BASIC STRUCTURES

by

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To my daughter

Gretel

PREFACE

Although much of the subject matter of this book is not new, the arrangement of material and the methods of development are unconventional and require some explanation. In the arrangement, it seemed logical to start with known forces and to develop, step by step, the methods of transmitting these forces through space. Consequently a treatment of the material usually found in the first quarter of a textbook on mechanics, with the omission of such subjects as friction and dynamics, was required. Special emphasis has been placed on methods of summation of forces (and moments) with reference to an established system of axes, instead of treating this under shear and bending moment diagrams.

From this point on, the arrangement has been dictated by the actual development and analysis of the structure as a means of force transmission, with particular emphasis on the physical action involved. At the proper points essential information on the mechanical properties of materials is presented in a manner which permits a direct utilization of the stress-strain diagram in strength calculations.

An attempt has been made to clarify the relationship between axial and transverse force transmission by showing that all force transmission over finite distances must be accomplished by axial forces or stresses. This treatment leads to new methods in the development of the classical equations for bending and shear stresses.

Problems which have been found important in the author's experience have been given special attention, particularly the subjects of unsymmetrical bending, shear flow, the tapered beam, and combined stresses. The principles essential to an understanding of instability and statically indeterminate structures have been included. Some of the more recent developments in column theory and interaction curves are presented because of their simplicity and basic importance.

To make the text as useful as possible, calculus and advanced mathematical methods have been entirely avoided. As a result the development of new methods of attack has been necessary in some cases, but in general these methods correspond quite closely to those actually used in engineering (most structures require summation methods rather than integration). The use of the terminology of calculus as a mathematical shorthand is explained in appropriate places.

It is hoped that the book will be found useful in developing a clear understanding of the basic principles of force transmission and in supplying new material needed in the analysis and design of aircraft structures. Suggestions for further improvement or correction of the text will be welcomed, as it has not been possible, under existing conditions, to devote a normal amount of time to the preparation and checking of the manuscript.

The author gratefully acknowledges the permission granted by Mr. Hall L. Hibbard, Vice President and Chief Engineer, to reproduce many Lockheed photographs and portions of the tables in Appendix 1 (from Lockheed Stress Memos). Mr. Randall Porter, of Lockheed Aircraft Corporation, was of much assistance in the preparation of the figures and problems. The authors of several other structural textbooks have kindly permitted the reproduction of certain useful material, as noted in the text.

F. R. Shanley

Los Angeles, California
January, 1944

NOTE TO STUDENTS

Most of the figures in this book have been reduced in size to conserve space. In engineering practice it is customary to use larger scales, particularly for vector diagrams. This should be kept in mind in preparing sketches and diagrams for the problems.

F. R. S.

CONTENTS

SYMBOLS

A = area
c = fixity coefficient for columns
D = diameter
E = modulus of elasticity for axial loading
e = strain
F = allowable stress
f = stress
G = modulus of elasticity (or rigidity) for transverse loading
H = shear flow increment
h = depth of a beam
I_x = moment of inertia (second moment of area) about axis X–X
I_{xy} = product of inertia with respect to axes X–X and Y–Y
I_p = polar moment of inertia
L = length
M = moment of a force or forces
n = effectiveness factor, for transformed sections
P = force
p = pressure
Q = static moment (first moment of area)
q = shear flow
R = radius of curvature; reaction or resultant
r = radius of a tube or shell
S = shear force
t = thickness
U = unsymmetrical factor for beams
w = distributed (running) loading
x, y, z = distances along reference axes X–X, Y–Y, Z–Z
y = distance from neutral axis of cross section
Z = section modulus, for bending
α (alpha) = angular measurement
β (beta) = angular measurement
Δ (delta) = prefix indicating change or increment
θ (theta) = slope due to bending; twist due to torsion
μ (mu) = Poisson's ratio
ρ (rho) = radius of gyration
Σ (sigma) = prefix indicating summation

CHAPTER 1

STRUCTURAL ANALYSIS METHODS

1·1. External Loads. The starting point for any problem in structural analysis is the determination of the external loads,* that is, the forces which the structure must transmit. This includes not only the magnitude of these loads but also their location, or their distribution over the structure. The methods used to determine the loading conditions for a given structure will depend on its nature and function. In an airplane, for instance, many of the loading conditions are obtained by considering certain flight speeds and attitudes and then by calculating the resulting distribution of air pressures over the lifting and balancing surfaces. In bridge structures the *pull of gravity* is the major cause of the forces which must be withstood by the structure; *dynamic* forces must also be considered. Dams are built to resist water *pressures*. In electrical machinery *magnetic* forces may predominate; in high speed machinery *centrifugal* forces must be provided for; in control systems the maximum force likely to be exerted by the hand or foot must be estimated.

Fortunately for the structural engineer the methods of load determination are usually well established in each branch of engineering. For an airplane the loading conditions are specified in considerable detail by the military or civil authorities [1, 2, 3] †, and new problems are attacked with the aid of the aerodynamic expert and the wind tunnel. In the design of buildings certain requirements are set up by safety authorities (Ref. 4, Chapt. 1). In electrical work the loads may usually be calculated with accuracy by means of highly developed theories and test data.

Because of the wide variation in the nature of the external forces that must be considered, a detailed discussion of such forces is beyond the scope of this text. It will be assumed that the magnitude and location of the external forces to be resisted by the structure are known, or that

* The term **load** is usually used with reference to a force *imposed on* something, while **force** is preferred as a general term which can refer to either external or internal forces.

† Superior numbers refer to bibliography on page 358.

they can be determined. The problem then becomes one of structural analysis and design.

1·2. Internal Forces. Structural analysis (or *stress analysis,* as it is often called) deals largely with the determination of the forces or stresses *within* a structure. If the general design of the structure is dictated by function rather than strength alone, the stress analyst has little to say about the arrangement of the material. His work then consists largely in analyzing the structure to determine the internal forces and stresses under the action of the external loads. If the computed stresses are dangerously high, measures must be taken to reduce the chances of failure, such as, more material, stronger material, higher heat treatment. Although very important from the standpoint of safety, this type of structural analysis is secondary to the functional design process.

Many examples of this type of analysis are found in mechanical engineering. For instance, the shape of a crankshaft of an automobile engine is determined primarily by the function which it performs. (The structural engineer would never choose such a complicated arrangement of material for the transmission of torque!) Nevertheless, it is possible to apply the principles of structural analysis to this problem to determine the actual dimensions needed to prevent failure under the applied forces.

On the other hand, the design of a bridge gives the engineer a chance to arrange the structural material to best advantage, with a minimum number of limitations. This type of problem may be thought of as a problem in **force transmission,** in which it is necessary to provide a structure which will transmit certain forces over a definite distance and without too much deflection.

Actually, most structural problems represent a combination of both methods of attack. The shape of an airplane wing, for instance, is determined largely by aerodynamic considerations. Nevertheless, the designer is relatively free to arrange the structure within the prescribed contours of the wing in such a manner as to carry the required loads with a minimum of structural material. In fact, in the airplane, weight is so important that aerodynamic efficiency, comfort, and ease of construction must often be compromised to some extent in favor of the structure, in order to obtain the best all-round design.

In summarizing the foregoing comments on *internal loads,* it is apparent that two different methods of attack are possible:

a. Stress analysis of a predetermined structure under the action of known forces.

b. The *design* of a structure for the specific purpose of transmitting known forces over definite distances.

Although the same basic principles apply to both these methods of attack, the method of approach differs considerably. Most treatments of this subject emphasize the stress analysis approach. In this text emphasis will be placed on the *force transmission* point of view, as this is of more direct interest and benefit to the designer. It is also of primary importance in airplane design, where it is essential to visualize, *from the beginning*, the most efficient method of force transmission.

1·3. Strength of Materials. Regardless of the method of attack employed, the designer usually wants to know one or more of the following things about the behavior of the structure.

a. Deflection of the structure at different stages of loading.

b. The intensity of loading which will produce the first appreciable *permanent set.*

c. The maximum load-carrying capacity of the structure, i.e., the *ultimate* or breaking load.

d. The intensity of loading which will produce the first appreciable *buckling* deformation (of particular interest in airplane work).

Depending on the type of structure involved, some of the above items may predominate or may be completely disregarded. Usually we are primarily concerned with the *ultimate strength* (item *c*). But if *stiffness* is the primary concern, the strength may not be the deciding factor. In heavy machinery *buckling* (item *d*) can usually be ignored entirely, but in airplane work it plays an important role in the design of the structure because of the slender members and thin sheets employed.

In special cases it may also be desirable or necessary to know one or more of the following:

e. Effect of repeated stresses (vibration, etc.).

f. Resistance to impact loading.

g. Effect of abnormally high or low temperatures.

h. Effect of sustained (long-time) loading (creep).

The behavior of materials under various loading conditions is commonly studied under the heading "Strength of Materials." This includes the mechanical properties of the materials themselves (strength, stiffness, etc.) and usually goes beyond this into actual structural analysis methods. In approaching the subject of structures from the *design* point of view, the study of material properties will be deferred until the basic principles of force transmission have been covered. At this point it will only be noted that the majority of mechanical properties needed by the structural engineer concern the relationship between *stress* and *strain*, that is, the manner in which a material deforms under load. The **stress-strain diagram** gives most of the needed

FIG. 1·1. Lockheed *Constellation.*

information and is therefore one of the most important tools of the structural engineer.

1·4. Factors of Safety. There is no general agreement on the exact definition and application of factors of safety. Sometimes the expected loads are multiplied by a suitable factor to obtain *design loads* which provide for discrepancies between the actual strength of the structure and the predicted strength. This practice is usually employed in airplane work. In other fields it is common practice to divide the actual strength properties by some factor of safety, to give reduced *allowable stresses,* or *working stresses.* Excess strength is sometimes indicated by a *margin of safety.* The structural engineer should be thoroughly familiar with the practice peculiar to the field in which he is working, as it is easy to become confused in the interpretation of *factors of safety, working stresses,* and *margins of safety.*

To avoid misunderstandings, the treatment of structural theory and design in this text will be based on the *actual* behavior of the structure, as it is always possible to correlate this method of analysis with any system of factors of safety or working stresses. In other words, the use of fictitious allowable stresses which have been arbitrarily reduced from the true ultimate or yield stresses will be avoided.

CHAPTER 2

FORCES AND MOMENTS *

2·1. Designation of Forces. Although forces cannot actually be visualized (only their *effects* are visible), it is convenient to create a mental picture of forces by means of **vectors** (lines supplied with arrowheads). This process becomes somewhat difficult when dealing with forces in three-dimensional space, as the textbook writer is confined to two-dimensional paper. For this reason, it is better to discuss fundamental principles first in terms of the two-dimensional (plane) system and then to extend this discussion to the three-dimensional (space) system. In either system, **reference axes** are required, as well as units of measurement for the *location, direction,* and *magnitude* of the forces. These units may take any convenient form. In general, in this text the engineering units of *feet* (or *inches*) and *pounds* will be used.

Figure 2·1 illustrates a force in the plane of the paper. The *location* of the point of application A is determined by the coordinates of the point, which have been designated z_A and x_A. (Note that z and x are used here as "vertical" and "horizontal" axes to agree with airplane practice. Any other reference system will, of course, serve equally well.) †

It is necessary to establish positive and negative directions along the reference axes and to adhere to these conventions throughout the solution of any problem. Thus, in Fig. 2·1, distances measured away from the origin in the direction of the axes Z or X will be called positive. Measurements in the opposite direction will therefore be negative.

The *direction* of the force is shown by means of the vector itself. Since

* This elementary chapter may be read as a review by those already familiar with the subject matter. The new student should be sure he understands this chapter thoroughly before proceeding with the subsequent ones.

† There is no well-established convention governing the use of subscripts in structural work. Letters and numbers are used indiscriminately for this purpose. A subscript may refer to a particular point, a reference axis, a loading condition, or almost anything else. The most important thing is to be *consistent* throughout a particular problem. In aircraft work, for instance, it has been found impracticable to devise one set of subscripts which can be used for all structural problems. But in analyzing a particular structure (such as a wing beam) the subscripts chosen must be adhered to and properly used until the problem is solved.

a force can act in either direction along a straight line, it is necessary to indicate *which* direction by means of an arrowhead. In general, the arrowhead should be directed away from the point of application, but

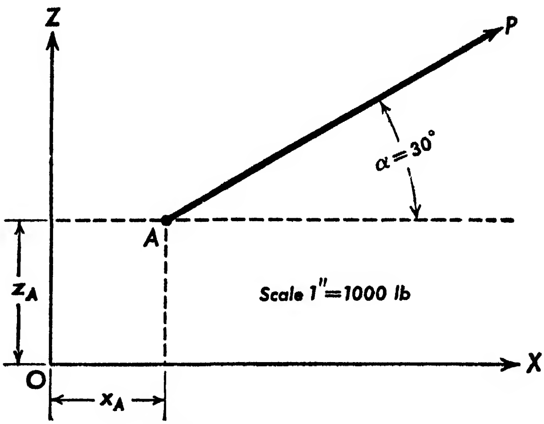


FIG. 2 1. Illustration of a force by means of a vector.

there is no established convention except in graphical addition, when it must be observed (see Sec. 2·5). The opposite procedure is usually followed in illustrating forces acting on a beam or column.

In Fig. 2·1 the magnitude of the force is indicated as 2000 lb by establishing an arbitrary scale of 1000 lb to the inch and making the vector 2 in. long.

2·2. Components. Although the vector picture may be used to convey information on both *magnitude* and *direction* of the force, it is often inconvenient to draw such pictures in extended structural calculations. The *magnitude* of the force can, of course, be specified numerically, i.e., $P = 2000$ lb. The direction can be given by specifying the angle made with reference to some known axis or plane (such as angle α in Fig. 2·1), but it is also possible to take care of this by dividing or "resolving" the force into **components.** This is illustrated in Fig. 2·2, in which the force P has been divided into the components P_z and P_x. These components can be thought of as representing the "effectiveness" of the force P along the axes Z and X, respectively. In Fig. 2·2, the effect of the force $P = 2000$ lb, acting as shown in Fig. 2·1, is obtained by the *simultaneous* application of forces, $P_z = 1000$ lb and $P_x = 1333$ lb.

The determination of the components of a force or, conversely, their resolution into the **resultant** force, may be done graphically or analytically. The components are actually the **projections** of the force vector on the new lines of action (represented in the illustration by lines drawn parallel to the Z and X axes, respectively). *When the components make a right angle with each other,* they may be determined analytically by the simple rule that the effectiveness of a force along a given line of action

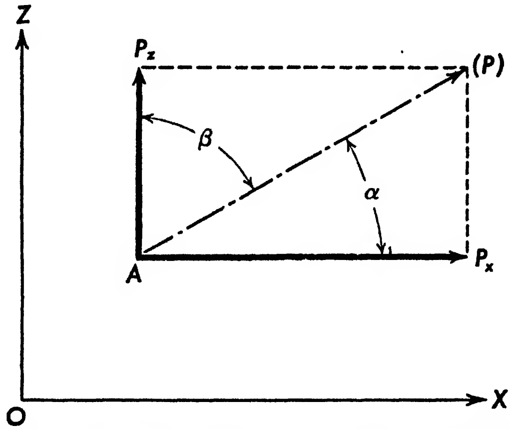


FIG. 2·2. Designation of a force by means of components.

is proportional to the cosine of the angle between the original line and the new line of action. Thus in Fig. 2·2 the components are given by the equations

$$P_z = P \cos \beta$$
$$P_x = P \cos \alpha \qquad [2\cdot 1]$$

It is sometimes more convenient to work with only one angle. This can be done by making use of the relationship between sines and cosines, giving the typical equations

$$P_z = P \sin \alpha$$
$$P_x = P \cos \alpha \qquad [2\cdot 1a]$$

or, alternatively,

$$P_z = P \cos \beta$$
$$P_x = P \sin \beta \qquad [2\cdot 1b]$$

These elementary relationships are constantly used in stress analysis and design work. If they are thoroughly understood there is little need for more advanced applications of trigonometry, except in special cases.*

* The student who has not had a formal course in trigonometry need not be discouraged by the use of trigonometric symbols such as sine and cosine. Actually these terms are derived from exactly the same relationships that we are now using for components, hence we are using them primarily as "shorthand" symbols because

2·3. Sign Conventions. In designating forces numerically (instead of vectorially) it becomes necessary to introduce definite conventions as to the *sense* of the forces. This is usually done by applying to the force or its components the same type of conventions that are used for *distances*. That is, certain axes are designated as positive and all force measurements along these axes in a direction *away from the origin* are called positive (+). Measurements in the opposite direction are, of course, negative (−).

If, in Fig. 2·2, the axes X and Z are assumed to indicate positive forces in the directions shown by the arrowheads, the components P_x and P_z are likewise positive (being directed away from the origin).

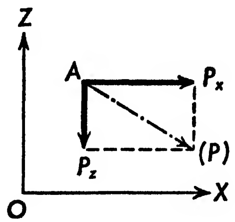


FIG. 2·3. Force with one negative component.

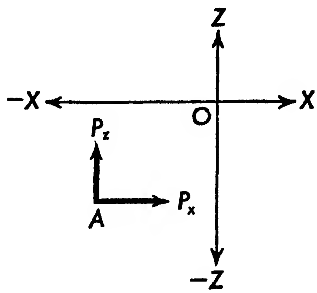


FIG. 2·4. Positive components located by negative distances.

Figure 2·3 shows one component (P_x) positive, and the other (P_z) negative. In Fig. 2·4 the point of application is located by negative distances, but the components of the force are both positive.

It is important to note that, although structural calculations may be carried out by working separately with each component, *the final result will not be obtained until the effects are combined.* Failure to recognize this principle may lead to serious errors or wrong conclusions.

2·4. Space Systems. In three-dimensional space the preceding principles are extended by the use of an additional axis and an additional component along this axis. This is illustrated in Fig. 2·5, which is, of course, diagrammatic in character, dimensions along the Y axis being out of scale. The point A is now located by the three coordinates x, y, and z. The components of the force P are P_x, P_y, and P_z. To help clarify the three-dimensional picture it is convenient to visualize com-

their values have been worked out and tabulated for all angles. Tables of trigonometric functions are found in nearly all engineering and mathematical handbooks and can also be purchased separately. (For instance, *Wiley Trigonometric Tables* and Ralph G. Hudson, *The Engineers' Manual.*)

ponents as defining the sides of a box, as shown in Fig. 2·5. The components of a force are then seen to be edges of the box, of which the force itself is a diagonal. As in plane systems, the *cosine* is used to determine the component analytically, the angles being measured between the

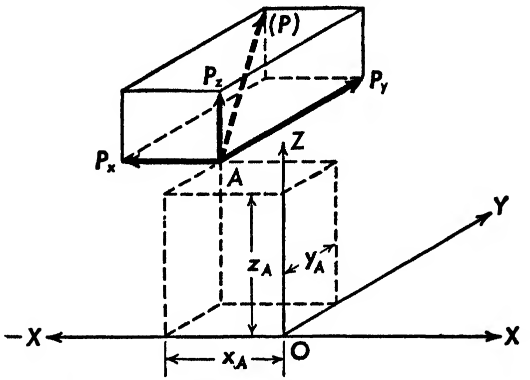


FIG. 2·5. Designation of a force in space (three-dimensional system).

force itself and the respective components. (In space systems these angles are seldom actually measured, but the cosines are determined directly from length ratios. When so used, the cosines are often called **direction cosines.** Their actual use will be taken up later.)

2·5. Vector Addition of Forces. Assume that two forces having different magnitudes and directions are acting simultaneously at the same point (Fig. 2·6). It is desired to know the **resultant force.** The addition may be done graphically by representing each force by an accurate

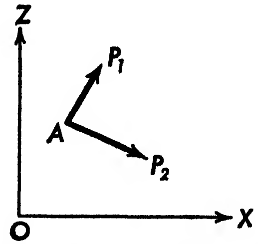


FIG. 2·6. Two forces acting at same point.

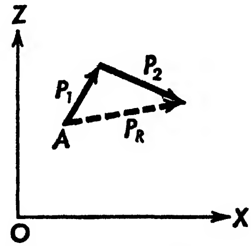


FIG. 2·7. Vector addition of two forces.

scale vector and "adding" these, as shown in Fig. 2·7. In adding vectors graphically the "tail" of one vector is placed at the arrowhead of another. The final result is obtained by drawing the vector from the original point of application to the head of the last added vector. In Fig. 2·7, the resultant force, P_r, is indicated by the dashed line. In this process the convention of directing the arrowhead away from the point of application *must* be followed. The *wrong* way to add vectors is shown in Fig. 2·8.

Figure 2·9 shows a common process used to add vectors by constructing a *parallelogram*, the sides of which are parallel to the vectors in

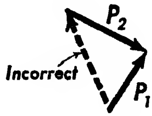


FIG. 2·8. Wrong method of adding vectors.

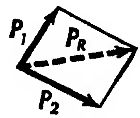


FIG. 2·9. Parallelogram method of vector addition.

question. The diagonal represents the resultant force. It can readily be seen that this agrees with the previous method, as either of the light lines used to construct the parallelogram can be thought of as one of the vectors being added to the other one. The main reason for using this construction is that the use of two lines eliminates the need for measuring off one of the vectors. The method shown in Fig. 2·7 is preferred as a means of illustrating the simplicity of vector addition.

The vector addition of *several different forces acting at the same point* is accomplished in the same manner as that previously described for two forces. The sequence of addition is immaterial, provided that the proper directions of the forces are observed and maintained. Figure 2·10 shows the addition of four forces by vector methods. The actual

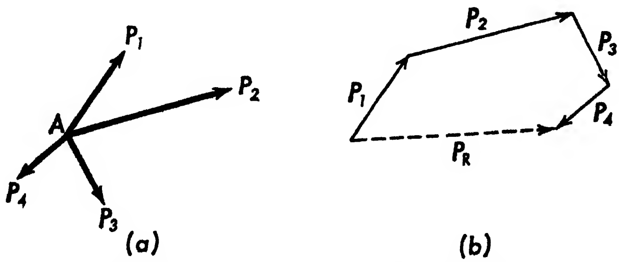


FIG. 2·10. Vector addition of four forces.

location of the forces in space is shown by (*a*) which is called the **space diagram.** The figure formed by the vector addition (*b*) is called the **force polygon.** The resultant force is given by the **closing line,** directed *away from* the starting point. This is the dashed line in Fig. 2·10.

If the vector diagram closes, the resultant force must obviously be zero. This is an important means of checking the equilibrium of a structure.

Forces whose lines of action intersect at the same point are called **concurrent forces** and may be treated in the same manner as forces which act at the same point, that is, they may be added vectorially or analytically. The resultant force will then act along a line through the common point of intersection. Three concurrent forces are illustrated in Fig. 2·11. The resultant of these forces will act through point O.

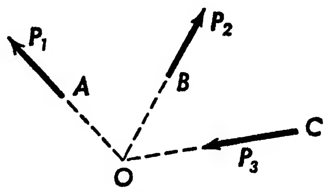


FIG. 2·11. Concurrent forces.

Forces not acting at the same point (or whose lines of action do not intersect at a common point) may be added vectorially to obtain the magnitude and direction of the resultant force. However, *this process alone will not determine the effective point of application* of the resultant force, which must be done by other methods to be described.

The graphical *subtraction* of vectors may best be considered as a *change of sign* of the vector to be subtracted, that is, the arrowhead is placed on the other end, and the process of addition is then carried out in the usual manner. This is, of course, analogous to algebraic addition in which the sign of the quantity determines whether it increases or decreases the total.

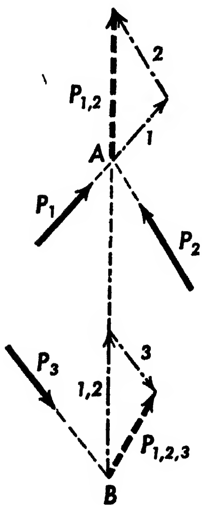


FIG. 2·12. Vector method of locating resultant force.

2·6. Location of Resultant Force (Graphical Methods). As previously noted, if the forces, or their lines of action, all pass through a single point, the resultant force will also pass through that point. In such a case vector addition of the forces as described in Sec. 2·5 will be sufficient to determine the resultant completely. But if all lines of action of the different forces do not coincide (non-concurrent forces), it becomes necessary to take additional steps in the graphical solution.

In a plane system it is possible to find the location of the resultant force graphically by successive steps, taking only two forces at a time. Figure 2·12 shows how this is done. The resultant of any two forces (such as P_1 and P_2) acts through the intersection of their lines of action (point A); hence these two forces may be replaced by a single resultant force passing through this intersection. In Fig. 2·12 force P_1 is moved to point A (along its line of action, of course) and force P_2 is added to it vectorially, to give the resultant force $P_{1,2}$. This process may be repeated with another force (such as P_3) until all the forces have been combined in this manner. The resultant $P_{1,2,3}$ not only has the proper magnitude but also acts along the proper line. Although perfectly sound and straightforward, this method may become cumbersome if the points of intersection are widely separated. The method cannot be used, of course, if any two forces are parallel, since their intersection is at infinity.

The resultant of two parallel forces may be located graphically by transforming them into non-parallel forces. This may be done by adding (vectorially) *equal and opposite forces acting along the same line.* The net effect of such a *pair* is zero, so far as the resultant is concerned. Figure 2·13 shows how this is done. Arbitrary forces P_0 are assumed

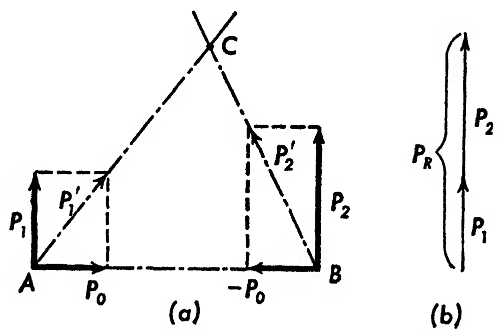


Fig. 2 13 Resultant of parallel forces.

to be acting at A and B, in opposite directions. The resultant forces at points A and B have lines of action which intersect at C, which determines a point on the line of action of the resultant force. The direction and magnitude of this resultant may be determined by direct vector addition of P_1 and P_2. It is usually convenient to handle this separately by drawing a force polygon as described in Sec. 2·5. The polygon for this case is shown in Fig. 2·13*b*.

The process of adding a pair of forces is analogous to adding equal quantities to each side of an algebraic equation. In dealing with vectors, however, it is not sufficient that these added forces be equal and opposite in sense, but they must also act along the same line in space; otherwise a *couple* would be added. (See Sec. 2·13.)

The process may also be carried out by working with the *lines of action* of the forces, rather than with their actual points of application. In Fig. 2·14*a* the line X–X has been arbitrarily drawn across the lines of action of P_1 and P_2 and a pair of forces P_0 has been added at the intersecting points M and N. The lines of action of the resulting forces P_1' and P_2' are found by drawing separate vector diagrams as indicated in (*b*). (The lines forming these diagrams must, of course, be parallel with the forces in the space diagram.) Note that the pair of forces P_0 must be directed along the same line, X–X.

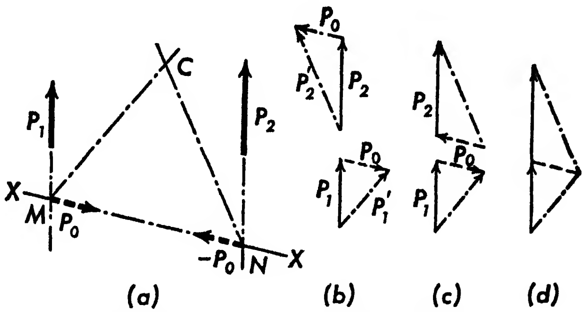


FIG. 2·14. Evolution of string polygon.

Sketch (*c*) shows a different sequence for addition of forces P_2 and P_0. Finally, in sketch (*d*) these two diagrams are superimposed to give a very simple vector diagram. This diagram combines the force polygon and the addition of the pair of forces P_0.

In practice, the intermediate steps are omitted and the construction is carried out as shown in Fig. 2·15, which shows that the method may also be used for non-parallel forces. The force polygon is first constructed as in (*b*). Then an arbitrary point O is chosen, as shown. This represents the addition of P_0 to force P_1 and the "subtraction" of this same force from force P_2. Point O is then connected with the ends of the force polygon. The line X–X is now drawn across the lines of action of P_1 and P_2, in the space diagram (*a*) parallel with the force vector P_0 in the force polygon. (Line X–X represents the common line of action for the added forces P_0.) Lines are drawn through the intersecting points parallel with the lines through O, thus establishing point C. The resultant force P_R, already found in the force polygon, acts through this point.

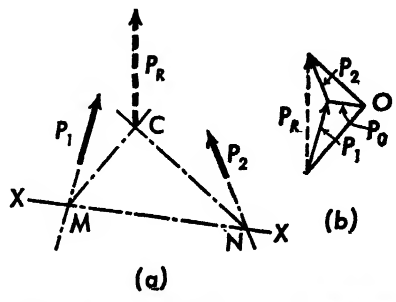


FIG. 2·15. Resultant of non-parallel forces, by string polygon.

The lines MC and NC, in Fig. 2·15, are called **strings** and the figure MNC is called a **string polygon.** The word string is used because the lines in question actually represent the positions which would be assumed by strings under the action of the force system in question. (In Fig. 2·15 forces P_1 and P_2 would have to be reversed in direction to put all the strings in tension.) Point O in Fig. 2·15*b* is called the **pole** and the lines originating at this point are called the **rays.**

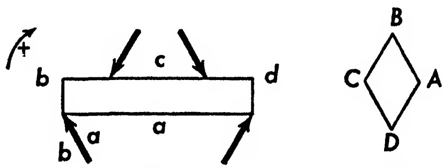
2·7. Bow's Notation. Before proceeding to the vector treatment of multiple forces it is convenient to adopt a method of force designation known as Bow's notation. This system is commonly used in the graphical analysis of trusses. It eliminates the need for arrowheads and establishes a relationship between the force designation in the force polygon and the force location in the space diagram.

Bow's notation is based on two rules:

a. Force vectors are represented in the force polygon by two capital letters, one at each end of the vector. The *sense* of the force is determined by the alphabetical order of the letters.

b. The *sequence* of application of the forces in space is denoted by lower-case letters, also in alphabetical order.

Thus in Fig. 2·16*a*, the lower-case letters are placed between forces and are arranged in a clockwise direction. In the vector diagram (*b*),



(a) (b)

FIG. 2·16. Bow's notation.

the direction of the force is indicated by the sequence of the corresponding capital letters. In establishing this sequence it is necessary to adhere to the convention that the progression around the space diagram be made in alphabetical order. The force acting between a and b, for instance, is properly designated by the vector AB, as the force actually acts in the direction from A to B. To avoid confusion the "space" letters may be placed on opposite sides of a force vector, as indicated for force AB, in Fig. 2·16.

2·8. Resultant of Multiple Forces (String Polygon). The method of the string polygon may be applied to more than two forces by adding more pairs of forces and thereby extending the vector diagram accordingly.

Thus in Fig. 2·17 the force polygon is drawn for forces AB, BC, and CD, determining the resultant AD. Point O is chosen and the rays are drawn as shown. At any convenient point on the line of action ab, line *bo* is drawn parallel with force *OB*, intersecting the line of action *bc*. Line *co* is now drawn from this point parallel with force vector *CO*, to intersect line of action *cd*. Lines *ao* and *do* are now drawn parallel with force vectors *AO* and *DO*, locating a point on the line of action of the resultant force, which is shown acting at this point.

The physical significance of this operation may be explained by considering each triangle formed by the rays to represent successive operations of adding and subtracting pairs of equal and opposite vectors, as

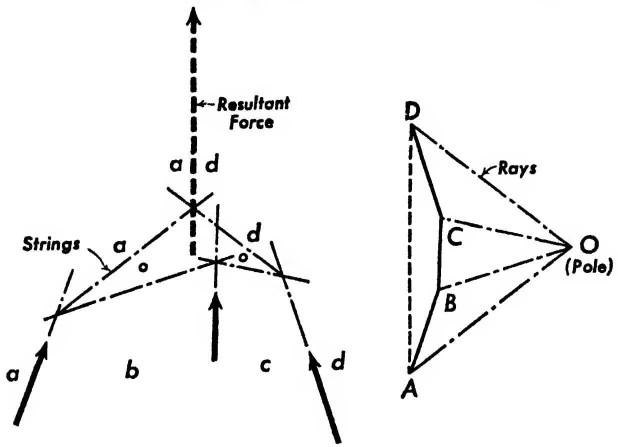


FIG. 2·17. String polygon for multiple forces.

previously described for two forces. The entire system of forces is thus reduced to two equivalent forces (*AO* and *DO*) the directions of which establish a point on the line of action of the resultant force. The intermediate rays represent the added and subtracted forces, whereas the strings in the space diagram represent the common lines of action for these forces.

It should again be noted that the space diagrams used in these illustrations are not in equilibrium, as the object is to obtain *resultant* forces, not *reactions*. If the reaction is desired, the direction of the resultant would be reversed, i.e., in the force polygon the vector would be read *DA* instead of *AD*, and the resultant would be shown in the opposite direction in the space diagram. (See Chapt. 4 for further discussion of reactions.)

2·9. Analytical Addition of Forces. Forces which act *along the same axis*, or *parallel to each other*, may be added algebraically. For the more general case illustrated in Fig. 2·18 (forces at an angle to each other), it is necessary to resolve the forces into components which *do* act parallel to each other. These may then be added directly, thereby determining the components of the resultant force. (Note that any system of axes will serve for this process, provided that all *added* components are taken parallel to the same axis.)

Figure 2·18 illustrates the analytical addition of forces P_1 and P_2. This is best done by tabulating the computations as shown.

Force	X Component	Z Component
P_1	+210	+400
P_2	+450	−190
Sum	+660	+210

The resultant P_R is given by components $P_x = +660$ and $P_z = +210$, as illustrated in Fig. 2·19. The use of rectangular* coordinates permits

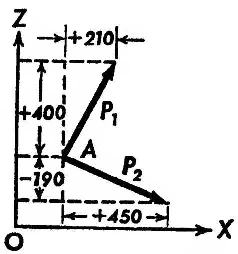


Fig. 2·18. Analytical addition of two forces.

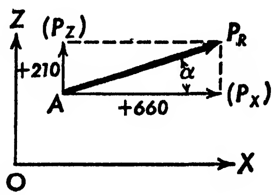


Fig. 2·19. Resultant of forces shown in Fig. 2·18.

the determination of the magnitude of the resultant force analytically by means of the well-known formula

$$P_R = \sqrt{P_x^2 + P_z^2} \qquad [2\cdot2]$$

In Fig. 2·19

$$P_R = \sqrt{(660)^2 + (210)^2} = 694$$

The *direction* of the resultant force may be determined from the magnitudes of its components. In the illustration, the angle of the resultant force (with respect to the X axis) is given by any one of the following formulas.

$$\tan \alpha = \frac{P_z}{P_x} = \frac{210}{660} = 0.318$$

or

$$\cos \alpha = \frac{P_x}{P_R} = \frac{660}{694} = 0.951$$

* Cartesian coordinates, in which the major reference axes are mutually perpendicular.

or

$$\sin \alpha = \frac{P_z}{P_R} = \frac{210}{694} = 0.303$$

From which,

$$\alpha = 17° 40'$$

Note that Figs. 2·18 and 2·19 are drawn only to illustrate the analytical process; the mathematical work will usually be carried out without the aid of the figures.

In three-dimensional space the same process is followed, with the addition of a third component (Y component in this case). Equation 2·2 then becomes

$$P_R = \sqrt{P_x^2 + P_z^2 + P_y^2} \qquad [2{\cdot}2a]$$

Assume that the forces in the previous example have components also along a third (Y) axis. The process of finding the resultant would then be shown by the tabulation, in which Y components have been added.

Force	X Component	Z Component	Y Component
P_1	+210	+400	+650
P_2	+450	−190	−50
Sum	+660	+210	+600

$$\begin{aligned} P_R &= \sqrt{P_x^2 + P_y^2 + P_z^2} \\ &= \sqrt{(660)^2 + (210)^2 + (600)^2} \\ &= 916 \end{aligned}$$

The vector picture is obtained by considering these three components to be acting at the same point in space, parallel to the three reference axes, as described in Sec. 2·4. Analytically, the direction of the resultant force would be completely described by the three *direction cosines*:

$$\frac{P_x}{P_R}, \quad \frac{P_y}{P_R}, \quad \text{and} \quad \frac{P_z}{P_R}$$

In three-dimensional work the analytical method becomes most useful, as the graphical method must be confined to the two-dimensional plane. (Special methods of applying two-dimensional graphical analysis to space frameworks have been developed, however.) As in vector addition, forces not acting at the same point may be added by the above process, but the effective *point of application* cannot be determined by this method alone.

2·10. Moments. Although structural theory and literature are replete with references to moments (bending moments, torsional moments, moment diagrams, etc.), it should be clearly understood that the term moment has no structural significance unless associated with a force or forces.* It is often convenient to designate, by some symbol or quantity, the *effect* of a force with relation to some point in space. If the point in question does not lie on the line of action of the force, there will be produced a *tendency to rotate* about this point. This tendency is directly proportional to the magnitude of the force (denoted by P) and the distance over which the full rotational effect is exerted. Figure 2·20 illustrates a force P, acting at a distance d from point A. The tendency to rotate about point A is proportional to P and also to d, and hence may be measured by the quantity Pd. Such a unit of measurement (force times distance) is often used to denote the turning effect, or moment, of a force. Strictly speaking, *turning* moment, *bending* moment, *torsional* moment, etc., should always be referred to since the word moment really means magnitude or effectiveness. In structural usage, however, the identifying adjective is often dropped and only the word "moment" is used.

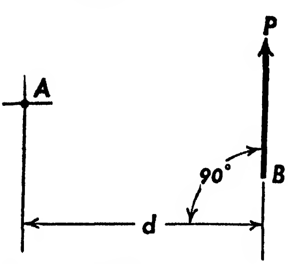


FIG. 2·20. Moment of a force.

The *distance* used in measuring turning moments must be taken normal to the *line of action* of the force (see Fig. 2·20). This is true regardless of the location or point of application of the force. In Fig. 2·20 it would be wrong to use the distance between A and B, even though the force acts at B. Whenever necessary the line of action must be extended to enable the correct distance to be measured.

An alternative method of measuring a turning moment is to resolve the force into two components, one along the line between the two points in question and the other normal to it (Fig. 2·21). The component P_h measured along the line AB has no turning effect about point A and can consequently be ignored in computing the moment. The moment of force P about point A would therefore be given by the quantity $P_n d$.

It should be noted that a *moment* is made up of two different quantities: *force* and *distance*. From this it follows that the same moment may represent different combinations of force and distance. For instance, the two forces shown in Fig. 2·22, each have the same moment about A,

* Webster's *Collegiate Dictionary* defines *moment* as: "tendency, or measure of tendency, to produce motion, especially about a point or axis."

as force P_2 is half as large as P_1 but acts over twice the distance (i.e., $d_2 = 2d_1$).

Another consequence of this dual nature of moments is that the units in which they are measured must contain the units of measurement for

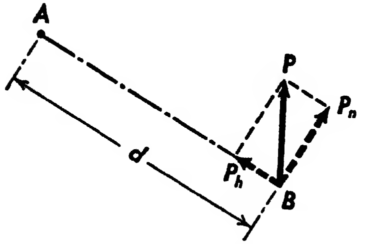


FIG. 2·21. Moment determined by components.

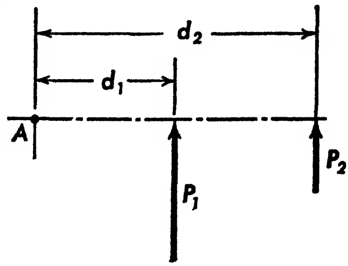


FIG. 2·22. Equivalent moments.

both force and distance. Typical units of measurement for moments are, therefore, the *inch-pound* and the *foot-pound.*

2·11. Moment Vectors. It is often convenient or necessary to designate a moment vectorially. This is done by showing the *axis* about which the moment acts and denoting the magnitude of the moment by the length of the vector. The *sense* of the moment, that is, the direction in which it tends to turn, is indicated by the arrowhead, using either the right-hand or the left-hand rule. (Thumb toward arrowhead, fingers indicating turning direction.) If the left-hand rule were employed, the motion of the hands of a clock, for instance, would be indicated by a vector normal to the clock face and directed toward the person viewing the clock. If the right-hand rule were used, the vector would be pointed in the opposite direction. There is no standard convention as to which rule should be used. For convenience, a *double* arrowhead is often used to distinguish a moment vector from a force vector. In Fig. 2·23, the vector M represents a turning effect about point A, indicating a tendency to rotate about the vector itself. The magnitude of the moment would be indicated by the length of the vector, using some previously established scale. In Fig. 2·23, a curved arrow has been added to help indicate the nature of the moment (using left-hand rule).

FIG. 2·23. Moment vector.

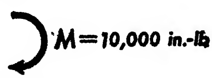


FIG. 2·24. Moment in a plane.

If the axis of rotation is normal to the plane of the paper, it becomes impossible to indicate the moment vectorially. (The axis would appear as a dot.) The curved arrow may be used to show the existence of such a moment, however, as in Fig. 2·24. To indicate the magnitude it would be necessary to quote the actual value, such as, $M = 10{,}000$ in.-lb.

Turning moments are often visualized as a tendency to rotate in a certain plane. The *plane of rotation* is established by the *axis of rotation*, to which it is perpendicular. Hence either system may be used in visualizing the effects of turning moments. However, the plane system does not lend itself to graphical or analytical solution, whereas the axis system does.

2·12. Addition of Moment Vectors. The greatest advantage of the axis type of moment vector is that it may be treated in the same manner as the force vector. Thus, the effect of two separate tendencies to rotate may be combined vectorially into a single resultant effect. All the rules previously given for the addition, subtraction, and analytical treatment of forces and force vectors may be applied to moment vectors, with the exception that *moment vectors may be added vectorially regardless of their point of application.* (This is because the point of application has no effect on the resultant moment, as will be explained later.)

For example, assume that in Fig. 2·25 there exist at a point A two turning moments about different axes. If the lengths of the vectors M_1 and M_2 are made proportional to these turning moments, the net effect will be indicated by the resultant vector M_R, which was obtained by adding vector M_1 and M_2 vectorially. The new axis of rotation will also be shown by the direction of M_R.

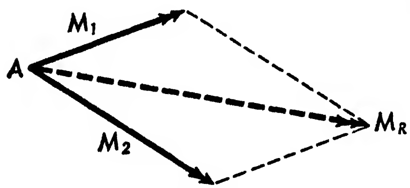


FIG. 2·25. Addition of moment vectors.

2·13. Couples. Turning moments are often indicated by means of couples. The significance of the couple is that the use of two parallel forces, oppositely directed, completely cancels out the *direct* effects of the forces, leaving only the turning effect. Thus in Fig. 2·26 the equal

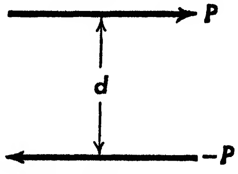


FIG. 2·26. Couple.

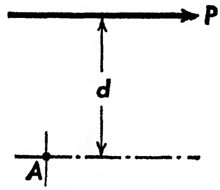


FIG. 2·27. Half couple.

forces acting in opposite directions and separated by a distance d represent a pure turning moment, Pd. The use of only *one* force, as shown in

Fig. 2·27, represents a turning moment (about point A) of exactly the same magnitude, but now the *direct* effect of force P must also be considered.

This can be illustrated by a simple example such as shown in Fig. 2·28, which represents a typical control column of an airplane. In (*a*) the turning moment exerted by the pilot is equal to P_1D (where D is the wheel diameter). Note that there is no direct force exerted on the vertical column. In (*b*) the moment about the wheel centerline is equal to P_2R (where R is the radius), and therefore has the same magnitude as

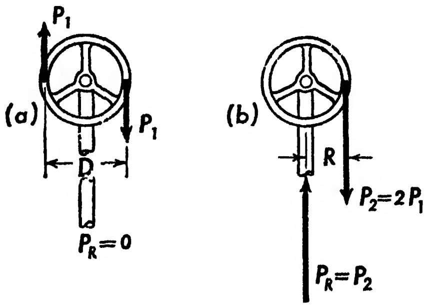


FIG. 2·28. Couple vs. direct force.

in (*a*), (since $P_2 = 2P_1$ and $D = 2R$). In (*b*), however, the vertical column must resist or "transmit" the entire force P_2 directly, as indicated by the reaction P_R. (Moment reaction not shown.)

2·14. Effectiveness of Force Systems. Since the structural engineer is usually more interested in the effect of the forces than in the forces themselves, it is useful to know under what conditions various force systems have identical effects. Disregarding local conditions at the point of application, the following general statements can be made.

a. Forces of the same magnitude and direction will have the same general effect if they are applied anywhere along the *same line of action.* This is illustrated in Fig. 2·29. The *general* effect on the object will be the same in both (*a*) and (*b*). In stress analysis it is therefore possible to "move" a force from one point to another without changing its effect, provided that such movement is along the line of action of the force.

b. The general effect at some point which does *not* lie on the line of action of the force may be represented by the force itself, plus a moment equal to the turning effect of the force about the point. Figure 2·30 shows two force systems which have the same general effect on the object.

c. Moments (couples) will have the same *general* effect regardless of where they are applied; that is, the couple may be moved to any point, provided that the magnitude of the moment and the direction of its axis

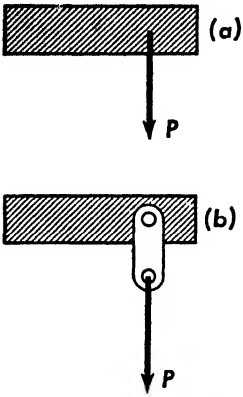


FIG. 2·29. Equivalence of forces.

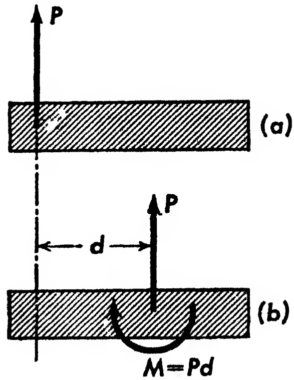


FIG. 2·30. Equivalent force systems.

are unchanged. This is shown by Fig. 2·31, in which the moments are shown by couples. In both (*a*) and (*b*) there will be the same tendency to rotate, although conditions *within* the beams (local conditions) will be different.

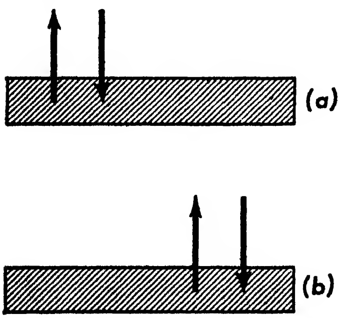


FIG. 2·31. Equivalent moments (couples).

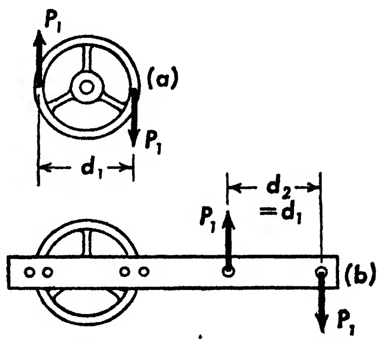


FIG. 2·32. Equivalent moments on steering wheel.

At first sight it may be difficult to realize that a couple has the same effect regardless of where it is applied. This can be further illustrated by a condition such as shown in Fig. 2·32. Here the effect on the steering wheel is the same in both (*a*) and (*b*), even though the turning moment in (*b*) is not applied directly to the wheel. (*Note.* The foregoing statements apply to *structures,* not mechanisms. Obviously a turning moment may be changed in magnitude during transmission, through the use of pulleys, gears, and other mechanical devices.)

The principles underlying the equivalence of force systems are of great value in computing *reactions* and in analyzing a certain portion of a structure without making a complete analysis of the entire structure. These principles are also frequently applied in setting up structural tests, as they often make it possible to substitute a simple loading condition for a more complicated one.

2·15. Forces Not Acting at Same Point (Non-Concurrent). Using the foregoing principles, it is possible to determine the net effect, at any point, of any system of forces, provided of course that the magnitude,

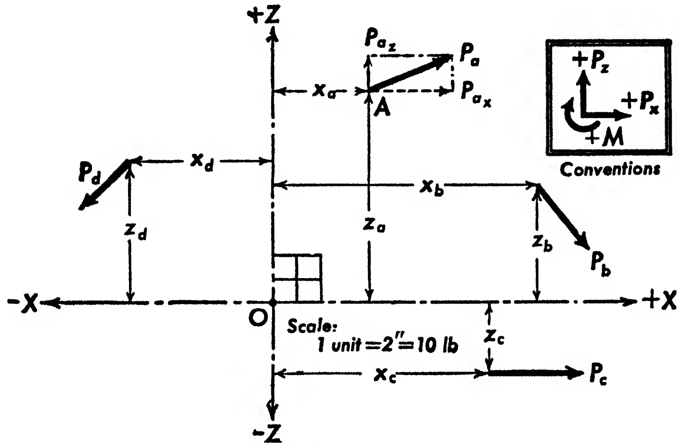


FIG. 2·33. Force system about point O.

direction, and location of the forces are known. (Location may refer to the *line of action only*; that is, it is not necessary to know the exact point of application in determining the net effect.) The process consists simply in adding all the forces (either vectorially or analytically) and then adding the turning moments of the forces about the point in question. The final result will consist of a single *resultant force* and a *resultant moment*, both acting at the point about which the summation is taken, or it may take the form of *components* of these resultants.

If all the forces act in the same plane the process is simple. In the analytical method the components of the forces along two selected axes are determined and added, as described in Sec. 2·9. It is usually convenient to determine the turning moments of these components, rather than the moment of the forces themselves, as each component is normal to one of the reference axes.

For example, assume that it is desired to know the effectiveness at point O, of the forces acting at points A, B, C, and D, in Fig. 2·33. Table 2·1 is constructed as shown, using inch-pound units.

TABLE 2·1

SUMMATION OF FORCES NOT ACTING AT A POINT

Point	Location		Forces			Moments	
	x	z	P	P_x	P_z	$P_x z$	$P_z x$
A	8	18	38 0	+35	15	+630	−120
B	22	10	35 6	+22	−28	+220	+616
C	18	−6	40 0	+40	0	−240	0
D	−12	12	28 2	−20	−20	240	−240
				+77	−33	+370	+256

$$P_{Rx} = +77 \qquad P_{Rz} = -33$$

$$P_R = \sqrt{(77)^2 + (-33)^2} = 83.8 \text{ lb}$$

$$M_0 = +370 + 256 \qquad = +626 \text{ in.-lb}$$

(These values are shown in Fig. 2·34.)

By comparing Ta' e 2·1 with the table in Sec. 2·9, it will be seen that additional columns have been included for the determination of the moments of the force components about point O. Two columns are for the tabulation of the moment arms of these components, which are equal to the coordinates of the points at which they act. All the moments due to the components are added algebraically, to determine the resultant moment at point O. (Algebraic addition is permissible because the moment vectors are parallel, since all the forces are in the same plane.)

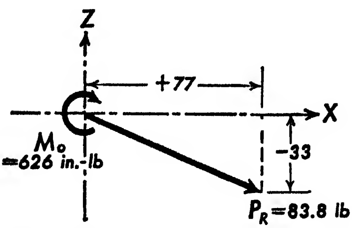


FIG. 2 34. Resultant of forces in Fig. 2 33.

The most important part of this operation is to establish a definite convention for positive turning moments and to be sure that each value of moment entered in the table has the proper sign. In a plane system such as shown in Fig. 2·33, it is possible to do this by inspection. For instance, having established positive turning moments as clockwise (shown by curved arrow) it is obvious that the moment of P_{az} about point O is negative. The value for $P_z x$ entered for point a is therefore shown as negative in the table.

2·16. General System. It is possible to establish a system of sign conventions such that the algebraic multiplication of the forces and distances automatically gives the proper signs for the moments, thereby making it unnecessary to determine the sign of the moment by inspection. Such a consistent system is indicated in Fig. 2·35. In this system, positive values for both force and distance will give positive values for turning moment. Therefore all cases are automatically covered by properly observing the signs in multiplying forces and distances.

Generally, however, it will not be convenient to start out with an entirely consistent system. For instance, in airplane work positive forces are usually measured *upward*, as shown in Fig. 2·33. It would be confusing to change this convention just to obtain consistency between force, distance, and moment conventions. It is still possible, however, to make the tabulation process "automatic," by the following scheme.

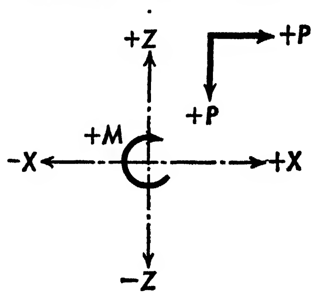


FIG. 2·35. Consistent conventions.

Taking first the z forces (Fig. 2·33), *positive* values will be assumed for both the force and its moment arm (x distance). This gives an upward-acting force and a moment arm measured to the right of point O, causing a counterclockwise turning moment. Hence the moment $P_z x$ will be negative (according to the convention established for positive moments in Fig. 2·33). It is only necessary to indicate this by attaching a negative sign to the term $P_z x$ at the top of the last column in Table 2·1. In filling in the table, the sign of the moment would then be correctly determined by using the signs of the forces and distances algebraically. For example, the operation for point d would be

$$-P_z x = -(-20) \times (-12) = -240$$

For the x forces, positive values for both P_x and z will cause a positive turning moment; hence the column for $P_x z$ may be left unchanged.

Once a table has been set up in this manner it is unnecessary to observe the direction of moments by inspection. It is advisable, nevertheless, to check the results by inspection of one or more typical operations, as the matter of keeping track of signs can easily become confusing.

This general method of handling signs algebraically is not only convenient but becomes absolutely necessary when setting up equations for the analytical solution of force problems. The entire procedure simply amounts to considering the general case of *positive* forces and distances and setting up equations and tables on this basis, including the proper signs for moments thus determined.

2·17. Location of Resultant Force. In the preceding example the forces were summed up about point O and it was found that, in addition to the resultant force, there was a resultant turning moment of 626 in.-lb about this point (Fig. 2·34). If it is desired to convert the entire force system into a single resultant force having the same net effect, this resultant turning moment must be eliminated. This may be done by moving the resultant force away from point O until its moment about that point equals the resultant moment of the force system. The required moment arm is obtained by dividing the moment by the resultant force

$$d = \frac{M}{P_R} = \frac{626}{83.8} = 7.48 \text{ in.}$$

The resultant force may be located graphically by drawing it tangent to an arc of radius d, as shown in Fig. 2·36. A force vector could be drawn tangent to either side of the circle: the correct location is, of course, determined so that the turning effect about point O is in the same direction as the resultant moment M_R, which is being replaced.

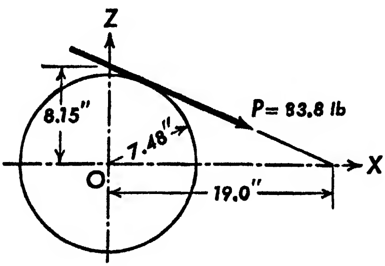


FIG. 2·36. Methods of locating resultant force.

The location of the line of action of the resultant force may also be obtained analytically by working directly with its two components P_x and P_z (Table 2·1). Let the resultant force be assumed to act at some unknown point having coordinates x_R and z_R. The total moment produced by the resultant force will then be

$$M = -(P_z x_R) + (P_x z_R) \qquad [2\cdot3]$$

(Note that in writing this equation a minus sign is placed in front of the term $P_z x$, as explained in Sec. 2·16.) Since we are to determine x_R and z_R so that the moment of the resultant force about point O equals that of the original force system, the value 626 determined in Table 2·1 is substituted directly for M in the above equation. Substituting also for the components P_z and P_x (from Table 2·1) we obtain

$$+626 = -(-33x_R) + (77z_R)$$

$$x_R = 19.0 - 2.33z_R$$

This equation determines a *line* rather than a specific set of values for x_R and z_R. This agrees with the fact that the effect of a force is the same when applied anywhere along its line of action. To determine the location of this line of action it is only necessary to substitute convenient values of x_R or z_R. By successively setting these values equal to zero we obtain the two *intercepts*, $x_0 = 19.0$ and $z_0 = 8.15$, as indicated in Fig. 2·36.

2·18. Progressive Summation. In structural work it is often necessary to sum up forces about a *successive* series of points, using only those forces to the right (or left) of the point in question. For example, assume that a cantilever beam is subjected to the force system shown in Fig. 2·37. The summation for this would be made from left to right, using only those forces acting on the *left* of the point in question.

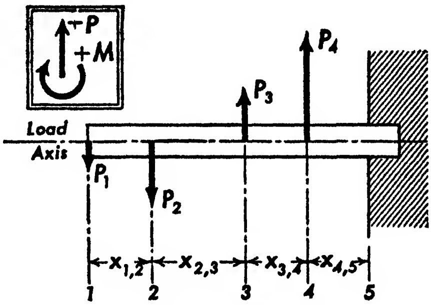


FIG. 2·37. Loaded beam.

