.05$ in.

Use $t_1 = 12.5$ in. for drop panel thickness.

In designing flat slab floors for light loadings, it will often be found that minimum t_1, as determined from Eq. (37), J.C.R., exceeds the maximum t_1 determined from slab thickness conditions. Such a result indicates that the drop panel conditions govern the slab thickness instead of Eq. (38), J.C.R. The slab thickness must then be revised, subject to the condition that $t_2 = \frac{2}{3}t_1$.

Moments

$$M_0 = 0.09W\left(1 - \frac{2}{3}\frac{c}{l}\right)^2 l. \quad \text{Eq. (36), J.C.R.}$$

$$W = 356 \times 20 \times 20 = 142{,}400 \text{ lbs.}$$

$$2c/3l = 9.0/60 = 0.15$$

$$M_0 = 0.09 \times 142{,}400 \times 0.85^2 \times 20 = 185{,}190 \text{ ft-lbs.}$$
$$= 2{,}222{,}300 \text{ in-lbs.}$$

Moments in Principal Moment Strips. Table VI, J.C.R.

Two Column Strips: M_5 of Fig. 70

Negative moment $= 0.54M_0 = -\ 1{,}200{,}000$ in-lbs.
Positive moment $= 0.19M_0 = +\ 422{,}250$ in-lbs.

Middle Strip: M_4 of Fig. 70

Negative moment $= 0.08M_0 = -\ 177{,}800$ in-lbs.
Positive moment $= 0.19M_0 = +\ 422{,}250$ in-lbs.

Design of Principal Moment Strips

Positive Moment Steel in Two Column Strips

$M = +\ 422{,}250$ in.-lbs. Moment to be carried by one layer of steel near bottom of slab, Band A_1 as shown in Fig. 70. Steel to have 1 in. of cover. Assume ½-in. round rods, placed 1¼ in. from bottom of slab; $d = 8.5 - 1.25 = 7.25$ in. Assume $j = 0.86$.

$$A_s = \frac{M}{f_s j d} = \frac{422{,}250}{18{,}000 \times 0.86 \times 7.25} = 3.76 \text{ sq. in.}$$

Use 20 – ½ in. ϕ rods, $A_s = 20 \times 0.196 = 3.92$ sq. in.
Check compressive stress in concrete. Eq. (41), J.C.R.

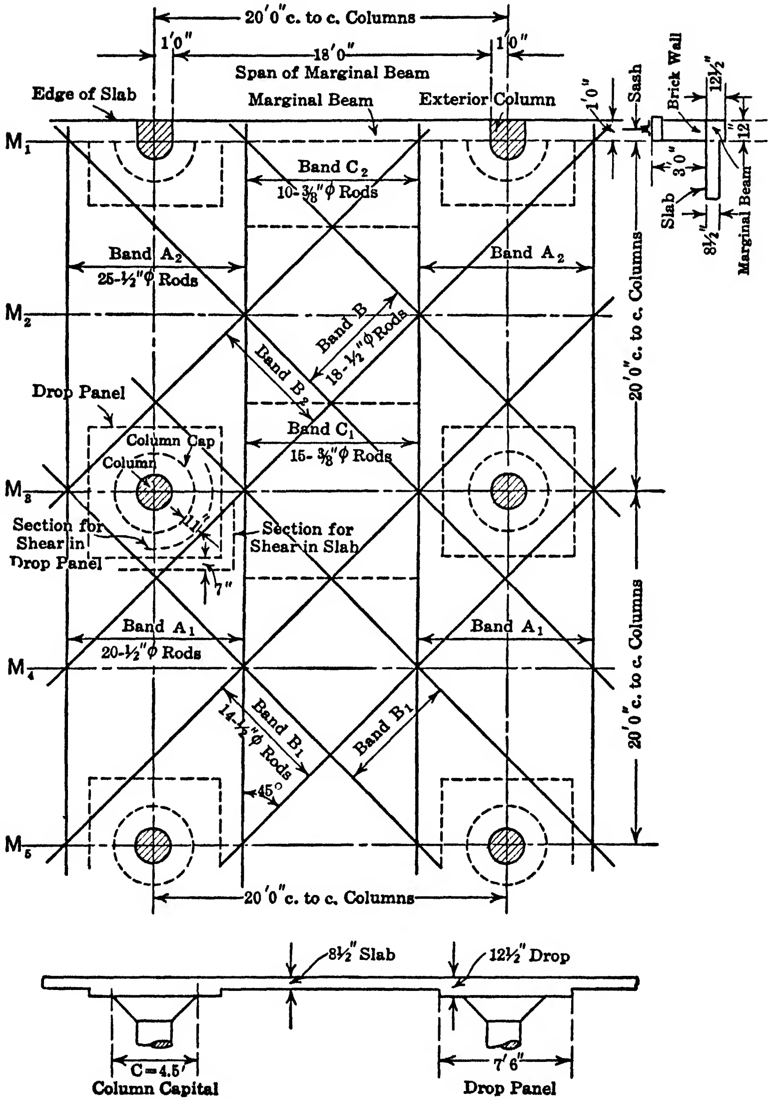
$$f_c = \frac{6RM_0}{0.67\sqrt[3]{pn}\,l_1 d^2}. \qquad RM_0 = 422{,}250 \text{ in.-lbs.}$$

$$p = \frac{3.92}{10 \times 12 \times 7.25} = 0.00452; \qquad pn = 0.00452 \times 12 = 0.0542;$$

$$\sqrt[3]{pn} = 0.379; \quad l_1 = 20 \text{ ft.} = 240 \text{ in.}; \quad d = 7.25 \text{ in.}$$

$$f_c = \frac{6 \times 422{,}250}{0.67 \times 0.379 \times 240 \times 52.6} = 790 \text{ lbs/in.}^2$$

Allowable = 1000 lbs/in.²



20'0" c. to c. Columns
1'0"
18'0"
1'0"
Span of Marginal Beam
Edge of Slab
Marginal Beam
Exterior Column
1'0"
Sash
Brick Wall
12½"
12"
3'0"
Slab
8½"
Marginal Beam
M1
M2
M3
M4
M5
Band C2
10-⅜"φ Rods
Band A2
25-½"φ Rods
Band A2
Band B
18-½"φ Rods
Band B2
Drop Panel
Column Cap
Column
Band C1
15-⅜"φ Rods
Section for Shear in Drop Panel
Section for Shear in Slab
11"
7"
20'0" c. to c. Columns
Band A1
20-½"φ Rods
Band A1
20'0" c. to c. Columns
Band B1
14-½"φ Rods
Band B1
45°
20'0" c. to c. Columns
8½" Slab
12½" Drop
C=4.5'
Column Capital
7'6"
Drop Panel

FIG. 70.

Positive Moment Steel in Middle Strip

$M = +422{,}250$ in-lbs. Moment to be carried by two bands of diagonal steel near bottom of slab, Bands B_1, Fig. 70. Steel to have 1 in. of cover. Average $d = 8.5 - 1.5 = 7.0$ in. Assume $j = 0.86$.

$$A_s = \frac{M}{f_s j d} = \frac{422{,}250}{18{,}000 \times 0.86 \times 7.0} = 3.90 \text{ sq. in.}$$

Since there are two diagonal bands of steel, one-half the required area, or 1.95 sq. in., must be provided by each band. Diagonal bands make an angle of 45° with the direction of moment. See Fig. 70. Hence effective area of band = cross-section area times cosine 45°.

$$\text{Area required per band} = \frac{1.95}{0.707} = 2.76 \text{ sq. in.}$$

Use 14 — ½ in. ϕ rods in each band.

$A_s = 14 \times 0.196 = 2.74$ sq. in.

Check compressive stress in concrete.

Eq. (41), J.C.R. $RM_0 = 422{,}250$ in-lbs.

$$p = \frac{2.74 \times 2 \times 0.707}{10 \times 12 \times 7.0} = 0.00463; \quad pn = 0.0556.$$

$$\sqrt[3]{pn} = 0.382; \quad l_1 = 20 \text{ ft.} = 240 \text{ in}; \quad d = 7.0 \text{ in.}$$

$$f_c = \frac{6 \times 422{,}250}{0.67 \times 0.382 \times 240 \times 49} = 842 \text{ lbs/in.}^2$$

Allowable 1000 lbs/in.^{2}

Shearing-stresses in Slab. See Art. 131 (*b*), J.C.R.

Shearing-stress in slab to be taken on a section $t_2 - 1.5 = 8.5 - 1.5 = 7.0$ in. from edge of drop panel. Section shown on Fig. 70.

$$\text{Load causing shear} = V = 356\left[20 \times 20 - \left(7.5 + \frac{2 \times 7}{12}\right)^2\right]$$

$$= 356(400 - 67) = 118{,}700 \text{ lbs.}$$

$$\text{Shear area} = \tfrac{7}{8}(t_2 - 1.5)4[b + 2(t_2 - 1.5)]$$

$$= \tfrac{7}{8} \times 7 \times 4(7.5 \times 12 + 2 \times 7) = 2400 \text{ sq. in.}$$

$$v = \frac{V}{\text{Shear area}} = \frac{118{,}700}{2400} = 49.4 \text{ lbs/in.}^2$$

Allowable shear $= 0.02f_c'(1 + r)$ but not to exceed 75 lbs/in.^{2} Eq. (33), J.C.R.

From Art. 131 (*b*), J.C.R., r shall be assumed as the proportional amount of the negative reinforcement, within the column strip, crossing entirely over the drop panel. Assuming the reinforcement to cover the entire strip $r = 0.75$ and allowable shear $= 0.02 \times 2500 \times 1.75 = 87.5$ lbs/in.^{2} Shearing-stress satisfactory.

Since the compressive and shearing-stresses in the slab are within allowable limits, the assumed slab is satisfactory. If in any case the allowable values are exceeded, the slab thickness must be revised.

Negative Moment Steel in Middle Strip

$M = -177{,}800$ in-lbs. Moment to be carried by one layer of steel near top of slab. Band C_1, Fig. 70. $d = 8.5 - 1.25 = 7.25$ in. Assume $j = 0.86$.

$$A_s = \frac{M}{f_s j d} = \frac{177{,}800}{18{,}000 \times 0.86 \times 7.25} = 1.58 \text{ sq. in.}$$

Use 15 – $\frac{3}{8}$ in. ϕ rods. $A_s = 15 \times 0.11 = 1.65$ sq. in.

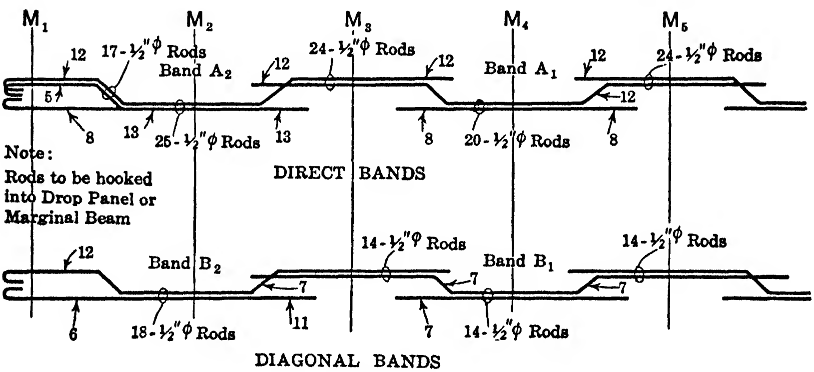


FIG. 71.

On substituting in eq. (41), J.C.R., it is found that the compressive stress in the concrete is 455 lbs/in.^{2}

Negative Moment Steel in Two Column Strips

$M = 1{,}200{,}000$ in-lbs. Moment to be carried by steel bent up from diagonal Band B_1 and direct Band A_1, Fig. 70. There will be four layers of steel near the stop of the slab at the drop panel. $d = 12.5 - 2.0 = 10.5$ in. Assume $j = 0.86$.

$$A_s = \frac{M}{f_s j d} = \frac{1{,}200{,}000}{18{,}000 \times 0.86 \times 10.5} = 7.38 \text{ sq. in.}$$

This steel is to be provided from steel in place in the positive moment strips, subject to the conditions of Art. 153, J.C.R. Many arrangements of steel are possible. Fig. 71 shows the arrangement adopted.

The area provided on the section of M_5, Fig. 71, is as follows:

From Direct Band

24 — ½ in. ϕ rods, $24 \times 0.196 = 4.70$ sq. in.

From Two Diagonal Bands

2 bands of 14 — ½ in. ϕ rods

$2 \times 14 \times 0.196 \times 0.707 = 3.88$ sq. in.

Total area provided 8.58 sq. in.

This area is slightly larger than required, but on checking concrete stresses later, it will be found that this steel area must be used in order to keep the concrete stresses within allowable limits.

In Art. 142, J.C.R., it is stated that the ratio of reinforcement for negative moment steel in the column strip shall not exceed 0.01. For the adopted arrangement, $p = \dfrac{8.58}{7.5 \times 12 \times 10.5} = 0.00908$, which is within allowable limits.

Check compressive stress in concrete, Column Section. Use eq. (40), J.C.R.

$$f_c = \frac{3.5RM_0}{0.67\sqrt[3]{pn}\,b_1 d^2}\left(1 - 1.2\frac{c}{l}\right). \quad RM_0 = 1{,}200{,}000 \text{ in-lbs.};$$

$$\sqrt[3]{pn} = \sqrt[3]{0.00908 \times 12} = 0.477;\ b_1 = 7.5\text{ ft.} = 90\text{ in.};\ d = 10.5\text{ in.}$$

$$\left(1 - 1.2\frac{c}{l}\right) = (1 - 1.2 \times 4.5/20) = 0.73.$$

$$f_c = \frac{3.5 \times 1{,}200{,}000 \times 0.73}{0.67 \times 0.477 \times 90 \times 10.5^2} = 968 \text{ lbs/in.}^2$$

Allowable = 1000 lbs/in.^{2}

Shearing-stresses in Drop Panel. Art. 131 (*a*), J.C.R.

Shear area shown on Fig. 70.

$$V = \text{load causing shear} = 356\left\{20^2 - \frac{\pi}{4}[c + 2(t_1 - 1\tfrac{1}{2})]^2\right\}$$

With $2(t_1 - 1\tfrac{1}{2}) = 2(12.5 - 1.5) = 22$ in. $= 1.83$ ft., and $c = 4.5$ ft.

$$V = 356[400 - 0.785(4.5 + 1.83)^2] = 131{,}200 \text{ lbs.}$$

$$\text{Shear area} = \tfrac{7}{8}(t_1 - 1\tfrac{1}{2})\pi[c + 2(t_1 - 1\tfrac{1}{2})]$$

$$= \tfrac{7}{8} \times 11 \times 3.1416 \times 76 = 2300 \text{ sq. in.}$$

$$v = \frac{V}{\text{Shear area}} = \frac{131{,}200}{2300} = 57.0 \text{ lbs/in.}^2$$

From Art. 131 (*a*), allowable shear $= 0.02 \times 2500\ (1 + r)$, where r = proportional amount of negative reinforcement within the column strip, crossing the column capital. From Fig. 70, $r = 4.5/7.5 = 0.6$ and allowable shearing-stress $= 0.02 \times 2500 \times 1.6 = 80$ lbs/in.^{2}, but not to exceed 75 lbs/in.^{2} Shearing-stress satisfactory.

The design of the interior panel is therefore satisfactory. Figs. 70 and 71 show the arrangement of steel and dimensions of the several parts.

Typical Exterior Panel

The J.C.R. does not make a definite recommendation regarding the moments to be used in exterior panels. An increase in moment over that required for interior panels is recommended for the negative moment at the column head section at the first interior row of columns and for the positive moment section between these columns and the edge of the panel. However, the moment to be provided for at the discontinuous edge of the panel is left to the judgment of the designer. Many designers assume that the moments at the discontinuous edge of the panel are 80% of those at an interior line of columns.

The Specifications of the American Concrete Institute make a definite recommendation regarding the moments to be used in an exterior panel. No increase in moments is recommended for moment sections on the first interior row of columns. Moments in the section half-way between this line of columns and the discontinuous edge of the panel are to be increased 25% over similar normal interior sections. At the wall, or discontinuous edge of the panel, it is recommended that the negative moment in the column strip be taken as not less than 90% and in the middle strip not less than 62½% of the corresponding moments for a normal interior panel. This recommendation will be adopted in the design under consideration.

In the design of a typical exterior panel which follows, it will be assumed that the column capitals and the drop panels are so arranged that the value of c in moment equation (36), J.C.R., will remain the same as for an interior panel. Fig. 70 shows the adopted arrangement. In case any marked change is made in the size and shape of the exterior column capital and the drop panel, this must be taken into account and the true value of c used in the calculation of the value of the moment M_0.

Based on the A.C.I. Specifications, the moments in the several moment strips, as determined from the corresponding moments in a typical interior panel, are as follows:

Two Column Strips

Negative Moment at Wall, M_1 of Fig. 70

$$= 0.90M_5 = 0.90(-1{,}200{,}000) = -\ 1{,}080{,}000 \text{ in-lbs.}$$

Positive Moment M_2, Fig. 70

$$= 1.25M_4 = 1.25(+422{,}250) = +\ 527{,}810 \text{ in-lbs.}$$

Middle Strip

Negative Moment at Wall, M_1 of Fig. 70

$$= 0.625M_5 = 0.625(-177{,}800) = -\ 111{,}000 \text{ in-lbs.}$$

Positive Moment, M_2 of Fig. 70

$$= 1.25M_4 = 1.25(+422{,}250) = 527{,}810 \text{ in-lbs.}$$

Note that the moments at the first line of interior columns, M_3 of Fig. 70, remain unchanged.

Design of the Principal Moments Strips

Positive Moment M_2 in the Middle Strip. From the calculations given for the typical interior panel, it can be seen that this moment will determine the slab thickness required for the exterior panel.

$M_2 = 527{,}810$ in-lbs. Moment to be carried by two diagonal Bands B_2 of Fig. 70. Assume same slab as for interior panel. Slab 8½ in. thick. Steel 1½ in. above bottom of slab, $d = 7.0$ in. Assume $j = 0.86$.

$$A_s = \frac{M}{f_s j d} = \frac{527{,}810}{18{,}000 \times 0.86 \times 7.0} = 4.87 \text{ sq. in.}$$

$$\text{Area required for each band} = \frac{1}{2} \times \frac{4.87}{0.707} = 3.44 \text{ sq. in.}$$

Use 18½ in. ϕ rods in each band, $A = 18 \times 0.196 = 3.53$ sq. in. Checking the compressive stress in the concrete, using eq. (41), J.C.R., we find $f_c = 968$ lbs/in.^{2} Allowable 1000 lbs/in.^{2}

If the concrete stress had exceeded the allowable value, the slab thickness would have to be increased. This would require a different slab thickness for interior and exterior panels. Some designers prefer to use a common slab thickness throughout the floor. In this case the slab thickness should be determined for the conditions in an exterior panel. Using this slab thickness the calculations can then be carried out as in a typical interior panel.

Positive Moment M_2 in a Column Strip

$M_2 = 527{,}810$ in-lbs. Moment to be carried by direct steel in Band A_2 of Fig. 70. Since the conditions are similar to those for the design of Band A_1, we can find the steel required for Band A_2 by increasing the steel for Band A_1 by 25%. Hence, use $20 \times 1.25 = 25 -$ ½ in. ϕ rods. A check on the compressive stress in the concrete shows that the allowable limit has not been exceeded.

Shearing-stress in Slab, Exterior Panel

Since the exterior slab and its loading are the same as for the interior panel, it can be readily seen that the shearing-stresses in the exterior panel slab are the same as those for the interior panel. In case the drop panel at the exterior column is not the same as for an interior panel, the shearing-stress must be determined for the true conditions.

Negative Moment M_1 in the Middle Strip

$M_1 = 0.625\ M_5 = 111{,}000$ in-lbs. Moment carried by Band C_2,

Fig. 70. Since the conditions are similar to those for Band C_1, the steel area required is $0.625 \times 15 = 10 - \frac{3}{8}$ in. ϕ rods.

Negative Moment M_1 in Two Column Strips

$M_1 = 0.90\ M_5 = 1{,}080{,}000$ in-lbs. Since it has been assumed that the exterior drop panel is similar to the interior drop panels, the steel area required can be determined by ratio from the calculations given for an interior panel. Steel area required $= 0.90 \times 7.38 = 6.64$ sq. in. Steel to be provided by bending up bars from Direct Band A_2 and Diagonal Bands B_2, Fig. 70. Adopting the arrangement at section M_1, Fig. 71, the steel area provided is as follows:

From Direct Band A_2

$17 - \frac{1}{2}$ in. ϕ rods, $A = 17 \times 0.196 = 3.33$ sq. in.

From Two Diagonal Bands B_2

$12 - \frac{1}{2}$ in ϕ rods in each band

$A = 2 \times 0.707 \times 12 \times 0.196 = 3.33$ sq. in.

Total area $= 6.66$ sq. in.

Check compressive stress in concrete, using eq. (40), J.C.R.

$$p = \frac{6.66}{90 \times 10.5} = 0.0074;$$

$$f_c = \frac{3.5 \times 1{,}080{,}000 \times 0.73}{0.67 \times 0.439 \times 90 \times 10.5^2} = 945 \text{ lbs/in.}^2$$

Allowable $= 1000$ lbs/in.^{2} Satisfactory.

Marginal Beams

Marginal beams are of two types: (*a*) marginal beams of greater depth than the drop panel, (*b*) marginal beams equal in depth to the drop panel.

Art. 148, J.C.R., recommends that the loadings for these beams be taken as follows:

(*a*) Marginal beams deeper than the drop panel shall carry, in addition to the wall load, at least one-fourth of the distributed load in the adjoining panel, and the column strip parallel to the beam shall be designed to resist a moment at least one-half as great as that specified for an interior column strip.

(*b*) Marginal beams in which the depth does not exceed the depth of the drop panel shall be designed to carry at least the load superimposed directly on it, exclusive of the panel load.

In the design under consideration, the depth of the marginal beam will be made equal to the depth of the drop panel. Fig. 70 shows the dimensions of the beam.

As shown in Fig. 70, the marginal beam carries a 12-in. curtain wall 3 ft. high. The superimposed load on the beam is: wall load, 3-ft. wall at 125 lbs/ft. = 375 lbs.; sash load, 30 lbs/ft.; beam load, 150 lbs/ft.; total load 555 lbs/ft. This beam is designed as a fixed beam of a span equal to the clear distance between faces of columns, which is 18 ft. (see Fig. 70). End and center moments are taken as $1/12wl^2$. This design contains no new features; the calculations will not be given.

Since it has been assumed that a negative moment of 111,000 in-lbs. is brought to the middle strip M_1 section of Fig. 70, the marginal beam will be investigated for torsional stresses set up in the beam by the above moment.

From eq. (11), Art. 266

$$v = T\frac{1.8b \times 3h}{b^2 h^2}$$

in which v = torsional shearing-stress at the center of the longer side of the beam; b = width and h = depth of marginal beam, and T = torsional moment. For the given conditions, $T = \frac{1}{2} \times 111{,}000 = 55{,}500$ in-lbs., and from Fig. 70, $b = 12$ in. and $h = 12.5$ in.

$$\text{Then } v = 55{,}500\frac{1.8 \times 12 + 3 \times 12.5}{12^2 \times 12.5^2} = 146 \text{ lbs/in.}^2$$

This shear is to be added to the shear in the beam due to vertical loading. Since the drop panel increases the shear area near the column, it will be assumed that the maximum combined shear-stress occurs at the edge of the drop panel. From Fig. 70, the shear at the edge of the drop panel is $555(9 - 3.75) = 2910$ lbs. The unit shear is

$$v = \frac{V}{bjd} = \frac{2910}{12 \times 0.86 \times 11} = 25.6 \text{ lbs/in.}^2$$

Hence the total shear is $146 + 25.6 = 171.6$ lbs/in.^{2} This shearing-stress is large, but it can be carried by carefully designed shear reinforcement of the type recommended in Art. 269.

Concluding Remarks. The above discussion has considered in detail the principal points in the design of typical interior and exterior panels. Fig. 70 shows the general details of the reinforcement of a flat slab floor. To simplify the reinforcement plan, only a portion of the steel has been shown. To complete the reinforcement, direct bands, similar to A_1, should be placed perpendicular to the direct bands shown. There should also be placed parallel to the marginal beam a direct band equal in width to one-half an A_1 band.

CHAPTER X

ARCHES

282. Introduction and Definitions.—The present chapter deals with the analysis of the straight arch without hinges and supported on abutments that may be considered as rigid. This is the most common problem encountered in reinforced-concrete arch design. For the analysis of the skew arch and of structures consisting of two or more arches supported on piers whose deformations must be considered, reference must be made to more complete works on arches.

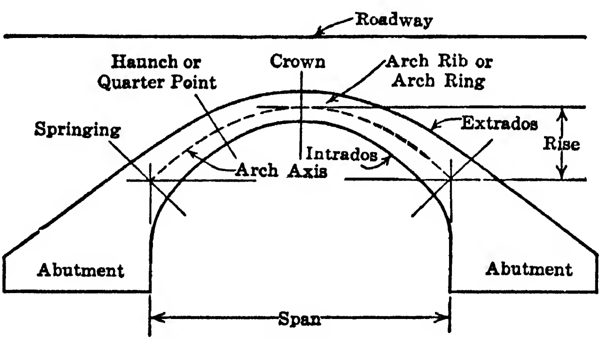


FIG. 1.

Figs. 1 and 2 illustrate the significance of the various common terms used in arch nomenclature, most of which are self-explanatory.

Arch Ring. Used when the arch is continuous transversely.

Arch Rib. Used when the arch consists of two or more separate rings of limited width transversely.

Springing. Junction of arch and abutment. Well defined in the old type of masonry arches where a relatively narrow ring rested on massive abutments. Less well defined where the arch ring gradually widens out into the abutment. Must in such cases be somewhat arbitrarily assumed as the section beyond which the deformation of the structure may be neglected.

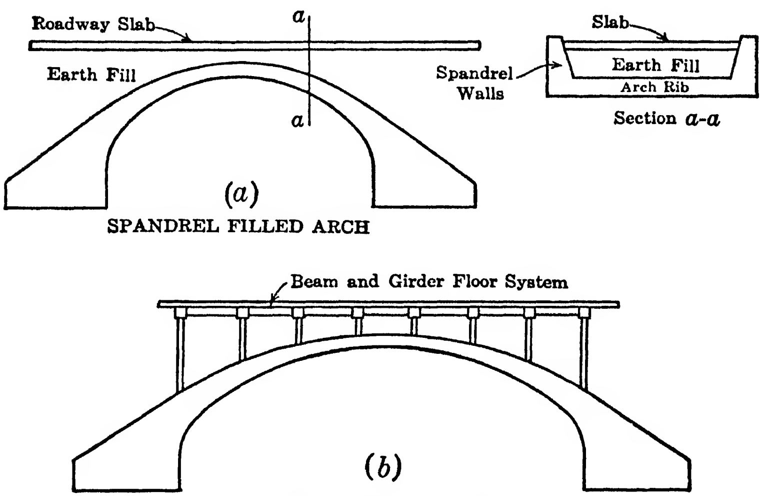
Crown. Highest point.

Haunch. A somewhat indefinite section at about the quarter-point.

Arch Axis. Line through centre of arch ring.

Extrados. Upper surface of arch ring.

Intrados. Lower surface of arch ring.



OPEN SPANDREL ARCH

FIG. 2.

Spandrel. Space between roadway and arch ring on either side of the centre.

Spandrel Filled Arch. Spandrel filled with earth to support roadway. Economical for short spans.

Open Spandrel Arch. Roadway supported by beams and columns or by small arches and piers. Economical for long spans.

283. Advantages of the Reinforced-Concrete Arch.—If all the loads on an arch were fixed loads, it would be possible in any case to construct an arch ring so that the resultant pressure at all sections would be at the centre of gravity of the section. The compressive stress at any section would then be uniformly distributed over the section, and the arch would be proportioned only for this uniform compression. The "line of pressure" would lie at the axis of the arch throughout. If, however, the arch ring is not made to fit the "line of pressure," or if part of the load is a live load, then the resultant pressure will not in general coincide with the axis of the arch. There will exist both bending and direct compression.

In ordinary masonry and concrete arches tensile stresses are not permissible. The arch ring must therefore be designed so that the line of pressure will not pass outside the middle third. In reinforced-concrete arches this limitation does not exist. The arch ring is a beam, and if properly reinforced may carry heavy bending moments involving tensile stresses in the steel.

Theoretically the gain in economy by the use of steel in a concrete arch is not great. If the pressure line does not depart from the middle third, the steel reinforces only in compression and in this respect is not as economical as concrete. If the line of pressure deviates farther from the centre, resulting in tensile stresses on the section, the conditions are such that those stresses must be provided for by the use of the steel at very low working values. That is to say, the direct compression in the arch is so large a factor that the limiting stresses in the concrete will always result in relatively small unit tensile stresses in the steel where any tension exists at all.

Practically the value of reinforcement is very considerable. It renders an arch a much more secure and reliable structure, it greatly aids in preventing cracks due to any slight settlement, and by furnishing a form of construction of greater reliability makes possible the use of working stresses in the concrete considerably higher than are usual in plain masonry. Consequently, in long-span arches where the dead load constitutes by far the larger part of the load, the possible increase in average working stress counts greatly toward economy. It affects not only the arch but also the abutments and foundations.

When a structure is desired which readily lends itself to artistic treatment, the arch has many advantages over any other type of structure. Its graceful curves blend into the landscape, and being a deck structure, a clear unobstructed view of the surrounding scenery may be secured from the roadway. In many cases artistic considerations may lead to the choice of an arch structure even though greater economy may be secured by the use of beam type of structure.

284. Reinforcement of Arches.—The most common type of reinforcement consists of longitudinal bars placed near extrados and intrados with a few transverse bars to prevent longitudinal cracks and to equalize the load on the arch. The outer and inner bars are connected by ties as in a "tied" column. The total amount of longitudinal steel at the crown section is usually about 1%, and a common practice is to use this same number and spacing of bars throughout, giving a reduced steel ratio at the springing section. This arrangement requires, for equal stresses at crown and springing, a relatively large ratio of depth at springing to depth at crown, usually from 2 to 2.5. If a smaller ratio is desired, the reinforcement may be increased towards the springing, a practice more commonly used for long spans. The same amount of steel is generally used in the two layers, giving a symmetrical section, but here again special conditions may make a different arrangement desirable.

DEVELOPMENT OF GENERAL FORMULAS FOR STRESS ANALYSIS

285. General Method of Procedure.—The method of analysis presented here is based on the elastic theory and is of general application to arches of variable moment of inertia and loaded in any manner. It is mainly an algebraic method, although, if desired, certain simple graphical aids may be used advantageously. It necessarily assumes that a preliminary design has been made by the aid of approximate or empirical rules or by reference to the proportions of existing arches. This arch is then exactly analyzed and the results used in correcting the design; the corrected design may then in turn be analyzed if it departs too greatly from the one first assumed. A discussion of the methods used in determining the preliminary arch is given in Art. 318.

The analysis of an arch consists in the determination of the forces acting at any section, usually expressed as the *thrust*, *bending moment*, and *shear* at such section. The thrust is the component of the resultant force parallel to the axis of the arch at the given point; the bending moment is the moment of the resultant forces about the gravity axis of the section; and the shear is the component of the thrust at right angles to the arch axis. The thrust and bending moment constitute a case of combined bending and compression, and the resulting stresses at the section are determined as explained in Chap. III. The shear causes stresses similar to those produced by the vertical shear in a simple beam. Shearing-stresses in an arch are generally very small and are usually neglected. The important stresses are those due to bending moment and thrust. In the arch analysis the arch axis is taken as the centre line which, for symmetrical reinforcement, is also

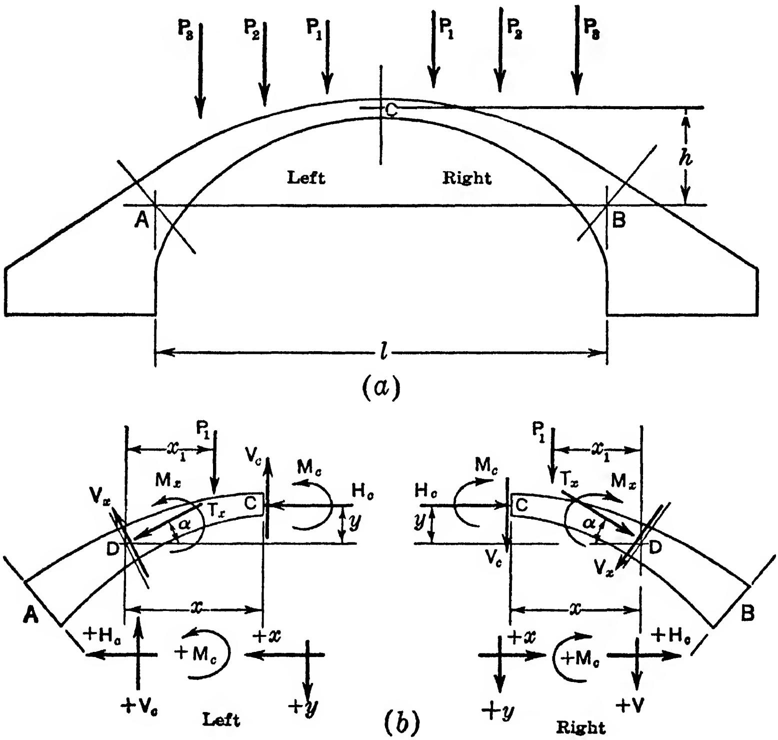


FIG. 3.

the gravity axis. For unsymmetrical reinforcement the gravity axis will deviate somewhat from the centre line, but such deviation may usually be neglected in the arch analyses. In getting fibre stress, however, the actual section must be considered.

The method of procedure in the analysis of an arch will be to determine, first, the thrust, bending moment, and shear at the crown. These being known, the values of similar quantities for any other section can readily be determined. Fig. 3(*a*) shows a symmetrical arch loaded in any manner with loads P_1, P_2, etc., and assumed as fixed at the abutments A and B. In Fig. (*b*) the arch is shown cut at the crown, each half forming a cantilever beam acted upon by the applied loads and the unknown forces at the crown, represented as H_c, V_c, and M_c, the same on each side but opposite in direction. The three equations necessary to determine the values of these unknowns are established by applying to each half the laws pertaining to the deflection of curved beams and the condition that the deflection of the point C on one side must correspond exactly with the deflection of this point on the other.*

286. Notation.—(See Fig. 3.)

H_c = thrust at the crown;
V_c = shear at the crown;
M_c = bending moment at the crown;
T_x, V_x, and M_x = thrust, shear, and moment at any other section;
ds = length of a division of the arch ring measured along the arch axis;
ds_1 = length of division at the crown;
N = number of divisions in one-half of the arch;
A = area of cross-section, $= A_c + n\,A_s$;
I = moment of inertia of any section $= I_c + n\,I_s$;
I_1 = moment of inertia at the crown;
P = any load on the arch;
x, y = coordinates of any point on the arch axis referred to the crown as origin, and all to be considered as positive in sign;
y_1 = vertical ordinate of any point on the axis referred to an X-axis through the elastic centre;
y_0 = distance from crown to elastic centre;
m = bending moment at any point in the cantilever due to the external loads P; these moments are negative;
α = inclination of arch axis at any point;
ρ = radius of curvature of arch axis at any point;
f = average compressive fibre stress in the arch at any section, $= T/A$;

* The same equations may be derived by the method of least work.

ω = coefficient of expansion;
t = change of temperature;
q = ratio of $d\,s/I$ for any small division $d\,s$ of the arch to the quantity $d\,s_1/I_1$ at the crown;
l = span length;
s = length of arch axis;
h = rise.

287. Deflection of Curved Beams.—Let AB, Fig. 4, represent any curved beam fixed at A and free at B. It is acted upon by various forces and changes of temperature, producing, in general, direct stress or thrust, shear, and bending moment at all sections, as a result of which the beam is deflected to the new position AB'. Let Δ_x, Δ_y be respectively the horizontal and vertical components of the movement BB', and Δ_ϕ the change in direction of the tangent at B, or the amount of rotation of a right section. The positive directions of motion are as shown. The origin is at B.

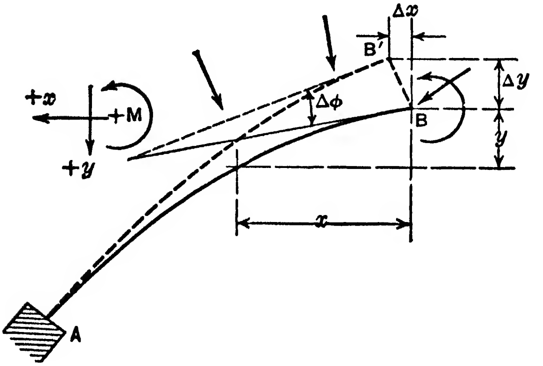


FIG. 4.

From mechanics the following relations are known to exist: *

$$\left.\begin{aligned}
\Delta_\phi &= \int_A^B \left[\frac{M\,d\,s}{E\,I} + \frac{f\,d\,s}{E\,\rho}\right] \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (a)\\
\Delta_y &= \int_A^B \left[-\frac{M\,x\,d\,s}{E\,I} + \frac{f\,d\,y}{E} - \frac{f\,x\,d\,s}{E\,\rho} - \omega\,t\,d\,y\right] \quad (b)\\
\Delta_x &= \int_A^B \left[\frac{M\,y\,d\,s}{E\,I} + \frac{f\,d\,x}{E} + \frac{f\,y\,d\,s}{E\,\rho} - \omega\,t\,d\,x\right] \; . \quad (c)
\end{aligned}\right\} \quad (1)$$

* Modern Framed Structures, Vol. II, p. 131.

In the foregoing expressions the terms containing M represent the effect of bending moments, those containing f the effect of the direct compression, called rib shortening, and those containing ω the effect of temperature changes.

288. Condition Equation for Determination of Thrust, Shear, and Moment at the Crown, H_c, V_c, and M_c.—Referring again to Fig. 3, the two parts of the arch are in static and elastic equilibrium and any movement or rotation of the crown section on the one side of the arch corresponds exactly with the movements on the other side. We may then write the condition equations, paying due attention to sign:

$$\left.\begin{array}{ll} \Delta y_L = \Delta y_R \quad \ldots\ldots & (A) \\ \Delta x_L = -\Delta x_R \quad \ldots\ldots & (B) \\ \Delta \phi_L = -\Delta \phi_R \quad \ldots\ldots & (C) \end{array}\right\} \quad \ldots\ldots \quad (2)$$

in which the subscripts L and R refer to the left and right halves of the arch.

These are the three needed condition equations for solving the problem, and it remains now to express the quantities in terms of the unknowns, M_c, H_c, and V_c, and other quantities derived from the given loads and dimensions.

289. Forces Acting at Any Section of the Arch Rib.—From Fig. 3(*b*) the moment, thrust, and shear at any section of the arch, expressed in terms of these values at the crown and the applied loading, are:

For the left side

$$M_x = M_c + H_c y + V_c x + m_L. \quad \ldots\ldots \quad (3a)$$

For the right side

$$M_x = M_c + H_c y - V_c x + m_R. \quad \ldots\ldots \quad (3b)$$

If the applied loads are vertical, we have also, for the left side

$$\left.\begin{array}{l} T_x = H_c \cos\alpha + (\Sigma_C^D P - V_c) \sin\alpha \\ V_x = H_c \sin\alpha - (\Sigma_C^D P - V_c) \cos\alpha \end{array}\right\} \quad \ldots \quad (4a)$$

For the right side

$$\left.\begin{array}{l} T_x = H_c \cos\alpha + (\Sigma_C^D P - V_c) \sin\alpha \\ V_x = -H_c \sin\alpha + (\Sigma_C^D P - V_c) \cos\alpha \end{array}\right\} \quad \ldots \quad (4b)$$

For other than vertical loads the thrust and shear are the components of the resultant force at the section taken parallel and perpendicular to the arch axis. They are readily found by means of a force diagram or algebraically.

