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LEONARD CHURCH URQUHART

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BY

LEONARD CHURCH URQUHART, C.E.
Professor in Charge of Structural Engineering
Cornell University

AND

CHARLES EDWARD O'ROURKE, C.E.
Professor of Structural Engineering
Cornell University

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PREFACE

In preparing the manuscript for this book, the authors have had two principal aims in mind: (1) to produce a text suitable for courses in structures which are given to non-civil engineers and architects; (2) to produce a compact guide book for all graduate engineers and architects, who must demonstrate a fundamental understanding of the principles of structural design in order to obtain a professional license or certificate of registration.

A brief explanation of the basic principles of structural mechanics has been included, to lead toward a deeper understanding of the subsequent design theories and methods. The more important properties of materials commonly used in simple structures have also been discussed in order to make it possible to use these materials more intelligently. A separate chapter describes the loads for which structures of various types should be designed in order to fulfill adequately the purpose for which each type is intended. Various methods of computing deflections of beams and trusses are outlined in another chapter, together with applications of these methods to typical practical problems. The remaining chapters explain and apply the theories involved in the design of the various structural elements, such as homogeneous beams of timber and steel; plate girders; reinforced-concrete beams; tension and compression members in steel and timber trusses; columns of timber, steel, and reinforced concrete; members subjected to bending and axial stresses; and footings.

No attempt has been made to assemble the individual structural elements into completed structures, inasmuch as such procedure is definitely beyond the scope of the book. However, a separate chapter explains in detail the fundamental principles involved in the design of connections and splices, and these principles can readily be applied by the student in building up some of the simpler structures.

The authors have tried to present the material clearly and concisely, and yet in such a way as to require careful thought and deliberation on the part of the student. It is their great hope that a study of these elementary principles of structural design will lead to further interest in the subject, and to more advanced study.

L. C. URQUHART,
C. E. O'ROURKE.

ITHACA, N. Y.,
January, 1941.

CONTENTS

ELEMENTARY STRUCTURAL ENGINEERING

CHAPTER I

STRUCTURAL MECHANICS

1. Conditions of Equilibrium. A system of forces acting upon a body is in *equilibrium* when the state of motion, size, or shape is not being changed.

In Fig. 1, P_1, P_2, and P_3 represent a system of concurrent forces applied at O. X-X and Y-Y are any two axes at right angles to each other. Each force may be resolved into components along the two axes. These components form a system equivalent to the original system. The algebraic sum of the horizontal components is indicated as ΣX, and that of the vertical components as ΣY. The resultant of ΣX and ΣY, which is R, is also equal to the resultant of the original system of forces. R is evidently equal in amount to $\sqrt{(\Sigma X)^2 + (\Sigma Y)^2}$, and its direction is determined, since $\Sigma X \div \Sigma Y = \tan \theta$, while its point of application is at O.

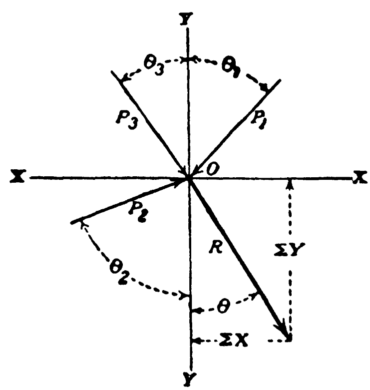


FIG. 1.

For equilibrium, R must be zero, or $\sqrt{(\Sigma X)^2 + (\Sigma Y)^2}$ must be zero, which requires that

$$\Sigma X = 0 \tag{1}$$

and

$$\Sigma Y = 0 \tag{2}$$

These equations express *the first two conditions of static equilibrium:* (1) *the sum of the horizontal components equals zero;* and (2) *the sum of the vertical components equals zero.* These two conditions must exist in a system of concurrent forces which are in equilibrium.

In Fig. 2, P_1, P_2, and P_3 represent a system of nonconcurrent forces. Each force may be resolved into components along the two axes. The amount of the resultant is $\sqrt{(\Sigma X)^2 + (\Sigma Y)^2}$, and its direction is determined, since $\Sigma X \div \Sigma Y = \tan\theta$. Its line of action is obtained by determining the distance from O that this

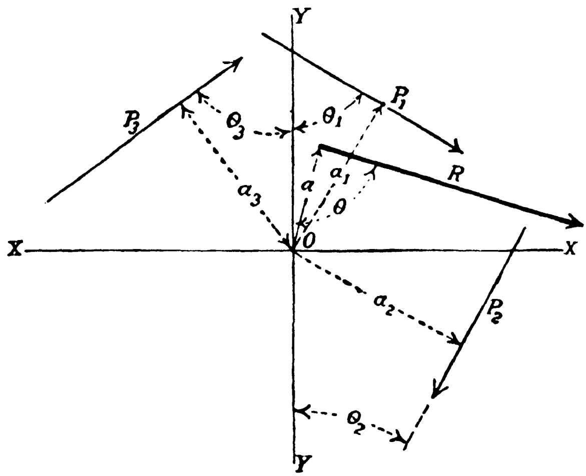


FIG. 2.

force must act in order to produce the same moment about O as the original system of forces; *i.e.*,

$$Ra = P_1a_1 + P_2a_2 + P_3a_3$$

In order to have equilibrium, a force equal and opposite to R and having the same line of action must be applied, so that about the point O

$$\Sigma M = 0 \qquad (3)$$

This equation expresses *the third condition* of *static equilibrium, namely, the sum of the moments equals zero.* This condition, together with the first two, must exist in a system of nonconcurrent forces which are in equilibrium.

2. Stresses. When a solid body is acted upon by external forces, (*a*) the body is deformed to some extent, and (*b*) within the body internal resisting forces are developed which balance the external forces. The deformation produced is referred to as either deformation or strain, and the internal forces acting between adjacent particles of the body are known as stresses. If the body remains at rest, the stresses on any imaginary section through the body must be in equilibrium with the external forces on one side of the section.

The stresses produced in a body may be (*a*) tensile stresses, (*b*) compressive stresses, (*c*) shearing stresses, (*d*) bending stresses, (*e*) torsional stresses, or a combination of two or more of those listed from (*a*) to (*e*). Tension may be defined as a pulling stress, and compression as a pushing or crushing stress. When these stresses act along the axis of a body, *i.e.*, through the center of gravity of a cross-section through the body perpendicular to its axis, they are known as direct stresses. Shearing stresses are produced by forces tending to slide one particle upon another. Tension, compression, and shear are the elementary stresses.

Bending stresses are produced by external forces so applied as to produce compression on one side of a body and tension on the other side, in which case shearing stresses exist through some or all portions of the body.

Torsional or twisting stresses are a combination of shearing, compressive, and tensile stresses.

Combined stresses are those resulting from a combination of direct and bending stresses, which produce a bending in the member and at the same time an elongation or shortening depending upon the character of the direct stress.

Impact stresses are the additional stresses caused by the sudden application of an external force.

3. Elasticity. When a body has been deformed by the action of external forces and the external forces are then withdrawn, the body tends more or less to return to its original form. The extent of the recovery is a measure of the elasticity of the body. Most structural materials are nearly completely elastic under small unit stresses; but as the intensity of the stresses increases, a point is reached beyond which, upon the withdrawal of the forces, the body does not return to its original form. The maximum unit stress within which complete recovery occurs is known as the *elastic limit.*

When a material is stressed beyond the elastic limit, greater deformation of more or less permanent character takes place and eventually rupture occurs. The unit stress causing rupture is known as the *ultimate strength.*

Within the elastic limit, the deformation is proportional to the stress, and the ratio of unit stress to unit deformation is known as the *modulus of elasticity* E of the material. A material has moduli of elasticity in tension, compression, and shear, the first two being of the same value for most materials.

4. Center of Gravity. The *center of gravity* of a body is the point through which the resultant of all the gravity forces acting upon the body will pass. Since all gravity forces act in parallel lines, the problem of determining the center of gravity is one of determining the resultant of a system of parallel forces. The term "center of gravity" is often applied to bodies which have no weight, such as areas or lines. Such a determination is frequently necessary in structural analyses.

The center of gravity of the shaded area of Fig. 3 is obtained as follows: It is divided into the three rectangles, as shown, whose centers are their respective centers of gravity. Taking moments of their respective areas about any plane OY, the distance $\overline{X}$ of the center of gravity of the whole area from OY is determined.

$$\overline{X} = \frac{A_1x_1 + A_2x_2 + A_3x_3}{A}$$

Similarly the distance

$$\overline{Y} = \frac{A_1y_1 + A_2y_2 + A_3y_3}{A}$$

5. Moment of Inertia and Radius of Gyration. The amount of tensile stress which a body can sustain depends mainly on the kind of material of which it is composed and its minimum cross-sectional area. On the other hand, the shape of the cross-section has a good deal to do with its ability to resist compressive, shearing, and bending stresses.

The measure of the resisting ability of a cross-section usually appears as the *moment of inertia*, I, of the area or some function

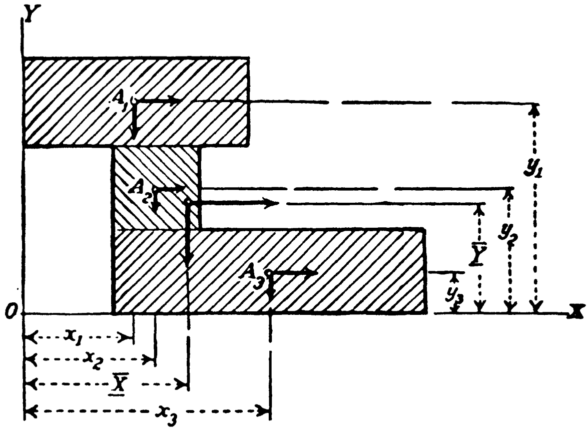


FIG. 3.

thereof. Strictly speaking, an area has no weight or inertia and therefore no moment of inertia; but it is customary to give the name *moment of inertia* to the quantity Σx^2dA or

$$I = \int x^2dA, \tag{4}$$

in which dA denotes any differential area, each part of which is the same distance x from the axis of reference.

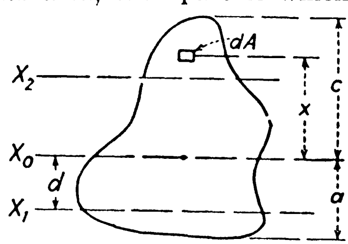


FIG. 4.

It is sometimes convenient to use a function of the moment of inertia, such as its ratio to the area. Thus, $I = \int x^2dA = R^2A$. The quantity R is known as the *radius of gyration* and is the distance from the axis at which all of the area could be concentrated and the moment of inertia remain the same, or R^2 is the mean value of x^2 for equal differential areas. As commonly determined,

$$R = \sqrt{I/A}. \tag{5}$$

The moment of inertia about an axis through the center of gravity of a member is the *least moment of inertia.* This can be shown as follows: In Fig. 4, I_0 about the axis X_0 through the center of gravity is $\int_{-a}^{+c} x^2 dA$. Considering any element dA, its moment of inertia about any other axis such as X_1 is $(x + d)^2 \times dA$ and if all the elements dA at variable distances x from the gravity axis are considered,

$$I_1 = \int (x + d)^2 dA = \int x^2 dA + 2d\int xdA + d^2\int dA$$

But $2d\int xdA = 0$ since $\int xdA = \overline{X}A$ and $\overline{X} = 0$ at the gravity axis. Also $d^2\int dA = Ad^2$. If d were taken as negative the same relation would hold, because the term $-2d\int xdA$ would still be zero and $d^2\int dA$ would still be positive.

In addition to showing that the least moment of inertia is about the gravity axis the foregoing also shows that the moment of inertia I_1 of an area about any axis in its plane is equal to its moment of inertia I_0 about the gravity axis plus the product of the area A and the square of the distance d between the axes, or,

$$I_1 = I_0 + Ad^2 \tag{6}$$

This formula is known as the "Transfer Formula."

Examples. I_0 (axis parallel to b) for a rectangle of width b and height h =

$$\int_{-\frac{h}{2}}^{+\frac{h}{2}} x^2 dA = \int_{-\frac{h}{2}}^{+\frac{h}{2}} bx^2 dx = \frac{bx^3}{3}\Bigg]_{-\frac{h}{2}}^{+\frac{h}{2}} = \frac{1}{12} bh^3$$

I about one base b =

$$\frac{1}{12} bh^3 + bh\left(\frac{h}{2}\right)^2 = \frac{bh^3}{3}$$

I_0 for a circle (Fig. 5) of diameter d and radius r =

$$\int x^2 dA$$

$$dA = \rho d\rho d\theta \qquad \text{and} \qquad x = \rho \sin \theta$$

Therefore

$$I_0 = \int_0^r \int_0^{2\pi} \rho^3 d\rho \sin^2 \theta d\theta$$
$$= \int_0^r \rho d\rho \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{2\pi}$$
$$= \pi \int_0^r \rho^3 d\rho = \frac{\pi r^4}{4}$$

The moments of inertia of other areas may be obtained in a similar manner. The least radius of gyration of the rectangle bh about an axis parallel to the side b is $\sqrt{\frac{bh^3}{12} \div bh} = \frac{h}{\sqrt{12}}$, and the least radius of gyration of the circle whose radius is r is $\sqrt{\frac{\pi r^4}{4} \div \pi r^2} = \frac{r}{2}$.

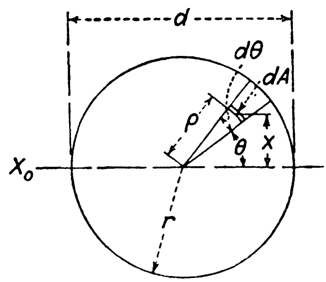


FIG. 5.

6. Bending Moment and Shear in Simple Beams with Fixed Loads. The bending moment at any section in a beam is the algebraic sum of the products of all the forces on one side of a section multiplied by the respective lever arms measured from the section normal to the forces. In Fig. 6 the bending moment at section A is the sum of the reaction R_1 times the distance from R_1 to A and the load P_1 times the distance from P_1 to A, due consideration being given to the direction in which the forces tend to rotate the beam about A. In order to determine the bending moment at A it is first necessary to compute the value of R_1. Taking moments about R_2:

$$-P_1(b + c + d) - P_2(c + d) - P_3 d + R_1 l = 0.$$

In writing this equation, clockwise rotation is considered positive and counterclockwise rotation negative. If the values of the loads are known, the value of R_1 is obtained from the above equation. The bending moment at A is then obtained by considering the portion of the beam to the left of A as a free body.

$$M_A = R_1(a + e) - P_1 e.$$

The same result could have been obtained by computing R_2 and taking the free body on the right of the section.

It is often convenient to represent graphically the variation in moment from one end of a beam to the other. Such a construction is known as a moment diagram. The moment diagram for the loads of the beam of Fig. 6 is shown directly beneath the beam.

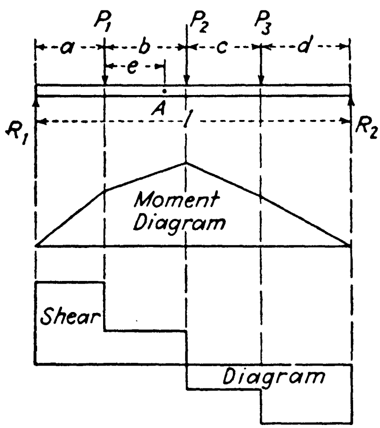


FIG. 6.

Referring to Fig. 6, at the left end an upward force is exerted on the beam which tends to slide adjacent particles of the beam upon each other since the beam is prevented from moving upward by the downward action of the loads. The force producing such an action is the external shear, and the shear at any section is the sum of all the forces on one side of the section, due consideration being given to the direction of action of the forces. In Fig. 6 the shear along the distance a is equal to R_1, and at the point A the shear is $R_1 - P_1$. The variation in shear from one end of the beam to the other is shown graphically in the shear diagram.

There is one important relation between bending moment and shear which may be developed as follows. In Fig. 7 the beam is loaded with both uniform and concentrated loads as shown. Assume that, for a given system of loads, the bending moment diagram shows that the maximum moment occurs at some section between P_2 and P_3. The bending moment at any section between these two loads, at a distance x from the left reaction, is

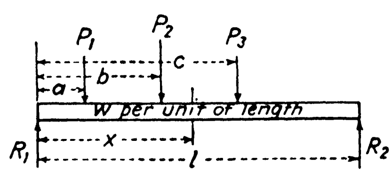


FIG. 7.

$$M = R_1x - wx \cdot \frac{x}{2} - P_1(x - a) - P_2(x - b).$$

The value of x which makes the bending moment a maximum is obtained by equating to zero the first derivative of M with respect to x,

$$\frac{dM}{dx} = R_1 - wx - P_1 - P_2 = 0$$

from which x may be determined for any system of known loads. But $R_1 - wx - P_1 - P_2$ is the expression for the shear between P_2 and P_3. Therefore, the maximum moment occurs at the section where the shear passes through zero.

Examples. 1. A beam with a span of 20 ft.-0 in. supports a uniform load of 500 lb. per lin. ft. and in addition two concentrated loads of 1000 lb. each,

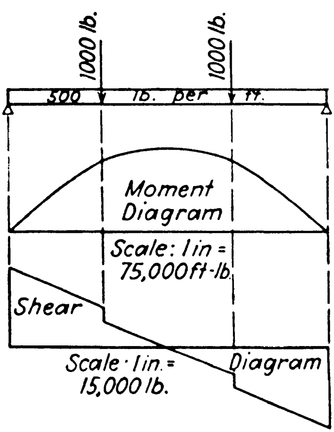


FIG. 8.

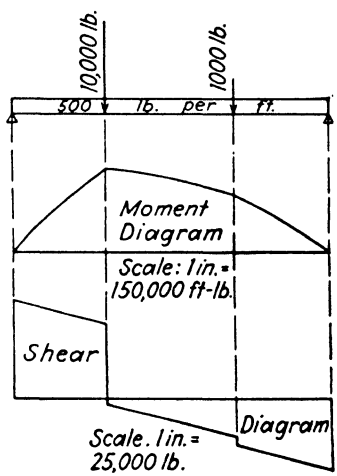


FIG. 9.

acting respectively 6 ft. from each support. Determine the maximum moment and shear in the beam, and draw the corresponding shear and moment diagrams.

The reactions are $\frac{500 \times 20}{2} + 1000 = 6000$ lb. each. The shear at the left of the left-hand concentrated load is $6000 - 6 \times 500 = +3000$ lb.; at the right of this same load the shear is $6000 - 6 \times 500 - 1000 = +2000$ lb., and it becomes zero at the center of the span. At this point the maximum moment is $6000 \times 10 - 500 \times \frac{10^2}{2} - 1000 \times 4 = 31{,}000$ ft.-lb. At the point of application of the left-hand concentrated load the moment is $6000 \times 6 - 500 \times \frac{6^2}{2} = 27{,}000$ ft.-lb.

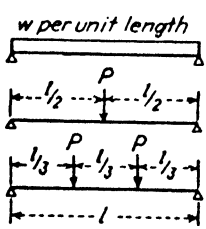
The moment and shear diagrams are shown in Fig. 8.

2. If the left-hand concentrated load in Example 1 were 10,000 lb., determine the maximum moment and shear, and draw the corresponding shear and moment diagrams.

The left reaction is $\frac{500 \times 20}{2} + \frac{10{,}000 \times 14}{20} + \frac{1000 \times 6}{20} = 12{,}300$ lb. At the left of the left-hand concentrated load the shear is $12{,}300 - 6 \times 500 = 9300$ lb., and at the right of this load the shear is $12{,}300 - 6 \times 500 - 10{,}000 = -700$ lb. Since the shear passes through zero under this concentrated load, the maximum moment occurs at this point and is equal to $12{,}300 \times 6 - 500 \times \frac{6^2}{2} = 64{,}800$ ft.-lb. The moment and shear diagrams are shown in Fig. 9.

3. A beam 30 ft. long rests on two supports 20 ft. apart, with 10 ft. of the beam overhanging the right support. The right half of the beam carries a uniform load of 10,000 lb. per lin. ft. The left half of the beam carries a uniform load of 5000 lb. per lin. ft. Draw the shear diagram and compute the locations and amounts of the maximum bending moments. The solution is left to the student. The maximum positive moment (a positive moment is arbitrarily considered as one which produces compression in the upper, and tension in the lower portions of the beam) occurs at a section 5.64 ft. from the left support, and the maximum negative moment (compression in the lower, and tension in the upper portions of the beam) occurs at the right support. The values are +79,500 ft.-lb. and −500,000 ft.-lb., respectively. The moment changes from positive to negative at a section 11.28 ft. from the left support.

Equations may be developed for common systems of loading from which the bending moments may easily be computed. The three more usual cases are as follows:



	Left reaction	Maximum moment
Uniform load	$\frac{wl}{2}$	$\frac{wl}{2}\cdot\frac{l}{2} - \frac{wl}{2}\cdot\frac{l}{4} = \frac{1}{8}wl^2$
Concentrated load at center	$\frac{P}{2}$	$\frac{P}{2} \times \frac{l}{2} = \frac{1}{4}Pl$
Equal concentrated loads at third points	P	$P \cdot \frac{l}{3} = \frac{1}{3}Pl$

7. Bending Moment and Shear in Simple Beams with Moving Loads. In construction, many examples occur of loads that move across the span of a beam. In such cases, though there are definite values of moment and shear produced by the weight of the beam itself as a fixed load, it is usually the moving load or loads which produce the greatest moments and shears. With a single load it is obvious that the maximum moment occurs when the load reaches the center of the span and the maximum shear is produced when the load is an infinitesimal distance from a support.

When two equal loads which are at a fixed distance apart move across a beam, it might seem that the maximum moment would occur when each load was the same distance from the center of the span. However, if in Fig. 10 the equal loads P which are a fixed distance a apart are placed as shown, the left reaction is

$$R_L = [P(l - x) + P(l - x - a)] \div l$$

and the moment under the left-hand load is

$$M = \frac{P}{l}(2lx - ax - 2x^2)$$

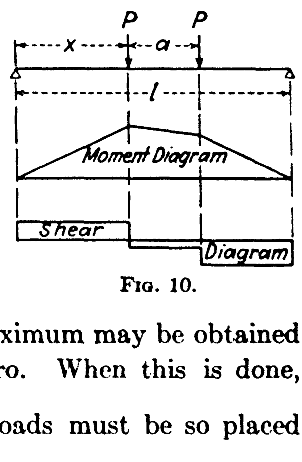


Fig. 10.

The value of x which will make M a maximum may be obtained as in Art. 5 by equating dM/dx to zero. When this is done, $x = \frac{l}{2} - \frac{a}{4}$, which indicates that the loads must be so placed that the center of the span bisects the distance between the center of gravity of the loads and one of the loads. In that position the shear will pass through zero under the load nearest to the center of the span, and the maximum moment will occur under that load. The moment and shear diagrams for the loads so placed are shown in Fig. 10. The *maximum shear* is produced when the loads are placed as near to one support as possible.

If two unequal loads a fixed distance apart move across a span, their position is determined in a similar manner, the only change in the final conclusion being that the loads must be so placed that the center of the span bisects the distance between the center of gravity of the loads and the *heavier* load, the maximum moment occurring under the heavier load. The maximum shear occurs with the heavier load at one support and the other at the fixed distance on the span.

Example. Determine the maximum moment and shear in a beam having a span of 30 ft.-0 in., due to two moving loads of 8000 and 2000 lb., respectively, which are a fixed distance apart of 14 ft.-0 in.

The center of gravity of the two loads is $(2000 \times 14) \div 10{,}000 = 2.8$ ft. from the larger load. This load must therefore be placed 1.4 ft. from the center of the span or 13.6 ft. from the left support. Taking moments about the right support, the left reaction is

$$(8000 \times 16.4 + 2000 \times 2.4) \div 30 = 4533 \text{ lb.},$$

and the bending moment under the heavier load is $4533 \times 13.6 = 61{,}653$ ft.-lb. It should be noted that with this system of loads on a shorter span (25.2 ft. or less) the smaller load will be off the span when the heavier load is placed 1.4 ft. from the center of the span. In such a case the maximum moment is obtained by placing the heavier load at the center of the span.

The maximum shear is obtained by placing the heavier wheel at the left support and taking moments about the right support; its value is

$$(8000 \times 30 + 2000 \times 16) \div 30 = 9067 \text{ lb.}$$

8. Continuous and Fixed Beams. The value of the external bending moment in a beam varies according to the method of support. The beams discussed in Arts. 6 and 7 were considered to be resting freely on two supports, and all bending moment was of positive character, *i.e.*, causing tension in the lower and compression in the upper portion of the beam. Beams are often continuous over one or more supports, and the ends may be restrained or fixed. Negative moments are then developed at the continuous supports and at the fixed ends. These moments may be determined by the application of the theorem of three moments. For beams of constant cross-section, with the supports all on the same level, the two fundamental equations of this theorem are (see Fig. 11):

For uniform loads,

$$M_1l_1 + 2M_2(l_1 + l_2) + M_3l_2 = -\tfrac{1}{4}w_1l_1^3 - \tfrac{1}{4}w_2l_2^3$$

For concentrated loads,

$$M_1l_1 + 2M_2(l_1 + l_2) + M_3l_2 = -\Sigma P_1l_1^2(k_1 - k_1^3) - \Sigma P_2l_2^2(2k_2 - 3k_2^2 + k_2^3)$$

If the beam has only two spans, with the ends freely supported, M_1 and M_3 are each zero, and M_2 can be computed directly from the proper equation (uniform or concentrated loads or a combination of both). If the beam has more than two spans,

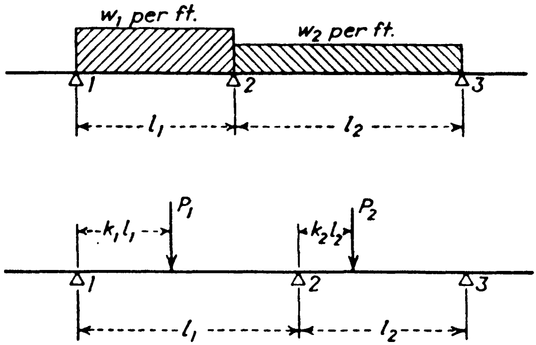


FIG. 11

write the full equation for spans l_1 and l_2, then for l_2 and l_3, etc., and solve the resulting equations simultaneously. If the beam is fixed at one or both ends, replace each condition of restraint by adding an imaginary span at that end, and then substitute a value of zero for the length of each imaginary span.

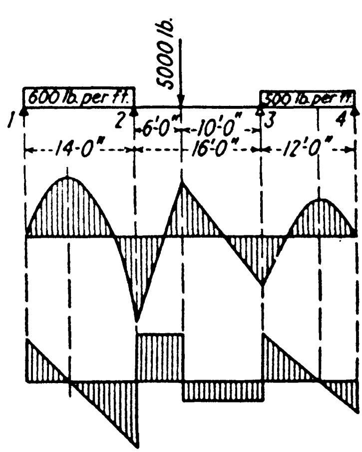


FIG. 12.

Examples. 1. Determine the moments and shears in the beam shown in Fig. 12. Applying the three-moment equations to supports 1, 2, and 3 (the two left-hand spans),

$$14M_1 + 2M_2(14 + 16) + 16M_3 = -\tfrac{1}{4} \times 600 \times 14^3 - 5000 \times 16^2\,[2(\tfrac{3}{8}) - 3(\tfrac{3}{8})^2 + (\tfrac{3}{8})^3]$$

and since $M_1 = 0$,

$$60M_2 + 16M_3 = -411,600 - 1,280,000(\tfrac{195}{512}) = -899,100 \text{ ft.-lb.}$$

For supports 2, 3, and 4 (the two right-hand spans),

$$16M_2 + 2M_3(16 + 12) + 12M_4 = -5000 \times 16^2[\tfrac{3}{8} - (\tfrac{3}{8})^3] - \tfrac{1}{4} \times 500 \times 12^3$$

and since $M_4 = 0$,

$$16M_2 + 56M_3 = -1,280,000(\tfrac{165}{512}) - 216,000 = -628,500 \text{ ft.-lb.}$$

Solving these two equations simultaneously,

$$M_2 = -12{,}981 \text{ ft.-lb.,} \qquad \text{and} \qquad M_3 = -7514 \text{ ft.-lb.}$$

The reactions may now be determined. Cutting a section at support 2, considering the left-hand portion of the beam as a free body, and taking moments about 2: $12{,}981 - 600 \times 14 \times 7 + 14R_1 = 0$ from which $R_1 = 3270$ lb. Similarly, moments about 3:

$$7514 + 3270 \times 30 - 600 \times 14 \times 23 - 5000 \times 10 + 16R_2 = 0$$

from which $R_2 = 8600$ lb. Similar moments about 3 and 2, considering the right-hand portion of the beam as a free body, give $R_4 = 2370$ lb. and $R_3 = 5160$ lb.

The maximum positive moments are computed as follows:

In the first span the shear passes through zero at a distance of $3270 \div 600 = 5.43$ ft. from the left support. At this point the bending moment is

$$3270 \times 5.43 - 600 \times \frac{(5.43)^2}{2} = 8911 \text{ ft.-lb.}$$

In the second span the shear passes through zero under the 5000-lb. load ($3270 - 8400 + 8600 = +3470$, and $3270 - 8400 + 8600 - 5000 = -1530$). At this point the bending moment is

$$3270 \times 20 + 8600 \times 6 - 14 \times 600 \times 13 = 7800 \text{ ft.-lb.}$$

In the third span the shear passes through zero at a distance of $2370 \div 500 = 4.74$ ft. from the right support. At this point the bending moment is

$$2370 \times 4.74 - 500 \times \frac{(4.74)^2}{2} = 5617 \text{ ft.-lb.}$$

The moment and shear diagrams are shown in Fig. 12. It is to be noted that the positive moment in each span is less than if each span were considered as a simple supported beam (first span 8911 against 14,700, second span 7800 against 18,750, and third span 5617 against 9000), while the reactions R_2 and R_3 are greater and the reactions R_1 and R_4 smaller. This latter variation causes an increase in the shear near supports 2 and 3 over that computed for simply supported spans.

2. Determine the moments and shears in the beam of Fig. 13. This beam is built into the walls at the supports and is thus fixed at the ends. The bending moments may be computed from the three-moment equations by considering a span of zero length to the left of support 1 and another span of zero length to the right of support 2. Applying the equations to the two left-hand spans,

$$M_0 \times 0 + 2M_1(0 + 15) + 15M_2 = -\tfrac{1}{4} \times 800 \times 15^3 - 6000 \times 15^2\,[2(\tfrac{1}{3}) - 3(\tfrac{1}{3})^2 + (\tfrac{1}{3})^3]$$

or

$$30M_1 + 15M_2 = -675{,}000 - 500{,}000 = -1{,}175{,}000 \text{ ft.-lb.}$$

For the two right-hand spans,

$$15M_1 + 2M_2(15 + 0) + M_3 \times 0 = -\tfrac{1}{4} \times 800 \times 15^3 - 6000 \times 15^2\,[\tfrac{1}{3} - (\tfrac{1}{3})^3]$$

or

$$15M_1 + 30M_2 = -675{,}000 - 400{,}000 = -1{,}075{,}000 \text{ ft.-lb.}$$

From these two equations,

$$M_1 = -28{,}333 \text{ ft.-lb.,} \qquad \text{and} \qquad M_2 = -21{,}667 \text{ ft.-lb.}$$

If a section is cut at the reaction R_2 and that portion of the beam to the left considered as a free body, as indicated in Fig. 13*a*, then by equating to zero the sum of the moments about 2, the following equation is obtained:

$$-28{,}333 + R_1 \times 15 - 6000 \times 10 - 12{,}000 \times 7.5 + 21{,}667 = 0$$

from which $R_1 = 10{,}444$ lb. Similarly (moments about 1) $R_2 = 7556$ lb.

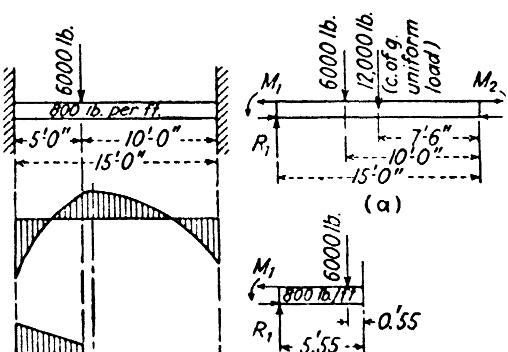


FIG. 13.

Since the shear passes through zero $(10{,}444 - 4000 - 6000) \div 800 = 0.55$ ft. to the right of the concentrated load, the positive moment will be maximum at that point. Considering that portion of the beam to the left of this point as a free body as indicated in Fig. 13*b*, the moment at the section is

$$M = -28{,}333 + 10{,}444 \times 5.55 - \frac{800 \times (5.55)^2}{2} - 6000 \times 0.55$$
$$= 14{,}013 \text{ ft.-lb.}$$

For approximate computations it is general practice to consider that the bending moment over the interior supports of a fully continuous, uniformly loaded beam is $\frac{1}{12}wl^2$ and for a partly continuous beam (continuous over one support only) $\frac{1}{10}wl^2$. Bending moments for concentrated loads may be reduced in proportion; for instance, the bending moment for a fixed load in the center of the span for a partly continuous beam would be taken as $\frac{8}{10} \times \frac{1}{4}Pl = \frac{1}{5}Pl$ and that for a fully continuous beam as $\frac{8}{12} \times \frac{1}{4}Pl = \frac{1}{6}Pl$. Similarly, the bending moments for third point loading as shown in Art. 6 would become $\frac{4}{15}Pl$ and $\frac{2}{9}Pl$, for partly and fully continuous beams, respectively.

GRAPHIC STATICS

9. The Force Polygon. Since a force is completely determined when it is known in amount, in direction, and in point of applica-

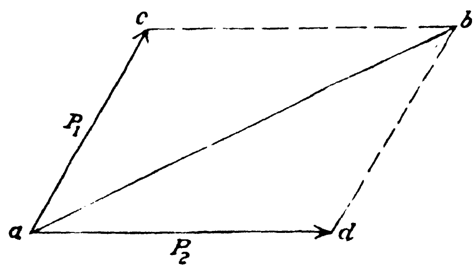


FIG. 14.

tion, any force may be represented by the length, direction, and position of a straight line. The length of the line is fixed by the relation between the amount of the given force and a prearranged unit of force. If the unit of force be taken as 1 lb. to the inch, a line 3 in. long represents a force of 3 lb.

In Fig. 14, P_1 and P_2 represent two forces in the same plane acting at the point *a*. The amount of the resultant of the two forces is given by the length of the diagonal of the parallelogram *acbd* constructed upon the two given forces as sides. A single force applied at the point *a*, acting in the direction of point *b*, of an amount represented by the length of the line *ab*, will replace P_1 and P_2 in so far as accomplishing work is concerned. A force of the same amount by acting in the opposite direction will hold the forces P_1 and P_2 in equilibrium, *i.e.*, at rest.

Reference to Fig. 14 shows that, in order to determine the amount and direction of the resultant of P_1 and P_2, it is not necessary to construct the complete parallelogram *acbd*. From *c* of the force P_1 a line *cb* is drawn parallel and equal to P_2. The triangle *acb* is called the *force triangle* and represents graphically the relation between the forces P_1, P_2, and *ab*.

Thus it is seen that, if in addition to forces P_1 and P_2 a third force *ba* is applied at point *a* in the direction from *b* to *a*, these three forces will be in equilibrium. The point *a* will then remain at rest. It may now be concluded that, *if three forces meeting in a point are in equilibrium, they will form a closed force triangle*, *i.e.*, not only one in which the three lines parallel and equal to the three forces, respectively, form a closed, three-sided figure, but one in which the arrows representing the direction of the forces all point in the same direction around the triangle.

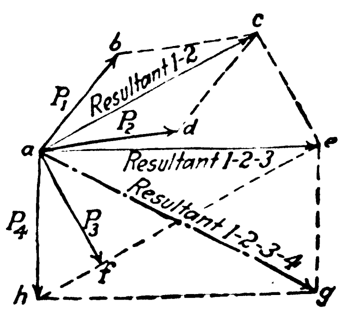


FIG. 15.

By applying this principle, any one of the three forces may be determined in amount and direction if the other two are fully known, or any two of the forces may be obtained in amount if their directions are given and the third force is fully known.

If, instead of two forces, three or more were acting at point *a*, the resultant of these forces could be determined by the successive application of the principle of the force triangle; two forces are first composed into one, and this one combined with the third force; this process is continued until the last force is combined with the resultant of all the other forces to determine the resultant of the entire group, as in Fig. 15.

A study of Fig. 15 shows that it is not necessary to complete all the parallelograms in order to obtain *ag*. The polygon *abceg*, of which four sides, respectively, are parallel and equal to forces P_1, P_2, P_3, and P_4, gives the required information. This polygon is repeated for clearness in Fig. 16. The line drawn from the beginning *a* to the end *g* of the last known line *eg* of the polygon, contrary to the order of the forces, gives the intensity and direction of the resultant of the four given forces. A force of equal intensity applied in the opposite direction, from g to a (in the direction of the forces), would hold the four given forces in equilibrium. Point a would remain at rest under the action of five forces P_1, P_2, P_3, P_4, and ga.

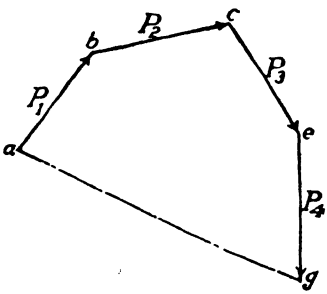


Fig. 16.

The polygon $abceg$, formed by the successive laying off of lines parallel and equal to the given forces, is called the *force polygon*—the force triangle expanded.

Application of the Force Polygon. The crane truss of Fig. 17 consists of a vertical post bd, supporting a boom dc, from which the external load P_1 is suspended. The boom is anchored to the post by the tie rod bc, and the post is in tu n anchored at its upper end by the backstay ab fastened to the ground. In constructing the force polygon, P_1 $(= mn)$ is laid off equal to the external load. By drawing mk and nk parallel to T_2 and T_1, respectively, the values of the stresses in these two members are

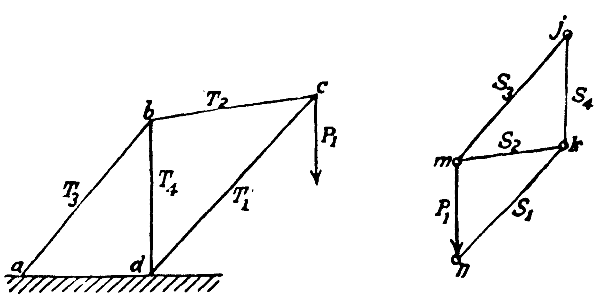


Fig. 17.

obtained. Similarly, mj and kj determine the stresses in T_3 and T_4. In the triangle mnk, the direction of P_1 is known. Following around the triangle in the direction indicated by P_1, S_1 acts from n to k; and, transferring this direction to the truss diagram, the stress in T_1 acts toward the joint c and is therefore compression. Proceeding in the direction from k to m, the stress in T_2

is tension. In the triangle *kmj*, since T_2 is in tension and acts away from the joint *b*, the direction to be taken around the triangle is from *m* to *k*. This indicates tension in T_3 and compression in T_4.

10. The Equilibrium Polygon. In Fig. 18*a* the forces P_1, P_2, P_3, and P_4 acting upon the given body are not in equilibrium. The amount and direction of their resultant are obtained from the force polygon *abcde* (Fig. 18*b*). The line of action of this resultant may be determined as follows:

From any point *O* in the force polygon, a line may be drawn to each of the vertices of the polygon. Since the lines *Oa* and *Ob* form a closed triangle with the force P_1, they represent two

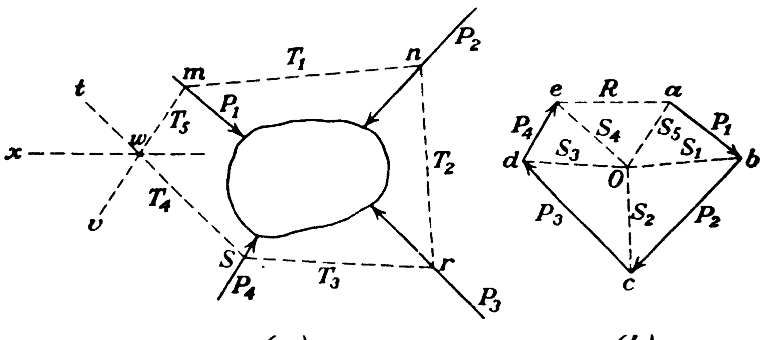


FIG. 18.

forces that will hold P_1 in equilibrium—two forces that may replace P_1 in the force diagram. In Fig. 18*a* at any point *m* on the line of action of P_1, lines *mn* and *mv* drawn parallel to S_1 and S_5, respectively, represent the lines of action of these two forces. Similarly, S_1 and S_2 represent two forces that may replace P_2. The line of action of S_1 has already been fixed by the line *mn*. From the point of intersection *n* of the line *mn* with the force P_2, *nr* is drawn parallel to S_2. From *r*, *rs* is drawn parallel to S_3; and from *s*, *st* is drawn parallel to S_4. Lines *mv* and *st*, which are parallel to S_5 and S_4, respectively, represent the lines of action of S_5 and S_4. These latter forces, together with the resultant *ae*, form a closed force triangle. S_5, S_4, and *ae* therefore represent a series of three forces in equilibrium. To fulfill this condition where three forces only are involved, the three forces must meet at a point. The line of action of the resultant *ae* must therefore pass through the point of intersection *w* of the lines *mv* and *st*. The resultant of the four given forces is thus fully determined. A force of equal magnitude but acting in the opposite direction, *i.e.*, from *e* to *a* (in the direction of the forces in the force polygon), will hold P_1, P_2, P_3, and P_4 in equilibrium and keep the body on which they act at rest.

The polygon *mnrsw* represents a jointed frame which, by means of the stresses of tension and compression in its various members, could hold the given forces in equilibrium. It is called, therefore, an *equilibrium polygon*. The amount and character of stress in each of the members of the jointed frame are given by the lines S_1 . . . S_5 in the force polygon. The point O is called the "pole," and the lines S_1 . . . S_5 are called the "rays" of the force polygon. Obviously, since an infinite number of poles may be selected and an infinite number of starting points may be used, an infinite number of equilibrium polygons may be drawn for a given group of forces; the final result will, however, be the same for all; *i.e.*, the line of action of the resultant of any given force group, determined by one equilibrium polygon, will concide with that found by any of the other possible polygons.

11. Resultant of Parallel Forces. In the case of a system of parallel forces, the force polygon becomes a straight line. Figure 19 represents a system of four such forces, not in equilibrium. The force polygon *abcde* is a straight line; and *ae*, its closing line, gives the magnitude and direction of the resultant. For convenience in the following construction, the forces represented by each of the lines of the force polygon in Fig. 19 are indicated at the right of the polygon. To hold the given forces in equilibrium, a force equal in amount to *ea*, acting upward from *e* to *a*, is necessary. The line of action of this force is determined from the equilibrium polygon.

The pole O is selected and the rays drawn to the vertices of the force polygon. From any point on the line of action of P_1, lines *mn* and *mt* are drawn parallel to *Ob* and *Oa* (the rays that form with P_1 a closed force triangle), respectively. From *n*, the point of intersection of *mn* and the line of action of P_2, a line *nr* is drawn parallel to *Oc*; from *r*, a line *rs* is drawn parallel to *Od*; and from *s*, a line *sv* is drawn parallel to *Oe*. The point of intersection *w* of the lines *mt* and *sv* is a point in the line of

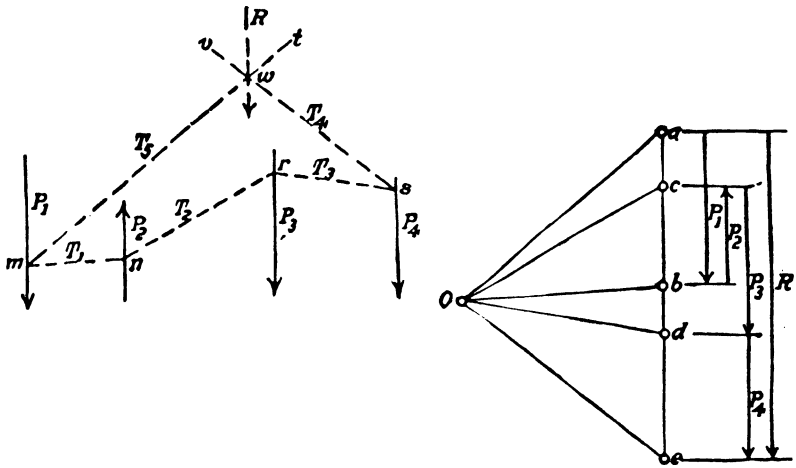


Fig. 19.

action of the resultant force. An upward force equal and parallel to *ea*, applied so that its line of action passes through point *w*, will hold the forces $P_1 \ldots P_4$ in equilibrium. A downward force of the same amount will replace the given forces.

CHAPTER II

STRUCTURAL MATERIALS

TIMBER

12. General Characteristics. Wood is composed of a series of cellulose cells cemented together with lignin. Closely packed thick-walled cells are indications of strength and general suitability for structural use. In most trees the growth is by annual rings, composed of two parts, the inner, or springwood, and the outer, or summerwood. The latter is much denser, and timber for structural use should have a large proportion of summerwood.

Defects are any irregularities in the structure of the grain of the wood caused by internal or external agencies. The principal defects which may make the timber unfit for use are decay, caused by vegetable organisms; cross grain, where the grain of the wood deviates from the direction parallel to the axis of growth; knots, portions of branches incorporated in the body of the trunk; shakes, a separation of the wood fibers along the grain; and checks, a separation of the wood fibers across the grain. Since most timber is liable to have one or more of these defects in some part of its length, timber is graded into several classes depending upon the percentage of the piece which is free from all defects.

13. Mechanical Properties. The strength, stiffness, hardness, and durability of wood determine its fitness for structural use. Owing to the shape of the cells, timber is much stronger in compression parallel to the grain than in compression perpendicular to the grain. In compression along the axis of growth, wood is, in fact, in proportion to its weight one of the strongest of structural materials. In tension, timber free from defects has about the same strength as in compression. Its ability to resist shear is not great on account of the tendency of one grain fiber to split away from the adjacent fiber.

The stiffness and hardness of the timber depend somewhat upon the density of its structure and the degree of seasoning. In general, there is not a great difference in stiffness among the various species of hardwoods or among the different kinds of softwoods.

The life of timber depends first upon the structure of the wood itself, its species, its condition when placed, and the general conditions surrounding it. Two woods of the same species from trees growing under different conditions may have entirely different structures, and it is common knowledge that some kinds of wood are more durable than others. Green wood, used before proper seasoning, is more liable to decay and deterioration; but any unprotected timber subjected to moisture, air, and moderate heat will deteriorate at a more or less rapid rate.

Variations in the moisture content of the cell walls cause large changes in the strength and stiffness of wood. As the green wood dries out, the tensile and compressive strength and the stiffness are increased, but defects arising from shrinkage stresses often cause a decrease in its ability to resist shearing stresses.

The working stresses given in the accompanying table may be used for timber which is reasonably free from defects, and which is to be continuously dry in the structure.

WORKING STRESSES IN TIMBER
(Continuously dry locations)

Kind of timber	Fiber stress in bending, or tension, p.s.i.	Horizontal shear, p.s.i.	Compression perpendicular to grain, p.s.i.	Compression parallel to grain, p.s.i.	Compression parallel to grain, short columns, p.s.i.	Modulus of elasticity, p.s.i.	Value of K for column design (see Art. 88)
Douglas fir	1500	100	350	1500	1200	1,600,000	23.4
Longleaf yellow pine	1400	120	300	1400	1200	1,600,000	23.4
Close-grained redwood	1400	80	250	1400	1200	1,200,000	20.3
Oak	1400	120	500	1500	1100	1,500,000	23.7
Red cypress	1400	120	300	1400	1200	1,200,000	20.3
Spruce	1200	100	250	1200	1100	1,200,000	21.2
Western red cedar	1000	100	200	1000	800	1,000,000	22.7

For bearing on surfaces inclined to the grain, the Hankinson formula may be used. It is

$$s = Cp$$

$$\text{where } C = \frac{1}{(p/q)\cos^2\theta + \sin^2\theta}$$

in which s = allowable unit stress in compression on the inclined surface.

p = allowable unit stress in compression parallel to the grain.

q = allowable unit stress in compression perpendicular to the grain.

θ = the angle made by the inclined surface with the fibers.

Values of C may be obtained from Fig. 20.

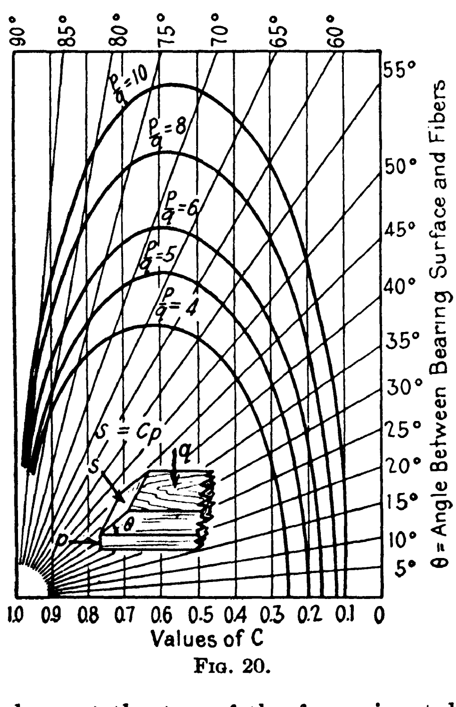


FIG. 20.

The two timbers at the top of the foregoing table are in most general demand for structural purposes. The source of the first is the northwestern states, and the second comes from the southern states. Together they form a large proportion of the structural lumber used in this country. Other species such as hemlock and native fir are often used where good specimens may be obtained locally.

14. Preservation. Various methods of impregnating wood with preservatives have been developed. A superficial coating applied by means of a brush or spray has been found to be inexpensive but also not too effective. A penetration method, in which the timbers are soaked in open containers, is cheap and of considerable benefit. Preservatives may consist of a zinc chloride solution or creosote. If the latter is used, the bath should be alternately heated and cooled. The time of immersion depends upon the kind and condition of the wood and the degree of absorption desired.

The pressure processes are far more effective but at the same time more expensive. In this method, hot creosote is forced into the wood under a pressure of about 150 p.s.i. When the timber is thoroughly impregnated a partial vacuum is used to remove the oil near the surface. The amount of oil varies with the kind of wood, being approximately 10 lb. per cu. ft.

STEEL

15. Raw Materials. The principal ores from which steel is produced are hematite (Fe_2O_3), magnetite (Fe_3O_4), siderite ($FeCO_3$), and limonite (hydrated Fe_2O_3). About 86 per cent of the 48,618,000 tons of ore produced in the United States in 1936 was hematite from the Lake Superior region. Ore is also mined in Tennessee and Alabama and in the Rocky Mountain region. A high-quality steel is produced from the extensive deposits of high-grade magnetite in Sweden.

Of the 1936 world output of 124,374,000 tons of steel ingots and castings, the United States produced 48,478,000 tons, Germany 19,160,000 tons, Japan 16,300,000 tons, and Great Britain 11,700,000 tons.

16. Production. The production of steel from the ore is generally accomplished in two principal steps. First the ores, which are primarily oxides, are reduced with the aid of carbon to an impure metallic iron with a high carbon content, known as *pig iron.* Then the pig iron is converted into the desired metal by lowering the amount of impurities, adjusting the carbon content to the desired value, and sometimes adding certain metals to produce desired properties.

17. Pig Iron. The pig iron is produced in a *blast furnace,* a tall cylindrical stack lined with firebrick. The ore, the fuel (chiefly coke or charcoal), and the flux (usually limestone) are introduced at the top of the blast furnace in alternate layers and descend slowly to the bottom of the furnace as the fuel is burned at the bottom. Air is forced in at the bottom of the furnace to provide an upward draft, heating the charge as it descends through the furnace. During the descent of the charge the ore is reduced to metallic iron by CO gas originating in the fuel, and the lime of the flux melts and combines with the earthy gangue (impurities) of the ore. The separation of the molten slag from the molten iron takes place in the crucible at the base of the stack because of the difference in specific gravities. The molten iron is saturated with carbon and contains small quantities of certain other metals.

18. Methods of Manufacture of Steel. The *Bessemer processes* convert molten pig iron to steel by oxidation of the impurities followed by recarburization to the desired carbon content. The oxidation is brought about by air blown through the molten iron from the bottom or from the side of the converter. The heat necessary to maintain fluidity of the charge is furnished by the oxidation of the impurities, chiefly silicon in the acid Bessemer and phosphorus in the basic Bessemer. The fundamental metallurgical difference between the acid and basic processes is the inability of the acid process to remove sulfur and phosphorus, while the basic process by the use of a lime flux removes the phosphorus to a percentage considered harmless. This is accomplished by the formation of calcium phosphate ($[CaO]_4.P_2O_5$) which enters the slag. The control of the quality of metal in the Bessemer process is largely up to the operator. Bessemer steels are believed by some to be inferior on account of occluded gases due to the air blown through the charge. Side-blown metal, however, furnishes a product particularly free from gases, since the charge is not agitated and only a small amount of air is actually blown through the metal. The Bessemer processes are used only to a limited extent.

The *open-hearth processes* produce steel by oxidation of the impurities from a charge consisting usually of molten pig iron and steel scrap. The oxidizing agent is principally iron ore, which is added to the charge as needed. A reverberatory furnace is used. The heat necessary for the process is obtained by the burning of gas or oil fuel. The differences between the acid and basic processes are the same as in the Bessemer processes. Control of the composition and quality of the charge is accomplished by fracture of small test ingots and by chemical analysis. On account of the better metallurgical control of the charge and the more quiescent state of the metal, during both the operation of the furnace and the pouring of the metal, open-hearth steels are generally considered superior to Bessemer steels. About 90 per cent of the steel made in the United States is produced by the basic open-hearth process.

19. Control of Properties. The properties of steel may be controlled by four methods:

a. Alteration of the carbon content.

b. Heat-treatment.

c. Shaping operations.

d. Addition of alloying elements.

In many instances, methods which increase the tensile strength tend to decrease the ductility and toughness. Which one, or which combinations, of these methods may be used to improve a given steel depends upon the composition of the metal and the *combination* of properties desired.

20. Shaping Operations. Most *steel castings* have a carbon content of less than 0.3 per cent. Casting steel affords opportunity for the formation of checks and blow-holes and for the segregation of impurities. Since castings are usually of complex shape, different parts may cool at different rates, and initial strains and nonuniformity of structure result. Therefore, cast steel tends to be weaker than rolled steel of the same composition.

Rolled steel is shaped by being passed through a series of rolls, each designed to bring it nearer the desired cross-section. Shapes of constant cross-section, such as channels, I-beams, rods, and angles are shaped by hot-rolling, or rolling the steel at a temperature above the upper critical range. The pressure exerted during the rolling process reduces the grain size, thus increasing the strength and durability of the steel. If the rolling is completed at a temperature well above the critical range, the grain size will increase as the steel cools to the critical range; but if the final passing through the rolls occurs at a temperature just above the critical range, there will be little opportunity for crystal growth, and the grain size will be a minimum, making for increased strength.

Steel rolled at atmospheric temperature is said to be *cold-rolled.* Cold-rolling increases the strength of steel, but decreases its ductility to a marked degree. A smooth, bright finish is produced by cold-rolling, whereas a hot-rolled bar is covered with a film of hard, dark oxide.

Forging consists in shaping the metal by hammering or pressing Small shapes are often formed to the desired dimensions by dropping a hammer upon the piece of metal, which rests on an anvil; the hammer and anvil together are shaped to form a mold. Pieces may also be shaped by pressing the metal with sufficient force to cause the metal to flow into the desired form. Forging may be done when the metal is either hot or cold and has the same general effect upon the material as rolling under the same conditions. The magnitude of the change in properties is dependent upon, among other factors, the amount of plastic deformation developed in the process.

Drawing consists in pulling the material through a die that has an opening of the desired cross-section. The process is used for shaping wire and small rods. It may be accomplished when the metal is hot or cold and has the same general effect upon the properties as the other methods of mechanical working. Wire which is cold-drawn must frequently be annealed between draws to prevent it from becoming too brittle.

21. Working Unit Stresses. The unit stresses for use in steel building design, as specified by the American Institute of Steel Construction (1936), are given below in p.s.i.:

Tension:

- Structural steel, net section 20,000
- Rivets, on area based on nominal diameter 15,000

Compression:

- Axially loaded columns, gross section
  - With values of $\frac{l}{R}$ not greater than 120 .... $17,000 - 0.485 \frac{l^2}{R^2}$
  - With values of $\frac{l}{R}$ greater than 120 .... $\dfrac{18,000}{1 + \dfrac{l^2}{18,000R^2}}$

  in which l is the unbraced length of the column and R is the corresponding radius of gyration of the section, both in inches
- Plate-girder stiffeners, gross section 20,000
- Webs of rolled sections at toe of fillet 24,000

Bending:

- Tension in extreme fibers of rolled sections, plate girders, and built-up members 20,000
- Compression in extreme fibers of rolled sections, plate girders, and built-up members, for values of $\frac{l}{b}$ not greater than 40 .... $\dfrac{22,500}{1 + \dfrac{l^2}{1800b^2}}$

  with a maximum of 20,000

  in which l is the laterally unsupported length of the member and b is the width of the compression flange, both in in.
- Stress in extreme fibers of pins 30,000

Shearing:

- Rivets 15,000
- Pins, and turned bolts in reamed or drilled holes 15,000
- Unfinished bolts 10,000
- Webs of beams and plate girders, gross section 13,000

Bearing:	Double Shear		Single Shear
Rivets	40,000		32,000
Turned bolts in reamed or drilled holes	40,000		32,000
Unfinished bolts	25,000		20,000
Pins		32,000	
Contact area			
Milled stiffeners and other milled surfaces		30,000	
Fitted stiffeners		27,000	
Expansion rollers and rockers, lb. per lin. in.		$600d$	
in which d is diameter of roller or rocker, in.			

Cast steel:

- Compression and bearing same as for structural steel. Other unit stresses, 75 per cent of those for structural steel

CONCRETE

22. Introductory. Since the beginning of the twentieth century concrete has taken its place as one of the most useful and important structural materials. Owing to the comparative ease with which it can be molded into any desired shape its structural uses are almost unlimited; therefore, wherever Portland cement and suitable aggregates can be obtained, it has, for certain classes of work, displaced older materials. This apparent ease with which concrete may be prepared has led to its being employed by anyone who feels that the material is suited to his particular purpose. In many instances, proper knowledge of the substance and skill in its manufacture are not available, so that the resultant concrete is little more than a bulky heavy material, lacking the strength and other properties which it should possess and often failing to fulfill the purpose for which it was intended.

To the individuals who obtain such results, concrete is merely a shoveled-together mass of cement, sand, stone, and water, which in a short time attains a varying degree of hardness and an uncertain strength. To the engineer who is more or less familiar with the many factors and variables entering into its manufacture, the process of making concrete does not appear quite so elementary. Experience shows that the quantity and quality of cement, aggregates, and water and the processes of mixing and curing are all involved in the production of concrete. Results are dependent upon all of these variables. It is, therefore, the problem of the engineer so to study and control these factors that a concrete of the desired quality may be obtained.

23. Portland Cement. Portland cement is the product resulting from finely pulverizing the clinker obtained by calcining to incipient fusion an intimate and properly proportioned mixture of argillaceous and calcareous materials with no additions subsequent to calcination except water and calcined or uncalcined gypsum. All Portland cement used in reinforced-concrete construction should pass such standard specifications as those of the American Society for Testing Materials.[1] The color and rapidity of hardening of different brands of cement vary considerably and may be elements requiring special attention for the particular work involved.

Portland cement is manufactured by either the dry process or the wet process.[2] In the dry process the calcareous material is usually limestone and in the wet process marl. In all cases its content is principally calcium carbonate. The argillaceous material may be shale, clay, argillaceous limestone, or blast-furnace slag which provides the silica and alumina.

In the dry process the raw materials are ground separately, mixed in the proper proportions, pulverized, and burned in a kiln. The resulting clinker is cooled and seasoned, materials are added to control the rate of setting, and the clinker is ground to a fine powder. In the wet process the procedure is similar except that the marl is stored in the form of a thin mud in vats and the argillaceous material in powder form is mixed with it before burning.

24. Fine Aggregate. Fine aggregate should consist of sand, stone screenings, or other inert materials with similar characteristics, or a combination thereof, having clean, hard, strong, durable, uncoated grains, and being free from injurious amounts of dust, lumps, soft or flaky particles, shale, alkali, organic matter, loam, or other deleterious substances. In general, all particles passing a No. 4 sieve (4 meshes per lin. in.) are considered as fine aggregate. Most specifications, however, allow some degree of latitude from this requirement. The report of the Joint Committee[3] recommends that not less than 95 per cent of the fine aggregate pass through a No. 4 sieve. Similarly, it is in

[1] A.S.T.M. Standard C9-38.

[2] For description of cement manufacture see "Materials of Construction," Johnson, Withey, and Aston, 8th ed., p. 310, John Wiley & Sons, Inc.

[3] The Joint Committee on Standard Specifications for Concrete and Reinforced Concrete consists of five representatives from each of the following: American Society of Civil Engineers, American Society for Testing Materials, American Railway Engineering Association, American Concrete Institute, Portland Cement Association, American Institute of Architects.

general advantageous that the fine aggregate be well graded from fine to coarse, and the report mentioned above recommends that not more than 30 or less than 5 per cent of the fine aggregate pass through a No. 50 sieve. Extremely fine particles, if present in any great amount, are not beneficial to the strength of the resultant concrete, since they furnish so great an excess of surface area to be covered by the cement. Specifications vary as to the amount of these allowed, but in general not more than 6 per cent of the fine aggregate should pass through a No. 100 sieve.[1]

25. Coarse Aggregate. Coarse aggregate should consist of crushed stone, gravel, or other approved materials of similar characteristics or combinations thereof, having clean, hard, strong, durable, uncoated particles free from injurious amounts of soft, friable, thin, elongated, or laminated pieces, alkali, organic, or other deleterious matter.

Crushed Stone. The quality of an aggregate of this type obviously depends upon the character of the original rock. The principal classes of rocks from which aggregates are derived are granites, traprocks, limestones, and sandstones. Granite is an igneous rock whose principal mineral constituents are quartz and feldspar, with varying amounts of mica, hornblende, and other materials. The structural qualities of granite vary greatly, but granites as a class rank among the hardest, strongest, and most durable stones. Traprock includes basalt, diabase, and a number of other igneous rocks possessing similar chemical and physical properties. The principal mineral constituents of most of these rocks are pyroxene and feldspar. They are generally rather fine-grained, hard, tough, and durable. Limestone is a sedimentary rock which contains carbonate of lime, calcite, or carbonate of lime together with a double carbonate of lime and

[1] The grading recommended by the A.S.T.M. is as follows:

	Per Cent, by Weight
Passing ⅜-in. sieve	100
Passing No. 4 sieve	95 to 100
Passing No. 16 sieve	45 to 80
Passing No. 50 sieve	5 to 30
Passing No. 100 sieve	0 to 8

magnesia, dolomite, as the essential constituent. Sand and clay are common impurities, some varieties of which contain large amounts of shells and other fossils. Limestones vary greatly in structure, strength, hardness, and durability; and although there are some limestones which have superior structural qualities, the average limestone is inferior to average granites and traprocks as a concrete aggregate. Sandstones consist of grains of varying sizes, chiefly quartz, bound together by various cementing agencies or binders. A siliceous binder produces a sandstone of the greatest structural strength; an iron oxide or lime carbonate is much less efficient. A sandstone whose binder is clay is the least valuable of all as a concrete aggregate.

The maximum size of coarse aggregate advisable depends upon the character of the work. Since the stone is one of the strongest constituents of concrete, it is desirable to have as many and as large particles as possible. The greater the size of the particles, the less surface area there is to be coated, and the smaller amount of cement required for a concrete of given strength. When, however, the maximum size is comparatively large, it is very important that it be well graded down to the minimum size in order to make a dense, compact mass. In small reinforced-concrete members the maximum size available is as small as 3/4 in. in diameter, and rarely in any reinforced work is a diameter greater than 1½ in. advisable. On the other hand, for large, massive work, with no structural reinforcement, much larger sizes may be used to advantage. The specification for minimum size recommended by the Joint Committee is that not more than 10 per cent by weight shall pass a No. 4 sieve, and not more than 5 per cent by weight a No. 8 sieve.

Gravel. Gravel of good quality makes a suitable concrete aggregate. Gravel is nothing more than pieces of natural rock broken away from the parent ledges and worn down by stream or glacial action. Its strength as an aggregate depends upon the rock from which it came, provided it has not become decayed or coated with objectionable organic matter. Too much emphasis cannot be placed on the necessity of determining the cleanness of the gravel. A clayey coating is easily detectable; but the transparent organic coatings which prevent adhesion are not so easily discerned, without chemical analysis, and many weak and inferior concretes result from the use of an apparently clean, but really "dirty" gravel as an aggregate.

Natural gravel may have a large proportion of particles so small as to be classed as "fine aggregate." These may be screened out before using; or, a sieve analysis of the natural gravel having been made, the proper amount of additional fine aggregate (if any) to add to obtain the desired proportions may be determined.

Slag. Slag from blast furnaces is a hard though porous material of high compressive strength, which in some localities can be obtained much more cheaply than stone of good quality. It offers a very rough surface for the adhesion of the cement, and, provided the sulphur content is low, it may make an excellent aggregate for massive concrete construction. Generally it should not be used in thin sections exposed to any action from water, on account of its porosity.

Cinders. Cinders as an aggregate have the advantage of making a concrete considerably lighter in weight than that made from stone or gravel. Formerly, it was thought that well-burned cinders made a more fire-resisting concrete than other aggregates, but more recent experiences have shown that cinder concrete is little, if any, better in this respect. Cinder concrete is inferior in strength to stone concrete; and, on account of the danger from the probable sulphur content, it is not used where any great structural strength is required. Its principal use occurs for filling where no great strength is necessary. When used, cinders should be free from unburnt coal or fine ashes.

26. Water. Water used for concrete should be clean and free from oil, acid, alkali, organic matter, or other deleterious matter. Sea water is not so desirable as fresh water, although where it has been used in structures subject to the weathering action of sea water no greater disintegration has resulted.

27. Design of Mortar Mixtures. In selecting the proportions of materials that are to go into a mortar it is often necessary and usually desirable to be able to predict the strength, permeability, and amount of mortar that will result from a given proportion of the ingredients. If there are no air voids in a wet mortar (and tests show that this is practically true for consistencies used in practice), the resultant volume of the mortar is equal to the absolute volume of the cement plus the absolute volume of the sand, plus the volume of water used. In the average cubic foot of cement (1 sack of 94 lb.) there is 0.487 cu. ft. of solids. In 1 cu. ft. of sand the amount of solids depends upon the grading and the size of the grains. In beach sand there is 0.60 to 0.63 cu. ft. and in commercial building sands about 0.68 cu. ft. of solids. The exact amount may be determined for any sand by obtaining its weight per cubic foot and its specific gravity.

The water in a mixture chemically reacts upon the cement and produces a colloidal coating on the particles. The finest particles of cement become completely colloidal, but some of the larger particles are never completely penetrated. This colloidal coating is the whole cementing action; and, once sufficient water is present to produce this reaction, additional water tends to weaken the cementing material. Therefore, it is reasonable to expect that the strength of a mortar at any given age will vary with the amount of mixing water per unit quantity of cement and the larger the amount of water the weaker the mortar. In Fig. 21 are shown the results of studies of several investigators. This figure brings out clearly the direct relation between strength and water-cement ratio. This relation between strength and water-cement ratio is known as the *water-cement ratio theory*, and applies both to mortars and to concretes. It is the basis of modern proportioning methods (see Arts. 28 and 31).

In the measurement of water in a mixture, all water outside of the solid particles is included. Most sand is delivered in a more or less damp condition, and the moisture that is carried must be determined and allowance made in determining the amount of water to add.

28. Determination of Yield and Strength. It is desired to supply 100 cu. yd. of 1:3 mortar which will reach a strength of 3000 p.s.i. in 28 days. The sand available has 35 per cent voids, weighs 100 lb. per cu. ft. when dry, and contains 2 per cent of moisture by weight, in which condition it weighs 83 lb. per cu. ft. What quantities of materials are required?

Reference to Abram's curve in Fig. 21 shows that a 3000-lb. mortar cured damp for 28 days should have 7.2 gal. of water per sack of cement. A 1-sack

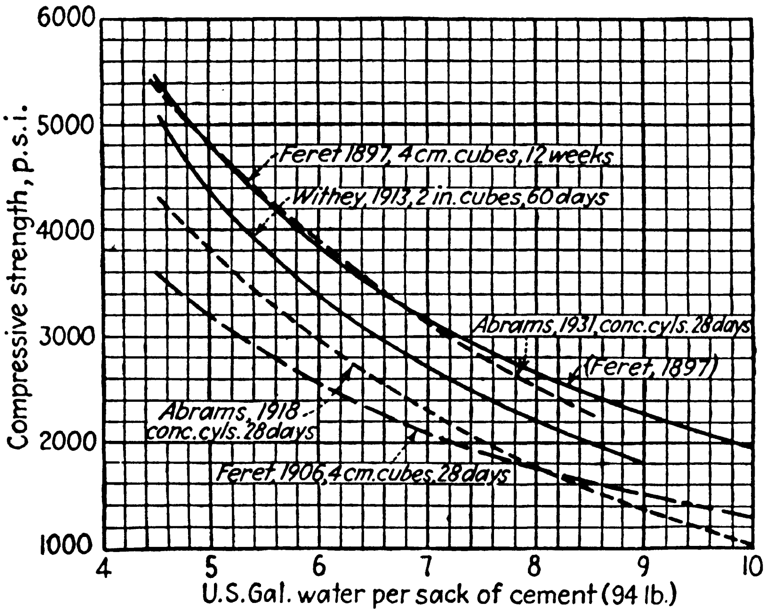


Fig. 21.—Relation between compressive strength and water-cement ratio.

batch will yield $0.487 + 3 \times 0.65 + 7.2 \times 0.134 = 3.40$ cu. ft. $= 0.126$ cu. yd. In 100 cu. yd. there will be $100 \div 0.126 = 793$ one-sack batches. Therefore, the quantities required are

$$\begin{aligned} \text{Cement} &= 793 \text{ sacks} \\ \text{Sand } 3 \times 793 \div 27 &= 88.1 \text{ cu. yd. dry} \\ \text{Water } 7.2 \times 793 &= 5710 \text{ gal.} \end{aligned}$$

However, since the sand contains 2 per cent moisture, the number of cubic yards in its moist condition is

$$\tfrac{100}{83} \times 88.1 = 106 \text{ cu. yd.}$$

If this mortar were to be mixed in approximately 1-cu. yd. batches (8 sacks), the sand and water for each batch would be as follows:

$$\begin{aligned} \text{Sand } 3 \times 8 \times \tfrac{100}{83} &= 28.9 \text{ cu. ft.} \\ \text{Water } 7.2 \times 8 - (0.02 \times 2400) \div 8.35 &= 52 \text{ gal.} \end{aligned}$$

In actual work some of the water in the mix is lost either by absorption by adjacent materials or by evaporation, so that the actual yield is reduced from 2 to 4 per cent. This should be considered in making estimates of quantities of materials required for a given volume of mortar.

29. Concrete Mixtures. Concrete may be considered as a mortar mixture into which the coarse aggregate particles have been mixed, or it may be considered as a combination of the separate ingredients. In any case the factors governing the strength of a mortar determine the strength of a concrete provided a suitable coarse aggregate is used. Since the water in the mixture has to wet the surfaces of the coarse-aggregate particles as well as those of the cement and sand grains, it follows that the larger the size of the coarse aggregate, the less the water that will be required for a given consistency; hence, the greater the strength and durability of the mixture.

Design specifications have not kept pace with the improvement in cement in recent years. For this reason, caution should be exercised in designing for low specified strengths. With fair curing conditions a compressive strength of 2000 p.s.i. in 28 days may be obtained with a very lean mixture, but the resulting concrete will be neither watertight nor capable of withstanding ordinary weather conditions. Experience has shown that concrete exposed to the elements should have a water content of not more than 7½ gal. per sack of cement.

A concrete must have a certain degree of workability; but, as for mortars, no method has yet been developed which is an accurate measure of workability. In the case of a concrete, this quality is influenced by the nature and amount of the coarse aggregate as well as by the amounts of cement and water and by the amount and nature of the fine aggregate. A method of measuring the workability, or more properly the consistency, is the *slump test.* A conical shell of 16-gage galvanized metal with a base 8 in. in diameter, a top 4 in. in diameter, and an altitude of 12 in. is filled to overflowing with concrete rodded into the shell in three separate layers, each receiving 25 strokes of a ⅝-in. bullet-end rod 24 in. in length. The excess is carefully struck off, and the mold is at once lifted slowly vertically. The amount of drop of the top of the mass below the original 12-in. height, measured in inches, is known as the slump. Mass concrete and highway mixtures are workable with a slump of 1 to 3 in. Concretes for reinforced beams and columns require a greater degree of workability, with slumps of 4 to 6 in.

30. Design of Concrete Mixtures. In the actual selection of the proportions that are to go into a given mix, the first step must be the selection of the water-cement ratio which will produce the strength required. In making this selection this

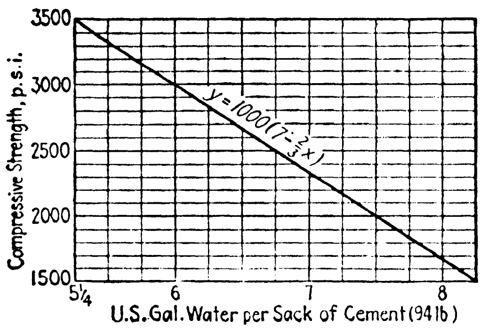


FIG. 22.

ratio must in no case exceed 7½ gal. per sack of cement, and if the concrete is to be subjected to severe weather conditions a decrease to 6 ot 6½ gal. will be worth-while in insuring a durable concrete.

Abram's curve shown in Fig. 21 presupposes ideal conditions of mixing, placing, and curing, which usually are not met on the average job. Therefore, the selection of the water-cement ratio for the desired strength should be taken from Fig. 22, which is a graphical representation of the specification for strength of the American Concrete Institute (1928).

Since part of the water will usually be contained in the aggregates, this amount must be determined. A portion of the moisture in the aggregates is in the interior of the particles. Such moisture, if it remains in the interior until after the cement has set, does not affect the water-cement ratio. If a bone-dry aggregate is used, a certain amount of water will be absorbed by the aggregates and the water-cement ratio will be changed. The average aggregate will absorb about 1 per cent of water by weight, traprock and granite about one-half that amount, and a very porous aggregate may absorb several per cent by weight. It may then be desirable to make up several trial batches, with variable quantities of fine and coarse aggregate, until the desired degree of workability is obtained. From the record of these trial batches, the amount of water, fine aggregate, and coarse aggregate for each batch may be selected.

From the above paragraph it is obvious that an arbitrary specification of the quantities of cement, sand, and coarse aggregate is not desirable. It is far better to specify the strength desired, for some types of work the maximum size of aggregate which may be used, and possibly a limiting ratio between the amounts of fine and coarse aggregate. The following example shows the procedure under such a specification.

31. Example of the Design of a Concrete Mixture. Certain aggregates have been approved for a specific job. They have the following characteristics: The sand has 35 per cent voids and weighs 104.4 lb. per cu. ft. dry and rodded. One cubic foot measured in its natural moist condition weighs 88.1 lb., and this same quantity when thoroughly dried weighs 84.4 lb. The percentage of moisture by weight is therefore $(88.1 - 84.4) \div 84.4 = 4.4$ per cent, and the bulking factor is $104.4 \div 84.4 = 1.24$. The coarse aggregate has 33 per cent voids and weighs 107.5 lb. per cu. ft. dry and rodded. One cubic foot measured in its natural moist condition weighs 105.0 lb., and this same quantity when thoroughly dried weighs 103.3 lb. The percentage of moisture by weight is therefore $(105.0 - 103.3) \div 103.3 = 1.6$ per cent, and the bulking factor is $107.5 \div 103.3 = 1.04$.

The specified strength of the concrete is 2000 p.s.i. at 28 days. From Fig. 22, 7½ gal. of water are to be used for each sack of cement. Trial batches using 5 lb. of cement and 3.3 lb. of water (which maintains the desired water-cement ratio) are made up using various proportions of surface-dry aggregates. The desired degree of workability, as measured by the slump, is obtained with 12.1 lb. of sand and 21.4 lb. of coarse aggregate.

For a 1-bag batch with the surface-dry aggregates the following quantities must be used:

$$\begin{aligned}
&\text{Cement} = 1 \text{ sack} && = 94.0 \text{ lb.} \\
&\text{Sand} = {}^{94}\!/_{5} \times 12.1 && = 227.5 \text{ lb.} \\
&\text{Coarse aggregate} = {}^{94}\!/_{5} \times 21.4 && = 402.3 \text{ lb.} \\
&\text{Water} && = 7.5 \text{ gal.}
\end{aligned}$$

However, the aggregates are to be used in the field in their natural condition as described above. It will be assumed that each of the aggregates naturally absorbs 1.0 per cent of moisture by weight. The amount of sand must then be increased by 4.4 − 1.0 = 3.4 per cent to 235.2 lb. The free water in the sand is 7.7 lb. = 0.92 gal. Likewise, the amount of coarse aggregate must be increased by 1.6 − 1.0 = 0.6 per cent to 404.7 lb., and the free water in the coarse aggregate is 2.4 lb. = 0.29 gal.

The field mix by weight using the natural aggregates is then:

Cement = 1 sack	= 94.0 lb.
Sand	= 235.2 lb.
Coarse aggregate	= 404.7 lb.
Water = 7.5 − 0.92 − 0.29	= 6.3 gal.

The field mix by volume is:

Cement = 1 sack	= 1.00 cu. ft.
Sand = 235.2 ÷ 88.1	= 2.67 cu. ft.
Coarse aggregate = 404.7 ÷ 105.0	= 3.85 cu. ft.

The yield for a 1-bag batch is:

Cement = 1 cu. ft. (1 − 0.513)	= 0.49 cu. ft.
Sand = 2.67 cu. ft.* 2.67(1 − 0.35) ÷ 1.24	= 1.40 cu. ft.
Coarse aggregate = 3.85 cu. ft. 3.85(1 − 0.33) ÷ 1.04	= 2.48 cu. ft.
Water = 7.5 gal.	= 1.00 cu. ft.
	5.37 cu. ft.

The yield from a 1-bag batch being known, it is possible to compute the quantities required for a given volume of concrete. As in mortars, 2 to 4 per cent should be added on account of the actual loss, after placing, of a portion of the water.

32. Mixing, Placing, and Curing. Practically all concrete is machine-mixed. The ordinary mixer consists of a rotating drum, in the interior of which are blades so placed that, as the drum rotates, they lift the ingredients, which in turn slide off the blades and drop to the mass then at the bottom of the drum. The time of mixing depends upon the character of the mixture and the speed of the mixer, but in general little is to be gained by continuing the operation beyond 2 min. The Joint Committee

* With the water within the particles assumed as 1 per cent, a given number of sand grains will occupy the same space whether this amount of moisture is actually present or not.

recommends that "the mixing of each batch shall continue not less than 1 min. after all the materials are in the mixer, during which time the mixer shall rotate at a peripheral speed of about 200 ft. per min."

The filling of the forms should be continuous where possible in order to prevent the formation of laitance or "day's work" planes. Laitance is a whitish substance consisting of the finest particles of the cement together with some of the silt and clay from the aggregates. It is brought to the surface of freshly mixed concrete where excess water is used (as it usually is in reinforced concrete). As it hardens very slowly and never acquires much strength, it constitutes a plane of weakness. Where such continuous deposition is impossible, the laitance should be scraped off and the surface of the old concrete roughened and wetted before placing is resumed.

Where forms have considerable height, with reinforcement continuous in either a vertical or a horizontal direction over the full height, some means should be provided of depositing the concrete without dropping it through too great a distance. In addition to the separation that is bound to take place, both forms and reinforcement become coated with hardened concrete long before they are completely filled, which may cause planes of weakness in the top of the structure. Properly constructed enclosed chutes or pipes extending down into the forms will insure against these weakness planes and also produce a satisfactory surface finish.

Spading, or puddling, of the concrete is necessary to cause it to spread laterally after it has been deposited in the forms. This puddling should be done in such a manner as to force the coarse aggregate against the forms. In a properly designed mix, sufficient mortar will follow and encase the coarse aggregate to insure a good surface finish; but if all the coarse aggregate is forced into the interior, away from the forms, the surface is less durable.

Vibrators have been used in the past few years to aid in the flow of the concrete in the forms. They are operated either by compressed air or by electricity. They are used both internally and against the outside of the forms. Since they allow the use of a very sluggish mixture, with its consequent lower water-cement ratio, higher strength concretes may be expected.

The principal variations in curing conditions which affect the process of hardening and the strength of the concrete are variations in moisture and temperature conditions. Though it is important that the amount of water used in mixing be controlled so that the consistency is as nearly normal as practical, it is just as important that the concrete be not allowed to dry out immediately, if the maximum strength obtainable is to be attained. All concrete should be protected against premature drying out for at least 1 week, and for a longer time if the temperature is near the freezing point. This may be done by sprinkling with water at intervals or by covering with damp or wet burlap. In road construction, water may be held over the entire surface by damming the edges with loose earth and forming a series of ponds. The importance of keeping the concrete moist while hardening cannot be too strongly emphasized. Tests show that a concrete allowed to dry out immediately will usually reach a strength of not more than 50 per cent of the strength of similar concrete kept moist over the entire period of curing.

The relation between the mean temperature during the curing period and the strength of concrete is illustrated in Fig. 23. The tests from which the curves were plotted covered a wide range of temperature conditions, and the results were fairly consistent. A knowledge of the effect of the mean temperature upon the strength is very necessary in determining the time when forms may be removed and loads applied, and a careful study of Fig. 23 will furnish the necessary information for determining the relative length of time the forms should be kept in place under different temperature conditions.

By combining high temperatures with a saturated condition of the atmosphere, it would follow that accelerated hardening of the concrete would be obtained. These conditions are brought into being by the application of live steam to concrete while hardening. This method is especially useful in the manufacture of concrete blocks, tile, small pipe, etc., where the saving in forms, storing space, and time is important. By placing the concrete products in a confined space and applying the steam under pressure, a still more rapid increase in strength will be attained. The steam should not be applied until after the concrete has obtained an initial set. Results of tests show that up to 80 p.s.i. gage pressure, steam has an accelerating action on the hardening of the concrete and that the compressive

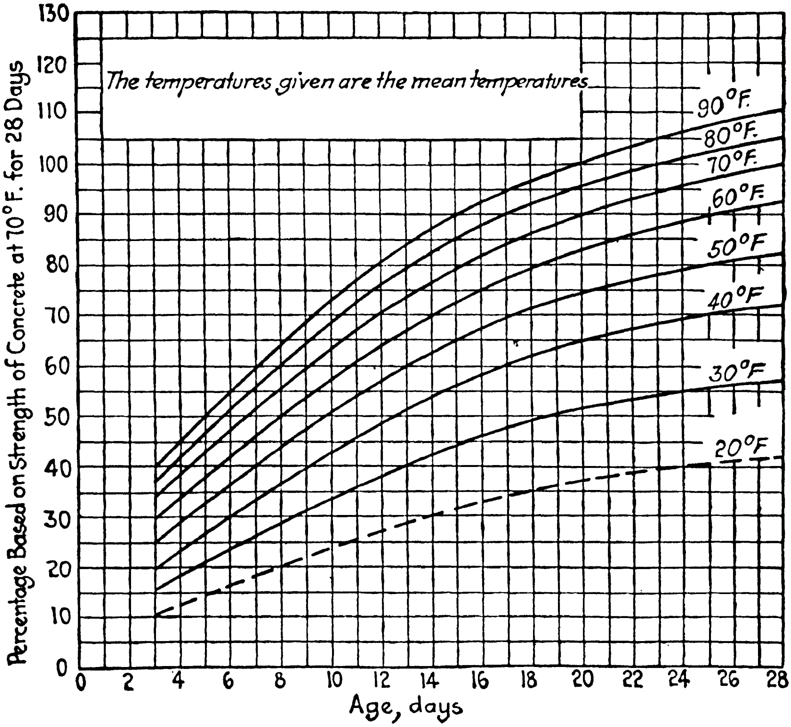


FIG. 23.

strength increases with the pressure and time of exposure. The application of the steam, too, permanently accelerates the hardening after the exposure to steam ceases. Concrete so treated has reached a compressive strength in 2 days (exposure to steam under pressure 24 hours) greater in some cases by 100 per cent than unsteamed concrete has reached in 28 days.

33. Freezing. The effect of low temperatures in delaying the hardening of concrete is shown in Fig. 23. When water reaches a temperature of 39 deg. Fahrenheit, some subtle change occurs which decreases its chemical ability to combine. This change becomes more marked as the freezing point is approached, and concretes placed with the temperature near the freezing point take several times as long to obtain a final set as concretes cured at normal temperatures. In case of dry atmospheric conditions much of the water may evaporate before the final set takes place, and insufficient water be left to combine chemically with the cement. In case the temperature falls below the freezing point before final set, the expansion of the water while freezing exerts a force sufficient to destroy the cohesion between the particles of the green concrete.

The injurious effect of freezing is lessened by two factors, namely, concrete is a very poor conductor of heat, and the chemical action of setting and hardening generates a certain amount of heat to combat the freezing action of the atmospheric conditions. Thus the serious injury is usually confined to the surface of the concrete and rarely penetrates more than an inch or two in depth. In massive members this may not seriously impair the strength but may be harmful only to the appearance. In the smaller members, however, a large percentage of the strength may be lost.

The best method to prevent the freezing of concrete is the heating of the materials and keeping the green concrete at a temperature several degrees above freezing for several days.