A typical method of summation for this case would be as follows. First, a reference axis for the structural member is chosen. This is sometimes a simple matter; at other times the choice will depend on the nature of the structure. However, it is always possible to choose any convenient axis and to transfer the results to some other structural axis as the analysis progresses. The axis along which the summation is conducted may be called the **load axis.** In Fig. 2·37 a line is drawn through the center of the beam.

Next, conventions for positive forces and moments are established, with respect to the chosen axis. This is indicated in Fig. 2·37. (The use of a sketch to indicate the sign conventions is always desirable, in order to avoid confusion.) It is now necessary to select the successive points at which the forces are to be summed up. It may be convenient to use the points of application of concentrated loads. On the other hand, an arbitrary division into equal lengths may be desirable.

Assume that the summations are to be made about the points of load application. A tabular method of computation is most convenient. Table 2·2 shows how this would be done for the example illustrated in Fig. 2·37.

The process is essentially the same as that described in Sec. 2·15, which deals with finding the resultant of a number of forces. In Fig. 2·37, however, it is desired to find the resultant at several *progressive* points and only for the forces which lie *outside* of these points. Hence,

TABLE 2·2

SUMMATION OF FORCES AND MOMENTS

Station	① Δx Distance between Stations (in)	② P Local Force (lb)	③ ΣP Summation at Each Station = Σ②	④ ΔM Moment Increment = ① × ③	⑤ M Total Moment = Σ④
1		−150	−150	0	0
	20				
2		−400	−550	−3,000	−3,000
	30				
3		+300	−250	−16,500	−19,500
	20				
4		+700	+450	−5,000	−24,500
	20				
5		0	+450	+9,000	−15,500

in Table 2·2, column ③ represents progressive summation of the direct forces, whereas column ⑤ represents progressive summation of the moments of these forces.

Table 2·2 illustrates some simple working rules which facilitate the computations and help avoid errors. In column ① the distances between stations are located *between* the lines of the table representing these stations. This avoids any question of whether the distance is measured from the *preceding* point or the *following* point.

The operations to be done in each column may be indicated symbolically by numerals or letters. This reduces the danger of looking at the wrong column when doing the arithmetical work. This practice is also very helpful when standard tabular forms are used in actual engineering work, as it permits the computations to be performed by inexperienced people who would otherwise be unable to understand the operations.

Another small point which is used in practice is the "blanking out" of spaces which need not be filled in. This is demonstrated in column ①.

Figure 2·38 will assist in understanding the steps taken in Table 2·2. The arithmetical operations have been placed beneath the station to which they apply. Graphs have been drawn to show the variation of the computed quantities. Note that the moment increments (ΔM)

added at each station are equivalent to the *area* under the curve of net load. This important relationship will be used constantly in dealing with beams and bending moments, which will be taken up later. Figure 2·38 also indicates that the moment curve will be a series of straight

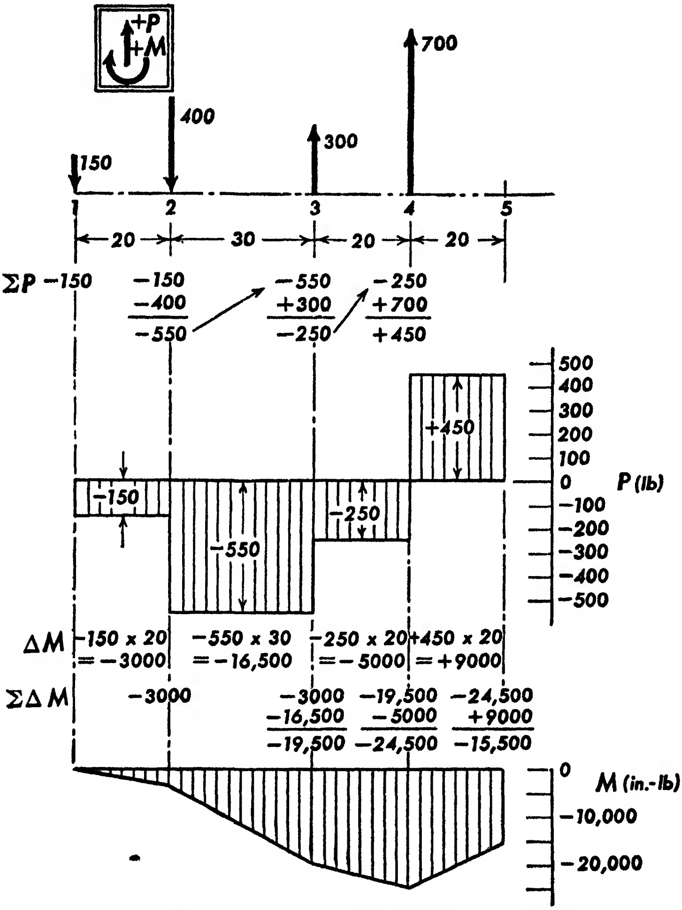


FIG. 2·38. Summation of forces and moments.

lines when the loading is produced by a series of concentrated loads normal to the load axis.

In Fig. 2·38 the only moment arms involved are x distances. This eliminates any need for a column of z or y distances and also reduces the moment computations to two columns, ④ and ⑤. In the more general cases additional columns will have to be included to take care of moments about the other axes.

For example, Fig. 2·37 may be made more general by including x force components and introducing z distances, as shown in Fig. 2·39.

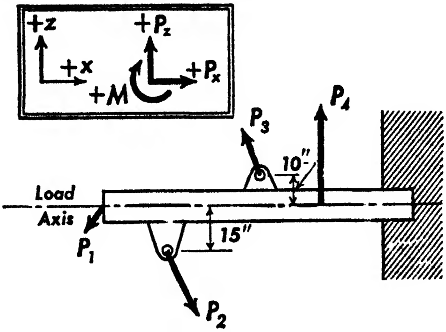


FIG. 2 39. Loaded beam.

Table 2·3 indicates the additional steps that must now be taken. One important point is that the x forces, once transferred to the load axis,

TABLE 2.3

SUMMATION OF FORCES AND MOMENTS

Station	Distances		Forces				Moments		
	①	②	③	④	⑤	⑥	⑦	⑧	⑨
	Δx Distance between Stations (in.)	z Distance to Load Axis	Force Component P_z	Force Component P_x	$\Sigma P_z =$ Σ③	$\Sigma P_x =$ Σ④	$P_x z =$ ④ × ②	$(\Sigma P_z \Delta x)$ ⑤ × ①	$\Sigma \Delta M$ Σ⑦ + Σ⑧
1		0	−150	−100	−150	−100	0	0	0
	20								
2		−15	−400	+200	−550	+100	−3,000	−3,000	−6,000
	30								
3		+10	+300	−100	−250	0	−1,000	−16,500	−23,500
	20								
4		0	+700	0	+450	0	0	−5,000	−28,500
	20								
5			0	0	+450	0	0	+9,000	−19,500

may be "moved" from station to station without an increase in moment. The only additional moments are those due to the z moment arms. These moment increments are computed in column ( ) and must be added into the final summation in column ⑨.

Finally, there could be forces or components along each of the three-dimensional reference axes (X, Y, and Z). As shown in Fig. 2·40, forces in the z direction, acting at y distances from the load axis, would cause moments about the load axis (X). These would be computed and summed up by using additional columns in the table. Similarly a force in the x direction acting at a y distance from the load axis would cause a moment about the Z axis and additional columns would have to be added to take care of this.

Forces in the y direction (such as P_y in Fig. 2·40) require the same treatment as forces in the z direction. As they are summed up and moved along the load axis (X) they produce moments about the Z axis.

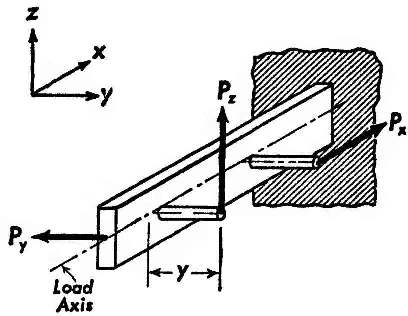


FIG. 2·40. General case.

2·19. Distributed Loading. Up to this point only concentrated loads acting at definite points have been considered. In structural work it is common to find distributed loads which are "spread out" over the length or surface of the structure. A load distributed over a length is usually specified as a **running load,** in units of *pounds per inch,* or *pounds per foot.* When distributed over the surface, the loading will obviously be in terms of **pressure,** in units of *pounds per square inch,* or *pounds per square foot.*

The general procedure is to convert pressure into running loads and running loads into concentrated loads, after which the procedure is the same as previously outlined. Pressures are converted into running loads by multiplying by a *width.* Thus, if an airplane wing is subjected to a uniform (upward) pressure of 100 lb per sq ft, the running load (pounds per foot) is obtained by multiplying this pressure by the width (chord) of the wing. If the pressure varies along the wing, the local pressure at any point is multiplied by the chord at that point. Figure 2·41 indicates how this is done. Suitable stations are selected and values of pc (pressure × chord) are computed. These are plotted as shown and the points connected to determine a smooth curve.

To obtain the equivalent concentrated loads the simplest approximation consists in multiplying the value of running load at each station by the distance over which it may be considered to act (usually half the distance to adjacent stations). The resulting concentrated loads are shown in Fig. 2·41. The dotted lines indicate the nature of this rough approximation. The accuracy obtainable is obviously increased by

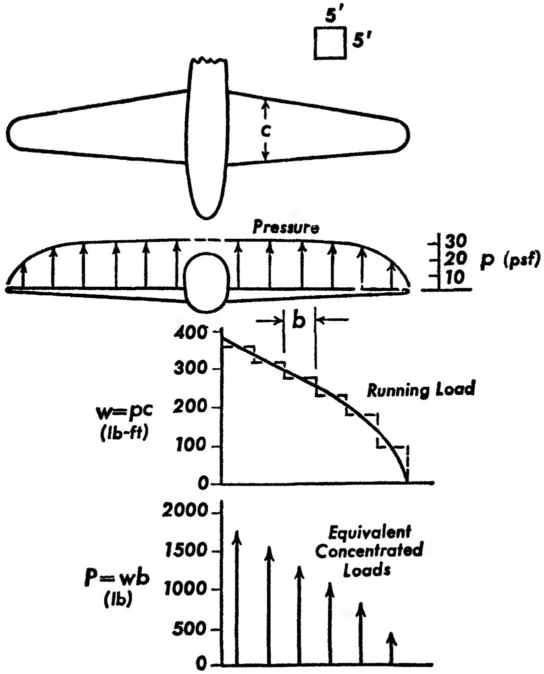


Fig. 2·41. Conversion of pressures to concentrated loads.

using more closely spaced stations. From this point on, the summation of loads and moments may be carried out as previously described.

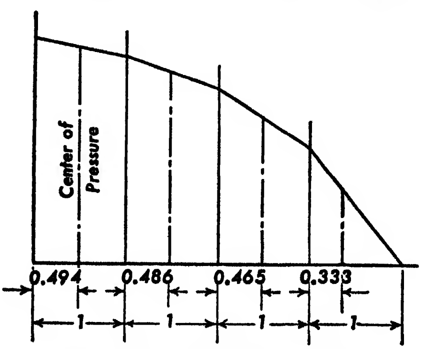


Fig. 2·42. Use of trapezoids.

Instead of representing the running load curve by a series of steps, it may be considered to be made up of a series of straight lines as indicated in Fig. 2·42. Then the area under the curve is formed by a series of trapezoids. Each trapezoid represents the load between stations and the centroid of the trapezoid locates the resultant force. The equivalent concentrated forces are now located at these centroids and the process proceeds as before, except that new stations must be assumed. The trapezoid method is slightly more accurate than the average load method previously described, as it takes into account the difference in effective moment arm between a trapezoid and a rectangle. However, if stations are selected at close intervals this refinement is unnecessary, as indicated by the values shown in Fig. 2·42. (Obtained from Table 1·1 in Appendix.)

The equivalent concentrated loads may also be obtained directly by multiplying each section of wing area (between stations) by the average pressure at that point. This method eliminates the computation of running load entirely.

In working with pressures and running loads care must be taken to use consistent units of measurement. For instance, the pressure on airplane wings is usually computed in terms of *pounds per square foot,* but the analysis is carried out on a *pound-inch* basis. Hence it is advisable to convert pressure to *pounds per square inch* before starting. If this is done, the chord must be measured in *inches.* It would, of course, be wrong to multiply pounds per square foot by the chord in inches, or vice versa.

2·20. Integration. The process of converting running loads to concentrated loads, summing these up along the load axis, and finally summing up the resulting moments is often described as **integration.** Actually this term refers to a mathematical process which is equivalent to assuming an infinitely large number of stations along the load axis. This eliminates all errors caused by approximating a curve with a series of straight lines, drawn between stations some distance apart. To apply this process it is necessary to know the equation of the curve of running load, in terms of the distance along the load axis. Furthermore, this equation must be fairly simple, or it will require an unreasonable amount of mathematical effort to integrate it.

In engineering work there is no assurance that the loading curve can be expressed by any reasonable mathematical equation. In an airplane wing, for instance, the variation of chord with span may or may not be linear, whereas the variation of pressure from tip to root, if accurately computed, will be found to be highly complex from a mathematical standpoint. In addition, there are likely to be concentrated local loads superimposed on the pressure loads, making the problem almost impossible for mathematical integration. It must be concluded, therefore, that integration will be useful only occasionally, and that more workable methods, such as those previously described, must be used for general engineering purposes.

Mathematical symbols are sometimes used as a sort of shorthand. The integral sign, $\int$, is often used in textbooks to indicate the step-by-step *summation*, although this operation should really be indicated by the Greek letter *sigma* (Σ). The engineer should be familiar with this shorthand usage of the integral sign and should realize that it usually does not require the actual application of integral calculus.

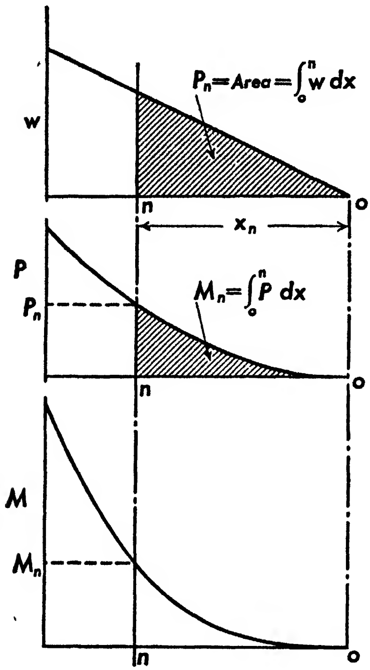


Fig. 2·43. Integration.

To show how the integral sign may be used to describe the summation of running loads, consider Fig. 2·43. If w represents the running load at any station x, the *total* load at any specific point x_n may be expressed by the equation

$$P_n = \int_0^n w\,dx \qquad [2\cdot4]$$

Figure 2·43 shows the operation graphically. The summation is carried out from station 0, to station n, as indicated by the small letters attached to the integral sign and by the subscript for P. The dx simply indicates the variable length over which the summation is made. Actually what is found is the *area* under the curve, expressed in terms of the product of the two variables (w and x). The area up to various stations is plotted in Fig. 2·43 as a curve of P against x. It represents the net or total load at each station. (*Pounds per inch* × *inches* = *pounds.*)

This curve may again be integrated to obtain the moment as indicated by the following shorthand formula.

$$M_n = \int_0^n P\,dx \qquad [2\cdot5]$$

The results are plotted as the third curve of Fig. 2·43, and are expressed in terms of *inch-pounds.*

Equation 2·5 may also be written in the general form

$$M = \iint w\,dx\,dx \qquad [2\cdot6]$$

This simply indicates *double integration*, exactly as already described. The process may be carried even further, as in beam deflections. At this point in the text it is necessary only to recognize the practical significance of the integral signs when they are encountered in these particular operations.

The area under a curve may be determined by means of a *planimeter*. In using this instrument, true integration is performed mechanically. A close approximation to the area under a curve may also be obtained by counting squares, if the curve in question is plotted on graph paper. Most of these methods are sufficiently accurate for structural work. (In measuring actual areas the results must of course be multiplied by the *scales* involved.)

2·21. Rotation of Axes. In working with curved or irregular members it is convenient to sum up loads and moments along a straight load axis, as previously described. Having done this it will eventually be necessary to transfer these values to some other axes, as indicated in Fig. 2·44. A direct shift of axes (without rotation) is accomplished by means of the rules and principles previously described (i.e., *moments* moved directly, *forces* moved axially without change.) If the new axes make an angle with the old ones, however, it is necessary to resolve the forces and moments into components along these new axes.

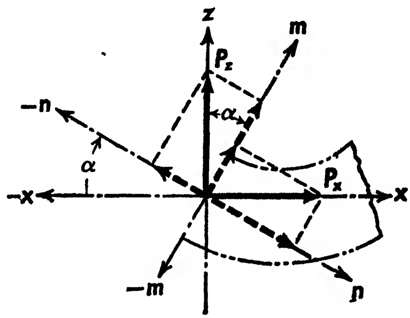


FIG. 2·44. Rotation of axes.

If conditions are such as shown in Fig. 2·44, the following equations may be used.

$$P_m = P_z \cos \alpha + P_x \sin \alpha$$
$$P_n = -P_z \sin \alpha + P_x \cos \alpha \qquad [2 \cdot 7]$$

where α is the angle through which the axes are rotated, and m and n are the new axes.

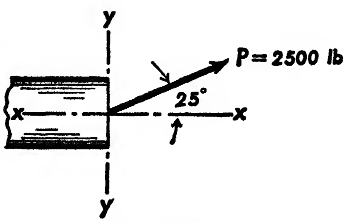
The same equations would be used for moment vectors. If the rotation of axes takes place about one of the three reference axes, no correction of forces or moment vectors along this axis need be made. Thus in Fig. 2·44, the rotation is about the y axis, and no corrections of y forces or moments about the y axis need be made.

It would be possible to write a set of general equations and rules which would take care of all cases, but it is considered better to understand the general method of making such corrections and to work out each specific case as it arises. A sketch should be drawn (as in Fig. 2·44) showing *positive* values of vectors along the original axes. The components of these values are then projected onto the new axes, indicating the sign to be employed. It will be noted that the *cosine* always applies to the two axes between which the angle of rotation is measured. Thus,

FIG. 2·45. Internal structure of horizontal tail surface for Lockheed *Constellation.*

in Fig. 2·44, the component of P_z along the m axis is given by cosine α, as α is measured between the z and m axes. The other component (n axis) will always be given by the sine of the angle.

PROBLEMS



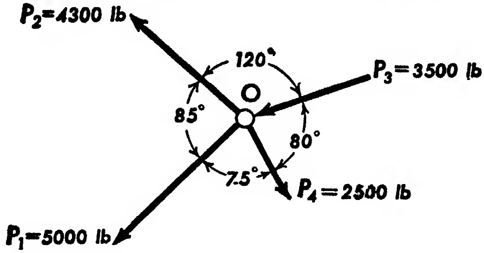
PROBLEM 2·1.

2·1. Determine the components of P acting along the X–X and Y–Y axes, (a) graphically and (b) analytically.

2·2. In Fig. 2·1 (p. 6) assume that P equals 10,000 lb and that the angle α is 50 degrees. Assume also that x_A and z_A are each equal to zero. Select a suitable scale and draw the vector picture to determine the components along the X and Z axes. Calculate the components by using equation 2·1. Repeat the calculations for Eqs. 2·1a and 2·1b.

2·3. Assume that a force of 1000 lb, acting at the origin, is rotated from the X axis to the Z axis (α varies from 0 to 90 degrees). Calculate the x and z components for each 10-degree interval (set up a table for this). Plot these values as curves of force against angle.

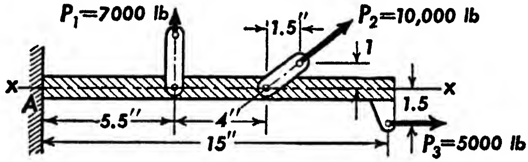
2·4. Determine the resultant (line of action, magnitude, and direction) of the following coplanar forces, (*a*) graphically and (*b*) analytically.



PROBLEM 2·4.

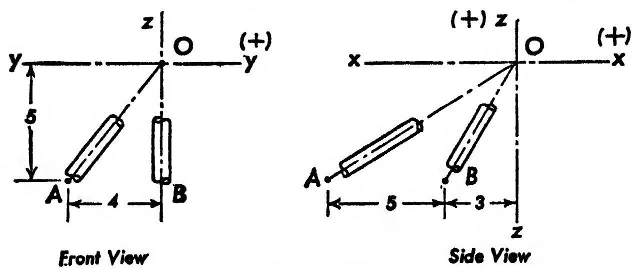
2·5. Select a suitable scale and draw two parallel forces 3 in. apart, one having a value of 50 lb and the other 75 lb. Find the location of the resultant force and draw the vector which represents it. (Use graphical method of adding a pair of forces.)

2·6. Find the resultant applied load on a beam loaded as shown, (*a*) graphically and (*b*) analytically.



PROBLEM 2·6.

2·7. What x, y, and z loads are applied at point O by members A and B when the load in A is 5000 lb compression and that in B is 6500 lb tension?



PROBLEM 2·7.

2·8. Draw four vectors representing non-concurrent forces of different magnitude acting in the plane of the paper (similar to Fig. 2·17). Find the location and magnitude of the resultant force by means of a string polygon. (Use Bow's notation.) Using the same construction, determine the resultant of any two adjacent forces.

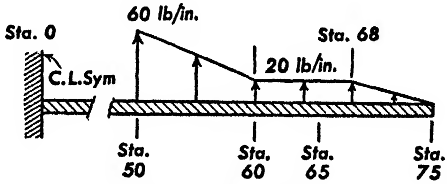
2·9. Using Fig. 2·38 as a basis, change the values of the distances and loads and carry through the summation of forces and moments as indicated in Table 2·2.

2·10. In Fig. 2·32 prove that the moment about the wheel axis is the same for case (*b*) as for case (*a*). (Subtract moment of one force from that of the other.) Then in case (*a*) assume that the right-hand force is doubled. By what percentage does this increase the total turning moment about the wheel axis?

2·11. Referring to Table 2·1, change all the values given for distances and components and redraw Fig. 2·33 to agree. Calculate the resultant force and its moment about the origin. Determine its location by the method of Fig. 2·36. Check analytically by drawing force and string polygons.

2·12. In Table 2·3 change the values of the force components (to represent a different loading condition) and carry through the summation of forces and moments. Draw a diagram, to scale, showing the new force vectors. Determine the magnitude and location of the resultant force vectorially. Determine, vectorially, the components of the resultant force and check against the summation of forces. Measure the moment arm of the resultant about the root of the beam (Sta. 5) and check the calculated summation of moments.

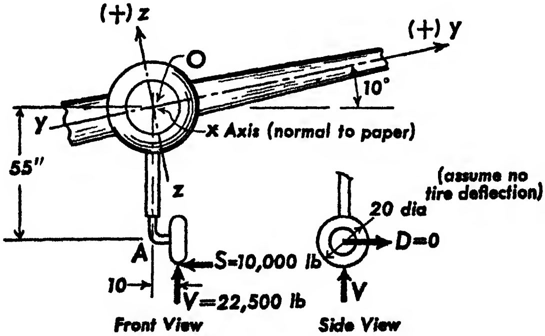
2·13. Find the summation of forces at stations 50, 60, 65, and 75 acting on the wing beam loaded as shown in the illustration.



PROBLEM 2·13.

2·14. Following the procedure shown in Fig. 2·41 draw a top view of an airplane wing of any desired size and shape (not rectangular). Assume that the net applied loading consists of a uniform upward pressure of 250 lb per sq ft. Carry out the operations shown in Fig. 2·41 and determine the progressive summation of forces and moments from tip to root (as indicated in Fig. 2·38).

2·15. Determine the equivalent forces and moments acting along the wing load axes at point *O*, due to the applied landing gear loads.

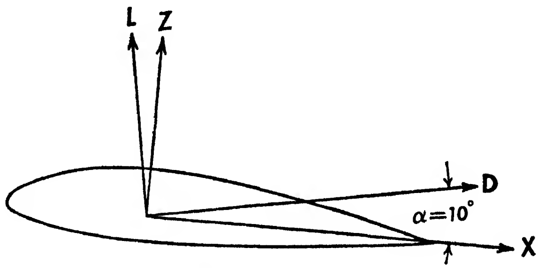


PROBLEM 2·15.

2·16. A flagpole is supported by four guy wires attached at a point 40 ft above the ground. The anchorage of the wires is made at the four corners of a 20-ft square, symmetrically located with respect to the flagpole. If the load in each wire is 300 lb, what is the total vertical load which the wires apply to the flagpole?

2·17. In Problem 2·16 assume that the load in one of the wires drops to zero, but that the other three continue to carry 300 lb each. Find the resultant vertical and lateral loads exerted by the wires at the attachment point.

2·18. In the figure the X and Z axes represent the structural axes of an airplane wing, while L and D represent lift and drag, respectively. Assume that for a



PROBLEM 2·18.

given flight condition the total air load on the wing is given by the components $P_L = 10{,}000$ lb and $P_D = 2000$ lb. Find the corresponding components along the structural axes, using Eq. 2·7. Check vectorially.

CHAPTER 3

FORCE TRANSMISSION

3·1. Structural Axes. In Chapt. 1 a brief discussion of force transmission was given. Chapter 2 described the methods by which forces and moments may be added, collected, and resolved into resultant or effective components with reference to any selected system of axes. The classification of these forces and moments with reference to a particular structural axis and with respect to the *effect on the structure itself* will now be studied.

The determination of the proper structural axis to use may be simply a matter of inspection (as in a round tube), or it may represent a stress analysis procedure in itself (as in an irregularly shaped box beam). The more complicated structures cannot be taken up until the basic principles of force transmission have been covered; hence the discussion will be confined to simple cases.

The simplest axis is a straight line between the point of application of the force and the point to which it is being transmitted. This type of axis is used in dealing with a pin-jointed structure, in which the members may be represented diagrammatically by lines connecting the centers of the pins. In determining the *internal* stresses in a structural member it is usually necessary to work with some axis about which the geometrical properties of the cross section may be computed. For a symmetrical section, such as a bar, a tube, or a rectangular beam, the centerline of the member itself is commonly used.

At this point it is necessary only to assume that a structural axis has been selected; the forces and moments acting on the structure may then be classified with respect to this axis and designated by the commonly used structural terms (*bending*, *torsion*, *shear*, etc.).

3·2. Axial and Transverse Forces. Let us assume that a force P is applied at point 1 and is to be transmitted to point 2, as illustrated in Fig. 3·1 (both points are assumed to be in the plane of the paper). The first step in clarifying the principles of force transmission is to establish two basic types under which all cases can be classified. As indicated in Fig. 3·1, this is done by resolving the applied force (P) into two components, one *along* the axis of transmission, the other *normal* to it. These two classes of forces are called **axial** and **transverse,** respectively, and they will be designated by the symbols P_a and P_s. The subscript s is used here for transverse forces because it is customary to call these *shear* forces; the term shear more accurately refers to a special type of transverse force transmission. In view of the common usage of the word shear, however, it will be used interchangeably with the word transverse when referring to the transmission of a force along an axis normal to its line of action.

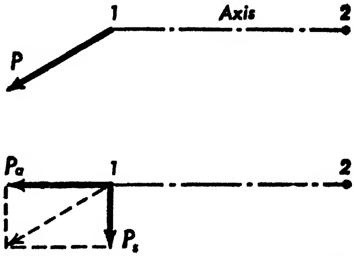


FIG. 3·1. Axial and transverse (shear) forces.

These two kinds of forces can now be dealt with separately, provided that the effects of separate components are recombined in obtaining the final answer.

3·3. Axial Forces. It is evident that axial forces (P_a) are so called because of their *direction*; that is, they are directed along the axis of transmission. They can be further classified with respect to the structure by considering whether the force is directed *away from* or *toward* the point to which it is being transmitted. These two sub-classes of force transmission are self-evident; one is called **tension,** the other **compression.** By established convention, tension forces are indicated by the *positive* (+) sign and compression forces by the *negative* (−) sign. Typical symbols used for these forces are P_t and P_c. The visual picture of this situation is given by Fig. 3·2, which shows both types of force transmission between points 1 and 2. If desired, the nature of the force transmission may be indicated by placing the plus or minus symbols along the axis of transmission, as shown.

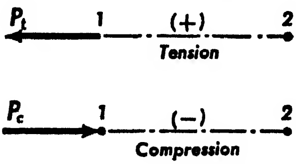


FIG. 3·2. Tension and compression forces.

It is particularly important to note that this system of force classification is independent of the absolute direction of the force in space, i.e., with reference to some predetermined coordinate system. Instead, the governing feature may be thought of as the *effect on the structure. Positive* axial forces (tension) will always tend to pull the member apart; *negative* axial forces (compression) will tend to force the elements closer together.

The use of two different systems of force classification, both employing plus and minus signs for identification, is bound to be confusing at times. The problem would be much easier to handle if, for instance, it were possible to introduce into the literature a new type of sign which would be used only for identification of forces with respect to their *effect* on the structure. For instance, the usual signs might be encircled thus: ⊕ and ⊖. When so used the circle would indicate that the signs referred to *tension* and *compression* forces, rather than to forces which were positive or negative with respect to some space convention. However, because of the difficulty of introducing new symbols not already found in existing textbooks this idea will not be adopted. The structural engineer must become used to the systems already in existence and should be able to recognize the special nature of the signs employed, without the aid of new symbols.

3·4. Transverse (Shear) Forces. Referring again to Fig 3·2, the axial component, P_a, has been classified as to *sense*,* i.e., tension or compression. It is difficult, however, to establish a single logical convention for the sense of the transverse (shear) component, P_s. This is, in fact, one of the most frequent causes of confusion and error in structural analysis It is easy to obtain a mental picture of tension and compression by visualizing the effect on the structure itself. In tests, direct measurement of the deformation of the structure under load will indicate the nature of the condition. In dealing with **transverse** (shear) forces it is impossible to distinguish between positive and negative conditions by reference to the structure alone. An additional convention as to direction must be included.

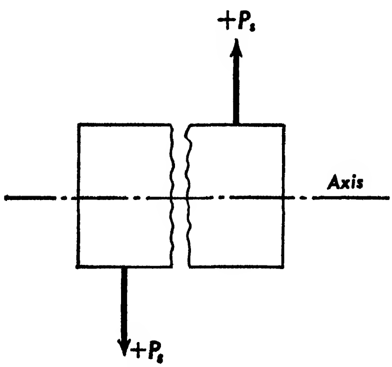


Fig. 3·3. Shear conventions.

The simplest method of doing this is to show positive shear by means of a sketch such as Fig. 3·3. Actually this amounts to assigning a positive value for the *rotational effect* of the transverse forces acting on either side of a given cross section. The directions chosen should agree with the *bending moment* conventions, to be described later, i.e., the question of whether to draw the force vectors in Fig. 3·3 as shown, or in the opposite directions, will be decided by the conventions adopted for bending moments.

The way in which the shear type of convention differs from the space type may be indicated by the fact that in the latter an upward force

* "Sense" is here used as defined in Webster's *Collegiate Dictionary:* "One of the two opposite directions in which a line . . . may be supposed to be described by the motion of a point . . ."

would always be positive (assuming it had been designated thus, of course), whereas in the shear type of convention, it would be positive or negative, depending on whether it acted to the left or right of the section in question.

Shear conventions are not actually needed except in special types of beam problems, hence they will not be discussed further at this point. It is necessary only to realize that such conventions exist and to recognize the way in which they differ from the regular space type of conventions.

3·5. Summary of Force Designations. Thus far forces, or their components, have been classified by two different systems. For clarification we may call these the *space* system and the *structural* system.

In the space system (Chapt. 2) the forces are located in space, and their magnitude and direction are specified. This is done by means of some system of reference lines (usually rectangular coordinates) and definite conventions as to *positive* and *negative* values.

In the *structural* system the nature of the force is classified with reference to the axis of transmission and the effect on the elements of the structure. The system is resolved into two classes: *axial*, in which the force is transmitted *along* its own axis, and *transverse*, in which the force is transmitted in a direction *normal* to its own axis. The latter is more commonly referred to as *shear*.

Axial forces are further subdivided, as to sense, into two classes: *tension* (+), and *compression* (−). *Transverse* forces have no inherent quality by which a convention as to sense may be set up, but certain arbitrary conventions are sometimes used in special beam problems.

It is most important to recognize that two different systems of classifications are employed and to know when to use each system. In analytical calculations (for instance, space frameworks) "space" and "structural" systems may be employed simultaneously, and it is therefore essential to have a clear understanding of each. To clarify the picture, the two systems have been shown diagrammatically on Fig. 3·4.

3·6. Examples of Force Transmission. A simple example of the basic types of force transmission is shown in Fig. 3·5, which represents an ordinary swing. Obviously the force *P* must eventually be resisted at the ground and is divided equally between the two sides of the structure. Taking either half of the structure and starting with the applied force *P*, the types of transmission involved are, successively: *transverse* (seat), *axial-tension* (ropes), *transverse* (crossbar), and *axial-compression* (uprights).

In Fig. 3·5 is an illustration of a simple "two-way" division of the applied force. Because of symmetry, no special computations were re-

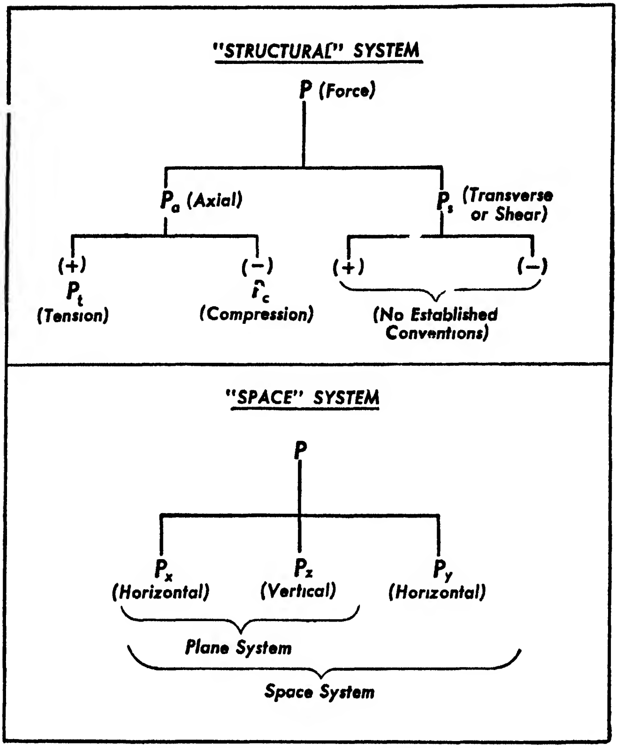


FIG 3·4. Systems of force classification

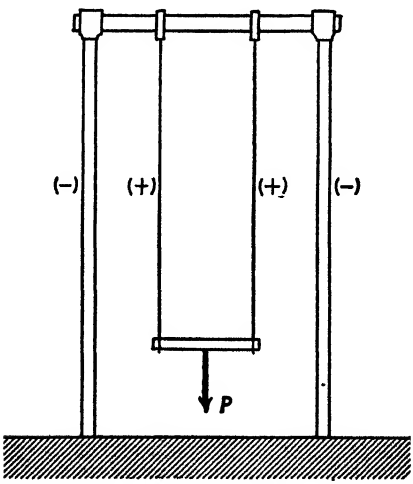


FIG. 3·5. Example of force transmission.

quired to find the distribution of force between the two sides of the structure.

Another somewhat different example is shown in Fig. 3·6, which represents a cantilever beam or strut to which the force P is applied at an angle. The solution of this problem is begun by separating the force P into its axial and transverse components P_t and P_s, respectively.

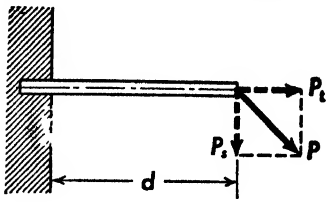


FIG. 3·6. Cantilever beam under axial and transverse loading.

3·7. Moment Transmission. In Chapt. 2, it was shown that one of the effects of a force acting at a distance could be represented by a *turning moment, M.* It was also shown how such moments could be designated by vectors, which indicated the axis of rotation and the magnitude of the turning effort. Structures are often required to transmit turning moments, and it is therefore necessary to classify the mode of transmission, just as was done for forces. This is covered in the following sections.

3·8. Torsion. By using the same principles, the transmission of turning moments can be classified by reference to the *axis* of the moment and that of the structure along which the moment is transmitted. Transmission of a moment *along its own axis* is called **torsion.** (Such turning moments may be referred to as **torque, torsional moment, twisting moment,** M_t, etc.) In torsion the axis of the turning moment coincides with that of the structural member along which it is transmitted. Torsion is therefore analogous to axial force transmission. Instead of shortening or elongating the member, however, the effect is to *twist* it, or rotate it about its axis, as shown in Fig. 3·7.

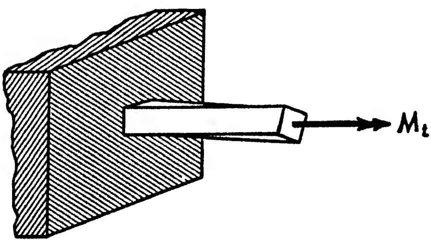


FIG. 3·7. Torsional moment, or torque.

The transmission of torsional moments is a very common and useful function of structures. The ordinary doorknob serves as an elementary example of torsion transmission. In the mechanical field the transmission of power by torque shafts is an outstanding example of this important form of structural loading. Torsion may also be a by-product of some other form of force transmission, because of eccentric loading or similar effects.

There is no established convention for the sense of torsional moments with respect to their *effect on the structure* and it will not be found necessary to set up such a convention. Space conventions, as described in Chapt. 2, will usually suffice. (Figure 3·7 would indicate that the left-hand rule had been used.)

3·9. Bending. If the turning moment is transmitted in a direction normal to its axis of rotation it is classed as a **bending moment.** This is because the effect of such a moment is to "bend" the member to which it is applied, i.e., to cause its axis to curve out of its original position. Bending moments are analogous to transverse forces. An example of bending is shown in Fig. 3·8. The top view indicates clearly that the moment is transmitted along a line *normal* to its own axis. The side view shows the deflected position exaggerated. (The *right-hand* rule has been used in drawing the moment vector.)

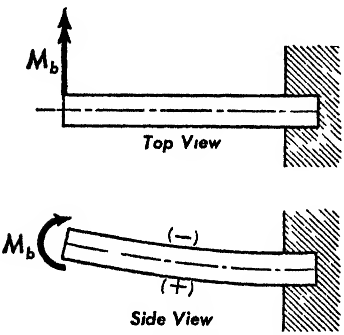


FIG. 3·8. Bending moment.

Pure bending is not usually found as a primary form of force transmission, but it is almost always the result of transverse forces. In the ordinary lever the bending moment is indirectly utilized to magnify the effect of an applied force. Usually, however, the bending moment serves no useful function, even though it is often found to be the most severe type of loading for a structure. In an airplane wing, for instance, bending moments account for the largest share of weight in the structure, yet the *useful* force transmission is done by transverse (shear) forces.

Bending moments can be identified with respect to their effect on the structure by using the conventions for tension and compression together with an arbitrary convention for the "positive" side of the structure. Thus if the bottom of a beam is considered as positive, the bending will be considered to be positive when the bottom portion of the beam is in tension. This convention is indicated in Fig. 3·8. This system of classification will be found useful in more advanced bending problems.

3·10. Combined Bending and Transverse Force (Shear). Figure 3·9 illustrates bending moment due to transverse force transmission. Sketch (*a*) shows the actual loading condition, consisting of a single transverse force acting at the end of the beam. Sketches (*b*) and (*c*) show the equivalent conditions at points 2 and 3. This is a direct application of the principles of Sec. 2·18, in which the progressive summation of forces was described. Sketches (*b*) and (*c*) illustrate a very common procedure in structural analysis, consisting in hypothetically cutting the member at the point in question and placing at the cut section the summation of all forces and moments determined at that point. The practice of cutting is a convenient way to illustrate the effective loading condition at any point. It will also be used later in determining reactions and internal loads.

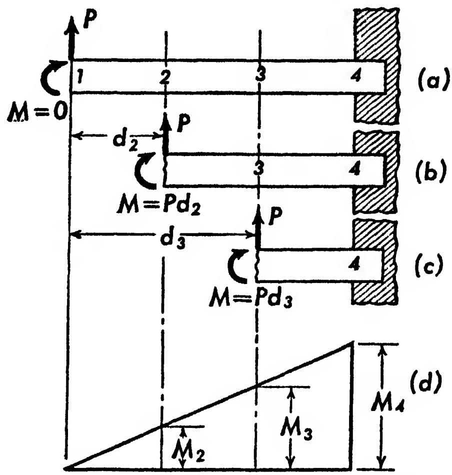


FIG. 3·9. Bending moment due to transverse force.

In Fig. 3·9 the assumed structural axis is not indicated, as it is clear that the centerline of the beam is a satisfactory axis.

Sketch (*d*) is a plot of the bending moment along the span of the beam. Since this moment is caused by a single force P and is directly proportional to d, the curve is a straight line.

The effective transverse force at any cut section is equal to the applied force P, for the case illustrated. This *effective* or *net* transverse force is usually called the shear, but the shear conventions described in Sec. 3·4 have not actually been used.

It will be seen that the processes of summation described in Chapt. 2 may be used directly in determining the net shear and bending moment for a member. If the *load axis* happens to be the proper *structural axis* for the member, the axial and transverse forces and the net moments may be considered to be *tension* or *compression*, *shear*, *bending moment*, or *torsion*, as the case may be. It may sometimes be necessary to *transfer* the net forces and moments to a different structural axis, as described in Sec. 2·21.

3·11. Combined Shear, Bending, and Torsion. Figure 3·10 illustrates a very common type of loading, such as found in the ordinary crank. Here the force P is applied at A and transmitted to B and then to C. At point A there is neither bending nor torsion, as the point lies on the line of action of the force. Between A and B the bending moment increases as the distance y is increased, reaching a maximum at point B. Member AB is therefore subjected to combined shear and bending. The turning moment at point B has the same axis as member BC; hence it represents a *torsional* moment for this member. The value of the torsion does not change between point B and C. (Moments or couples may be transferred without any change of effect, Sec. 2·14.) These conditions are indicated by the sketches in Fig. 3·10.

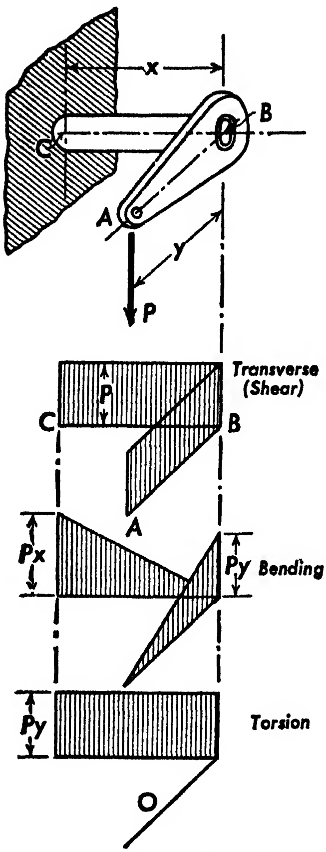


Fig. 3·10. Combined loadings.

In addition to the torsion acting on member CB there is also the shear

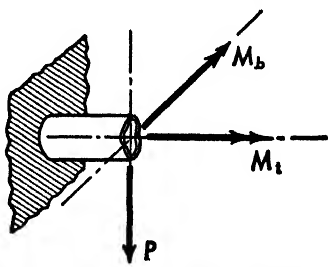


Fig. 3·11. Combined loading at cut section.

force P, which must be transmitted transversely through the distance x. Hence a bending moment equal to Px is built up and is a maximum at point C. Member BC therefore is subjected simultaneously to *shear* (P), *torsion* (Py), and *bending* (Px). It would also be possible to apply an axial force at point B, thus adding another type of force transmission to the member BC.

The loading conditions at any section through BC could be represented by space vectors such as shown in Fig. 3·11. Note that the double-headed vectors used to indicate turning moments must point in the proper direction to be consistent with the loading conditions (right-hand rule used here).

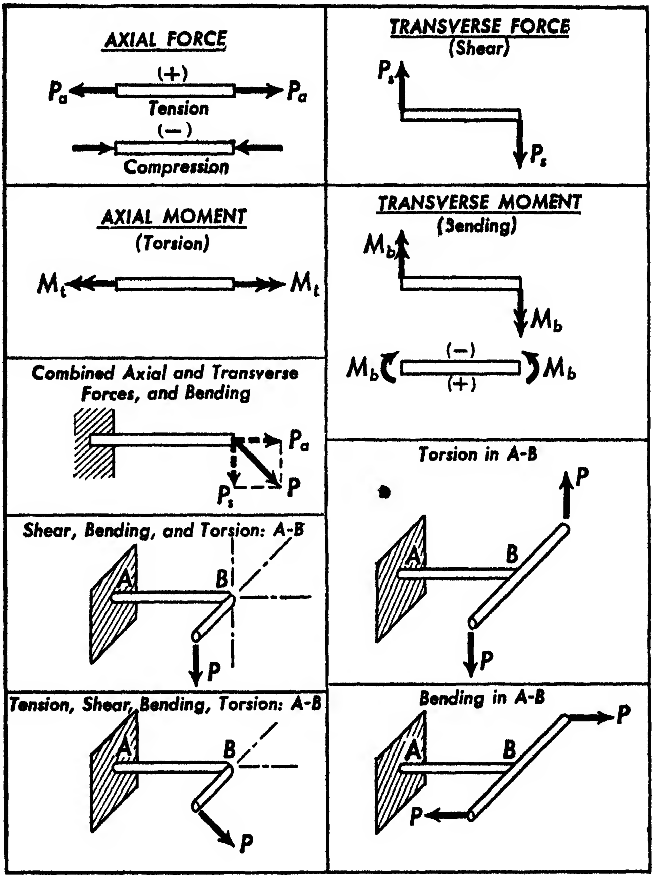


Fig. 3·12. Examples of force transmission.

3·12. Summary of Force Transmission. Figure 3·12 contains a number of examples which will serve to summarize the general types of force transmission. The main points brought out by this chapter are the classification of forces and moments with respect to a structural axis and the introduction of a structural system of loading conventions which is independent of space conventions. Both structural and space systems are used in analysis work, and it is therefore important to distinguish between them.

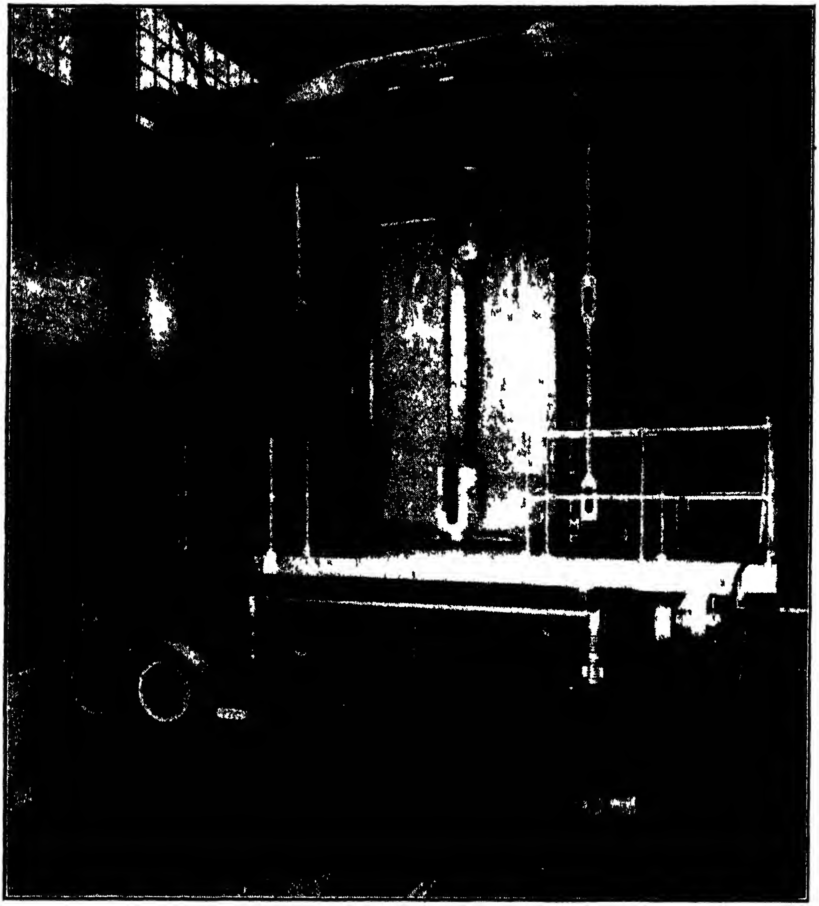
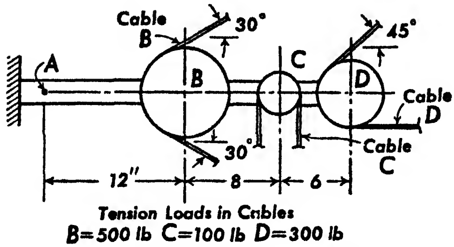


Fig. 3·13. Large testing machine at Aluminum Research Laboratories. Capacity 1,000,000 lb in tension, 3,000,000 lb in compression. (Courtesy of Aluminum Company of America.)

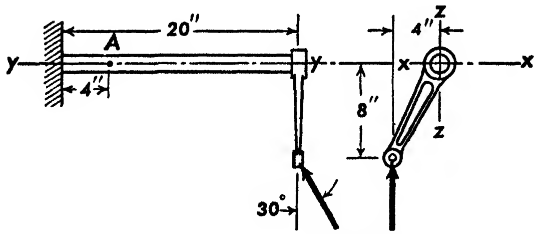
PROBLEMS

3·1. What is the *axial* load at *A* due to the pulley loads shown in the diagram?



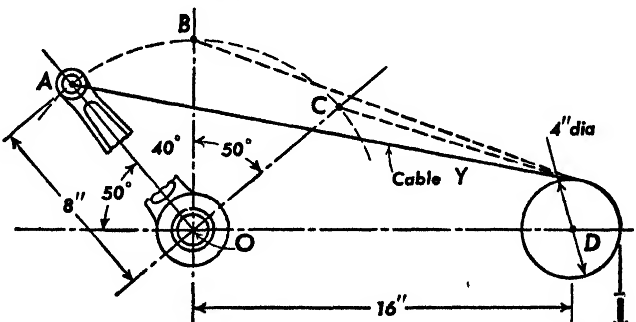
PROBLEM 3·1.

3·2. Determine the axial and transverse forces and the bending and torsional moments acting at point *A*.



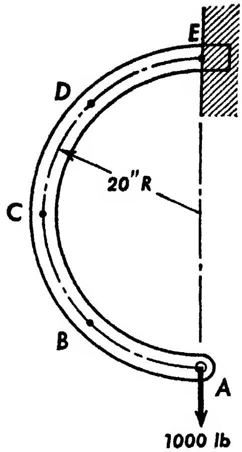
PROBLEM 3·2.

3·3. Lever *AO* is located at the extremity of a cantilever shaft and perpendicular to it. The shaft is supported 20 in. from *O*. Cable *Y* has a constant tension



PROBLEM 3·3.

load of 500 lb as shown. Find graphically the torsional moment and bending moment in the cantilever shaft at the support when the lever is at *A*, *B*, and *C*.



PROBLEM 3·4.

3·4. Determine the axial and transverse loads and the bending moments at points A, B, C, D, and E, which are equally spaced along the centerline of the half ring. (Neglect deflections and assume pin at E.)

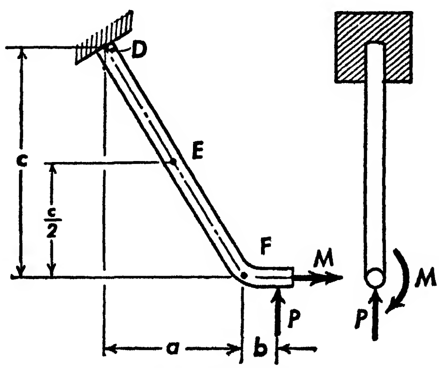
3·5. In Problem 3·4 assume that the 1000-lb load acts horizontally and to the right, and calculate the same quantities.

3·6. In Problem 3·4 assume that the load at point A has a downward component of 500 lb and a horizontal component of 900 lb (acting to the right). Using the results from Problems 3·4 and 3·5, calculate the axial and transverse loads and the bending moments, at the designated points. Check the loads by means of Eq. 2·7 (Sec. 2·21).

3·7. In Problem 3·4 assume that the 1000 lb load acts normal to the plane of the member (plane of paper). Calculate the bending and torsional moments at the designated points. (*Note.* Problems 3·4 to 3·7 illustrate a common procedure of analyzing a member for unit loads—in this case 1000 lb—acting separately along the three reference axes. The results may then be used for any combination of loads or components along these axes.)

3·8. In Fig. 3·10 assign any desired values to x, y, and P and compute the loading condition at point C (transverse load, torsion, and bending). Repeat the process with an additional load P acting at point A in the plane of the structure and directed outward, away from the wall. (In this example the axial force must be included. Also the maximum bending moment must be found by vector addition.)

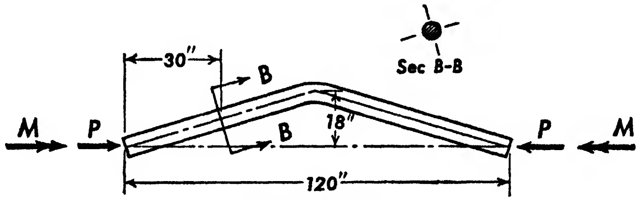
3·9. Draw a sketch similar to the figure and measure the distances a, b, and c, to a suitable scale (c should be between 40 and 60 in.). Select a value for P (between 1200 and 1900 lb) and a value for M (between 3200 and 3900 in.-lb). Find the loading conditions along the diagonal strut at points D, E, and F) (axial force, transverse force, bending, and torsion).



PROBLEM 3·9.

3·10. In the example used for Problem 3·9 assume that the moment vector is acting normal to the plane of the bent strut at point F, and that it is pointed away from the reader. Using the same vertical load, rework the problem.

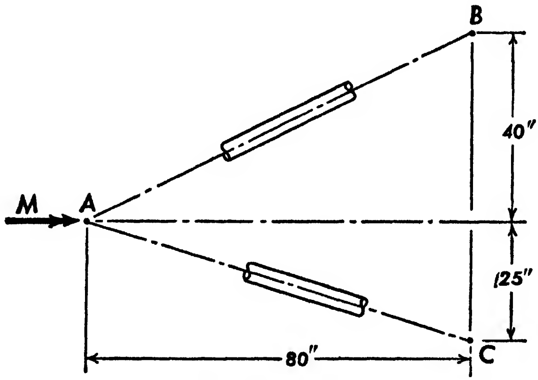
3·11. Select a value for P and find the maximum bending moment for the bent strut, neglecting deflections. Select another value for M, and determine the location and value of the maximum torsional moment.



PROBLEM 3·11.

3·12. In Problem 3·11 assume that the force P and the moment M are applied simultaneously. Find the maximum bending moment at section B–B. Show its location by means of a vector drawn on a view of the cross section at point B. (Use right-hand rule.)

3·13. Assume that the tubes shown in the sketch are welded together at A and that the ends B and C are restrained in such a manner that the tubes can resist only *torsion* (could be done by using universal joints at B and C). Find the torsional moment in each tube.



PROBLEM 3·13.

3·14. In Problem 3·13 assume that the tubes can resist only *bending* (could be done by using "frictionless" screwed joints at B and C.) Find the bending moment in each tube. (*Hint.* The bending moment vectors at point A must be normal to the centerlines of the tubes.)

(*Note.* Problems 3·13 and 3·14 represent two simplifying assumptions which could be made if the tubes were rigidly attached to their supports at B and C, as they would be if they were welded. The true answer will lie somewhere between the results obtained for the two simple cases. A direct solution would require a knowledge of redundant structures, which cannot be taken up at this point.)

CHAPTER 4

EQUILIBRIUM AND REACTIONS

4·1. One-Way Force Transmission. Up to this point all illustrations of force transmission have been of the one-way type. For clarity these were illustrated by sketches showing a "wall" type of support (see Fig. 2·37). Beams so supported are usually called **cantilever** beams. In these the net effect at any point on the structure may be obtained by direct summation of all the forces and moments acting outside of that point. The conditions at the point of support are obtained in the same manner and represent the summation of *all* the forces and moments acting on the structure in question.

Cantilever beams always have a free end, just outside of which the applied forces and bending moments are zero. This permits the summation for forces and moments to be started from zero. Another feature is that the conditions at the point of support have no effect on the shear and bending moment "outboard" of this point (except for localized effects confined to an area near the point of support).