290. Equations for the Crown Stresses, H_c, V_c, and M_c, for Symmetrical Arches.—The required equations are obtained by applying the conditions represented in eq. (2) to eq. (1) and substituting values of M obtained from (3). Inasmuch as the terms containing f are relatively small, we can assume in general that $T_x = H_c$; hence $f = H_c/A$. Furthermore, referring to eq. (1), it will be found that the terms $\int \frac{f\,x\,d\,s}{E\,\rho}$ and $\int \frac{f\,y\,d\,s}{E\,\rho}$ are very small and may be neglected. The term $\int \frac{f\,d\,y}{E}$ does not appear in the final equations. Hence, the only term involving f that needs to be considered is the term $\int \frac{f\,d\,x}{E}$ of eq. 1 (c). Then since $d\,x = d\,s \cos\alpha$, we have $\int f\,d\,x = \int f \cos\alpha\,d\,s = H_c \int \frac{\cos\alpha\,d\,s}{A}$. For dead load alone a more precise expression is $H_c \int \frac{d\,s}{A}$.

With these modifications the various substitutions may be made and the resulting equations solved for M_c, H_c, and V_c. In general, however, the equations representing the shape of the arch axis and the variation in the moment of inertia of the arch are of such form that direct integration is not practicable and the method of summation must be used.

For this purpose the arch rib is divided into short sections $d\,s$ in length over which the moment of inertia and the bending moment may be assumed as uniform. Referring to eq. (1), it will be noted that (excluding terms containing ρ) the quantity $d\,s$ always appears with I as the ratio $d\,s/I$. The lengths $d\,s$ may be of any small magnitude, equal or unequal, but they are usually made either equal or of such value that the ratios $d\,s/I$ are equal. In general, it will be convenient to express the quantity $d\,s/I$ as $q\,d\,s_1/I_1$, where $d\,s_1/I_1$ is the value at the crown. (If equal $d\,s$'s are used, then q is a variable. If $d\,s/I$ is made constant then $q = 1$.)

Proceeding with the substitutions, we have from condition A, eq. (2),

$$\sum_A^C M\,x\,q\frac{d\,s_1}{I_1}+\sum_A^C \omega\,t\,E\,d\,y=\sum_B^C M\,x\,q\frac{d\,s_1}{I_1}+\sum_B^C \omega\,t\,E\,d\,y.$$

Substituting M from (3), and noting that the terms involving ω for the two sides are equal, and that the terms involving M_c and H_c are also equal, we have

$$\sum_A^C m_L\,x\,q\frac{d\,s_1}{I_1}+\sum_A^C V_c\,x^2\,q\frac{d\,s_1}{I_1}$$

$$=\sum_B^C m_R\,x\,q\frac{d\,s_1}{I_1}-\sum_B^C V_c\,x^2\,q\frac{d\,s_1}{I_1}.$$

From which

$$V_c=\frac{\sum_A^C (m_R-m_L)\,x\,q}{2\sum_A^C x^2\,q}. \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

In this expression $m_R - m_L$ is the difference in moments due to the applied load at corresponding points of the two halves of the arch, and the summation is for the half arch.

From condition C, eq. (2), we get, by similar process, neglecting the term $\frac{f\,d\,s}{E\,\rho}$,

$$M_c=-\frac{\sum_A^C (m_R+m_L)\,q+2\,H_c\sum_A^C y\,q}{2\sum_A^C q}. \quad . \quad . \quad . \qquad (6)$$

From condition B, neglecting the term $\frac{f\,y\,d\,s}{E\,\rho}$, we get an equation containing M_c and H_c, from which by substitution from (6) we derive

$$H_c=\frac{\sum_A^C (m_R+m_L)q\sum_A^C yq-\sum_A^C (m_R+m_L)yq\sum_A^C q+\frac{\omega tlE}{ds_1/I_1}\sum_A^C q}{2\left[\left(\sum_A^C q\right)\left(\sum_A^C y^2q+\frac{I_1}{ds_1}\sum_A^C\frac{ds\cos\alpha}{A}\right)-\left(\sum_A^C yq\right)^2\right]}.(7)$$

291. Transfer of X-axis to the Elastic Centre; Simplified Formulas for Crown Stresses.—If the X-axis be shifted a certain distance below the crown it will be found that the equations for M_c and H_c will be considerably simplified. In Fig. 5 the proposed centre of coordinates is at O, a distance y_0 from C, this distance to be determined. Let y_1 be the vertical ordinate of any point on the arch axis to the new axis. Then $y = y_1 + y_0$, and eqs. (6) and (7) will be adjusted to the new axis by substituting $y_1 + y_0$ for y. The desired value of y_0 is such that will make $\Sigma\, y_1\, q = 0$. Then the quantity $\Sigma\, y\, q$ which appears in (6) and (7) becomes $y_0 \Sigma\, q$. The desired value of y_0 is found from the stated relation, $\Sigma\, y\, q = \Sigma\, y_1\, q + y_0 \Sigma\, q = 0 + y_0 \Sigma\, q$, whence $y_0 = \Sigma\, y\, q / \Sigma\, q$. This value is calculated at the outset and all ordinates y_1 measured from the new axis. When $d\,s/I$ is made constant, the quantity y_0 is simply the average value of y.

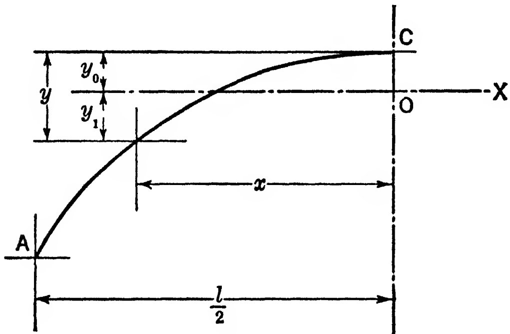


FIG. 5.

Substituting $y_1 + y_0$ for y in eqs. (6) and (7), we get the simplified equations

$$M_c = -\frac{\sum_A^C (m_R + m_L)\, q}{2\sum_A^C q} - H_c\, y_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6a)$$

$$H_c = \frac{-\sum_A^C (m_R + m_L)\, y_1\, q + \dfrac{\omega\, t\, l\, E}{d\, s_1/I_1}}{2\left[\sum_A^C y_1^2\, q + I_1 \sum_A^C \dfrac{\cos\alpha}{A}\right]} . \quad . \quad . \quad . \qquad (7a)$$

The equation for V_c, eq. (5), will not be changed, as y does not enter.

The point O is called the " elastic centre"; it will be found from eq. (1a) that a horizontal force applied to the arch at this elevation will cause no change in angle at the crown; the plus and minus moments just balance.

The simplified formulas (6*a*) and (7*a*) will be used in the example which follows.

In all the foregoing equations the summations are for one-half the arch only. Where m_R and m_L are involved, the values at corresponding points on the two sides are first combined and then the summation taken for one-half the arch.

Note that all terms of the denominators are functions of the dimensions of the arch only. The effect of applied loads is included in the terms containing m, and that of temperature in the term containing the coefficient ω. The effect of rib shortening is in the term containing $\cos \alpha$, and its relative importance can readily be observed in any particular case.

292. Temperature Stresses.—It is sometimes desirable to have separate expressions for crown stresses due to temperature changes. These may be had from eqs. (5), (6*a*), and (7*a*) by placing m_L and m_R, the terms representing the applied load, equal to zero. If V_t, M_t, and H_t represent respectively the crown shear, moment, and thrust due to a rise in temperature, we have

$$\left.\begin{aligned} V_t &= 0 \\ M_t &= H_t y_0 \\ H_t &= \frac{\omega t \dfrac{l E}{d s_1 / I_1}}{2\left[\sum_A^C y_1^2 q + I_1 \sum_A^C \dfrac{\cos \alpha}{A}\right]} \end{aligned}\right\} \quad . \; . \; . \quad (8)$$

293. Deflection of the Crown.—The deflection of the crown can be determined from eq. (1*b*) of Art. 287. Omitting the third term as before and denoting the deflection by D_c, we have

$$D_c = -\left[\sum_A^C M_x x q \frac{d s_1}{E I_1} - \sum_A^C f \frac{d y}{E} + \sum_A^C \omega t \, d y\right]$$

Substituting for M_x its value as given by eq. (3*a*), we have finally

$$D_c = -\frac{d\,s_1}{E\,I_1}\left\{(M_c + H_c\,y_0)\sum_A^C x\,q + H_c\left[\sum_A^C x\,y_1\,q - \frac{1}{d\,s_1/I_1}\sum_A^C \frac{d\,y}{A}\right] + V_c\sum_A^C x^2\,q + \sum_A^C m_L\,x\,q\right\} - \omega\,t\,h \quad . \quad . \quad . \quad (9)$$

A minus sign indicates an upward movement. The term

$$\frac{1}{d\,s_1/I_1}\sum_A^C \frac{d\,y}{A}$$

represents the effect of rib shortening. Since $d\,y/d\,s = \sin\alpha$. it may be written

$$\frac{1}{d\,s_1/I_1}\sum_A^C \frac{\sin\alpha}{A}\,d\,s,$$

and if $d\,s$ is constant it is

$$I_1\sum_A^C \frac{\sin\alpha}{A}.$$

For ordinary ratios of rise to span this term is too small to be considered. It can readily be calculated if desired.*

294. Determination of *d s* Divisions.—As stated in Art. 290, the arch rib is divided into short sections over which the moment of inertia and the bending moment may be considered as uniform. Generally these $d\,s$ divisions should not be less than about 8 in number for each half, nor exceed about 4 ft. in length. In fixing the length of the element $d\,s$, there are two general methods of procedure. The $d\,s$ sections may be made of uniform length, or the lengths may be so taken that $d\,s/I$ for all sections is a constant.

When the $d\,s$ sections are made constant in length, the ratio $d\,s/I$ for the several sections will vary. This method of division is fully illustrated by the problem which follows.

When the ratio $d\,s/I$ is made constant for all sections, the formulas for crown stresses are much simplified, as the ratio q becomes unity. A disadvantage, noticeable particularly in arches in which the ratio of springing to crown thickness is large, say greater than 2.0, is that the $d\,s$ sections near the springing are very long. This tends to intro-

* Rib shortening effect is also included in the value of H_c.

duce considerable error into the calculations because the assumption that the bending moment is constant for the entire length of a section is not realized.

295. Division of an Arch Ring into Sections with ds/I Constant.— The value of ds/I to adopt so that there will be no fractional ds division may be determined as follows:

Let $i = 1/I$.

Also, let i_a = mean value of i;

$s/2$ = half length of the arch ring measured along the arch axis;

N = desired number of divisions in one-half of the arch.

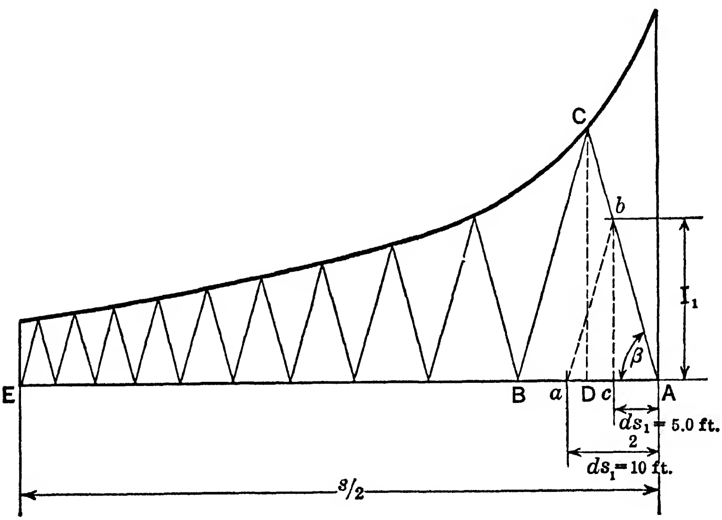


FIG. 6.

Calculate first the mean value of i for the half arch ring by determining several values at equal intervals along the arch. Then the desired value of ds/I is

$$\frac{ds}{I} = \frac{s\,i_a}{2N}. \qquad (10)$$

To illustrate the application of the method described above, the half-span of the arch shown in Fig. 7 will be divided into 10 divisions for which ds/I is a constant. Fig. 6 shows the I-curve plotted for values calculated in Table A of Art. 302. For the values of I shown, it will be found that $i_a = 6.135$. From Art. 304,

$$s/2 = 38.97 \text{ ft.}$$

Then from eq. (10), with $N = 10$, we have

$$\frac{d\,s}{I} = \frac{(38.97)(6.135)}{10} = 23.80.$$

Fig. 6 shows a graphical method by means of which the 10 $d\,s/I$ divisions can be determined. It consists of constructing 10 similar isosceles triangles between the I-curve and the base line such that the ratio of base to altitude is the desired value of $d\,s/I = 23.8$. The necessary slopes of the sides of these triangles can be found by assuming any value of $d\,s$, say 10 ft., and lay it off as $A\,a$, Fig. 12. Then at mid-point c, erect a perpendicular $c\,b$ equal to $10/23.8 = 0.420$ in height. Then $A\,b$ and $b\,a$ give the required slopes of the sides of the several triangles.

To complete the diagram, produce $A\,b$ to an intersection with the I-curve at C. At C, draw $C\,B$ parallel to $b\,a$. Continue the construction for 10 divisions, which should bring it to the centre point E. Any deficiency or overrun can be corrected by proportionate distribution of the error. In this case the first $d\,s$ length next to the springing line would be more than twice the length of a $d\,s$ division if made of uniform length.

296. Formulas for ds/I a Constant.—In this case the ratio q is unity; and if N = number of divisions for each half of the arch, $\Sigma\, q = N$, and eqs. (5), (6a), and (7a) become

$$V_c = \frac{\sum_A^C (m_R - m_L)\,x}{2\sum_A^C x^2} \quad \ldots \ldots \ldots \quad (11)$$

$$M_c = -\frac{\sum_A^C (m_R + m_L)}{2\,N} - H_c\,y_0 \quad \ldots \ldots \quad (12)$$

$$H_c = \frac{-\sum_A^C (m_R + m_L)\,y_1 + \dfrac{\omega\,t\,l\,E}{d\,s_1/I_1}}{2\left[\sum_A^C y_1^2 + \sum_A^C \cos\alpha \dfrac{I_1}{A}\right]} \quad \ldots \ldots \quad (13)$$

The equations for temperature stresses and deflection are similarly simplified.

297. Shrinkage and Plastic Flow.—The foregoing formulas have been developed on the basis of the elastic theory, namely, that concrete acts as a perfectly elastic material. In two important respects this is incorrect. As pointed out in Chap. II and other chapters, the effect of shrinkage and plastic flow act to modify considerably the results obtained by the elastic theory. A consideration of these factors is especially important in the case of the arch where the stresses are a function of distortion as well as of load.*

Shrinkage.—The effect of shrinkage may be considered as both direct and indirect. Its direct effect is like that in columns, as discussed in Chap. VII. It acts to increase the stress in the steel and decrease that in the concrete. The result on total distortion is a shortening of the column. This shortening effect in the case of the arch ring also produces bending and direct stress throughout the arch in a manner similar to a change of temperature.

The direct effect can readily be analyzed as in the case of the column. Assume the arch ring free from the abutments.

Let m = shrinkage coefficient of the concrete;
m' = actual net unit contraction of the reinforced rib.

Under the shrinkage action the steel will be under compression and the concrete under tension.

Then the relation of stresses and distortions will be as follows:

$$m' = m - \frac{f_c}{E_c} = \frac{f_s}{E_s}.$$

Also, since total steel stress equals total concrete stress, we have,

$$f_s = \frac{f_c}{2p}$$

where $2p$ = total steel ratio.

* See progress report on Arches by C. S. Whitney, Journ. Am. Concrete Inst., March, 1932, p. 479.

From these relations, we derive

$$m' = \frac{m}{1 + 2pn} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

$$f_c = (m - m')\,E_c = E_c\,m\,\frac{2pn}{1 + 2pn} \quad . \quad . \quad . \quad . \quad (15)$$

$$f_s = m'E_s = E_s\,\frac{m}{1 + 2pn}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

The total change in span length will be $m'l$, and the indirect effect will be found by substituting $m'l$ for $\omega\, t\, l$ in eq. (8) for H. This effect is exactly similar to that caused by a reduction of temperature.

The shrinkage coefficient m will depend upon the design of the arch and the manner of its construction. For a spandrel-filled arch it will be less than for an open spandrel arch. Shrinkage may be reduced by pouring the rib in sections and allowing considerable shrinkage to occur before the closing section is poured. In the construction of some long-span arches in France, Freyssinet * has made use of jacks to introduce initial compression before closing and thus reduce the effect of shrinkage and rib shortening. However, as pointed out later, the effect of plastic flow is such that the benefit secured by such a process cannot be very great.

As stated in Chap. II, the shrinkage of concrete when thoroughly dried out is likely to be as much as 0.04%. Probably a value of 0.02% for m would represent average arch conditions sufficiently well. Before attempting, however, to evaluate eqs. (15) and (16) it is necessary to consider the influence of plastic flow.

Plastic Flow.—For long-continued stresses, concrete will compress an amount much greater than represented by the value of E_c as usually determined. This compression is roughly proportional to stress, so that in the analysis of the effects of permanent loads it may be added to the usual elastic distortion. The coefficient of plastic flow may be taken at about 0.000001 (Chap. II) and if the elastic distortion is 0.0000005, corresponding to $E_c = 2{,}000{,}000$, the total

* Eng. News-Record, Sept. 18, 1924.

distortion will be 0.0000015, and the ratio of stress to total distortion is $\frac{1}{0.0000015} = 667{,}000$ lbs/in.^{2} Inasmuch as the effect of shrinkage is largely a permanent one and the dead-load compressive stresses are certain to be greater than the tensile shrinkage stress f_c of eq. (15), we may arrive at approximate results by using the modified value of $E_c = 667{,}000$, or a similar value, in the foregoing equations. For an elastic value of $E_c = 3{,}000{,}000$, the modified value to take account of plastic flow will be 750,000. The value of n will be modified accordingly, being equal to 45 and 40 respectively for the two values here suggested (equal to $15 + 30$ and $10 + 30$).

For illustration, let $m = 0.0002$, $p = 0.005$, $n = 45$; then from the foregoing equations we get $m' = \frac{0.0002}{1 + 0.45} = 0.000138$; $f_c = 667{,}000 \times 0.000062 = 41.4$ lbs/in.^{2} tension; $f_s = 0.000138 \times 30{,}000{,}000 = 4140$ lbs/in.^{2} compression. The net shrinkage coefficient of 0.000138 is equivalent to a temperature change of 23°, using $\omega = 0.000006$.

The foregoing gives some notion of the direct effect of shrinkage; it places a considerable initial compression in the steel and tension in the concrete. These stresses may be added to those determined from other causes.

The indirect effect of shrinkage is found by substituting the coefficient 0.000138 for ωt in eq. (8) and using for E in that equation the value of 667,000 so as to include plastic flow. It will be found that the value of H will be quite small under these assumed conditions. In the problem analyzed later on, it is only 410 lbs., producing a crown moment of $410 \times 1.78 = 730.0$ ft-lbs. This is in contrast to a crown thrust and moment due to combined dead and live loads of 34,000 lbs. and 7400 ft-lbs. respectively. The same general result is obtained from temperature effects that are long continued, such as those due to cooling of the concrete after pouring. Seasonal changes of temperature act too quickly to get much benefit from plastic flow, and for such changes the value of E in eq. (8) must be taken more nearly equal to the usual elastic value.

From the foregoing it is seen that the effect of plastic flow is greatly to minimize the effect of arch shortening due to shrinkage or slow and permanent changes of temperature. The effect of plastic flow upon dead- and live-load stresses remains to be considered.

Referring to the equations for thrust and moment, it will be noted that the value of E does not appear except in the quantity n in calculating I (omitting the temperature term). It will be further noted that, if the values of I for the entire arch are increased or decreased proportionately, the values of q will not be changed and the values of H and M will remain the same. We may then conclude that small proportionate changes in I, such as would be caused by a change in n, would not affect the values of H and M appreciably. Hence the dead-load moments and thrusts may be taken as correct when determined by using the ordinary value of n of 8 to 15. In getting fibre stresses from the moments and thrusts the correct value of $p\,n$ (including plastic flow) should be used in reading values from the diagrams. In the case of live-load stresses, plastic flow is of little consequence, and the usual values of E_c and n should be employed.

But unless we can assume that Case I of bending and compression (compression over the entire section) prevails, we cannot determine total stresses by calculating dead-load and live-load stresses separately and adding. The total stress must be determined for the total load. This difficulty can be met satisfactorily in the following manner: Calculate first the stresses due to dead load, using a value of n to include plastic flow. Then recalculate stresses for elastic conditions, using a corresponding value of n. The difference between these two sets of stresses will represent the effect of plastic flow; the steel stresses will be compressive and the concrete stresses tensile. Then with the usual value of E_c and n (elastic values), the stresses due to combined live and dead load (and seasonal temperature changes if desired) are calculated. The total stresses will then be equal to those due to dead and live load (and temperature), plus those due to plastic flow. To these the shrinkage stresses may be added if desired.

The foregoing method of considering the effect of plastic flow may be visualized by considering the application of dead and live load to take place as follows: First, the application of dead load until the plastic flow has taken place. Second, a removal of dead load for a short time, leaving the permanent deformation in steel and concrete. Third, reapplication of dead load with the live load and temperature effect under the assumption of elastic behavior. Although this method is not strictly correct for Case II, it gives results which represent the facts fairly well and as closely as the conditions of the problems warrant.

For application of this process to a particular case, see Art. 313.

Conclusion.—The direct effect of shrinkage is to increase steel stresses and reduce concrete stresses. The indirect effect from shortening of the arch is small.

The effect of plastic flow is to minimize greatly the effect of slow changes of temperature. Its effect on dead-load stresses is to increase steel stresses and reduce concrete stresses. The total effect on fibre stresses is marked.

298. General Observations.—The method of analysis here given is simple in theory, and easily followed in the numerical work. It will be noted that the loads and their points of application have been considered apart from the division of the arch ring into $d\,s$ sections, as the two things are in no wise related. Where spandrel filled arches are used and the entire load is applied continuously along the arch ring, the load is generally divided into concentrations equally spaced along the horizontal. Where open spandrel arches are used, the live load and a large part of the dead load will be applied at the centres of the columns which support the floor system. The weight of the main arch ring may also be considered as concentrated at these same points. However, if the arch ring is very heavy, more accurate results will be secured by dividing the arch ring into short sections as in the spandrel filled arch.

If calculations are to be made for more than one loading condition, it will be noted that the denominators for values of H_c, M_c, and V_c do not change, being dependent upon the form and shape of the arch ring and not upon the loadings. The quantities involving m, the moment due to loadings, are the only ones requiring recalculation. If the load on but one-half of the arch is changed, then the values of m for that half only need be recalculated. In the case of a symmetrical loading, or a load on one-half of the arch, the calculation of m is also necessary for one-half of the arch only. For symmetrical loads $V_c = 0$.

In view of the uncertainty regarding the allowance to be made for shrinkage, plastic flow, and temperature stresses in arches, no great refinement in the calculations is warranted. It will be found, however, that in the calculation of crown stresses the formulas involve differences between two quantities. Hence, in determining these values, a sufficient number of significant figures must be used so that these differences will have the desired precision.

299. The Unsymmetrical Arch.—When the arch is unsymmetrical, explicit formulas for the crown stresses H_c, M_c, and V_c are very cumbersome. It will be found best to carry the solution in general form only as far as the development of three simultaneous equations involving the unknowns H_c, M_c, and V_c. These can readily be written out from the fundamental equations, (1), (2), and (3), keeping in mind the fact that the two sides of the arch are unlike. For a given arch subjected to stated loading conditions, the numerical values of the coefficients of the unknowns can be found, and a solution of the resulting simultaneous equations will yield the values of the unknowns.

THE INVESTIGATION OF AN ARCH

300. Methods of Procedure.—In the application of the foregoing formulas to the analysis of an arch there are two general methods of procedure. The *influence-line method* and the *direct-load method.*

By the influence-line method the values of moment and thrust at the crown and other critical sections are determined for a unit load placed at successive intervals along the arch and sufficiently near together to give the desired accuracy. From these values the effect of dead load and any arrangement of live load can be calculated and the maximum stresses determined. Plotted influence lines can also be used to advantage in deriving general rules for placement of live load in the direct-load method.

By the direct-load method close approximations to the maximum stresses can be obtained by loading the arch in accordance with general rules determined by experience and the study of influence lines. The accuracy is not so great as obtained by the influence-line method but the process is somewhat briefer and is sufficiently exact for moderate span lengths and ordinary proportions.

If a critical investigation of an arch is to be made, the influence-line method will give the desired results with the least amount of work.

If stresses are desired for a single set of loads, as for example, the dead load, the direct-load method is more expeditious.

On examining the general formulas for crown stresses given in Art. 290, it will be noted that, in all equations, the denominators consist of terms whose values depend only upon the form of the arch. This is true also of certain factors in the numerators. It will then be found convenient in analyzing a given arch to determine first those terms which depend upon the form of the arch. Simplified formulas for crown stresses can then be written which are functions of load only. These formulas can then be used in subsequent calculations by either the influence-line or the direct-load method. In the articles which

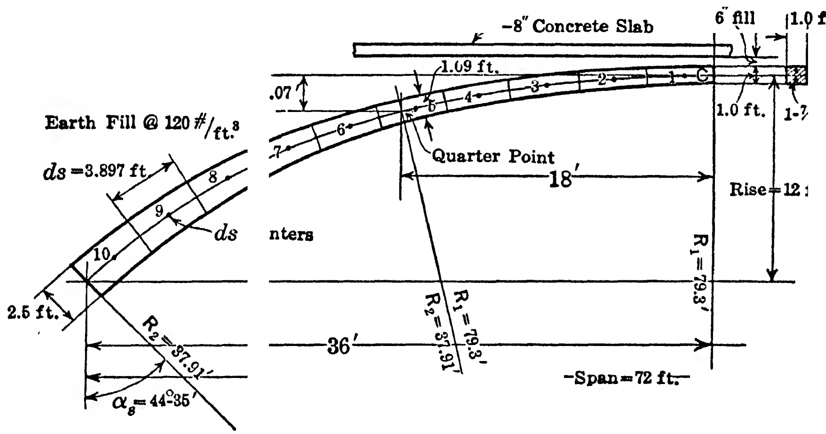


FIG. 7.

follow, such simplified formulas for a given arch will first be calculated, after which the effect of loads will be considered.

301. Dimensions of the Arch to be Investigated.—Fig. 7 shows the principal dimensions of the arch to be analyzed. The span of the arch is 72 ft. and the rise 12 ft. At the crown there is an earth fill 6 in. deep which is covered by an 8-in. concrete roadway slab. The arch ring thickness at the crown is 1 ft.; at the springing line, 2½ ft.; and at the quarter point 1.09 ft.

The arch rib is reinforced by ⅞-in. round bars placed 1½ in. from extrados and intrados and 12 in. apart. A length of arch ring of 1 ft. will be considered. E_c = 2,000,000 lbs/in.^{2}

302. Properties of the Arch Ring.—In the calculations which follow, each half of the arch ring has been divided into 10 equal $d\,s$ sections. From measurements made on a large-scale layout of the arch ring, or from calculations which can readily be made from the information given on Fig. 7, the half-length of the arch axis is found to be 38.97 ft. Each $d\,s$ section then has a length of 3.897 ft. Fig. 7 shows the half arch ring divided as indicated and the several sections numbered from crown towards springing.

Table A gives the calculations of the quantities I, A, q, and $\cos\alpha/A$. The values of I are calculated for the entire section. While under certain conditions of loading the bending moment is such that there will be tension on one side, giving rise to Case II of bending and compression in which the tensile stress in the concrete is ignored, the moment of inertia of the entire cross-section will best represent the facts as regards deformations. (See discussion in Chap. VI.) The quantities in columns 2 to 8 relate to the ends of each division; in columns 9 to 14 they relate to the $d\,s$ centres, or are averages of values given for the ends of the divisions.

Table B gives the various needed functions of x, y, y_1, and q. Values of x and y were scaled from a layout of the arch ring, then $\Sigma\,y\,q$ calculated and y_0 found from the equation $y_0 = \Sigma\,y\,q/\Sigma\,q = 11.047/6.212 = 1.78$. Then $y_1 = y - y_0$. Columns 11 to 15 of this table contain summations of the respective functions from the springing line, or tenth division, to the division in question. These quantities are used in eqs. (7) and (8) of Art. 304.

303. Formulas for Crown Stresses.—On substituting values of the several summations given in Tables A and B in the general formulas for crown stresses as given by eqs. (5), (6*a*), and (7*a*) Arts. 290 and 291, neglecting the temperature terms which will be evaluated separately, we have

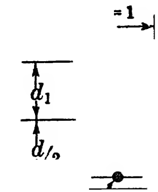
$$V_c = \frac{\sum_A^C (m_R - m_L)\,x\,q}{3153} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$M_c = -\frac{\sum_A^C (m_R + m_L)}{12.42} - 1.78\,H_c \quad . \quad . \quad (2)$$

TABLE A

PROPER1 ES OF E ARCH 'G

Valu of I, ς A, ε $\frac{\alpha}{A}$



End of Section (1)	Radial Depth, d (2)	d_1 (3)	d_1^2 (4)	I_s $2(n-1)A_s d_1^2$ (5)	I_c $1/12\, b d^3$ (6)	I (5)+(6) (7)	A $bd + 2(n-1)A_s$ (8)	Ave. I (9)	q I_1/I (10)	Ave. A (11)	α (12)	$\cos\alpha$ (13)	$\frac{\cos\alpha}{A}$ (14)	ds Centre (15)
Crown	1.00	0.375	0.141	0.0165	0.0833	0.0998	1.117	0.1013	1.000	1.122	1°–25′	1.000	0.89	1
1	1.01	0.380	0.144	0.0169	0.0859	0.1028	1.127	0.1042	0.972	1.132	4°–14′	0.997	0.88	2
2	1.02	0.385	0.148	0.0173	0.0884	0.1057	1.137	0.1073	0.944	1.142	7°–03′	0.992	0.87	3
3	1.03	0.390	0.152	0.0178	0.0911	0.1089	1.147	0.1138	0.890	1.162	9°–52′	0.985	0.84	4
4	1.06	0.405	0.164	0.0192	0.0994	0.1186	1.177	0.1270	0.798	1.202	12°–41′	0.976	0.81	5
5	1.11	0.430	0.185	0.0216	0.1139	0.1355	1.227	0.1571	0.645	1.282	18°–05′	0.951	0.74	6
6	1.22	0.485	0.235	0.0275	0.1512	0.1787	1.337	0.2259	0.448	1.432	23°–58′	0.914	0.64	7
7	1.41	0.580	0.336	0.0393	0.2338	0.2731	1.527	0.3760	0.269	1.677	29°–52′	0.867	0.52	8
8	1.71	0.730	0.533	0.0624	0.4166	0.4790	1.827	0.6456	0.157	1.997	35°–45′	0.812	0.41	9
9	2.05	0.900	0.810	0.0948	0.7175	0.8123	2.167	1.132	0.089	2.392	41°–38′	0.747	0.31	10
Spring.	2.50	1.125	1.270	0.1485	1.3021	1.4506	2.617							

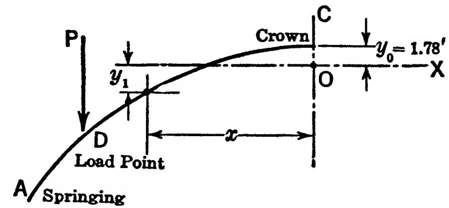
Σq $\quad\quad \Sigma \quad = 6.9$

$$A_s = \frac{2 \times 14 \times 0.601}{144} \quad\quad 3.897 \quad \frac{d s_1}{1} = \quad 897 \quad = 38.47;\ q = \quad = \frac{I_1}{I}$$

TABLE B

PROPERTIES OF THE ARCH RING

Values of x, y_1, q, and Products



Section	Values of Functions at Centre of Each Section										Summations from Springing to Given Section = $\sum_A^D$					Section
	q (1)	y (2)	$y\,q$ (3)	y_1 (4)	x (5)	$x\,q$ (6)	$x^2\,q$ (7)	$y_1\,q$ (8)	$y_1^2\,q$ (9)	$x\,y_1\,q$ (10)	$\Sigma\,q$ (11)	$\Sigma\,x\,q$ (12)	$\Sigma\,x^2\,q$ (13)	$\Sigma\,y_1\,q$ (14)	$\Sigma\,x\,y_1\,q$ (15)	
1	1.000	0.01	0.010	−1.77	1.95	1.95	3.8	−1.77	3.14	− 3.45	6.212	83.29	1576	0	107.5	1
2	0.972	0.20	0.194	−1.58	5.88	5.72	33.6	−1.53	2.43	− 9.00	5.212	81.34	1573	1.77	111.0	2
3	0.944	0.58	0.547	−1.20	9.75	9.20	89.7	−1.13	1.36	− 11.01	4.240	75.62	1539	3.30	120.0	3
4	0.890	1.18	1.050	−0.60	13.58	12.09	164.2	−0.53	0.317	− 7.20	3.296	66.42	1449	4.43	131.0	4
5	0.798	1.92	1.532	+0.14	17.40	13.88	241.5	+0.11	0.156	+ 1.91	2.406	54.33	1285	4.96	138.2	5
6	0.645	2.92	1.883	+1.14	21.18	13.66	289.3	+0.74	0.840	+ 15.68	1.608	40.45	1043	4.85	136.3	6
7	0.448	4.33	1.942	+2.25	24.79	11.12	275.6	+1.14	2.92	+ 28.26	0.963	26.79	754	4.11	120.6	7
8	0.269	6.11	1.646	+4.33	28.28	7.62	215.5	+1.17	5.05	+ 33.1	0.515	15.67	479	2.97	92.3	8
9	0.157	8.22	1.290	+6.44	31.56	4.95	156.3	+1.01	6.51	+ 31.9	0.246	8.08	263	1.80	59.2	9
10	0.089	10.65	0.953	+8.87	34.60	3.10	107.1	+0.79	7.01	+ 27.4	0.089	3.10	107	0.79	27.4	10
$\sum_1^{10}$	6.212	......	11.047	.....	....	83.29	1576.6	0.0	29.73	+107.5	....	....	....	....	.....	$\sum_1^{10}$

and

$$H_c = \frac{\sum_A^C (m_R + m_L)\, y_1\, q}{60.8} \quad . \; . \; . \; . \; . \quad (3)$$

These simplified formulas will be further evaluated in the articles which follow on the influence-line and the direct-load methods.

The denominator of eq. (3) in general form, as given by eq. (7*a*) is

$$2\left[\sum_A^C y_1^2\, q + I_1 \sum_A^C \frac{\cos\alpha}{A}\right].$$

Substituting the values from Tables A and B, this becomes equal to 2 [29.73 + 0.101 × 6.916] = 60.8.

Note that in this calculation the effect of rib shortening is included in the term containing A, and appears in the numerical quantity 0.101 × 6.916 = 0.70, which affects the final value of H_c about 2.3%. This produces considerable dead-load moment at the crown and springing but has little relative effect on live-load moments. For thicker or flatter arches the effect would be greater.

The crown stresses due to temperature changes are given by eq. (8), Art. 292. For this particular case they reduce to

$$M_c = -\,1.78\, H_c \quad . \; . \; . \; . \; . \; . \; . \; . \quad (4)$$

$$H_c = \frac{\dfrac{\omega\, t\, l\, E}{d\, s_1/I_1}}{60.8} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (5)$$

Calculation of Stresses by the Influence-Line Method

304. Values of Crown Stresses for a Unit Load at Any Point on the Arch Rib.—Fig. 8 shows an arch rib with a unit load placed at any point D on the left, at a distance a from the crown C. For such a load we have, for points between A and D, $m_L = -(x - a)$; for points between D and C, $m_L = 0$; and

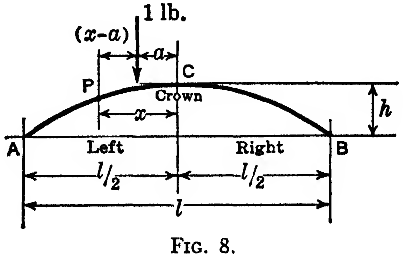


FIG. 8.

for the right half $m_R = 0$. Substituting these values in eq. (1) of Art. 303, we have

$$V_c = \frac{\sum_A^D x^2 q - a \sum_A^D x q}{3153} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Likewise from eq. (2)

$$M_c = -\frac{-\left[\sum_A^D x q - a \sum_A^D q\right]}{12.42} - 1.78\, H_c \quad . \quad . \quad (7)$$

Also from eq. (3)

$$H_c = \frac{\sum_A^D x y_1 q - a \sum_A^D y_1 q}{60.8} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

In these equations the several summations Σ_A^D are summations of the quantities indicated, taken from the left end of the arch rib at A to the load point D. These summations are given in Table B.

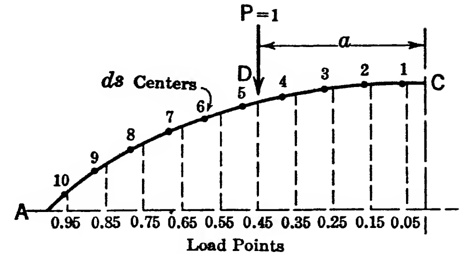
305. Influence Lines for M_c, H_c, and V_c, the Moment, Thrust, and Shear at the Crown.—The necessary values of M_c, H_c, and V_c are obtained by applying unit load successively at closely spaced intervals along the arch, and calculating the resulting crown stresses by the use of the equations of Art. 304. In this case the spacing between load points will be taken as $1/10$ the half span, the first load point being $1/20 \times 36 = 1.8$ ft. from the crown. The positions of the load points with reference to the $d\,s$ centres are shown on the sketch accompanying Table C. Such a sketch is useful in showing the number of $d\,s$ divisions to be included in the several summations Σ_A^D appearing in the equations. (It will be noted that a strictly precise calculation would involve fractional values of $d\,s$ in these summations, but the error involved in using whole units in each case is negligible.)

Table C contains all the calculations for the values of H_c, M_c, and V_c for unit loads as described. Fig. 9 shows the influence lines drawn from these values. The totals of the $+M_c$ and $-M_c$ areas are indicated on the moment influence line. On the thrust influence line, the areas corresponding to load positions for $+M_c$ and $-M_c$ are also indicated.