"In freezing weather suitable means shall be provided for maintaining a temperature of at least 50 deg. Fahrenheit for not less than 72 hours after placing, or until the concrete has thoroughly hardened." The heating of aggregates on large jobs can be accomplished by placing perforated pipes under the storage piles and forcing steam through the pipes. On small jobs, wood fires built in sheet-iron pipes buried in storage piles prove satisfactory.

34. Placing under Water. When concrete is placed under water, it is essential that the water be still and not flowing and that the concrete be dumped as a batch in its final position so as to avoid the segregation that would take place if it were allowed to fall freely through several feet of water. For underwater placement, drop-bottom buckets or, preferably, the tremie method may be used. A *tremie* consists of a vertical pipe, with the lower end resting in the freshly deposited concrete, and with a hopper at the upper end which is above the water. The tremie is kept at least partly filled with fresh concrete during the entire placing operation. From time to time the tremie is lifted slightly to allow part of the concrete in the pipe to flow out of the bottom. A stiff but thoroughly plastic mix should be used.

35. Compressive Strength. The ultimate strength of a concrete normally increases with age. This increase proceeds very rapidly for the first few days after the concrete is placed but becomes more gradual as time goes on, though continuing at a more reduced rate for an indefinite period. The compressive strength of concrete at the age of 28 days is generally used as a measure of the quality of the concrete. This is based on the assumption of proper mixing and placing and suitable curing conditions. The compressive strength of the concrete is based on tests of 6- by 12-in. or 8- by 16-in. cylinders made in accordance with the Standard Methods of Making and Storing Specimens of Concrete in the Field, of the American Society for Testing Materials,[1] and tested in a well-equipped laboratory by a competent operator.

The ultimate compressive strength expressed in pounds per square inch is used as a basis for determining the unit stresses to be used in design, for it has been found that practically all the other structural properties of a concrete are proportional to the compressive strength.

36. Tensile Strength. The tensile strength of concrete is a property of little importance, for it is so low in comparison with the compressive strength that it is usually neglected altogether in the design of reinforced-concrete structures. It may roughly be estimated as having a value of about 10 per cent of the compressive strength.

[1] A.S.T.M. Standard C31-39.

37. Transverse Strength. The transverse, or flexural, strength of concrete is low as compared with its compressive strength but much greater than the strength in pure tension. The transverse strength is measured by the stress developed in beam action. In a reinforced-concrete member this strength is usually disregarded, and steel reinforcement is placed in the member to develop the flexural stresses on the tension side. Load tests, however, on reinforced-concrete structures have shown that the transverse strength of the concrete contributes to a marked degree in increasing the capacity of the structure. Tests made by Duff A. Abrams indicate that, at 28 days, the transverse strength varies from 26 per cent of the compressive strength for a 1000 p.s.i. concrete to 15 per cent for a 4000 p.s.i. concrete.

38. Shearing Strength. The shearing strength of concrete is important in that failure by shear on a diagonal plane often occurs in short compression specimens. The direct shear must not be confused with the combination of shear and diagonal tension that occurs in the web of a beam. The resistance of concrete to direct shear is difficult to determine, for it is almost impossible to eliminate the effect of bearing, diagonal tension, and other stresses, so that different series of tests show a considerable variation. For most concrete the shearing strength is at least 60 per cent of the compressive strength and will need to be considered in design only in exceptional cases.

39. Elasticity. Concrete is not a perfectly elastic material, there being a slight decrease in the ratio of stress to strain as the stress increases. Concrete also shows a permanent set under the smallest loads, but within working limits there is a fairly constant relation between temporary stress and strain, which may be considered as the modulus of elasticity of the concrete. The value of the modulus of elasticity of concrete (in pounds per square inch) for use in the design of reinforced-concrete members is generally taken as $1000 f'_c$, where f'_c is the ultimate compressive strength determined as described in Art. 35. The modulus of elasticity of reinforcing steel is taken as 30,000,000 p.s.i.

40. Plastic Flow. Tests have shown that all concrete under a sustained load continues to deform or "plastically flow" for a long period of time. This deformation is independent of that due to shrinkage caused by the decrease in moisture content.

The effect of this plastic flow in a reinforced-concrete column is, under sustained load, to gradually relieve the compressive stress in the concrete until the yield point of the steel is reached. Tests of reinforced-concrete columns, sponsored by the American Concrete Institute, made at Lehigh University and at the University of Illinois, have shown increases in the steel stress of two to four times that obtained under the initial loading.

In beams and slabs the effect of the shrinkage and plastic flow is to move the neutral axis toward the tensile reinforcement, thus decreasing the stress in the concrete and increasing that in the steel.

In all cases the removal of the load causes some recovery or backward flow, which often continues for an appreciable period, but such values are always less than the deformations under the sustained load.

41. Contraction and Expansion. Concretes expand as the temperature is raised and contract as the temperature is lowered. The coefficient of expansion per degree of temperature change increases somewhat with the richness of the mix, but the range of values is small. Tests made in the laboratories of Cornell University gave a range of values of from 0.00000677 for a 1:1½:3 concrete to 0.00000537 for a 1:3:6 concrete, with an average for all tests of 0.00000604. Other tests have shown a close agreement. The value generally used is 0.000006 per degree Fahrenheit.

Concretes expand in volume if kept wet or immersed in water and contract if exposed to air. This property is not confined to freshly placed concrete but is characteristic of concretes of many years' service. A concrete which dries out in air may be expected to contract 0.02 to 0.05 per cent and when immersed in water may expand at least one-half of this amount.

This tendency to change in volume with different moisture conditions and changes in temperature does, of course, set up stresses of both tension and compression in a restrained reinforced-concrete structure. The tensile stresses often exceed the amount that the concrete can sustain, and cracks result.

42. Weight. The weight of a concrete varies somewhat with the proportions of the mix, the consistency, and the character of the aggregate. The richer concretes are slightly heavier, and the wetter consistencies are lighter, except when cinders are used as the coarse aggregate. A stone or gravel concrete will usually weigh between 140 and 150 lb. per cu. ft., with an average of about 145 lb. per cu. ft. In reinforced concrete the steel adds 3 to 5 lb. per cu. ft., and the weight of the reinforced concrete (including the steel) is usually taken as 150 lb. per cu. ft. The weight of cinder concrete may be taken as 115 lb. per cu. ft.

Lightweight concrete (averaging about 100 lb. per cu. ft.) is sometimes made from artificial aggregates made from burned clay, which are known commercially as Haydite, Gravelite, etc. Still lighter weight concretes (as low as 50 lb. per cu. ft.) are made by adding certain chemicals to ordinary concrete, which generate gases and cause the concrete to fluff up. The process is known as aerating.

43. Working Unit Stresses. The following tables give the unit stresses for design recommended by the Standard Building Code Committee of the American Concrete Institute. f'_c is the value obtained for ultimate compressive strength as described in Art. 35. n is the ratio $E_s \div E_c$ (see Art. 39).

ALLOWABLE UNIT STRESSES IN REINFORCEMENT

a. Tension:

(f_s = tensile unit stress in longitudinal reinforcement)

(f_v = tensile unit stress in web reinforcement)

20,000 p.s.i. for rail-steel concrete reinforcement bars, billet-steel concrete reinforcement bars (of intermediate and hard grades), axle-steel concrete reinforcement bars (of intermediate and hard grades), and cold-drawn steel wire for concrete reinforcement.

18,000 p.s.i. for billet-steel concrete reinforcement bars (of structural grade), and axle-steel concrete reinforcement bars (of structural grade).

b. Compression, vertical column reinforcement:

(f_s = nominal working stress in vertical column reinforcement)

20,000 p.s.i. for rail or hard-grade steel.

16,000 p.s.i. for intermediate-grade steel.

WORKING UNIT STRESSES FOR REINFORCED CONCRETE MEMBERS

Description		Allowable unit stresses, p.s.i.				
		For any strength of concrete as fixed by test $n = \frac{30,000}{f_c}$	$f'_c =$ 2000 p.s.i. $n = 15$	$f'_c =$ 2500 p.s.i. $n = 12$	$f'_c =$ 3000 p.s.i. $n = 10$	$f'_c =$ 3750 p.s.i. $n = 8$
Flexure: f_c						
Extreme fiber stress in compression	f_c	$0.40f'_c$ †	800	1000	1200	1500
Extreme fiber stress in compression adjacent to supports of continuous or fixed beams or of rigid frames	f_c	$0.45f'_c$	900	1125	1350	1688
Shear: v						
Beams with no web reinforcement and without special anchorage of longitudinal steel	v_c	$0.02f'_c$	40	50	60	75
Beams with no web reinforcement but with special anchorage of longitudinal steel	v_c	$0.03f'_c$	60	75	90	113
Beams with properly designed web reinforcement but without special anchorage of longitudinal steel	v	$0.06f'_c$	120	150	180	225
Beams with properly designed web reinforcement and with special anchorage of longitudinal steel	v	$0.12f'_c$	240	300	360	450
Flat slabs at distance d from edge of column capital or drop panel	v_c	$0.03f'_c$	60	75	90	113
Footings	v_c		60	75	75	75
Bond: u						
In beams and slabs and one-way footings:*						
Plain bars	u	$0.04f'_c$	80	100	120	150
Deformed bars	u	$0.05f'_c$	100	125	150	188
In two-way footings:						
Plain bars	u	$0.045f'_c$	90	113	135	169
Deformed bars	u	$0.056f'_c$	112	140	168	210
Bearing: f_c						
On full area	f_c	$0.25f'_c$	500	625	750	938
On one-third area	f_c	$0.375f'_c$	750	938	1125	1405

* Where special anchorage is provided, 1½ times these values in bond may be used in beams, slabs, and one-way footings. The values given for two-way footings include an allowance for special anchorage.

† The revised Joint Code (1941) specifies $0.45f'_c$ instead of $0.40f'_c$. Since this revision was adopted subsequent to the first printing of this book, problems in Chapter X are based on $f_c = 0.40f'_c$.

CHAPTER III

LOADS ON STRUCTURES

44. Types of Loads. Loads on structures may be divided into two general classes, *dead load* and *live load*.

The dead load is the weight of the structure itself and the weight of any permanent fixtures which are carried by the structure. It is a fixed load which remains in position during the life of the structure and acts in a vertical direction.

The live load includes all other loads or forces acting on the structure. In a building it includes people, furniture, machinery, or any other movable object which may or may not be present at any particular time. On a bridge it includes the trains, trucks, cars, wagons, or people which move across the bridge. It also includes snow loads, wind loads, impact forces, and other special loads, although these are often referred to under separate classifications.

45. Dead Loads. The dead-load weights of single members may be assumed, the design made for this load and the specified live loads, and the true weight computed, after which the original assumption may be revised if necessary. This is probably the simplest method of designing beams and columns and is to be preferred over a more complicated solution which includes in the original design equation a term having the unknown size of the beam or column multiplied by the unit weight of the material.

For structures composed of several members such as roof trusses, bridge trusses, plate girders, and arches, the labor involved in arriving at the selection of the size and weight of each member makes it desirable that the original assumptions of dead-load weight be sufficiently close to the final weight so that no revision is necessary. Several empirical formulas have been developed by computing the weights of previously designed trusses of different span and carrying different classes of loading, and adapting a formula to the results.

The weight of a roof truss will vary with the span and rise of the truss, the distance between trusses, the material used in the truss, the type of roof covering, and the geographical location of the structure as affecting the snow and wind pressures, as well as other obvious design factors. No practical formula can be obtained which includes all these variables. The span and spacing of trusses cause the greatest variation in weight; and since the dead weight of the truss usually forms but a small part of the total design load, the error involved in neglecting the other factors is not great. Either of the following formulas gives sufficiently close results for the design of either timber or steel trusses:

$$w = \frac{l}{25} + \frac{l^2}{6000}$$

or

$$w = K(1 + 0.1l)$$

in which w = weight of truss per square foot of horizontal covered surface supported.

l = span of truss, in feet.

K = a constant depending upon the material of which the truss is constructed: 0.5 for timber trusses and 0.75 for steel trusses.

The dead load supported by a bridge truss consists of the weight of the trusses and their bracing together with the weight of the floor system. The latter can be designed and the weight computed before the design of the trusses. The weight of the trusses and their bracing varies with the length of span, type of truss, panel-length, character of details, load carried, type of floor, etc. The principal factors are the span and the class of live load. For approximate weights of steel highway bridges the following formula is suggested:

$$w = \frac{w_1}{g} + l$$

in which w = weight per foot of one truss and one-half of the bracing.

w_1 = total superimposed load per foot, in pounds, brought to each truss (weight of floor, live load, and impact).

l = span, in feet.

For single-track steel railroad bridges, the weight per foot of one truss and one-half the bracing *and* floor system may be taken as

$$w = 2(2l + 5E)$$

in which l = span, in feet.

E = the number representing the class of Cooper's live loading to be carried (see Art. 48).

The use of a solid floor with ballast materially increases the weight of the floor and adds somewhat to the weight of the trusses over that determined by the preceding formula. The weight of a double-track bridge is 80 to 90 per cent greater than that of a corresponding single-track structure.

46. Live Loads for Buildings. The minimum live loads for which the floors and the roof of any building must be designed are always specified in the building code that governs the site of the construction.

The range of minimum live-load values in pounds per square foot of floor or roof area, as given in several typical building codes, is as follows:

Apartments	40
Auditorium and theaters:	
With fixed seats	50–80
Without fixed seats	100
Dwellings	40
Hospitals	40
Hotels:	
Rooms	40
Corridors, lobbies, dining rooms	100
Manufacturing buildings:	
Light manufacturing	75–125
Heavy manufacturing	125–200
Office buildings:	
Office space	50–60
Corridors, public spaces	100–125
Roofs, flat	30–40
School buildings:	
Classrooms	50–60
Corridors, public spaces	100
Garages:	
All types of vehicles	100–175
Passenger cars only	75–125
Store buildings:	
Retail	75–125
Wholesale	100–125
Warehouses:	
Light storage	75–150
Heavy storage	200–250

The specified minimum live loads cannot always be used. The type of occupancy should be considered, and the probable loads should be computed as accurately as possible. Warehouses for heavy storage may be designed for loads as high as 500 lb. or more per sq. ft.; unusually heavy operations in manufacturing buildings may require a large increase in the 200-lb. maximum specified above; and special provision must be made for all definitely located heavy concentrated loads.

Some building codes provide for occasional probable concentrated loads, such as the weight of a heavy safe, by requiring that in addition to being adequate to support the specified minimum uniform live load, the floor system shall also be capable of supporting a single concentrated load of 2000 lb. (or more) on an area $2\frac{1}{2}$ ft. square when this load is placed in any position on the floor. The concentrated load and the uniform load are, however, not assumed to act simultaneously.

47. Live Loads for Highway Bridges. *Truck Loadings.* For loaded lengths up to 60 ft. the live load specified by the American Association of State Highway Officials consists of a series of trucks as shown in Fig. 24. The wheel spacing, weight distribution, and amount of clearance required for the individual trucks are shown in the figure. A width of 9 ft. is required for each line of trucks; this width is called the lane width.

The highway live loads of the above specifications are divided into three classes, $H20$, $H15$, and $H10$. The number of the loading indicates the gross weight in tons of the heaviest truck in the series, and each other truck of the series has a gross weight of three-quarters of that amount. The gross weight of each truck is divided between the front and rear axles in the proportions shown.

Long-span Loadings. For loaded lengths of 60 ft. or more a uniform live load plus a concentrated load as shown in Fig. 25 is used.

Electric-railway Loadings. For bridges carrying electric-railway traffic the loading is determined on the basis of the class of traffic which the bridge may be expected to carry. Such a loading, for design purposes, consists of a train of two cars, followed by and/or preceded by a uniform load as shown in Fig. 26.

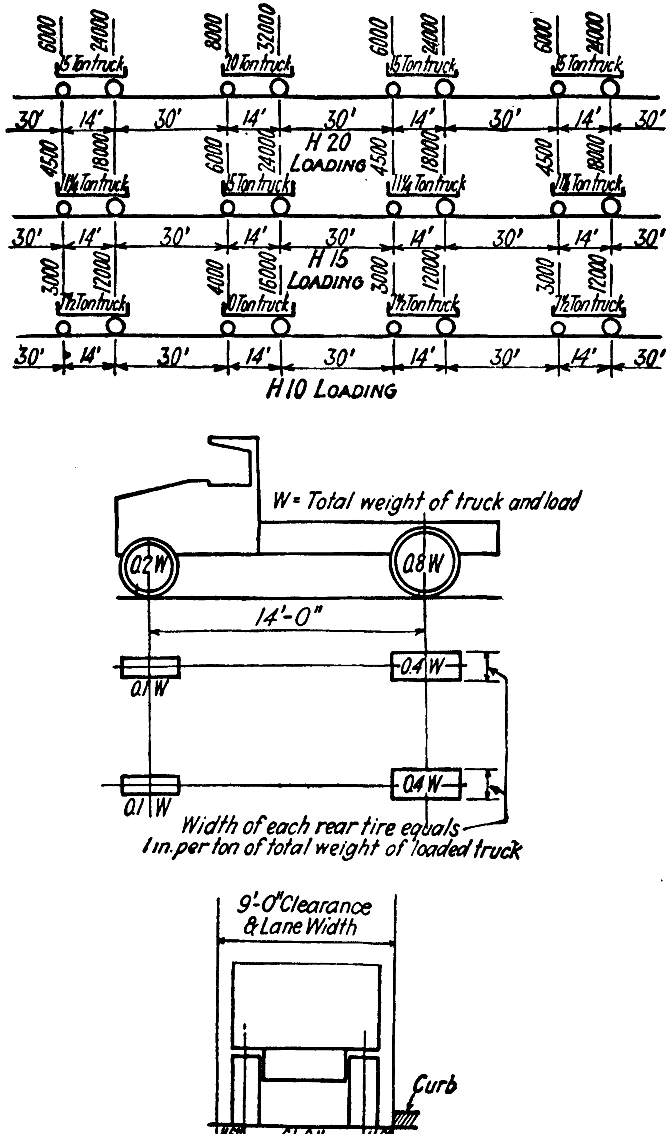


FIG. 24.—Standard truck-train loadings.

Sidewalk Loadings. Sidewalk floors, stringers, and their immediate supports are usually designed for a live load of 100 lb. per sq. ft.

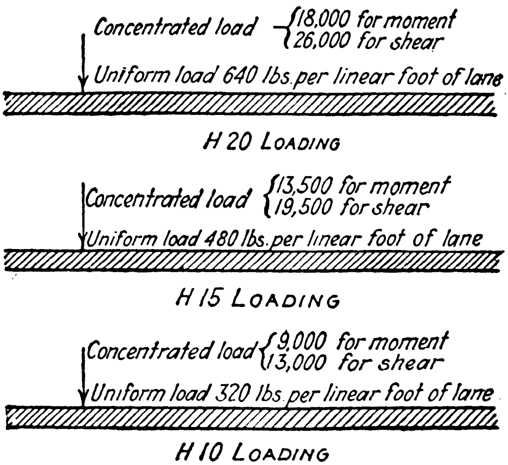


FIG. 25.—Equivalent highway loadings.

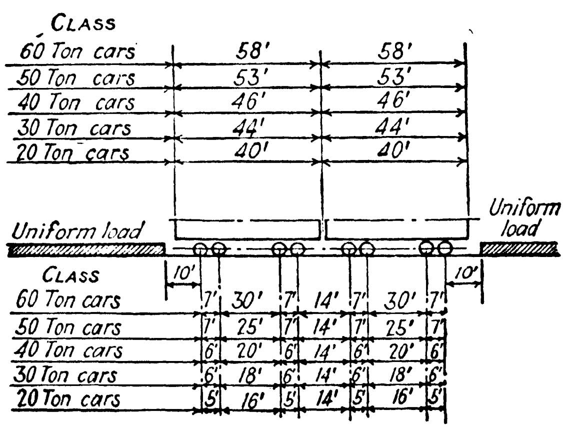


FIG. 26.—Electric railway loadings.

Selection of Loadings. Bridges of the different classes are designed for loadings as follows:

Class of Bridge	Loading
AA	*H*20
A	*H*15
B	*H*10

48. Live Loads for Railroad Bridges. For railroad bridges the live load consists of the moving locomotives and the train and acts as a system of concentrated rolling loads. The maximum load usually specified consists of two locomotives coupled together followed by a uniform load representing the train load. A great variety of engine loadings has been used by the various railroad companies, each road, as a rule, selecting a loading based on the weights of the heaviest locomotive in service or to be anticipated during the life of the structures under consideration. With such great divergence in specifications the calculation of stresses by exact methods produced variable and questionable results. This resulted in the proposal of several typical compromise loadings in which the wheel weights and spacings were modified to secure greater simplicity. The

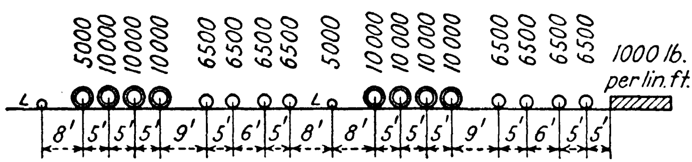


FIG. 27

system of loads proposed by Theodore Cooper in 1894, consisting of two locomotives followed by a uniform train load, is the only one which has come into any extensive use. This system is now more universally used by the railroads of this country than any other type of load.

In Cooper's loadings, the total load on one driving-wheel axle corresponds with the designation of the class of loading. In Fig. 27 each driving-wheel axle carries a load of 10,000 lb., or 10 kips, and hence this loading is known as Cooper's E-10 loading. Any other class of Cooper's loading may be obtained directly from this one; for all axle loads bear the same relation to the weights on the driving-wheel axles for all classes of loads, and the axle spacing is the same for each class. For example, each load in the E-50 class is equal to $50/10$ times the corresponding axle load in the E-10 class. At the present time most specifications which include Cooper's loadings are specifying classes from E-65 to E-75.

In order to reduce the numerical work involved in the computations of stresses due to a specified system of concentrated

MOMENT TABLE FOR COOPER'S E-10 LOADING

LOADS: 2.5 | 5 | 5 | 5 | 5 | 3.25 | 3.25 | 3.25 | 3.25 | 2.5 | 5 | 5 | 5 | 5 | 3.25 | 3.25 | 3.25 | 3.25 | UNIFORM LOAD 500 LB. PER FT.

SPACING: 8 – 5 – 5 – 5 – 9 – 5 – 6 – 5 – 8 – 8 – 5 – 5 – 5 – 9 – 5 – 6 – 5 – 5

WHEEL No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	
DISTANCES	0	8	13	18	23	32	37	43	48	56	64	69	74	79	88	93	99	104	109
LOADS (1)	2.5	7.5	12.5	17.5	22.5	25.75	29.0	32.25	35.5	38.0	43.0	48.0	53.0	58.0	61.25	64.50	67.75	71.0	71.0
MOMENTS	0	20	58	120	208	410	539	713	874	1158	1432	1677	1917	2182	2704	3010	3397	3736	4091
DISTANCES		0	5	10	15	24	29	35	40	48	56	61	66	71	80	85	91	96	101
LOADS (2)		5.0	10.0	15.0	20.0	23.25	26.5	29.75	33.0	35.5	40.5	45.5	50.5	55.5	58.75	62.0	65.25	68.5	68.5
MOMENTS		0	25	75	150	330	446	605	764	1018	1302	1505	1732	1985	2484	2778	3160	3459	3819
DISTANCES			0	5	10	19	24	30	35	43	51	56	61	66	75	80	86	91	96
LOADS (3)			5.0	10.0	15.0	18.25	21.50	24.75	28.0	30.5	35.5	40.5	45.5	50.5	53.75	57.0	60.25	63.5	63.5
MOMENTS			0	25	75	210	301	430	554	778	1022	1200	1402	1630	2084	2353	2695	2996	3314
DISTANCES				0	5	14	19	25	30	38	46	51	56	61	70	75	81	86	91
LOADS (4)				5.0	10.0	13.25	16.50	19.75	23.0	25.5	30.5	35.5	40.5	45.5	48.75	52.0	55.25	58.5	58.5
MOMENTS				0	25	115	182	240	329	563	767	920	1097	1300	1709	1953	2265	2541	2834
DISTANCES					0	9	14	20	25	33	41	46	51	56	65	70	76	81	86
LOADS (5)					5.0	8.25	11.5	14.75	18.0	20.5	25.5	30.5	35.5	40.5	43.75	47.0	50.25	53.5	53.5
MOMENTS					0	45	86	155	229	373	537	665	817	995	1359	1578	1860	2111	2329
DISTANCES						0	5	11	16	24	32	37	42	47	56	61	67	72	77
LOADS (6)						3.25	6.5	9.75	13.0	15.5	20.5	25.5	30.5	35.5	38.75	42.0	45.25	48.5	48.5
MOMENTS						0	16	55	104	208	332	435	562	715	1034	1228	1460	1706	1949
DISTANCES							0	6	11	19	27	32	37	42	51	56	62	67	72
LOADS (7)							3.25	6.5	9.75	12.25	17.25	22.25	27.25	32.25	35.5	38.75	42.0	45.25	45.25
MOMENTS							0	20	52	130	228	314	426	562	852	1030	1262	1472	1698
DISTANCES								0	5	13	21	26	31	36	45	50	56	61	66
LOADS (8)								3.25	6.5	9.0	14.0	19.0	24.0	29.0	32.25	35.5	38.75	42.0	42.0
MOMENTS								0	16	68	140	210	305	425	686	848	1061	1254	1464
DISTANCES									0	8	16	21	26	31	40	45	51	56	61
LOADS (9)									3.25	5.75	10.75	15.75	20.75	25.75	29.0	32.25	35.5	38.75	38.75
MOMENTS									0	26	72	126	205	308	540	685	879	1056	1250
DISTANCES										0	8	13	18	23	32	37	43	48	53
LOADS (10)										2.5	7.5	12.5	17.5	22.5	25.75	29.0	32.25	35.5	35.5
MOMENTS										0	20	58	120	208	410	539	713	874	1052

TWO 35.5 TON ENGINES + 1000 LB. PER FT. MOMENTS IN THOUSAND FOOT POUNDS FOR ONE RAIL. LOADS IN THOUSANDS OF POUNDS FOR ONE RAIL.

loads, it is desirable to prepare a table or diagram giving weights, distances, and moments. Many forms for these tables have been devised by different designers to suit their respective needs. Such a table for Cooper's E-10 loading is given on page 57. In that table the first three horizontal lines give, respectively, the amount of each wheel load, the spacing between wheels, and the number designation of each wheel. The next three lines contain, respectively, the summations of distances, loads, and moments from wheel 1 to any given wheel. Thus, under wheel 5, the distance of wheel 5 from wheel 1 is given as 23 ($= 8 + 5 + 5 + 5$), the summation of wheel loads 1 to 5 as 22.5 ($= 2.5 + 5 + 5 + 5 + 5$), and the summation of the moments of all the wheel loads to left of the wheel 5 about wheel 5 as 208 [$= 2.5 \times 23 + 5(15 + 10 + 5 + 0)$]. Under certain conditions wheels may pass off the structure as the loads are moved to the left, so that the summations of distances, loads, and moments are given in the table with each wheel of the first locomotive successively omitted. For instance, under wheel 12 in the horizontal triple line 6, the distance from wheel 6 to wheel 12 is given as 37, the summation of wheel loads 6 to 12 as 25.5, and the summation of the moments of wheel loads 6 to 12 about wheel 12 as 435.

49. Impact. The live load on a bridge moves at a high velocity, and the track or roadway has considerable irregularity even under the most favorable conditions. On railroad bridges, unbalanced locomotive drivers and the vibration of the machinery add to the general effect. The increase in stress over that caused by the live load at rest is known as the impact stress.

Many investigations with varying results have been made to determine the amount of this stress. These stresses are usually computed by empirical formulas. For highway bridges the specifications of the American Association of State Highway Officials give the formula $I = \frac{50}{l + 125}$, where l is the loaded length in feet and I the impact coefficient, by which the live-load stress is multiplied to obtain the impact stress. For railroad bridges the American Railway Engineering Association specifies (*a*) a percentage of live-load stress equal to $100/S$, where S is the spacing in feet between centers of trusses, plus (*b*) a percentage of the live-load stress equal to $100 - 0.60l$ for spans less than 100 ft. and a percentage of $\frac{1800}{l - 40} + 10$ for spans greater than 100 ft., where l is the span of the truss in feet.

50. Lateral Forces on Bridge Trusses. A bridge structure is subjected to lateral forces due to wind pressure and the vibration caused by the impact of the moving loads.

The wind is regarded as blowing horizontally at right angles to the structure and exerting a pressure of about 50 lb. per sq. ft. on the unloaded bridge, the exposed area being taken as the exposed surface of the trusses and floor system, as seen in elevation. When the bridge is loaded, this pressure is reduced to 30 lb. per sq. ft. on the loaded structure and its load. The pressure acting on the live load is always considered as a moving load, but specifications differ as to whether the pressure on the structure itself is to be considered as a moving or a static load.

On account of the vibration caused by the impact of the moving load, it is usual to specify lateral forces considerably greater than would be obtained by the use of the unit pressures mentioned above in order to secure the desired rigidity.

For highway bridges, the American Association of State Highway Officials specifies that:

a. The wind force on the structure shall be assumed as a moving horizontal load equal to 30 lb. per sq. ft. on $1\frac{1}{2}$ times the area of the structure as seen in elevation, including the floor system and railings, and on one-half the area of all trusses or girders in excess of two in the span.

b. The lateral force due to the moving live load and the wind pressure against it shall be considered as acting 6 ft. above the roadway and shall be as follows:

Highway bridges, 200 lb. per lin. ft.

Highway bridges carrying electric railway traffic, 300 lb. per lin. ft.

c. The total assumed wind force shall be not less than 300 lb. per lin. ft. in the plane of the loaded chord and 150 lb. per lin. ft. in the plane of the unloaded chord on truss spans, and not less than 300 lb. per lin. ft. on girder spans.

d. A wind pressure of 50 lb. per sq. ft. on the unloaded structure, applied as specified in paragraph *a*, shall be used if it produces greater stresses than the combined wind and lateral forces of paragraphs *a* and *b*.

For railroad bridges, the American Railway Engineering Association specifies that:

a. The wind force on the structure shall be a moving load of 30 lb. per sq. ft. on 1½ times its vertical projection on a plane parallel with its axis, but not less than 200 lb. per lin. ft. at the loaded chord or flange and 150 lb. per lin. ft. at the unloaded chord or flange.

b. The wind force on the train shall be a moving load of 300 lb. per lin. ft. on one track, applied 8 ft. above the top of rail.

c. The lateral force to provide for the effect of the sway of the engines and train, in addition to the wind loads specified in paragraphs *a* and *b*, shall be a moving load of 20,000 lb. applied at the top of rail, in either horizontal direction at any point of the span.

When a train with its brakes set crosses a bridge, a horizontal force is exerted upon the track through the friction of the braked wheels. The amount of this force is usually taken as 20 per cent of the vertical live load. The stress produced in the chords by this force increases uniformly from the free to the fixed end of the bridge. The effect of this force on the structure is materially reduced by the continuity of the track over the bridge, and usually no stresses of any importance are produced in any members except the loaded chords, and these are often neglected in the design.

Where bridges occur on curves, the spacing between trusses must be made greater than where the track is straight, in order to provide sufficient clearance. The centrifugal force developed and the eccentricity of the track both produce additional stresses in the main members of the truss as well as in the members of the lateral system. Where the curvature is slight, these stresses may be neglected.

51. Snow and Wind Loads. The *snow loads* to be used in the design of roofs vary with the geographical location and altitude of the structure and with the slope of the roof. Where snow is likely to occur, a minimum value of 25 lb. per sq. ft. of horizontal covered surface should be used for all slopes up to 20 degrees; this load may be reduced 1 lb. for each degree of slope above 20 degrees. Snow loads are not considered in the design of bridges, for the snow must be cleared before the maximum live load can move across the structure.

Wind pressures vary with the geographical location and the slope of the roof. The wind is assumed to blow horizontally with a velocity sufficient to cause a pressure of 30 to 40 lb. per sq. ft. on a vertical surface, which represents a velocity of approximately 86 to 100 m.p.h. Since the friction of air on comparatively smooth surfaces is very slight, it may be assumed without appreciable error that only the normal effect of this pressure need be considered. The amount of normal pressure for varying slopes is given by Duchemin in the following formula:

$$P_N = P_H\left(\frac{2 \sin A}{1 + \sin^2 A}\right)$$

in which P_H = horizontal pressure per square foot on a vertical surface.

P_N = normal pressure per square foot of sloping surface.

A = angle of inclination of the sloping surface.

For a horizontal pressure of 30 lb. per sq. ft., the values of P_N for different slopes as computed from this formula are tabulated below:

Slope A, deg.	Normal pressure P_N, lb. per sq. ft.	Slope A, deg.	Normal pressure P_N, lb. per sq. ft.
5	5.19	35	25.90
10	10.11	40	27.29
15	14.55	45	28.28
20	18.37	50	28.97
25	21.51	55	29.41
30	24.00	60	29.69

For intermediate angles of inclination the normal pressures may be obtained by interpolation. For horizontal pressures other than 30 lb. per sq. ft., the values given above change in proportion.

CHAPTER IV

HOMOGENEOUS BEAMS

52. Types of Beams. A beam is a structural element subjected to transverse loads and reactions. A homogeneous beam is one that is made of one material such as steel, plain concrete, or timber. A reinforced-concrete beam, which is a combination of concrete and steel, is nonhomogeneous. Design theories and methods for reinforced-concrete beams are given in Chap. X.

Plain concrete is unsuited for use in beams because of its very low tension strength. Timber beams are usually square or rectangular in cross section. Steel beams are of two general types: (1) rolled sections, (2) built-up sections. The rolled steel shapes that are most commonly used for beams are: the standard I-beam, Fig. 28*a*; the wide-flange beam, Fig. 28*b*; the channel section, Fig. 28*c*. Built-up sections are combinations of plates, angles, and other rolled shapes, riveted or welded together in such a manner as to insure unity of action between the individual elements of the section. Beams of the latter type are known as plate girders, box girders, lattice girders, etc., depending upon the arrangement of the component elements. The theory and the details of the design of built-up beam sections are given in Chap. VIII.

(a) (b) (c)

Fig. 28.

Beams must be designed to give adequate strength in flexure and shear, and the material must be so arranged in the cross-section as to prevent buckling and excessive deflection.

53. Assumptions in the Theory of Flexure. The common theory of flexure assumes: (1) A plane cross-section before loading remains a plane cross-section after loading; (2) Stress is proportional to deformation. The first of these two assumptions implies that the unit deformations of the fibers in any short element of the beam are proportional to their distances from the neutral axis, or plane of zero deformation. The second assumption implies that the unit stresses in the fibers at any section vary as the distances of the fibers from the neutral axis.

54. Flexure Stresses in a Rectangular Beam. If the rectangular beam *ab* in Fig. 29 is cut in a vertical plane through any point *c* and the piece *ac* is separated from the piece *cb*, the piece *ac* can be kept in equilibrium provided the internal forces which

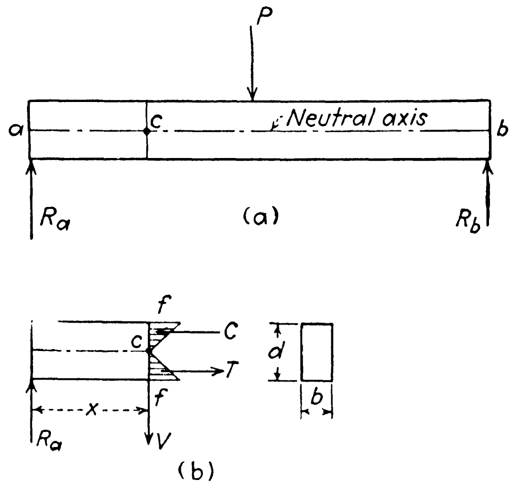


Fig. 29.

existed in the original beam at this section are replaced by similar external forces acting on the cut face. These original forces consist of horizontal fiber stresses of compression in the upper portion of the beam and tension in the lower portion, and a vertical shearing force V which, with the loading shown, is equal to the left reaction R_a.

According to assumption 1 in Art. 53 the horizontal fiber stresses vary uniformly from a maximum value f at the extreme fibers to zero at the neutral axis. The neutral axis of any homogeneous beam coincides with the gravity axis of the section. Since the assumed section is symmetrical about this gravity axis, the actual value of the unit stress on the extreme top fiber is equal to that on the extreme bottom fiber. This condition exists in any rectangular homogeneous beam whether it be of timber, steel, or concrete and also in any symmetrical rolled steel section. If the depth of the beam is d and the width is b, then the total compression C in the upper half of the beam is equal to the area of the triangle with base f and altitude $d/2$, multiplied by the constant width b, or $C = \frac{1}{2} f \frac{d}{2} b = f\left(\frac{1}{4} bd\right)$. The total tension T is likewise equal to $f(\frac{1}{4}bd)$. For purposes of computation C and T can be concentrated at the centers of gravity of the respective triangular wedges, so that the lever arm of the couple which is formed by these two equal and opposite forces is $2 \times \frac{2}{3}(d/2) = \frac{2}{3}d$. If in the free body shown in Fig. 29*b* the sum of the moments of all forces about point c is equated to zero, it follows that $R_a x = \frac{1}{4}fbd \times \frac{2}{3}d = \frac{1}{6}fbd^2$. But $R_a x$ is the external bending moment M at section c; hence the relation above can be written as follows:

$$M = f(\tfrac{1}{6}bd^2) \qquad (1)$$

The right side of equation (1) represents what is termed the internal moment of resistance at section c in the rectangular beam ab, or the moment that is exerted by the portion of the beam on the right of section c in resisting the applied load and reactions. Equation (1) states that the internal moment at any section of a beam is equal to the external moment at that section.

The physical significance of the above discussion can be demonstrated by considering that a fixed pin is placed at c in the piece ac, about which the piece is free to rotate. The external moment tends to rotate the piece ac in a clockwise direction about the imaginary pin c; and if this rotation is to be prevented, an equal moment in the opposite direction must be furnished. This moment is furnished by the resisting couple, consisting of the two equal and opposite forces C and T as described above. With this arrangement of the resisting forces, horizontal movement of the pin c is prevented ($\Sigma H = 0$). Vertical movement is resisted by the shearing force V which is necessary to satisfy the third requirement for equilibrium, namely, that the summation of vertical forces on a free body must equal zero. As mentioned above, this force V in this case is a downward force of the same magnitude as R_a.

55. Flexure Stresses in Any Homogeneous Beam. The preceding article applies specifically to homogeneous beams of rectangular cross-section. In order to obtain an expression for the internal moment of resistance in a beam of any cross-section, consider the beam in Fig. 30, the cross-section of which is purposely indicated as an undimensioned area of indefinable shape. The gravity axis is assumed to be at a distance e from the extreme bottom fiber and at a distance e' from the top fiber. Let f

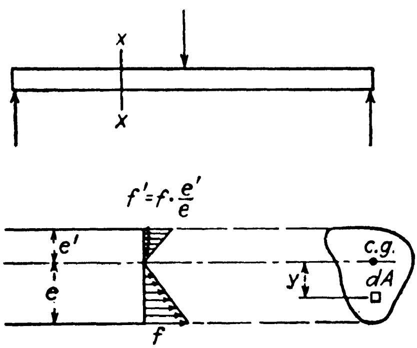


FIG. 30.

be the unit fiber stress on the extreme fiber (the lower fiber in this case) at section x-x. The value of the unit stress on an elementary area dA, at a distance y below the gravity axis (the neutral axis), is $f \cdot \frac{y}{e}$; and the total stress on the area dA is $f \cdot \frac{y}{e} \cdot dA$. Since the unit deformation of two fibers, one at a distance y below the neutral axis and one at the same distance above the neutral axis, are equal, the stresses producing these changes in length are equal, provided the modulus of elasticity of the material in tension is the same as that in compression. Hence, the preceding equation is also true for elementary areas above the neutral axis, the stresses, however, being of opposite character to those on areas below this axis.

The moment of resistance of the cross-section is equal to the sum of the moments, about the neutral axis, of the stresses on each elementary area dA in the entire cross-section, or

$$\text{Moment of resistance} = \int \left(\frac{y}{e}\right) \cdot f \cdot dA \cdot y = \frac{f}{e} \int y^2 dA$$

By definition (see Art. 5), $\int y^2 dA$ is the moment of inertia I of the cross-section about the neutral axis. The foregoing equation may therefore be written as follows:

$$\text{Moment of resistance} = f\frac{I}{e} \qquad (2)$$

The moment of resistance at any section is at all times equal to the bending moment M at that section caused by the external loads and reactions, so that the fundamental equation that expresses the relation between the external moment, the simultaneous extreme fiber stress, and the physical dimensions of the beam is

$$M = f\frac{I}{e} \qquad (3)$$

For rolled steel shapes, the quantity I/e is given, for various types, sizes, and weights of section, in the steel manufacturers' handbooks. This quantity is called the "section modulus" and is denoted by the letter S. Hence, equation (3) may be written in the form

$$M = fS \qquad (3a)$$

which is of advantage particularly in the design or investigation of rolled steel beams.

If the section x-x is that section at which the moment caused by the external loads and reactions is a maximum, if f is taken as the maximum allowable unit fiber stress in flexure, and if I/e is variable, then equation (3) represents the condition that the maximum moment of resistance (the resisting moment of the beam) is equal to the maximum bending moment M.

With proper interpretation equation (3) is perfectly general, and it applies both to design and review. If the cross-section is to be determined, the maximum moment and the maximum allowable fiber stress are known, and the required value of I/e may be computed. If the maximum moment M and the cross-section are known, the actual maximum extreme fiber stress f (which may be greater or less than the allowable stress) can be determined. If the cross-section and maximum allowable fiber stress f are given, the maximum allowable bending moment M may be obtained.

Equation (1) for rectangular beams could be obtained directly from equation (3). Since, for a rectangular section $I = \frac{1}{12}bd^3$ and $e = d/2$, it follows that $I/e = \frac{1}{6}bd^2$ and $M = f(\frac{1}{6}bd^2)$.

56. Variation of Flexure Stress in a Given Cross-section. Since unit stresses on the fibers in a given cross-section vary directly as the distances from the neutral axis, the unit stress f_y on a fiber at a distance y from the neutral axis is

$$f_y = f \cdot \frac{y}{e}.$$

Hence, from equation (3),

$$M = f_y \left(\frac{e}{y}\right) \cdot \frac{I}{e} = f_y \cdot \frac{I}{y} \tag{4}$$

In Fig. 30 the unit stress f' on the extreme top fiber is therefore given in the equation

$$M = f' \frac{I}{e'} \tag{5}$$

Equation (5) is just as important in the design of beams of unsymmetrical cross-section as equation (3) because of the fact that quite frequently the *allowable* unit stress on the extreme compression fiber is less than that on the extreme tension fiber, owing to the possible tendency of the cross-section to buckle sidewise under large compression stresses.

In Fig. 30 the bottom fiber is assumed to be farther from the neutral axis than the top fiber. It should be remembered that, if the shape of the beam is changed, the reverse condition might well exist, in which case, with the same nomenclature as indicated in Fig. 30, e' would be larger than e and the actual stress f' would be greater than the actual stress f.

It is not necessary to consider equations (3) and (5) as two separate equations. Either one could be used in investigating both the extreme tension and the extreme compression fibers, provided that the nomenclature is changed so that f (or f') is the unit stress on one or the other of these extreme fibers and e (or e') is the distance from the gravity axis *to the extreme fiber in question.* This follows from equation (4) with proper interpretation of the distance y and the stress f_y.

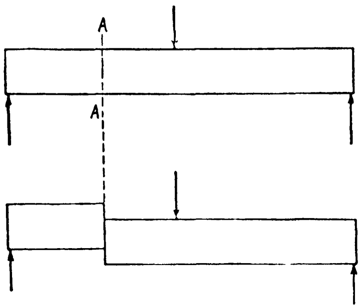


FIG. 31.

57. Shearing Stresses. All beams are subjected to shearing stresses in addition to flexural stresses. Shearing stresses are of two kinds, vertical shear and longitudinal shear. The former represents the tendency for one segment of the beam to move up (or down) with respect to the other segment. In Fig. 31, the portion to the left of section AA is being pushed upward by the left reaction, while the right portion is being pushed downward. Actual motion is prevented by the shearing resistance of the material in the beam. The total vertical shear at any section

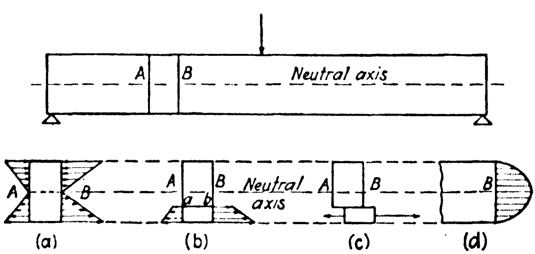


FIG. 32.

is equal to the algebraic sum of the loads and reactions on one side of the section.

Longitudinal shear is caused by the difference in flexural stresses at various sections along the beam. In Fig. 32*a*, the fiber stresses of compression (above the neutral axis) and tension (below the neutral axis) are greater on the right face of the segment AB than on the left, because (1) the bending moment is greater on the right than on the left and (2) the fiber stresses are proportional to the bending moment. At any horizontal plane ab in the segment AB (Fig. 32b) there is a tendency for the portion between the plane and the nearest surface of the beam to move horizontally (to the right in this case) with a force equal to the total difference in the flexural stresses on the two sides of this portion of the segment (Fig. 32c). The longitudinal shear is obviously equal to zero at the top and bottom surfaces and is a maximum at the neutral surface. At any point in any cross-section of the beam, the intensity of the longitudinal shear is equal to the intensity of the vertical shear.

For a rectangular beam, the value of the maximum unit longitudinal shear is obtained as follows: In Fig. 33, all of the forces which act on the portion AB of the beam in Fig. 32 are shown. The length x of AB is considered to be very small, so that the total vertical shear V on the face A can be assumed to be the same as that on the face B. For purposes of computation the compression forces C, C' and the tension forces T, T' are assumed to be concentrated at the centers of gravity of the respective triangular stress wedges. The sum of the moments about point m (in the line of action of C and C' and also in the vertical plane B) must be equal to zero; hence,

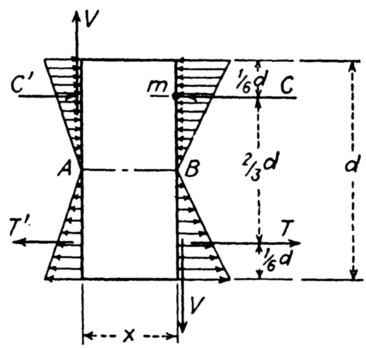


FIG. 33.

$$\tfrac{2}{3}d(T - T') = Vx$$

and

$$(T - T') = \frac{3}{2}\frac{Vx}{d}$$

But, at the neutral axis, the total longitudinal shear is $(T - T')$, and the unit shear v' on the horizontal plane AB, the length of which is x and the width b, is

$$v = \frac{T - T'}{bx}$$

Hence,

$$v = \frac{3}{2}\frac{V}{bd} \tag{6}$$

At horizontal planes other than the neutral plane, the longitudinal shear varies as the ordinates of a parabola, as shown in Fig. 32*d*.

For a rolled steel shape such as an I-beam, in establishing the design procedure for shear, it is assumed that the centers of gravity of the compression and tension stresses are located at the extreme top and bottom fibers. The equations corresponding to those used in the preceding derivation for rectangular sections are then as follows:

$$(T - T')d = Vx$$

$$(T - T') = \frac{Vx}{d}$$

$$v = \frac{T - T'}{bx}$$

Therefore,

$$v = \frac{V}{bd} \tag{7}$$

Since the majority of the flexure stresses are in the flanges, the assumption made above, *i.e.*, that the lever arm of each of the stress couples T, C and T', C' is equal to the full depth of the beam, is not seriously in error, especially for fairly deep beams, and the maximum unit shear v as obtained from equation (7) is not materially lower than the actual value. Whatever error does exist is compensated for in the selection of the allowable unit design stress in shear.

Equations (6) and (7) indicate that, on any beam, the absolute maximum unit shearing stress occurs in the section at which the total vertical shear V is a maximum, usually at one of the supports.

58. Diagonal Tension Stresses. Combined with the bending stresses, the two shearing stresses (longitudinal and vertical) cause inclined stresses of tension T, T (Fig. 34) in the lower portion of the beam, and compression in the upper portion. Diagonal tension stresses as a rule need not be investigated in timber or steel beams because of the relatively large tensile strength of the materials. Diagonal compression stresses are of no importance in rectangular timber beams, for the same reason; but in very deep steel beams, especially in the built-up sections, these stresses are apt to cause buckling of the webs. This tendency to buckle is overcome by stiffening the webs of deep built-up beams, at proper intervals, with vertical angles which are riveted or welded to either side of the web (see Art. 127). The standard rolled-beam sections are so proportioned that the

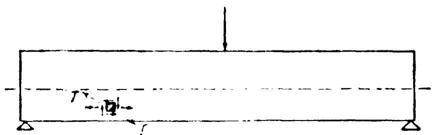


FIG. 34.

danger of such buckling is eliminated if the usual safe bending and shearing unit stresses are not exceeded, *i.e.*, the ratio of web thickness to total height is such as to provide sufficient inherent resistance to those inclined stresses.

59. Deflection. Beams which support tile or concrete floors, plastered ceilings, etc., should be designed to avoid excessive deflections in order to prevent the formation of unsightly cracks in the floor or ceiling. A deflection of $\frac{1}{360}$ of the span is the generally accepted maximum for such beams, provided that they are reasonably free from vibration and impact. Experience has shown that plaster will crack if the deflection materially exceeds the foregoing value; also, the structure as a whole will not have sufficient inherent rigidity to prevent giving an impression of undue flexibility. The theory and formulas for deflection are given in Chap. IX. Equations for the actual maximum deflection of a few of the more common types of beams and loadings are given in Fig. 126, page 193.

After the design for bending and shear is made, the theoretical deflection of the beam can be obtained from the proper deflection equation. Comparison with the allowable deflection will indicate whether or not the tentative section is sufficiently rigid. If not, the value of I which is required to reduce the actual deflection to a value not exceeding the allowable deflection may be computed and a new section selected.

Beams which are subject to much vibration or shock should have greater rigidity than is furnished by the procedure outlined above. This greater rigidity can be secured by lowering the value of the maximum allowable deflection, but more generally in steel design it is assured by specifying a depth-span ratio which has been shown by experience to be adequate for the type of structure under consideration. Typical ratios are given in the various standard specifications (see Art. 116).

60. Design of Rectangular Timber Beams. After the span and loads have been established and the species and grade of timber have been selected, the design procedure for rectangular timber beams is as follows:

1. Compute the maximum bending moment M, and the maximum total vertical shear V.
2. Determine from equation (1) the value of bd^2 required to furnish a resisting moment equal to the maximum bending moment, using the proper value of f as given in Art. 13.
3. Determine from equation (6) the required product bd to furnish adequate shearing resistance. The allowable value of v is given in Art. 13.
4. Select values of b and d to satisfy both requirements 2 and 3. In making this selection, the depth should be greater than the width in order to secure maximum economy as far as bending is concerned, but not so much greater that the beam will tend to topple over or buckle sidewise because of insufficient width. Ordinarily, a width one-half to three-fourths of the depth will be satisfactory, although in many cases square cross-sections are necessary or desirable. An exception occurs in the case of floor joists and other similar members, which are supported laterally by bridging, flooring, or transverse struts. For such members the depth may be as much as 7 times the width (*i.e.*, a 2 by 14 joist). If the depth-width ratio is less than 3, no lateral support is necessary. For depth-width ratios greater than 3 the distance between points of lateral support should be not greater than about 40 times the width of the beam or joist. Standard sizes of timbers should be used, to prevent waste. Nominal standard dimensions are 2, 3, 4, 6 in., and greater in multiples of 2 in. Timbers with either

dimension greater than 12 in. are more expensive per board foot[1] than smaller sizes, and the unit cost of the longer timbers increases with the size. The cost should be considered where possible, in making the final selection.

Where the timber is to be dressed on four sides (and most structural timbers are so dressed), the nominal size is reduced accordingly, and the dressed sizes should be used in the computations. Nominal dimensions less than 6 in. are reduced $\frac{3}{8}$ in. in dressing, and nominal dimensions 6 in. or greater are reduced $\frac{1}{2}$ in., so that a nominal 12 by 12 timber should be considered as $11\frac{1}{2}$ by $11\frac{1}{2}$, a nominal 2 by 4 shouid be considered as $1\frac{5}{8}$ by $3\frac{5}{8}$, etc.

The weight of the beam should be considered in computing design moments and shears. The weight of structural timbers varies from about 36 to 48 lb. per cu. ft. (3 to 4 lb. per board foot). In all the following problems a weight of 40 lb. per sq. ft. is assumed, and beam weights are based on nominal sizes.

5. Compute the theoretical maximum deflection, compare with the allowable, and increase the cross-section if necessary.

6. Compute the bearing area required at the support by dividing the reaction by the allowable unit compressive stress perpendicular to the fibers (Art. 13), and make the bearing length sufficient to furnish the required area. A minimum length of 4 in. should be used.

Example. A timber beam with a span of 16 ft.-0 in. is to support a total uniform load (including the weight of the beam) of 1000 lb. per lin. ft. The beam is to be of Douglas fir, for which the allowable unit fiber stress in bending (Art. 13) is 1500 p.s.i., the allowable unit longitudinal shear is 100 p.s.i., and the modulus of elasticity is 1,600,000 p.s.i. Design the beam.

$$M = \tfrac{1}{8} \times 1000 \times 16^2 \times 12 = 384{,}000 \text{ in.-lb.}$$
$$V = 1000 \times \tfrac{16}{2} = 8000 \text{ lb.}$$

From equation (1),

$$bd^2 \text{ (required)} = \frac{6 \times 384{,}000}{1500} = 1536 \text{ in.}^3$$

Let $b = 9\frac{1}{2}$ in. (10 in. nominal); then d (required) $= \sqrt{\frac{1536}{9.5}} = 12.7$ in.

A nominal 10 × 14-in. beam is selected ($b = 9\frac{1}{2}$ in., $d = 13.5$ in.)

A 12 × 12 beam ($bd^2 = 11.5 \times 11.5^2 = 1520$ in.^{3}) furnishes about 1 per cent less strength than is required. A deficiency up to about 2 per cent is considered safe in ordinary design practice, and the 12 × 12 could be considered satisfactory for flexure stresses if the 10 × 14 were not available.

[1] A board foot is $\frac{1}{12}$ cu. ft. and is the equivalent of a board 1 ft. long, 1 ft. wide, and 1 in. thick. The number of board feet in a given timber is obtained by multiplying the width in feet by the length in feet and by the thickness in inches.

From equation (6),

$$bd \text{ (required)} = \frac{3}{2} \times \frac{8000}{100} = 120 \text{ sq. in.}$$

Either the 10 × 14 beam (bd = 9.5 × 13.5 = 128 sq. in.) or the 12 × 12 beam (bd = 11.5 × 11.5 = 132 sq. in.) would furnish adequate shearing resistance.

The moment of inertia of the 10 × 14 beam is $\frac{1}{12} \times 9.5 \times 13.5^3 = 1945$ in.^{4} From Fig. 126, page 193, the maximum theoretical deflection δ is

$$\delta = \frac{5 \times 16{,}000(16 \times 12)^3}{384 \times 1{,}600{,}000 \times 1945} = 0.473 \text{ in.}$$

For the 12 × 12 beam, I = 1460 in.^{4} and

$$\delta = \frac{5 \times 16{,}000(16 \times 12)^3}{384 \times 1{,}600{,}000 \times 1460} = 0.630 \text{ in.}$$

The allowable deflection is $\frac{1}{360}(16 \times 12) = 0.534$ in. The 10 × 14 beam must be used, since the actual deflection of the 12 × 12 exceeds the allowable value.

The bearing area required at the support is $\frac{8000}{350} = 22.8$ sq. in., and the bearing length must be at least 22.8/9.5 = 2.4 in. A minimum of 4 in. should be used.

61. Investigation of Rectangular Timber Beams. If the size and span of a beam are known, together with the species and grade of timber, the beam may be investigated to determine the maximum unit stresses for a given loading, or the maximum loading for stated maximum allowable unit stresses, by substituting known quantities in equations (1) and (6) and solving for the quantities required.

Example. A nominal 8 × 12 timber beam with a span of 12 ft.-0 in. supports a concentrated load P at the mid-span. The beam is of longleaf yellow pine, with allowable unit stresses as follows: extreme fiber stress in bending, 1400 p.s.i.; shearing, 120 p.s.i.; modulus of elasticity, 1,600,000 p.s.i. Allowing for the weight of the beam, what is the maximum safe value of the load P?

The maximum allowable moment M, from equation (1), is

$$M = 1400 \times \tfrac{1}{6} \times 7.5 \times (11.5)^2 = 231{,}000 \text{ in.-lb.}$$

The weight of the beam is $\frac{8 \times 12}{144} \times 40 = 27$ lb. per ft., and the maximum moment M_D due to this weight is

$$M_D = \tfrac{1}{8} \times 27 \times 12^2 \times 12 = 5800 \text{ in.-lb.}$$

The moment due to the load P cannot exceed 231,000 − 5800 = 225,200 in.-lb. Hence, since the maximum bending moment (see page 10) is $\frac{1}{4} Pl$,

$$\tfrac{1}{4}P \times 12 \times 12 = 225{,}200$$
$$P = 6240 \text{ lb.}$$

The maximum allowable total shear V, from equation 6, is

$$V = \tfrac{2}{3} \times 120 \times 7.5 \times 11.5 = 6900 \text{ lb.}$$

The maximum dead-load shear is $27 \times \frac{12}{2} = 162$ lb.; hence, the shear caused by the load P cannot exceed $6900 - 162 = 6740$ lb. Since $V_P = P/2$, $P = 2 \times 6740 = 13{,}480$ lb.

The preceding maximum values of P show that the bending strength governs in this case, and the maximum safe load P cannot exceed 6240 lb. The preceding computations show also the comparatively small effect of the weight of the beam, in view of which it may be stated that, in the design of timber beams, the weight of the beam itself may ordinarily be neglected.

If the deflection is limited to $\frac{1}{360} \times 12 \times 12 = 0.4$ in., the maximum load P that could be placed on the beam without exceeding the allowable deflection (neglecting the effect of the weight of the beam) is obtained from the proper deflection equation (page 193), as follows:

$$I = \tfrac{1}{12} \times 7.5 \times (11.5)^3 = 954 \text{ in.}^4$$
$$\delta = 0.4 = \frac{P(12 \times 12)^3}{48 \times 1{,}600{,}000 \times 954}$$
$$P = 9800 \text{ lb.}$$

Since this is greater than 6240 lb., the latter value is still the governing load.

62. Design of Timber Floor Joists. Timber floor joists are designed in the same manner as any other rectangular timber beam. Floor joists are usually 2 in. (nominal) in width, although 3-in. and occasionally 4-in. widths may be necessary for heavily loaded floors. The depths vary usually from 6 to 14 in., in multiples of 2 in. The spacing of joists varies generally from 12 to 24 in., 16 in. being the most common spacing for residence construction. The maximum spacing is governed by the strength of the flooring which is supported by the joists and may be as much as 6 ft. or more in certain types of construction where heavy plank flooring is used. In the design of floor joists, therefore, the load, the span, the spacing of joists, and the size of the joists are the significant factors. The spacing and the size of the joists are both variable; hence, one or the other must be assumed in order to compute the other. Revision of the assumed variable must be made if the value of the computed variable is unreasonable or impractical. In general, it is necessary to design joists for deflection as well as for moment and shear. Transverse bracing in the form of bridging is generally placed at a maximum spacing of about 7 ft., as shown in Fig. 35.

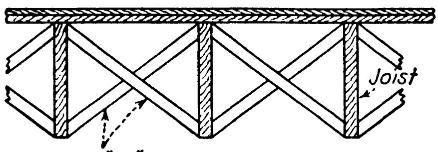


FIG. 35.

Example. Longleaf yellow pine floor joists with a span of 14 ft.-0 in. are spaced 18 in. center to center. Using the allowable unit stresses as given in Art. 13 and assuming that the deflection cannot exceed $\frac{1}{360}$ of the span, or 0.48 in., what size joists are required to support a total load (including the weight of the flooring and the weight of the joists) of 100 lb. per sq. ft.?

$$\text{Load per ft.} = \frac{18}{12} \times 100 = 150 \text{ lb.}$$
$$M = \frac{1}{8} \times 150 \times 14^2 \times 12 = 44{,}100 \text{ in.-lb.}$$
$$V = 150 \times \frac{14}{2} = 1050 \text{ lb.}$$

From equation (1),

$$bd^2 \text{ (required)} = \frac{6 \times 44{,}100}{1400} = 189 \text{ in.}^3$$

If the nominal width is 2 in. ($1\frac{5}{8}$ actual), the depth required is $\sqrt{189/1.62} = 10.8$ in., and 2×12 joists ($1\frac{5}{8} \times 11\frac{1}{2}$ actual) are satisfactory for moment.

From equation (6),

$$bd \text{ (required)} = \frac{\frac{3}{2} \times 1050}{120} \doteq 13.1 \text{ in.}^2$$

With $b = 1\frac{5}{8}$ in., d must be at least $13.1/1.62 = 8.1$ in., and the 2×12 joist which was necessary for moment is also adequate for shear.

The theoretical deflection of the 2×12 joist is computed from the proper formula on page 193, as follows: With $I = \frac{1}{12} \times 1.62 \times (11.5)^3 = 206$ in.^{4} and $W = 150 \times 14 = 2100$ lb.,

$$\delta = \frac{5 \times 2100 \times (14 \times 12)^3}{384 \times 1{,}600{,}000 \times 206} = 0.39 \text{ in.}$$

This is less than the allowable value of 0.48 in. If it had been greater, a value of 0.48 could have been substituted for δ in the foregoing equation, with the moment of inertia I as the unknown variable. The required value of I could then have been computed and the minimum depth d determined from the equation $I = \frac{1}{12}bd^3$, assuming that the width b would remain $1\frac{5}{8}$ in. (actual).

63. Investigation of Timber Floor Joists. Quite frequently it is necessary to determine the safe load that can be carried by an existing floor. In investigating the strength of the joists, (1) compute the resisting moment of one joist; (2) equate the resisting moment to the maximum external bending moment, expressed in terms of the unknown load and the known span, and solve the equation for the load that can be carried safely by the joists; (3) divide the load that can be carried by one joist by the floor area (square feet) supported by that joist to obtain the load per square foot of floor area; (4) compute the maximum total shear that can be carried by one joist; (5) divide the maximum total shear by one-half the floor area supported by one joist, to obtain the load per square foot of floor area. If necessary, the investigation for deflection is carried out as explained in Art. 62.

Example. What live load per square foot may be placed on a warehouse floor which has 4 × 12 longleaf yellow pine joists spaced 24 in. on centers? The span of the joists is 16 ft.-0 in.

The allowable unit stress in bending is 1400 p.s.i., the allowable unit shear is 120 p.s.i., and $E = 1{,}600{,}000$ p.s.i. From equation (1), the resisting moment of one joist is

$$M = 1400 \times \frac{1}{6} \times 3\frac{5}{8} \times (11\frac{1}{2})^2 = 112{,}000 \text{ in.}^4$$

The maximum bending moment is

$$M = \frac{1}{8}Wl = \frac{1}{8} \times W \times 16 \times 12 = 24W \text{ in.-lb.}$$

in which W = the total load on one joist, in pounds.
Equating the bending and resisting moments, $24W = 112{,}000$ from which $W = 4660$ lb.
The equivalent total load per square foot of floor area is then

$$w = \frac{4660}{16 \times 2} = 146 \text{ lb. per sq. ft.}$$

From equation (6), the maximum allowable total shear is $V = \frac{2}{3} \times 120$

$\times 3\frac{5}{8} \times 11\frac{1}{2} = 3340$ lb., and

$$w = \frac{3340}{\frac{1}{2}(16 \times 2)} = 208 \text{ lb. per sq. ft.}$$

The strength of the joist is governed, therefore, by its resistance to bending, and the safe load is 146 lb. per sq. ft. of floor area. The weight of one joist per linear foot, at 40 lb. per sq. ft., is approximately

$$\frac{4 \times 12}{144} \times 40 = 14 \text{ lb.},$$

which is equivalent to $\frac{14}{2} = 7$ lb. per sq. ft. of floor area. If the flooring weighs 6 lb. per sq. ft. (1-in. subfloor and $\frac{25}{32}$-in. hardwood finish floor), the total dead load is 13 lb. per sq. ft. The safe live load is then $146 - 13 = 133$ lb. per sq. ft.

If deflection were an important consideration and if the maximum allowable deflection were $\frac{1}{360} \times 16 \times 12 = 0.53$ in., the following investigation would be necessary. The moment of inertia of the joist cross-section is $\frac{1}{12} \times 3\frac{5}{8} \times (11\frac{1}{2})^3 = 460 \text{ in.}^4$ The maximum actual deflection (page 193) is

$$\delta = \frac{5 \times W \times (16 \times 12)^3}{384 \times 1{,}600{,}000 \times 460} = 0.000125W \text{ in.}$$

Equating the actual and the allowable deflections, $0.53 = 0.000125W$ from which the maximum total load is $W = 4240$ lb.

The load per square foot is then $w = \frac{4240}{16 \times 2} = 133$ lb. per sq. ft. Since this is less than 146 lb. per sq. ft., the safe load would be governed by the deflection requirement, and the safe live load on the floor would be $133 - 14 = 119$ lb. per sq. ft.

64. Design of Rolled Steel Beams.

The design procedure for steel beams is as follows:

1. Compute the maximum bending moment M and the maximum total vertical shear V.

2. Determine from equation (3a) the section modulus required, and select, from the Steel Handbook, that shape which will furnish this section modulus with the least weight. The allowable unit stress on the extreme compressive fiber is less than that on the extreme tensile fiber, because of the tendency of the compression flange to buckle sidewise, much as a long column does. This stress is based on the ratio between the unsupported length l of the compression flange and its overall width b. The specifications of the American Institute of Steel Construction (Art. 21) permit, for the compression flange, an allowable unit stress f (in pounds per square

inch) of

$$f = \frac{22,500}{1 + \frac{l^2}{1800b^2}} \text{ (with maximum of 20,000)}$$

in which l = unsupported length of the compression flange, in inches.
b = the width of that flange, in inches.

Therefore, unless the compression flange is braced laterally throughout its length, the allowable unit stress to be used in equation (3a) is that specified for the compression flange.

3. From equation (7), compute the maximum unit shearing stress on the web of the selected section to show that this is less than the allowable unit stress. As previously stated, for rolled sections with ordinary spans the shearing strength is usually not a governing factor.

4. Compute the maximum theoretical deflection and compare with the allowable deflection, to determine whether or not the assumed section is adequate as far as deflection is concerned.

Example. A steel beam with a span of 30 ft.-0 in. carries a uniform live load of 2000 lb. per lin. ft. The top flange is supported laterally at the mid-point by transverse beams on either side. Using the specifications of the American Institute of Steel Construction, what rolled section is required?

Assume that the weight of the beam is 90 lb. per ft.

$$M = \tfrac{1}{8} \times 2090 \times 30^2 \times 12 = 2,820,000 \text{ in.-lb.}$$

The allowable unit stress on the extreme tension fiber is 20,000 p.s.i. With $l = \frac{30}{2} \times 12 = 180$ in. and b (assumed) = 7 in., $l/b = \frac{180}{7} = 25.7$, and from the formula at the top of this page the allowable unit stress on the extreme compression fiber is 16,460 p.s.i. From equation (3a), the section modulus required is

$$S = \frac{2,820,000}{16,460} = 171 \text{ in.}^3$$

From the Steel Handbook, a standard I-beam 24 in. deep, weighing 79.9 lb. per ft., with a flange width of 7 in. is satisfactory. (S = 173.9 in.^{3}). The actual flange width is the same as the assumed width, and no revision of the allowable stress is necessary. The weight is sufficiently close to the assumed weight so that no revision of the bending moment is necessary. The revised total load per foot is 2080 lb., and the maximum shear is $V = 2080 \times \frac{30}{2} = 31,200$ lb. The thickness of the web is 0.5 in., and the unit shearing stress, from equation (7), is

$$v = \frac{31,200}{24 \times 0.5} = 2600 \text{ p.s.i.}$$

The allowable unit shearing stress (page 29) is 13,000 p.s.i.

A more economical design could be made by using one of the wide-flange beam sections. A 24 × 9 wide-flange section, weighing 24 lb. per ft., has a section modulus of 170.4 in.^{3} With a 9-in. flange width, $l/b = 20$, the allowable unit stress is 18,410 p.s.i., and the section modulus required is 153 in.^{3} The saving in weight, as compared with the standard I-beam section, is 5.9 lb. per ft., or 177 lb. per beam.

For the wide-flange section, $I = 2033.8$ in.^{4}, $W = 2074 \times 30 = 62{,}200$ lb., and the maximum theoretical deflection is

$$\delta = \frac{5 \times 62{,}200 \times (20 \times 12)^3}{384 \times 30{,}000{,}000 \times 2033.8} = 0.62 \text{ in.}$$

The usual allowable deflection is $\frac{1}{360} \times 30 \times 12 = 1$ in.

65. Web Crippling of Rolled Steel Beams. When a steel beam rests on top of its support or when a concentrated load is placed on top of the beam at some point in the span, the web in the region of the support or under the interior load acts as a column, and, as such, is likely to buckle or be crippled by the reaction or load. The usual method of designing to prevent such a failure is to assume that the web acts as a column, with one dimension of the cross-section of the assumed column equal to the thickness t of the web and the other dimension equal to slightly more than the bearing length. For interior loads the American Institute of Steel Construction specifications consider the latter dimension to be equal to the bearing length plus two times the distance k (see Fig. 36) from the outer face of the flange to the point where the fillet between the web and the flange joins the web (*i.e.*, at the web toe of the fillet). For end reactions, this effective column dimension is specified as the bearing length plus k. The actual unit stress f_b on the assumed column is then equal to the concentrated load divided by the effective column area. The maximum allowable unit stress is 24,000 p.s.i.

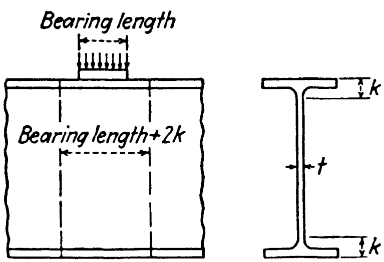


FIG. 36.

The formulas for the required length of bearing, N, according to the foregoing analysis are derived as follows:

For an interior load P,

$$f_b = \frac{P}{t(N + 2k)} = 24{,}000 \text{ (max.)}$$

from which

$$N \text{ (min.)} = \frac{P}{24{,}000t} - 2k$$

For an end reaction R,

$$f_b = \frac{R}{t(N + k)} = 24{,}000 \text{ (max.)}$$

from which

$$N \text{ (min.)} = \frac{R}{24{,}000t} - k$$

In case the required length of bearing is not available, a beam with a thicker web should be used, or the thinner web should be stiffened by means of plates or vertical angles riveted or welded on either side.

Example. For the wide-flange beam used in the example in Art. 64 the web thickness is 0.430 in., $k = 1\frac{1}{4}$ in., $R = V = 31{,}100$ lb., and the required length of bearing to prevent web crippling at each support is

$$N = \frac{31{,}100}{24{,}000 \times 0.430} - 1.25 = 2.75 \text{ in.}$$

Actually, a bearing length of less than 4 in. is not considered satisfactory, and a 4-in. length would be used.

66. Beam Bearing on Wall. If the beam in the preceding example were to rest on a brick wall, for which the allowable unit bearing stress were 300 p.s.i., the bearing area required on the wall would be 31,100/300 = 103.7 sq. in. A plate would be required under the beam; the dimension parallel to the wall might be made 12 in., and the dimension parallel to the beam 9 in., giving an area of 108 sq. in.

The required thickness of the plate would be determined by treating the portion which projects beyond the edge of the flange of the beam as a cantilever beam. The length of the cantilever is $\frac{12 - 9}{2} = 1.5$ in., and the unit upward pressure is

31,100/108 = 288 p.s.i. Considering a plate 1 in. wide, the maximum moment in the cantilever is

$$M = \tfrac{1}{2} \times 288 \times (1.5)^2 = 324 \text{ in.-lb.}$$

Since the cross-section of the cantilever is a rectangle 1 in. wide and t in. thick, equation (1) can be used to determine the required thickness.

$$324 = 20{,}000 \times \tfrac{1}{6} \times 1 \times t^2$$
$$t = 0.31 \text{ in.}$$

A ⅜-in. plate could be used, but it would be better to use a ½-in. plate in order to obtain a more rigid bearing.

Obviously, if the wall were not thick enough to accommodate the 9-in. plate width, a narrower plate would be chosen, with a correspondingly greater dimension parallel to the wall. If the wall were of concrete with an allowable bearing pressure of 600 p.s.i., the bearing area required would be 51.8 sq. in. Since the flange width is 9 in., the bearing length required is

$$51.8/9 = 5.8 \text{ in.}$$

No plate would be required in this case; the beam would be made long enough to bear on the wall for a length of about 6 in.

67. Beams with Inclined Loads. In all the preceding discussions it has been assumed that the load is perpendicular to one of the principal gravity axes of the beam. If, as in Fig. 37, the load is not perpendicular to one of these axes, the usual procedure is to resolve it into two components normal and parallel to axis 1-1. A beam section is assumed, the extreme fiber stresses due to the two load components are computed, and these are added together to determine the actual maximum fiber stress. For the loads shown in Fig. 37, this maximum fiber stress will exist only on the fibers in the upper right-hand corner (maximum compression) and in the lower left-hand corner (maximum tension) of the cross-section. If the computed maximum stress is greater than the allowable stress, a larger cross-section must be assumed and the investigation repeated. A typical example is given in the following article.

68. Design of a Timber Purlin. The beam in Fig. 37 is a purlin resting on the upper chord of a roof truss and supporting rafters spaced 2 ft.-0 in. on centers. The normal component of the load on each rafter is 1100 lb., and the tangential component is 300 lb. The upper chord of the truss makes an angle θ of 26°34′ with the horizontal, and the trusses are spaced 12 ft.-0 in. on centers. Design the purlin, using Douglas fir, for which the allowable unit fiber stress is 1500 p.s.i.

Assuming an 8 by 10 purlin, at 40 lb. per cu. ft. the weight of the purlin is $\frac{8 \times 10}{144} \times 40 \times 12 = 270$ lb., the normal component is $270 \times 0.8944 = 240$ lb., and the tangential component is $270 \times 0.4412 = 120$ lb. The moment M_N due to the normal weight component is $\frac{1}{8} \times 240 \times 12 \times 12 = 4330$ in.-lb., and the moment M_T due to the tangential component is $\frac{1}{8} \times 120 \times 12 \times 12 = 2170$ in.-lb.

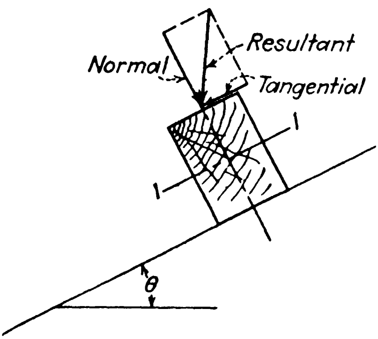


Fig. 37.

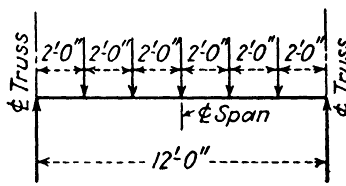


Fig. 38.

Assuming that one rafter is placed at the center of the span (see Fig. 38), the moment M_N due to the normal components of the rafter loads is

$$M_N = [2\tfrac{1}{2} \times 1100 \times 6 - 1100(2 + 4)]12 = 119{,}000 \text{ in.-lb.}$$

Similarly,

$$M_T = [2\tfrac{1}{2} \times 300 \times 6 - 300(2 + 4)]12 = 32{,}400 \text{ in.-lb.}$$

The total M_N is 123,330 in.-lb., and the total M_T is 34,570 in.-lb. The maximum unit fiber stress due to M_N (equation 1) is

$$f_N = \frac{6 \times 123{,}330}{7.5 \times (9.5)^2} = 1090 \text{ p.s.i.}$$

Similarly,

$$f_T = \frac{6 \times 34{,}570}{9.5 \times (7.5)^2} = 386 \text{ p.s.i.}$$

The total maximum fiber stress is then $1090 + 386 = 1476$ p.s.i., which is but slightly less than the allowable stress of 1500 p.s.i. No revision of the assumed section is necessary.

If it were desired to design the purlin so that the deflection did not exceed a given amount (usually 1/360 of the span), only the normal components of the rafter loads and the purlin weight would be used. To compensate for the fact that the modulus of elasticity of timber for long-continued loads is only about one-half of the usual value, the dead loads should be doubled in computing the effective total load on the purlin. Since the rafter loads are closely spaced, the deflection formula for uniform load could be used. One-half of a full rafter load should be considered as applied to each end of the purlin, so that the total load W in the deflection equation would include six full rafter loads (normal components only, with permanent parts of the load doubled) and two times the normal component of the weight of the purlin.

CHAPTER V

STRESSES IN TRUSSES

ROOF TRUSSES

69. Types of Trusses. Several of the more common types of roof trusses are shown in Fig. 39. The distance between the centers of the supports is the span, and the distance from the

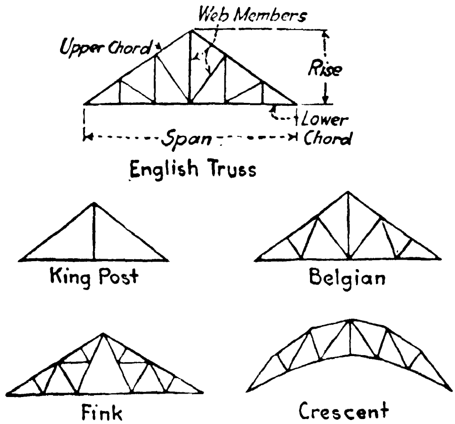


FIG. 39.

highest point (the peak) to the line on which the span is measured is the rise. The uppermost line of members extending from one support to the other, through the peak, is the upper chord; and the lowermost line of members is the lower chord. The diagonal and vertical elements which furnish the necessary bracing for the chords at the joints[1] are the web members.

The total load for which a roof truss must be designed consists of the weight of the truss, the weight of the roof covering, the probable weight of the snow, and the pressure of the wind. In addition, the truss may be called upon to support some special load, such as a suspended ceiling or floor or heavy machinery.

[1] The intersection of the center lines of the web members with the center lines of the chords.

The weight of the truss is usually divided equally between the panel points of the upper chord; for large trusses, however, a part of this load should be applied at the lower-chord panel points.

The type of roof covering depends on the character of the building, whether monumental, public, residence, or mill. The most common materials used for sloping roofs are slate, tile, tin, and wooden shingles resting on 1-in. wooden sheathing or on a precast slab of gypsum or concrete supported directly on the purlins. The wooden sheathing is supported on rafters of fairly close spacing placed parallel to the upper chord, and these

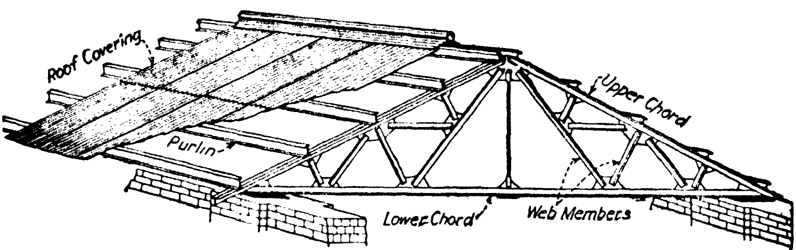


FIG. 40.

in turn are supported by purlins resting on the upper chords at the panel points. The purlins are either wooden beams or steel shapes. For mill and shop buildings, the roof covering usually consists of corrugated metal sheathing, with insulating material on the underside (see Fig. 40), supported directly on the purlins. For preliminary computations and estimates, the following values of the weights in pounds per square foot of roof surface will prove satisfactory:

Tin shingles	1
Wooden shingles	2
Slates	10
Tiles	10–20
1-in. wooden sheathing	3
Gypsum or concrete slabs	10–25
Wooden rafters	1½–3
Wooden purlins	2–4
Steel purlins	3–5
Corrugated steel	1–2

Example. Let it be required to determine the dead, snow, and wind panel loads for the wooden roof truss shown in Fig. 41. The span of the truss is 60 ft.-0 in., the rise 15 ft.-0 in., and the distance between adjacent trusses 12 ft.-0 in. The roof covering consists of slates laid on 1-in. wooden sheathing supported on 2 × 6-in. rafters, 24 in. on centers; the rafters in turn are supported on 8 × 10-in. purlins resting on the upper chords at the panel points.

From the second equation on page 51, the probable weight of the truss per square foot of horizontal covered surface is

$$w = 0.5(1 + 0.1 \times 60) = 3.5 \text{ lb.}$$

and the total weight of one truss 3.5 × 12 × 60 = 2520 lb. The proportion of the total weight assumed as concentrated at each intermediate upper-

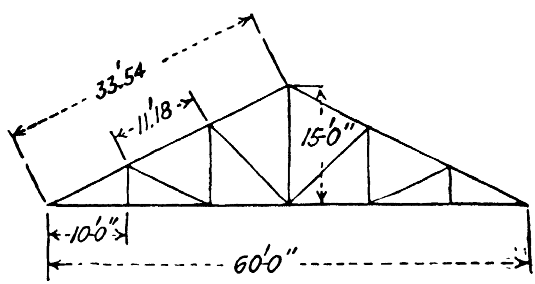


Fig. 41.

chord panel point is $\frac{1}{6} \times 2520$, or 420 lb. One-half of this amount, or 210 lb., is assumed as acting at each end panel point.

Each intermediate panel point supports 11.18 × 12, or 134.0, sq. ft. of roof surface and 10.0 × 12, or 120.0, sq. ft. of horizontal projection of roof surface. The weight of 1-in. sheathing supported by each panel point is 134.0 × 3 = 402 lb., and the weight of slate 134.0 × 10 = 1340 lb. Since each purlin, and therefore each panel point, supports $\frac{12}{2}$ or 6 rafters, the panel load due to the rafter weight, the weight of timber being assumed as 36 lb. per cu. ft., is $6 \times \frac{2 \times 6}{12} \times 11.8 \times 3 = 201$ lb. The weight of one purlin is $\frac{8 \times 10}{12} \times 12 \times 3 = 240$ lb.

The angle of inclination of the roof surface is 26°34′ ($\tan^{-1} \frac{15}{30}$); and, from the table on page 61, the normal wind pressure is 22.3 lb. per sq. ft. The wind panel load is therefore 134.0 × 22.3, or 2990 lb. The snow panel load is (25 − 6) × 120.0, or 2280 lb. (see page 60).

The total panel loads computed above, taken to the nearest 10 lb., are summarized in the following tabulation:

	Lb.
Slate	1340
Sheathing	400
Rafters	200
Purlins	240
Truss, per panel	426
Total dead panel load	2600
Snow panel load	2280
Wind panel load	2990

The end panel load in each case is one-half of the amounts shown above, and the wind panel load at the peak is one-half of the value computed for the intermediate panel points.

70. Stresses Due to Vertical Loads. Stresses in roof trusses due to dead and snow loads may be computed analytically, or graphical methods may be used. For all trusses except those having a simple regular form with few web members, the latter method usually furnishes the readier solution.

Graphical Analysis. The stresses in the truss of Fig. 41 due to dead and snow loads are to be determined graphically. The combined dead and snow panel load is 4880 lb. The diagram of the truss is repeated in Fig. 42. In this diagram, the trusses are lettered in the spaces between the members and the loads, and each member or load is designated by means of the letters in the adjoining spaces; *i.e.*, the first vertical is *EF*, the left lower chord member is *ER*, and the left reaction *AR*.

For equilibrium, the external forces on the truss, consisting of the downward panel loads and the upward reactions, must form a closed force polygon. On account of the symmetry of the loads, both in amount and in position, each reaction equals one-half of the total downward load on the truss. Since the external forces are parallel, the sides of the force polygon lie in a straight line *a . . . a′ra* (Fig. 42), constructed by laying off the panel loads in regular order from left to right and the reactions equal to *a′r* and *ra* at the right and the left ends, respectively, point *r* being midway between *a* and *a′*.

At the left support, the external forces *AB* and *AR* are held in equilibrium by the internal stresses in the members *BE* and *ER*. These four forces therefore form a closed force polygon.

The vertices of the force polygon are lettered with the lower-case letters corresponding to the capital letters of the truss diagram. Through the point *b*, a line *be* is drawn parallel to *BE*; through *r*, a line *re* is drawn parallel to *RE*. The lengths of the lines *be* and *re*, measured to the scale of the load line, give

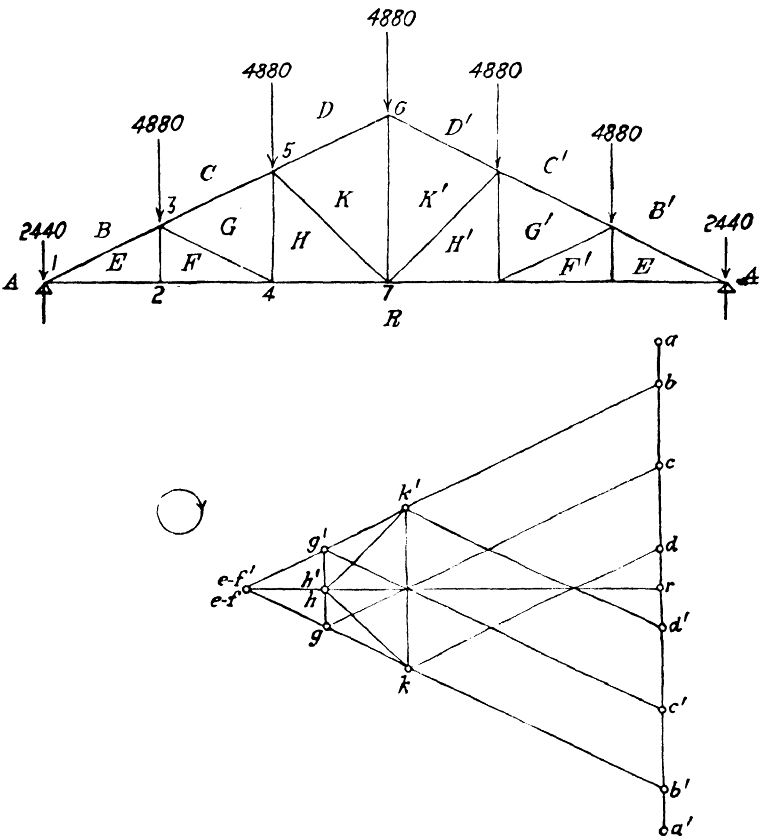


Fig. 42.

the magnitudes of the stresses in *BE* and *RE*, respectively. To determine the character of the stresses, it is necessary only to trace the forces around the polygon in regular order. For equilibrium, the forces must act in the same direction around the polygon. The reaction *ra* acts upward, and the load *ab* downward; hence, the stress *be* must act from *b* to *e*, and the stress *er* from *e* to *r* to close the polygon. Transferring these two directions to the truss diagram, the stress in BE acts downward toward the joint under consideration and the stress in ER acts away from this joint. BE is therefore in compression and ER in tension.