4·2. Two-Way Force Transmission. Another common type of force transmission for which structures are designed might be called two-way transmission. A simple example is found in the ordinary footbridge (Fig. 4·1). The force P, acting at point A, must be transmitted to points B and C. Obviously part of the force goes in one direction and the remainder in the other. Unless the force happens to be applied midway between points B and C, it is not immediately obvious how it is divided between the two supports. It is impossible to make an analysis of such a structure until the division of force between the two paths is determined. The method of determining this division involves the determination of the *reactions*, which in turn requires the use of the principles of *static equilibrium.*

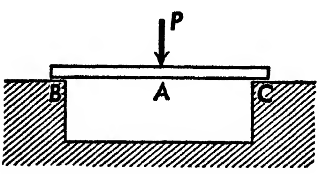


FIG. 4·1. Two-way force transmission.

Before taking up reactions it should be noted also that in Fig. 4·1 the bending moment at point A could not be zero; hence it would be impossible to start the summation of moments from zero at this point.

If, however, the points of support are incapable of applying a bending moment, i.e., pin-jointed or roller support, it is immediately evident that the bending moments at each end (points B and C) *are* zero. This indicates that the summation of forces should be started from the points of support. The process may be thought of as reversing the force transmission picture by applying, at B and C, the forces which are needed to resist the force at point A.

4·3. Hydraulic Analogy. The mental picture of this process may be clarified by considering a hydraulic analogy. Figure 4·2 illustrates a

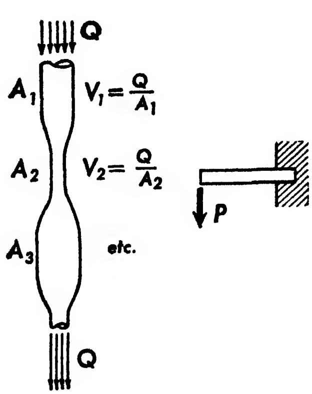


FIG. 4·2. Hydraulic analogy for one-way force transmission.

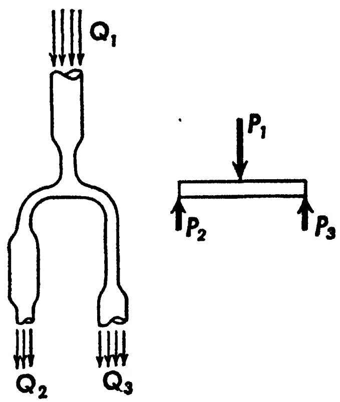


FIG. 4·3. Hydraulic analogy for two-way force transmission.

tube of variable cross section through which a liquid is flowing. If the rate of flow (gallons per minute) is known, the velocity at any point may be determined by dividing by the cross-sectional area. It is not necessary to determine the rate of flow at the outlet, as this is obviously the same as that at the inlet. This picture corresponds to one-way force transmission. Figure 4·3 represents an analogy for the two-way transmission. A knowledge of the flow at the inlet Q_1 is *not* sufficient to determine the local conditions throughout the tube. It is necessary to know the values of Q_2 and Q_3 before this can be done. Note that if *either* of the latter values is determined, the other may be found by subtracting the known value from Q_1.

4·4. Reactions. In any structure the points of support may be thought of as imposing external forces on the structure. Such forces are called reactions. Once the reactions have been determined, they may be regarded as any other external forces; the term reaction generally implies that the forces in question are not known at the beginning of the problem. More specifically, reactions are those forces which will be developed at the points of support or restraint when the structure is subjected to loading. The reactions *resist* the applied forces, and their net effect is always *opposite* to that of the applied forces.

4·5. Free Body. At this point it is important to obtain a clear conception of the common structural practice of "isolating" a structure (or a part of it) in space and placing it in *static equilibrium* by applying the reactions as external forces. A structure so isolated is called a free body. In the hydraulic analogy used in Sec. 4·3, the flow of a fluid through a tube was compared with the transmission of force through a structure. Note that in Figs. 4·2 and 4·3 the volume of the flow Q was indicated as leaving the tube in *the same direction* that it entered, which is, of course, in accordance with the facts. In dealing with forces, however, we adopt the opposite procedure and show, *not* the forces that are leaving the structure at the reaction points, but those that are *resisting* the forces at these points.

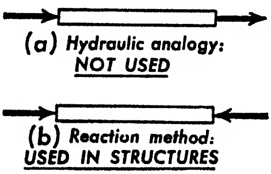


FIG. 4·4. Method of isolating a free body.

This is an application of Newton's third law of motion, which states that every action is opposed by an equal and opposite reaction. It would of course be possible to use the hydraulic analogy and sketch the forces as shown in Fig. 4·4*a*, instead of as in Fig. 4·4*b*. But this would not permit us to apply the conception of static equilibrium, which is one of the most useful structural analysis tools.

As previously noted, a structure (or a part of one) which has been thus isolated in space by applying the *reactions* as external forces is referred to as a free body. Physically, this implies that the structure has hypothetically been disconnected from its supports (or adjacent structure) and that the supports have been replaced by the forces which they exerted. If this were actually done, the body would be free to move in space and to change its velocity. But since we know that it is *not* moving (with respect to the supporting points) it follows that the reactions applied must satisfy the physical laws of *static equilibrium.* Thus, through the application of the simple classical laws of motion, it is possible to determine the reactions as those forces which are required, at the points of assumed disconnection, in order to prevent motion with respect to these points.

4·6. Equations of Equilibrium. Newton's first law of motion tells us that if the free body is to remain at rest there must be *no unbalanced external force* acting on it. Actually there may be many external forces, but the *summation* of all these must be zero, i.e., taken as a whole the forces must balance each other. The same statements may be made about turning moments. If there is an unbalanced turning moment the free structure will not remain at rest, but will rotate.

These two statements may be expressed mathematically by the equations:

For forces $$\Sigma P = 0 \qquad [4 \cdot 1]$$

For moments $$\Sigma M = 0 \qquad [4 \cdot 2]$$

If all the external forces are acting along the same line, or have parallel lines of action, Eq. 4·1 may be used directly. If not, the equation must be written for each assumed reference axis, as it is impossible to sum up algebraically forces which do not act parallel to the same axis (Sec. 2·9). If reference axes X, Y, and Z are used, for instance, the equations of equilibrium are written (for forces):

Equilibrium Equations for Forces

$$\Sigma P_x = 0$$
$$\Sigma P_y = 0 \qquad [4 \cdot 3]$$
$$\Sigma P_z = 0$$

The same equations may be applied to turning moments, by using the subscript to designate the axis about which the turning effect is exerted. Equations 4·3 then become:

Equilibrium Equations for Moments

$$\Sigma M_x = 0$$
$$\Sigma M_y = 0 \qquad [4 \cdot 4]$$
$$\Sigma M_z = 0$$

One or more of Eqs. 4·3 and 4·4 may be automatically satisfied, hence need not be used. For instance, if all forces act along the same reference axis only one of the Eqs. 4·3 need be used, and there is no turning moment about any point on this line. If the *summation* of forces (resultant force) equals zero, the system can have no turning moment about any point. Hence all three moment equations are satisfied automatically and need not be used.

If all forces are in the *same plane* (such as X–Y plane), all turning moments will act in this plane about axes perpendicular to it (Z axis). Hence only *one* equation of equilibrium for moments need be used.

Using these principles, the following set of equations of equilibrium may be set up.*

4·7. Specific Equations of Equilibrium. *a. All forces acting along same line.*

$$\Sigma P = 0 \qquad [4\cdot1]$$

Taking four forces, for example,

$$\Sigma P = P_1 + P_2 + P_3 + P_4 = 0$$

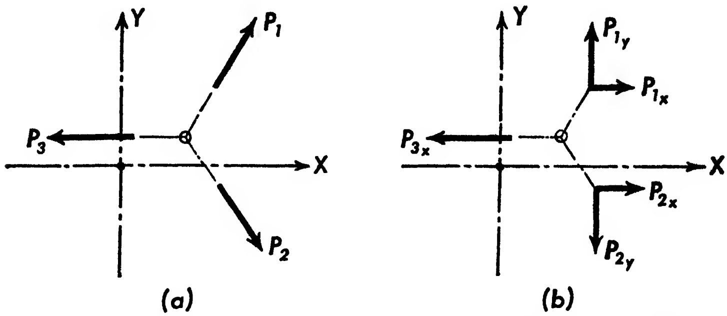


FIG. 4·5. (*a*) Concurrent forces in equilibrium. (*b*) Same system resolved into components.

b. All forces acting in same plane and through same point (concurrent, coplanar forces).

$$\Sigma P_x = 0$$
$$\Sigma P_y = 0 \qquad [4\cdot5]$$

For the example shown in Fig. 4·5:

$$\Sigma P_x = P_{1x} + P_{2x} - P_{3x} = 0$$
$$\Sigma P_y = P_{1y} - P_{2y} = 0$$

Then because lines of action of all forces intersect at a common point and hence have no moment about this point,

$$\Sigma M = 0$$

c. Forces acting in any direction but through common point (concurrent forces). Same as *b* except third component added.

$$\Sigma P_x = 0$$
$$\Sigma P_y = 0 \qquad [4\cdot6]$$
$$\Sigma P_z = 0$$

* In Chapt. 2 the X–Z plane was used for plane problems. The X–Y plane is used here for the same purpose. This is done to accustom the student to the use of different systems of symbols.

d. All forces acting in same plane and parallel to same line.

$$\Sigma P = 0$$
$$\Sigma M = 0 \qquad [4\cdot7]$$

For the example shown in Fig. 4·6,

$$\Sigma P_x = P_1 + P_2 - P_3 - P_4 = 0$$
$$\Sigma M_z = M_1 + M_2 + M_3 + M_4 = 0$$
$$= P_1y_1 + P_2y_2 - P_3y_3 + P_4y_4 = 0$$

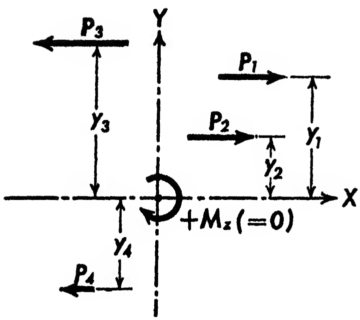


FIG. 4·6. Parallel forces in equilibrium.

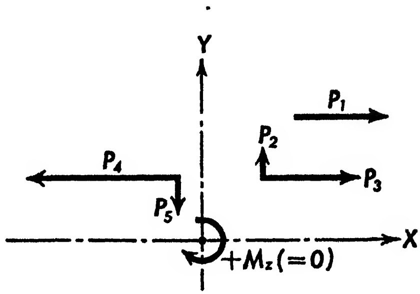


FIG. 4·7. Coplanar forces in equilibrium.

e. All forces acting in same plane.

$$\begin{matrix} \Sigma P_x = 0 \\ \Sigma P_y = 0 \\ \Sigma M_z = 0 \end{matrix} \quad \text{or} \quad \left\{\begin{matrix} \Sigma P_x = 0 \\ \Sigma P_z = 0 \\ \Sigma M_y = 0 \end{matrix}\right. \quad \text{or} \quad \left\{\begin{matrix} \Sigma P_y = 0 \\ \Sigma P_z = 0 \\ \Sigma M_x = 0 \end{matrix}\right. \qquad [4\cdot8]$$

For the example shown in Fig. 4·7,

$$\Sigma P_x = P_1 + P_3 - P_4 = 0$$
$$\Sigma P_y = P_2 - P_5 = 0$$
$$\Sigma M_z = P_1y_1 - P_2x_2 + P_3y_3 - P_4y_4 - P_5x_5 = 0$$

Note. In the above example it may be assumed that forces P_2 and P_3 represent *components* of a single force. P_4 and P_5 likewise represent a single force.

f. Forces acting in any direction and at any point, i.e., having components along and moments about all three reference axes.

$$\left.\begin{array}{ll} \text{Forces} & \Sigma P_x = 0 \\ & \Sigma P_y = 0 \\ & \Sigma P_z = 0 \\ & \Sigma M_x = 0 \\ \text{Moments} & \Sigma M_y = 0 \\ & \Sigma M_z = 0 \end{array}\right\} \text{General case} \quad \text{(Eqs. 4·3 and 4·4)}$$

This general case includes all the foregoing special ones, which were obtained by omitting those equations which were obviously satisfied by the stated conditions of the problem. If there is any doubt, it is always possible to use these general equations, as no error is introduced through the use of the "extra" equations.

4·8. Determination of Reactions (Axial Loads). The equations of equilibrium just described may be used to determine the *reactions.* First it is necessary to "disconnect" the structure (or part being analyzed) at the reaction points, hypothetically replacing the connecting fittings by the unknown forces (or moments) which they were exerting on the structure. The structure now may be regarded as a free body, and the equations of equilibrium may be written. *All* the known external forces and moments must be included in the equations; the reactions may then be regarded as the unknown quantities for which the equations are to be solved.

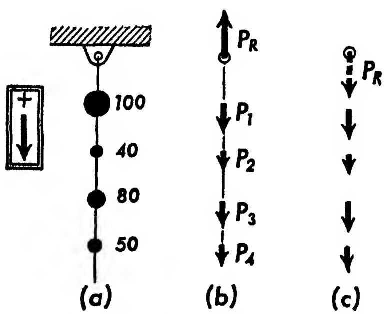


FIG. 4·8. Reaction to axial loads

The simplest equilibrium equation is, of course, that of a series of forces which all act along the same line (*a*, Sec. 4·7). A physical conception of this may be obtained by imagining a string supporting a number of different weights (Fig. 4·8*a*). The problem is to find the load acting on the supporting fitting. Obviously this load is equal to the *sum* of the weights. To solve by the method of equilibrium the equation would be written

$$[\Sigma P = 0] \qquad P_1 + P_2 + P_3 + P_4 - P_R = 0$$

$$P_R = P_1 + P_2 + P_3 + P_4$$

$$= 100 + 40 + 80 + 50 = 270$$

It is most important to note the manner in which the sign conventions were used in this elementary case. In drawing the free-body sketch in Fig. 4·8*b* the reaction P_R was shown acting in a direction *opposite* to that of the applied forces. Hence, if the latter forces are regarded as positive, the reaction, as sketched, must be given a negative sign in the equation of equilibrium. This practice of predicting the direction of the reaction is quite satisfactory, if properly used, but it becomes confusing in the more complicated cases.

In general, it is better to assume all reactions to be *positive* in making the free-body sketch and in writing the equations of equilibrium. The solution of the equations will then indicate the actual direction or sense of the reaction by the algebraic sign.

The problem of the weighted string, if handled in this manner, would appear as in Fig. 4·8*c*, and the equations would be

$$[\Sigma P = 0] \qquad P_R + P_1 + P_2 + P_3 + P_4 = 0$$

$$P_R = -P_1 - P_2 - P_3 - P_4$$

$$= -100 - 40 - 80 - 50$$

$$= -270$$

The negative sign now shows that the reaction P_R actually acts in a negative direction, that is, opposite to the direction shown in Fig. 4·8*c*.

The most important rule to be learned from these elementary examples is that of *consistency*. The unknown reactions may be assumed to act in either a positive or negative sense and the algebraic solution will indicate whether the assumption was correct or not (correct if sign comes out positive). But unless an unknown reaction is arbitrarily assumed to be positive to start with, the sign obtained from solving the equations of equilibrium will *not* mean anything with respect to the general conventions set up for the known forces. It is therefore most important that the free-body sketch and the equations agree with respect to signs.

For example, assume that the sketch of Fig. 4·8*b* had been used (showing the reaction acting opposite to the known forces). Assume also that the correct equations were used, as shown. The answer for P_R comes out positive. It would obviously be wrong to interpret this as meaning that the force P_R is positive as to *sense* and therefore directed downward.

The point which must always be kept in mind is that the signs obtained from solving the equations of equilibrium indicate *whether the initial assumptions were right or wrong*. The signs of the solution may be used to indicate the sense directly *only when the initial assumptions agree with the positive conventions.*

An illustration may be given of another elementary example in which it is more difficult to predict the direction of the resultant force. Assume that a cable is fastened to a permanent support (such as a post) and that a number of known forces are applied on opposite sides of this support (Fig. 4·9) At first glance it is not possible to predict the direction of the reaction at the support. It is therefore best to make no attempt to do so, but to assume that the reaction is in a positive direction (shown acting to left). If the support is replaced by its reaction on the cable the equation of equilibrium becomes

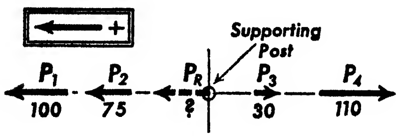


FIG. 4·9.

$$P_1 + P_2 + P_R - P_3 - P_4 = 0$$

$$P_R = -P_1 - P_2 + P_3 + P_4$$

$$= -100 - 75 + 30 + 110 = -35$$

The negative sign indicates that the assumption as to the direction of the reaction was wrong; hence the reacting force actually acts in the direction opposite to that shown in the sketch.

This simple example also illustrates that the structure may be "cut" or isolated at any convenient point in order to obtain the desired force. For instance, if the fitting to which the *left-hand* cable is attached is to be designed, the cable is cut at this fitting, and the reaction is found as for Fig. 4·8. The reaction for the *right-hand* cable is likewise found separately. The summation of these two reactions obviously gives the same answer as that obtained by cutting the support itself.

4·9. Action and Reaction. While still dealing with elementary examples it is advisable to clarify the relationship between action and reaction. In accordance with Newton's third law of motion the only difference between these two quantities is in the *sign* or *sense*—they are always equal and oppositely directed. The only point likely to cause trouble in structural analysis is to know which force acts on which part of the structure. When the reaction is obtained by solving the equations of equilibrium, this force (or moment) must always be considered as acting on the free body, that is, on the part of the structure on which the known external loads (used in the equations) are acting. Conversely, the *action* of this structure on the point or points to which it is attached is always represented by the reactions *reversed in direction.*

For example, in Fig. 4·9 the reaction of 35 lb acting to the right (indicated by negative sign in solution) represents the net force exerted *on the cable* by the supporting post. To obtain the force on the post the sign of the reaction is reversed, the negative sign becoming positive, indicating that the force acts to the *left* on the post.

At first sight it might seem that this process of writing the equations of equilibrium in order to obtain the reactions is unnecessarily complicated and that a straightforward summation of forces would give the same result more directly. This would be true for force transmission of the one-way type such as used for illustration up to this point. As previously noted, however, the need for obtaining reactions arises when there is more than one path for the force transmission. A direct summation cannot then be made, and it is necessary to use the equations of equilibrium to determine the reactions, which in turn show how the forces are divided between the various paths. The simple one-way examples were used to help create a clear picture of the problem before taking up the more general cases.

4·10. Reaction to Forces in Same Plane, Acting through Common Point. Assume that the problem is modified by having cables pulling on the post in various directions. To find the net reaction exerted by the post, the equations of equilibrium for (*b*) Sec. 4·7 are applied. It is first necessary to obtain the components of the forces along two mutually perpendicular axes. (These axes may be established in any convenient manner.) Both the *magnitude* and *direction* of the reaction must be found. The latter will be established by the relationship between the *components* of the reaction. These unknown components are sketched in Fig. 4·10*a*, using *positive* force conventions. Assume that the following values are known, for the applied forces.

$$P_{1x} = +10; \quad P_{1y} = +10$$

$$P_{2x} = +15; \quad P_{2y} = -5$$

$$P_{3x} = +10; \quad P_{3y} = -15$$

$$[\Sigma P_x = 0] \qquad 10 + 15 + 10 + P_{Rx} = 0$$

$$P_{Rx} = -35$$

$$[\Sigma P_y = 0] \qquad 10 - 5 - 15 + P_{Ry} = 0$$

$$P_{Ry} = +10$$

$$P_R = \sqrt{35^2 + 10^2} = 36.4$$

Figure 4·10*b* may now be drawn to show the true free body in equilibrium. The direction of the reaction is easily obtained graphically by laying out the values of its components parallel to the reference axes and in the direction indicated by the signs obtained in the solution.

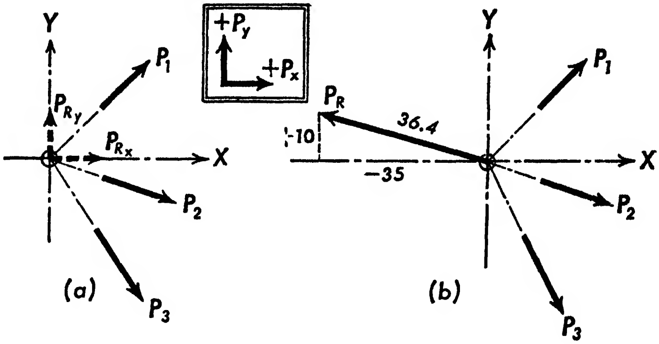


FIG. 4·10. Determination of reaction.

4·11. Reaction to Forces in Any Direction, through Common Point. The equations of equilibrium for this kind of force transmission are given in Sec. 4·7, Eq. 4·6. Referring to the previous example (Fig. 4·10), imagine that, in addition to the forces acting parallel with the ground, there is another force which acts normal to the ground, i.e., downward or upward. Assume, for instance, that Fig. 4·10*a* shows the post as it would be seen from above, and that an upward force of 30 lb is acting simultaneously with the other forces. The reaction will now have a component in a vertical direction, which may be denoted by P_{Rz}. One more equation must be added to those of Sec. 4·10 to take care of the *Z* direction. If the upward direction is established as positive for loads normal to the ground, and the reaction is assumed to have a positive *Z* component, the third equation of equilibrium becomes

$$[\Sigma P_z = 0] \qquad 30 + P_{Rz} = 0$$

$$P_z = -30$$

This indicates, of course, that the reaction is equal and opposite to the applied load. But since this is only a component of the total reaction, the value of the latter must be obtained from the equation

$$P_R = \sqrt{P_{Rx}^2 + P_{Ry}^2 + P_{Rz}^2}$$

$$= \sqrt{35^2 + 10^2 + 30^2} = 47.2$$

The procedure would be the same regardless of the number of loads acting in a vertical direction. If the applied loads were neither parallel nor normal to the ground, it would be necessary to resolve them into components in the X, Y, and Z directions. The solution would then proceed as before, treating these components as separate loads.

4·12. Reactions to Forces in Same Plane, Acting Parallel to Same Line. This condition is expressed by (*d*), Sec. 4·7. To apply it to the example of the post, it is necessary to add another member, as indicated in Fig. 4·11. Equations 4·7 may now be applied. These show that the post must have not only a reacting *force* (P_R) but also a reacting *turning moment* (M_R).

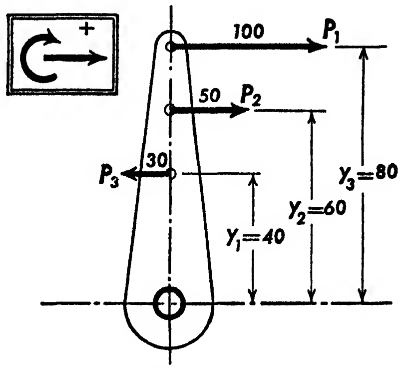


FIG. 4·11. Parallel forces (reactions at post not shown).

The equations become

$[\Sigma P_x = 0] \qquad 100 + 50 - 30 + P_R = 0$

$$P_R = -120 \text{ lb}$$

$[\Sigma M = 0]$

$$100 \times 80 + 50 \times 60 - 30 \times 40 + M_R = 0$$

$$8000 + 3000 - 1200 + M_R = 0$$

$$M_R = -9800 \text{ in.-lb}$$

It must be kept in mind that these values represent the *reactions* on the free body, which is really the arm used to transmit the forces to the post. The effect on the post will, of course, be exactly opposite, i.e.,

$$P = +120 \text{ lb}$$

$$M = +9800 \text{ in.-lb}$$

Figure 4·11 shows, by the sign conventions, which way these forces and moments act, although they are not actually shown in the figure.

4·13. Reactions to Forces in Same Plane (Not Parallel to Each Other). In Fig. 4·12 it is first necessary to resolve the loads P_1, P_2, and P_3 into components along suitable reference axes, as shown in (*b*).

The conventions adopted are indicated also. It is now necessary to write three equations of equilibrium, as follows. (See Eq. 4·8.)

$$\Sigma P_x = 0$$

$$\Sigma P_y = 0$$

$$\Sigma M_z = 0$$

The solution of these equations will give the reactions on the *arm*. The resultant force will be obtained by vector addition of the two

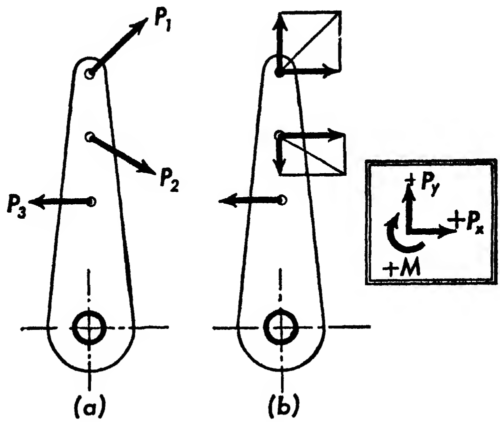


FIG. 4·12. (Reactions not shown.)

resultant components, P_{Rx} and P_{Ry}. The forces acting on the *post* will, of course, be reversed in sign, with the resultant in the opposite direction.

Note that Figs. 4·11 and 4·12 are *not* shown in equilibrium, i.e., as free bodies.

4·14. Reactions to Forces in Any Direction. Assume that in Fig. 4·12 there are also forces or components acting normal to the plane of the paper at the loading points. These components would be designated P_z. It is now necessary to use the general equations of equilibrium (Sec. 4·7) and to adopt definite conventions for moment axes, as indicated.

Note that none of the applied forces would have any moment about the Y axis; hence the equation for moment equilibrium about this axis could be omitted. It is always safe, however, to write *all* the equations of equilibrium, as any unnecessary equations will drop out when an attempt is made to evaluate them.

4·15. Reactions vs. Resultants. These simple examples should have made it obvious that the processes of finding reactions and resultants are basically identical, as the reaction is really nothing more than the exact opposite of the resultant. Hence the general rules and procedures outlined for resultants in Chapt. 2 may be used in computing reactions. The main reason for working with reactions rather than resultants is that their use makes it more convenient to apply the equations of equilibrium. The use of these equations permits the application of algebraic methods and sign conventions, which will be found to be essential in dealing with more complicated examples.

4·16. Reactions for Two-Way Transmission. As previously noted, the main reason for computing reactions is to determine the manner in which force transmission is divided between several paths. In a simple beam, such as shown in Fig. 4·13, there are two paths for the forces to follow, as the beam is supported at points 1 and 2. Since it is impossible to tell immediately how much of each force goes to each support, it is necessary to assume some unknown reactions at these supports and solve the equations of equilibrium to find these reactions.

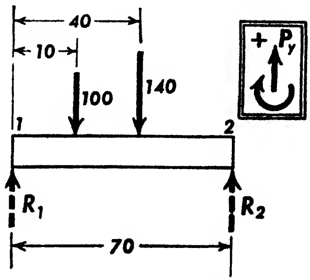


Fig. 4·13. Simple beam with two supports.

Accordingly, Fig. 4·13 is supplied with two force vectors, R_1 and R_2 (shown dotted to indicate that their magnitudes are unknown). The *direction* (but not the *sense*) of these vectors is determined by the nature of the support. Thus if the beam is lying on two frictionless rollers, it is obvious that the reactions must be *normal* to the beam, i.e., there can be no component along the beam itself. It is therefore satisfactory to assume the reactions to be vertical.

The *sense* (up or down) could here be determined by inspection, but this is not always possible. It is therefore better to use the general rule of assuming the reactions to be positive and letting the algebraic solution determine whether the assumption was correct or not. (In Fig. 4·13 it happens that the positive conventions coincide with the obvious directions of the reactions.)

Since all the forces and reactions are parallel to the Y axis, the equations of equilibrium for forces reduce to

$$[\Sigma P_y = 0] \qquad R_1 - 100 - 140 + R_2 = 0$$

or

$$R_1 + R_2 = 240 \qquad [4\cdot9]$$

This solution gives only the *sum* of the reactions. It is therefore necessary to apply the equation of equilibrium for moments. (This could have been foreseen by referring to Sec. 4·7.) Since the free body is in equilibrium, the net moment of the external loads about any point should equal zero; hence it should make no difference which point is selected in summing up moments. It is usually convenient to select a point on the line of action of one of the reactions, as this eliminates this reaction from the moment equation.

If point 1 is selected, the equation of equilibrium for moments becomes

$$[\Sigma M = 0] \quad (100 \times 10) + (140 \times 40) - (R_2 \times 70) = 0$$

$$1000 + 5600 - 70R_2 = 0$$

$$R_2 = \frac{6600}{70} = 94.3$$

The other reaction could be found by taking moments about R_2, but it is simpler to substitute the above value of R_2 in Eq. 4·9, giving

$$R_1 = 240 - R_2$$
$$= 240 - 94.3$$
$$= 145.7$$

Both methods can be used, giving two independent computations for R_1, which should check. This type of check is extremely valuable in structural work and should be employed whenever possible.

4·17. Use of Fictitious Support. The physical action can perhaps be followed more easily by starting out as if the beam were fixed at one end, as shown in Fig. 4·14*a*. The equilibrium equations then become

$$[\Sigma P = 0] \quad R_1 - 100 - 140 = 0$$

$$R_1 = 240$$

$$[\Sigma M = 0] \quad (100 \times 10) + (140 \times 40) + M_R = 0$$

$$M_R = -6600$$

Actually there can be no resisting moment at point 1, but there can be a vertical force at point 2. The fictitious moment M_R must therefore be supplied by a couple, acting at points 1 and 2, on a 70-in. moment arm. The value of each of the forces composing this couple is

$$P_M = \frac{M_R}{70} = \frac{6600}{70} = 94.3$$

Since the computed value of M_R was found to be negative, the reacting couple must act in a counterclockwise direction (clockwise assumed to be positive in Fig. 4·14a). This couple is now substituted for the moment vector M_R, as shown in Fig. 4·14b, and the net reaction at point 2 is obtained by algebraic addition. The results are, of course, identical to those obtained by the conventional method.

Actually the numerical work is exactly the same in both methods. The physical conception of the couple is a very useful tool, and the analyst should attempt to visualize reactions in this manner. Examples will be given later to show how this conception may be used to simplify the work.

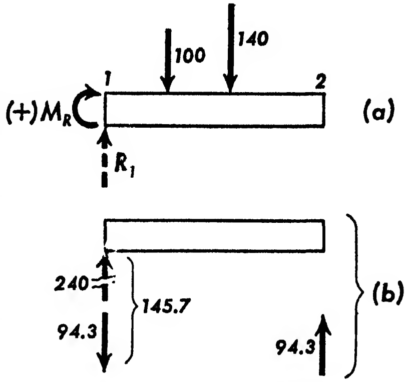


Fig. 4·14. Use of fictitious fixed end support.

4·18. Simple Beam: Non-Parallel Forces. As an example of non-parallel forces consider a bicycle held stationary on a sloping surface by means of a rear-wheel brake. Instead of using actual forces, symbols will be used. The solution will then be general, and it can be used for any similar example by direct substitution of forces and distances.

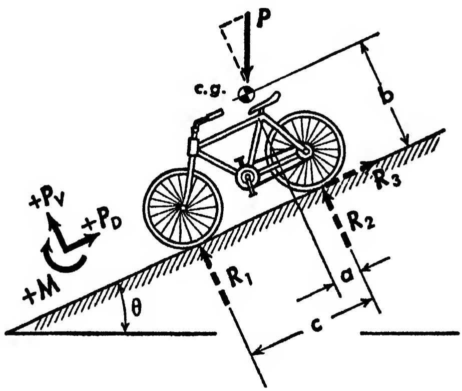


Fig. 4·15. Reactions on a bicycle.

First, suitable reference axes and conventions are established, as noted in Fig. 4·15. (*Note.* Symbols V and D, for vertical and drag, are purposely used to illustrate the point that *any* system is satisfactory, provided that it is adhered to during the problem.) The next step is to examine the points of support (reactions) to see whether the direction of any of the reactions can be determined in advance. Since the brake is on the rear wheel only, it is obvious that the reaction on the front wheel must be normal to the ground line.

The direction of the rear-wheel reaction cannot be determined by inspection, as this reaction includes a drag force as well as a vertical force; this unknown reaction must be split into two components, the directions of which are known, but the magnitude of which must be determined. These are labeled R_2 and R_3.

The load acting on the bicycle (its own weight plus that of the rider) is assumed to be concentrated at the center of gravity of the combination, which is located by the dimensions a and b. This load must be resolved into components acting along the assumed reference axes. These components are

$$P_V = -P\cos\theta$$

$$P_D = -P\sin\theta$$

where θ is the slope of the ground.

The equations of equilibrium may now be written (Sec. 4·7):

(1) $[\Sigma P_V = 0] \quad R_1 + R_2 - P\cos\theta = 0$

(2) $[\Sigma P_D = 0] \quad R_3 - P\sin\theta = 0$

(3) $[\Sigma M = 0] \quad R_1 c - aP\cos\theta - bP\sin\theta = 0$

Note. Moments are summed up about contact point of rear wheel. This eliminates two reactions from the equation.

The second equation immediately gives the reaction $R_3 = P\sin\theta$. The third equation may be solved for R_1 giving

$$R_1 = P\frac{a\cos\theta + b\sin\theta}{c} \qquad [4\cdot10]$$

The first equation yields R_2, after substituting for R_1 as follows.

$$\begin{aligned} R_2 &= P\cos\theta - R_1 \\ &= P\left[\cos\theta - \left(\frac{a\cos\theta + b\sin\theta}{c}\right)\right] \qquad [4\cdot11] \end{aligned}$$

Equations 4·10 and 4·11 will give the answer for any specific case, simply by substituting numerical values for the symbols used. For instance, if the bicycle were on level ground, θ would be zero and Eqs. 4·10 and 4·11 would become

$$R_1 = P\frac{a}{c} \qquad [4\cdot10a]$$

$$R_2 = P\left(1 - \frac{a}{c}\right) = P\frac{b}{c} \qquad [4\cdot11a]$$

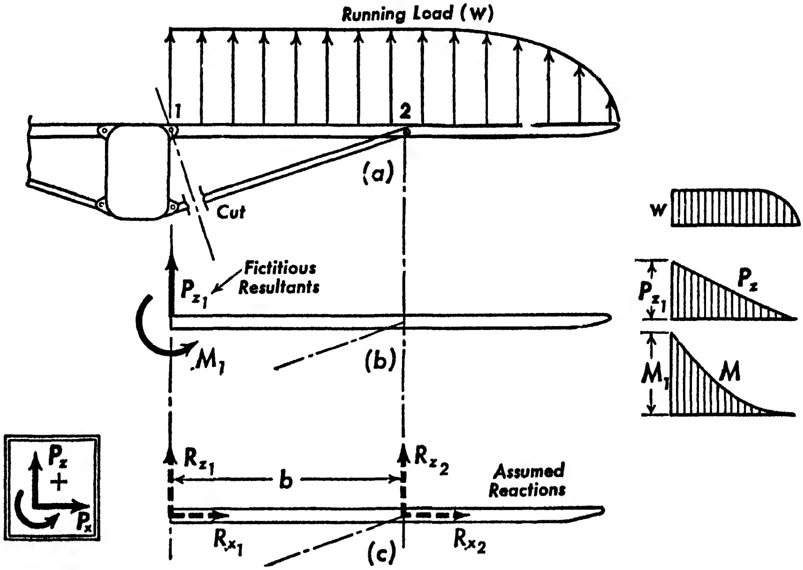
since $\cos\theta = 1$, $\sin\theta = 0$, and $a = c - b$.

If the brake had been applied at the *front* wheel only, the conditions of Fig. 4·15 would be reversed, i.e., the drag component R_3 would act on the front wheel. The question then arises, "What if *both* wheels are braked." Obviously the answer may lie anywhere between the two extremes already described, depending on the degree of braking applied to each wheel. The problem is therefore *indeterminate*, at least so far as the simple laws of static equilibrium are concerned, for additional information is necessary. It is most important to be able to recognize a *statically indeterminate* (or *redundant*) structure. This is more fully covered in Chapt. 5. From this point on it must be kept in mind that only *statically determinate* structures are dealt with, unless otherwise noted.

4·19. Reactions for Distributed Loads. The use of the equations of equilibrium in computing reactions is relatively simple when the external loads take the form of a few concentrated forces. For *distributed* loads, however, it is difficult to handle the problem in this way, and a somewhat different approach may be used to simplify the work. In Sec. 2·18 it was shown how distributed loads could be progressively summed up along arbitrary axes, giving resultant forces and moments at any desired point. This method can be employed to reduce the external loads to equivalent resultants, which can be more easily handled in the equilibrium equations.

An externally braced airplane wing offers an excellent example of the procedure, as shown in Fig. 4·16. It may be assumed that the external loads have been accumulated into a curve of *running load* (w) in the plane of the paper. By following the procedure given in Sec. 2·19, these running loads may be collected (summed up or integrated) into a curve of net vertical force (P_z), which is again integrated to give a curve of net moment at any station. These processes are indicated by the small sketches on Fig. 4·16.

The first step consists of this summation process, under the assumption that the diagonal "lift" strut does not exist. Sketch (a) illustrates the assumed cut. Sketch (b) shows the fictitious resultants thus obtained at point 1. So far as *reactions* are concerned, the original loads may now be assumed to be replaced entirely by these *equivalent* resultants, making the problem much easier to solve.



Equilibrium at point 1

$$\begin{cases} \Sigma P_z: & R_{z_1} + R_{z_2} + P_{z_1} = 0 & [a] \\ \Sigma P_x: & R_{x_1} + R_{x_2} + 0 = 0 & [b] \\ \Sigma M_{y_1}: & M_1 + R_{z_2} b = 0 & [c] \end{cases}$$

Lift strut $R_{x_2} = KR_{z_2}$ [d]

Procedure

1. Solve (*c*) for R_{z_2}.
2. Substitute R_{z_2} in (*a*) and find R_{z_1}.
3. Substitute R_{z_2} in (*d*) and find R_{x_2}.
4. Substitute R_{x_2} in (*b*) and find R_{x_1}.
5. Combine components of R_2 to obtain resultant,

$$R_2 = \sqrt{R_{x_2}^2 + R_{z_2}^2}$$

FIG. 4·16. Computation of reactions for airplane wing.

Now if the connection at point 1 is made by means of a pin joint, it is impossible for the computed moment to be resisted at this point. The only thing that keeps the beam from rotating upward is the lift strut, which must therefore resist such rotation by means of a tension load, pulling down on the beam. Since the direction of the lift strut is known, it can be replaced by an assumed reaction R_2, the magnitude of which remains to be computed.

According to the standard procedure for computing reactions, the components of the unknown reactions at points 1 and 2 are assumed to be *positive*, as shown in sketch (*c*). (Sketches *b* and *c* are separated for clarity; they must be combined to put the free body into equilibrium.) The equations of equilibrium are then written, combining the fictitious applied loads of sketch (*b*) with the assumed reactions of sketch (*c*). (This process is the mathematical counterpart of superimposing these two sketches.) Since the values of P_1, and M_1 are already known, the equations can be solved for the unknown resultants, as indicated in Fig. 4·16.

Note that the *direction* of the lift strut has an effect on the results. This is introduced mathematically by means of Eq. *d*, which gives the relationship between the two components of the lift strut force. The signs obtained from the solution of the equations will indicate which way the reactions act on the beam. It will be found in the illustration that all the signs of R_{z1}, R_{z2}, and R_{x2} come out negative, indicating that the directions assumed for the forces in sketch (*c*) were wrong.

There are many possible variations in this procedure, but the method used here contains most of the important principles involved. It would have been possible to assume *tension* in the lift strut and to draw the vectors R_{x2} and R_{z2} in the opposite directions from those shown. The solution would then indicate that the assumption of *tension* had been correct.

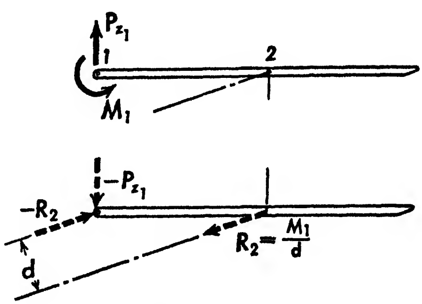


FIG. 4·17. Computation of lift strut reaction.

4·20. Short-Cut Method. It is frequently possible to shorten the computations by direct application of the principle outlined in Sec. 2·13, that of the *couple*. In the preceding example it could be seen that the reaction to the vertical resultant P_{z1} would be an equal and opposite force acting at point 1, while the moment M_1 could be replaced by an equivalent couple. Since the couple *must* act between the two points of support, and since one of the forces *must* act along the line of the lift strut, it is obvious that the reacting couple will appear as shown in Fig. 4·17. Only the magnitude of the reaction R_2 is unknown, and this is very easily found by dividing the moment by the moment arm of the couple. This distance is, of course, the perpendicular distance *d* from point 1 to the line of action of R_2, as shown. The final resultant at point 1 is obviously obtained by combining the two vectors which act there. The *axial* and *transverse* forces applied at point 2 may be obtained from the components of R_2.

This principle of taking moments is of great value in structural work, as it obviously reduces the mathematical work required and it gives a better physical conception of the problem. It is particularly useful if only the load in the cut member is to be found. But a method of this type must be used with caution, as it does not have the automatic features of the standard method, and it is therefore susceptible to mistakes in signs. The best engineering procedure in the example used here would be to go through the standard method and then check it by computing the reaction at point 2 by the short-cut method of taking moments.

4·21. Distributed Loads. Although the foregoing methods take care of the reactions, they do not give the final conditions between the points of support. It would, of course, be possible to apply the computed reactions to the free body, putting it in equilibrium, and then start all over again to obtain the *net* forces and moments by the method of progressive summation. This is not really necessary, however, as this summation has already been done on the basis that there is no reaction at point 2. It should therefore be satisfactory to make a summation for the reactions at point 2 *only* and superimpose the results on those already obtained. This process is almost completely explained by Fig. 4·18, which shows, side by side, the two separate summations, together with the final conditions obtained by combining them.

In using the process of superposition it is obviously essential that exactly the same sign conventions be used. To avoid confusion, the plotted curves should adhere to these conventions also, as shown. However, it is convenient to reverse the signs of the loads caused by the reactions and to plot them on the diagram for the external forces. The net values are then obtained by direct measurement of the differences between curves. This is indicated by the dotted lines in the figures.

The following notes are of interest in connection with Fig. 4·18.

a. The axial (x) component of the lift strut reaction has no direct effect on the shear or moment curves. (Secondary effects are discussed in Chapt. 19.)

b. The final net vertical force (shear) at station 1 should be equal to the computed value for the vertical reaction at that point.

c. The final moment at station 1 should turn out to be zero. (The last two points afford an excellent check on the work.)

Since it is known that the net moment at station 1 must be zero (because of the pin joint) and that the moment at 2, due to the lift strut, is also zero, the net moment curve can be obtained directly by drawing a dotted line as shown in Fig. 4·18.

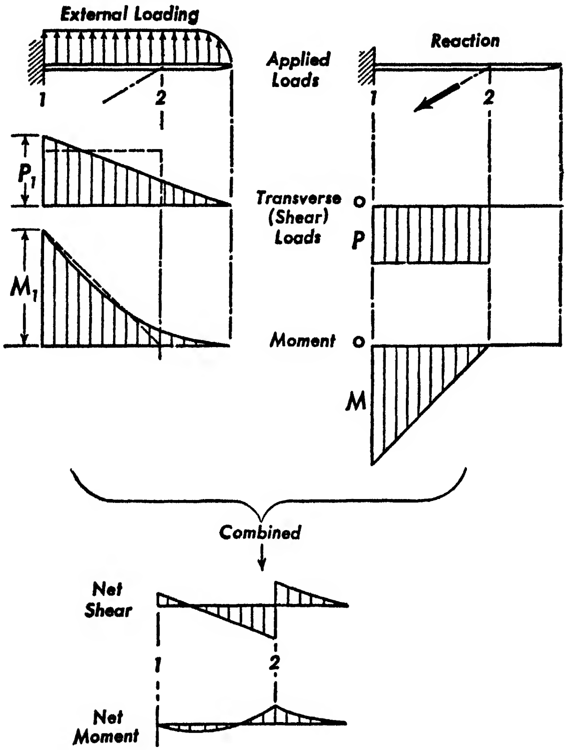


Fig. 4·18. Computation of net loads and moments.

4·22. Graphical Methods of Determining Reactions. Chapter 2 described various methods for determining the resultant of several forces graphically. The reaction is, of course, equal and opposite to the resultant and is given by the closing line of the vector diagram. If Bow's notation is used (Sec. 2·7), the direction of the reaction is determined by continuing around the force polygon, instead of reading from the starting point (see Sec. 2·8). By means of the *string polygon* the resultant of a system of forces may be located on the space diagram, and the *moment* of this resultant about any point may be readily determined by measuring the moment arm. In one-way force transmission the reactions may be found by this process.

The principle of the string polygon may also be employed to find the reactions in two-way force transmission. The force polygon is constructed in the usual manner, a *pole* is selected, and *rays* are drawn as described in Sec. 2·8 and shown in Fig. 4·19. Instead of using the two outer rays (OA and OD) to determine a point on the line of action of the resultant force, they are used to determine a closing line for the string polygon. This is done by extending the string polygon to meet the lines of action of the unknown reactions. In Fig. 4·19, lines *oa* and *od* are thus drawn and determine the position of closing line *oe*. (Two dotted lines have been included to show the alternate construction for a *single* resultant or reaction, but they are not used here.) The string *oe* corresponds to a *ray* OE in the force polygon, which may now be drawn parallel to *oe*. Point E determines reactions DE and EA.

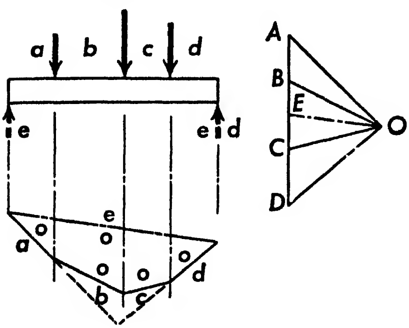


FIG. 4·19. Graphical determination of reactions.

The use of the string polygon may be extended to include cases in which the direction of only one of the reactions and the point of application of the other are known to start with.

Figure 4·20 shows such a case, in which the direction of R_1 is predetermined, but only the point of application of R_2 is known. The polygon for the known forces is constructed as usual, including the rays

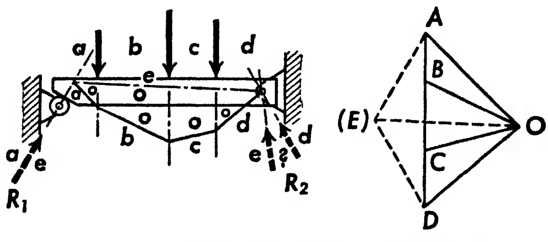


FIG. 4·20. Graphical method—general case.

OA, OB, etc. The line AE may be drawn in the proper direction, but the *length* of the line cannot immediately be determined. It will be found when the final ray is drawn parallel with the closing line of the string polygon. Since the moment arm of reaction R_1 about the other reaction point must be preserved, the string polygon cannot be started at any arbitrary point, but must pass through the point of application of R_2. Hence the construction is started at this point and line *od* is drawn parallel to OD, *oc* parallel to OC, etc. The closing line *oe* gives the direction for force vector OE in the polygon and thereby determines point E and vectors DE and EA. The dotted lines in the force polygon are those found during the solution of the problem.

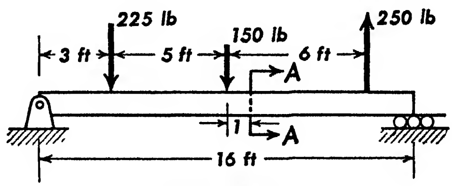
Many useful applications of the string polygon in determining reactions and bending moment curves are to be found in structural handbooks and textbooks. References 4 and 7 contain numerous interesting examples and problems.

4·23. "Handbook" Curves. The foregoing examples have been used mainly to give a clear understanding of the processes involved, and so they have been made quite general. It is obviously possible to select a large number of types of loading conditions and to work out the *reactions*, *shears*, and *bending moments* for each one in terms of the load intensity, dimensions, and type of support. If the assumed conditions are simple (uniform loading, triangular loading, etc.) the work can be done entirely by mathematics.

General formulas and charts have been prepared for many simple types of loading and are very useful to the structural engineer. They may be found in almost any structural textbook or engineering handbook and may be used for routine calculations. Appendix 3 covers most of the common loading conditions.

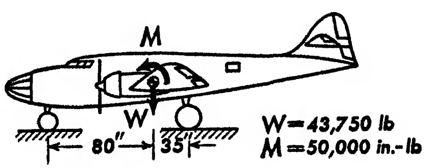
PROBLEMS

4·1. Draw a free body of the portion of the beam to the right of section A–A showing all the forces and moments acting at the cut. (Indicate the reaction as an external force.)



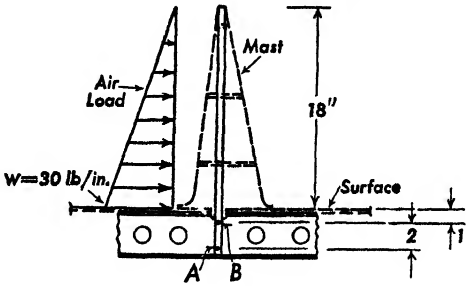
PROBLEM 4·1.

4·2. Find the vertical reactions required at each wheel to balance the applied landing loads.



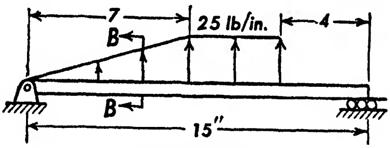
PROBLEM 4·2.

4·3. A radio mast must resist a uniformly varying air load as shown. Find the forces required at pins A and B, which attach the mast to the structure.



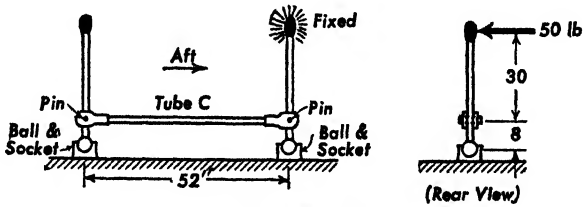
PROBLEM 4·3.

4·4. Make a free-body diagram of the portion of the beam to the left of section B–B, and indicate the shear and moment at this section. (Assume B-B at 5 in. from left end.)



PROBLEM 4·4.

4·5. A dual control system for a light plane is illustrated. Considering the handgrip on the aft stick to be fixed, what is the torque in tube C due to an applied side force of 50 lb acting at the forward handgrip? (*Note:* No side force can be transmitted through the connecting tube, due to the ball-and-socket arrangement. Likewise no torque may be transmitted through the base structure.)



PROBLEM 4·5.

4·6. Draw a number of concurrent force vectors as shown in Fig. 4·10, but of different magnitude and directions. Find the value and direction of the reaction analytically. Check graphically.

4·7. To the forces used in Problem 4·6 add another acting normal to the plane of the paper (along Z axis). Compute the reaction analytically. Check its value graphically by adding the reaction to the new force, to the reaction obtained in Problem 4·6. (The plane in which this addition takes place will actually be normal to the paper, but the vectors must be drawn in the plane of the paper.)

4·8. Change the values of the forces shown in Fig. 4·11 and find the reactions. Check by writing the equations of equilibrium, using the point of application of force P_1 as a reference point.

4·9. Draw a sketch of a cantilever beam loaded by three transverse forces of different magnitude. Assign values to the forces and to the dimensions. Calculate the reactions (shear and bending moment) at the fixed end and draw the shear and moment curves for the beam. Then assume that the method of support is changed to that of a simple beam and find the reacting couple required to replace the fixed-end moment. Show how this affects the shear and moment diagrams.

4·10. Work out Eqs. 4·10 and 4·11 for a bicycle (Fig. 4·15) braked at the front wheel only.

4·11. Set up a problem such as shown in Fig. 4·20 and obtain the reactions graphically. Check by computing the bending moment and shear force at the right end, assuming the beam to be fixed at this end. Then replace the fixed support by a pin and apply the required couple to supply the reacting moment. Find the right reaction by graphical addition.

4·12. Make a scale drawing of an externally braced airplane wing beam, as shown in Fig. 4·16. Draw a curve of running load (w) similar to that shown and having a value of 100 lb/in. at the inboard end of the wing. Draw the shear and bending moment curves, assuming the wing to be fixed at the inboard end, with lift strut cut. Determine the reactions and draw the net shear and bending moment curves for the actual conditions. What is the axial force in the beam between stations 1 and 2?

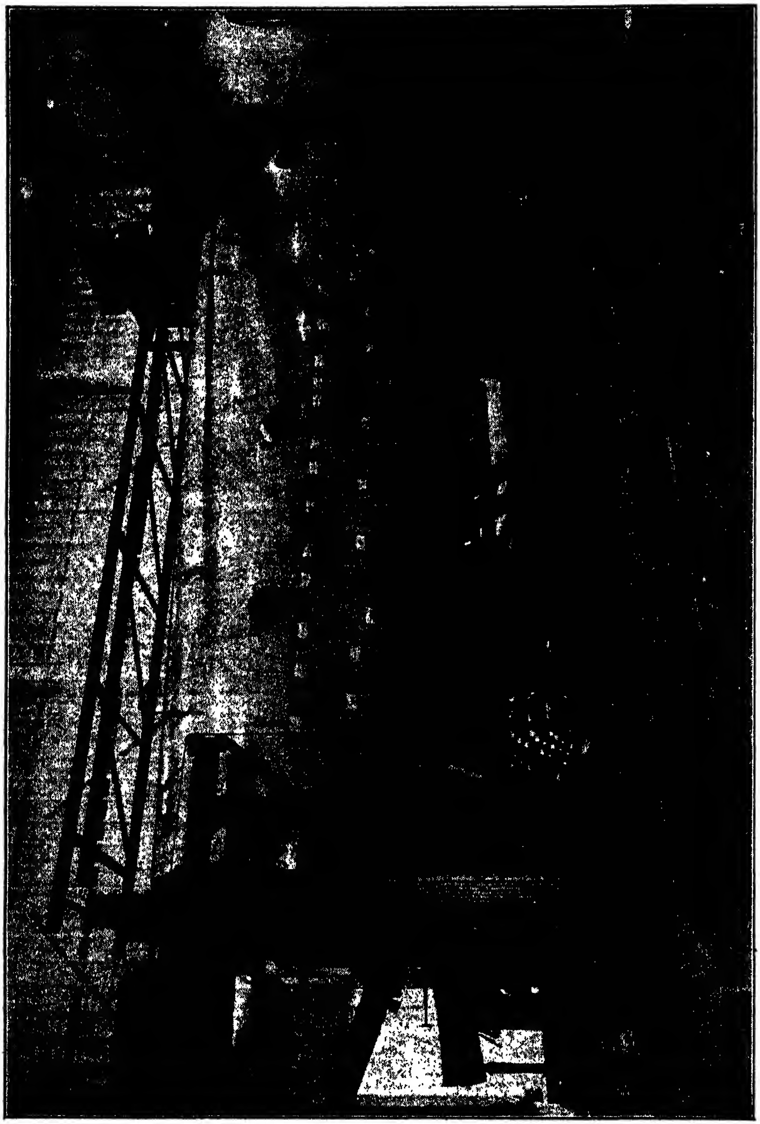


FIG. 4·21. Static test of a full-sized portion of an airplane fuselage. Various combinations of transverse load, bending, and torsion were applied by means of hydraulic jacks, some of which can be seen in the photograph.

CHAPTER 5

STABILITY, CONSTRAINTS, AND REDUNDANCIES

5·1. Statically Indeterminate Structures. The example of the bicycle (Sec. 4·18) introduced one of the most important principles of structural analysis, the distinction between statically determinate and statically indeterminate structures. In the first class of structures the reactions may be obtained by treating the structure as a free body and solving the equations of static equilibrium, from which the term statically determinate is derived. Statically indeterminate structures (also called **redundant**) cannot be solved by this process alone; it is necessary to consider other factors such as deflection or strain energy. The methods involved are considerably beyond the state of knowledge corresponding to this point in the development of structural analysis methods; they must therefore be deferred until the necessary background has been covered.

Even in elementary structural analysis, however, the engineer must be able to recognize statically indeterminate structures, otherwise he may waste many hours trying to solve an impossible problem. It is also useful to know how to obtain *approximate* answers by making assumptions that will reduce the indeterminate structure to a determinate one.

The bicycle example illustrated two extreme conditions which included all intermediate solutions. This method is often applicable in structural work. In general, the procedure is to reduce the problem to a statically determinate one by making one or more simplifying assumptions. After obtaining a solution, the assumptions can be changed and another solution obtained. This is often referred to as a method of *overlapping assumptions.* Other examples of this procedure will be given later, but at this point it is important to realize that the simple principles of static equilibrium can often be applied to indeterminate problems in this manner.

5·2. Unnecessary Members. Another way to describe a statically indeterminate structure is to state that it has more members (or conditions of constraint) than are necessary for the transmission of the force. Thus, only one wheel of the bicycle needed to be constrained by the brake: the use of both brakes caused one *extra* constraint and made it impossible to solve the problem by the simple laws of equilibrium. Stated mathematically, in a redundant structure the conditions of equilibrium do not supply as many equations as there are unknowns.