TABLE C

CALCULATION OF H_c, M_c, AND V_c FOR UNIT LOADS AT VARIOUS POINTS

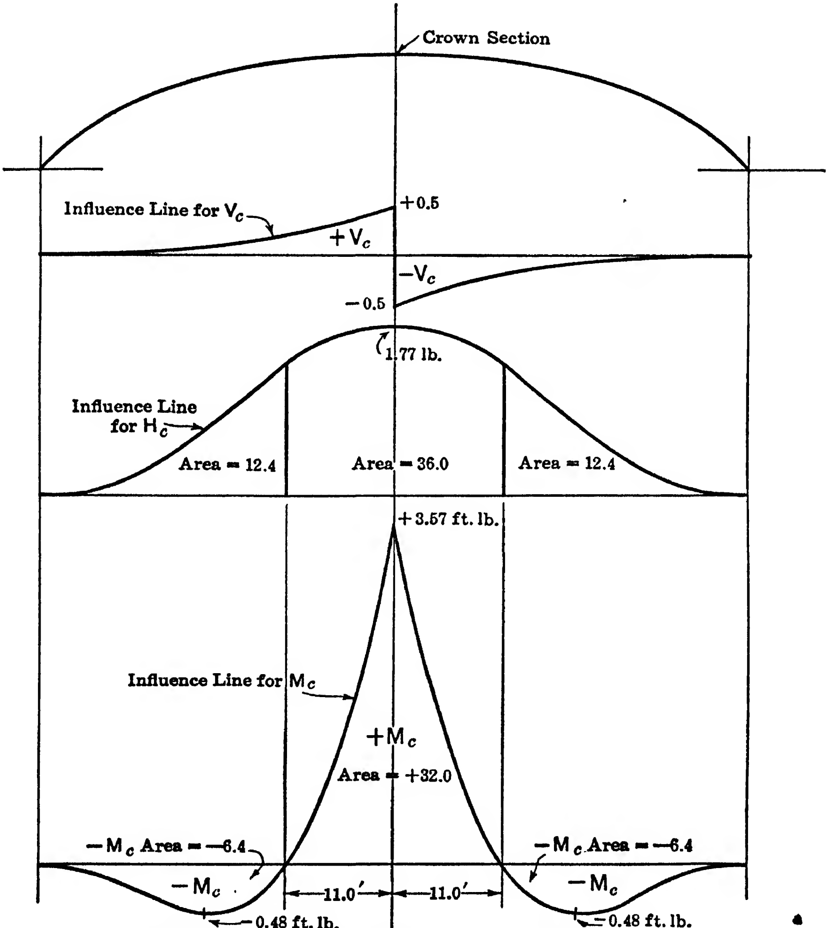
Influence-Line Values for Crown Section



Load Point $\frac{a}{\frac{1}{2}l}$ (1)	a (2)	Sections Included in $\sum_A^D$ (3)	$\sum_A^D xy_1q$ (4)	$a\sum_A^D y_1q$ (5)	H_c $\frac{(4)-(5)}{60.8}$ (6)	$\sum_A^D xq$ (7)	$a\sum_A^D q$ (8)	$\frac{(7)-(8)}{12.42}$ (9)	$1.78 H_c$ (10)	M_c (9)−(10) (11)	$\sum_A^D x^2q$ (12)	$a\sum_A^D xq$ (13)	V_c $\frac{(12)-(13)}{3153}$ (14)	Load Point
Crown	0	1–10	107.5	0	1.766	83.29	0	6.71	3.140	+3.57	1576	0	+0.500	Crown
0.05	1.8	1–10	107.5	0	1.766	83.29	11.18	5.80	3.140	+2.66	1576	150	+0.452	0.05
0.15	5.4	2–10	111.0	9.6	1.667	81.34	28.15	4.28	2.968	+1.31	1573	439	+0.359	0.15
0.25	9.0	3–10	120.0	29.7	1.485	75.62	38.17	3.015	2.643	+0.372	1539	680	+0.272	0.25
0.35	12.6	4–10	131.0	55.8	1.237	66.42	41.54	2.002	2.200	−0.198	1449	836	+0.194	0.35
0.45	16.2	5–10	138.2	80.4	0.950	54.33	38.99	1.235	1.690	−0.455	1285	880	+0.128	0.45
0.55	19.8	6–10	136.3	96.1	0.661	40.45	31.84	0.693	1.176	−0.483	1043	800	+0.077	0.55
0.65	23.4	7–10	120.6	96.2	0.401	26.79	22.53	0.343	0.714	−0.371	754	627	+0.040	0.65
0.75	27.0	8–10	92.3	80.2	0.199	15.67	13.91	0.142	0.354	−0.212	479	423	+0.018	0.75
0.85	30.6	9–10	59.2	55.1	0.067	8.05	7.53	0.042	0.120	−0.078	263	246	+0.005	0.85
0.95	34.2	10	27.4	27.0	0.006	3.10	3.04	0.005	0.011	−0.006	107	106	+0.0	0.95

$$H_c = \frac{\sum_A^D xy_1q - a\sum_A^D y_1q}{60.8};\quad M_c = \frac{\sum_A^D xq - a\sum_A^D q}{12.42} - 1.78\,H_c;\quad V_c = \frac{\sum_A^D x_2q - a\sum_A^D xq}{3153}.$$

In Fig. 9 the influence line for shear is given to complete the analysis. It will be found, however, that the shearing-stresses are so small that they may be neglected. In this case, for example, the total



INFLUENCE LINES FOR MOMENT, THRUST, AND SHEAR AT THE CROWN SECTION

FIG. 9.

shearing-stress at crown for the half span loaded is about 1120 lbs., or an average shearing-stress on the section of 7.8 lbs/in.[2] This is about 3.5% of the fibre stress due to bending, which indicates its small magnitude. For the other cases discussed the shear will be neglected.

The final result desired in these stress calculations is the maximum fibre stress at various sections along the arch. Such fibre stress is a function of both bending moment and thrust, and hence the maximum fibre stress will not in general occur for the particular position of loading giving maximum moment. For a true maximum the loading should extend somewhat beyond the limits indicated by the moment line, but the increased effect is very small, so that the position for maximum moment can safely be used for fibre stress. In this particular problem, for example, it was found that, for maximum fibre stress at the crown, the load should extend about 2 ft. farther each way than for maximum moment, but the effect of such additional load was to add less than 1% to the live-load stress, an amount of no consequence.

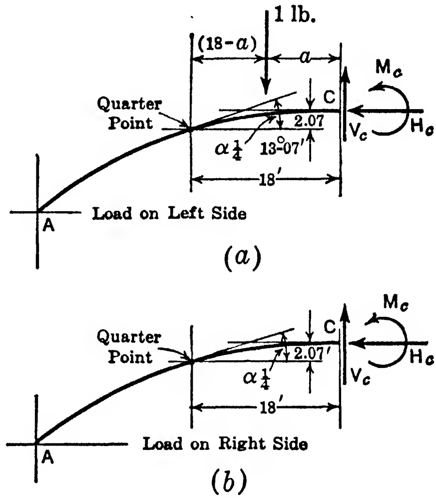


FIG. 10.

306. Influence Lines for Moment and Thrust at the Quarter Point.— Fig. 10 shows the conditions at the quarter point. For a load between the crown and the quarter point we have

$$\left.\begin{aligned} M_{1/4} &= M_c + 2.07\,H_c + 18\,V_c - (18 - a) \\ T_{1/4} &= H_c \cos\alpha_{1/4} + (1 - V_c)\sin\alpha_{1/4} \end{aligned}\right\} \quad . \quad . \quad (9)$$

When the load is between A and the quarter point or on the right side of the arch:

$$\left.\begin{aligned} M_{1/4} &= M_c + 2.07\,H_c + 18\,V_c \\ T_{1/4} &= H_c \cos\alpha_{1/4} - V_c \sin\alpha_{1/4} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

For a load on the right half, V_c will be negative. In Table D, all calculations are given for moments and thrusts at the quarter point. Fig. 11 shows the influence lines plotted from the values given in Table D.

TABLE D

CALCULATION OF $M_{1/4}$ AND $T_{1/4}$

Influence-Line Values for Quarter Point

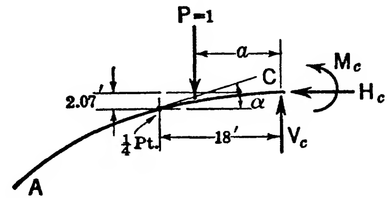
Loads A to ¼ Pt. and on Right Half

$$M_{1/4} = M_c + 2.07\,H_c + 18\,V_c$$
$$T_{1/4} = H_c \cos\alpha - V_c \sin\alpha$$

Loads ¼ Pt. to C

$$M_{1/4} = M_c + 2.07\,H_c + 18\,V_c - (18 - a)$$
$$T_{1/4} = H_c \cos\alpha + (1 - V_c)\sin\alpha$$

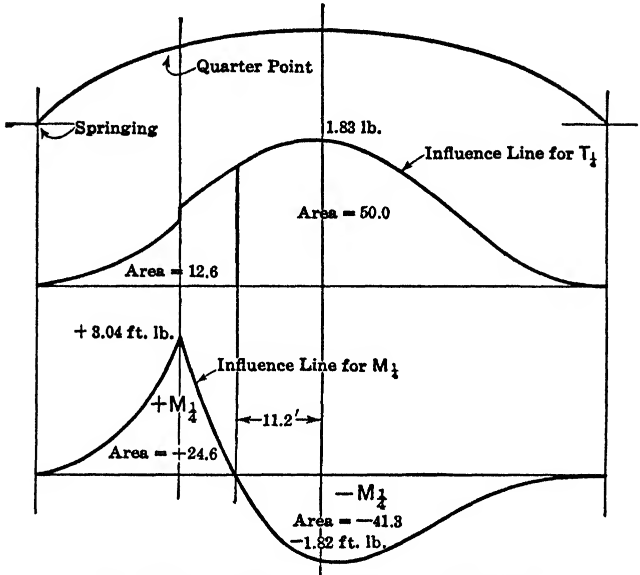
$\alpha = 13°\text{–}07'$
$\cos\alpha = 0.974$
$\sin\alpha = 0.227$



	Load Point	M_c	$2.07\,H_c$	V_c	$18\,V_c$	$-(18 - a)$	$M_{1/4}$	$H_c \cos\alpha$	$(1 - V_c)$	$(1 - V_c)\sin\alpha$	$V_c \sin\alpha$	$T_{1/4}$	Load Point
A	0.95	−0.006	0.012	+0.0003	+0.005	....	+0.011	0.006	.....	....	.....	0.006	0.95
	0.85	−0.078	0.139	+0.005	+0.090	....	+0.151	0.065	.....	....	+0.001	0.064	0.85
	0.75	−0.212	0.412	+0.018	+0.324	....	+0.524	0.194	.....	....	+0.004	0.190	0.75
	0.65	−0.371	0.830	+0.040	+0.720	....	+1.179	0.390	.....	....	+0.010	0.380	0.65
LEFT	0.55	−0.483	1.37	+0.077	+1.38	....	+2.27	0.645	.....	....	+0.018	0.627	0.55
¼ Pt.	0.50	−0.470	1.67	+0.102	+1.84	0	+3.04	0.785	0.898	0.204	+0.023	0.762 } 0.989 }	¼ Pt.
	0.45	−0.455	1.97	+0.128	+2.30	− 1.8	+2.02	0.925	0.872	0.198	.....	1.123	0.45
	0.35	−0.198	2.56	+0.194	+3.50	− 5.4	+0.46	1.205	0.806	0.182	.....	1.387	0.35
	0.25	+0.372	3.08	+0.272	+4.90	− 9.0	−0.65	1.45	0.728	0.165	.....	1.62	0.25
	0.15	+1.31	3.45	+0.359	+6.46	−12.6	−1.38	1.62	0.641	0.145	.....	1.76	0.15
	0.05	+2.66	3.66	+0.452	+8.14	−16.2	−1.74	1.72	0.548	0.125	.....	1.84	0.05
C	Crown	+3.57	3.66	+0.500	+9.00	−18.0	−1.77	1.72	0.500	0.113	.....	1.83	Crown
	0.05	+2.66	3.66	−0.452	−8.14	....	−1.82	1.72	.....	....	−0.102	1.82	0.05
	0.15	+1.31	3.45	−0.359	−6.46	....	−1.70	1.62	.....	....	−0.081	1.70	0.15
	0.25	+0.372	3.08	−0.272	−4.90	....	−1.45	1.45	.....	....	−0.062	1.51	0.25
RIGHT	0.35	−0.198	2.56	−0.194	−3.50	....	−1.14	1.205	.....	....	−0.044	1.249	0.35
	0.45	−0.455	1.97	−0.128	−2.30	....	−0.79	0.925	.....	....	−0.029	0.954	0.45
	0.55	−0.483	1.37	−0.077	−1.38	....	−0.49	0.645	.....	....	−0.017	0.662	0.55
	0.65	−0.371	0.830	−0.040	−0.720	....	−0.26	0.390	.....	....	−0.009	0.399	0.65
	0.75	−0.212	0.412	−0.018	−0.324	....	−0.124	0.194	.....	....	−0.004	0.198	0.75
	0.85	−0.078	0.139	−0.005	−0.090	....	−0.029	0.065	.....	....	−0.001	0.066	0.85
B	0.95	−0.006	0.012	−0.0003	−0.005	....	0	0.006	.....	....	.....	0.006	0.95

307. Influence Lines for Moment and Thrust at the Left Springing Section.—Referring to Fig. 12, we have, for a load on the left:

$$\left.\begin{aligned} M_s &= M_c + 12\,H_c + 36\,V_c - (36 - a) \\ T_s &= H_c \cos \alpha_s + (1 - V_c) \sin \alpha_s \end{aligned}\right\} \quad . \quad (11)$$



INFLUENCE LINES FOR THRUST AND MOMENT AT THE QUARTER POINT

FIG. 11.

For a load on the right:

$$\left.\begin{aligned} M_s &= M_c + 12\,H_c + 36\,V_c \\ T_s &= H_c \cos \alpha_s - V_c \sin \alpha_s \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; (12)$$

The calculations are given in Table E, and the influence lines in Fig. 13.

308. Dead-Load Stresses.—The dead load on a spandrel filled arch is represented by the weight of the roadway slab, if any; the earth fill above the arch rib; and the weight of the arch rib. As this is a variable load, it is best represented by a series of closely spaced concentrated loads. For this purpose the half span of the arch will be divided into 10 equal parts, measured along a horizontal line, and the weight of all material in each of these 10 sections estimated. This is a convenient division, as it corresponds to the divisions used in calculating the influence lines, but this arrangement is not necessary.

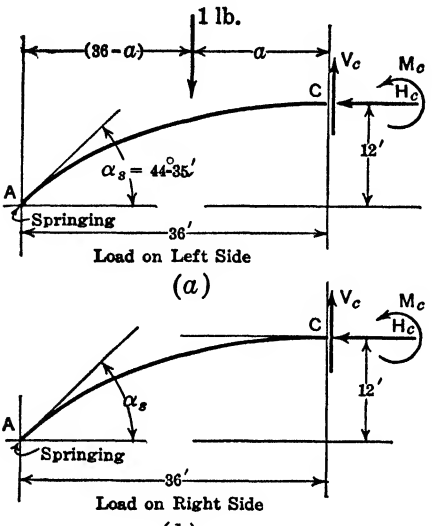


FIG. 12.

Table F gives the calculations of the dead-load concentrations. Fig. 14(a) shows the general dimensions of the arch under consideration, and Fig. 14(b) shows the half span of the arch divided into 10 equal horizontal sections 3.6 ft. long. To simplify the determination of the load due to slab and fill, the slab is reduced to an equivalent earth depth. Assuming the earth fill to weigh 120 lbs/ft.^{3}, the equivalent slab depth is $8/12 \times 150/120 = 0.83$ ft. Hence the total fill at the crown is $0.50 + 0.83 = 1.33$ ft., as shown in Fig. 14(b).

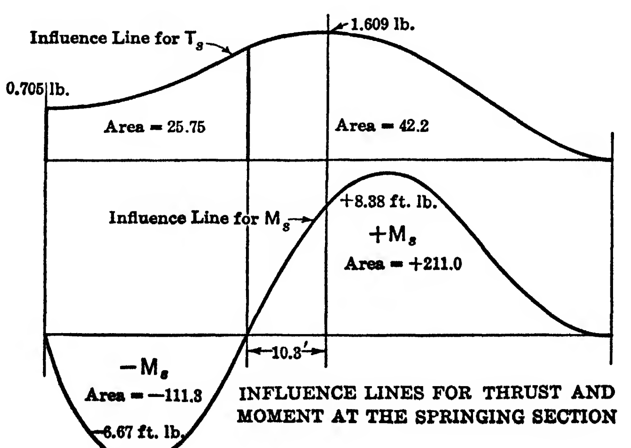


FIG. 13.

TABLE E

CALCULATION OF M_s AND T_s

Influence-Line Values for Left Springing

Loads on Left

$M_s = M_c + 12\,H_c + 36\,V_c - (36 - a)$

$T_s = H_c \cos\alpha + (1 - V_c)\sin\alpha$

Loads on Right

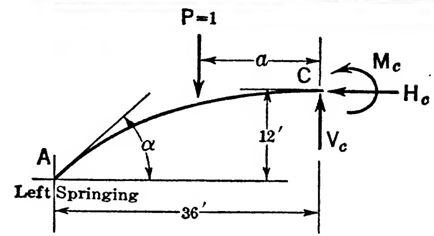
$M_s = M_c + 12\,H_c + 36\,V_c$

$T_s = H_c \cos\alpha - V_c \sin\alpha$

$\alpha = 44°\text{–}35'$

$\cos\alpha = 0.712$

$\sin\alpha = 0.702$



		Load Point	M_c	$12\,H_c$	V_c	$36\,V_c$	$-(36-a)$	M_s	$H_c \cos\alpha$	$(1-V_c)$	$(1-V_c)\sin\alpha$	$V_c \sin\alpha$	T_s	Load Point
LEFT	A	0.95	−0.006	0.072	+0.0003	+ 0.01	−1 8	−1 73	0.004	1.000	0.702	.....	0.706	0.95
		0.85	−0.078	0.80	+0.005	+ 0.18	− 5.4	−4.50	0.048	0.995	0.699	.....	0.747	0.85
		0.75	−0.212	2.40	+0.018	+ 0.65	− 9.0	−6.14	0.142	0.982	0.690	.....	0.832	0.75
		0.65	−0.371	4.81	+0.040	+ 1.45	−12.6	−6.67	0.286	0.960	0.673	.....	0.959	0.65
		0.55	−0.483	7.93	+0.077	+ 2.77	−16.2	−6.00	0.471	0.923	0.647	.....	1.118	0.55
		0.45	−0.455	11.40	+0.128	+ 4.61	−19 8	−4.24	0.677	0.872	0.612	.....	1.289	0.45
		0.35	−0.198	14.84	+0.194	+ 6.99	−23.4	−1.77	0.880	0.806	0.565	.....	1.445	0.35
		0.25	+0.372	17.82	+0.272	+ 9.80	−27.0	+0.99	1.058	0.728	0.511	.....	1.569	0.25
		0.15	+1.31	20.00	+0.359	+12.93	−30.6	+3.64	1.188	0.641	0.450	.....	1.638	0.15
		0.05	+2.66	21.20	+0.452	+16.27	−34.2	+5.93	1.258	0.548	0.384	.....	1.642	0.05
	C	Crown	+3.57	21.20	+0.500	+18.00	−36.0	+6.67	1.258	0.500	0.351	0.351	1.609	Crown
RIGHT		0.05	+2.66	21.20	−0.452	−16.27	....	+7.59	1.258	.....	....	−0.317	1.575	0.05
		0.15	+1.31	20.00	−0.359	−12.93	....	+8.38	1.188	.....	....	−0.252	1.440	0.15
		0.25	+0.372	17.82	−0.272	− 9.80	....	+8.39	1.058	.....	.....	−0.191	1.249	0.25
		0.35	−0.198	14.84	−0.194	− 6.99	....	+7.65	0.880	.....	.....	−0.136	1.016	0.35
		0.45	−0.455	11.40	−0.128	− 4.61	....	+6.34	0.677	.....	....	−0.090	0.767	0.45
		0.55	−0.483	7.93	−0.077	− 2.77	....	+4.68	0.471	.....	....	−0.054	0.525	0.55
		0.65	−0.371	4.81	−0.040	− 1.45	....	+2.99	0.286	.....	....	−0.028	0.314	0.65
		0.75	−0.212	2.40	−0.018	− 0.65	....	+1.54	0.142	.....	.....	−0.013	0.155	0.75
		0.85	−0.078	0.80	−0.005	− 0.18	....	+0.54	0.048	.....	.....	−0.003	0.051	0.85
	B	0.95	−0.006	0.072	−0.0003	− 0.01	....	+0.05	0.004	.....	.....	.....	0.004	0.95

All other depths are scaled from Fig. 14(*b*), using the equivalent fill level as a base. Vertical depths of the arch ring, as given in column 4 of Table F, are also scaled from Fig. 14(*b*). After the average depths of concrete have been calculated, the corresponding load is equal to the average depth times the weight of concrete per foot of

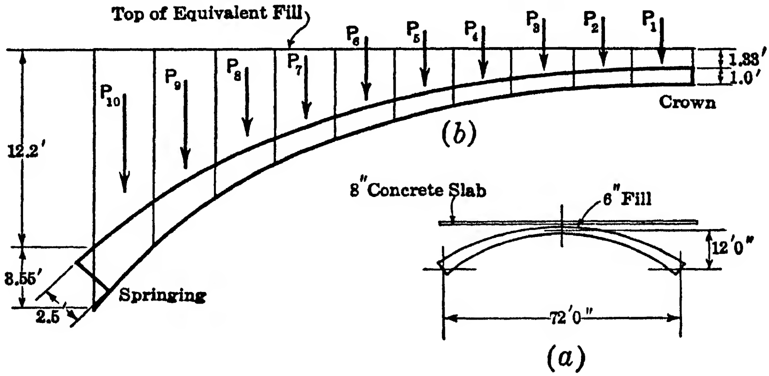


FIG. 14.

depth, which is $3.6 \times 150 = 540$ lbs. These loads are given in column 7 of the table.

TABLE F

DEAD-LOAD CONCENTRATIONS

Section (1)	Depth of Fill in Feet (2)	Depth of Fill in Terms of Concrete in Feet (3)	Depth of Arch Ring in Feet (4)	Total Depth in Terms of Concrete in Feet (5)	Average Depth in Feet (6)	Load in Pounds (7)	Load (8)
Spring	12.2	9.75	3.55	13.30	11.71	6,320	P_{10}
9	9.50	7.60	2.52	10.12	8.93	4,830	P_9
8	7.35	5.88	1.85	7.73	6.86	3,710	P_8
7	5.62	4.50	1.50	6.00	5.33	2,880	P_7
6	4.28	3.42	1.25	4.67	4.22	2,280	P_6
5	3.32	2.66	1.12	3.78	3.47	1,870	P_5
4	2.60	2.08	1.08	3.16	2.90	1,570	P_4
3	2.00	1.60	1.05	2.65	2.48	1,340	P_3
2	1.61	1.29	1.02	2.31	2.21	1,190	P_2
1	1.38	1.10	1.01	2.11	2.08	1,120	P_1
Crown	1.33	1.06	1.00	2.06			

$\Sigma P = 27{,}110$ lbs.

TABLE G

DEAD-LOAD STRESSES AT CROWN, QUARTER POINT, AND SPRINGING

Influence-Line Method

	Load Point	Load		Crown: Influence-Line Ordinate for M_c	Crown: M_c	Crown: Influence-Line Ordinate for T_c	Crown: T_c	Quarter Point: Influence-Line Ordinate for $M\frac{1}{4}$	Quarter Point: $M\frac{1}{4}$	Quarter Point: Influence-Line Ordinate for $T\frac{1}{4}$	Quarter Point: $T\frac{1}{4}$	Left Springingg: Influence-Line Ordinate for M_s	Left Springingg: M_s	Left Springingg: Influence-Line Ordinate for T_s	Left Springingg: T_s
LEFT HALF	0.95	P_{10}	6,320	−0.006	− 36	0.006	40	+0.011	+ 70	0.006	40	−1.73	−10,950	0.706	4,470
	0.85	P_9	4,830	−0.078	− 377	0.067	320	+0.151	+ 730	0.064	310	−4.50	−21,700	0.747	3,610
	0.75	P_8	3,710	−0.212	− 787	0.199	740	+0.524	+1,940	0.190	700	−6.14	−22,800	0.832	3,090
	0.65	P_7	2,880	−0.371	−1,070	0.401	1,154	+1.179	+3,400	0.380	1,100	−6.67	−19,200	0.959	2,760
	0.55	P_6	2,280	−0.483	−1,100	0.661	1,510	+2.27	+5,170	0.627	1,430	−6.00	−13,680	1.118	2,550
	0.45	P_5	1,870	−0.455	− 852	0.950	1,780	+2.02	+3,780	1.123	2,100	−4.24	− 7,930	1.289	2,410
	0.35	P_4	1,570	−0.198	− 311	1.237	1,940	+0.46	+ 720	1.387	2,170	−1.77	− 2,780	1.445	2,270
	0.25	P_3	1,340	+0.372	+ 498	1.485	1,990	−0.65	− 870	1.62	2,170	+0.99	+ 1,330	1.569	2,100
	0.15	P_2	1,190	+1.31	+1,560	1.667	1,980	−1.38	−1,640	1.76	2,090	+3.64	+ 4,330	1.638	1,950
	0.05	P_1	1,120	+2.66	+2,980	1.766	1,980	−1.74	−1,950	1.84	2,060	+5.93	+ 6,650	1.642	1,840
RIGHT HALF	0.05	P_1	1,120		+ 505		13,430	−1.82	−2,040	1.82	2,040	+7.59	+ 8,500	1.575	1,760
	0.15	P_2	1,190					−1.70	−2,020	1.70	2,020	+8.38	+ 9,960	1.440	1,710
	0.25	P_3	1,340					−1.45	−1,940	1.51	2,020	+8.39	+11,230	1.249	1,670
	0.35	P_4	1,570					−1.14	−1,790	1.25	1,960	+7.65	+12,000	1.016	1,590
	0.45	P_5	1,870					−0.79	−1,480	0.954	1,780	+6.34	+11,860	0.767	1,430
	0.55	P_6	2,280					−0.49	−1,120	0.662	1,510	+4.68	+10,700	0.525	1,200
	0.65	P_7	2,880					−0.26	− 750	0.399	1,150	+2.99	+ 8,600	0.314	900
	0.75	P_8	3,710					−0.124	− 460	0.198	730	+1.54	+ 5,710	0.155	570
	0.85	P_9	4,830					−0.029	− 140	0.066	320	+0.54	+ 2,610	0.051	250
	0.95	P_{10}	6,320					0	0	0.006	40	+0.05	+ 320	0.004	30
				Totals	+1,010		26,860		− 390		27,740		− 5,240		38,160

To determine stresses by influence-line methods, each of these dead loads must be multiplied by the corresponding influence-line ordinate. The sum of all such values will represent the total moment or thrust desired. In this case, the influence-line ordinates have been calculated for points which coincide with the load positions. If this condition did not exist, the desired influence-line ordinates could be scaled from the influence lines of Figs. 9, 11, and 13, or calculated by interpolation.

Table G gives in convenient form all calculations for the determination of dead-load moment and thrust at the crown, the quarter point, and the left springing section. Since crown values are symmetrical about the centre line, the total is twice the summation for one-half of the arch.

If the arch axis follows exactly the dead-load equilibrium polygon, there would be no dead-load moments except for rib shortening, and these could be calculated from the ratio of the second term in the denominator of eq. (7*a*), Art. 291, to the entire denominator. From Art. 303 this ratio is 2.3%, hence the value of H_c for dead load will be decreased 2.3%, or by 630 lbs., resulting in moment of $630 \times 1.78 = +1220$ ft-lbs. at the crown and $630 \times (12 - 1.78) = -6400$ ft-lbs. at springing. Compare these with the values given in Table G.

309. Live-Load Stresses.—Most specifications for arches state that they shall be designed for a uniform load of from 80 to 120 lbs/ft.^{2} of roadway, or for a truck weighing from 15 to 18 tons. For spandrel filled arches, the roadway slab and the earth fill tend to distribute these concentrated truck loads so that they may be assumed to reach the arch rib as a uniformly distributed load over the length covered.

Assuming that the arch is to be designed for an 18-ton truck, which occupies a roadway area 10 ft. by 30 ft., this will give an equivalent uniform load of $36{,}000/300 = 120$ lbs/ft.^{2} Generally from ⅔ to ¾ of the total load is carried on the rear axles. Assuming this load distribution and adding about 25% for impact, the uniform load under the rear axles may be as much as 200 lbs/ft.^{2} In the calculations which follow, a uniform live load of 200 lbs/ft.^{2} will be assumed.

The live-load moments and thrusts for uniform loading can be obtained by multiplying the influence line area by the load per square foot. Such areas for maximum positive and negative moments, together with the corresponding area for simultaneous thrust, are shown on the influence lines of Figs. 9, 11, and 13. Table H contains all information necessary for the calculation of live-load moments and the simultaneous thrusts at the crown, quarter point, and the springing sections.

310. **Temperature Stresses.**—Considerable uncertainty exists regarding the exact nature of temperature changes in an arch and their relation to external temperature conditions. In spandrel filled arches only the lower surface of the arch ring is directly exposed to full temperature effect. Temperature changes in the concrete naturally lag behind atmospheric changes. It has been observed that the temperature changes in an arch ring will be equal to about ⅔ of the annual variation between the seasonal maximum and minimum. Thus in the North Central states, where the variation from maximum summer to minimum winter temperature is about 120° F., the arch ring is subjected to about an 80° F. change.

TABLE H

LIVE-LOAD MOMENTS AND THRUSTS

Influence-Line Method

$w = 200$ lbs/ft.

Section	Maximum Moments		Thrusts Corresponding to Maximum Moments	
	Influence-Line Area	Moment, Ft-lbs.	Influence-Line Area	Thrust, Lbs.
Crown	+32.0 −12.8	+6,400 −2,560	36.0 24.8	7,200 4,960
Quarter Point	+24.6 −41.3	+4,920 −8,260	12.6 50.0	2,520 10,000
Left springing	+211.0 −111.3	+42,200 −22,260	42.2 25.75	8,440 5,150

It is generally assumed that this temperature change in the arch ring takes place as a 40° F. plus or minus change from an assumed

TABLE I

COMBINED MOMENTS AND THRUSTS

Dead Load, Live Load and Temperature

Moments in foot-pounds; thrusts in pounds

			Dead Load	Live Load	Temperature Condition	Temperature Effect	Total
Crown........	Positive moment	$+M_c$ T_c	+ 1,010 26,860	+ 6,400 7,200	Fall	+ 3,800 − 2,130	+11,200 31,900
	Negative moment	$-M_c$ T_c	+ 1,010 26,860	− 2,560 4,960	Rise	− 3,800 + 2,130	− 5,350 33,900
Quarter point..	Positive moment	$+M_{1/4}$ $T_{1/4}$	− 390 27,740	+ 4,920 2,520	Rise	+ 610 + 2,080	+ 5,140 32,300
	Negative moment	$-M_{1/4}$ $T_{1/4}$	− 390 27,740	− 8,260 10,000	Fall	− 610 − 2,080	− 9,260 35,700
Springing......	Positive moment	$+M_s$ T_s	− 5,240 38,160	+42,200 8,440	Rise	+21,800 + 1,520	+58,800 48,100
	Negative moment	$-M_s$ T_s	− 5,240 38,160	−22,260 5,150	Fall	−21,800 − 1,520	−49,300 41,800

normal condition at which the arch was closed. This ignores the temperature condition prevailing at the time of closing of the arch, which is likely to be considerably above the normal. However, as pointed out in Art. 297, the effect of a permanent change in temperature is small, and hence only the seasonal changes need be considered. A 40° F. plus or minus change will therefore be assumed in this problem.

The value of ω, the coefficient of linear expansion per degree Fahrenheit, depends upon the density of the concrete, varying from 0.000005 to 0.000006. In the calculations which follow, we will assume $\omega = 0.000006$. Then from eqs. (4) and (5), for a rise of 40°,

$$H_c = 53.3t = 2130 \text{ lbs., compression.}$$

$$M_c = -1.78\, H_c = -3800 \text{ ft-lbs.}$$

Quarter Point Values

$$M_{1/4} = -3800 + (2132)2.07 = +610 \text{ ft-lbs.}$$

$$T_{1/4} = 2132 \times 0.974 = 2080 \text{ lb., compression.}$$

Springing Values

$$M_s = -3800 + (2132)12 = +21{,}800 \text{ ft-lbs.}$$

$$T_s = 2132 \times 0.712 = 1520 \text{ lb., compression.}$$

For a fall in temperature, all values have opposite signs.

311. **Total Moments and Thrusts.**—Table I gives the total stresses at the crown, quarter point, and springing sections of the arch rib, due to dead load, live load, and temperature. In making the several combinations, the temperature conditions, rise or fall, are so chosen as to increase the moment values.

312. **Fibre Stresses in the Arch Rib.**—Maximum fibre stresses at the crown, quarter point, and springing sections due to the moments and thrusts recorded in Table I can be determined by the methods given in Chap. III on Bending and Direct Stress.

Table J presents all necessary calculations, making use of diagrams 12 to 22.

TABLE J

MAXIMUM FIBRE STRESSES

Dead Load, Live Load, and Temperature

Section		Moment, Ft-lbs.	Thrust Lbs.	e	h	e/h	h/e	p	pn or $p(n-1)$	d'/h	Case	C	f_c Lbs./in.²	f_s Lbs./in.²
Crown	$+M$	+11,200	31,900	0.351	1.0	.....	2.85	0.0042	0.063	0.125	II	8.4	654	3,500
	$-M$	− 5,350	33,900	0.158	1.0	0.158	....	0.0042	0.059	0.125	I	1.72	405	......
Quarter Point	$+M$	+ 5,140	32,300	0.159	1.09	0.146	....	0.0038	0.053	0.115	I	1.64	337	......
	$-M$	− 9,260	35,700	0.259	1.09	.....	4.23	0.0038	0.057	0.115	II	9.0	486	200
Springing	$+M$	+58,800	48,100	1.223	2.5	.....	2.04	0.0017	0.025	0.050	II	10.0	652	12,200
	$-M$	−49,300	41,800	1.179	2.5	.....	2.12	0.0017	0.025	0.050	II	9.8	535	8,800

It will be noted that the maximum compressive stresses in the concrete at the crown and springing are practically equal, while the quarter point stresses are relatively low. The steel stresses are also relatively low, except at the springing section.

If in any case the fibre stresses at any section are somewhat above the allowable values, these stresses may be reduced by changing the steel reinforcement. Any small change in the reinforcement will not greatly affect the moments and thrusts, as these values are a function of the deformation of the arch as a whole, which is not greatly affected by local changes in the reinforcement. Hence the reinforcement can be revised considerably without the necessity of recalculation.

313. Effect of Shrinkage and Plastic Flow on Fibre Stresses.—Calculations will be made for the crown section in accordance with the method of Art. 297.

The following value will be assumed:

Shrinkage coefficient $m = 0.0002$;

Coefficient of plastic flow $= 0.000001$;

Corresponding value of $n = 30$;

$n = 15$ for elastic deformations;

$n = 45$ for both plastic and elastic deformations;

$p = 0.0042$.

Effect of Shrinkage.—Referring to Art. 297, eqs. (14) to (16), we have

$$m' = \frac{0.0002}{1 + 45 \times 0.0084} = 0.000145;$$

$$f_c = 667{,}000 \times 0.0000555 = 37 \text{ lbs/in.}^2 \text{ tension;}$$

$$f_s = \frac{37}{0.0084} = 4400 \text{ lbs/in.}^2 \text{ compression.}$$

This is the direct effect. Substituting the coefficient of 0.000145 for ω in eq. (5), Art. 303, gives a value of $H = -430$ lbs., producing a crown moment of $430 \times 1.78 = 770$ ft-lbs., a positive moment. This can be added to the dead-load moment as it is of a permanent nature.

Effect of Plastic Flow.—Referring to Table I, the dead-load crown moment is $+1010$ and thrust 26,860. Combining these with shrinkage effect gives $M_c = +1780$, $T_c = 26{,}430$. Then, using a value of $n = 45$, $p\,n = 0.19$, we proceed to find the fibre stresses in the usual way. The results are:

$$f_c = 180;\quad f'_c = 88;\quad f'_s = 7600;\quad f_s = 4500, \text{ all compression.}$$

Then with the same moments and thrusts, using $n = 15$, $p\,n = 0.063$, recalculate the stress values, getting

$$f_c = 223;\quad f'_c = 103;\quad f'_s = 3120;\quad \text{and } f_s = 1700.$$

Taking the differences between these values gives the following stresses, which may be considered as the combined effect of shrinkage and plastic flow.

$$\begin{aligned} \text{Concrete}\quad f_c &= -37 + 180 - 223 = -80 \text{ (tension);} \\ f'_c &= -37 + 88 - 103 = -52 \text{ (tension).} \\ \text{Steel}\quad f'_s &= 4400 + 7600 - 3100 = +8900 \text{ (compression);} \\ f_s &= 4400 + 4500 - 1800 = +7100 \text{ (compression).} \end{aligned}$$

The stresses due to dead and live load and temperature are as given in Table J, namely: $f_c = 654$ compression and $f_s = 3500$ tension. The concrete stress will be reduced about 80 lbs/in.^{2}, as above noted. The total steel stress f_s will be about $7100 - 3500 = 3600$ lbs/in.^{2} compression. The value of f'_s (not shown in Table J) is, from dead and live load, 7800; and the total value will be $8900 + 7800 = 16{,}700$ lbs/in.^{2} compression.

314. Influence-Line Methods Applied to the Open Spandrel Arch.—In the open spandrel type of arch, the floor loads, such as the dead load from the roadway and the live load applied to the roadway, are brought to the arch rib as concentrations. In calculating stresses for such arches, the concentrations at each column should be determined and the stress then found by multiplying each concentration by the corresponding influence-line ordinate, or by the value calculated by interpolation from the values given in the tables. In calculating dead-load stresses due to the arch rib, it will be best to divide the arch rib into short sections, and proceed as in Art. 308. The position of live load for maximum stresses may be taken from the influence lines.

Calculation of Stresses by the Direct-Load Method

315. General Method of Procedure.—As stated in Art. 300, in the direct-load method the applied loads are placed in position on the arch rib and the crown stresses determined from the equations of Arts. 290 and 291.

In calculating dead-load stresses by this method, the concentra-

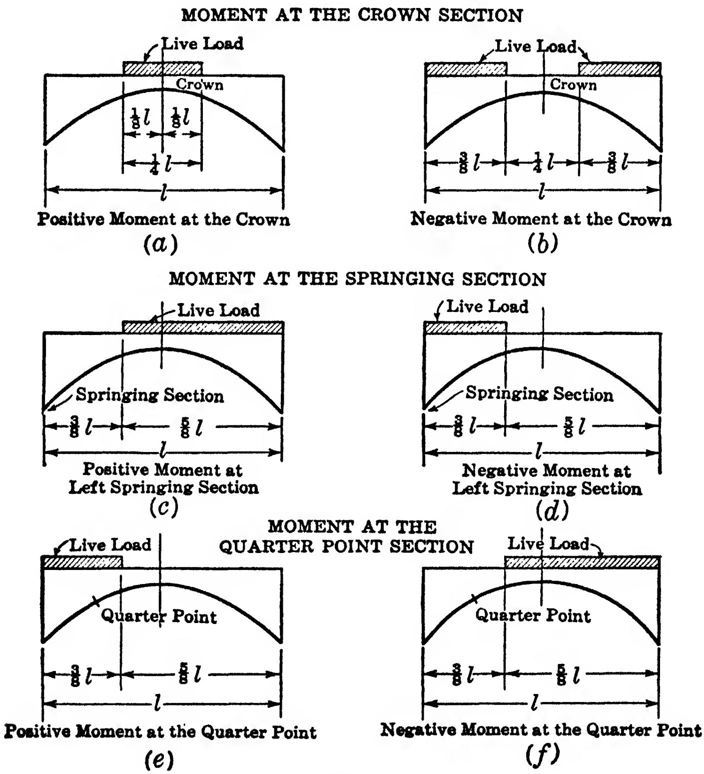


FIG. 15.

tions due to dead load must be determined, as in Table F of Art. 308. Values of m_L and m_R must then be calculated and substituted in the general equations.

For accurate results the live-load stresses must be determined by cut-and-try methods, using several trial positions and comparing results for maximum values. If influence lines are available, the correct live-load positions can readily be determined. In most cases, the live-load stresses can be found with sufficient accuracy from approximate load positions which represent average conditions for the several centres. Fig. 15 shows such load positions for the crown, springing, and quarter-point sections. Compare these with the exact positions as shown by the influence lines.

TABLE K

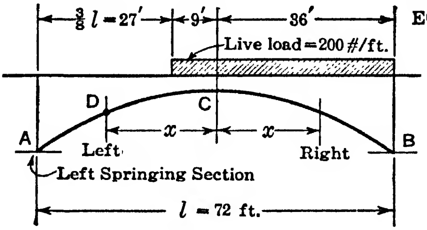
VALUES OF m_L FOR DEAD LOAD

Direct-Load Method

Point	Distance x from Crown	Load at Point	Sum of Loads	Distance between Points	Increment of Moment	Total Moment, m_L	Point
Crown	0	....	....	....	....	....	Crown
P_1	1.80	1,120	1,120	1.80	....	....	P_1
1	1.95	....	1,120	0.15	170	170	1
P_2	5.40	1,190	2,310	3.45	3,870	4,040	P_2
2	5.88	....	2,310	0.48	1,110	5,150	2
P_3	9.00	1,340	3,650	3.12	7,210	12,360	P_3
3	9.75	....	3,650	0.75	2,740	15,100	3
P_4	12.60	1,570	5,220	2.85	10,400	25,500	P_4
4	13.58	....	5,220	0.98	5,120	30,620	4
P_5	16.20	1,870	7,090	2.62	13,680	44,300	P_5
5	17.40	....	7,090	1.20	8,500	52,800	5
¼ Pt.	18.00	....	7,090	0.60	4,250	57,050	¼ Pt.
P_6	19.80	2,280	9,370	1.80	12,750	69,800	P_6
6	21.18	....	9,370	1.38	12,930	82,730	6
P_7	23.40	2,880	12,250	2.22	20,800	103,530	P_7
7	24.79	....	12,250	1.39	17,030	120,560	7
P_8	27.00	3,710	15,960	2.21	27,100	147,660	P_8
8	28.28	....	15,960	1.28	20,430	168,090	8
P_9	30.60	4,830	20,790	2.32	37,000	205,090	P_9
9	31.56	....	20,790	0.96	20,000	225,090	9
P_{10}	34.20	6,320	27,110	2.64	54,900	279,990	P_{10}
10	34.60	....	27,110	0.40	10,850	290,840	10
Spring	36.00	....	27,110	1.40	38,000	328,840	Spring

316. Dead-Load Stresses.—In Table K are given the dead-load concentrations (from Table F) and the calculation of the values of m_L for the left half. Table L gives the calculation of the terms required in the equations for H_c and M_c and the final values of these quantities. Compare these values with those given in Table G. The discrepancy between the values for M_c is of no significance as the moments are small.