The stress diagram is completed for the given truss by constructing a force polygon for each of the remaining joints in regular order. Joint 3 must be considered before joint 4 in order to determine the stress in FG; otherwise, there would be three unknowns at joint 4, and the solution of this joint would be impossible. The complete stress diagram is shown in Fig. 42. When measured to the scale of the load line aa', each of the lines in this diagram represents the magnitude of the combined dead- and snow-load stress in the corresponding member of the truss.

It is possible to determine the character of the stress in each member of the truss without reference to any other member or load and thus save considerable time in the stress determination. In Fig. 42, the loads and reactions were laid off on the load line in the order AB, BC . . . $A'R$, RA—clockwise around the truss. This is indicated by the circular arrow on the stress diagram. In passing around joint 3 in this same direction, the members and external load occur in the order BC, CG, GF, FE, EB. The force polygon for this joint is $bcgfeb$ and must be followed in this same order, since the load bc acts downward and is laid off from b to c. Hence, the stress in GF acts from g to f; when transferred to the truss diagram, this direction is toward the joint, and GF is in compression. In like manner, the character of the stress in any member may be obtained without reference to any other. This method of determining the character of the stresses as obtained from the stress diagram is known as the *principle of the circular arrow.*

Analytical Analysis. The net reaction at A is $2\frac{1}{2} \times 4880 = 12{,}200$ lb. Applying the first two conditions of static equilibrium, *i.e.*, $\Sigma X = 0$ and $\Sigma Y = 0$, and taking successive joints as free bodies, the stresses are determined as follows:

$$\text{At joint } 1 \ldots \begin{cases} BE \times \dfrac{15}{33.54} + 12{,}200 = 0 & \qquad BE = -27{,}300 \\ ER - \dfrac{30}{33.54} \times 27{,}300 = 0 & \qquad ER = +24{,}400 \end{cases}$$

At joint 2 . . . . $EF = 0$

At joint 3 . .
$$\begin{cases} +27{,}300 \times \dfrac{15}{33.54} + CG \times \dfrac{15}{33.54} - FG \times \dfrac{15}{33.54} - 4880 = 0 \\ -27{,}300 \times \dfrac{30}{33.54} - CG \times \dfrac{30}{33.54} - FG \times \dfrac{30}{33.54} = 0 \end{cases}$$

$FG = -5500$

$CG = -21{,}800$

At joint 4 . . $GH - 5500 \times \dfrac{15}{33.54} = 0$ $\qquad GH = +2500$

At joint 5 . .
$$\begin{cases} +21{,}800 \times \dfrac{15}{33.54} + DK \times \dfrac{15}{33.54} - HK \times \dfrac{10}{14.14} - 4880 - 2500 = 0 \\ -21{,}800 \times \dfrac{30}{33.54} - DK \times \dfrac{30}{33.54} - HK \times \dfrac{10}{14.14} = 0 \end{cases}$$

$HK = -6900$

$DK = -16{,}300$

At joint 6 . . $+2 \times 16{,}300 \times \dfrac{15}{33.54} - 4880 - KK' = 0$

$KK' = +9700$

71. Reactions Due to Wind Loads. Since the wind pressure on sloping roofs act normal to the roof surface, the load line is not vertical, and the reactions are not equal. If both ends of the truss are anchored to the walls in such a manner that each wall furnishes resistance to a horizontal thrust, both reactions are inclined. Since there are four unknowns (amount and direction of each reaction) and only three equations available for the solution ($\Sigma X = 0$, $\Sigma Y = 0$, and $\Sigma M = 0$), an assumption must be made as to the distribution of the horizontal thrust between the two reactions. One assumption often made is that the horizontal components of the two reactions are equal. Another assumption is that both reactions are parallel to the resultant wind loads on the truss. The latter assumption is as reasonably accurate as the former and permits of a somewhat easier graphical solution.

In steel trusses, especially for the longer spans, provision is usually made for expansion and contraction by allowing one end to rest on rollers in such a manner that the truss is free to move horizontally at this end. In such cases, the reaction at the free end must be vertical, since no provision is made for resisting a horizontal pressure at this end. The entire horizontal component of all the wind panel loads must be resisted by the other support, which is fixed rigidly to the wall in order to furnish the required horizontal resistance. The reaction at the fixed end will therefore be inclined at an angle, unknown until the complete analysis of the external forces is made.

In steel trusses of shorter spans, the provision for expansion and contraction is made by merely allowing one end of the truss to rest freely on a steel plate securely fastened to the wall.

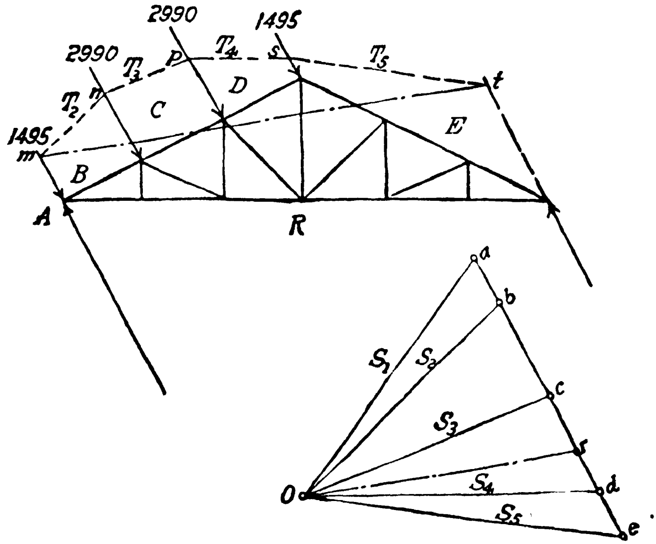


FIG. 43.

Friction being neglected, the reaction at the free end in this type of construction will be vertical, as in the preceding type.

Both Ends Fixed. Ordinarily the analytical solution for determining truss reactions is somewhat easier than the graphical solution and also more exact. The values of the reactions are determined by taking moments first about one reaction point and then about the other. The graphical solution, however, may prove more desirable if the lever arms of the given loads about the centers of moments are not easily obtained.

The graphical solution for determining wind-load reactions for trusses with both ends fixed, each reaction being assumed parallel to the resultant wind load, is illustrated in Fig. 43. The truss is that of Fig.. 41, whose panel loads were computed on page 87.

Considering the truss as a free body acted upon by the loads and reactions as shown, the load line ae is constructed by laying off the loads AB . . . DE in order.

A convenient pole O is selected, and the rays Oa . . . Oe are drawn to the load line. The equilibrium polygon *mnpst* is constructed for the pole O by drawing, on the truss diagram, lines parallel to the rays S_1, S_2 . . . S_5 in order. Each of these lines terminates on the lines of action of the wind panel loads

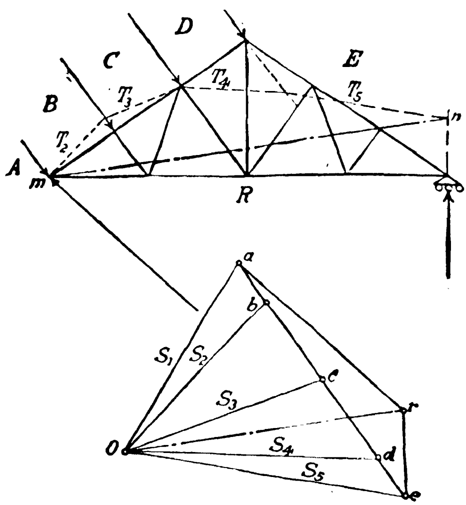


FIG. 44.

corresponding to the loads on either side of the parallel ray in the force polygon—for example, line T_2, which is parallel to the ray S_2, is drawn from the line of action of AB to that of BC. This polygon may be commenced anywhere on the line of action of the force AB. Since the lines of action of AB and AR coincide, the line of the equilibrium polygon which is parallel to the ray S_1 becomes a point. This explains the apparent omission of the parallel to S_1 from the construction. The line Or, parallel to the closing line mt of the equilibrium polygon, divides the load line into the required reactions, cr representing the right reaction and ra the left reaction. The scale of the original drawing was

100 lb. to the inch. Line *er* measured 2.80, and line *ra* 6.17 in. The right reaction is therefore 2800, and the left reaction 6170 lb.

One End Free. With one end of the truss free to move horizontally, the reaction at the free end is vertical, and the entire horizontal component of the wind pressure is resisted at the fixed end. Since the wind pressure may be exerted in either direction and the stresses in the truss members must be determined for each condition, the reactions for wind on the fixed side and on the free side must alike be determined. In a truss with unbroken chords and equal panels, the vertical reaction may be obtained readily by the method of moments, the center of moments being the fixed end of the truss. With this reaction known, as well as the value of the resultant wind load, the construction of a force triangle determines the amount and direction of the fixed reaction.

A graphical method is illustrated in Fig. 44. Here, since the reaction at the right end is known to be vertical, the equilibrium polygon is constructed between its vertical line of action and the inclined lines of action of the wind loads. Since the line of action of the reaction at A is unknown, *the equilibrium polygon must be commenced at the point* A, which is the only known point on the line of action of that reaction. The ray *or* in the force polygon, drawn parallel to the closing line *mn* of the equilibrium polygon, intersects a vertical through *e* and determines the amount and direction of the reaction AR and the amount of the reaction ER.

72. Wind-load Stresses. The reactions being known, the wind-load stresses are determined by the application of the principle of the force polygon to each joint in succession, in a manner similar to that used for stresses due to vertical loads. A broken upper chord or an inclined lower chord or both do not materially complicate the internal stress determination. Figure 45 shows the complete construction necessary for the determination of the wind-load stresses in a truss of this type. In this truss, all the members are stressed by the wind, while in trusses similar to those of Fig. 44 the web members on the leeward side have no wind-load stresses. A check on the accuracy of the construction of wind-load diagrams is that the last line drawn in the stress diagram must be parallel to the corresponding member of the truss diagram. In Fig. 45, *jr*, connecting the points *j* and *r* previously located in the construction, must be parallel to the member *JR* in the truss diagram to show absolute accuracy of construction.

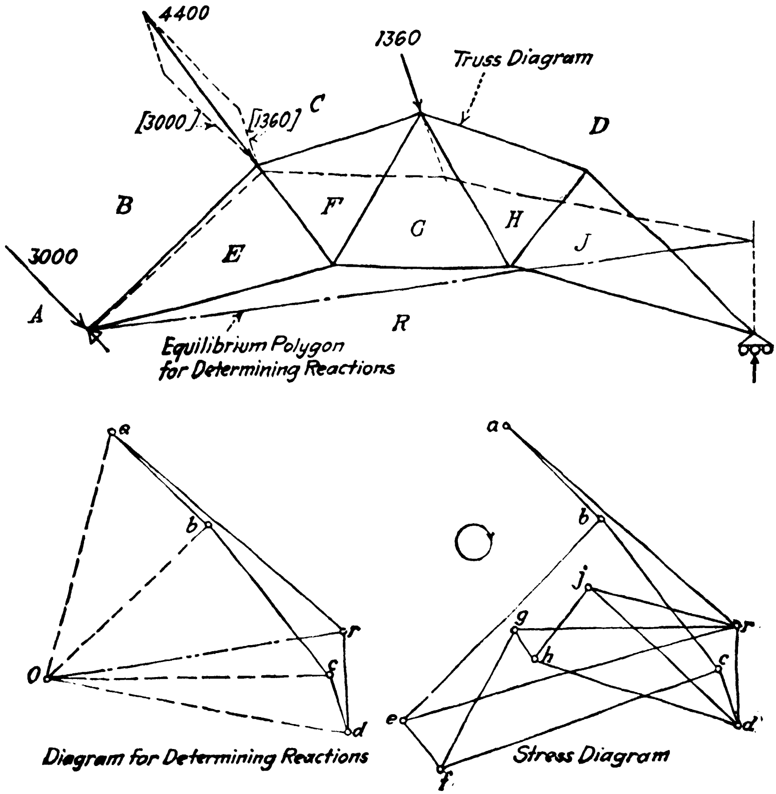


FIG. 45.

73. Maximum Stresses. The greatest stress that occurs in any member of a roof truss is the algebraic sum of the individual stresses caused by the simultaneous action of the dead load, snow load, wind load, ceiling load, or special loads. Since the wind may come from either side, the character of the stress in any one member may be variable, so that a complete analysis of any roof truss requires that the stresses due to the wind be determined for both conditions.

It is generally assumed that full snow-load stresses are not likely to occur simultaneously with full wind-load stresses. The exact pitch of the roof has an effect on the proportion of snow-load stresses to be used with the full wind-load stresses. One-half of the snow-load stresses are commonly used in such a combination.

The value of the probable snow load, varying as it does in different latitudes, indicates that there are three possible combinations of loads that may produce the maximum stresses. These are: (1) dead load plus snow load; (2) dead load plus wind load; (3) dead load plus one-half snow load plus wind load.[1] In the middle latitudes, combination (3) usually produces the maximum stresses.

BRIDGE TRUSSES

74. The Component Parts of a Bridge. A bridge usually consists of two parallel trusses or girders supporting the bridge floor or roadway. A railroad track may rest directly upon the upper chords of a pair of trusses or girders which are braced together to form a *deck bridge*. Where it becomes desirable or necessary to reduce the vertical distance from the bottom of the truss or girder to the top of rail, the trusses or girders are spread,

[1] In computing combinations (2) and (3) it is obviously necessary that the dead-load stresses and the snow-load stresses be determined separately instead of being combined as in Art. 70. Snow-load stresses can be computed from the dead-load stresses by multiplying the latter by the ratio of the snow panel load to the dead panel load. Or, if the combined stresses have been determined, as in Art. 70, the amount of stress caused by each of the two types of load can be obtained by multiplying the combined stress by the ratio of the panel load for one type to the sum of the panel loads for both types. For example, in Art. 70, the dead-load stress in BE is equal to

$$-27{,}300 \times \frac{2600}{4880} = -14{,}550 \text{ lb.},$$

and the snow-load stress is

$$-27{,}300 \times \frac{2280}{4880} = -12{,}750 \text{ lb.}$$

and the track is carried by a floor system at or near the lower chord panel points. This type is known as a *through bridge*, since the traffic is through the bridge between the trusses rather than on top of them. Short-span highway bridges of the through type with trusses not deep enough to permit overhead bracing are known as *pony truss bridges.*

An open floor system is one in which the ties rest directly upon longitudinal beams, known as *stringers*, supported by transverse *floor beams* connected to the trusses at the panel points. A solid floor system supports a ballasted track and consists of a concrete slab or steel plates, supported by a series of transverse

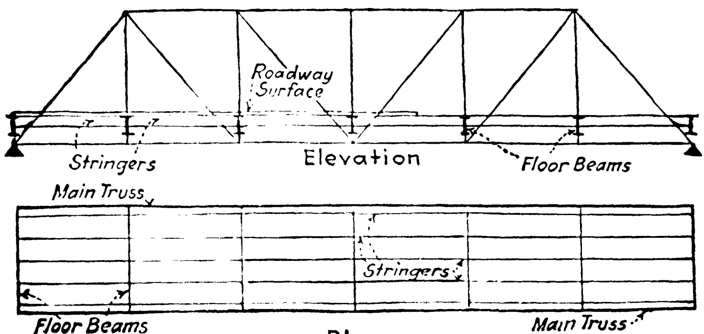


FIG. 46.

beams resting on the chords of the trusses or by stringers and floor beams. The floor of a highway bridge is somewhat similar; usually a concrete slab either furnishes the roadway surface itself or supports wood-block, granite-block, brick, or some type of bituminous surfacing. In the usual cases, in both highway and railroad bridges, the loads are brought more or less directly to the stringers and transferred from them to the floor beams, which in turn deliver them to the trusses at the panel points.

In addition to the bracing effect of the floor beams, a through bridge usually has its trusses braced together by upper and lower lateral bracing, portal bracing, and sway bracing (see Fig. 47). In a deck bridge, the portal and sway bracing are replaced by cross-frames which may extend the full depth of the trusses. A deck bridge is more economical for short spans, especially for

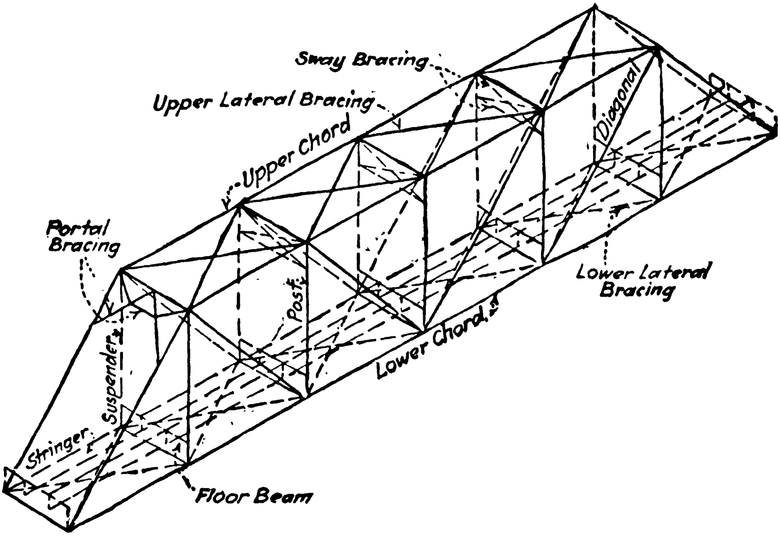

Sway Bracing
Upper Lateral Bracing
Upper Chord
Portal Bracing
Diagonal
Post
Suspender
Stringer
Lower Chord
Lower Lateral Bracing
Floor Beam

Fig. 47.

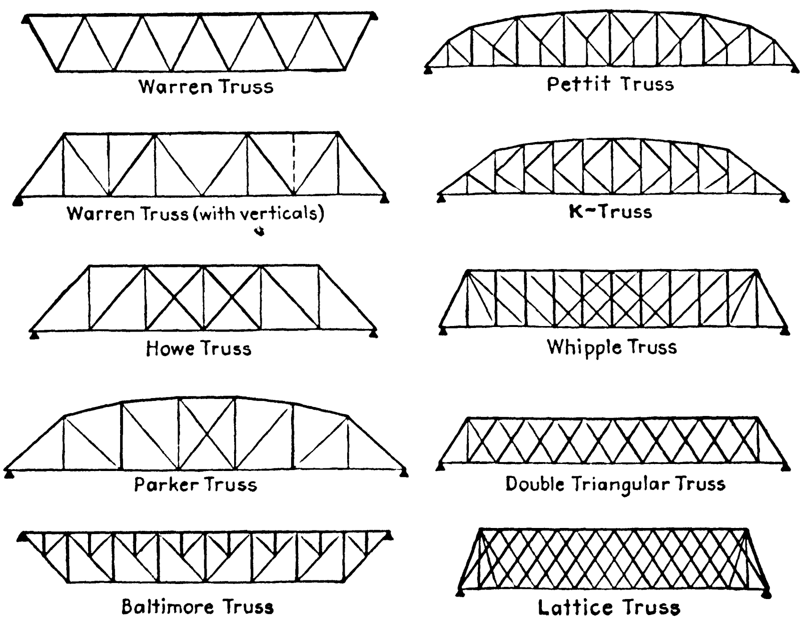

Warren Truss
Pettit Truss
Warren Truss (with verticals)
K-Truss
Howe Truss
Whipple Truss
Parker Truss
Double Triangular Truss
Baltimore Truss
Lattice Truss

Fig. 48.

railroad bridges, for the trusses may be placed closer together, and a smaller amount of floor system and bracing is required.

75. Types of Truss. The two most common types of truss for ordinary spans are the Pratt truss, illustrated in Fig. 46, with vertical posts and tension diagonals sloping downward toward the center; and the Warren truss (see Fig. 48), with inclined compression and tension diagonals altering to form isosceles triangles with the chords. The Warren truss is often built with verticals added at the intersections of the diagonals and chords The Howe truss is used only where steel is difficult to obtain and then most generally for temporary structures. The verticals are the only members where steel is required.

The curved-chord Pratt, or Parker, truss is a common type for intermediate span lengths, whereas where it becomes necessary to subdivide the panel, as mentioned above, the Baltimore or the Pettit (or Pennsylvania) type is used. As an alternative to subdivided panels, the K-truss offers a desirable form of a multiple-web system. It has comparatively low secondary stresses and is particularly well adapted for long riveted spans.

The Whipple truss is in reality two Pratt trusses superimposed upon one another. The double triangular, or double Warren, and the lattice truss are other types of multiple-web system. The indeterminate character of the stresses and the consequent uncertainty involved have resulted in the gradual abandonment of the last three types as desirable forms of construction, the simpler and more readily analyzed types having taken their place.

DEAD-LOAD STRESSES IN BRIDGE TRUSSES

76. Stresses in Web Members. *Parallel Chords and Single-web Systems.* In this type of truss, there are two classes of web members. The first and simplest class is illustrated by member *Bb* of Fig. 49. Such a member is known as a *suspender* or *sub-vertical.* From an inspection of the section *MM*, it is seen that the stress is dependent only upon, and is equal to, the load P_1.

The other class of web member is illustrated by the member *cD.* Since the internal stresses in any section of a truss hold in equilibrium the external forces on either side of the section, a summation of the forces acting on that portion of the truss to the left of the section NN determines the vertical component of the stress in the member cD. Taking that portion of the

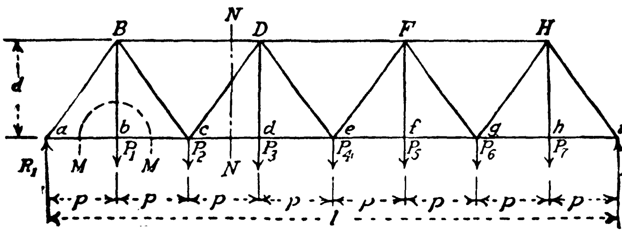


FIG. 49.

truss as a free body, as shown in Fig. 50, with θ as the acute angle between cD and a vertical,

$$R_1 - P_1 - P_2 + S \cos \theta = 0$$

But $R_1 - P_1 - P_2$ is the algebraic sum of all the external vertical forces on the left of the section and is the *vertical shear* in the section and may be designated as V. Therefore,[1]

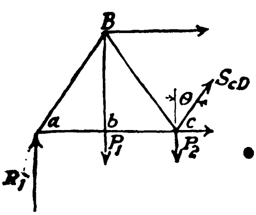


FIG. 50.

$$V + S \cos \theta = 0 \qquad \text{or} \qquad S = -V \sec \theta$$

From this it follows that for trusses with horizontal chords and single-web systems, the stress in any web member, other than subverticals, is equal to the vertical shear in the member multiplied by the secant of the angle which the member makes with the vertical.

Inclined Chords and Single-web Systems. In trusses of this type, since the inclined chord has a vertical component, the method developed above cannot be used without modification, and the method of moments offers a simpler solution.

Figure 51 represents a portion of a Parker truss. To determine the stress in Bc, the section is cut as shown. The chord

[1] If the unknown stress is always assumed to act away from the section under consideration, a positive result indicates a tensile stress and a negative result a compressive stress.

BC produced intersects the chord bc produced at the point O, which is therefore the center of moments for the member Bc. The lever arm is Om, the perpendicular distance from O to Bc

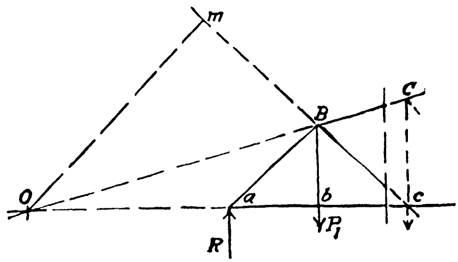


FIG. 51.

produced. From the similar triangles Omc and Bbc,

$$Om = \frac{Oc \times Bb}{Bc}$$

With the net reaction at a, and the panel loads at B and b known, taking moments about O,

$$S_{Bc} \times Om - R \times Oa + P_1 \times Ob = 0$$

from which the stress in Bc may be determined. Although there is no difficulty involved in this method, it can be somewhat

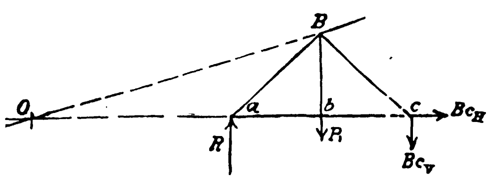


FIG. 52.

simplified by determining first the vertical component of the diagonal and from it the stress. The lever arm is then the same as that for the adjacent vertical and often a multiple of the panel length. In Fig. 52 consider the unknown stress in Bc resolved into two components applied at c, one horizontal Bc_H, and the other vertical Bc_V. The line of action of the horizontal component passes through the center of moments O. Taking moments about O, as before,

$$Bc_V \times Oc - R \times Oa + P_1 \times Ob = 0$$

from which Bc_V may be determined. The actual stress in Bc is Bc_V multiplied by the secant of the angle that Bc makes with the vertical. The stresses in verticals, such as Cc, are determined in a manner similar to that used for the diagonals, except that the stress is determined directly, since there is no horizontal component to be considered.

77. Stresses in Chord Members. Chord stresses in simple trusses may always be obtained by the application of the principle of moments. For example, in Fig. 49, in section NN, the stress in the chord BD may be determined by taking the center of moments at c, *the intersection of the other members cut by the section NN*, and solving the equation of moments for all the forces on one side of the section; *i.e.*, $R_1 \times 2p - P_1 \times p - BD \times d = 0$.

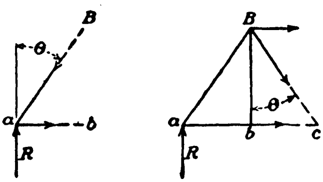


FIG. 53.

For trusses with parallel chords, the stresses may be determined more readily by the resolution of forces. In Fig. 53*a*, which represents the left end of the truss of Fig. 49, the stresses in aB and ab are in equilibrium with the reaction R. A summation of horizontal components gives $-aB \sin \theta + ab = 0$; and since $aB = V \sec \theta$, $ab = +V \tan \theta$. By inspection, $ab = bc$; and in Fig. 53*b*, the summation of horizontal forces gives $BD + Bc \sin \theta + bc = 0$ or $BD = -(ab + Bc \tan \theta)$. In a similar manner, by passing successive sections, the stress in each chord member may be determined by *adding* the horizontal stress component ($V \tan \theta$) of the diagonal cut by the section to the chord stress previously obtained.

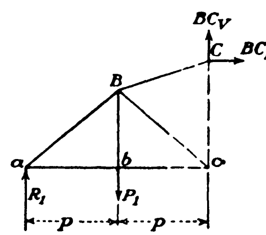


FIG. 54.

In trusses with inclined chords, the stress in the inclined chord may be determined by resolving it into its horizontal and vertical components where its line of action cuts a vertical plane through the center of moments. For example, in Fig. 54, with the center of moments at c, $BC_H = \frac{-R \times 2p + P_1 \times p}{\text{distance } Cc}$. The stresses in the horizontal chords are determined directly by similar analysis.

78. Examples. *The Pratt Truss.* In the Pratt truss of Fig. 55, the weight of the track is assumed as 500 lb. per ft. of bridge. From the equation on page 52 for Cooper's E-60 loading, $w = 2(2 \times 150 + 5 \times 60) = 1200$ lb.

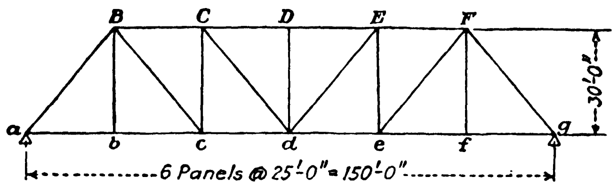


FIG. 55.

The upper panel load is taken as $\frac{1}{3}(25 \times 1200) = 10.0$ kips, and the lower panel load as $\frac{2}{3}(25 \times 1200) + 250 \times 25 = 26.2$ kips.

$$\text{Sec } \theta = (\sqrt{25^2 + 30^2}) \div 30 = 1.30, \text{ and } \tan \theta = 25 \div 30 = 0.83.$$

WEB STRESSES

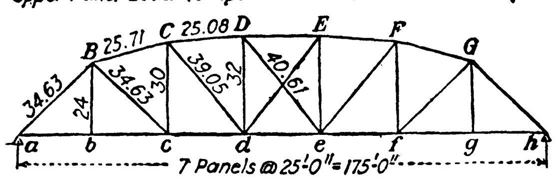
Member	Shear	Stress
aB	90.5	−117.7
Bc	54.3	+ 70.6
Cc	28.1	− 28.1
Cd	18.1	+ 23.5
Bb	....	+ 26.2
Dd	....	− 10.0

CHORD STRESSES

Diagonal	Shear	Horizontal component	Stress	Chord
aB	90.5	75.1	75.1	*ab* = *bc*
Bc	54.3	45.1	120.2	*BC* = *cd*
Cd	18.1	15.0	135.2	*CD*

The Parker Truss. Figure 56 shows all the essential computations for the dead-load stresses in a truss of this type. The numbers on the truss diagram are the lengths of the members in feet.

Upper Panel Load 10 kips Lower Panel Load 30 kips



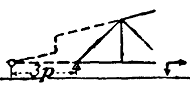
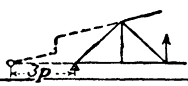
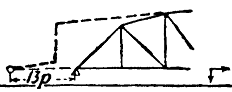
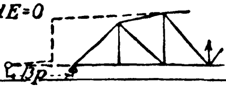
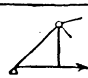
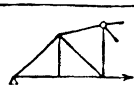
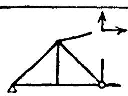
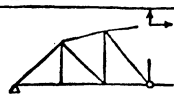
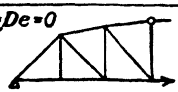
Member	Section	Moment Equation	Vert. Comp.	Length / Vert. Proj.	Stress
aB			−120	34.63 / 24	−173.2
Bb			+30	1	+30.0
Bc		$Bc_V \times 5p - 120 \times 3p + 40 \times 4p = 0$	+40	34.63 / 24	+57.7
Cc		$-Cc \times 5p - 120 \times 3p + 40 \times 4p + 30 \times 5p = 0$	−10	1	−10.0
Cd		$Cd_V \times 16p - 120 \times 13p + 40 \times 14p + 40 \times 15p = 0$	+25	39.05 / 30	+32.6
Dd	dE = 0 	$-Dd \times 16p - 120 \times 13p + 40 \times 14p + 40 \times 15p + 30 \times 16p = 0$	+5	1	+5.0
			Hor. Comp.	Length / Hor. Proj.	Stress
ab = bc		$-ab \times 24 + 120 \times 25 = 0$	+125.0	1	+125.0
cd		$-cd \times 30 + 120 \times 50 - 40 \times 25 = 0$	+166.7	1	+166.7
BC		$BC_H \times 30 + 120 \times 50 - 40 \times 25 = 0$	−166.7	25.71 / 25	−171.4
CD		$CD_H \times 32 + 120 \times 75 - 40 \times 50 - 40 \times 25 = 0$	−187.5	25.08 / 25	−188.1
de = −DE	dE = De = 0 	$-de \times 32 + 120 \times 75 - 40 \times 50 - 40 \times 25 = 0$	+187.5	1	+187.5

FIG. 56.

LIVE-LOAD STRESSES IN BRIDGE TRUSSES

79. Uniform Live-load Stresses. *Trusses with Horizontal Chords.* The general effect on a bridge structure produced by a live or moving load is the same whether the load is considered as uniform or as a system of concentrated loads. The former type presents fewer complications to the beginner and furnishes an approach to the apparently more exact method of calculation. It seems advisable, therefore, to consider, first, bridge trusses under uniform live loads.

A uniform load per linear foot is selected of such an amount as will produce stresses in the truss members about equal to the stresses that would be produced by the actual wheel loads. This uniform load per foot is multiplied by the panel length to give the equivalent load per panel. It is then assumed that this panel load may be applied at any number of consecutive panel points of the truss.

The value of the uniform load selected to replace a series of wheel concentrations varies with the span of the bridge and the shear and moment calculations. Numerous charts have been prepared for the selection of equivalent uniform loads, one of the best appearing in Vol. 86, page 606, of the *Transactions* of the American Society of Civil Engineers. An approximate value may be obtained by dividing the sum of the wheel loads by the span of the bridge. The position of the uniform load for maximum chord stresses is obvious; for since a load at any point on the span produces a positive moment, the whole span should be loaded for maximum moments, from which maximum chord stresses are obtained.

For web members, the conventional method of calculation of the stresses due to uniform load is to assume all panel points on one side of the panel cut by the section through the member fully loaded, while no load is considered on the other side. The maximum positive and maximum negative shears are obtained under these two possible loadings. For instance, in Fig. 57, if all panel points to the right of the section are loaded with a load P, moments being taken about the right support, the reaction at a

is $15P/8$, and, since there is no load between a and the section, the shear in the section is $15P/8$. If a load were added at c, the reaction would become $^{21}/_8P$, but the shear in the section would be $^{21}/_8P - {}^8/_8P = {}^{13}/_8P$, which is less than the value obtained with only the panel points to the right of the section loaded. It follows, therefore, that the largest positive shear occurs when the live load extends from the section to the right support

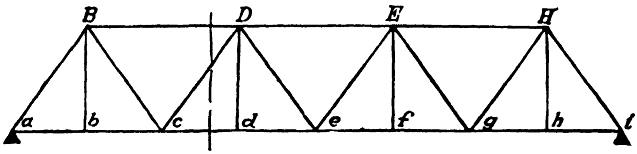


Fig. 57.

and the largest negative shear when the live load extends from the section to the left support. The maximum positive shear produces the greatest stress of the same character as the dead-load stress. The resulting stress is therefore called the maximum live-load stress. The maximum negative shear produces the greatest stress of opposite character to the dead-load stress, and the resulting stress is called the minimum live-load stress. The

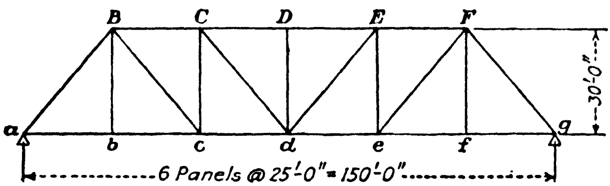


Fig. 58.

minimum live-load stress neutralizes some of the dead-load stress, and if it is greater than the dead-load stress the resultant stress in the member will change character as the live load crosses the structure, requiring then that the member be designed for both types of stress, tension and compression.

Example. The stresses in the Pratt truss of Fig. 58 are to be obtained for a uniform live load of 3775 lb. per ft. of truss. The panel load is 3775 × 25 = 94 kips. Sec θ = 1.30, tan θ = 0.83.

STRESSES IN WEB MEMBERS

Web member	Panel points loaded for maximum stress	Shear	Stress	Panel points loaded for minimum stress	Shear	Stress
aB	*b, c, d, e, f*	235.5	−305.5	0	0	0
Bb	*b*	.....	+ 94.0	0	......	0
Bc	*c, d, e, f*	156.7	+203.7	*b*	−15.7	−20.4
Cc	*d, e, f*	94.0	− 94.0	*b, c*	−47.0	+47.0
Cd	*d, e, f*	94.0	+122.2	*b, c*	−47.0	−61.1
Dd	0	.....	0	0	......	0

LIVE-LOAD STRESSES IN CHORD MEMBERS

Diagonal	Shear	Horizontal component	Stress	Chord
aB	235	195.8	+195.8	*ab* = *bc*
Bc	141	117.5	+313.3	*BC, cd*
Cd	47	39.2	−352.5	*CD*

80. Uniform Live-load Stresses. *Trusses with Inclined Chords.* In general, the same conditions of loading are applied to trusses with inclined chords to obtain the maximum and minimum stresses as were used for trusses with horizontal chords. As in the dead-load stress calculation, the method of moments is used.

As an example let the truss whose dead-load stresses were computed in Art. 78 be taken, with a live panel load of 98 kips (Fig. 59).

With live load on every panel point of the truss, the left reaction is... 294

With 5 panel loads from the right, $R_L = \frac{98}{7}(1 + 2 + 3 + 4 + 5)$... 210

With 4 loads.......... 140
With 3 loads.......... 84
With 2 loads.......... 42
With 1 load.......... 14

The stresses in the web members on the right of the center are computed with the load brought on from the right. The values thus obtained correspond to the stresses in the symmetrical members with the load brought on from the left. Note that the left reaction is used for all computations, but that for members to the right of the center, the center of moments is on the right of the truss.

The two diagonals in the center panel are to be designed to resist either tension or compression, and it is sufficiently accurate to assume that, under all loading conditions, each will resist one-half of the shear in the panel, one diagonal being in tension and the other in compression.

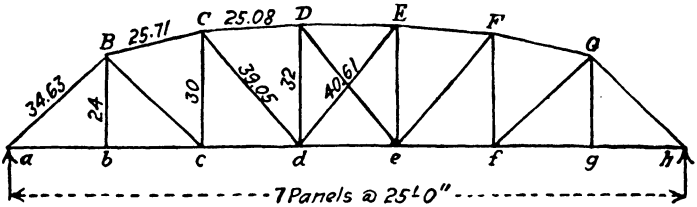


FIG. 59.

The stresses in aB and Bb are determined as for a Pratt truss.

$$aB = -294 \times \frac{34.63}{24} = -424.2 \text{ kips, and } Bb = +98.0 \text{ kips.}$$

BC produced intersects the lower chord produced 3 panel lengths to the left of a. With this intersection as the center of moments and the live load extending from c to h,

$$Bc = \frac{210 \times 3 \times 25}{5 \times 25} \times \frac{34.63}{24} = +181.6 \text{ kips}$$

The center of moments for Cc is the same as for Bc, but the load extends only from d to h.

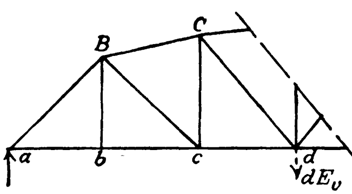


FIG. 60.

$$Cc = -\frac{140 \times 3 \times 25}{5 \times 25} = -84.0 \text{ kips}$$

CD produced intersects the lower chord produced 13 panel lengths to the left of a. With this intersection as the center of moments and the load extending from d to h,

$$Cd = \frac{140 \times 13 \times 25}{16 \times 25} \times \frac{39.05}{30} = +148.1 \text{ kips}$$

In the center panel the maximum shear occurs with the panel points e, f, and g loaded and has a value of 84 kips. Under the assumed conditions of equally divided shear

$$De = \frac{84 \times 40.61}{32} \times \frac{1}{2} = +53.3 \text{ kips}$$

Since, with any unsymmetrical live loading, both of the center diagonals are stressed, it is impossible to cut a section through the member Dd without also cutting a diagonal. The section shown in Fig. 60 offers as little difficulty as any; and, with the center of moments the same as for Cd

and the panel points e, f, and g loaded,

$$Dd = \frac{42 \times 16 \times 25 - 84 \times 13 \times 25}{16 \times 25} = -26.3 \text{ kips}$$

The maximum tension in Dd is obtained by computing that for the symmetrical member Ee. Here again, a diagonal must also be cut by the

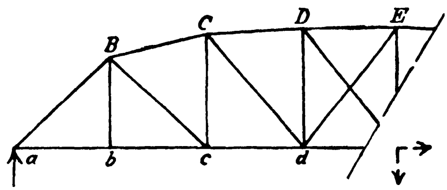


FIG. 61.

section, and the section of Fig. 61 may be used with the center of moments 13 panel lengths to the right of h and the loading the same as for Dd.

$$Ee = \frac{84 \times 20 \times 25 - 42 \times 16 \times 25}{16 \times 25} = +63.0 \text{ kips}$$

The maximum stresses in eF, Ff, and fG, computed in order to furnish the minimum stresses for the symmetrical members, Cd, Cc, and Bc, respectively, are all obtained by taking the center of moments on the right side of the truss and are

$$eF = -\frac{42 \times 20 \times 25}{16 \times 25} \times \frac{39.05}{30} = -68.3 \text{ kips}$$

$$Ff = \frac{42 \times 10 \times 25}{5 \times 25} = +84.0 \text{ kips}$$

and

$$fG = -\frac{14 \times 10 \times 25}{5 \times 25} \times \frac{34.63}{24} = -40.4 \text{ kips}$$

The maximum and minimum stresses due to dead and live load are as follows:

Member	Dead-load stress	Maximum live-load stress	Minimum live-load stress	Maximum	Minimum
aB	−173.2	−424.2	0	−597.4	−173.2
Bb	+ 30.0	+ 98.0	0	+128.0	+ 30.0
Bc	+ 57.0	+181.8	−40.4	+238.8	+ 16.6
Cc	− 10.0	− 84.0	+84.0	− 94.0	+ 74.0
Cd	+ 32.6	+148.1	−68.3	+180.7	− 35.7
Dd	+ 5.0	+ 63.0	−26.3	+ 68.0	− 21.3
De	0	+ 53.3	−53.3	+ 53.3	− 53.3

Since the maximum chord stresses are obtained with live load at every panel point, the live-load stresses bear the same relation to the dead-load stresses as the live panel load to the dead panel load. In this case the two panel loads are 98 kips and 40 kips, respectively, so that the stresses in the chords due to live load are $^{98}\!/_{40}$ of the dead-load stresses.

81. Concentrated Live-load Stresses. The live-load stresses may be computed with more accuracy if the exact position of each load is considered. In a system of loads such as Cooper's, as shown in Fig. 27, whenever the wheels are so placed that a maximum stress is produced, each individual load has a definite position on the span and its contribution to the stress being computed can be determined. It is possible to develop criteria that can easily be solved, which will indicate the exact position of any system of loads producing the maximum stress in any member.[1] It is beyond the scope of this book to carry out such a development. It will be sufficient to realize that for web members a load is always placed at the right end of the panel cut by the imaginary section through the member whose stress is to be determined and that the length of load in the panel bears about the same relation to the panel length as the length of load on the bridge bears to the span of the bridge. With locomotive loadings one of the heavy driving wheel loads is always placed at the right-hand end of the panel. For chord members the span should normally be fully loaded, but the influence of the heavy driving wheels causes the position for maximum chord stresses to be slightly different, resulting in the first wheel load being some distance from the end of the span.

In the truss of Fig. 62, the maximum live-load stress in the member *Cd* due to Cooper's *E*-70 loading is desired. With wheel 3 (see table on page 57) placed at *d*, wheels 1, 2, and 3 are in the panel *cd* and distribute their load to the truss through the stringers and floor beams at panel points *c* and *d*. The moment of wheels 1 and 2 about 3 equals the moment of 1 and 2 about the panel point *d* and is 58 kip-ft. Therefore, the stringer reaction at *c* is $58 \div 25 = 2.32$ kips. The distance from panel point *d* to the right support is 75 ft., and the distance from wheel 1 to the right support is 88 ft. In the table the moments of wheels 1 to 14 about wheel 15 is given as 2704 kip-ft.

[1] Such criteria are developed in the author's "Stresses in Simple Structures."

Therefore the left reaction of the truss is 2704 ÷ 150 = 18.03 kips. The shear in the panel *cd* is the left reaction minus any loads transferred to the truss to the left of a section cut through the panel and is

$$18.03 - 2.32 = 15.71 \text{ kips.}$$

This value is for Cooper's *E*-10 loading. For *E*-70 it is 7 × 15.71 = 109.97 kips, and, since sec θ = 1.30, the stress is 109.97 × 1.30 = 143.0 kips.

In the truss of Fig. 62 the maximum live-load stresses in members *Bb* and *BC* due to the *H*20 loading (see page 53) are desired. It will be considered that the bridge has a two-lane width so that each truss will sustain the load at one line of trucks.

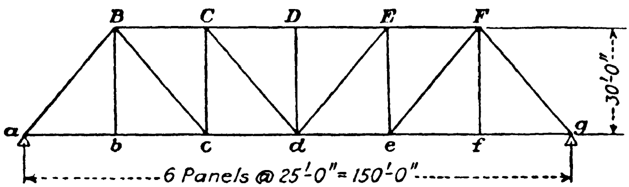


FIG. 62.

The stress in the member *Bb* depends only upon the loads in the panels *ab* and *bc*. The distance *ac* is 50 ft. which is less than 60 ft. so that the truck loading will be used. With the rear wheel placed at *b*, the load at that point is 32 + (11 × 8) ÷ 25 = 35.5 kips, which is the stress in member *Bb*.

The stress in the member *BC* depends upon the moment at the point *c*. This is maximum when the 18,000-lb. load is placed at *c* and the remainder of the span loaded with the uniform load of 640 lb. per ft. of lane. With this loading the left reaction is

$$(18 \times 100 + 0.64 \times 150 \times 75) \div 150 = 60 \text{ kips,}$$

the moment at $c = 60 \times 50 - 0.64 \times 50 \times 25 = 2200$ kip-ft., and the stress in $BC = 2200 \div 30 = 73.3$ kips.

82. Maximum Stresses. The maximum stresses for which the bridge is designed are the sum of the dead-load, live-load, and impact stresses. The latter are obtained by using the formulas given in Art. 49 for the class of bridge under consideration. In simple trusses the additional stresses due to the lateral forces rarely control the design of the main members; for when they are added to the three stresses mentioned above, all specifications allow an increase in the working unit stresses.

It is important that minimum live-load stresses (see Art. 79) be computed in order to determine whether a reversal in stress may be expected as the live load moves across the structure. If, for any member, the algebraic sum of the dead-load, minimum live-load, and impact stresses is of opposite sign to the sum of the dead-load, maximum live-load, and impact stresses, a reversal of stress is apparent in that member, and it must be designed for both of the resulting stresses.

CHAPTER VI

DESIGN OF TENSION AND COMPRESSION MEMBERS

83. Tension Members of Timber. Structural members whose main stress is tension are not constructed of timber, unless the structure must of necessity be wholly of wood. Where possible a combination of steel bars for tension with timber members for compression makes a more economical design. However, very often the lower chords and the tension web members of trusses are constructed of timber.

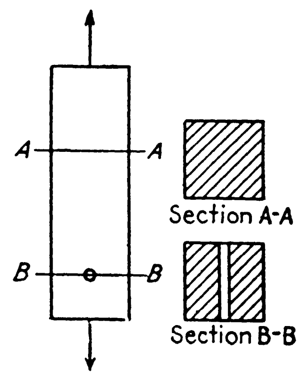


FIG. 63.

If a piece of timber such as is shown in Fig. 63 is subjected to a tensile force, barring imperfections, final failure will occur at the section *BB* because of the smaller amount of material there. A section such as *BB* is known as the *net section*, the critical section for tensile stress. In the design of tension members the net section of area required is determined by dividing the total stress in the member by the allowable unit stress.

Tension members of timber trusses are often built up of 2- or 3-in. planks, held together by various types of connectors, but in such cases as well as for single sticks the net section is usually found in the joints, where the member is connected to the other members of the truss. The details of such joints will be explained in the next chapter.

84. Tension Members of Steel. Tension members of steel may be divided into three general classes: (1) eye bars; (2) square or round bars; (3) sections made up of a single elementary structural shape or a group of such shapes fastened together so as to act as a unit.

Eye bars are used mainly for the tension members of pin-connected truss bridges. They are made by first upsetting, in a die, each end of a bar of rectangular cross-section to an approximately round shape, and then boring holes of the proper size in these enlarged ends, the distance center to center of holes being as required in the finished structure (Fig. 64*a*). The heads are made of the same thickness as the rest of the bar and of a width sufficient to prevent failure at the ends. The tensile stress is transmitted from the eye bar to the other members of the structure by means of a pin which passes through the eye in the bar and through holes in the other members. The dimensions of standard eye bars are given in the handbooks of the steel manufacturers.

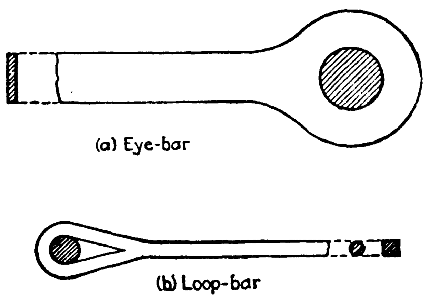


FIG. 64.

Square or round bars are sometimes made suitable for use as tension members in unimportant structures by bending each end of the bar back upon the bar and welding it to form a loop (Fig. 64*b*). The loops are made stronger than the main body of the bar, so that final failure will occur at the weakest point between the loops. The tensile stress is transmitted from the loop bar in a manner similar to that in the eye bar.

Round bars are also prepared for use as tension members by threading the ends and attaching nuts on the threaded ends after the bar is in its proper place in the structure. The bar may be threaded on the normal ends, or the ends may first be upset so as to form a larger cross-section into which the threads are cut. In the former case failure will occur near the ends of the bar, owing to the smaller effective cross-section there, whereas in the latter case failure will occur at the weakest point between the threaded ends, provided the amount of upset is sufficient to give a cross-section at the root of the thread greater than that of the main body of the bar.

Plates are sometimes used for small tension members, but their tendency toward vibration renders them generally less desirable than single angles or channels. They are, however, quite satisfactory in small riveted structures which are comparatively free from vibration.

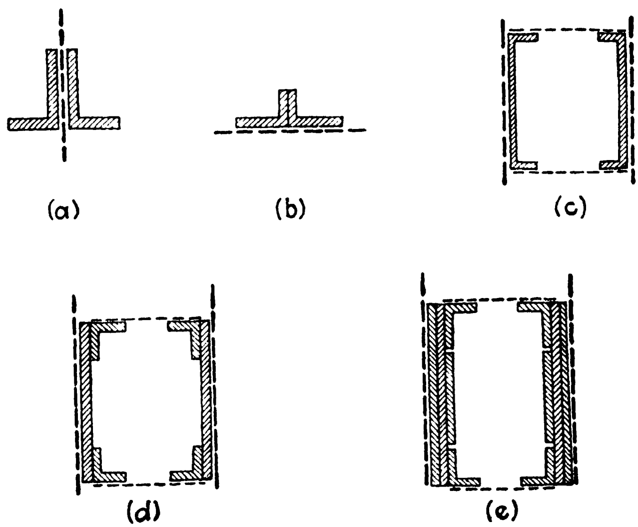


FIG. 65.

Single angles are stiffer than plates, but the difficulty of arranging the end connections, so as to prevent eccentricity of stress and bending in the angle, limits their use to small and unimportant members. Single channels are more satisfactory in this respect than single angles, and they can be used for larger members; but the undesirable eccentricity still exists to a certain extent.

Built-up sections, consisting of two or more angles or channels, or of a combination of angles and plates or channels and plates, are used for the tension members of the larger riveted structures and for some of the members of the pin-connected structures. In such members, most of the metal should be concentrated in plates parallel to the plane of the truss so as to make the transmission of stress at the joint connections as direct as practicable. Some of the more common sections of built-up tension members are shown in Fig. 65.

The individual members comprising a riveted steel structure are joined together by means of connection or gusset plates, and the stress in one member is transmitted through the gusset plates to the other members by means of rivets, as shown in Fig. 66. The connection plates at the joints for the cross-sections shown in Fig. 65 are indicated by the heavy dotted lines, and the tie plates which are used at intervals to prevent

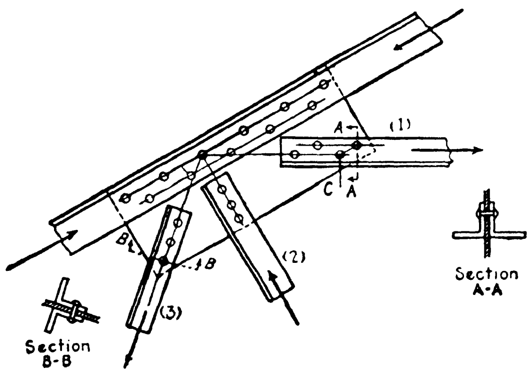


Fig. 66.

vibration of the individual elements are indicated by the light dotted lines. The sections *c*, *d*, *e* of Fig. 65 can be used in pin-connected trusses as well as in riveted trusses, the pin in each case passing through the webs of the members.

The Net Section. The strength of a tension member is dependent upon the net cross-sectional area of the member. The net section of a member which is riveted to connection plates at its ends is usually a section which passes through the holes for the rivets connecting the member to the plates. In general it might be concluded that the weakest section of an ordinary tension member is a cross-section through the largest number of rivet holes. There are several modifications of this general rule which are made necessary by the various special arrangements of the members and the rivets connecting these members.

The net section of member 3 (Fig. 66) is equal to the gross section of the two angles minus the area which is removed from the cross-section by two rivet holes, one in each angle. This latter area for each angle is a rectangle, one side of which is equal to the thickness of the angle and the other side equal to the effective diameter of the rivet hole. The diameter of the hole as made is usually $\frac{1}{16}$ in. greater than the nominal diameter of the rivet, in order to allow for placing of the heated rivet in the hole. Since, however, during the punching process the fibers around the hole are torn to some extent by the punch, most specifications require that the effective diameter of the hole be

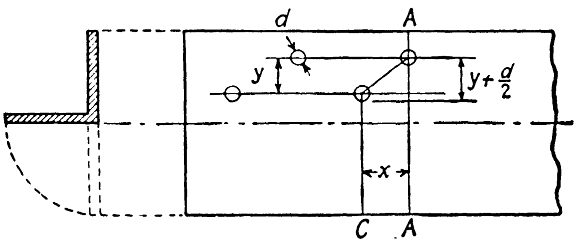


FIG. 67.

taken as $\frac{1}{8}$ in. greater than the nominal diameter of the rivet in computing the net section.

In member 1, which has rivets in two gage lines, final failure may occur along the line AA, or along the line AC, depending upon the distance in the line of stress between the two rivets cut by the line AC. Figure 67 shows one of the angles of member 1, with the two legs bent to the same plane. In order that the two sections AA and AC may have equal strength, the distance between the two rivets, measured in the line of stress, (the pitch) must be of such an amount as to make the length of AC minus the effective diameter of two rivet holes equal to the length of AA minus the effective diameter of one rivet hole. Let the effective diameter of one rivet hole be d (*i.e.*, d = diameter of rivet plus $\frac{1}{8}$ in.). Then, for equal strength,

$$y + \frac{d}{2} - \frac{d}{2} \quad \text{must equal} \quad \sqrt{x^2 + y^2} - d$$

Equating and solving for x,

$$x = \sqrt{2yd + d^2}$$

If the actual pitch of the rivets, x, is greater than the value obtained from the above equation, probable failure may be assumed along the line AA, and vice versa.

The American Institute of Steel Construction specifies that, in the case of a chain of holes extending across a part in any diagonal or zigzag line, the net width of the part shall be obtained by deducting from the gross width the sum of the diameters of

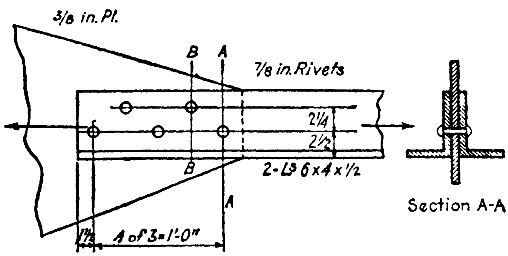


FIG. 68.

all the holes in the chain, and adding, for each gage space in the chain, the quantity $S^2/4g$, where

S = longitudinal spacing (pitch), in inches, of any two successive holes (corresponding to x in Fig. 67).

g = transverse spacing (gage), in inches, of the same two holes (corresponding to y in Fig. 67).

Applying this specification to the member shown in Fig. 68,

Cross-section of two angles $= 2 \times 4.75 \qquad = 9.50$ sq. in.

Area deducted for two 1-in. holes each angle (one in section A, one in section B) $= 2(2 \times 1 \times \frac{1}{2}) = 2.00$ sq. in.

7.50 sq. in.

$$\frac{S^2}{4g} = \frac{3^2}{4 \times 2\frac{1}{4}} = 1.00 \text{ sq. in.}$$

Net section $= 8.50$ sq. in.

This value is the same as if the section A-A alone were considered. If the pitch in this case were greater than 3 in., section *A-A* would be the critical net section; if the pitch were less than 3 in., the net section as determined above would govern.

The specifications of the American Railway Engineering Association require that for a single angle in tension, connected

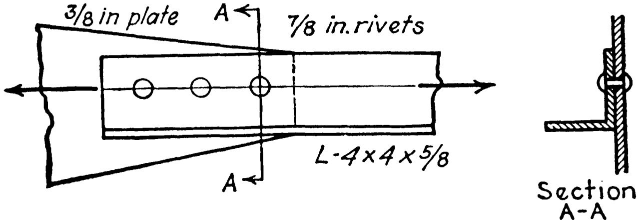


Fig. 69.

by one leg only, the net section in tension be taken as the net area of the connected leg plus 50 per cent of the area of the unconnected leg. Since the distribution of stress throughout the cross-section of an angle connected by one leg only is not

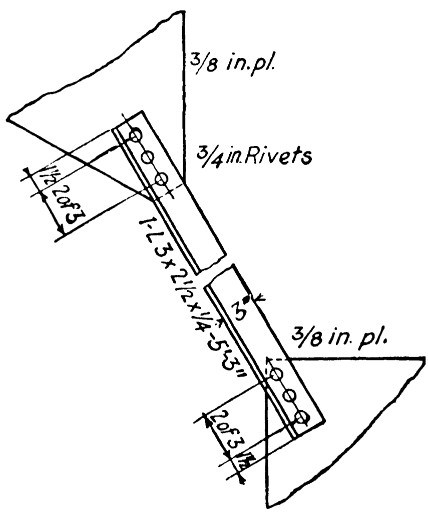


Fig. 70.

uniform, this specification has been written to prevent the overstressing of the connected leg. Applying this specification to the angle shown in Fig. 69 the net section is

$$(4 - 1) \times \tfrac{5}{8} + \tfrac{1}{2} \times 3\tfrac{3}{8} \times \tfrac{5}{8} = 2.93 \text{ sq. in.}$$

85. Examples. In all the following problems the strength of the connection at each end of a member is assumed greater than the strength of the member itself. The elements of all sections used are taken from the 1939 edition of "Steel Construction" of the American Institute of Steel Construction. The working unit stress in tension is 20,000 p.s.i. (see Chap. II).

1. How much tensile stress can be supported safely by the single angle of Fig. 70?

The effective net section of the angle is

$$(3 - \tfrac{7}{8}) \times \tfrac{1}{4} + \tfrac{1}{2}(2\tfrac{1}{4} \times \tfrac{1}{4}) = 0.81 \text{ sq. in.}$$

The safe tensile stress is $0.81 \times 20{,}000 =$ 16,200 lb.

2. What tensile stress may be transmitted by the two angles shown in Fig. 71?

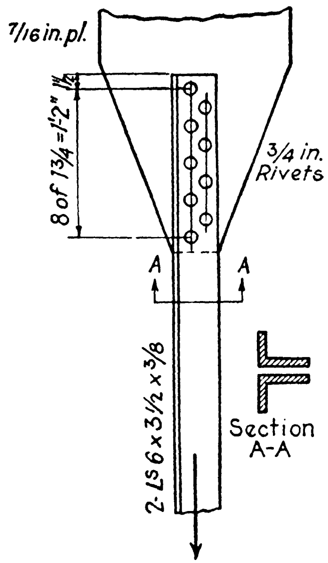


Fig. 71.

Gross section of two angles $= 2 \times 3.42 = 6.84$ sq. in.

Area deducted for two rivet holes in each angle $2(2 \times \tfrac{7}{8} \times \tfrac{3}{8}) = 1.31$ sq. in.

5.53 sq. in.

$$\frac{S^2}{4g} = \frac{(1\tfrac{3}{4})^2}{4 \times 2\tfrac{1}{2}} = 0.31 \text{ sq. in.}$$

5.84 sq. in.

The safe tensile stress is $5.84 \times 20{,}000 =$ 116,800 lb.

3. The angles of Fig. 72 are to transmit a tensile stress of 72,000 lb. The size of each angle is $3\tfrac{1}{2} \times 3$ in. Determine the required thickness.

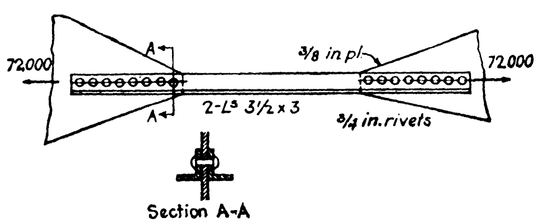


Fig. 72.

The net section required is

$$\frac{72{,}000}{20{,}000} = 3.60 \text{ sq. in.}$$

Assuming two $3\tfrac{1}{2} \times 3 \times \tfrac{3}{8}$ angles, the gross area $= 2 \times 2.30 = 4.60$ sq. in. The net area $= 4.60 - 2 \times \tfrac{7}{8} \times \tfrac{3}{8} = 3.94$ sq. in. A pair of angles

$\frac{5}{16}$ in. in thickness furnish a net section of only 3.31 sq. in.; therefore, a thickness of $\frac{3}{8}$ in. is adopted.

4. The built-up member of Fig. 73 is to sustain a tensile stress of 690,000 lb. The size of the angles, the width of the plates, and the arrangement of the rivets are shown in the figure. The spacing of the rivets along the member in the joint is 4 in. What is the required thickness of the plates?

The net section required is

$$\frac{690,000}{20,000} = 34.5 \text{ sq. in.}$$

In the net section A-A, allowance must be made for four rivet holes in each plate and one in each angle. The gross area of each angle is 3.75 sq. in., and

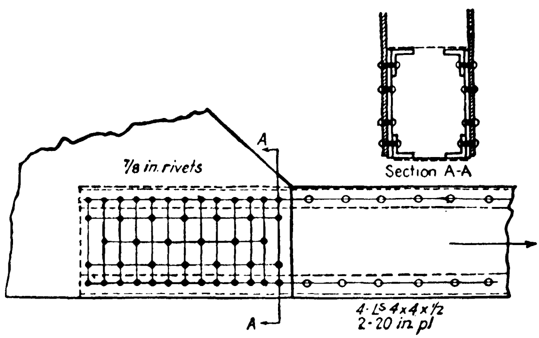


FIG. 73.

the net area of four angles is $4(3.75 - 1 \times \frac{1}{2}) = 13.0$ sq. in. This leaves $34.5 - 13.0 = 21.5$ sq. in. to be furnished by the plates. The net depth of each plate is $20 - 4 \times 1 = 16$ in. Therefore, the required thickness of each plate is $21.5 \div (2 \times 16) = 0.67 = \frac{11}{16}$ in.

86. Tension Members of Concrete. Since concrete has so little strength in direct tension, tension members of concrete are not good design. Where apparently such members exist, they are usually structural steel shapes or steel bars encased in concrete. In such cases they are designed in the same manner as steel tension members, the concrete being merely a protective covering.

COMPRESSION MEMBERS

87. Rankine's Formula. The compression members or columns usually employed in engineering practice fail under stresses caused by combined flexure and compression. The ultimate unit stress for these members is less than that for short prisms. When such a member is perfectly straight, an axial load produces the same unit stress on all parts of the section. When any bending occurs, due to imperfections of the material or lack of straightness, the unit stress on the concave side becomes greater than that on the convex side. Figure 74*a* shows the flexure exaggerated, and Fig. 74*d* shows the distribution of stress on a cross-section at the middle of the column, since it is clear that

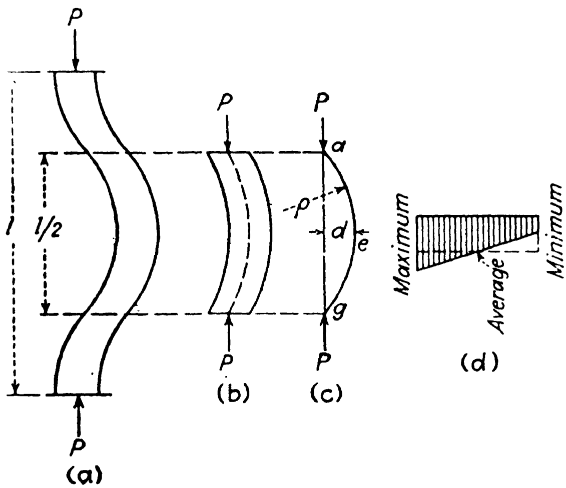


Fig. 74.

the flexure formula (see Art. 55) will apply in cases of lateral bending.

The following notation will be used:

f' = ultimate unit strength in compression.
f_1 = average unit stress due to direct compression.
f_2 = maximum unit stress due to flexure.
f = maximum allowable unit stress in compression.
A = area of cross-section.
I = least moment of inertia of cross-section.
R = least radius of gyration of cross-section.
l = unsupported length of column.
c = distance from gravity axis of column to extreme fiber.
P = load on the column.
d = maximum lateral deflection of column.

From the notation given above and from Fig. 74*d* it is seen that failure will occur when $f_1 + f_2 = f'$. Since $f_1 = P/A$ and $f_2 = Mc/I$, the ultimate unit stress at failure is

$$f' = \frac{P}{A} + \frac{Mc}{I} \qquad (a)$$

Figure 74*a* shows a column fixed at both ends and supporting a load P. Due to the lateral bending a moment is produced in the column which tends to produce tension in the left-hand side of the column at the ends and compression on that side at the center. The reverse is true of the right-hand side. This shows that the moment changes in character or passes through a zero value. The points at which zero moment occurs are called the points of inflection, and they are located at a distance of $l/4$ from each end of the column. Figure 74*b* shows the length of the column between the points of inflection. Since the moment is zero at the ends of this section, the greatest moment developed is at the point of maximum lateral bending which is at the center of the column. Figure 74*c* represents the axis of the center portion of the column, and the maximum bending moment is Pd.

Substituting Pd for M,

$$f' = \frac{P}{A} + \frac{Pcd}{I} \qquad (b)$$

If acg is considered to be a circular arc of radius ρ, then from similar triangles $\rho = l^2/32d$; and since[1] $M = EI/\rho = f_2I/c$, it follows that $cd = l^2\left(\frac{f_2}{32E}\right)$. At failure the quantity $f_2/32E$ is a constant for any material. Let this constant be known as C. Then,

$$f' = \frac{P}{A} + \frac{PCl^2}{I}$$

and substituting AR^2 for I

$$f' = \frac{P}{A}\left(1 + C\frac{l^2}{R^2}\right)$$

[1] See Art. 134.

or

$$\frac{P}{A} = \frac{f'}{1 + C\dfrac{l^2}{R^2}}$$

If f be substituted for f', the equation becomes

$$\frac{P}{A} = \frac{f}{1 + C\dfrac{l^2}{R^2}} \tag{1}$$

and is now in usable form for design purposes for various slenderness ratios (l/r) provided that C is known. As C depends upon the stress due to flexure at failure and the modulus of elasticity of the material, an approximate value of C is easily arrived at. For timber the usual value is $1/3000$ and for steel 1/25,000.

The formula as developed above is for columns fixed at both ends, a condition often referred to as "square ends." In modern construction the ends of columns generally are partly or wholly fixed. However, there are cases where one or both ends are free, or "pin-ended." In such a case no moment is developed at the free end of the column. With both ends pinned, there is no point of inflection, and the formula for safe load becomes

$$\frac{P}{A} = \frac{f}{1 + 4C\dfrac{l^2}{R^2}} \tag{2}$$

With one end pinned and one end square,

$$\frac{P}{A} = \frac{f}{1 + \dfrac{16}{9}C\dfrac{l^2}{R^2}} \tag{3}$$

Though Rankine's formula has often been used for the design of compression members, modern practice has developed other formulas for the various materials. These formulas are not wholly theoretical but are based partly on analysis and partly on test data.

88. Compression Members of Timber. Many simple straight-line column formulas have been used for design. As a result of tests made by the Forest Products Laboratory of the U.S. Forest Service, two formulas somewhat more complicated than those hitherto in use have been proposed. They have been incorporated in the specifications of the National Lumber Manufacturers' Association. The tests mentioned above demonstrated that up to a certain point, as the slenderness ratio increased, the detrimental effect of defects on the strength of columns decreased. It was therefore desirable to provide formulas for design for two classes of columns, *intermediate columns* and *long columns.* In these specifications, columns are divided into three classes: short columns, intermediate columns, and long columns.

A short column is a column in which the ratio of the unsupported length to the least dimension (l/d) is not greater than 11. The allowable unit stress f is the allowable unit stress parallel to the grain.

An intermediate column is a column in which the l/d ratio is greater than 11 and less than K. The allowable unit stress is

$$f_i = f\left[1 - \frac{1}{3}\left(\frac{l}{Kd}\right)^4\right]$$

A long column is a column in which the l/d ratio is greater than K and less than 50. The allowable unit stress is

$$f_i = \frac{\pi^2 E}{36(l/d)^2} = \frac{0.274E}{(l/d)^2}$$

In the foregoing, $K = \frac{\pi}{2}\sqrt{\frac{E}{6f}}$. The values of K for some of the species of timber in general use are given on page 23.

89. Examples. 1. A Douglas fir column with a nominal cross-section of 8 × 8 in. (actual size 7.5 × 7.5 in.) has a length of 12 ft. What safe load will it support according to the specifications of the N.L.M.A.?

From page 23, $f = 1200$ and $K = 23.4$. $\frac{l}{d} = \frac{12 \times 12}{7.5} = 19.2$ which is less than 23.4; thus, this column is classed as an intermediate column, and

$$f_i = 1200\left[1 - \frac{1}{3}\left(\frac{144}{23.4 \times 7.5}\right)^4\right] = 1020 \text{ p.s.i.}$$

The safe load is 1020 × 7.5 × 7.5 = 57,400 lb.

2. A longleaf yellow pine column is 18 ft.-0 in. in length and is to support a load of 125,000 lb. What size column is required?

From page 23, $f = 1200$ and $K = 23.4$. The minimum dimension cannot be less than $(18 \times 12) \div 23.4 = 9.2$ for an intermediate column, so that a 10 × 10 in. (actual size $9\frac{1}{2} \times 9\frac{1}{2}$ in.) is selected for trial.

$$\frac{l}{d} = \frac{12 \times 18}{9.5} = 22.7 \text{ and } f_c = 1200\left[1 - \frac{1}{3}\left(\frac{216}{23.4 \times 9.5}\right)^4\right] = 840 \text{ p.s.i.},$$

and the column as selected can sustain $840 \times 9.5 \times 9.5 = 75{,}800$ lb.

Since this is not sufficient, a 12 × 12 in. (11.5 × 11.5 in. actual) will be tried.

$\frac{l}{d} = \frac{12 \times 18}{11.5} = 18.8$ and $f_c = 1050$, so that a 12 × 12-in. column can sustain 138,900 lb. and this size is adopted.

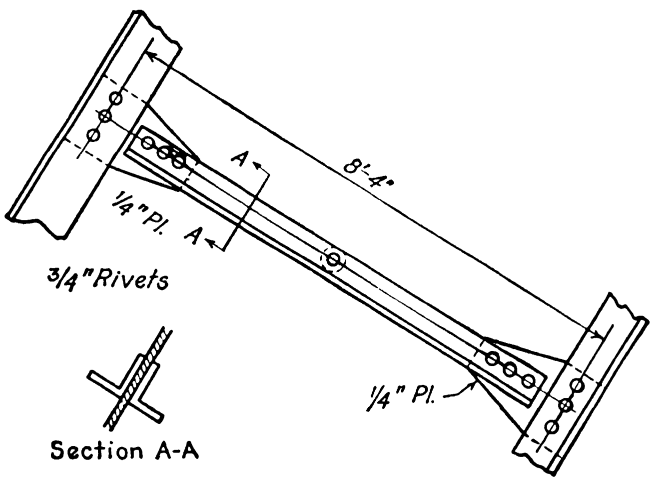


FIG. 75.

90. Compression Members of Steel. The column formulas of the American Institute of Steel Construction are given on page 29. For values of l/r less than 120, the allowable unit stress is

$$f_s = 17{,}000 - 0.485\frac{l^2}{R^2}$$

and for l/r greater than 120

$$f_s = \frac{18{,}000}{1 + \frac{l^2}{18{,}000R^2}}$$

In designing compression members the gross section may be considered as effective; for even though holes are punched in the members having riveted connections, these holes are later filled by rivets and compressive stresses may be transmitted by this filling metal.

91. Examples. 1. A two-angle member, the details of which are shown in Fig. 75, is subjected to a compressive stress of 20,000 lb. Design the member. The ratio l/r is not to exceed 120.

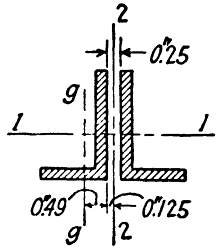


FIG. 76.

The unsupported length of the member is taken as the distance between the points of intersection of the gage lines at the joints, 8 ft.-4 in. Therefore, the minimum value of $R = \frac{8.33 \times 12}{120} = 0.83$ in. From the A.I.S.C. "Steel Construction" (1939, page 43) the value of $I_{g\text{-}g}$ (Fig. 76) for a single 3 × 2 × ¼-in. angle = 0.39 in.^{4} The gross area = 1.19 sq. in., and the distance from the gravity axis to the back of the angle = 0.49 in. For the entire section $I_{2\text{-}2} = 2[0.39 + 1.19 \times (0.49 + 0.125)^2] = 1.68$ in.^{4}, and $R_{2\text{-}2} = \sqrt{\frac{1.68}{2 \times 1.19}} = 0.84$ in. The value of $R_{1\text{-}1}$ is the same as for a single angle, *i.e.*, 0.95 in.

$f_s = 17{,}000 - 0.485(100/0.84)^2 = 10{,}100$ p.s.i., and the total gross area required is 20,000 ÷ 10,100 = 1.98 sq. in. The two angles furnish 2.38 sq. in., but no smaller angles could be used without the l/R ratio becoming greater than 120.

2. Determine the safe load that can be supported by a column 20 ft.-0 in. in length, the cross-section of which is shown in Fig. 77*a*.

Area of Cross-section

$$\begin{aligned} 2 \text{ plates} &= 2 \times 12 \times 1 = 28.00 \text{ sq. in.} \\ 2 \text{ channels} &= 2 \times 8.79 = \underline{17.58} \text{ sq. in.} \\ & \qquad\qquad\quad 45.58 \text{ sq. in.} \end{aligned}$$

Moment of Inertia about Axis 1-1 (Fig. 77*b*)

$$\begin{aligned} 2 \text{ plates} &= 2(\tfrac{1}{12} \times 14 \times 1^3 + 14 \times 6.5^2) &= 1186.0 \text{ in.}^4 \\ 2 \text{ channels} &= 2 \times 161.2 (\text{A.I.S.C. Handbook}) &= \underline{322.4} \text{ in.}^4 \\ & & I_{1\text{-}1} = 1508.4 \text{ in.}^4 \end{aligned}$$

Radius of Gyration about Axis 1-1

$$R_{1\text{-}1} = \sqrt{\frac{1508.4}{45.58}} = 5.75 \text{ in.}$$

Moment of Inertia about Axis 2-2 (Fig. 77*c*)

$$I_{g\text{-}g} \text{ for 1 channel } = 5.2 \text{ in.}^4$$
$$2 \text{ channels } I_{2\text{-}2} = 2[5.2 + 8.79(0.68 + 3.5)^2] = 317.4 \text{ in.}^4$$
$$2 \text{ plates } I_{2\text{-}2} = 2 \times \tfrac{1}{12} \times 1 \times 14^3 = 458.0 \text{ in.}^4$$
$$I_{2\text{-}2} = 775.4 \text{ in.}^4$$

Radius of Gyration about Axis 2-2

$$R_{2\text{-}2} = \sqrt{\frac{775.4}{45.58}} = 4.13 \text{ in.}$$

Safe Load

The value of $l/R = 240 \div 4.13 = 58.1$, the allowable unit stress is

$$f_s = 17{,}000 - 0.485 \times (58.1)^2 = 15{,}400,$$

and the safe load is $45.58 \times 15{,}400 = 702{,}000$ lb.

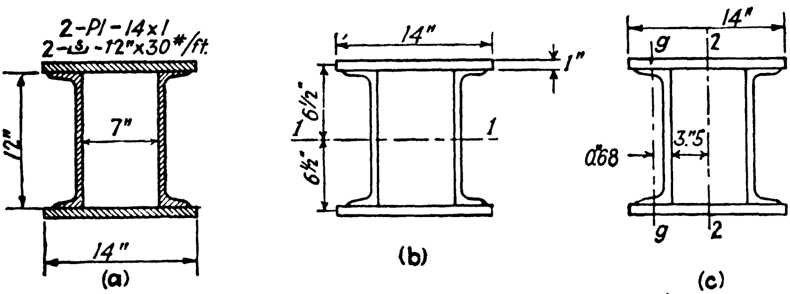


Fig. 77.

92. Members Subject to Bending and Direct Stress. Various examples are encountered in structures, where a member whose primary stress is direct tension or compression also acts as a beam. Some typical examples are: (*a*) horizontal or inclined members of bridge trusses which carry a direct stress of tension or compression and a bending stress due to their own weight; (*b*) chord members of trusses, which support direct vertical loads such as ties or joists of a floor system; (*c*) upper chords of roof trusses which support purlins between panel points; (*d*) columns with eccentrically applied loads.

The maximum unit fiber stress in a member of this type is equal to the sum of the unit stresses due to the direct stress and the bending stress, and as shown in Art. 87

$$f = f_1 + f_2 = \frac{P}{A} + \frac{Mc}{I}$$

Substituting AR^2 for I and solving for A,

$$A = \frac{P}{f} + \frac{Mc}{fR^2}$$

The first term on the right-hand side of the above equation is the area required to resist the direct stress and the second term the area required to resist the bending stress. If the direct stress is tension, the value of f is the same in both terms of the equation. If the direct stress is compression and the member is rigidly supported against possible bending due to direct stress alone, the value of f is the same in both parts of the equation for steel members, and nearly the same for timber members. If the direct stress is compression and the member is not braced against possible lateral displacement due to the direct stress, then the values of f are not the same in the two parts of the equation. To cover all cases, the equation may be written

$$A = \frac{P}{f_1} + \frac{Mc}{f_2R^2}$$

where f_1 = the unit working stress allowed for the direct stress.
f_2 = the unit working stress allowed in bending.

93. Examples. 1. A simple horizontal member is to consist of two $5 \times 3\frac{1}{2}$-in. angles with the short legs horizontal and below the long legs. It has a direct tensile stress of 80,000 lb. and in addition supports a load of 2400 lb. at the center of its 10 ft.-0 in. length. Neglecting the small moment caused by the weight of the member itself, determine the thickness of angles required.

The maximum bending moment is at the center of the member where the only rivets would be the stitch rivets (rivets placed through the two legs nearly in contact with washers between the backs of the angles to prevent vibration), so that only one $\frac{7}{8}$-in. hole (for $\frac{3}{4}$-in. rivets) need be subtracted from the gross section of each angle. Assuming $\frac{7}{16}$-in. angles the net section is (A.I.S.C. Handbook) $2(3.53 - 1 \times \frac{7}{8} \times \frac{7}{16}) = 6.30$ sq. in. The value of R (about the axis of bending) = 1.59 in. and the required area is

$$A = \frac{80,000}{20,000} + \frac{(\frac{1}{4} \times 2400 \times 10 \times 12) \times 1.63}{20,000 \times (1.59)^2} = 6.32 \text{ sq. in.}$$

This is sufficiently close to the area furnished, and the $\frac{7}{16}$-in. thickness is considered satisfactory.

2. An oak column with an unsupported length of 15 ft.-0 in. sustains a direct stress of 60,000 lb. and a bending moment of 50,000 in.-lb. Design the column.

The minimum value of d in order that this column may be considered an intermediate column (see Art. 88) is $(15 \times 12) \div 23.7 = 7.6$ in. Assuming a 10×10 in. (actual $9\frac{1}{2} \times 9\frac{1}{2}$ in.) column, $\frac{l}{d} = 180 \div 9\frac{1}{2} = 18.95$ and

$$f_i = f_1 = 1100\left[1 - \frac{1}{3}\left(\frac{18.95}{23.7}\right)^4\right] = 950 \text{ p.s.i.}$$

$$A = 9.5 \times 9.5 = 90.25 \text{ sq. in.}$$

$$I = \tfrac{1}{12} \times (9.5)^4 = 679 \text{ in.}^4$$

$$R = 2.74$$

The area required is

$$A = \frac{60,000}{950} + \frac{50,000 \times 4.75}{1400 \times (2.74)^2} = 85.8 \text{ in.}$$

This is slightly less than that furnished, and a 10- × 10-in. column will be adopted.

CHAPTER VII

CONNECTIONS AND SPLICES

CONNECTING ELEMENTS

94. Rivets. Rivets are used to hold together two or more pieces of steel and to transfer stress from one piece to another piece. They are made with one head; the other head is formed after the rivet is in place. Numerous shapes are given to the

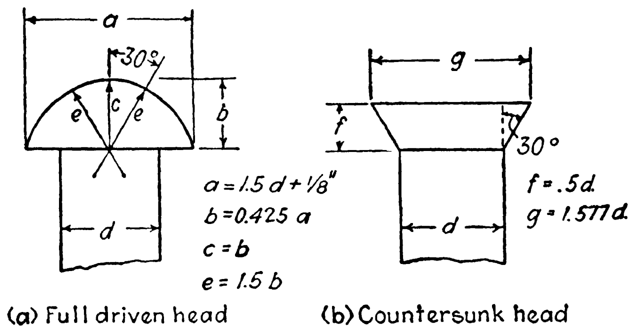


Fig. 78.

heads. The usual head for structural rivets (a buttonhead) is shown in Fig. 78*a*; where clearance requirements will not permit the full head depth, it may be flattened as necessary. Flattened heads may be formed by using a properly shaped die in the riveting machine, or the head may be flattened with a sledge after driving. Countersunk heads (Fig. 78*b*) are necessary where no clearance (or a very small clearance) is available, such as is the case where a plate, riveted to an angle or other shape, must rest directly on (yet not be connected to) another plate. Holes for countersunk rivets are reamed by special reaming tools; the projecting portions are chipped off if necessary by means of a cutting tool or chisel held in a pneumatic hammer. Rivets are ordered of a length sufficient to pass through the plates which they are to connect and to project far enough beyond the face of the last plate to furnish enough material to form the second head. The combined thickness of the plates is called the grip of the rivet. The portion of the rivet between the heads is called the shank. The nominal size of a rivet is the diameter of the shank before it is heated and driven. Nominal shank diameters for structural rivets vary from about ½ to 1¼ in., the most common diameters being ⅝, ¾, and ⅞ in.

The amount of stress that can be transferred is dependent upon the shearing strength of the rivet and on the bearing strength of the plates through which the rivet passes. If three plates are connected as shown in Fig. 79*a*, there will be a tendency

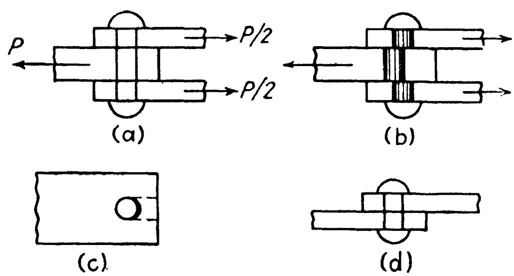


FIG. 79.

to shear the rivet along two planes, one at either surface of the middle plate, as shown in *b*. This is called double shear. The safe total stress P in the middle plate cannot exceed two times the shearing strength of the rivet bar, or two times the cross-sectional area of the rivet in square inches multiplied by the safe shearing strength of the rivet steel in pounds per square inch.

If the middle plate is too thin, there is the possibility that the pressure of the rivet will be sufficient to crush the plate, thus enlarging the hole as shown in Fig. 79*c* and allowing the plates to slip. A failure of this type is called a compression or bearing failure, and it is guarded against in design by keeping the unit stress on the projected bearing area of the rivet below a safe working value. The projected bearing area is equal to the product of the thickness of the plate and the nominal diameter of the rivet. The safe bearing value of a rivet is equal to the allowable unit bearing stress multiplied by the projected bearing area.

When a rivet is placed near the end of a plate, as in Fig. 79*c*, there is a tendency for the rivet to pull out of the plate by shearing off the metal along the dotted lines shown in the figure. This type of failure can be prevented by placing the rivet far enough away from the end of the plate so that the resistance of the plate in shear is greater than the shearing or bearing value of the rivet. With the general relation of unit stresses used in design this is accomplished by specifying a minimum end distance of about 1½ times the diameter of the rivet, measured from the center of the rivet to the end of the plate.

Assuming that the outer plates are of equal thickness, the stress in each outer plate is $P/2$; and if, as is usually the case,

TABLE 1.—RIVETS, PINS, AND TURNED BOLTS IN DRILLED OR REAMED HOLES
(Allowable loads in kips)

Shear and tension .... 15,000 p.s.i.
Bearing: Single Shear .... 32,000 p.s.i.
Double Shear .... 40,000 p.s.i.