Another simple example is the ordinary table. A three-legged table has just enough legs to support a load; the addition of a fourth leg must therefore make the problem statically indeterminate. This is literally true, as the load on each leg will now depend not only on the laws of equilibrium but also on the length of the legs, the condition of the floor, or both. The method of *overlapping assumptions* would be a logical one for a table, as it would be sensible to design each leg on the basis that any one of the others might be inactive

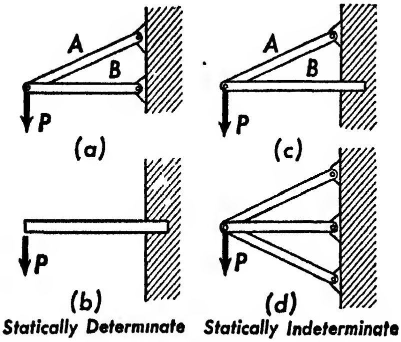


FIG. 5 1. Classes of plane structures.

The minimum number of members required to transmit a force will depend on the type of structure and on the number of space dimensions involved. For instance, transmission of a force by axial tension involves only "one space," and so only one member is required. A practical application of this rule is found in certain types of airplanes which use double tie rods (side by side). It is customary to use overlapping assumptions in computing the force in each rod, as the exact value will depend on the relative rigging loads. (For instance, each rod might be designed to carry 60 per cent of the total load, instead of 50 per cent.)

In "two space" (plane) a minimum of two members, or one member and one constraint, are required. Thus, in Fig. 5·1, (*a*) shows two members, *A* and *B*, whereas (*b*) shows one member constrained at one end. Both of these problems are statically determinate. But if one of the members of (*a*) were also fixed at the end, as in (*c*), or if a third member were added as in (*d*), the problem would be classed as statically indeterminate.

The table, previously discussed, illustrates conditions in three space. A *tripod* is also an excellent example which is widely used in structures. If in Fig. 5·1*d* the third member does not lie in the plane of the other two, the structure again becomes determinate, even though the load is still applied in the plane of the original two members. This will be explained under stability.

5·3. Stability. Before proceeding farther with the determination of reactions it is advisable to discuss briefly the problem of stability. In advanced structural work this is one of the most important subjects, but it also has an elementary application in connection with reactions and indeterminate structures.

If a weight is supported by a pin-ended vertical compression member as in Fig. 5·2a, according to the foregoing rules this one member should be enough to transmit the force. But anyone would recognize the practical impossibility of doing the job in this manner and would either drive the member into the ground (as in b) or attach some additional members (as in c). The problem is apparently made statically indeterminate by thus adding a constraint or a member, but actually it may still be treated as a determinate one.

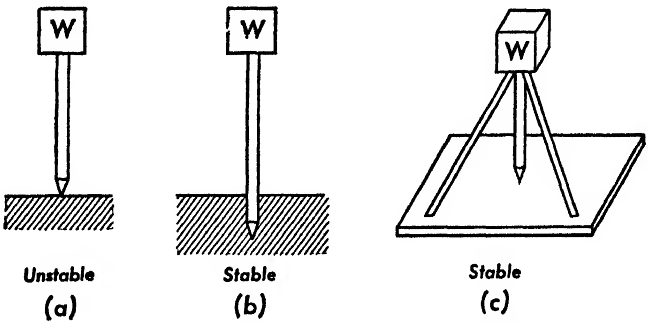


Fig. 5·2. Examples of stability and instability.

The answer is, of course, that we can neither escape from the reality of living in three-space nor can we load a structure perfectly. In Fig. 5·2a the slightest eccentricity of the load, or slant of the stake, would produce a force component in another space direction, and there would be nothing to resist it. Hence we must provide for these very small departures from perfection, which means that all *practical* structures must have sufficient members or constraints to take care of a three-space type of loading. If the applied force does not theoretically require these additional space members (or constraints), they may be neglected in determining whether the structure is statically determinate, even though they are required from a practical standpoint.

The bicycle (Sec. 4·18) may be used as another example. As shown in Fig. 4·15, the problem is statically determinate, when considered two dimensional. Actually, if the bicycle were standing still it would tend to fall over sidewise and would require an additional member or constraint to *stabilize* it in the third dimension. Similarly, if it were on level ground the constraint afforded by the brake would theoretically be unnecessary. However, the application of the brake would not make the problem indeterminate, and from a practical standpoint it would take care of any slight departure from the assumption that the ground was level.

One apparent exception to the above reasoning is the simple *tension* member. If the weight of Fig. 5·2 had been suspended by a cable, as in Fig. 5·3, it would be unnecessary to add lateral supports or constraints. This is because a tension member is self-stabilizing. Thus, if some disturbance caused the weight to move sidewise, as shown by the dash line, the weight component acting normal to the cable would pull it back into line again. This is not really an exception, however, as the addition of just enough members to keep the weight from swaying would not change the load in the cable, and the problem would remain statically determinate. Without these members the weight could be set in motion and the combination should then be classed as a **mechanism** (specifically, a pendulum).

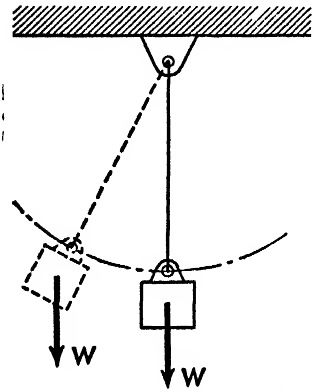


FIG. 5·3. Tension member.

5·4. Mechanisms. It has been shown that a statically determinate structure should have *just enough* but not too many members or constraints. If there are *not enough*, there is no structure at all (except in tension members, as noted above). Such an arrangement of members is called a **mechanism.** From the standpoint of force transmission a mechanism is useless, as it will not transmit a force but will move as soon as the force is applied.

Since it is of no practical use as a means of static force transmission, the mechanism is extremely important as something to be recognized and avoided. It is quite possible to find that a proposed design is not a structure but a mechanism, hence incapable of performing its function. Cases have even occurred in which this was not found out until the "structure" was built, only to cause extreme embarrassment to the designer! It is essential that the structural engineer be able to recognize a mechanism, so that he will waste no time trying to analyze it as a structure and will know what to do to correct it.

An arrangement of members may constitute a *structure* with respect to one type of loading and a *mechanism* with respect to another. Thus in Fig. 5·1*a* the two members form a sound structure with respect to forces acting in their own plane, but if the attachments at the wall were made with ball-and-socket joints, the structure would be unable to transmit a load *normal* to its own plane. With respect to such loads it would therefore be a mechanism. (Such an arrangement is actually used in hoists and brackets.)

5·5. Constraints. It has been shown that the direction and nature of the reactions will depend on the conditions of constraint or support at the points in question. This is so important that some additional types of support will be examined.

The simplest form of support is of course the **roller.** Such supports were assumed in Figs. 4·15 and 4·20, as they simplified the problem by compelling the reaction to be exerted normal to the contact surface. The frictionless roller may therefore be thought of as an idealized form of support which is incapable of transmitting a *moment* and which can transmit a *force* along only one line of action. (In some structures rollers are actually used to produce exactly these conditions.)

The **pin** is an even more common type of constraint, represented in practice by bolts, rivets, axles, etc. It is usually assumed that the pin can transmit no turning moment about its own axis, but that it can transmit a force in any direction in a plane normal to its axis. The combination of pin and roller is, of course, a **wheel.** Assuming no friction, the direction of the force on the pin will be determined by the point of contact of the wheel with the supporting surface.

Pins may be installed so as to transmit moments about an axis normal to the axis of the pin by means of a couple or its equivalent. Thus a pin may be used to transmit torsion into a member without causing a bending constraint in one plane. To eliminate bending constraints completely, *two* pins may be used at right angles, making a **universal joint.** This type of joint is widely used in machinery (see Fig. 5·4).

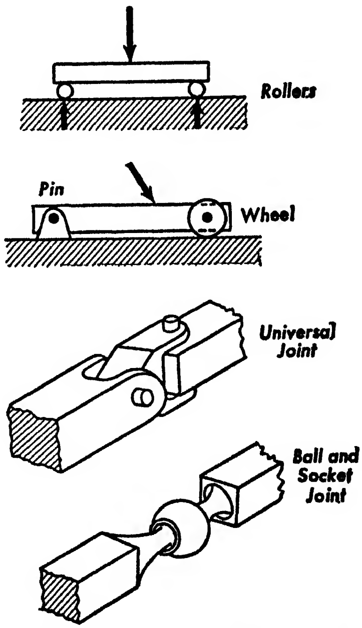


Fig. 5·4. Various types of joints.

The **ball-and-socket** joint (Fig. 5·4) can transmit an axial force in any direction but is incapable of transmitting any moment about any axis.

Various other types of joints may be used for specific purposes, such as the **spline** (transmits torsion and sometimes bending but no axial force), **sleeve** (transmits bending but no torsion or axial force, as in airplane shock absorbers), **threaded** connections (may transmit axial forces and bending but no torsion, neglecting friction).

5·6. Simplifying Assumptions for Constraints. In actual practice it will be unusual to find a structure that does not have at least some slight degree of redundancy. It is theoretically impossible, for instance, to consider a pin to be frictionless; this is particularly true if the pin takes the form of a tight bolt or rivet. *Welded* connections are certainly capable of transmitting both torsion and bending, as well as axial forces. But if every factor of this type were taken into account, the analysis would become almost impossible It is most important to recognize

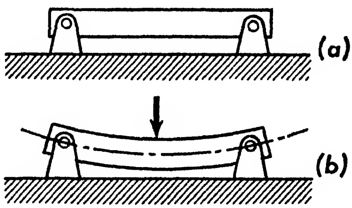


FIG. 5·5. Pinned beam.

those forms of constraint which may be regarded as *secondary* and to eliminate them from the problem. This should not be done blindly, however, as it may be necessary to investigate such constraints more carefully if they appear to be relatively important.

Although no specific rules can be set up at this point, common sense will usually suffice to show what simplifying assumptions can be made. A few typical ones will be discussed below to illustrate the reasoning involved.

One of the most common assumptions is the hypothetical replacement of a pin joint by a roller supporting the beam. Except in special cases, beams are likely to be bolted, riveted, or welded in place. Even a single pin through each end of a beam (as in Fig. 5·5) theoretically makes it redundant, as each pin is capable of carrying "side" loads along the axis of the beam. As the beam bends under the vertical load P, it tends to pull the two supporting pins closer together, as indicated in Fig. 5·5*b*. If the beam is very flexible in bending, but strong in tension (as in a spring or cable), this process may go on until most of the load is carried by tension in the beam, instead of by bending. If, however, the beam is quite deep as compared with its length, the amount of deflection will not be enough to produce appreciable tension loads; it may consequently be assumed that one of the supports is replaced by a roller, and the problem becomes statically determinate. If one of the holes should be "sloppy," i.e., considerably larger than the pin, a condition equivalent to roller support is obtained.

The *tightrope* represents the other extreme, in which the bending stiffness of the "beam" may be neglected. The *suspension bridge* is also based on this principle. In *flat plates* under normal pressure, conditions may vary from one extreme to the other, depending on the dimensions of the plate. Thick plates may be treated as beams (bending), whereas very thin plates behave like the suspension bridge (tension). Intermediate conditions involve both bending and tension.

In taking up *trusses*, later, it will be found that the analysis is greatly simplified by assuming all members of the truss to be *pin jointed,* even though they may be attached at the ends with gusset plates, or welded. This simplifying assumption may sometimes have to be modified to account for the bending moments introduced at the ends of the members. However, the simple pin-jointed assumptions are usually made first, after which more refined methods may be applied to take care of end fixity conditions.

From these examples it can be seen that the underlying principle to be followed in simplifying the conditions of constraint is that of *comparison.* In the beam pinned at each end a comparison was made between the two extreme assumptions of pure bending and pure tension (suspension-bridge action). Common sense usually will determine which assumption is the more reasonable.

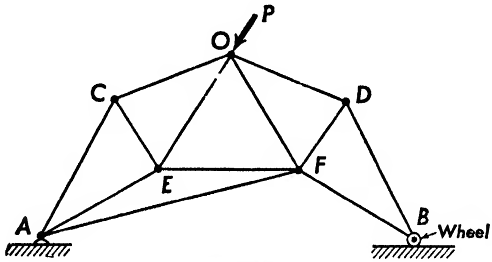
This method can be applied to obtain an approximate solution for Fig. 5·1*c*. It would first be assumed that the two members A and B are pinned. In the alternative assumption member B would be assumed to act by itself as a cantilever beam. Usually the first assumption will be closer to the true conditions, and an approximate analysis may therefore be made on that basis, by static methods.

5·7. Summary. The foregoing discussion of redundancies and constraints has been included at this early stage so that there will be no misconception as to the accuracy of structural analysis methods. The engineer must realize that much depends on intelligent assumptions and common sense, and he should start developing these indispensable tools as soon as possible. Many seemingly complicated structural problems can be solved quickly and with sufficient accuracy by simple methods based on reasonable assumptions. On the other hand, much time can be wasted by applying rigorous methods in order to take care of minor redundancies and secondary constraints. Finally, a clear conception of constraints and their effects will be of great value in connection with modern relaxation methods of analysis, which are built up along the lines discussed in this chapter.[46, 47]

PROBLEMS

5·1. A pin-jointed truss is loaded as shown.

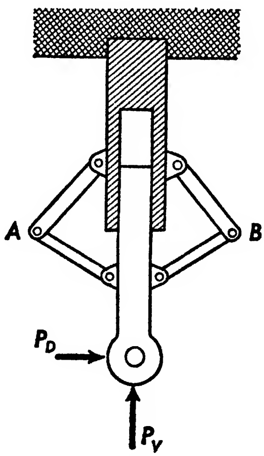
(*a*) Is this structure determinate or indeterminate?
(*b*) If member *AF* is removed, will this truss collapse?
(*c*) If member *DF* (only) is removed, will this truss collapse?



PROBLEM 5·1.

5·2. The structure illustrated represents an elementary shock absorber installation for an airplane. The cylinder may be regarded as capable of resisting axial load, through internal pressure.

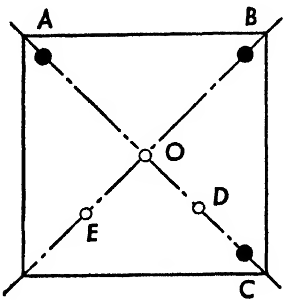
(*a*) Is the structure determinate under vertical load, P_V?
(*b*) Is it determinate for drag load, P_D?
(*c*) Assume that the piston is free to move in the cylinder and that an additional member *AB* is installed. Is the arrangement now redundant? (*Note.* Member *AB* not shown.)



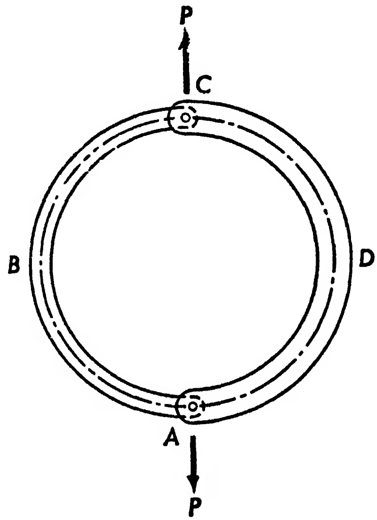
PROBLEM 5·2.

5·3. The figure shows a top view of a square table having only three legs (A, B, C). Classify the structure (legs) as *indeterminate*, *determinate*, or a *mechanism*, for the following conditions and indicate how much of the weight will be resisted by leg C in each case. (Neglect weight of table itself.)

(*a*) Weight at exact center (O).
(*b*) Weight directly over leg B.
(*c*) Weight at D (halfway between O and C).
(*d*) Weight at E.



PROBLEM 5·3.

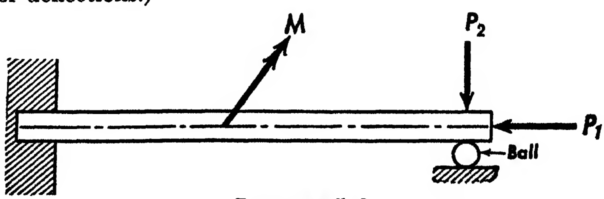


PROBLEM 5·4.

5·4. In the structure shown, either of the members may be removed without destroying the load-carrying capacity. Member ADC is considerably more rigid than member ABC. Is the structure redundant? Could it be analyzed by the principles of static equilibrium if both members were of the same size and stiffness?

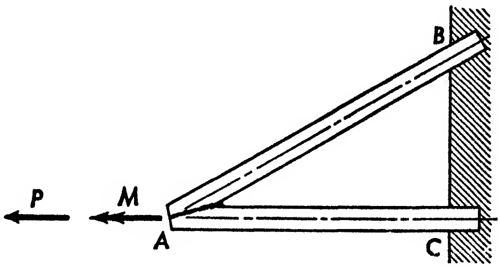
5·5. In Problem 5·4 assume that each of the curved members has the same stiffness and that a straight member AC is added. Is the structure statically indeterminate? Classify the structure with one curved member removed, i.e., one straight and one curved member remaining.

5·6. Is the structure determinate under the loading condition shown? (Neglect effect of deflections.)



PROBLEM 5·6.

5·7. The tubes are welded together at A and are completely fixed at B and C. To what extent is the structure statically indeterminate when loaded as shown?

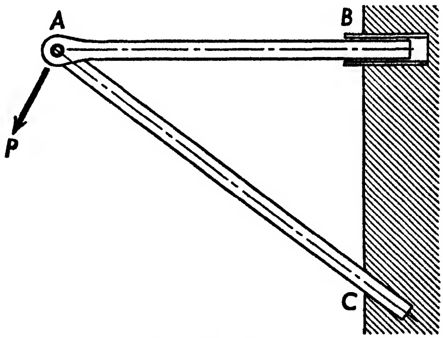


PROBLEM 5·7.

5·8. In Problem 5·7 assume that joint C is changed to a ball-and-socket joint. Is the structure statically indeterminate, determinate, or a mechanism? Describe the loading condition for member AB.

5·9. Assume that the structure of Problem 5·7 has a universal joint at B, a ball-and-socket joint at C, and no support at A. What will happen under the loading condition shown?

5·10. Member AB is free to slide at B but can carry bending. Is the structure determinate?



PROBLEM 5·10

CHAPTER 6

BEHAVIOR OF MATERIALS UNDER LOAD * (PHYSICAL PROPERTIES)

6·1. Tension Stress. To the structural engineer, the behavior of materials under tension is the most useful source of information available. This information is usually obtained by pulling a test specimen and measuring the applied forces and resulting stretch.† Since structural literature is replete with information on the tension test, stress-strain diagrams, and other similar subjects, the following discussion will be presented as a brief review of material which is essential to further understanding of structural phenomena.

To arrive at a standard basis for comparing different materials engineers have adopted the terms *stress* and *strain*. Since there is often some confusion as to the exact meaning of these terms, the following definitions will be strictly adhered to throughout this text.

Stress ‡ may be defined as the **unit force, or force per unit of cross-sectional area.** It is obtained by dividing the axial (tension) force by the area of the cross section *normal* to the force, as indicated by the following equation:

Axial stress

$$\boxed{f = \frac{P}{A}} \qquad [6\cdot1]$$

where f = stress.§
P = force.
A = cross-sectional area.

* This chapter is included only as a basis for further development of force transmission principles. It is not intended to cover the entire subject of failure of materials.

† The technique involved in such tests has been highly refined and standardized. For further information see the publications of the American Society for Testing Materials (ASTM). For a complete discussion of the tension test see Ref. 8.

‡ In some texts stress is more broadly used to include the *total* force; here its use is confined to *unit* force.

§ The Greek letter *sigma* (σ) is usually used in the theory of elasticity; in engineering work the letter s is sometimes used.

Since the force is usually measured in pounds and the area in square inches, stress will usually be measured in *pounds per square inch*, which will be abbreviated in this text to psi.*

For example, if the tension member of Fig. 6·1*a* has a cross-sectional area of 2 sq in. and is subjected to a force of 40,000 lb, the axial stress will be

$$f = \frac{P}{A} = \frac{40,000}{2} = 20,000 \text{ psi} \quad (\text{or } 20 \text{ ksi})$$

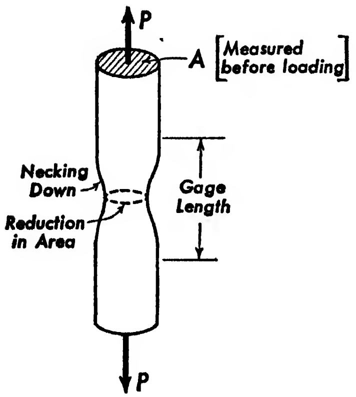


FIG. 6·1*a*. Tension specimen under load.

Some observations on tension† stresses and the tension test should be carefully noted.

a. The cross section for which A is measured must be *normal* to the direction of the axial force (see Fig. 6·1*a*).

b. The *distribution* of the force over the cross section is assumed to be *uniform*, i.e., Eq. 6·1 gives the *average* stress.

c. The area A is usually measured *before* loading; reductions or "necking down" during loading are then not accounted for.

d. The *length* of the member does not enter into the calculation.

6·2. Ultimate Tension Stress. Most tension tests are carried to the point where the specimen breaks. If the *maximum* force exerted (P) is divided by the *original* cross-sectional area (A) at the point of failure, the resulting stress is a measure of the strength of the material in tension. In fact, this maximum stress is often referred to specifically as the tensile strength, but it is more accurate to call it the *ultimate tensile stress* (or *ultimate tension stress*). Tensile strength might be misunderstood to be the total strength of a member in pounds.

The *ultimate tension stress* is used directly in determining the required size of a tension member. It is also considered to be the most useful indication of the *general* strength characteristics of a material. However, modern structural design requires a much more extensive knowledge of the material properties, and it is therefore necessary to establish

* Because of the large numbers usually involved in measuring loads and stresses it is convenient to use a larger unit of measurement. In British practice the *ton* is used sometimes. Recent U.S. trend is toward the *kilopound*, abbreviated kip, which is a unit of 1000 lb. The corresponding stress unit is *kilopounds per square inch*, abbreviated ksi.

† For consistency the word *tension* will be used in place of the more common *tensile* and *compression* will be used instead of *compressive*.

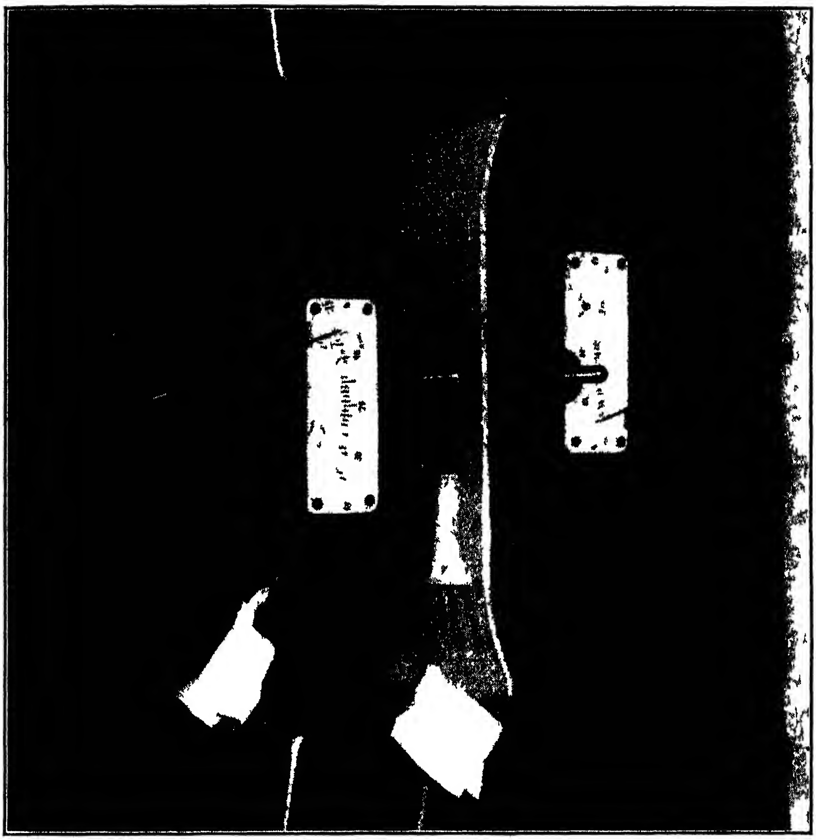


FIG 6 1*b* Tension test of a wood specimen Note the strain gages for measuring elongation. The wires lead to a resistance type of gage cemented to the specimen.

other criteria that may also be obtained from the tension test. Most of these are associated with the *stress-strain diagram,* which is of utmost importance in structural engineering.

6·3. Tension Strain. Under tension loading, a member will stretch or elongate. Since the total amount of stretch will depend on the length of the member, it is necessary to establish a unit of measurement that is independent of the length. This unit is the **strain,** defined as the *elongation per unit length of the member.*

Axial strain $$e = \frac{\Delta L}{L} \qquad [6\cdot 2]$$

where ΔL = the change in length (read "delta L").
L = the length over which ΔL is measured.

For example, if the member shown in Fig. 6·1 is 10 in. long (between points of measurement) and elongates 0.01 in. in this distance, the *strain* would be

$$e = \frac{\Delta L}{L} = \frac{0.01}{10} = 0.001$$

Some observations on *strain* follow.

a. Strain is non-dimensional; it is expressed as a fraction or a percentage, as it is obtained by dividing quantities having the same units of measurement. However, it is sometimes written *inches per inch.*

b. Although the primary effect of length is eliminated by using non-dimensional strain units, *local* effects often make it necessary to know over what actual length the elongation measurements were taken. This length is called the **gage length.**

c. For uniformity, a 2-in. gage length has been adopted as standard in comparing materials. However, other gage lengths are often employed.

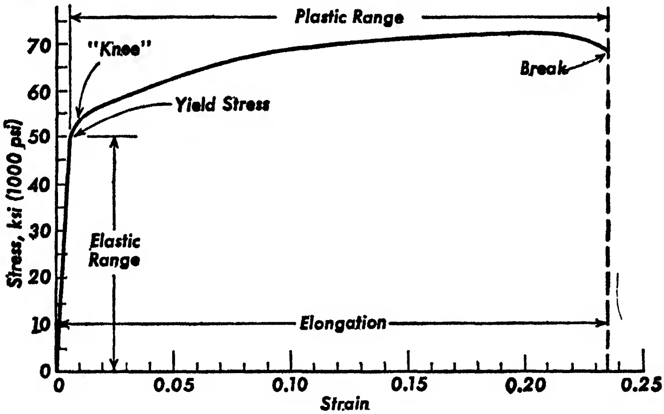


FIG. 6·2. Tension stress-strain diagram for aluminum alloy (24S-T).

6·4. The Stress-Strain Diagram. If simultaneous readings of stress and strain are taken during a tension test it becomes possible to illustrate the relationship between these two quantities by means of a graph. Such a graph is called a **stress-strain diagram.** A typical diagram is shown in Fig. 6·2. (Stress is usually plotted vertically.) The most important feature to note is that the diagram is divided into two fairly well-defined parts. The first part consists of a portion in which the stress increases very rapidly with strain. This portion is usually referred to as the *elastic* range and is characterized by the substantially constant ratio between stress and strain, i.e., by a *straight line.* The fact that most strong materials show a constant ratio between load and deflection was first established by Hooke and is commonly referred to as *Hooke's law.* The *elastic* range is of greatest interest to structural engineers because it is only within this range that a structure may be loaded and unloaded without permanent deformation.

The remaining portion of the diagram is referred to as the *plastic* or *inelastic* range. In this portion the material undergoes a great deal of elongation (strain) without a corresponding increase in stress. The

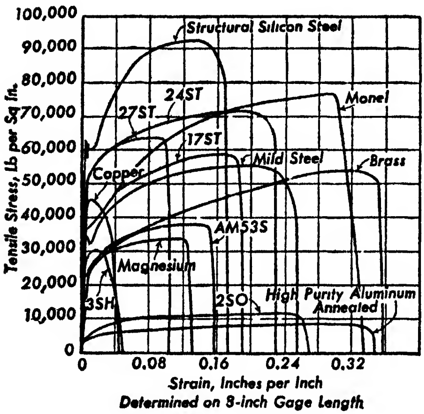


FIG. 6·3. Typical complete stress-strain curves for several metals. (From Templin and Sturm, "Some Stress-Strain Studies of Metals," *Journal of the Aeronautical Sciences*, March, 1940.)

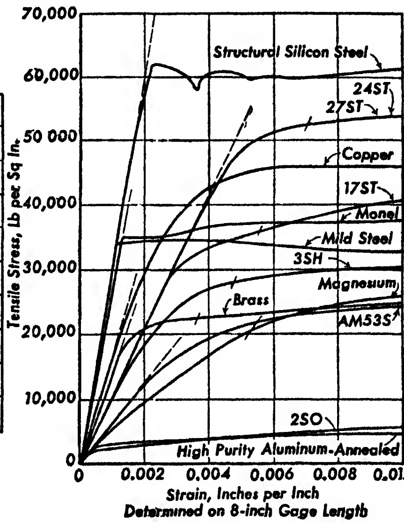


FIG. 6·4. Typical stress-strain curves for metals shown in Fig. 6·3 (from Templin and Sturm).

proportionality between stress and strain does not hold in this range, i.e., the diagram departs from a straight line. The plastic range is of greatest interest in operations involving *forming* of the material (bending, drawing, etc.), but it is also important to the structural engineer in dealing with *columns, stress concentrations, energy absorption,* and similar matters. Stress-strain diagrams for various materials are shown in Figs. 6·3 and 6·4, from Ref. 10.

6·5. Modulus of Elasticity. In the *elastic* range the most important characteristic of a material is the ratio between stress and strain. This is called the modulus of elasticity and is expressed by the equation

Modulus of elasticity $$\boxed{E = \frac{f}{e}} \qquad [6\cdot3]$$

E is also called Young's modulus, after the scientist who is credited with determining it for the first time. The modulus of elasticity does not refer to elasticity in the everyday sense but rather to the *stiffness*. Thus a so-called elastic material like rubber has a very low value of E, as compared with steel. E is in fact a measure of a material's *resistance to deformation*. The term elastic implies that the deformation is *not permanent*. High values of E are obviously desirable in structural work to reduce deflections under load.

Since f is in terms of *stress* (psi) and e is non-dimensional, the value of E is also expressed in terms of stress. Another interesting observation can be made by assuming that a member has been stretched to *twice* its original length. Then the value of e would be 1.0, and E would equal the stress required to produce this elongation. Therefore E can be thought of as the stress which would be required to double the length of the member, assuming that the entire action is *elastic*. Actually, no metal will meet this condition, as the plastic range is reached at a relatively low value of strain, usually less than 1 per cent, and the specimen will break at a strain seldom exceeding 20 per cent. However, the idea will help explain the very high values of stress represented by E (about 30,000,000 psi for steel; 10,000,000 psi for aluminum alloy).

This conception is also useful in calculating elongations. Assume that a member 60 in. long is subjected to a stress of 20,000 psi. How much does it stretch? To get the answer we must know E. If the member is steel, we can use a value of about 30,000,000 psi for E. Thus, if a stress of 30,000,000 psi were applied, the (fictitious) elastic elongation would be equal to the original length, or 60 in. Since the actual applied stress is only 20,000 psi, the elongation is equal to

$$\frac{20{,}000}{30{,}000{,}000} \times 60 = 0.04 \text{ in.}$$

In more general terms, the equations for deflection *within the elastic range* are expressed as

$$\Delta L = eL \qquad [6\cdot4a]$$

Substituting for E from Eq. 6·3:

$$\Delta L = \frac{f}{E} L \qquad [6\cdot4b]$$

Substituting for f from Eq. 6·1:

$$\Delta L = \frac{PL}{AE} \qquad [6\cdot4c]$$

These are simply different forms of the same equation and express mathematically the conception just given above. If a clear understanding of *stress, strain,* and *modulus of elasticity* is obtained, it is unnecessary to remember any particular equation for deflection, but Eq. 6·4*c* is so convenient that it is well to memorize it.

6·6. Proportional Limit (Elastic Limit). It is important to know at what value of stress or strain the material begins to enter the plastic range, i.e., at what point the stress-strain diagram begins to depart from a straight line. From a scientific viewpoint this is a difficult problem, as much depends on the sensitivity of the measuring instruments. From an engineering viewpoint the *exact* point of departure, if there is one,* is not too important. The structural engineer is therefore not very much concerned with the exact value of the *proportional limit,* although it is well to have a good understanding of the *shape* of the curve in this vicinity. The proportional limit is shown on Fig. 6·5. This point is sometimes referred to as the **elastic limit,** which, strictly speaking, means the maximum stress that can be withstood without causing *permanent* deformation. It can be seen that these two terms are almost synonymous, from an engineering point of view.

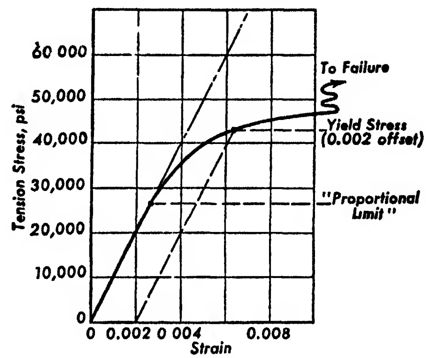


FIG. 6·5. Stress-strain diagram (partial) for Alclad 24S-T aluminum alloy.

6·7. Yield Stress. In certain types of steels there is a definite stress at which the material abruptly begins to yield, that is, to depart from the straight line stress-strain diagram (see Fig. 6·4). But in most materials this departure is so gradual that the *proportional limit* is not practical as an engineering quantity. To take care of this, an arbitrary yield stress † has been established. This is done by selecting some standard value for the departure of the curve from the straight line. A common value used in engineering work is a strain of 0.002. This determines a definite point on the curved part of the stress-strain diagram (see Fig. 6·5). The stress so determined is called the **yield stress.**

* For some materials there seems to be no true elastic range, i.e., the stress-strain diagram has a slight curvature from the very beginning.

† The term *yield stress* is used here in preference to *yield strength,* as the latter could refer to the *total* strength. The term *yield point* should be reserved for materials which yield abruptly.

It is located on the curve by drawing a straight line through a strain value of 0.002 and parallel with the straight line portion of the curve.

From Fig. 6·5 it may appear that the 0.002 yield stress permits a considerable portion of the *plastic* range to be included in the *elastic* range. Actually, the figure is quite misleading, as the scale for strain is greatly magnified. To give a better picture of this it is necessary to show the entire curve, to failure, as in Fig. 6·2. Note how the details of the curve near the yield stress become relatively insignificant when the curve is so plotted.

6·8. Permanent Set. Most materials exhibit permanent set after having been loaded beyond the proportional limit. The return or unloading portion of the stress-strain diagram tends to be parallel with the original straight portion. For instance, if the material represented in Fig. 6·5 were stressed up to the yield stress and no farther, the return to zero stress would be very closely represented by the straight dashed line. Thus it is reasonable, but not strictly accurate, to consider the 0.002 *yield stress* as that stress at which a permanent set of 0.002 (inches per inch) is obtained.

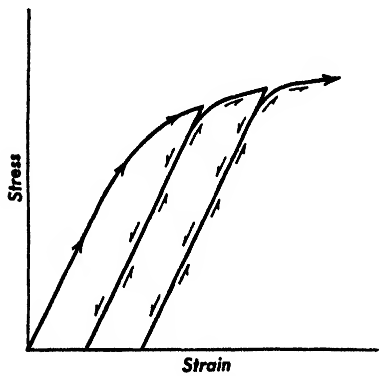


FIG. 6·6. Effect of successive loadings.

Even after successive loadings this relationship for permanent set appears to hold, as indicated by Fig. 6·6. This shows the stress-strain diagram for a specimen which has been loaded and unloaded several times, the stress being made higher each time.[10]

Note that a metal that has once been permanently stretched, i.e., stressed beyond its proportional limit, has a new stress-strain diagram. The new proportional limit and yield stress are both higher than before, and the knee of the diagram becomes more acute. This is of considerable importance in dealing with cold-worked materials. In Fig. 6·6 each successive repetition of loading creates a new stress-strain diagram.

6·9. Elongation. Figure 6·2 illustrates what is meant by **ultimate elongation.** It is simply the *total strain* at failure. In general, the term elongation, used by itself, means *ultimate* elongation; the term *strain* is used when discussing conditions prior to failure. Elongation is important to the engineer, as it is a measure of the amount of deformation that can be withstood before failure occurs. It also plays an important part in the ability of the material to absorb energy, as the area under the stress-strain diagram is a measure of the energy required to produce failure.

It should be remembered that elongation, being a strain, must be measured over some definite gage length and then reduced to a *unit* value by dividing by this gage length. Much depends on the length used, as the distribution of strain, beyond the yield stress, may not be uniform over the length of the member. For instance, a certain metal might exhibit an elongation of 12 per cent, measured over a 2-in. gage length. If the measurement were made over a length of ¼ in. at the point of incipient failure, the elongation might be as high as 30 per cent. This phenomenon is closely associated with necking down, which is indirectly measured by the *reduction of area*. (Shown in Fig. 6·1.)

6·10. Tangent Modulus and Secant Modulus. Modern structural theory (particularly as applied to airplanes) makes considerable use of the tangent modulus of elasticity. The secant modulus is also employed occasionally. These two modifications of the classical modulus of elasticity apply in the *plastic* range and are illustrated in Fig. 6·7. The elastic modulus E is, of course, illustrated by the slope of the straight portion of the diagram. The tangent modulus E_t is obtained by drawing a tangent to the diagram at the point under consideration. This gives the *local* or *instantaneous* rate of change of stress with strain.

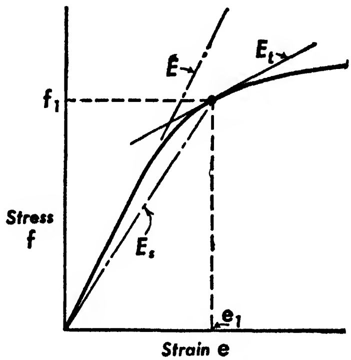


FIG. 6·7. Tangent and secant moduli.

The secant modulus E_s is illustrated by drawing a secant (straight line) from the origin to the point in question. This modulus measures the ratio between stress and actual strain. Mathematically, the two moduli are given by the equations

$$E_t = \frac{df}{de}^{*} \qquad [6\cdot 5a]$$

$$E_s = \frac{f}{e} \qquad [6\cdot 5b]$$

Note that E, E_t, and E_s are all identical below the proportional limit. The tangent and secant moduli may be thought of as extensions of the

* Read "derivative of f with respect to e." This is a symbol borrowed from calculus to indicate rate of change.

elastic modulus E into the plastic range; the two different forms result from different ways of defining E. It is apparent that the values of E_s and E_t will vary with stress, both becoming lower (in general) as the stress is increased beyond the proportional limit. By selecting successive values of stress (or strain) on a given stress-strain diagram and determining the moduli at each point it is possible to obtain data from which curves of *tangent* and *secant* modulus can be plotted against stress. Typical curves are shown in Refs. 11 and 12. (See also Chapt. 19.)

6·11. Lateral Contraction. One more physical characteristic of a material is needed to complete the general engineering picture. When a tension specimen is stretched, it undergoes a lateral contraction, even before the necking down stage is reached. Conversely, when the material is compressed it expands laterally. There is usually a constant relationship between the lateral and axial deformation and it is expressed as Poisson's ratio, usually denoted by μ (Greek letter mu). If e is used to denote the strain in the direction of loading and if e_x denotes the strain at right angles to the direction of loading, Poisson's ratio is given by the equation

Poisson's ratio

$$\boxed{\mu = \frac{e_x}{e}} \qquad [6\cdot 6]$$

It should be remembered that tension causes contraction and vice versa. The value of μ might therefore be thought of as negative, but it is ordinarily given as positive. Most metals have about the same value (approximately 0.30). Some materials, such as cork, have a very low value. (In fact, corks are used as stoppers largely because they can be compressed without undergoing any appreciable lateral expansion.)

Poisson's ratio applies only in the elastic range; in the *plastic* range the ratio of lateral to axial strain is higher.[13]

6·12. Endurance Limit. This important physical property of a material pertains to the effect of *repeated* stresses and indicates the maximum allowable stress under such conditions. At this point it will only be noted that the other properties already described apply primarily to *static* conditions or relatively slow rates of loading. If the structure is subjected to vibratory loads or impact conditions it may be necessary to use considerably lower strength properties in the design. This is an important branch of structural analysis which is beyond the scope of this text. Reference 14 contains much valuable information.

6·13. Compression Stress. Most of the discussion of tension stress and strain may also be applied to compression stresses. The same basic equation for stress may be used ($f = P/A$) and a similar type of stress-strain diagram is obtained from actual tests.* The modulus of elasticity (E) is basically the same as for tension. For metals, the stress-strain diagram will have a knee and a plastic range, the values for *proportional limit* and *yield stress* being practically the same as for tension. (Any differences are usually due to cold-working effects, rather than to

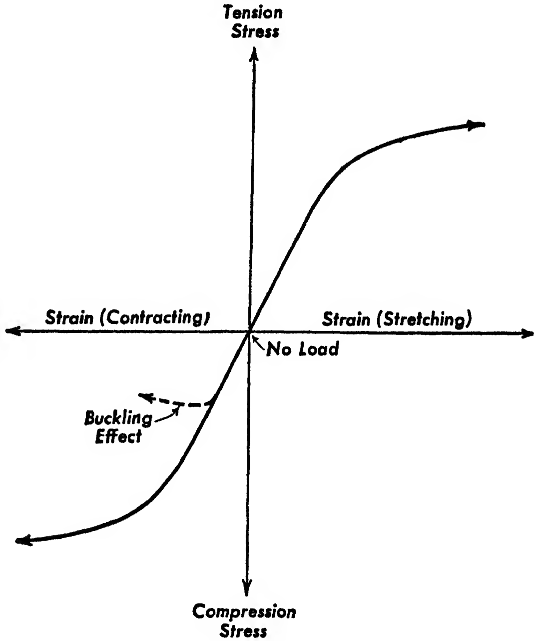


FIG. 6·8. Tension-compression stress-strain diagram.

fundamental differences in the behavior of the metal under tension and compression forces.)

The most important difference between tension and compression members is to be found in the manner of failure. In tension the material first *yields* and finally *breaks*, by actual separation of the part. In compression the type of failure depends on the proportions of the specimen. A long slender rod, for instance, may buckle before the stress even exceeds the *proportional limit*. Short heavy members may exceed the *yield stress* appreciably before failure as a column occurs. Blocks, in which no column action can occur, do not have any well-defined failing stress, as the material tends to spread out or pancake as the load is increased.

* Because of the difficulty of making compression tests on thin sheets, most physical tests of materials have been made in tension.

In the weaker materials, such as wood and concrete, compression failures are usually caused by shearing action, not by direct compression.

A compression member may also fail by local buckling. Such failures are usually experienced in members having thin walls which collapse before the member as a whole begins to fail.

Compression failures will be discussed in greater detail in a subsequent chapter. At this time it is necessary only to recognize the fact that several different types of failure are possible and that it is therefore impossible to assign a single value of allowable stress for a material in compression.

By considering compression stresses as negative and plotting them below the horizontal axis it is possible to show, on a single diagram, the entire stress range. Such a diagram is illustrated in Fig. 6·8. Note that the *elastic* range for both tension and compression is given by a single straight line (same value of E).

On Fig. 6·8 a dotted line has been included in the compression diagram to show a possible effect of buckling. This line might occur at almost any point on the diagram, depending on the dimensions of the part. However, no such limitation exists in the *tension* range, as buckling under tension is impossible. (Wrinkles in thin tension members are due to compression strains acting normal to the tension strain.)

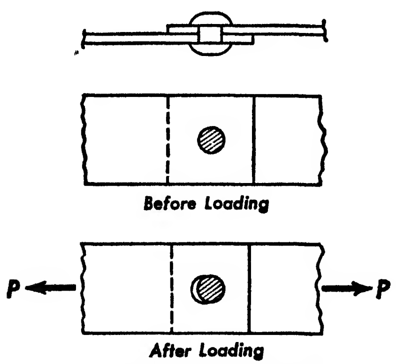


FIG. 6·9. Bearing failure.

6·14. Bearing Stress. Joints and fittings subjected to highly localized loads offer a special problem in compression loading. Figure 6·9 indicates how a bolt or rivet may elongate a hole. To prevent this, the local pressures must be kept below certain values, which are called *allowable bearing stresses.* This stress is really the *average* intensity of loading on the projected area of the member, normal to the direction of loading. Thus, in Fig. 6·9, the area would be equal to the product of rivet diameter and sheet thickness (one sheet). The bearing stress is calculated from the basic formula

Bearing stress $$f_{br} = \frac{P}{A} \quad [6\cdot7]$$

where P = the load on the member.

A = the projected area normal to the line of action of the load.

The allowable bearing stress is usually at least as high as the ultimate tension stress and frequently is considerably higher. Since the failure is not a true failure but a rather complex combination of axial yielding, buckling, etc., the allowable bearing stress is not a true strength property and will depend to some extent on the sheet thickness, rivet or bolt diameter, and other variables.

In wood the allowable bearing stress is greatly affected by the angle which the load makes with the grain.[24, 43]

6·15. Shear Stress. The failure of a material under load is actually quite complex and involves two basic types of failure, which may be thought of as *separation* and *sliding*. Every solid material has a resistance to being pulled apart; likewise it resists sliding action within itself. The true nature of internal failure cannot be taken up in detail at this point, but it should be noted that the ultimate shear stress is an important mechanical property which is a measure of a material's *sliding resistance*.

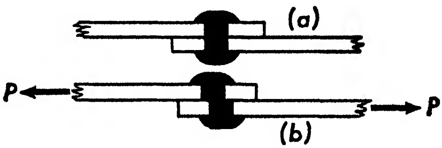


FIG. 6·10. Shear failure of rivet.

The type of loading involved is illustrated by a rivet or bolt, as in Fig. 6·10. The loading on the rivet is *transverse* (see Chapt. 3) and the area involved is that of a cross section *parallel* with the load, instead of normal to it. Figure 6·10*b* shows how the rivet is forced to fail by sliding, under a force P. If the cross-sectional area of the rivet is A, the *average* shear stress over the area of failure is given by the basic formula:

Average shear stress $$f_s = \frac{P_s}{A} \qquad [6\cdot8]$$

where f_s = the average shear stress.*
P_s = the force *parallel* with the cross section in question.
A = the area of the cross section in question.

By testing materials under conditions approximating those of Fig. 6·10 values for the *ultimate shear stress* are obtained. They are generally about half the magnitude of the ultimate tension stress, but this ratio may vary considerably. (For instance, the ultimate tension stress for a typical aluminum alloy is 62,000 psi and its ultimate shear stress is 35,000 psi.) †

* In the theory of elasticity the Greek letter τ (tau) is used for shear stress.

† The shear stress distribution for this type of loading is not actually uniform; hence the *true* ultimate shear stress is somewhat higher than the *average* value obtained in such tests.[15]

It should be noted particularly that no *length* is involved in true shear failure. In Fig. 6·10, for instance, the length, if any, would be the distance between the two plates, which are actually in contact with each other. If the plates were separated, so that the force P had to be transmitted (transversely) over a finite distance, other types of stress would be introduced and the failure would no longer be that of pure shear.

Similarly, since no length is involved, *there can be no stress-strain diagram for pure shear,* as thus defined. It follows that there is no true modulus of elasticity in shear. However, a *modulus of rigidity* (G) is often employed to calculate deformation under transverse loading or torsion. It will be shown later that this has no connection whatever with the phenomenon of *sliding* action and is in fact entirely dependent on the values of modulus of elasticity (E) and Poisson's ratio (μ), the two basic stiffness characteristics under *axial* loading. It is also possible to draw a shear stress-strain diagram on this same basis, but it is not a truly independent characteristic of a material.

For engineering strength calculations it is sufficient to know the *ultimate shear stress* for the material in question. This applies only when the failure occurs by sliding between two planes which are zero distance apart. Typical values for ultimate shear stress are given in Appendix 2. (See Chapt. 10 for further discussion of shear.)

6·16. Temperature Effects. Temperature affects both the strength and the strain characteristics of a material. Obviously the strength disappears entirely when the *melting point* is reached; hence there must be an intermediate range in which the strength characteristics become poorer as the temperature rises. Structural materials are, of course, chosen so that they are very little affected by normal variations in temperature. (Most of them have high melting points.) The normal strength characteristics may therefore be assumed to be satisfactory for most applications. If the structure must function at relatively high temperatures, however, a special investigation should be made for the particular material involved.

The combination of high temperature and constant loading may cause a flow of the material, defined as **creep.** This should be kept in mind in structural design. For instance, certain *plastics* may appear to be suitable for structural use, but if their *creep* characteristics are poor the structure might gradually acquire a permanent set in hot weather.

A change in temperature causes a change in dimensions; increases in temperature cause *expansion* and decreases cause *contraction.* This change may be measured in units of strain (inches per inch). The strain caused by a one degree change in temperature is called the **coefficient of expansion.** In Fahrenheit units the following values are typical.

Steel	0.0000065 (inch per inch per degree Fahrenheit)						
Aluminum alloy	0.000012	"	"	"	"	"	"
Magnesium alloy	0.000016	"	"	"	"	"	"

6·17. Stresses Caused by Constraint. To produce a given axial strain in a material usually requires the application of a corresponding *stress.* But there are conditions under which a strain is produced without actually applying a stress in the direction of the strain. One of these is the lateral contraction effect, measured by Poisson's ratio (Sec. 6·11). The other is the *thermal* effect, measured by the *coefficient of expansion.* Poisson's ratio is effective in two dimensions (normal to the direction of the applied tension), whereas the thermal effect acts in all three dimensions, i.e., a *volume* change.

If for any reason these strains are prevented from occurring, the effect will be to induce stresses corresponding to the amount of strain which was not permitted. The physical action is clarified by assuming first that the constraint does not exist and then applying the stresses necessary to bring the structure back to the dimensions to which it is really held. The two most common cases are discussed below.

6·18. Edge Constraint. If a long flat sheet is loaded in tension in one direction, it will tend to become narrower in width. Assume that before loading occurs, the sheet is rigidly clamped along the edges by some means which will not interfere with the tension loading, but which will prevent this narrowing from taking place. The strain due to the tension load may be calculated and multiplied by Poisson's ratio to get the normal lateral strain. Now if this is to be entirely prevented, it will require a lateral stress corresponding to the lateral strain. In the elastic range this is determined by multiplying by E.

Since E was also used to determine the axial stress it cancels out, which means that the lateral stress may be determined directly by multiplying the axial stress by Poisson's ratio. In doing this, however, it should be kept in mind that Poisson's ratio does not refer to stress directly, but is a measure of strain effects.

6·19. Thermal Effects. The same general principles may be applied to determine stresses caused by changes in temperature. If a member is held rigidly so that it cannot change one of its dimensions, any change in the temperature will cause a corresponding change of stress. As in lateral constraint, however, the phenomenon is based on *strain* rather than stress. It is always safer to calculate the induced strains and then convert them into stress, using the stress-strain diagram if necessary, to take care of conditions in the plastic range.

For example, assume that an aluminum-alloy bar is rigidly clamped at each end so that its length cannot change. Referring to Sec. 6·16, the coefficient of expansion is about 0.000012 in./in./deg. F. Since E is approximately 10,000,000 psi for aluminum alloy, the change of stress per degree (F) will be

$$\Delta f = 0.000012 \times 10{,}000{,}000$$

$$= 120 \text{ psi/degree}$$

Therefore a change in temperature of 10 degrees would cause a stress of 1200 psi.

The above method may be summarized by the following formulas.

Change in strain $\quad \Delta e = K(\Delta t)$ [6·9*a*]

Change in length $\quad \Delta L = L(\Delta e) = LK(\Delta t)$ [6·9*b*]

Change in stress (rigidly restrained) $\quad \Delta f = E(\Delta e) = EK(\Delta t)$ [6·9*c*]

where Δ signifies change.
K = coefficient of expansion.
Δt = change in temperature.

Note that the change of stress is independent of length; also that the formulas do not give the actual strain or stress but only the *change* involved. To get the actual stress at a given temperature it would be necessary to know at what temperature the stress was zero (usually the temperature at which the member was installed).

Thermal effects are put to practical use in shrinking operations and have many other important applications.

6·20. Material Variations. The structural engineer should not make use of arbitrary allowable stresses without some knowledge of their relationship to the actual allowable stresses for the material involved. This is particularly true because of the different ways in which *factors of safety* are handled in different branches of structural engineering. As noted in Chapt. 1, it is possible to avoid this confusing subject to some extent by dealing with the actual behavior of the structure under load.

If the engineer were called on to predict the behavior of a specific structure he might attempt to obtain an actual sample of the material used and run a physical test to determine its properties. Usually, however, we are not dealing with an individual structure but with a whole series of them; furthermore, a certain structure might have parts made up from various runs of the material, each of which might be slightly different from the other in quality. To obtain some control over this

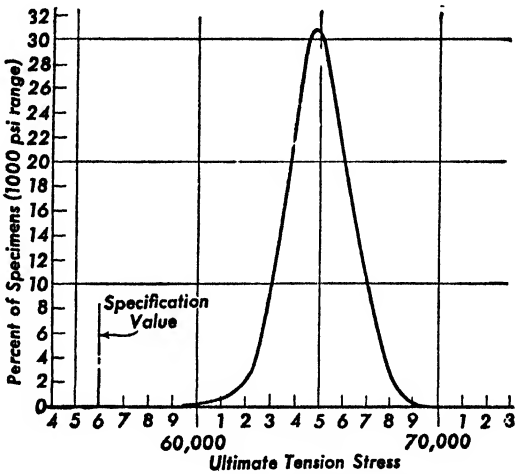


FIG. 6·11. Frequency distribution curve (Alclad 24S-T aluminum alloy, from Lockheed tests).

situation *specifications* have been adopted for all widely used structural materials. These specifications usually require the producer of the material to guarantee specific *minimum* properties, such as ultimate tension stress.

The minimum guaranteed value set up by a specification is usually selected so that most of the material produced at the mill will be accepted. The *actual* properties will therefore tend to be higher than the *specification* properties and will also vary from specimen to specimen. This situation may be shown graphically by means of a *frequency distribution* chart, such as shown in Fig. 6·11.

Such a chart shows the probability of getting a piece of material having a certain actual strength value. The chart is constructed from data obtained by making many tests over a wide range of samples and over a considerable period of time. The number of specimens falling within a certain stress range is plotted against the stress. Figure 6·11 is based on actual values obtained for Alclad 24S-T material (Lockheed tests). Note that the specification value was about 13 per cent lower than the most probable value.

It is interesting to note that the actual values of modulus of elasticity (E) show very little variation for a given material. In the strong materials, E appears to be controlled mainly by the *density*. Light metals, therefore, have relatively low values of E as compared with heavier ones. In fact the ratio between E and the density is substantially constant for many different materials (steel, aluminum alloy, magnesium, etc.).

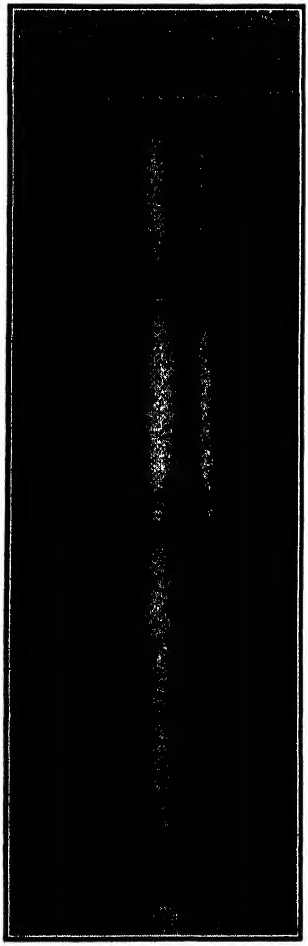


FIG. 6·12. Tension test of a flash-welded tube. Note effects of lateral contraction at point of failure.

The *yield stress* appears to be the property most severely affected by variations in fabrication processes, heat treatment, or fabrication. This is indicated also by Fig. 6·6, showing the effect of cold working (stretching) on the yield stress.

6·21. Summary of Strength Characteristics. The most important characteristics of a material under load are summarized as follows. The letter F here indicates *allowable* stress, not actual stress. (See Ref. 16.)

Modulus of elasticity, E.
Ultimate tension stress, F_u.
Yield stress, F_y.
Proportional (elastic) limit.
Elongation.
Tangent modulus.
Secant modulus.
Poisson's ratio, μ.
Endurance limit.
Allowable bearing stress, F_{br}.
Allowable shear stress, F_s.
Coefficient of expansion.

Typical values for the more important properties listed above are given in Appendix 2.

PROBLEMS

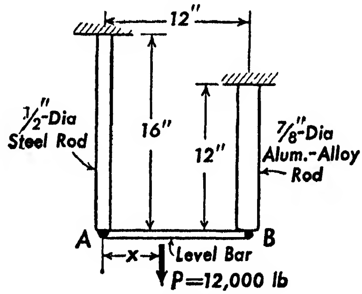
6·1. A 25-in. steel rod is subjected to a tension stress of 50,000 psi. Assuming $E = 30,000,000$ psi, find the increase in length due to this stress.