317. Live-Load Stresses.—As an example of the calculation of live-load stresses by the direct-load method, the maximum live-load



EQUATIONS FOR VALUES OF m_L AND m_R

Left Side:—

C to D $x < 9$ ft.

$m_L = -100\,x^2$

D to A $x > 9$ ft.

$m_L = -1800\,(x - 4.5)$

Right Side:—

C to B

$m_R = -100\,x^2$

FIG. 16.

positive moment and the corresponding thrust at the left springing section will be determined. Fig. 16 shows the assumed live-load posi-

TABLE L

VALUES OF M_c AND H_c FOR DEAD LOAD

Direct-Load Method

Point, ds Center	q	$y_1 q$	$2m_L = (m_R + m_L)$	$(m_R + m_L)\,q$	$(m_R + m_L)\,y_1 q$
1	1.000	−1.77	− 340	− 340	+ 600
2	0.972	−1.53	− 10,300	− 10,000	+ 15,900
3	0.944	−1.13	− 30,200	− 28,500	+ 34,100
4	0.890	−0.53	− 61,240	− 54,500	+ 32,500
5	0.798	+0.11	−105,600	− 84,300	− 11,600
6	0.645	+0.74	−165,460	−106,800	−122,500
7	0.448	+1.14	−241,120	−108,000	−275,000
8	0.269	+1.17	−336,180	− 90,500	−393,000
9	0.157	+1.01	−450,180	− 70,700	−454,000
10	0.089	+0.79	−581,680	− 51,800	−460,000
Totals				−605,440	−1,633,000

$$H_c = \frac{+1{,}633{,}000}{60.8} = 26{,}900 \text{ lbs.}$$

$$M_c = \frac{605{,}440}{12.42} - 26{,}900 \times 1.78 = +940 \text{ ft-lbs.}$$

tion as in Fig. 15(*c*), and formulas for *m* for various positions of the arch.

Tables M and N give the calculations of the necessary functions of *m* and the resulting values of M_c, H_c, and V_c. Then for the springing point

$$M_s = 4960 + 9100 \times 12 - 427 \times 36 - 200 \times 9 \times 31.5$$
$$= +42{,}000 \text{ ft-lbs.}$$

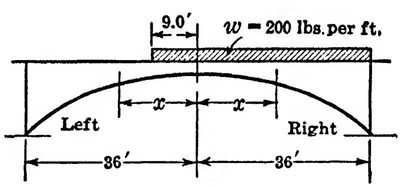
$$T_s = 9100 \cos \alpha_s + (1800 + 427) \sin \alpha_s = 8040 \text{ lbs.}$$

TABLE M

VALUES OF m_L AND m_R FOR MAXIMUM LIVE-LOAD MOMENT AT LEFT SPRINGING

Direct-Load Method

$w = 200$ lbs/ft.



For $x = 0$ to 9 ft., $m_L = 200 \times x^2$

For $x = 9$ to 36 ft., $m_L = 200 \times 9 \times (x - 4.5)$

$m_R = 200 \times x^2$

Point	x	x^2	$(x - 4.5)$	m_L	m_R	Point
1	1.95	3.80	....	− 380	− 380	1
2	5.88	34.60	....	− 3,460	− 3,460	2
3	9.75	95.0	5.25	− 9,450	− 9,500	3
4	13.58	184.4	9.08	−16,340	− 18,440	4
5	17.40	303	12.90	−23,200	− 30,300	5
6	21.18	449	16.68	−30,000	− 44,900	6
7	24.79	615	20.29	−36,500	− 61,500	7
8	28.28	800	23.78	−42,800	− 80,000	8
9	31.56	996	27.06	−48,700	− 99,600	9
10	34.60	1200	30.10	−54,200	−120,000	10

TABLE N

VALUES OF M_c, H_c, AND V_c FOR MAXIMUM LIVE-LOAD MOMENT AT LEFT SPRINGING

Direct-Load Moment

Point	q	xq	$y_1 q$	(m_R+m_L)	(m_R-m_L)	$(m_R+m_L)q$	$(m_R+m_L)y_1q$	$(m_R-m_L)xq$
1	1.000	1.95	−1.77	− 760	0	− 760	+ 1,350	0
2	0.972	5.72	−1.53	− 6,920	0	− 6,730	+ 10,600	0
3	0.944	9.20	−1.13	− 18,950	− 50	− 17,900	+ 21,400	− 460
4	0.890	12.09	−0.53	− 34,780	− 2,100	− 30,900	+ 18,400	− 25,400
5	0.789	13.88	+0.11	− 53,500	− 7,100	− 42,700	− 5,880	− 98,500
6	0.645	13.66	+0.74	− 74,900	−14,900	− 48,300	− 55,400	− 203,500
7	0.448	11.12	+1.14	− 98,000	−25,000	− 44,000	−111,800	− 278,000
8	0.269	7.62	+1.17	−122,800	−37,200	− 33,000	−143,700	− 283,000
9	0.157	4.95	+1.01	−148,300	−50,900	− 23,300	−149,800	− 252,000
10	0.089	3.10	+0.79	−174,200	−65,800	− 15,500	−137,800	− 204,000
					Totals	−263,090	−552,700	−1,344,860

$$H_c = \frac{552{,}700}{60.8} = 9{,}100 \text{ lbs.}$$

$$M_c = \frac{263{,}090}{1{,}242} - 9{,}100 \times 1.78 = + 4{,}960 \text{ ft-lbs.}$$

$$V_c = \frac{-1{,}344{,}800}{3{,}153} = - 427 \text{ lbs.}$$

The Design of an Arch

318. The Problem of Design.—*General Conditions.*—The foregoing articles have presented a method of analysis of an arch the dimensions of which have been assumed. There remains to be considered the question of the selection of a tentative design and the manner of making modifications in proportions which the stress analysis shows to be necessary or desirable. A common method of procedure is to select an arch ring whose dimensions are obtained by comparison with existing arches of similar span and loading conditions. This arch is then analyzed, and if the stresses are found to be within the allowable limits the design is considered satisfactory. If the stresses are too high or too low, changes are made until finally a design is obtained in which the stresses are satisfactory.

Owing to the general nature of the problem and the many variables involved, it is impracticable to formulate definite rules for the determination of the proper dimensions of the arch ring and its reinforcement. In the process of arriving at a tentative design much assistance may be obtained from the elaborate papers by Cochrane * and Whitney.† In each of these papers are given formulas and tables relating to the form of arch axis and influence lines for moment and thrust and temperature effects for a wide variety of proportions. In the former the arches are classified according to ratio of rise to span and ratio of depth at springing to depth at crown, and also as to whether the arch is of open spandrel or filled spandrel type; in the latter the classification is based on relative position of arch axis at quarter point and the ratio of moment of inertia at crown to moment of inertia at springing multiplied by cosine of slope at springing. The work of Mr. Cochrane is the better adapted to the type of arch in which a constant amount of reinforcement is used throughout with a relatively thick arch at springing; Mr. Whitney's, to the type where there is less variation in thickness and the reinforcement is increased towards the end. His curves do not cover a very wide range of depth ratios between crown and springing. For designs corresponding to the types assumed, the diagrams and formulas of these papers can be used to arrive at a final design without going through the detailed process of analysis, but for the purposes of this chapter, a brief table of coefficients (Table 25) from Cochrane's work will be used only for the purpose of tentative design. For more detailed use the complete papers must be referred to.

319. Order of Procedure.—The various factors involved can best be discussed by taking up the problem of a tentative design in its natural order, and the arch already analyzed will be used for purposes of illustration. The order of procedure will be as follows, assuming span, rise and character of loading to be already determined.

(*a*) Estimate of dead load per foot at crown and springing.
(*b*) Form of arch axis.
(*c*) Dead-load thrust at crown and springing.

* Proc. Engr. Soc. West. Penn., Vol. 32, 1916.
† Trans. Am. Soc. C.E., Vol. 88, 1925.

(*d*) Assumption of relative depth at crown and springing.
(*e*) Approximate calculation of live load and temperature moment and thrust at crown and springing for the assumed arch proportions.
(*f*) Calculation of necessary section at crown for the thrusts and moments thus found and for the allowable unit stresses. Check stresses at springing. If the stresses at crown or springing are out of balance, as is likely, then
(*g*) Change ratio of depths at springing to crown to bring stresses more nearly in balance.
(*h*) Recompute dead load from the information now available relative to weight of arch ring and revise dead-load thrusts.
(*i*) Recalculate live-load and temperature effects and repeat (*f*) and (*g*) if necessary.

These operations are quickly performed and will result in a tentative design that will not need to be greatly modified after a detailed analysis.

The reinforcement may be varied according to the ideas of the designer. For very long spans the dead load is of such great influence that a high grade of concrete and relatively large amount of reinforcement will be desirable. For spans of short or moderate length a relatively small amount of reinforcement will produce an economical structure.

An examination of the formulas for moment and thrust due to applied loads will show that so far as the moment of inertia of the arch rib is concerned these quantities depend not upon the actual value of the moment of inertia at any point but upon the relative values along the rib, that is, upon the variation in the ratio q. And if the variation of I from crown to springing is made to follow a certain law, then the moment and thrust will vary with the ratio of the values of I at crown and springing. This principle is used by Mr. Whitney in his calculations. In Mr. Cochrane's work the ratio of depths is used instead. This method gives results not precisely comparable on account of the influence of the reinforcement on the values of I. For small amounts of steel the error is negligible, but for large amounts the actual values of I at crown and springing should be calculated and the equivalent depth of an unreinforced section determined by the formula $I = \frac{1}{12} bd^3$. The ratio of these modified depths should then be used in selecting coefficients from the table.

320. Dead Load at Crown and Springing.—An approximate estimate must be made of the dead load per foot at crown and springing. This will include the weight of the arch rib itself, as yet unknown, but reasonably close values for this purpose can be obtained from existing designs.

In the arch analyzed in this chapter the final values are, from

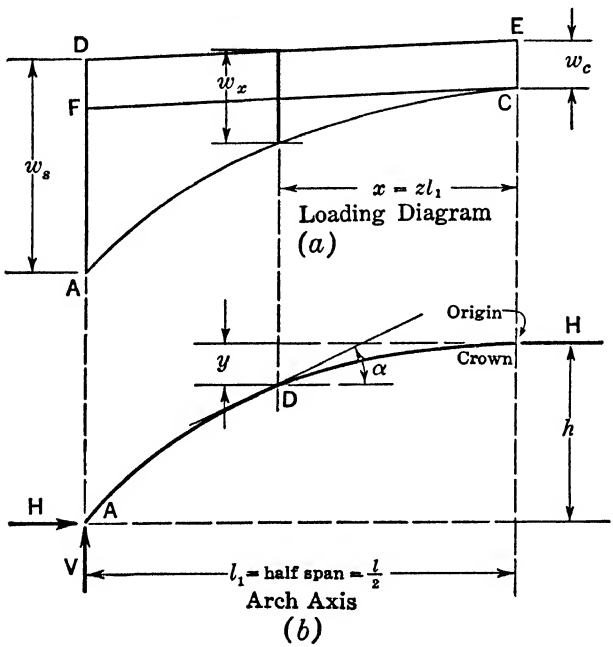


FIG. 17.

Table F, $w_c = 2.06 \times 150 = 309$ lbs/ft.; $w_s = 13.30 \times 150 = 2000$ lbs/ft.

321. The Form of the Arch.—Experience has shown that in the ordinary case the best form for the arch axis is such that the dead-load line of pressure follows the axis, thus making the dead-load moment zero at all sections. The axis will be the equilibrium polygon for dead load. For arches of relatively large rise, say one-fourth the span length or more, more satisfactory results are secured if one-half the live load be added to the dead load.

In Fig. 17, suppose the dead load to be represented by the ordinates to the line DE. The line DE will generally vary somewhat from a straight line, but assuming it to be such the dead load per foot at any distance x from C will be

$$w_x = w_c\left[1 + (g - 1)\frac{y}{h}\right]$$

where $g = w_s/w_c$. Substituting this value of w_x in the general differential equation for the linear arch, which is $d^2y/dx^2 = w_x/h$, and differentiating, we have finally

$$y = \frac{h}{(g - 1)}(\cosh cz - 1) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

which is the equation of the arch axis.* In this equation,

y = ordinate to arch axis;
x = distance from crown;
h = rise of arch axis;
$c = \cosh^{-1} g$ where $g = w_s/w_c$;
z = proportionate distance from crown to ordinate y in terms of the half span l_1.

Eq. (1) may also be used for open spandrel arches. In this case the several concentrations should be reduced to equivalent loads per foot. A curve can then be drawn similar to Fig. 17 and the desired values of w_s and w_c scaled from the diagram.

Mr. Cochrane has derived somewhat similar formulas which do not contain hyperbolic functions. These equations are as follows:

For spandrel filled arches

$$y = z^2 h\left[\frac{1 + \frac{1}{10}(g - 1)z^3}{1 + \frac{1}{10}(g - 1)}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

---

* See Strassner, Neuere Methoden zur Statik der Rahmentragwerke, Band II, Berlin, 1927.

For open spandrel arches

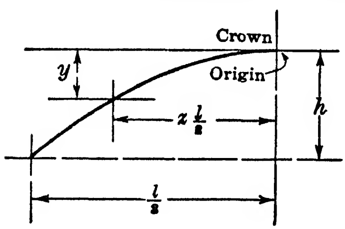
$$y = z^2 h \left[\frac{1 + \frac{1}{6}(g - 1) z^2}{1 + \frac{1}{6}(g - 1)} \right] \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Values of y given by eqs. (1) and (2) agree closely.

In the foregoing equations the values of w_s and w_c can be closely estimated after the selection of tentative values for depth of arch ring under (324) below. Table 22 gives values of $A = y/h$ for various values of g. The arch axis actually used in the example was a three-centred arch with the ordinate (2.07) at the quarter point taken from this table. The other ordinates vary slightly from the tabular values.

TABLE 22

ORDINATES TO ARCH AXIS SPANDREL FILLED ARCHES



(After Cochrane)

$$y = h z^2 \left[\frac{1 + \frac{1}{10}(g - 1) z^3}{1 + \frac{1}{10}(g - 1)} \right] = A h$$

Values of A

g	Value of z										
	0 Crown	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0 Spring
1	0	0.01	0.04	0.09	0.16	0.25	0.36	0.49	0.64	0.81	1.0
2	0	0.00910	0.0364	0.0821	0.1464	0.2301	0.3343	0.4607	0.6116	0.7900	1.0
3	0	0.00847	0.0339	0.0754	0.1350	0.2136	0.3129	0.4364	0.5880	0.7734	1.0
4	0	0.00771	0.0308	0.0698	0.1254	0.1995	0.2949	0.4157	0.5679	0.7594	1.0
5	0	0.00717	0.0286	0.0651	0.1172	0.1875	0.2793	0.3980	0.5508	0.7472	1.0
6	0	0.00669	0.0277	0.0608	0.1101	0.1771	0.2660	0.3827	0.5359	0.7368	1.0
7	0	0.00628	0.0251	0.0572	0.1038	0.1680	0.2542	0.3693	0.5229	0.7277	1.0
8	0	0.00592	0.0237	0.0539	0.0983	0.1598	0.2438	0.3575	0.5112	0.7196	1.0
9	0	0.00559	0.0224	0.0511	0.0934	0.1528	0.2346	0.3469	0.5012	0.7124	1.0
10	0	0.00530	0.0212	0.0485	0.0891	0.1464	0.2263	0.3375	0.4920	0.7060	1.0

322. Dead-Load Thrust.—Referring to Fig. 16, and considering the arch as hinged at A (no bending moment at this point), the dead-load crown thrust can be obtained by taking moments about A, after the curve $A\,C$ has been fixed upon. A closely approximate formula is

$$H_c = [\tfrac{1}{8}\, w_c + 0.015\,(w_s - w_c)]\frac{l^2}{h} \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

A more exact value can be had later on, if desired, by dividing the dead load into short sections and calculating the moment in detail.

In the example used we have

$$H_c = \left[\frac{309}{8} + 0.015(1693)\right]\frac{72^2}{12} = 27{,}500 \text{ lbs.}$$

Using the exact moments given in Table K we would have

$$H_c = \frac{328{,}840}{12} = 27{,}400.$$

The thrust at the springing section will be

$$\frac{H_c}{\cos \alpha_s} = \frac{27{,}500}{0.712} = 38{,}600 \text{ lbs.}$$

The effect of rib shortening is not included here.

323. Relative Depth at Springing and Crown.—Experience will indicate approximately what this should be. In the given arch it is 2.5, but for purposes of illustration a ratio of 2.0 will first be assumed.

324. Calculation of Moments and Thrusts from Live Load, Temperature, and Rib Shortening.—*Determination of Sections.*—Table 23 gives coefficients for approximate values for a wide range of rise-span ratios and ratios d_s/d_c. In the arch under consideration $h/l = 0.167$ and $d_s/d_c = 2.0$. Then from the table we get for a live load of 200 lbs/ft., the following crown stresses; positive moment $= 0.0065 \times 200 \times 72^2 = 6600$ ft-lbs., thrust $= 0.48 \times 200 \times 72 = 6900$ lbs.

To determine temperature and rib-shortening effects a crown section must be assumed. Try a depth of 10 in. = 0.833 ft. Then, neglecting reinforcement, $I_c = 0.051$ ft.[4] The temperature coefficient is 29, and for positive moment

$$H_c = -29 \times \frac{0.000006 \times 40 \times 2{,}000{,}000 \times 144}{12^2} \times 0.051$$

$$= -29 \times 480 \times 0.051 = -700 \text{ lbs.}$$

$$M_c = -20 \times \frac{700 \times 12}{100} = 1700 \text{ ft-lbs.}$$

Total dead-load, live-load, and temperature thrust = 27,500 + 6900 − 700 = 33,700 lbs.

Rib-shortening thrust,

$$H_R = -0.93 \times \frac{33{,}700}{0.83 \times 69{,}000} \times 700 = 0.55 \times 700 = 380 \text{ lbs.}$$

Rib-shortening moment = 0.55 × 1700 = 930 ft-lbs.

Total positive moment = 6600 + 1700 + 930 = 9230 ft-lbs.

Total thrust = 33,700 − 380 = 33,300 lbs.

Assuming $p = 0.005$, we find $f_c = 1100$ lbs/in.^{2}, an excessive value.

Try a depth of 12 in. as actually used, with $p = 0.0042$, $A_c = 1.12$, $I_c = 0.1$. Temperature thrust $H_t = -29 \times 480 \times 0.1 = -1400$ lbs. Total thrust = 33,000 lbs.

Temperature moment

$$= 20 \times \frac{1400 \times 12}{100} = 3350 \text{ ft-lbs.}$$

Rib-shortening thrust

$$= H_R = -0.93 \times \frac{33{,}000}{1.12 \times 69{,}000} \times 1400$$

$$= -0.40 \times 1400 = -560 \text{ lbs.}$$

Rib shortening moment = 0.40 × 3350 = +1340 ft-lbs. Total positive moment = 6600 + 3350 + 1340 = 11,290 ft-lbs. Total thrust = 33,000 − 560 = 32,440 lbs. With these values we find $f_c = 660$ lbs/in.^{2}

For the springing section, live-load positive moment = 0.036 ×

TABLE 23

COEFFICIENTS FOR MOMENTS AND THRUSTS

Live Load and Temperature

(After Cochrane)

$\frac{h}{l}$	$\frac{d_s}{d_c}$	Open Spandrel Arches						Spandrel Filled Arches						Rib Short, D. L.
		Live Load				Temperature		Live Load				Temperature		
		Crown		Springing				Crown		Springing				
		$+M_c$	T_c	$+M_s$	T_s	M_c	T_c	$+M_c$	T_c	$+M_s$	T_s	M_c	T_c	
	1.5	0.0051		0.023		26	18	....		......		23	20	0.94
0.10	2.0	.0045	0.65	.027	0.90	22	26	0.0055	0.72	0.034	1.10	20	30	.90
	2.5	.0041		.030		18	35	.0049		.037		17	40	.86
	3.0	.0039		.032		16	43	.0044		.040		15	50	.82
	1.5	.0054		.023		27	17	....		......		23	19	.96
.15	2.0	.0048	.43	.027	.60	22	25	.0062	.52	.036	.68	20	29	.92
	2.5	.0044		.030		19	33	.0056		.040		17	38	.88
	3.0	.0041		.032		17	40	.0051		.043		15	47	.84
	1.5	.0058		.023		27	16	....		......		23	18	1.01
.20	2.0	.0052	.33	.027	.42	23	23	.0070	.41	.037	.50	20	27	.96
	2.5	.0047		.030		19	31	.0064		.041		17	35	.91
	3.0	.0044		.032		17	37	.0058		.044		15	43	.87
	1.5	.0062		.023		28	15	....		......		23	16	1.06
.25	2.0	.0055	.27	.027	.38	23	22	.0079	.35	.037	.39	20	24	1.01
	2.5	.0050		.030		20	28	.0073		.041		17	31	.96
	3.0	.0047		.032		18	34	.0067		.045		15	38	.91
	1.5	.0066		.023		29	14	....		......		23	14	1.12
.30	2.0	.0059	.23	.027	.31	24	20	.0089	.30	.037	.32	20	20	1.06
	2.5	.0054		.030		20	26	.0083		.041		17	26	1.01
	3.0	.0050		.032		18	31	.0077		.044		15	33	.96
	1.5	.0070		.023		30	13	....		......		23	12	1.20
.35	2.0	.0063	.20	.028	.29	25	18	.0100	.26	.035	.26	20	17	1.14
	2.5	.0058		.031		21	23	.0094		.039		17	22	1.08
	3.0	.0054		.033		19	28	.0087		.043		15	26	1.02

h = rise
l = span
d_s = depth of arch ring at springing
d_c = depth of arch ring at crown

Rib-shortening. Crown thrust H_R for any given crown thrust H is

$$H_R = - \text{coef.} \times \frac{H}{A_c\, \omega\, t\, E} \times H_c \text{ (temp.)}$$

Live load: Moment = coef. $\times\ p\, l^2$; Thrust = coef. $\times\ p\, l$

Temperature:

$$T_c = \text{coef.} \times \frac{\omega\, t\, E\, I_1}{h^2}$$

$$M_c = - \text{coef.} \times \frac{T_c\, h}{100}$$

$$M_s = M_c + T_c\, h$$

$200 \times 72^2 = 38{,}000$ ft-lbs.; live-load thrust $= 0.62 \times 200 \times 72 =$ 6400 lbs. Temperature crown thrust for positive moment $= +1400$, and $M_s = 1400 \times 12 - 3350 = 13{,}500$ ft-lbs. Thrust at springing $= 1400 \times \cos\alpha = 1400 \times 0.712 = 1000$ lbs. Rib-shortening effect $= 40\%$ of temperature effect as above calculated, giving a moment (negative) of 7400 ft-lbs. and a thrust of -400 lbs. Total + moment $= 38{,}000 + 13{,}500 - 5400 = 46{,}100$ ft-lbs. Total thrust $= 38{,}600 + 6400 + 1000 - 400 = 45{,}600$ lbs. For these values, and $p = 0.0021$, $pn = 0.031$, we find, using diagram 17, $f_c = 780$ lbs/in.²

Thus at crown $f_c = 660$ and at springing $f_c = 780$, an unbalanced condition. To increase the depth at springing will increase the moments at that point somewhat and reduce those at the crown; but the fibre stress at springing will be reduced. For like moments the fibre stress is nearly inversely proportional to d^2, while the moments increase comparatively little. From Table 23 it is noted that for an increase in depth from 2.0 to 2.5 ft. the live-load moment is increased about 11%; at the crown it is reduced about 10%.

Adopting a ratio of d_s/d_c, of 2.5 as actually used, recalculation from Table 23 gives the following results:

At crown

Live-load moment	= 6100 ft-lbs.
Live-load thrust	= 6900 lbs.
Temperature thrust	= −1780 lbs.
Temperature moment	= 3600 ft.-lbs.
Dead-load thrust	= 27,500 lbs.
Rib shortening	= 38% of temperature effect.
Thrust	= −670 lbs.
Moment	= +1370 ft-lbs.

Total thrust = 31,900 lbs., total moment = 11,100 ft-lbs. These values are almost identical with those resulting from the detailed analysis given in Table I. The value of f_c will be 650 lbs/in.²

At springing

Live-load moment	= 42,000 ft-lbs.
Live-load thrust	= 6,400 lbs.
Temperature thrust	= 1,270 lbs.

Temperature moment = 17,800 ft-lbs.
Dead load thrust = 38,600 lbs.
Rib shortening = 38% of temperature effect.
Thrust = −480 lbs.
Moment = −6800 ft-lbs.

Total thrust = 45,800 lbs., total moment = 53,000 ft-lbs., f_c = 585 lbs/in.[2] These values differ from those obtained in the detailed analysis chiefly in the temperature moment, which appears to be too small.

Note that in these calculations the dead-load moments are considered to be zero, except for rib-shortening effect. The results compare closely with those shown in Table G.

325. Variation in Thickness of Arch between Crown and Springing. —Table 24, after Cochrane, gives recommended variations in thickness of the arch rib from crown to springing for ratios of springing to crown thickness from 1.5 to 3.25.

TABLE 24

PROPORTIONATE THICKNESS OF TYPICAL ARCHES

(After Cochrane)

Proportionate Distance along the Arch Axis from Crown.	Ratio of Crown to Springing Thickness							
	1.5	1.75	2.0	2.25	2.50	2.75	3.0	3.25
0	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.05	1.007	1.006	1.005	1.004	1.003	1.002	1.001	1.000
0.15	1.021	1.018	1.015	1.012	1.009	1.006	1.003	1.000
0.25	1.035	1.030	1.025	1.020	1.015	1.010	1.005	1.000
0.35	1.049	1.042	1.035	1.028	1.023	1.021	1.023	1.030
0.45	1.063	1.054	1.048	1.048	1.057	1.070	1.083	1.101
0.55	1.077	1.072	1.085	1.105	1.133	1.165	1.193	1.231
0.65	1.095	1.125	1.168	1.215	1.269	1.328	1.385	1.455
0.75	1.145	1.223	1.311	1.403	1.508	1.625	1.737	1.865
0.85	1.245	1.393	1.547	1.700	1.862	2.025	2.185	2.355
0.95	1.406	1.621	1.837	2.055	2.277	2.495	2.709	2.932
1.00	1.500	1.750	2.00	2.250	2.500	2.750	3.00	3.250

The arch rib thicknesses given in Table 24 are measured on sections normal to the arch axis. To determine the angle which a normal section makes with the vertical, differentiate eqs. (1) and (2) of Art. 321. For spandrel filled arches,

$$\tan \alpha = \frac{4hz}{l}\left[\frac{1 + \frac{1}{4}(g-1)z^3}{1 + \frac{1}{10}(g-1)}\right]$$

and for open spandrel arches

$$\tan \alpha = \frac{4hz}{l}\left[\frac{1 + \frac{1}{3}(g-1)z^2}{1 + \frac{1}{6}(g-1)}\right]$$

where α is the angle between the normal section and the vertical.

On comparing the arch rib thicknesses given in Table 24 with those obtained from the Strassner formulas and used by Mr. Whitney it will be found that arches determined by Strassner's formulas are somewhat thicker than those given by Cochrane. Also, Strassner seems to prefer an arch rib in which the crown to springing thickness varies from about 1.5 to 2.0, whereas Cochrane allows a much wider range.

Fibre stresses calculated at critical sections of Strassner and Cochrane arches in which the crown and springing thickness ratios and the reinforcement are identical give a means of comparing the two types of arches. It will be found that the fibre stresses at the crown and springing sections in the Strassner arch due to applied loads are less than those in the Cochrane arch. At the haunch the stresses in the Cochrane arch are the smaller. On account of temperature, the stresses in the Cochrane arch are less than those in the Strassner arch. This is due to the fact that the Cochrane arch, being thinner in the haunch, is more flexible than the Strasner arch, and therefore temperature stresses are somewhat smaller.

However, the total, or combined dead-load, live-load, and temperature stresses in the Strassner arch are slightly less than those in the Cochrane arch.

326. Concrete Stresses in Arches.—The arch rib is a compression member subjected to bending stresses. It acts in a sense like a column but is well supported against buckling action by the earth fill or the spandrel construction, so that long column action need not be considered. Between expansion joints the spandrel construction affords a very considerable amount of support to the arch ring, as the entire structure acts as a unit. The allowable stresses may then be taken about the same as for the tied column, or about three-tenths the ultimate strength for combined stresses due to load and temperature. Stresses somewhat above this value are commonly allowed when full temperature effect is included. Considering the reduction in concrete stress due to plastic flow this is perhaps admissible, but the same effect is present in columns, and until more is known concerning the effect on ultimate strength a conservative practice should be followed.

327. Arch Details.—The various details of spandrel construction, either of retaining walls or of open spandrel construction, are carried out on the same principles as in other forms of reinforced work. Special consideration must be given to expansion joints. The rise and fall of the arch ring due to temperature changes is marked, and to avoid damage to the superstructure ample expansion joints must be provided in spandrel walls or the floor of the open spandrel construction. These should be provided at points above the abutments and, where the rise is large, at about the quarter points also. Spandrel columns should be thoroughly attached to the arch rib so as to resist the bending stresses due to temperature deformations.

328. Abutments.—The design of abutments and foundations for arches requires great care. A concrete arch is a relatively rigid structure where stresses will be greatly changed by small settlements of foundations. Pressures on foundations must take into account the most unfavorable direction and amount of thrust at springing together with minimum pressure of back fill. The resultant pressure should be nearly perpendicular to the foundation surface and inclined piling used if necessary to take care of the horizontal component of the abutment pressure.

CHAPTER XI

RETAINING-WALLS

329. Advantages of Reinforced Concrete.—Retaining-walls, dams, bridge abutments, and the like constitute a class of structures in which the outside forces acting are mainly horizontal, and in which, therefore, the question of stability is largely a question of safety against overturning. Where ordinary masonry is used in these structures the weight of the material must be depended upon to balance the overturning forces, for though the structure be anchored to the foundation no tensile stresses can be allowed in the masonry. As a consequence of these limitations the maximum compressive stresses in such structures are not high, except in extreme cases, so that generally the dimensions are determined by the weight of the material. The application of reinforced concrete in such cases enables the design to be so modified as to utilize the weight of the material to be retained as part of the resisting weight and to calculate the sections to develop more nearly the full strength of the concrete. A very considerable gain in economy therefore results.

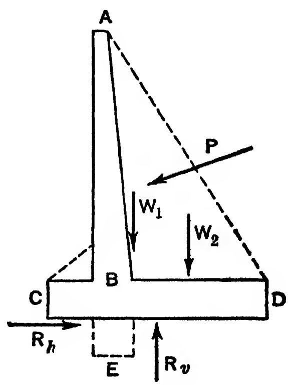


FIG. 1.

330. Form of Reinforced-Concrete Retaining-Wall.—Fig. 1 illustrates the usual form of reinforced-concrete wall. It consists of a vertical wall *A B* connected to a floor *C D*. For low walls the upright part, *A B*, and the two sections of the floor, *C B* and *B D*, are each designed as cantilever beams. For high walls it is economical to connect the wall *A B* to the floor *B D* at intervals by means of back stays of reinforcing bars embedded in concrete cross walls and thoroughly attached to *A B* and *B D*. These transverse walls with their reinforcement are called "counterforts," although their action is quite unlike the typical counterfort in masonry. The toe *C B* may likewise be supported at intervals by small buttress walls joining the toe to the vertical wall. This general arrangement changes the action of the several parts of the wall from simple cantilevers to continuous slabs, each panel of which is supported on three sides. The principal reinforcement will be horizontal and parallel to the axis of the wall, but some transverse reinforcement should be used near the junction of wall and floor to take the negative moments along this line.

The forces acting on the wall are the earth pressure *P* on the wall *A B*, the weight of the wall W_1, the weight of the earth W_2 on the floor *B D*, and the reactions R_v and R_h of the foundation. The horizontal force R_h may be supplied wholly by frictional resistance or in part by the horizontal earth pressure in front of the wall, increased if necessary by the use of a vertical projection (as *B E*) extending into the foundation material.

331. Stability of the Retaining-Wall.—The first problem to be solved in the design of a retaining-wall is to determine such proportions that the pressure under the toe *C* will not be excessive and that there will be no danger of sliding on the foundation. Inasmuch as the provision against sliding can readily be made in the manner suggested, this question will not be further considered here. The provisions against undue pressure at the toe, giving rise to unequal settlement and undesirable or dangerous tipping of the wall, is the main problem. Where the foundation material is relatively hard the wall is generally designed so as to bring the resultant pressure on the foundation at about the edge of the middle third, thus giving a maximum unit pressure of twice the average, with zero pressure at the heel *D*. If such a large variation in pressure is likely to cause undesirable settlement then the floor *C D* must be made wider to equalize more nearly the pressures at toe and heel. For relatively soft foundations these pressures should be nearly equal and the centre of pressure brought approximately to the centre of foundation.

332. The Earth Pressure.—Many factors affect earth pressure, and no theoretical analysis can be expected to take into account some of the most important, such as variation in cohesiveness of the material, effect of moisture conditions and of frost action. Notwithstanding the inadequacy of theoretical analyses the Rankine theory of earth pressure, based on the assumption of dry granular material, is of considerable value in giving a measure of relative pressures for different materials and establishing certain general relations between pressure and other variables. Theory can, however, serve as only a rough guide, as the successful design of retaining walls depends largely upon experience with materials of a character similar to those under consideration.

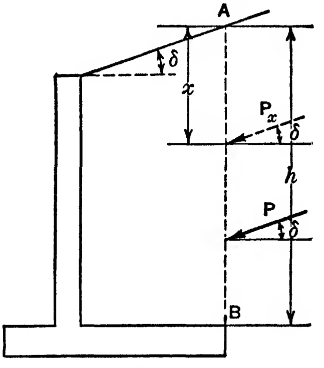


FIG. 2.

According to the Rankine theory, the unit pressure p_x upon a vertical plane $A\,B$, Fig. 2, at a distance x below the surface is

$$p_x = C\,w\,x \quad . \quad . \quad . \quad (1)$$

where w = unit weight of earth and C is a coefficient depending upon the slope δ of the surface, and the friction angle ψ for the material, or the angle of maximum slope at which the material will stand. The value of C is

$$C = \cos\delta\,\frac{\cos\delta - \sqrt{\cos^2\delta - \cos^2\psi}}{\cos\delta + \sqrt{\cos^2\delta - \cos^2\psi}} \quad . \quad . \quad . \quad . \quad (2)$$

Since by (1) the value of p increases with x, the total pressure P on the plane $A\,B$ will be, per unit length of wall,

$$P = \frac{C\,w\,h^2}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and its point of application will be $h/3$ above the base. According to the theory, also, the direction of P is parallel to the surface.

Table 25 gives values of C for various values of δ and ψ. For $\delta = 0$ the values of C multiplied by w give the equivalent weight of a fluid producing the same pressures as the earth.

333. **Width of Base.**—Consider the forces acting upon the body $A\,B\,C\,D$, Fig. 3. The earth pressure from the right is P, the weight of the earth W_2, and the weight of wall W_1, all per foot of length. The force P is given by eq. (3) and Table 25; it acts at a slope δ. Its horizontal component $P\cos\delta$ is the overturning force; the vertical component $P\sin\delta$ will aid in resisting this overturning. Sub-

TABLE 25

VALUES OF C IN FORMULA $p_x = Cwx$

Angle of Internal Friction	Slope with the Horizontal							
	1 : 1	1 : 1½	1 : 2	1 : 2½	1 : 3	1 : 4	Level	
	Corresponding Slope Angle							
ϕ	45°	33°–40′	26°–34′	21°–50′	18°–30′	14°–0′	0	ϕ
55°	0.18	0.13	0.12	0.11	0.11	0.10	0.10	0.57
50°	0.29	0.18	0.16	0.15	0.14	0.14	0.13	0.64
45°	....	0.26	0.22	0.20	0.19	0.18	0.17	0.71
40°	....	0.36	0.29	0.26	0.24	0.22	0.22	0.77
35°	....	0.58	0.38	0.33	0.31	0.29	0.27	0.82
30°	....	....	0.54	0.44	0.40	0.37	0.33	0.87
25°	....	....	....	0.60	0.52	0.46	0.40	0.91
20°	....	....	....	....	0.72	0.58	0.49	0.94

Angles of Repose and Weight of Materials

	Angle of Repose	Weight, lbs/ft.³
Dry sand	30°	100
Moist sand	40°	110
Ordinary earth	40°	100
Gravel	45°	120
Gravel, sand and clay	30°	110

stantially correct results, and quite as accurate as is warranted by the theory, will be obtained if the value of P be calculated for the height h and then the vertical component $P \sin \delta$ be ignored.

Fig. 4 shows the simplified conditions. The necessary width of

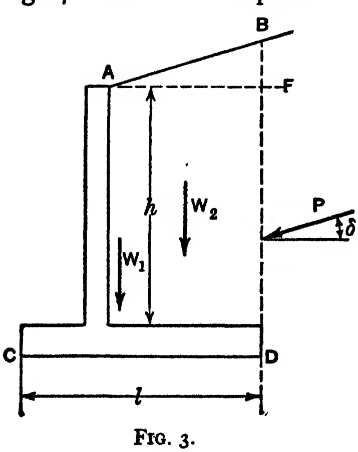


Fig. 3.

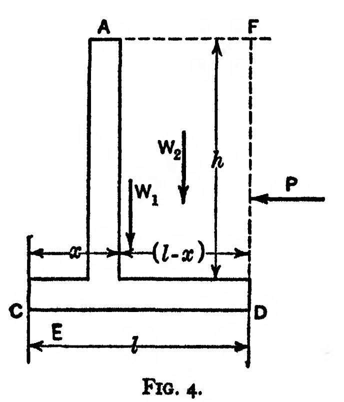


Fig. 4.

base l will be calculated under the assumptions: (*a*) that the resultant pressure on the foundation cuts the edge of the middle third; (*b*) that the resultant pressure cuts the centre of the base; and (*c*) a width l which will result in a given unit pressure at the toe.

(*a*) *Resultant Pressure to Cut the Edge of the Middle Third.*—The weight of the wall itself will be neglected. Its effect will be small on the present problem unless the length of toe $C\,E$ is relatively small.

By the assumed relation we have

$$P \cos \delta \times \frac{h}{3} = W_2 \left[\frac{2}{3} l - \left(\frac{l - x}{2}\right)\right] = W_2 \left(\frac{l}{6} + \frac{x}{2}\right).$$

Substituting value of P from (3) and letting $x = k\,l$, we derive the equation

$$l = h \sqrt{\frac{C \cos \delta}{(1 - k)(1 + 3\,k)}}. \qquad (4)$$

For a minimum value of l, $k = 1/3$ or $x = 1/3\,l$.

(*b*) *Resultant Pressure to Cut Centre of Base.*—Equating moments about the centre of the base we derive in a similar manner

$$l = h \sqrt{\frac{C \cos \delta}{3\,(1 - k)\,k}}. \qquad (5)$$

For a minimum value of l, $k = 1/2$ or $x = 1/2\,l$.

(*c*) *Width of Base Determined by Allowable Pressure on Foundation.*—If the resultant pressure cuts the edge of the middle third, the maximum unit pressure at the toe will be twice the average and the pressure at the heel will be zero. If the allowable pressure is less than the toe pressure found in this way, but greater than the average pressure, the necessary width of base will be somewhere between the values calculated under assumptions (*a*) and (*b*), and the pressure area on the base will be trapezoidal in form. If e = distance from centre of base to point where the resultant pressure cuts the base (e/l = eccentricity) then the equation of moments gives

$$P \cos \delta \times \frac{h}{3} = W_2 \left[\frac{l}{2} + e - \left(\frac{l - x}{2}\right)\right] = W_2 \left(e + \frac{x}{2}\right).$$

The maximum pressure at toe for an eccentricity e will be

$$p = \frac{W_2}{l}\left(1 + \frac{6\,e}{l}\right)$$

whence

$$e = \frac{l}{6}\left(\frac{p\,l}{W_2} - 1\right).$$

Substituting as before we derive the expression

$$l = h\sqrt{\frac{C \cos \delta}{p/w\,h - 1 + 4\,k - 3\,k^2}} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

in which p = allowable pressure per square foot.