Rivet diameter, in.	½		⅝		¾		⅞		1		1⅛	
Tension	2.94		4.60		6.63		9.02		11.78		14.91	
Single shear	2.94		4.60		6.63		9.02		11.78		14.91	
Double shear	5.89		9.20		13.25		18.05		23.56		29.82	
Thickness of plate, in.	Bearing		Bearing		Bearing		Bearing		Bearing		Bearing	
	S.S.	D.S.	S.S.	D.S.	S.S.	D.S.	S.S.	D.S.	S.S.	D.S.	S.S.	D.S.
⅛	2.00	2.50	2.50	3.13	3.00	3.75	3.50	4.38	4.00	5.00	4.50	5.63
3/16		3.75	3.75	4.69	4.50	5.63	5.25	6.56	6.00	7.50	6.75	8.44
¼		5.00		6.25	6.00	7.50	7.00	8.75	8.00	10.00	9.00	11.25
5/16				7.81		9.38	8.75	10.94	10.00	12.50	11.25	14.06
⅜						11.25		13.13		15.00	13.50	16.88
7/16						13.13		15.31		17.50		19.69
½								17.50		20.00		22.50
9/16										22.50		25.31
⅝												28.13
11/16												
1	16.0	20.0	20.0	25.0	24.0	30.0	28.0	35.0	32.0	40.0	36.0	45.0

the thickness of each of these plates is equal to at least one-half that of the middle plate, the outer plates will be safe in bearing.

When only two plates are connected by a rivet, the rivet may shear off along only one plane, as shown in Fig. 79*d*. The rivet is then said to be in single shear, and the shearing strength is equal to the area of the rivet bar times the allowable unit shearing stress. The bearing value of the rivet is obtained as in the double-shear connection, using the thickness of the thinner plate in the computation. The unit bearing stress in a single-shear connection is generally specified as about 25 per cent less than the corresponding value for a double-shear connection, for the bearing of the rivet on the plates is not apt to be uniform throughout the thickness of the plates in the single-shear connection, thus resulting in a bearing stress somewhat greater than the average along the contact surface.

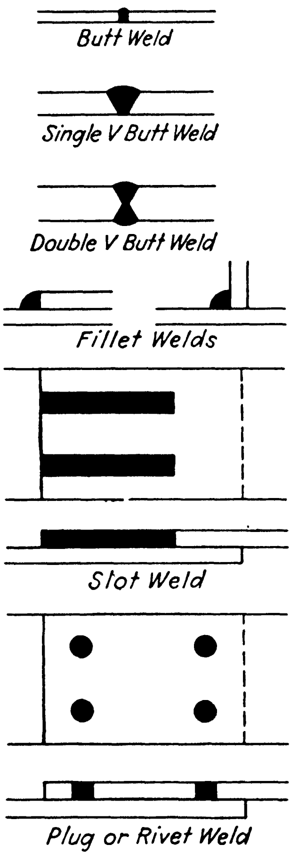


FIG. 80.—Types of welds.

Table 1 gives the total allowable single-shear and double-shear values of rivets and the total bearing values for various plate thicknesses, as recommended by the American Institute of Steel Construction for use in buildings.

95. Welds. The use of welded connections in steel structures is increasing rapidly. Welds are made by heating the surfaces to be joined to a plastic or fluid state and allowing them to flow together. Additional molten steel must be added to preserve the full sections of the members joined. The various types of welds in general use are shown in Fig. 80. The fillet weld is used more than any other type in the welding of structural steel. Two types of fillet-weld connections are shown in Fig. 81. In (*a*) the strength of the connection depends upon the shearing strength of the fillets along the planes of the "throat" of the fillet (see Fig. 82). In (*b*) the strength depends upon the shear along one contact plane and the tension on the other. Tests

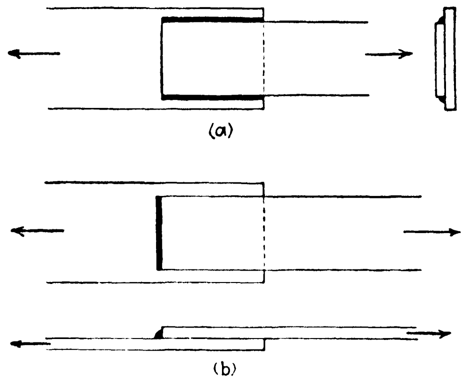


FIG. 81.

have shown greater strength per unit of length of fillet for connections made in the manner shown in (*b*); but when the two methods of connection have been combined, failure in the weld normal to the line of stress occurred first. Working stresses

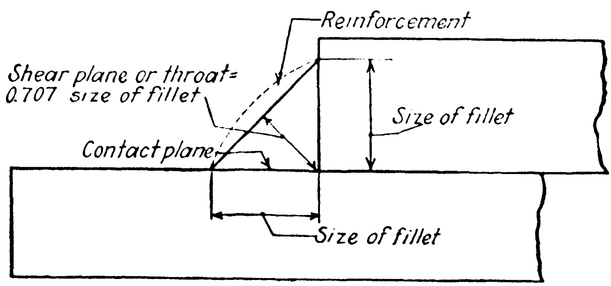


FIG. 82.

for fillet welds, placed in either of these two ways, are given in Table 2.

TABLE 2.—WORKING STRESSES FOR FILLET WELDS

Size of fillet, in.	1/4	5/16	3/8	1/2	5/8	3/4
Strength, lb. per lin. in.	2000	2500	3000	4000	5000	6000

For structural connections the minimum size of fillet recommended is ¼ in., and no fillet should have a length less than four times its size. The maximum size of fillet is governed by the thickness of the plate connected, although quite frequently the size is less than the plate thickness.

96. Bolts and Nuts. Bolts are used to connect two or more pieces of steel or timber in much the same way that rivets are used to connect steel elements. The shearing strength of the bolts and the bearing strength of the connected pieces must be investigated, as in the design of riveted connections. Bolts, in the form of long rods threaded at one or both ends, are also used as tension members in timber trusses and sometimes in light steel trusses. In such cases, the strength of the bolt is dependent upon the cross-sectional area at the root of the thread.

The short diameter of square or hexagonal heads and nuts is 1½ times the bolt diameter plus ⅛ in. The areas at the root of the thread (net areas) for bolts of various diameters are given in Table 3.

TABLE 3.—BOLT DIAMETERS AND AREAS

Diameter, in.		Area, sq. in.		Diameter, in.		Area, sq. in.	
Total	Net	Total	Net	Total	Net	Total	Net
¼	0.185	0.049	0.027	1⅜	1.158	1.485	1.054
⅜	0.294	0.110	0.068	1½	1.283	1.767	1.294
½	0.400	0.196	0.126	1⅝	1.389	2.074	1.515
⅝	0.507	0.307	0.202	1¾	1.490	2.405	1.744
¾	0.620	0.442	0.302	1⅞	1.615	2.761	2.049
⅞	0.731	0.601	0.419	2	1.711	3.142	2.300
1	0.838	0.785	0.551	2¼	1.961	3.976	3.021
1⅛	0.939	0.994	0.693	2½	2.175	4.909	3.716
1¼	1.064	1.227	0.890	2¾	2.425	5.940	4.619

The shearing stresses to be used in the design of turned bolts are the same as for rivets (see Table 1), and the bearing values on steel plates are also the same. The bearing value of a bolt on a timber member is computed in the same manner as for a rivet on a steel plate, using for the allowable unit stress the value specified in compression in Art. 13, due consideration being given to the angle between the projected bearing area and the fibers of the timber. Long bolts should be investigated for bending, since the bending strength is quite apt to be the governing factor.

97. Washers. Where steel rods transfer their stresses to timber members and where bolts are used to fasten two or more timber pieces, washers must be placed under the bolt heads and nuts to prevent crushing of the fibers of the timbers. Three common types of washers are the plate washer, the cast-iron ogee washer (Fig. 83*a*), and the beveled washer (Fig. 83*b*). Plate washers are usually square or rectangular and can be obtained in any size and thickness; in the smaller sizes circular washers are available. Ogee washers are round and are made in standard diameters and thicknesses, which vary somewhat with different manufacturers. In one standard form the diameter of the top or face is two times the bolt diameter plus ½ in., the diameter of the bottom or bearing surface is four times the bolt diameter, and the thickness is ⅛ in. less than the bolt diameter. The beveled washer shown in Fig. 83*b* is a solid casting, consisting essentially of a cylinder and a plate, the axis of the cylinder being at any required angle with the surface of the plate, with diaphragms connecting the cylinder to the four corners of the plate. Beveled washers are necessary when there is a marked inclination of the bolt or rod relative to the member to which the rod is to be connected. The diameter of the hole in a washer is usually made 1/16 in. greater than the bolt diameter.

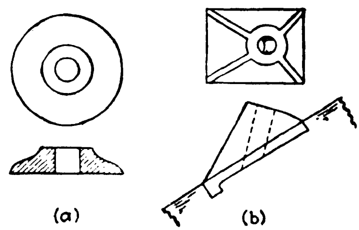


FIG. 83.—Ogee and beveled washers

For plate or ogee washers, the area of washer required (exclusive of the hole area) is equal to the stress in the rod divided by the allowable unit compression against the fibers of the timber. Washers of this type should be placed at right angles to the bolt or rod; therefore, if the bolt is inclined with respect to the axis of the timber member to which it connects, the timber must be notched accordingly, and the pressure of the washer will be neither parallel nor perpendicular to the fibers. The allowable unit bearing stress will depend upon the angle between the bearing surface and the fibers; recommended values for various angles are given in Art. 13.

The required thickness of a plate washer is determined by passing a vertical plane through the washer at the center line of the bolt, computing the moment at the section formed by this plane, and applying equation (1), page 64, to determine the thickness, using for b the net width of the washer at the given section and for f the allowable unit stress in bending (Art. 21). For a rectangular washer, the required thickness is governed by the bending in the longer direction.

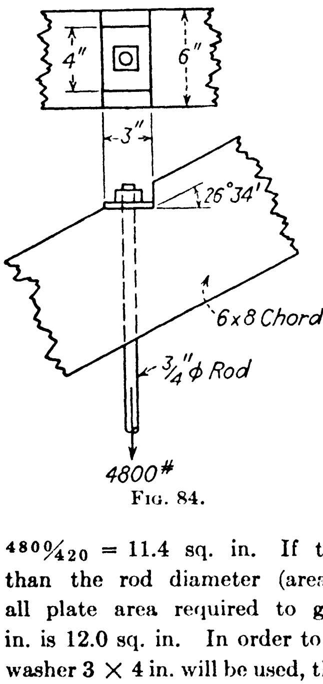


FIG. 84.

Example. In Fig. 84, the stress in the $\frac{3}{4}$-in. rod is 4800 lb., the timber is Douglas fir, and the angle between the bearing surface of the washer and the fibers of the timber is 26°34′. Determine the required size and thickness of the washer.

With $p/q = {}^{1500}/_{350} = 4.28$, the allowable unit bearing stress, from the equation on page 24, and Fig. 20, is $0.28 \times 1500 = 420$ p.s.i., and the bearing area required $= {}^{4800}/_{420} = 11.4$ sq. in. If the hole in the timber is $\frac{1}{8}$ in. greater than the rod diameter (area of $\frac{7}{8}$-in. hole = 0.60 sq. in.), the overall plate area required to give an actual bearing surface of 11.4 sq. in. is 12.0 sq. in. In order to avoid cutting too deeply into the timber, a washer 3×4 in. will be used, the 4-in. dimension being at right angles to the chord. In the long direction, with a nut diameter of $1\frac{1}{2} \times \frac{3}{4} + \frac{1}{8} = 1\frac{1}{4}$ in.,

$$M = \frac{4800}{2}\left(\frac{2}{2} - \frac{1.25}{4}\right) = 1650 \text{ in.-lb.}$$

The net width of the washer, assuming the hole in the washer to be $\frac{1}{16}$ in. larger than the rod diameter, is $3 - {}^{13}/_{16} = 2\frac{3}{16}$ in. From equation (1), page 64,

$$1650 = 20{,}000 \times \frac{1}{6} \times 2.19 \times t^2$$
$$t = 0.476 \text{ in., or } \frac{1}{2} \text{ in.}$$

Beveled washers are indented into the timber, as shown in Fig. 85. The indent area required (in the plane *ab*) is equal to the component of the stress in the rod in the direction of the axis of the timber member, divided by the allowable unit compressive stress parallel to the fibers. The base area required, (in the plane *bc*, Fig. 85), is equal to the component of stress normal to the base divided by the allowable unit compressive stress perpendicular to the fibers of the wood. In order to avoid bending stresses, the depth of indent and length of base are so adjusted that lines perpendicular to the indent and the base at their centers of gravity intersect on the axis of the cylindrical hole through which the rod passes, as shown in Fig. 85.

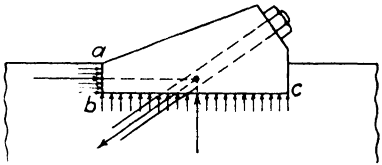


FIG. 85.

98. Timber Connectors. Special types of timber connectors are frequently used in fastening timber members which are to transfer stress from one to the other or others. The most widely known connectors are manufactured by the Timber Engineering Company of Washington, D.C., and are known as Teco connectors. Two common types are: (1) toothed-ring connectors; (2) split-ring connectors. The toothed-ring connector (Fig. 86*a*) is a corrugated circular band of 16-gage steel with sharp teeth on each edge. The split-ring connector (Fig. 86*b*) is a smooth circular band of mild steel, with a tongue-and-groove break or split. The timbers which are to be fastened together are lapped and the connector is embedded for half its depth in each timber. A bolt is passed through the timbers at the center of the connector to keep them from spreading. Toothed-ring connectors are placed between the lapped timbers and forced into place by pulling the timbers together with a high-strength bolt which is later replaced by the regular bolt. Split-ring connectors are set in circular grooves cut in the timbers by

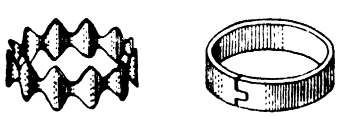


FIG. 86.

means of a special tool before the joint is put together. At least one connector is required at each surface of contact between the timbers, so that, in a connection such as is shown in Fig. 87, at least two connectors are required at the joint. More than the one pair shown may be used if necessary and if there is sufficient space in which to place them.

The safe load that can be transmitted by one toothed-ring connector is dependent primarily upon the size of the ring and the angle between the fibers of the timber and the direction of the load; recommended values for each available size are given in Table 4. Where the angle between the load and the fibers of the timber (angle θ, Fig. 87) is less than 45 degrees, the allowable load is directly proportional to the angle, interpolation being made between the two values given in the table. Where two or more rings are used in one joint, the total safe load on the joint is the product of the tabulated value for one connector times the number of connectors. For example, in Fig. 87, if $\theta = 40$ degrees and if $2\frac{5}{8}$-in. toothed rings are used, the safe load on one connector is 1400 lb., and the safe load P on the joint is $2 \times 1400 = 2800$ lb.

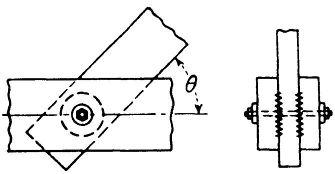


FIG. 87.

TABLE 4.—MINIMUM SIZES OF BOLTS AND WASHERS AND ALLOWABLE LOADS FOR TOOTHED-RING CONNECTORS

Diameter of ring, in.	Diameter of bolt, in.	Plate washer, in.	Ogee washer, in.	Allowable load, lb. Load parallel to grain	Allowable load, lb. Angle of load with grain 45 deg. or more
2	½	2 × 2 × 3/16	½ × 2	1100	825
2⅝	⅝	2½ × 2½ × ¼	⅝ × 2½	1800	1350
2⅜	¾	3 × 3 × ⅜	¾ × 3	2600	1950
4	¾	3½ × 3½ × ⅜	⅞ × 3⅝	3150	2350

Safe loads for split-ring connectors are given in Table 5. Where the angle between the fibers of the timber and the direction of the load is between 0 degrees (load parallel to fibers) and 90 degrees (load perpendicular to the fibers), the safe load is obtained by simple proportion, assuming a uniform variation between the two tabulated values. Where more than one split ring is used at a joint, the safe load on the joint is the product of the tabulated load for one connector times the number of connectors.

TABLE 5.—MINIMUM SIZES OF BOLTS AND WASHERS AND ALLOWABLE LOADS FOR SPLIT-RING CONNECTORS

Diameter of ring, in.	Diameter of bolt, in.	Plate washer, in.	Ogee washer, in.	Douglas fir, southern pine, tamarack, or western larch: Dense grades, Load parallel to grain	Dense grades, Load perpendicular to grain	Nondense grades, Load parallel to grain	Nondense grades, Load perpendicular to grain
2½	½	2 × 2 × 1/8	½ × 2½	3,300	2,350	2,850	2,000
4	¾	3 × 3 × 3/16	¾ × 3¼	7,000	4,900	6,000	4,200
6	¾	3 × 3 × ¼	¾ × 3¼	10,500	6,300	9,000	5,400
8	1	3½ × 3½ × 3/8	1 × 4	13,500	6,700	11,500	5,750

99. Pins. Pins are sometimes used to connect groups of bars, plates, or structural shapes between which a stress is to be transmitted. A pin is a solid steel bar which is passed through holes in the connected members. The pin is threaded on the ends, and recessed nuts of pressed or cast steel are placed at each end. Figure 88 shows a pin connecting three eye bars, two of which are pulling in one direction and one in the opposite direction. Washers are placed between the eye bars on the pin to hold the bars in their proper positions. Pins must be designed for shearing and bending stresses, and the members through which they pass must be sufficiently thick to provide adequately for the bearing stresses. The investigations for shearing and bearing are made in the same manner as for rivets. The investigation for bending is made by computing the maximum bending moment in the pin, which is treated as a simple beam with the fixed loads, two of which are considered as reactions, then equating the bending moment and the resisting moment, and solving for the diameter required. The resisting moment M of a circular section is

$$M = f\frac{I}{e} = f\frac{\pi d^4/64}{d/2} = f\frac{\pi d^3}{32}$$

in which f = the allowable extreme fiber stress in bending.

d = the diameter of the pin.

The specifications of the American Institute of Steel Construction permit a unit shearing stress of 15,000 p.s.i., a unit bearing stress of 32,000 p.s.i., and a unit extreme fiber stress in bending of 30,000 p.s.i.

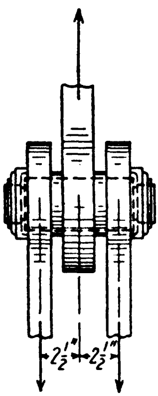


FIG. 88.

Example. If the stress in the middle member in Fig. 88 is 80,000 lb. and if the distance between the outer members (the supports of the pin) is 5 in., what size pin is required to resist shearing and bending stresses, and what thicknesses of plates are necessary to provide for bearing stresses?

$$V = \frac{80,000}{2} = 40,000 \text{ lb.}$$

$$M = \frac{80,000}{2} \times 2.5 = 100,000 \text{ in.-lb.}$$

$$100,000 = \frac{30,000 \times \pi \times d^3}{32}$$

from which

$$d = 3.23 \text{ in., or a } 3\tfrac{1}{4}\text{-in. pin}$$

The unit shearing stress on the 3¼-in. pin (area = 8.296 sq. in.) is 40,000/8.296 = 4830 p.s.i., which is well below the allowable stress of 15,000 p.s.i.

The bearing area required for the middle plate is 80,000/32,000 = 2.5 sq. in., and the thickness of the plate must be 2.5/3.25 = 0.77 in., or ⅞ in. Each outer plate must be at least ½ × ⅞ = 7⁄16 in. thick.

TIMBER CONNECTIONS AND SPLICES

100. Types of Joints. Some of the more common types of joints which are used in places where one timber member rests upon or frames into another are shown in Fig. 89. A butt joint in which there is an appreciable amount of stress should be investigated for bearing perpendicular to the fibers. The bearing area required is equal to the stress in the noncontinuous member divided by the allowable unit stress. If the area of the member is less than this, a steel plate of the proper area

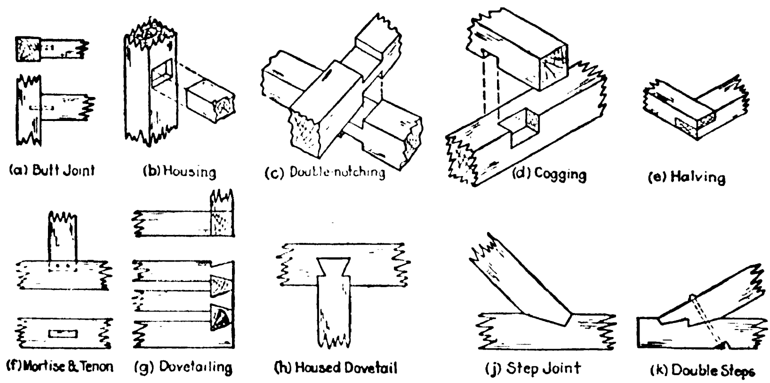


FIG. 89.—Typical timber joints.

must be placed between the members. The required thickness of the plate is obtained by treating the projecting portion of the plate as a cantilever. For a 1-in. width the maximum moment is $\frac{1}{2}wl^2$ in which w is the unit pressure against the plate and l is the projecting length; the thickness is obtained from equation (1), page 64, with $b = 1$ in.

In designing a step joint, such as that in Fig. 90, the depth of indent must be sufficient to provide enough bearing area to prevent crushing of the fibers of both timbers. The depth of the notch is assumed; the angle at the tip of the notch is generally made 90 degrees, partly to facilitate the cutting or framing and partly to avoid secondary stresses. The components of the stress at right angles to the two surfaces of the notch are computed, the allowable bearing pressures for the proper angles of inclination with the fibers (Art. 13) are determined, and the bearing areas required along the two surfaces are obtained by dividing the component of stress by the allowable unit pressure, for each surface.

Example. Figure 90 shows the details of joint c in the timber roof truss shown in Fig. 91. Design the indent for the inclined strut, which has a stress of 13,000 lb. Assume that the allowable unit bearing stress parallel

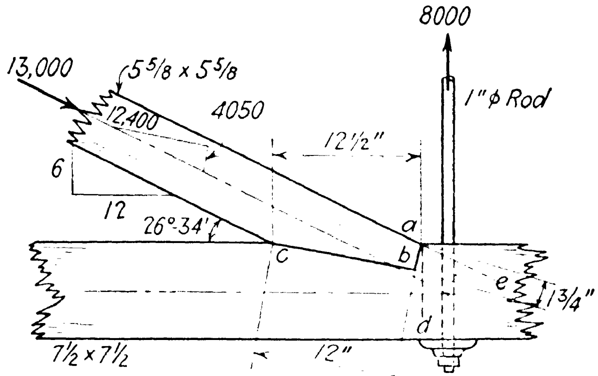


FIG. 90.

to the fibers is 1800 p.s.i. and that the allowable unit bearing stress perpendicular to the fibers is 465 p.s.i.

Assume the length of the short side ab of the notch as $1\frac{3}{4}$ in. Then, with a 90-degree angle at b, the length cb = 12 in.; angle $ead = 90° - 26°34' = 63°26'$; angle dab = angle $acb = \tan^{-1}(1.75/12) = 8°20'$; angle $eab = 71°46'$. The components of the stress in the strut at right angles to the faces ab and bc, determined graphically, are shown in the figure. Face ab makes an angle with the chord $= 90° - 8°20' = 81°40'$. The strength of the strut therefore governs. The allowable unit pressure on the fibers (using Fig. 20 with angle $= 71°46'$ and $p/q = {}^{1800}/_{465} = 3.87$) is $0.77 \times 1800 =$ 1380 p.s.i. and the length ab required $= \dfrac{12{,}400}{1380 \times 5\frac{5}{8}} = 1.6$ in. The assumed length of

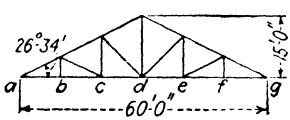


FIG. 91.

$1\frac{3}{4}$ in. is therefore satisfactory, and yet it could not be reduced enough to warrant a redesign. Face bc makes an angle with the fibers of the strut of 18°14′ and with the fibers of the chord of 8°20′. The latter governs; the allowable pressure is 470 p.s.i., and the required length $bc = \dfrac{4050}{470 \times 5\frac{5}{8}} =$ 1.53 in., which is much less than the length furnished. Since the length bc is governed by the length ab, which in turn is the critical dimension as far as cutting into the chord is concerned, no revision of the design is necessary.

If the horizontal length from point *b* to the right end of the horizontal timber were quite short, the shearing strength of the chord along the horizontal plane through *b* to the end would have to be investigated. The shearing area required is equal to the horizontal component of the stress in the inclined strut divided by the allowable unit longitudinal shearing stress in the horizontal timber. The area furnished is the product of the length from *b* to the right end of the horizontal timber multiplied by the width of that timber. The length from *b* would have to be made sufficient at least to furnish the required area.

101. Timber Splices. The three general methods of splicing timbers in the direction of their length are (1) fishing, (2) lapping, and (3) scarfing. Fishing consists in bolting two plates of steel

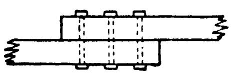


FIG. 92.—Lap joint.

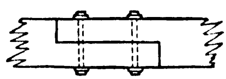


FIG. 93.—Half-lap joint.

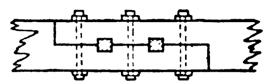


FIG. 94.—Half-lap joint with keys.

or timber across the joint and on opposite faces of the timbers which are to be spliced. The extra plates, called fishplates, may have projections or tables which engage corresponding notches in the main timbers, in which case the joint is called a *tabled fishplate joint* (Fig. 99). This is a very satisfactory type of tension joint. A *plain fishplate joint* (without notches and tables, Fig. 98) is suitable for tension joints with comparatively small stresses, and for compression joints.

Lap joints (Fig. 92) are sometimes used to join two tension timbers, but this type of joint is not satisfactory because of the large bending moment in the bolts and should not be used for large stresses or in permanent members. They are quickly made and inexpensive. *Half lap joints* (Fig. 93) are satisfactory in compression; the length of the lap should be twice the depth of the timber and the bolts placed at the quarter points of the lap. *Half lap joints with keys* (Fig. 94) may be satisfactory for slightly stressed tension splices.

Oblique scarf joints (Fig. 95) have more flexural strength than half lap joints (note: the latter is also a form of scarfing). The depth at the ends of the timbers is usually made one-fourth of the total and the length of the joint at least twice the total depth. *Tabled scarf joints* (Fig. 96) are fairly efficient in tension and compression. They are more generally used for compression, and especially where small reversals of stress are probable. The depth and length of tables are determined in somewhat the same manner as described above for the tabled fish plate joint. *Oblique tabled scarf joints* (Fig. 97) are uneconomical because

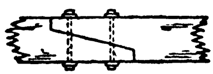


FIG. 95.—Oblique scarf joint.

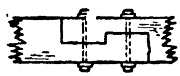


FIG. 96.—Tabled scarf joint.

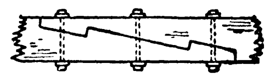


FIG. 97.—Oblique tabled scarf joint.

of the great length required to secure sufficient shearing surface parallel to the fibers; the eccentric application of the stress also tends to cause considerable secondary bending in the splice.

102. Design of a Plain Fishplate Joint. Figure 98 illustrates a plain fishplate joint in which two fishplates (one on either side of the abutting main timbers) are bolted through the main timbers. The number of bolts required may be determined as follows: (1) Assume the size of bolt. (2) Compute the safe load P that one bolt can support in flexure, treating the bolt as a restrained beam with a span equal to the distance center to center of fishplates and supporting a uniform load P. If h is the thickness of the main timber and if the thickness of each fishplate is $h/2$, then $M = Ph/8$. The resisting moment of a bolt of diameter d is $f\pi d^3/32$. Equating bending and resisting moments and solving for P,

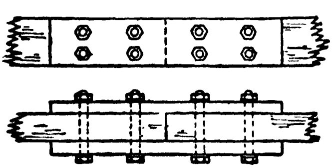


FIG. 98.—Plain fishplate joint.

$$P = \frac{0.785fd^3}{h}$$

If $f = 18{,}000$,

$$P = \frac{14{,}130d^3}{h}$$

If $f = 20{,}000$,

$$P = \frac{15{,}700d^3}{h}$$

If $f = 22{,}000$,

$$P = \frac{17{,}270d^3}{h}$$

(3) Find the safe load P that can be placed on one bolt without crushing the ends of the fibers of the timber (P = diameter of bolt × depth of main timber × allowable unit compression stress on ends of fibers). (4) Determine the number of bolts required by dividing the total tension stress in the main timber by the smaller value of P from (2) and (3). The bolts must be spaced far enough apart to prevent shearing of the timber parallel to the fibers. Because of the splitting tendency caused by the curved surface of the bolt, only three-fourths of the value specified in Art. 13 for shearing stresses parallel to the

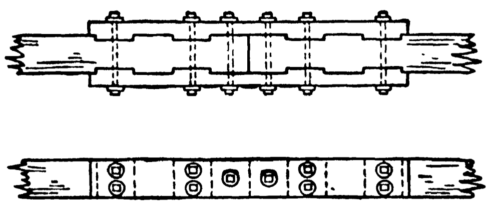


FIG. 99.—Tabled fishplate joint.

fibers should be used in this computation. If two lines of bolts are used, two bolts (one in each line) should be placed in the same cross-section. Plain fishplates are more satisfactory than tabled fishplates in compression splices.

103. Design of a Tabled Fishplate Joint. Figure 99 illustrates a *tabled fishplate joint*, in which tables on the fish plates engage corresponding notches in the main timbers. A tabled fishplate joint is more efficient in tension than a plain fishplate joint because of the effect of the tables in resisting separation of the main timbers. The fishplates may be made of steel instead of wood, in which event steel flats are riveted to them to form the tables. The principles involved in the design of a tabled fishplate joint are given in the following example.

Design a tabled fishplate joint, using wooden fishplates, to transmit a tensile stress of 60,000 lb. For the material in the main timbers and fishplates, allowable unit stresses (in p.s.i.) are to be assumed as follows: tension 1950, compression parallel to fibers 1950, compression perpendicular to fibers 390, shear parallel to fibers 270, tension in bolts 16,000. Required areas (square inches) are: tension 60,000/1950 = 30.8; compression at ends of tables 60,000/1950 = 30.8; shear along plane of tables 60,000/270 = 222.

Assuming four tables on each side of the joint (for maximum economy a design should also be made using six tables and the two compared as to cost), the required area of each table = $^{222}/_{4}$ = 55.5 sq. in. If the width of main timber is not fixed by other details, make dimensions of the table preferably so that the length is $\frac{2}{3}$ to 1 times the width. Hence with $7\frac{1}{2}$-in. dressed width (8 in. nominal), the length required = 55.5/7.5 = 7.4 or $7\frac{1}{2}$ in. Depth of table = 30.8/(4 × 7.5) = 1 in. Assume $\frac{3}{4}$-in. bolts in $\frac{7}{8}$-in. holes. Net width of main timber = 7.5 − 2 × $\frac{7}{8}$ = 5.75 in. Net depth required = 30.8/5.75 = 5.3 in. Gross depth required = 5.3 + 2 × 1 = 7.3 in. Use 8 × 8 main timber, dressed to $7\frac{1}{2} \times 7\frac{1}{2}$, and 8 × 4 fishplates dressed to $7\frac{1}{2} \times 3\frac{5}{8}$.

Considering only the pressure against one table (Fig. 100), assuming the tension in the fishplate concentrated at mid-point of net depth and the compression against the table concentrated at mid-point of table depth, the moment of the couple thus formed = $(60{,}000/4)(1\frac{5}{16} + \frac{1}{2})$ = 27,200 in.-lb. With two bolts in each table (Fig. 99) and with a lever arm for the stress in each bolt about the center of rotation at a (Fig. 100) equal to 7.5/2 = 3.75 in., the stress in each bolt = $\frac{1}{2}$(27,000/3.75) = 3640 lb. The area required at the root of the thread = 3640/16,000 = 0.227 sq. in. From Table 3, page 136, $\frac{3}{4}$-in. bolts, net area = 0.302 sq. in., are required. Bearing area required under washer = $^{3640}/_{390}$ = 9.3 sq. in. Allowing for $\frac{7}{8}$-in. hole, the gross area of washer = 9.3 + 0.60 = 9.9 sq. in. A $3\frac{3}{4}$-in. ogee washer is selected, two of which will just fit on the $7\frac{1}{2}$-in. dressed width of the fishplate. If the two ogee washers were too large to bear on the fishplate, a plate washer should be used, with a length of $7\frac{1}{2}$ in. and width equal to (2 × 9.9)/7.5 = $2\frac{3}{4}$ in.

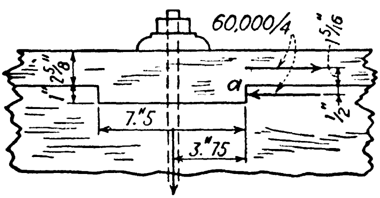


FIG. 100.

With steel fishplates the bolts should be placed just beyond the nonbearing edge of the flat, *i.e.*, away from the joint between the spliced timbers.

104. Timber Truss Bearings. Figure 101 shows a typical end bearing of a timber roof truss. A bolster is inserted between the lower chord and the wall plate in order to provide sufficient bearing area to prevent crushing of the fibers of the chord. The bearing area required between the bolster and the chord is equal to the maximum vertical reaction divided by the allowable unit stress perpendicular to the fibers of the chord. The width of the bolster is generally made the same as that of the chord, and the length is made sufficient to furnish the required bearing area. The bearing area required between the bolster and the wall plate is equal to the vertical reaction divided by the allowable unit stress perpendicular to the fibers of the wall plate or bolster, both of which are made of the same timber. Since the width of the bolster is fixed, the width of the wall plate must be sufficient to furnish the required bearing area. The length of the wall plate must be sufficient to provide enough bearing area on the wall to prevent crushing of the brick, stone, or concrete in the wall. The area required is obtained by dividing the

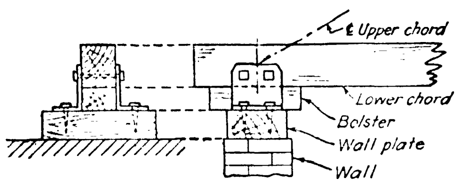


FIG. 101.—Timber roof-truss end bearing.

vertical reaction by the allowable unit bearing stress on the masonry. The thickness of the bolster and the wall plate are obtained by treating the projecting portions of these pieces as cantilevers, computing the maximum moment in the cantilever, and applying equation (1), page 64. A bent steel plate is placed on either side of the bolster and chord; the two plates are bolted through the chord and fastened to the wall plate by means of lag screws, as shown in Fig. 101, in order to prevent sidewise movement of the truss. The wall plate is generally bolted to the wall.

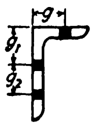
JOINTS IN STEEL FRAMES

105. Types of Joints. The three fundamental types of joints commonly encountered in steel structures are (1) gusset-plate connections, (2) beam connections, and (3) bracket connections. The stresses are transferred from one piece to another by means of rivets, bolts, or welds. Bolts are generally used only for connections which must be made in the field, and then only for structures which are practically free from vibration.

106. Riveted Gusset-plate Connections. A typical gusset-plate connection is shown in Fig. 102, which illustrates a lower chord joint of a steel roof truss. A steel plate (gusset plate) is placed between the pairs of angles which meet at the panel point or joint, and rivets are placed through the angles to transfer the stresses to the plate. The number of rivets required in each inclined member is equal to the stress in the member divided by the strength of one rivet (see Art. 94). If, as in Fig. 102, the lower chord is continuous across the joint, the number of rivets required in the chord through the gusset plate is equal to the difference in the chord stresses on either side of the joint divided by the strength of one rivet. If the chord is cut at or near the panel point, the full stress in each chord member (*db* and *be*) would be used in determining the number of rivets required in each chord. If the number of rivets is excessive, necessitating a very long gusset plate, part of the stress in each chord could be transmitted across the joint through a splice plate riveted to the bottom of the horizontal legs of the chord angles, and then only the remainder of the stress in each chord member would have to be taken by the rivets in the vertical legs through the gusset plate. The splice plate must be of sufficient thickness and width to furnish an adequate net section to carry in tension at least as much stress as the rivets.

The size and shape of the gusset plate are determined by the number of rivets and the arrangement of the connected members. Where relatively small stresses are involved, ¼-in. plates are usually satisfactory; but in cases where the stresses are comparatively large, thicker plates will be more economical in that they reduce the required number of rivets and consequently the size of the plate. Intersecting members should be arranged wherever possible so that the lines on which the rivets are placed in each member (the gage lines) meet at a point, in order to avoid any eccentricity in the connection. Where rivets are placed in two lines in an angle, the gage line nearest the back of the angle should be the intersecting line. Where symmetrical steel shapes are connected to a gusset plate, the center lines of the members should intersect. Standard gages for angles and the maximum-size rivets that can be used in the various sizes of legs are given in Table 6.

TABLE 6.—STANDARD GAGES FOR ANGLES



Size of leg	2	2½	3	3½	4	5	6	7	8
g	1⅛	1⅜	1¾	2	2½	3	3½	4	4½
g_1	...	...	...	...	...	2	2¼	2½	3
g_2	...	...	...	...	...	1¾	2½	3	3
Max. rivet.	⅝	¾	⅞	⅞	⅞	⅞	⅞	1	1⅛

In order to reduce to a minimum the size of the gusset plate, the rivets in each member should be placed as close together as possible. In order to make it possible to form the head on a rivet which is next to one already placed, the distance between the centers of adjoining rivets must be at least 3 diameters of the rivet. Also, to provide for an adequate amount of metal around a hole, minimum distances from the center of the hole to any edge of a plate or other shape are required as follows:

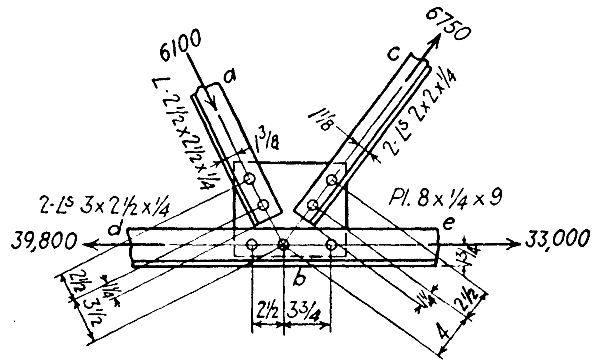
Rivet diameter, in.	To sheared edge, in.	To rolled edge, in.
⅝	1⅛	1
¾	1¼	1⅛
⅞	1½	1¼
1	1¾	1½
1⅛	2	1¾

Example. Determine the number of rivets required in each member at the joint shown in Fig. 102, using ⅝-in. rivets and a ¼-in. gusset plate.

From Table 1, page 133, working stresses for one rivet are as follows:

Single shear = 4600 lb.
Double shear = 9200 lb.
Bearing on ¼-in. plate = { Single shear 5000 lb. / Double shear 6250 lb. }

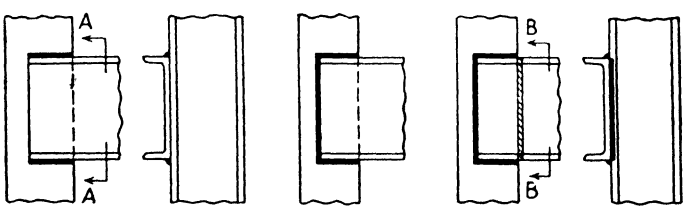
In member *ab* the rivets are in single shear, and the number of rivets required is $6700/4600 = 2$. In member *cb* the rivets are in double shear, so that the bearing value on the ¼-in. plate governs the strength of one rivet. The number required is $6750/6250 = 2$. Since the lower chord *de* is continuous across the joint, only the difference in stress between the parts *db* and *be* need be provided for by the rivets; since these are in double shear, the strength of one rivet is equal to the bearing value on the ¼-in. plate, and $\frac{39{,}800 - 33{,}000}{6250} = 2$ rivets are required. Not less than two rivets should



Joint 6
FIG. 102.

ever be used in any one member. Also, whenever possible, a rivet should be placed at the intersection of the members which meet at the joint. The required rivets in this design are placed as shown in Fig. 102, the extra rivet in the lower chord being used to avoid too great a distance from the edge of the plate to the nearest rivet in the lower-chord angles.

107. Welded Gusset-plate Connections. In designing a welded gusset-plate connection all that is necessary is to divide



(a) (b) (c)
FIG. 103.

the stress in each member by the strength of the fillet per linear inch to determine the total length of fillet required. For a symmetrical member the total length of fillet is placed symmetrically with respect to the center line of the member, as shown in Fig. 103, which shows the connection of a channel to a plate.[1] If the plate is wide enough, one-half of the required fillet is placed along either flange of the channel, as shown in *a*; in this case there is no limit (other than practical considerations in laying the fillet) on the thickness of the fillet to be used. If the arrangement shown in (*a*) does not give sufficient fillet length, the fillet can be run down the end of the channel, as shown in (*b*); here the size of the end fillet is limited by the thickness of the channel web. If the arrangement shown in (*b*) still does not give sufficient length of fillet, an additional fillet can be laid against the back of the web of the channel and against the edge of the plate as shown in (*c*); the size of the latter fillet cannot be greater than the thickness of the plate.

If the member to be connected is not symmetrical, the total fillet length is so arranged as to make the center of gravity of the fillets coincide with the center of gravity of the member, in order to avoid eccentricity in the connection. Figure 104 shows the connection of an angle to a plate. The total length of fillet required is

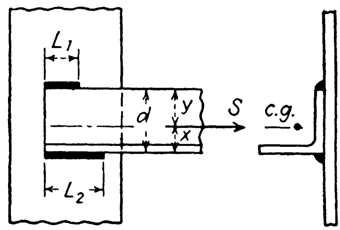


Fig. 104.

$$L = \frac{S}{f} \qquad (1)$$

in which f = the allowable stress in the fillet per linear inch.
S = the total stress in the angle.

The thickness of the fillet is limited by the thickness of the angle. The total required length L is divided between the two edges in two lengths L_1 and L_2 in such a way that the center of gravity of these lengths coincides with the center of gravity of the angle. The stress taken by the weld of length L_1 is L_1f and by the weld L_2 is L_2f. Taking moments about the back of the angle,

$$L_1fd = Sx = Lfx$$

$$L_1 = L\frac{x}{d} \qquad (2)$$

[1] In Fig. 103 the plate is one flange of a rolled column section.

Similarly,

$$L_2 = L\frac{y}{d} \tag{3}$$

The lengths actually used should be at least ½ in. greater than the theoretical, in order to allow for the crater that forms at the ends of the welds. In many trusses, gusset plates may be eliminated by lapping the members to be joined at any joint. Thus in the joint shown in Fig. 102, by making the lower chord of an inverted T shape, the diagonal angles can be welded directly to the stem of the T. The two diagonals will have to be spread a little to make room for them and their welds on the T, but the slight eccentricity that results at the joint is not serious where comparatively small stresses are involved.

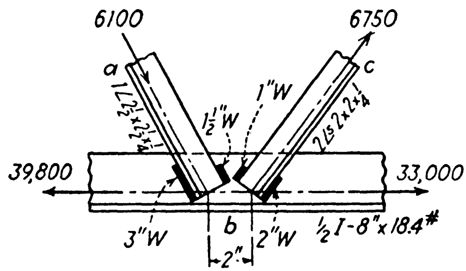


FIG. 105.

Example. Design a welded connection for the joint shown in Fig. 102, assuming that the diagonal members remain the same but that the lower chord consists of a half of an I-beam as shown in Fig. 105. The size of all fillets will be ¼ in., as limited by the thickness of the angles. From Table 2, page 135, the working strength of a ¼-in. fillet is 2000 lb. per lin. in.

From equations (1), (2), and (3), the following weld lengths are computed:

Member	Stress in each angle	L, in.	d, in.	x, in.	y, in.	Theoretical		Actual	
						L_1	L_2	L_1	L_2
ab	6100	3.05	2½	0.72	1.78	0.88	2.17	1½	3
bc	3400	1.70	2	0.59	1.41	0.50	1.20	1	2

The arrangement of the members and the details of the welds are shown in Fig. 105.

108. Riveted Beam Connections. The most common type of connection which is used where a simple beam frames into the web of a girder is shown in Fig. 106. This is called a *web connection.* The number of rivets required in the web of the beam is equal to the maximum end reaction or shear in the beam divided by the allowable stress for one rivet in bearing on the web of the beam. These rivets are also in double shear, so that the shearing value of one rivet is usually much greater than the bearing value. In Fig. 106 the rivets in the outstanding legs of the connection angles through the web of the girder are in single shear and in bearing on the girder web or on the leg of the angle. The number of rivets required here is equal to the beam shear divided by the governing rivet value.

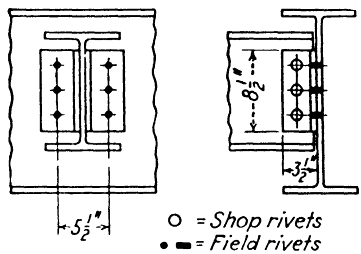


Fig. 106.

If there is a beam on each side of the girder web, in the same location, the number of rivets required through the girder web should be computed first by dividing the individual beam shears by the single-shear value of one rivet and then by dividing the combined beam shears by the bearing value of one rivet on the girder web; the larger of these two numbers of rivets should be used. If the two beams are of different depths, as shown in Fig. 107, the rivets passing through the web of the girder must be arranged so that: (1) there are enough rivets in each pair of connection angles to satisfy the single-shear requirement for the particular beam which is connected by each pair of angles; (2) there are enough rivets in the entire connection to satisfy the bearing requirement for the girder web; (3) the rivets are spaced so that no rivet will be too close to the lower flange of the shallower beam, which would interfere with the placing and driving of that rivet.

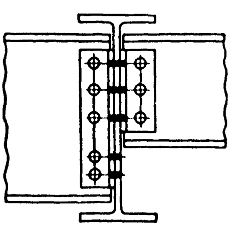


Fig. 107.

If the upper surface of the top flange of the beam is above, at, or near the same elevation as the top flange of the girder, the top flange of the beam must be cut away, or coped, as shown in Fig. 108. A shorter connection is then necessary, and a check should be made to insure that there is sufficient length in which to place the required number of rivets.

Connections of the type described above (web connections) may also be used where a beam frames into either the flange or the web of a column. If the beam which frames into the web of a column has a wider flange than the clear depth of the column web, the corners of the beam flanges can be coped to clear the column flanges. If the beam frames into the flange of the column, the arrangement of rivets must be made so that the web of the column will not interfere with the placing and driving of the rivets.

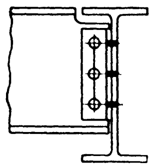


FIG. 108.

The size of the angles used in web connections varies somewhat with the size of the beam. The legs of the angles must be large enough to permit placing and driving the rivets in the outstanding legs after those in the legs which rest against the beam web have been placed. If only one row of rivets is required in each leg, for rolled beams greater than 10 in. in depth 4- by 3½-in. angles are satisfactory, with a thickness of ⅜ or 7/16 in. For beams 10 in. or less in depth, it is generally necessary to place the rivets in the beam web in two gage lines, and 6- by 4- by ⅜-in. angles, with the 6-in. leg along the web of the beam, are usually used.

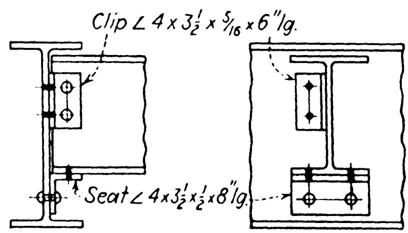


FIG. 109.

Another type of connection which is sometimes used where a beam frames into a girder or column is shown in Fig. 109. This is called a *seated connection*. The seat angles are designed to carry the entire end reaction of the beam. Light clip angles are used either along the web of the beam as shown in Fig. 109 or above the top flange of the beam merely to steady the beam and resist lateral movement. This type of connection can be used in beam-to-girder connections only when there is sufficient web depth in the girder to provide for the seat angle; it is of particular advantage in beam-to-column connections where shallow column sections or narrow-flange column sections are used, in which cases it might be difficult or impossible to use a web connection. For the larger beam reactions it is generally necessary to stiffen the seat angle by a pair of short vertical angles as shown in Fig. 110.

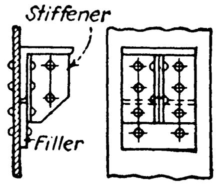


FIG. 110.

Examples. 1. The beam in Fig. 106 is a 12 × 8 wide-flange section, weighing 40 lb. per lin. ft., with an end shear of 14,000 lb. The girder into which it frames is an 18 × 7½ wide-flange section weighing 47 lb. per ft. Design a web connection for the beam, using ⅞-in. rivets and allowable stresses as given on page 133.

The thickness of the web of the beam is 0.294 in., and that of the girder is 0.350 in. The bearing value of one rivet on the web of the beam, at 40,000 p.s.i., is 0.294 × 40,000 × ⅞ = 10,300 lb. The number of rivets required is 14,000/10,300 = 2. The bearing value of one rivet on the web of the girder, at 32,000 p.s.i., is 0.350 × 32,000 × ⅞ = 9800 lb., and the single-shear value is 9020 lb. The number of rivets required through the web of the girder is 14,000/9,020 = 2. At least two rivets should be placed in each connection angle; hence, a minimum of four rivets should be placed through the girder web. A standard connection for a 12-in. beam, as given in the steel manufacturers' handbooks, uses three shop rivets through the web of the beam and three field rivets in the outstanding leg of each 4 × 3½ × ⅜-in. connection angle. The standard connection is detailed in Fig. 106.

2. Design a riveted seated connection for the beam in Example 1. The only rivets that are required are those in the vertical leg of the seat angle through the web of the girder. The value of one rivet is governed by the strength of the rivet in single shear, and as in Example 1 two rivets are required. Two rivets would also be placed in the outstanding leg of the seat angle, one passing through the flange of the beam on each side of the beam web. A 4 × 3½ × ½ seat angle 8 in. long will be satisfactory. There is just about enough depth in the web of the girder to allow for the placing of the seat angle without having to cope the top flange of the beam, provided that the beam does not have to be set at any given distance below the top flange of the girder. The length of bearing necessary to prevent crippling of the web of the beam (see Art. 65) is $\frac{14,000}{24,000 \times 0.294} - \frac{3}{4} = 1.23$ in., the distance k being obtained from the steel manufacturers' handbooks. Much

more than this is furnished by the 3½-in. leg of the connection angle, even though the full width of this leg is not considered effective in bearing because of its lack of stiffness. A single clip angle 4 × 3½ × 5/16 in., 6 in. long, with two rivets or bolts in each leg, will be sufficient to steady the beam.

109. Welded Beam Connections. Welded beam connections are mostly seated connections, because of certain practical difficulties (caused by required erection clearances) in making a web connection. The total length of fillet required to connect the seat angle to the web of the girder or column is determined by dividing the beam shear by the strength of the fillet per linear inch. The fillet can be placed in several ways: (1) All of it can be placed along the bottom of the vertical leg of the seat angle against the web of the girder; (2) half of it can be placed along each end of the seat angle; (3) part of it can be placed along the bottom of the vertical leg of the seat angle and the remainder divided equally between the two ends of the angle. A short tack weld should be placed along each edge of the beam flange against the outstanding leg of the seat angle in order to steady the beam, and a short clip plate should be welded to the beam web with its end welded to the girder web for additional lateral support. Figure 111 shows a typical welded connection in which all the required fillet is placed along the bottom of the vertical leg of the seat angle.

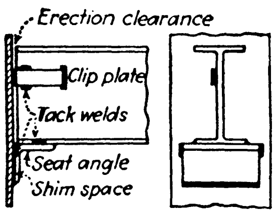


FIG. 111.

110. Eccentric Riveted Connections. As stated in the preceding paragraphs, if uniform stress in a joint is to be obtained it is necessary that the lines of action of the various members connected pass through the center of gravity of the rivets used for the joint. This condition is not, however, always practically possible to obtain, and it is necessary in such cases to determine the actual distribution of stress.

Figure 112*a* represents a simple eccentric riveted connection. The line of action of the force P produces a bending moment about the center of the connection, rivet B, equal to Pe, and the outside rivets A and C at the connection are each subjected to a bending stress of $Pe \div 2l$ in addition to their proportional part of the direct stress due to the load P. The stress due to bending acts normal to the radii from the center of the group of rivets, downward for rivet A and upward for rivet B (see Fig. 112c).

Assuming that all the rivets are ⅞-in. power-driven rivets in single shear at 12,000 p.s.i., that the distance l is 6 in., and that the angle between the two members is as indicated on the

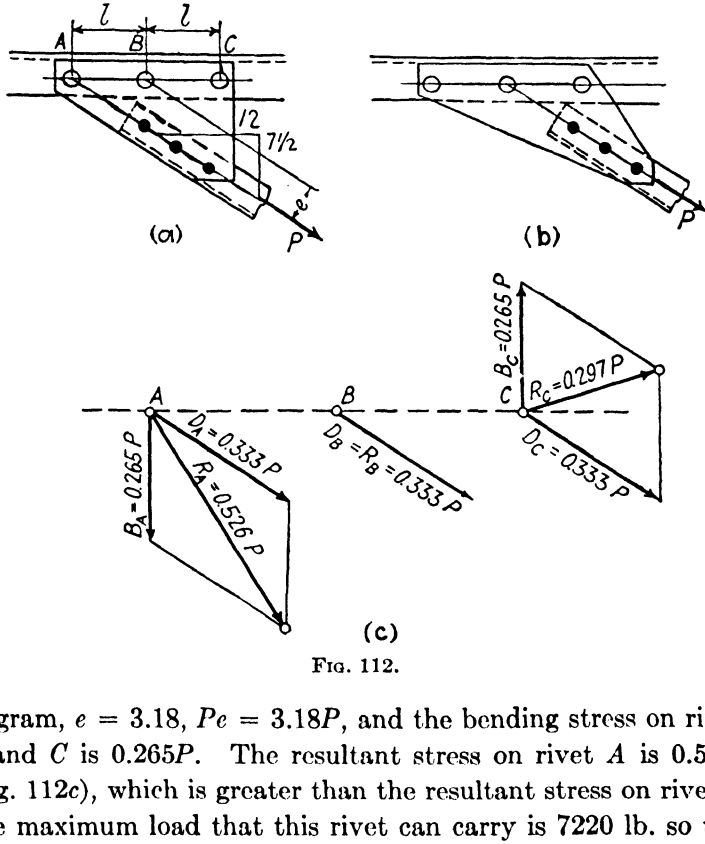


FIG. 112.

diagram, $e = 3.18$, $Pe = 3.18P$, and the bending stress on rivets A and C is $0.265P$. The resultant stress on rivet A is $0.526P$ (Fig. 112c), which is greater than the resultant stress on rivet C. The maximum load that this rivet can carry is 7220 lb. so that the force P cannot exceed 13,730 lb. In a connection such as is shown in Fig. 112b, where no bending stresses are present, the allowable force P is $3 \times 7220 = 21{,}660$ lb. The increased capacity of a joint where moment is eliminated at once becomes apparent.

In Fig. 113 the plate is riveted to a large, relatively stiff member. The line of action and amount of stress in the other member connected to the plate are represented by P. The rivets connecting this member to the plate are assumed to be in the line of action of the member and are not shown. If C represents the center of gravity of the group of rivets and e the normal distance from C to the line of action of the force P, then the plate is subjected to a turning moment equal to Pe. This moment causes an additional bending stress on each rivet which is proportional to the distance of the rivet from the center of gravity of the group. Let r_0 represent the bending stress on any rivet at a unit distance from C, and r the bending stress on any rivet at a distance z from C. Then $r = r_0 z$, and the moment of resistance of this stress is $r_0 z^2 = r_0(x^2 + y^2)$ where x and y are the corresponding components of z.

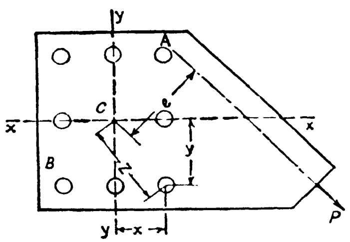


FIG. 113.

Since the total resisting moment is equal to the turning moment,

$$r_0(\Sigma x^2 + \Sigma y^2) = Pe$$

and

$$r_0 = \frac{Pe}{\Sigma x^2 + \Sigma y^2}$$

When r_0 is known, the value of r for any rivet may readily be determined from the relation $r = r_0 z$. The direction of this stress is normal to the radius from the center of gravity to the rivet. The bending stress may be combined with the direct stress to determine the resultant stress on the rivet. In Fig. 113, A is obviously the most stressed rivet, for the two forces acting upon it are nearly parallel and in the same direction. The bending stress on B has the same value as that on A but it acts in the opposite direction, so that unless the bending stress is greater than the direct stress it will be the least stressed rivet of the whole group.

Example. **Determine the maximum stress on the critical rivet of the group of 10 rivets which connects the bracket shown in Fig. 114 to the column.**

The center of gravity of the group is midway between the two central rivets. The value of x is $2\frac{3}{4}$ in. for each rivet, and the value of y is zero for the two central rivets, 8 in. for the four outer rivets, and 4 in. for the remaining four rivets. The maximum stress occurs on one of the corner rivets in the group.

$$r_0 = \frac{20{,}000 \times 10}{10 \times (2\frac{3}{4})^2 + 4(4^2 + 8^2)} = 505 \text{ lb.}$$

When the load on the bracket is vertical, as in this case, it is simpler to compute the vertical and horizontal components of r, because the direct stress, which is vertical, can be added to the vertical component of stress on the critical rivet. This vertical component can then be combined analytically with the horizontal component of r to determine the resultant stress. The horizontal component r_H is equal to $r_0 y$ and the vertical component r_V is equal to $r_0 x$.

$$r_H = 505 \times 8 = 4040 \text{ lb.}$$
$$r_V = 505 \times 2\tfrac{3}{4} = 1390 \text{ lb.}$$

The total vertical component of stress on the critical rivet is $1390 + \frac{20{,}000}{10} = 3390$ lb., and the resultant stress is $\sqrt{4040^2 + 3390^2} = 5270$ lb.

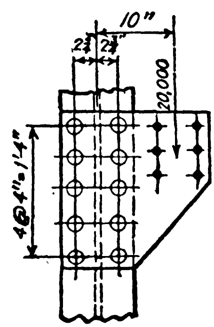


FIG. 114.

If $\frac{3}{4}$-in. rivets are used and if the allowable unit shearing stress on the rivet is 13,500 p.s.i., the strength of one rivet in single shear is 5960 lb., and the arrangement of rivets as shown in Fig. 114 is satisfactory. Actually, a slightly smaller rivet spacing could have been used if it were desired to reduce the depth of the bracket.

111. Steel Truss Bearings. Very light roof trusses could rest directly on the wall, the bearing plate being fastened to the lower chord angles. This type of bearing creates a bending stress in the chord member and results in an unequal distribution of the pressure on the wall. A more satisfactory type of bearing for medium-size trusses is shown in Fig. 115. The gusset plate is extended below the lower chord, and a pair of angles (shoe angles) is riveted to the lower end of the plate. A horizontal plate (sole plate) is riveted to the outstanding legs of the angles, the lower heads of the rivets being countersunk to permit this plate to rest smoothly on another plate (bed plate) beneath it, which transfers the load to the wall. The bed, or wall, plate is anchored to the wall by means of anchor bolts. The anchor bolts project up through the sole plate, but the holes in the sole plate for these bolts are slotted to permit the whole truss to move slightly as the length of the truss changes owing to temperature variations. The design of the component parts of this bearing is in accordance with principles already explained.

Heavier trusses, such as bridge trusses, and girders should be provided with pin bearings to secure uniform pressure on the

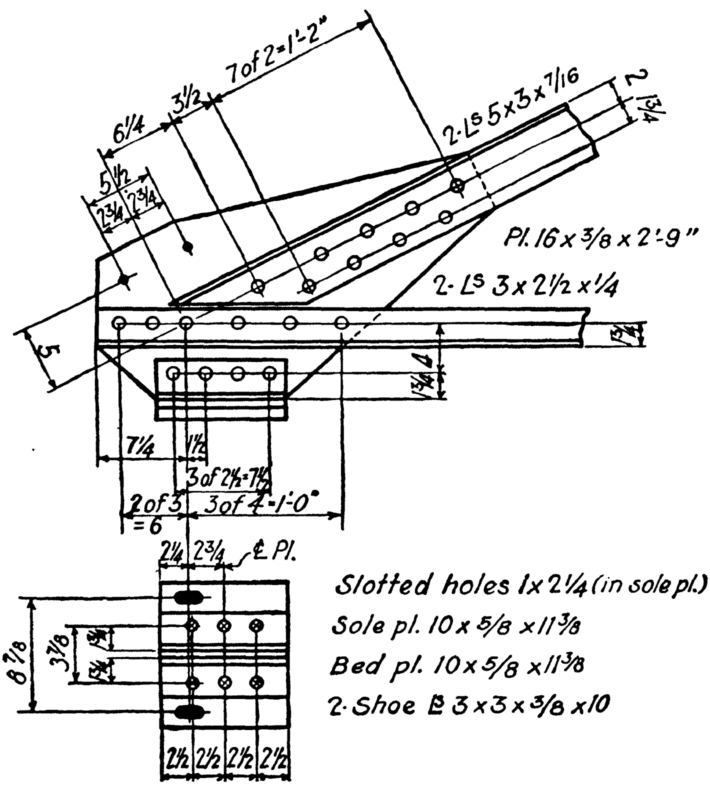


Fig. 115.

walls or piers, and at one end large rollers or rockers should be placed under the lower part of the bearing to provide for movement caused by expansion and contraction. The details and design of such bearings are beyond the scope of this book.[1]

[1] See the authors' "Design of Steel Structures," McGraw-Hill Book Company, Inc.

STEEL SPLICES

112. Splices for Maximum Efficiency. When two steel plates in tension are to be spliced, they can be lapped at the ends, as shown in Fig. 116, or the ends can be butted together and two splice plates placed across the joint, one on either surface of the main plates, as shown in Fig. 117. If the width of the splice plates is the same as that of the main plates, obviously the thickness of each splice plate must be at least one-half that of the main plates, since each splice plate carries across the joint one-half the stress in the main plates. Actually, each splice plate should be slightly thicker than one-half the thickness of the main plates, as the following analysis will show. The number of rivets required is determined by dividing the stress in the plates by the strength of one rivet.

A tension joint might fail because of inadequate riveting or because of overstressing the net section of the plates. Failure due to inadequate riveting can be avoided by placing enough rivets in the splice so that the rivets can take more stress than the net section of the plates can take. The main plates are stronger beyond the splice than they are at the splice, where holes are cut out for rivets. The rivets in the splice should be so arranged, therefore, as to cause the strength of the plates at the splice to approach as nearly as possible the strength of the plates beyond the splice. The strength of the splice (in the way it will fail), divided by the strength of the gross section of the main plates beyond the splice, is called the efficiency of the splice.

For the lap splice shown in Fig. 116, the net width of plate *a* at section 1 is the gross width minus 1 rivet-hole diameter. Plate *a* is carrying the full stress at this section, and the maximum safe value of this stress is the net width times the thickness times the allowable unit stress in tension. At section 2 the net width of plate *a* is the gross width minus 2 rivet-hole diameters. The stress in plate *a* at section 2 is less than the stress at section 1 by the amount of stress that the rivet in section 1 has removed from this plate and transferred to plate *b*. Hence, the maximum safe stress in plate *a*, as governed by the strength of the plate at section 2, is equal to the strength of the net section at 2 plus the value of the rivet in section 1. Similarly, the maximum safe stress in plate *a*, as governed by the strength of the plate at section 3, is equal to the strength of the net section at 3 plus the values of the three rivets in sections 1 and 2. The safe strength of the plate *a* will be governed usually either by section 1 or by section 2. Section 4 need not be investigated since the net area of the plate is the same as in section 3, and the total stress in the plate is less by the value of the three rivets in section 3. The same analysis holds for plate *b*, with the sections numbered from the right end of the splice. The strength of this splice is obviously greater than if the required number of rivets had been placed in transverse rows with the same number of rivets in each row; hence, the efficiency is greater. Both plates are trimmed at the ends as shown in Fig. 116, to avoid curling of the corners.

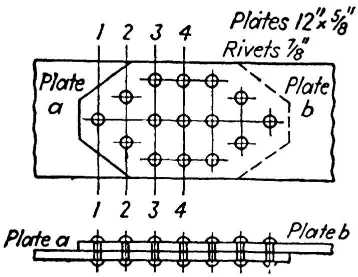


FIG. 116.

A similar analysis can be made for a butt splice. Here, however, the critical net section of the splice plates is in the row of rivets nearest the center of the splice, because the splice plates are carrying the full stress and their net width is a minimum at this section. Hence, in order that the splice plates will not govern the strength of the splice, the thickness of each splice plate should be slightly greater than one-half the thickness of the main plates.

Examples. 1. A tension member consisting of a single plate 12 in. wide and 5/8 in. thick is to be spliced to a similar section with 7/8-in. rivets. Design and sketch a joint of maximum efficiency, using the following unit working stresses: tension 18,000 p.s.i.; shear in rivets 12,000 p.s.i.; bearing 24,000 p.s.i.

The effective diameter of one rivet hole is $\frac{7}{8} + \frac{1}{8} = 1$ in. The bearing value of one rivet is $24{,}000 \times \frac{7}{8} \times \frac{5}{8} = 13{,}100$ lb., and the single-shear value of one rivet is $\frac{\pi \times (\frac{7}{8})^2}{4} \times 12{,}000 = 7230$ lb. Placing the rivets as shown in Fig. 116, the following investigations are made:

Section 1: Net area of plate a = ⅝(12 − 1) = 6.88 sq. in.
Safe stress in plate a = 6.88 × 18,000 = 124,000 lb.
Section 2: Net area of plate a = ⅝(12 − 2) = 6.25 sq. in.
Safe stress in plate a = 6.25 × 18,000 + 7230 = 119,730 lb.
Section 3: Net area of plate a = ⅝(12 − 3) = 5.62 sq. in.
Safe stress in plate a = 5.62 × 18,000 + 3 × 7230 = 122,690 lb.

The safe stress in the plates is therefore governed by section 2, the value being 119,730 lb. The number of rivets required is 119,730/7230 = 17. With one additional row of 3 rivets, the arrangement in Fig. 116 provides a total of 18 rivets. The safe stress in the plates beyond the splice is 12 × ⅝ × 18,000 = 135,000 lb., and the efficiency of the joint is 119,730/135,000 = 0.885 or 88½ per cent.

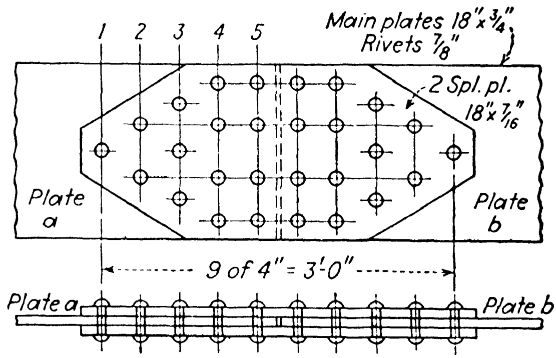


FIG. 117.

2. Two tension plates 18 in. wide and ¾ in. thick are to be spliced by two plates 18 × 7/16 in., using ⅞-in. rivets. Design and sketch a splice for maximum efficiency, using allowable unit stresses as follows: tension 16,000 p.s.i.; shear in rivets 12,000 p.s.i.; bearing 24,000 p.s.i.

The bearing value of one rivet on the main plates is 24,000 × ⅞ × ¾ = 15,700 lb., and the double-shear value is $2 \times \frac{\pi \times (7/8)^2}{4} \times 12,000 = 14,440$ lb. For the arrangement of rivets shown in Fig. 117, the following analyses are made:

Section 1: Net area of plate a = ¾(18 − 1) = 12.75 sq. in.
Safe stress in plate a = 12.75 × 16,000 = 204,000 lb.
Section 2: Net area of plate a = ¾(18 − 2) = 12.00 sq. in.
Safe stress in plate a = 12.00 × 16,000 + 14,400 = 206,440 lb.
Section 5: Net area of 2 splice plates = 2 × 7/16(18 − 4) = 12.25 sq. in.
Safe stress in 2 splice plates = 12.25 × 16,000 = 196,000 lb.

The safe stress in the main plates is governed, therefore, by the strength of the splice plates, the value being 196,000 lb. The number of rivets required in the splice on either side of the joint is 196,000/14,400 = 14; with the arrangement shown in Fig. 117, 14 rivets are furnished. The safe strength of the main plates beyond the splice is 18 × ¾ × 16,000 = 216,000 lb., and the efficiency of the joint is 196,000/216,000 = 0.908, or 90.8 per cent.

113. Boiler and Tank Riveting. Work requiring riveted joints is of two general kinds, structural work and boiler or tank work, each of which has different specifications for the riveting. Rivet holes are made 1⁄16 in. larger than the nominal size of the rivet. In structural work, on account of the variation permitted in the fit of the holes in any two plates joined by rivets, the *nominal* size of the rivet is used in computing bearing and shearing strengths. In boiler, tank, and pipe riveting, specifications require that the holes in the plates shall be concentric; thus, the size of the *finished* rivet (diameter equal to diameter of hole) is used in the computation of bearing and shearing strengths. In structural work, because of the tearing of the fibers around punched holes, in the computation of net areas of plates the diameter of the hole is considered to be ⅛ in. larger than the nominal diameter of the rivet, whereas in boiler work, since the holes are usually drilled or reamed, the actual diameter of the hole (1⁄16 in. larger than the nominal size of the rivet) is used in net-area computations.

For boiler-plate riveting the American Society of Mechanical Engineers Boiler Code gives allowable unit stresses as follows: tension, net section, 11,000 p.s.i.; shear on rivets 8800 p.s.i.; bearing 19,000 p.s.i. For tanks, somewhat larger stresses are used; but these are generally smaller than the stresses ordinarily specified for structural riveting. The principles outlined in the preceding article may be adapted to the design or investigation of a splice in boiler or tank plates, these splices usually being made with fewer rivets in the outer rows of the splice in order to increase the efficiency of the joint.

Examples. 1. Two ⅜-in. plates of a cylindrical tank are joined longitudinally by a lap joint with three rows of ¾-in. rivets. The pitch of the rivets in the outer rows is 4½ in., and in the center row 2¼ in. It is required to find the strength per linear inch of joint if the allowable stresses are 12,000 p.s.i. in shear, 24,000 p.s.i. in bearing, and 16,000 p.s.i. in tension. There are four rivets in a 4½-in. length of joint (two in the center row and one in each of the outer rows). The bearing strength of one rivet on the

$\frac{3}{8}$-in. plate is $\frac{13}{16} \times \frac{3}{8} \times 24{,}000 = 7300$ lb., and the shearing strength of one rivet is $\frac{1}{4} \times \pi \times (\frac{13}{16})^2 \times 12{,}000 = 6240$ lb. The strength of the four rivets is therefore $4 \times 6240 = 24{,}960$ lb., and, so far as the rivets are concerned, the stress in the plate cannot exceed $24{,}960/4.5 = 5550$ lb. per lin. in.

The net section of the plate in the outer row of rivets, for a $4\frac{1}{2}$-in. width, is $(4\frac{1}{2} - \frac{13}{16})\frac{3}{8} = 1.38$ sq. in., and the allowable stress in the plate per linear inch of splice is $(1.38 \times 16{,}000)/4.5 = 4920$ lb. The net section of the plate in the center row of rivets, for a $4\frac{1}{2}$-in. width, is $(4\frac{1}{2} - 2 \times \frac{13}{16})\frac{3}{8} = 1.08$ sq. in., and the allowable stress in the plate for a $4\frac{1}{2}$-in. width is $1.08 \times 16{,}000 = 17{,}250$ lb.

However, the actual stress in the plate at this section for the $4\frac{1}{2}$-in. width is less than the stress in the same width of plate beyond the joint by the amount of stress which one rivet in the outer row has transferred to the other plate (6240 lb., as governed in this case by the strength of the rivet in single shear). The total stress in the plate beyond the splice can therefore be $(17{,}250 + 6240)/4.5 = 5220$ lb. per lin. in., without overstressing the net section of the plate at the center row of rivets. The total stress in the plate is governed, therefore, by the strength of the net section in the outer row of rivets (compare 5550, 4920, and 5220 lb. per lin. in.) and is 4920 lb. per lin. in.

2. It is required to find the safe unit pressure in pounds per square inch which can be applied to the inside of the tank in Example 1, if the inside diameter of the tank is 60 in. For a 1-in. length of tank the total tension in the plate cannot exceed 4920 lb. The actual tension per linear inch which is caused by a unit pressure p (pounds per square inch) is equal to $\frac{1}{2} \times p \times d$, in which d is the diameter of the tank in inches. Hence,

$$4920 = \frac{1}{2} \times p \times 60$$

from which $p = 164$ p.s.i.

114. Joints in Concrete Structures. Concrete structures are generally poured as a unit, in which case the individual members are continuous at the joints and provision is made by proper placing of the reinforcement for the negative moments at these joints. The arrangement of the reinforcement is a part of the design of the members themselves, and no special joint investigation is necessary. Occasionally, some members are poured after the main structural framework is completed, in which case grooves or recesses are formed in the main frame which serve as supports for the latter members. Dowels are placed in the main frame, extending across and projecting beyond the grooves to serve as ties or anchors for the new members.

CHAPTER VIII

PLATE GIRDERS[1]

115. Types. Plate girders are used extensively in every form of steel construction because of their adaptability. They resist transverse bending like beams, but they are used for heavier loads, for longer spans, or for conditions for which single rolled beams are not so well adapted. Each is made with a web plate to which flanges are riveted or welded at the top and bottom edges; the most common forms of cross-section are as shown in Fig. 118, type *a* being welded, the others being either riveted or welded.

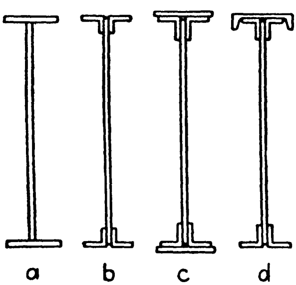


FIG. 118.

116. Depth. The depth of a plate girder is often predetermined by specific requirements. Within practical limits, flanges may be designed for any depth of girder; the deeper the girder for a given load and span, the lighter the flanges will be. The most practical depth varies from one-seventh the length for short spans to one-twelfth the length for long spans; in the absence of other data, an average depth of one-tenth the length may be chosen. The maximum depth is limited to about 10 ft.-6 in. or 11-ft.-0 in. by shipping clearances. The depth of the web plate is usually a multiple of 2 in. and preferably a multiple of 6 in. Flange angles are usually placed so that they project $\frac{1}{4}$ in. beyond the edge of the web to avoid the necessity of chipping projections in the plate resulting from imperfect shearing at the mill. In case such projection might leave a rain pocket exposed to the elements, as in an outdoor girder without cover plates, the angles are made flush

[1] Most of this chapter has been taken from "Civil Engineering Handbook," L. C. Urquhart, editor-in-chief, Sec. 6, Steel Design, by Carlton T. Bishop, Associate Professor of Civil Engineering, Yale University.

with the *top* of the web. Thus, the depth from back to back of flange angles is either 1/2 or 1/4 in. more than the nominal depth of the web plate.

117. Web Thickness. The usual thickness of the web of a plate girder is 5/16 to 5/8 in., with a minimum of 1/170 or 1/160 of the clear distance between flange angles (or sometimes 1/20 the square root of this distance). The web plate must have sufficient area to resist the maximum shear. The maximum shear intensity at the neutral axis is substantially equivalent to the value found by dividing the maximum shear by the gross area of cross-section of the web plate, the flange angles and cover plates being neglected; this shear intensity should not exceed 12,000 p.s.i. or other value specified (see Art. 21). The web plate should also be thick enough to furnish sufficient bearing for the flange rivets and sufficient strength between rivets, as explained later.

118. Flange Angles. The angles most used for plate girders are 6 by 4, 6 by 6, 8 by 6, and 8 by 8; smaller sizes are not used so much as they were before the larger rolled girder beams were available. The unequal-legged angles preferably should be placed with the shorter leg against the web and the longer leg outstanding, because the strength of the girder is greater that way both vertically and horizontally. Occasionally, however, it is necessary to have the longer leg against the web to provide for adequate riveting. When angles are used in the compression flange without cover plates, the thickness of each must not be less than 1/10 or 1/12 the length of the outstanding leg.

119. Cover Plates. Cover plates are used to give additional strength in case 6 by 4 by 3/4 or 6 by 6 by 3/4 angles are insufficient; 8 by 8 angles are seldom used without cover plates; 14- (or 13-in.) plates are used with 6-in. angles, and 18-in. plates with 8-in. angles. The area in the cover plates should not exceed that in the angles, unless 6 by 6 by 3/4 or larger angles are used, and no cover plate should have a thickness greater than the angles. For convenience, the plates of each flange are treated as a single plate in the design; but if the total thickness exceeds the maximum punching thickness in riveted work (generally 3/4 in.), it is subdivided so that each plate is at least $\frac{3}{8}$ in. and not more than the punching maximum; usually the smallest number of plates conforming to these limits is used, in order to save the cost of extra handling and punching. The plates should be of the same thickness, unless metal may be saved by making one plate $\frac{1}{16}$ in. less than the others, in which case the thicker plates should be next to the angles. If two plates are used in each flange of a welded girder, one should be made at least 1 in. wider than the other to facilitate welding.

120. Methods of Design. Girders may be designed by the moment-of-inertia method or by the flange-area method; either method is permitted by some specifications, but only the former method by others. Unless exhaustive tables of moments of inertia of plate girders are available, this method of design is likely to be tedious; it is often quicker to use the flange-area method first and then, if necessary, verify the size thus determined by the moment-of-inertia method. The weight of the girder is the weight of the main component parts plus about 10 per cent to allow for stiffeners, fillers, rivet heads, and other details. The weight of the web plate can be found at once, but in the preliminary design the weight of the angles must be assumed; if cover plates are used, they may be assumed to weigh about the same as the angles. In the final design, the weight of the girder should be found from the sizes resulting from the preliminary design.

121. Moment-of-inertia Method. This is an application of the formula for flexure, $M = fI/e$ (see Art. 55). Holes for rivets in the tension half of the girder weaken that half and must be considered in finding the moment of inertia, unless the girder is to be welded or is designed by specifications that permit the use of the gross section. For an unsymmetrical girder, as type *d* (Fig. 118), it is necessary to find the center of gravity of the cross-section in order to locate the neutral axis, and to consider the strength of the girder as determined both by the tension half and by the compression half. Gross area of the compression half must be not less than that of the tension half, but may be greater since the allowable unit stress is less. For a girder with equal flanges, the strength as determined by the tension half is substantially the same as if holes were deducted from both halves, the neutral axis being thus kept midway between the angles, no attention being paid to lack of symmetry when the top angles are flush with the edge of the web and the bottom angles project $\frac{1}{4}$ in. Similarly, the strength as determined by the compression half is found by assuming both halves alike, with no rivet holes deducted.

The moment of inertia of the gross section of the entire girder about the neutral axis is found by combining the moments of inertia of the different component parts about the same axis. The moment of inertia of the net section is found by deducting the moments of inertia of the holes from the moment of inertia of the gross section. The moment of inertia of the web plate is $\frac{1}{12}td_w^3$ where d_w is the depth of the web. The moments of inertia of the other component parts are found by transfer from one axis to another (see Art. 5). For a given area, the moment of inertia about an axis through the center of gravity is less than about any parallel axis; the moment of inertia about a parallel axis I_a is found from the moment of inertia about the gravity axis I_g by adding the product of the area and the square of the distance x between the two axes, or $I_a = I_g + Ax^2$. The moment of inertia of one angle about its own center of gravity may be found from the usual table of properties. The moment of inertia of the cover plates may be taken as if all the plates of each flange were combined; the moment of inertia about the center of gravity of this plate may be neglected, because the $I_g = bd^3/12$ is relatively small when d is the smaller dimension; the Ax^2 is important.

The holes for the rivets which fasten the angles to the web are placed on the standard gage lines, two lines being used for 6-in. legs or over. It is assumed that rivets in two lines will be staggered; near the point of maximum bending moment, the rivets will usually be far enough apart so that only the one nearest the back of the angle need be considered. The area of this hole is the combined thickness of the web and the two angles multiplied by the diameter of the hole, which is taken as

⅛ in. larger than the nominal size of the rivet. The moment of inertia is simply this area times the square of the distance from its center to the neutral axis, the moment of inertia about its own gravity axis being negligible. Similarly, two additional holes in each flange are deducted when cover plates are used, regardless of whether one or two rows are used in each angle; the combined thickness of angle and cover plate is used in finding the area, and the distance from the neutral axis to the center of the hole may be found by subtracting one-half this thickness from the distance e. In case stiffening angles or splice plates are to be used near the section of maximum bending moment, holes should be taken out of the web plate at an average distance of about 4 in. apart, a little variation having relatively little effect. Stiffening angles are required if the clear distance between flange angl s is more than sixty times the web thickness, and at points of concentration.

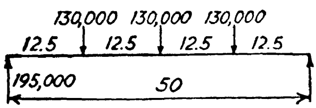


FIG. 119*a*.

122. Design of a Girder by Moment-of-inertia Method. Design a plate girder with a span of 50 ft.-0 in. to support three concentrated loads of 130,000 lb. each placed at the center and quarter points of the span (Fig. 119*a*). The web is to be 72 in. deep, flange angles are to be 6 × 6 in., cover plates 14 in. wide, rivets ⅞ in. Allowable unit stresses are to be assumed as 12,000 p.s.i. in shear on the gross section of the web and 18,000 p.s.i. in bending on the extreme tension fiber instead of the values given in Art. 21. The assumed specifications further stipulate that, if the distance l between lateral supports (the concentrated loads) exceeds fifteen times the width b of the flange, the unit stress in the compression flange must be not greater than $\dfrac{20{,}000}{1 + \dfrac{l^2}{2000b^2}}$, considering the moment of inertia of the gross section. In this case 12.5 × 12 does not exceed 15 × 14, and so the compression flange may be made the same as the tension flange. This specification differs somewhat from the corresponding value in Art. 21.

In designing a girder by the moment-of-inertia method, the angle and cover-plate sizes must be assumed and the investigation carried out to determine if the extreme fiber stresses are satisfactory. In this case, it will be considered that a preliminary design by the flange-area method described in Art. 123 was first made (see Art. 124) and that the resulting section consisted of a web 72 × ⅜, four angles 6 × 6 × ¾, and four cover plates

$14 \times \frac{5}{8}$. This section will now be investigated to see whether or not it is satisfactory. The distance back to back of angles will be made 72.5 in., thus making the angles project $\frac{1}{4}$ in. beyond the web plate to allow for proper placing of the cover plates.

R_L (concentrated loads) = 1.5 × 130,000 = 195,000 lb.
M (concentrated loads) = 195,000 × 25 − 130,000 × 12.5 = 3,250,000 ft.-lb.

The weight of the assumed section, allowing 10 per cent for details, is 1.1(92 + 4 × 28.7 + 2 × 59.5) = 360 lb. per ft.

M (weight) = $\frac{1}{8} \times 360 \times 50^2$ = 112,000 ft.-lb.
Total moment M = 3,362,000 ft.-lb.
= 40,340,000 in.-lb.

$$e = \frac{72.5}{2} + 2 \times \frac{5}{8} = 37.5 \text{ in.}$$

The net I required is obtained from the equation

$$M = f\frac{I}{e},$$

$$40{,}340{,}000 = 18{,}000 \times \frac{I}{37.5}$$

from which I = 84,000 in.[4]

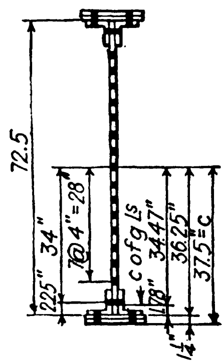


FIG. 119*b*.

I (web) = $\frac{1}{12} \times \frac{3}{8} \times 72^3$ = 11,660 in.[4]
I (4 angles, see Fig. 119*b*) = 4(28 + 8.44 × 34.47²) = 40,220 in.[4]
I (4 cover plates) = 2 × 14 × 1¼(36.25 + 0.62)² = 47,590 in.[4]
Total I of gross section = 99,470 in.[4]

I of holes through angles and cover plates = 4 × 1 × 2(37.5 − 1)²
= 10,660 in.[4]
I of holes through angles and web = 2 × 1 × 1⅞ × 34² = 4,330 in.[4]
I of other holes in web (see Fig. 119*b*) = 2 × 1 × ⅜(28² + 24² + 20² + 16² + 12² + 8² + 4²) = 1,680 in.[4]
Total I of holes = 16,670 in.[4]
Total I of net section = 99,470 − 16,670 = 82,800 in.[4]

This is somewhat less than the 84,000 required, and so the girder should be strengthened by increasing the thickness of one cover plate in each flange by $\frac{1}{16}$ in. This would increase the total net moment of inertia to 84,910 in.[4] The first design is only 1.5 per cent under the required strength and would be considered sufficiently close by most designers. If the girder were welded instead of riveted, no deduction would be made for rivet holes.

123. Flange-area Method. In this method, the stress in each flange is assumed to act at the center of gravity of the flange, and a web equivalent is counted as flange area; thus the resisting moment is the moment of these two resultant forces, which form a couple. Different unit stresses are used in the two flanges as in the method just described, the tension flange being designed first for net section and the compression flange being made like it unless the distance between lateral supports is such that a larger compression flange must be used, considering gross section. In the tension flange of a riveted girder, one hole is deducted from each angle for the rivets which connect the angles to the web even if two rows of rivets are used, for they would be staggered and usually far enough apart in the vicinity of the point of maximum bending moment so that adjacent holes need not be considered. In case cover plates are used, one additional rivet in each angle is assumed to be in the same cross-section. In a welded girder, there are no rivets connecting the component parts, but any holes provided for other connections should be deducted.

The lever arm of the couple is the distance from center to center of gravity of the gross areas of the flanges and is called the "effective depth," d_g. The gross area is used for convenience, for it is simpler and gives substantially the same results. In a girder without cover plates, usually the effective depth can be determined with sufficient accuracy from assumed values, for the position of the center of gravity of the angles does not change greatly with different thicknesses; if the lengths of legs are known or assumed correctly, the effective depth may be assumed (to the nearest $\frac{1}{10}$ in.) as the distance back to back of angles minus twice the distance from the back of the outstanding leg to the center of gravity of an angle of intermediate thickness, say, $\frac{1}{2}$ or $\frac{9}{16}$ in. When cover plates are used, a preliminary design is made with an effective depth equal to the nominal depth of the web plate d_w if the vertical legs of the angles are shorter than the others, or 1 in. less than the depth of the web if the legs are equal. From the sizes of the angles and cover plates required in the trial design, the distance from the backs of the angles to the center of gravity of the gross areas of the angles and the cover plates may be determined and the effective depth to be used in the redesign obtained. A depth greater than the distance back to back of angles should not be used.