6·2. A square bar 30 in. long and ¾ in. on a side elongates 0.0533 in. under an applied tension load of 10,000 lb. Of what material is it probably made?

6·3. A 1-in. diameter steel rod 20 in. long is elongated so as to produce a tension stress of 50,000 psi at a temperature of 65° F.

(*a*) What is the new length of the member?
(*b*) If the temperature of the rod should rise to 80° F while the new length is maintained constant, what is the tension stress in the rod?

6·4. Neglecting bending in bar *AB*, where must the load *P* be located so that *AB* remains level?



PROBLEM 6·4.

6·5. Draw, on graph paper, a stress-strain diagram which will satisfy the following conditions:

a. $E = 6,000,000$ psi.
b. Yield stress (at 0.002 strain) = 25,000 psi.
c. Ultimate stress = 40,000 psi.
d. Ultimate elongation = 15 per cent.

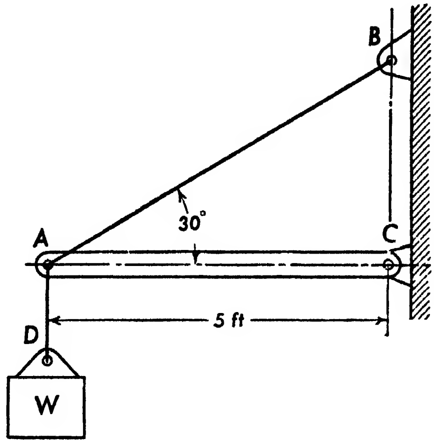
(Use Fig. 6·2 as a guide.)

6·6. Using the diagram drawn for Problem 6·5, find the stress required to produce a permanent strain of 5 per cent.

6·7. Draw a new stress-strain diagram for the material after being stretched as in Problem 6·6.

6·8. Using the stress-strain diagram of Problem 6·5, select 10 points in the plastic range and determine the tangent modulus at each point. Plot these values as a curve of tangent modulus against stress.

6·9. Select a value for *W* between 21,000 and 29,000 lb. Assume that the cables *DA* and *AB* must provide a factor of safety of 5 against failure and a factor of safety of 2 against yielding. Select a material from Appendix 2 and determine the cross-sectional area required for each cable.

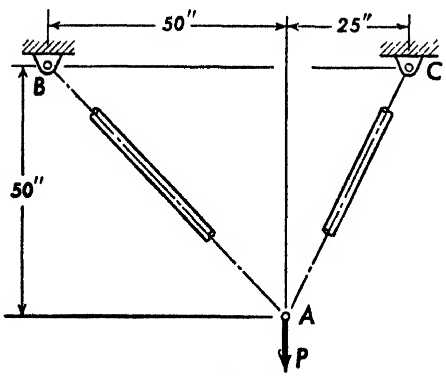


PROBLEM 6·9.

6·10. In Problem 6·9 assume that member AC is a fixed-end (cantilever) beam and that the cable DAB passes over a pulley at A. Find the size of the cable AB, for the same design conditions. Also find the bending moment, shear, and axial force at the fixed end C.

6·11. In Problem 6·10 compute the vertical deflection of the weight, neglecting the deflection of the beam AC.

6·12. Select a value for P (between 60,000 and 90,000 lb) and determine the cross-sectional areas required for the steel rods AB and AC, such that the stress in each rod will be 50,000 psi under this load. Assuming that this stress does not exceed the proportional limit, find the total elongation of each rod. (*Note.* These elongations may be used to find the deflection of point A, by finding the point of intersection of arcs drawn with the extended lengths as radii. Because of the small deflections involved, however, special graphical or analytical methods must be used.)



PROBLEM 6·12.

6·13. A heavy steel jig is used to assemble an aluminum-alloy tension member. Assume that the member is bolted rigidly to the jig at two points 30 ft apart, at a temperature of 80° F, and that the temperature later drops to 20° F. Find the stress induced in the member, neglecting the effect of its load on the jig.

CHAPTER 7

AXIALLY LOADED MEMBERS

7·1. Tension Members. Tension members are of basic importance in structural design, as they represent the most efficient method of force transmission. The most important characteristic of a tension member is its *stability*. This is illustrated by Fig. 7·1, in which the straightening effect of the tension load is shown. Figure 7·1*a* shows a tension member (such as a string) before the load is applied. Assuming that it is flexible, it will become straight under load, due to the fact that bending moments caused by eccentricities (such as e) are in such a direction as to decrease these eccentricities. This may seem to be an obvious and trivial point, but it explains the widespread use of ropes, wires, cables, and tie rods for transmitting forces. In contrast, the *compression* member tends to be *unstable*, requiring special analysis methods and excess material to prevent buckling.

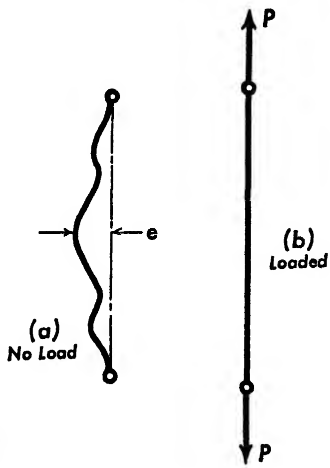


FIG. 7·1. Tension member.

7·2. Design of Straight Tension Members (Ties). The procedure of designing a *straight* tension member (tie, tie rod, cable, etc.) is so simple that very little space will be devoted to it here. (This, of course, does not apply to the design of the end connections, which is ordinarily classed under fitting design.) For a given force, P, the required cross-sectional area is determined from Eq. 6·1, as follows.

$$A = \frac{P}{F} \qquad [7\cdot 1]$$

where F is the allowable stress.

The term allowable stress requires some further explanation. If we consider it to be the *ultimate tension stress* and use it in Eq. 7·1, we will obtain a member which will break approximately when the load P is applied. (Actually there is always some variation in material properties, so it is impossible to predict the failing load with absolute accuracy.) If, however, we wish to design a member which will not exhibit any appreciable yielding under the load P, we must substitute the tension yield stress, F_{ty}, in Eq. 7·1.

For example, assume that a tension force of 180,000 lb is to be transmitted. If steel is to be used, we must decide on some particular type of steel and then determine its strength properties. Let us assume that X4130 steel (SAE designation for chrome-molybdenum steel) is to be used. From some standard reference book (such as Ref. 16) it can be found that the allowable tension properties for X4130 steel are:

$$F_{tu} = 90{,}000 \text{ psi}$$

$$F_{ty} = 70{,}000 \text{ psi}$$

The cross-sectional area required, if P is considered as the *breaking* load, would be

$$A = \frac{180{,}000}{90{,}000} = 2 \text{ sq in.}$$

If the tie rod is to be capable of carrying the load of 180,000 lb without appreciable *yielding*, we must use the yield stress to determine the required area.

$$A = \frac{180{,}000}{70{,}000} = 2.57 \text{ sq in.}$$

Assume that the same force of 180,000 lb is to be carried by an aluminum-alloy tension member for which the allowable stresses are

$$F_{tu} = 56{,}000 \text{ psi}$$

$$F_{ty} = 37{,}000 \text{ psi}$$

The cross-sectional areas required would be

(1) For no failure $A = \dfrac{180{,}000}{56{,}000} = 3.22$ sq in.

(2) For no yielding $A = \dfrac{180{,}000}{37{,}000} = 4.87$ sq in.

7·3. Curved Tension Members. Under certain conditions a *curved* member may be subjected to pure tension. A common example is found in the circular *hoop* subjected to uniform outward-acting radial loads, Fig. 7·2. Such a hoop may be found in pressure vessels of circular cross section by considering a section of unit length (1 in.) as shown in Fig. 7·2*b*. The tension force in such a strip can be easily calculated by cutting the hoop into two halves, as shown.

The total force, in a given direction, caused by a uniform pressure acting over a given area is equal to the product of the pressure and the *projection* of the area on a plane normal to the line of action of the force. In Fig. 7·2 the projection plane is shown by the dotted lines and has an area equal to D (because unit length was assumed). The total force is therefore equal to pD, where p is the unit pressure. Since this is equally divided between the two sides, the load in the strip is one-half this value, i.e.,

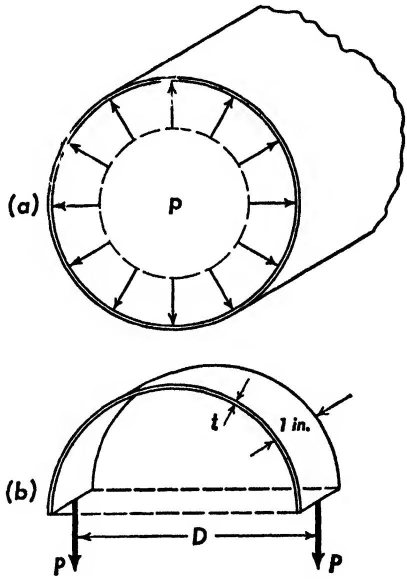


Fig. 7·2. Hoop tension.

$$P = \frac{pD}{2} = pR \qquad [7\cdot2]$$

where P = load in a strip of unit width.
D = diameter.
R = radius.
p = internal pressure.

The tension stress is now computed by applying the elementary stress equation (in which the area equals t).

$$f = \frac{P}{A} = \frac{pR}{t} \qquad [7\cdot3]$$

where f = the circumferential or hoop tension stress.
t = the thickness at the point being investigated.

The *hoop tension* formula, like most other structural theories, is based on an approximate assumption, which is that the thickness of the tension member is small in comparison with the radius. This assumption in turn permits the assumption of *uniform* tension stress in the member. If the wall is thick and the radius small, more advanced methods of analysis must be employed if accuracy is desired.

The hoop tension formula may also be used for circular *arcs*, or segments of circles, provided that the thickness t is relatively small as compared with the arc length. The basis for this is indicated by Fig. 7·3, which gives an exaggerated comparison between the condition for a complete hoop and a portion of one, i.e., an *arc*. Figure 7·3*a* indicates the expanded position of the hoop (greatly exaggerated) by a dotted line. Since the amount of expansion involved is actually very small, it is sufficiently accurate to assume that the radius of curvature is unchanged. Hence there will be no appreciable *bending* action.

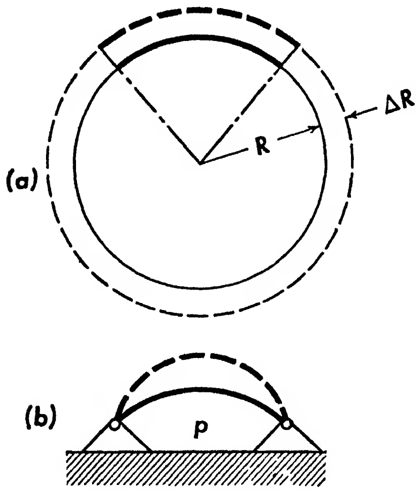


FIG. 7·3. Circular arc.

In Fig. 7·3*b* a portion of the hoop is removed and attached to fixed supports at each end. The hoop tension formula may be applied by using the radius of the arc. Deflection of the arc now causes a *decrease* in the radius of curvature, as shown. This is usually quite small and will therefore produce negligible bending moments, unless the thickness of the strip is large, in which case part of the pressure load is taken by the strip acting as a beam.

7·4. Strings (Cables). The foregoing example of *hoop tension* is a specific example of a more general class of structures in which the member transmitting the load has *no resistance to bending* (thin string, cable, chain, etc.). Such structures may be classified by the general term *string* (or cable), which is simply a member capable of transmitting only axial force. (This force must, of course, be *tension,* as the absence of bending resistance would make it impossible for the member to transmit *compression.*)

Under any given system of applied loads the string must automatically assume a configuration which will permit it to carry only tension forces. Several loading conditions are shown in Fig. 7·4. In the thin shell the cross section tends to become *circular* under uniform internal pressure (*a*). A string subjected to a constant running load (with respect to a line joining its two ends) will assume the form of a *parabola* (*b*). A cable carrying only its own weight will assume the shape of a *catenary* (*c*). A system of localized loads applied to a string will make it take the form of a *polygon,* often called a *string polygon* (*d*). The *suspension bridge* is, of course, an outstanding example of the practical application of the string theory.

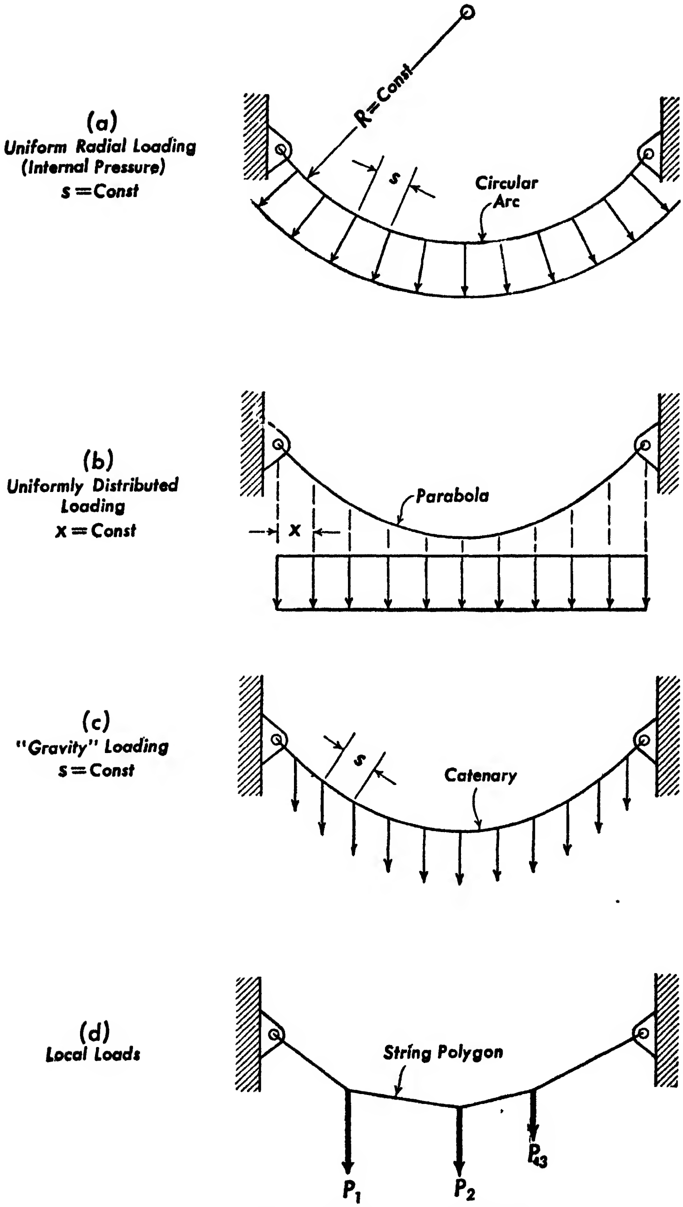


FIG. 7·4. Examples of strings or cables.

It can be seen from Fig. 7·4 that for relatively flat cables (large ratio of radius to span) the loading conditions for (*a*), (*b*), and (*c*) tend to become identical as the curvature becomes less, and the hoop tension theory may be used to give an approximate answer for them. Unless the sag is large the *parabola* will give a good approximation to the *catenary* curve of (*c*).

Any of these conditions may be approximated by a series of local loads by selecting a suitable number of points along the cable. It is then possible to solve the problem graphically by the methods described in Secs. 2·8 and 4·22. As illustrated in Fig. 7·4*d*, any system of local forces will cause the cable to assume the shape of a string polygon. (In solving such a problem the closing line between the two supports may not come out at the correct angle. This can be corrected by relocating the ordinates of the polygon, measuring from the proper base line. If a certain length of cable is involved the graphical solution requires a trial-and-error method.)

Analytical solutions have, of course, been worked out for the parabolic curve and the catenary.[6,7] For an advanced treatment of suspension bridges see Ref. 19.

7·5. Thin Shells. The *sphere* under internal pressure is similar to the *hoop* in principle. If a sphere of radius R is cut into two halves, the total load due to pressure equals $p\pi R^2$. This acts over a circumferential length of $2\pi R$. Hence the load per unit width is given by

$$P = \frac{pR}{2} \qquad [7\cdot4]$$

This is only half the value obtained for the cylinder. The physical explanation is that instead of working with a projected strip of constant width (Fig. 7·2), the projection is now *triangular,* hence has only half the area of the strip. Note that each unit square of material will be equally loaded in tension in both directions.

The remarks made for *arcs* will apply also to portions of spheres. Such structural members are often used as *bulkheads* in pressure vessels.

The *longitudinal* unit loading for a cylindrical shell under internal pressure may be obtained by the same general process. It will be found to have half the value of the circumferential (hoop tension) loading, i.e.,

$$P = \frac{pD}{4} = \frac{pR}{2} \qquad [7\cdot5]$$

It should be noted that all the formulas developed for tension will, theoretically, apply equally well for compression. Thus it is possible to use them to calculate the compression stresses in a sphere subjected to *external* pressure. In general, however, stability requirements make it necessary to use relatively thick walls, and the formulas are therefore less accurate than for internal pressure (tension stresses).*

7·6. Centroid of Area. The fundamental axial stress formula ($f = P/A$) gives the *average* stress over a cross section of the member. If the actual stress is not to exceed this value it must be *uniform*, which in turn implies a *uniform strain*. This leads to the basic assumption (often overlooked in the tension theory) that *plane cross sections remain plane and parallel during loading*. This is the physical basis for the axial stress formula and a clear understanding of it will help greatly later on when comparable assumptions are made for bending. It will also clarify problems involving different materials.

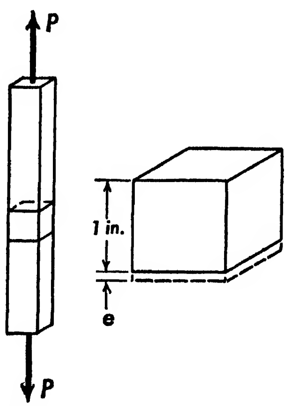


FIG. 7·5. Plane cross sections.

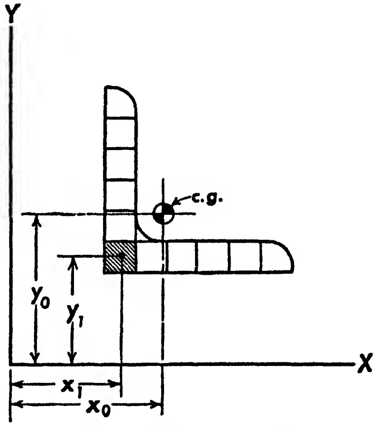


FIG. 7·6. Centroid of area.

Figure 7·5 shows a member under *tension* and gives an exaggerated picture of the change in length of a unit 1 in. long. The dotted lines indicate the new position of the cross-sectional plane. The strain e is determined by the distance between the two planes (before and after loading). Obviously the cross-sectional surface must remain plane and parallel with the original (unstressed) one, otherwise e will not be uniform. If e is not uniform, the stress f, being proportional to e, will vary over the cross section, and the basic formula will not apply.

Conversely, if the axial stress *is* uniform it will have a certain resultant, which acts at the *centroid* or *center of gravity* (c.g.) of the cross section. Methods of determining the centroids of areas are available in any engineering handbook, but it should be noted that the basic method is

* See Ref. 18 for an advanced treatment of spheres under pressure.

the same as that of determining the resultant of a number of parallel forces. Because it is so often used it will be given here in detail.

The area is divided into a number of small units, as in Fig. 7·6, and the distances from the centers of these units to two selected reference axes are tabulated. The moment of each unit about each axis is obtained and these are summed up and then divided by the total area, as shown in Table 7·1. The resulting distances locate the centroid, as shown in Fig. 7·6. (Note that if a uniform stress of unity is assumed to be acting, the areas may be regarded as forces.)

The quantity ΣAx (or ΣAy) is called the *first moment of area.* Its value will be zero for any axis drawn through the centroid of area. This is, in fact, one way of defining the centroid.

TABLE 7.1

CENTROID OF AREA

Unit Number	Area A	x	y	Ax	Ay
1 2 3 Etc.					
Sum	ΣA			ΣAx	ΣAy

$$A = \Sigma A \qquad x_0 = \frac{\Sigma Ax}{A} \qquad y_0 = \frac{\Sigma Ay}{A}$$

Since the assumption of a uniform axial stress causes the resultant axial force to act at the centroid of the cross section, it follows that the *external* force must be applied so as to act along a line through this centroid. Otherwise the axial stress will not be uniform. This is important in designing end fittings, particularly for *compression.* Note that the centroid of area will not necessarily lie within the cross section itself. (See Fig. 7·6, for example.)

7·7. Different Materials in Combination. If two or more members of different materials are compelled to act together under axial load the distribution of force between them is easily determined by applying the *parallel cross-section* assumption. If the *strain* is uniform, the *stress* (in the elastic range) will be in proportion to the moduli of elasticity E for the two materials. Thus, in Fig. 7·7, the stress in the steel portion will be approximately three times as great as that in the aluminum-alloy portion (E = approximately 30,000,000 psi for steel; 10,000,000 psi for aluminum alloy). Therefore, if one unit of aluminum-alloy area is considered as representing a unit force, the steel units must be multiplied by 3 in determining the *effective area* and *center of resistance.*

The *effective area* is the summation of all areas after multiplying the unit areas by the proper *effectiveness factor.* This transforms the composite structure to a hypothetical equivalent one of uniform material (in this figure, aluminum alloy). The stress in the *transformed* section is now obtained by the regular axial formula ($f = P/A$), where A is the *effective* area. The actual stress in each portion is then obtained by multiplying this stress by the *effectiveness factor* (in this case, 3).

The *center of resistance* will no longer be the geometric centroid of the cross section, but will be the centroid of the *transformed* section. It is readily found by the usual process, after weighting the unit areas as described above. The location of each unit of area must not be changed, however. The *center of resistance* could be compared with the *center of gravity* of a section through the structure, assuming each portion to have a weight proportional to its modulus of elasticity. Composite structures should be loaded through the center of resistance, if the axial stress in each portion is to be kept uniform. This is indicated in Fig. 7·7.

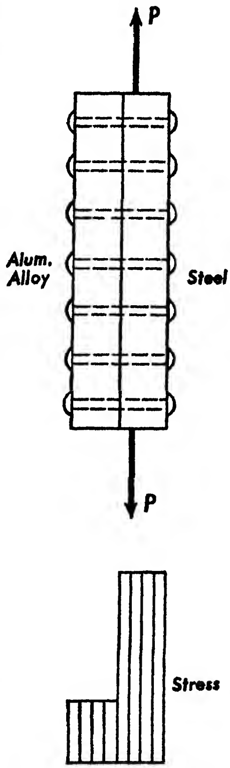


FIG. 7·7. Composite member.

7·8. Load-Deflection Curves. A more general method of handling problems involving relative deflections is that of *load-deflection,* or *load-strain* curves. Since load equals stress times area, the stress-strain diagram may be converted into a *load-strain* diagram by multiplying it by the cross-sectional area of a member. If several members are working together (compelled to have the *same strain*) the composite load-strain diagram is readily obtained by *adding* the diagrams of the various members, as shown in Fig. 7·8

This process not only takes care of different areas and values of E, but also gives the true picture in the *plastic range.* It will also show which member will break first (if the *complete* stress-strain diagram is known). At any given total load the load in each member may be found directly from the diagram, as indicated. If the stress-strain diagrams are also included on the chart, the stress can be determined in the same manner, or the load in a member may be divided by its cross-sectional area, to obtain the stress.

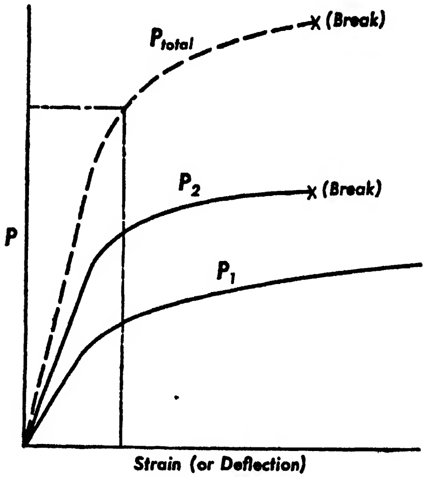


FIG. 7·8. Composite load-deflection curves.

7·9. Local Conditions. If axial stresses are not applied uniformly over a cross section, the basic assumption that adjacent cross-sectional planes remain parallel will not be true near the point of application. The determination of the actual stress distribution involves the *theory of elasticity* and often represents an advanced mathematical problem.

For instance, if a concentrated load is applied at the end of a member as shown in Fig. 7·9, the theoretical stress distribution at three different cross sections is given by (*a*), (*b*), and (*c*). (From Ref. 18, p. 50.) It can be seen that the distribution becomes substantially uniform at a distance equal to the width of the member.

This is an application of *Saint-Venant's principle*, which states, in effect, that plane cross sections remain plane even under localized loadings, *provided* that the distance away from the point of load application is at least equal to the width of the member (maximum width over which the load is to be distributed). Stated somewhat differently, the local effects of concentrated loads may be considered to vanish in a distance approximately equal to the width over which these effects take place.

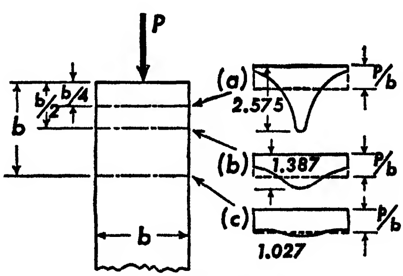


FIG. 7·9. Local effects (from Ref. 18).

One of the practical lessons to be learned from Saint-Venant's principle is that forces should be introduced into structural members in a manner reasonably consistent with the way in which they will tend to distribute themselves. Otherwise the member will tend to be overstressed at the point of application and hence cannot be as efficient as it would be otherwise. Alternatively, the regions of concentrated load application should be made stronger (heavier) than the major portions of the member. (In aircraft work it is customary to apply fitting factors for this purpose.)

An exact application of the theoretical local stress distribution is not often required in engineering work, as the materials of construction usually have a sufficiently large plastic range to permit considerable readjustment of stress before actual failure occurs. For brittle materials, or under conditions of repeated stress or impact, the elimination of local effects is of primary importance.

7·10. Stress Concentration. The foregoing discussion of local effects applies also when the *dimensions* of the member are varied locally, as in a notched bar, or a plate with a hole in it. Any abrupt change in cross section will cause a concentration of stress at the point of change. Although these increases are highly localized they are potential sources of failure. Under repeated or alternating stresses a crack may eventually form and enlarge until complete failure occurs. An abrupt change of section is often referred to as a *stress raiser*.

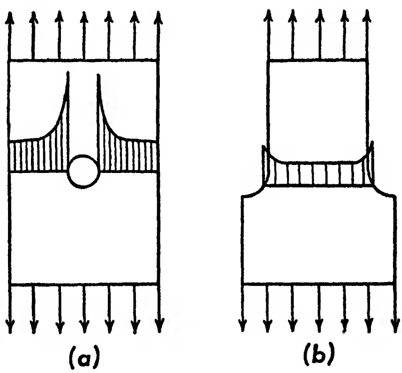


Fig. 7·10. Stress concentration.

Figure 7·10 shows the nature of stress concentration around a hole and at an abrupt change of cross section. In an infinitely wide plate a round hole increases the local stress to three times the average value. If the hole is near the edge of a plate the *stress concentration factor* (maximum local stress divided by average stress) is even greater. Figure 7·10*b* shows the increase in local stress at a fillet. The stress concentration factor may be reduced by using larger fillets.

A crack across the direction of tension stress represents an extremely powerful stress raiser; hence if a small crack once forms it is likely to grow rapidly under repeated stress. (The practice of drilling a hole at the end of a crack can be seen to be a matter of substituting a relatively low stress raiser for a much higher one.)

Further detailed discussion of stress concentration would be out of place at this point. The foregoing material has been included at this early stage to show the limitations of the common formulas for *average* stress and to emphasize the dangers of abrupt changes of cross section.

Photoelasticity is a powerful experimental method for determining local stresses in models of actual parts. Figure 7·11 is reproduced from Frocht's excellent text on this subject.[25]

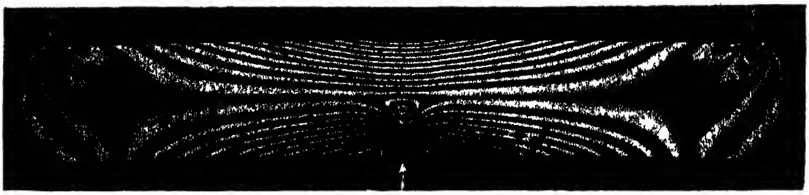


FIG. 7·11. Photoelastic stress pattern of simply supported centrally loaded beam. (Reproduced from Frocht, Vol. I, *Photoelasticity*.)

A thorough discussion of stress concentration will be found in Refs. 21 and 22. Reference 23 contains an extensive collection of stress concentration factors for various cases. Reference 14 is highly recommended to designers, as it gives many examples of the results of stress concentration and tells how these dangers may be avoided.

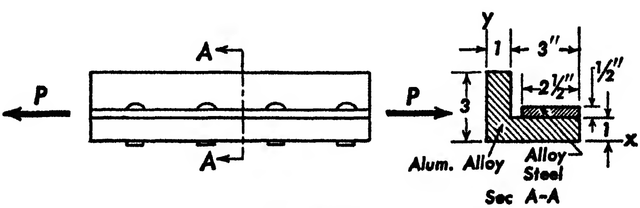
PROBLEMS

7·1. A 1-in. O.D. × 0.049 wall alloy-steel tube (not heat treated) is subjected to a 12,000 lb axial tension load. Will this tube yield? Will it break? (Use Appendix 2.)

7·2. What diameter high-strength aluminum-alloy rod is required to carry a load of 14,500 lb before breaking? (Use Appendix 2.)

7·3. For the composite member shown (use Appendix 2):

(*a*) Where must *P* be applied to produce uniform axial tension stress in each part?
(*b*) Which material will fail first?
(*c*) What load will cause the member to break?

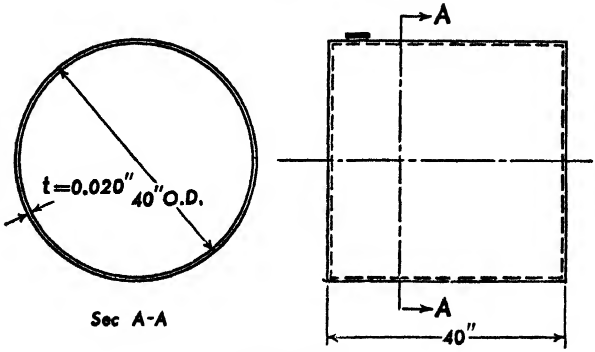


PROBLEM 7·3.

7·4. The cylindrical tank shown is subjected to an internal pressure of 12 psi.

(*a*) What is the hoop tension stress?

(*b*) What is the longitudinal tension stress?



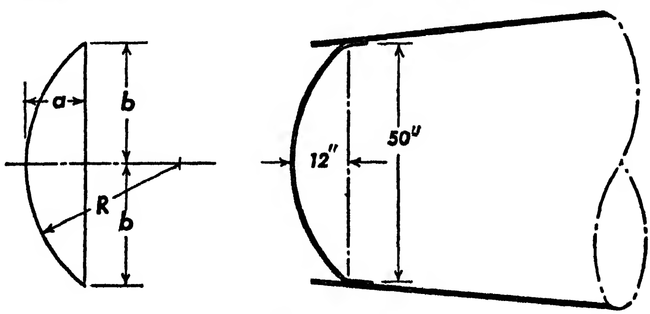
Problem 7 4.

7·5. Draw, to any suitable scale, a series of ten parallel, equal, and equally spaced forces, as shown in Fig. 7·4*b*. Construct a relatively flat string polygon (depth approximately one-fifth span), representing a cable carrying these forces. Calculate several points on a parabola passing through the lowest point on the polygon (the curves should agree). Draw a circular arc through this point to show the degree of approximation which this involves.

7·6. Repeat Problem 7·5, using a relatively deep polygon (depth approximately equal to half span).

7·7. A spherical tank 32 in. in diameter will be subjected to a working pressure of 5 psi. For safety reasons it is desired that the stress in the shell, at that pressure, should not exceed one-third of the tension yield stress for the material. Select a material from Appendix 2 and determine the minimum wall thickness that may be used.

7·8. The figure shows a cross section through a pressure bulkhead in an airplane fuselage of circular cross section. The bulkhead is of spherical shape and has a thickness of 0.050 in. Find the tension stress developed by an internal



Problem 7·8.

pressure of 6 psi. Also determine the tension force per inch width of the fuselage shell at this point and calculate the tension stress for a sheet thickness of 0.032 in.

Note. A convenient formula for the radius of curvature of a circular arc is:

$$R = \frac{a^2 + b^2}{2a}$$

where a = the maximum offset (or sag).
b = one-half the chord (or span).

7·9. Assume that a cable is stretched across a span of 100 ft and that it sags 18 ft at the midpoint. The safe tension load in the cable is assumed to be 30,000 lb. Find the corresponding safe value of uniformly distributed loading (Fig. 7·4*b*) in pounds per foot, assuming that the hoop-tension formula is sufficiently accurate. (*Note.* The linear weight of the cable would have to be subtracted from the above answer to determine the *net* loading which the cable will support.)

7·10. Assume that the distributed load found for Problem 7·9 is concentrated into a single force at mid-span and that the cable is shortened so that the sag at this point remains 18 ft. Neglecting the weight of the cable, find the tension load in the cable and compare with that used in Problem 7·9. (Concentrated loading will be more severe.)

7·11. Assume that two concentric tension members, each 100 in. long, are constructed as follows:

Member *A*	2 in. × 0.035	steel tube
Member *B*	1 in. × 0.058	aluminum-alloy tube

Draw, to the same scale and on the same chart, approximate stress-strain diagrams for the two materials. Convert these into load deflection curves and plot the composite load-deflection curve for the two tubes loaded simultaneously. From this show the distribution of load at some point in the plastic range (beyond yield stress). Refer to Appendix 1·2 for tube data.

CHAPTER 8

TRANSVERSE FORCES

8·1. The Basic Problem. Up to this point the discussion has been confined to the transmission of *axial* forces, for which *tension* or *compression* members are employed. The transmission of forces in a direction *normal* to their line of action (designated as **transverse**) is a more complicated problem. In fact, this is found to be the basic problem of almost all stress analysis because it is seldom possible to find a simple case of axial loading; almost every structural problem will be found to involve *transverse* forces of some description.

A very simple illustration of transverse forces is found in railroading. It may be desired to move a railroad car which is on a track adjacent to the main train or engine. This is done as shown in Fig. 8·1 by inserting a bar (column) diagonally between the two cars. The engine then transmits its force *axially* through the string of cars, and it is transmitted *sidewise* (transversely) to the adjacent track by the diagonal member. The important point to note is that this sidewise transmission of the force really is *axial* with respect to the bar itself.

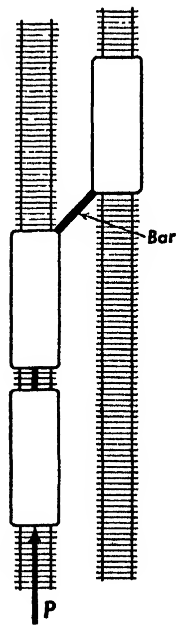


FIG. 8·1. Example of transverse force transmission.

This seemingly minor point is essential to a clear understanding of force transmission. In fact, it can be stated that the *only possible* form of force transmission through a finite distance is *axial*. This may seem contrary to previous conceptions, but it will become clearer with further examples. (Compare Sec. 6·15, on *shear stresses*.)

Another lesson to be learned from Fig. 8·1 is that it is impossible to transmit a force sidewise (transversely) without, at the same time, transmitting it some distance along its own line of action (axially). Imagine trying to move the car by using a bar that went *straight across* the tracks (normal to the cars). Even if the ends were pinned, nothing would them also to transfer the force in the desired direction. This develops the **Warren** truss, as illustrated by (*c*).

If the force to be resisted acts always in the same direction (such as a gravity load), it is most efficient to place the diagonal members of the Pratt truss so that they will be in tension. This puts the vertical members in compression. The advantage of having the shorter members in compression (instead of the longer ones) is obvious. It would then be possible to use a simple diagonal tension member such as a tie rod, wire, or cable. This type of construction is shown in (*d*), in which the dot-dash lines are used to indicate tension members, that is, members incapable of carrying compression.

If there is a possibility of the force P changing its direction at any time during the life of the structure, the construction shown in (*d*) becomes inadequate, owing to the inability of the diagonal tie rods to resist compression forces. Under these conditions it is necessary either to use diagonals which will act as columns (regular Pratt truss) or to provide another set of tension diagonals (counters) which will resist forces applied in the opposite direction. The latter form of construction is shown in (*e*). In airplane work it is often referred to as the wire-braced type. (It was formerly employed extensively in the side trusses of the airplane fuselage and in the drag bracing of the wing.)

The wire-braced type of truss may be thought of as two different trusses superimposed on each other and having common horizontal and vertical members. When the external force is reversed, the opposite set of wires or tie rods (called counter-wires) comes into action, and the others become ineffective. In the analysis of this type of structure only those diagonal members subjected to tension forces are considered. (Some refinements of this method are sometimes necessary, to account for initial rigging forces, but usually these forces may be neglected. Their effect disappears as soon as the counter-wires become slack.)

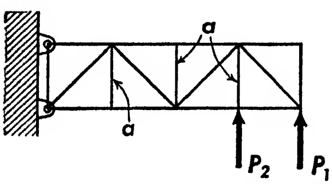


FIG. 8·5. Modified Warren truss.

Figure 8·5 shows a Warren truss, to which have been added vertical members (*a*). These members serve no useful purpose as far as transmission of the force P_1 is concerned. They may therefore be neglected in making the analysis. (The resemblance of this truss to the Pratt truss may be confusing, but the action is essentially that of the Warren truss.) The extra vertical members are usually added to reduce the unsupported length of the horizontal (flange) members, or to transmit local forces into the structure (as indicated by P_2).

8·4. Two-Way Trusses. All the trusses discussed so far have been built up from triangles. Another significant feature is that the transverse force was being transmitted in *one direction only*, that is, there was only one possible path for the force to follow. Another class of problems is encountered when the force may be transmitted in more than one direction. A simple example, which may be thought of as the bridge problem, is shown in Fig. 8·6. This shows a *symmetrical* case. It is obvious from inspection that if a symmetrical structure is used, half of the load will go to the left and half to the right To design such a structure a double triangle can be used, as shown in Fig. 8·7. The vertical member A transmits the force P to the two diagonal (tension)

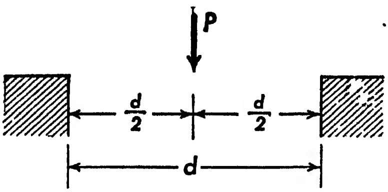


FIG. 8·6. Two-way transmission.

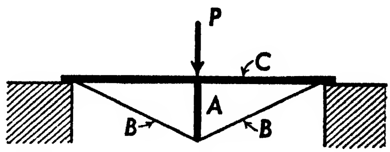


FIG. 8·7. Double triangle (king-post).

members B, each of which transmits half the force to the supporting points. A significant feature of this type of truss, called the **king-post** truss, is that the horizontal components of the two diagonal members resist each other at the apex of the triangle. At the other ends of the diagonals these components are resisted through the horizontal compression member c.

8·5. The Suspension Truss. An even simpler version of this method of force transmission may be obtained by an arrangement in which the horizontal components of the diagonal forces are resisted by the supporting points (hence transmitted through the ground or structure to which these points are attached). As shown in Fig. 8·8, this is the simplest possible method of transverse force transmission. The basic principles have already been covered. (Sec. 7·4.) The high efficiency of this structure is due to the minimum number of members, use of *tension* members for transmitting the force, and the use of the earth itself as an equalizing compression member.

The success of this structure is well demonstrated by the common use of suspension bridges for very long spans. The suspension bridge utilizes the same basic principle that is illustrated in Fig. 8·8, the difference in shape being due only to the distribution of the applied forces over the entire span, as explained in Sec. 7·4.

If Fig. 8·7 were to be inverted, the diagonal members would have to carry compression and the vertical member tension. Such a form of construction (Fig. 8·9) is less efficient but often more convenient than that shown in Fig. 8·7. (It is frequently seen in small bridges along country roads and is the basic roof truss for most small houses.)

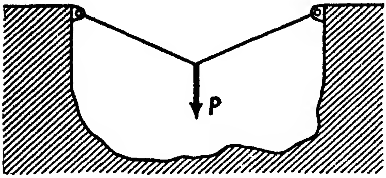


FIG. 8·8. Suspension truss.

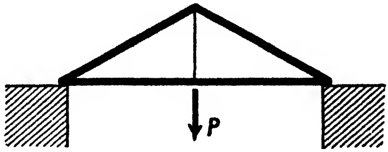


FIG. 8·9. Double triangle (compression diagonals).

8·6. The Roof Truss. By combining more than one double-triangle element it is possible to develop various other forms of trusses. One of the most useful is the Fink type, shown in Fig. 8·10. Careful examination of this will show that the force P is transmitted by the compression member A as in Fig. 8·9, while the forces P' are taken care of by two king-post trusses superimposed on the first truss, as indicated by the sketches. This idea may be carried even farther, as shown in Fig. 8·11 (compound Fink type). The efficiency of this type of truss is explained by the minimum number of compression members (shown by heavy lines in Fig. 8·10) and their short length. This, incidentally, illustrates a fundamental principle of design for light weight, which is, *compression members should be reduced to a minimum in number and length.* It follows that *tension* members should be used as much as possible.

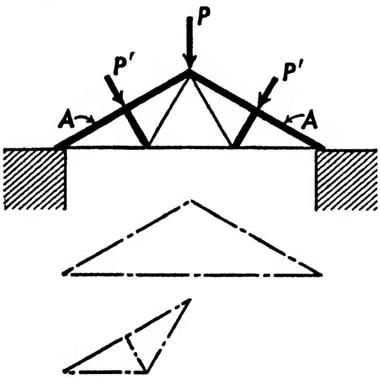


FIG. 8·10. Roof truss (Fink type).

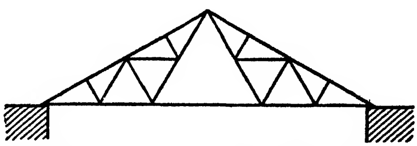


FIG. 8·11. Compound Fink truss.

8·7. Summary of Trusses. The foregoing examples show how axially loaded members may be combined in various manners to form a structure capable of transmitting forces *transversely,* i.e., in a direction normal to the line of action of the force. The analysis of trusses is based on the principles outlined in Chapt. 2. Before this subject is taken up in detail, however, other methods of transmitting forces transversely will be described.

8·8. Shear Webs. The diagonal brace wires of Fig. 8·4*e* can be replaced by a solid plate or web. Such a web will act as a series of diagonal members. (Its action will be described in Chapt. 10.) A typical web type of beam or girder is illustrated in Fig. 8·12. Vertical *stiffeners* are employed to prevent the web from buckling or to resist forces produced after buckling. If the web is relatively thick the structure is usually called a *plate girder*. The design of such beams involves structural principles not yet discussed but which will be taken up later.

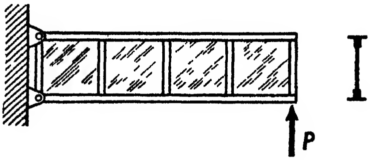


Fig. 8·12. Plate girder

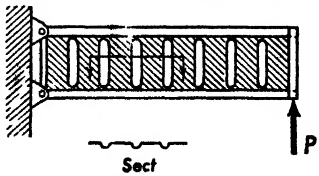


Fig. 8·13. Beaded web.

Another method of stiffening a web is to employ *beads*, as shown in Fig. 8·13. These take the place of separate vertical stiffeners.

If the beads are placed close enough together the sheet becomes *corrugated* (Fig. 8·14). Such shear webs are quite efficient for moderately heavy loadings, but require special attention in the design of the edge attachment.

A type of web often used in aircraft work is the web with *lightening holes* (Fig. 8·15). Actually the use of holes makes the web relatively inefficient as compared with a solid web of the same weight. The term lightening hole can properly be applied only when the holes are used to remove excess material in a web which is so lightly loaded that even the minimum practical thickness of material is much too strong. However, the real purpose of such holes is usually to provide access for assembly or inspection. When so used the holes should be called access holes. The analysis of a web containing holes is quite complicated; in fact, tests are usually resorted to as a means of determining the strength of such webs.

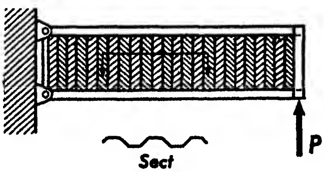


Fig. 8·14. Corrugated web.

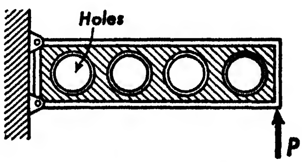


Fig. 8·15. Lightening or access holes.

The idea of lightening or access holes may be combined with the basic principle of the wire-braced truss, as shown in Fig. 8·16. Here the web is cut away so as to leave diagonal straps which act the same way as brace wires. This type of web is relatively simple to analyze and is quite efficient for lightly loaded structures.

The sheetmetal *stamping* or *hydropress type* shear web is widely used in aircraft work. In general it is one of the foregoing basic types in which all the members are formed from a single sheet in one or more pressing operations.

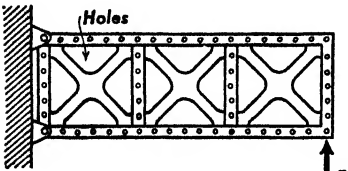


FIG. 8·16. Routed web.

Obviously, the simplest possible shear web is the *flat plate*, without vertical stiffeners or beads. The plate serves as diagonal members (carrying both tension and compression) and as verticals. If the intensity of loading is high, such a type of construction may be used; the web must be relatively thick compared with its depth, however. When so used the web is often designated as *shear resistant*, indicating that it requires no additional stiffening to be able to carry the transverse (shear) loads.

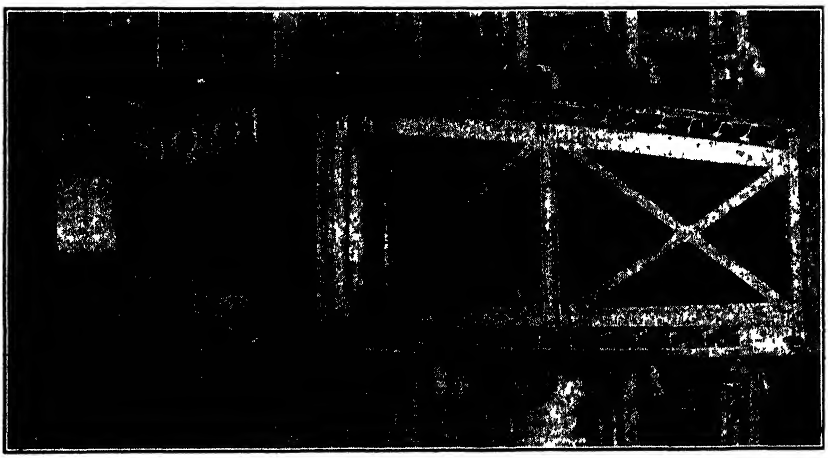


FIG. 8·17. Static test of wing rib. Note the types of shear webs employed. The loading is applied by hydraulic jacks through sponge rubber pads to simulate air loads.

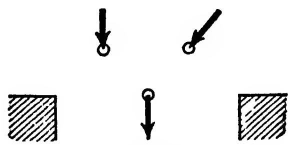
Various other combinations and ideas have been employed in the design of sheetmetal shear webs, but the basic principles of design are not essentially different from those already described.

PROBLEMS

8·1. Prove that the most efficient angle for a diagonal truss member (as in Fig. 8·4*b*) is 45°, assuming that the allowable stress for the diagonal is unaffected by the length of the member. *Procedure:* Write equations for the length of the diagonal and for the diagonal force in terms of the angle. Assume that weight of member is directly proportional to its length and cross section.

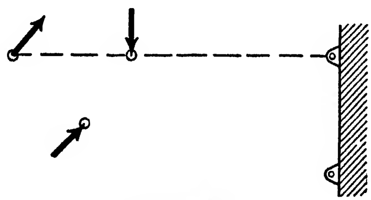
8·2. Assuming constant allowable stresses throughout, calculate the relative weights of the shear members (diagonals and verticals) for the Pratt and Warren trusses, with diagonals at 45°. (For instance, what percentage weight increase in shear material is involved in changing from Warren to Pratt truss?)

8·3. Design a truss which will efficiently carry the loads shown in the figure, using roller support at one end.



PROBLEM 8·3.

8·4. Design a cantilever truss for the loads shown, to give good efficiency for the diagonal members. (Omit consideration of flange member efficiency.)



PROBLEM 8·4.

CHAPTER 9

TRUSS ANALYSIS

9·1. Conditions at a Joint. In Chapt. 8 it was shown that the basic element of a truss is the triangle. The analysis of any truss is based on the behavior of the members of the triangle under load. The loads in the individual members are in turn influenced by the conditions at the joints. It is therefore necessary to examine the physical behavior of a joint in order to get a clear understanding of truss analysis.

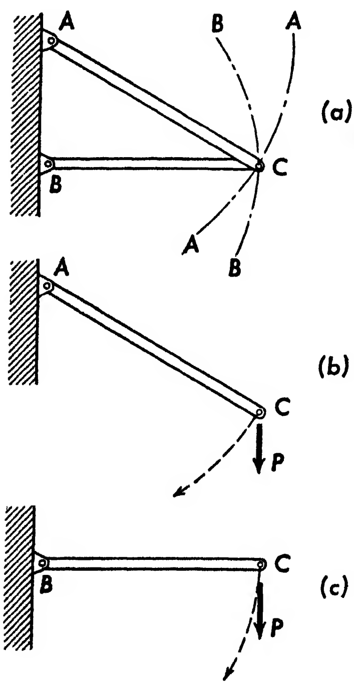


FIG. 9·1. Elements of truss.

In Fig. 9·1 the two members of the triangle (*a*) are shown separately at (*b*) and (*c*). In this condition they obviously fall into the class of *mechanisms* (see Sec. 5·4) and are unable to resist the application of a force, *P*. Since they are pinned at the ends, they will describe arcs in their motion under the action of the force. These arcs will coincide only at one point, *C*, and will diverge on either side of this point, as shown in (*a*).

Now if a pin is inserted at *C* and the force *P* is again applied the members will attempt to diverge as before, but will be prevented from doing so by the pin. More accurately, the pin, acting as a *joint*, requires the ends of the two members to move in the same path. This can take place only if one or both members *change in length*. It is this action that induces the axial forces in the members.

For example, if a rubber band were substituted for member *AC*, rotation of *BC* under the force *P* would simply *stretch* member *AC* and very little axial force could be built up in it. The arrangement would therefore be useless for transmitting large forces, as it would not hold its shape, but would rotate about point *B*. In an actual structure there will be only a slight change of length of the members and point B will therefore deflect only a small amount. (The actual deflection may be found by considering the lengths of the members under load and drawing new arcs to determine their point of intersection, as will be shown later.)

The action may be further illustrated by considering both members to be hinged at the *same point* on the wall. Their outer ends will then describe the same arc and the insertion of a pin will have no effect. Such an arrangement is obviously incapable of resisting any force which would tend to rotate the members about their point of attachment. It could resist only a tension force and hence is a *mechanism* with respect to transverse forces.

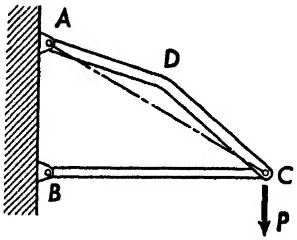


FIG. 9·2. Curved member.

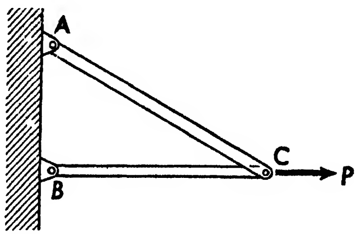


FIG. 9·3. One member inactive.

It is now obvious that the development of forces in members of a triangle or truss depends on the manner in which they are compelled to *change their length* under the applied loading condition. This in turn depends primarily on the location of the points about which the opposite ends of the members rotate. To show that the *direction* of the member at the joint is not the primary factor Fig. 9·2 has been included. A superficial examination of joint C might lead to an analysis based on the direction of the member between D and C. Actually, however, point C will tend to rotate about point A just as if the member had been straight. The analysis must therefore be started as if member ADC had been straight. This will give the correct force in member BC. The straight line AC is sometimes called a *phantom member.* (The analysis of the curved member ADC is a more advanced problem that will be taken up later.)

One more example is given in Fig. 9·3. Here the force P is applied along the axis of member BC. Since the line of action of the force passes through the hinge at point B, there will be no tendency for member BC to rotate. Consequently, it will resist the entire force P without any assistance from member AC. (Compare Sec. 5·3.)

A question may be raised about the effect of the elongation of member BC. It is true that point C will tend to move outward under load, as member BC must elongate slightly. It might be supposed that member AC would have to elongate similarly. Actually, however, this change of position of point C is accomplished much more easily by a very slight rotation of members AC and BC, involving no elongation of member AC, hence no load in it. This rotation is usually so small as to be negligible, i.e., the *original geometry is assumed to be unaffected by actual distortion of the members.* This basic assumption should be kept in mind, as occasionally it may cause appreciable errors.

9·2. Joint Analysis. To determine the distribution of forces between two or more members meeting at a common joint the members may be considered as being replaced by the forces which they exert on the joint (actually, on the pin or bolt which forms the connecting link). Usually the value of the force in one of the members is known, and the forces in the others must be determined. The *direction* of the forces in the unknown members is fixed by their location, or, as already discussed, by the location of the points about which they rotate. Figure 9·4 is typical. The force in member C is known and the forces in members A and B must be determined. If it is assumed that members A and B are straight and pinned at the opposite ends, they can transmit only axial forces; therefore the line of action of forces P_A and P_B must be along the center-lines of members A and B.

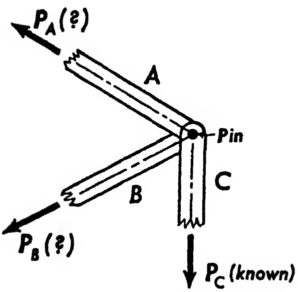


FIG. 9·4. Conditions at joint.

The principles of static equilibrium are now applied, as outlined in Chapt. 4. The pin connecting all three members is in a state of equilibrium, as it is not being accelerated in any direction by the action of unbalanced forces. Hence the net effect of all the forces acting on the pin must be zero. (This must hold for any direction.)

If the net result of three forces is zero, the vector diagram must close. The joint problem can therefore be stated as follows. Given one of the forces acting at the joint (including its direction) and the lines of action of the other two forces, find the magnitude of the two unknown forces such that the vector sum of all three is zero.

9·3. Graphical Solution. The graphical solution is readily obtained by constructing the vector diagram as follows (illustrated in Fig. 9·5).

First a scale picture of the joint must be drawn, as in (*a*), showing the lines of action of the known force and the two unknown forces. These lines are usually the centerlines of the three members intersecting at the joint. (See Sec. 9·1 for exception.) Next the vector diagram is started by selecting a suitable force scale and laying out the vector for the known force P_c, parallel with its lines of action and in the proper direction. This is indicated in (b).

Next, a line is drawn through one end of the vector, parallel to one of the members (line AA). A line parallel to the other member is drawn through the other end of the vector P_c (line BB). The intersection of these two lines determines a triangle, which is the closed vector diagram desired. The arrows indicating the direction of the forces are now placed in accordance with the convention for adding vectors (Sec. 2·5), as three forces are being added to obtain the vector sum of zero. If desired,

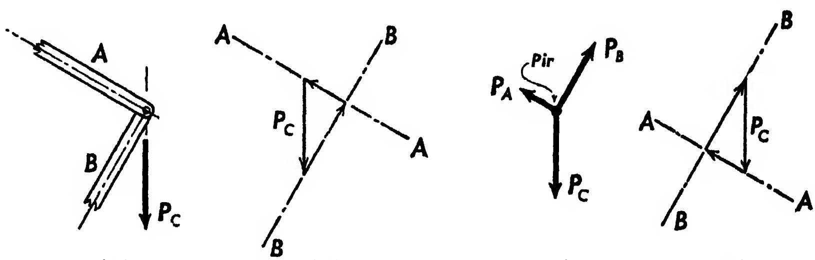


Fig. 9·5. Graphical analysis of joint.

the forces may then be shown as acting on the pin, as in (c). This assists in determining the nature of the loading in each member. Since these forces are being exerted *by* and not *on* the members, (c) indicates tension in member A and compression in member B.

Note that the same result would have been obtained if line BB had been placed at the top of the vector P_c and line AA at the bottom (Fig. 9·5d). It is necessary only to remember to place the head of each vector away from the head of the adjacent one. (Heads should never be together in a closed diagram.)

The foregoing simple analysis represents the basic process used in the graphical analysis of trusses and plane frames. More specific general methods will be given later; at this point it is most important to obtain a clear mental picture of the physical action and the significance of the vector operations.