For

$$p = \frac{2\,W_2}{l} = \frac{2\,w\,h\,(l - x)}{l}$$

this reduces to case (a), and for

$$p = \frac{w\,h\,(l - x)}{l}$$

it reduces to case (b).

For a fixed value of p the value of l is a minimum for $k = \frac{2}{3}$.

Table 26 gives values of l/h for cases (a) and (b) for various values of $C \cos \delta$ and for various values of k.

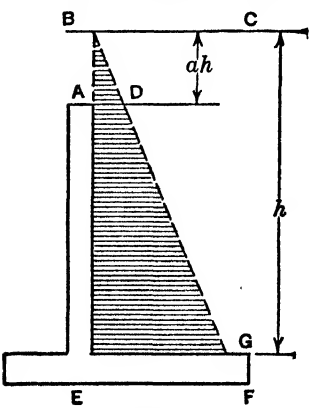


FIG. 5.

334. Effect of Surcharge.—The effect of any given load above the level of the wall is taken account of by adding an equivalent depth of filling $a\,h$, Fig. 5, and considering the pressure on the wall $A\,E$ to be represented by the trapezoid $A\,D\,E\,G$, the pressure at any point being proportional to the distance below $B\,C$. The moment of this pressure about the base is equal to the moment of triangle $B\,E\,F$, minus moment of triangle $A\,B\,D$. The moment of triangle $B\,E\,F = \frac{1}{6}\,C\,w\,h^3$. Area $B\,A\,D = \frac{1}{2}\,C\,w\,(a\,h)^2$. Moment of area about base $= \frac{1}{2}\,C\,w\,(a\,h)^2\,(h - \frac{2}{3}\,a\,h)$. Total moment about base =

$$M = C\,w\,h^3\left[\tfrac{1}{6} - \tfrac{1}{2}\,a^2\,(1 - \tfrac{2}{3}\,a)\right] \quad . \quad . \quad . \quad . \quad . \quad (7)$$

TABLE 26

VALUES OF l/h, CASES (a) AND (b)

Case (a); $\frac{l}{h} = \sqrt{\frac{C \cos \delta}{(1-k)(1+3k)}}$. Case (b); $\frac{l}{h} = \sqrt{\frac{C \cos \delta}{3k(1-k)}}$.

k	Case (a) Values of $C \cos \delta$											k
	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.0	
0	0.22	0.32	0.45	0.55	0.63	0.71	0.77	0.84	0.89	0.95	1.00	0
0.05	0.21	0.31	0.43	0.52	0.61	0.68	0.73	0.80	0.85	0.90	0.96	0.05
0.10	0.20	0.30	0.42	0.51	0.59	0.66	0.71	0.78	0.82	0.87	0.93	0.10
0.15	0.20	0.29	0.40	0.50	0.56	0.64	0.69	0.76	0.80	0.85	0.90	0.15
0.20	0.19	0.29	0.39	0.49	0.55	0.63	0.68	0.75	0.79	0.84	0.88	0.20
0.25	0.19	0.28	0.39	0.48	0.55	0.62	0.67	0.73	0.78	0.83	0.87	0.25
0.30	0.19	0.28	0.39	0.48	0.54	0.62	0.67	0.72	0.77	0.82	0.86	0.30
0.35	0.19	0.28	0.39	0.48	0.54	0.62	0.67	0.72	0.77	0.82	0.86	0.35
0.40	0.19	0.28	0.39	0.48	0.55	0.62	0.67	0.73	0.78	0.83	0.87	0.40

k	Case (b) Values of $C \cos \delta$											k
	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.0	
0.05	0.59	0.84	1.19	1.45	1.67	1.88	2.06	2.22	2.38	2.52	2.66	0.05
0.10	0.43	0.61	0.86	1.05	1.22	1.36	1.49	1.61	1.72	1.82	1.92	0.10
0.15	0.36	0.51	0.73	0.88	1.02	1.14	1.29	1.35	1.44	1.53	1.62	0.15
0.20	0.32	0.46	0.65	0.79	0.89	1.02	1.23	1.21	1.29	1.37	1.44	0.20
0.25	0.30	0.42	0.60	0.73	0.84	0.94	1.03	1.12	1.19	1.27	1.33	0.25
0.30	0.28	0.40	0.56	0.69	0.80	0.89	0.97	1.05	1.13	1.19	1.26	0.30
0.35	0.27	0.38	0.54	0.66	0.76	0.86	0.94	1.03	1.08	1.15	1.21	0.35
0.40	0.26	0.37	0.53	0.64	0.74	0.83	0.91	0.99	1.05	1.12	1.18	0.40
0.45	0.26	0.37	0.52	0.63	0.73	0.82	0.90	0.97	1.04	1.10	1.16	0.45
0.50	0.26	0.37	0.52	0.63	0.73	0.82	0.89	0.97	1.03	1.10	1.15	0.50
0.55	0.26	0.37	0.52	0.63	0.73	0.82	0.90	0.97	1.04	1.10	1.16	0.55
0.60	0.26	0.37	0.53	0.64	0.74	0.83	0.91	0.99	1.05	1.12	1.18	0.60

Note that the moment $\frac{1}{6} C w h^3$ is the moment on a wall of the full height h. In the case of the surcharged wall this is reduced by $C w h^3 [\frac{1}{2} a^2 (1 - \frac{2}{3} a)]$. The proportionate reduction is equal to $\frac{1}{2} a^2 (1 - \frac{2}{3} a)/\frac{1}{6} = a^2 (3 - 2 a)$. Thus if the height of wall is 20 ft., surcharge = 5 ft., then $a = 0.2$ and the proportionate reduction in overturning moment below that for a wall 25 ft. high is $0.04 (3 - 0.4) = 0.104$, or 10.4%. The proportionate reduction in width of base will be about 5%.

335. **Comparison of the Stability of Plain Masonry and Reinforced Concrete Walls.**—Retaining-walls of masonry or plain concrete have commonly been designed in accordance with certain rules of practice of long standing which represent the experience of engineers extending over many years. These rules are not expressed in terms of earth pressure but give the ratio of width of base to height for certain standard forms and for various conditions of earth fill, surcharge, etc. It will be of some value to compare the stability of a standard form of wall of plain masonry, proportioned according to such rules, with the reinforced type discussed in the preceding articles.

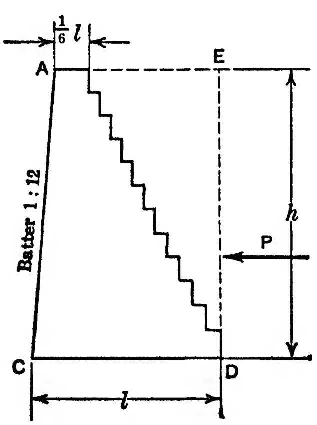


Fig. 6.

The most common type of ordinary masonry wall is shown in Fig. 6. The batter of the front face will be taken at 1 : 12 and the top width at $\frac{1}{6}$ of the bottom width. The weight of masonry will be assumed at 150 and that of the earth filling at 100 lbs/ft.^{3} As in Art. 333, the horizontal pressure P acting in the vertical plane $E D$ will be $C w h^2/2$ and will be applied a distance of $h/3$ above the base. Calculating the stability by equating moments about the edge of the middle third, we get $l = 0.934\, h\sqrt{C}$. For a reinforced-concrete wall, eq. (4), with $k = \frac{1}{3}$ and $\delta = 0$, gives $l = 0.865\, h\sqrt{C}$. The reinforced wall of the same stability as the type of solid wall shown in Fig. 6 will therefore have a base width of $\frac{0.865}{0.934} = 93\%$ of the width of the solid wall. To adapt various rules of practice for base widths established for the solid wall, as shown in Fig. 6, the width of the reinforced wall will be about 93% of the indicated width of the solid wall.

336. **Examples.**—The various problems arising in the design will be illustrated by examples.

1. **Cantilever Retaining-wall.**—A reinforced-concrete retaining-wall will be designed to support a bank of earth 16 ft. high. Top of bank level with no surcharge loading. Angle of internal friction of earth will be taken as 35°.

The limiting stresses are:

$f_s = 16{,}000$ lbs/in.^{2}; $f_c = 800$ lbs/in.^{2}; $n = 15$; $v = 40$ lbs/in.^{2}; $u = 80$ lbs.

Allowable soil pressure 10,000 lbs/ft.^{2}

Weight of earth 100 lbs/ft.^{3}; weight of concrete 150 lbs/ft.^{3}; coefficient of friction against sliding 0.4.

To allow for frost action, the bottom of the footing will be placed 4 ft. below the surface of the ground. Over-all height of wall = 16 + 4 = 20 ft.

General Method of Procedure.—A trial section is assumed and tested to make certain that the foundation pressure is compression across the full width of the base. Certain rules of practice and the general relations established in the preceding articles are used in making up the trial section.

Thickness of Base Slao.—Assume a base section whose thickness is equal to one-tenth of the total height of wall. Hence base section is 1/10 × 20 = 2 ft. thick. The vertical cantilever wall will then be 18 ft. high, as shown in Fig. 7.

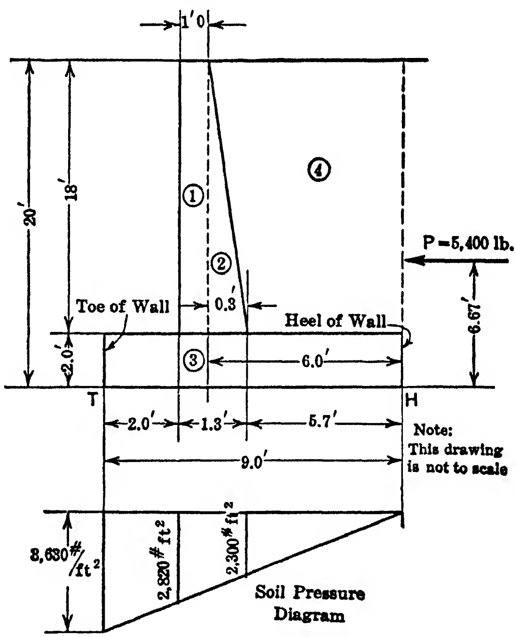


FIG. 7.

Thickness of Vertical Cantilever Wall.—Assume top thickness of wall as 1 ft. Since the moment and shear at the bottom of the vertical wall can be calculated as soon as the height of the wall is known, the thickness of this wall will be determined for the existing moments and shears.

Moment at base of wall $= M = P\frac{h}{3} = \frac{1}{6} C w h^3$ ft-lbs. $= 2 C w h^3$ in-lb.

From Table 25 for $\phi = 35°$, surface of ground level, $C = 0.27$. With $w = 100$ lbs/ft.², and $h = 18$ ft., $M = 2 \times 0.27 \times 100 \times 18^3 = 315,000$ in-lbs.

From Diagram 4, $n = 15$, $f_s = 16,000$, $f_c = 800$, $R = 147$. Then $d = \sqrt{\frac{M}{b R}} = \sqrt{\frac{315,000}{12 \times 147}} = 13.4$ in.

Shear at base of vertical wall $= P = \frac{1}{2} C w h^2 = \frac{1}{2} \times 0.27 \times 100 \times 18^2 = 4370$ lbs. Then $d = \frac{V}{v b j} = \frac{4370}{40 \times 12 \times 0.86} = 10.6$ in.

Moment conditions determine the thickness of the wall. Adding 2 in. for imbedment of steel, total thickness of wall at base $= 13.5 + 2 = 15.5$ in. $= 1$ ft. $3\frac{1}{2}$ in.

Length of Base.—Since the foundation conditions are very good, as indicated by the high allowable soil pressure, the base width will be determined on the assumption that the resultant force cuts the base at the edge of the middle third. Use Case (*a*), Art. 333, and Table 26. For $C = 0.27$ and $\phi = 35°$, $l/h = 0.45$. Bottom width of base $= 0.45 \times 20 = 9.0$ ft.

Position of Vertical Wall.—It is shown in Art. 333 that for minimum base width for Case (*a*) the vertical wall should be placed with its back at a distance from the toe equal to one-third of the base width. Place back of wall $9/3 = 3$ ft. from toe. Fig. 7 shows the adopted arrangement.

Position of Resultant Pressure on Base.—Before proceeding to the design of reinforcement it is necessary to make certain that the resultant pressure on the base falls inside or near the edge of the middle third of the base. This can be done by taking moments about the heel of the wall, point H of Fig. 7.

CALCULATIONS FOR POSITION OF RESULTANT

Area	Load	Arm	Moment at H
1	$1 \times 18 \times 150 =$ 2,700 lbs.	6.5 ft.	17,800 ft-lbs.
2	$0.3 \times 18 \times \frac{1}{2} \times 50 =$ 135	5.9	795
3	$9 \times 2 \times 150 =$ 2,700	4.5	12,150
4	$6 \times 18 \times 100 =$ 10,800	3.0	32,400
	Vertical load = 16,335 lbs.		
	Horizontal load P $= \frac{1}{2} C w h^2 = \frac{1}{2} \times 0.27 \times 100 \times 20^2 =$ 5,400 lbs.	6.67	36,000
		Total moment about H	= 98,845 ft-lbs.

Note that by considering the concrete of area 2 to weigh 50 lbs/ft.^{3} area 4 can be considered as a rectangle with a 6-ft. base width, and a weight of 100 lbs/ft.^{3}

Distance from H to resultant on base $= \frac{98,845}{16,335} = 6.04$ ft. Distance from H to far edge of middle third = 6.0 ft. Hence the resultant falls slightly outside the middle third. However, since the soil conditions are excellent, we can assume that the resultant falls at the edge of the middle third.

Where the soil conditions are not favorable, the dimensions of the wall should be changed so that the resultant will fall inside the middle third of the base. It will be found that changing the position of the vertical wall has very little effect on the position of the resultant. Changing the width of the base, keeping the vertical wall at the same distance from the heel, will be found to be the most effective means of bringing the resultant inside the middle third of the base.

Soil Pressure on Base.—Assuming that the resultant pressure on the base cuts the edge of the middle third, the soil pressure diagram is a triangle, as shown in Fig. 7. Hence the toe pressure is twice the average base pressure and the heel pressure is zero. Toe pressure $= 2 \times \frac{16,335}{9} = 3630$ lbs/ft.^{2} This is far below the allowable value. Fig. 7 shows the soil pressure diagram. Values of soil pressure at the front and back of the vertical wall are also shown.

Design of Beam Sections with Faces which Are Not Parallel.—It will be noted in Figs. 7 to 9 that reinforced-concrete retaining-walls contain beam sections in which the two faces are not parallel. An exact analysis of such sections is very complex. The best available material is given by Professor Cain.* For the case under consideration and for the haunched beam of Art. 263, an approximate method of analysis, based on a modification of values obtained for ordinary rectangular sections, will be found to give satisfactory values.

When the angle between the two faces of the beam section is not greater than 15°, the formulas for ordinary sections can be used without modification. When the angle between the two faces varies from 15° to 30°, the fibre stress values determined for normal sections are to be modified as follows:

Steel parallel to sloping face of section, as in Figs. 8 and 9:

Concrete stress f_c = same as for normal sections.
Steel stress $f_s = f_{s\ (normal)} \sec \beta$ where β = angle between faces of section.

Steel parallel to vertical face of section in Figs. 8 and 9:

Steel stress f_s = same as for normal section.
Concrete stress $f_c = f_{c\ (normal)} \sec^2 \beta$.

* Earth Pressure, Retaining Walls and Bins. Wiley & Sons, New York.

Results obtained by this approximate method are somewhat smaller than those given by the method of Professor Cain.

Design of Toe Slab.—The toe slab must be designed to resist the upward soil pressure and the downward load due to the weight of the base slab. It is usual to neglect the effect of the earth load above the toe slab. From Fig. 7, the moment and shear at the front edge of the vertical wall are as follows:

Direction	Load	Arm	Moment
Upward.....	3630 × 2 × ½ = 3630 lbs.	2 × ⅔ ft.	4,840 ft-lbs.
Upward.....	2820 × 2 × ½ = 2820	2 × ⅓	1,880
	6450 lbs.	....	6,720 ft-lbs.
Downward...	2 × 2 × 150 = 600	1	600
		....	
	Total shear = 5850 lbs.	.... Total moment	= 6,120 ft-lbs.
			= 73,500 in-lbs.

Steel to be placed near bottom of slab with 3-in. cover. Art. 67, J. C. R. Place steel 3½ in. above bottom of slab, $d = 24 - 3\frac{1}{2} = 20.5$ in. Assume $j = 0.86$.

$$A_s = \frac{M}{f_s j d} = \frac{73,500}{16,000 \times 0.86 \times 20.5} = 0.261 \text{ in}^2/\text{ft}.$$

$$\Sigma_0 = \frac{V}{u j d} = \frac{5850}{80 \times 0.86 \times 20.5} = 4.15 \text{ in/ft}.$$

From Table 28, Chap. XII, ½-in. ϕ rods at 4½-in. centres provide $A_s = 0.52$ in²/ft., and $\Sigma_0 = 4.19$ in/ft. Note that the bond stress conditions determine the required steel area. To develop the full strength of the rods, they must be extended $\frac{f_s}{4u} = \frac{16,000}{4 \times 80} = 50$ diameters or $50 \times \frac{1}{2} = 25$ in. beyond the point of maximum stress, as shown in Fig. 8.

The shearing-stress in the base slab at the edge of the vertical wall is $v = \frac{V}{bjd} = \frac{5850}{12 \times 0.86 \times 20.5} = 27.6$ lbs/in.² Allowable shearing-stress = 40 lbs/in.²

Design of the Heel Slab.—The heel slab is to be designed for the upward load due to the soil pressure and the downward loads due to the weight of the slab and the earth fill. From Fig. 7, the moment and shear at the back edge of the vertical wall are as follows:

Direction	Load	Arm	Moment
Downward	300 × 5.7 = 1,710 lbs.	½ × 5.7 ft.	4,870 ft-lbs.
	18 × 100 × 5.7 = 10,250	½ × 5.7	29,200
	11,960 lbs.	....	34,070 ft-lbs.
Upward...	2300 × 5.7 × ½ = 6,560	⅓ × 5.7	12,480
	Total shear = 5,400 lbs.	.... Total moment =	21,590 ft-lbs.
		=	259,000 in-lbs.

Moment to be carried by steel placed 3½ in. from top of slab, $d = 24 - 3.5 = 20.5$ in. Assume $j = 0.86$.

$$A_s = \frac{M}{f_s j d} = \frac{259{,}000}{16{,}000 \times 0.86 \times 20.5} = 0.92 \text{ in}^2/\text{ft}.$$

$$\sum{}_0 = \frac{V}{u j d} = \frac{5400}{80 \times 0.86 \times 20.5} = 3.83 \text{ in/ft}.$$

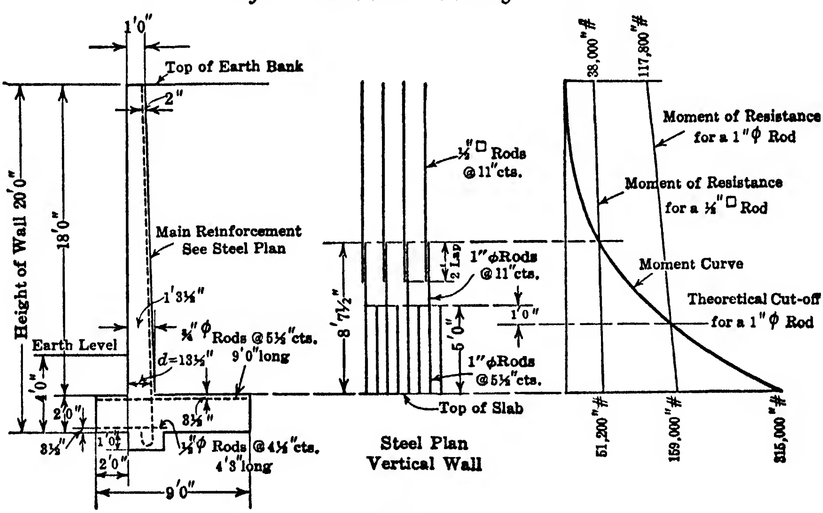


FIG. 8.

From Table 28, Chap. XII, ¾-in. ϕ rods at 5½-in. centres provide A_s = 0.96 in²/ft. and $\Sigma_0 = 5.14$ in/ft. of slab. Imbedment required for full strength = 50 diameters = 50 × ¾ = 37.5 in. Rods must extend completely across the footing slab, as shown in Fig. 8.

Shearing stress in footing slab $= \dfrac{V}{bjd} = \dfrac{5400}{12 \times 0.86 \times 20.5} = 25.5$ lbs/in.² Allowable shearing stress = 40 lbs/in.²

Design of Vertical Wall.—As determined in the early part of this design, $M = 315{,}000$ in-lbs.; $V = 4370$ lbs.; and $d = 13.5$ in. for a balanced design.

Then
$$A_s = \frac{M}{f_s j d} = \frac{315{,}000}{16{,}000 \times 0.86 \times 13.5} = 1.70 \text{ in}^2/\text{ft.}$$

and
$$\Sigma_0 = \frac{V}{u j d} = \frac{4370}{80 \times 0.86 \times 13.5} = 4.70 \text{ in/ft.}$$

From Table 28, 1-in. ϕ rods at 5½-in. centres provide $A_s = 1.71$ in²/ft. and $\Sigma_0 = 6.85$ in/ft.

These rods are required for only a short distance above the base slab. Fig. 8 shows the bending moment curve due to earth pressure against the wall. The curve is plotted from the equation $M = 2\,C\,w\,x^3$ where $C = 0.27$, $w = 100$ lbs/ft.³; and x = distance in feet from top of wall. To determine where one of the 1-in. ϕ rods can be cut off, calculate the moment of resistance for a 1-in. rod at 11-in. spacing for sections at the top and bottom of the wall. Plot this line in Fig. 8 and locate its intersection with the moment diagram. A 1-in. ϕ rod at 11-in. spacing furnishes 0.855 in²/ft. At the bottom of the wall $M_R = A_s f_s j d = 0.855 \times 16{,}000 \times 0.86 \times 13.5$ $= 159{,}000$ in-lbs. At the top of the wall $M_R = 159{,}000 \times \frac{10}{13.5} = 117{,}800$ in-lbs. Fig. 8 shows the required length of rod.

In order to save steel, the top portion of the reinforcement will be supplied by small rods. Assume ½-in. square rods spaced 11 in. centres so as to match the lower rods. From Table 28, ½-in. square rods at 11-in. centres furnish 0.275 in²/ft. At the base of the wall, $M_R = A_s f_s j d = 0.275 \times 16{,}000 \times 0.86 \times 13.5 = 51{,}200$ in-lbs., and at the top of the wall $M_R = 51{,}200 \times \frac{10}{13.5} = 38{,}000$ in-lbs. Fig. 8 shows the arrangement of rods selected to meet the moment requirements.

Proper anchorage in the base slab must be provided for the main reinforcement. The necessary imbedment for full strength of the steel is 50 diameters or $50 \times 1 = 50$ in. Since the required imbedment is greater than the depth of the base slab, a projecting lug 1 ft. deep will be placed on the under side of the base slab, as shown in Fig. 8. By placing a hook at the end of the rod, the required anchorage is provided.

Stability of Retaining-wall against Horizontal Sliding.—As shown in the preceding calculations, the total horizontal force acting on the wall is 5400 lbs., and the total vertical force is 16,335 lbs. The coefficient of friction between the base slab and the earth is 0.4. Hence, resistance to sliding is $16{,}335 \times 0.4 = 6534$ lbs. Since the force tending to cause sliding, 5400 lbs., is less than the resisting force, there is no danger that the retaining-wall will slide along the base.

Longitudinal Reinforcement.—Art. 184 (*h*) J. C. R. requires not less than 0.25 sq. in. of horizontal reinforcement in order to prevent temperature and shrinkage cracks. Place ½-in. square rods at 12-in. centres on the front face of the vertical wall. This steel is not shown in Fig. 8.

Expansion Joints.—Art. 184 (*i*) J. C. R. requires that grooved lock joints be placed not over 60 ft. apart to care for temperature. This detail is not shown on Fig. 8.

2. Counterforted Retaining-wall.—Counterforted retaining-walls of the general type shown in Fig. 9 are used for walls of a total height greater than about 25 ft. This type of wall differs from the one shown in Fig. 8 in that the vertical wall is supported at intervals by triangular-shaped braces called counterforts. The spacing of counterforts depends upon the loads to be supported, being so spaced as to allow the use of front and base slabs of reasonable thickness. Generally, the adopted spacing varies from about 7 ft. for heavy loads to about 10 or 12 ft. for lighter loadings. Comparative designs are generally required before the most economical arrangement

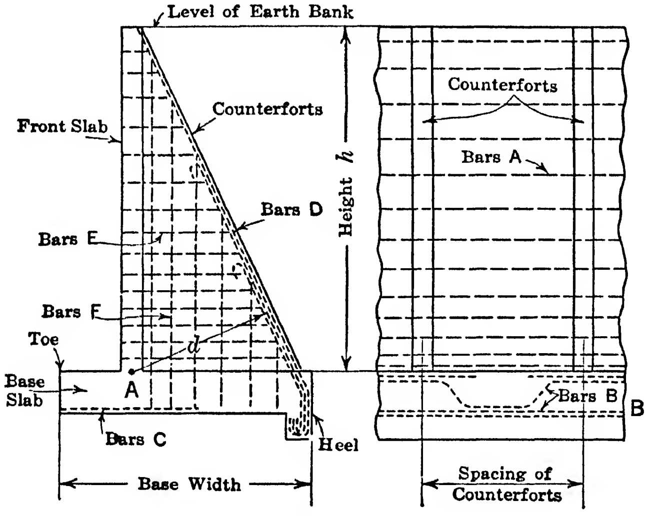


FIG. 9.

can be secured. In the discussion which follows, no attempt will be made to design a definite structure. General methods of procedure will be stated and a design for any given case will be left for the student.

The width of base is determined by the same method as used in Example 1. In testing the section in order to determine the position of the resultant force on the base with respect to the middle third, a section of wall equal in length to the counterfort spacing should be used. As before, the resultant should fall inside the middle third of the base.

In designing the front slab, it is assumed to act as a series of horizontal continuous beams supported by the counterforts. Moment values are as recommended in the J. C. R. for continuous slabs. It is assumed that the slab carries the horizontal earth pressure, which acts outward on the front slab. Since the earth pressure is a maximum at the base of the front slab, balanced design conditions are generally assumed in calculating the lower

1-ft. section of the slab. This same thickness is then used throughout the slab, varying the steel to meet the moment conditions. Fig. 9 shows a gradual increase in rod spacing of the bars lettered *A*.

The toe section is designed by the same methods as used for the cantilever wall. Bars *C* provide the necessary reinforcement.

In designing the counterfort portions of the wall, it is assumed that the counterfort stem and the front wall form a T-beam. It is assumed that the front wall forms the flange of the T-beam and that it carries all the compression, the concrete stress being very small. The steel in the back of the counterfort, furnished by bars *D*, is assumed to take all the tension. It is assumed that the lever arm of the steel, corresponding to $j\,d$ of T-beam formulas, is given by the distance d in Fig. 9, measured from the centre *A* of the front slab to the centre of the steel. The moment to be carried by bars *D* is taken as the bending moment due to earth pressure above the top of the base slab. Bars *D* are cut off and hooked where no longer needed.

The rear portion of the base slab is designed as a continuous beam supported by the counterforts. These beams carry the downward load due to the earth and slab load minus the upward soil pressure on the base. Since the resultant load is a maximum at the heel of the slab, the design should be made for this section of the slab. Bars *B* carry this moment.

Bars *E* and *F* serve to tie the front and base slabs to the counterforts. These bars are designed by the same methods as used for stirrups. Since the pressure on the front and base slabs are outward, bars *E* and *F* are in tension.

It is to be noted that the design methods outlined above are approximate in nature, for it is assumed that each part of the structure is an independent unit. In reality, the parts act together. For example, the bottom part of the front slab is rigidly connected to the base slab so that the front slab and base form a slab supported on three sides. Also, this same slab may be considered as a horizontal beam between counterforts. Hence, the distribution of loads between the two possible systems is indeterminate. However, the general methods outlined above will yield a structure which is safe and can be depended upon to carry the earth loads safely.

CHAPTER XII

TABLES AND DIAGRAMS

TABLE 27

Areas, summations of perimeters and weights of bars

ΣA = Total Area of Bars in Square Inches
Σo = Total Perimeter of Bars in Inches

Bar Size	Weight Lbs./Ft.		Number of Bars									
			1	2	3	4	5	6	7	8	9	10
3/8" Round	0.376	ΣA	0.11	0.22	0.33	0.44	0.55	0.66	0.77	0.88	0.99	1.10
		Σo	1.18	2.36	3.53	4.71	5.89	7.07	8.25	9.42	10.60	11.78
1/2" Round	0.668	ΣA	0.20	0.39	0.59	0.79	0.98	1.18	1.38	1.57	1.77	1.96
		Σo	1.57	3.14	4.71	6.28	7.85	9.42	10.99	12.57	14.14	15.71
1/2" Square	0.850	ΣA	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.25	2.50
		Σo	2.00	4.00	6.00	8.00	10.00	12.00	14.00	16.00	18.00	20.00
5/8" Round	1.043	ΣA	0.31	0.61	0.92	1.23	1.53	1.84	2.15	2.45	2.76	3.07
		Σo	1.96	3.93	5.89	7.85	9.82	11.78	13.74	15.71	17.67	19.64
3/4" Round	1.502	ΣA	0.44	0.88	1.33	1.77	2.21	2.65	3.09	3.53	3.98	4.42
		Σo	2.36	4.71	7.07	9.42	11.78	14.14	16.49	18.85	21.21	23.56
7/8" Round	2.044	ΣA	0.60	1.20	1.80	2.41	3.01	3.61	4.21	4.81	5.41	6.01
		Σo	2.75	5.50	8.25	11.00	13.74	16.49	19.24	21.99	24.74	27.49
1" Round	2.670	ΣA	0.79	1.57	2.36	3.14	3.93	4.71	5.50	6.28	7.07	7.85
		Σo	3.14	6.28	9.42	12.57	15.71	18.85	21.99	25.13	28.27	31.42
1" Square	3.400	ΣA	1.00	2.00	3.00	4.00	5.00	6.00	7.00	8.00	9.00	10.00
		Σo	4.00	8.00	12.00	16.00	20.00	24.00	28.00	32.00	36.00	40.00
1 1/8" Square	4.303	ΣA	1.27	2.53	3.80	5.06	6.33	7.59	8.86	10.12	11.39	12.66
		Σo	4.50	9.00	13.50	18.00	22.50	27.00	31.50	36.00	40.00	45.00
1 1/4" Square	5.313	ΣA	1.56	3.12	4.69	6.25	7.81	9.38	10.94	12.50	14.06	15.62
		Σo	5.00	10.00	15.00	20.00	25.00	30.00	35.00	40.00	45.00	50.00

TABLE 28

AREAS AND SUMMATIONS OF PERIMETERS OF BARS FOR VARIOUS SPACINGS

Areas ΣA given in Square Inches per Foot of Slab Width
Perimeters Σo given in Inches per Foot of Slab Width

Bar Size		Spacing of Bars in Inches													
		2	2½	3	3½	4	4½	5	5½	6	7	8	9	10	12
¼" Round	ΣA	0.29	0.24	0.20	0.17	0.15	0.13	0.12	0.11	0.10	0.08	0.07	0.07	0.06	0.05
	Σo	4.71	3.77	3.14	2.69	2.36	2.09	1.88	1.71	1.57	1.35	1.18	1.05	0.94	0.79
⅜" Round	ΣA	0.66	0.53	0.44	0.38	0.33	0.29	0.26	0.24	0.22	0.19	0.17	0.15	0.13	0.11
	Σo	7.07	5.66	4.71	4.04	3.53	3.14	2.83	2.57	2.36	2.04	1.78	1.57	1.41	1.18
½" Round	ΣA	1.18	0.94	0.78	0.67	0.59	0.52	0.47	0.43	0.39	0.34	0.29	0.26	0.24	0.20
	Σo	9.42	7.54	6.28	5.38	4.71	4.19	3.77	3.43	3.14	2.69	2.36	2.09	1.88	1.57
½" Square	ΣA	1.50	1.20	1.00	0.86	0.75	0.67	0.60	0.55	0.50	0.43	0.37	0.33	0.30	0.25
	Σo	12.00	9.60	8.00	6.86	6.00	5.33	4.80	4.36	4.00	3.43	3.00	2.67	2.40	2.00
⅝" Round	ΣA	1.84	1.47	1.23	1.05	0.92	0.82	0.74	0.67	0.61	0.53	0.46	0.41	0.37	0.31
	Σo	11.78	9.42	7.85	6.73	5.89	5.23	4.71	4.28	3.92	3.36	2.94	2.62	2.35	1.96
¾" Round	ΣA	2.65	2.12	1.77	1.51	1.32	1.18	1.06	0.96	0.88	0.76	0.66	0.59	0.53	0.44
	Σo	14.13	11.30	9.42	8.08	7.06	6.28	5.65	5.14	4.71	4.04	3.53	3.14	2.83	2.36
⅞" Round	ΣA	3.61	2.88	2.40	2.06	1.80	1.60	1.44	1.31	1.20	1.03	0.90	0.80	0.72	0.60
	Σo	16.49	13.18	10.98	9.42	8.24	7.32	6.59	6.00	5.50	4.71	4.12	3.66	3.30	2.75
1" Round	ΣA	.....	3.77	3.14	2.69	2.36	2.09	1.88	1.71	1.57	1.35	1.18	1.05	0.94	0.78
	Σo	.....	15.07	12.56	10.76	9.42	8.38	7.54	6.85	6.28	7.38	4.71	4.19	3.77	3.14
1" Square	ΣA	.....	4.80	4.00	3.43	3.00	2.67	2.40	2.18	2.00	1.71	1.50	1.33	1.20	1.00
	Σo	.....	19.20	16.00	13.71	12.00	10.67	9.60	8.73	8.00	6.86	6.00	5.33	4.80	4.00
1⅛" Square	ΣA	.....	.....	5.06	4.34	3.80	3.37	3.04	2.76	2.53	2.17	1.89	1.69	1.52	1.27
	Σo	.....	.....	18.00	15.43	13.50	12.00	10.80	9.82	9.00	7.72	6.85	6.00	5.40	4.50
1¼" Square	ΣA	.....	.....	6.25	5.36	4.69	4.17	3.75	3.41	3.12	2.68	2.34	2.08	1.87	1.56
	Σo	.....	.....	20.00	17.14	15.00	13.33	12.00	10.91	10.00	8.57	7.50	6.67	6.00	5.00

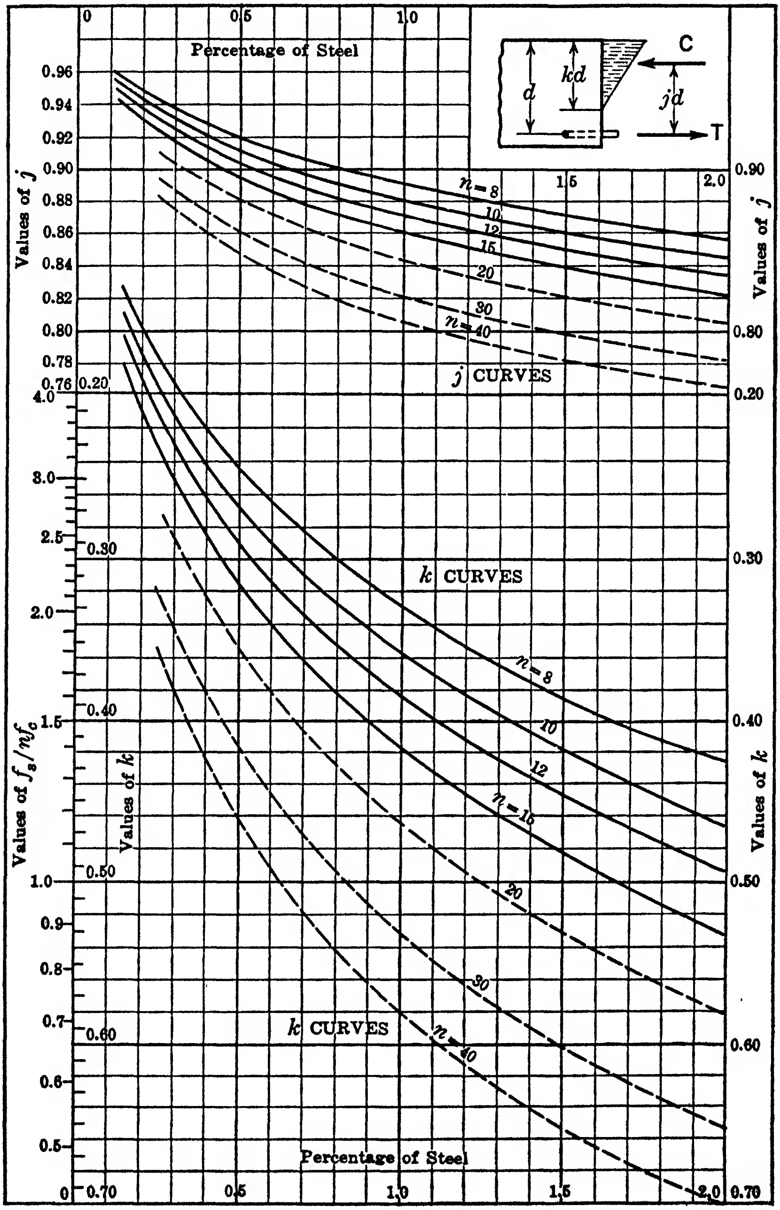


Diagram 1.—Values of k and j for Rectangular Beams. Working Stress Values.

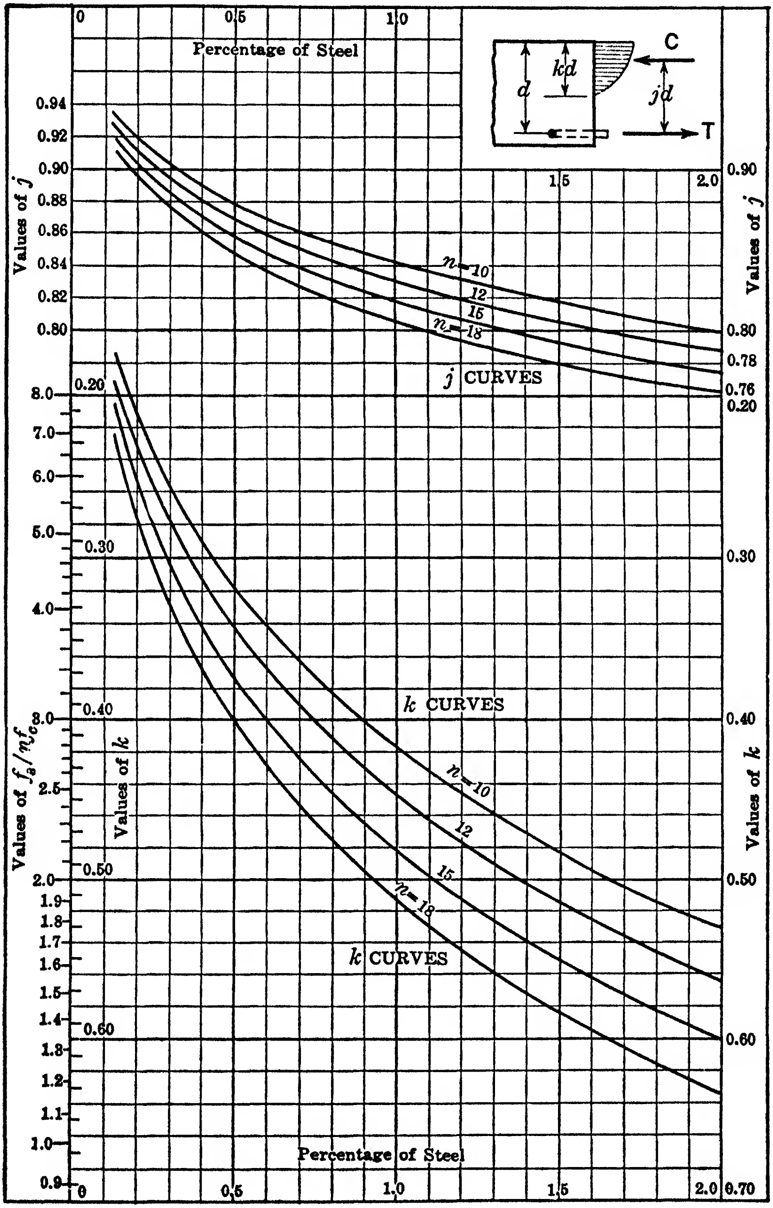


DIAGRAM 2.—Values of *k* and *j* for Rectangular Beams. Ultimate Stress Values.