One-eighth the gross area of the web plate is counted as flange area. This takes into account the strength of the web acting as a rectangular beam. A plate without holes would have a resisting moment of $\frac{1}{6}ftd_w^2$, and this would be used in a welded girder; by changing $\frac{1}{6}$ to $\frac{1}{8}$, it is assumed that the strength of the net section is $\frac{3}{4}$ that of the gross section, which allows for 1-in. holes for flange rivets and for rivets about 4 in. apart in the stiffeners. By dividing by the effective depth d_g and by the unit stress, the result is $\frac{1}{8}td_w$, on the assumption that $d_w/d_g = 1$. It is safe to make this assumption, because d_g is usually less than d_w and can exceed it only in heavy girders, where its effect is relatively small.

The flange-area method may be summarized as follows: Let A' = area of angles and cover plates in one flange (net area for investigation of tension flange, and gross area for compression flange).

$$M = fd_g(A' + \tfrac{1}{8}td_w) \qquad (1)$$

or

$$\frac{M}{fd_g} - \frac{1}{8}td_w = A' \qquad (2)$$

To design the tension flange: Divide the bending moment in inch-pounds by the allowable unit stress in bending (see Art. 21) and by the effective depth in inches (see page 174); subtract one-eighth the gross area of the web, leaving the *net* area A' required in the angles and cover plates [see equation (2)]; find two angles the net area of which will equal or exceed the required area A' or one-half this area in case cover plates are to be used; in taking net areas, one hole per angle is deducted when no cover plates are used and two holes per angle when cover plates are used, each hole being the product of the thickness of the angle by the diameter of the hole, which is taken as $\frac{1}{8}$ in. greater than the rivet diameter; determine the net area required in the cover plates of one flange by subtracting from A' the net area of the angles used; find the net width of cover plate by subtracting the diameters of two holes from the plate width; find the total thickness of cover plates of one flange by dividing the net-area required by the net width, expressing the result in a multiple of $\frac{1}{16}$ in. If cover plates are used, a revised design is necessary, the corrected effective depth and weight being used; in the final design, the total thickness of cover plates may be subdivided into the smallest number of plates of equal thickness (or differing only $\frac{1}{16}$ in.), so that no plate shall be less than $\frac{3}{8}$ or more than the maximum punching thickness for riveted work. This punching limit does not apply to welded girders. The punching limit is usually $\frac{3}{4}$ or sometimes $\frac{7}{8}$ in.

To design the compression flange: The method of design for the compression flange is essentially the same as for the tension flange, except that (1) gross areas of angles and cover plates are used in place of net areas; (2) the allowable unit stress in bending for the extreme compression fiber (see Art. 21) is used instead of that for the extreme tension fiber; (3) the equivalent flange area of the web is taken as $\frac{1}{6}td_w$ instead of $\frac{1}{8}td_w$ [see equation (2)]. A larger section may be required for the compression flange than the tension flange, but nothing smaller should be used.

124. Design of a Girder by Flange-area Method. Design a plate girder with a span of 50 ft.-0 in. to support three concentrated loads of 130,000 lb. each, placed at the center and quarter points of the span (Fig. 119*a*). The web is to be 72 in. deep, flange angles are to be 6 × 6 in., cover plates 14 in. wide, rivets $\frac{7}{8}$ in. Allowable unit stresses are to be the same as specified in the design in Art. 122. The distance back to back of angles will be made 72.5 in., as explained in Art. 122.

Assuming that the web will be $\frac{3}{8}$ in. thick, the angles $\frac{3}{4}$ in., and the area of the covers the same as the angles, and adding 10 per cent for details (rivet heads, stiffener angles, etc.,) the weight per foot is $1.1(92 + 2 \times 4 \times 28.7) = 350$ lb.

$$R_L \text{ (concentrated loads)} = 1.5 \times 130{,}000 = 195{,}000 \text{ lb.}$$

The minimum allowable web thickness is $\dfrac{72.5 - 2 \times 6}{160} = \dfrac{3}{8}$ in. The web thickness required for shear is $\dfrac{195{,}000 + 350 \times 25}{12{,}000 \times 72} = 0.235$ in. A $\frac{3}{8}$-in. web will be used.

$$M \text{ (concentrated loads)} = 195{,}000 \times 25 - 130{,}000 \times 12.5 = 3{,}250{,}000 \text{ ft.-lb.}$$

$$M \text{ (weight)} = \tfrac{1}{8} \times 350 \times 25^2 = 110{,}000 \text{ ft.-lb.}$$
$$M \text{ (total)} = 3{,}360{,}000 \text{ ft.-lb.} = 40{,}320{,}000 \text{ in.-lb.}$$

Assuming that d_g is 1 in. less than the depth of the web, or 71 in., the total net area required in the tension flange is

$$\frac{40{,}320{,}000}{18{,}000 \times 71} = 31.6 \text{ sq. in.}$$

The equivalent flange area of the web is $\frac{1}{8} \times 72 \times \frac{3}{8} = 3.4$ sq. in.; therefore, the angles and covers of the tension flange must furnish a net area of $31.6 - 3.4 = 28.2$ sq. in. A $\frac{3}{4}$-in. plate is the maximum that most shops will punch full size, and so the angles will be made $\frac{3}{4}$ in. thick. The net area of two angles, allowing for one hole (diameter $\frac{7}{8} + \frac{1}{8} = 1$ in.) in each leg of each angle, is then $2(8.44 - 2 \times \frac{3}{4} \times 1) = 13.9$ sq. in. The net area required in the cover plates is $28.2 - 13.9 = 14.3$ sq. in. The net width of each cover plate, allowing for two holes in the same section, is $14 - 2 \times 1 = 12$ in., and the total thickness of cover plates must be at least $14.3/12 = 1.19$ in. The revised weight of the girder is $1.1(92 + 4 \times 28.7 + 2 \times 59.5) = 360$ lb. per ft.; the revised M (weight) is 112,000 ft.-lb.; the revised M (total) is 3,362,000 ft.-lb. = 40,340,000 in.-lb.

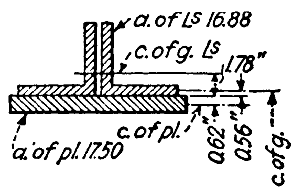


FIG. 120.

The distance from the back of the angles to the center of gravity of the flange section (see Fig. 120) is

$$\frac{2 \times 8.44 \times 1.78 - 17.5 \times 0.62}{16.88 + 17.5} = 0.56 \text{ in.}$$

The revised effective depth is $72.5 - 2 \times 0.56 = 71.4$ in., and the revised total net flange area required is

$$\frac{40{,}340{,}000}{18{,}000 \times 71.4} = 31.4 \text{ sq. in.}$$

This is 0.2 sq. in. less than the original area, and it will be deducted from the area of the cover plates, making 14.1 sq. in. required in these plates. The revised thickness of the cover plates is then $14.1/12 = 1.17$ in. or $1\frac{3}{16}$ in. This will be subdivided into one plate $14 \times \frac{5}{8}$ and one plate $14 \times \frac{9}{16}$. If the girder were welded instead of riveted, no deduction would be made for holes and $\frac{1}{6}$ of the web area would be used instead of $\frac{1}{8}$.

The compression flange will be made the same as the tension flange; no investigation is necessary in this case, since the distance between lateral supports ($12.5 \times 12 = 150$ in.) does not exceed fifteen times the flange width (14 in.). If the investigation were required, it would be carried out as for the tension flange, using, however, gross areas and an allowable unit stress of $\dfrac{20,000}{1 + \dfrac{l^2}{2000b^2}}$.

125. End-connection Angles. The connection of a girder to the face of a supporting column or girder by means of connection angles is like a beam connection (see Art. 108), except that the connection angles are not in contact with the web but are separated from it by the vertical legs of the flange angles and by fillers of the same thickness. The rivets which pass through both connection angles and flange angles are usually fully developed in transmitting flange stress, as explained later, so it is well not to count upon them to carry part of the vertical reaction. The rivets which pass through loose fillers are not so effective as those which connect parts in direct contact, for the parts are more liable to slide and to bend the rivets. Specifications commonly require that if rivets carrying stress pass through fillers, the fillers shall be extended beyond the connected member and the extension secured by enough additional rivets to develop the value of the filler. The number of rivets in the angles, not counting those through the flange angles, is determined by the double-shear value, and the total number in the angles and fillers is at least the number determined by the bearing value on the web. This provides for no increase in number beyond those which would be used if the fillers did not extend beyond the angles, but the arrangement is better because the fillers are fastened to the web and the rivets are less liable to bend. The extra rivets are placed opposite the rivets in the connection angles, although sometimes every other rivet is omitted. This spacing will usually provide extra rivets and is considered satisfactory.

Example. Determine the number and arrangement of ¾-in. rivets required to connect the plate girder shown in Fig. 121 to a column, if the maximum reaction is 50,000 lb. and the allowable unit stresses are 12,000 p.s.i. in shear and 24,000 p.s.i. in bearing.

The number of rivets required in the angles through the web of the girder is 50,000/10,600 = 5, as determined by double shear. The number of rivets required in the angles and fillers, determined by bearing on the 5⁄16-in. web is

50,000 ÷ 5630 = 9. The number needed in the fillers outside the angles is 9 − 5 = 4, and five are used to provide excess strength and to simplify the spacing. The number required in the outstanding legs of the angles, through the flange of the column, determined by single shear, is 50,000/5300 = 10. The details of the connection are shown in Fig. 121.

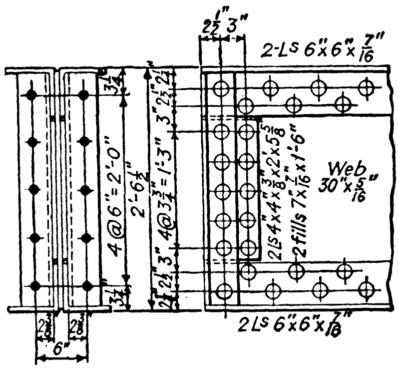


Fig. 121.

126. End Stiffener Angles. When a girder rests on a supporting column, pedestal, or masonry, the shearing stresses in the web plate are transmitted to the support by means of end stiffening angles or stiffeners. These angles act as columns restrained in one direction by the web; but this is not an important factor, since the stresses from the web are cumulative through the rivets and reach the full load only at the bottom rivet. The entire reaction is transmitted through the bearing of the stiffening angles on the outstanding legs of the flange angles. The required bearing area is found by dividing the reaction by the allowed unit stress in bearing; this is the same as the bearing value allowed for rivets in single shear unless a different value is specified. Since the ends of the stiffeners must be cut to clear the curved fillets of the flange angles, this portion cannot be counted in bearing, because contact cannot be assured; and even if it could, the bearing on a curved surface would not be effective. The radius of the flange-angle fillets is ½ in. (⅝ in. for 8 by 8 angles); so unless the stiffeners are thicker than this amount, no part of the web legs can be counted. If the stiffeners are thicker than the curve fillet, the effective bearing area can be taken as the gross area minus the area in contact with the fillet. The web legs are 3, 3½, or 4 in., unless a double row of rivets is required; then 6-in. legs are used. The outstanding legs are usually the largest commercial size which will not extend beyond the flange angles, but sometimes the same size is used even though the legs project beyond the flange angles or have to be planed off. No part of the stiffeners which may project beyond the flange angles can be counted in bearing. The thickness of the stiffeners must be sufficient to give the required effective area; the minimum thickness is 3/8 in., or one-twelfth the length of the outstanding leg. Stiffeners are used in pairs, more than one pair being used if necessary. The outstanding legs are placed at the center of bearing preferably, as in Fig. 122*a* or *b*. Sometimes they are placed at the extreme end of the girder, as in *c*, but full bearing cannot be assured with ordinary shop methods. When two pairs of stiffeners are placed at opposide ends of the bearing plate, as in *d*, the distri-

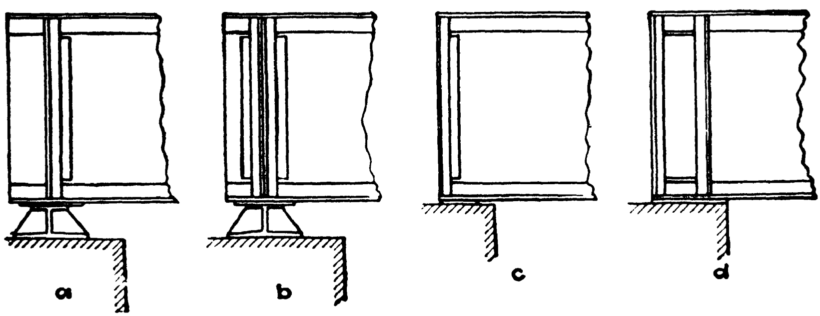


FIG. 122.

bution of stress is problematical; owing to the deflection of the girder, the inner pair is often assumed to take twice as much of the reaction as the pair at the extreme end.

The number of rivets to be used is determined as in the preceding article; fillers are made wide enough to extend under all stiffeners, allowing projections for additional rows of rivets. In a welded girder, plates are used instead of angles for stiffeners, the edges being welded to the web. The size of the plates is determined by bearing like the outstanding legs of the stiffening angles, and the number of linear inches of intermittent welds must be enough to carry the reaction.

Example. Design the end stiffeners for the plate girder shown in Fig. 123, on the assumption that the maximum reaction is 360,000 lb. and the allowable unit stresses are 13,500 p.s.i. in shear, 24,000 p.s.i. in bearing (single shear), and 30,000 p.s.i. in bearing (double shear). Use 7/8-in. rivets. The flange angles of the girder are 6 × 6 × 3/4.

The maximum outstanding leg of each stiffener angle, for 6-in. flange angles, is 5 in., and four 5 × 3½ × ¾ angles will be assumed. These are placed in pairs, each pair back to back, one pair on either side of the web plate. Since the radius of the curve fillet in a 6 × 6 angle is ½ in., the effective bearing area of each stiffener angle is the gross area minus the area in contact with the fillet (3.5 × ½ = 1.75 sq. in.). The effective bearing area of the four angles is then 4(5.81 − 1.75) = 16.24 sq. in. The bearing area required is 360,000/24,000 = 15.0 sq. in., and the assumed angles are satisfactory provided they are milled at the bottom to fit tightly against the outstanding legs of the flange angles.

The number of rivets required in the angles, not counting those through the flange angles, is 23 = 360,000/16,240, determined by double shear. The number required in the angles and fillers is 32 = 360,000/11,480, determined by the bearing in the 7/16-in. web. With the same number of rivets in each pair of angles, at least 24, or 12 in each pair, must be used; as a matter of detail, it is better to use an odd number, and so 13 are used, which allows the omission of alternate rivets in the extended fillers and intermediate stiffeners while symmetrical spacing is still maintained, so that the stiffeners are interchangeable instead of being rights and lefts. If the web plate is to be spliced, it is desirable to place the rivets closer together near the flange angles than in the central portion, so that the same vertical spacing can be used as in the splices where the rivets farthest from the neutral axis are most effective, as explained later. The final spacing is shown in the figure.

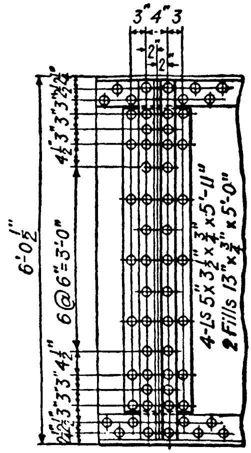


FIG. 123.

127. Intermediate Stiffener Angles.

Stiffeners with fillers are used at points of concentration to transmit the loads to the web. Stiffeners with fillers or crimped stiffeners without fillers are used at other points of deeper girders to stiffen the web plate against buckling when the clear vertical distance between flange angles exceeds sixty (or sometimes fifty) times the web thickness. The spacing of intermediate stiffeners in the clear should not exceed 6 ft. or the distance in inches determined by a formula in the specifications, as for example, $\frac{255{,}000t}{v}\sqrt[3]{\frac{vt}{a}}$, in which t is the web thickness in inches, a the clear depth in inches of the web plate between flanges, and v the shear intensity in pounds per square inch found by dividing the shear for the section under consideration by the gross area of the web plate. Standard angles with unequal legs are used, with the outstanding longer legs just slightly less than the outstanding legs of the flange angles; the minimum allowed thickness is usually sufficient. The rivets line up with those in the end stiffeners, but alternate rivets may be omitted if the resulting spaces do not exceed 6 in.; sometimes 1 ft.-0 in. is allowed. Stiffeners and rivets which support concentrated loads must be designed for those loads, additional angles often being used on one side of the web. Plates are used instead of angles in welded girders.

128. Flange Rivets. The rivets or welds which fasten the flange angles to the web must be designed to transmit the stresses in the angles and cover plate to the web. The equal and opposite total stresses in the top and bottom flanges form a couple, the moment of which equals the bending moment. Since the stresses are transmitted from the flanges to the web through the rivets, the lever arm of the couple is equal to the depth d_r between rivet lines, the mean depth being used if there are two lines of rivets in each flange (*i.e.*, for 6-in. legs or larger). Since $dM/dx = V$ (see Art. 6), the increase in bending moment between two sections a unit distance apart is equal to the average shear at these two sections, and the corresponding increase in the flange stress, which is called the horizontal increment of flange stress, H.I., is V/d_r. Since part of the flange stress is resisted by the web, the rivets transmit only the remainder, or that proportion of the total flange stress which is actually resisted by the angles and cover plates. This proportion is $\frac{V}{d_r} \times$ ratio. If the girder is designed by the moment-of-inertia method, the ratio is the ratio of the moment of inertia of the angles and covers of *both* flanges to the moment of inertia of the entire cross-section including the web plate. If the girder is designed by the flange-area method, the ratio is the ratio of the area of the angles and covers of *one* flange to the area of the angles and covers of one flange plus one-eighth the gross area of the web plate. The difference in the pitches computed for the gross section and for the net section is not great if consistent values are used; hence, either gross or net areas and moments of inertia can be used in computing the ratio, depending upon which is more readily obtained from the preceding computations.

If the value of one rivet in bearing on the web is r, the maximum pitch, or *longitudinal* distance p between rivets (regardless of whether they are in the same gage line or in two gage lines), is

$$p = \frac{r}{\frac{V}{d_r} \times \text{ratio}} = \frac{rd_r}{V \times \text{ratio}} \tag{3}$$

The pitch should be computed at sections where the shear V changes and at sections where the ratio changes, the latter change being caused by the cutting off of one or more cover plates. If the shear changes continuously, as in a girder with a uniform load, the pitch should be computed at each stiffener and that pitch maintained to the next stiffener where it can be increased to the next computed value. If the only uniform load is the weight of the girder itself, the variation in pitch between the concentrated loads will be very small, and the pitch required at one concentrated load can be used between loads without adding materially to the cost. Moving loads should be placed to give the maximum shear for each section. Pitches are usually expressed to the nearest $\frac{1}{4}$ in., with a maximum of 6 in. A pitch may have to be increased next to a stiffener in order to provide sufficient driving clearance, but otherwise no pitch should exceed the calculated value.

Example. Compute the required pitches of $\frac{7}{8}$-in. rivets, with an allowable bearing stress of 30,000 p.s.i. in the 50-ft. girder which was designed in Arts. 122 and 124. The weight of the girder is 360 lb. per ft.

The bearing value of one rivet on the $\frac{3}{8}$-in. web plate is 9840 lb. The distance d_r is $72.5 - 2 \times 2\frac{1}{4} - 2\frac{1}{2} = 65.5$ in. The maximum shear in the first panel (*i.e.*, at the support) is $195{,}000 + 360 \times 25 = 204{,}000$ lb. The maximum shear in the second panel (just to the right of the first load) is $204{,}000 - 130{,}000 - 360 \times 12.5 = 69{,}500$ lb.

$$\text{Total net } I = 84{,}910 \text{ in.}^4$$

$$\text{Net } I \text{ of web plate} = 11{,}660 - 4330 \times \frac{3/8}{1\frac{7}{8}} - 1680 = 9110 \text{ in.}^4$$

$$\text{Ratio (moment-of-inertia method)} = \frac{84{,}910 - 9110}{84{,}910} = 0.89$$

$$\text{Ratio (flange-area method)} = \frac{13.9 + 12 \times 1\frac{3}{16}}{13.9 + 12 \times 1\frac{3}{16} + 3.4} = 0.89$$

$$p \text{ (in first panel)} = \frac{9840 \times 65.5}{204{,}000 \times 0.89} = 3\tfrac{1}{2} \text{ in.}$$

$$p \text{ (in second panel)} = \frac{9840 \times 65.5}{69{,}500 \times 0.89} = 6 \text{ in. (max.)}$$

129. Effect of Flange Loads on Rivet Pitch. Loads which rest directly or indirectly on the top flange are transmitted to the web through the rivets, subjecting them to vertical components as well as the same horizontal components which they would have if the loads were transmitted to the web by other means; the resultant stress must not exceed the limiting value r. Both components must be in equivalent units before the resultant is taken, the stress per linear inch being most convenient. The rivet pitch in any panel is found by dividing the value of one rivet by the resultant per linear inch. The horizontal component is $\frac{V}{d_r} \times$ ratio, which varies with the shear. The vertical component is the stress per inch which tends to shear the rivets vertically and is constant; it includes the weight of the top flange angles and cover plates, any rail, floor, or other dead load, and the live load. Weights of continuous material per foot may be combined and divided by 12; weights of rails per yard may be divided by 36; crane wheel loads may be considered to be distributed by the rail and top flange over 30 in.; concentrated engine loads may be considered to be distributed over three ties (from 32 to 36 in.). In order that the vertical component may be distributed properly among the rivets, the pitch should not exceed $4\frac{1}{2}$ in. (4 in. for crane girders).

Example. Compute the required pitches of $\frac{3}{4}$-in. rivets at the support and at a section 6 ft. from the support in a 40-ft. girder composed of a $48 \times \frac{7}{16}$ web, four angles $6 \times 4 \times \frac{3}{4}$, and four cover plates $14 \times \frac{1}{2}$. The angles are placed with the 6-in. leg outstanding and 48.5 in. back to back. The girder carries on the top flange a fixed uniform load of 8000 lb. per lin. ft. The allowable bearing stress is 24,000 p.s.i. Both cover plates of each flange are assumed to extend the full length of the girder.

The limiting value of one rivet in bearing on the 7/16-in. web is 7880 lb. The weight of the girder, allowing 10 per cent for details, is 290 lb. per ft. The end shear is 8290 × 20 = 165,800 lb., and the shear at a section 6 ft. from the end is 165,800 − 6 × 8290 = 116,100 lb. The weight of the two flange angles and cover plates is approximately 100 lb. per ft., so that the total vertical load on this flange is 8100 lb. per ft. or 675 lb. per in.

Using gross sections,

$$I \text{ (web)} = \tfrac{1}{12} \times \tfrac{7}{16} \times 48^3 = 4{,}030 \text{ in.}^4$$
$$I \text{ (4 angles)} = 4(8.7 + 6.94 \times 23.17^2) = 14{,}920 \text{ in.}^4$$
$$I \text{ (4 cover plates)} = 2 \times 14 \times 1 \times (24.75)^2 = 17{,}100 \text{ in.}^4$$
$$\text{Total } I = 36{,}050 \text{ in.}^4$$

$$\text{Ratio} = \frac{14{,}920 + 17{,}100}{36{,}050} = 0.89$$

The distance d_r is 48.5 − 2 × 2½ = 43.5 in., and the horizontal increments of flange stress to be resisted by the rivets, using the moment-of-inertia method, are

$$\text{At the support, } \frac{165{,}800}{43.5} \times 0.89 = 3390 \text{ lb. per in.}$$

$$\text{6 ft. from the support, } \frac{116{,}100}{43.5} \times 0.89 = 2380 \text{ lb. per in.}$$

The resultant stresses on the rivets and the required pitches are

$$\text{At the support, } \sqrt{(3390)^2 + (675)^2} = 3470 \text{ lb. per in.}$$

$$p = \frac{7880}{3470} = 2\tfrac{1}{4} \text{ in.}$$

$$\text{6 ft. from the support, } \sqrt{(2380)^2 + (675)^2} = 2480 \text{ lb. per in.}$$

$$p = \frac{7880}{2480} = 3\tfrac{1}{4} \text{ in.}$$

It should be noted that the pitch of 2¼ in. at the support is the smallest that will give the required minimum distance (3 diameters) between adjacent rivets, for only one gage line can be used in a 4-in. leg. If the calculated pitch were less than the minimum pitch, the design would have to be modified by increasing the web thickness or the rivet diameter or in some other way. If the rivets could be placed in two gage lines, the pitch could be less than 3 rivet diameters, as long as the diagonal distance between adjacent rivets was at least 3 diameters.

130. Cover-plate Rivets. Two rivets are always placed in the same cross-section to fasten the cover plates to the angles, one through each angle. The strength of the two rivets (as governed by single shear) is generally large enough so that the maximum pitch permitted by specifications (6 in.) will give more than the required strength. Hence the pitch of cover-plate rivets need not be computed, except in very heavy girders. These rivets may be placed in one line in each angle; but usually if the angle leg is 6 in. or more, they are staggered in two gage lines in each angle.

If the pitch were to be computed, the part of the total horizontal increment of flange stress per linear inch that must be resisted by the rivets is obtained by an analysis similar to that used for computing the pitch of the rivets connecting the angles to the web (Art. 128). The ratio in the expression $(V/d_r) \times$ ratio is equal to ratio of the moment of inertia of the cover plates of *both* flanges to the moment of inertia of the entire cross-section, including the web, or the ratio of the area of the cover plates of *one* flange to the area of the cover plates plus angles of one flange plus one-eighth the area of the web. The pitch is then equal to the single-shear value of one rivet divided by the quantity $(V/d_r) \times$ ratio. A close spacing of 4 diameters or 3 in. is used at the end of each plate for a distance equal to about $1\frac{1}{2}$ times the width of the plate. These rivets are not affected by vertical loads on the top flange, except in so far as these loads affect the value of V in the above expression.

131. Lengths of Cover Plates. Beams and girders are designed to resist the maximum bending moments, and unless cover plates are used they are generally made of uniform cross-section throughout their lengths, regardless of the fact that less metal is needed near the ends where the bending moments are much less. It is not practical to reduce the sections of rolled beams or plate girders without cover plates; for the cost of splicing would offset the saving in material, and the resulting girders would not be so satisfactory. When cover plates are used to furnish part of the flange area, they may be placed only where needed and be discontinued beyond the points where the remaining section is sufficient to meet the requirements. If a girder is exposed to the weather, one cover plate of the top flange should extend the full length in order to keep water from getting between the angles and the web. Similarly a crane-runway girder should have the top cover plate or channel extend full length to give uniform bearing for the rail.

In order to determine the point at which each cover plate can be discontinued, the bending moment curve should be drawn by computing the maximum bending moment at various critical sections. The curve of bending moments for a series of concentrated loads is a series of straight lines with a break at each point of concentration. If the loads are symmetrical about the center, only one-half need be plotted. If the uniform load is relatively small, as when there is no uniform load other than the weight of the girder itself, the combined bending moment may be found at each point of concentration and the curve plotted accordingly as a series of straight lines. When enough uniform load is added to make appreciable curves in the diagram, separate curves should be plotted for the bending moments due to the uniform and concentrated loads and the two curves combined graphically to obtain the moment curve for the total load.

Next, the resisting moments of the various sections which remain after each successive cover plate is removed are computed by the reverse process to that explained either in Art. 121 or 123. These resisting moments are then plotted as a series of horizontal lines on the bending-moment curve; points at which the horizontal lines intersect the bending-moment curve represent points at which the corresponding cover plates are no longer required. The plates should be extended beyond the theoretical points of cutoff for a distance of $1\frac{1}{2}$ to 3 ft. to allow for possible inaccuracies in the graphical analysis or possible variations in the behavior of the loads as affecting the computed moments.

Example. Determine the required lengths of the outer and middle cover plates of the girder shown in Fig. 124. The span of the girder is 70 ft.-0 in.; the flange angles are $6 \times 6 \times \frac{3}{4}$; each of the three cover plates in each flange is $14 \times \frac{3}{4}$; the web plate is $84 \times \frac{7}{16}$; and the rivets are $\frac{7}{8}$ in. The allowable bending stress on the net section of the tension flange is

16,000 p.s.i., and the flange-area method (Art. 123) is to be used. The maximum bending-moment curve is assumed to be as shown in Fig. 124.

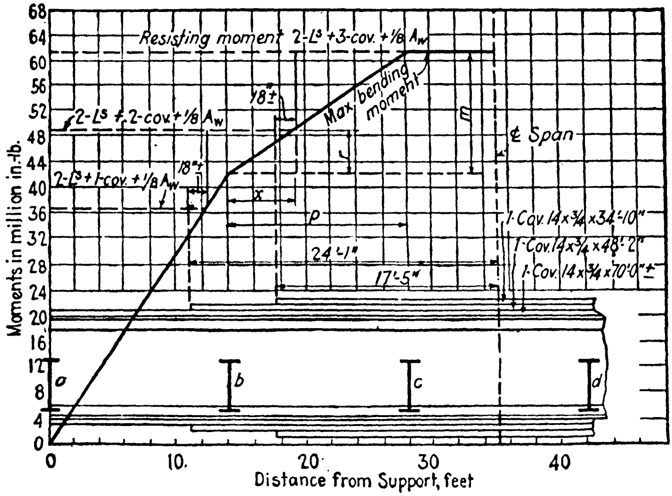


FIG. 124.

Resisting moments of the various sections are computed from the formula in Art. 123,

$$M = fd_o(A' + \tfrac{1}{8}td_w)$$

in which A' = the net area of the angles and covers in the lower flange, with 3 covers, 2 covers, and 1 cover.

Flange section	A'	$\frac{1}{8}td_w$	$A' + \frac{1}{8}td_w$	d_o	M, in.-lb.
2 angles, 3 covers......	40.88	4.60	45.48	84.5	61,600,000
2 angles, 2 covers......	31.88	4.60	36.48	83.7	48,800,000
2 angles, 1 cover.......	22.88	4.60	27.48	82.6	36,200,000

The resisting-moment values are plotted as horizontal lines on the moment diagram (Fig. 124), and the covers are discontinued 18 in. nearer the support than the sections at which the corresponding resisting-moment lines intersect the bending-moment curve. The required lengths are shown diagrammatically on the side elevation of the girder in the lower part of the figure. Points *a*, *b*, *c*, and *d* represent panel points at which floor beams frame into the girder.

132. Web Splices. The web plates of all but the shorter plate girders must be spliced, since the wider plates are not rolled long enough to extend the entire length, and for convenient handling no single plate should weigh over 3000 lb. The flange angles usually extend full length. When more than one splice is used, they are spaced symmetrically about the center line; the splice for maximum stress is designed first, and the other splices are usually made like it. A splice is preferably located under a pair of stiffeners; the angles stiffen the splice, there is one less line of rivets to be driven, and metal is saved by the use of thinner fillers. A single pair of splice plates, one plate resting against either side of the web plate in the clear depth of the web, as in Fig. 125, may be designed to provide both shear and moment requirements; but in some of the heavier girders a pair of plates is placed near each flange to transmit stresses due to bending moment, and another pair is placed between them to transmit the stresses due to shear.[1] In order to develop fully the web plate in shear, the gross area of the two splice plates must equal or exceed the gross area of the web plate; this condition will be met when, in order fully to develop the web plate in moment, the section modulus of the two splice plates equals or exceeds the section modulus of the web plate; thus, the square of the depth of the splice plates multiplied by their combined thickness must be not less than the square of the depth of the web plate multiplied by its thickness. The thickness of each splice plate must be not less than the minimum thickness of metal allowed by the specifications, usually $3/8$ or $5/16$ in.

Two to four rows of rivets are used on each side of the splice; the resultant shearing stress acts upward at the center of gravity of the rivets in one half, and an equal resultant shearing stress acts downward at the center of gravity of the rivets in the other half; these two forces form a couple, one-half the moment of which, as well as the direct stress, must be resisted by the rivets in each half. The rivets in each half must also resist that part of the bending moment carried by the web plate, and this depends on the method used in design. Theoretically, if the moment-of-inertia method is used, the bending moment to be

[1] For this type of splice see Bishop, "Structural Design," p. 113, John Wiley & Sons, Inc., 1938.

resisted bears the same relation to the total bending moment at the point of splice as the net moment of inertia of the web plate bears to the net moment of inertia of the whole cross-section at that point; and if the net-area method is used, the bending moment to be resisted bears the same relation to the total bending moment at the point of splice as one-eighth the gross area of the web bears to one-eighth the gross area of the web plus the net area of the angles and cover plates of one flange at that point. It is more convenient and quite accurate to develop the web plate. Thus, in the moment-of-inertia method, the moment to be resisted is the net moment of inertia of the web plate times the unit stress in bending divided by the distance from the neutral axis to the extreme fiber in the flange; and, in the net-area method, the moment to be resisted is one-eighth the gross area of the web times the unit stress in bending times the effective depth from center to center of gravity of the flanges. The horizontal spacing between rivets is usually 3 in. except that the inner rows are separated by twice the gage distance of the stiffeners; this allows ample edge distance on the web segments and a space between them. Three rows of rivets in each half are first assumed, with the spacing the same as in the end stiffeners, and then these rivets are tested by the method of eccentric connections explained in Art. 110; if the number is found excessive, alternate rivets may be omitted from the outer row, or the entire row may be omitted, and the splice tested for each revision until the stress in the critical rivet is just below the limiting value of one rivet bearing in the web. Since the rivets near the flanges are more effective in resisting moment than those near the neutral axis, it is better to use a few 3-in. spaces near each flange, unless that results in an excessive number in the stiffeners.

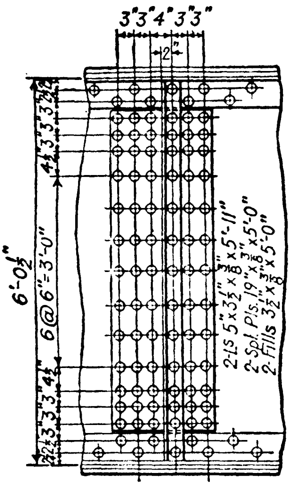


Fig. 125.

Example. Design the web splice at a section for which the shear is 100,000 lb. in the girder for which the end stiffeners were designed in Art. 126. Assume three rows of rivets in each half, as shown in Fig. 125, with the vertical spacing like that in the end stiffeners. Assume that the girder was designed by the moment-of-inertia method and that two ⅝-in. cover plates are required in each flange at the section taken. The allowable unit stress in bending on the extreme tension fiber is to be taken as 18,000 p.s.i.

Allowing ¼-in. clearance at each end, the depth of the splice plates is $72.5 - 2(6 + \frac{1}{4}) = 60$ in. The required thickness of each splice plate is $(72^2 \times \frac{7}{16}) \div (2 \times 60^2) = \frac{5}{16}$ in.; the specified minimum thickness of ⅜ in. will be used.

$$\text{Eccentric moment due to shear} = \frac{1}{2} \times 100{,}000(3 + 4 + 3) = 500{,}000 \text{ in.-lb.}$$

$$\text{Gross } I \text{ of web} = 13{,}610 \text{ in.}^4$$

$$I \text{ of holes} = 1 \times \frac{7}{16} \times 2(6^2 + 12^2 + 18^2 + 22.5^2 + 25.5^2 + 28.5^2 + 34^2) = 3180 \text{ in.}^4$$

$$\text{Net } I \text{ of web} = 13{,}610 - 3180 = 10{,}430 \text{ in.}^4$$

$$e = \frac{1}{2} \times 72.5 + 2 \times \frac{5}{8} = 37.5 \text{ in.}$$

The moment resisted by the web plate is

$$f\frac{I}{e} = \frac{18{,}000 \times 10{,}430}{37.5} = 5{,}020{,}000 \text{ in.-lb.}$$

The combined moment to be resisted is 5,020,000 + 500,000 = 5,520,000 in.-lb.

$$\Sigma x^2 + \Sigma y^2 = 2 \times 13 \times 3 + 6(6^2 + 12^2 + 18^2 + 22.5^2 + 25.5^2 + 28.5^2) = 15{,}070$$

$$r_0 \text{ (see Art. 110)} = \frac{5{,}520{,}000}{15{,}070} = 366 \text{ lb.}$$

The horizontal component of the stress on the critical rivet is $366 \times 28.5 = 10{,}430$ lb., and the vertical component is $366 \times 3 + \frac{100{,}000}{3 \times 13} = 3660$ lb. The resultant stress is 11,050 lb., which is less than the 11,480 lb. allowed, but not enough less to permit a reduction in the number of rivets.

CHAPTER IX

DEFLECTION

133. Methods Used in Computing Deflections. When any beam or truss is loaded, it will be deformed. If the load or loads are vertical, the axis of the beam or the joints of the truss will be deflected to a new position. If the unloaded beam or truss has a straight horizontal axis, the deflected axis will take the form of a curve. The amount of deflection at any point on the axis of the beam or of any joint of the truss depends not only upon the amount and position of the load, but also upon the elastic properties of the material, and in a truss upon the relative stiffness of the several members.

The deflection of beams may be obtained by any one of three general methods; (1) by the direct use of the differential equation of the elastic curve $d^2y/dx^2 = M/EI$, (2) by the moment-area method, or (3) by the principle of work.

The deflection of trusses may be obtained by the application of the principle of work or by the construction of a displacement diagram. When the deflection of only a single point in the truss is desired, the former method often furnishes the shorter solution; but if the simultaneous displacements of several points in a truss are desired, the construction of the displacement diagram is advisable.

Maximum Deflection of Beams. The maximum deflection of a simply supported beam occurs at or near the center of the beam unless the load is extremely unsymmetrical. In the usual case it is the deflection at the center that is desired, and such a value is sufficient for most cases encountered. The same is true for a beam fixed at both ends. The maximum deflection of a beam fixed at one end and free at the other does not occur at the center; it is obvious that the maximum deflection of a cantilever beam is at the free end.

Figure 126 gives the maximum deflection for the types of loading and reaction conditions usually encountered. The following articles give methods of determining the deflection for any condition of loading for a simply supported beam. By comparing the values given in the table for the same types of loading but with different reaction conditions, an approximate maximum deflection for any type of loading for beams other than simply supported may be obtained.

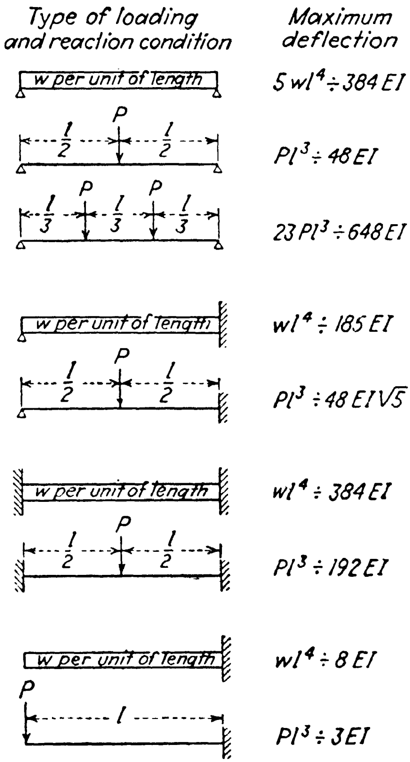
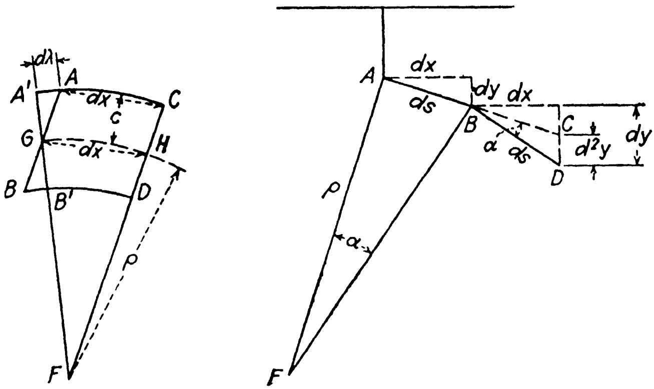


FIG. 126.

134. Deflection of Beams Using the Differential Equation of the Elastic Curve. The neutral axis of any beam in flexure was originally straight. Owing to flexure it becomes curved with a radius of curvature ρ. In Fig. 127(a), the two surfaces $A'B'$ and CD of the elementary length ds were originally parallel. Since the radius of curvature is large, ds may be assumed equal to dx. The center of curvature F is located by producing $A'B'$ and CD to their intersection. In the similar triangles $A'AG$ and GHF, $d\lambda : dx = c : \rho$, in which $d\lambda$ is the elongation of the portion originally dx of the outer fiber. Within the elastic limit, $d\lambda : dx = f : E$. Therefore, $f : E = c : \rho$; and since $fI/c = M$, it follows that $EI/\rho = M$.

In Fig. 127(b) are shown two consecutive elementary lengths ds with equal dx's. Produce the first ds to intersect the dy of the second, thus cutting off d^2y, the difference between two consecutive dy's. At the left end of each ds, perpendiculars are constructed. Their intersection F locates the center of curvature and thus the radius of curvature ρ. The two triangles *FAB* and *BCD* have their small angles equal; and since the curvature is very flat, d^2y is nearly perpendicular to the prolonged ds, and ds may be taken equal to dx. Hence, $\rho : dx = dx : d^2y$ or $\rho = dx^2/d^2y$. Substituting this value of ρ in the equa-



(a) (b)

Fig. 127.

tion for M of the preceding paragraph,

$$EI\frac{d^2y}{dx^2} = M \qquad \text{or} \qquad \frac{d^2y}{dx^2} = \frac{M}{EI}$$

In the equation $d^2y/dx^2 = M/EI$, M is written in terms of x and the resulting equation is integrated twice, so that the value of y, or the deflection, is obtained in terms of x. In the first integration the limits of the slope dy/dx are unknown, and in the second integration the limits of y are unknown; hence, constants of integration must be employed. The values of these constants are determined by substituting corresponding known values of the variables.

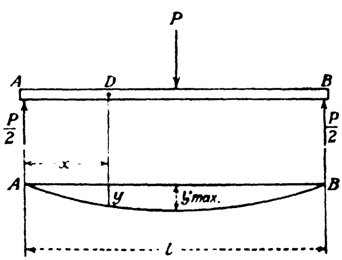


Fig. 128.

Examples. 1. In Fig. 128, the beam *AB* is loaded at mid-span with a single concentrated load *P*. The maximum deflection is required.

With the origin at A, $M_D = Px/2$ and

$$EI\frac{d^2y}{dx^2} = \frac{Px}{2}.$$

By integration

$$EI\frac{dy}{dx} = \frac{Px^2}{4} + C_1$$

Since $\frac{dy}{dx} = 0$ at mid-span, where $x = \frac{l}{2}$,

$$C_1 = -\frac{Pl^2}{16}$$

$$EI\frac{dy}{dx} = \frac{Px^2}{4} - \frac{Pl^2}{16}$$

$$EIy = \frac{Px^3}{12} - \frac{Pl^2x}{16} + C_2$$

Since $y = 0$ where $x = 0, C_2 = 0$,

$$EIy = \frac{Px^3}{12} - \frac{Pl^2x}{16}$$

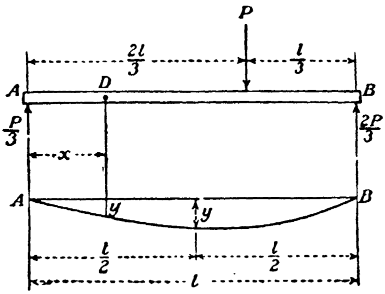


FIG. 129.

The maximum value of y, or the maximum deflection of the beam, is at the center where $x = l/2$ and is

$$y = \frac{Pl^3}{48EI}$$

2. In Fig. 129 the beam is loaded at the right third point of the span with a load P, and it is required to find the deflection at mid-span.

With the origin at A the moment at any point D, on the left of the load P, is $Px/3$.

$$EI\frac{d^2y}{dx^2} = \frac{Px}{3}$$

$$EI\frac{dy}{dx} = \frac{Px^2}{6} + C_1 \qquad (a)$$

The moment at any point D on the right of the load P is

$$\frac{Px}{3} - P\left(x - \frac{2l}{3}\right).$$

$$EI\frac{d^2y}{dx^2} = P\left[\frac{x}{3} - \left(x - \frac{2l}{3}\right)\right]$$

$$EI\frac{dy}{dx} = \frac{Px^2}{6} - \frac{P\left(x - \frac{2l}{3}\right)^2}{2} + C_2 \qquad (b)$$

The slope dy/dx at the concentrated load is the same no matter which portion of the beam (left of the load or right of the load) is considered; and for $x = 2l/3$, equating (a) and (b),

$$\frac{4Pl^2}{54} + C_1 = \frac{4Pl^2}{54} - 0 + C_2$$

from which $C_1 = C_2$.

For the left portion, integrating (a),

$$EIy = \frac{Px^3}{18} + C_1x + C_3 \qquad (c)$$

and for the right portion, integrating (b),

$$EIy = \frac{Px^3}{18} - \frac{P(x - \tfrac{2}{3}l)^3}{6} + C_1x + C_4 \qquad (d)$$

For the left portion $y = 0$ when $x = 0$, and $C_3 = 0$. Since the deflection must be the same for both portions when $x = 2l/3$, equating (c) and (d), $C_4 = 0$.

For the right portion $y = 0$ when $x = l$; hence

$$0 = \frac{Pl^3}{18} - \frac{Pl^3}{162} + C_1l$$

from which

$$C_1 = \frac{Pl^2}{162} - \frac{Pl^2}{18} = -\frac{4Pl^2}{81}$$

Then, from equation (c), for any point on the left of the concentrated load,

$$EIy = \frac{Px^3}{18} - \frac{4Pl^2x}{81}$$

and for $x = l/2$, the deflection is

$$y = \frac{23Pl^3}{1296EI}$$

The point of maximum deflection may be located by equating $EI\frac{dy}{dx} = 0$ and solving for x.

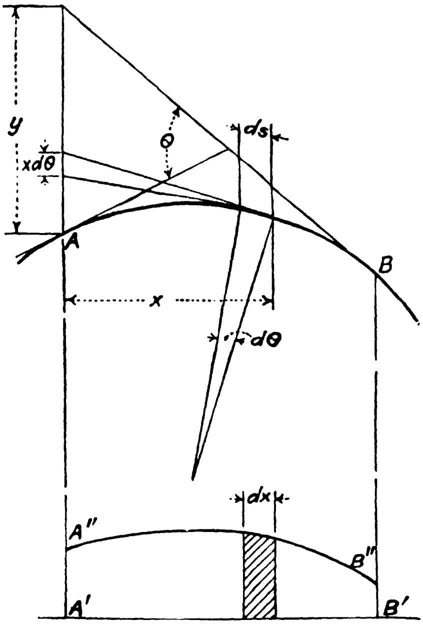


Fig. 130.

135. Deflection of Beams by the Moment-area Method. The line AB of Fig. 130 represents a portion of the elastic curve of a member in flexure. An elementary length ds of the member is shown in Fig. 131. The angle between the radii at the ends of ds is denoted by $d\theta$. The linear deformation of a fiber at a distance c from the neutral surface is $cd\theta$, and the unit deformation of the same fiber is $cd\theta/ds$. The unit stress in the fiber is $f = Mc/I$, in which M is the resisting moment and I the moment of inertia of the section.

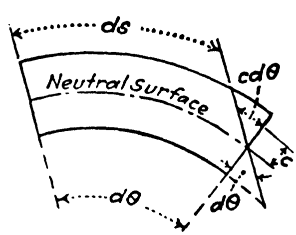


Fig. 131.

Since the modulus of elasticity is the ratio of unit stress to unit deformation,

$$E = \frac{Mc}{I} \div \frac{cd\theta}{ds}$$

from which

$$d\theta = \frac{M}{EI}\,ds$$

In a well-designed beam the curvature and slope are small, so that dx may be substituted for ds without appreciable error. Then,

$$d\theta = \frac{M}{EI}\,dx$$

In Fig. 130 an ordinate measured between the curve $A''B''$ and the straight line $A'B'$ at any point between A' and B' represents, to some scale, the moment in the member AB at that point, divided by EI, or $A''B''B'A'$ is the M/EI diagram for the member AB. The area of the diagram for the length dx is $\frac{M}{EI}\,dx$, and the area of the diagram $A''B''B'A'$ is $\int_A^B \frac{M}{EI}\,dx$. But $d\theta = \frac{M}{EI}\,dx$, and the angle between the tangents to the elastic curve at A and B is $\theta = \int_A^B d\theta = \int_A^B \frac{M}{EI}\,dx$. Hence,

The change in the slope of the elastic curve between any two points is equal to the area of the M/EI diagram for the portion of the member between these two points. (*First moment-area principle.*)

In Fig. 130 the tangents at the extremities of the elementary length ds are extended until they intersect the vertical line through A. Since the angles are small, the intercept between these tangents is practically equal to $xd\theta$. The total vertical distance y is the algebraic sum of all the intercepts between the tangents to the curve between A and B. That is,

$$y = \int_A^B x\,d\theta$$

Substituting for $d\theta$ its value as previously determined,

$$y = \int_A^B \frac{M}{EI}\,x\,dx$$

In the $\frac{M}{EI}$ diagram of Fig. 130, $\frac{M}{EI}dx$ is the area of the diagram for the length dx, and $\frac{M}{EI}dx$ times x is the moment of this area about the point A. The moment of the entire area of the M/EI diagram between the points A and B may be expressed as

$$\int_A^B \frac{M}{EI} xdx$$

which is equal to the expression developed above for y. Hence,

The distance of any point on the elastic curve from a tangent to the curve at any other point measured in a direction normal to the initial position of the member is equal to the moment of the area of the M/EI diagram, included between the two points, about the first point. (*Second moment-area principle.*)

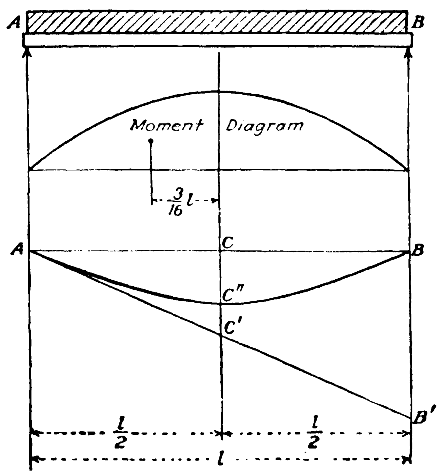


FIG. 132.

Examples. 1. In Fig. 132, the beam AB is loaded with a uniform load per foot equal to w. The maximum (mid-span) deflection is required.

The area of the whole moment diagram is $\frac{1}{8}wl^2 \times \frac{2}{3}l = \frac{1}{12}wl^3$.

$$BB' = \frac{1}{12}wl^3 \times \frac{l}{2} \div EI = \frac{wl^4}{24EI}, \qquad CC' = \frac{wl^4}{48EI}$$

$$C'C'' = \frac{1}{24} wl^3 \times \frac{3l}{16} \div EI = \frac{wl^4}{128EI}$$

The maximum deflection is

$$CC'' = \frac{8wl^4}{384EI} - \frac{3wl^4}{384EI} = \frac{5wl^4}{384EI}$$

2. In Fig. 133, the beam AB is loaded with equal loads P at the third points; the deflection at the point C and the maximum deflection are required.

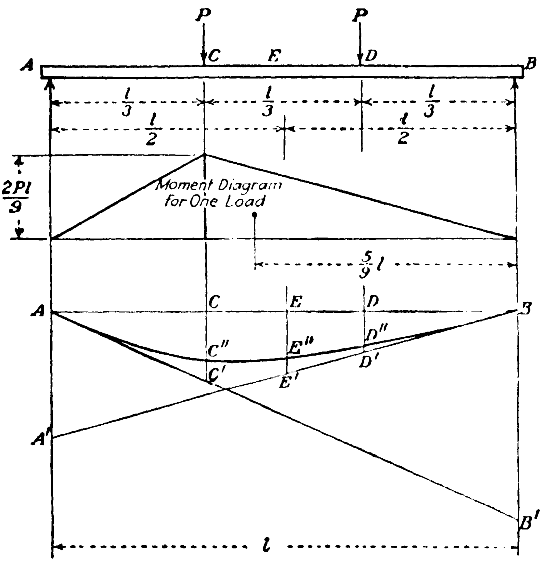


FIG. 133.

The area of the whole moment diagram for one load only is $Pl^2/9$.

$$BB' = \frac{Pl^2}{9} \times \frac{5l}{9} \div EI = \frac{5Pl^3}{81EI}, \qquad CC' = \frac{5Pl^3}{243EI},$$

$$C'C'' = \frac{Pl^2}{27} \times \frac{l}{9} \div EI = \frac{Pl^3}{243EI}$$

and

$$CC'' \text{ (for one load)} = \frac{4Pl^3}{243EI}$$

Similarly,

$$AA' = \frac{4Pl^3}{81EI}, \qquad DD' = \frac{4Pl^3}{243EI}, \qquad D'D'' = \frac{Pl^3}{486EI},$$

and

$$DD'' \text{ (for one load) } = \frac{7Pl^3}{486EI}$$

Since a load at C produces the same deflection at D as a load at D produces at C (see Art. 137),

$$\text{the total deflection at } C = \frac{5Pl^3}{162EI}$$

$$AA' \text{ (for one load) } = \frac{4Pl^3}{81EI}, \qquad \text{hence,} \qquad EE' = \frac{2Pl^3}{81EI}$$

Applying the second moment-area principle to the portion EB,

$$E'E'' = \frac{Pl}{6} \times \frac{l}{2} \times \frac{1}{2} \times \frac{l}{6} \div EI = \frac{Pl^3}{144EI}$$

Therefore, since $EE'' = EE' - E'E''$,

$$EE'' \text{ (for one load) } = \frac{23Pl^3}{1296EI}$$

and, for the two loads,

$$\text{the total deflection at } E = \frac{23Pl^3}{648EI}$$

136. Deflection of Beams by the Principle of Work. In Fig. 134, the beam AB is loaded in any manner, the load being represented by the force P. It is desired to determine the deflection at any section FF' due to this load.

At the section FF' a force of unity is applied in the direction in which the deflection will take place. It may be assumed that this force of unity and the force P are gradually and simultaneously applied.

Let S = the unit stress in the outer fiber at C due to the force P, and s = the unit stress in the outer fiber at C due to the force unity. Considering any fiber YY' of length dx, a distance y from the neutral surface, the stress in such a fiber due to the force P is

$$S' = \frac{y}{c} S$$

and that due to the force unity in the same fiber is

$$s' = \frac{y}{c} s$$

Hence, the total unit stress in the fiber YY' is $S' + s'$, and the elongation of the fiber is

$$\lambda = \frac{(S' + s')dx}{E}$$

Owing to the force unity alone the total stress in the fiber YY', whose area is dA, is $s'dA$, and the work done by the force unity

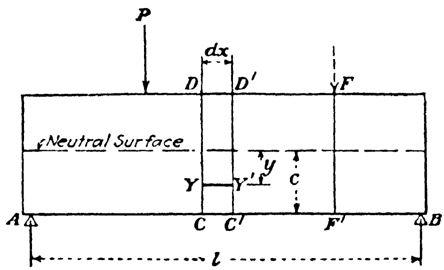


FIG. 134.

(which is considered gradually applied, *i.e.*, increasing from zero to unity) on the fiber is $\frac{1}{2} s'\lambda dA = \frac{\frac{1}{2}s'(S' + s')dAdx}{E}$. Substituting the values of S' and s', as first determined above, the work done on the fiber YY' is

$$\frac{1}{2} \cdot \frac{s(S + s)}{Ec^2} y^2 dAdx$$

Integrating this expression with respect to dA for the full depth of the beam, S, s, and dx remaining constant, since $\int y^2 dA = I$, the work done on all of the fibers of the slice $DD'C'C$ is

$$\frac{1}{2} \cdot \frac{s(S + s)Idx}{Ec^2}$$

but since $S = Mc/I$ and $s = mc/I$, where M = the bending moment due to the force P and m = the bending moment due to the force unity, the expression for the total work done on all the fibers of the section may be reduced to

$$\frac{1}{2}\frac{mc}{I}\left(\frac{Mc}{I}+\frac{mc}{I}\right)\frac{Idx}{Ec^2}=\frac{1}{2}\frac{m(M+m)dx}{EI}$$

For the whole length of the beam the internal work done by the force unity is

$$\frac{1}{2}\int_A^B \frac{m(M+m)dx}{EI}$$

The external work done by the force unity must equal the internal work done by the same force.

Let δ be the deflection at F due to the force P, and Δ the deflection at F due to the force unity. Since P and the force unity are gradually and simultaneously applied, the point F moves through a distance of $\delta + \Delta$, and the external work done by the force unity is

$$\tfrac{1}{2}(\delta + \Delta)$$

Therefore,

$$\delta + \Delta = \int_A^B \frac{m(M+m)dx}{EI}$$

If $P = 0$, δ and $M = 0$, and with the force unity alone acting

$$\Delta = \int_A^B \frac{m^2dx}{EI}$$

Subtracting this equation from the previous one

$$\delta = \int_A^B \frac{Mmdx}{EI}$$

Examples. 1. Referring to Fig. 129 determine the deflection at the center of the span by the method of this article.

Assume a load of unity applied at the center of the span (the point at which the deflection is desired). With the origin at A, from $x = 0$ to $x = l/2$,

$$M = \frac{Px}{3} \qquad \text{and} \qquad m = \frac{1}{2}x$$

$$\int_0^{\frac{l}{2}} \frac{Mmdx}{EI} = \int_0^{\frac{l}{2}} \frac{Px^2dx}{6EI} = \left[\frac{Px^3}{18EI}\right]_0^{\frac{l}{2}} = \frac{Pl^3}{144EI}$$

With the origin at B, from $x = l/3$ to $x = l/2$,

$$M = \frac{2Px}{3} - P\left(x - \frac{l}{3}\right) = \frac{P}{3}(l - x) \qquad \text{and} \qquad m = \frac{1}{2}x$$

$$\int_{\frac{l}{3}}^{\frac{l}{2}} \frac{P}{6} x \frac{(l - x)dx}{EI} = \int_{\frac{l}{3}}^{\frac{l}{2}} \left(\frac{Plx}{6EI} - \frac{Px^2}{6EI}\right) = \left[\frac{Plx^2}{12EI} - \frac{Px^3}{18EI}\right]_{\frac{l}{3}}^{\frac{l}{2}}$$

$$\frac{Pl^3}{48EI} - \frac{Pl^3}{144EI} - \frac{Pl^3}{108EI} + \frac{Pl^3}{486EI} = \frac{13Pl^3}{1944EI}$$

With the origin at B, from $x = 0$ to $x = l/2$,

$$M = \frac{2Px}{3} \qquad \text{and} \qquad m = \frac{1}{2}x$$

$$\int_0^{\frac{l}{3}} \frac{Mmdx}{EI} = \int_0^{\frac{l}{3}} \frac{Px^2dx}{3EI} = \left[\frac{Px^3}{9EI}\right]_0^{\frac{l}{3}} = \frac{Pl^3}{243EI}$$

The total deflection at mid-span is

$$\int_0^{\frac{l}{2}} \frac{Mmdx}{EI} \text{(origin at } A) + \int_{\frac{l}{3}}^{\frac{l}{2}} \frac{Mmdx}{EI} \text{(origin at } B) + \int_0^{\frac{l}{3}} \frac{Mmdx}{EI}$$

$$\text{(origin at } B) = \frac{Pl^3}{144EI} + \frac{13Pl^3}{1944EI} + \frac{Pl^3}{243EI}$$

$$= \frac{(27 + 26 + 16)}{3888} \cdot \frac{Pl^3}{EI} = \frac{23Pl^3}{1296EI}$$

2. In Fig. 135, the beam AB is loaded over the central half with a load of w per linear foot. The deflection at the center C is required.

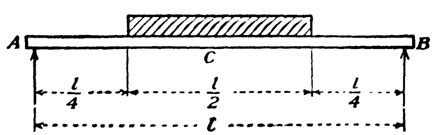


FIG. 135.

With the origin at A, from $x = 0$ to $x = \frac{l}{4}$, $M = \frac{wlx}{4}$ and $m = \frac{x}{2}$

$$\int_0^{\frac{l}{4}} \frac{Mmdx}{EI} = \int_0^{\frac{l}{4}} \frac{wlx^2}{8EI} = \left[\frac{wlx^3}{24EI}\right]_0^{\frac{l}{4}} = \frac{wl^4}{1536EI}$$

From $x = \frac{l}{4}$ to $x = \frac{l}{2}$, $M = \frac{wlx}{4} - \frac{w}{2}\left(x - \frac{l}{4}\right)^2$ and $m = \frac{1}{2}x$

$$\int_{\frac{l}{4}}^{\frac{l}{2}} Mmdx = \int_{\frac{l}{4}}^{\frac{l}{2}} \left(\frac{wlx^2}{8EI} - \frac{wx^3}{4EI} + \frac{wlx^2}{8EI} - \frac{wl^2x}{64EI}\right) = \int_{\frac{l}{4}}^{\frac{l}{2}} \left(\frac{wlx^2}{4EI} - \frac{wx^3}{4EI} - \frac{wl^2x}{64EI}\right) = \left[\frac{wlx^3}{12EI} - \frac{wx^4}{16EI} - \frac{wl^2x^2}{128EI}\right]_{\frac{l}{4}}^{\frac{l}{2}} = \frac{wl^4}{96EI}$$

$$- \frac{wl^4}{256EI} - \frac{wl^4}{512EI} - \frac{wl^4}{768EI} + \frac{wl^4}{4096EI} + \frac{wl^4}{2024EI} = \frac{49wl^4}{12{,}288EI}$$

The total deflection, considering the symmetry of the loading, is

$$2\left(\frac{wl^4}{1536EI} + \frac{49wl^4}{12{,}288EI}\right) = \frac{19wl^4}{2048EI}$$

137. Reciprocal Deflections. In Fig. 136, let equal loads P be applied at any two points c and d. When the load P at d is

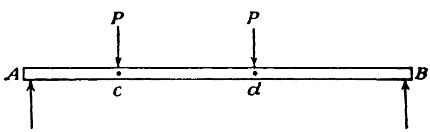


FIG. 136.

removed, the deflection at d due to the load P at c is

$$\delta_c = \int_0^l \frac{M_c m_d dx}{EI}$$

where M_c = the moment at any section due to the load P at c.
m_d = the moment at any section due to unit load at d.
Similarly, with the load P at c removed and the load P at d alone acting,

$$\delta_d = \int_0^l \frac{M_d m_c dx}{EI}$$

Since $M_c = Pm_c$ and $M_d = Pm_d$, it follows that $\delta_c = \delta_d$, *i.e., that, given any two points in a beam, the deflection at the first point due to a given load acting at the second point is equal to the deflection at the second point due to the given load acting at the first point.*

This principle, known as "Maxwell's law of reciprocal deflections," is most useful and important, not only in computing deflections but in analyzing indeterminate structures and in the mechanical analysis of structures developed in recent years. It applies to trusses as well as to beams.

138. Deflection Due to Shear. In the preceding articles only the deflection due to bending has been considered. The deflection due to shear may be determined in a similar manner. In Fig. 137, the cantilever beam AB is loaded at B with a load P. Assume the section dx only as elastic. The external work done is $\frac{1}{2}P\delta$ and the internal work is $\frac{1}{2}\,Vf$. Hence, $\delta = \frac{V}{P}f = vf$, where v is the shear when P is unity.

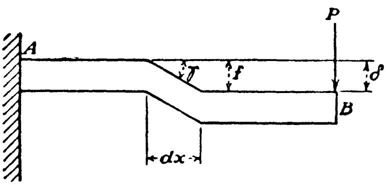


Fig. 137.

Let G = the shearing modulus of elasticity, and γ the angle of shear. Then,

$$G = \frac{V}{A} \div \gamma \qquad \text{or} \qquad \gamma = \frac{V}{AG}$$

$f = \gamma dx$ since f and γ are small. Therefore, $f = \frac{Vdx}{AG}$ and

$$\delta = \int_0^l \frac{Vvdx}{AG}$$

Only in exceptional cases where the ratio of depth to span is large will the shearing deflection exceed 5 per cent of the moment deflection, so that in general it may be neglected.

139. Deflection of Trusses by the Principle of Work. In Fig. 138, the truss is loaded in any manner, the load being represented by the force P. It is desired to determine the deflection of any point A due to this load.

At A a force of unity may be applied as shown. Let S_1, S_2 . . . , etc., be the stresses in the corresponding members of the truss due to the load P, and u_1, u_2 . . . , etc., be the stresses in the corresponding members due to the force of unity applied at A. The stress in any member n due to both loads is $\frac{(S_n + u_n)l_n}{AE}$.

Assuming that the forces P and unity are applied gradually and simultaneously, the internal work done by the force unity in changing the length of the member n is $\frac{u_n}{2}\left(\frac{S_n + u_n}{AE}\right) l_n$, and the work done on all the members by the force unity is

$$\frac{1}{2}\sum \frac{u(S + u)l}{AE}$$

The external work done by the force unity must equal the internal work done by this same force. Let δ be the deflection of A due to the force P, and Δ the deflection of A due to the force

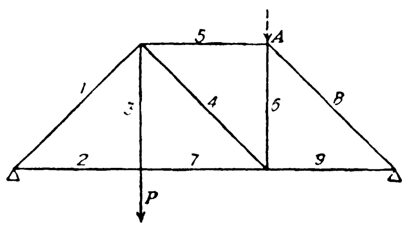


FIG. 138.

unity. Since P and the force unity are gradually and simultaneously applied, the point A moves through a distance $\delta + \Delta$, and the external work done by the force unity is $\frac{1}{2}(\delta + \Delta)$. Hence,

$$\delta + \Delta = \sum \frac{u(S + u)l}{AE}$$

When $P = 0$, δ and $S = 0$, and with the force unity alone acting,

$$\Delta = \sum \frac{u^2 l}{AE}$$

and

$$\delta = \sum \frac{Sul}{AE}$$

Example. In Fig. 139 which is the skeleton diagram of a Pratt truss whose dead-load stresses were computed in Art. 78, it is desired to determine the maximum deflection due to dead load. In the tabulation on page 208 the stresses are those computed in Art. 78.

The areas of the various members are the gross sectional areas (in integral square inches) required to take care of a live load equivalent to Cooper's E-60 loading, impact stresses, and lateral stresses in addition to those stresses caused by the dead load. The modulus of elasticity E is taken as 30,000,000 p.s.i.; sec θ = 1.30 and tan θ = 0.83. The column headed u contains the

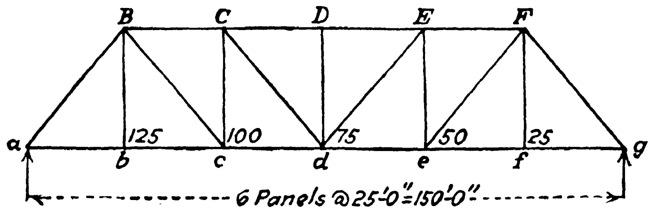


FIG. 139.

stresses in the several members of the truss due to a load of unity applied at d.

Member	Length, in.	Stress, lb.	Area, sq. in.	$\frac{Sl}{AE}$	u	$\frac{Sul}{AE}$
aB	469	−118,000	46	0.040	−0.65	0.023
BC	300	−120,000	46	0.026	−0.83	0.022
CD	300	−135,000	52	0.026	−1.25	0.033
ab	300	+ 75,000	29	0.026	+0.42	0.011
bc	300	+ 75,000	29	0.026	+0.42	0.011
cd	300	+120,000	45	0.027	+0.83	0.022
Bc	469	+ 70,000	30	0.036	+0.65	0.023
Cd	469	+ 24,000	20	0.019	+0.65	0.012
Bb	360	+ 26,000	21	0.015	0	0
Cc	360	− 28,000	20	0.017	−0.50	0.009
Dd	360	0	20	0	0	0

The sum of all the values in the last column of the table is 0.166 in. Since the summation was made for one-half the truss only, the total deflection of the point d under dead load only is 0.332 in.

If it were desired to determine the horizontal movement at g (a roller bearing being furnished at g) due to a change of temperature of 100 deg. Fahrenheit, the movement would be $\Sigma\omega tlu$, where ω is the coefficient of expansion or contraction of the steel, t is the number of degrees of temperature change, and l and u have the same significance as above. Applying a horizontal force of unity at g, only the lower chord members are stressed. The value of ω is 0.0000065, and ωtlu for each lower chord member is 0.0000065 × 100 × 300 × 1 = 0.195 in. and the total movement of g due to 100 deg. Fahrenheit temperature change is 6 × 0.195 = 1.17 in.

DISPLACEMENT DIAGRAMS

140. The Displacement of a Joint. If any member of a framed structure be subjected to a stress or a change in temperature, its length is changed. When the change is caused by stress, its amount is

$$\lambda = \frac{Sl}{AE}$$

in which λ = change in length.
S = total stress in the member.
l = its length.
A = its cross-section.
E = modulus of elasticity of the material.

When the change in length is caused by a change in temperature, its amount is

$$\lambda_t = \omega t l$$

in which ω = the coefficient of linear expansion per degree of temperature change.
t = the number of degrees of change.

In Fig. 140, consider the points *a* and *c* fixed as to horizontal or vertical movement and the frame *abc* hinged at its three joints. The effect of a load applied at *b*, as shown, is to shorten the member *ab* and lengthen the member *bc* due to the respective compressive and tensile stresses produced in them. Let their lengths under this load be *aa′* and *cc′*, respectively. With *a* as a center and *aa′* as a radius, let an arc be described. The point *b* must lie somewhere on this arc. Similarly, an arc is described with *c* as a center and *cc′* as a radius. The intersection of the two arcs *b′* is the position of the point *b*; and the frame *abc*, under the action of the load shown, assumes the shape *ab′c*.

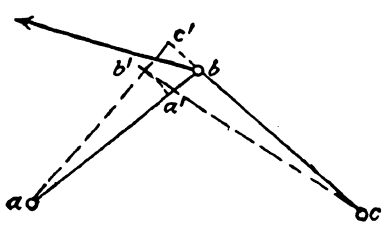


FIG. 140.

The deformations, as shown in the figure, are very much exaggerated, for if they were laid off to the same scale as the lengths they would not be visible. Since they are so small in comparison with the lengths of the members, the tangents to the arcs may be substituted for the arcs themselves without appreciable error. Therefore, the determination of the position of the point b' may be made by erecting perpendiculars to ab and bc at a' and c', respectively, their intersection being the position of the point b'.

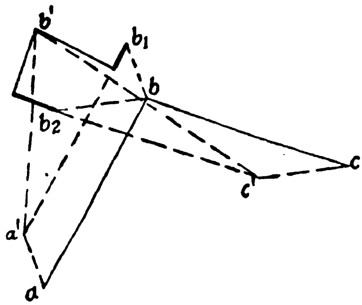


FIG. 141.

141. The Displacement Diagram. In the previous discussion two points of the frame abc were considered fixed. If, however, this frame were considered merely as a portion of a larger frame, the stresses in which cause a shifting in the positions of the points a and c, the final position of b would also be affected. In Fig. 141, the magnitude and direction of the displacements of the points a and c, respectively, are represented by the lines aa' and cc'. Considering first the position of b as affected by the change in position of a, it is seen that b takes the position b_1; but the

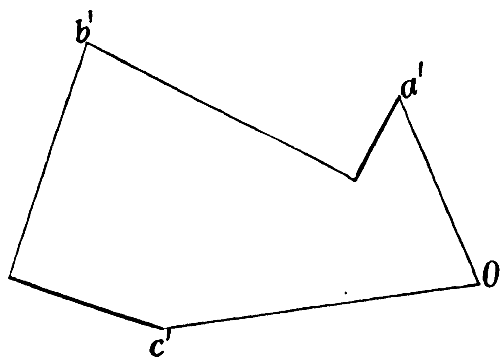


FIG. 142.

shifting in the position of c to c' considered alone would cause b to assume the position b_2. The deformation in the members ab and bc causes a further shifting of b. Proceeding as in the previous article, the change in length of $ab = a'b_1$ is laid off from b_1 *toward* a', since the change in length is a shortening; its magnitude is represented by the heavy line. Similarly, from b_2 the elongation of $cb = c'b_2$ is laid off *away from* c'. At the extremities of these deformations perpendiculars are erected, their intersection locating the point b'. The final displacements of the three points a, b, and c are bb_1, bb', and bb_2, respectively.

The displacements of the points a and c, the deformations of ab and bc, together with the perpendiculars erected at the extremities of the deformations, form a closed polygon. Therefore, that portion of the diagram may be constructed separately in order to determine the final displacements. This is desirable on account of the exceedingly small values of the deformations compared with the lengths of the members. Figure 142 is the necessary diagram drawn to a larger scale. Such a diagram is called a *Williot displacement diagram.*

142. The Displacement Diagram for a Truss. The changes in length of the several members of the truss of Fig. 143(a) due to a single load applied at d are marked on the diagram. The support a is fixed while the other support c is free to move in the inclined direction indicated.

The displacement diagram may be constructed by considering any joint fixed in position, and one of the members making the joint may be considered fixed in direction. In the following construction, a and the direction of ad are regarded as fixed.

Beginning at a' in Fig. 143(b), the elongation of the member ad is laid off in the direction of a toward d, since in the final position of the deformed truss d is pulled away from a. With d' thus determined, the displacement of b is found by regarding a and d in the triangle abd as fixed. The elongation of bd is laid off in the direction of d toward b from d', and the shortening of ab in the direction of b toward a from a'. The intersection of the perpendiculars, erected at their extremities, is the point b'. In the same manner, c' is located by laying off the shortening of bc from b' and the elongation of dc from d' and erecting the perpendiculars. The lines $a'b'$ and $a'c'$ are the displacements of the points b and c, respectively.

The deformation of the truss, under the assumed conditions of a and the direction of ad being fixed, are shown in Fig 143(c)

where the displacements obtained in Fig. 143(*b*) are laid off to a smaller scale and the corresponding panel points joined by broken lines. The deformation is greatly exaggerated in order to show the general effect.

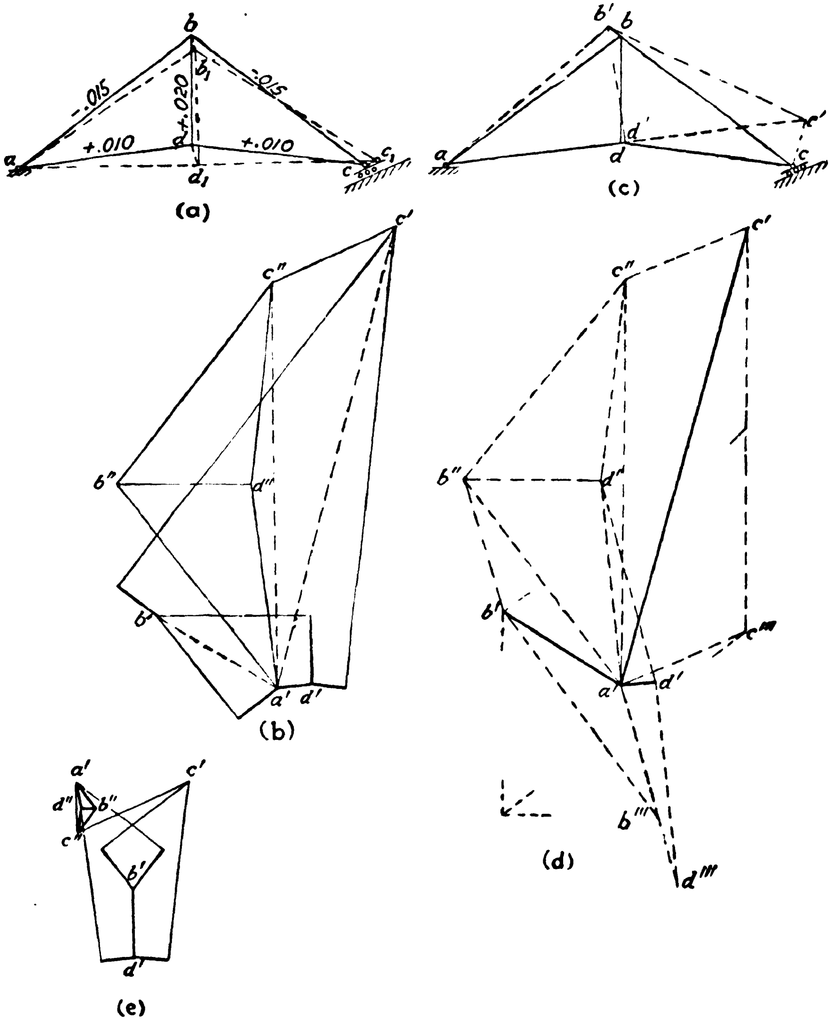


FIG. 143.

The original conditions, however, require that c shall move only on the inclined plane of the support; and, therefore, the whole truss must be revolved about a as a center until c' falls into a line drawn through c parallel to the constrained line of motion. As the arc thus described by c' is very small compared with the radius ac' and as the direction of ac' is practically the same as that of ac, a perpendicular from c' to ac may be substituted for the arc.

In Fig. 143(d), in which the displacements $a'b'$, $a'c'$, and $a'd'$ are shown without the construction lines of Fig. 143(b), the corresponding path of rotation of c is represented by $c'c'''$, which is drawn perpendicular to ac of Fig. 143(a) to an intersection with a line drawn through a' parallel to the constrained line of motion of c. The actual displacement of c is $a'c'''$. The displacement of b caused by the rotation of the truss about a is $b'b'''$, which is drawn perpendicular to ab of Fig. 143(a) and whose length bears the same relation to $c'c'''$ as their respective radii of rotation bear to one another. That is, $b'b'':c'c''' = ab:ac$. The length of $b'b'''$ may be determined by similar triangles as follows: Through the mid-point of $c'c'''$ draw a line through b' and from c''' another parallel line to its intersection with one drawn from b' perpendicular to ac of Fig. 143(a). From this intersection a line perpendicular to bd of Fig. 143(a) locates b''' at its intersection with the line drawn from b' perpendicular to ab of Fig. 143(a). The construction is indicated on the diagram. The displacement of d caused by the rotation of the truss is determined in a similar manner. The resultant displacements are then represented in amount and direction by $a'b'''$, $a'c'''$, and $a'd'''$. Since a is fixed, there is no displacement of that joint, and a' and a''' coincide.

In Fig. 143(a) the final position of the deformed truss is shown in broken lines, the resultant displacements $a'b'''$, $a'c'''$, and $a'd'''$ being laid off to a smaller scale from the points b, c, and d, respectively. As in Fig. 143(c) the actual displacements are greatly exaggerated.

For the purpose of simplifying the construction, the parallelograms of Fig. 143(d) are completed; *i.e.*, $b'b''$ is drawn parallel to $a'b'''$, $a'b''$ parallel to $b'b'''$, $c'c''$ parallel to $a'c'''$, $a'c''$ parallel to $c'c'''$, $d'd''$ parallel to $a'd'''$, and $a'd''$ parallel to $d'd'''$. Since $b''b' = a'b'''$, $c''c' = a'c'''$, and $d''d' = a'd'''$, $b''b'$, $c''c'$, and $d''d'$ represent the respective final displacements of the points b, c, and

d. Also, since $a'b''$, $a'c''$, and $a'd''$ are, respectively, perpendicular to ab, ac, and ad of Fig. 143(a), if the points a', b'', c'', and d'' are joined, they form a truss similar to the original truss of Fig. 143(a). It follows, therefore, that the actual displacements may be obtained from the diagram of Fig. 143(b) as follows: From c' draw $c'c''$ parallel to the constrained line of motion of the point c to an intersection with a perpendicular from a' to the radius of rotation of c, *i.e.*, to ac of Fig. 143(a). On $a'c''$ as a base construct a truss diagram similar to the truss of Fig. 143(a). The required displacements are then given by the directions and distances of b', c', and d' from b'', c'', and d'', respectively. The necessary construction is shown in Fig. 143(b) in full lines. The truss diagram $a'c''b''$ is called *Mohr's correction diagram.*

Since the displacement diagram may be drawn, assuming any joint fixed and the direction of any one of the members making that joint also fixed, it is advisable to choose the joint and the direction which will make the diagram as compact as possible. This allows the use of a larger scale and reduces the probability of error. Such a result may be secured by selecting a member which suffers the minimum change in direction under the applied load. In a simple truss one of the chords in the center panel or the middle vertical should be chosen. In the truss whose displacement diagram has been constructed as described above, if the direction of the middle vertical and either one of its extremities are considered fixed, the resultant diagram is as shown in Fig. 143(e). If the constrained line of motion were horizontal, the truss diagram $a'b''c'd''$ would be reduced to a point, since a' and c'' would coincide.

143. The Deflection of a Bridge Truss. When live load covers the entire span of a simple truss bridge, the elongation and shortening of the various members of the trusses are due to the stresses caused by the live and dead load, and the several panel points are displaced an appreciable amount from their theoretical positions with no stresses in the members. If this displacement were not considered in the design of the trusses, there would be, in any but the shortest spans, a noticeable sag at the mid-point of the span. In order to prevent any point of the lower chord falling below a line joining the two supports, the tension members are shortened by an amount equal to or greater than their elongation as computed with full live load on the bridge. Similarly, the compression members are lengthened a corresponding amount. Since full live load does not cause the maximum stresses in most of the web members of a simple truss, it is not

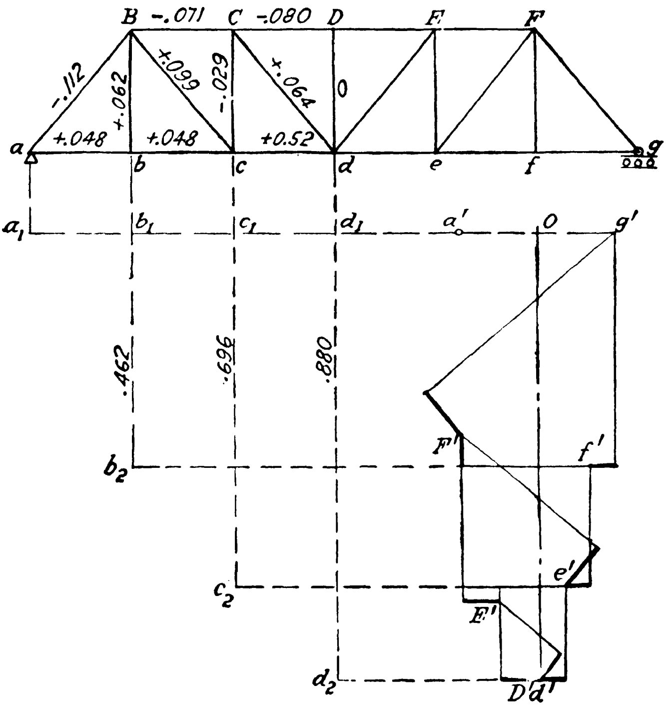


FIG. 144.

necessarily the maximum stresses that are used in determining the various elongations and shortenings.

With the sections and lengths of the members known, together with their stresses under full live load, the change in length in each member may be computed by the equation given in Art. 140, and the displacement diagram constructed. Generally, the vertical components of the displacements of the lower chord panel points are all that are required. These components are the deflections.

Example. Figure 144 shows the necessary construction to determine the deflections of the lower chord panel points of a six-panel through Pratt truss when the live load covers the entire span. The left support a is fixed, and the right support g rests on a roller bearing and is free to move horizontally. The changes in the lengths of the several members are given on the truss diagram.

In constructing the displacement diagram, the direction of Dd is considered fixed. Since the stress in Dd is due to dead load only and its amount inconsiderable, the shortening in Dd may be neglected; therefore, D' and d' coincide. In a symmetrical truss sustaining symmetrical load, if the center member is chosen as the fixed member for the construction of the displacement diagram, the diagram itself is symmetrical about this member as was shown in Fig. 143(e). Therefore, for the truss under consideration it is necessary to construct only one-half of the displacement diagram in order to determine the deflection of the lower chord panel points. The construction for the left half is shown in Fig. 144.

The displacements are measured from a' ($a'O = Og'$) since g'' and a' coincide (see Art. 142) and a' is also a'', b'', c'', d'', etc. The deflections or vertical components of the displacements for the panel points b, c, and d are easily determined graphically by drawing horizontal lines from a', f', e', and d' and scaling the distances b_1b_2, c_1c_2, and d_1d_2. Their values are given on the figure. The deflections of the symmetrical panel points on the right side of the truss are of the same amount.

CHAPTER X

REINFORCED-CONCRETE BEAMS AND SLABS

144. Reinforced Concrete. The tensile and transverse strength of plain concrete is very low and unreliable (see Art. 36), and its practical uses are limited to structures or parts of structures in which no tensile stresses are induced, *i.e.*, to arches, piers, and certain massive constructions. In order to make concrete available for use in structural members involving tension, such as beams, for example, steel bars are embedded in the tension side of the beam. It is, of course, assumed that the bars are embedded so that the union between the steel and concrete is sufficient to make the two materials act as one. The purpose of the steel is to carry the tensile stresses. The concrete sustains the compressive and shearing stresses, because its resistance to these is comparatively large. Concrete in which steel bars are so embedded is called reinforced concrete. Steel reinforcement is also sometimes used in the compression areas of beams; because of its greater strength, the steel will resist more stress than the concrete that it replaces, and so, for a given total stress, the size of the beam can be reduced when compression reinforcement is used. Compression members, such as columns, are invariably reinforced, partly to reduce the size and partly to add to the security of the member.

145. Types of Reinforcement. The reinforcing steel in reinforced-concrete construction must be of such form and size that it may easily be incorporated as a part of the structure and provide sufficient surface to bond the two materials thoroughly together. Square and round bars, varying in size from ¼ to 1¼ in., are used for reinforcing beams, slabs, walls, and columns; and wire fabric and expanded metal in various forms are used to a considerable extent in slabs, walls, and other thin sections.