9·4. Analytical Method. A joint may be analyzed mathematically by working with linear components and the equations of equilibrium. Since vectors cannot be added and subtracted algebraically unless they all lie along the same line, or are parallel, all forces (both known and unknown) must be resolved into components which *can* be added and subtracted directly. In plane frameworks (two-dimensional) it is necessary to establish two independent reference axes. These are chosen perpendicular to each other to facilitate the computations. All forces acting on the joint (pin) along each reference axis are then added algebraically and equated to zero. If there are two unknown forces, this yields two equations in two unknowns, which may then be solved algebraically.

In the algebraic solution it is necessary to make an initial assumption as to the *sense* of the unknown forces in the members. It is most convenient to make the arbitrary assumption that all unknown forces are tension, as shown in Fig. 9·6, using the "structural" convention that tension is positive (Sec. 3·3). Then if a negative sign is obtained in the first solution we know that the direction of the force is reversed, i.e., the member is under compression. (Compare Sec. 4·8.)

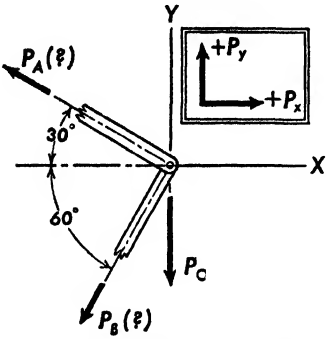


FIG. 9·6. Analytical method.

Since the unknown forces must remain as symbols until the equations of equilibrium are solved, these symbols must be multiplied by factors which will give their components along the reference axes. The *direction cosines* of the members are used for this purpose (Sec. 2·4). The direction cosines of a member may be thought of as the *components of a unit tension force* which the member is exerting on the joint. Thus the direction cosines of member A could be found by drawing a vector of unit length for the assumed tension force P_A and measuring its components along the X and Y axes. (In Fig. 9·6 the y direction cosine of member B would be negative with respect to the force conventions used.)

For Fig. 9·6 the data may be tabulated as follows.

MEMBER OR FORCE	x DIRECTION COSINE	y DIRECTION COSINE
P_C	0.0	1.000
P_A	−0.866	+0.500
P_B	−0.500	−0.866

The equations of equilibrium are now written:

$$[\Sigma P_x] \qquad (0 \times P_C) - 0.866P_A - 0.500P_B = 0 \qquad [a]$$

$$[\Sigma P_y] \qquad 1.000P_C + 0.500P_A - 0.866P_B = 0 \qquad [b]$$

From Eq. a,

$$P_A = \frac{-0.500}{0.866}P_B = -0.577P_B$$

Substituting in Eq. *b*:

$$P_C = 0.288P_B - 0.866P_B = 0$$

$$P_B = +0.866P_C \quad \text{(tension)}$$

$$P_A = -0.577\ (0.866P_C) = -0.500P_C \quad \text{(compression)}$$

9·5. Solution by Taking Moments. Another useful method of solving a triangle involves taking moments about one of the hinges or joints. (This method was described in Sec. 4·20.) Assume that one of the members is replaced by the force it is transmitting, and that this force acts on the common joint. The remaining member must still be in equilibrium; consequently there must be no tendency to rotate about any point.

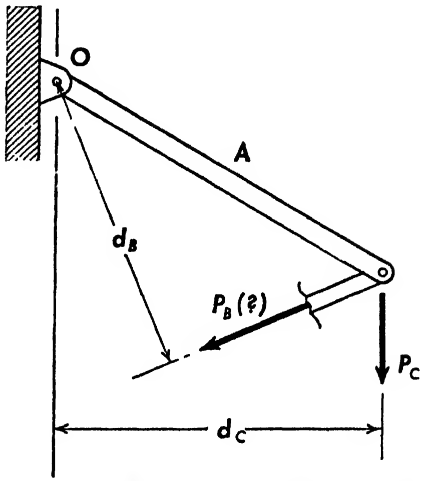


FIG. 9·7. Solution by taking moments.

Taking Fig. 9·7, for example, the turning moment of P_B about point *O* must exactly counteract the moment of P_C about the same point. (The axial force in member *A* will, of course, have no turning moment about point *O* and can therefore be neglected.)

Expressed mathematically,

$$[\Sigma M = 0] \qquad P_B d_B + P_C d_C = 0$$

Solving,

$$P_B = -\frac{d_C}{d_B} P_C$$

Note that the solution gives a negative value for P_B. This is because P_B was assumed to represent a *tension* force in Fig. 9·7. The negative sign indicates that it is actually a compression force and that it acts in a direction opposite to that shown in the sketch.

To obtain the force P_A by this method, moments may be taken about the lower hinge point. Or P_A may be obtained by vector addition of P_C and P_B, to give a closed diagram. Both methods may be used, to obtain a check.

9·6. Summary of Triangle and Joint Analysis. The foregoing brief discussion of joint analysis brings out the three basic methods of determining the axial forces in a plane framework. Certain fundamental principles were also developed and emphasized. Each method will be more fully described later, but the following summary should be thoroughly understood.

a. The basic element of the plane truss or framework is the *triangle* (or Vee).

b. *Pinned ends* must be assumed in order to permit the direction of the forces acting on the joint to be determined.

c. The joint must be in *static equilibrium.*

d. The *graphical* method of solution involves the construction of a closed diagram in which one force is completely known but only the lines of action of the other two are known.

e. The *analytical* method requires all forces to be resolved into components along arbitrary axes, after which the equations of equilibrium along each axis are set up and solved.

f. In the analytical method it is best to assume all unknown forces to be *tension.* The sign of the solution then shows the real sense of the force.

g. In the solution by *taking moments* the principle involved is to write the equation of equilibrium for turning moments about one of the joints, thereby eliminating one axial force and permitting the solution of the equation to determine the other.

9·7. Truss Analysis. Since a plane truss is composed of a series of triangles having common joints the foregoing methods may be used to determine the loads in all the members. The process is started at a joint which has only two unknown loads (such as the free end of a cantilever beam). By properly choosing successive joints it is usually possible to proceed through all the members. If two-way force transmission is involved, however, it is necessary first to determine the reactions at the points of support (Sec. 4·2).

For plane trusses the vector method is usually more rapid than the analytical method and, if suitably large scales are used, the results are sufficiently accurate for engineering purposes.

It should be noted that trusses may contain members which are not theoretically essential to truss action. From the example shown in Fig. 9·8*a* it is apparent that before starting the analysis the truss can be simplified as in (*b*). The additional members do not make the truss statically indeterminate, as might be supposed, even though two of the verticals are under load. One way to check this is to attempt removing any of the *other* members which attach to the member in question. If none of these can be removed without destroying the continuity of the truss, the truss is not indeterminate.

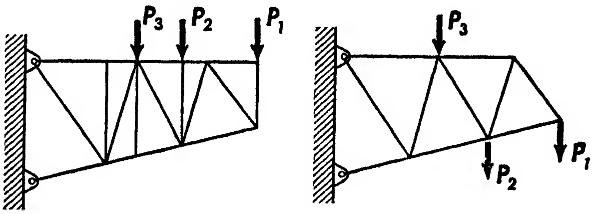


FIG 9 8. Removal of superfluous members.

In Fig. 9·9*a* an extra member *does* make the truss statically indeterminate. As shown in (*b*) and (*c*) either of two members may be removed without reducing the truss to a mechanism. Therefore, (*a*) is statically indeterminate, and it is impossible to analyze the members in question by any of the methods given up to this point. The remainder of the truss can be analyzed, however, by assuming one of the redundant members to be removed, as in (*b*) or (*c*). This will give conservative

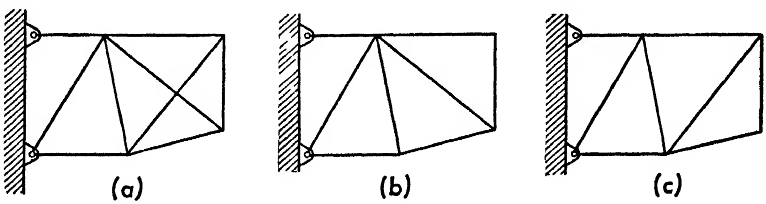


FIG. 9 9. Redundant truss.

(too high) loads in the redundant members and in those to which they attach.

9·8. Vector Analysis of Trusses. The use of *Bow's notation*, already described in Sec. 2·7, greatly simplifies the analytical solution of a truss. The solutions of the various joints can be done by means of a single diagram, similar to that used in determining the *string polygon*. In Sec. 2·8 the *force polygon* was arbitrarily drawn and the lines of action on the *space diagram* were determined from it. In the truss the centerlines of the members (or phantom members) define the space diagram, and the problem is to find the force polygon for the members of the truss.

Even if one-way transmission is involved, it is best to determine the reactions before starting, using one of the methods given in Chapt. 4. These reactions are then regarded as applied forces, and the force polygon representing them and the known forces must close. Therefore the force polygon for the truss members (internal forces) must also close, which affords a check on the accuracy of the work.

Figure 9·10 shows the graphical analysis of a simple truss. The reaction R_1 was found by taking moments. R_2 was found from the vector diagram (dotted line AC). Point D on the force polygon was

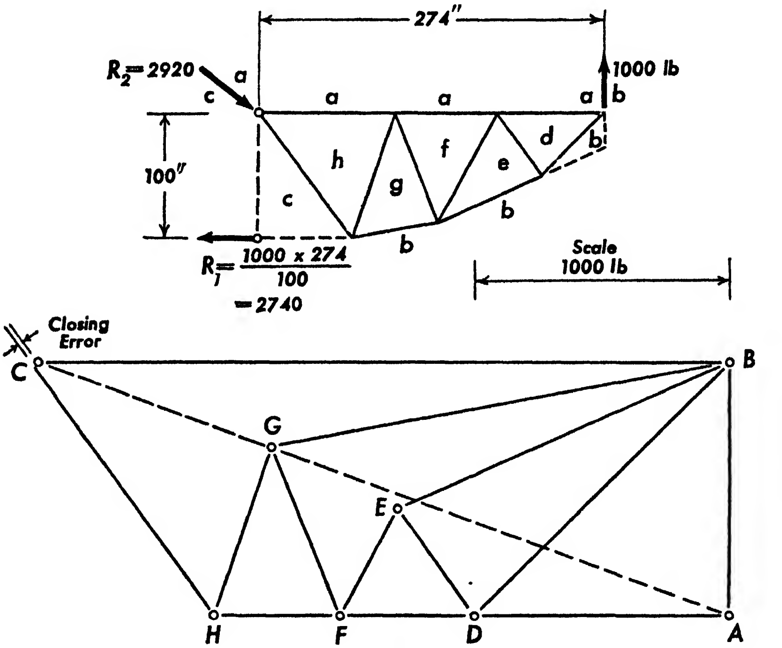


FIG. 9·10. Vector analysis of truss.

first determined, by drawing a line through B parallel with *bd* and another line through A parallel with *ad*. Since point D must lie on both of these lines, their intersection obviously determines its location. The lengths AD and BD, multiplied by the force scale, give the loads in *ad* and *bd*. The direction of these loads is determined by reading clockwise around a joint in the space diagram and following the same sequence of letters on the force polygon. Thus the load in member *gf* will be found to be compression, as the force acts *toward* the joints, according to the foregoing rule.

The remaining steps are similar and should require no further explanation. Note that the last line drawn (HC) failed to pass through the previously determined point C, indicating that the construction was not absolutely accurate. For extreme accuracy the diagrams should be drawn to a relatively large scale, and care should be taken in keeping force lines parallel with their respective space lines.

9·9. Other Methods. The method of taking moments about one of the joints (described in Sec. 9·5) may be used for various members in a truss. It is only necessary to cut the truss in such a manner that the "cut" passes through one joint and one member. An equation may then be written for the moment of all forces on the free body, acting about the cut joint. The force in the cut member will be the only unknown in the equation, and may therefore be determined directly. This method is convenient when it is not desired to analyze the truss completely. An example is shown in Fig. 9·11.

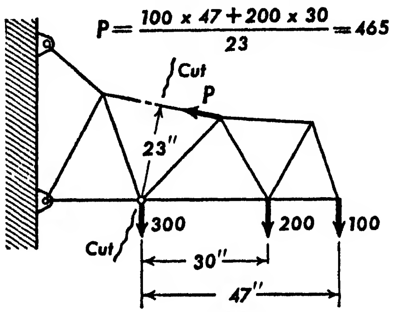


Fig. 9·11. Solution by taking moments.

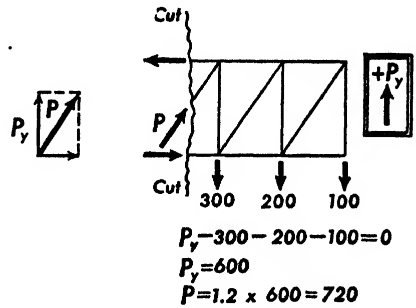


Fig. 9·12. Solution by section.

If the cut passes through more than one member, a single equation of equilibrium is not sufficient to obtain the loads in the cut members. Additional equations of equilibrium for horizontal or vertical forces, or both, must be written and solved. This is usually referred to as taking a section through the structure.

This process is particularly useful in determining the loads in diagonal members of parallel-chord trusses, such as shown in Fig. 9·12. Since the chords in the illustration have no vertical (y) components, it is unnecessary to include them in writing the equation of equilibrium for vertical forces. The solution of the equation gives the vertical *component* in the diagonal member; the total load is then determined graphically or analytically. (In Fig. 9·12, $P = 1.2\ P_y$.)

9·10. Space Frameworks. Although this volume is confined largely to *plane* problems, the analysis of *space* (three-dimensional) frameworks is such a common procedure that a brief discussion will be presented. The basic principles are essentially the same as those used in plane problems. If the analytical procedure is followed, each joint may be analyzed by writing *three* equations of equilibrium, instead of two. The *direction cosines* for the members being solved must, of course, be known, or computed from available dimensional data. The unknown forces should be assumed to be *tension*. The sign of the solution will tell whether this assumption was correct or not.

Graphical methods can also be applied to three-dimensional frameworks by determining the true view of various portions and analyzing them as plane trusses. This procedure is sometimes more convenient and more rapid than the analytical procedure. Examples of both methods will be given, for illustration.

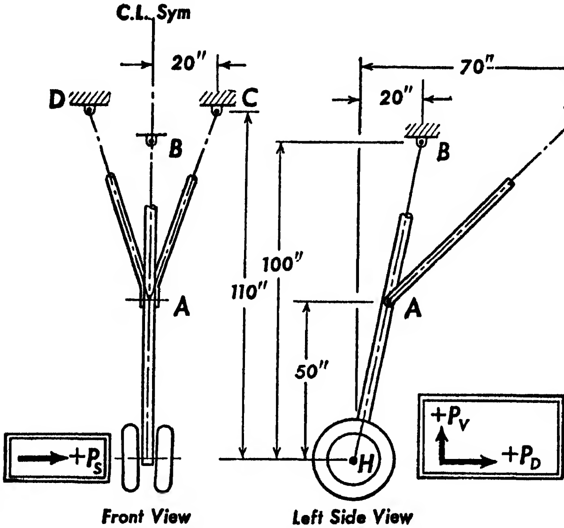


FIG. 9·13. Airplane nose gear.

9·11. Airplane Nose Gear (Analytical Method). Figure 9·13 shows the nose landing gear of a large airplane. In designing such a gear it is customary to consider various loading conditions, one of which is selected for this example. In this side load condition it is assumed that the resultant load acting at the wheel hub is composed of two components, a vertical component ($V_H = +15{,}000$ lb) and a side component ($S_H = -22{,}500$ lb). The sign conventions to be used for applied loads are shown in Fig. 9·13.

The loads applied at the hub H are transmitted up to point A entirely by the shock strut HB. The structure may therefore be thought of as being made up of two separate units, a *beam* supported at two points, (A and B) and *tripod* ($ABCD$). The analysis will be made accordingly.

The first step is to calculate the *direction cosines* and lengths of the tripod members in Table 9·1.

TABLE 9.1

MEMBER	V	D	S	V^2	D^2	S^2	L^2	L	$\frac{V}{L}$	$\frac{D}{L}$	$\frac{S}{L}$
AB	50	10	0	2500	100	0	2600	50.99	0.982	0.196	0
AC	60	60	20	3600	3600	400	7600	87.18	0.688	0.688	0.2295
AD	60	60	20	3600	3600	400	7600	87.18	0.688	0.688	0.2295

Next the shock strut is analyzed as a beam to find the reactions at A and B. The analysis for side load is shown in Fig. 9·14a. The reaction at A is determined by taking moments about point B, from which

$$R_{AS} = \frac{22{,}500 \times 100}{50} = 45{,}000 \text{ (lb)}$$

where subscript A refers to point A,
subscript S indicates side load.

Note. The true lengths involved are 50.99 and 101.98 in., but the same *ratio* is obtained by using the vertical components only.

The value of 45,000 lb is the load exerted *on* member HB by the tripod which supports it. Hence, when this load is applied to the tripod it must be reversed, requiring a negative sign (by inspection of Fig. 9·14a).

Before this same process is carried out for the side view (Fig. 9·14b), the applied load of 15,000 lb at H is resolved into *axial* and *transverse* components, using the direction cosines from Table 9·1. Since the axial component of 14,720 lb (compression) is carried directly through the shock strut to B, it may be dropped out of the subsequent tripod analysis. It must be added to the loads determined for member AB, however. (Compare Sec. 5·3 and Fig. 9·3.)

The transverse component of 2940 lb is now treated in the same manner as the side component, to determine the reaction at A.

$$R_{AN} = \frac{2940 \times 101.98}{50.99} = 5880$$

Note. True lengths were used here. The same result would be obtained by using 50 and 100 in.

This reaction acts *normal* to the shock strut, as point A tends to move in such a direction when rotated about point B. The resisting force of

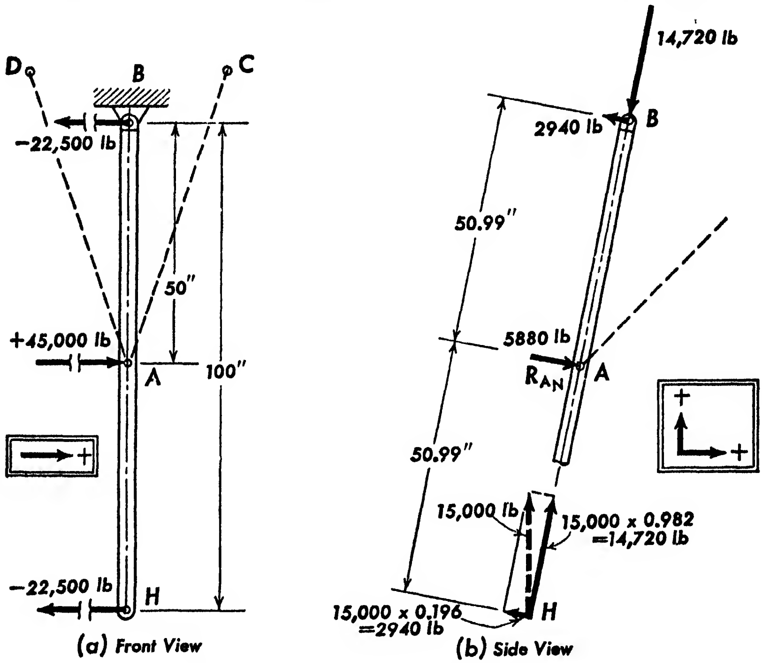


FIG. 9·14. Analysis of shock strut.

5880 lb must be supplied by the tripod, hence this force will be reversed and applied at point A in the tripod analysis.

Figure 9·15 shows the loads for which the tripod must be analyzed. Note that the directions of the loads have been reversed with respect

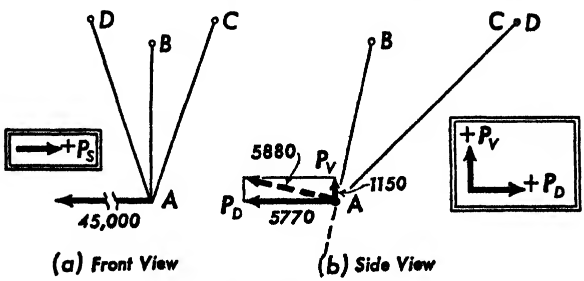


FIG. 9·15. Loads on tripod.

to Fig. 9·14. Before writing the equations of equilibrium the normal force at A is resolved into vertical and drag components, using the direction cosines for member AB (reversed, since the force is normal to the member).

$$P_V = 0.196 \times 5880 = 1150$$

$$P_D = -0.982 \times 5880 = -5770$$

The equations of equilibrium may now be written for point A. To determine the signs in the equations the unknown forces are assumed to be tension, i.e., they all pull away from point A. The proper sign is then determined with respect to the positive conventions already established. The letters used to designate members are also used to indicate the forces in them.

$$[\Sigma P_V = 0] \quad 0.982AB + 0.688AC + 0.688AD + 1150 = 0 \qquad [a]$$

$$[\Sigma P_D = 0] \quad 0.196AB + 0.688AC + 0.688AD - 5770 = 0 \qquad [b]$$

$$[\Sigma P_S = 0] \quad 0 + 0.2295AC - 0.2295AD - 45{,}000 = 0 \qquad [c]$$

Subtracting Eq. b from Eq. a gives

$$0.786AB + 6920 = 0$$

$$AB = -\frac{6920}{0.786} = -8800$$

(The negative sign shows that this is a compression force.)

Multiplying Eq. c by $\frac{0.688}{0.2295}$ [= 3.0] gives

$$0 + 0.688AC - 0.688AD - 135{,}000 = 0 \qquad [d]$$

Adding Eq. b gives

$$0.196AB + 1.376AC + 0 - 140{,}770 = 0$$

Substituting for AB and solving,

$$AC = \frac{140{,}770 + 1730}{1.376} = +103{,}600 \quad \text{(tension)}$$

Substituting this result in Eq. c

$$0 + 23{,}700 - 0.2295AD - 45{,}000 = 0$$

$$AD = \frac{-45{,}000 + 23{,}700}{0.2295} = -93{,}000 \quad \text{(compression)}$$

In summarizing, the forces obtained from the *tripod* analysis are:

$$AB = -8{,}800 \quad \text{(compression)}$$

$$AC = +103{,}600 \quad \text{(tension)}$$

$$AD = -93{,}000 \quad \text{(compression)}$$

The total force in member AB is found by adding the axial force obtained from the analysis of the shock strut.

$$AB = -8800 - 14{,}720 = -23{,}520 \quad \text{(compression)}$$

It should be noted that this member transmits bending moments and shear forces in two planes, as well as the axial force just determined. Methods of analysis for such conditions are given in subsequent chapters.

The foregoing method of analyzing the landing gear structure is only one of many possible variations. A method employing vector diagrams will be described and used as a check.

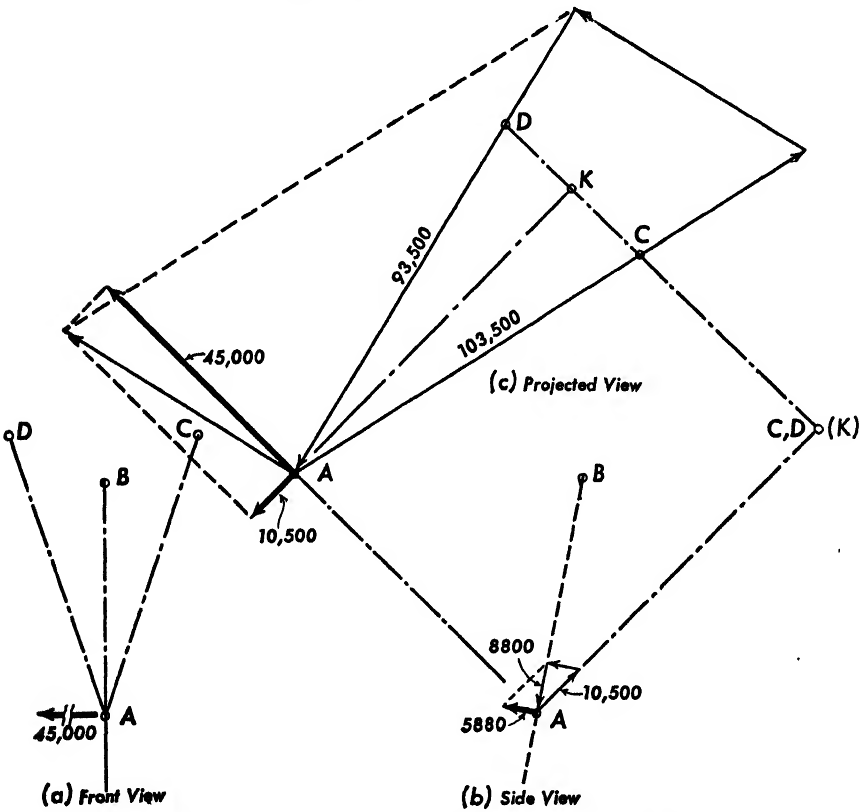


FIG. 9·16. Graphical analysis of landing gear tripod.

9·12. Airplane Nose Gear (Graphical Method). Figure 9·16 shows a graphical analysis of the tripod, using the loads that were applied at point A in the previous analytical method. Sketch (a) shows the side load of 45,000 lb, while (b) shows the normal load of 5880 lb. By using the centerlines of the members, the forces in AB and AK are found by forming a closed vector diagram of which one side represents the force of 5880 lb. The force in AB is found to be 8800 lb, acting toward the joint; hence AB is in compression. (This value checks the value obtained analytically in the previous section.)

The force of 10,500 lb acts in the hypothetical member AK. This force is the net force exerted by the Vee members AC and AD on the joint, and it must therefore be reversed in direction in considering it as an applied force in (c), which is a projected or true view of the members.

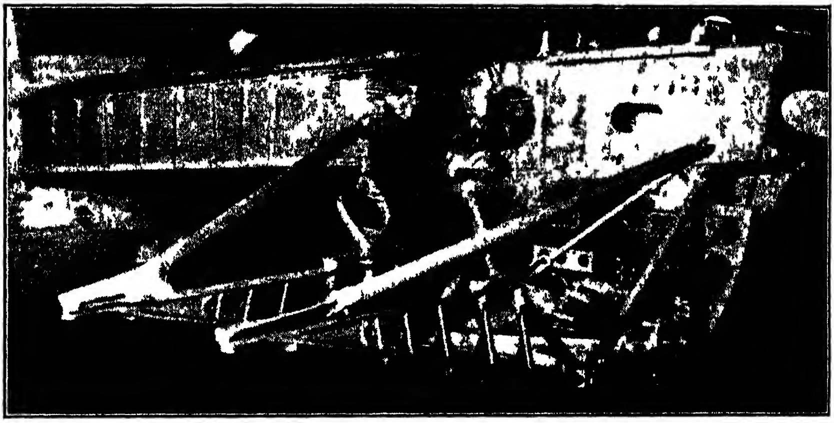


FIG 9·17. Lockheed *Lightning* (P–38) engine mount. Note combination of space framework and forging

The side force of 45,000 lb is also drawn to scale in this view. The vector sum is found, as shown, and resolved into two components in members AC and AD, using the centerlines of these members as a basis for the vector diagram. The force of 93,000 lb (scaled from the diagram) acts toward the joint, showing that AD is in compression. The force in AC is obviously tension, as it acts away from the joint. These two forces check very closely the values found by the analytical method.

PROBLEMS

9·1. Draw a truss of the type shown in Fig. 9·8*b*, using any desired proportions and dimensions. Assign values to loads P_1, P_2, and P_3. Use Bow's notation.

(*a*) Solve each joint analytically to obtain member loads.
(*b*) Solve the truss graphically.
(*c*) Find the load in lower left member by taking moments.

(All three solutions should agree.)

9·2. Using Fig. 8·4*b* as a basis, assume that diagonals are placed at 45° and that a load of 100 lb acts at each joint along the lower flange.

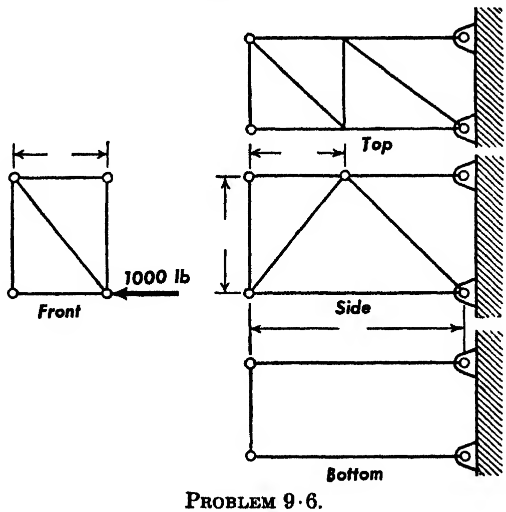
(*a*) Calculate the forces in the diagonal and vertical members. (Use method of sections.)
(*b*) Draw the vector diagram for the truss.
(*c*) Check the lower flange member loads at any two places, by taking moments.

9·3. Draw Fig. 9·9 to a larger scale and apply an upward-acting vertical force of 1000 lb at the lower end joint. Analyze graphically for the two cases shown in (*b*) and (*c*) and show which members are affected by the redundancy.

9·4. Extend Problem 9·3 by arbitrarily assuming one of the redundant members to be acting with a known force equal to half of the value determined in the first solution. (Apply the forces from this member as external forces at the joints to which it attaches.)

9·5. Draw a roof truss such as shown in Fig. 8·10, assigning arbitrary dimensions. Assume any desired values for the loads shown and solve the truss (assuming roller support at one end). Check load in lower horizontal member by taking moments.

9·6. Find the loads in the members of this structure by solving each joint analytically.



PROBLEM 9·6.

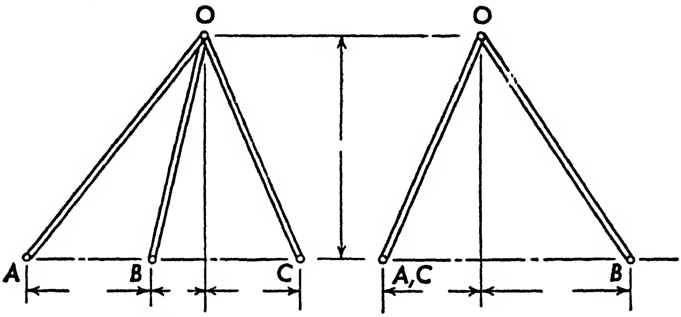
(*a*) Assign dimensions and draw to scale.
(*b*) Letter each joint and assign positive conventions for forces.
(*c*) Note any members which are superfluous for the loading shown.
(*d*) Solve equations of equilibrium for each joint, starting at point of load application.
(*e*) Check load in rear top truss diagonal by method of sections.

(*Note.* If the 1000-lb load had been applied at the top, the top truss would have carried the load directly, without loading the side truss members. If diagonal members were added to the bottom truss the structure would become redundant unless the front diagonal were removed.)

9·7. Using the structure of Problem 9·6, assume that the 1000-lb load acts vertically at the point shown and determine the loads in the members.

9·8. Draw to scale a tripod similar to that shown in the figure and measure the dimensions indicated. Establish positive space conventions and find the load in each member for each of the following loading conditions at point *O*:

(*a*) 1000 lb, acting horizontally to right (sketch *a*).
(*b*) 1000 lb, acting directly upward.
(*c*) 1000 lb, acting directly to rear (to the right in sketch *b*).

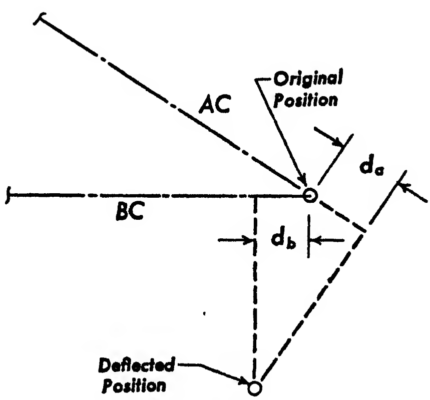


PROBLEM 9·8.

9·9. Check the forces in *OA* and *OC* graphically, for loading condition (*a*) in Problem 9·8. (Draw a true view of triangle *OAC*.)

9·10. Draw a pair of members, such as shown in Fig. 9·1*a*, to a suitable scale (making *BC* about 60 in. long). Assume that force *P* is 10,000 lb. Select tube sizes (Appendix 1·2) such that the stress developed in each tube is about 40,000 psi. Calculate the total axial deflection of each tube (for steel).

9·11. Using the data from Problem 9·10 find the magnitude and direction of motion of point *C* under the assumed load. Note that this can be done graphically by swinging new arcs about points *A* and *B*, using the deflected lengths. An attempt to do this will show that the deflections are too small to be handled to the scale of the original drawing of the structure. The conditions at point *C* must therefore be redrawn on a greatly expanded scale. In this construction the arcs may be considered as straight lines drawn normal to the original centerlines of the members, as indicated in the figure.



PROBLEM 9·11.

CHAPTER 10

SHEAR WEBS

10·1. The Frame. In Fig. 8·4e a *wire-braced* truss was shown. The essential feature of this structure was the use of two crossed diagonals one of which was assumed to be out of action. The development of the shear theory for webs and solid sections may be started from this simple structure by assuming that *both* of the diagonals are acting.

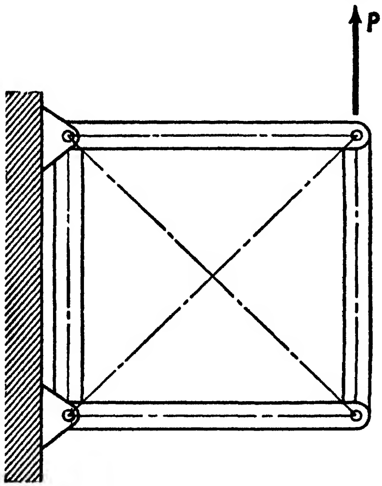


FIG. 10·1. Shear frame.

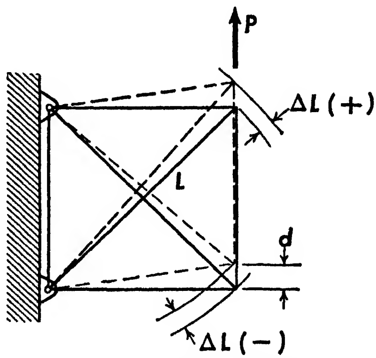


FIG. 10·2. Deflection of shear frame.

Figure 10·1 shows an element of such a truss, which may be called a **frame.** The following important assumptions are made in developing conditions for pure shear.

a. All members of the frame are pinned at the corners (free to rotate).

b. The *edge* members are assumed to be entirely *rigid* (no change in length, no bending).

c. Opposite edge members are *parallel,* i.e., the frame is rectangular.

To start with, it will be further assumed that the frame is *square,* but this is not a necessary assumption.

Figure 10·2 shows the deflected position of the frame under a load P (greatly exaggerated). The *tension* diagonal must undergo an increase in length, ΔL. The *compression* diagonal undergoes a similar *decrease* in length.

If the deflections involved are assumed to be small, it can be proved that these two deflections are equal. Hence the *strain* in each diagonal member is equal and opposite. If the two diagonals are made from the same material and are of the same size, i.e., if their *load-strain* curves are identical, they will exert equal forces against the action of the load P, provided that the *compression* diagonal does not buckle (as it would if it were a wire or cable).

The total load in one diagonal acting alone (for a square frame) is obviously,

$$P_{\text{diag}} = \frac{P}{\sin 45^\circ} = \frac{P}{0.707} = P\sqrt{2} = 1.41P \qquad [10\cdot1]$$

For *crossed* diagonals which are equally effective the load in *each* is half this value.*

$$P_{\text{diag}} = \pm\frac{P}{2\sin 45^\circ} = \pm\frac{P}{1.41} = \pm\frac{P}{\sqrt{2}} = \pm 0.707P \qquad [10\cdot2]$$

10·2. Deflection of Frame. The relationship between the frame deflection and the deflection of the diagonal can be easily determined by examining conditions at a corner of the frame. Figure 10·3 shows these conditions in enlarged view. The dot-dash lines are arcs drawn with centers at the other ends of members A and D. (Since A is assumed to undergo no change in length only one arc is drawn for it.) If the deflections are very small, as in structures, these arcs may be regarded as *straight lines* drawn perpendicular to their respective members. With this assumption, the small shaded triangle becomes similar to the triangle represented by one-half of the frame (since the lines composing it are normal to the lines of the frame). Hence the following relationship holds for the *square* frame.

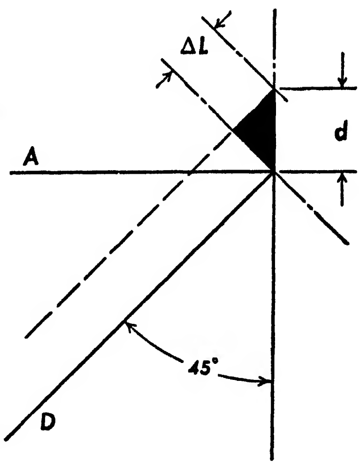


FIG. 10·3. Corner of frame (enlarged).

$$d = \Delta L\sqrt{2} = 1.41\Delta L \qquad [10\cdot3]$$

* These equations indicate the various forms in which this relationship may be expressed.

The same methods may be used to find the relationships for a *rectangular* frame. Figure 10·4 shows that the small shaded triangle is similar to the frame triangle; hence the following relationships hold.

$$\frac{d}{\Delta L} = \frac{D}{B} = \frac{1}{\sin \beta} \qquad [10 \cdot 4]$$

$$d = \frac{\Delta L}{\sin \beta} \qquad [10 \cdot 4a]$$

Equations 10·1 and 10·2 may also be generalized in this manner, to give the loads in the diagonals as a function of their angle.

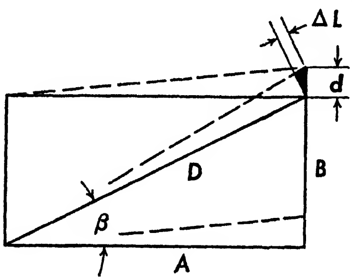


FIG. 10·4. Rectangular frame.

For one effective diagonal

$$P_{\text{diag}} = \frac{P}{\sin \beta} \qquad [10 \cdot 5]$$

For two equally effective diagonals

$$P_{\text{diag}} = \frac{P}{2 \sin \beta} \qquad [10 \cdot 6]$$

These simple relationships are sufficient to analyze any rectangular truss with diagonal bracing. They apply to a test frame such as shown in Fig. 10·26. They will also enable the *shear* deflections to be determined, but they do not account for deflections of the other frame members which have so far been assumed to be rigid. (It will be shown later that the shear deflections usually are relatively small as compared with those of the flange members.)

10·3. Maximum Strain. Figure 10·5 shows another member attached to the frame at some angle other than 45°. Since the frame is assumed to be rigid this member will be elongated through a distance ΔL_1, as shown, if the frame is deflected through the distance d_0.

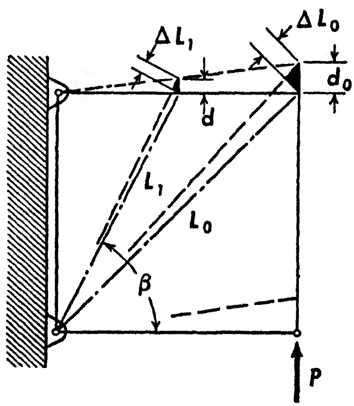


FIG. 10·5. Variation in angle of diagonal.

It can be proved that for a given frame deflection the *maximum* strain $(\Delta L/L)$ will take place when the angle $\beta = 45°$. Obviously, the strain will be zero when $\beta = 0°$ or $90°$. Between these limits the strain varies in proportion to $\sin \beta \cos \beta$ ($= \frac{1}{2} \sin 2\beta$). The nature of this variation is shown by Fig. 10·6. This curve may also be used to show the relationship between the strain e at any angle β and the maximum strain e_0 at 45°.

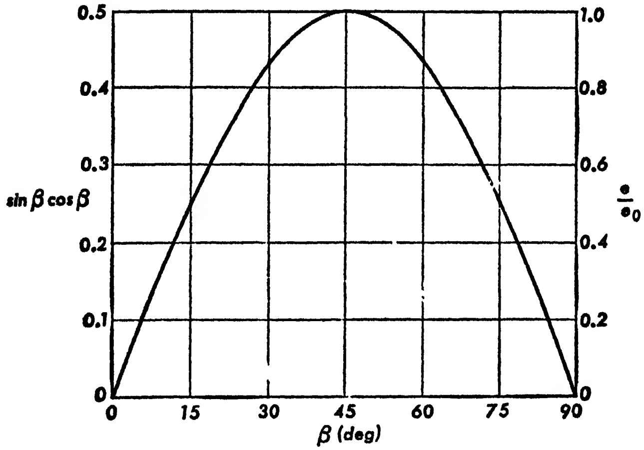


FIG. 10·6. Variation of strain with angle.

In polar coordinates the resulting curve is a **leminiscate.** Figure 10·7 shows such a curve, in which the vectors indicate the relative magnitude of the axial strain. These relationships hold also for the *rectangular* frame, even though the angle β is less than 45°. They do not apply, however, to the *single* diagonal, as a decrease in the angle β will also require an increase in the load being carried by the diagonal.

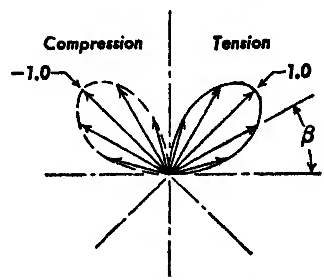


FIG. 10·7. Relative strain at different angles.

10·4. Multiple Diagonals. It can also be proved that *any* element placed at 45° will undergo the same strain as that of the main diagonal. This should be evident from Fig. 10·8, which shows that the ratio d/L remains constant. The main points covered so far may be summarized as follows.

a. The maximum strain occurs at 45°.

b. The strain is the same for any element at this angle, i.e., for all parallel elements.

c. One family of elements will undergo tension, while the other one (at right angles) will undergo compression.

d. Elements at angles other than 45° undergo strains in proportion to the function $\sin\beta\cos\beta$, where β is the angle which the element makes with one of the frame sides.

e. Elements parallel with the sides of the frame undergo no strain (follows from *d*).

f. The above rules hold for any *rectangular* frame, provided the frame can deflect only at the pinned corners.

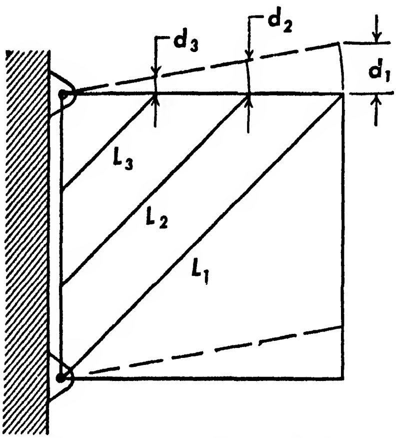


Fig. 10·8. Parallel diagonals.

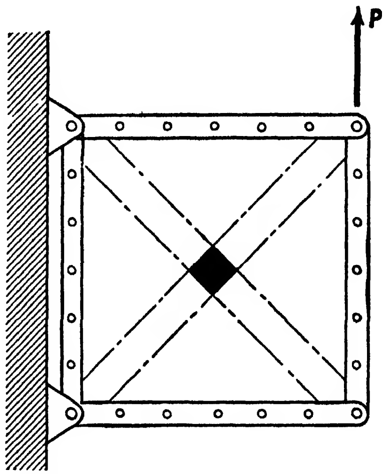


Fig. 10·9. Flat plate shear web.

10·5. Flat Plate in Shear. It may now be assumed that a flat plate is attached to the frame along the centerlines of the frame members, replacing the diagonals used for the truss. This plate can be hypothetically cut into strips which will conform with the foregoing analysis. The only real difference is that any element of the plate acts in *both directions*, i.e., the same element forms a part of the tension and compression diagonals, as indicated in Fig. 10·9.

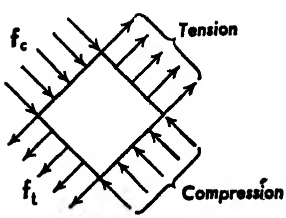


Fig. 10·10. Axial stresses on an element in pure shear.

If the compression strip does not buckle, there will be a compression stress corresponding to the compression strain. Since the tension and compression strains are equal, the element will be subjected to equal tension and compression *stresses*, acting at 90° to each other as shown in Fig. 10·10. This condition is designated as **pure shear** and is associated with the **shear-resistant** web described in Sec. 8·8.

A sheet so thin that it has practically no resistance to compression is the other extreme of a plate in shear. The compression strains will cause *buckling* or *wrinkling* as indicated in Fig. 10·11, instead of building up a resisting stress. If it is assumed that zero compression stress is devel-

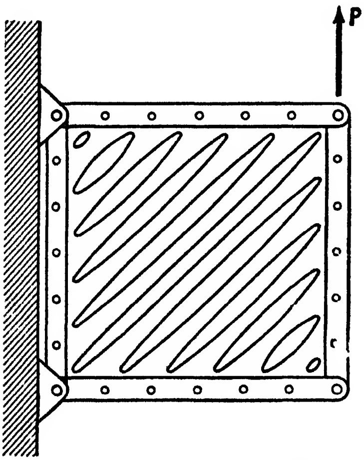


FIG. 10·11. Tension-field shear web.

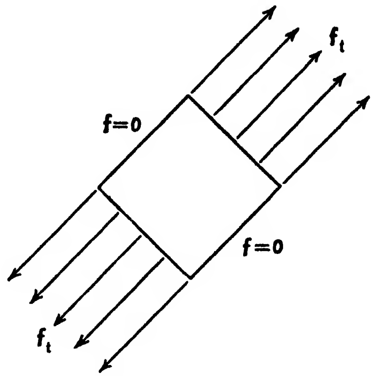


FIG. 10·12. Element of tension field.

oped, the element will appear as in Fig. 10·12. This idealized type of shear web is referred to as the **tension-field** web. It is very nearly approached in some forms of light sheetmetal construction.

10·6. Shear Stress. The shear stresses acting on an element in pure shear may be visualized by making a vertical slice through the element giving a picture such as indicated by Fig. 10·13, in which the two halves are shown separated.*

If the element is assumed to have unit width, i.e., 1 in. on a side, and unit thickness, the stresses shown may be thought of as *forces.*† By drawing vector diagrams as in Fig. 10·14 it will be seen that there is no unbalanced *lateral* component to cause any horizontal axial stress (which

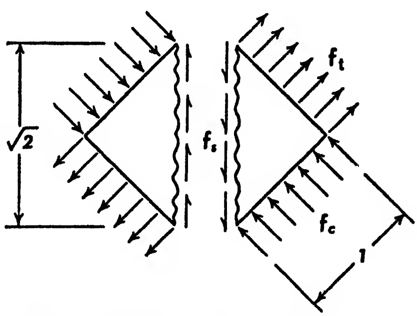


FIG. 10·13. Shear stress in element.

* *Half* arrows are often used to indicate shear stresses or forces, as shown in Fig. 10·13.

† *Stresses* cannot be added vectorially, as the area on which they act changes as the angle of action varies. It is always necessary to convert stresses into forces by multiplying by the proper area, before vector addition is attempted.

agrees with previous findings). The reaction R is equal to $\sqrt{2}f_t$ or $\sqrt{2}f_c$. It acts over the vertical face of the cut element, which has a length equal to $\sqrt{2}$. The shear stress is therefore given by

$$f_s = \frac{R}{\sqrt{2}} = \frac{\sqrt{2}f_t}{\sqrt{2}}$$

from which:

Pure shear

$$\boxed{f_s = f_t = f_c} \qquad [10\cdot7]$$

Equation 10·7 is very important, as it gives the relationship between shear stress and diagonal axial stresses for a state of *pure shear*. It should be kept in mind that the shear stress is always measured on a section 45° away from the section on which the maximum axial stresses act.

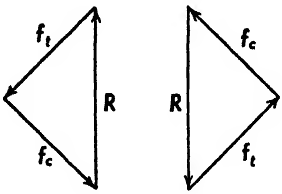


Fig. 10·14.

Assume that a square element is cut from a shear-resistant web so that its sides are parallel with the frame, i.e., parallel with the direction of the transverse force. Such an element is shown in Fig. 10·15. Since there are no axial stresses in strips parallel with the sides of the frame there will be none acting normal to the sides of the element. Shear stresses will act on the edges of the element, as found in Fig. 10·13. It is incorrect, however, to show the shear stresses on the vertical edges *only*, as this would indicate an unbalanced couple (see Fig. 10·15*b*).

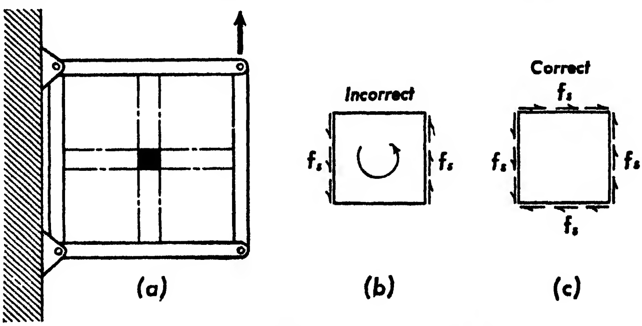


Fig. 10·15. Element in pure shear.

It is necessary to resist this turning moment by shear stresses on the other two free edges, as shown in Fig. 10·15*c*.

The presence of these stresses could have been proved independently by cutting the element of Fig. 10·13 in a *horizontal* plane, instead of a vertical one.

Another way to indicate the need for shear vectors on *all* sides is to show the diagonal axial stresses acting on the square element, as in Fig. 10·16. Note that at every point there is a *pair* of tension and compression vectors which cancel out their components normal to the cut surface. The resultant of each pair acts parallel with the cut surface and may be represented by the shear vector already illustrated. (Note that Fig. 10·16 is for illustration only. *Stress* vectors cannot be added vectorially without first converting to force.)

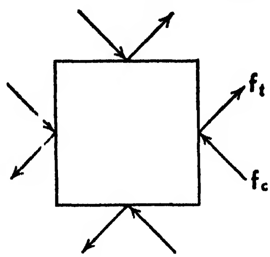


FIG. 10·16

10·7. Magnitude of Shear Stress. Since every parallel strip in the shear web undergoes the same strain (Sec. 10·4) it follows that the distribution of axial stress across a diagonal section is constant. Hence the distribution of shear stress across a vertical or horizontal cut must be constant. This permits the use of the equation for average shear stress (Sec. 6·15).

Shear stress $$f_s = \frac{P}{A_s} = \frac{P}{ht} \qquad [10\cdot8]$$

where A_s = the shear area
h = the depth.
t = the thickness.

Note that the area is that of a cross section *parallel with* the line of action of the force instead of normal to it.

The magnitude of the diagonal tension and compression stresses will also be given by the above equation, since all are equal for pure shear (Eq. 10·7).

10·8. Shear Flow. It is often more convenient to work with a running shear load, which is usually called the shear flow. This is given by the equation

Shear flow $$q = \frac{P}{h} \qquad [10\cdot9]$$

where h is the height or width in direction of loading.

This gives the value of the shear load over each inch of the cut section, as indicated in Fig. 10·24. The shear *stress* is obviously obtained from the equation

Shear stress $$f_s = \frac{q}{t} \qquad [10\cdot10]$$

where t is the web thickness.

10·9. Axial Deflections in Pure Shear. An element in pure shear will deflect as indicated by the dotted lines in Fig. 10·17, in which (*a*) and (*b*) show the two equivalent ways to illustrate a shear condition. It is important to realize that all the deflection (in the elastic range) is caused by the *axial* stresses. This is difficult to see if the shear condition is shown as in (*b*), but sketch (*a*) should make it clear.

In Sec. 6·5 it was shown that the axial strain e, in the elastic range, is equal to f/E. In pure shear, however, the compression stress acting at right angles to the tension stress causes an additional expansion

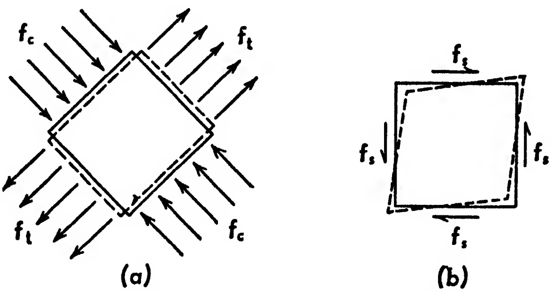


FIG. 10·17. Deflections in pure shear.

measured by Poisson's ratio, μ. (See Sec. 6·11.) The total tension strain is therefore composed of two parts.

$$e = \frac{f_t}{E} + \mu \frac{f_c}{E} \qquad [10\cdot11]$$

Since $f_t = f_c$

Axial strain in pure shear $$e_L = (1 + \mu)\frac{f_t}{E} \qquad [10\cdot12]$$

where subscript L indicates that the strain is measured in direction of axial stress (tension or compression).

Since $f_t = f_s$, in pure shear, the equation may also be written

$$e_L = (1 + \mu)\frac{f_s}{E} \qquad [10\cdot13]$$

Equation 10·12 shows that the presence of a compression stress normal to a tension stress and equal to it causes a reduction in the effective value of E, which could be shown by writing Eq. 10·12 in the form

$$e_L = \frac{f_t}{E/(1 + \mu)}$$

where $E/(1 + \mu)$ may be regarded as the effective modulus of elasticity for such a condition. If μ is taken as 0.30, for instance, the effective value of E is $E/1.30$ or $0.77E$.

10·10. Transverse Deflections in Pure Shear. It must be kept in mind that the shear web functions as a means of transmitting a force transversely, i.e., normal to its line of action. It is of interest to know the ratio between the deflection in the direction of the force and the *normal* distance through which the force is transmitted. In axial loading the unit deflection, e, was found by dividing the total deflection by the *axial* distance through which the force was transmitted. In pure shear the same principle will be used, except that the two quantities involved are measured at right angles to each other, as shown in Fig. 10·18.

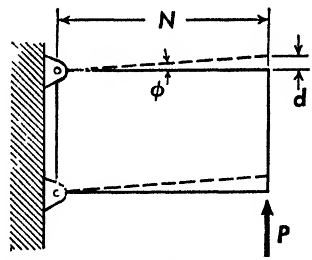


FIG. 10·18. Measurement of shear deflection.

The *shear strain* is given by the equation

Shear strain
$$e_s = \frac{d}{N} \qquad [10\cdot 14]$$

From Fig. 10·18 it is evident that this quantity is the tangent of the angle ϕ; hence the shear strain is not just a ratio (as for axial strain) but is a measure of an *angular* deflection. In fact, for the small angles usually involved, the strain may very accurately be considered as a direct measure of the angle in radians. This can be expressed by the equations

$$\begin{aligned} e_s &= \tan^{-1} \phi \\ &= \phi \quad \text{(radians)} \qquad [10\cdot 15] \\ &= \frac{\phi}{57.3} \quad \text{(degrees)} \end{aligned}$$

10·11. Shear Modulus (Modulus of Rigidity). In computing shear deflections (particularly in torsion) it is convenient to work directly with shear stress and shear strain by using an effective modulus of elasticity which gives the ratio between them. This modulus is often called the **modulus of rigidity** and is usually denoted by G. Its value may be found by considering the unit element of Fig. 10·19, which is shown deflected under a pure shear loading. From Eq. 10·3

$$e_s = \Delta L\sqrt{2}$$
$$\Delta L = e_L L = e_L\sqrt{2}$$

By substituting for e_L from Eq. 10·13,

$$\Delta L = (1 + \mu)\frac{f_s}{E}\sqrt{2}$$

Therefore

$$e_s = 2(1 + \mu)\frac{f_s}{E} \qquad [10\cdot 16]$$

The modulus of rigidity is defined as the ratio of shear stress to shear strain.

$$G = \frac{f_s}{e_s} \qquad [10\cdot 17]$$

$$= \frac{f_s}{2(1 + \mu)f_s/E}$$

Modulus of rigidity

$$\boxed{G = \frac{E}{2(1 + \mu)}} \qquad [10\cdot 18]$$

The foregoing basic shear formulas were derived for a *square* element; hence their use for a *rectangular* frame might be questioned. But a

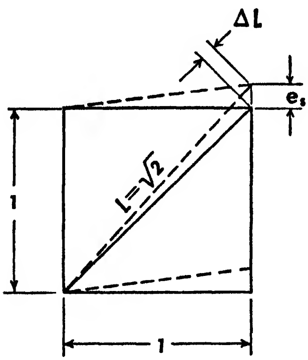


FIG. 10·19.

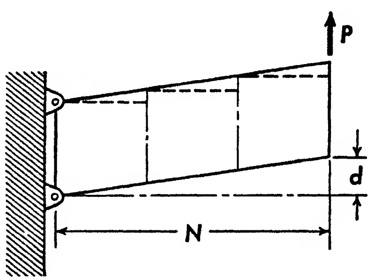


FIG. 10·20. Deflection of rectangular web.

rectangle may be thought of as being made up of a series of squares as shown in Fig. 10·20. If the shear stress remains constant the deflections of the squares are equal and the total deflection will be proportional to N. It has already been stated (Sec. 10·3), that the maximum axial strain will occur at 45° for a rectangular panel as well as for a square panel, so that the formulas may be used for a panel of any length (provided the assumptions as to frame rigidity are complied with).