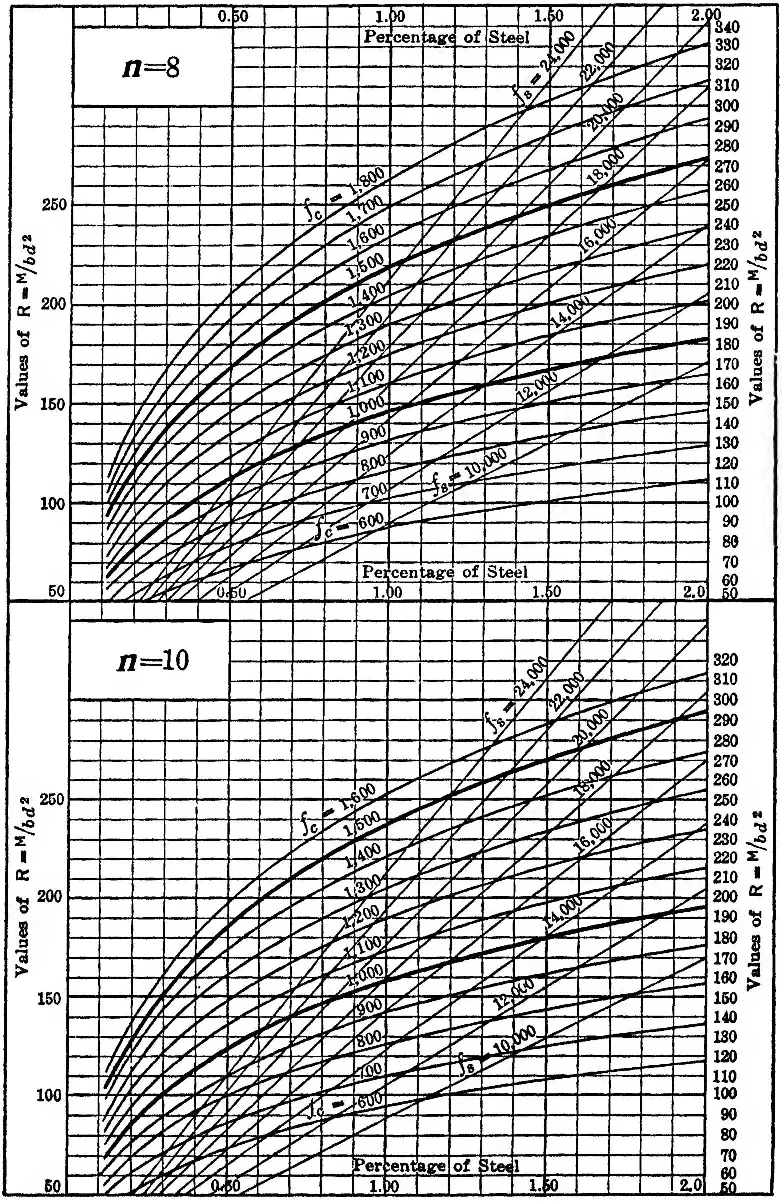


DIAGRAM 3.—Coefficients of Resistance of Rectangular Beams. $M = R\,b\,d^2$

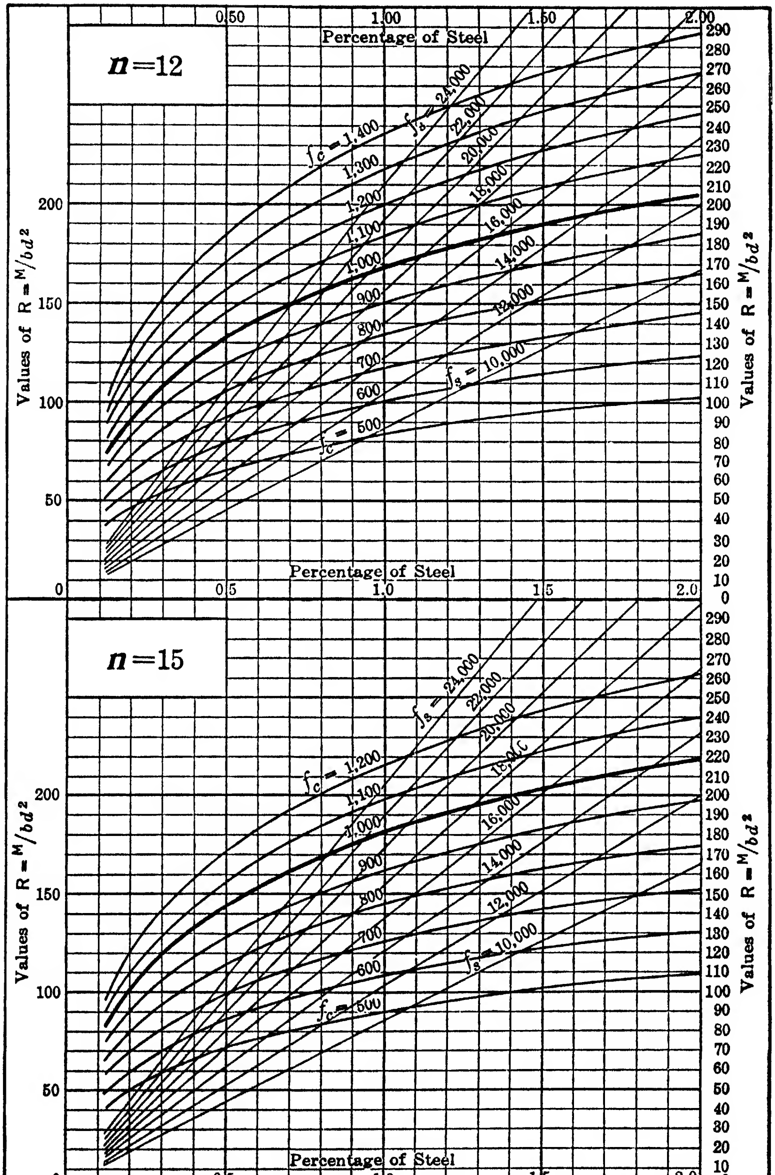


DIAGRAM 4.—Coefficients of Resistance of Rectangular Beams. $M = R b d^2$

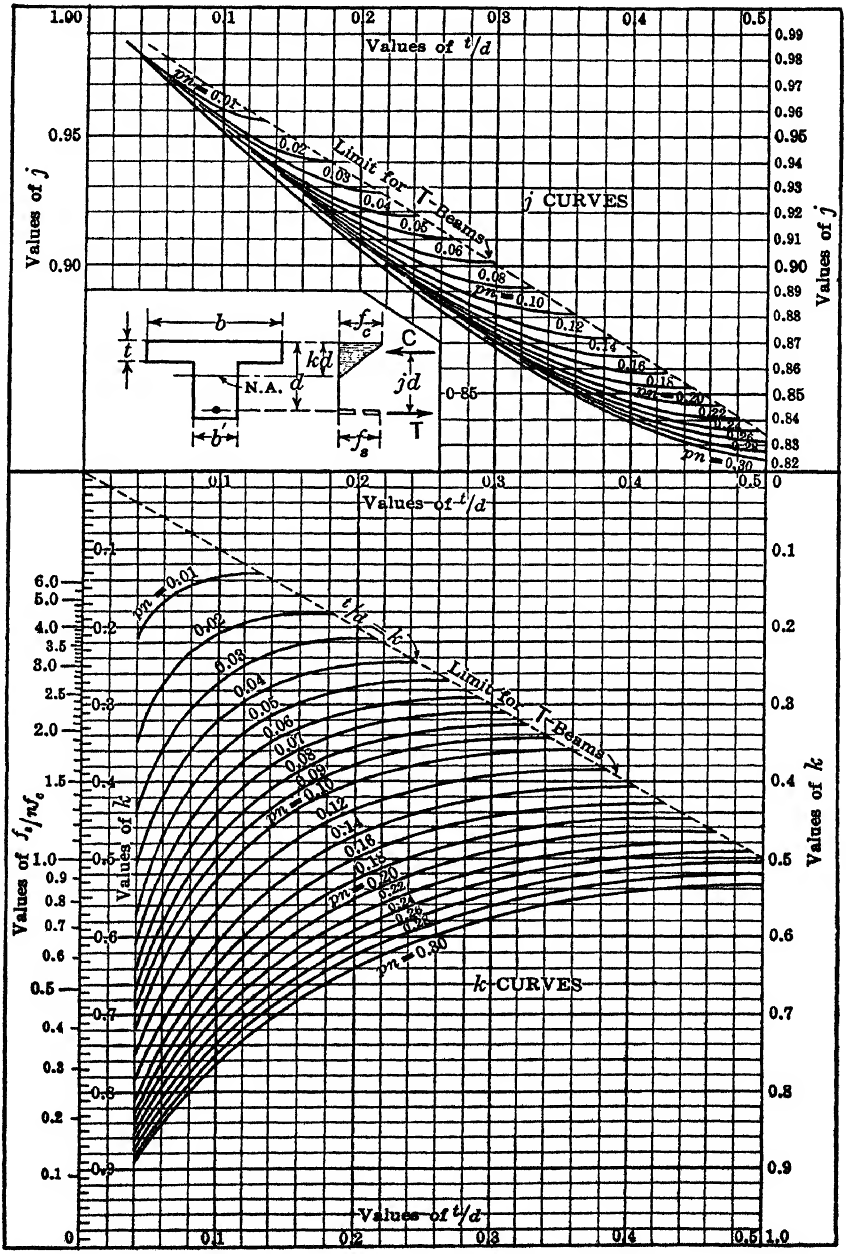


Diagram 5.—Values of *k* and *j* for T-Beams.

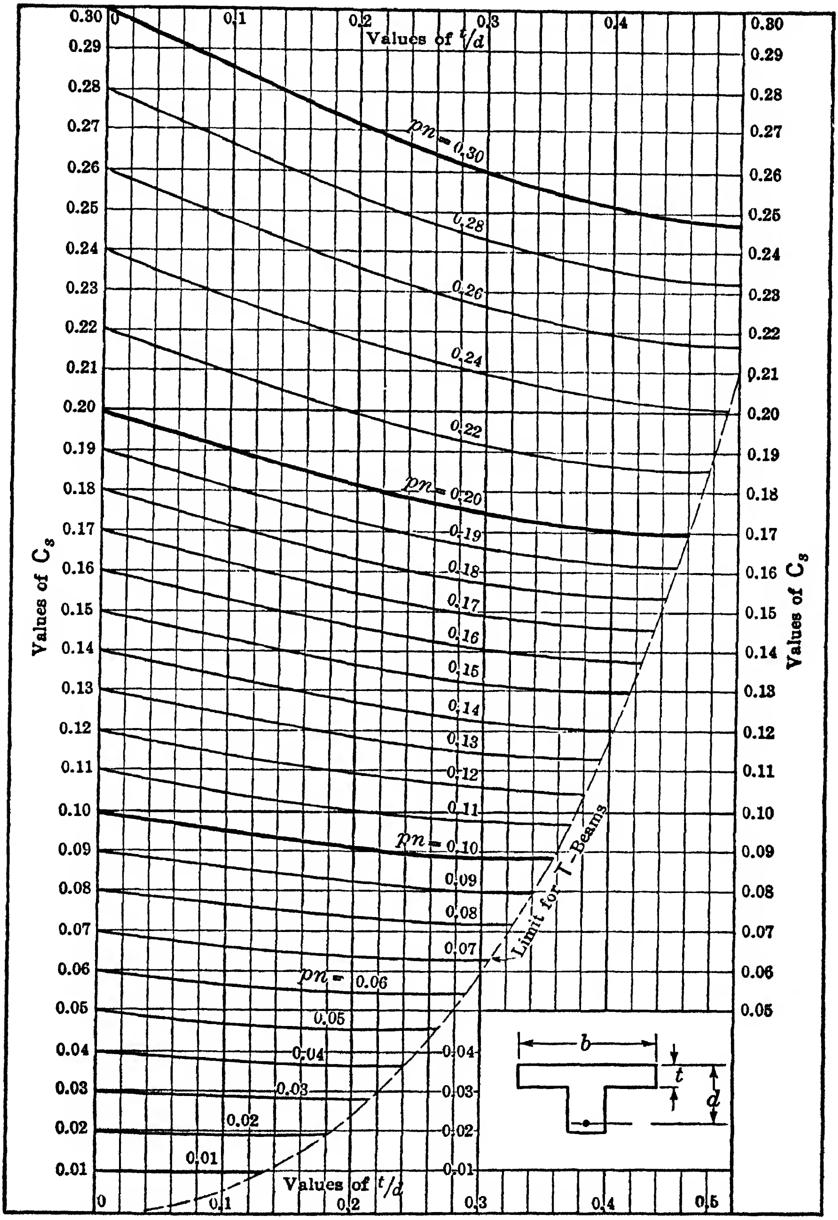


Diagram 6.—Coefficients of Resistance of T-Beams with Respect to the Steel.

$$M_s = C_s \frac{f_s}{-} b d^2.$$

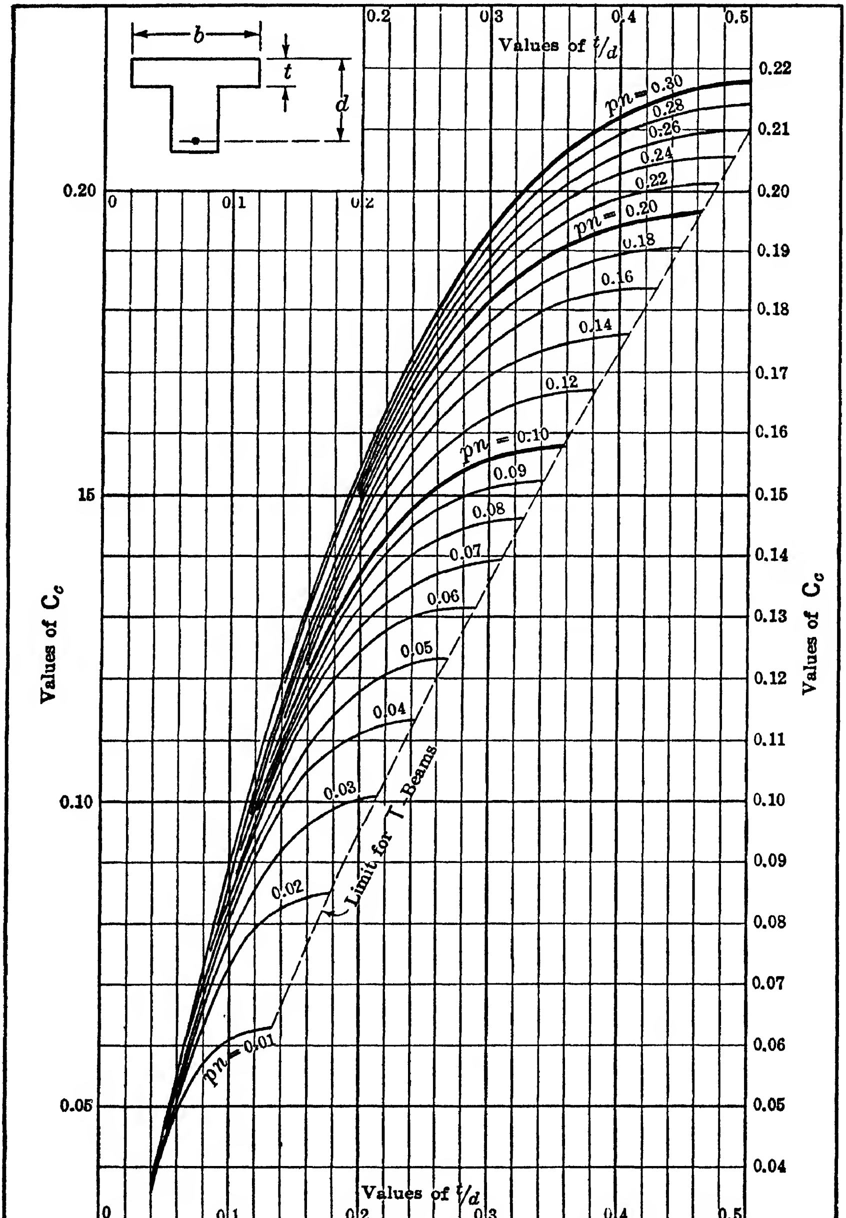


DIAGRAM 7.—Coefficients of Resistance of T-Beams with Respect to the Concrete.

$M_c = C_c f_c b d^2.$

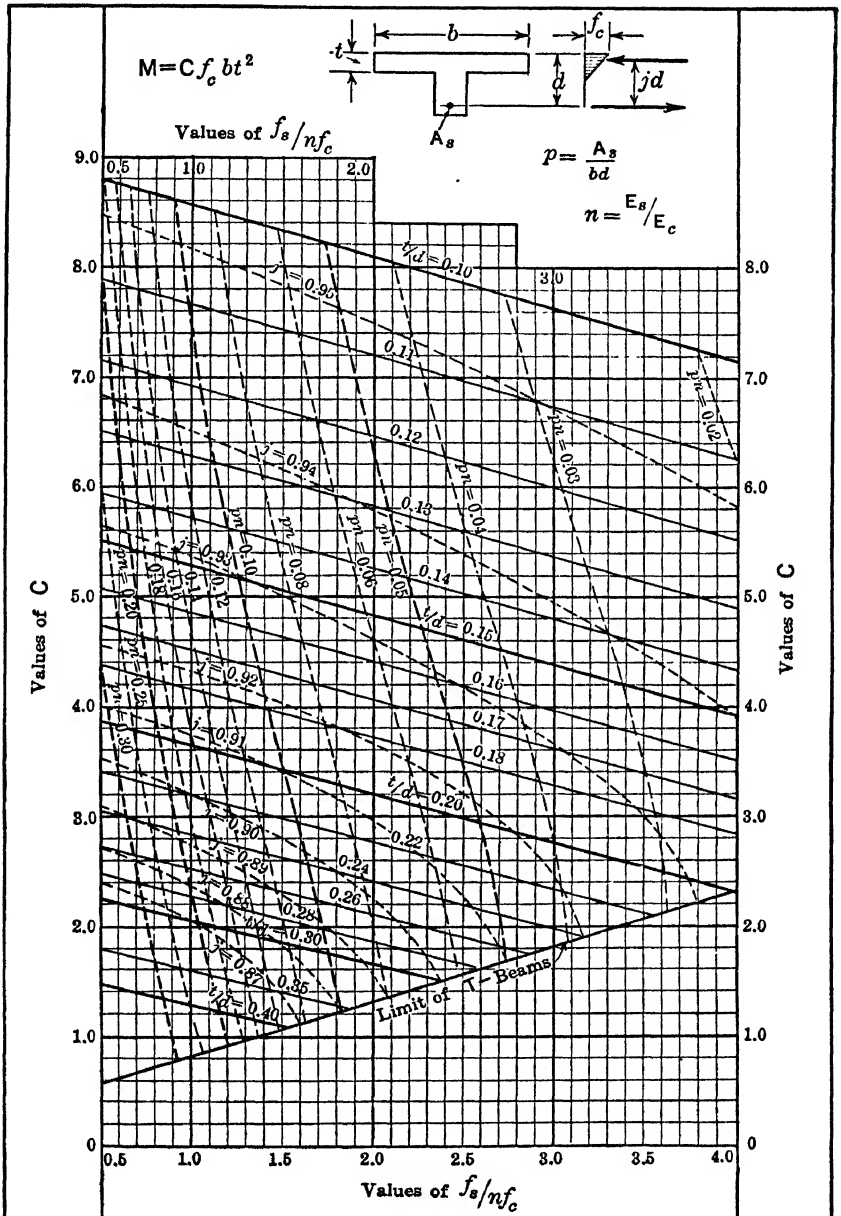


Diagram 8.—Coefficients of Resistance of T-Beams for a Balanced Design.
$M = C f_c b t^2$.

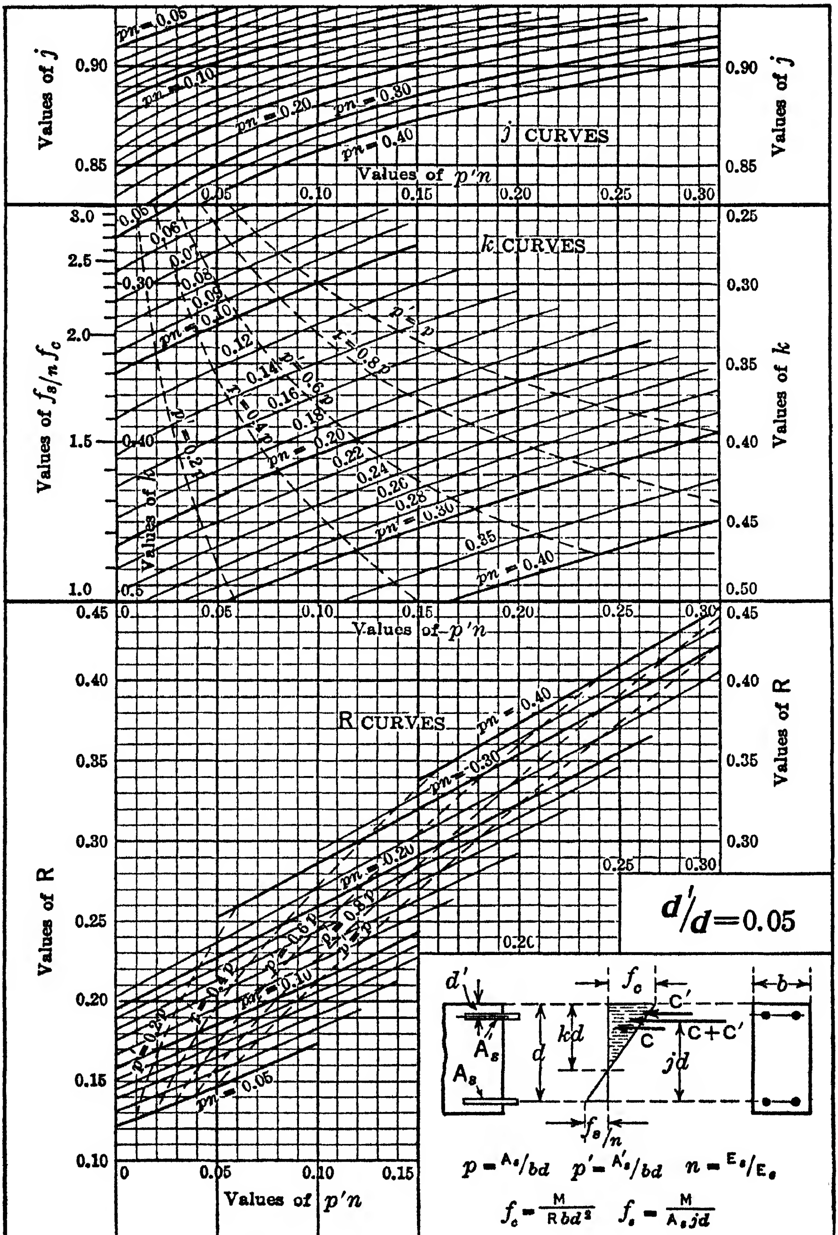


DIAGRAM 9.—Rectangular Beams Reinforced for Compression. $M = f_c R b d^2$.

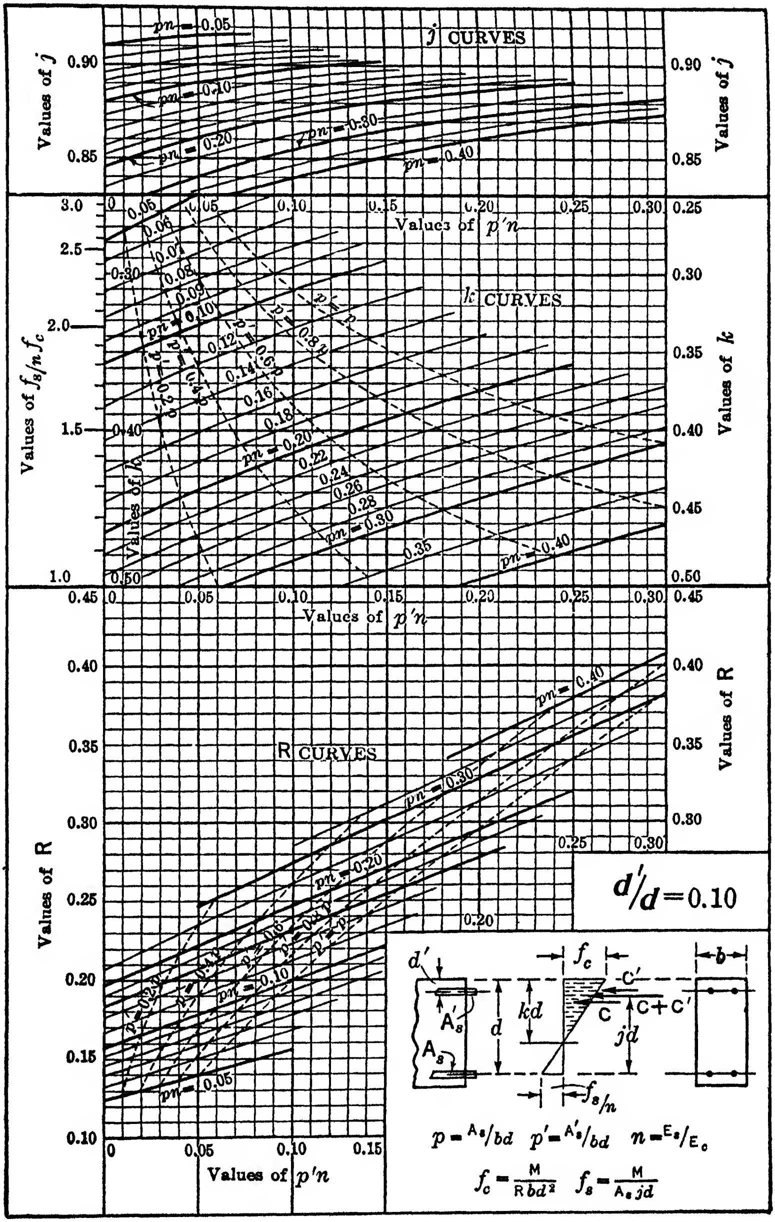


Diagram 10.—Rectangular Beams Reinforced for Compression. $M = f_c R b d^2$.

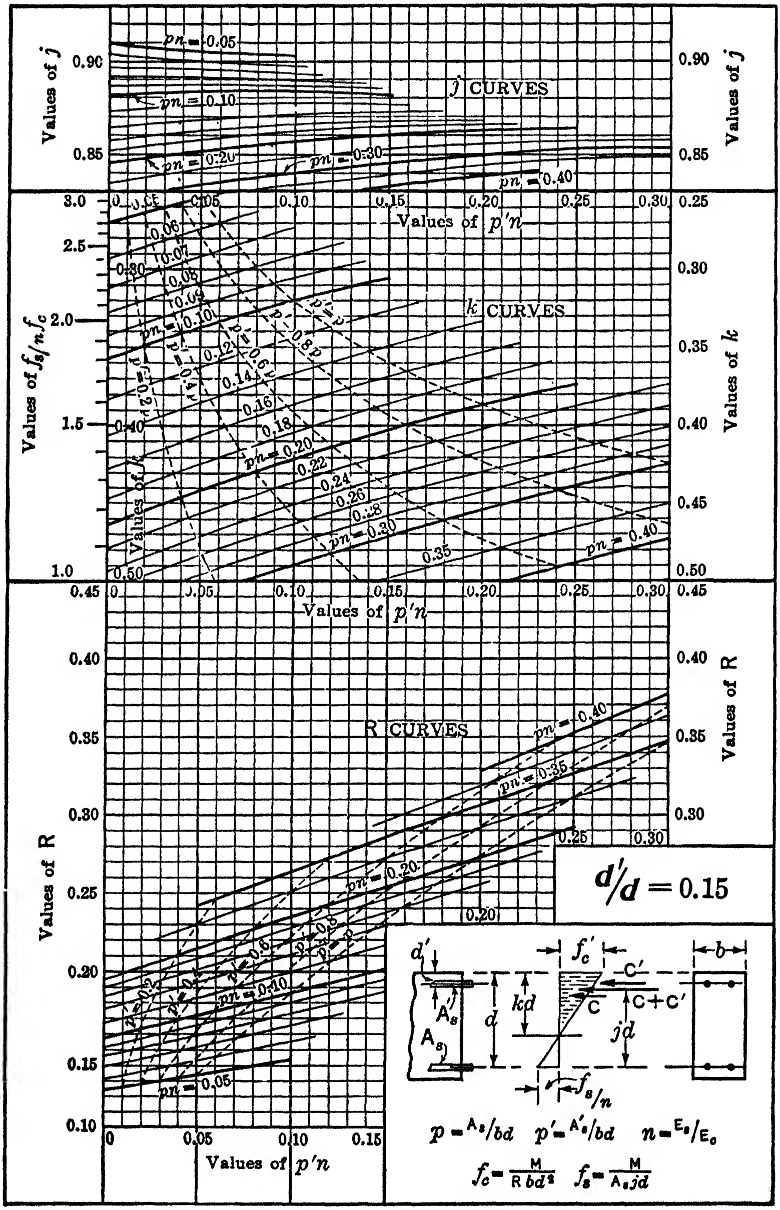
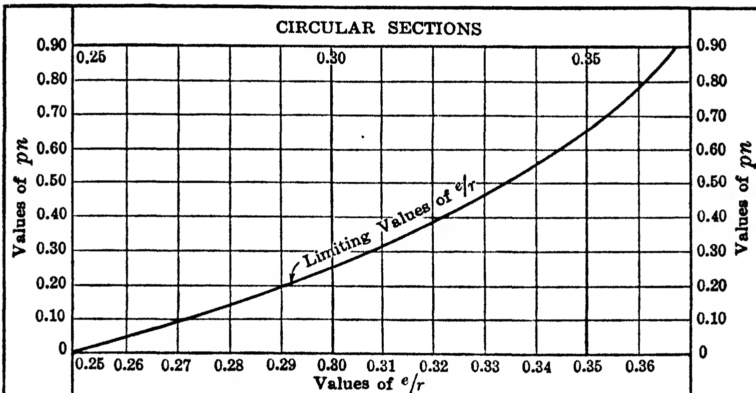
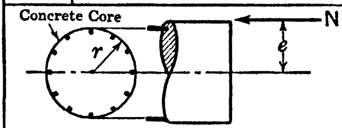


Diagram 11.—Rectangular Beams Reinforced for Compression. $M = f_c R b d^2$.

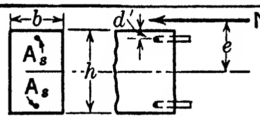




A_s = total steel area
r = radius of concrete core
p = steel ratio = $A_s/\pi r^2$
n = E_s/E_c

General Note:—When e/h or e/r for the given conditions is *less* than the value shown by the curves, *CASE I governs*.

When the actual e/h or e/r is *greater* than the value shown by the curves, *CASE II governs*.



p = steel ratio for one face of section = A_s/bh

n = E_s/E_c

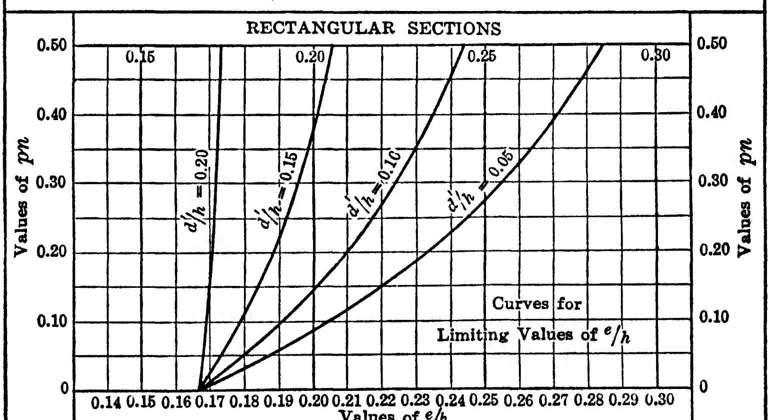
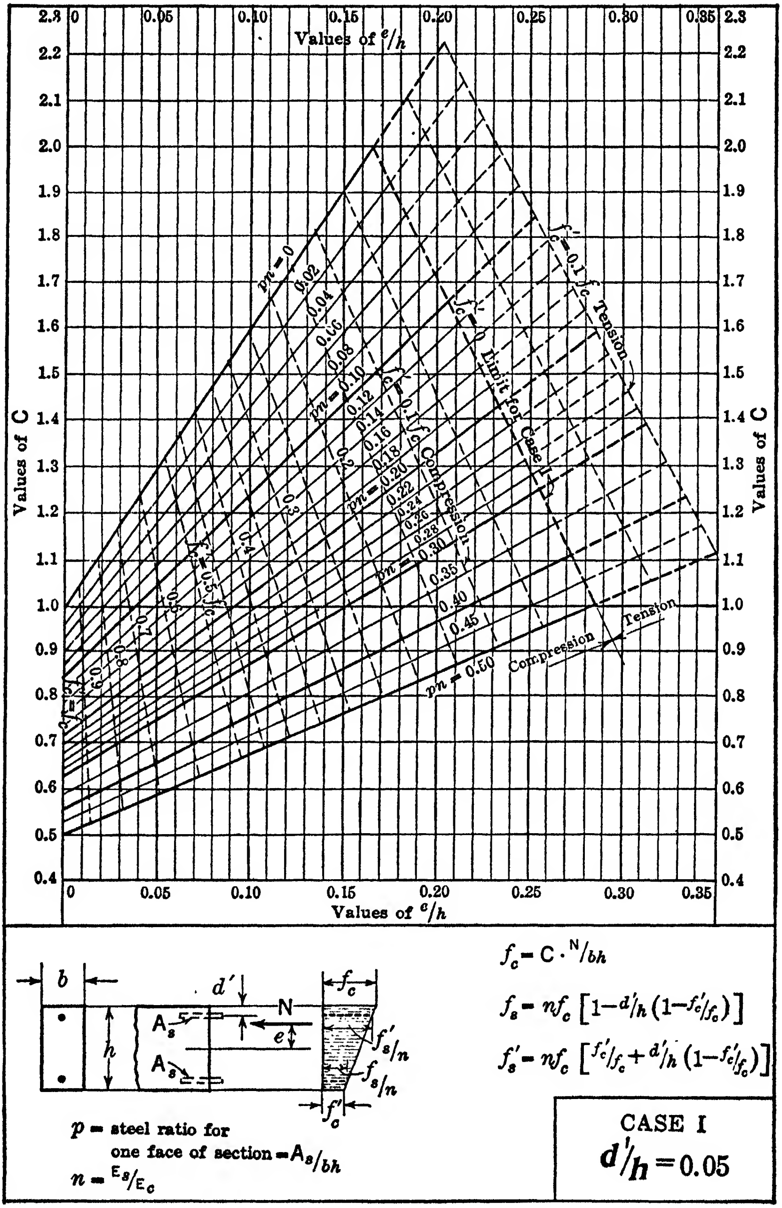


DIAGRAM 12.—Bending and Direct Stress; Limiting Conditions for Cases I and II.



$f_c = C \cdot N/bh$

$f_s = nf_c\left[1 - d'/h\,(1 - f_c'/f_c)\right]$

$f_s' = nf_c\left[f_c'/f_c + d'/h\,(1 - f_c'/f_c)\right]$

Diagram 13.—Bending and Direct Stress. Case I. Rectangular Sections.

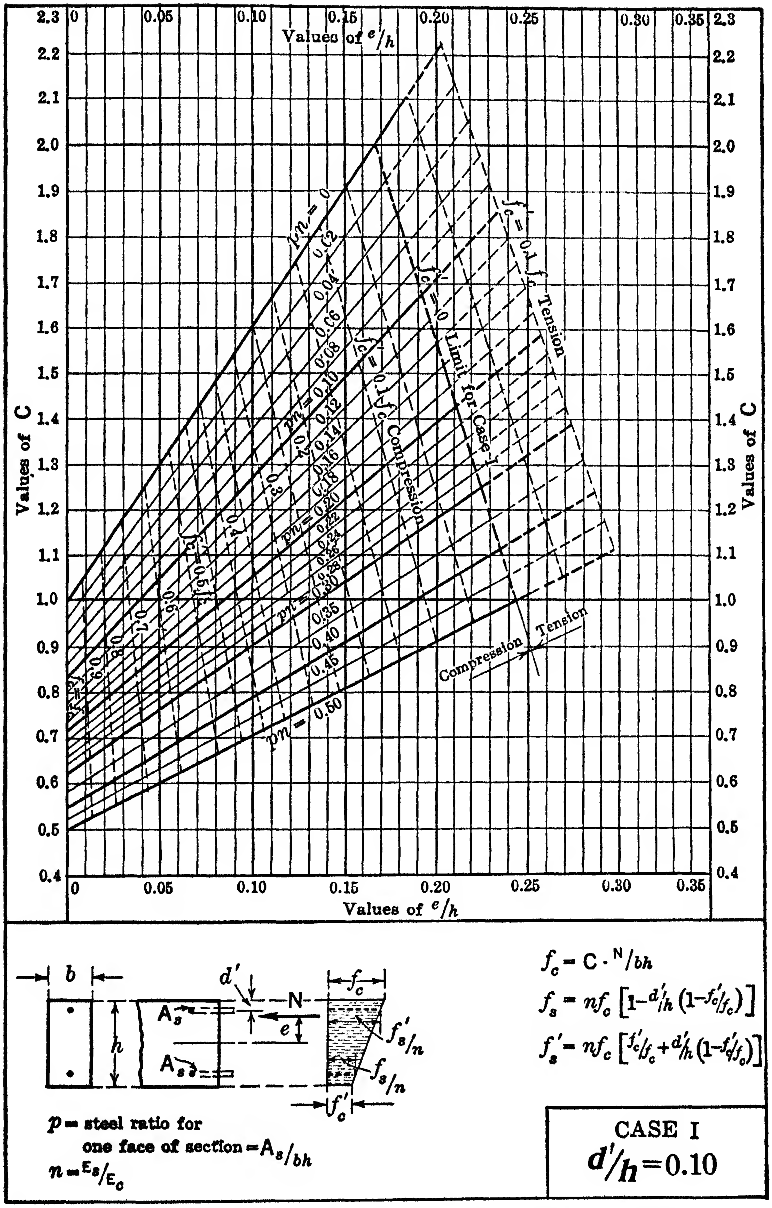
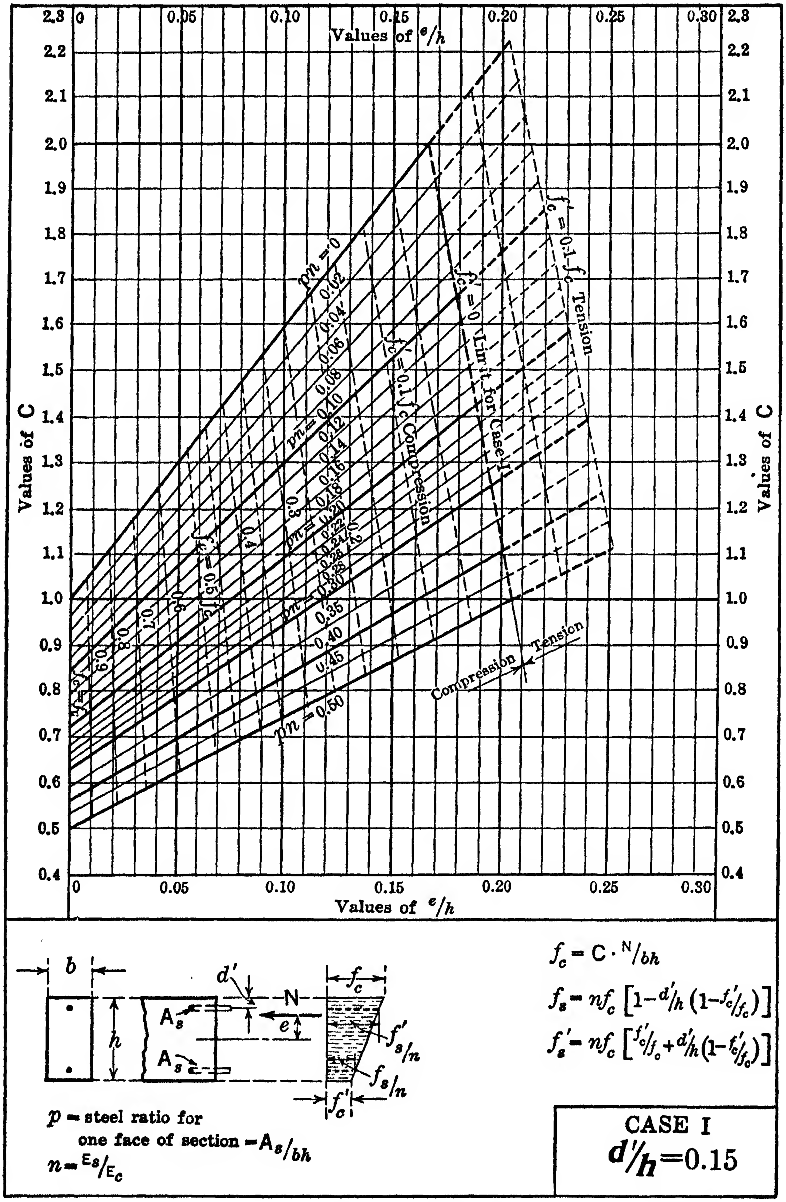


DIAGRAM 14.—Bending and Direct Stress. Case I. Rectangular Sections.