Bars are almost invariably made with projections or indentations, or both, on the surfaces to increase the adhesion between them and the concrete. These bars are called deformed bars, several types of which are shown in Fig. 145. Some of the

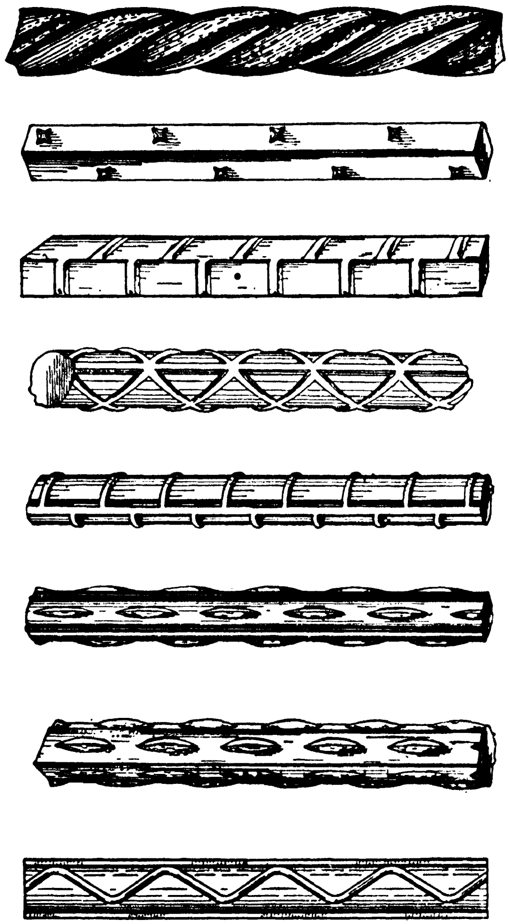


FIG. 145.

types of mesh reinforcement are shown in Fig. 146. Three grades of steel are used for bars rolled from new billet steel: (1) structural, (2) intermediate, and (3) hard. Structural-grade steel should have an ultimate strength of 55,000 to 70,000 p.s.i.; intermediate, from 70,000 to 90,000 p.s.i.; and the hard grade, 80,000 p.s.i. or greater. Hard or high-carbon steel is likely to be too brittle for safe use in light members subject to sudden impact stresses. In the past, the structural grade was preferred by most designers, but the present tendency is toward the use of the intermediate grade, with higher design stresses than were formerly allowed. Bars are sometimes rolled from old steel

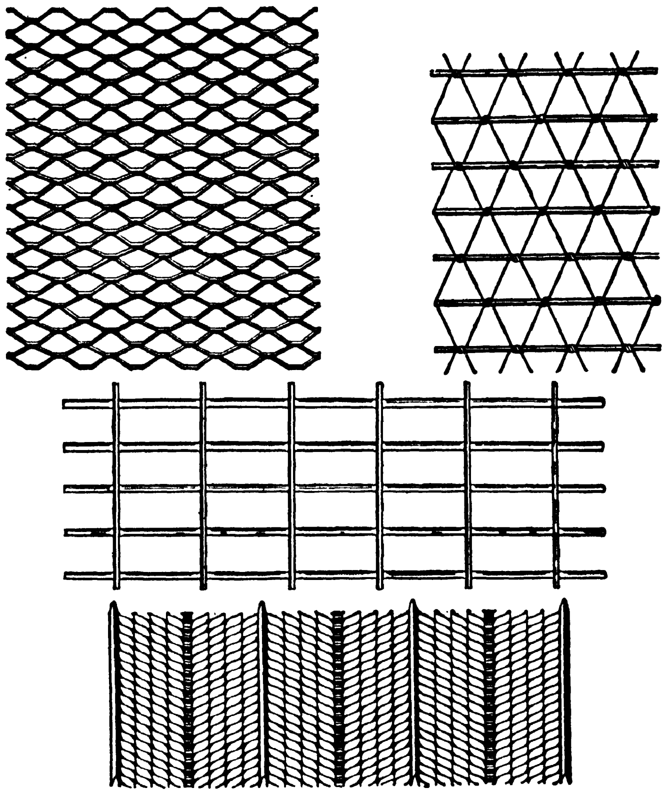


Fig. 146.

rails, but many specifications prohibit the use of such "rerolled steel" for important structures.

Reinforcing bars are sold by weight. The base price applies only to bars ¾ in. in size, or larger. Bars smaller than these are charged for at the base price plus a "size extra," which varies with the size of the bars. Bars are available only in certain standard sizes, as listed, with their areas, perimeters, weights, and size extras, in Table 1.

TABLE 1.—STANDARD REINFORCING BARS

Round bars					Square bars				
Size, in.	**Area, sq. in.**	Perimeter, in.	Weight, lb. per lin. ft.	Size extra, cts. per 100 lb.	Size, in.	Area, sq. in.	Perimeter, in.	Weight, lb. per lin. ft.	Size extra, cts. per 100 lb.
1/4	0.0491	0.785	0.17	100	1/2	0.2500	2.00	0.85	20
3/8	0.1104	1.178	0.38	40	1	1.0000	4.00	3.40	0
1/2	0.1963	1.571	0.67	20	1 1/8	1.2656	4.50	4.30	0
5/8	0.3068	1.964	1.04	10	1 1/4	1.5625	5.00	5.31	0
3/4	0.4418	2.356	1.50	0					
7/8	0.6013	2.749	2.04						
1	0.7854	3.142	2.67						

146. Placing the Reinforcement. In placing the reinforcement, four general requirements must be fulfilled: (1) There must be sufficient space between the bars to permit proper placing of the concrete around them. (2) There must be sufficient concrete in the plane of the bars properly to transmit the stresses of tension and shear. (3) There must be sufficient concrete covering the bars to afford ample protection for the steel against moisture and fire damage. (4) The reinforcement must be accurately placed and adequately secured in position by concrete or metal chairs and spacers.

Most specifications require that the minimum clear distance between parallel bars shall be 1½ times the diameter for round bars and 2 times the side dimension for square bars, but in no case shall the clear spacing between bars be less than 1 in. or less than 1⅓ times the maximum size of the coarse aggregate. The reinforcement of footings and other principal structural members in which the concrete is deposited against the ground shall have not less than 3 in. of concrete between it and the ground contact surface. At surfaces of concrete exposed to the weather, the reinforcement shall be protected by not less than 2 in. of concrete. In structures in which the fire hazard is limited, reinforcement at surfaces not exposed directly to ground or weather shall be protected by not less than ¾ in. of concrete for slabs and walls, 1½ in. for beams and girders, and 2 in. for columns. In structures with an abnormal fire hazard, the minimum covering for slab reinforcement shall be 1 in., for beam and girder reinforcement 2 in., and for column reinforcement 3 in. The insulations commonly used are shown in Fig. 147.

147. Allowable Unit Stresses. The unit stresses recommended by the Standard Building Code Committee of the American Concrete Institute (hereafter referred to as the Joint Code) are given on page 49. These are stated in terms of the ultimate unit compressive strength f'_c of the concrete at the age of 28 days, as determined by tests of 6- by 12-in. or 8- by 16-in. cylinders. Concretes used in ordinary constructions have ultimate compres-

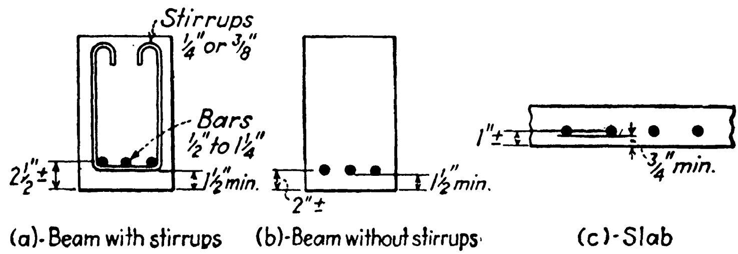


Fig. 147.

sive strengths varying, with the type of member involved, from 1500 to 4000 p.s.i., the most common values ranging from 2000 to 3000 p.s.i. The required strength of the concrete is obtained by the proper proportioning of the cement, aggregate, and water (see Art. 30).

RECTANGULAR BEAMS WITH TENSION REINFORCEMENT

148. Flexure Formulas. Figure 148 represents a portion of a rectangular concrete beam with a width of b in. and a total height of a in., and with tension reinforcement of area A_s sq. in. placed at a distance of d in. from the compression surface. The distance d is called the effective depth of the beam. Let AB represent any cross-section before the load is applied to the beam and $A'B'$ the same cross-section after the load is applied. The upper fibers of the beam (the compression fibers) will tend to shorten, and the lower fibers will lengthen. According to assumption 1, Art. 53, the deformation at any horizontal plane in the beam is proportional to the distance from that plane to the neutral axis. With proper interpretation, the distance AA' may be considered to represent the shortening of the extreme upper fibers for a unit length of beam, and BB' the unit elongation of the steel.

The tensile strength of the concrete in a beam is disregarded because of its low value, and hence all the tension force is concentrated in the steel; the amount of this tension T is equal to the area of the steel, A_s, in square inches multiplied by the unit stress in the steel, f_s, in pounds per square inch. The unit compression stress is a maximum, f_c, at the extreme fiber; and it is assumed that this stress decreases uniformly to zero at the

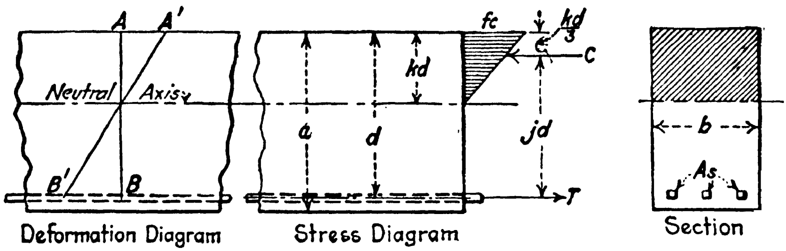


FIG. 148.

neutral axis. Let k be the ratio of the distance between the compression surface and the neutral axis to the effective depth d, so that kd is the distance from the compression surface to the neutral axis. The total compression C is equal to the area of the shaded triangle in Fig. 148 multiplied by the width of the beam, or $C = \frac{1}{2}f_c \times kd \times b$. For purposes of computation this total compression is assumed to be concentrated at the center of gravity of the triangle, which is one-third of the distance kd from the compression surface. Let j be the ratio of the lever arm of the resisting couple C, T, to the effective depth d, so that jd is the distance between the centers of gravity of the compression and tension forces. The internal moment M_c in the beam, as determined by the compression forces, is equal to the total compression C times the lever arm jd; hence,

$$M_c = (\frac{1}{2}f_ckd \cdot b)jd = \frac{1}{2}f_ckjbd^2 \qquad (1)$$

Similarly, the internal moment M_s as determined by the tension forces is

$$M_s = A_s f_s jd \tag{2}$$

If the ratio between the area of the steel and the effective area bd of the beam is called p (*i.e.*, $p = A_s/bd$), then equation (2) may be written

$$M_s = pbdf_s jd = pf_s jbd^2 \tag{2a}$$

The actual internal moments as expressed by equations (1) and (2), or (1) and (2*a*), are each equal to the external bending moment at all stages of loading (see Art. 54). But if the maximum allowable moment M_c [equation (1) with f_c taken at maximum allowable value] is reached before the maximum allowable moment M_s [equation (2) with f_s its maximum allowable value], this means that the beam will be overstressed on the compression side before the steel is stressed to its limit; *i.e.*, the beam has more steel than is theoretically required—it is over-reinforced. In designing a beam it is desirable to place in the beam an amount of steel such that the limiting values of f_c and f_s will be reached simultaneously. If this ideal steel ratio is obtained,

$$M_c = M_s = \tfrac{1}{2} f_c kjbd^2 = A_s f_s jd = pf_s jbd^2 \tag{a}$$

or

$$M = Kbd^2 \tag{3}$$

in which $K = \tfrac{1}{2} f_c kj$ or $pf_s j$.

f_c and f_s = the unit working stresses as given in Art. 43.

The value of k to be used in designing a beam is obtained as follows: From the assumption that deformations vary as the distance from the neutral axis,

$$\frac{AA'}{BB'} = \frac{kd}{d - kd} \tag{b}$$

Since E = unit stress/unit deformation, it follows that

$$AA' = \frac{f_c}{E_c} \qquad \text{and} \qquad BB' = \frac{f_s}{E_s}$$

Hence,

$$\frac{AA'}{BB'} = \frac{E_s}{E_c} \cdot \frac{f_c}{f_s} = \frac{nf_c}{f_s} = \frac{n}{r} \qquad (c)$$

in which $n = E_s/E_c$

$r = f_s/f_c$

Equating equations (b) and (c),

$$\frac{nf_c}{f_s} = \frac{n}{r} = \frac{kd}{d - kd} \qquad (d)$$

from which

$$k = \frac{n}{n + r} \qquad (4)$$

The formula for the ideal steel ratio p which is required to make the maximum moment that can be resisted by the compression forces (*i.e.*, the resisting moment of the concrete) equal to that of the steel is obtained as follows: From equation (a), $\frac{1}{2}f_ckj = pf_sj$, and from this relation it follows that $p = k/2r$. Since for any given simultaneous values of f_c and f_s, $k = \dfrac{n}{n + r}$ [equation (4)], the required value of p, which will be called p_i so as to distinguish it from the actual steel ratio A_s/bd, is

$$p_i = \frac{n}{2r(n + r)} \qquad (5)$$

The maximum allowable values of M_c and M_s will be equal to each other only when the amount of steel in the beam is such that the actual steel ratio, $p = \dfrac{A_s}{bd}$, is equal to the specific value p_i obtained from equation (5).

The relation between simultaneous values of f_s and f_c is obtained directly from equation (d):

$$f_c = \frac{f_sk}{n(1 - k)} \qquad (6)$$

$$f_s = \frac{nf_c(1 - k)}{k} \qquad (7)$$

The formula for k which applies to the review of a beam is derived as follows: At any stage of loading, for equilibrium the total compression stress $C = \frac{1}{2}f_ckbd$ must equal the total tension stress $T = A_sf_s = pbdf_s$. Hence,

$$\frac{1}{2}f_ckbd = pbdf_s \qquad (e)$$

Substituting in equation (*e*) the value of f_s from equation (7),

$$\frac{1}{2}f_ckbd = pbd\,\frac{nf_c(1 - k)}{k}$$

from which

$$k = \sqrt{2pn + (pn)^2} - pn \qquad (8)$$

The equation for j is obtained by inspection from Fig. 147; *i.e.*, $jd = d - \frac{1}{3}kd$, and hence

$$j = 1 - \frac{k}{3} \qquad (9)$$

149. Method of Design. Compute the maximum bending moment caused by the external loads (see Art. 6). Assume the weight of the beam and add the bending moment due to this assumed weight to that caused by the external loads, to get the total bending moment in the beam. This must be revised later, if the actual weight of the beam does not agree with the assumed weight. Determine k [equation (4)], j [equation (9)], cross-section of beam [equation (1)], area of steel [equation (2)]. Unless it is governed by architectural or other limitations, make b from ½ to ¾ of d, and keep b in multiples of 2 in.

150. Method of Review. Determine k [equation (8)], j [equation (9)], M_c or f_c [equation (1)], M_s or f_s [equation (2)]. If the safe load is required, use the smaller of the values M_c or M_s, deduct the moment due to the weight of the beam, and compute the live load which will produce the remaining moment.

151. Tables for Values of k and j. Values of k, j, K, and p_i, based on equations (4), (9), (3), and (5) for various combinations of n, f_s, and f_c, are given in Table 2. Values of k and j, based on equations (8) and (9), for various combinations of n and p, are given in Table 3. The computations involved in actual problems can be simplified to some extent by reading the values of k and j from these tables. The values of the ideal steel ratio p_i in Table 2 are particularly useful in review problems in which

TABLE 2.—DESIGN OF RECTANGULAR BEAMS AND SLABS

$$k = \frac{n}{n + r} \qquad j = 1 - \frac{k}{3} \qquad p_i = \frac{n}{2r(n + r)} \qquad K = \frac{1}{2} f_c k j \text{ or } p f_s j$$

n and f'_c	f_s	f_c	k	j	p_i	K
8 (3750)	18,000	1200	0.348	0.884	0.0116	185
		1400	0.384	0.872	0.0149	234
		1500	**0.400**	**0.867**	**0.0167**	**260**
		1688	*0.429*	*0.857*	*0.0201*	*310*
	20,000	1200	0.324	0.892	0.0097	173
		1400	0.359	0.880	0.0126	221
		1500	**0.375**	**0.875**	**0.0141**	**246**
		1688	*0.403*	*0.866*	*0.0170*	*294*
10 (3000)	18,000	1000	0.357	0.881	0.0099	157
		1100	0.379	0.874	0.0116	183
		1200	**0.400**	**0.867**	**0.0133**	**208**
		1350	*0.428*	*0.857*	*0.0161*	*248*
	20,000	1000	0.333	0.889	0.0083	148
		1100	0.355	0.882	0.0098	172
		1200	**0.375**	**0.875**	**0.0113**	**197**
		1350	*0.403*	*0.866*	*0.0136*	*235*
12 (2500)	16,000	750	0.360	0.880	0.0084	119
		800	0.375	0.875	0.0094	131
		900	0.403	0.866	0.0113	156
		1000	**0.429**	**0.857**	**0.0134**	**184**
		1125	*0.457*	*0.848*	*0.0161*	*218*
	18,000	750	0.333	0.889	0.0069	111
		800	0.348	0.884	0.0077	123
		900	0.375	0.875	0.0094	148
		1000	**0.400**	**0.867**	**0.0111**	**173**
		1125	*0.428*	*0.857*	*0.0134*	*207*
	20,000	750	0.310	0.897	0.0058	104
		800	0.324	0.892	0.0065	116
		900	0.351	0.883	0.0079	140
		1000	**0.375**	**0.875**	**0.0094**	**164**
		1125	*0.403*	*0.866*	*0.0113*	*196*
15 (2000)	16,000	650	0.379	0.874	0.0077	108
		700	0.397	0.868	0.0087	121
		750	0.414	0.862	0.0097	134
		800	**0.429**	**0.857**	**0.0107**	**147**
		900	*0.457*	*0.848*	*0.0129*	*175*
	18,000	650	0.351	0.883	0.0063	101
		700	0.368	0.877	0.0072	113
		750	0.385	0.872	0.0080	126
		800	**0.400**	**0.867**	**0.0089**	**139**
		900	*0.429*	*0.857*	*0.0107*	*165*
	20,000	650	0.328	0.891	0.0053	94
		700	0.344	0.885	0.0060	106
		750	0.359	0.880	0.0067	118
		800	**0.374**	**0.875**	**0.0075**	**131**
		900	*0.403*	*0.866*	*0.0091*	*157*

Boldface for $0.4f'_c$; italics for $0.45f'_c$. See ftnt. †, pg. 49.

TABLE 3.—REVIEW OF RECTANGULAR BEAMS AND SLABS

$$k = \sqrt{2pn + (pn)^2} - pn \qquad j = 1 - \tfrac{1}{3}k$$

p	n = 8		n = 10		n = 12		n = 15	
	k	j	k	j	k	j	k	j
0.0010	0.119	0.960	0.132	0.956	0.145	0.952	0.158	0.947
0.0020	0.164	0.945	0.181	0.940	0.196	0.935	0.217	0.928
0.0030	0.196	0.935	0.217	0.928	0.235	0.922	0.258	0.914
0.0040	0.223	0.926	0.246	0.918	0.266	0.911	0.292	0.903
0.0050	0.246	0.918	0.270	0.910	0.291	0.903	0.320	0.893
0.0054	0.254	0.915	0.279	0.907	0.300	0.900	0.329	0.891
0.0058	0.262	0.913	0.287	0.904	0.309	0.897	0.337	0.888
0.0062	0.269	0.910	0.296	0.901	0.317	0.894	0.348	0.884
0.0066	0.276	0.908	0.304	0.899	0.325	0.892	0.356	0.881
0.0070	0.283	0.906	0.311	0.896	0.334	0.889	0.365	0.878
0.0072	0.286	0.905	0.314	0.895	0.338	0.887	0.369	0.877
0.0074	0.290	0.903	0.318	0.894	0.342	0.886	0.372	0.876
0.0076	0.293	0.902	0.321	0.893	0.345	0.885	0.376	0.875
0.0078	0.297	0.901	0.325	0.892	0.349	0.884	0.380	0.873
0.0080	0.300	0.900	0.328	0.891	0.353	0.882	0.384	0.872
0.0082	0.303	0.899	0.332	0.889	0.356	0.881	0.387	0.871
0.0084	0.306	0.898	0.336	0.888	0.360	0.880	0.390	0.870
0.0086	0.309	0.897	0.338	0.887	0.363	0.879	0.394	0.869
0.0088	0.312	0.896	0.341	0.886	0.366	0.878	0.398	0.867
0.0090	0.314	0.895	0.344	0.885	0.370	0.877	0.402	0.866
0.0092	0.317	0.894	0.347	0.884	0.373	0.876	0.405	0.865
0.0094	0.320	0.893	0.350	0.883	0.376	0.875	0.407	0.864
0.0096	0.322	0.893	0.353	0.882	0.379	0.874	0.411	0.863
0.0098	0.325	0.892	0.356	0.881	0.381	0.873	0.414	0.862
0.0100	0.328	0.891	0.358	0.881	0.385	0.872	0.418	0.861
0.0104	0.333	0.889	0.363	0.879	0.391	0.870	0.423	0.859
0.0108	0.339	0.887	0.369	0.877	0.396	0.868	0.429	0.857
0.0112	0.343	0.886	0.375	0.875	0.402	0.866	0.434	0.855
0.0116	0.348	0.884	0.380	0.873	0.407	0.864	0.440	0.853
0.0120	0.353	0.882	0.384	0.872	0.412	0.863	0.446	0.851
0.0124	0.357	0.881	0.389	0.870	0.417	0.861	0.451	0.850
0.0128	0.362	0.879	0.394	0.869	0.422	0.859	0.457	0.848
0.0132	0.366	0.878	0.398	0.867	0.427	0.858	0.461	0.846
0.0136	0.370	0.877	0.403	0.866	0.432	0.856	0.466	0.845
0.0140	0.374	0.875	0.407	0.864	0.436	0.855	0.471	0.843
0.0144	0.379	0.874	0.412	0.863	0.440	0.853	0.475	0.842
0.0148	0.383	0.872	0.416	0.861	0.444	0.852	0.479	0.840
0.0152	0.386	0.871	0.420	0.860	0.449	0.850	0.483	0.839
0.0156	0.390	0.870	0.424	0.859	0.453	0.849	0.487	0.838
0.0160	0.394	0.869	0.428	0.857	0.457	0.848	0.493	0.836
0.0170	0.403	0.866	0.437	0.854	0.467	0.845	0.502	0.833
0.0180	0.412	0.863	0.446	0.851	0.476	0.841	0.513	0.829
0.0190	0.420	0.860	0.455	0.848	0.485	0.838	0.522	0.826
0.0200	0.428	0.857	0.463	0.846	0.493	0.836	0.531	0.823

the resisting moment of a given beam is required. The actual steel ratio ($p = A_s/bd$) can be compared with the tabulated value of p_i; if it is greater than p_i, obviously the beam is over-reinforced and the strength of the beam is governed by its compressive resistance [equation (1)], and vice versa.

152. Design of a Rectangular Beam with Tension Reinforcement. Determine the required cross-section and area of steel for a simply supported rectangular beam with a span of 18 ft.-0 in. which is to carry a uniform live load of 1000 lb. per lin. ft. A 2500-lb. concrete and reinforcing bars of structural-grade steel are to be used.

Assume weight of beam to be 220 lb. per lin. ft. Total load = 1220 lb. per ft.

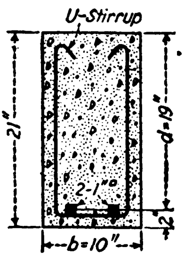


FIG. 149.

$$M = (\tfrac{1}{8} \times 1220 \times 18^2) \times 12 = 593{,}000 \text{ in.-lb.}$$

From Table 2, $k = 0.400$, $j = 0.867$, and $K = 173$.

$$593{,}000 = \tfrac{1}{2} \times 1000 \times 0.400 \times 0.867 \times bd^2 = 173bd^2$$

$$bd^2 \text{ (required)} = 3420 \text{ in.}^3$$

Select $b = 10$; then $d = 18.5$ or 19 in.

Total cross-section of beam = 10×21 (see Fig. 149), and weight = $(10 \times 21/144) \times 150 = 220$ lb. per lin. ft., as assumed.

$$593{,}000 = A_s \times 18{,}000 \times 0.867 \times 19$$

$$A_s \text{ (required)} = 2.00 \text{ sq. in.}$$

Two 1-in. square bars will be used, as shown in Fig. 149. The purpose of the stirrups shown in Fig. 149, and the method of determining their size and spacing are explained in Arts. 163 to 165.

153. Review of a Rectangular Beam with Tension Reinforcement. A rectangular reinforced-concrete beam has a total cross-section of 8 by 14 in. and a span of 20 ft.-0 in. It is reinforced with four ½-in. square bars in one row, the center of which is 1½ in. above the lower surface of the beam. Assuming a 2000-lb. concrete and structural-grade steel, what is the greatest concentrated load that can be placed on the beam at mid-span?

$$p = \frac{4 \times 0.25}{8 \times 12.5} = 0.0100$$

From Table 3, $k = 0.418$, and $j = 0.861$.

$$M_c = \tfrac{1}{2} \times 800 \times 0.418 \times 0.861 \times 8 \times (12.5)^2 = 180{,}000 \text{ in.-lb.}$$

$$M_s = 1.00 \times 18{,}000 \times 0.861 \times 12.5 = 194{,}000 \text{ in.-lb.}$$

Hence, this beam is slightly weaker in compression than it is in tension, and the maximum bending moment cannot exceed 180,000 in.-lb. Owing to its own weight (117 lb. per lin. ft.) the bending moment is

$$\tfrac{1}{8} \times 117 \times 20^2 \times 12 = 70{,}200 \text{ in.-lb.}$$

The maximum moment available for the concentrated load is then $180{,}000 - 70{,}200 = 109{,}800$ in.-lb. The maximum bending moment caused by the concentrated load (see Art. 6) is

$$\tfrac{1}{4}Pl = \tfrac{1}{4}P \times 20 \times 12 = 60P \text{ in.-lb.};$$

hence, $60P = 109{,}800$ and P (max.) $= 1830$ lb.

The fact that for this beam M_c is less than M_s could have been shown by comparing the actual steel ratio 0.0100 with the ideal steel ratio for the given stresses, which, from Table 2, is 0.0089. The comparison shows that the beam is overreinforced, and the above computation for M_s is unnecessary.

ADDITIONAL PROBLEMS

1. A simply supported rectangular beam has a total cross-section of 10×16 in. and a length of 20 ft.-0 in. It is reinforced with four $\tfrac{5}{8}$-in. round bars in one row. The distance from the centers of the bars to the lower surface of the beam is $2\tfrac{1}{2}$ in. With 2500-lb. concrete and intermediate grade steel, what is the resisting moment of the beam?

2. If a concentrated load of 3500 lb. were placed on the beam of Example 1, at a distance of 7 ft.-0 in. from the support, what would be the maximum unit stress in the concrete and the maximum unit stress in the steel?

3. A simply supported rectangular beam with a span of 18 ft.-0 in. supports a uniform live load of 975 lb. per lin. ft. and a concentrated load of 3000 lb. at the middle of the span. With $f'_c = 3000$ p.s.i. and with intermediate-grade reinforcing steel, determine the required cross-section and steel area.

4. A simply supported rectangular beam with a span of 17 ft.-0 in. supports a live load which varies in amount from zero at the left support uniformly to an amount of 1000 lb. per lin. ft. at the right support. A 2500-lb. concrete and structural-grade reinforcing bars are to be used. Determine the required cross-section and steel area.

5. A rectangular beam, simply supported, has a width of 14 in. and an effective depth of 26 in. If $n = 12$ and if the allowable unit stresses are 1000 p.s.i. and 20,000 p.s.i. for the concrete and steel, respectively, what steel area must be used in order that the resisting moment with respect to the strength of the concrete is the same as the resisting moment with respect to the steel?

6. What would be the resisting moment of the beam in Example 5 if the reinforcement consisted of four $\tfrac{7}{8}$-in. round bars?

SLABS

154. Types of Slabs. Slabs may be supported on two sides only, or they may rest on beams along all four edges. A slab which is supported only on two sides is essentially a rectangular beam of comparatively large ratio of width to depth. There are, however, certain modifications entering into the design and review of such slabs which it was not necessary to consider in the solution of rectangular beams. Slabs which are supported on four sides, with reinforcement in two directions, present the additional problems of determining the proportion of the total load that is transmitted in each direction to the supporting beams and also allocating this proportion of the total load to the various strips into which the slab is assumed to be divided. These problems are considered in Art. 160.

Floor slabs in buildings are usually designed for a uniform live load covering the entire slab area. The following discussions are intended to apply only to the analysis of such uniformly loaded slabs. Concentrated loads on concrete slabs are supported by a greater width of slab than the mere contact width. Methods of computing the probable distribution of concentrated loads are generally given in the specifications governing the design of the structure of which the slab is a part. Multiples of ¼ in. may be used in selecting the total thickness of slabs.

155. Slabs Supported on Two Sides Only. The simplest form of slab is one of indefinite width, supported only by two beams, one at each edge of the slab. If a 12-in. strip of slab were cut out at right angles to the supporting beams, such as either of the shaded areas in Fig. 150, a rectangular beam 12 in. in width would result, with a depth equal to the thickness of the slab, and a length equal to the distance between supports. This strip could then be analyzed by the same formulas which were used in problems dealing with rectangular beams, the bending moment being computed for a width of 1 ft. The load per square foot on the slab would then be the load per linear foot on the imaginary beam. Since all the load on the slab must be transmitted to the two supporting beams, it follows that all the reinforcing steel should be placed at right angles to these beams, with the exception of any bars that may be placed in the other direction to take care of shrinkage and temperature stresses. A slab which is supported on two sides only thus consists (in theory) of a series of rectangular beams side by side.

The ratio of steel in a slab may be determined by dividing the sectional area of one bar by the area of concrete between two

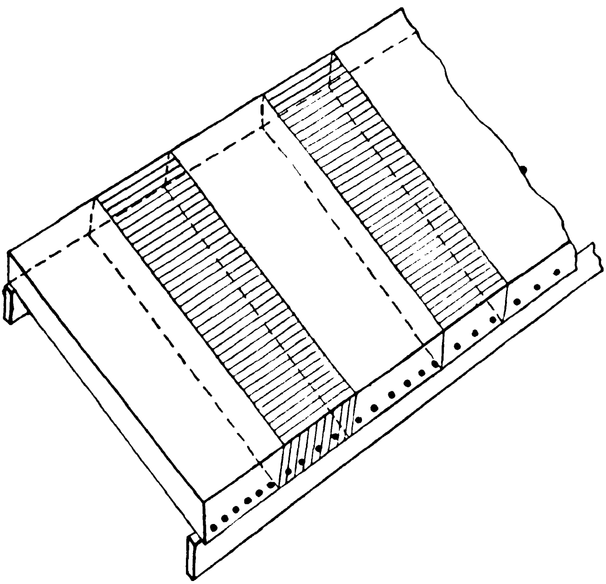


Fig. 150.

successive bars, the latter area being the product of the depth to the center of the bars and the distance between them, center to center. Thus, if a 5-in. slab with an effective depth d of 4 in. is reinforced with ½-in. round bars spaced 4½ in. on centers, the ratio $p = \dfrac{0.1963}{4\frac{1}{2} \times 4} = 0.0110.$

The spacing of bars which is necessary to furnish a given area of steel per foot of width is obtained by dividing the number of bars required to furnish this area into 12. For example, to furnish an average area of 0.444 sq. in. per ft., with ½-in. round bars, requires 0.444/0.1963 = 2.3 bars per foot; the bars must be spaced not more than 12/2.3 = 5.2 in. center to center. The required spacing of bars could also be obtained from Table 4.

TABLE 4.—AREAS OF BARS IN SLABS, IN SQUARE INCHES PER FOOT

Spacing, in.	Size of **round** bar, in.							Size of **square** bar, in.			
	¼	⅜	½	⅝	¾	⅞	1	½	1	1⅛	1¼
3	0.20	0.44	0.78	1.23	1.77	2.40	3.14	1.00	4.00	5.06	6.25
3½	0.17	0.38	0.67	1.05	1.51	2.06	2.69	0.86	3.43	4.34	5.36
4	0.15	0.33	0.59	0.92	1.32	1.80	2.36	0.75	3.00	3.80	4.69
4½	0.13	0.29	0.52	0.82	1.18	1.60	2.09	0.67	2.67	3.37	4.17
5	0.12	0.26	0.47	0.74	1.06	1.44	1.88	0.60	2.40	3.04	3.75
5½	0.11	0.24	0.43	0.67	0.96	1.31	1.71	0.55	2.18	2.76	3.41
6	0.10	0.22	0.39	0.61	0.88	1.20	1.57	0.50	2.00	2.53	3.12
6½	....	0.20	0.36	0.57	0.82	1.11	1.45	0.46	1.85	2.34	2.89
7	....	0.19	0.34	0.53	0.76	1.03	1.35	0.43	1.71	2.17	2.68
7½	....	0.18	0.31	0.49	0.71	0.96	1.26	0.40	1.60	2.02	2.50
8	....	0.17	0.29	0.46	0.66	0.90	1.18	0.37	1.50	1.89	2.34
9	....	0.15	0.26	0.41	0.59	0.80	1.05	0.33	1.33	1.69	2.08
10	....	0.13	0.24	0.37	0.53	0.72	0.94	0.30	1.20	1.52	1.87
12	....	0.11	0.20	0.31	0.44	0.60	0.78	0.25	1.00	1.27	1.56

If the slab is of one span only and if it rests freely on its supports, the maximum positive moment M, assuming a uniform load of w lb. per sq. ft., is $M = \frac{1}{8}wl^2$. The span length l is taken as the distance center to center of supports, but it need not exceed the clear span plus the depth of the slab. If a single-span slab is built monolithically with the supporting beams, provision must be made for the negative moment which is developed at the supports by the condition of restraint there. The maximum positive moment would be less than for a corresponding freely supported slab. Positive and negative moments of $\frac{1}{10}wl^2$ may be used.

If the slab is of more than one span, built monolithically with the supporting beams or walls, both positive and negative moments exist, which should be computed by the principles of continuity. The span length l of such slabs may be taken as the clear distance between faces of supports. Consideration should be given to the relative lengths of adjoining spans, the comparative stiffness of the supports, and the relation between the magnitudes of the dead and live loads. The Joint Code recommends the following maximum moments and shears where the spans are approximately equal (actually, where the longer of two adjacent spans does not exceed the shorter by more than 20 per cent) and where the intensity of the live load does not exceed three times the intensity of the dead load:

Negative moment at face of first interior support:

For spans greater than 10 ft.

Two spans, $M = \frac{1}{8}wl^2$
More than two spans, $M = \frac{1}{10}wl^2$

For spans less than 10 ft.

Two spans, $M = \frac{1}{10}wl^2$
More than two spans, $M = \frac{1}{12}wl^2$

Negative moment at face of other interior supports:

$M = \frac{1}{12}wl^2$

Positive moment at center of span:

End spans, $M = \frac{1}{10}wl^2$
Interior spans, $M = \frac{1}{12}wl^2$

Shear in end spans at first interior support:

$$V = 1.20\frac{wl}{2}$$

Shear at other supports:

$$V = \frac{wl}{2}$$

156. Placing the Reinforcement. The insulation at the bottom should follow the recommendation of the Joint Code unless conditions warrant some change (see Art. 146). In the average slab, a depth of 1 in. below the center of the steel may be used.

The lateral spacing of bars, except those which are used only to prevent shrinkage and temperature cracks, should not exceed three times the thickness of the slab; the minimum spacing is given in Art. 146.

157. Temperature Reinforcement. Reinforcement for shrinkage and temperature stresses normal to the principal reinforcement shall be provided in slabs where the principal reinforcement extends in one direction only. The Joint Code specifies the following minimum ratios of reinforcement area to effective concrete area, but in no case shall such reinforcement bars be placed farther apart than five times the slab thickness or more than 18 in.

Floor slabs where plain bars are used	0.0025
Floor slabs where deformed bars are used	0.0020
Floor slabs where wire fabric is used, having welded intersections not farther apart in the direction of stress than 12 in.	0.0018
Roof slabs where plain bars are used	0.0030
Roof slabs where deformed bars are used	0.0025
Roof slabs where wire fabric is used, having welded intersections not farther apart in the direction of stress than 12 in.	0.0022

In general, ¼-in. round bars at 12 in. on centers or ⅜-in. round bars at 18 in. on centers will be satisfactory.

158. Design of a One-way Slab. Design a fully continuous reinforced-concrete slab, supported on two sides only, to sustain a live load of 120 lb. per sq. ft. The span of the slab is 11 ft.-0 in. A 2000-lb. concrete is to be used; $f_s = 18{,}000$ p.s.i.

Assuming a 5-in. slab and considering a 12-in. strip at right angles to the supporting beams, the maximum external bending moment on this strip, which may be considered as a rectangular beam 12 in. in width, is

$$M = \tfrac{1}{12} \times 182 \times 11^2 \times 12 = 22{,}000 \text{ in.-lb.}$$

Since $M = Kbd^2$, in which $K = 139$ (Table 2), the required effective cross-section of the imaginary beam is

$$bd^2 = \frac{22{,}000}{139} = 159.0 \text{ in.}^3$$

Since $b = 12$ in., $d = 3.6$ in. Selecting d as a multiple of ½ in., the depth to the center of the steel is made 4 in., and the total thickness 5 in. This agrees with the assumed value, and no revision is necessary.

From Table 2, $j = 0.867$. The area of steel per foot of slab width is obtained from equation (2) of Art. 148 as follows:

$$22{,}000 = A_s \times 18{,}000 \times 0.867 \times 4$$
$$A_s = 0.353 \text{ sq. in.}$$

Selecting ½-in. round bars, 0.353/0.1963 = 1.8 bars are required per foot of width. The maximum spacing is then 12/1.8 = 6.7 in. In order to simplify the construction, a spacing of 6½ in. is used throughout the slab. Since this gives a suitable arrangement, the ½-in. round bars are satisfactory. To provide for the negative moments at the supports alternate bars will be bent up at the quarter points of the span and continued across the supports into the adjacent spans to the quarter points of these spans, thus furnishing the same steel area near the top of the slab over the supports as is furnished near the bottom of the slab at the middle of the span. Temperature and shrinkage stresses in the direction perpendicular to the main reinforcement will be provided for by placing ¼-in. round bars at 12 in. on centers, or ⅜-in. round bars at 18 in. on centers, at right angles to the main reinforcement.

159. Review of a One-way Slab. Review the slab which was designed in Art. 158, to determine the maximum unit stresses in the steel and in the concrete, when the slab is loaded with its full live load of 120 lb. per sq. ft.

The maximum bending moment on a 12-in. strip of slab, as computed in Art. 158, is 22,000 in.-lb. The average steel ratio is

$$p = \frac{0.1963}{6.5 \times 4} = 0.0075$$

From Table 3, $k = 0.374$ and $j = 0.875$; and substituting in equation (2*a*), the unit stress in the steel is obtained as follows:

$$22{,}000 = 0.0075 \times f_s \times 0.875 \times 12 \times 4^2$$
$$f_s = 17{,}500 \text{ p.s.i.}$$

The unit stress in the concrete is obtained from equation (6), Art. 148.

$$f_c = \frac{17{,}500 \times 0.374}{15(1 - 0.374)} = 697 \text{ p.s.i.}$$

ADDITIONAL PROBLEMS

1. Design a fully continuous slab, supported on two sides only, to support a uniform live load of 200 lb. per sq. ft. The span of the slab is 10 ft.-0 in., and a 2500-lb. concrete is to be used, with structural-grade reinforcing steel.

2. A slab of one span, freely supported on two sides only, has a span of 11 ft.-0 in. and a total thickness of 5 in. and is reinforced with ½-in. round bars 7 in. on centers, the centers of the bars being 1 in. above the lower surface of the slab. If $f'_c = 2500$ p.s.i. and $f_s = 20{,}000$ p.s.i., what is the safe uniform live load, in pounds per square foot, that can be placed upon the slab?

160. Slabs Supported on Four Sides. When a slab is square or nearly so and there are beams at the four edges of the panel, the slab should be reinforced in two directions so as to transmit the total load to all four beams. If the panel is square and if the construction is such that the same degree of restraint exists at each edge, one-half the total load will be transmitted to each pair of beams. If the panel is longer in one direction than in the other, more than one-half the load will be transmitted in the shorter direction, and the remainder will be transmitted in the longer direction. If, however, one side of the panel is very much longer than the other, such a large proportion of the total load will be transmitted in the shorter direction (owing to the relatively large stiffness of the slab in this direction) that reinforcement parallel to the longer side would be of little practical value.

The distribution recommended in the 1928 Report[1] of the Joint Code Committee is that the part w_1 of the total load w which is transferred in the short direction (see Fig. 151) is represented by the equation

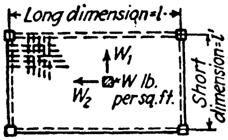


Fig. 151.

$$w_1 = \left(\frac{l}{l'} - \frac{1}{2}\right) w \qquad (10)$$

in which l = the longer dimension, in feet.
l' = the shorter dimension, in feet.

Then, if w_2 is the part of the total load that is transferred in the longer direction,

$$w_2 = w - w_1 \qquad (11)$$

Two strips, each with a width of 12 in., must be designed (or investigated), one strip parallel to the short dimension of the slab and one parallel to the long dimension. The value of d that is established for one set of bars definitely fixes the corresponding value to be used for the other set. The two sets of bars are placed one above the other, the upper resting directly on the lower. In a square slab it is customary to use the effective depth for the upper row in all computations and to place the same reinforcement in the lower row. In rectangular slabs, it will generally be found economical to place the shorter bars, which carry the larger part of the load, underneath the longer bars.

161. Design of a Two-way Slab (1928 Joint Code). A typical interior (fully continuous) floor panel is to be 9 ft.-0 in. by 10 ft.-0 in. in plan. The slab is to be reinforced in two directions and is to carry a uniform live load of 300 lb. per sq. ft. Assume $f_c = 1000$, $f_s = 18{,}000$, and $n = 12$. Design the slab.

Assume weight of slab as 50 lb. per sq. ft. (4-in. slab); then

$$w = 300 + 50 = 350 \text{ lb. per sq. ft.}$$

[1] See Art. 162.

For a 12-in. strip in the short direction,

$$w_1 = (10/9 - 1/2)350 = 215 \text{ lb. per ft.}$$
$$M = 1/12 \times 215 \times 9^2 \times 12 = 17{,}500 \text{ in.-lb.}$$

From Table 2, $k = 0.400$ and $j = 0.867$.

$$17{,}500 = 1/2 \times 1000 \times 0.40 \times 0.867 \times 12 \times d^2$$
$$d \text{ (required)} = 2.9 \text{ in.}$$

Use $d = 3$ in.; with 1-in. insulation, total thickness = 4 in., as assumed.

For a 12-in. strip in the long direction,

$$w_2 = 350 - 215 = 135 \text{ lb. per ft.}$$
$$M = 1/12 \times 135 \times 10^2 \times 12 = 13{,}500 \text{ in.-lb.}$$
$$13{,}500 = 1/2 \times 1000 \times 0.400 \times 0.867 \times 12 \times d^2$$
$$d \text{ (required)} = 2.55 \text{ in.}$$

With a 4-in. slab the d furnished for the long bars (assumed as ½ in. in diameter) is only $4 - 1 - 1/2 = 2.50$ in. Though sufficiently close for practical purposes, theoretically the slab should be made thicker, to furnish at least 2.55 in. of effective depth for the long (upper) bars. A 4¼-in. slab will be used. Then, for the short bars, $d = 3.25$ in., and for the long bars, $d = 2.75$ in.

Short bars:

$$17{,}500 = A_s \times 18{,}000 \times 0.867 \times 3.25$$
$$A_s = 0.345 \text{ sq. in.}$$

From Table 4, ½-in. round bars 6½ in. center to center are satisfactory for the short direction.

Long bars:

$$13{,}500 = A_s \times 18{,}000 \times 0.867 \times 2.75$$
$$A_s = 0.314 \text{ sq. in.}$$

From Table 4, ½-in. round bars 7½ in. center to center are satisfactory for the long direction.

161a. Review of a Two-way Slab (1928 Joint Code). A typical interior floor panel (fully continuous), 10 ft.-0 in. by 12 ft.-0 in., is reinforced in two directions with ½-in. square bars 8 in. center to center, the center of the lower row of bars being placed 1 in. above the lower surface of the slab. The total thickness of the slab is 5 in. Allowable unit stresses are $f_s = 20{,}000$, $f_c = 800$, and $n = 15$. What live load per square foot will the panel sustain?

Investigation of Short Direction. The bars in the short direction are placed beneath the others, and $d = 4$ in.

$$p = \frac{0.25}{8 \times 4} = 0.0078$$

Comparing this with the steel ratio required for balanced design in Table 2 (0.0077), it is seen that the slab is slightly overreinforced in the short direction. From Table 3, $k = 0.380$ and $j = 0.873$; hence, for a 12-in. strip,

$$M_c = \tfrac{1}{2} \times 800 \times 0.380 \times 0.873 \times 12 \times 4^2 = 25{,}500 \text{ in.-lb.}$$
$$25{,}500 = \tfrac{1}{12} \times w_1 \times 10^2 \times 12$$
$$w_1 = 255 \text{ lb. per ft.}$$
$$255 = (\tfrac{12}{10} - \tfrac{1}{2})w = 0.7w$$
$$w = 364 \text{ lb. per sq. ft.}$$

Investigation of Long Direction. The effective depth of the long bars is 3½ in.

$$p = \frac{0.25}{8 \times 3.5} = 0.0089$$

Table 2 shows that the slab is overreinforced in the long direction, and Table 3 gives $k = 0.400$ and $j = 0.866$; hence, for a 12-in. strip,

$$M_c = \tfrac{1}{2} \times 800 \times 0.400 \times 0.866 \times 12 \times 3.5^2 = 20{,}400 \text{ in.-lb.}$$
$$20{,}400 = \tfrac{1}{12} \times w_2 \times 12^2 \times 12$$
$$w_2 = 142 \text{ lb. per ft.}$$
$$142 = 0.3w$$
$$w = 473 \text{ lb. per sq. ft.}$$

From the two investigations above, it is seen that a total load of 473 lb. per sq. ft. could be placed on the slab without overstressing it in the long direction. This load, however, would considerably overstress the slab in the short direction. Hence, the maximum load per square foot that can be placed on the slab is governed by the strength of the short direction and equals 364 lb. as computed above. Since the slab weighs 62 lb. per sq. ft., the safe live load is 364 − 62 = 302 lb. per sq. ft.

162. Analysis of Two-way Slabs According to 1940 Joint Code. The method of load distribution described in Art. 160 is no longer considered a generally accepted means of analyzing slabs supported on four sides. However, the design is easily made, the resulting details are known to be on the safe side, and, if only a small floor area is involved, the added cost of the construction is not prohibitive. The use of the method may therefore be justified in some cases, and the problems in Arts. 161 and 161*a* are given to illustrate the fundamental principles involved.

A reasonably simple rational method of two-way slab analysis has been proposed in the Joint Committee Report of June, 1940, which takes into consideration the effect of discontinuity at one or more edges of the panel. In this method, the slab is considered to be built monolithically with the supporting walls or beams, so that a negative moment is developed on all exterior edges.

The panel is divided into middle strips and outer strips. If the panel is nearly square, the middle strip has a width of one-half panel, and each outer strip has a width of one-quarter panel. In panels where the ratio of short span L_S to long span L_L is less than 0.5, the width of the middle strip that extends in the short direction is equal to the difference between the long and short spans, the remaining area being divided equally between the two outer strips.

The span lengths are taken as the distance between the centers of supports or as the clear span plus twice the slab thickness, whichever value is the smaller. The critical sections for bending moment are along the center lines of the panel for positive moment and along the faces of the supporting beams or walls for negative moment.

The bending-moment coefficients are shown in the code for a strip of slab 1 ft. wide and are in terms of wL_S^2, where w is the uniform load per square foot on the panel and L_S is the short-span length. Moments, whether for strips in the long or short direction, are computed in terms of the short-span length. The recommended coefficients are based partly on analysis and partly on test data. These coefficients vary, according to whether all edges of the panels are continuous, or whether one or more edges are discontinuous.

The loads on the supporting beams are assumed to be uniformly distributed throughout the spans of the beams, but the amount of load supported by each beam is considered to be all the load on an area bounded by the intersection of a 45-degree line from the corners of the panel with the center line of the panel parallel to the long side. The moments and shears in these beams are then computed in the usual way.

A complete discussion of this method, together with the recommended coefficients and numerical applications, is given in the authors' "Design of Concrete Structures."

DIAGONAL TENSION, SHEAR, AND BOND

163. Shear and Diagonal Tension. The preceding articles contain an outline of the methods of calculating the maximum fiber stresses in the concrete and steel of a reinforced-concrete beam or slab and of so proportioning the amounts of steel and concrete that the working strength of any part of the beam in flexure is not exceeded.

There are other internal stresses existing in a concrete beam which, if not properly cared for, may in themselves cause failure of the beam. These stresses are: (1) shearing stresses, or those

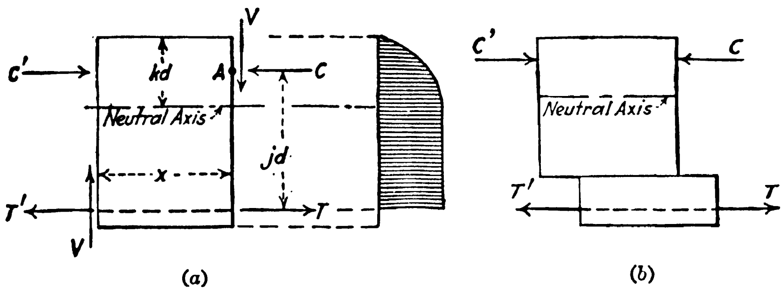


Fig. 152.

tending to make one plane of concrete, either vertical or horizontal, slide along an adjacent plane; (2) diagonal-tension stresses, or those which cause cracks in the concrete along inclined planes near the points of maximum shear; and (3) in reinforced beams, bond stresses, or those tending to cause the steel to pull away from the concrete when under stress and thus destroy the unity of the beam.

The formula for the maximum unit longitudinal shear is derived as follows: Let Fig. 152*a* represent a short piece of a reinforced-concrete beam, with the internal stresses of compression, tension, and vertical shear acting on both the cut faces. As explained in Art. 57, if this piece is taken from the left portion of the beam, the forces T and C are greater than T' and C'; hence, there is a tendency for the portion below any horizontal plane between the steel and the neutral axis to move toward the right as shown in *b*. This movement is resisted by the longitudinal shearing stresses acting on the plane. The unit shear is

$$v = \frac{T - T'}{bx} \qquad (a)$$

in which bx = the area of the plane under consideration. Taking moments about point A, following the same procedure as for a homogeneous beam (Art. 57), $(T - T')jd = Vx$, from which

$$T - T' = \frac{Vx}{jd} \qquad (b)$$

Substituting in equation (a),

$$v = \frac{V}{bjd} \qquad (12)$$

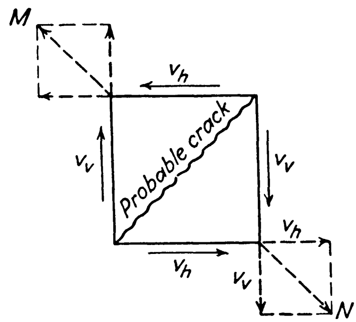


FIG. 153.

In all computations involving shearing stresses, the value of j may be taken as $\frac{7}{8}$ inasmuch as a slight variation in j will not materially affect the ultimate result. Above the neutral axis, the unit shear is less than the value given in equation (12), varying to zero in a parabolic curve, as shown in the shaded area in Fig. 152*a*. In any beam the absolute maximum unit shear occurs at the section where the total shear V is a maximum.

Diagonal tension stresses are inclined tension stresses which are caused by a combination of shearing stresses and tension due to bending. Let Fig. 153 represent an infinitely small cube taken from a beam at any section along the neutral axis. Two pairs of shearing forces, horizontal (v_h) and vertical (v_v), of equal intensity, are acting on the cube as shown. When these shearing forces are combined in pairs as shown in Fig. 153, tension stresses are produced in the direction MN, and compression stresses at right angles. The inclined tension stresses tend to produce a crack across the cube as shown, because of the low strength of concrete in tension. For a cube below the neutral axis, the forces M and N would be further affected by the tension fiber stresses in the concrete. An exact solution for the maximum value of the inclined tension stress is not possible, but it can be seen that the inclined stress varies with the shearing stress. By limiting the shearing stress to a value which has been found by actual tests to be low enough to insure against failure by diagonal tension, it may be considered that the danger of such failure has been eliminated. The crack which is shown in Fig. 153 can be prevented by placing vertical or inclined bars in the beam in the region where the crack is expected. Hooking the main tensile reinforcement will also tend to prevent the formation of inclined cracks. Bars which are placed in the web of the beam are called web reinforcement and are discussed in the following article.

Where no web reinforcement is used in the beam the allowable unit shear, as computed from equation (12), is $0.02f'_c$ if the longitudinal bars are not hooked and $0.03f'_c$ if the bars are hooked at each end. In all portions of the beam where the unit shear is greater than either of these values, web reinforcement must be used. If adequate web reinforcement is used, the allowable unit shear is generally taken as $0.06f'_c$, although a higher value (up to $0.12f'_c$) can be used if certain precautions are taken in placing and anchoring the reinforcement. In general, then, the beam must be designed so that the unit shear v is not greater than $0.06f'_c$, even though it is intended that web reinforcement be provided where necessary.

Equation (12) can be used: (1) to determine the maximum unit shearing stress in a given beam, which, when compared with the allowable stress, will show whether web reinforcement is necessary or whether even with adequate web reinforcement the given beam is or is not safe for the loads which it is expected to carry; (2) to determine the cross-section bd required to support the given loads without exceeding the allowable shearing stress.

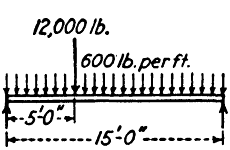


FIG. 154.

Example. A rectangular beam with an overall cross section of 12 × 22 in. supports live loads as shown in Fig. 154. The beam is reinforced with four ¾-in. round bars in one row, the center of which is 2 in. above the lower surface of the beam. A 2000-lb. concrete is assumed. Determine the maximum unit shearing stresses in sections at the left and right supports, and in sections just to the left and right of the concentrated load.

The dead weight of the beam is 275 lb. per lin. ft.; hence, the total uniform load is 875 lb. per ft. At the left support,

$$V = (\tfrac{2}{3} \times 12{,}000) + (\tfrac{15}{2} \times 875) = 14{,}560 \text{ lb.}$$

$$v = \frac{14{,}560}{12 \times \tfrac{7}{8} \times 20} = 70 \text{ p.s.i.}$$

At the right support,

$$V = (\tfrac{1}{3} \times 12{,}000) + (\tfrac{15}{2} \times 875) = 10{,}560 \text{ lb.}$$

$$v = \frac{10{,}560}{12 \times \tfrac{7}{8} \times 20} = 50 \text{ p.s.i.}$$

Just to the left of the concentrated load,

$$V = 14{,}560 - (5 \times 875) = 10{,}180 \text{ lb.}$$

$$v = \frac{10{,}180}{12 \times \tfrac{7}{8} \times 20} = 49 \text{ p.s.i.}$$

Just to the right of the concentrated load,

$$V = 14{,}560 - (5 \times 875) - 12{,}000 = -1820 \text{ lb.}$$

$$v = \frac{1820}{12 \times \tfrac{7}{8} \times 20} = 9 \text{ p.s.i.}$$

164. Web Reinforcement. The regions over which web reinforcement is required are determined by computing the unit shearing stress at critical points, drawing the unit shear diagram, and noting the parts of the beam where the unit shearing stress is greater than $0.02f'_c$ (if the longitudinal bars are not hooked) or $0.03f'_c$ (if the longitudinal bars are hooked). Separate vertical (or, more rarely, inclined) bars or stirrups may be used for web reinforcement, or some of the longitudinal bars may be bent up where they are no longer required for the bending stresses, so as to cross the planes of the probable inclined cracks and thus prevent the formation of these cracks, as shown in Fig. 155. Stirrup bars are usually bent in the form of the letter U and placed as shown in Fig. 147*a*. The effective area A_v of the stirrup is then twice the area of the bar. For very wide beams, more legs than two might be used, in which case the area A_v is equal to the number of vertical legs times the area of the bar.

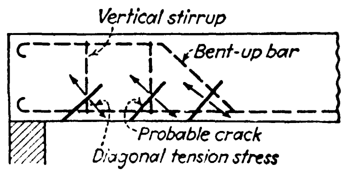


FIG. 155.

Example. For the beam in the example at the end of Art. 163, determine the regions over which web reinforcement is required.

If the longitudinal bars are not hooked, the allowable unit shear that can be resisted by the concrete is $0.02 \times 2000 = 40$ p.s.i. A study of the unit-shear computations in Art. 163 shows that web reinforcement is required from the left support to the concentrated load and from the right support for a distance x, to the section at which the unit shear is 40 p.s.i. This distance is obtained as follows: The total shear V at a distance x ft. from the right support is equal to $10{,}560 - 875x$; hence, from equation (12),

$$40 = \frac{10{,}560 - 875x}{12 \times \frac{7}{8} \times 20}$$
$$x = 2.47 \text{ ft.}$$

165. Spacing and Size of Vertical Stirrups. The unit shearing stress (which is a measure of the diagonal tension), from equation (12), is $v = V/bjd$. If vertical stirrups are spaced s in. apart, the total shearing stress on any horizontal plane between the longitudinal steel and the neutral axis over the distance s is then

$$\frac{V}{bjd} \times bs = \frac{Vs}{jd}$$

Since the unit horizontal and vertical shearing stresses at any point in a beam are equal to each other and since the vertical shearing stress is equal to the vertical component of the diagonal tension (see Fig. 153), the quantity Vs/jd is the amount of stress that normally would be resisted by each stirrup. However, tests show that the concrete resists a part of this shearing force, and in designing stirrups the amount of shear V_c that can safely be resisted by the concrete is deducted from the total shear V in the foregoing expression. Hence, the stress in one stirrup is $\frac{(V - V_c)s}{jd}$. The strength of one stirrup is $A_v f_v$, in which f_v is the allowable unit stress in the stirrup (Art. 43) and A_v is the effective area of the stirrup (Art. 164). The maximum allowable spacing is then obtained by equating stress to strength, or

$$A_v f_v = \frac{(V - V_c)s}{jd}$$

from which

$$s = \frac{A_v f_v jd}{V - V_c} \qquad (13)$$

The quantity V_c is obtained from equation (12) substituting for v a value of $0.02f'_c$ or $0.03f'_c$, as recommended in Art. 163; designating this specific value of v as v_c, the formula for V_c is then

$$V_c = v_c bjd \tag{14}$$

An arbitrary maximum value of s is specified to avoid having the distance between stirrups so great that cracks may form between them. The arbitrary maximum is generally taken as one-half the effective depth of the beam. Stirrup bars are hooked at the free ends (see Fig. 147) in order to prevent the bars from pulling out of the concrete.

Example. Determine the required arrangement of ¼-in. round U-stirrups for the beam in the example at the end of Art. 163, assuming $f_v = 16{,}000$ p.s.i. The maximum spacing of stirrups at the left support, from Eq. (13), is

$$s = \frac{2 \times 0.0491 \times 16{,}000 \times \frac{7}{8} \times 20}{14{,}560 - (40 \times 12 \times \frac{7}{8} \times 20)} = 4.4 \text{ in.}$$

Just at the left of the concentrated load, the total shear is 10,180 lb., and

$$s = \frac{2 \times 0.0491 \times 16{,}000 \times \frac{7}{8} \times 20}{10{,}180 - (40 \times 12 \times \frac{7}{8} \times 20)} = 15.5 \text{ in.}$$

Remembering that the spacing of vertical stirrups should not exceed $0.5d$, or 10 in. in this case, by simple proportion the following selection of spacings may be made for the region between the left support and the concentrated load, placing the first stirrup about 2 in. from the edge of the support: three at 4 in., two at 6 in., and four at 10 in.

At the right support the total shear is 10,560 lb., and $s = 12.7$ in.; four stirrups will be used at this end, spaced 10 in. on centers, which will bring the web reinforcement beyond the section where such reinforcement is no longer required, as determined in the example at the end of Art. 164.

166. Spacing of Inclined Bars. Inclined bars are more effective as web reinforcement than vertical stirrups, because they cross more of the probable planes of rupture for a given length of bar. Where bars are bent at an angle of 45 degrees with the horizontal, the maximum spacing is obtained by inserting in the denominator of equation (13) a factor equal to the cosine of 45 degrees (approximately 0.7); hence,

$$s = \frac{A_v f_v jd}{0.7(V - V_c)} \tag{15}$$

The arbitrary maximum horizontal distance over which such bent-up bars can take diagonal tension is generally specified as the effective depth of the beam, measured from the point at which the bars are bent. For the sizes of bars which are usually used for longitudinal reinforcement, the spacing obtained from equation (15) is almost always much greater than the arbitrary maximum, and the latter distance governs in most cases. The points at which the bars may be bent up, usually in pairs, are found by drawing the moment diagram for the beam and noting the point or points at which the bending moment is equal to the proper percentage of the maximum. For example, if there are 8 bars in a given beam and 2 are bent up in one place, 6 bars remain in the bottom; and two bars may be bent up where the moment is equal to $6/8$ of the maximum. If 2 more bars are bent at another place, 4 bars then remain in the bottom, and this second pair of bars may be bent at the section where the moment is equal to $4/8$ of the maximum. All the bars in a beam cannot be bent up. Enough bars must remain in the bottom at the support to furnish adequate bond resistance, as explained in Art. 167.

167. Bond Stresses. Referring to Fig. 152, since T is greater than T' there is a tendency for the bars to pull out of the concrete, moving toward the right. This tendency is resisted by the bond between the concrete and the surface of the bars. In Art. 163 it was shown that $T - T' = Vx/jd$. But, if the total perimeter of the bars is Σ_0, the surface area in length x is $\Sigma_0 x$; hence, the bond stress u on a unit area is equal to $T - T'$ divided by $\Sigma_0 x$, or

$$u = \frac{T - T'}{\Sigma_0 x} = \frac{V}{\Sigma_0 jd} \tag{16}$$

In all bond-stress computations, the value of j may be taken as $7/8$, for the reason given in Art. 163.

The maximum bond stress occurs at the section where the total shear V is a maximum, *i.e.*, at one of the supports. In order to prevent pulling out of the bars, the value of u should not exceed a safe working limit which has been established from a study of tests on beams in which such failure has occurred. The Joint Code specifies a maximum value of $u = 0.04f'_c$ for plain bars and $0.05f'_c$ for deformed bars. If the bars are hooked at the ends, the foregoing values for the allowable unit bond stress may be increased 50 per cent.

Example. Determine the maximum unit bond stress on the bars in the beam shown in Fig. 154, details of which are given in the problem at the end of Art. 165. The maximum value of u occurs at the left support, since the shear is a maximum at this point. Assuming that none of the bars are bent up (note that vertical stirrups were used as web reinforcement in Art. 165) the maximum value of u, from equation (16), is

$$u = \frac{14,560}{4 \times 2.35 \times \frac{7}{8} \times 20} = 89 \text{ p.s.i.}$$

168. Embedment of Bars. The length of embedment, l_1, required to furnish sufficient bond resistance to enable the full tensile strength of a bar to be developed is obtained by equating the strength of the bar, A_sf_s, to the bond resistance, which is equal to the product of the contact area, $l_1\Sigma_0$, and the allowable bond stress u. Thus,

$$A_sf_s = l_1\Sigma_0 u$$

If the diameter of a round bar is i,

$$\frac{\pi i^2 f_s}{4} = l_1 \pi i u$$

from which

$$l_1 = \frac{f_s}{4u} i \qquad (17)$$

For square bars the same formula results, in which i is the side dimension of the bar.

T-BEAMS

169. Types of T-beams. When a reinforced-concrete floor slab is constructed as a monolith with the supporting beam, part of the slab may be assumed to assist the upper part of the beam in resisting compressive stresses. These two acting together constitute what is known as a T-beam (Fig. 156). The slab is called the *flange*, and the portion of the beam below the slab is called the *web* or *stem.* The exact width of slab that can be assumed effective in resisting compressive stresses depends upon the thickness of the slab, the span of the beam, and the spacing of beams. This effective width b is determined as follows:

a. It shall not exceed one-fourth the span length of the beam, except that, for beams with a flange on one side only, it shall not exceed one-tenth the span.

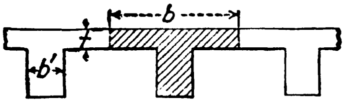


FIG. 156.

b. Its overhanging width on either side of the stem shall not exceed eight times the thickness of the slab.

c. It shall not exceed the distance center to center of beams.

Another form of T-beam, which is of infrequent occurrence, is one which does not form a part of a floor system, the flange being provided merely to furnish sufficient area in compression. Since the concrete in the lower part of the beam is assumed as taking no tension, its only purpose is to bind the tensile steel and the compressive concrete together. This involves mainly shearing stresses; all the rectangular section is not required in large beams, and so a saving in concrete results when the T-form is used. It is, however, usually more satisfactory to use a rectangular beam with compressive reinforcement to care for cases requiring an excessive amount of concrete rather than to resort to the T-section. A saving in cost of forms and certain evident structural advantages of the rectangular beam will, in general, counteract the saving in concrete in the T-beam. All the following discussions will be based only on T-beams which are part of a floor system.

170. Shearing Stresses. Owing to the relatively large width of flange, it is safe to say that the compressive strength of a T-beam which is part of a floor system will seldom (if ever) govern the design. The effective cross-section $b'd$ is determined from the shearing requirement, [equation (12)], with $v = 0.06f'_c$, assuming always that web reinforcement will be provided as necessary and in accordance with the methods outlined in Art. 164. In all computations involving shearing strength of

T-beams, the width of the stem b' is used, instead of the width of flange b, because of the fact that the unit shear involved is the unit shear along any horizontal plane between the tension steel and the neutral axis.

171. Bending Stresses. Formulas for the resisting moments and for k and j are derived in a manner similar to that used for similar functions of rectangular beams. The small amount of compression in the stem (between the neutral axis and the bottom

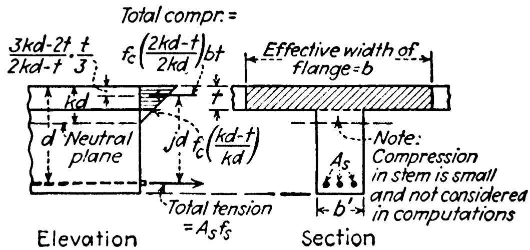


FIG. 157.

of the slab) is disregarded, so that the total compression is considered to be a trapezoid with bases equal to f_c and $f_c\,\dfrac{(kd - t)}{kd}$, an altitude equal to t, and a length equal to the effective width of the flange b as shown in Fig. 157. Approximate resisting-moment formulas for use in design are obtained by assuming that $jd = d - \frac{1}{2}t$ and that the average compression stress in the trapezoid of depth t is $\frac{1}{2}f_c$. The essential equations for T-beam design and review are tabulated below:

$$p = \frac{A_s}{bd} \tag{18}$$

$$M_s = A_sf_sjd \tag{19}$$

$$M_s = A_sf_s(d - \tfrac{1}{2}t) \text{ (approximate, for design only)} \tag{20}$$

$$M_c = f_c\left(1 - \frac{t}{2kd}\right) bt \cdot jd \tag{21}$$

$$M_c = \tfrac{1}{2}f_cbt(d - \tfrac{1}{2}t) \text{ (approximate, for design only)} \tag{22}$$

$$b'd = \frac{V}{vj} \tag{23}$$

$$f_c = \frac{f_sk}{n(1 - k)} \tag{24}$$

$$k = \frac{np + \frac{1}{2}(t/d)^2}{np + (t/d)} \tag{25}$$

$$j = \frac{6 - 6(t/d) + 2(t/d)^2 + (t/d)^3(\frac{1}{2}pn)}{6 - 3(t/d)} \tag{26}$$

$$b = \text{not to exceed } \tfrac{1}{4} \text{ span or } 16t + b' \tag{27}$$

172. Method of Design. Determine $b'd$ [equation (23)], and select proper values for each (b is usually $\frac{1}{3}$ to $\frac{1}{2}d$); A_s [equation (20)]. If a more exact design is required, review the beam (see following paragraph) to determine f_s and f_c. If f_c is too great, the beam must be deepened; if f_s is too small, less steel may be used and the review repeated. In rare cases only will the cross-section be governed by the bending stress in the concrete.

173. Method of Review. Determine k [equation (25)], j [equation (26)], M_c or f_c [equation (21)], M_s or f_s [equation (19)]. In all computations $p = A_s/bd$ [equation (18)], in which b is the width of the flange [equation (27)]. After the resisting moment has been determined, the safe live load may be computed as in Art. 153.

174. Effect of Position of Neutral Axis. If the slab is relatively thick, it is possible that the neutral axis will be above the

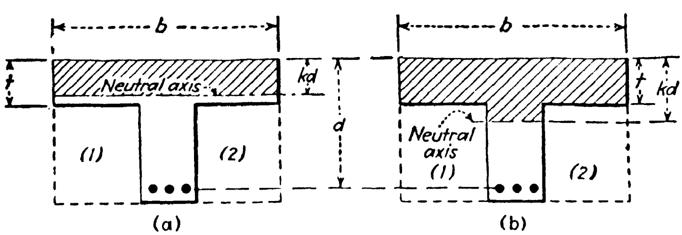


FIG. 158.

bottom of the slab, as in Fig. 158(a). In this event, the values of k and j and the resisting moments should be obtained from the rectangular-beam equations (1) to (9), using the flange width b in all computations. The reason for this is obvious from a study of Fig. 158(a): If the additional concrete necessary to make a rectangular beam of width b and effective depth d [areas (1) and (2)] had been added when the beam was poured, the resulting rectangular beam would have the same strength in bending as the actual T-beam, since the added areas (1) and (2) are all in the tension portion of the cross-section. In Fig.

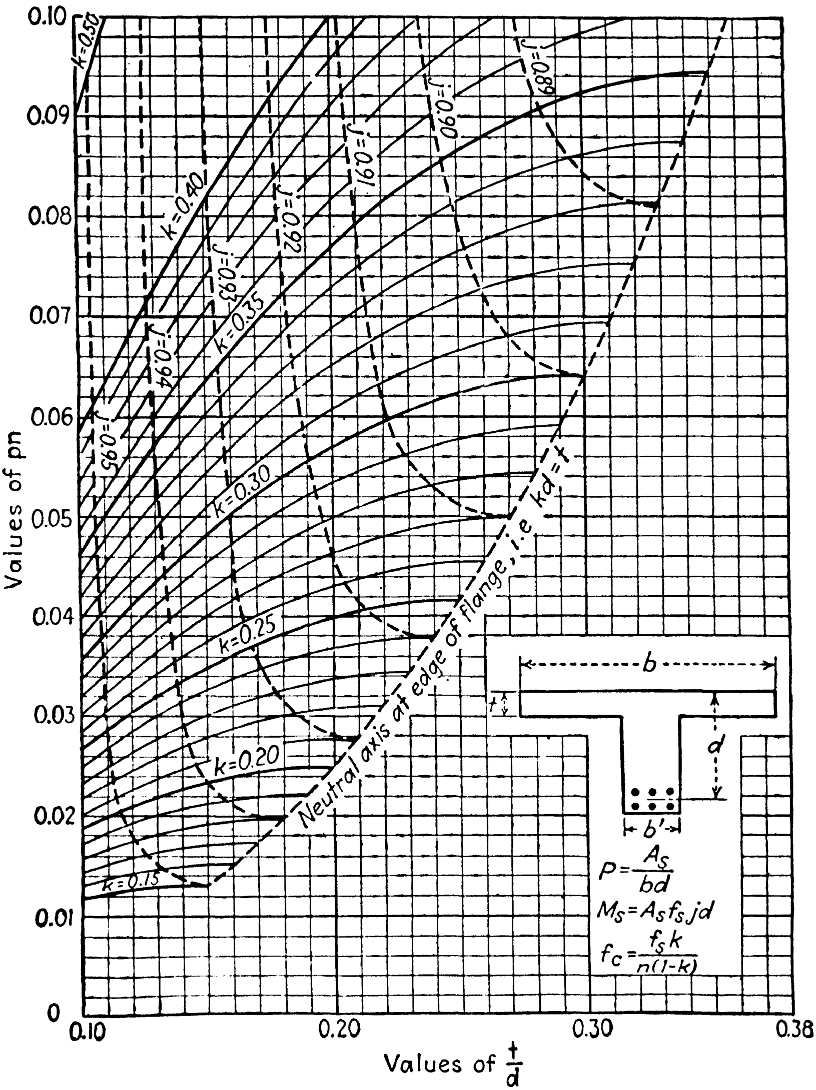


FIG. 159.—T-beam review.

158(b), however, the neutral axis is assumed to be below the bottom of the flange; hence, if areas (1) and (2) had been added, the resulting rectangular beam would have been stronger than the actual T-beam, because part of the added areas is above the neutral axis. In the latter case, the T-beam equations (18) to (27) must be used.

175. Diagrams for k and j. Values of k and j as given in equations (25) and (26) may be obtained from Fig. 159. If the intersection of the entering factors pn and t/d falls outside of the limits of the curves for k, it is an indication that the neutral axis is above the bottom of the slab, and the rectangular beam equations (8) and (9) should be used.

176. Design of a T-beam. A series of parallel continuous floor beams are spaced 11 ft.-0 in. on centers, supporting a monolithic 5-in. slab which

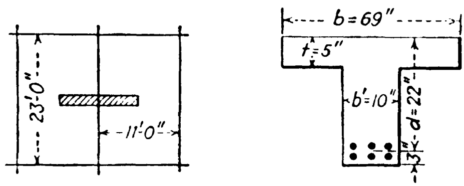


FIG. 160.

sustains a live load of 100 lb. per sq. ft. The span of the beams is 23 ft.-0 in. Design one of the beams, assuming $f_s = 20{,}000$ and $f'_c = 2000$.

Weight of slab = 62 lb. per sq. ft. Load on beam from slab = 11(100 + 62) = 1780 lb. per ft. Assume weight of stem = 200 lb. per ft.; then the total load on the beam is 1780 + 200 = 1980 lb. per ft.

$$M = \tfrac{1}{12} \times 1980 \times 23^2 \times 12 = 1{,}045{,}000 \text{ in.-lb.}$$

$$V = 1980 \times \tfrac{23}{2} = 22{,}700 \text{ lb.}$$

$$b'd = \frac{22{,}700}{\tfrac{7}{8} \times 120} = 215 \text{ sq. in.}$$

Select $b' = 10$ in. and $d = 22$ in. Allowing for two rows of bars, the total height of the beam must be 22 + 3 = 25 in. (see Fig. 160). The cross-section below the slab is 10 by 20 in., and the weight of the stem is 210 lb. per ft., approximately as assumed.

$$A_s = \frac{1{,}045{,}000}{20{,}000 \times (22 - 2.5)} = 2.68 \text{ sq. in.}$$

Six ¾-in. round bars (area 2.65 sq. in.) will be used. The required size and spacing of web reinforcement bars are obtained as explained in Arts. 164 and 165.

177. Review of a T-beam. Determine the unit stresses in the extreme compression fiber of the concrete and in the tension steel of the beam which was designed in the preceding article (Fig. 160), when the floor is fully loaded with the specified live load of 100 lb. per sq. ft.

The flange width b is limited in this case to $\frac{1}{4}$ span or $\frac{1}{4}(23 \times 12) = 69$ in.

$$p = \frac{2.65}{69 \times 22} = 0.0017, \qquad \frac{t}{d} = \frac{5}{22} = 0.227$$

Figure 159 shows that the neutral axis is in the flange, and so the values of $j = 0.934$ and $k = 0.199$ are taken from Table 3, page 227, or computed from equations (8) and (9). Then, from equations (2) and (6),

$$1{,}045{,}000 = 2.65 \times f_s \times 0.934 \times 22$$
$$f_s = 19{,}200 \text{ p.s.i.}$$
$$f_c = \frac{19{,}200 \times 0.199}{15(1 - 0.199)} = 318 \text{ p.s.i.}$$

BEAMS REINFORCED FOR TENSION AND COMPRESSION

178. Formulas for Design. Quite frequently a beam is limited by certain structural conditions or architectural limitations to a size which is not adequate to furnish the required compression resistance. Additional compression resistance is furnished in such cases by placing reinforcing bars near the extreme compression surface of the beam, of sufficient area to provide for the moment in excess of the carrying capacity of the concrete. Formulas for the design of a beam reinforced for tension and compression are derived by determining first the amount of moment M_1 which the given concrete area can develop in compression [equations (1), (4), and (9)], and computing the area of tension steel A_{s_1} required to develop this moment [equation (2)].

Thus,

$$M_1 = \tfrac{1}{2} f_c kjbd^2 \tag{28}$$

and

$$A_{s_1} = \frac{M_1}{f_s jd} \tag{29}$$

If the total moment M in the beam is greater than M_1, the amount M_2 still to be taken care of by the compression steel A'_s and additional tension steel A_{s_2} is $M - M_1$. If d' is the distance from the compression surface of the beam to the center of the compression steel, the lever arm of this additional stress couple (Fig. 161) is $d - d'$. Hence,

$$M_2 = A_{s_2} f_s (d - d') \tag{30}$$

from which

$$A_{s_2} = \frac{M_2}{f_s(d - d')} \tag{31}$$

The total tension steel area required to develop the moment M is, then,

$$A_s = A_{s_1} + A_{s_2} \tag{32}$$

The area of compression steel required is determined from the assumptions that, for equilibrium, the additional tension resist-

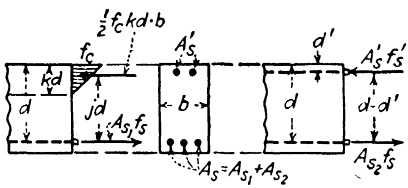


FIG. 161.

ance must be equal to the added compression resistance and that unit stresses vary as the distance from the neutral axis. Thus, if f'_s is the unit stress in the compression steel,

$$A'_s f'_s = A_{s_2} f_s$$

$$\frac{f_s}{f'_s} = \frac{d - kd}{kd - d'}$$

Hence,

$$A'_s = A_{s_2} \times \frac{1 - k}{k - (d'/d)} \tag{33}$$

Vertical ties (¼ in. minimum, spaced 12 in. maximum) are usually specified, primarily to stiffen the compression bars.

179. Design of a Doubly Reinforced Beam. A simply supported rectangular beam with a span of 20 ft.-0 in. is limited in cross-section to 8 by 18 in. The beam supports a live load of 580 lb. per lin. ft. over its entire length. Using 2 in. of insulation and assuming $f_s = 18{,}000$, $f_c = 1000$, and $n = 12$, determine the area of steel required for tension A_s and for compression A'_s.

The weight of the beam is 150 lb. per ft.; hence,

$$M = \tfrac{1}{8}(580 + 150) \times 20^2 \times 12 = 438{,}000 \text{ in.-lb.}$$

From equations (4) and (9) or Table 2, $k = 0.400$ and $j = 0.867$.

$$M_1 = \tfrac{1}{2} \times 1000 \times 0.400 \times 0.867 \times 8 \times 16^2 = 355{,}000 \text{ in.-lb.}$$
$$M_2 = 438{,}000 - 355{,}000 = 83{,}000 \text{ in.-lb.}$$
$$A_{s_1} = \frac{355{,}000}{18{,}000 \times 0.867 \times 16} = 1.42 \text{ sq. in.}$$
$$A_{s_2} = \frac{83{,}000}{18{,}000(16 - 2)} = 0.33 \text{ sq. in.}$$
$$A_s = 1.42 + 0.33 = 1.75 \text{ sq. in.}$$
$$A'_s = 0.33 \times \frac{1 - 0.44}{0.400 - \frac{2}{16}} = 0.72 \text{ sq. in.}$$

Four ¾-in. round bars in tension and two ¾-in. round bars in compression are selected; each set is placed in one row, the center of which is 2 in. from the bottom and top surfaces of the beam, respectively.

180. Formulas for Review. The equations for k, j, and the resisting moments to be used in reviewing a doubly reinforced rectangular beam are derived in a manner similar to that used in deriving the corresponding equations for rectangular beams with tensile reinforcement only. The essential difference in the derivation is that here two separate compressive resistances must be considered, one contributed by and located in the plane of the compression steel ($C' = A'_s f'_s$) and the other furnished by the concrete in the compression area ($C = \tfrac{1}{2} f_c kdb$), as shown in Fig. 162. The total compressive resistance $C + C'$ is assumed to be concentrated at the center of gravity of these two forces, the lever arm jd being measured from this point to the center of the tension steel. The resulting equations are as follows:

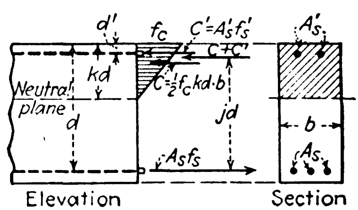


FIG. 162.

$$p = \frac{A_s}{bd} \tag{34}$$

$$p' = \frac{A'_s}{bd} \tag{35}$$

$$M_s = A_s f_s jd \tag{36}$$

$$M_c = \frac{1}{2} f_c k \left(1 - \frac{k}{3}\right) bd^2 + f'_s p' bd(d - d') \tag{37}$$

$$f_c = \frac{f_s k}{n(1 - k)} \tag{38}$$

$$f'_s = \frac{f_s \left(k - \frac{d'}{d}\right)}{1 - k} \tag{39}$$

$$k = \sqrt{2n\left(p + p' \cdot \frac{d'}{d}\right) + n^2(p + p')^2} - n(p + p') \tag{40}$$

$$j = \frac{k^2 - \frac{1}{3}k^3 + 2p'n\left(k - \frac{d'}{d}\right)\left(1 - \frac{d'}{d}\right)}{k^2 + 2p'n\left(k - \frac{d'}{d}\right)} \tag{41}$$

If the unit stresses for a given loading are required, determine k [equation (40)], j [equation (41)], f_s [equation (36)], f'_s [equation (39)], f_c [equation (38)]. If the resisting moment of a given beam is required, determine k [equation (40)], j [equation (41)], f_c [equation (38)], (substituting f_s equal to its allowable value); if f_c is less than its allowable value, the resisting moment of the beam is given by equation (36), substituting f_s equal to its limiting value; if f_c, as determined above, is greater than the allowable value, determine f_s [equation (38)], substituting for f_c its allowable value; then use this value of f_s in equation 36 to determine the resisting moment of the beam. The two resisting moments involved could be obtained directly from equations (36) and (37), and the smaller value selected. However, the foregoing analysis obviates the necessity of solving the cumbersome equation for M_c [equation (37)] and yet gives the identical ultimate information.

181. Diagrams for k and j for Use in Review Problems. Diagrams can be drawn from which the values of k and j as given in equations (40) and (41) for various values of pn and $p'n$ may be obtained. Such a diagram is shown in Fig. 163, for beams in which $d'/d = 0.1$ and in Fig. 164 for beams in which $d'/d = 0.15$. For other values of d'/d, interpolate or extrapolate from Figs. 163 and 164. Diagrams for other values of d'/d can be found in the authors' "Design of Concrete Structures."

182. Review of a Doubly Reinforced Beam. A rectangular beam with a cross-section of 12 by 30½ in. is reinforced as follows: for tension, eight ⅞-in. round bars in two rows, 2 in. center to center, the center of the lower row being 2½ in. above the lower surface of the beam, and for compression four ⅞-in. round bars in one row, the center of which is 2 in. below the upper surface of the beam. Assuming $f_s = 18{,}000$, $f_c = 800$, and $n = 15$, what is the safe resisting moment of the beam?

$$\frac{d'}{d} = \frac{2}{27} = 0.07$$

$$p = \frac{8 \times 0.6013}{12 \times 27} = 0.0148$$

$$p' = \frac{4 \times 0.6013}{12 \times 27} = 0.0074$$

From equations (40) and (41), $k = 0.422$ and $j = 0.881$. When f_c is a maximum, the corresponding value of f_s [equation (38)] is

$$\frac{800 \times 15(1 - 0.422)}{0.422} = 16{,}450 \text{ p.s.i.}$$

Therefore, the strength of the beam depends upon the resistance in compression, and the safe resisting moment occurs when the tension steel is stressed only to 16,450 p.s.i., so that, from equation (36),

$$M = 8 \times 0.6013 \times 16{,}450 \times 0.881 \times 27 = 1{,}880{,}000 \text{ in.-lb.}$$

Values of k and j could have been obtained from Figs. 163 and 164.