10·12. Significance of G. Equation 10·18 shows that G is not really an independent mechanical property or characteristic of a material but that it depends entirely on the values of E and μ. If μ is taken as 0.30,

$$G = \frac{E}{2(1 + 0.30)} = 0.385E \qquad [10 \cdot 19]$$

This is approximately the relationship which holds for most metals. The method of deriving G also shows that the deflections which it measures are really caused by *axial* stresses, not by the shear stresses

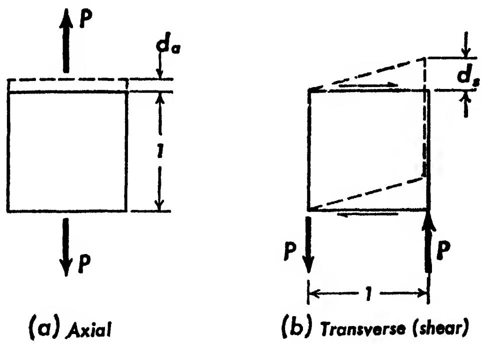


FIG. 10·21. Relative deflections under same load.

themselves. As noted in Sec. 6·15, shear stresses can cause no deflection by sliding action until the plastic range is reached. Conversely, the relationships between G, E, and μ just derived hold *only in the elastic range.*

The relationship between G and E indicates that an element of material has considerably greater resistance to *axial* deformation than to *shear* (transverse) deformation. This ratio is given by $2(1 + \mu)$, or about 2.6 for most materials. In Fig. 10·21, the value of d_s would then be 2.6 times as great as that for d_a, if the same element and same force were used in each case. Note that if μ were zero, axial transmission would be exactly *twice* as efficient as transverse. The lateral contraction effect simply makes matters worse for transverse loading.

10·13. The Tension-Field Web. In Sec. 10·5 it was shown that if the sheet buckled or wrinkled at negligible loads, it would be nearly correct to assume that no compression stresses existed. (This is equivalent to placing a number of tension diagonals in parallel.) Although the design of a tension-field beam is beyond the state of knowledge up to this point, it is of interest to consider what happens to the basic shear formulas.

Since it has already been shown that for crossed diagonals the compression diagonal resists half the load, it can be inferred that the absence of a compression field will require the diagonal tension stresses to be twice as great. This is actually true, so that Eq. 10·7 becomes

Tension-field web
$$f_t = 2f_s \qquad f_c = 0 \tag{10·20}$$

The shear stress, f_s, is computed as for a shear-resistant web, using Eq. 10·8 or Eq. 10·10.

The *axial* strain along a diagonal would be found as in Sec. 10·9, except that in Eq. 10·12, f_t would be replaced by $2f_s$ and μ would be considered equal to zero, giving (instead of Eq. 10·13),

$$e_L = \frac{2f_s}{E} \tag{10·21}$$

If this equation is used in the derivation of the modulus of rigidity (Sec. 10·11) the value of G will be found to be

$$G = \frac{E}{4} \tag{10·22}$$

This indicates that the resistance to deformation is still further reduced by the fact that the diagonal compression resistance of the material is not utilized. In fact, if Poisson's ratio were neglected, we would find that in going from axial to transverse loading the *stiffness* is reduced by one-half, and that it is reduced again by one-half if the diagonal compression field does not act. Actually the relationship between the tension-field web and the shear-resistant web is in the order of 0.25/0.385 or about 0.65, instead of 0.50. In other words, the tension-field beam is about 65 per cent as stiff as the shear-resistant beam, or, conversely, the deflections would be approximately 50 per cent greater.

10·14. Calculation of Deflection. The deflection of a flat plate or web in pure shear may be calculated by the following steps.

a. Compute shear stress.

b. Divide by G to obtain shear strain.

c. Multiply by length through which force is transmitted to obtain deflection.

Example. Assume that the frame of Fig. 10·22 has attached to it a sheet which is ¼ in. thick and that a load of 10,000 lb is applied as shown. Assume that the shear modulus for the material is 3,850,000 psi (about right for aluminum alloy). Then,

$$f_s = \frac{P}{ht} = \frac{10{,}000}{10 \times 0.25} = 4000 \text{ psi}$$

$$\phi = e_s = \frac{f_s}{G} = \frac{4000}{3{,}850{,}000} = 0.00104$$

$$d = N\phi = 20 \times 0.00104 = 0.01208 \text{ in.}$$

If the plate were considered as a tension-field web (which here would not be true), a value of G equal to $E/4$, or about 2,500,000 psi should be used

10·15. Strength of Shear Webs. A true *shear-resistant* web would not fail by buckling, hence the only types of failure possible would be *tension* and *shear.* Since the maximum tension and shear stresses in the web are equal and since for most materials the ultimate tension stress is much higher than the ultimate shear stress, a shear failure would result. (This, of course, neglects the possibility of failure at the edges due to rivet holes, etc.) This could take place on either of the planes of maximum shear stress (vertical or horizontal). (See Fig. 10·23*a*.)

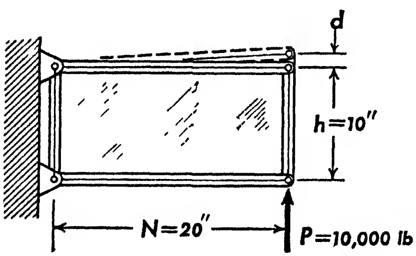


FIG. 10·22. Example of shear web.

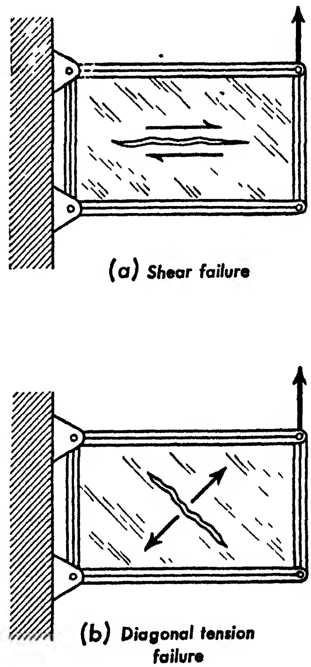


FIG. 10·23. Failures of shear web.

For a *tension-field* web the tension stress is twice as high as the shear stress, hence failure usually occurs in tension, at approximately 45° (Fig. 10·23*b*.)

In the web shown in Fig. 10·22, for instance, assume that the allowable shear stress is 30,000 psi and the allowable tension stress 50,000 psi. The *shear-resistant* web would then be good for a maximum load of

$$P = htF_s$$
$$= 10 \times 0.25 \times 30{,}000 = 75{,}000 \text{ lb}$$

As a *tension-field* web, the maximum load would be

$$P = \tfrac{1}{2}htF_t$$
$$= \tfrac{1}{2} \times 10 \times 0.25 \times 50{,}000$$
$$= 62{,}500 \text{ lb}$$

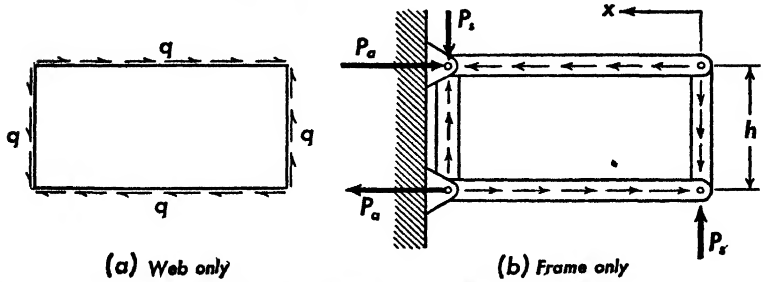


FIG 10·24 Loads on frame.

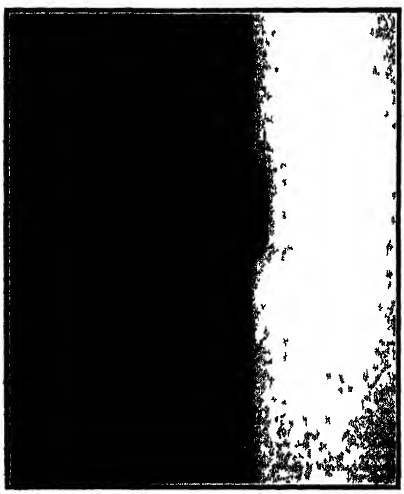


FIG. 10·25. Compression failure of plastic material Note slippage along shear planes at 45°.

10·16. Function of the Frame. All the preceding developments have been based on the assumption that the shear web is attached to a frame. The forces exerted on the frame by the shear web are of special interest, as they build up the *bending moment*; they are also of interest in designing the attachment of the shear web at its edges.

Figure 10·24 shows the frame and the shear web separated. The vectors represent the running shear load, or *shear flow*, acting respectively on the web and the frame. They are, of course, equal and opposite. It is important to note that the *shear flow* is constant on *all sides* of a rectangular web which transmits a constant shear force. This can be understood by considering that each unit square has the same shear flow acting on all its sides.

Figure 10·24 also shows the effect of this shear flow on the horizontal or *flange* members of the frame. This effect is *cumulative*, causing an increasing axial force in the flange members. For the case shown this force is compression in the top and tension in the bottom.

Fig. 10·26. Tension failure of web under shear loading. Note wrinkles at approximately 45°

The value of the axial force thus produced is obviously given by

$$P_a = \Sigma q = qx = \frac{P_s}{h} x \qquad [10\cdot23]$$

(See Fig. 10·24 for symbols)

This simple formula could also have been derived by computing the moment of the force P and dividing by the depth to obtain the equivalent couple. The *shear flow* derivation has the advantage of showing how these so-called bending forces are produced.

Conditions are slightly different for a *tension-field* web and considerably different for a frame which is *tapered* in depth. These modifications will be taken up later.

PROBLEMS

10·1. A 10-in. square sheet is carrying a pure shear stress of 10,000 psi. Assuming that $E = 10{,}000{,}000$ psi and $\mu = 0.30$, find the increase in length of a diagonal. Check by calculating G and from it derive the shear deflection and convert into diagonal deflection (using Eq. 10·3).

10·2. In Fig. 10·22 assign values as follows:

N between 30 and 40 in.; h between 12 and 18 in.
P between 15,000 and 25,000 lb.
Assume $E = 10{,}000{,}000$ psi, $\mu = 0.30$

Calculate:

(*a*) Shear flow.
(*b*) Sheet thickness for which shear stress will be 12,000 psi.
(*c*) Deflection, d, for shear-resistant web.
(*d*) Deflection, d, for pure tension-field web.
(*e*) Axial reactions shown in Fig. 10·24, based on shear flow. (Check by taking moments.)

10·3. A steel plate is carrying a pure shear stress of 20,000 psi.

(*a*) What is the value of the maximum axial strain? (Assume $E = 30{,}000{,}000$ and $\mu = 0.30$.)
(*b*) What is the axial strain at an angle of 22.5° from the direction of zero strain?

10·4. A shear load of 5000 lb is to be transmitted through an available depth of 1 ft. Using an ultimate tension stress of 55,000 psi determine the web thickness required to provide a factor of safety of 2.0 against failure. (Assume that the web acts as a tension-field web.)

10·5. Assume that the aluminum alloy shear web of Problem 10·4 is 48 in. long and calculate:

(*a*) The angular deflection in degrees (due to shear only).
(*b*) The total deflection (due to shear only).

10·6. A shear web 50 in. long deflects 0.2 in. due to shear only. What is the shear strain?

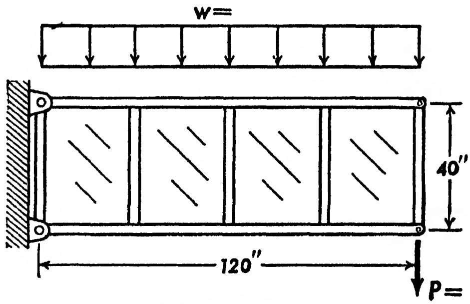
(*a*) If G is 11,000,000 psi, what is the shear stress? (Assume shear-resistant web.)

(*b*) Assume that the web acts under a pure tension field and that the material is aluminum alloy (E = 10,000,000 psi). What is the shear stress?

10·7. Select values for P and w as follows:

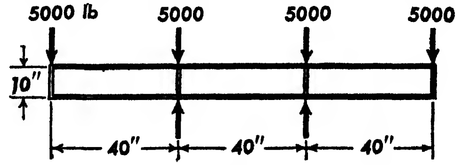
P between 31,000 and 39,000 lb.
w between 160 and 190 lb per in.

Find the thickness required for the shear web, assuming that the maximum allowable shear stress is 8000 psi. Plot the required thickness against the length of the beam. Assume that the design permits only one sheet splice, at the middle of the beam, and find the required thickness for the two sheets, to the nearest $\frac{1}{32}$ in. (*Note.* The design of the stiffeners is a more advanced problem.)



PROBLEM 10·7.

10·8. Assume that, in addition to the concentrated loads shown, a uniformly distributed load, w, acts downward over the entire beam. Select a value for w between 120 and 150 lb per in. Assume uniform shear flow. Draw a curve showing the variation in web thickness for the condition that the allowable shear stress equals 10,000 psi. Select a single thickness for each bay, to the



PROBLEM 10·8.

nearest one-hundredth inch. Indicate on the sketch the direction in which wrinkles would form if the web buckled. (Neglect the design of vertical stiffeners and flanges.)

10·9. An airplane wing (cantilever type) has a total area of 200 sq ft and is 20 in. deep at the centerline of the airplane. The net average wing loading (air pressure less weight of wing) is 16 lb per sq ft. Assume that all the shear load is carried through a single thin-webbed beam and is resisted at the centerline of the airplane. Find the shear flow across the web at this point, for a loading condition in which the net wing loading is multiplied by a load factor of 6.0. If the allowable shear stress is 12,000 psi, what web thickness would be required at this point?

10·10. Assuming that the same ultimate tension stress could be used in each case, determine which would be the lighter design for the shear-resisting member of a square frame: (*a*) a single diagonal tension member, (*b*) a tension-field web. (Neglect weight of attachment to frame.)

Assume that a single tension member can resist no compression and that the frame must be able to resist the shear force in either direction. How does this affect the design and the weight comparison?

10·11. Using Fig. 10·24 as a basis, assume that the web cannot resist compression strains and therefore acts as a *tension-field* web. If the maximum tension stresses act at 45°, prove that the inward pull on the frame members is a distributed normal force having a value equal to the shear flow, q. (*Note.* Assume unit web thickness and work with a diagonal strip of unit width. Convert stress into force before finding the component normal to the frame.)

10·12. From the relationship developed in Problem 10·11 prove that the vertical frame members are being pulled together (by the tension field) with a total lateral force equal to P, hence they exert compression forces on each horizontal frame member equal to $P/2$. Show also that the horizontal frame members are being pulled together with a force equal to Px/h, where x is the distance between verticals.

(*Note.* The preceding three problems represent an introduction to the design of elementary thin-web beams. Further information on the design of stiffeners and joints will be found in texts on aircraft structural design, such as Refs. 2, 3, and 44.)

CHAPTER 11

CURVED SHEAR WEBS

11·1. Curved Shear Flow. It has been shown that a shear force may be transmitted by means of a flat plate or web and that the shear flow q across a section of the web is equal to the force divided by the depth of

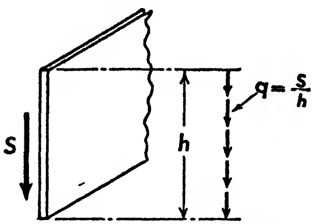


FIG. 11·1. Shear flow (straight).

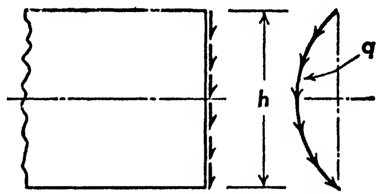


FIG. 11·2. Curved shear web.

the web (Eq. 10·9). Such a shear flow may be visualized as a series of unit forces all acting in a straight line, as shown in Fig. 11·1. By definition, the resultant S of a constant *straight line* shear flow over the distance h equals

$$S = qh$$

It will now be assumed that the web is *curved* and that the shear flow follows the curvature of the cross section, as shown in Fig. 11·2. The *resultant* of such a shear flow may be found by representing the flow as a series of consecutive straight force vectors of unit length and magnitude q. If the lengths are considered to be very short the resulting curve will closely approximate the shape of the web, as in Fig. 11·3. This may now be treated as a vector diagram and the resultant will be equal to the closing line, the length of which is h. Since the *scale* used is equal to q pounds per inch, the resultant shear force is given by

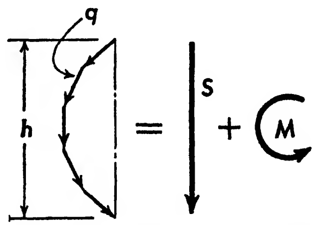


FIG. 11·3. Resultant of curved shear flow.

Resultant of curved shear flow $S = qh$ [11·1]

where q = the shear flow along the curved line.
h = the length of the closing line.

This is obviously the same equation as that for a straight flow, a fact which is very convenient in dealing with curved shear webs. The following general statements can be made.

a. The resultant of a constant shear flow over a curved path has the same magnitude as the resultant of the same shear flow acting in a straight line between the two end points of this path.

b. The shear flow for a *curved* web transmitting a shear load is the same as that for a hypothetical *straight* web between the two end points of the curve (assuming constant shear flow in both the curved and straight web).

In determining the shear stress in a curved web it is therefore unnecessary to take into account the actual path of the web, provided it can be assumed that the shear flow is constant. For example, assume that the leading edge of an airplane wing (Fig. 11·4) must carry a total shear load of 5000 lb. The shear flow in the web is then found from Eq. 11·1.

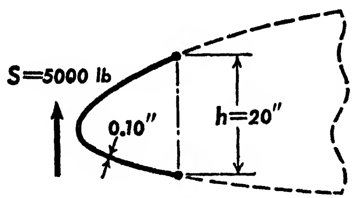


FIG. 11·4. Leading edge of airplane wing.

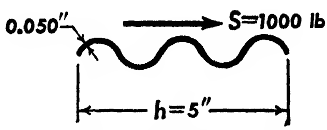


FIG. 11·5. Corrugated shear web.

$$q = \frac{S}{h} = \frac{5000}{20} = 250 \text{ lb/in.}$$

If the web were 0.10 in. thick, the shear stress would be

$$f_s = \frac{250}{0.10} = 2500 \text{ psi}$$

Another typical example is found in the *corrugated* shear web, as shown in Fig. 11·5, where the shear stress would be

$$f_s = \frac{q}{t} = \frac{S}{ht} = \frac{1000}{5 \times 0.05} = 4000 \text{ psi}$$

It is important to note that the equation for shear stress in the corrugated shear web is not the usual one of load divided by *actual* cross-sectional area. In fact, the area used is that of a fictitious straight web of the same thickness. This point would indicate that the straight web is more efficient than the corrugated one. (Actually the curved web usually has a higher *allowable* stress, in thin sheet; hence corrugations are often used in such structures.)

11·2. Moment of a Shear Flow. Figure 11·3 indicates that a curved shear flow must be represented by a resultant shear force and a *moment.* Or, alternatively, the location of the resultant shear force is not actually *on* the line of closure but is displaced from it. This could have been predicted from the appearance of Fig. 11·3, which shows that the vectors have moment arms about any point on the line of closure.

To determine the effective location of the resultant shear force it is necessary to know the turning moment of the shear flow about some convenient point. To develop the simple rule which gives this answer, start with a single force, as shown in Fig. 11·6*a*. The moment of this force about point O is obviously equal to the rectangular area represented by Pd. (Note that if this area is the actual area in square inches it must be multiplied by the scale for the force vector. Also, if the drawing is not to actual size the area must be converted to full size by multiplying again by the square of the drawing scale.) Sketch (*b*) shows that a triangular area may be substituted, if it is then multiplied by 2 to obtain the turning moment.

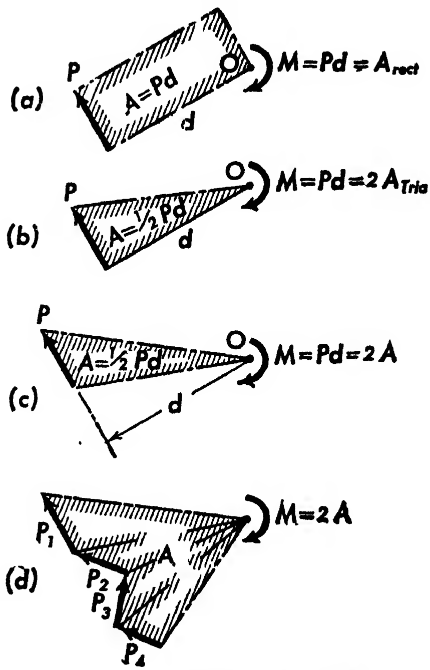


FIG. 11·6. Moment of forces.

Sketch (*c*) shows that the principle still holds regardless of the location or direction of the force, as the area of the triangle is equal to $\frac{1}{2}Pd$ whether it is a right triangle or not. This important principle may be stated as follows.

The moment of a force about a point is equal to twice the actual enclosed area of the triangle formed by the point and the force vector, multiplied by the force scale.

Finally sketch (*d*) shows that the moment of a consecutive series of force vectors drawn to the same scale is proportional to twice the area enclosed by the vector diagram.* This simply amounts to adding the triangles for the different vectors. Since a constant shear flow, q, can be considered as a series of force vectors drawn to a scale, q, this principle can be applied to find the resulting moment.

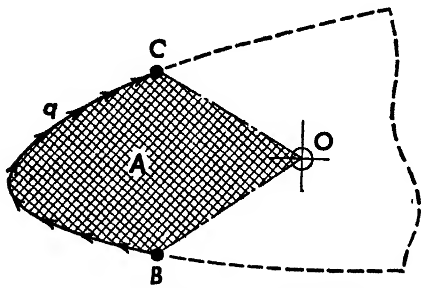


Fig. 11·7. Moment of shear flow.

Assume, for instance, that it is necessary to determine the moment of a constant shear flow in the leading edge of an airplane wing, about some reference point, O (Fig. 11·7). Lines OB and OC are drawn, and the shaded area is measured or computed. If the actual area turned out to be 800 sq in., and the shear flow had a value of 100 lb/in., the moment would be

$$M = 2qA = 2 \times 100 \times 800 = 160{,}000 \text{ in.-lb}$$

Note that if the drawing were actually laid out quarter-size and the measured area had been 50 sq in., it would have had to be multiplied by 16.

The foregoing rules may be summarized by the basic equation,

Moment of shear flow $$M = 2qA \qquad [11\cdot 2]$$

where M = the moment about a point in the plane of the cross section.
q = the constant shear flow between two points on the cross section.
A = the actual area of cross section enclosed by the triangle formed by the three points, plus the area enclosed by the contour over which the shear flow acts.

Equation 11·2 may also be used for a variable shear flow by dividing the shear web into segments over each of which the shear flow is assumed to be constant. The area for each segment must then be determined and used in Eq. 11·2. The total moment is of course obtained by summation.

11·3. Location of Resultant Shear. It is now apparent that although the magnitude and direction of the resultant shear force in a curved web may be based on the closing line, the location of the resultant will not be on this line itself. To determine the effective location it is necessary

* Note that in Fig. 11·6*d* the *space* diagram and *force polygon* are the same.

to find the moment of the shear flow. If this is done with respect to any point on the closing line, the resulting moment will equal twice the area between the web and the line. (Note that only *one* line is needed to enclose the area in this case.)

Figure 11·8 shows the equivalent conditions for a typical curved web. The effective moment arm of the resultant shear, often called the **shear center**, is obtained from the equation

$$x = \frac{M}{S} = \frac{2qA}{qh}$$

Shear center $$x = \frac{2A}{h} \qquad [11 \cdot 3]$$

From Fig. 11·8 it can be seen that the term A/h represents the average height b of the rectangular shape having the base h and the area A. The location of the resultant shear force may therefore be estimated quite accurately by this simple rule.

It is now evident that the resultant will lie outside of the web itself. This may seem to be irrational, but it is actually true and has an important bearing on the behavior of curved shell structures.

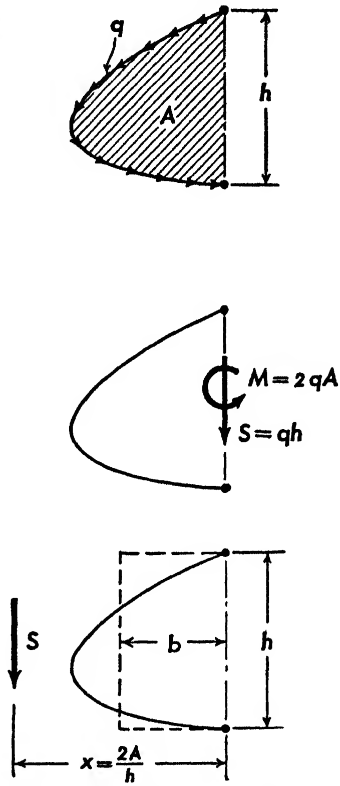


FIG. 11·8. Location of resultant shear (shear center).

11·4. The Single Curved Web. In Chapt. 10 the two-flange type of beam was used in discussing the straight shear web. If this beam is made with a curved web, it will be able to transmit a shear force, but only if the force is applied at the *shear center* and parallel with the plane of the two flanges, as shown in Fig. 11·9. If it is applied along any other line of action, it will be equivalent to loading a straight web beam off center. This is illustrated by the sketches in Fig. 11·10. Either of these two loadings will cause an unbalanced turning moment wnich the beam alone is unable to resist.

The *shear center* determined by the foregoing methods is based on constant shear flow, which occurs only in the thin-web, two-flange type of beam. For other types of beams (such as channels), the location of the shear center will be influenced by the variation of shear flow across the beam. This will be taken up later.

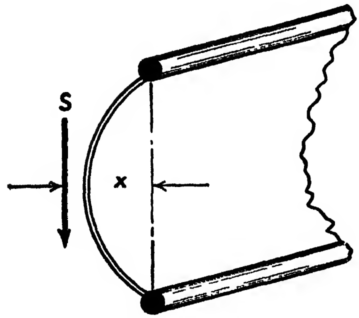


FIG. 11·9. Proper method of loading curved web beam.

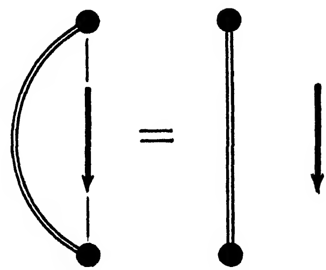


FIG. 11·10. Effect of loading in plane of flanges.

11·5. Shear Flow vs. Axial Force Flow. It will be noted that in treating shear flow in curved webs the shear forces were apparently regarded as axial forces. There is an important distinction, however, which must be kept in mind. The shear forces shown in the plane of the paper are actually being transmitted *normal* to this plane, whereas axial forces thus shown are transmitted and resisted *in* the plane.

11·6. Diagonal Stresses. Section 7·4 might indicate that the diagonal tension and compression fields in a curved shear web would require additional reactions normal to the sheet. Actually, however, these two fields are equal and opposite in pure shear, as explained in Chapt. 10 and shown in Fig. 11·11, and the tendency of the compression forces to bow the sheet outward is exactly offset by the tendency of the tension forces to pull it inward. It is this phenomenon which makes it possible to transmit forces "around a corner" in pure shear without requiring normal (radial) reactions.

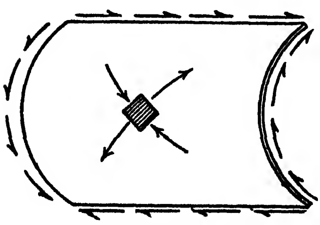


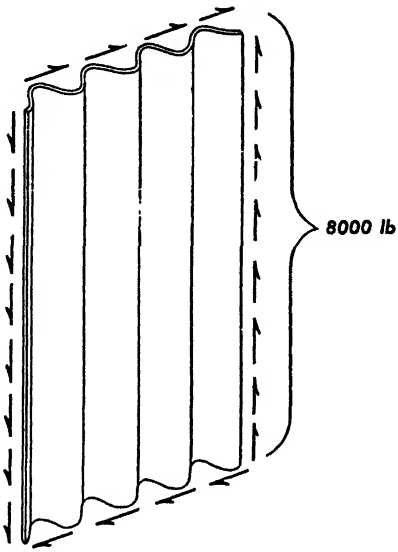
FIG. 11·11. Diagonal stresses in curved shear web.

As in the flat plate, the diagonal compression stresses may cause buckling under certain conditions. In general, the curved plate has a higher buckling stress than the flat one of the same thickness and dimensions. After buckling occurs the tension field will predominate, and the balance will be disturbed; the tension field will pull the sheet inward along the diagonal, tending to make the web flatten out.

The calculation of the critical buckling stress for curved plates in shear is beyond the scope of this volume. For additional information see Refs. 2, 3, 16, 26, 41, or 44.

PROBLEMS

11·1. A rectangular corrugated shear web is 20 in. long on the straight side and 10 in. across the corrugation. The total shear load to be transmitted across the short dimension is 8000 lb, and this load is uniformly distributed over the length of the web. If the web thickness is 0.040 in., what is the shear stress acting on a cross section normal to the corrugations?



PROBLEM 11·1.

11·2. Draw a curved web such as shown in Fig. 11·9, using any convenient depth and radius. Assume that a shear load S of 5000 lb is to be transmitted.

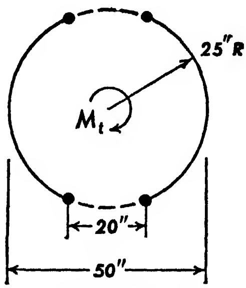
(a) Calculate the shear flow.
(b) Find the proper location for the application of the load (dimension x).

11·3. Referring to Fig. 11·4, assume that the shear web has a depth h equal to 18 in. and that the cross-sectional area of the web itself is 3.0 sq in. If the web is 0.10 in. thick, what shear stress is produced by a shear load of 9000 lb, applied so as to produce constant shear flow?

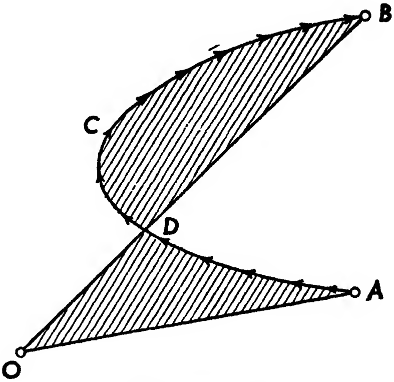
11·4. Draw a curved shear web representing the nose of an airplane wing (as in Fig. 11·4).

(a) Determine the enclosed area and the location of the resultant of a constant shear flow.
(b) What unbalanced torsional moment would be induced by applying a shear force of 5000 lb in the plane of the flanges, i.e., across the free edges of the web?

11·5. Assume that a torsional moment, M_t, of 100,000 in.-lb is applied to the shell structure shown and that this moment is resisted by constant shear flow in the curved webs. (Dotted lines represent cut-out material.) What is the value of the shear flow?



PROBLEM 11·5.

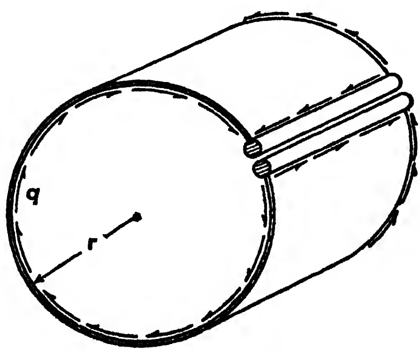


PROBLEM 11·6.

11·6. The sketch shows the determination of the moment of a shear flow about point O. Explain why it is unnecessary to draw a line OC. Do the areas OAD and BCD add or subtract? (Explain by following procedure of Fig. 11·6 of the text.)

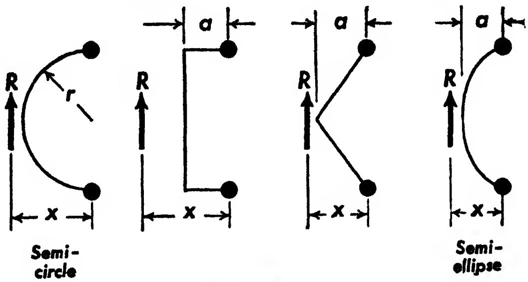
11·7. Make a scale drawing of a curved web and determine the location of the resultant shear flow as shown in Fig. 11·8. Check this by finding the moment of the curved shear flow about any point on the line of action of the resultant, using the construction shown in Problem 11·6.

11·8. Assume that the two flanges of a curved web beam are brought together as shown, so that the web forms a complete circle. Find the value of the resultant shear flow and its moment about the center of the circle, in terms of q and r. (Neglect the slight gap that would actually be required if two separate flanges were used.)



PROBLEM 11·8.

11·9. Find, analytically, the location of the resultant of a constant shear flow for the cases shown, expressing x in terms of the dimensions given.



PROBLEM 11·9.

CHAPTER 12

TORSION

12·1. Thin Shells. The thin shell in torsion may be regarded as a curved shear web in which the free edges are joined, making a closed section. If the wall thickness of the shell is small in relation to the enclosed cross section, the shear stress may be assumed to be constant over the thickness of the sheet. The resultant shear flow will then be located at the median line of the sheet.

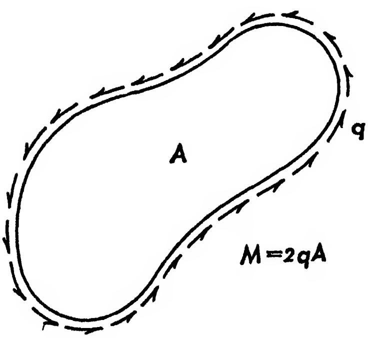


FIG. 12·1. Moment of closed shear flow.

If it is further assumed that the shear flow is constant around the section, the following statements will be true.

a. The resultant force for a constant shear flow around any closed section will be zero. (This is because the ends of the vector diagram close.)

b. The resultant moment will be equal to twice the enclosed area, multiplied by the shear flow. (This follows from Eq. 11·2, for the curved web.)

The second point is expressed mathematically by the equation:

Moment of shear flow $$M = 2qA \qquad [12\cdot 1]$$

where M = the torsional moment, or torque.*
q = a constant shear flow around the section.
A = the area enclosed by the median line of the shell (Fig. 12·1).

Since it is more often required to find the shear flow caused by a given torsional moment the equation is usually found in the form

Shell torsion $$q = \frac{M}{2A} \qquad [12\cdot 2]$$

* Other symbols sometimes used: T, M_t.

This will be called the shell torsion equation. It may be regarded as the basic equation for the torsion of thin-walled shells and tubes. It will be found under various names, such as the *membrane analogy*, *Batho's equation*, and *Bredt's formula*. Actually it is elementary in nature since it is derived without using the more general equations and theories developed for torsional stresses. The only assumption involved is that of a *constant shear flow*. The accuracy of the results depends entirely on how nearly this assumption is realized in the structure.

Dividing Eq. 12·2 by t gives the formula for shear stress.

Torsional shear stress (shells)

$$f_s = \frac{M}{2At} \qquad [12\cdot3]$$

The value of the stress at any point around the cross section will be found by using the thickness t at that point, since q is assumed to be constant.

12·2. Stresses in Circular Shells (Tubes). The shear flow in a thin-walled circular shell is readily derived from Eq. 12·2 by substituting πr^2 for A.

$$q = \frac{M}{2A} = \frac{M}{2\pi r^2} \qquad [12\cdot4]$$

$$f_s = \frac{q}{t} = \frac{M}{2\pi r^2 t} \qquad [12\cdot5]$$

Example. The tube of Fig. 12·2 has an average radius (to median line of shell) of 10 in. and is subjected to a torque of 100,000 in.-lb. The shear flow is found from Eq. 12·4.

$$q = \frac{M}{2\pi r^2} = \frac{100{,}000}{2\pi \times 10 \times 10} = 159 \text{ lb/in.}$$

The shear stress is found by dividing by the thickness.

$$f_s = \frac{159}{0.08} = 1990 \text{ psi}$$

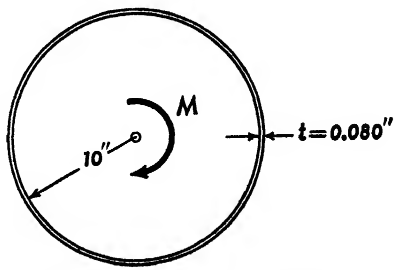


FIG. 12·2. Torsion in thin-walled tube.

If the wall thickness of the tube should vary around its circumference, the stress at any point is obtained by dividing the shear flow q by the thickness at the point in question. Thus if a portion of the tube of Fig. 12·2 were made out of 0.020-in. sheet the stress would be, over that portion,

$$f_s = \frac{q}{t} = \frac{159}{0.020} = 7950 \text{ psi}$$

Equation 12·5 shows that the maximum torque which may be carried by a thin cylindrical shell equals $2\pi r^2 t f_s$. If the allowable stress f_s is assumed to be constant, the strength is proportional to $2\pi r^2 t$. The cross-sectional area of the shell wall is given by $2\pi r t$. Hence the ***strength-weight*** factor, for constant allowable stress, is

$$\frac{\text{Torsional strength}}{\text{Weight}} = \frac{2\pi r^2 t}{2\pi r t} = r$$

This shows that large diameter tubes are basically more efficient in carrying torsion. However, the allowable stress tends to decrease as the ratio r/t is increased, thus offsetting this influence to a considerable extent. It should also be noted that the strength must be based on the

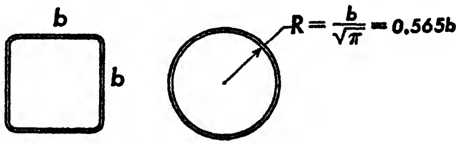


FIG. 12·3. Comparison of square and round tubes (equal enclosed area).

smallest wall thickness, if the thickness varies. The most efficient design for torsion, therefore, is that of *constant wall thickness.*

For a given wall thickness, t, and enclosed area, A, the minimum weight will occur when the developed length of the cross section is a minimum. Since the *circle* has the minimum ratio of developed length to enclosed area, it is the most efficient shape for carrying torsion.

For example, compare a square tube with a circular one having the same enclosed area, as in Fig. 12·3. The radius of the circle will be $r = b/\sqrt{\pi} = 0.565b$. For the same torsional moment, each of these tubes will develop the same shear flow. If they have the same wall thickness, the square one will be heavier in proportion to the periphery, or developed length,

$$\text{Weight} \qquad \frac{\text{Square}}{\text{Circular}} = \frac{4b}{2\pi r} = \frac{4b\sqrt{\pi}}{2\pi b} = \frac{2}{\sqrt{\pi}} = 1.13$$

12·3. Torsional Deflection of Circular Shell. In Sec. 10·10 it was shown that the shear deflection of a flat plate is measured in terms of an angle, ϕ, representing the deflection divided by the distance over which the shear force is being transmitted. In Fig. 12·4 this angle is shown for a circular shell. Its value is found from Eqs. 10·15 and 10·14:

$$\phi = \frac{d}{L} = \frac{f_s}{G} \qquad [12\cdot6]$$

where ϕ is measured in radians.

In calculating torsional deflections it is more convenient to use the angle of *twist*, indicated as θ in Fig. 12·4. This angle is equal to d/r (in radians). Using the shear deflection equation to determine d, the angle of twist in a *unit* length is found to be (since $L = 1.0$)

Unit torsional deflection (circular shell)

$$\theta = \frac{f_s}{Gr} \qquad [12\cdot7]$$

where θ is measured in radians.
f_s = the shear stress.
G = the modulus of rigidity (Sec. 10·11).
r = the radius of the shell.

Note that this compares directly with the equation for unit *axial* deformation, $e = f/E$ (Eq. 6·3).

The total twist over a length L is given by

$$\theta_L = \frac{f_s L}{Gr} \qquad [12\cdot8]$$

If θ varies over the length L, the total twist will be obtained by summation or integration (area under curve of θ plotted against L).

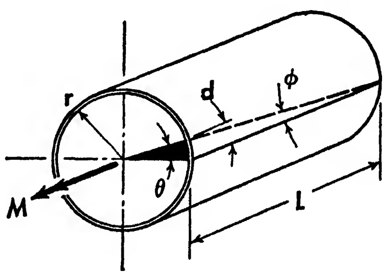


Fig. 12·4. Torsional deflection (shells).

It is interesting to express Eq. 12·7 in terms of the torsional moment. This may be done by substituting for f_s from Eq. 12·3, giving

$$\theta = \frac{M}{G\,2Atr} = \frac{M}{G\,2\pi r^3 t} \qquad [12\cdot9]$$

This equation shows that the *torsional rigidity* of a thin-walled circular shell is proportional to the wall thickness and the cube of the radius, that is, the torque required to twist a unit length through a unit angle equals $2\pi r^3 tG$. The cross-sectional area of the wall is given by the term $2\pi rt$. For shells of the same material G is constant; hence the *stiffness-weight* factor for such a structure is

$$\frac{\text{Torsional stiffness}}{\text{Weight}} = \frac{2\pi r^3 t}{2\pi rt} = r^2$$

This relationship indicates the great importance of using large diameter tubes or shells where high torsional rigidity is required, as in airplanes. In an airplane fuselage, for instance, maximum torsional rigidity is obtained by locating the structural material as near the surface as possible.

12·4. Torsional Deflection of Non-Circular Shell. Although the exact derivation of the general torsional deflection equation is beyond the scope of this text, an analogy can be presented which will give a clear picture of the problem and enable its solution in a simple manner. Consider any arbitrary cross section, such as shown in Fig. 12·5. If an equivalent *circular* shape can be found which has the same torsional rigidity, its radius $\bar{r}$ may be used in the deflection equation for the circular shell.

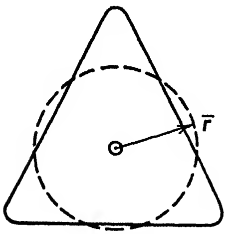


FIG. 12·5. Equivalent cross sections (shells).

The value of the *effective radius* may be determined from the following equation.

$$\bar{r} = \frac{2A}{s} \qquad [12\cdot10]$$

where A = the enclosed area.
s = the developed length (perimeter) of the cross section.

For a variable cross section the shear stress will vary inversely as the wall thickness. An *effective stress,* $\bar{f}_s$*,* must therefore be used. This is obtained as the integral or summation of stress over the periphery of the cross section (average height of area obtained by plotting stress against developed length around section).

Using the "bar" notation to denote "effective" values, the basic equation for torsional deflection of *any shell having constant shear flow* may be written

Unit torsional deflection
$$\theta = \frac{\bar{f}_s}{G\bar{r}} \qquad [12\cdot11]$$

where θ = the angle of twist (radians) per unit length.
$\bar{f}_s$ = effective shear stress (average over periphery of section).
$\bar{r}$ = effective radius (Eq. 12·10).

Equation 12·11 usually appears in the following form.

$$\theta = \frac{M}{4A^2G}\int\frac{ds}{t} \qquad [12\cdot12]$$

This can be shown to be identical with Eq. 12·11 by substituting $2A/s$ for $\bar{r}$ and expressing f_s as $M/2At$ (Eq. 12·3). The integral term can be thought of as the area under a curve of $1/t$ plotted against s. For *constant wall thickness* Eq. 12·12 becomes

$$\theta = \frac{Ms}{4A^2Gt} \qquad [12\cdot13)$$

12·5. Warping of Cross Section. As noted in Sec. 12·1 the accuracy of the shell torsion theory depends entirely on how nearly the actual shear flow approaches the assumed constant value around the cross section. If the wall thickness is small in relation to the radius (or equivalent dimension) the *radial* variation in shear flow is negligible. For thick shells or solid sections this is not true and special methods must be used.

Another type of error is caused, sometimes, by *constraint* of the cross section. This can be illustrated by assuming that a shell, under torsion,

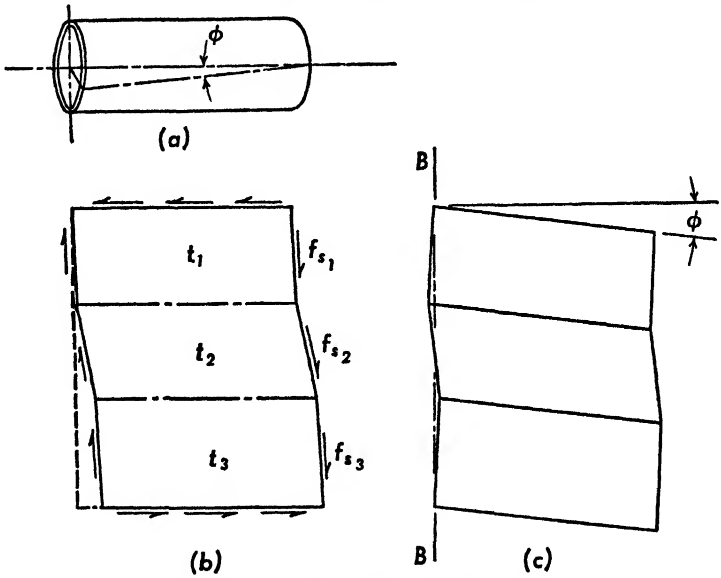


FIG. 12·6. "Warping" of cross section.

is cut open and flattened out, at the same time holding the distorted shape of the wall material. If the wall thickness varies the shear stress will vary. This causes a variation in shear strain. The result might appear as shown in Fig. 12·6, which represents a *circular* shell of varying wall thickness. If it is assumed that the shell is composed of three separate sheets of different thickness, the shear deflection of each will be different, as indicated in (*b*). Since the free edges must be brought together again in the actual shell, it is necessary to rotate the picture as shown in (*c*) so that the ends are normal to the centerline of the shell. If this distorted sheet is rolled back into a tube, it is obvious that the ends will be *warped*, that is, they will not lie in the original plane. The degree of warping is indicated by the departure from the straight line *B-B* which represents the unwarped cross section.

If the shell has a non-circular cross section the *unwarped* line will not be straight but will depend on the shape of the section. Even a constant shear stress will then require some warping of the cross section.

It can be seen, then, that for constant shear flow to be maintained the cross section must be *free to warp*, except when the deflected shape of the sheet agrees with that of the unwarped, but twisted, shell. Such a special case is found in the circular shell of constant wall thickness.

If one of the ends of a shell is rigidly attached to a flat surface, the cross section will be forced to remain plane. The tendency to warp will induce *axial* stresses, the nature of which can be predicted by the flat pattern method previously described. The shear flow distribution around the cross section will no longer be constant where this occurs. Fortunately, the effects of constraint are not usually large and are confined mainly to the local region near the plane of constraint. It is necessary only to know that this effect exists and to realize the conditions under which it might be serious.*

12·6. The Solid Round Bar. The general solution of the torsion problem for all cross-sectional shapes is one of the basic problems in the theory of elasticity, and a large amount of literature is available on it.† The reason for this is that the exact solution for any shape except a few simple ones becomes most complex mathematically.

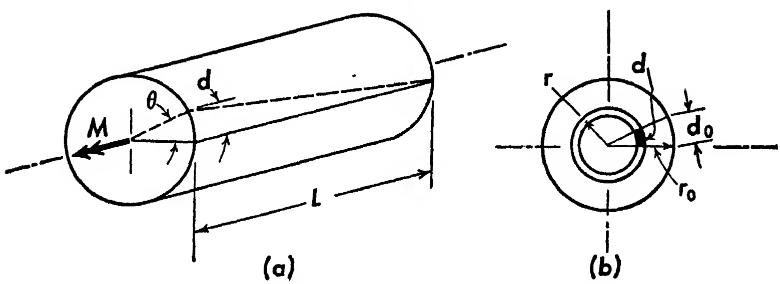


FIG. 12·7. Torsion in solid bar.

The most common shape used to transmit torsion is that of the round bar or tube. It is also one of the few shapes for which the problem has a simple solution. Since the thin-walled tube has already been discussed, it is convenient to make use of it in deriving the equations for the solid bar.

For a solid bar such as shown in Fig. 12·7*a* it will be assumed that all points rotate through the same angle, θ; i.e., radial lines drawn on the

* The root of an airplane wing sometimes requires special checking on this point because of the end plate effect of the plane of symmetry.

† The general solution was first obtained by Saint-Venant.

cross section would remain straight during twisting. Hence, for any tubular element such as indicated in (*b*), the value of d will be directly proportional to the radius, i.e.,

$$d = d_o \frac{r}{r_o}$$

Figure 12·4 and Eq. 12·6 show that the shear *stress* is proportional to d/L (in the elastic range). It follows that the stress is directly proportional to r, giving a linear variation of shear stress from the center to the outer fiber.

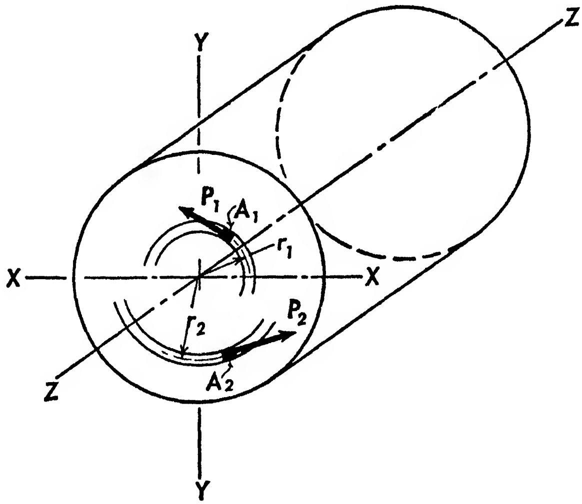


FIG. 12·8. Computation of torsional moment.

The total resisting moment is composed of the individual moments of each unit area about the center of rotation. The resisting moment will be calculated by arbitrarily assuming that the shear stress equals the radius (compare Sec. 13·6, on bending). Refer to Fig. 12·8 and assume that

$$f_s = r \qquad [12\cdot 14]$$

Then

$$P_1 = A_1 f_1 = A_1 r_1$$

$$P_2 = A_2 f_2 = A_2 r_2, \text{ etc.}$$

The individual moments are found by multiplying by the radii.

$$M_1 = P_1 r_1 = A_1 r_1^2$$

$$M_2 = P_2 r_2 = A_2 r_2^2, \text{ etc.}$$

The total resisting moment than equals the sum of all these, for the entire cross section.

$$M_0 = \Sigma A r^2 \qquad [12 \cdot 15]$$

where M_0 is the resisting moment corresponding to the assumption $f_s = r$.

The quantity so obtained is called the *polar moment of inertia* (*or moment of area*), and will be designated by I_p (other symbols are I_ρ, J, etc.) Equation 12·15 may therefore be written

$$M_0 = I_p \qquad [12 \cdot 16]$$

For any other torsional moment the stresses will be increased by the factor M/M_0 or M/I_p, and the assumed stresses (Eq. 12·14) must be multiplied by this factor, giving

Torsion in circular sections $$f_s = \frac{Mr}{I_p} \qquad [12 \cdot 17]$$

where f_s = the shear stress caused by the torsional moment M.
r = the radius measured from the axis of twist (centerline of bar).
$I_p = \Sigma A r^2$ = polar moment of inertia of cross section.

The polar moment of inertia is similar to the moment of inertia used in computing *bending* stresses (see Chapt. 13). In analyzing bending the axis about which the second moments of area are taken lies *in* the place of the cross section, whereas in torsion the turning axis is *normal to* the plane of the cross section. Because of this the polar moment of inertia is larger than the moment of inertia about any axis in the plane of the cross section. It is, in fact, equal to the sum of the moments of inertia about any two mutually perpendicular axes.

Polar moment of inertia $$I_p = I_x + I_y \qquad [12 \cdot 18]$$

For a doubly symmetrical cross section (such as a circular one) $I_x = I_y$. Therefore

$$I_p = 2I_x$$

where I_x is the second moment of area about any axis through the center of gravity.

For a *solid round bar* the value of the polar moment of inertia is

$$I_p = \tfrac{1}{2}\pi r^4 \qquad [12 \cdot 19]$$

The maximum shearing stress occurs in the outer fiber and is equal to

$$f_{s\,\max} = \frac{Mr}{I_p} = \frac{Mr}{\frac{1}{2}\pi r^4} = \frac{2M}{\pi r^3} \qquad [12\cdot 20]$$

Or, since the cross-sectional area $= \pi r^2$,

$$f_{s\,\max} = \frac{2M}{Ar} \qquad [12\cdot 21]$$

12·7. Thick-Walled Tubes. Equation 12·17 also applies to *thick-walled tubes*, for which the polar moment of inertia is given by

$$I_p = \tfrac{1}{2}\pi(r_o^4 - r_i^4) \qquad [12\cdot 22]$$

where r_o = the outer radius.
r_i = the inner radius.

For a thin-walled circular section such as discussed in Secs. 12·1 to 12·3, the polar moment of inertia is equal to

$$I_p = A_s r^2 = 2\pi r t r^2$$
$$= 2\pi r^3 t$$

where A_s = the cross-sectional area of the shell wall.
r = the radius to the median line of the shell wall.
t = the shell thickness.

By substituting this value in Eq. 12·17,

$$f_s = \frac{Mr}{I_p} = \frac{Mr}{2\pi r^3 t} = \frac{M}{2\pi r^2 t}$$

This checks Eq. 12·5, which was derived on the basis of constant shear flow. It can be seen that for circular sections Eq. 12·17 is the more general formula, as it can also be used for the very thin wall. The shell equation, however, is applicable to any cross-sectional shape for which the shear flow can be assumed to be constant along a median line, and is therefore more general, in this respect, than either Eq. 12·5 or Eq. 12·17.

12·8. Non-Circular Solid Bars. The use of the polar moment of inertia in Eq. 12·17 implies that the resisting force in each element acts normal to the radius. If this reasoning were applied to a non-circular cross section, it would be found that shear forces exist which have components normal to the free outside surface, as indicated in Fig. 12·9. This is clearly impossible, and the problem is therefore complicated by a boundary condition requiring the shear stresses in the outer fibers to be parallel with the surface.*

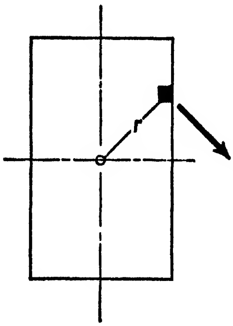


Fig. 12·9.

Although the solution of the torsion problem for non-circular cross sections is beyond the scope of this text, it is useful to know in general how the stresses vary over such a section. One method of showing this is to consider the section to be made up from a number of concentric tubes or shells, each of which is carrying the *same shear flow.* If the lines defining these tubes could be determined, it would be a simple matter to compute the total resisting moment for a given shear flow. From this relationship and the known wall thickness of the tubes, the shear stress at any point could be determined.

12·9. The Membrane Analogy. The location of these lines of constant shear flow can be visualized by the aid of the *membrane analogy.*† It was shown by Prandtl that the equation for the deflection of a membrane under a uniformly distributed normal loading was basically the same as that developed by Saint-Venant for the torsion problem. Assume that a thin membrane is stretched across an opening having the shape of the cross section, and loaded by a uniform load, as shown in Fig. 12·10. Contour lines are obtained by passing planes through the deflected membrane parallel with the plane of the cross section and at constant spacing. It can be shown that the lines so obtained define tubes, which all have the *same shear flow.* Since shear stress equals q/t, the stress is proportional to the *slope* of the membrane.

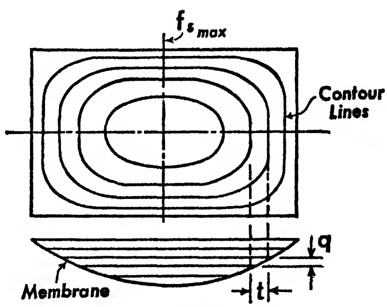


Fig. 12·10. Rectangular bar in torsion (membrane analogy).

The contour lines so obtained also define the *direction* of the maximum shear stresses, but do not represent lines of constant shear stress (except

* There will, of course, be shear stresses acting normal to those which appear on the cut section, but they are normal to the plane of the paper and cannot be shown in the same view.

† See Ref. 18 for complete description.

for circular sections). For any given tube the shear stress will be a maximum where the thickness (distance between contour lines) is a minimum. If the tubes are closely spaced, i.e., thin-walled, the shear stress may be assumed to be constant from one contour line to the next.

From Fig. 12·10 it will be seen that the tubes become narrower as the edges of the cross section are approached. In accordance with the linear shear stress distribution found for the circular section, the thickness of the tubes tends to vary inversely as the distance from the center of twist. This center is, of course, located where the shear stress is zero, determined by the point of zero slope (maximum deflection) of the membrane.

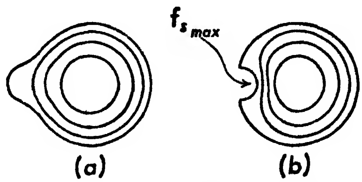


FIG. 12·11. Local effects in torsion.

Since the moment of any thin-walled tube equals the shear flow multiplied by twice the enclosed area, the total resisting moment of the solid section is proportional to twice the *volume* represented by the deflection of the membrane—each slice in Fig. 12·10 represents the product of q and the area enclosed by the corresponding contour line.