$f_c = C \cdot N/bh$

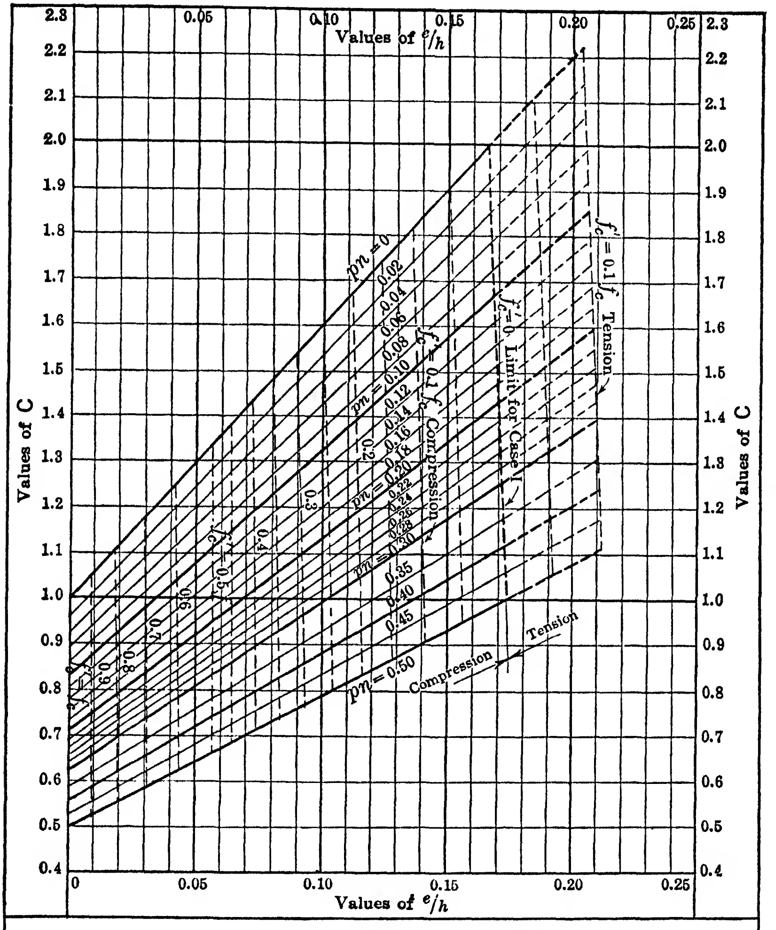
$f_s = nf_c\left[1 - d'/h\,(1 - f_c'/f_c)\right]$

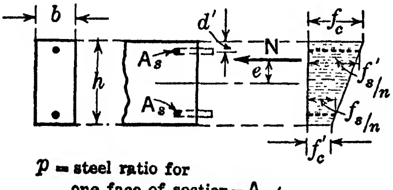
$f_s' = nf_c\left[f_c'/f_c + d'/h\,(1 - f_c'/f_c)\right]$

CASE I

$d'/h = 0.15$

DIAGRAM 15.—Bending and Direct Stress. Case I. Rectangular Sections.





$f_c = C \cdot N/bh$

$f_s = nf_c \left[1 - d'/h\,(f_c'/f_c)\right]$

$f_s' = nf_c \left[f_c'/f_c + d'/h\,(1 - f_c'/f_c)\right]$

p = steel ratio for one face of section $= A_s/bh$

$n = E_s/E_c$

CASE I

$d'/h = 0.20$

DIAGRAM 16.—Bending and Direct Stress. Case I. Rectangular Sections.

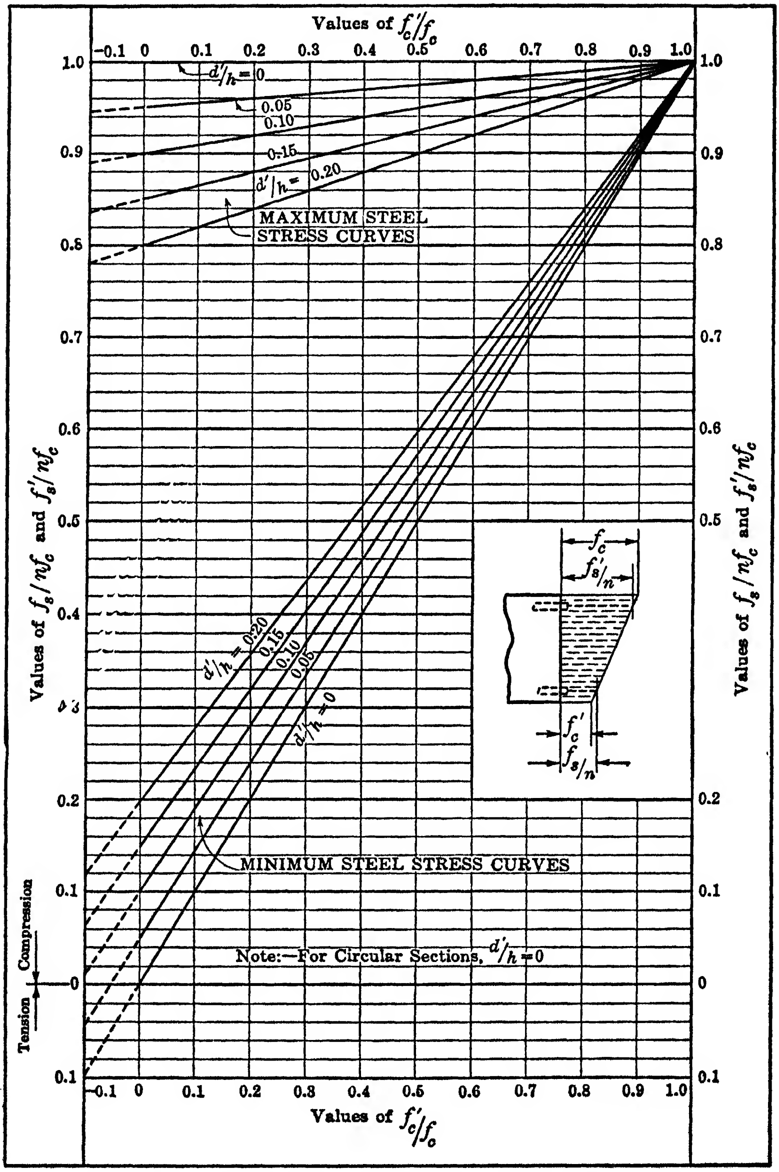


Diagram 17.—Bending and Direct Stress. Steel Stresses, Case I.

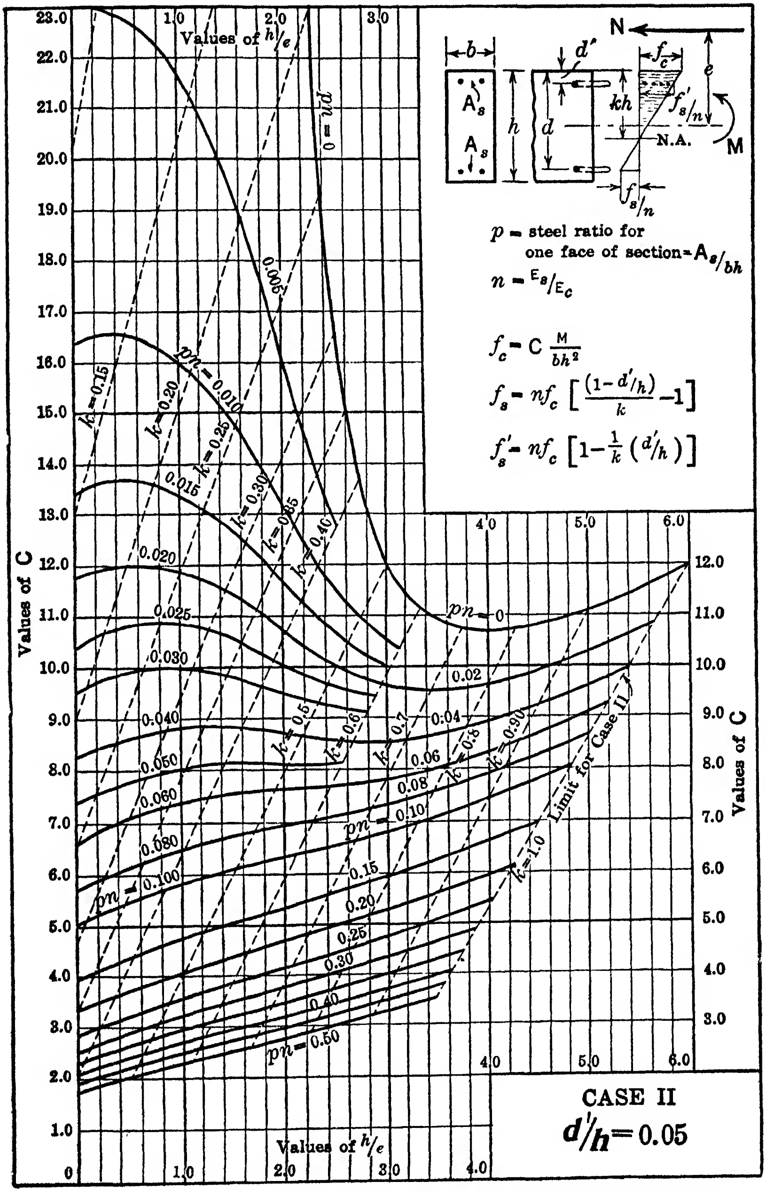


Diagram 18.—Bending and Direct Stress. Case II. Rectangular Sections.

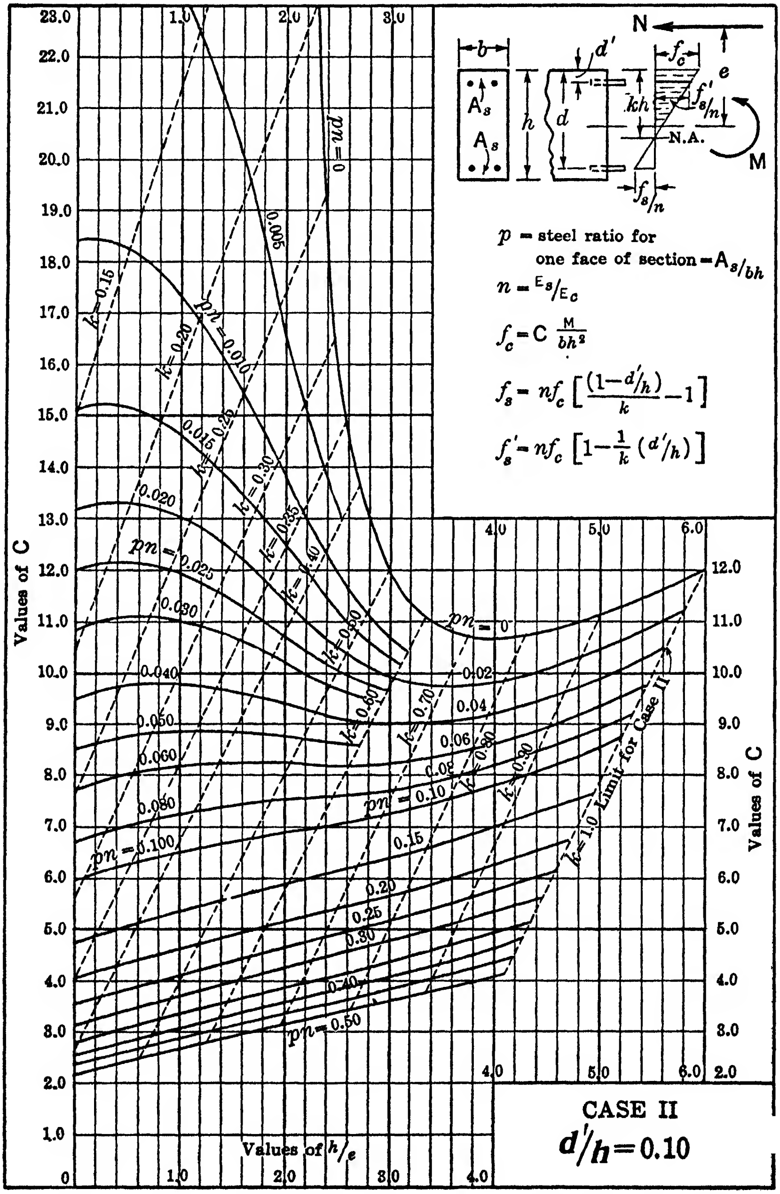


DIAGRAM 19.—Bending and Direct Stress. Case II. Rectangular Sections.

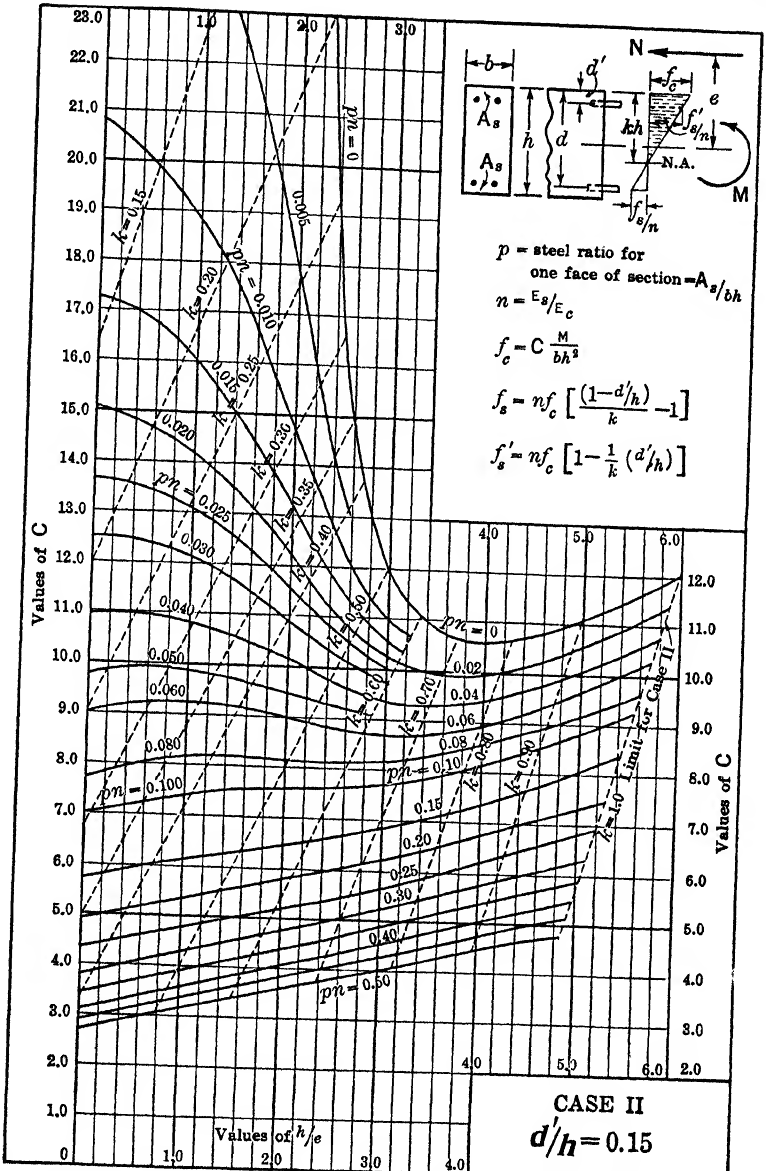


DIAGRAM 20.—Bending and Direct Stress. Case II. Rectangular Sections.

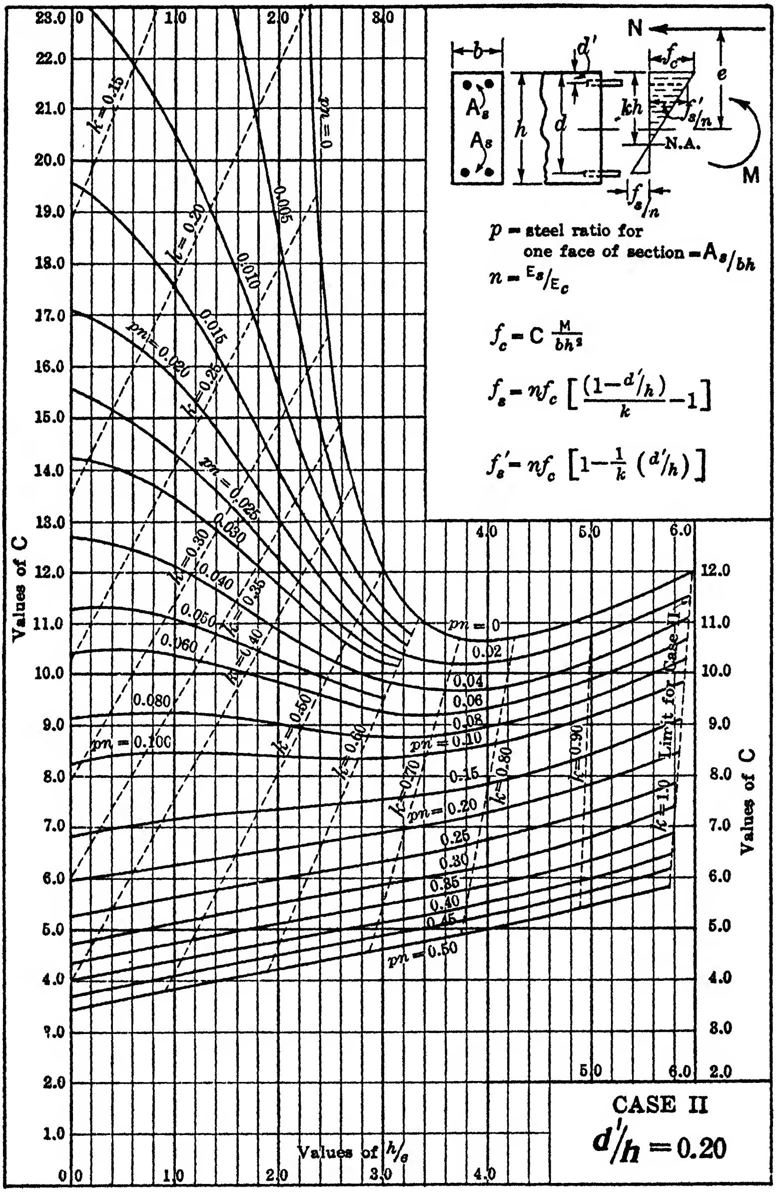


DIAGRAM 21.—Bending and Direct Stress. Case II. Rectangular Sections.

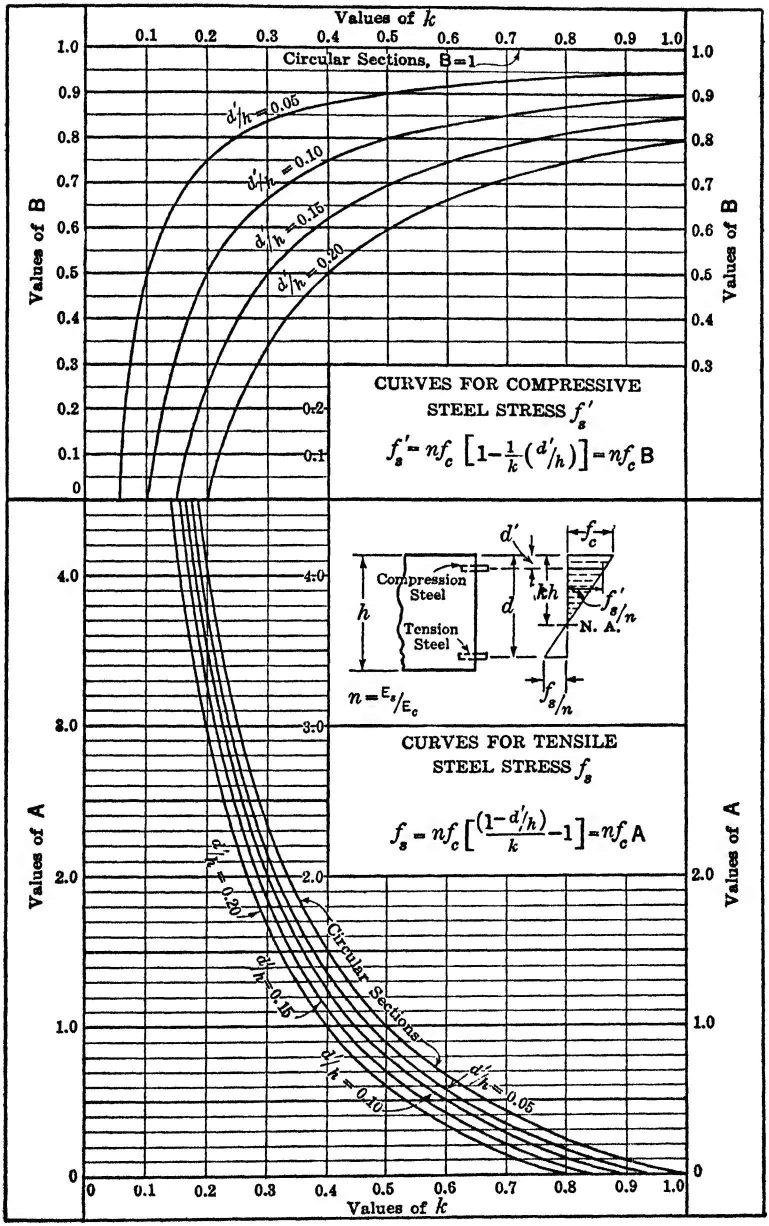


DIAGRAM 22.—Bending and Direct Stress. Steel Stresses, Case II.

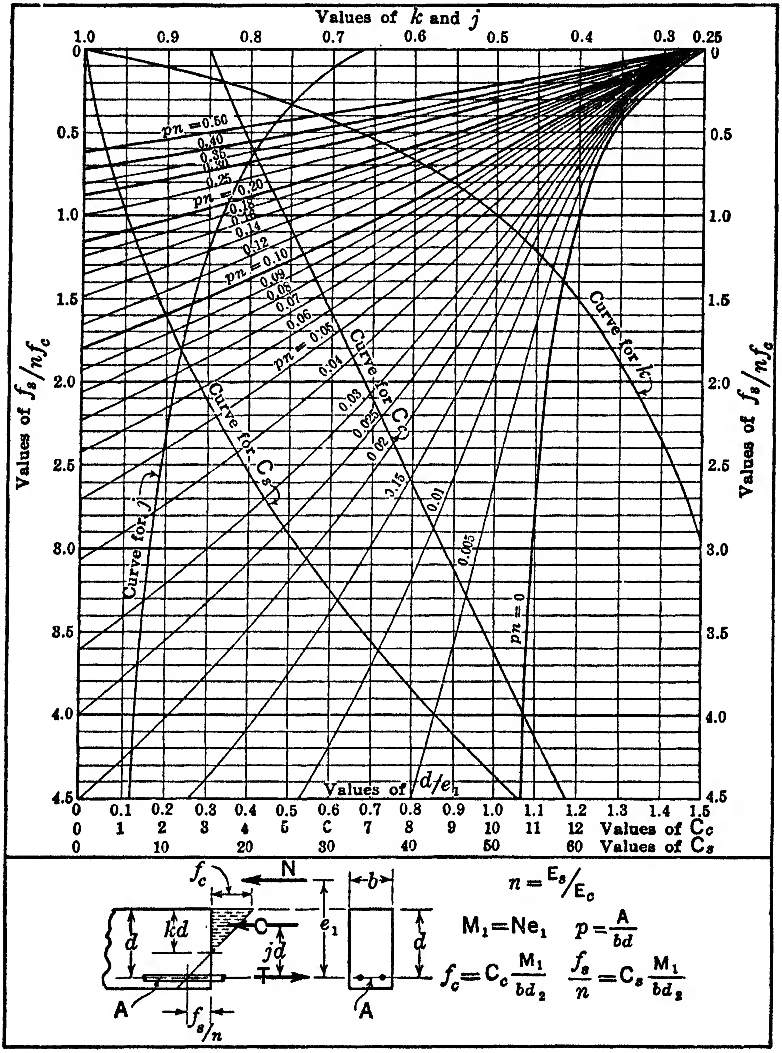


DIAGRAM 23.—Bending and Direct Stress. Reinforcement on Tension Face Only.

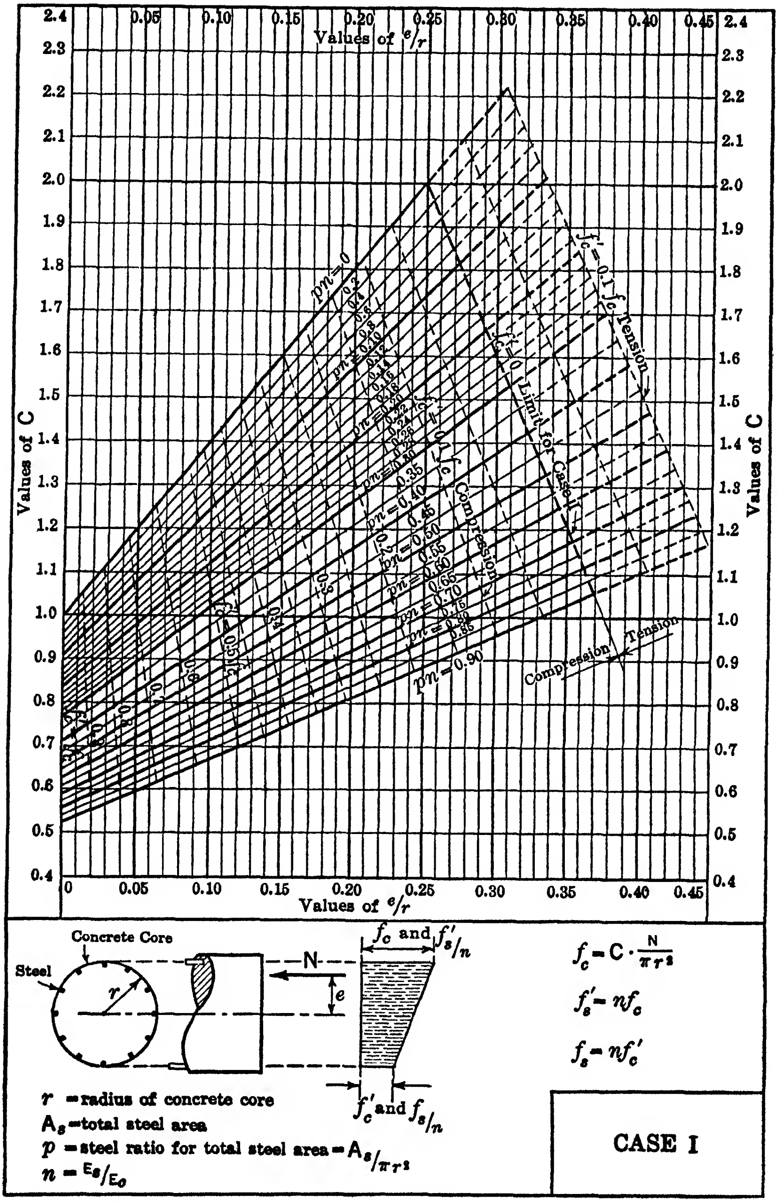


DIAGRAM 24.—Bending and Direct Stress. Case I. Circular Sections.

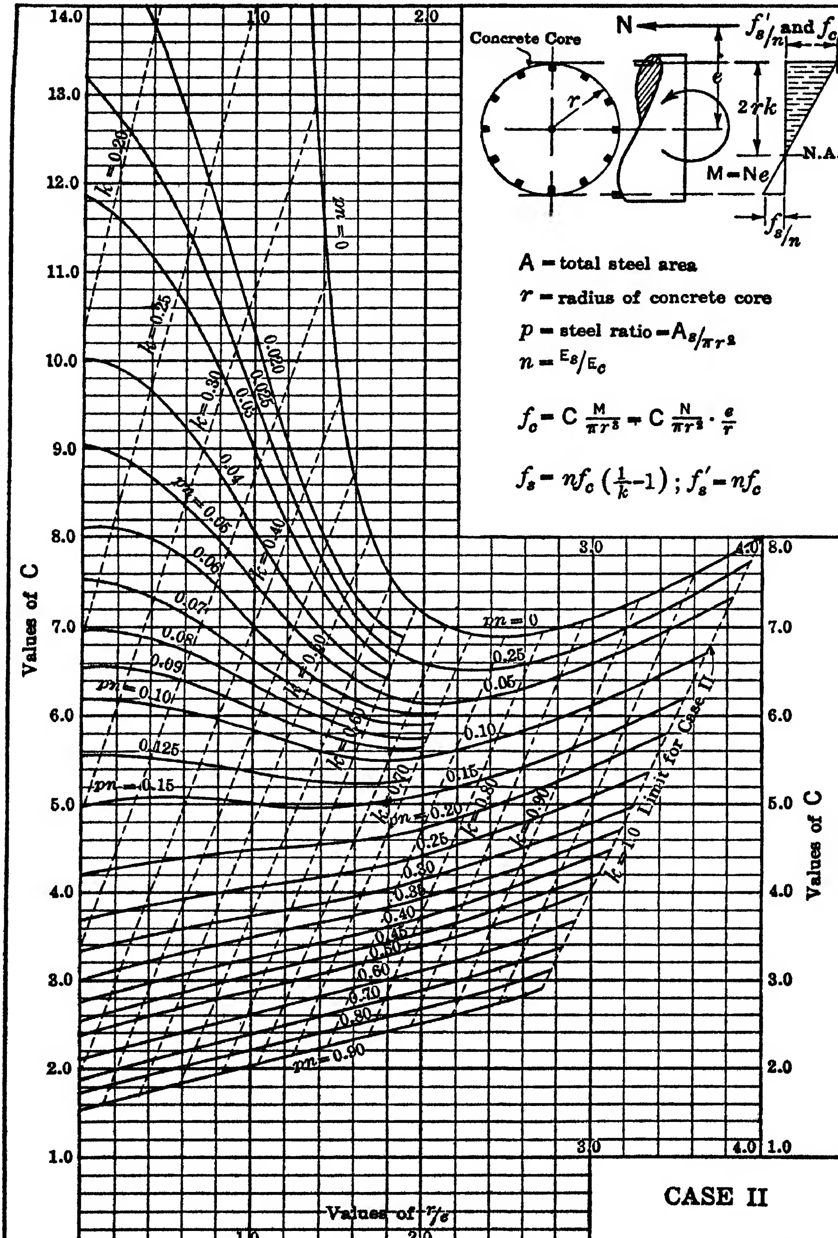


Diagram 25.—Bending and Direct Stress. Case II. Circular Sections.

APPENDIX

Report of Joint Committee on Standard Specifications for Concrete and Reinforced Concrete, 1924

CHAPTER XI. DESIGN

A. General Assumptions

103. General Assumptions.—The design of reinforced-concrete members under these specifications shall be based on the following assumptions:

(*a*) Calculations are made with reference to working stresses and safe loads rather than with reference to ultimate strength and ultimate loads.

(*b*) A plane section before bending remains plane after bending, shearing distortions being neglected.

(*c*) The modulus of elasticity of concrete in compression is constant within the limits of working stresses, and the distribution of compressive stress in beams is rectilinear.

(*d*) The moduli of elasticity of concrete in computations for the position of the neutral axis, for the resisting moment of beams, and for compression of concrete in columns, are as follows: *

(1) $\frac{1}{15}$ that of steel, when the compressive strength of the concrete at 28 days exceeds 1,500 and does not exceed 2,200 lbs/in.^{2};
(2) $\frac{1}{12}$ that of steel, when the compressive strength of the concrete at 28 days exceeds 2,200 and does not exceed 2,900 lbs/in.^{2};
(3) $\frac{1}{10}$ that of steel, when the compressive strength of the concrete at 28 days is greater than 2,900 lbs/in.^{2}

(*e*) In calculating the moment of resistance of reinforced-concrete beams and slabs the tensile resistance of the concrete is neglected.

(*f*) The bond between the concrete and the metal reinforcement remains unbroken throughout the range of working stresses. Under compression the two materials are therefore stressed in proportion of their moduli of elasticity.

(*g*) Initial stress in the reinforcement due to contraction or expansion of the concrete is neglected, except in the design of reinforced-concrete columns.

*A.C.I. Specifications: $n = 30{,}000/f'_c$, where f'_c = ultimate strength of concrete at 28 days.

B. Flexure of Rectangular Reinforced-concrete Beams and Slabs

104. **Flexure Formulas.**—Standard formulas as in Chapter III.

105. **Notation.**—Standard notation as in Chapter III.

106. **Span Length.**—The span length, l, of freely supported beams and slabs, shall be the distance between centres of the supports, but shall not exceed the clear span plus the depth of beam or slab. The span length for continuous or restrained beams built to act integrally with supports shall be the clear distance between faces of supports. Where brackets having a width not less than the width of the beam and making an angle of 45 deg. or more with the horizontal axis of a restrained beam are built to act integrally with the beam and support, the span shall be measured from the section where the combined depth of the beam and bracket is at least one-third more than the depth of the beam, but no portion of such a bracket shall be considered as adding to the effective depth of the beam. Maximum negative moments are to be considered as existing at the ends of the span, as defined above.

107. **Slightly Restrained Beams of Equal Span.**—Beams and slabs of equal spans built to act integrally with beams, girders, or other slightly restraining supports and carrying uniformly distributed loads shall be designed for the following moments at critical sections:

(*a*) Beams and slabs of one span,

Maximum positive moment near centre,

$$M = \frac{w\,l^2}{8}. \qquad (12)$$

(*b*) Beams and slabs continuous for two spans only,

(1) Maximum positive moment near centre,

$$M = \frac{w\,l^2}{10}. \qquad (13)$$

(2) Negative moment over interior support,

$$M = \frac{w\,l^2}{8}. \qquad (14)$$

(*c*) Beams and slabs continuous for more than two spans,

(1) Maximum positive moment near centre and negative moment at support of interior spans,

$$M = \frac{w\,l^2}{12}. \qquad (15)$$

(2) Maximum positive moment near centres of end spans and negative moment at first interior support,

$$M = \frac{w\,l^2}{10}. \qquad (16)$$

(*d*) Negative moment at end supports for cases (*a*), (*b*), (*c*) of this section,

$$M = \text{not less than } \frac{w\,l^2}{16}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16a)$$

108. Beams Built into Brick or Masonry Walls.—Beams and slabs built into brick or masonry walls in a manner which develops partial end restraint shall be designed for a negative moment at the support of

$$M^* = \text{not less than } \frac{w\,l^2}{16}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

109. Freely Supported Beams of Equal Span.—Beams and slabs of equal spans freely supported and assumed to carry uniformly distributed loads shall be designed for the moments specified in Section 107, except that no reinforcement for negative moment need be provided at end supports where effective measures are taken to prevent end restraint. The span shall be taken as defined in Section 106 for freely supported beams.

110. Restrained Beams of Equal Span.—Beams and slabs of equal span built to act integrally with columns, walls, or other restraining supports and assumed to carry uniformly distributed loads, shall (except as provided in Section 107) be designed for the following moments at critical sections:

(*a*) Interior spans:

(1) Negative moment at interior supports except the first,

$$M = \frac{w\,l^2}{12}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

(2) Maximum positive moment near centres of interior spans,

$$M = \frac{w\,l^2}{16}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

(*b*) End spans of continuous beams and beams of one span in which I/l is less than twice the sum of the values of I/h for the exterior columns above and below which are built into the beams:

(1) Maximum positive moment near centre of span and negative moment at first interior supports,

$$M = \frac{w\,l^2}{12}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

(2) Negative moment at exterior supports,

$$M = \frac{w\,l^2}{12}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

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*A.C.I. Specifications: M = not less than $\frac{w\,l^2}{24}$.

(*c*) End spans of continuous beams, and beams of one span, in which I/l is equal to or greater than twice the sum of the values of I/h for the exterior column above and below which are built into the beams:

(1) Maximum positive moment near centre of span and negative moment at first interior support,

$$M = \frac{w\,l^2}{10}. \qquad (22)$$

(2) Negative moment at exterior support,

$$M = \frac{w\,l^2}{16}. \qquad (23)$$

111. **Continuous Beams of Unequal Spans or with Non-uniform Loading.**—Continuous beams with unequal spans, or with other than uniformly distributed loading, whether freely supported or restrained, shall be designed for the actual moments under the conditions of loading and restraint.

Provision shall be made where necessary for negative moment near the centre of short spans which are adjacent to long spans, and for the negative moment at the end supports, if restrained.

112. **Unsupported Flange Length.**—The distance between lateral supports of the compression area of a beam shall not exceed 24* times the least width of compression flange.

C. Flexure of Reinforced-concrete T-Beams

113. **Flexure Formulas.**—Standard formulas as in Chapter III.

114. **Notation.**—Standard notation as in Chapter III.

115. **Flange Width.**—Effective and adequate bond and shear resistance shall be provided in beam-and-slab construction at the junction of the beam and slab; the slab shall be built and considered an integral part of the beam; the effective flange width to be used in the design of symmetrical T-beams shall not exceed one-fourth of the span length of the beam, and its overhanging width on either side of the web shall not exceed 8 times the thickness of the slab nor one-half the clear distance to the next beam.

For beams having a flange on one side only, the effective flange width to be used in design shall not exceed one-tenth of the span length of the beam, and its overhanging width from the face of the web shall not exceed 6 times the thickness of the slab nor one-half the clear distance to the next beam.

116. **Transverse Reinforcement.**—Where the principal slab reinforcement is parallel to the beam, transverse reinforcement, not less in amount than 0.3% of the sectional area of the slab, shall be provided in the top of the slab and shall extend across the beam and into the slab not less than

*A.C.I. Specifications: 32 times.

two-thirds of the width of the effective flange overhang. The spacing of the bars shall not exceed 18 in.

117. Compressive Stress at Supports.—Provision shall be made for the compressive stress at the support in continuous T-beam construction.

118. Shear.—The flange of the beam shall not be considered as effective in computing the shear and diagonal tension resistance of T-beams.

119. Isolated Beams.—Isolated beams in which the T-form is used only for the purpose of providing additional compression area shall have a flange thickness not less than one-half the width of the web and a total flange width not more than 4 times the web thickness.

D. Diagonal Tension and Shear

120. Notation.—Standard notation as in Chapter IV and the following:

A_v = total area of web reinforcement in tension within a distance s, that is $s_1, s_2, s_3, \ldots s_n$, or the total area of all bars bent up in any one plane;

f'_c = ultimate compressive strength of concrete at age of 28 days;

f_v = tensile unit stress in web reinforcement;

r = ratio of cross-sectional area of negative reinforcement which crosses entirely over the column capital of a flat slab or over the dropped panel, to the total cross-sectional area of the negative reinforcement in the two column strips;

t_1 = thickness of flat slab without dropped panels or thickness of a dropped panel;

t_2 = thickness of flat slab with dropped panels at points away from the dropped panel;

α = angle between web bars and longitudinal bars.