183. Continuous T-beams. Analysis at Supports. The required cross-section of a T-beam which is continuous over the supports, and the steel area required at the center of the span, are determined as explained in Art. 176. At the supports the bending moment is negative, so that the upper surface becomes the tensile surface, while the lower portion of the beam is in compression. Since in reinforced-concrete design the steel is assumed to resist all the tensile forces, sufficient steel must be placed near the upper surface of the beam over the support to develop the negative moment at that point. When, as is usually the case, the moment at the support is assumed numerically equal to the moment at mid-span, the tensile steel required near the upper surface over the support is approximately equal

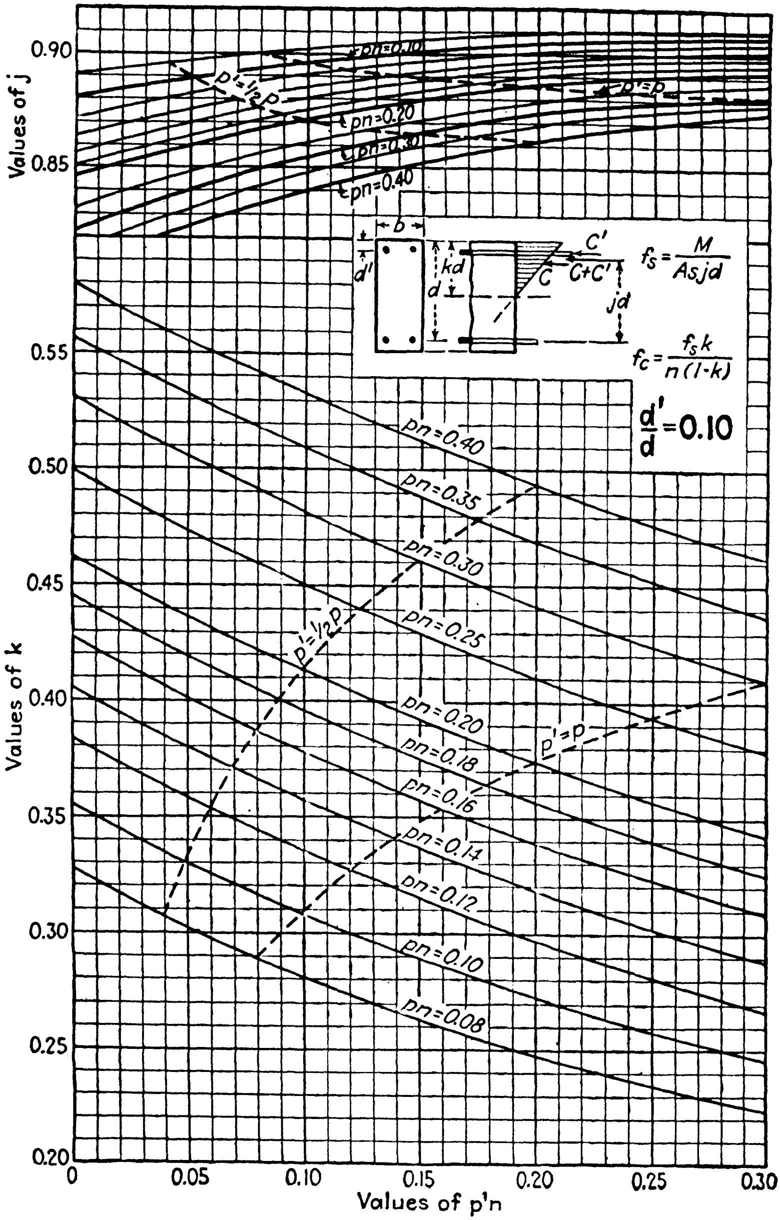


FIG. 163.—Rectangular beams reinforced for compression.

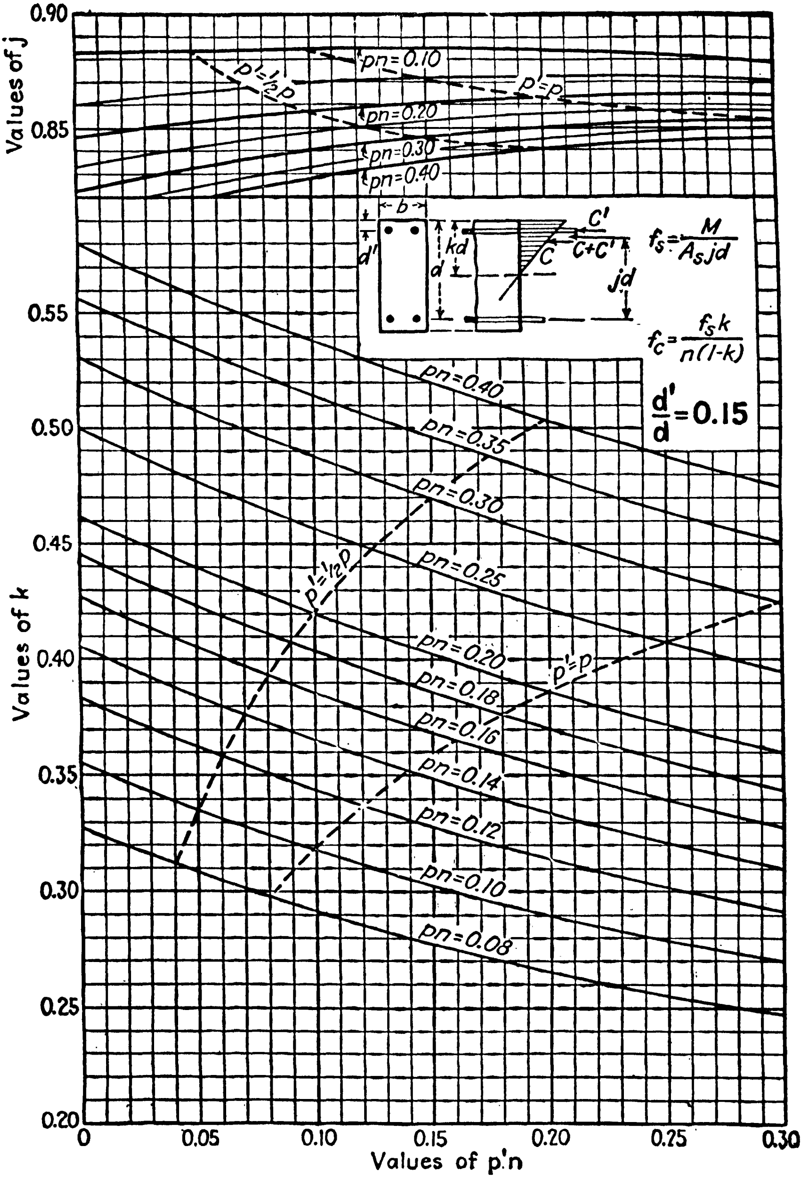


FIG. 164.—Rectangular beams reinforced for compression.

to that required in the lower section of the beam at the center of the span. The usual method of furnishing the required tension steel at the support is to bend up about one-half of the bars from each adjacent beam and to continue them near the top surface to the assumed point of inflection in the adjoining span. This point is generally considered to be at distances from the support varying from one-fifth to one-third of the span, depending upon the conditions of continuity and loading. The points at which the bars can be bent up depend upon the variation in the positive moment; *i.e.*, if four bars are used in the beam, two of these bars may be bent up at the section where the positive moment is one-half of the maximum.

Since the tension side is uppermost at the support, the flange of the T-beam cannot be considered as effective in resisting bending stresses at this point. Hence, the form of the beam, as far as stress analysis is concerned, becomes rectangular at the support, the width being equal to the width of the stem. On account of the small compressive area of concrete (now below the neutral axis), a failure by compression would probably occur if steel were not added in the compressive area to assist the concrete. Since only a part of the horizontal bars are being bent up over the support, the remaining bars may be brought straight through and extended about 30 or 40 diameters into the adjoining span and thus furnish the added compressive resistance required.

Where one-half of the longitudinal reinforcement which is furnished at the center of the span is bent up from each of two adjacent beams and the remainder of the reinforcement is continued straight through the support far enough to develop its compressive strength in bond, the amounts of tensile and compressive reinforcement are equal. If less compressive reinforcement is required, the bars from the adjacent beams need to be carried beyond the support only far enough to develop a lap splice. With such an arrangement the compressive reinforcement is equal in amount to one-half the tensile reinforcement. Other arrangements can be made to suit any individual case. For example, if there are seven bars at the center, four bars can be bent up and continued over the support, leaving three from each side at the bottom of the section. The tension steel at the support would then consist of eight bars, and the compression steel of six bars. In any case, the effective section of the beam at the support is an inverted rectangular beam reinforced for compression, and it may be analyzed according to the principles of Art. 180.

Since the negative moment decreases very rapidly and only a short section is under maximum stress, a higher compressive stress is allowed in the concrete at the support than at mid-span. The Joint Code specifies an allowable unit compression stress of $0.45f'_c$ at the supports of continuous beams, as compared with $0.40f'_c$ at the center of the span of such beams.

The details of a typical continuous T-beam are shown in Fig. 165, together with cross-sections at the center of the span and at the support. It should be noted that, in reviewing these beams at the support, since the compression surface is at the bottom of the beam, the effective depth d is measured from this lower surface upward to the center of the tension steel near the top of the beam and that the distance d' is measured from the lower surface upward to the center of the compression steel.

Example. The cross-section of a fully continuous T-beam below the 4-in. slab which forms the flange of the beam is 8 × 19 in. The span is 20 ft.-0 in., and the distance between adjacent beams is 10 ft.-0 in. At the support the reinforcement near the top of the beam consists of four ⅞-in. round bars in two rows, 2 in. center to center, the center of the upper row being 2 in. from the top surface of the slab, and the reinforcement near the bottom of the beam consists of two ⅞-in. round bars with their centers 2 in. from the lower surface. If the slab supports a uniform live load of 150 lb. per sq. ft., what are the values of f_c and f_s at the support of the beam? Assume $n = 15$.

The weight of the slab is 50 lb. per sq. ft., and the weight of the stem of the beam is 160 lb. per ft. The total load on the beam is (150 + 50)10 + 160 = 2160 lb. per lin. ft., and the maximum bending moment is $\frac{1}{12} \times 2160 \times 20^2 \times 12 = 864{,}000$ in.-lb.

$$np = 15 \times \frac{4 \times 0.6013}{8 \times 20} = 0.2255$$

$$np' = 15 \times \frac{2 \times 0.6013}{8 \times 20} = 0.1127$$

$$\frac{d'}{d} = \frac{2.0}{20} = 0.1$$

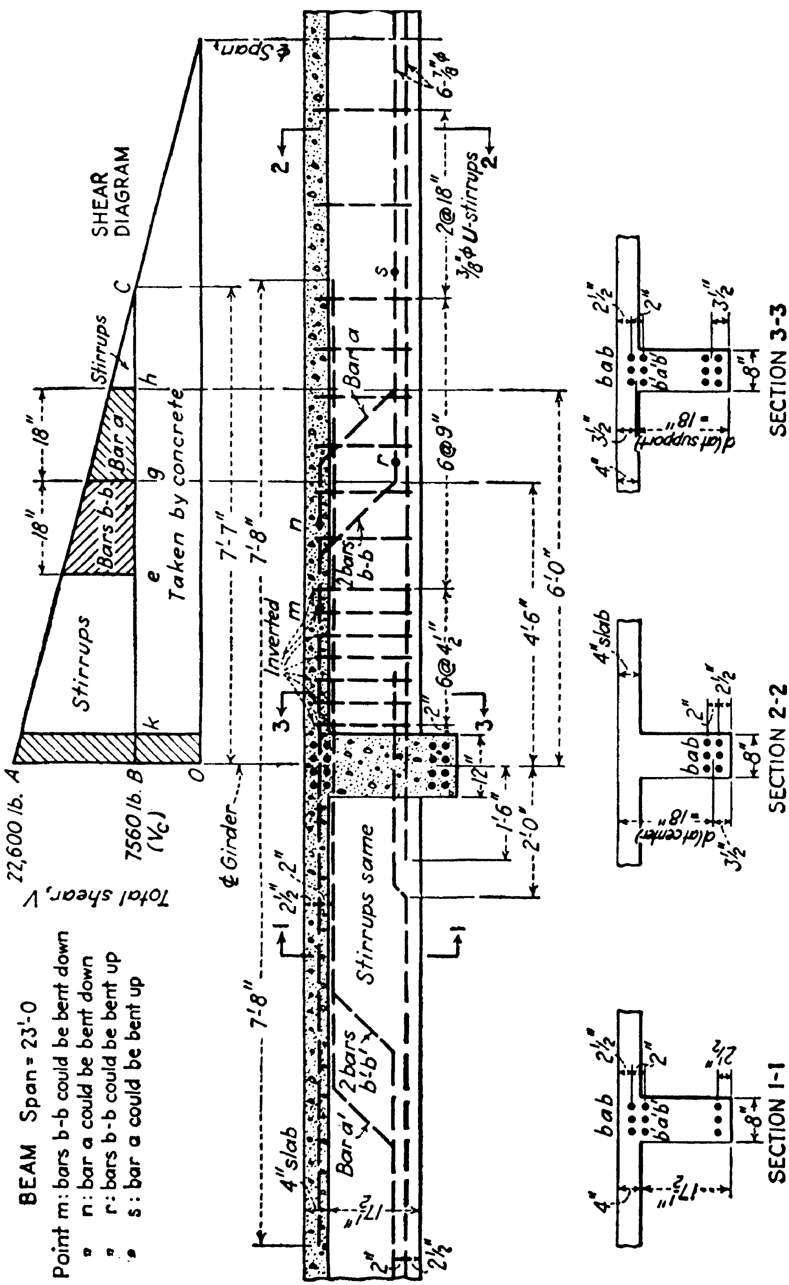


Fig. 165.—Details of beam.

From Fig. 163, $k = 0.430$ and $j = 0.870$.

$$864{,}000 = (4 \times 0.6013) \times f_s \times 0.870 \times 20$$

from which

$$f_s = 20{,}600 \text{ p.s.i.}$$

$$f_c = \frac{20{,}600 \times 0.430}{15(1 - 0.430)} = 1030 \text{ p.s.i.}$$

ADDITIONAL PROBLEM

If the allowable unit stresses at the support of the beam in the preceding problem were $f_s = 20{,}000$ p.s.i. and $f_c = 900$ p.s.i., what safe uniform live load could be placed on the slab without overstressing the beam at the support? Assume $n = 15$.

CHAPTER XI

REINFORCED-CONCRETE COLUMNS

184. Types of Columns. Concrete compression members whose unsupported length is more than four times the least dimension of the cross-section are classified as columns. Such

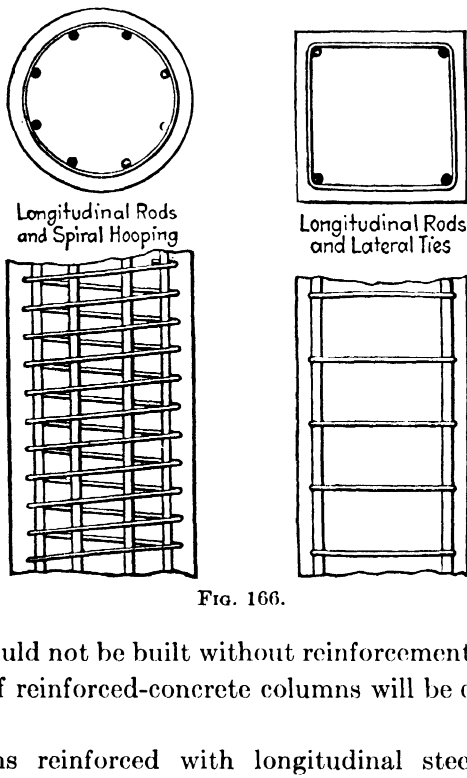


FIG. 166.

members should not be built without reinforcement of some type. Two types of reinforced-concrete columns will be considered (see Fig. 166).

1. Columns reinforced with longitudinal steel and closely spaced spirals.

2. Columns reinforced with longitudinal steel and lateral ties.

185. Unsupported Length and Limiting Dimensions. The reinforced-concrete column, as it is commonly used in ordinary construction, may be classified as a short column. In specifications it is usual to establish a ratio of length to diameter or of length to least radius of gyration above which the column can no longer be considered as a short column. Tests have shown that, as long as the ratio of length to diameter is less than 20 or the ratio of length to least radius of gyration is less than 60, there is little variation in the actual strength of columns of the same cross-section for variations in length. In practice it is customary to specify a somewhat smaller ratio. The Joint Code limits a short column to one whose unsupported length is not greater than ten times the least lateral dimension.[1] This same code limits the minimum diameter to 12 in. for principal members or, in the case of rectangular columns, a minimum thickness at 10 in. and a minimum gross area of 120 sq. in.

The unsupported length h of a column is the distance between those points at either end where lateral support is present in at least two directions, making an angle of 90 degrees or nearly 90 degrees with one another.

186. Columns with Spiral and Longitudinal Reinforcement. Whenever a material is subjected to compression in one direction, there will be an expansion in the direction perpendicular to the compression axis. Where this expansion is resisted, lateral compressive stresses are developed, which tend to neutralize the effect of the longitudinal compressive stress and thus to increase the resistance against failure. This is the principle involved in the use of spiral or hooped reinforcement. Within the limit of elasticity the hooped reinforcement is much less effective than longitudinal reinforcement. Such reinforcement, however, raises the ultimate strength of the column, because the hooping delays ultimate failure of the concrete. Thus a somewhat higher working stress may be employed on the concrete

[1] For long columns, the Joint Code gives the working load on columns whose length is greater than ten times the least dimension as

$$P\left(1.3 - 0.03\frac{h}{d}\right)$$

in which P = the total safe load on a column of the same section where the h/a (or h/d) ratio is less than 10, where a or d is the least dimension of the column and h the unsupported length of the column.

contained within such hooping than on a concrete not so confined. Tests show that about 1 per cent of closely spaced spiral hooping increases the resistance to ultimate failure sufficiently to allow a reasonable increase in the working stress in the concrete.

As long as the bond between the steel and the concrete is effective, the two materials will deform equally, and the intensities of the stresses will be proportional to their moduli of elasticity. That is, since the deformation λ_c of the concrete must be equal to the deformation λ_s of the steel,

$$\lambda_c = \frac{f_c}{E_c} = \lambda_s = \frac{f_s}{E_s}$$

Therefore,

$$f_s = nf_c$$

Let

A_c = net area of concrete in the cross-section.
A_s = area of longitudinal steel in this section.
A_g = overall or gross area of concrete section = $A_c + A_s$.
A_t = area of transformed section = $A_c + nA_s$
= $A_g + (n - 1)A_s$.
p_g = steel ratio A_s/A_g.
f_s = unit compressive stress in steel.
f_c = unit compressive stress in concrete.
P = total strength of reinforced column.

Then,

$$\begin{aligned} P = f_cA_c + f_sA_s &= f_c(A_g - p_gA_g) + f_cnp_gA_g \\ &= f_cA_g[1 + (n - 1)p_g] \\ &= f_c[A_g + (n - 1)A_s] \end{aligned}$$

In the above analysis the stress in the steel does not exceed nf_c. Tests conducted by the American Concrete Institute at Lehigh University and at the University of Illinois showed that the steel reinforcement actually was capable of withstanding much higher stresses without the bond between steel and concrete being destroyed. In general these tests showed that the strength of a reinforced-concrete column is actually the sum of the strengths of the concrete and the longitudinal reinforcement, regardless of the ratio of the moduli of elasticity.

For this type of column, the Joint Code (1940) specifies for the safe axial load

$$P = 0.225f'_cA_g + f_sA_s \tag{1}$$

where f'_c = the ultimate strength of the concrete, in pounds per square inch.

f_s = the working stress in the longitudinal reinforcement (16,000 p.s.i. for intermediate grade and 20,000 p.s.i. for hard grade).

The longitudinal reinforcement should consist of at least six bars of a minimum diameter of ⅝ in., and its effective cross-sectional area should be not less than 1 per cent or more than 8 per cent of the gross area of the column section. The ratio of spiral reinforcement p' must be not less than

$$p' = 0.45(R - 1)\frac{f'_c}{f'_s}$$

where p' = ratio of volume of spiral reinforcement to the volume of spiral core (out to out of spirals).

R = ratio of gross area of column to core area.

f'_s = useful limit stress of spiral reinforcement, 40,000 p.s.i. for hot-rolled bars of intermediate grade, 50,000 p.s.i. for hard grade, and 60,000 p.s.i. for cold-drawn wire.

The center-to-center spacing of spirals shall not exceed one-sixth the core diameter, and the clear spacing between spirals shall not exceed 3 in. or be less than 1⅜ in. The spirals should be continuous and held firmly in place and true to line by at least three vertical spacer bars. The spiral bars or wire shall be not less than ¼ in. in diameter for columns up to 18 in. in diameter or less than ⅜ in. above 18 in. The thickness of concrete outside the spiral reinforcement shall be not less than 1½ in.

Where the reinforcing bars are spliced by lapping, the length of the lap should be 24 bar diameters for intermediate-grade steel and 30 bar diameters for rail steel provided that the concrete has a strength of 3000 p.s.i. or more. With concrete of lesser strength the lengths given above should be increased one-third. The values above are for deformed bars; lap lengths of plain bars should be increased by 25 per cent. Where changes in the cross-section of a column occur, the offset of the bars should be made where there is lateral support, such as a column capital, floor slab, or metal ties or spirals. The slope of the inclined portion of the bars should not exceed 1 in 6, and the bars above and below should be parallel with the axis of the column.

187. Columns with Longitudinal Reinforcement and Lateral Ties. Tests show that columns without spirals develop lower stresses in both the concrete and the steel, and the Joint Code (1940) specifies the safe axial load for this type of column as 0.8 of that for a spirally reinforced column, or

$$P = 0.18f'_cA_g + 0.8A_sf_s \qquad (2)$$

The longitudinal reinforcement should consist of at least four bars of a minimum diameter of ⅝ in., and its effective cross-

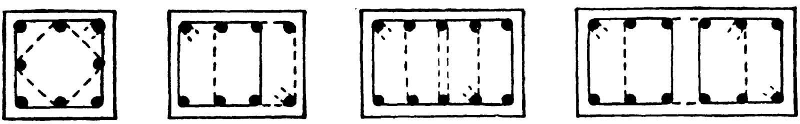


FIG. 167.

sectional area should be not less than 1 per cent or more than 4 per cent of the gross area of the column section. There should be a clear distance between the longitudinal bars and the face of the column of 1½ in. plus the diameter of the tie.

The longitudinal bars are held in alignment during construction by lateral ties as illustrated in Figs. 166 and 167. These ties would be made of wire at least ¼ in. in diameter,[1] and the vertical distance between ties or sets of ties should not exceed 16 bar diameters, 48 tie diameters, or the least dimension of the column. When the number of bars in a column exceeds four, the ties should be so detailed as to prevent the outward bending of every bar at each tie interval. The methods of accomplishing this are illustrated in Fig. 167.

188. Flexural Stresses in Columns. Bending moments are produced in columns: (*a*) by reactions from eccentrically placed

[1] There is no rational method of determining the size of wire that should be used for a lateral tie. A safe rule to follow is to use wire of such diameter that the area of its section is not less than 2 per cent of the section of the longitudinal reinforcement held in place by the tie.

beams; (*b*) by the loads on brackets or cantilevers; (*c*) by the eccentricity of the columns themselves, a condition which often occurs in the wall columns of a building where the sections of the columns are changed at some floor levels while the exterior faces of the columns are kept in line throughout the height of the structure; (*d*) by the application of a direct horizontal force or of a force having a horizontal component; or (*e*) by the transfer from slabs or girders built monolithic with the columns of unbalanced moments due to the loads on the slabs or girders.

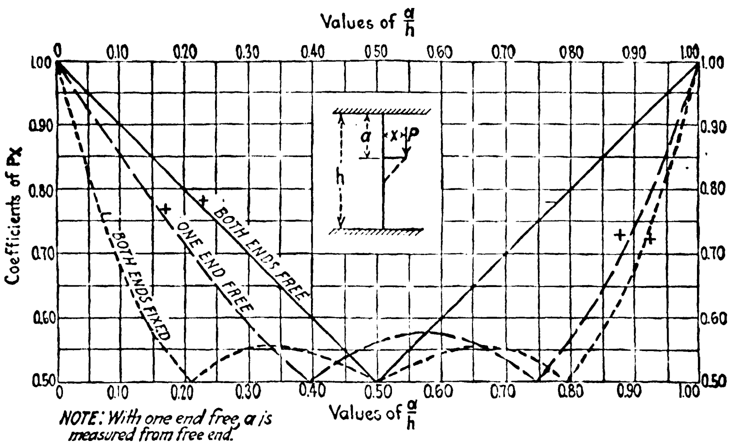


FIG. 168.

An eccentric load applied to a column at any point will produce a maximum moment at the point of application. The distribution of the moment to the column depends upon the height at which the load is applied and the end conditions of the column. When the load is applied at the top or bottom of the column, the bending moment has its maximum possible value and is equal to Px (Fig. 168). For other positions of the load the moment is less, the minimum moment being $Px/2$. The coefficient of Px for different end conditions and different positions of the load may be obtained from Fig. 168. The values of the extreme fiber stresses on either side of the column are obtained by the general method for combined bending and axial stress explained in Arts. 192 to 196. The value of e is obtained by dividing the moment M by the total load (not necessarily P alone) supported by the column at the point where the eccentric load is applied.

189. Working-unit Stresses in Columns Subject to Bending. The maximum permissible stress in the concrete may be increased when the combined effect of bending and axial stress is considered.[1] The maximum stress occurs only on one side of the column and rapidly decreases toward the axis of the column. Tests have shown that a much higher unit stress can be developed on the extreme fiber in flexure than when the stress is uniformly distributed over the cross-section.

The Joint Code (1940) allows a stress on the compressive side of the column of

$$f_c = f_a \left(\frac{1 + \frac{ec}{R^2}}{1 + C\frac{ec}{R^2}} \right) \tag{3}$$

where f_a = average permissible stress on an equivalent axially loaded concrete column

$= \dfrac{0.225f'_c + f_s p_g}{1 + (n - 1)p_g}$ for spiral columns

$= \dfrac{0.18f'_c + 0.8f_s p_g}{1 + (n - 1)p_g}$ for tied columns.

C = ratio of f_a to the permissible fiber stress for members in flexure ($f_a \div 0.45f'_c$).

c = distance from the gravity axis to the extreme fiber in compression.

e = eccentricity of the resultant load on the column, measured from the gravity axis.

R = least radius of gyration of the column section.

As the principal variable in the above equation is the eccentricity e, it involves but a slight error if the transformed steel area is neglected. Figures 169 to 171 are plotted on this basis, for

[1] The unincreased unit stress must, of course, not be exceeded when the maximum axial load is sustained or when the total load is considered as an axial load.

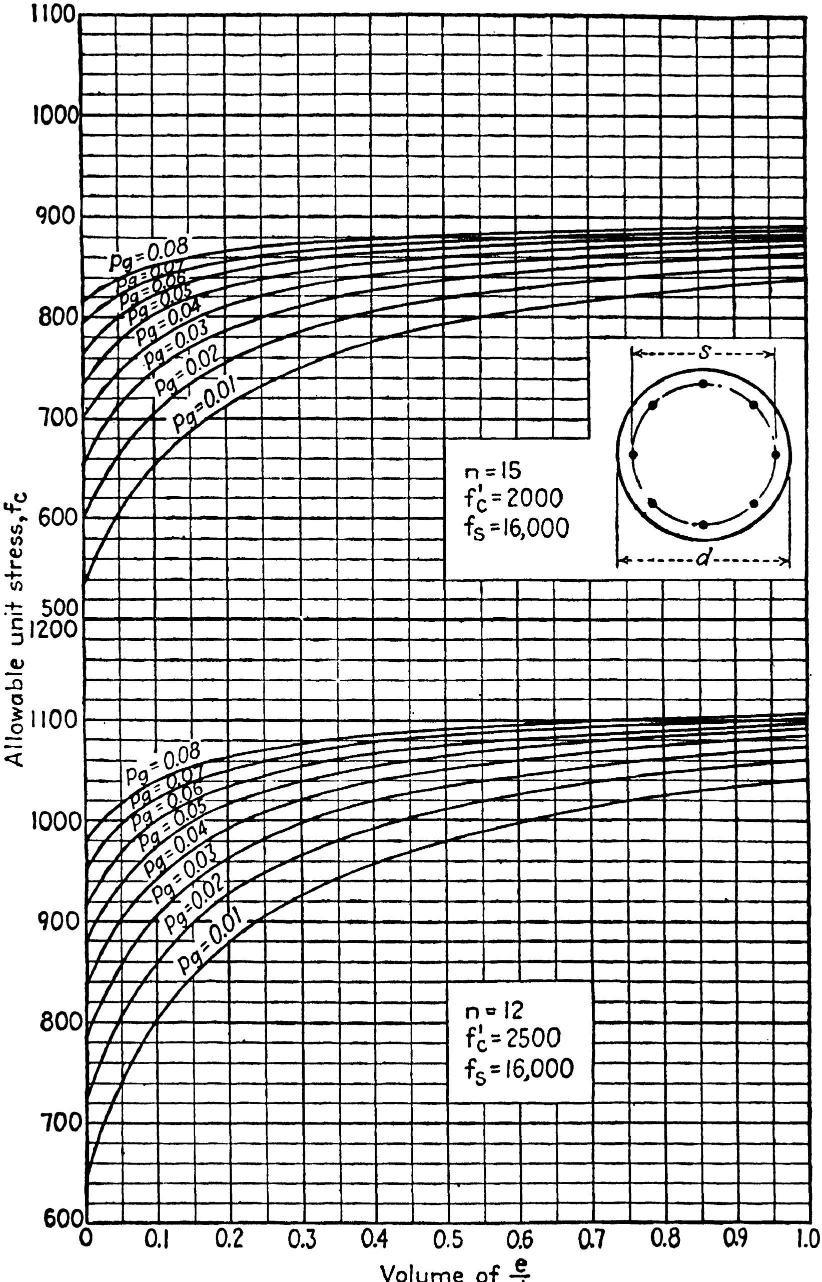


FIG. 169.—Allowable unit stresses in eccentrically loaded circular spiral columns.

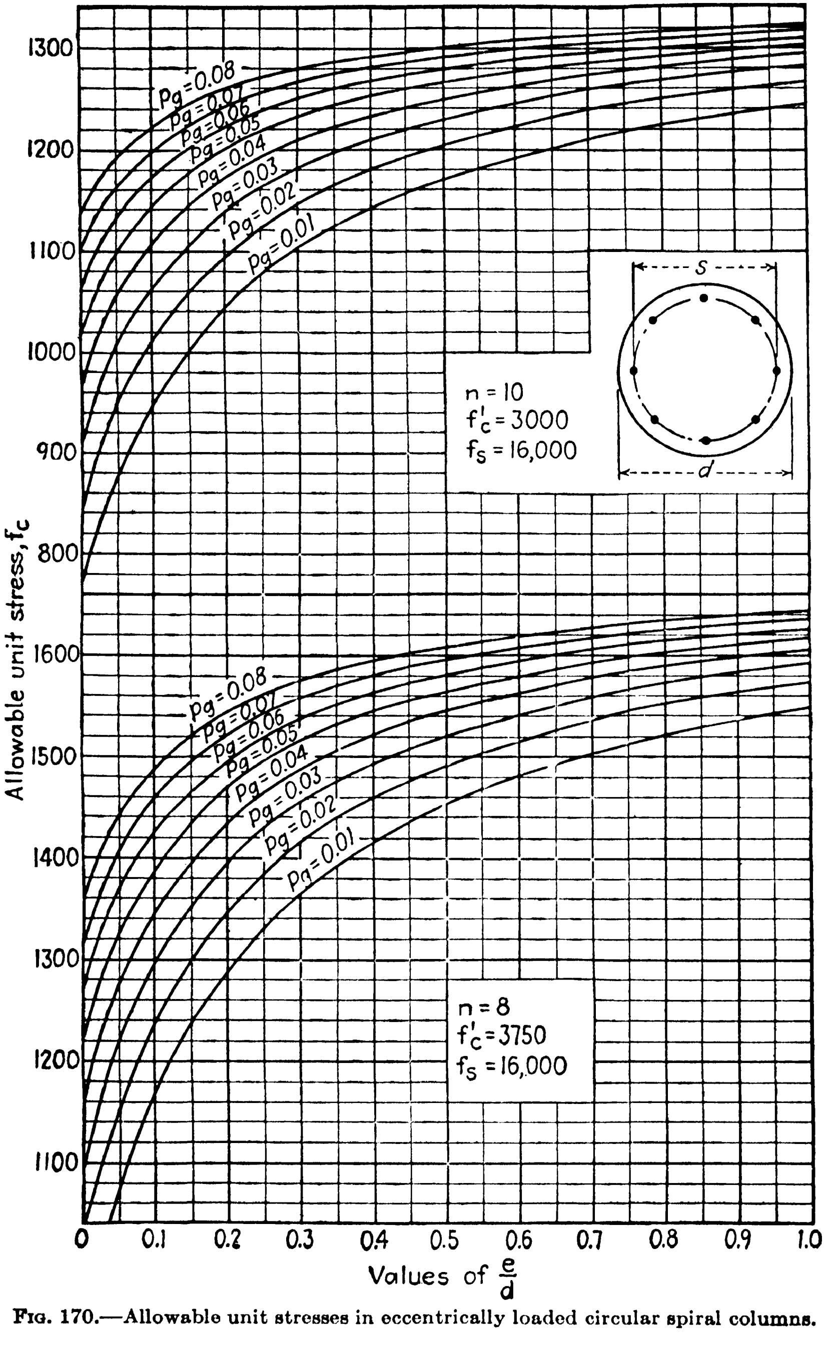


FIG. 170.—Allowable unit stresses in eccentrically loaded circular spiral columns.

tied columns with e/a and p_g as arguments and for spiral columns with e/d and p_g as arguments, respectively.

190. Column Tables. Tables 1 and 2 give the safe concentric loads on the concrete and the longitudinal bars of reinforced-

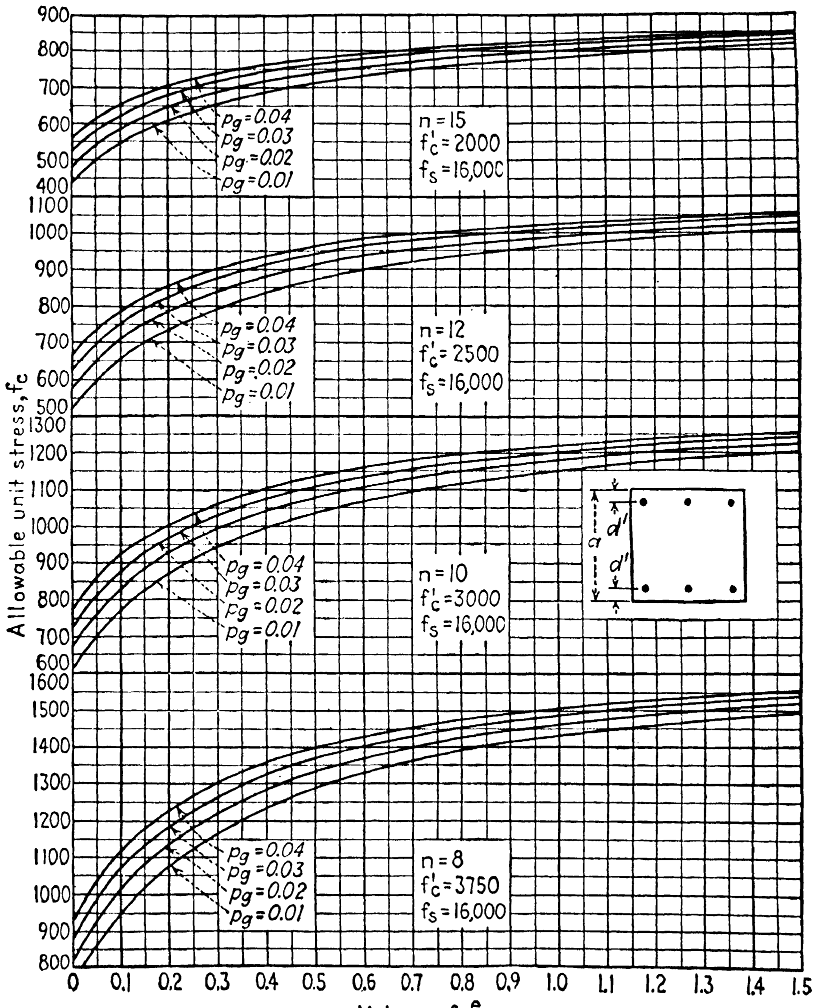


FIG. 171.—Allowable unit stresses in eccentrically loaded tied columns.

concrete columns. Values are given for the minimum amount of longitudinal steel. Loads for greater percentages of steel are proportional.

Table 3 gives the area of section, weight per foot, and moment of inertia of circular and octagonal sections. It also gives these same functions for rectangular sections 1 in. in width.

TABLE 1.—COLUMNS WITH LONGITUDINAL BARS AND SPIRALS
(Loads in thousand pounds)
$P = 0.225f'_cA_g + f_sA_s$

Diameter of column, in.	Gross area A_g, sq. in.	Load on bars, f_sA_s for $p_g = 0.01$		Load on concrete, $0.225f'_cA_g$			
		f_s		f'_c			
		Intermediate grade 16,000	Hard grade 20,000	2000	2500	3000	3750
12*	113	14	18	41	51	61	76
13*	133	17	22	48	60	72	90
14	154	25	31	69	87	104	130
15	177	28	35	80	99	119	149
16	201	32	40	91	113	136	170
17	227	36	45	102	128	153	192
18	255	41	51	114	143	172	215
19	284	45	57	128	159	191	239
20	314	50	63	141	177	212	265
21	346	55	69	156	195	234	292
22	380	61	76	171	214	257	321
23	416	66	83	187	234	280	350
24	452	72	90	204	254	305	382
25	491	79	98	221	276	331	414
26	531	85	106	239	299	358	448
27	573	92	115	258	322	387	483
28	616	98	123	277	346	416	519
29	661	106	132	297	372	446	557
30	707	113	141	318	398	477	596

* Spirals for these are excessive and seldom available. The loads given in the table are for circular columns with lateral ties.

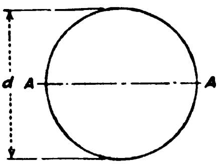
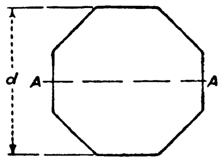
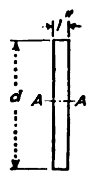
Table 4 gives the moment of inertia of the longitudinal reinforcement where the bars composing it are arranged in the form of a circle. The circle is assumed to be 5 in. less in diameter than

TABLE 2.—COLUMNS WITH LONGITUDINAL BARS AND LATERAL TIES
(Loads in thousand pounds)
$P = 0.18f'_cA_g + 0.8f_sA_s$

Dimensions of column, in.		Gross area A_g, sq. in.	Load on bars, $0.8f_sA_s$ for $p_g = 0.01$		Load on concrete $0.18f'_cA_g$			
			Intermediate grade 16,000	Hard grade 20,000	f'_c 2000	2500	3000	3750
10	12	120	15	19	43	54	65	81
	14	140	18	22	50	63	76	95
	16	160	20	26	58	72	86	108
12	12	144	18	23	52	65	78	97
	14	168	22	27	60	76	91	113
	16	192	25	31	69	86	104	130
14	14	196	25	31	71	88	106	132
	16	224	29	36	81	101	121	151
	18	252	32	40	91	113	136	170
16	16	256	33	41	92	115	138	173
	18	288	37	46	104	130	156	194
	20	320	41	51	115	144	173	216
	22	352	45	56	127	158	190	238
18	18	324	41	52	117	146	175	219
	20	360	46	58	130	162	194	243
	22	396	51	63	143	178	214	267
	24	432	55	69	156	194	233	292
20	20	400	51	64	144	180	216	270
	22	440	56	70	158	198	238	297
	24	480	61	77	173	216	259	324
	26	520	67	83	187	234	281	351
22	22	484	62	77	174	218	261	327
	24	528	68	84	190	238	285	356
	26	572	73	92	206	257	309	386
	28	616	79	99	222	277	333	416
24	24	576	74	92	207	259	311	389
	26	624	80	100	225	281	337	421
	28	672	86	108	242	302	363	454
26	26	676	87	108	243	304	365	456
	28	728	93	116	262	328	393	491
28	28	784	100	125	282	353	423	529
30	30	900	115	144	324	405	486	608

the diameter of the column. With 1-in. bars, ½-in. spirals, and 1½ in. of concrete outside the spirals, this is exactly true. The possible variation (as long as the minimum 1½ in. is used) either

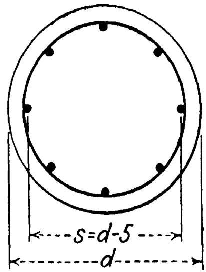
TABLE 3.—AREAS, WEIGHTS, AND MOMENTS OF INERTIA
(Moments of inertia about the axis A-A)

d									
	Area, sq. in.	Weight per ft., lb.	I, in.^{4}	Area, sq. in.	Weight per ft., lb.	I, in.^{4}	Area, sq. in.	Weight per ft., lb.	I, in.^{4}
12	113	118	1,018	119	124	1,136	12.0	12.5	144
13	133	138	1,402	140	146	1,565	13.0	13.5	183
14	154	160	1,886	162	169	2,105	14.0	14.6	229
15	177	184	2,485	186	194	2,775	15.0	15.6	281
16	201	210	3,217	212	221	3,591	16.0	16.7	341
17	227	237	4,100	239	249	4,577	17.0	17.7	409
18	255	265	5,153	268	280	5,753	18.0	18.8	486
19	284	295	6,397	299	312	7,142	19.0	19.8	572
20	314	327	7,854	331	345	8,768	20.0	20.8	667
21	346	361	9,547	365	381	10,658	21.0	21.9	772
22	380	396	11,499	401	418	12,837	22.0	22.9	887
23	416	433	13,737	438	457	15,335	23.0	24.0	1,014
24	452	471	16,286	477	497	18,181	24.0	25.0	1,152
25	491	511	19,175	518	539	21,406	25.0	26.1	1,302
26	531	553	22,432	560	583	25,042	26.0	27.1	1,465
27	573	597	26,087	604	629	29,123	27.0	28.1	1,640
28	616	642	30,172	650	677	33,683	28.0	29.2	1,829
29	661	688	34,719	697	726	38,759	29.0	30.2	2,032
30	707	736	39,761	746	777	44,388	30.0	31.2	2,250

way is negligible. The values are given for a steel ratio p_g of 1 per cent. Other values are proportional.

Table 5 gives the moment of inertia of single bars about an axis at varying distances from the center of the bars. The values obtained from this table, multiplied by the number of bars and

TABLE 4.—MOMENT OF INERTIA OF COLUMN VERTICALS ARRANGED IN A CIRCLE OF DIAMETER 5 IN. LESS THAN DIAMETER OF COLUMN
[Values of $(n - 1)I_s$ in inches[4] for $p_g = 0.01$]



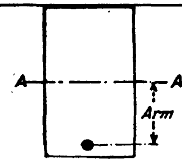
Diameter of column d, in.	Diameter of steel circle s, in.	A_s where $p_g = 0.01$	f'_c			
			2000	2500	3000	3750
12	7	1.13	97	76	62	49
13	8	1.33	149	117	96	74
14	9	1.54	218	172	140	109
15	10	1.77	310	243	199	155
16	11	2.01	426	335	274	213
17	12	2.27	572	449	368	286
18	13	2.55	754	593	485	377
19	14	2.84	974	765	626	487
20	15	3.14	1166	916	750	583
21	16	3.46	1550	1218	996	775
22	17	3.80	1922	1510	1236	961
23	18	4.16	2359	1853	1516	1179
24	19	4.52	2856	2244	1836	1428
25	20	4.91	3437	2701	2210	1719
26	21	5.31	4098	3220	2635	2049
27	22	5.73	4853	3813	3120	2427
28	23	6.16	5703	4481	3666	2852
29	24	6.61	6663	5235	4283	3331
30	25	7.07	7733	6076	4971	3867

The bars are assumed transformed into a thin-walled cylinder having the same sectional area as the bars. Then $I_s = A_s s^2 \div 8$.

For other values of p_g multiply the value from the table by $100p_g$.

$n - 1$, may be used in conjunction with Table 3 in obtaining the moment of inertia of rectangular columns.

TABLE 5.—MOMENTS OF INERTIA OF BARS IN INCHES[4]
(For various distances from an axis *A-A*)



Arm, in.	Round bars, inches					Square bars, inches			
	½	⅝	¾	⅞	1	½	1	1⅛	1¼
2	1	1	2	2	3	1	4	5	6
2½	1	2	3	4	5	2	6	8	10
3	2	3	4	5	7	2	9	12	14
3½	2	4	5	7	10	3	12	16	19
4	3	5	7	10	13	4	16	20	25
4½	4	6	9	12	16	5	20	26	32
5	5	8	11	15	20	6	25	32	39
5½	6	9	13	18	24	8	30	38	47
6	7	11	16	22	28	9	36	46	56
6½	8	13	19	25	33	11	42	54	66
7	10	15	22	29	39	12	49	62	77
7½	11	17	25	34	44	14	56	71	88
8	13	20	28	39	50	16	64	81	100
8½	14	22	32	43	57	18	72	92	113
9	16	25	36	49	64	20	81	103	127
9½	18	28	40	54	71	23	90	114	141
10	20	31	44	60	79	25	100	127	156
10½	22	34	49	66	87	28	110	149	172
11	24	37	53	73	95	30	121	153	189
11½	26	41	58	80	104	33	132	168	207
12	28	44	64	87	113	36	144	182	225
13	33	52	75	102	133	42	169	214	264

Table 6 gives the size and pitch of spirals for various column diameters which satisfy the provisions of the Joint Code given in Art. 186. These values are based on a thickness of con-

TABLE 6.—SIZE AND PITCH OF SPIRALS, JOINT CODE (1940)
Hot-rolled, intermediate grade

Diameter of column, in.	Out to out spiral, in.	f'_c 2000	2500	3000	3750
14	11	3/8-1 3/4	3/8-1 3/4	3/8-1 3/4	1/2-2
15	12	3/8-2	3/8-2	3/8-1 3/4	1/2-2
16	13	3/8-2	3/8-2	3/8-1 3/4	1/2-2
17	14	3/8-2 1/4	3/8-2 1/4	3/8-1 3/4	1/2-2 1/4
18	15	3/8-2 1/2	3/8-2 1/4	3/8-1 3/4	1/2-2 1/2
19	16	3/8-2 1/4	3/8-2 1/4	3/8-1 3/4	1/2-2 1/2
20	17	3/8-2 1/4	3/8-2 1/4	3/8-2	1/2-2 3/4
21	18	3/8-2	3/8-2	3/8-2	1/2-2 3/4
22	19	3/8-2	3/8-2	3/8-2	1/2-2 3/4
23	20	3/8-1 3/4	3/8-1 3/4	3/8-1 3/4	1/2-2 3/4
24	21	1/2-3 1/4	1/2-3 1/4	1/2-3 1/4	1/2-2 3/4
25	22	1/2-3 1/4	1/2-3 1/4	1/2-3 1/4	1/2-2 3/4
26	23	1/2-3	1/2-3	1/2-3	1/2-2 3/4
27	24	1/2-2 3/4	1/2-2 3/4	1/2-2 3/4	1/2-2 3/4
28	25	1/2-2 3/4	1/2-2 3/4	1/2-2 3/4	1/2-2 3/4
29	26	1/2-2 3/4	1/2-2 3/4	1/2-2 3/4	1/2-2 3/4
30	27	1/2-2 1/2	1/2-2 1/2	1/2-2 1/2	1/2-2 1/2

Values below dotted line are based on the minimum value for p' of 0.0112.

Above the dotted line the formula $p' = 0.45(R - 1)\frac{f'_c}{f'_s}$ governs.

Cold-drawn

Diameter of column, in.	Out to out spiral, in.	2000	2500	3000	3750
14	11	1/4-1 3/4	3/8-1 3/4	3/8-1 3/4	3/8-1 3/4
15	12	1/4-1 3/4	3/8-2	3/8-2	3/8-2
16	13	1/4-1 3/4	3/8-2	3/8-2	3/8-2
17	14	1/4-1 3/4	3/8-2 1/4	3/8-2 1/4	3/8-2 1/4
18	15	1/4-1 3/4	3/8-2 1/2	3/8-2 1/2	3/8-2 1/4
19	16	3/8-2 1/2	3/8-2 1/2	3/8-2 1/2	3/8-2 1/4
20	17	3/8-2 3/4	3/8-2 3/4	3/8-2 3/4	3/8-2 1/4
21	18	3/8-3	3/8-3	3/8-3	3/8-2 1/4
22	19	3/8-3	3/8-3	3/8-3	3/8-2 1/4
23	20	3/8-2 3/4	3/8-2 3/4	3/8-2 3/4	3/8-2 1/4
24	21	3/8-2 3/4	3/8-2 3/4	3/8-2 3/4	3/8-2 1/4
25	22	3/8-2 1/2	3/8-2 1/2	3/8-2 1/2	3/8-2 1/4
26	23	3/8-2 1/2	3/8-2 1/2	3/8-2 1/2	3/8-2 1/4
27	24	3/8-2 1/4	3/8-2 1/4	3/8-2 1/4	3/8-2 1/4
28	25	3/8-2 1/4	3/8-2 1/4	3/8-2 1/4	3/8-2 1/4
29	26	3/8-2 1/4	3/8-2 1/4	3/8-2 1/4	3/8-2 1/4
30	27	3/8-2	3/8-2	3/8-2	3/8-2

Values below dotted line are based on the minimum value for p' of 0.0075.

Above the dotted line the formula $p' = 0.45(R - 1)\frac{f'_c}{f'_s}$ governs.

crete outside the spirals of 1½ in. A greater thickness of concrete requires slightly more spiral reinforcement.

191. Examples. 1. A circular column reinforced with longitudinal steel and spiral hooping has an unsupported length of 14 ft.-0 in. and sustains a direct axial load of 250,000 lb. The ultimate strength of the concrete is specified as 3000 p.s.i. and hard-grade steel reinforcement is to be used. Design the column. From Table 1 the following selections are made:

Diameter of column	Load carried by concrete	Load by steel	Weight of column (Table 3)	Net load, kips	p_g	A_s, sq. in.	Bars selected
20	212	63	5	270	0.010	3.14	8-¾ ϕ
19	191	63	4	250	0.011	3.12	8-¾ ϕ
18	172	82	4	250	0.016	4.08	10-¾ ϕ
17	153	100	3	250	0.022	4.99	9-⅞ ϕ
16	136	117	3	250	0.029	5.83	8-1 ϕ
15	119	134	3	250	0.038	6.72	9-1 ϕ
14	104	148	2	250	0.048	7.39	10-1 ϕ

All these selections satisfy the specifications of the Joint Code. The 20-in. column has the minimum amount of reinforcement allowed by the specifications; but since it requires no more steel in obtainable sizes than the 19-in. column, the latter is theoretically more economical. However, metal column forms are usually available only in diameters of even integral inches, so that the 20-in. column would usually be chosen.

If it were desirable to have the column as small as possible, the 14-in. column could be selected.[1] Care must be taken when large percentages of steel are used that sufficient space is left between the bars. Usually about 2 per cent of longitudinal steel furnishes the most desirable column. For the 20-in. column, ⅜-in. spiral with a pitch of 2 in. (hot-rolled) or 2¾-in. (cold-drawn) is selected.

2. Select a column reinforced with longitudinal steel and lateral ties with an unsupported length of 14 ft.-0 in. to carry an axial load of 240,000 lb. The concrete strength is 3000 p.s.i., and intermediate-grade steel is to be used. Furthermore, the column is to be as small as possible, and its least dimension cannot exceed 14 in.

[1] Such a column would, however, be a long column, since $h/d = 12$, and the safe load would be $(1.3 - 0.03 \times 12) = 0.94$ of that for a short column. $252 \div 0.94 = 268$. This leaves 164 kips to be carried by the reinforcing bars, requiring $164 \div 31 = 5.3$ per cent, or an A_s of $0.053 \times 154 = 8.16$ sq. in. This requires additional steel, or eleven 1-in. round bars.

This column is a long column since $h/a = 12$ and its safe load is $(1.3 - 0.03 \times 12) = 0.94$ times that of a short column. From Table 2 it appears that a 14 × 18 in. column will be required, which weighs (Table 3) $18.8 \times 14 \times 14 = 2700$ lb., or approximately 3 kips. The total load to be carried is 243 kips, and a short column must be selected which is capable of sustaining $243 \div 0.94 = 259$ kips. From Table 2, the concrete can sustain 136 kips, leaving 123 kips for the steel, which requires a percentage of $123 \div 32 = 3.8$. This is slightly under the maximum of 4 per cent and requires an A_s of $0.038 \times 14 \times 18 = 9.58$ sq. in. This is furnished by ten 1-in. square bars. The bars must be held in place by ties. The arrangement in Fig. 172 shows

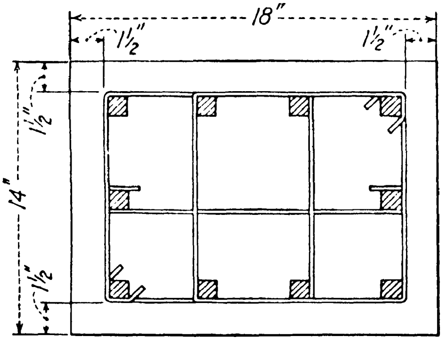


Fig. 172.

three sets of ties. Applying the rule given in the footnote on page 268, each tie should have a sectional area of $0.02 \times 4.00 = 0.08$ sq. in. Wire 5⁄16 in. in diameter furnishes approximately this cross-section and is selected. Ties are placed 14 in. center to center according to the specifications given in Art. 187.

ADDITIONAL PROBLEMS

1. Design the column of the preceding Example 2 as a square column using the minimum amount of reinforcement.

2. Design a column with an unsupported length of 20 ft.-0 in. to sustain a direct axial load of 300,000 lb. $f'_c = 2500$; $f_s = 16{,}000$.

a. As a spirally reinforced column with $p_g = 0.02$.

b. As a tied column with $p_g = 0.02$.

3. What safe load can be sustained by a spiral column 22 in. in diameter reinforced with twelve 7⁄8-in. round bars?

a. If $f'_c = 3750$, $f_s = 16{,}000$, and $h = 15$ ft.

b. If $f'_c = 2000$, $f_s = 20{,}000$, and $h = 20$ ft.

4. What safe load can be supported by a tied column 12 × 16 in. reinforced with four 1-in. round bars?

a. If $f'_c = 2500$, $f_s = 16{,}000$, and $h = 16$ ft.

b. If $f'_c = 3000$, $f_s = 20{,}000$, and $h = 10$ ft.

BENDING AND AXIAL STRESS

192. General Theory. In a reinforced-concrete member it is presupposed that the bond between the steel and the concrete remains intact under stress. Therefore, the steel in the compression side of a member subject to combined bending and axial stress can withstand a stress only sufficient to make it deform equally with the concrete, or n times the stress in the concrete. This steel might then be replaced by n times the amount of concrete at the same distance from the axis of the section. Such a section is known as the transformed section (or equivalent homogeneous section), to which the fundamental principles of bending and direct stress, as outlined in Art. 92, may be applied.

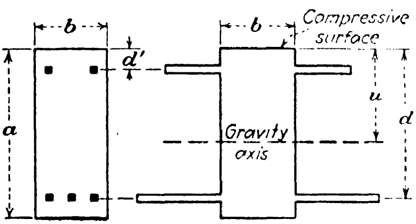


FIG. 173.

The following additional notation will be used. The face of the member most highly stressed will be called the "compressive surface," and the opposite face, the "tension surface."

R = resultant of all forces on the section.

N = resultant of all forces acting normal to the section, *i.e.*, the normal component of R.

e = eccentric distance of N.

M = bending moment = Ne.

u = distance from compressive surface to gravity axis of transformed section.

A_s = area of the bars near the tension surface.

A'_s = area of the bars near the compressive surface.

A_t = area of transformed section.

I_c = moment of inertia of concrete about gravity axis.

I_s = moment of inertia of steel about gravity axis.

$$p = \frac{A_s}{ba}, \qquad p' = \frac{A'_s}{ba}, \qquad p_0 = \frac{A_s + A'_s}{ba}$$

By referring to Fig. 173 it may be seen that

$$A_t = ba + (n - 1)(A_s + A'_s)$$
$$I = I_c + (n - 1)I_s$$
$$u = \frac{\frac{a}{2} + p(n - 1)d + p'(n - 1)d'}{1 + p(n - 1) + p'(n - 1)}$$
$$I_c = \frac{1}{3}b[u^3 + (a - u)^3]$$

Neglecting I_s about the axis of the bars,

$$I_s = A_s(d - u)^2 + A'_s(u - d')^2$$

If the reinforcement is symmetrical, then $u = a/2$, $A_s = A'_s$, and

$$I_c = \frac{1}{12}ba^3 \qquad \text{and} \qquad I_s = 2A'_s\left(\frac{a}{2} - d'\right)^2$$

If the eccentricity e/a is within certain limits, then compression exists over the whole section. For greater eccentricities there will be tension over a part of the section. If it be assumed that the concrete takes no tension, the analyses for these two cases are quite different. The value of e/a which results in zero stress on the tension surface is dependent upon the relative amounts of steel and concrete, and the ratio of the moduli of elasticity of the two materials.

193. Case I. Rectangular Sections. Compression over the Whole Section. (Fig. 174). *Only symmetrical reinforcement will be considered hereafter, and the total steel area will be referred to as* A_s. The maximum unit stress in the concrete may be computed as though the member were homogeneous and is

$$f_c = \frac{N}{A_t} + \frac{Mc}{I} \tag{4}$$

where c = distance from the gravity axis to the extreme fiber in compression (= $a/2$ for symmetrical sections).

This equation may be written

$$f_c = \frac{N}{ba + (n - 1)A_s} + \frac{Mc}{I_c + (n - 1)I_s} \tag{5}$$

On the compressive side of the member the unit flexural stress in the plane of the reinforcement is $\frac{M(c - d')}{I}$, and on the other side $\frac{M(d - c)}{I}$. The maximum unit stress in the steel is then

$$f'_s = n\left(\frac{N}{A_t}\right) + \frac{nM(c - d')}{I}$$

which is less than nf_c and is therefore always within the limits of a reasonable value for f'_s, provided f_c has a safe value. Since $f_s = n\left(\frac{N}{A_t}\right) - \frac{nM(d - c)}{I}$, it will always be less than f'_s. The investigation of columns under Case I is therefore completed when the value of f_c is computed from equation (5).

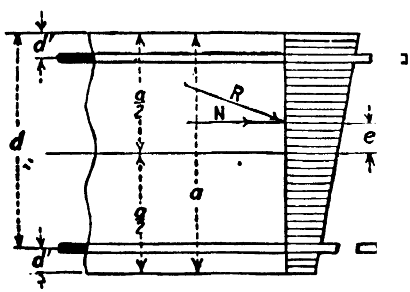


Fig. 174.

194. Case II. Rectangular Sections. Tension over Part of the Section (Fig. 175). When the second term of equation (4) is greater than the first, it indicates tension over part of the section. Unless this tension is so small that the concrete can take its proportionate part, the analysis of Case I is not applicable. With any appreciable tension on the tension surface of the member, it is usual to neglect the tension taken by the concrete and assume the full stress to be taken by the steel.

By reference to Fig. 175,

$$f'_s = nf_c\left(1 - \frac{d'}{ka}\right) \qquad (a)$$

and

$$f_s = nf_c\left(\frac{d}{ka} - 1\right) \qquad (b)$$

Since the total resultant stress $= N$,

$$\frac{1}{2}f_cbka + \frac{A_s}{2}f'_s - \frac{A_s}{2}f_s = N$$

Substituting the values of f'_s and f_s from equations (*a*) and (*b*),

$$N = \frac{f_c ba}{2} \times \frac{k^2 + 2np_g k - np_g}{k} \qquad (c)$$

or

$$f_c = \frac{N}{ba}\left[\frac{2k}{k^2 + 2np_g k - np_g}\right] \qquad (d)$$

Since the moment of the stresses about the gravity axis $= M$,

$$\frac{1}{2} f_c bka\left(\frac{a}{2} - \frac{ka}{3}\right) + \frac{f'_s A_s}{2}\left(\frac{a}{2} - d'\right) + \frac{f_s A_s}{2}\left(d - \frac{a}{2}\right) = M$$

and by eliminating f'_s and f_s as before,

$$\frac{M}{ba^2 f_c} = \frac{np_g(a - 2d')^2}{4ka^2} + \frac{k}{12}(3 - 2k) \qquad (e)$$

Since $M = Ne$, equation (*c*) may be multiplied by *e* and this

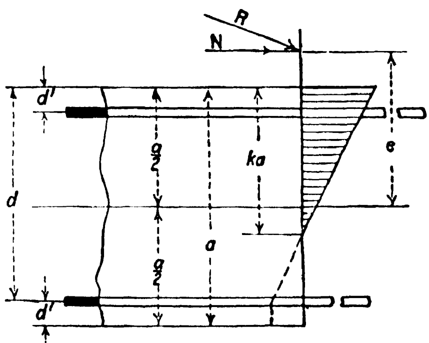


FIG. 175.

value substituted for *M* in equation (*e*). The following equation results:

$$k^3 - 3\left(\frac{1}{2} - \frac{e}{a}\right)k^2 + 6np_g\frac{e}{a}k = 3np_g\left[\frac{e}{a} + \frac{(a - 2d')^2}{2a^2}\right] \qquad (f)$$

In equation (*d*) the expression within the brackets is designated as *K*, and

$$f_c = \frac{NK}{ba} \qquad (6)$$

For constant values of d'/a, values of *k* and np_g may be substituted in equation (*f*) and values of e/a obtained. Substituting these same values of k and np_g in equation (*d*) the corresponding

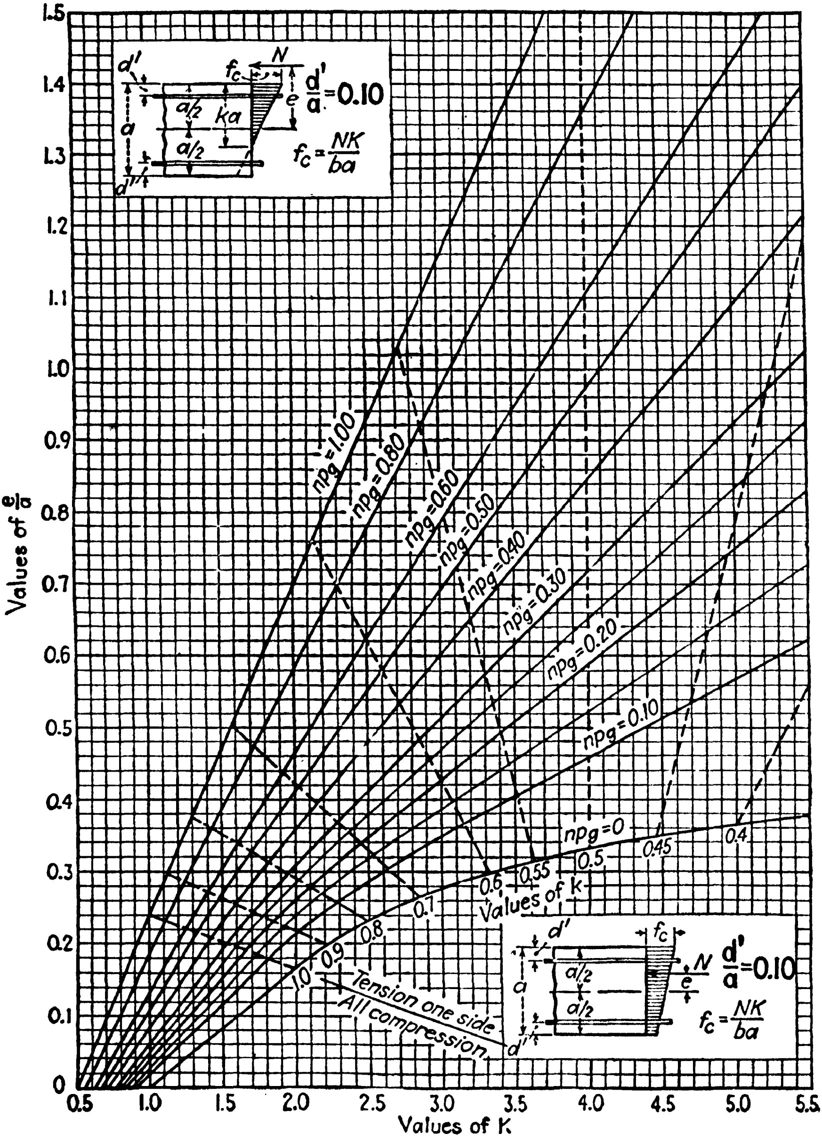


FIG. 176.—Bending and axial stress, rectangular sections.

values of K are obtained. Figures 176 to 178 have been plotted from such computations.

The investigation of columns under Case II is completed when the value of f_c is computed from equation (6), with the aid of

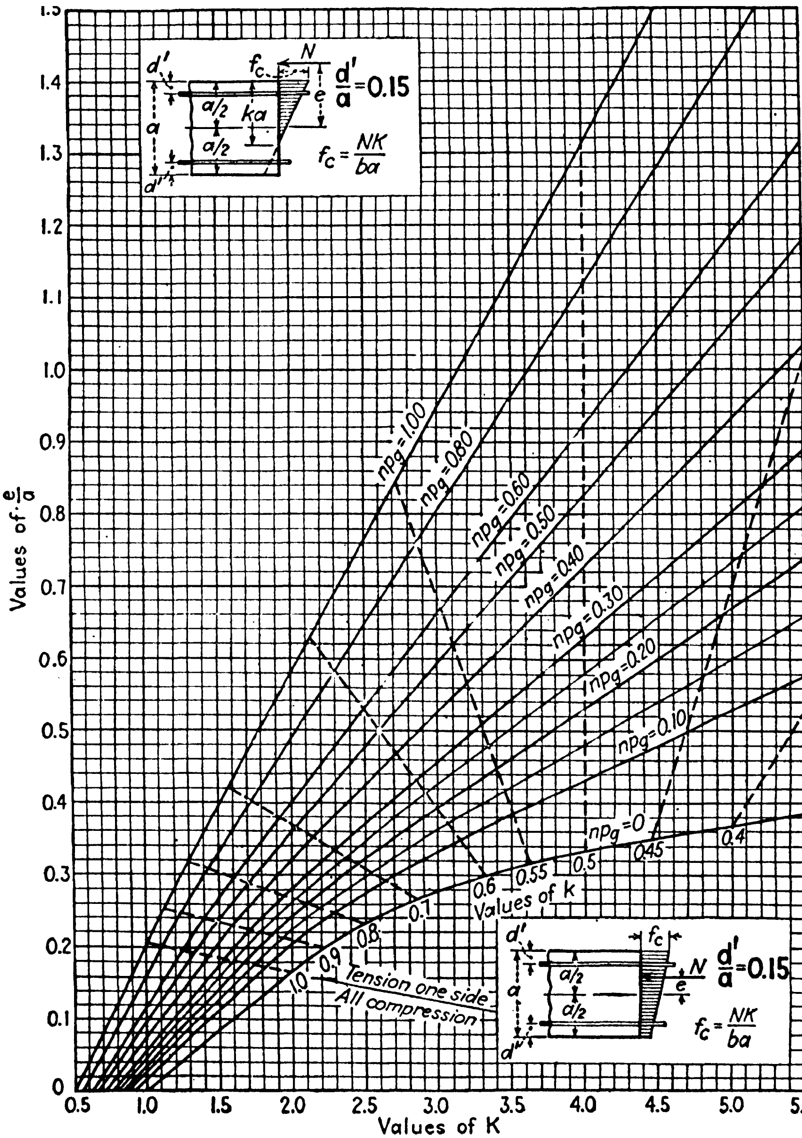


Fig. 177.—Bending and axial stress, rectangular sections.

Figs. 176 to 178. For either Case I or Case II, comparison between the computed f_c and the allowable unit stress, as given in Art. 189, will determine whether or not any revision of the column cross-section (or steel area) is necessary.

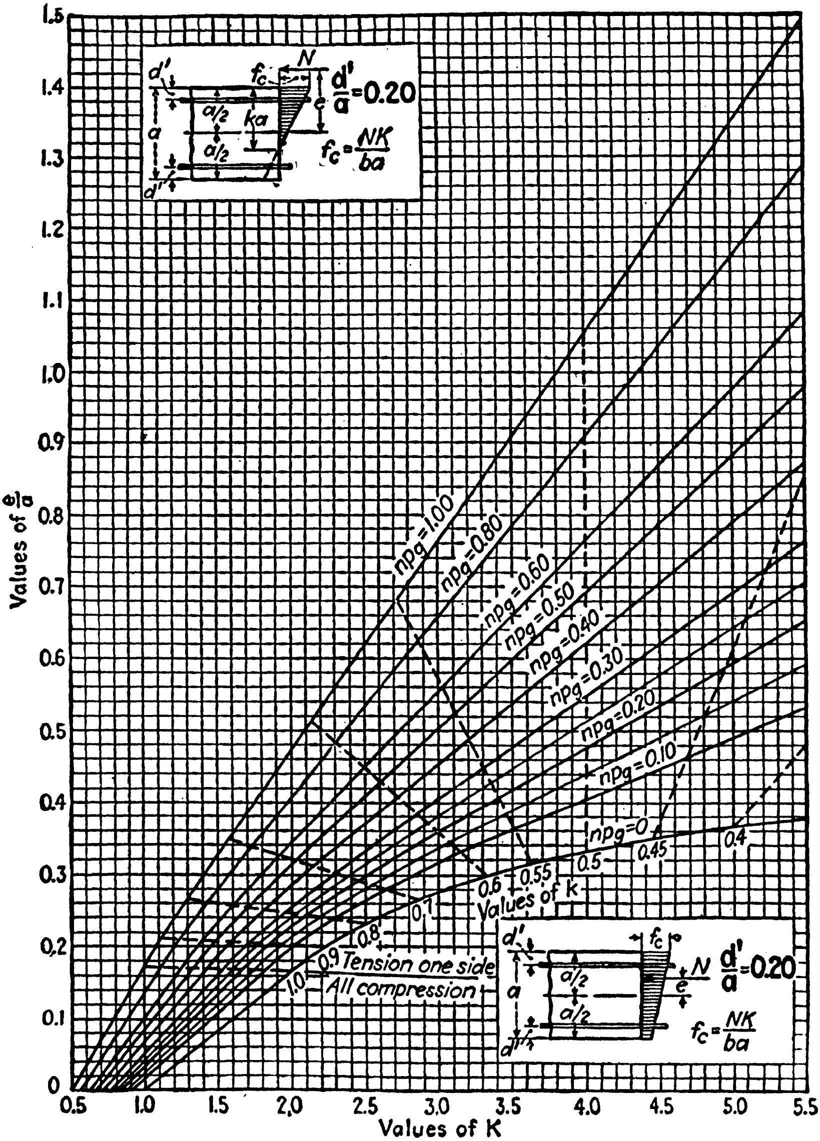


FIG. 178.—Bending and axial stress, rectangular sections.

195. Circular Sections. Compression over the Whole Section. The reinforcement in a circular section is practically always symmetrical; and although it is in the form of bars, it may be considered to be in the form of a cylinder whose average diameter is equal to the diameter of the circle on which the bars are placed and whose cross-sectional area is equal to the area of the bars (see Fig. 179).

Following the same procedure as for rectangular sections, with c equal to the radius r of the column,

$$f_c = \frac{N}{A + (n - 1)A_s} + \frac{Mr}{I_c + (n - 1)I_s} \tag{7}$$

Considering the longitudinal bars replaced by a cylinder of sectional area A_s, $I_s = A_sR^2 = A_ss^2/8$, where R is the radius

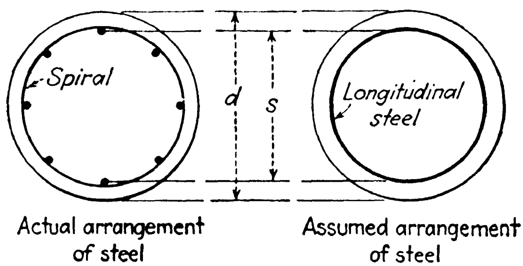


Fig. 179.

of gyration of the equivalent cylinder, whose average diameter s is the diameter of the circle on which the bars are placed.[1] Then the above equation becomes

$$f_c = \frac{N}{\pi r^2 + (n - 1)A_s} + \frac{Mr}{\frac{\pi r^4}{4} + (n - 1)A_s\frac{s^2}{8}} \tag{8}$$

These equations give an exact solution where there is no tension on the section. Equation (7) can be used in conjunction with the values given in Tables 3 and 4, or equation (8) can be used by making the proper numerical substitutions.

196. Circular Sections. Tension over Part of the Section. A direct solution by equations similar to those used for rectangular sections cannot be made for tension over part of a circular section. The expression of relations becomes very complicated. How-

[1] $R = \frac{A_ss^2}{8}$.

ever, by carrying the derivation partly to completion, values of some of the variables may be assumed, and from these other values determined. In such a manner Figs. 180 and 181 have been plotted. When the diagrams are entered with values of s/d

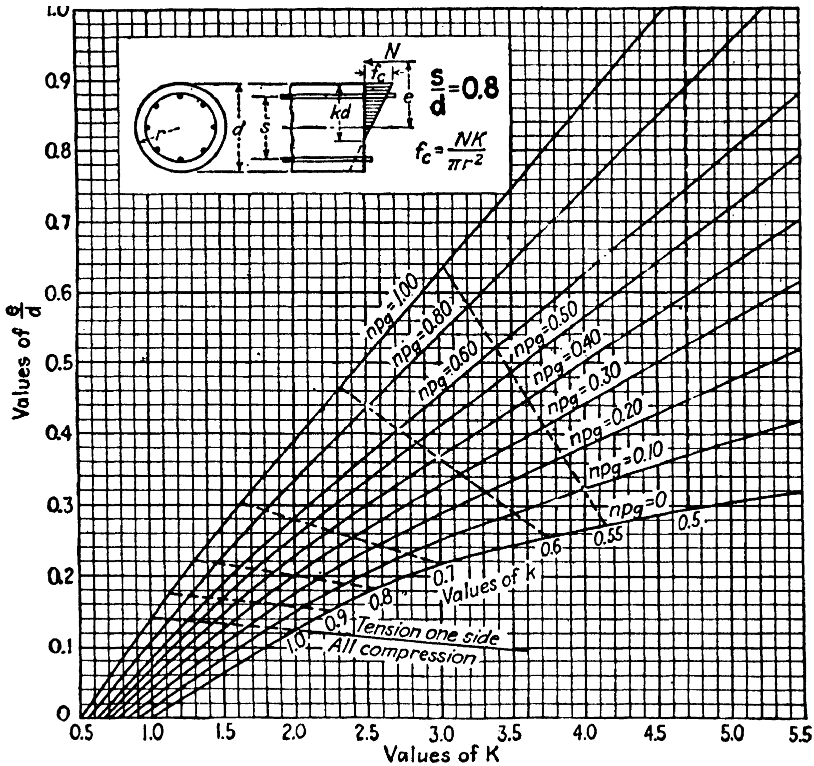


FIG. 180.—Bending and axial stress, circular sections.

and $p_g n$ as arguments, values of K are obtained for use in the equation

$$f_c = \frac{NK}{\pi r^2} \tag{9}$$

197. Examples. 1. A column reinforced with longitudinal steel and lateral ties is to support a direct axial load of 200,000 lb., and in addition an eccentric load of 25,000 lb. on a bracket whose center is 10 ft.-0 in. above the base of the column. The center of bearing of the 25,000-lb. load is 8 in. from the outside face of the column. The unsupported height of the column is 17 ft.-0 in. The ultimate strength of the concrete is specified as 2500 p.s.i., and intermediate-grade steel is to be used. Design the column.

From Table 2 a column 18 × 20 in. is selected for trial. This column with 2 per cent of steel will sustain 162 + 2 × 46 = 254 kips as a direct axial load as a short column. From Table 3, its weight is 20 × 18.8 = 376 lb. per ft., and the total load on the column is 200 + 25 + 17 × 0.376 = 231 kips. The value of h/d is $\frac{17 \times 12}{18} = 11.3$ so that as a long column it can sustain 1.3 − 0.03 × 11.3 = 0.96 of the safe load of a short column. 254 × 0.96 = 244 kips, showing that the column is safe for direct load.

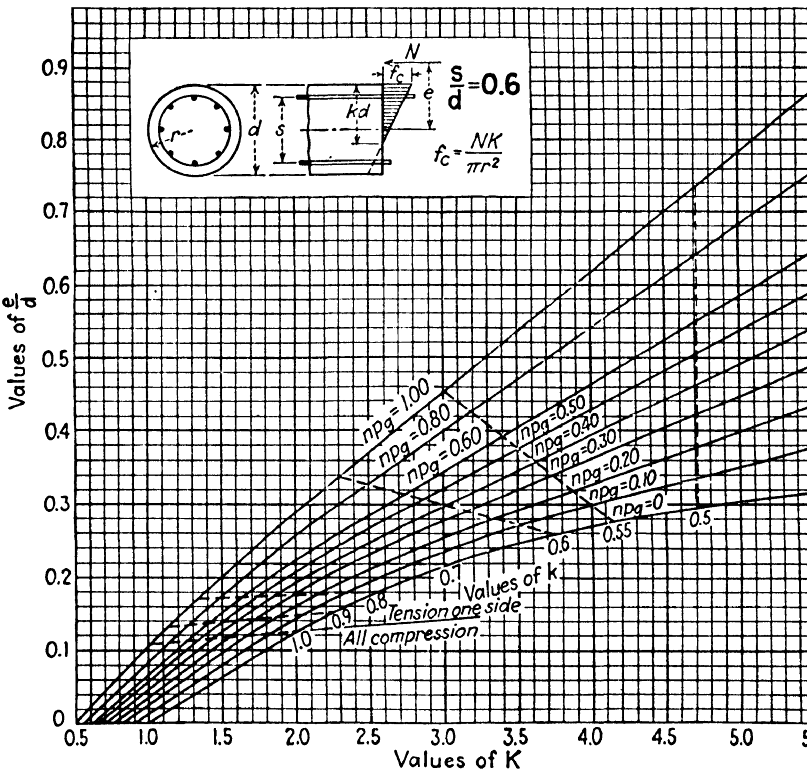


FIG. 181.—Bending and axial stress, circular sections.

Assuming that the bracket is on the 18-in. face of the column, if the eccentric load were applied at either the top or the bottom of the column the moment would be 25,000(10 + 8) = 450,000 in.-lb. Since, however, the load is applied 10 ft.-0 in. from the base of the column, from Fig. 168 (considering both ends fixed) the actual moment at the point of application is 0.54 × 450,000 = 243,000 in.-lb. Eight 1-in. square bars are selected for reinforcement. From Tables 3 and 5, the moment of inertia of the column section is 18 × 667 + 11 × 6 × 56 = 15,700 in.^{4}, and, from equation (4),

$$f_c = \frac{7 \times 376 + 225{,}000}{360 + 11 \times 8} + \frac{243{,}000 \times 10}{15{,}700} = 508 + 155 = 663 \text{ p.s.i.}$$

$$e = \frac{M}{N} = \frac{243{,}000}{227{,}600} = 1.07 \quad \text{and} \quad \frac{e}{a} = 1.07 \div 20 = 0.054$$

$$p_g = 8.00 \div 360 = 0.0222$$

From Fig. 171, the allowable unit stress f_c is 665 p.s.i., and the column as selected is satisfactory.

2. The column of Fig. 182, which is the same as the column designed in Example 1, supports a total load of 125,000 lb. This load is an eccentric load and produces a moment at the top of the column of 750,000 in.-lb. Compute the maximum value of f_c.

By comparison with Example 1, it is seen that the first term of equation (4) is decreased, but the second term is greatly increased, so that the latter becomes greater than the former, indicating tension over a part of the section.

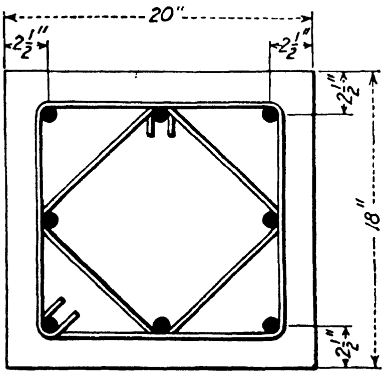


FIG. 182.

$$e = \frac{M}{N} = \frac{750{,}000}{125{,}000} = 6.0 \text{ in.}$$

$$\frac{e}{a} = \frac{6.0}{20} = 0.30$$

$$\frac{d'}{a} = 0.22$$

$$np_g = 0.27$$

From Fig. 178, $K = 2.35$, and

$$f_c = \frac{125{,}000}{360} \times 2.35 = 820 \text{ p.s.i.}$$

From Fig. 171, the allowable stress f_c is 845 p.s.i.

ADDITIONAL PROBLEMS

1. A square column reinforced with longitudinal steel and lateral ties supports a load on a bracket of 100,000 lb., the center of bearing of the load being 4 in. from the face of the column and the top of the bracket being 8 ft. above the base of the column whose unsupported length is 14 ft.-0 in. In addition, it supports a direct axial load of 75,000 lb. Design the column for $f'_c = 3000$ and $f_s = 16{,}000$.

2. A circular column with spirals sustains a bending moment at its base of 50,000 ft.-lb. In addition, at the top of the column, whose unsupported length is 15 ft.-0 in., there is a direct axial load of 250,000 lb. Design the column for $f'_c = 2500$ and $f_s = 20{,}000$.

3. A circular column with spirals has a diameter of 20 in. and is reinforced with ten 1-in. round bars. It sustains a direct load of 400,000 lb. $f'_c = 3750$; $f_s = 16{,}000$. What bending moment can it sustain at the top, its unsupported length being 12 ft.-0 in.?

CHAPTER XII

FOOTINGS

198. Types of Footings. A footing is required under columns, walls, and other similar structural members which rest on soil, in order to distribute the load over an area sufficiently large to prevent overstressing of the soil. Footings for structures are divided into three main classes: (1) wall footings; (2) single-column footings; (3) multiple-column footings, or those which support more than one column. Footings are mostly made of concrete, either plain or reinforced, although quite frequently grillages consisting of two or more tiers of steel beams encased in concrete are used. Occasionally similar grillages made of timber may prove satisfactory.

Single-column footings should be centered under the columns which they support, in order to avoid unsymmetrical resisting pressures, which would result in uneven settlement of the footing. Footings supporting more than one column should be so proportioned that any settlement which might take place will be uniform, in order to prevent the obvious damage to the superstructure which would otherwise occur.

199. Bearing Capacity of Soils. The sustaining power of earth formations depends mainly upon the composition, the amount of moisture contained, and the degree of confinement in the mass. No definite values can be given to the safe bearing capacity of different classes of soils because of the many variables which of necessity must be considered. Unless the bearing capacity of the material at a given site is already known, it should be determined by direct tests, if this is at all feasible. A satisfactory test for ordinary structures can be made by erecting a heavy post on a bearing plate of known area (1 to 4 sq. ft.), placed preferably at the same elevation as that of the proposed foundation, then applying loads to the top of the post, and measuring the settlement, if any. Some fraction (usually one-fourth to one-half) of the load which produces appreciable settlement is taken as the maximum pressure to be used in the subsequent footing design. Just what is an appreciable settlement is dependent somewhat upon the type of structure under consideration. A settlement of ¼ in. may be considered large for an ornamental structure, such as a cathedral, whereas a settlement of 1 in. or more may be of no consequence in a heavy concrete warehouse. Very heavy or unusually tall structures should be supported on rock, if this is economically possible, in order to avoid any settlement; in spite of all precautions, any settlement is apt not to be uniform, and uneven settlement is a serious matter in tall buildings or structures with much ornamental trim.

In the absence of any satisfactory test, the following limiting values taken from the Building Code of the National Board of Fire Underwriters may be used as a guide in selecting the bearing capacity of any given foundation bed.

RECOMMENDED BEARING CAPACITIES, TONS PER SQ. FT.

Soft clay	1	Coarse sand	4
Firm clay	2	Gravel	6
Wet sand	2	Soft rock	8
Sand and clay, mixed or in layers	2	Hardpan	10
Fine dry sand	3	Medium rock	15
		Hard rock	40

200. Plain-concrete Footings. The area of the base of a plain-concrete footing is obtained by dividing the total load on the footing, including its own weight, by the allowable unit soil pressure. The top area must be large enough to provide for proper distribution of the load from the wall or column (see Art. 212). Where there is a great difference between the area of the top and base, the upper surface of the footing is usually sloped or stepped.

The depth of the footing must be sufficient to keep the tension in the concrete within the allowable limit, which is given in the Joint Code as $0.03f'_c$. The depth must also be sufficient so that the unit shearing stress, computed at sections as prescribed in Art. 208 for reinforced-concrete footings, shall not exceed $0.02f'_c$. The bending moment is computed in the manner outlined in

Art. 203 for wall footings and in Art. 206 for column footings. The critical shear is computed as in Art. 203 for wall footings and in Art. 208 for column footings. In general, a plain concrete footing will be safe structurally if the length of any projection is not more than one-half the thickness of the projection.

201. Pedestals. Quite frequently a footing is required to be placed at some distance below the column that it supports. This is particularly true of footings for structural-steel columns, where the footing has to be located at some distance below the basement floor in order to rest on a suitable bearing stratum. In such cases, a pedestal is placed on the footing, and the column rests on top of the pedestal. The allowable compressive unit stress on the gross area of a plain concrete pedestal, as specified in the Joint Code, is $0.25f'_c$. Where this stress is exceeded, vertical reinforcement must be provided, and the pedestal designed as a reinforced-concrete column.

Short pedestals are also frequently required on top of the footings supporting reinforced-concrete columns, in order to provide sufficient length of embedment for the dowels that are used to complete the transfer of stress from the columns to the footing (see Art. 212). Short pedestals of this nature are required to be poured monolithically with the footings of which they are a part.

202. Reinforced-concrete Footings. In the majority of cases reinforced concrete is preferable to plain concrete for footing construction owing to the saving in excavation, in material, and in weight of the foundation itself. This is the result of the smaller depth required to provide for the existing bending and shearing stresses. The remainder of this chapter will be devoted to analysis of the various types of reinforced-concrete footings.

203. Analysis of Wall Footings. The principles of beam action are, in general, applicable to wall footings. Figure 183 shows a wall footing and a typical set of external forces acting upon it. For footings under concrete walls the critical section for moment is taken at the face of the wall. For footings supporting masonry walls (cement blocks, brick, or stone) the critical section is taken halfway between the middle and edge of the wall.

Assuming that the wall in Fig. 183 is of concrete, the critical moment for a 1-ft. length of footing is

$$M = \tfrac{1}{8}w(l - a)^2 \qquad (1)$$

The effective depth and the area of steel required to provide for this moment are computed from the same formulas as for rectangular reinforced-concrete beams (Art. 148), using a value of $b = 12$ in.; the resulting value of A_s is the area of steel required in a 12-in. length of wall.