121. Formula for Shear.—The shearing unit stress, v, in reinforced-concrete beams shall be taken as not less than that computed by Formula 29.

$$v = \frac{V}{bjd}. \qquad (29)$$

122. Variation of Shear in Beams with Uniform Load.—For purpose of design of beams carrying uniform loads, not less than one-fourth of the total shearing resistance required at either end of span shall be provided at the section where the computed shearing-stress is zero; from that section to the ends of span the required shearing resistance shall be assumed to vary uniformly.

123. Width of Beams in Shear Computations.—The shearing unit stress shall be computed on the minimum width of rectangular beams and on the minimum thickness of the web in beams of I or T-section.

124. Shear in Beam-and-Tile Construction.—The width of the effective section for shear as governing diagonal tension shall be assumed as the thickness of the concrete web plus one-half the thickness of the vertical webs of the concrete or clay tile in contact with the beam.

125. Types and Spacing of Web Reinforcement.—Web reinforcement may consist of:

(*a*) Vertical stirrups or web reinforcing bars;
(*b*) Inclined stirrups or web reinforcing bars forming an angle of 30 deg. or more with the longitudinal bars;
(*c*) Longitudinal bars bent up at an angle of 15 deg. or more with the direction of the longitudinal bars.

Stirrups or bent-up bars which are not anchored at both ends, according to the provisions of Section 141, shall not be considered effective as web reinforcement. When the shearing-stress is not greater than $0.06 f'_c$, the distance s measured in the direction of the axis of the beam between two successive stirrups, or between two successive points of bending up of bars, or from the point of bending up of a bar to the edge of the support, shall not be greater than

$$s = \frac{45\,d}{\alpha + 10} \qquad (30)$$

where the angle α is in degrees.

When the shearing-stress is greater than $0.06 f'_c$, the distance s shall not be greater than two-thirds of the values given by Formula 30.

126. Anchorage of Web Reinforcement.—See Section 141.

127. Beams without Special Anchorage of Longitudinal Reinforcement.—The shearing unit stress computed by Formula 29 in beams in which the longitudinal reinforcement is without special anchorage shall not exceed the values given by Formulas 31 and 32 and in no case shall it exceed $0.06 f'_c$.

When α is between 45 and 90 deg.,

$$v^* = 0.02 f'_c + \frac{f_v A_v}{b\,s \sin\alpha}. \qquad (31)$$

When α is less than 45 deg.,

$$v^* = 0.02 f'_c + \frac{f_v A_v}{b\,s} (\sin\alpha + \cos\alpha). \qquad (32)$$

128. Beams with Special Anchorage of Longitudinal Reinforcement.—The shearing unit stress computed by Formula 29 in beams in which longitudinal reinforcement is anchored by means of hooked ends or otherwise, as specified in Section 140, shall not exceed the value given by Formulas 31 and 32, when $0.03 f'_c$ is substituted for $0.02 f'_c$ in those formulas; in no case shall the shearing unit stress exceed $0.12 f'_c$.*

129. Beams with Bars Bent up at a Single Point.—Where the web reinforcement consists of bars bent up at a single point, the point of bending shall be at a distance s from the edge of the support, not greater than that

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* These formulas assume the concrete to carry a shearing-stress equal to $0.02 f'_c$, the reinforcement the remainder.

*A.C.I. Specifications limit the total shearing unit stress to $0.09 f'_c$, except that $0.12 f'_c$ may be used under special conditions of design and supervision.

given in Section 125, and the value of the quantity $\frac{f_v A_v}{b s}(\sin \alpha + \cos \alpha)$ used in the design shall not exceed 75 lbs/in.[2]

130. Combined Web Reinforcement.—Where two or more types of web reinforcement are used in conjunction, the total shearing resistance of the beam shall be assumed as the sum of the shearing resistances computed for the various types separately. In such computations the shearing resistance of the concrete (the term $0.02 f'_c$ or $0.03 f'_c$ in Formulas 31 and 32) shall be included only once. In no case shall the maximum shearing-stresses be greater than the limiting values given in Sections 127 and 128.

131. Shearing-stress in Flat Slabs.—The shearing unit stress in flat slabs shall not exceed the value of v as given by Formula 33,

$$v = 0.02 f'_c (1 + r). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

and shall not in any case exceed $0.03 f'_c$.

The shearing unit stress shall be computed on:

(*a*) A vertical section which has a depth in inches of $\frac{7}{8}(t_1 - 1\frac{1}{2})$ and which lies at a distance in inches of $t_1 - 1\frac{1}{2}$ from the edge of the column capital; and

(*b*) A vertical section which has a depth in inches of $\frac{7}{8}(t_2 - 1\frac{1}{2})$ and which lies at a distance in inches of $t_2 - 1\frac{1}{2}$ from the edge of the dropped panel.

In no case shall r be less than 0.25. Where the shearing-stress computed as in (*a*) is being considered, r shall be assumed as the proportional amount of the negative reinforcement, within the column strip, crossing the column capital. Where the shearing-stress computed as in (*b*) is being considered, r shall be assumed as the proportional amount of the negative reinforcement, within the column strip, crossing entirely over the dropped panel.*

132. Shear and Diagonal Tension in Footings.—The shearing-stress shall be taken as not less than that computed by Formula 29. The stress on the critical section shall not exceed $0.02 f'_c$ for footings with straight reinforcement bars, nor $0.03 f'_c$ for footings in which the reinforcement bars are anchored at both ends by adequate hooks or otherwise as specified in Section 140.

133. Critical Section for Soil Footings.—The critical section for diagonal tension in footings on soil shall be computed on a vertical section through the perimeter of the lower base of a frustum of a cone or pyramid which has a base angle of 45 deg., and which has for its top the base of the column or pedestal and for its lower base the plane at the centroid of longitudinal reinforcement.

134. Critical Section for Pile Footings.—The critical section for diagonal tension in footings on piles shall be computed on a vertical section at the inner edge of the first row of piles entirely outside a section midway between the face of the column or pedestal and the section described in Section 133

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* In special cases, where supported by satisfactory engineering analysis, diagonal tension reinforcement may be used and increased shearing-stresses allowed in accordance with Sections 127 to 130.

for soil footings, but in no case outside of the section described in Section 133. The critical section for piles not arranged in rows shall be taken midway between the face of the column and the perimeter of the base of the frustum described in Section 133.

E. Bond and Anchorage

135. Bond Stresses by Beam Action.—Where bar reinforcement is used to resist tensile stresses developed by beam action, the bond stress shall be taken as not less than that computed by Formula 34,

$$u = \frac{V}{\Sigma o j d} \quad \text{. . . . .} \quad (34)$$

For continuous or restrained members, the critical section for bond for the positive reinforcement shall be assumed to be at the point of inflection; that for the negative reinforcement shall be assumed to be at the face of the support, and at the point of inflection. For simple beams or freely supported end spans of continuous beams, the critical section for bond shall be assumed to be at the face of the support.

Bent-up longitudinal bars which, at the critical section, are within a distance $\frac{d}{3}$ from horizontal reinforcement under consideration, may be included with the straight bars in computing Σo.

In footings only the bars specified in Section 177 as effective in resisting bending moment shall be considered as resisting bond stresses. Special investigation shall be made of bond stresses in footings with stepped or sloping upper surface, as maximum bond stresses may occur at the vertical plane of the steps or near the edges of the footing.

136. Bond Stress for Ordinary Anchorage.—In beams where the ordinary anchorage described in Section 139 is provided, the bond stress computed by Formula 34 at any section shall not exceed the following values:

For plain bars. . . . . $u = 0.04 f'_c$
For deformed bars meeting the requirements of Section 23. $u = 0.05 f'_c$

137. Bond Stresses for Special Anchorage.—In beams where special anchorage of the bars is provided as specified in Section 140, bond stresses exceeding those specified in Section 136 may be used, provided the total tensile stress at a point of abrupt change in stress, or at the point of maximum stress, does not exceed the value of F given by Formula 35,

$$F = Q u \Sigma o y + u \Sigma o x \quad \text{. . . . . . .} \quad (35)$$

where F = total tension in the bar;
Σo = the perimeter of the bar under consideration;
Q = ratio of the average to the maximum bond stress computed by Formula 34 within the distance y;
u = permissible bond stress = $0.04 f'_c$ for plain and $0.05 f'_c$ for deformed bars meeting the requirements of Section 23;

x = the length of bar added for anchorage, including the hook, if any;

y = distance from the point at which the tension is computed to the point of beginning of anchorage.

The length of bar added for anchorage may be either straight or bent. The radius of bend shall not be less than 4 bar diameters.*

138. Bond Stress for Reinforcement in Two or More Directions.—The permissible bond stress for footings and similar members in which reinforcement is placed in more than one direction shall not exceed 75% of the values in Sections 136 and 137.

139. Ordinary Anchorage Requirements.—In continuous, restrained, or cantilever beams, anchorage of the tensile negative reinforcement beyond the face of the support shall provide for the full maximum tension with bond stresses not greater than those specified in Section 136. Such anchorage shall provide a length of bar not less than the depth of the beam. In the case of end supports which have a width less than three-fourths of the depth of the beam, the bars shall be bent down toward the support a distance not less than the effective depth of the beam. The portion of the bar so bent down shall be as near to the end of the beam as protective covering permits. In continuous or restrained beams, negative reinforcement shall be carried to or beyond the point of inflection. Not less than one-fourth of the area of the positive reinforcement shall extend into the support to provide an embedment of 10 or more bar diameters.

In simple beams or freely supported end spans of continuous beams, at least one-fourth of the area of the tensile reinforcement shall extend along the tension side of the beam and beyond the face of the support to provide an embedment of 10 or more bar diameters.

140. Special Anchorage Requirements.—Where increased shearing-stresses are used as provided in Sections 128 and 132 or increased bond stresses as provided in Section 137, special anchorage of all reinforcement in addition to that required in Section 139 shall be provided as follows:

(*a*) In continuous and restrained beams, anchorage beyond points of inflection of one-third the area of the negative reinforcement, and beyond the face of the support of one-third the area of the positive reinforcement, shall be provided to develop one-third of the maximum working stress in tension, with bond stresses not greater than those specified in Section 136.

(*b*) At the edges of footings, anchorage for all the bars for one-third the maximum working stress in tension shall be provided within a region where the tension in the concrete, computed as an unreinforced beam, does not exceed 40 lbs/in.^{2}

(*c*) In simple beams or freely supported end spans of continuous beams, at least one-half of the tensile reinforcement shall extend along the tension side of the beam to provide an anchorage beyond the face of the support for one-third of the maximum working stress in tension.

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*A.C.I. Specifications allow double bond stress values where special anchorage is used. This assumes a somewhat more uniform distribution of bond stress along the bar than Section 137.

141. Anchorage of Web Reinforcement.—Web bars shall be anchored at both ends by:

(*a*) providing continuity with the longitudinal reinforcement; or

(*b*) bending around the longitudinal bar; or

(*c*) a semi-circular hook which has a radius not less than 4 times the diameter of the web bar.

Stirrup anchorage shall be so provided in the compression and tension regions of a beam as to permit the development of safe working tensile stress in the stirrup at a point 0.3 *d* from either face.*

The end anchorage of a web member not in bearing on the longitudinal reinforcement shall be such as to engage an amount of concrete sufficient to prevent the bar from pulling out. In all cases the stirrups shall be carried as close to the upper and lower surfaces as fireproofing requirements permit.

F. Flat Slabs

(*Two-way and Four-way Systems with Rectangular Panels*)

142. Moments in Interior Panels.—The moment coefficients, moment distribution, and slab thicknesses specified herein are for slabs which have three or more rows of panels in each direction, and in which the panels are approximately uniform in size. Slabs with paneled ceiling or with depressed paneling in the floor shall be considered as coming under the requirements herein given. The symbols used in Formulas 36 to 41 are defined in Section 105 except as indicated in Sections 142, 145, and 155.

In flat slabs in which the ratio of reinforcement for negative moment in the column strip is not greater than 0.01, the numerical sum of the positive and negative moments in the direction of either side of the panel for which tension reinforcement must be provided, shall be assumed as not less than that given by Formula 36,

$$M_0 = 0.09\, W\, l\left(1 - \frac{2}{3}\frac{c}{l}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

where M_0 = sum of positive and negative bending moments in either rectangular direction at the principal design sections of a panel of a flat slab;

c = base diameter of the largest right circular cone, which lies entirely within the column (including the capital) whose vertex angle is 90 deg. and whose base is 1½ in. below the bottom of the slab or the bottom of the dropped panel;

l = span length of flat slab, centre to centre of columns in the rectangular direction in which moments are considered;

l_1 = span length of flat slab, centre to centre of columns perpendicular to the rectangular direction in which moments are considered; and

W = total dead and live load uniformly distributed over a single panel area.

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* Generally a properly anchored stirrup whose diameter does not exceed 1/50 of the depth of the beam will meet these requirements.

TABLE VI

MOMENTS TO BE USED IN DESIGN OF FLAT SLABS

Strip	Flat Slabs without Dropped Panels		Flat Slabs with Dropped Panels	
	Negative	Positive	Negative	Positive
Slabs with 2-way Reinforcement				
Column strip	$0.23\ M_0$	$0.11\ M_0$	$0.25\ M_0$	$0.10\ M_0$
2 Column strips	$0.46\ M_0$	$0.22\ M_0$	$0.50\ M_0$	$0.20\ M_0$
Middle strip	$0.16\ M_0$	$0.16\ M_0$	$0.15\ M_0$	$0.15\ M_0$
Slabs with 4-way Reinforcement				
Column strip	$0.25\ M_0$	$0.10\ M_0$	$0.27\ M_0$	$0.095\ M_0$
2 Column strips	$0.50\ M_0$	$0.20\ M_0$	$0.54\ M_0$	$0.190\ M_0$
Middle strip	$0.10\ M_0$	$0.20\ M_0$	$0.08\ M_0$	$0.190\ M_0$

143. **Principal Design Sections.**—In computing the critical moments in flat slabs subjected to uniform load the following principal design sections shall be used:

(*a*) *Section for Negative Moment in Middle Strip:* The section beginning at a point on the edge of the panel $l_1/4$ from the column centre and extending in a rectangular direction a distance $l_1/2$ toward the centre of the adjacent column on the same panel edge.

(*b*) *Section for Negative Moment in Column Strip:* * The section beginning at a point on the edge of the panel $l_1/4$ from the centre of a column and extending in a rectangular direction toward the column to a point $c/2$ therefrom and thence along a one-quarter circumference about the column centre to the adjacent edge of the panel.

(*c*) *Section for Positive Moment in Middle Strip:* The section of a length $l_1/2$ extending in a rectangular direction across the centre of the middle strip.

(*d*) *Section for Positive Moment in Column Strip:* The section of length $l_1/4$ extending in a rectangular direction across the centre of the column strip.

144. **Moments in Principal Design Sections.**—The moments in the principal design sections shall be those given in Table VI, except as follows:

(*a*) The sum of the maximum negative moments in the two column strips may be greater or less than the values given in Table VI by not more than $0.03\ M_0$.

*A.C.I. Specifications: Width of two columns strips taken equal to width of dropped panel where used, or one-half width of whole panel where not used.

(*b*) The maximum negative moment and the maximum positive moments in the middle strip and the sum of the maximum positive moments in the two column strips may each be greater or less than the values given in Table VI by not more than 0.01 M_0.

145. Thickness of Flat Slabs and Dropped Panels.—The total thickness, t_1, of the dropped panel in inches, or of the slab if a dropped panel is not used, shall be not less than:

$$t_1{}^* = 0.038\left(1 - 1.44\frac{c}{l}\right)l\sqrt{R\,w'\,\frac{l_1}{b_1}} + 1\tfrac{1}{2} \quad . \; . \; . \quad (37)$$

where R = ratio of negative moment in the two column strips to M_0;
w' = uniformly distributed dead and live load per unit of area of floor; and
b_1 = dimension of the dropped panel in the direction parallel to l_1.

For slabs with dropped panels the total thickness in inches at points beyond the dropped panel shall be not less than

$$t_2 = 0.02\,l\sqrt{w'} + 1. \quad . \; . \; . \; . \; . \; . \; . \; . \quad (38)$$

The slab thickness t_1 or t_2 shall in no case be less than $l/32$ for floor slabs, and not less than $l/40$ for roof slabs. In determining minimum thickness by Formulas 37 and 38, the value of l shall be the panel length centre to centre of the columns on long side of panel, l_1 shall be the panel length on the short side of the panel, and b_1 shall be the width or diameter of dropped panel in the direction of l_1, except that in a slab without dropped panel b_1 shall be 0.5 l_1.

146. Minimum Dimensions of Dropped Panels.—The dropped panel shall have a length or diameter in each rectangular direction of not less than one-third the panel length in that direction, and a thickness not greater than 1.5 t_2.

147. Wall and Other Irregular Panels.—In wall panels and other panels in which the slab is discontinuous at the edge of the panel, the maximum negative moment one panel length away from the discontinuous edge and the maximum positive moment between shall be increased as follows:

(*a*) Column strip perpendicular to the wall or discontinuous edge, 15% greater than that given in Table VI;

(*b*) Middle strip perpendicular to wall or discontinuous edge, 30% greater than that given in Table VI.

In these strips the bars used for positive moments perpendicular to the discontinuous edge shall extend to the edge of the panel at which the slab is discontinuous.

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*A.C.I. Specifications: $t_1 = 0.038\left(1 - 1.44\frac{c}{l}\right)l\sqrt{w'} + 1\tfrac{1}{2}$. Also where concrete of a higher ultimate strength than 2000 lbs/in.^{2} is used the thickness may be reduced by multiplying by the factor $\sqrt[3]{\frac{2000}{f'_c}}$.

148. Panels with Marginal Beams.—In panels having a marginal beam on one edge or on each of two adjacent edges, the beam shall be designed to carry at least the load superimposed directly upon it, exclusive of the panel load. A beam which has a depth greater than the thickness of the dropped panel into which it frames shall be designed to carry, in addition to the load superimposed upon it, at least one-fourth of the distributed load for which the adjacent panel or panels are designed, and each column strip adjacent to and parallel with the beam shall be designed to resist a moment at least one-half as great as that specified in Table VI for a column strip.*

Each column strip adjacent to and parallel with a marginal beam which has a depth less than the thickness of the dropped panel into which it frames shall be designed to resist the moments specified in Table VI for a column strip. Marginal beams on opposite edges of a panel and the slab between them shall be designed for the entire load and the panels shall be designed as simple beams.

149. Discontinuous Panels.—The negative moments on sections at and parallel to the wall, or discontinuous edge of an interior panel, shall be determined by the conditions of restraint.†

150. Flat Slabs on Bearing Walls.—Where there is a beam or a bearing wall on the centre line of columns in the interior portion of a continuous flat slab, the negative moment at the beam or wall line in the middle strip perpendicular to the beam or wall shall be taken as 30% greater than the moment specified in Table VI for a middle strip. The column strip adjacent to and lying on either side of the beam or wall shall be designed to resist a moment at least one-half of that specified in Table VI for a column strip.

151. Point of Inflection.—The point of inflection in any line parallel to a panel edge in interior panels of symmetrical slabs without dropped panels shall be assumed to be at a distance from the centre of the span equal to three-tenths of the distance between the two sections of critical negative moment at opposite ends of the line; for slabs having dropped panels, the coefficient shall be 0.25.

152. Reinforcement.—The reinforcement bars which cross any section and which fulfil the requirements given in Section 153 may be considered as effective in resisting the moment at the section. The sectional area of a bar multiplied by the cosine of the angle between the direction of the axis of the bar and any other direction may be considered effective as reinforcement in that direction.

153. Arrangement of Reinforcement.—The design shall include adequate provision for securing the reinforcement in place so as to take not only the critical moments but the moments at intermediate sections. Provision shall be made for possible shifting of the point of inflection by carrying all bars in rectangular or diagonal directions, each side of a section of critical moment, either positive or negative, to points at least 20 diameters beyond the point of inflection as specified in Section 151. Lapped splices

* In wall columns, brackets are sometimes substituted for capitals or other changes are made in the design of the capital. Attention is directed to the necessity for taking into account the change in the value of c in the moment formula for such cases.

† The committee is not prepared to make a more definite recommendation at this time.

shall not be permitted at or near regions of maximum stress except as described above. At least four-tenths of all bars in each direction shall be of such length and shall be so placed as to provide reinforcement at two sections of critical negative moment and at the intermediate section of critical positive moment. Not less than one-third of the bars used for positive reinforcement in the column strip shall extend into the dropped panel not less than 20 diameters of the bar, or in case no dropped panel is used, shall extend to a point not less than one-eighth of the span length from the centre line of the column or the support.

155. Tensile Stress in Reinforcement.—The tensile stress f_s in the reinforcement in flat slabs shall be taken as not less than that computed by Formula 39,

$$f_s = \frac{R\,M_0}{A_s\,j\,d} \qquad (39)$$

where $R\,M_0$ = moment specified in Section 144 for two column strips or for one middle strip; and

A_s = effective cross-sectional area of the reinforcement which crosses any of the principal design sections and which meets the requirements of Section 153.

The stress so computed shall not at any of the principal design sections exceed the values specified in Section 194.

156. Compressive Stress in Concrete.—The compressive stress in the concrete in flat slabs shall be taken as not less than that computed by Formulas 40 and 41, but the stress so computed shall not exceed $0.4 f'_c$.

Compression due to negative moment, $R\,M_0$, in the two column strips,

$$f_c = \frac{3.5\,R\,M_0}{0.67\sqrt[3]{p\,n}\,b_1\,d^2}\left(1 - 1.2\frac{c}{l}\right) \qquad (40)$$

where b_1 is as specified in Section 145.

Compression due to positive moment, $R\,M_0$, in the two column strips, or negative or positive moment in the middle strip,

$$f_c = \frac{6\,R\,M_0{}^*}{0.67\sqrt[3]{p\,n}\,l_1\,d^2} \qquad (41)$$

In special cases where supported by satisfactory engineering analysis, approved by the engineer, compression reinforcement may be used to increase the resistance to compression in accordance with other provisions of these specifications.

157. Shearing-stress.—See Section 131.

158. Unusual Panels.—For structures having a width of one or two panels, and also for slabs having panels of markedly different sizes, an analysis shall be made of the moments developed in both slab and columns, and the values given in Sections 142 to 157 modified accordingly.

159. Bending Moments in Columns.—See Section 171.

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* For all ordinary values of $p\,n$ the value of $k\,j$ is very closely given by $0.67\sqrt[3]{p\,n}$.

G. Reinforced-concrete Columns

160. Limiting Dimensions.—The following sections on reinforced-concrete columns are based on the assumption of a short column. Where the unsupported length is greater than 40 times the least radius of gyration (40 R), the safe load shall be determined by Formula 47. Principal columns in buildings shall have a minimum diameter or thickness of 12 in. Posts that are not continuous from story to story shall have a minimum diameter or thickness of 6 in.

161. Unsupported Length.—The unsupported length of reinforced-concrete columns shall be taken as:

(*a*) In flat slab construction the clear distance between the floor and under side of the capital.

(*b*) In beam-and-slab construction, the clear distance between the floor and the under side of the shallowest beam framing into the column at the next higher floor level;

(*c*) In floor construction with beams in one direction only, the clear distance between floor slabs;

(*d*) In columns supported laterally by struts or beams only, the clear distance between consecutive pairs (or groups) of struts or beams, provided that to be considered an adequate support, two such struts or beams shall meet the column at approximately the same level and the angle between the two planes formed by the axis of the column and the axis of each strut respectively is not less than 75 deg. nor more than 105 deg.

When haunches are used at the junction of beams or struts with columns, the clear distance between supports may be considered as reduced by two-thirds of the depth of the haunch.

162. Safe Load on Spiral Columns.—The safe axial load on columns reinforced with longitudinal bars and closely spaced spirals enclosing a circular core shall be not greater than that determined by Formula 42.

The symbols used in Formulas 42 to 49 are defined in Section 105, except as indicated in Sections 162, 165, 168, 170, 176, and 182.

$$P = A_c f_c + n f_c p A \qquad (42)$$

where P = total safe axial load on column whose h/R is less than 40;

A = area of the concrete core enclosed within the spiral; the diameter of the core (or of the spiral) shall be taken as the distance centre to centre of the spiral wire;

p = ratio of effective area of longitudinal reinforcement to area of the concrete core;

$A_c = A(1 - p)$ = net area of concrete core; and

f_c = permissible compressive stress in concrete =

$$300 + (0.10 + 4p) f'_c. \qquad (43)$$

The longitudinal reinforcement shall consist of at least six bars of minimum diameter of ½ in., and its effective cross-sectional area shall not be less than 1% nor more than 6% of that of the core.

163. Spiral Reinforcement.—The spiral reinforcement shall be not less than one-fourth the volume of the longitudinal reinforcement. It shall consist of evenly spaced continuous spirals held firmly in place and true to line by at least three vertical spacer bars. The spacing of the spirals shall be not greater than one-sixth of the diameter of the core and in no case more than 3 in. The spiral reinforcement shall meet the requirements of the Tentative Specifications for Cold-Drawn Steel Wire for Concrete Reinforcement.

164. Protection of Spirally Reinforced Column.—Reinforcement shall be protected everywhere by a covering of concrete cast monolithic with the core, which shall have a minimum thickness of 1½ in. in square columns and 2 in. in round or octagonal columns.

165. Safe Load on Columns with Lateral Ties.—The safe axial load on columns reinforced with longitudinal bars and separate lateral ties shall be not greater than that determined by Formula 44,

$$P = (A'_c + A_s n) f_c \quad . \; . \; . \; . \; . \; . \; . \; . \quad (44)$$

where A'_c = net area of concrete in the column (total column area minus area of reinforcement);

A_s = effective cross-sectional area of longitudinal reinforcement; and

f_c = permissible compressive stress in concrete and shall not exceed $0.20 f'_c$.

The amount of longitudinal reinforcement considered in the calculations shall be not more than 2% nor less than 0.5% of the total area of the column. The longitudinal reinforcement shall consist of not less than four bars of minimum diameter of ½ in., placed with clear distance from the face of the column not less than 2 in.

166. Lateral Ties.—Lateral ties shall be not less than ¼ in. in diameter, spaced not more than 8 in. apart.

167. Bending in Columns.—Reinforced-concrete columns subject to bending stresses shall be treated as follows:

(*a*) *With Spiral Reinforcement.*—The compressive unit stress on the concrete within the core area under combined axial load and bending shall not exceed by more than 20% the value given for axial load by Formula 43.

(*b*) *With Lateral Ties.*—Additional longitudinal reinforcement may be used if required, and the compressive unit stress on the concrete under combined axial load and bending may be increased to $0.30 f'_c$. The total amount of reinforcement considered in the computations shall be not more than 4% of the total area of the column.

Tension in the longitudinal reinforcement due to bending of the column shall not exceed 16,000 lbs/in.^{2}

168. Composite Columns.—The safe load on composite columns in which a structural steel or cast-iron column is thoroughly encased in a circumferentially reinforced concrete core shall be based on a certain unit stress for the steel or cast-iron core plus a unit stress of $0.25 f'_c$ on the area within the spiral core.

The unit compressive stress on the steel section shall be not greater than that determined by Formula 45,

$$f_r = 18{,}000 - 70\,h/R \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (45)$$

but shall not exceed 16,000 lbs/in.^{2}

The unit stress on the cast-iron section shall be not greater than that determined by Formula 46,

$$f_r = 12{,}000 - 60\,h/R \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (46)$$

but shall not exceed 10,000 lbs/in.^{2}

In Formulas 45 and 46,

f_r = compressive unit stress in metal core, and
R = least radius of gyration of the steel or cast-iron section.

The diameter of the cast-iron section shall not exceed one-half of the diameter of the core within the spiral. The spiral reinforcement shall be not less than 0.5% of the volume of the core within the spiral and shall conform in quality, spacing, and other requirements to the provisions for spirals in Section 163.

Ample section of concrete and continuity of reinforcement shall be provided at the junction with beams or girders. The area of the concrete between the spiral and the metal core shall be not less than that required to carry the total floor load of the story above on the basis of a stress in the concrete of $0.35 f'_c$, unless special brackets are arranged on the metal core to receive directly the beam or slab load.

169. Structural Steel Columns.—The safe load on a structural steel column of a section which fully encases an area of concrete, and which is protected by an outside shell of concrete at least 3 in. thick, shall be computed in the same manner as for composite columns in Section 168, allowing $0.25 f'_c$ on the area of the concrete enclosed by the steel section. The outside shell shall be reinforced by wire mesh, ties, or spiral hoops weighing not less than 0.2 lb/ft.^{2} at the surface of the mesh and with a maximum spacing of 6 in. between strands or hoops. Special brackets shall be used to receive the entire floor load at each story. The safe load in steel columns calculated by Formula 45 shall not exceed 16,000 lbs/in.^{2}

170. Long Columns.—The permissible working load on the core in axially loaded columns which have a length greater than 40 times the least radius of gyration of the column core (40 R) shall be not greater than that determined by Formula 47,

$$\frac{P'}{P} = 1.33 - \frac{h}{120\,R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (47)$$

where P' = total safe axial load on long column;
P = total safe axial load on column of the same section whose h/R is less than 40, determined as in Sections 162 and 165; and
R = least radius of gyration of column core.

171. **Bending Moments in Columns.**—The bending moments in interior and exterior columns shall be determined on the basis of loading conditions and end restraint, and shall be provided for in the design. The recognized methods shall be followed in calculating the stresses due to combined axial load and bending. In spiral columns the area to be considered as resisting the stress is the area within the spiral.

H. Footings

172. **General.**—The requirements for tension, compression, shear, and bond in Sections 103 and 141, inclusive, shall govern the design of footings, except as hereinafter provided.

173. **Soil Footings.**—The load per unit of area on soil footings shall be computed by dividing the column load by the area of base of the footing.

174. **Pile Footings.**—Footings on piles shall be treated in the same manner as footings on soil, except that the load shall be considered as concentrated at the pile centres.

175. **Sloped or Stepped Footings.**—Footings in which the thickness has been determined by the requirements for shear as specified in Sections 133 and 134 may be sloped or stepped between the critical section and the edge of the footing, provided that the shear on no section outside the critical section exceeds the value specified, and provided further that the thickness of the footing above the reinforcement at the edge shall not be less than 6 in. for footings on soil nor less than 12 in. for footings on piles. Sloped or stepped footings shall be cast as a unit.

176. **Critical Section for Bending.**—The critical section for bending in a concrete footing which supports a concrete column or pedestal shall be considered to be at the face of the column or pedestal. Where steel or cast-iron column bases are used, the moment in the footing shall be computed at the middle and at the edge of the base; the load shall be considered as uniformly distributed over the column or pedestal base.

The bending moment at the critical section in a square footing supporting a concentric square column shall be computed from the load on the trapezoid bounded by one face of the column, the corresponding outside edge of the footing, and the portions of the two diagonals. The load on the two corner triangles of this trapezoid shall be considered as applied at a distance from the face equal to six-tenths of the projection of the footing from the face of the column. The load on the rectangular portion of the trapezoid shall be considered as applied at its centre of gravity. The bending moment is expressed by Formula 48,

$$M = \frac{w}{2}(a + 1.2\,c)c^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

where M = bending moment at critical section of footing;
a = width of face of column or pedestal;
c = projection of footing from face of column; and
w = upward reaction per unit of area of base of footing.

For a round or octagonal column, the distance a shall be taken as equal to the side of a square of an area equal to the area enclosed within the perimeter of the column.

177. Reinforcement.—The reinforcement in each direction in the footing shall be determined as for a reinforced-concrete beam; the effective depth shall be the distance from the top of the footing to the plane of the reinforcement. The sectional area of reinforcement shall be distributed uniformly across the footing unless the width is greater than the side of the column or pedestal plus twice the effective depth of the footing, in which case the width over which the reinforcement is spread may be increased to include one-half the remaining width of the footing. In order that no considerable area of the footing shall remain unreinforced, additional reinforcement shall be placed outside of the width specified, but such reinforcement shall not be considered as effective in resisting the calculated bending moment. For the extra reinforcement a spacing double that within the effective belt may be used.

178. Concrete Stress.—The extreme fibre stress in compression in the concrete shall be kept within the limits specified in Section 189. The extreme fibre stress in sloped or stepped footings shall be based on the exact shape of the section for a width not greater than that assumed effective for reinforcement.

179. Irregular Footings.—A rectangular or irregularly shaped footing shall be computed by dividing it into rectangles or trapezoids tributary to the sides of the column, using the distance to the centre of gravity of the area as the moment arm of the upward forces. Outstanding portions of combined footings shall be treated in the same manner. Other portions of combined footings shall be designed as beams or slabs.

180. Shearing-stresses.—See Sections 132 to 134.

181. Bond Stress.—See Sections 135 to 141.

182. Transfer of Stress at Base of Column.—The compressive stress in longitudinal reinforcement at the base of a column shall be transferred to the pedestal or footing by either dowels or distributing bases. When dowels are used, there shall be at least one for each column bar, and the total sectional area of the dowels shall be not less than the sectional area of the longitudinal reinforcement in the column. The dowels shall extend into the column and into the pedestal or footing not less than 50 diameters of the dowel bars for plain bars, or 40 diameters for deformed bars.

When metal distributing bases are used, they shall have sufficient area and thickness to transmit safely the load from the longitudinal reinforcement in compression and bending. The permissible compressive unit stress on top of the pedestal or footing directly under the column shall be not greater than that determined by Formula 49,

$$r_a = 0.25 f'_c \sqrt[3]{\frac{A}{A'}} \qquad (49)$$

where r_a = permissible working stress over the loaded area;
A = total area at the top of the pedestal or footing;
A' = loaded area at the column base;
f'_c = ultimate compressive strength of concrete. (See Section 120.)

In sloped or stepped footings A may be taken as the area of the top horizontal surface of the footing or as the area of the lower base of the largest frustum of a pyramid or cone contained wholly within the footing and having for its upper base the loaded area A', and having side slopes of 1 vertical to 2 horizontal.

183. Pedestals without Reinforcement.—The allowable compressive unit stress on the gross area of a concentrically loaded pedestal or on the minimum area of a pedestal footing shall not exceed $0.25 f'_c$, unless reinforcement is provided and the member designed as a reinforced-concrete column.

The depth of a pedestal or pedestal footing shall be not greater than 3 times its least width, and the projection on any side from the face of the supported member shall be not greater than one-half the depth. The depth of a pedestal whose sides are sloped or stepped shall not exceed 3 times the least width or diameter of the section midway between the top and bottom. A pedestal footing supported directly on piles shall have a mat of reinforcing bars having a cross-sectional area of not less than 0.20 in^2/ft. in each direction, placed 3 in. above the top of the piles.

I. Reinforced-concrete Retaining Walls

184. Loads and Unit Stresses.—Reinforced-concrete retaining walls shall be so designed that the permissible unit stresses specified in Sections 186 to 197 are not exceeded. The heels of cantilever, counterforted, and buttressed retaining walls shall be proportional for maximum resultant vertical loads, but when the foundation reaction is neglected the permissible unit stresses shall not be more than 50% greater than the normal permissible stresses.

185. Details of Design.—The following principles shall be followed in the design of reinforced-concrete retaining walls:

(*a*) The supported toe and heel of the base slabs shall be considered as cantilever beams fixed at the edge of the support.

(*b*) The vertical section of a cantilever wall shall be considered as a cantilever beam fixed at the top of the base.

(*c*) The vertical sections of counterforted and buttressed walls and parts of base slabs supported by the counterforts or buttresses shall be designed in accordance with the requirements for a continuous slab in Section 110.

(*d*) The exposed faces of walls without buttresses shall preferably be given a batter of not less than ¼ in/ft.

(*e*) Counterforts shall be designed in accordance with the requirements for T-beams in Sections 113 to 115. Stirrups shall be provided in the counterforts to take the reaction when the tension reinforcement of the face walls and heels of bases is designed to span between the counterforts. Sitrrups shall be anchored as near the exposed face of the longitudinal wall and as close to the lower face of the base as the requirements for protective covering permit.

(*f*) Buttresses shall be designed in accordance with the requirements specified for rectangular beams.

(*g*) The shearing-stress at the junction of the base with counterforts or buttresses shall not exceed the values specified in Sections 120 to 130.

(*h*) Horizontal metal reinforcement shall be of such form and so distributed as to develop the required bond. To prevent temperature and shrinkage cracks in exposed surface not less than 0.25 $in.^2$ of horizontal metal reinforcement per foot of height shall be provided.

(*i*) Grooved lock joints shall be placed not over 60 ft. apart to care for temperature changes.

(*j*) Counterforts and buttresses shall be located under all points of concentrated loading, and at intermediate points, as may be required by the design.

(*k*) The walls shall be cast as a unit between expansion joints, unless construction joints formed in accordance with Sections 69 and 73 are provided.

(*l*) Drains or "weep holes" not less than 4 in. in diameter and not more than 10 ft. apart, shall be provided. At least one drain shall be provided for each pocket formed by counterforts.

J. Summary of Working Stresses *

186. General.—The following working stresses shall be used:

where f'_c = ultimate compressive strength of concrete at age of 28 days.

Direct Stress in Concrete

187. Direct Compression.—(*a*) Columns whose length does not exceed 40 R:

(1) With spirals . . . varies with amount of longitudinal reinforcement. (See Section 162.)

(2) Longitudinal reinforcement and lateral ties. (See Section 165.)

(*b*) Long columns. (See Section 170.)

(*c*) Piers and pedestals. . . . . 0.25 f'_c
(See Section 183.)

188. Compression in Extreme Fibre.—(*a*) Extreme fibre stress in flexure . . . . 0.40 f'_c

(*b*) Extreme fibre stress in flexure adjacent to supports of continuous beams . . . . 0.45 f'_c

189. Tension.—In concrete members. . . . . None

* For wind loads the A.C.I. Specifications allow an increase of 50% of the specified working stresses for combined dead load, live load, and wind stresses.

Shearing Stresses in Concrete

190. **Longitudinal Bars without Special Anchorage.**—(*a*) Beams without web reinforcement . . . . $0.02 f'_c$
(*b*) Beams with stirrups or bent-up bars or combination of the two . . . . $0.06 f'_c$

191. **Longitudinal Bars Having Special Anchorage.**—(*a*) Beams without web reinforcement . . . . $0.03 f'_c$
(*b*) Beams with stirrups or bent-up bars or a combination of the two . . . . $0.12 f'_c$

192. **Flat Slabs.**—(*a*) Shear at distance *d* from capital or dropped panel . . . . $0.03 f'_c$
(*b*) Other limiting cases in flat slabs. (See Section 131.)

193. **Footings.**—(*a*) Longitudinal bars without special anchorage . . $0.02 f'_c$
(*b*) Longitudinal bars having special anchorage . . . . $0.03 f'_c$

Stresses in Reinforcement *

194. **Tension in Steel.**—(*a*) Billet-steel bars:
(1) Structural steel grade . . . . 16,000 lbs/in.^{2}
(2) Intermediate grade . . . . 18,000 "
(3) Hard grade . . . . 18,000 "
(*b*) Rail-steel bars . . . . 18,000 "
(*c*) Structural steel . . . . 16,000 "
(*d*) Cold-drawn steel wire
(1) Spirals. (Stress not calculated.)
(2) Elsewhere . . . . 18,000 "

195. **Compression in Steel.**—(*a*) Bars. (Same as Section 194 (*a*) and (*b*).)
(*b*) Structural steel core of composite column . . . . . . 16,000 "
Reduced for slenderness ratio. (See Section 168.)
(*c*) Structural steel column . . . . 16,000 "
Reduced for slenderness ratio. (See Section 169.)

196. **Compression in Cast Iron.**—Composite cast-iron column . . . . 10,000 "
Reduced for slenderness ratio. (See Section 168.)

197. **Bond between Concrete and Reinforcement.**—(*a*) Beams and slabs, plain bars . . . . $0.04 f'_c$
(*b*) Beams and slabs, deformed bars . . . . $0.05 f'_c$
(*c*) Footings, plain bars, one-way . . . . $0.04 f'_c$
(*d*) Footings, deformed bars, one-way . . . . $0.05 f'_c$
(*e*) Footings, bars two-ways . . . (*c*) or (*d*) reduced by 25%

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* A.C.I. Specifications: Structural grade, 18,000 lbs/in.^{2}; intermediate and hard grade billet steel and rail steel, 20,000 lbs/in.^{2}

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