The maximum bond stress on the bars in the bottom of the footing is computed as for ordinary rectangular concrete beams (see Art. 167), assuming that the critical section is at the face of the wall. Bars in footings are invariably hooked at each end, so that the allowable unit bond stress is the ordinary stress for beams (Art. 167) increased by 50 per cent because of the extra anchorage. The resulting values are $0.075f'_c$ for deformed bars and $0.06f'_c$ for plain bars.

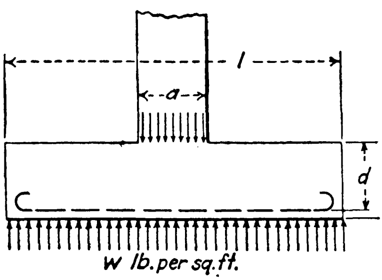


FIG. 183.

Test results and certain analytical considerations indicate that it is satisfactory to consider that the critical section for diagonal tension is at a distance d from the face of the wall (d being the distance from the center of the reinforcing bars to the top of the footing at the edge of the wall). The unit shear at this section is computed as for ordinary rectangular reinforced-concrete beams (Art. 163), and this is compared with the allowable shearing stresses to determine whether or not the footing has adequate resistance to diagonal tension. Web reinforcement is difficult to place in a footing, and its effect is open to question; hence, it is customary to design the footing so that the unit shearing stress (which is a measure of the diagonal tension) is less than the limit of the working stress permitted in beams without web reinforcement. This limit, which includes an allowance for the added resistance furnished by the hooks on the bars, is $0.03f'_c$.

In all design computations, except those for determining the required bearing area, the weight of the footing is neglected since this weight has no effect on values of moments and shears; *i.e.*, the extra upward pressure caused by the weight of the footing is exactly counterbalanced by the downward force of this weight.

204. Design of a Typical Wall Footing. A 16-in. concrete wall supports a total load of 23,100 lb. per lin. ft. and rests on soil whose safe bearing power is 2 tons per sq. ft. Design a footing for this wall that will satisfy the requirements of the Joint Code. A 2000-lb. concrete is to be used; $f_s = 18{,}000$ p.s.i.

Assuming the weight of the footing as 900 lb. per lin. ft., the total width of footing required = 24,000/4,000 = 6.0 ft. The net upward pressure on the footing = 23,100/6 = 3850 lb. per sq. ft. The moment at the critical section, for a strip 12 in. wide, is

$$M = 3850 \times \frac{(6 - 1.33)^2}{8} \times 12 = 126{,}000 \text{ in.-lb.}$$

With allowable unit stresses of 18,000 and 800, Table 2, page 226, gives $K = 139$ and $j = 0.867$. The required effective depth is

$$d = \sqrt{\frac{126{,}000}{139 \times 12}} = 8.7 \text{ in.}$$

An effective depth of 9 in. is used, which, with 3-in. insulation, gives a total thickness of 12 in. The weight per linear foot is then 900 lb. as assumed.

$$A_s = \frac{126{,}000}{18{,}000 \times 0.867 \times 9} = 0.90 \text{ sq. in. per ft.}$$

Assuming deformed bars anchored at both ends by means of hooks,

$$\Sigma_0 = \frac{2.33 \times 3850}{150 \times 7/8 \times 9} = 7.6 \text{ in. per ft.}$$

These requirements are satisfied by using ½-in. square bars, 3 in. center to center. Investigating for diagonal tension, the total external shear on a section 9 in. from the face of the wall is $\frac{23{,}100}{6} \times \frac{19}{12} = 6100$ lb. and the unit shear, which is a measure of diagonal tension, is

$$v = \frac{6100}{12 \times 7/8 \times 9} = 64 \text{ p.s.i.}$$

This is but 4 lb. in excess of the allowable $(0.03 \times 2000 = 60)$, and the design may be considered satisfactory.

SINGLE-COLUMN FOOTINGS

205. Computation of Bearing Area. Ordinarily, in single-column footings the load may be considered as applied uniformly over the bearing area of the column, and if the footing is symmetrically placed under the column the upward pressure is usually considered to be uniformly distributed over the base of the footing. The footing is then analogous to a cantilever slab supported at the top over a central area and loaded with a uniform upward load. As the projecting portion of the footing deflects upward, its surface assumes the shape of a bowl. Steel is required in the bottom of the footing to resist the resulting tension stresses. This steel is placed in two directions, as shown in Fig. 186,[1] the bars in one direction resting directly on top of those in the other direction.

The required area of the footing is obtained by dividing the total load on the footing (including its own weight) by the allowable unit soil pressure. The weight of a single-column footing will generally vary from 6 to 10 per cent of the column load. For maximum economy, a footing supporting a square or a round column should be square, whereas the sides of a footing supporting a rectangular column should be somewhat longer on the side parallel to the long side of the column than in the other direction, the sides of the footing being approximately in the same ratio as the sides of the column.

206. Bending Moment. The moment at any section is determined by passing through the section a vertical plane, which extends completely across the footing, and computing the moment of the forces acting over the entire area of the footing on one side of the plane. The critical section for moment is at the face of the column (or pedestal) for footings supporting a

[1] Occasionally footing reinforcement is placed in four directions. This is unusual in modern practice, however. The method of design is the same as for two-way footings, except that the steel area required in each direction is divided into one band parallel to the axis of the footing and two half bands in the diagonal direction, with practically all of the bars passing under the column.

single concrete column (or pedestal), or halfway between the face of the column and the edge of the metallic base for footings under metallic bases. The width resisting compression at any section is taken as the full width of the top of the footing at the section under consideration.

The effective depth required to prevent overstressing the concrete in compression is computed from the rectangular-beam equation [equation (3) Art. 148], $M = Kbd^2$. The allowable extreme fiber stress in compression is the same as for ordinary beams. Because of the large effective width, except for sloped-top footings (see Art. 211) the effective depth obtained from this computation would not be sufficient to provide adequate resistance to diagonal tension, and usually the depth selected is obtained from the latter requirement, as explained in Art. 208.

After the effective depth has been selected, the steel area required in each direction is obtained from equation (2), Art. 148, $M = A_s f_s jd$, an approximate value of $j = 0.9$ being used. Further refinement in computing the value of j is not justified, because of the assumptions made in computing the moment and in selecting the effective width b. The moment to be used in the computation of A_s in each direction is specified as 85 per cent of the moment obtained in the manner described at the beginning of this article.

207. Placing the Reinforcement. In square footings the required reinforcement in each direction is distributed uniformly across the full width of the footing. In rectangular footings the reinforcement in the long direction is distributed across the full width of the footing; in the short direction a portion ΔA_s of the total required reinforcement A_s is uniformly distributed in a band having a width equal to the length of the short side of the footing and centered with respect to the center line of the column. The portion ΔA_s is computed from the following equation:

$$\Delta A_s = A_s \left(\frac{2}{S + 1}\right) \qquad (2)$$

in which S = the ratio of the long to the short side of the footing. The remainder of the required reinforcement in the short direction is uniformly distributed in the outer portions of the footing.

The Joint Code requires that the reinforcement in footings or other principal structural members in which the concrete is deposited against the ground shall have not less than 3 in. of concrete between it and the ground contact surface. In ordinary computations, the effective depth is taken as 4 in. less than the total thickness of the footing, and this value is used in computations for both directions, even though actually the d in one direction is less than that in the other direction by an amount equal to the diameter of the bars.

208. Diagonal Tension. Tests indicate that diagonal tension develops in critical amounts at a distance from the face of the column equal to the effective depth d of the footing. Hence, the critical section for shear is assumed as a vertical section obtained by passing a series of vertical planes through the footing, each of which is parallel to a corresponding face of the column or pedestal and located a distance therefrom equal to the effective depth of the footing. According to the Joint Code, each face of the critical section for shear shall be considered as resisting an external shear equal to the net upward pressure on an area bounded by that face, two diagonal lines drawn from the corners of the column or pedestal at angles of 45 degrees with the principal axes of the footing, and that portion of the corresponding edge or edges of the footing intercepted between the two diagonals. Thus, in Fig. 184, the critical section for shear is the vertical section $ABCD$; the shear resisted by the face AB is the net upward load on the shaded area ab; and the shear resisted by the face BC is the net upward load on the shaded area bc.

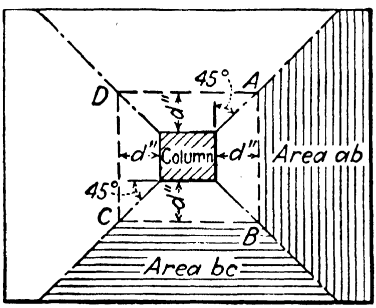


FIG. 184.

The unit shear, which is a measure of diagonal tension, is computed from equation (12), Art. 163, $v = \frac{V}{bjd}$, using for b the width of the face of the critical section and for d the depth at the face of the critical section above the plane of the reinforcement. For single block footings the effective depth d is the same for all sections in the footing, but for footings with a sloping top the effective depth d obviously varies with the location of the section. The allowable unit shear is specified in the Joint Code as $0.03f'_c$, with a maximum of 75 p.s.i.

The depth of the footing is usually governed by the shearing strength required; hence, in design, the above equation is used to determine the unknown effective depth for the known values of v, b, and V. This is in reality a "cut-and-try" process, since the values of b and V depend upon the value of d, which must therefore be assumed before the computations can be completed.

Web reinforcement is ordinarily undesirable in a single-column footing because of the uncertainty of its effectiveness and the difficulty of placing it. The specified maximum value of v, as given above, is therefore the value specified for ordinary beams without web reinforcement but with the bars hooked, and with an added arbitrary limit of 75 p.s.i.

209. Bond Stresses. Bond resistance is one of the most important factors governing the strength of concrete footings. The critical sections for bond are assumed at the face of the column or pedestal and at vertical planes where changes in section occur. The critical unit bond stress for each band of reinforcing bars is computed from equation (16), Art. 167, $u = V/\Sigma_0 jd$, considering the circumference of all of the bars in the band and assuming V as 0.85 of the same total net upward load that was used in computing the bending moment on the section under consideration. This load is defined in Art. 206.

The allowable unit bond stress as specified for ordinary beams with hooked bars is reduced by 25 per cent because of the loss in bond area at the intersections of the bars. The resulting allowable unit stress is $0.045f'_c$ for plain bars and $0.056f'_c$ for deformed bars.

210. Footings Supporting Round Columns. The preceding discussions have been based upon the assumption that the footings are supporting square or rectangular columns or metallic bases. In computing the stresses (bending, shear, and bond) in footings which support a round or octagonal concrete column or pedestal, the "face" of the column or pedestal shall be taken as the side of a square having an area equal to the area enclosed within the perimeter of the column or pedestal. Computations are carried out in the same manner as for footings supporting square or rectangular columns, the equivalent square column being used in place of the actual round column.

211. Stepped and Sloped Footings. If the depth required for moment is large, the footing may be sloped between the edge of the column (preferably 3 or 4 in. from the edge of the column) and the edge of the footing, provided that the actual depth to the steel at the critical section for diagonal tension is made sufficient to provide fully for this diagonal tension in accordance with the method outlined in Art. 208, and further provided that the top area of the footing is made sufficiently large to satisfy other requirements as given in Art. 212.

The Joint Code specifies that in sloped footings the thickness above the reinforcement at the edge of the footing shall be not less than 6 in. for footings on soil or 12 in. for footings on piles. The top of the footing may be stepped instead of sloped, provided the steps are so placed that the footing will have at all sections a depth at least as great as that required for a sloping top. Stepped footings must be cast monolithically. The value of b (see Art. 206) to be used in analyzing sloped or stepped footings for bending shall not exceed the width of the flat top of the footing, since this is the maximum width over which the full effective depth d, for moment, is available.

212. Transfer of Stress at Base of Column. The compressive stress in the longitudinal reinforcement at the base of a reinforced-concrete column is transferred to the pedestal or footing by means of dowels. There should be at least one dowel for each column bar, and the total sectional area of the dowels should be not less than the sectional area of the longitudinal reinforcement in the column. If f'_c is 3000 p.s.i. or greater, the dowels should extend into the column and into the pedestal or footing not less than 30 diameters of the dowel bars for plain bars or 24 diameters for deformed bars. If f'_c is less than 3000 p.s.i., both of these distances must be increased 25 per cent. If the footing thickness is not sufficient to furnish the required embedment, plus insulation, a pedestal must be used; the height of the pedestal must be sufficient to provide for the deficiency in the footing thickness, and the area must be sufficient to satisfy the requirements of the following paragraph.

The Joint Code specifies that the permissible compressive unit stress r_a on top of the pedestal or footing directly under the column shall be not greater than that determined by the formula

$$r_a = 0.25f'_c \sqrt[3]{\frac{A}{A'}} \tag{3}$$

in which A = total area of top of pedestal or footing.
A' = loaded area of pedestal or footing at the column base.

In sloped or stepped footings, A may be taken as the area of the top horizontal surface of the footing or as the area of the lower base of the largest frustum of a pyramid or cone contained wholly within the footing and having for its upper base the loaded area A', and having side slopes of 1 vertical to 2 horizontal. The Code further specifies that the unit stress on the gross area of a concentrically loaded pedestal shall not be greater than $0.25f'_c$. Pedestals used for the purpose of furnished depth for embedment of dowels must be poured monolithically with the footing.

213. Design of a Two-way Block Footing. A column 24 in. square supports a total load of 400,000 lb. Design a single-slab concrete footing, reinforced in two directions, to support this column on a soil which has a safe bearing capacity of 5000 lb. per sq. ft. A 2500-lb. concrete and intermediate-grade steel are to be used.

Assuming the weight of the footing as 25,000 lb., the bearing area required is 425,000/5000 = 85 sq. ft. A base 9 ft.-3 in. square, furnishing 85.5 sq. ft., is selected. The unit pressure due to the load on the column only is 400,000/85.5 = 4680 lb. per sq. ft.

The depth of the footing is governed by the shearing stresses; but, in order to locate the critical section for shear, the depth must be known or assumed. An approximate value can be obtained by first computing the depth required for the bending moment.

The maximum moment (see Fig. 185) is

$$M = 4680 \times 9.25 \times 3.625 \times \frac{3.625}{2} \times 12 = 3{,}420{,}000 \text{ in.-lb.}$$

From Table 2, page 226, $K = 164$; and from equation (3), Art. 148,

$$d = \sqrt{\frac{3{,}420{,}000}{9.25 \times 12 \times 164}} = 13.7 \text{ in.}$$

According to preceding discussions, the effective depth required for shear will be greater than 13.7 in. Assume that 18 in. will be adequate. The critical section for shear (Fig. 185) is then 18 in. from the face of the column;

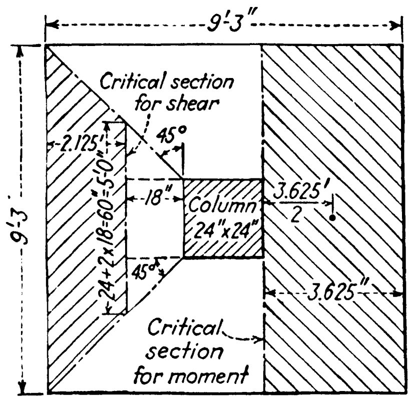


FIG. 185.

the width of this section is $24 + 2 \times 18 = 60$ in.; and the total external shear on the section is

$$V = \frac{5.0 + 9.25}{2} \times 2.125 \times 4680 = 71{,}000 \text{ lb.}$$

The allowable unit shear is 75 p.s.i. Hence,

$$v = \frac{71{,}000}{60 \times 0.9 \times d} = 75$$

from which $d = 17.6$ in., or 18 in., as assumed.

Any required revision in the assumed value of d would change not only the total shear V but also the effective shearing width b in the above equations. The total height of the footing, allowing 4 in. of insulation below the center of the steel, as recommended in Art. 207, is 22 in., and the weight of the footing is 23,600 lb., approximately as assumed. No revision of the bearing area is necessary.

The moment to be used in computing the steel area required in each direction is 0.85 × 3,420,000 = 2,910,000 in.-lb.

$$A_s = \frac{2,910,000}{20,000 \times 0.9 \times 18} = 9.0 \text{ sq. in.}$$

The allowable unit bond stress is 0.056 × 2500 = 140 p.s.i. The maximum shear to be used in the equation for unit bond stress is 0.85 of the net upward load on the rectangular area on one side of a vertical section extend-

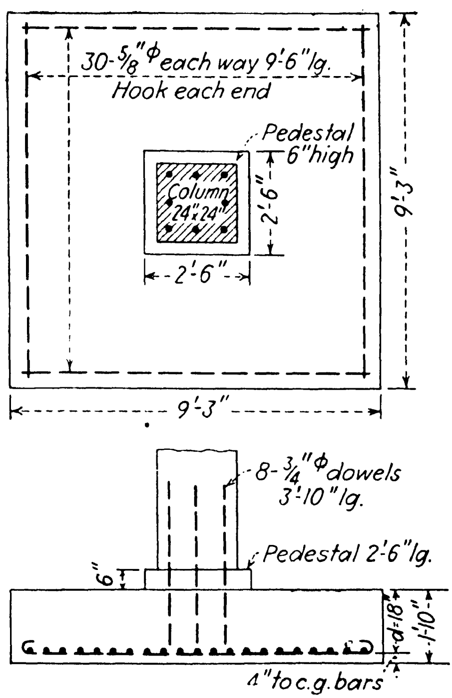


FIG. 186.

ing across the footing in the plane of one face of the column, or

$$0.85 \times 9.25 \times 3.625 \times 4680 = 133,000 \text{ lb.}$$

$$\Sigma_0 \text{ (required)} = \frac{133,000}{140 \times 0.9 \times 18} = 58.6 \text{ in.}$$

Thirty 5/8-in. round bars furnish an area of 9.21 sq. in. and a total perimeter of 58.9 in., which satisfy both the above requirements. Allowing about 4 in. of insulation to the edges of the footing, the spacing of the bars is approximately 3½ in., which is satisfactory.

Dowels will be placed in the footing to transfer the stress from the column bars to the footing. According to Art. 212, these dowels should be of the same size and the same number as the bars in the column. They should extend into the footing and into the column a distance of 30 diameters. Assuming that the column is reinforced with eight ¾-in. round bars, eight ¾-in. round dowels 3 ft.-10 in. long will be required. In order to furnish sufficient depth for embedment in the footing, a pedestal 6 in. high is required. The required area of the pedestal (see Art. 201) is $\frac{400{,}000}{625} = 640$ sq. in. The pedestal will be made 30 in. square (area 900 sq. in.), which will give a 3-in. projection beyond the column. The unit stress on the loaded area is $\frac{400{,}000}{24 \times 24} = 692$ p.s.i., and the allowable stress on this area (see Art. 212) is $r_a = 0.25 \times 2500 \sqrt[3]{900/640} = 700$ p.s.i. Complete details of the footing are shown in Fig. 186.

MULTIPLE-COLUMN FOOTINGS

214. Types of Multiple-column Footings. It is sometimes desirable and quite frequently necessary to support more than one column on the same footing. A multiple-column footing is necessary when two columns with fairly large loads are so close together that there is not sufficient space between the columns for two independent footings of economical proportions, or when the face of an exterior column coincides with, or is close to, the building line, thus making it impossible to center an independent footing under this column. In the latter case, an eccentric footing would result in unequal distribution of the pressure on the soil, with the possibility of uneven settlement, and bending in the columns; a single footing supporting the exterior column and the adjacent interior column may be so proportioned as to make the center of gravity of the footing area coincide with the center of gravity of the column loads and thus theoretically secure the uniform soil pressure which is essential to a satisfactory footing design. A footing of this type is called a combined footing.

A combined footing may be rectangular in plan or trapezoidal. The former shape is suitable if the interior column has the greater load and if the footing may be extended beyond this column as far as necessary. The trapezoidal shape is required if the column loads are unequal and if for any reason the footing cannot extend any appreciable distance beyond *either* column. The critical sections for diagonal tension in combined footings are the same as for single-column footings (see Art. 208), and the transfer of the stress at the base of the columns is accomplished in the same manner. The methods of computing critical moments are explained in Arts. 215 and 216.

215. Design of a Combined Footing Supporting Two Equal Column Loads. Two columns, each 24 in. square and each carrying a total load of 310,000 lb., are spaced 10 ft.-0 in. center to center, as shown in Fig. 187. Design a rectangular combined footing to support these two columns. The allowable soil pressure is 4000 lb. per sq. ft.; a 2000-lb. concrete is to be used; $f_s = 18{,}000$ p.s.i.

The proposed method of design considers that the load from each column will be carried by a beam in the direction of the width of the footing (*i.e.*, at right angles to the line of columns) and that this load will then be distributed longitudinally throughout the length of the footing by a band of steel in the longitudinal direction. Assuming the weight of the footing as $0.06 \times 2 \times 310{,}000 = 37{,}000$ lb., the bearing area required is

$$\frac{2 \times 310{,}000 + 37{,}000}{4000} = 164.3 \text{ sq. ft.}$$

The proper selection of width and length is governed by several factors. If the footing is too long, the moment in the portions projecting beyond the columns in the longitudinal direction will be excessive and the unit shearing and bond stresses in the transverse beam will be unduly large. If the footing is too short, the negative moment between the columns in the longitudinal direction will be too great for an economical design. A short footing will necessitate an excessive width, which will cause an unduly large moment at the edges of the columns in the transverse direction. Two or three trial designs may be necessary to secure the proportions which will give the most economical footing, considering excavation, concrete, and steel. The size finally selected in the present case is 18 ft.-9 in. by 8 ft.-9 in. (area = 164.1 sq. ft.).

The effective width of the longitudinal band (for both tension and compression) is equal to the full width of the footing. The effective compression width of the transverse beam can be assumed equal to $1\frac{1}{2}$ to 2 times the width of the column, whereas the steel in this direction can be spread out over a width somewhat greater than the column width plus twice the effective depth.

Design of Transverse Direction. In concrete combined-footing designs it is customary to consider, as in single-footing designs, that the critical sections for moment in the cantilever portions of the footing are at the edges of the columns or pedestals. At the edge of each column, the moment in the transverse beam is

$$M = \frac{310{,}000}{8.75} \times \frac{(3.375)^2}{2} \times 12 = 2{,}420{,}000 \text{ in.-lb.}$$

Assuming $b = 2 \times 24 = 48$ in., the required effective depth is

$$d = \sqrt{\frac{2{,}420{,}000}{139 \times 48}} = 19 \text{ in.}$$

Later investigation shows that an effective depth of 21 in. is required to avoid the necessity of using web reinforcement in the transverse beam. This increased depth will not only avoid the necessity of stirrups but also reduce the steel area required.

$$A_s = \frac{2{,}420{,}000}{18{,}000 \times 0.9 \times 21} = 7.12 \text{ sq. in.}$$

At the edge of the column, $V = 310{,}000/8.75 \times 3.375 = 119{,}500$ lb. Since the transverse bars will be in contact with the bottom longitudinal bars, the allowable unit bond stress, assuming proper anchorage at each end of each bar, as recommended in Art. 209, is 112 p.s.i.

$$\Sigma_0 = \frac{119{,}500}{112 \times 0.9 \times 21} = 56.4 \text{ in.}$$

Twenty-nine ½-in. square bars are selected and placed in a width of 72 in., a spacing of about 2½ in. being thus the result.

The total shear at a distance of 21 in. from the edge of the column is

$$V = \frac{310{,}000}{8.75}\left(3.375 - \frac{21}{12}\right) = 57{,}500 \text{ lb.}$$

$$v = \frac{57{,}500}{48 \times 0.9 \times 21} = 63 \text{ p.s.i.}$$

If no web reinforcement is used, the allowable unit shear, with anchored bars, is $0.03 \times 2000 = 60$ p.s.i. The above value may be considered satisfactory.

Design of Longitudinal Direction. The moment at the outer edge of the transverse beam is

$$M = \frac{620{,}000}{18.75} \times \frac{(2.375)^2}{2} \times 12 = 1{,}120{,}000 \text{ in.-lb.}$$

At the center of the footing,

$$M = \left(310{,}000 \times \frac{9.375}{2} - 310{,}000 \times 5\right) 12 = 1{,}200{,}000 \text{ in.-lb.}$$

The latter value governs, and the effective depth required is

$$d = \sqrt{\frac{1{,}200{,}000}{139 \times 8.75 \times 12}} = 9.1 \text{ in.}$$

Though a total thickness of $9\frac{1}{2} + 3 = 12\frac{1}{2}$ in. would be satisfactory for this part of the footing, the required steel area would be reduced by a greater thickness, and the shearing stresses would also be lowered. An effective depth of 12 in. will be selected. The bottom longitudinal bars will be placed underneath the transverse bars. Allowing 3 in. of insulation below the center of the lower bars, the total thickness of the longitudinal slab is 15 in.

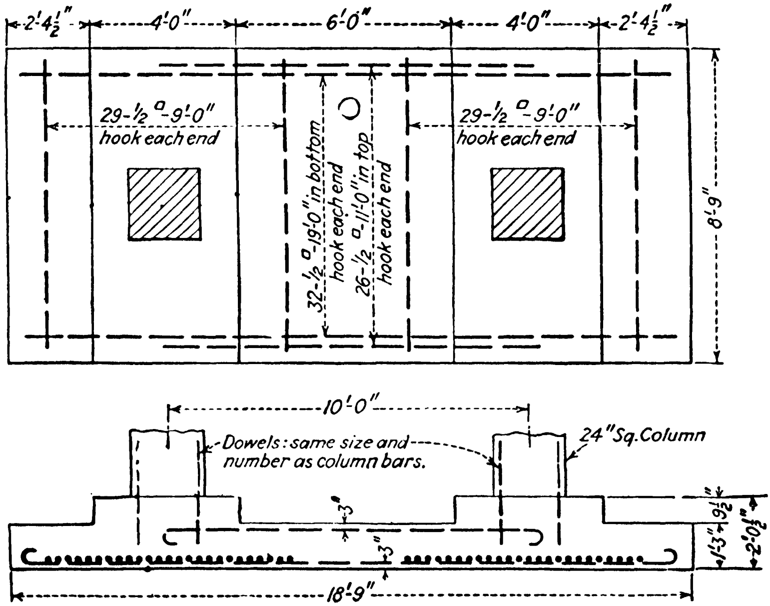


Fig. 187.—Combined footing with two equal column loads.

At the center of the footing,

$$A_s = \frac{1{,}200{,}000}{18{,}000 \times 0.9 \times 12} = 6.18 \text{ sq. in.}$$

An examination of the bending-moment diagram for the longitudinal direction shows that the maximum bond stress on these bars (the top bars between the columns) occurs at the point of inflection. Assuming this point to be at a distance of $\frac{1}{4} \times 10 = 2\frac{1}{2}$ ft. from the center line of the column, the shear at this point is

$$V = \frac{620{,}000}{18.75}(4.375 + 2.5) - 310{,}000 = -83{,}000 \text{ lb.}$$

The allowable unit bond stress, assuming hooked ends, is 150 p.s.i., since these bars are not in contact with bars at right angles to them.

$$\Sigma_0 = \frac{83,000}{150 \times 0.9 \times 12} = 51.1 \text{ in.}$$

Twenty-six ½-in. square bars are selected. These are placed 3 in. from the top of the footing and will extend between the column center lines, with standard hooks at each end.

At the outer edge of the transverse beam,

$$A_s = \frac{1,120,000}{18,000 \times 0.9 \times 12} = 5.78 \text{ sq. in.}$$

$$V = \frac{620,000}{18.75} \times 2.375 = 78,500 \text{ lb.}$$

The allowable unit bond stress here is 112 p.s.i., since there are two intersecting layers of bars.

$$\Sigma_0 = \frac{78,500}{112 \times 0.9 \times 12} = 64.8 \text{ in.}$$

At the outer edge of the column,

$$M = \frac{620,000}{18.75} \times \frac{(3.375)^2}{2} \times 12 = 2,250,000 \text{ in.-lb.}$$

$$d \text{ (required)} = \sqrt{\frac{2,250,000}{139 \times 8.75 \times 12}} = 12.4 \text{ in.}$$

The effective depth furnished at this section, assuming ½-in. bars, is ½ in. greater than that of the transverse beam, or 21½ in.

$$A_s = \frac{2,250,000}{18,000 \times 0.9 \times 21.5} = 6.45 \text{ sq. in.}$$

$$V = \frac{620,000}{18.75} \times 3.375 = 111,500 \text{ lb.}$$

$$\Sigma_0 = \frac{111,500}{112 \times 0.9 \times 21.5} = 51.4 \text{ in.}$$

The above computations show that, for the bottom longitudinal bars, the required area is governed by the section at the outer edge of the column (A_s = 6.45 sq. in.) and the required total perimeter is governed by the section at the outer edge of the transverse beam (Σ_0 = 64.8 in.). Thirty-two ½-in. square bars, hooked at the ends, are required. The approximate spacing is about 3¼ in. These bars should extend from the end of the footing to a section about 20 diameters beyond the point of inflection. Since this would leave but a short length between the bars near the center of the footing, in order to simplify the placing without adding materially to the weight of the steel the bottom longitudinal bars will be detailed to extend the full length of the footing.

For the longitudinal direction, the critical sections for shear are at distances of 21½ in. from the faces of the columns and 12 in. from the edges of the transverse beams. The total shears at the former sections are greater than those at the latter, and hence the diagonal-tension investigation is made at a distance of 21½ in. from the faces of the columns, which sections are 9½ in. from the edges of the transverse beams. Between the columns,

$$V = \frac{620{,}000}{18.75}\left(2.375 + 4.0 + \frac{9.5}{12}\right) - 310{,}000 = -73{,}000 \text{ lb.}$$

$$v = \frac{73{,}000}{8.75 \times 12 \times 0.9 \times 12} = 64 \text{ p.s.i.}$$

This is but slightly higher than the allowable unit stress for beams without web reinforcement (60 p.s.i.). Stirrups are unnecessary:

Outside of the columns,

$$V = \frac{620{,}000}{18.75}\left(2.375 - \frac{9.5}{12}\right) = 53{,}800 \text{ lb.}$$

Stirrups are obviously unnecessary in this region.

Complete details are given in Fig. 187. The actual weight of the footing is 39,000 lb. The soil pressure is $\frac{620{,}000 + 39{,}000}{164.1} = 4020$ lb. per sq. ft. which is but 0.5 per cent greater than the specified maximum and may be considered satisfactory.

216. Design of a Rectangular Combined Footing Supporting Two Unequal Column Loads. An exterior column, 24 by 18 in. in cross-section (see Fig. 189), supports a total load of 200,000 lb. The adjacent interior column is 24 by 24 in. in cross-section and carries a total load of 300,000 lb. The two columns are to be supported on one rectangular combined footing, one end of which cannot extend beyond the outer face of the exterior column. The distance center to center of columns is 18 ft.-0 in. The allowable soil pressure is 4000 lb. per sq. ft. A 2000-lb. concrete and structural-grade reinforcing bars are to be used in the footing.

Assuming the weight of the footing[1] as 0.12(300,000 + 200,000) = 60,000 lb., the bearing area required is 560,000/4000 = 140.0 sq. ft.

In order to secure uniform soil pressure, the center of gravity of the footing must coincide with the center of gravity of the column loads. The latter is $\frac{300{,}000}{500{,}000} \times 18 = 10.8$ ft. from the center of the exterior column. The length of the footing must be 2(10.8 + 0.75) = 23.1 ft. A length of 23 ft.-

[1] The weight of a combined footing of this type will usually vary from 8 to 15 per cent of the sum of the column loads, the distance between the columns being an important factor, as a careful analysis of the following design will indicate.

3 in. is selected. The width required is then 140.0/23.25 = 6.0 ft. The proposed method of design assumes the loads from the columns to be carried by transverse beams under the columns, these beams in turn distributing the loads to the longitudinal beam.

Design of Longitudinal Direction. The net upward pressure per linear foot is 500,000/23.25 = 21,500 lb. The maximum negative moment between the columns occurs at the section of zero shear. Let x be the dis-

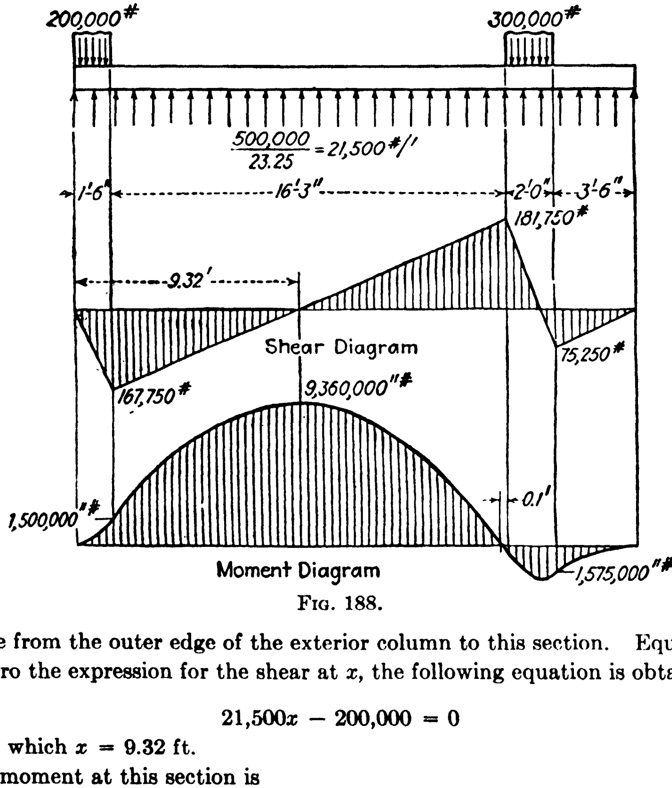


FIG. 188.

tance from the outer edge of the exterior column to this section. Equating to zero the expression for the shear at x, the following equation is obtained:

$$21{,}500x - 200{,}000 = 0$$

from which $x = 9.32$ ft.

The moment at this section is

$$M = \left[-200{,}000(9.32 - 0.75) + 21{,}500 \times \frac{(9.32)^2}{2} \right] \times 12$$

$$= -9{,}360{,}000 \text{ in.-lb.}$$

The moment at the right edge of the interior column (see Fig. 188) is

$$M = 21{,}500 \times \frac{(3.5)^2}{2} \times 12 = 1{,}575{,}000 \text{ in.-lb.}$$

For a 2000-lb. concrete, $f_c = 800$ p.s.i., and $n = 15$. From Table 2, page 226, $K = 139$ and $j = 0.867$. The effective width of the longitudinal beam, for both compression and tension, is equal to the full width of the footing, or 6.0 × 12 = 72 in.

$$d = \sqrt{\frac{9{,}360{,}000}{139 \times 72}} = 30.6 \text{ in.}$$

An effective depth of 31 in. is selected; with 3-in. insulation, the total thickness of the footing is 34 in. and the weight is 59,500 lb., which agrees closely with the assumed value.

Between the columns[1] the area of steel required is

$$A_s = \frac{9{,}360{,}000}{18{,}000 \times 0.867 \times 31} = 19.3 \text{ sq. in.}$$

The critical sections for bond, for these bars, are at the inner edge of the exterior column and at the point of inflection (see Fig. 188), which is approximately 0.1 ft. from the edge of the interior column. The latter section governs in this design, and the total shear is

$$V = 181{,}750 - \frac{21{,}500}{10} = 179{,}600 \text{ lb.}$$

With anchored bars, the allowable unit bond stress (steel in one direction only) is 1.5 × 0.05 × 2000 = 150 p.s.i.

$$\Sigma_0 = \frac{179{,}600}{150 \times 0.9 \times 31} = 43 \text{ in.}$$

The area requirement governs, and twenty 1-in. square bars are selected. The spacing is approximately 3½ in.

For the portion of the longitudinal beam which projects beyond the interior column,

$$A_s = \frac{1{,}575{,}000}{18{,}000 \times 0.9 \times 31} = 3.13 \text{ sq. in.}$$

The critical section for bond is at the edge of the column, at which section $V = 21{,}500 \times 3.5 = 75{,}300$ lb. Since these bars are in the bottom of the footing and hence are in contact with the transverse bars, the allowable unit bond stress, with anchored bars, is 0.75 × 150 = 112 p.s.i.

$$\Sigma_0 = \frac{75{,}300}{112 \times 0.9 \times 31} = 24.1 \text{ in.}$$

Thirteen ⅝-in. round bars are selected.

[1] In computing the area of longitudinal steel at this section, the use of the theoretical value of j is justified, since the design is analogous to that of an ordinary rectangular beam of definite width and ideal steel ratio. At other sections in the longitudinal beam, and in the transverse beam, an approximate value of j = 0.9 will be used.

The critical section for diagonal tension is at a distance of d in. from the columns. Near the inner edge of the exterior column,

$$V = 167{,}750 - 21{,}500 \times {}^{31}\!/_{12} = 112{,}300 \text{ lb.},$$

and the unit shear is

$$v = \frac{112{,}300}{72 \times 0.9 \times 31} = 56 \text{ p.s.i.}$$

With anchored bars the allowable unit shearing stress for beams without web reinforcement is $0.03f'_c = 60$ p.s.i., and no web reinforcement is required in this portion of the footing.

At a distance of 31 in. from the left edge of the interior column, the total shear is $181{,}750 - 21{,}500 \times {}^{31}\!/_{12} = 126{,}300$ lb., and the unit shear is 63 p.s.i. This is sufficiently close to the allowable value of 60 p.s.i. so that no web reinforcement need be placed in this region.

At a distance of 31 in. from the right edge of the interior column the total shear is 16,800 lb., and the unit shear is 9 p.s.i. No stirrups are required in this region.

Design of Transverse Beam under Interior Column. The moment at the edge of the interior column, in the transverse direction, is

$$M = \frac{300{,}000}{6} \times \frac{2^2}{2} \times 12 = 1{,}200{,}000 \text{ in.-lb.}$$

Assuming that the effective width for compression is equal to $1\frac{1}{2}$ times the column width, or $1\frac{1}{2} \times 24 = 36$ in., the required effective depth is

$$d = \sqrt{\frac{1{,}200{,}000}{36 \times 139}} = 15.5 \text{ in.}$$

The actual value of d which is furnished (see Fig. 189) is $31 - \frac{5}{8} = 30\frac{3}{8}$ in.

$$A_s = \frac{1{,}200{,}000}{18{,}000 \times 0.9 \times 30\frac{3}{8}} = 2.45 \text{ sq. in.}$$

At the edge of the column, $V = \dfrac{300{,}000}{6} \times 2 = 100{,}000$ lb. With anchored bars, $u = 112$ p.s.i., and

$$\Sigma_0 = \frac{100{,}000}{112 \times 0.9 \times 30\frac{3}{8}} = 32.8 \text{ in.}$$

Sixteen $\frac{5}{8}$-in. round bars are selected and placed arbitrarily in a width of 60 in. The spacing is about 4 in., and all bars fall well within a 45-degree line from the edge of the column.

No investigation for diagonal tension is necessary, since a section d in. from the edge of the column falls outside of the footing.

Design of Transverse Beam under Exterior Column. The moment at the edge of the exterior column is

$$M = \frac{200,000}{6} \times \frac{2^2}{2} \times 12 = 800,000 \text{ in.-lb.}$$

Assuming that the effective width for compression (concrete on one side of

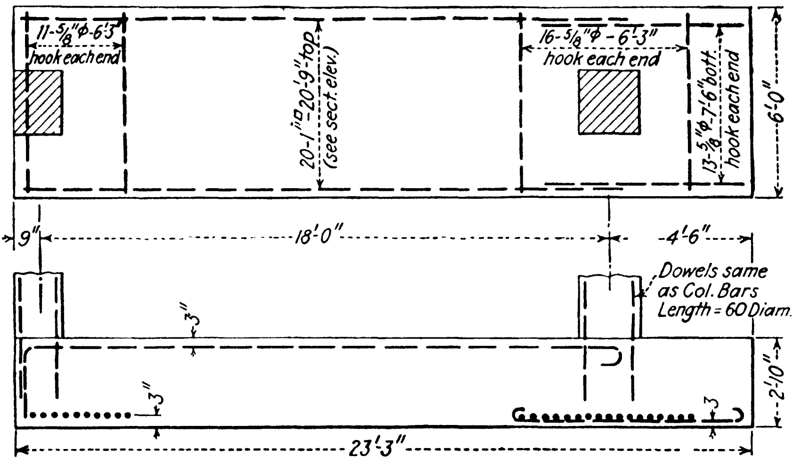


FIG. 189.—Combined footing with two unequal column loads.

the column only) is $1\frac{1}{4} \times 18 = 22$ in., the effective depth required is

$$d = \sqrt{\frac{800,000}{139 \times 22}} = 16.2 \text{ in.}$$

The actual d furnished is $30\frac{3}{8}$ in.

$$A_s = \frac{800,000}{18,000 \times 0.9 \times 30\frac{3}{8}} = 1.68 \text{ sq. in.}$$

At the edge of the column, $V = \frac{200,000}{6} \times 2 = 66,700$ lb.

$$\Sigma_0 = \frac{66,700}{112 \times 0.9 \times 30\frac{3}{8}} = 21.8 \text{ in.}$$

Eleven $\frac{5}{8}$-in. round bars are selected and placed arbitrarily in a width of 44 in. For the same reason as given in the design of the transverse beam under the interior column, no investigation for diagonal tension is necessary.

Complete details are shown in Fig. 189.

217. Miscellaneous Foundations. In designing foundations to rest on soil, the safe bearing power of which is very small, it sometimes becomes necessary to extend the footings to cover practically all of the area of the building, one connected to the other. Such foundations may consist of a solid flat slab of concrete, a series of beams with slabs at the top or bottom, or a series of connected multiple-column footings. In some cases it may be necessary to support the individual footings on piles.

PILE FOUNDATIONS

218. Concrete Foundations on Piles. Where a soil of a highly compressible nature is encountered, and where the amount of excavation which would be required to reach a firm stratum would be excessive, economy might dictate the use of a foundation supported on piles, the latter being long enough either to reach the firm substratum or to develop sufficient skin friction to overcome the loads to which they are subjected.

This type of foundation consists essentially of a concrete slab, usually reinforced, supported directly on the piles. The heads of the piles are allowed to project a short distance above the ground so that the concrete may encase these portions of the piles and form with them a solid unit. A minimum embedment of 6 in. is considered satisfactory in most cases. If desirable, the material around the piles may be excavated, the depth of excavation depending upon soil conditions, and the space thus made filled in with gravel or other solid material, on which the concrete is laid as stated above. Such procedure utilizes the increased bearing power of the earth surrounding the piles.

The essential difference between the design of a concrete foundation supported on piles and the design of a footing resting directly on the soil is in the manner in which the load on the footing is resisted by the foundation bed. In the former case, a series of concentrated upward loads must be considered; in the latter case, uniform distribution under the entire concrete area is assumed.

219. Bearing Power of Piles. When piles are supported entirely by the friction between their sides and the earth, the load is transmitted to a deep ground level in a conoid of pressure through the earth above it. Such piles should be driven so far apart, or to such a depth, that the increased area of bearing developed by the conoid of pressure, which has the required altitude to contain the frictional resistance, reaches a level at which the material will afford the required support before it intersects the corresponding conoid of an adjacent pile. In good practice, bearing piles are never spaced closer than 2½ ft. center to center, and preferably not closer than 3 ft.

The bearing value of a pile which is supported by the friction of the earth into which it is driven will depend, among other things, upon the soil conditions, the size and spacing of the piles, and the depth to which they are driven. The absolute maximum loads usually permitted on timber and concrete piles are 20 tons and 40 tons, respectively. The actual bearing power can be determined by a loading test, in which a pile or group of piles is loaded with a static load placed in increments until appreciable settlement is noted, or by a driving test, in which the measured penetration caused by several blows of the hammer is used to compute the probable bearing power.

The most widely used formulas for computing the bearing power from penetrations observed in driving are the *Engineering News* formulas. A suitable factor of safety is included in these formulas, so that the resulting values are the safe bearing capacities rather than the ultimate. When a drop hammer is used, with a free fall not exceeding 20 ft., the equation for the safe bearing power is

$$P = \frac{2WH}{s + 1} \tag{4}$$

When a single-acting steam hammer is used, the safe load per pile is

$$P = \frac{2WH}{s + 0.1} \tag{5}$$

When a double-acting steam hammer is used, the safe load is

$$P = \frac{2H(W + Am - b)}{s + 0.1} \tag{6}$$

in which P = safe bearing power per pile, in pounds.

W = weight of a drop hammer or the weight of the striking part of a steam hammer, in pounds.

H = height of fall of a drop hammer, or the length of the stroke of a steam hammer, in feet.
s = average penetration, in inches per blow, for at least three consecutive blows of the hammer.
A = effective area of piston, in square inches.
m = mean effective pressure of the steam on the downward stroke, in pounds per square inch.
b = total back pressure, in pounds.

A constant of 0.3 instead of 0.1 in equations (5) and (6) is sometimes used.

When a pile is driven through soft material to a hard substratum, the safe bearing value is computed by treating the pile as a column, with an unsupported length equal to two-thirds of the penetration in distinctly soft material.

PROPORTIONING FOOTING AREAS FOR UNIFORM PRESSURE

220. Methods of Securing Uniform Settlement. In all the preceding discussions and problems, the bearing areas of footings have been selected merely with the idea of keeping the unit soil pressures within specified limits; thus, to a great extent, the possibility of slight settlements and the effect of such settlements on the integrity of the structure above the footings have been ignored.

A safe soil pressure is usually considered to be some value smaller than the unit load at which appreciable settlement takes place. All soils are compressible to some extent; and, with unit pressures such as are commonly used in practice, a moderate amount of settlement under each footing must be expected. If the amount of settlement of each footing in a large building is the same as that of the other footings and if all the settlements are simultaneous, no particular harm is done even if, in time, the total settlement is noticeable. The problem confronting the designer is, therefore, not so much to prevent settlement as to make sure that whatever settlement takes place will be uniform throughout the entire structure.

If the footings rest on a fairly compressible soil, it is reasonable to expect that a great part of the settlement will be realized before the structure is made to serve its useful purpose, *i.e.*, before the live load is placed on it. Hence, in some cases, the areas of the footings are proportioned to give equal unit pressures under dead load alone. With a properly selected unit pressure, it is considered better practice, however, to assume that a part of the live load is necessary to produce noticeable settlement and to proportion the footings for uniform pressures under dead plus partial live loads. It is seldom desirable to consider the full live load in the proportioning of footing areas, (1) because of the reason given above; (2) because the full design live load is rarely realized except occasionally in such structures as warehouses, etc.; and (3) because the live load is generally a spasmodic load, and hence the settlement that may be caused by it is not in direct proportion to the amount of the load. The most generally adopted practice is to proportion footing areas so that the unit pressures under all footings will be the same when the footings are loaded with the dead load plus one-third or one-half of the probable live load.

221. Probable Live Load on Footings. The probable live load is the live load which is used in the design of the basement column. This is not necessarily the sum of the full live panel loads on each floor above the basement floor. Each floor panel must, of course, be designed for the full live load that may at some time be placed on it; but in most structures, especially office buildings, hotels, apartment houses, etc., the full live load will never exist on all floors simultaneously. To allow for this condition, in computing the design loads for columns and footings, some reduction in the specified live load is permitted, the amount of reduction depending primarily on the number of stories in the building. The New York City building code is a typical example of conservative practice. This code contains the following:

a. In structures intended for storage purposes all columns, piers or walls, and foundations may be designed for 85 per cent of the full assumed live load.

b. In structures intended for other uses the assumed live load to be used in the design of all columns, piers or walls, and foundations may be as follows:

100 per cent of the live load on the roof.
85 per cent of the live load on the top floor.
80 per cent of the live load on the next floor.
75 per cent of the live load on the next floor below.

On each successive lower floor, there may be a corresponding decrease in percentage, provided that in all cases at least 50 per cent of the live load shall be assumed.

222. Method of Proportioning Footing Areas. A method for determining the bearing area A which is required under each individual footing to give uniform pressures under dead plus partial live load is as follows: Select one footing as the critical footing. If a given bearing pressure is not to be exceeded, the critical footing will be that footing with the greatest percentage of live load. The area of the base of the critical footing is A_c, the total load on it is T_c, the live load L_c, and the dead load D_c. The live load on any other footing is L, the dead load D, and the area A. Then, in order not to exceed the given value for the safe bearing capacity of the soil, B, the required area of the critical footing is

$$A_c = \frac{T_c}{B} \tag{7}$$

The bearing pressure B_c at the base of the critical footing, when that footing is loaded with its full dead load, D_c, plus a definite part of its live load, ΔL_c, is

$$B_c = \frac{D_c + \Delta L_c}{A_c} \tag{8}$$

In order to secure equal unit pressures at the bases of all the footings in the given structure when these footings are loaded with their full dead loads D plus the stated part of their respective live loads ΔL, the area A of each footing must be equal to

$$A = \frac{D + \Delta L}{B_c} \tag{9}$$

If the footings are to be proportioned for equal unit pressures under dead load plus one-half live load, the value of Δ in the foregoing discussion is ½; for dead load plus one-third live load, $\Delta = \frac{1}{3}$. With areas computed from this equation the maximum unit pressure under the critical footing will be equal to B, the maximum unit pressure under any other footing will be less than B, and the unit pressure under dead load plus one-half (or one-third or any other specified fraction) of the live load will be the same under all footings. Once the area of each individual footing is determined as above, the remainder of the design of each footing must be based on the total load.

Example. Four typical footings in a given building support dead and probable live loads as tabulated below. If the maximum safe bearing power of the soil on which these footings are to rest is 5000 lb. per sq. ft., what must be the area of each of these footings in order that uniform settlement may be expected, when the footings are loaded with the full dead load plus one-third of the live load?

Footing	Dead load, lb.	Live load, lb.	Live load, per cent	$DL + \frac{1}{3}LL$, lb.
A	500,000	300,000	37	600,000
B	400,000	210,000	34	470,000
C	400,000	300,000	42	500,000
D	500,000	180,000	26	560,000

The critical footing is C, and the required area is

$$\text{Footing } C = \frac{400{,}000 + 300{,}000}{5000} = 140 \text{ sq. ft.}$$

A footing 12 ft. square is selected, the area of which is 144 sq. ft.

The unit pressure B_c under this footing when it is loaded with the full dead load plus one-third of the probable live load is

$$\frac{500{,}000}{144} = 3470 \text{ lb. per sq. ft.}$$

The required areas of the other footings are, therefore,

$$\text{Footing } A = \frac{600{,}000}{3470} = 173 \text{ sq. ft. (13 ft.-3 in. square)}$$

$$\text{Footing } B = \frac{470{,}000}{3470} = 135 \text{ sq. ft. (11 ft.-9 in. square)}$$

$$\text{Footing } D = \frac{560{,}000}{3470} = 161 \text{ sq. ft. (12 ft.-9 in. square)}$$

CHAPTER XIII

RETAINING WALLS

223. Introductory. A pile of earth, cinders, or other material possessing more or less frictional stability will, when deposited loosely in an unrestrained position, assume a definite slope. The steepness of this depends upon the internal friction of the material and other conditions, such as moisture content. A mound of earth whose sides are permitted to assume this natural slope will, when thoroughly compacted, maintain its own integrity and support external loads to a maximum amount which depends, among other things, upon the bearing qualities of the soil.

In engineering construction it frequently becomes necessary to prevent the sides of such a pile of earth from assuming this natural slope. Such a condition occurs when the width of a cut or embankment is limited either by restrictions of economy or right of ownership. The most common examples of the latter limitation are found in railway and highway construction where the width of the right of way is fixed. In such cases it is essential that the earth be held in position by means of a wall capable of resisting the lateral pressure caused by the conditions of restraint.

A wall whose express purpose is to hold in position a bank of earth or similar material is termed a retaining wall. The first step toward the design of a retaining wall is to determine its location. If the wall is to run along a fixed property line, such as a highway or a railroad, this provides definite placing. As is often the case, the amount of land available for the construction of a given cut or fill may be unlimited, but the cost of cutting or filling sufficiently to allow the natural slope of the earth to obtain may be excessive. Wherever it is found that a retaining wall of the necessary height and section is cheaper than the additional cut or fill that it replaces, economy favors the construction. In Fig. 190, the wall replaces the shaded volume of fill. A few trials will show at what point the wall should be placed to obtain the minimum cost.

The section of wall to be chosen will be determined by a consideration of economy, ease of construction, and other factors interposed by existing conditions.

224. Types. Masonry retaining walls may be divided into two general classes: (1) the gravity wall (Fig. 191), which retains the bank of earth entirely by its own weight; (2) the reinforced-concrete wall, which utilizes the weight of the earth behind it in resisting the overturning moment of the retained material. In this latter class are included the cantilever wall (Fig. 203), a type of construction consisting of a vertical arm supported upon a horizontal base slab, the vertical arm acting as a free cantilever in overcoming the pressure from the earth; and the counterfort wall (Fig. 192), the vertical slab of which is anchored or tied to the base slab by means of counterforts or buttresses—triangular cross walls extending from the top of the vertical slab to the extreme point of the base slab at regular intervals throughout the length of the wall. The vertical slab of the reinforced walls may be placed at the front, at the rear, or at any point along the base slab, the exact location depending upon limitations of economy and construction. Where conditions permit, a toe extension of $\frac{1}{3}$ to $\frac{1}{2}l$ will produce a more economical design for a cantilever wall than would result if the vertical arm were placed at the front edge of the base slab.

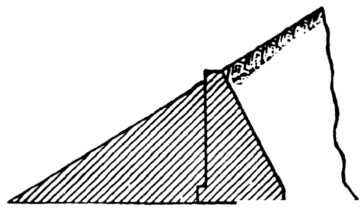


FIG. 190.

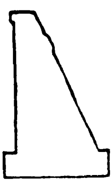


FIG. 191.—Gravity wall.

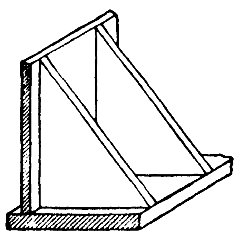


FIG. 192.—Counterfort wall.

The section of wall to be chosen will be determined by a consideration of economy, ease of construction, foundation requirements, and other factors imposed by existing conditions. In comparing the relative economy of gravity walls and reinforced-concrete walls, the added cost of construction in the case of the latter must be included. A careful study of the different types leads to the conclusion that, unless affected by unusual conditions, the gravity type will prove most economical for low walls, the cantilever type for walls of medium height, and the counterfort type for the higher walls. The critical height, or height of separation between the various types, is not clearly defined, since it depends upon too many economic as well as constructive conditions. In general it is found to be uneconomical to use the counterfort construction for walls that are less than 18 ft. in height.

The width of the base of the wall will vary from 0.45 to 0.60 of the height of the wall, depending upon the type of loading supported by the wall. This value will in general average 0.5 of the total height of the equivalent earth column at the heel of the wall, measured from the bottom of the wall, *i.e.*, $0.5h$ in Fig. 195*a*; $0.5\ (h + h')$ in Fig. 195*b*; or 0.5 of the height of the shaded pressure triangle in Fig. 196.

225. Conditions of Loading. Four general cases are possible: (1) Walls with no surcharge, the top surface of the fill being horizontal and level with the top of the wall. (2) Walls with an inclined surcharge, the top surface of the fill extending upward and back from the top of the wall; the angle of inclination of the sloping surcharge is usually taken as the angle of repose of the material in the fill (Art. 226). (3) Walls with a horizontal surcharge extending some distance above the top of the wall. (4) Loadings in which the actual surface of earth does not extend above the top of the wall, but supports an external load such as a building or railroad tracks. Such loads are converted into an equivalent height of earth above the top of the wall by dividing the weight of an additional load per square foot by the weight of the earth per cubic foot. The pressure against the wall is then computed for this equivalent height of earth surcharge, as in Case 3.

The American Railway Engineering Association recommends that the equivalent height of earth for Case 4 be obtained as follows: (*a*) In calculating the surcharge due to railroad tracks, the entire load shall be assumed to be uniformly distributed over a width of 14 ft. for a single track or for tracks more than 14 ft. on centers, and the distance center to center of tracks where tracks are spaced less than 14 ft. (*b*) In calculating the pressure on a retaining wall where the filling carries permanent tracks or structures, the full effect of the loaded surcharge shall be considered where the edge of the distributed load or structure is

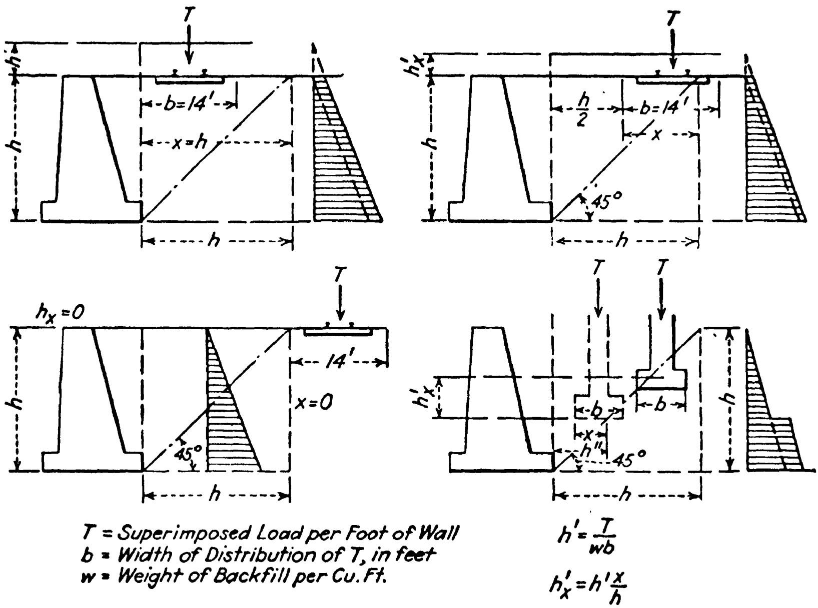


FIG. 193.

vertically above the back edge of the heel of the wall. The effect of the loaded surcharge may be neglected where the edge of the distributed load or structure is at a distance from the vertical line through the back edge of the heel of the wall equal to h, the height of the wall. For intermediate positions the equivalent uniform surcharge load may be interpolated between the two cases quoted above (see Fig. 193).

226. Determination of Earth Thrust. The first essential in any design is the determination of the force to be resisted. The principal force governing the dimensions of a retaining wall is the pressure exerted by the retained material in its attempt to assume its natural slope. In order fully to determine the pressure of the filling against the wall, the resultant must be known in amount, in line of action, and in point of application.

Many theories have been advanced which lead to a purely academic determination of earth thrust. These mathematical discussions of the action of earth masses premise an ideal, incompressible, homogeneous material, without cohesion, possessing frictional resistance between its particles, and of indefinite extent in the mass. Such a fill is rarely found in practice. The degree of exactness of the thrust as determined by any of the theoretical methods will depend upon the difference between the actual conditions and the theoretical.

Rankine's development of earth pressure, which starts out with an infinitesimal prism and leads to an expression for the thrust of an entire earth mass upon a given surface, results in the general equation

$$P = C \times \frac{wh_1^2}{2} \qquad (1)$$

in which P = the total thrust upon the back of the wall.
w = the weight of the earth per cubic foot.
h_1 = the height of the earth column, in feet.
C = a constant depending upon the angle of inclination of the back of the wall, the conditions of loading, and the physical properties of the earth fill. Values of P are given in Fig. 194 for various conditions of loading.

The amount of the pressure on any given horizontal strip 1 ft. in height at a distance x ft. below the surface of the earth is given by the equation

$$P_1 = Cwx \qquad (2)$$

The pressure distribution along the back of the wall for Case 1 of Art. 225 is shown in Fig. 195*a*, for Case 2 in Fig. 196, and for Cases 3 and 4 in Fig. 195*b*.

In determining foundation pressures in a wall of a cross-section as shown in Fig. 196 the earth fill vertically above the base slab is considered as a resisting pressure equivalent to the same weight of masonry, and the total overturning pressure is the total earth thrust on a vertical plane at the heel of the wall. The height

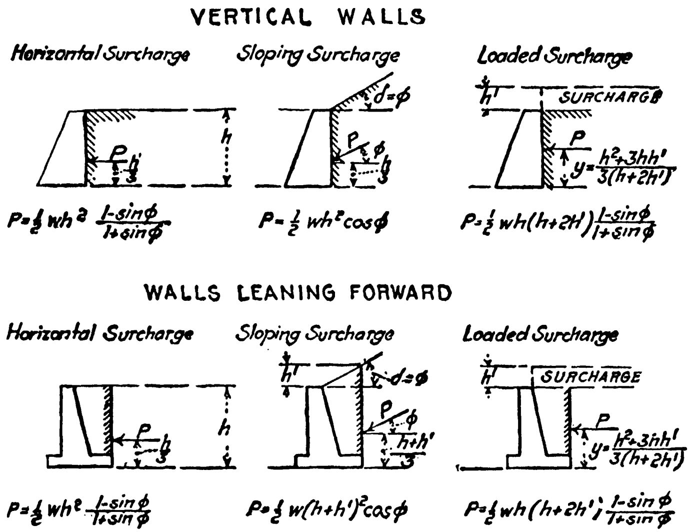


FIG. 194.—Pressures on retaining walls.

CD is therefore used in equation (1) for determining the earth thrust. This applies also to walls of gravity section in which the back slopes away from the fill. In determining the bending

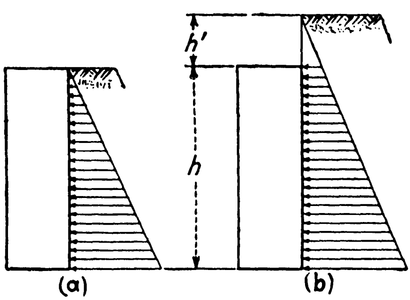


FIG. 195.

moment on the vertical arm *AB* (Fig. 196), the total thrust is computed for a column of earth of a height equal to *AB*.

227. Line of Action and Point of Application of Earth Pressure. The line of action of the total thrust upon a wall with a vertical back exposed to the action of the earth is parallel to the top surface of the filling. In a wall whose back slopes away from the fill the total thrust upon a vertical plane through the heel of the wall acts parallel to the top surface of the earth.

For walls with no surcharge or a sloping surcharge, the point of application of the total earth thrust is usually assumed at a point in the plane against which the earth is acting and at a distance of one-third its height, measured upward from the base of the plane. For walls with a loaded surcharge, the point of application is taken at the center of gravity of the pressure quadrilateral

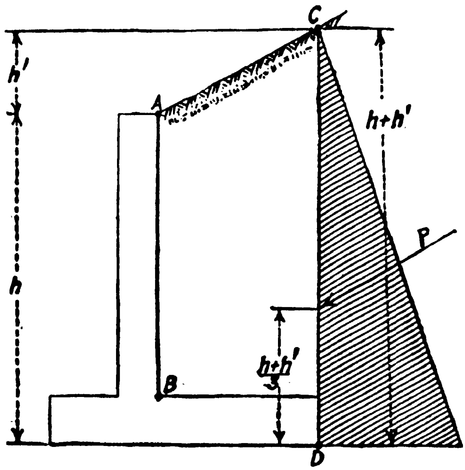


Fig. 196.

shown in Fig. 195*b*. The location of the point of application of the resultant thrust for the various conditions of loading is given in Fig. 194.

228. Factors Affecting the Design. Following the determination of the earth thrust, an investigation must be made of all possible modes of failure, and each element of the construction must be so proportioned as to make such failures impossible. A gravity wall may fail by sliding along the plane of the base, by overturning, or by settlement at the toe caused by the crushing of the soil there. An extreme case of this will also cause overturning. A reinforced-concrete wall may fail in any of the ways mentioned above. In addition, any of the thin sections which together furnish the necessary strength and rigidity might yield in a manner similar to a corresponding element in other constructions.

229. Overturning and Crushing. In order to prevent overturning of the wall, the resisting moment of the weight of the wall and the earth above the base of the wall about the toe must be greater than the overturning moment of the earth pressure about the same point. A factor of safety of 2 is generally required; *i.e.*, in Fig. 197, $W_1x_1 + W_2x_2 + W_3x_3$ must be greater than $2Py$. The factor of safety n against overturning may be determined from the approximate equation

$$n = \frac{l}{l - 2a}$$

in which l = length of base of wall.
a = distance from toe of wall to intersection of the resultant of P and ΣW with the base of the wall (see Fig. 198).

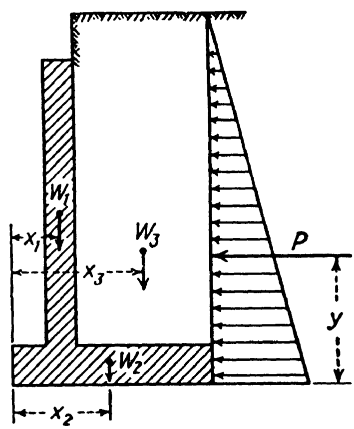


FIG. 197.

In order to prevent crushing of the soil at the toe of the wall, the soil pressure at that point should be not greater than the allowable pressure for the soil under the wall (see Art. 199). Formulas for the toe and heel pressures are given in Fig. 198, in which

E = resultant of the total earth pressure and the resisting weight (concrete and earth above base of wall) on a 1-ft. strip of wall, in pounds.
F = vertical component of E.
l = length of base, in feet.
a = distance, in feet, from toe to intersection of E with the base.

These formulas are derived by assuming that the column of earth directly under the wall supports an eccentric load F at a distance of $\frac{l}{2} - a$ from the gravity axis of the assumed earth column and applying the principles of flexure and direct stress as outlined in Chap. VI.

Figure 198 illustrates the desirability of having the resultant pressure E intersect the base of the wall within the middle third, in order that the base will bear on the soil for its full length.

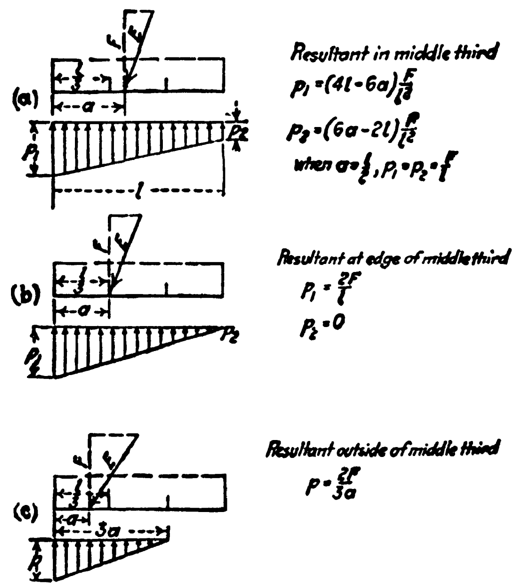


FIG. 198.—Pressures on foundation.

230. Sliding. In order to prevent sliding, the frictional resistance of the base against the foundation material must be greater than the horizontal component of the earth pressure P. A safety factor of 1½ is usually specified; *i.e.*, $f(\Sigma W)$ must be greater than 1½ P_H, in which

f = coefficient of friction, concrete on earth.

ΣW = total vertical load on base of wall = F in Figs. 198 and 200.

P_H = horizontal component of earth pressure P.

Values of f may be assumed as follows: on dry clay 0.5; on wet clay 0.33; on sand 0.4; on gravel 0.6. In case an adverse condition of sliding exists, the base may be widened, the weight of the wall being thus increased; narrow shallow trenches may be dug in the foundation, forming projections which will materially increase the resistance to sliding; the base may be inclined upward toward the toe; or the forward portion of the trench may be filled with masonry so that the wall butts directly against the original earth.

231. Details of Construction. The front of the wall is usually built with a batter of ½ to 1 in. in 12 in. A coping, projecting a short distance beyond the wall, adds to the architectural appearance and, to a certain extent, protects the masonry in the body of the wall from dripping water. A top width of less than 10 in. is generally undesirable; heavy gravity walls should preferably have a top width not less than 18 to 24 in. The base of the foundation should be a sufficient distance below the surface of the ground to prevent damage by frost, a minimum of about 3½ ft. being satisfactory in ordinary climates. Expansion joints should be provided at intervals along the wall, preferably not further apart than 30 ft.

In a cantilever wall where cracks would be not only unsightly but also detrimental to the integrity of the wall, additional horizontal steel should be placed at right angles to the main reinforcement to provide for temperature and shrinkage stresses. An amount varying from 0.10 to 0.33 per cent of the cross-section is usually specified. In general, in the vertical arm, ⅜- or ½-in. bars, placed 12 in. on centers near the exposed face and 24 in. on centers near the buried face, will be satisfactory. To support the horizontal bars near the exposed face, vertical ⅝- or ¾-in. bars, about 3 ft. on centers, are used, and the horizontal bars are wired to these. Longitudinal bars are also desirable in the base slab in order to prevent the formation of large cracks due to shrinkage of the concrete during the hardening process. In a counterfort wall, the steel that is required for stress is in the proper direction and position to provide also for the critical temperature and shrinkage stresses, although conservative designers may place additional longitudinal bars near all faces which do not already have such required steel.

Proper drainage of the fill behind the wall may be effected by inserting 4-in. drain tiles through the wall near the bottom, at intervals of 10 to 15 ft., and piling crushed stone, gravel, or other coarse material around the holes. At least one drain should be provided for each pocket formed by counterforts.

232. Design of Gravity Wall. A gravity wall 16 ft.-0 in. high is to sustain a bank of earth with a loaded horizontal surcharge equivalent to 4 ft. of

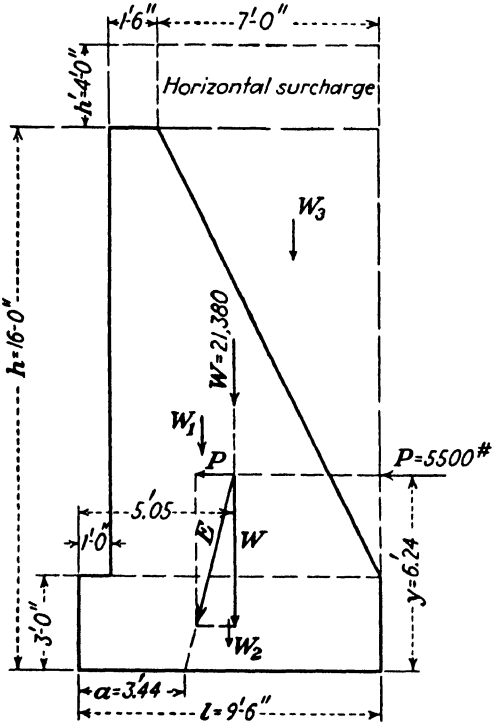


FIG. 199.—Details of gravity wall.

filling above the top of the wall. The safe bearing pressure on the clay foundation bed is 2 tons per sq. ft. The weight of the retained fill is 100 lb. per cu. ft., and the angle of repose 33°42'. Determine the required section of wall.

The ordinary procedure in the design of a gravity-type wall is to select a tentative section, the dimensions of which are governed by the judgment and experience of the designer. This tentative section is then analyzed in accordance with the principles outlined above, and modifications in the assumed dimensions made where necessary.

In the present case, the tentative dimensions are shown in Fig. 199. Investigation must be made with and without the portion of surcharge directly over the base included in the resisting weight W_3. Assuming that the former condition obtains, the analysis is as follows:

From Fig. 194, the total pressure per foot of wall against the vertical plane through the heel of the wall is

$$P = \frac{1}{2} wh(h + 2h') \times \frac{1 - \sin \phi}{1 + \sin \phi}$$

$$= \frac{1}{2} \times 100 \times 16(16 + 8)(0.286) = 5500 \text{ lb.}$$

The distance of the point of application of P above the bottom of the plane is

$$y = \frac{h^2 + 3h'h}{3(h + 2h')} = \frac{16^2 + 3 \times 4 \times 16}{3(16 + 8)} = 6.24 \text{ ft.}$$

$$W_1 = \frac{8.5 + 1.5}{2} \times 13 \times 150 = 9{,}750 \text{ lb.}$$

$$W_2 = 9.5 \times 3 \times 150 = 4{,}280 \text{ lb.}$$

$$W_3 = \frac{17 + 4}{2} \times 7.0 \times 100 = 7{,}350 \text{ lb.}$$

$$W = \overline{21{,}380} \text{ lb.}$$

By taking moments about the toe of the wall, the point of application of the total resisting load W is found to be 5.05 ft. from that point.

The resultant of P and W = 22,100 lb. and intersects the base 3.44 ft. from the toe, or 0.27 ft. inside the forward edge of the middle third.

$$p_1 = (4 \times 9.5 - 6 \times 3.44) \frac{21{,}380}{9.5^2} = 4100 \text{ lb. per sq. ft.}$$

$$p_2 = (6 \times 3.44 - 2 \times 9.5) \frac{21{,}380}{9.5^2} = 389 \text{ lb. per sq. ft.}$$

Investigation of the same section, assuming that the surcharge directly over the base is not included in the resisting load W_3, shows that the total load W = 18,580 lb. and its point of application is 4.90 ft. from the toe of the wall. The point of application of the resultant of P and W is 3.05 ft. from the toe. This is but 0.12 ft. outside the middle third and will be assumed satisfactory. The toe pressure for this case is

$$p_1 = \frac{2 \times 18{,}580}{3 \times 3.05} = 4050 \text{ lb. per sq. ft.}$$

For the latter loading condition, which is the most severe condition, the overturning moment is 5500 × 6.24 = 34,300 ft.-lb. and the resisting moment is 18,580 × 4.90 = 91,200 ft.-lb. The factor of safety against overturning = 91.200/34,300 = 2.66.

The force producing sliding = 5500 lb., and the force resisting sliding, assuming the coefficient of friction as 0.5, is 0.5 × 18,580 = 9290 lb. The factor of safety against sliding = 9290/5500 = 1.69.

233. Design of a Cantilever Wall. Design a cantilever wall 20 ft. high to support an earth fill with an inclined surcharge, the slope of which is 1½ : 1. The foundation material is a firm dry clay for which the coefficient of friction (concrete on clay) is 0.5 and the allowable soil pressure 3 tons per sq. ft. A 2000-lb. concrete is to be used in the wall (see Art. 43 for

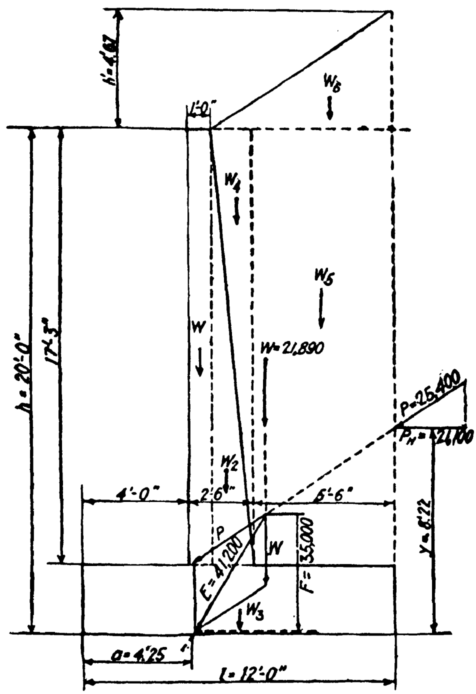


FIG. 200.

allowable unit stresses). The allowable unit stress in the steel is to be taken as 16,000 p.s.i. The assumed dimensions are shown in Fig. 200.

Overturning and Crushing. For a strip of wall 1 ft. in length, with earth at 100 and concrete at 150 lb. per cu. ft., W_1 = 2580, W_2 = 1940, W_3 = 4950, W_4 = 1290, W_5 = 9500, W_6 = 1630 lb. The lever arms about the toe of the wall are 4.5, 5.5, 6.0, 6.0, 9.25, and 9.67 ft., respectively. The total resisting load $W(= W_1 + W_2 + \ldots W_6)$ = 21,890 lb., and its center of gravity is 7.03 ft. from the toe.

From Fig. 194, $P = \frac{1}{2} \times 100(20 + 4.67)^2 \times 0.832$ = 25,400 lb.

$$y = \frac{20 + 4.67}{3} = 8.22 \text{ ft.}$$

Combining P and W graphically (see Fig. 200), $a = 4.25$ ft., and the factor of safety against overturning (Art. 229) $n = 12/(12 - 2 \times 4.25) = 3.4$. From Fig. 198:

$$p_1 = (4 \times 12.0 - 6 \times 4.25)\frac{35,000}{12.0^2} = 5460 \text{ lb. per sq. ft.}$$

$$p_2 = (6 \times 4.25 - 2 \times 12.0)\frac{35,000}{12.0^2} = 360 \text{ lb. per sq. ft.}$$

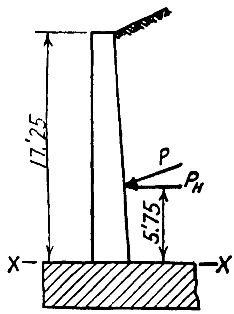


Fig. 201.

Sliding. Assume that the concrete is poured against the undisturbed earth at the toe and in the 9-in. trench beneath the base (see Fig. 203). The force resisting sliding $= 2.75 \times 6000 + \frac{3}{4} \times 6000 + 0.50 \times 35,000 = 38,500$ lb. The factor of safety against sliding (Art. 230) $= 38,500/21,100 = 1.8$.

Design of Vertical Arm. Figure 201 represents the forces acting on this member.

$$P = \tfrac{1}{2} \times 100 \times 17.25^2 \times 0.832 = 12,400 \text{ lb.}$$

$$P_H = 12,400 \times 0.832 = 10,300 \text{ lb.}$$

$$M_{XX} = 10,300 \times 5.75 \times 12 = 710,000 \text{ in.-lb.}$$

$$d \text{ (for moment)} = \sqrt{\frac{710,000}{147 \times 12}} = 20.1 \text{ in. [equation (3), Art. 148]}$$

$$V_{XX} = P_H = 10,300 \text{ lb.}$$

$$d \text{ (for shear)} = \frac{10,300}{40 \times 12 \times \frac{7}{8}} = 24.6 \text{ in. [equation (12), Art. 163]}$$

The thickness of 30 in. as assumed gives a value of $d = 27$ in., allowing 3 in. for insulation. A thinner wall would require excessive steel.

$$A_s = \frac{710,000}{16,000 \times 0.87 \times 27} = 1.88 \text{ sq. in. per ft. of wall [equation (2), Art. 148]}$$

1-in. round bars 5 in. center to center are selected

$$u = \frac{10,300}{1\frac{2}{5} \times 3.14 \times \frac{7}{8} \times 27} = 58 \text{ p.s.i. [equation (16), Art. 167]}$$

The bars must continue below section XX a distance [equation (17), Art. 168] equal to $[16,000/(4 \times 100)] \times 1 = 40$ in. This distance is furnished with the addition of the 9-in. key below the base slab, allowing for 2 in. clear insulation below the bars.

Alternate bars are discontinued at a point 5.25 ft. above the top of the base slab, and again at a point 10.25 ft. above the top of the base slab.*

* The theoretical points of cutoff are determined by computing the required steel area at two or more intermediate points and plotting required

Base Slab—Heel. The forces acting on the base slab are shown in Fig. 202. The load W_1 = vertical component of P, assumed as uniformly distributed over the heel slab. The load W_2 = weight of earth and concrete above base of heel slab.

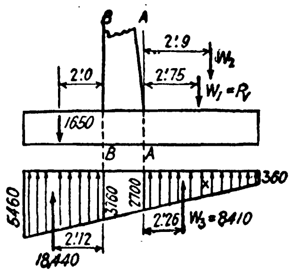


FIG. 202.

W_1 = 25,400 × 0.554 = 14,050 lb. Its center of gravity is 5.5/2 = 2.75 ft. from section AA.

W_2 = 9500 + 2270 + 1560 = 13,330 lb. Its center of gravity is 2.9 ft. from section AA.

W_3 (shaded area X) = [(2700 + 360)/2] × 5.5 = 8410 lb. Its center of gravity is 2.26 ft. from section AA.

$$M_{AA} = 14{,}050 \times 2.75 + 13{,}330 \times 2.9 - 8410 \times 2.26 = 58{,}300 \text{ ft.-lb.} = 700{,}000 \text{ in.-lb.}$$

$$V_{AA} = 14{,}050 + 13{,}330 - 8410 = 18{,}970 \text{ lb.}$$

$$d \text{ (for moment)} = \sqrt{\frac{700{,}000}{147 \times 12}} = 20.0 \text{ in.}$$

$$d \text{ (for shear)} = \frac{18{,}970}{60 \times 0.875 \times 12} = 30.0 \text{ in.}$$

With the bars 3 in. below the top of the slab, the d furnished with the assumed section is 30.0 in.

$$A_s = \frac{700{,}000}{16{,}000 \times 0.87 \times 30} = 1.68 \text{ sq. in. per ft. of wall}$$

1-in. round bars 5½ in. center to center are selected.

$$u = \frac{18{,}970}{(12/5.5) \times 3.14 \times 0.875 \times 30} = 105 \text{ p.s.i.}$$

These bars must extend to the left of section AA a distance [equation (17), Art. 168]

$$x_1 = \frac{16{,}000}{4 \times 100} \times 1 = 40 \text{ in.}$$

Base Slab—Toe.

$$M_{BB} = 18{,}440 \times 2.12 - 1650 \times 2.0 = 35{,}800 \text{ ft.-lb.} = 429{,}600 \text{ in.-lb.}$$

$$A_s = \frac{429{,}600}{16{,}000 \times 0.87 \times 30} = 1.03 \text{ sq. in. per ft. of wall}$$

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steel areas against heights of vertical arm. The investigations are as above, using as the height h the distance from the point considered to the top of the wall.

$\frac{3}{4}$-in. round bars 5 in. center to center are selected.

$$u = \frac{18{,}440 - 1650}{1\frac{2}{5} \times 2.356 \times 0.875 \times 30} = 112 \text{ p.s.i.}$$

This is satisfactory if the bars are thoroughly anchored (see Art. 167). The anchorage in this case at the front is furnished by hooks, and at the rear by extending the bars somewhat in excess of 40 diameters [equation (17), Art. 168] to the right of section *BB*.

Temperature Steel. In order to provide against possible cracks caused by changes in temperature and moisture conditions, the wall will be reinforced with vertical and horizontal steel on the front face of the vertical arm, additional horizontal steel on the back face, and longitudinal steel in the base slab, as shown in Fig. 203.

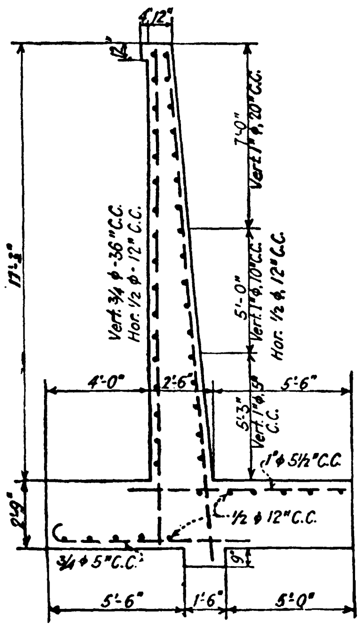

FIG. 203.—Cantilever wall.

234. Counterfort Walls. A spacing of about 10 ft. for the counterforts for walls of medium height will usually prove economical. The ordinary range of counterfort spacing is from 8 ft. for the low walls to 12 ft. for the higher walls. The thickness of counterforts varies from 12 to 18 in. The face wall is built with a vertical back; the front might be battered slightly for appearance. This wall is designed as a simple slab, supported by the counterforts; the steel is horizontal, on the outside of the wall, the spacing increasing from bottom to top. Some designers prefer to consider this wall as a partially continuous or fully continuous slab, in which event the moments would be decreased; but provision must be made for developing the negative moments at the counterforts by placing additional straight bars near the back face through the supports or by bending back some of the positive-moment steel. In the design, the wall is considered as a series of horizontal beams, 12 in. wide, which support a horizontal load per foot equal to the horizontal component of the earth pressure.

The base slab is designed as a simple slab supported by the counterforts, or sometimes as a partly or fully continuous slab. The steel is placed parallel to the wall and may be in the top or in the bottom of the slab, depending upon whether the result-

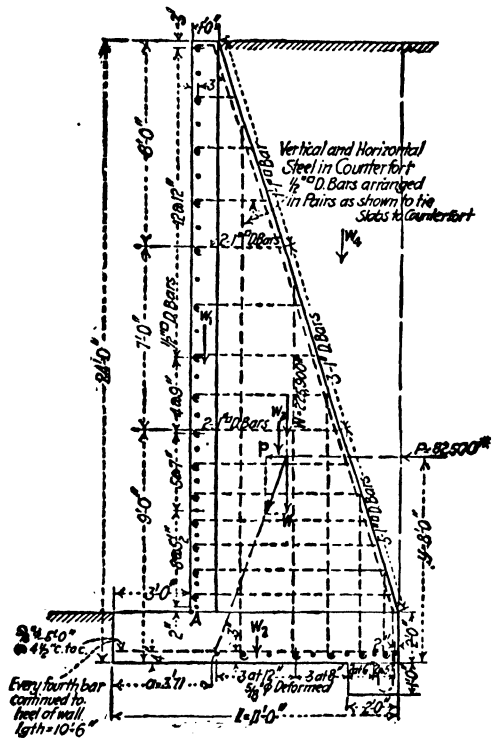

FIG. 204.—Counterfort wall.

ant moment causes tension on the upper or lower portions of the slab. The resultant moment is the algebraic sum of the moments caused by the upward earth pressure and the downward load of the filling, including in the latter the vertical component of the earth thrust as in the previous cantilever-wall design. A series of strips are investigated, the width of each strip usually being taken as about 2 ft. If 1-ft. strips were used, the thickness required near the heel of the wall would be excessive.

This procedure is justified by the monolithic nature of the construction.

The counterforts are designed as wedge-shaped, vertical cantilever beams, supporting a horizontal load equal to the total horizontal component of the earth pressure on the face wall between two counterforts. The effective depth of the counterfort is the perpendicular distance to the steel near the rear edge of the counterfort from the point of intersection of the counterfort, the front face of the vertical wall, and the upper face of the base slab. The counterforts must be tied to the vertical wall and to the base slab, with horizontal and vertical bars, respectively. The main steel in the counterforts is placed along the back or sloping edge. Some bars may be discontinued as the top is approached; the analysis for determining the theoretical points of cutoff is similar to that mentioned in the footnote on page 335. Figure 204 shows a typical counterfort wall.

INDEX

E

F

G

H

I

J

L