The *membrane analogy* is most useful in visualizing the effects of various modifications in cross-sectional shape. For instance, a projection on a circular bar will not be highly stressed in torsion, as the membrane would remain relatively flat over this projection (Fig. 12·11*a*). A depression, or slot, would tend to have the opposite effect, causing high local shear stresses as indicated by Fig. 12·11*b*.

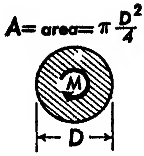
It will be noted from Fig. 12·10 that the contour lines tend to have an elliptical shape near the center and that as they approach the outside contour of the cross section they conform more closely to the external shape. Since the tubes must be continuous around the cross section, the contour lines will be forced closer together in the narrower portions. This means that the maximum shear stress will tend to occur across the shortest dimension.

12·10. Special Cases. Although the solution of the torsion problem has been worked out for many different cross sections, only a few of the more common ones will be given here. Table 12·1 gives the equations for shear stress and angle of twist in engineering form (based on Ref. 18).

It is interesting to note that for long slender rectangles the constant in the shear stress equation (Eq. *f*) approaches a value of 3. This equation may be used for calculating the shear stress in *open sections,* such as indicated in Table 12·1. If the ratio of width to thickness is large, the shape of the cross section will have little effect on either stress or tor-

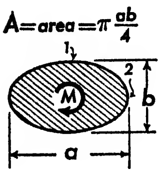
TABLE 12·1

TORSION OF SOLID BARS



$$f_{s\,max} = \frac{4M}{AD} = \frac{16M}{\pi D^3} \qquad [a]$$

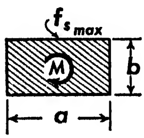
$$\theta = \frac{8M}{AD^2G} = \frac{32M}{\pi D^4 G} \quad \text{(radians per unit length)} \qquad [b]$$



$$f_{s1} = \frac{4M}{Ab} = \frac{16M}{\pi ab^2} \quad \text{(max. value)} \qquad [c]$$

$$f_{s2} = \frac{4M}{Aa} = \frac{16M}{\pi a^2 b} \qquad [d]$$

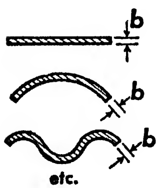
$$\theta = \frac{4(a^2 + b^2)M}{Aa^2b^2G} \quad \text{(rad. per unit length)} \qquad [e]$$



$$f_{s\,max} = \frac{K_1M}{Ab} = \frac{K_1M}{ab^2} \qquad [f]$$

$$\theta = \frac{K_2M}{Ab^2G} = \frac{K_2M}{ab^3G} \qquad [g]$$

(See Fig. 12·12 for values of K)



$$f_{s\,max} = \frac{3M}{Ab} \qquad [h]$$

$$\theta = \frac{3M}{Ab^2G} \qquad [i]$$

sional rigidity. This reveals the fact that a *split* tube is very inefficient in carrying torsional moments.*

The torsional rigidity of a solid section can be estimated with considerable accuracy by assuming that it is an elliptical section having the same area (according to Saint-Venant). For built-up sections composed of several bulky portions interconnected by relatively thin webs the torsional rigidity may be estimated by considering the parts to act separately, through the same angle of twist.

12·11. Warping. As in the shell theory, it is assumed that the cross sections are free to warp. The extent of such warping depends on how

* It must be remembered that the cross sections are assumed to be free to warp. If a split tube is prevented from warping at the ends, it will be able to carry a much greater amount of torsion and will be much more rigid. This problem, however, involves bending and axial stresses and is beyond the scope of this text.

radically the shape departs from that of a circle, for which the warping is zero. For long slender bars the effects of end restraint disappear at a reasonably short distance from the end. (Saint-Venant's principle, see Sec. 7·9.) For short bars of irregular cross section (such as I-beams) the effects of end constraint may be so great as to change the behavior completely. This applies also for short discontinuities or cut-outs. The

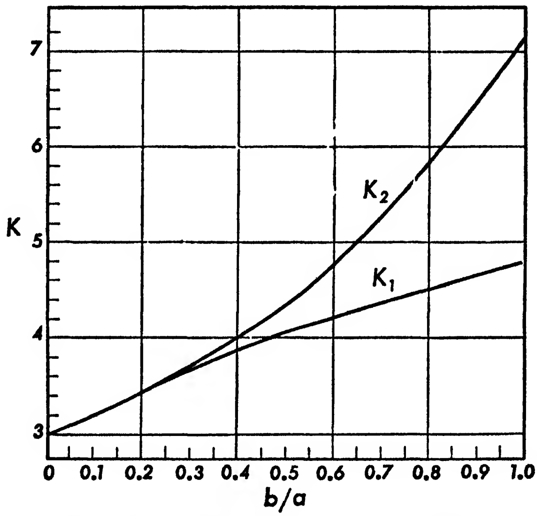


Fig. 12·12. Constants for rectangular bar in torsion, from Ref. 18. (See Table 12·1.)

classical torsion theory must therefore be used with caution under such conditions.

12·12. Plastic Range. Since the moment transmission is actually accomplished through diagonal *axial* stresses, it can be seen that high values of shear stress may cause these axial stresses to exceed the proportional limit. This will cause a reduction in E and hence a reduction in the effective modulus of rigidity G (since this depends on E, see Eq. 10·18). Further increases in shearing strain beyond this point will no longer cause the shear stress to increase proportionately. A plastic region will than be developed which increases in size as the applied torsional moment is increased. In this region the shear stress is not proportional to shear strain. In the extreme case the entire cross section would be in the plastic region.

If it is arbitrarily assumed that the shear stresses are constant over the whole cross section, the slope of the membrane (in the membrane analogy) would have to be constant. Nádai [18] has shown that this can be accomplished experimentally by replacing the membrane with a *sand* *heap* in which the force of gravity acting on the sand replaces the assumed normal pressure on the membrane. The angle of repose, being constant, represents the constant shear stress over the whole cross section.

Nádai has gone further and combined the membrane and sand heap analogies to determine experimentally the stress distribution in the plastic region. For additional information on this interesting subject see his book *Plasticity*.[13] See also Ref. 15 for other experiments in this field.

12·13. Axial Stresses in Torsion. Although it has been convenient to deal with shear stresses in developing the equations for torsion, it must

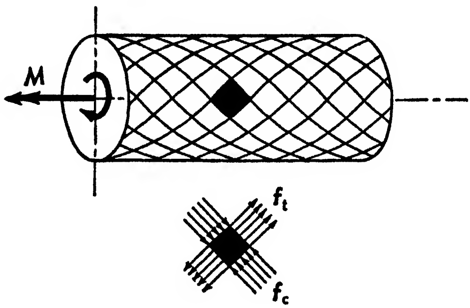


FIG. 12·13. Axial stresses in torsion.

be recognized that axial forces are required to transmit any sort of loading over a finite distance (see Chapt. 8). In torsion these take the form of diagonal tension and compression stresses which "spiral" around the member at an angle of approximately 45°, as shown in Fig. 12·13.

This conception will help to explain why the torsional shear stresses are low in projecting portions or sharp external corners (as on a square bar). The diagonal axial stresses tend to take a short cut in their spiral path through the member. Conversely, a sharp depression or internal angle will tend to cause these axial stresses to pile up, giving rise to increased shear stresses at such points.

As noted in Sec. 11·6, the equality of the diagonal tension and compression fields results in an exact balance between the inward and outward radial forces produced.* In thin shells, however, the compression field may cause buckling, after which the shell will either fail completely or continue to resist the torsion through a diagonal tension field. Special stiffening members would then be required to resist the radial and longitudinal components of the tension field.

* The *geodetic* form of construction employs individual members arranged spirally along lines such as shown in Fig. 12·13. Such structures have been successfully used in aircraft employing fabric covering.

12·14. Failure in Torsion. Torsional failures may occur in several different ways. Obviously, there is a possibility of true *shear* failure by sliding action, and for this reason the maximum allowable shear stress should not be exceeded. Owing to plastic action, however, the assumption of a linear stress distribution in a solid bar may give fictitiously high stresses in the outer fibers, as previously noted, and the torsional modulus of rupture may at times exceed the allowable shear stress

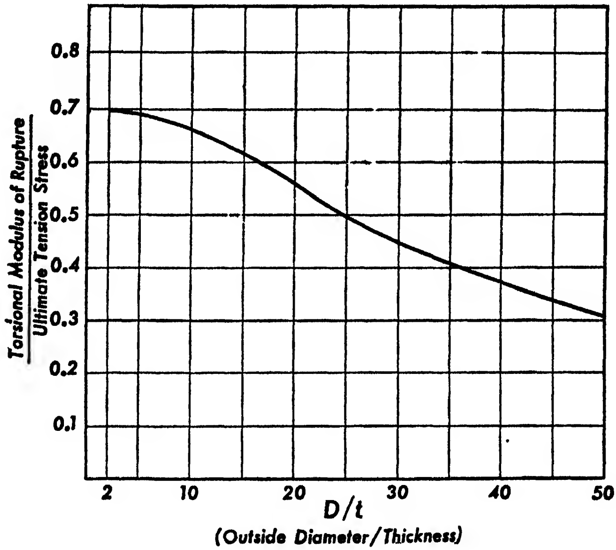


FIG. 12·14. Approximate torsional modulus of rupture for metal tubes.

determined for uniform stress distribution (compare Sec. 15·5, on plastic bending). Thus the modulus of rupture obtained by twisting solid round bars may be too high for use in designing a thick-walled tube.

The most probable type of failure in relatively thin tubes or shells is by buckling. The critical buckling load may be expressed as a shear stress (although it is actually caused by diagonal compression), and it depends to a large extent on the ratio r/t. (D/t is more convenient to use for tubes.) The location of stiffening members or bulkheads is also important. Although a detailed discussion of allowable stresses is beyond the scope of this volume, Fig. 12·14 has been included to show approximately how the torsional strength is affected by the D/t ratio.

It is possible also for the diagonal *tension* stresses to cause failure in certain cases. This can be true, however, only when the walls of the tube are very thick (or solid) and when the material is a brittle type, i.e., weaker in direct tension than in shear. Such brittle conditions are sometimes reproduced in impact, or failure under repeated stress (fatigue).

For members subjected to rapidly repeated or alternating torsional loads of considerable magnitude the *endurance limit* in shear must be used as an allowable stress. This is usually quite low, as compared with the ultimate shear stress.*

The design criterion is sometimes *absence of permanent set.* The most common example of this is the coil spring, which is loaded primarily in torsion. To prevent permanent set the allowable shear stress must not exceed the proportional limit in torsion. This is related to the proportional limit in tension and is usually between 60 and 80 per cent of that value (for steel and aluminum alloy). The actual value depends entirely on the nature of the material, some materials being quite unsuitable for springs because of their low proportional limit.

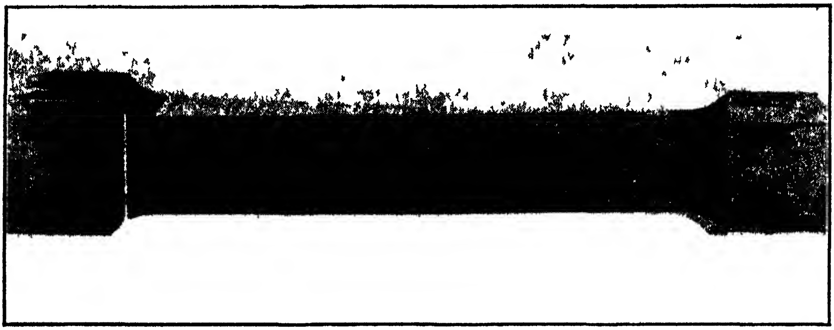


FIG. 12·15. Square specimen after torsion test. Note shear failure along mid-plane.

PROBLEMS

12·1. Check the shell torsion equation for a rectangle by calculating the moment of the shear flow about any convenient point. (Use symbols to represent quantities.)

12·2. Assuming the same allowable shear stress to apply and using the shell torsion equation, calculate the relative weight (cross-sectional area) of circular and square shells of equal torsional strength, for the following cases:

(*a*) Side of square = diameter of tube.
(*b*) Diagonal of square = diameter of tube.

(*Note.* Usually the allowable shear stress will be higher for a circular shell than for a square one.)

* Reference 16 gives a value of 9000 psi for the torsional endurance limit of 24ST aluminum-alloy bar, as compared with 38,000 psi for ultimate shear stress.

12·3. Let a and b represent two sides of a rectangular cross section. By varying the ratio a/b show that the square cross section gives maximum torsional strength for a given amount of material, assuming constant thickness and allowable stress. (*Note.* Use shell torsion equation and plot, against a/b, a curve of some quantity proportional to weight. The problem may also be solved directly by differential calculus.)

12·4. A thin-walled shell has a mean radius of 17 in. and transmits a torsional moment of 189,000 in.-lb. Calculate the shear flow. If the allowable shear stress is 5000 psi, what wall thickness is required?

12·5. A thin-walled shell of elliptical cross section has maximum and minimum dimensions of 20 in. and 10 in. The allowable shear stress is 12,000 psi and the wall thickness is 0.040 in. What is the allowable torsional moment?

12·6. If the circular shell of Problem 12·4 is 20 ft long, what total twist (in degrees) is produced? (Assume G = 3,850,000 psi.)

12·7. Calculate the twist for the elliptical shell of Problem 12·5 under a torsional moment equal to two-thirds of the allowable moment. Assume shell to be 30 ft long. (Assume G = 3,850,000 psi.)

12·8. A circular shell of 20-in. radius is made in three segments of equal width but which have the following thicknesses: 0.020, 0.040, and 0.080 in. Assume that a torsional moment of 500,000 in.-lb is applied.

(*a*) Calculate shear stress in each segment.
(*b*) Determine the maximum warpage, i.e., departure of actual cross section from a plane (refer to Fig. 12·6).

12·9. A round shaft must transmit a torsional moment of 10,000 in.-lb. A factor of safety of 2.0 against failure is desired.

(*a*) Find the diameters required for the various materials listed in Appendix 2.
(*b*) Compare relative areas and weights.
(*c*) Draw the cross sections to scale.

12·10. In Problem 12·9 assume that the shaft is to be made from a thick-walled tube having an inside diameter of 1 in.

(*a*) Find the outside diameters required for the various materials.
(*b*) Compare relative areas and weights with those of Problem 12·9.
(*c*) Draw cross sections to scale.

(*Note.* A trial-and-error method of solution may be used if necessary.)

12·11. Work out Problem 12·9 for a solid square shaft.

12·12. A solid shaft has a rectangular cross section 1 in. by 4 in. Assume that it transmits a torsional moment of 25,000 in.-lb through a distance of 30 ft. Find the maximum shear stress in the shaft and compute its total twist for:

(*a*) Alloy steel.
(*b*) Aluminum alloy.
(*c*) Magnesium alloy.

12·13. Using Fig. 12·14, calculate the ultimate torsional strength of a tube having an outside diameter of 2 in. and a wall thickness of 0.080 in. for the materials listed in Appendix 2.

12·14. A control wheel 16 in. in diameter must resist a couple of 80-lb forces, applied at opposite sides. Find the size of aluminum alloy tube required to transmit this torque with a factor of safety of 2.0 against failure. *Procedure:* Obtain approximate thicknesses required by using thin-shell formula for following average tube diameters: 1.5 in., 1.625 in., 1.75 in., assuming ultimate shear stress equal to one-half ultimate tension stress. Select lightest tube having a D/t value less than 50, and work out the wall thickness more accurately by using Fig. 12·14 and the thick-wall tube formula.

CHAPTER 13

BENDING—SYMMETRICAL

13·1. Pure Bending. Although bending is almost always induced by the transmission of *shear* or *transverse* forces it is convenient to develop the theory for pure bending before considering its combination with shear. As noted in Chapt. 3, pure bending is caused by the transmission of a turning moment or couple in its own plane, that is, along a line normal to its own axis. The term **beam** is used as a general name for a

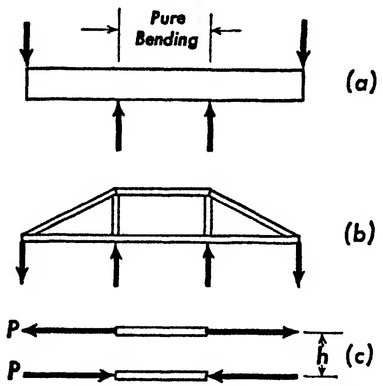


FIG. 13·1. Illustration of pure bending.

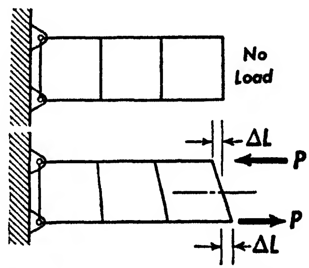


FIG. 13·2. Effect of vertical members.

structure transmitting a bending moment, but there is no well-defined rule on this point (other names: *truss*, *girder*, etc.).

A practical application of pure bending is illustrated in Fig. 13·1. (This symmetrical type of loading is often used in testing materials for bending strength.) Figure 13·1*b* shows a truss which will transmit these forces. In (*c*) the couple representing the bending moment is transmitted axially through the two flange members. In analyzing the action of pure bending it is convenient to use the *frame* analogy by dividing the truss into a number of units. If this is done by the addition of vertical members only, the structure will behave as indicated in Fig. 13·2. The axial deflection of the flanges will cause the vertical members to slant, but no curvature of the truss will be produced.

13·2. Frame in Pure Bending. Figure 13·3 shows a single rectangular frame to which a diagonal member has been added, and which is transmitting a bending moment in the form of a couple. The term *flange* will be used to denote the members which transmit the axial forces of the couple. Instead of assuming the frame members to be rigid, as in Chapt. 10, the *diagonal* member will be assumed to be inextensible. For a simple couple this assumption is actually true, in effect, as there is no shear force to load the diagonal. Likewise there is no vertical force to cause a loading of the vertical frame members. Hence the only deflections will be found in the horizontal flange members. These deflections are indicated in Fig. 13·3, greatly exaggerated.

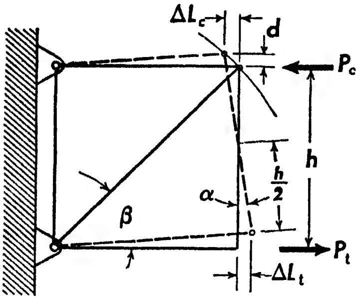


FIG. 13.3. Frame in bending

13·3. Bending Deflection of Frame. By following the process used for the shear frame in Chapt. 10, it can be shown that the contraction and expansion of the two flange members will cause an apparent shear deflection, indicated by d in Fig. 13·3. The relationship between the deflections is given by

$$d = \frac{\Delta L}{\tan \beta} \qquad [13\cdot 1]$$

(Compare Eq. 10·4*a*, for the *shear* frame.)

Figure 13·3 shows that one of the flanges shortens while the other becomes longer. This causes the vertical member to acquire an angle α. This angle, which is measured with respect to the other vertical member, will depend on the depth of the frame and on the relative deflections of the flanges. If the latter are equal in size, i.e., stiffness, they will deflect equally, and the vertical member will rotate about its midpoint. The angle α is then given by the equation *

$$\alpha = \frac{\Delta L}{(h/2)} \qquad [13\cdot 2]$$

where α is measured in radians.

It is this change in angle that is most important in the bending deflection of beams, as it causes each successive frame to start out at a different angle. Bending therefore produces deflections in *two* ways: first,

* Because of the small angles involved it is quite accurate to use this simplifying assumption.

by the direct deflection of the frame itself, and second, by the rotation of the other frames attached to it. This is shown by Fig. 13·4*b*, which is greatly exaggerated.

This development also indicates the origin of the *curvature* which is found in members subjected to bending. Such curvature is, of course, caused by non-linear conditions; i.e., the total deflection of a beam cannot be found simply by adding all the individual deflections of the frames, as in shear.

Figure 13·4*c* shows, for comparison, the type of distortion that would be produced by deflection of the diagonals *only*. Note that the verticals remain parallel, even though one diagonal may deflect more than another (This is illustrated by showing a greater deflection for member 2 than for the other two diagonals.)

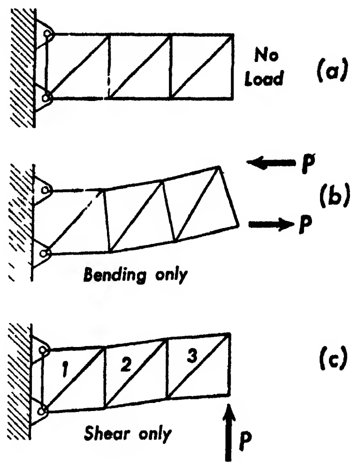


FIG. 13·4. Truss deflections.

Deflections of trusses usually involve both bending and shear and are computed by special methods which will not be taken up at this point. The foregoing simple principles, however, are sufficient as a basis for the development of any method of computing truss deflections.

13·4. Flange Forces. In a truss having *parallel flanges* it is a simple matter to compute the axial forces which they must resist under an applied bending moment, or couple (see Chapt. 9). It is necessary only to divide the bending moment at any point by the distance between the centroids of the flanges.

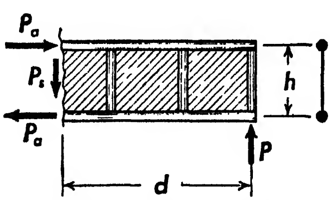


FIG. 13·5. Thin-web beam.

The principles developed for the truss may also be applied to the shear web or plate girder in which the diagonal members are replaced by a plate or sheet, acting as a shear web (see Chapt. 10). If the flange members are parallel and relatively large in cross section, as compared with the web, the problem may be divided into two parts. The *flanges* may be assumed to resist all the bending moment, while the *shear web* resists all the shear (transverse) force. A beam of this type is shown in Fig. 13·5. The depth of the beam (*h*) is taken as the distance between centroids of the flange cross sections.

The flange forces are then obtained from the equation

$$P_a = \frac{M}{h} \tag{13·3}$$

$$= \frac{Pd}{h} \quad \text{for a single shear force } P \tag{13·3a}$$

The *shear flow* and *stress* in the web are given by Eqs. 10·9 and 10·10, Sec. 10·8.

This type of analysis is very useful in thin-web types of structures. It is important to note, however, that the following assumptions are implied.

a. The flange members transmit all the bending moment in the form of an axial force couple.

b. The web transmits all the transverse (shear) force.

c. The flanges are *parallel.*

Assumption (*a*) neglects the fact that the shear web must be fastened to the flanges and therefore must elongate and contract with them. If the web is thin as compared with the flange, the amount of axial force carried in the adjacent web material is correspondingly small and may be neglected without serious error. If the web is relatively thick it will transmit part of the bending moment, acting as a solid beam. An approximate method of correcting for this consists in replacing the web by an *equivalent* pair of flanges, the areas of which may be added to those of the actual flanges. For this purpose an area of *one-sixth* of the total web area should be added to each flange. This is based on the theory of bending of solid sections (to be discussed later).

If the flanges themselves are deep, they may carry an appreciable portion of the shear. Hence Eq. 10·9 would not be strictly applicable and the rule for measuring h would require modification.

The flanges must be parallel; otherwise some of the transverse force will be resisted by the axial forces in the flanges, which would have a vertical component. The analysis of a beam with non-parallel flanges will be taken up later.

13·5. Examples of Flange-Type Beams. The method of assuming concentrated flanges may be used on many different types of beams and is particularly useful in preliminary design calculations. Several examples are shown in Fig. 13·6. The most important step is the estimation of the centroid of the flange material, which, of course, represents the location of the axial couple. In preliminary work this may usually be done by eye with satisfactory accuracy.

In the built-up type (*a*) a portion of the web is included as flange material. Sketch *b* shows a section through a tubular fuselage (airplane body). Note that the side truss members (verticals and diagonals) are not included; they are indicated in the sketch by dotted lines. Sketch *c* represents a more complicated type of structure often used for airplane wings, in which two or more shear webs may be employed.

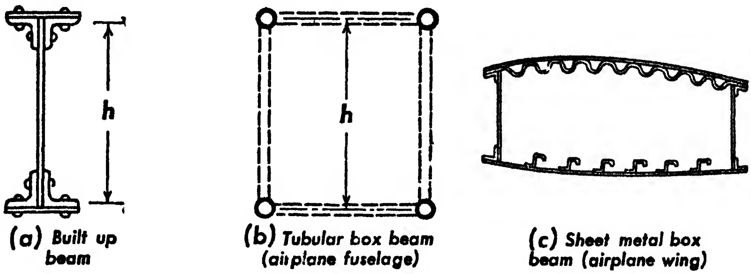


FIG. 13 6. Flange-type beams.

13·6. Development of the Bending Theory. The development of the classical beam theory will be started by assuming that the simple, parallel, two-flange beam is modified by adding additional flanges, as shown in Fig. 13·7. The two intermediate members will have some effect in resisting a bending moment, but their effectiveness is obviously not so great as that of the outer members. To obtain a measure of their effectiveness it is necessary to make some assumption as to the relative deflection of the members under load. The assumption made for this purpose forms the actual basis for the bending theory and it should therefore be thoroughly understood.

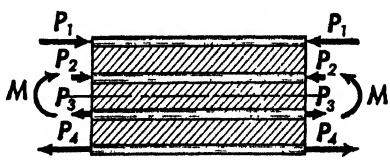


FIG. 13·7. Multi-flange beam.

In Chapt. 7 it was stated that the basic assumption in the *axial load* theory is that *plane cross sections remain plane and parallel* during loading. (This implies that there is relative translatory motion of adjacent cross sections during loading.) In the bending theory the corresponding assumption is that *plane cross sections remain plane* during loading, but they do not remain parallel. For *pure* bending it may be further assumed that the cross sections *rotate* about some axis such that there is no net translatory motion causing a resultant *axial* load.

Thus the two ends of the beam shown in Fig. 13·7 would remain straight, but not parallel, and would rotate about an axis normal to the paper. Such a condition is closely approximated in most actual cases, as the presence of the shear web prevents the flanges from acting independently.* A vertical slice 2 in. wide is now assumed to be cut from the bent beam, giving a picture as shown in Fig. 13·8 (deflections greatly exaggerated).

The deflections of each flange unit over a unit (1 in.) length are shown by e_1, e_2, etc. These also represent the *strains*, since a unit length was used as a basis (strain = deflection per unit length). If the values of these strains were known it would be possible to determine the *stress* in each flange, using the *stress-strain* diagram for the material. Each flange area would then be multiplied by its stress to find its axial force, and then by its distance from the axis of rotation to give its resisting moment. The summation of these moments for all flanges would give the total resisting moment.

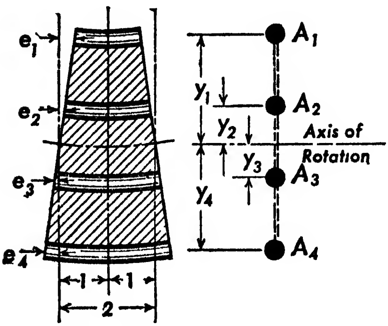


FIG. 13·8. Section of deflected beam.

This process may actually be used in certain cases, but it is not adapted to stress analysis work, in which the problem usually consists in finding the stresses corresponding to a given bending moment. It is possible, however, to obtain a *general* solution which will apply to nearly all types of beams. The equation which results is known as the **flexure formula** or the **bending equation.** It will be developed here by elementary methods in order to emphasize the physical significance of the formula.

To simplify the problem it is further assumed that the *strains remain within the elastic range,* so that the ratio between stress and strain is constant and equal to the modulus of elasticity, E; that is, *stress is proportional to strain.* Since strain has already been assumed to be proportional to the distance from the axis of rotation, it follows that *stress* is also proportional to this distance, i.e., the stress distribution is linear.† This may be expressed mathematically by the equation

$$f = Cy \qquad [13 \cdot 4]$$

where C is a constant.

* The assumption of plane cross sections may introduce appreciable errors when the flange units are widely separated and connected by relatively weak shear members, or at points of local loading where distortion of the cross section occurs. These refinements are not usually necessary, however, as Saint-Venant's principle applies to bending as well as to axial loading.

† It is important to note that this is not the fundamental assumption of the bending theory, but that it results from *two other* independent assumptions.

Since the value of C is unknown, its value will be taken initially as 1.0 and the resisting moment will be computed. From this result the stresses for any other condition may be obtained by multiplying by the ratio of moments. The initial assumption is, therefore, that the stress equals the distance from the axis of rotation:

$$f_1 = y_1 \qquad f_2 = y_2, \text{ etc.} \tag{13·5}$$

By multiplying stress by area, the flange loads are obtained.

$$P_1 = A_1 f_1 = A_1 y_1 \qquad P_2 = A_2 f_2 = A_2 y_2 \tag{13·6}$$

Similarly, the moment of each flange load may be obtained by multiplying but its moment arm.

$$M_1 = P_1 y_1 = A_1 y_1^2 \qquad M_2 = P_2 y_2 = A_2 y_2^2, \text{ etc,} \tag{13·7}$$

The total resisting moment is therefore given by the sum of these quantities

$$M_0 = \Sigma A y^2 \tag{13·8}$$

where M_0 is the resisting moment corresponding to the assumption $f = y$.

The quantity $\Sigma A y^2$ is the *moment of inertia* of the cross section, commonly denoted by the symbol I. Equation 13·8 may therefore be written

$$M_0 = I \tag{13·9}$$

It can now be seen that if an arbitrary bending moment, M, is applied, the assumed stresses will be increased by the factor M/M_0 or M/I. Hence Eq. 13.5 must be multiplied by this factor, which is actually the value to be used for C in Eq. 13.4.

Flexure formula

$$\boxed{f = \frac{My}{I}} \tag{13·10}$$

where f is the axial stress caused by the bending moment M.
y is the distance from the axis of rotation.*
$I = \Sigma A y^2$ = moment of inertia of cross section about axis of rotation.

* The symbol c is sometimes used in place of y. Usually c refers to the *maximum* value of y, which would give maximum stress.

This equation, often called the **flexure formula,** is one of the standard formulas used in structural analysis work and is of the same order of importance as the axial stress equation ($f = P/A$). These two equations cover two of the three basic loading conditions into which almost all stress problems may be divided (axial loading, bending, and torsion).

It is very important that the physical significance of the bending equation be realized (which is one reason why the development was carried out in such an elementary manner). In particular, the engineer should realize the limitations on its accuracy which were introduced by the assumptions made during the development.

13·7. Moment of Inertia. The term I, in Eq. 13·10, is usually called the moment of inertia, but this name has no direct significance in connection with the bending theory. A more proper name for the quantity is **second moment of area.** In Chapt. 7 it was shown that for pure axial loading (uniform axial stress) the resultant force must pass through the *centroid of area,* which was located by computing the *first moment of area* (ΣAy). The *first moment of area* is therefore associated with *axial* stress, while the *second moment of area* (I) is associated with bending stress.

Since the term moment of inertia is so well known, it will be used in further discussion of the beam theory, but it should be thought of as second moment of area.

TABLE 13.1

CALCULATION OF MOMENT OF INERTIA

(Refer to Fig. 13.8)

Flange Number	A sq in.	y in.	Ay^2
1	3	4	48
2	2	2	8
3	2	2	8
4	3	4	48
Sum	10		112

$A = 10$ sq in. $I = 112$ in.^{4}

One of the most common procedures in stress analysis work is the computation of the moment of inertia of a cross section. Table 13·1 shows the usual procedure, as applied to a simple case such as shown in Fig. 13·8. Values for the flange areas were selected so that symmetry of the cross section makes it possible to locate the axis of rotation by inspection.

Note that the procedure requires the tabulation of areas, thereby permitting the calculation of total area as one of the steps. (This will be useful when direct axial loading is combined with bending.) Note also that the units of measurement for *moment of inertia* are given as inches to the fourth power, since square inches have been multiplied by the square of a distance. The web areas have been neglected in this example.

13·8. Neutral Axis (Neutral Plane). Up to this point it has been assumed that the bending occurs about a known axis of rotation. In fact, Eq. 13·10 could be used for *any* assumed axis of rotation, even though there is only one axis which will correspond to pure bending in a given plane. Thus if a certain member is *forced* to bend about a definite axis the moment of inertia about this axis could be calculated and used directly in Eq. 13·10. But if this were done, the calculated internal forces would very likely have a net *axial* component along the axis, indicating that the condition is not one of pure bending.

In dealing with *pure* bending the internal forces must have no resultant axial component, as the definition of such bending implies that the external loading condition is a *couple*, or its equivalent. To determine the stresses for bending moments alone it is therefore necessary to select an axis of rotation that fulfils this requirement. Such an axis is called the **neutral axis.** On this basis the neutral axis may be defined as the *axis of zero stress in pure bending.**

If several adjacent cross sections of a beam are considered, the neutral axes through each section will define a plane, which may be called the **neutral plane.** If the beam is not of uniform cross section, or if the axis of the applied bending moment changes, this plane may be warped. The trace of the plane on a side of the beam is sometimes called the **neutral line** (see Fig. 13·9).

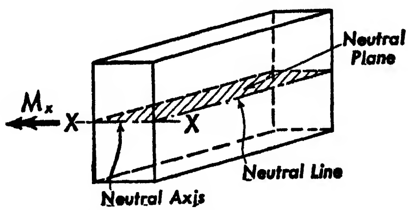


FIG. 13·9. Bending terminology.

In the development of the bending formula it was shown that the axial *force* in each flange unit is proportional to its area and its distance

* The term neutral axis is sometimes used to describe the axis of zero stress for *combined* axial load and bending. If so used, the term loses its direct connection with the geometric properties of the cross section. The author suggests the use of the terms *axis of zero stress*, or *plane of zero stress*, for the more general condition of combined loading (see Chapt. 19).

from the axis of rotation (Eq. 13·6). It can also be seen from Fig. 13·7 that flange forces on opposite sides of the axis of rotation will have different directions; on one side all elements will be in tension, and on the other side they will all be in compression. This can be indicated mathematically by letting the sign of y indicate which side of the axis of rotation is being considered. Refer to Fig. 13·8 and assume that stress equals the distance y.

$$\left.\begin{aligned} P_1 &= +y_1A_1 \\ P_2 &= +y_2A_2 \end{aligned}\right\}\text{positive values measured upward}$$

$$\left.\begin{aligned} P_3 &= -y_3A_3 \\ P_4 &= -y_4A_4 \end{aligned}\right\}\text{negative values measured downward}$$

$$\Sigma P = y_1A_1 + y_2A_2 - y_3A_3 - y_4A_4$$

This quantity, which may be indicated by Q, is called the first moment of area or the static moment of the cross section. It must equal zero if ΣP is to equal zero. Hence the *axis of rotation* must be located so that the following condition is satisfied:

$$Q = \Sigma Ay = 0 \qquad [13\cdot 11]$$

where Q is the *static moment*, or first moment of area.

This equation is identical with the equation representing an axis through the *centroid of area*, or *center of gravity* of the cross section (see Sec. 7·6). It can therefore be stated that *in pure bending the axis of rotation* (neutral axis) *will pass through the centroid of the cross-sectional area.**

This statement is not just an assumption or a working rule—it is a physical fact. If pure moments are applied at the ends of a beam the bending will actually occur about the neutral axis through the centroid of area. Bending about any other axis would require the *additional* application of axial force, which does not agree with the assumed external loading conditions.

13·9. General Method for Determining Neutral Axis and Moment of Inertia. When the neutral axis cannot be located by inspection, either of the following procedures may be followed.

a. The location of the axis through the centroid of area may be determined and this new axis may then be used to calculate I (second moment of area).

b. The first and second moments of area may be summed up about any convenient axis and the results transferred to the neutral axis.

* This statement is, of course, based on the assumptions mentioned in Sec. 13·6 in deriving the flexure formula.

Procedure (*a*) is covered by Secs. 7·6 and 13·8. Although easier to understand, it is generally less rapid than procedure (*b*), which is widely used in routine computations. Although the process of transferring moments of inertia from one axis to another is well covered in mechanics textbooks, it will be described here in terms of the physical action which it actually represents.

In Fig. 13·10 all the distances y are measured from an arbitrary reference line, XX, chosen at any convenient location. The computations are then carried out as in Table 13·2 (actual values inserted for illustration).

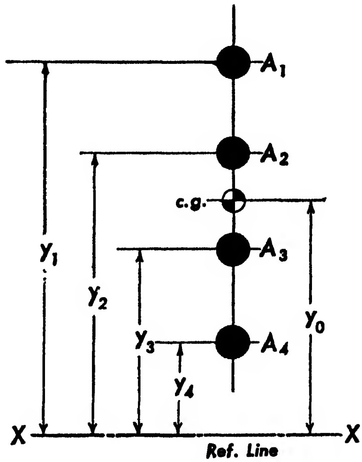


FIG. 13·10. General method.

The value of 12.0 obtained for the static moment Q_x indicates that the chosen axis does not fulfil the requirement that $Q = \Sigma Ay = 0$. In fact this quantity may be thought of as a measure of the resultant *axial* force which would be obtained at the reference axis if the beam were required to bend about this axis. If it is further assumed that $f = y$, the value

TABLE 13·2

CALCULATION OF CENTER OF GRAVITY, Q, AND I

(Refer to Fig. 13·10)

Flange Number	A (sq in.)	y (in.)	Ay	Ay^2
1	0.5	12	6	72
2	0.3	8	2.4	19.2
3	0.4	6	2.4	14.4
4	0.6	2	1.2	2.4
Sum	1.8		12.0	108.0

$A = 1.8 \quad Q_x = 12.0 \quad I_x = 108$

of Q gives the net axial force directly. The value of 108.0 obtained for I_x may similarly be thought of as the moment which resists bending about the reference axis, assuming that $f = y$.

The location of the *neutral axis* is found from the equation for the centroid of area.

$$y_0 = \frac{Q}{A} = \frac{\Sigma Ay}{A} \qquad [13 \cdot 12]$$

In the example,

$$y_0 = \frac{12.0}{1.8} = 6.67 \text{ in.}$$

This locates the point at which an axial force may be applied without causing an internal bending moment; conversely, the axis about which bending will produce no net axial force.

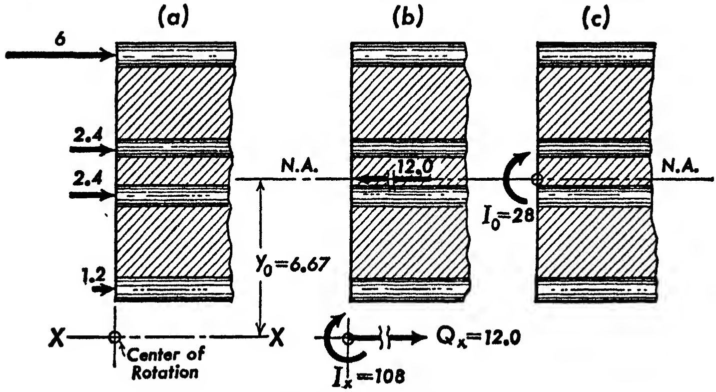


FIG. 13·11. Significance of static moment.

The physical significance of these quantities is shown in Fig. 13·11. It can be seen from (*b*) that bending about the *assumed* reference axis is resisted by a moment I_x and an axial force Q_x. For pure bending the axial force must be eliminated, which can be done by superimposing another hypothetical force $-Q_x$, acting in the opposite direction. This force must be applied at the neutral axis, hence its moment arm equals y_0, and its moment about the reference axis equals $-Q_x y_0$. From Eq. 13·12, $Q_x = Ay_0$, so that the moment produced by the force $-Q_x$ is equal to $-Ay_0^2$. The final moment is therefore given by

Transfer formula $$I_0 = I_x - Ay_0^2 \qquad [13 \cdot 13]$$

This is nothing more than the well-known *transfer formula* for moment of inertia, but here it has a special physical significance which makes it easier to understand. As applied to the values found in Table 13·2,

$$I_0 = 108 - 1.8\,(6.67)^2$$
$$= 28 \text{ in.}^4$$

The final condition is indicated by Fig. 13·11*c*.

13·10. The Solid Beam. The foregoing analysis was based on the assumption that the bending is resisted by a finite number of *flanges*, or members capable of resisting axial forces. The cross section of the beam is often solid, as in a bar or a channel. The procedure to be followed is basically the same as for the multi-flange beam and, in fact, consists in arbitrarily dividing the cross section into elements which are then treated as individual flanges.

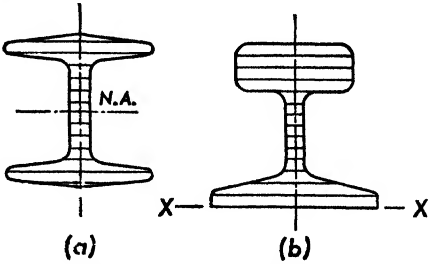


FIG. 13·12. Solid beams.

Thus in Fig. 13·12*a* the beam could be divided into narrow horizontal strips and a form such as Table 13·1 would be used to compute the moment of inertia about the horizontal axis through the c.g. For a shape such as shown in Fig. 13·12*b*, the *general* procedure would be followed, using some arbitrary axis such as X–X and employing a table such as 13·2.

The use of strips or flange elements in this manner introduces a small error, depending on the number of subdivisions made in the process. As the number of elements is increased and their size is correspondingly decreased the error approaches zero. The physical significance of this

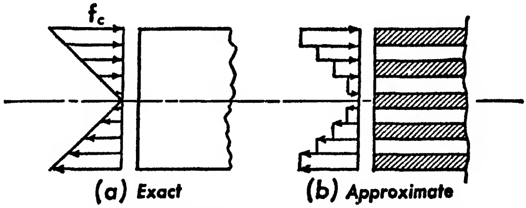


FIG. 13·13. Significance of strip method.

is simple; the computation of I in this manner neglects the *bending* resistance of each individual element and considers only its *axial* resistance. If the element is made thin enough its bending resistance becomes negligible.

Another way to illustrate this point is shown in Fig. 13·13. The *exact* application of the assumption of linear stress distribution would require a variable stress over each element, as in (*a*). The use of several strips would imply a stepped stress distribution as shown in (*b*).

13·11. Integration Methods. For any mathematically simple shape (rectangle, circle, etc.) the value of I and the location of the neutral axis may be exactly computed by *integration*, a mathematical operation which is equivalent to the summation of infinitely small elements.

Appendix 1 gives several standard formulas which are derived in this manner. Many more can be found in handbooks and in textbooks on mechanics.

It is often possible to compute the total moment of inertia of a cross section by dividing it into simple parts for which the c.g. and moment

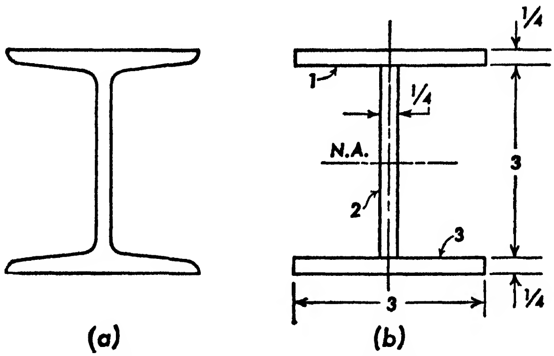


FIG. 13·14. Equivalent sections.

of inertia may be readily computed. The moments of inertia of all the parts are added together (which takes care of the bending resistance of each part acting by itself), and the *transfer formula* is used to compute the *axial* effect of each part about the desired axis.

Figure 13·14 shows how this system may be applied to an I-*beam*, for example. The actual cross section in (*a*) is first modified as in (*b*), so as to consist only of rectangles. The following table shows the method of computation.

TABLE 13·3

COMPUTATION OF I

Unit Number	A (sq in.)	y (in.)	I_0	Ay^2
1	0.75	1.625	0.0039	1.98
2	0.75	0	0.5630	0
3	0.75	1.625	0.0039	1.98
Sum	2.25		0.5708	3.96

$$I = 0.571 + 3.960 = 4.531 \text{ in.}^4$$

A clear understanding of these principles will aid in deciding to what extent the calculations for moment of inertia need be carried. For instance, a shell structure may be composed of many small units and a few large ones. If the c.g. of each unit is used in the calculations, no error in the location of the neutral axis will be caused by neglecting the individual moments of inertia. A trial calculation will usually show that the I of a small unit is negligible in comparison with that of the whole structure. It might be found desirable, however, to include the moments of inertia of the larger units. Appendix 1 contains tables which may be used in computing the moment of inertia of various elements.

13·12. Section Modulus. The maximum axial stress due to bending will occur at the point on the cross section which is farthest away from the neutral axis, i.e., at the maximum value of y. If the letter c is used to indicate the maximum value of y the bending equation may be written

$$f_{\max} = \frac{Mc}{I} \qquad [13\cdot14]$$

From this equation it can be seen that for a given bending moment the stresses produced will be inversely proportional to the quantity I/c, which may therefore be regarded as a measure of the strength in bending. This quantity is called the **section modulus.** It is usually identified by the symbol Z, thus,

$$Z = \frac{I}{c} \qquad [13\cdot15]$$

from which

$$f_{\max} = \frac{M}{Z} \qquad [13\cdot16]$$

13·13. Efficiency in Bending. The section modulus has the same relationship to bending moment that the cross-sectional area has to axial load. It is a measure of the *structural efficiency* in bending. Various cross-sectional shapes may be compared on this basis. Such a comparison will show that the best possible distribution of material in bending is that which will produce the largest value of the section modulus.

Figure 13·15 shows several shapes which have the same cross-sectional area (hence same weight) but which have quite different values for Z. Obviously the most efficient shape is that in which the material is concentrated as far from the neutral axis as possible. This is approached in the common I-beam. Sketch (*c*) shows a type of cross section which is to be avoided if possible. Here the outstanding flange gives a large value for c without a corresponding increase in I.

A comparison of (*d*) and (*e*) shows why it is highly desirable to use hollow tubes and shells rather than solid bars, if bending loads predominate. Hollow rectangles are also employed to some extent. Box beams have been widely used in aircraft work.

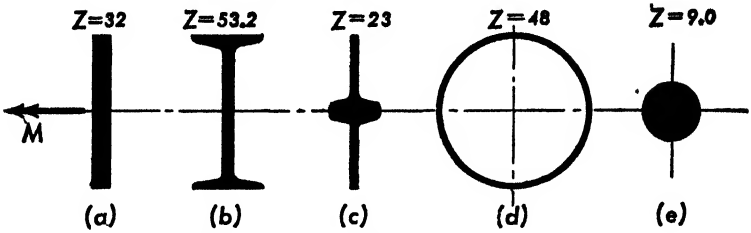


FIG. 13·15 Sections of same area.

13·14. Unbalanced Sections. If the cross section is not symmetrical about the neutral axis the value of c will not be the same on each side. The stresses on the side having the smaller value of c will therefore exceed those on the other side. This may sometimes be an advantage,

FIG. 13·16. Test jig for applying constant bending moment over mid-portion. (For maximum accuracy roller support would be used.)

as the allowable stresses in tension and compression are not usually equal. In airplane wings, for instance, it is customary to use more cross-sectional material in the top portion of the wing structure than in the bottom. This is because the maximum bending moment is usually produced by upward-acting air loads, causing compression stresses in the top portion. Since the allowable compression stresses for a shell structure are generally lower than the allowable tension stresses (because of buckling), it is necessary to place more material in the region subjected to compression. Conversely, the addition of material in the lower portion, although it increases I, will not appreciably reduce the compression stresses in the top, as this causes an increase in c for this part of the beam.

The same conclusions may be reached by treating the beam as an equivalent two-flange type.

PROBLEMS

13·1. It is estimated that a certain tube will have an ultimate bending strength of 60,000 in.-lb. The longest length available for testing is 5 ft. Sketch a loading method which will put the middle third of the tube in pure bending and calculate the applied loads and reactions at the estimated ultimate strength of the tube. (*Note.* In actual practice the tube would have to be reinforced at the points of load application, to prevent local failure.)

13·2. The two flanges of a square frame such as shown in Fig. 13·3 are composed of alloy-steel tubes, each 36 in. long and having a cross-sectional area of 0.60 sq in. A couple is applied as shown, in which the load P equals 50,000 lb. Find:

(*a*) The deflection, d.
(*b*) The angular deflection, α (degrees).

13·3. Assume that three frames, such as described in Problem 13·2, are joined together to make a truss such as shown in Fig. 13·4*a*. The same couple is now to be transmitted through all three frames. Find:

(*a*) The total deflection at the free end.
(*b*) The angular deflection at the free end.

(*Note.* Since the angular deflections are very small, the deflection corresponding to a given angular rotation may be found by multiplying the angle, in radians, by the length involved.)

13·4. A bending moment of 100,000 in.-lb is to be transmitted by a parallel-flange type of truss, 1 ft deep. Assume that the allowable stresses are 80,000 psi for the tension flange and 50,000 psi for the compression flange. Find the cross-sectional areas required for the flanges.

13·5. In Fig. 13.6*b* assume that h is 40 in. and that the cross section is that of an airplane fuselage, located 12 ft forward of the tail. A *down* load of 1000 lb is assumed to be acting at the tail (no other applied loads).

(*a*) Find the loads in the flanges (longerons).
(*b*) Assume that the allowable ultimate stresses are 68,000 psi and 40,000 psi for tension and compression, respectively, and determine the cross-sectional area required for each longeron in order to provide a factor of safety of 1.5 against failure.

13·6. Calculate the bending moment resisted by a rectangular cross section 1 in. wide and 5 in. deep, assuming that axial stress equals the distance from the neutral axis ($f_{max} = \pm 2.5$). Use the following approximations and compare answers.

(*a*) Divide into five units (1 in. by 1 in.) and assume stress to be uniform over each unit and equal to the value at the center of the unit.
(*b*) Do the same for 10 units (each ½ in. deep).
(*c*) Check against formula for moment of inertia.

13·7. Draw an irregular unsymmetrical solid cross section of any desired shape (approx. 2 in. by 3 in.). Divide into units approximately ½ in. square and set up a table (similar to Table 13·2) to find the area, static moments, and moments of inertia about any two mutually perpendicular axes (not through c.g.). (*Note.* By drawing the section on graph paper the areas of odd shapes may be determined by counting squares.)

13·8. Find the centroid of the shape used in Problem 13·7 and determine the moments of inertia about axes through the c.g., using a similar tabulation. Check by means of the transfer formula. (*Note.* The values of Q should be computed as a check on the location of the c.g.)

13·9. Draw a rectangular cross section of any desired dimensions. Assume that the maximum allowable axial stress (in bending) is 9000 psi. Find the bending strength of a beam of such proportions, about each axis of symmetry.

13·10. Using the same maximum stress as in Problem 13·9, find the moment represented by bending about the smaller edge of the beam (instead of the center). What total axial force is thus induced?

13·11. Calculate the maximum axial stress in a 2 in. O.D. by 0.083 in. tube subjected to a bending moment of 20,000 in.-lb. (Calculate A, I, and Z.)

13·12. Draw a cross section of a circular shell similar to Fig. 13·15*d*, assigning arbitrary values for diameter and wall thickness. Convert into an equivalent solid section of same height and note resemblance to an I-beam.

CHAPTER 14

UNSYMMETRICAL BENDING

14·1. Principal Axes. It has been shown that bending about any axis other than one through the c.g. of the cross section will require the application of an *axial* force in addition to a pure bending moment. Hence it is always necessary to use the neutral axis in dealing with pure bending.

It will now be shown that unless the section has equal bending resistance (constant moment of inertia) about *all* axes through the c.g. there will be only *two* locations for which the axis of rotation and the axis of the applied bending moment will coincide. These two axes are called the **principal axes** and are perpendicular to each other. They are the axes for maximum and minimum moment of inertia. Methods of finding these axes are covered in mechanics textbooks, but it is desirable to understand their physical significance as applied to the bending of beams.

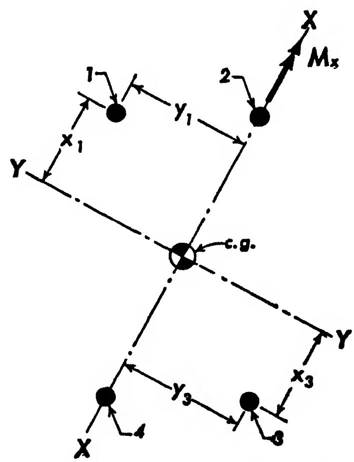


FIG. 14·1. Unsymmetrical bending.

A simple example is shown in Fig. 14·1, which might represent a cross section through the longerons of an airplane fuselage. (The shear members are not shown.) Assume that the loading condition causes a bending moment about the axis X–X which passes through the c.g. of the section. Another axis Y–Y is drawn normal to it. The logical method of attack might appear to consist in dividing the moment M_x by the distance h ($=y_1 + y_3$). If this were done, the axial loads obtained would be

$$P_1 = -P_3 = \frac{M_x}{h}. \qquad [14\cdot1]$$

Taking moments about the axis Y–Y,

$$M_y = P_1x_1 + P_3x_3 \neq 0 \qquad [14\cdot2]$$

This shows that something must be wrong with the method used, for although starting with an applied bending moment about the X–X axis *only*, a resisting moment was obtained which not only equals the applied moment M_x but also has an extra component of considerable magnitude about the Y–Y axis. This is a physical impossibility, as the Y–Y axis is normal to the X–X axis; there can, therefore, be no y component of the moment M_x. It will be evident later that the error lies in assuming that the beam will actually rotate or bend about the same axis as that of the applied moment.

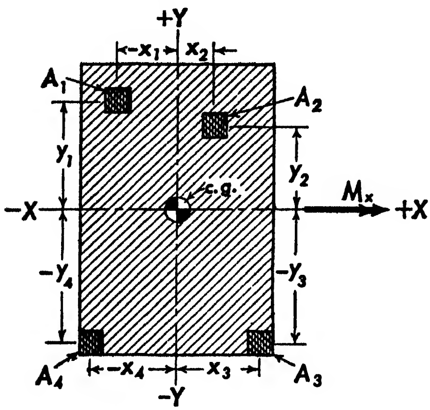


Fig. 14·2. Elements in unsymmetrical bending.

To find the correct loads it is necessary to determine the axes for which the bending action *does* agree with the applied moment. These are called the principal axes, as previously noted. The condition which must be satisfied is as follows. The axial forces produced by bending rotation of the cross section about one principal axis must produce *zero* resultant bending moment about the other axis at right angles to it. This is illustrated in Fig. 14·2, which shows two assumed axes and several elements of area. Now assume that a moment M is applied about the X–X axis, such that the top portion of the beam is in tension. Since the actual values are of no interest, but only the condition for zero moment about the Y–Y axis, it is permissible to make the arbitrary assumption used in Sec. 13·6, that *stress equals distance from the axis of rotation.*

For the four elements illustrated, this gives values for stress and force as shown in Table 14·1.

TABLE 14·1

Computation of Product of Inertia

Stresses	Forces	Moments
$f_1 = +y_1$	$P_1 = +y_1A_1$	$M_1 = P_1(-x_1) = -A_1x_1y_1$
$f_2 = +y_2$	$P_2 = +y_2A_2$	$M_2 = P_2(+x_2) = +A_2x_2y_2$
$f_3 = -y_3$	$P_3 = -y_3A_3$	$M_3 = P_3(+x_3) = -A_3x_3y_3$
$f_4 = -y_4$	$P_4 = -y_4A_4$	$M_4 = P_4(-x_4) = +A_4x_4y_4$

The moment of all the forces about the Y–Y axis must equal zero. Since the signs of the x distances indicate the sense of the moment arm, the moment of each force is obtained by multiplying by the algebraic value of x, giving typical values such as shown in the third column of Table 14·1. If this process is done for all the elements and the values of moment are summed up and equated to zero, the equation is

$$\Sigma Axy = 0 \qquad [14\cdot3]$$

This quantity is called the **product of inertia** and may be denoted by the symbol I_{xy}, as in the following equation.

Product of inertia $$I_{xy} = \Sigma Axy \qquad [14\cdot4]$$

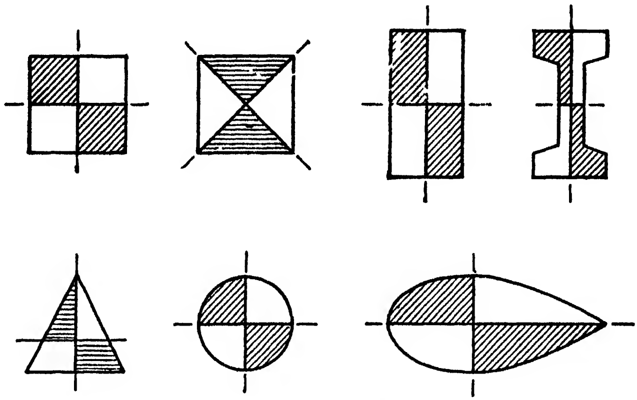


FIG. 14·3. Symmetrical sections.

A principal axis may, therefore, be defined as an axis through the c.g. *for which the product of inertia is zero.* From the nature of the operations in Table 14·1 it can be seen that if the *product of inertia* is zero for one axis, it will also be zero for the axis normal to it.

From Fig. 14·2 and Table 14·1 it is evident that there will always be two diagonally opposite quadrants in which the individual values of Axy are positive and two more in which they are negative. Hence, if the section is *symmetrical* about two perpendicular axes, the effects cancel out, and the product of inertia becomes zero. From this it can be seen that *axes of symmetry are principal axes.* Figure 14·3 illustrates this by showing several sections in which the positive quadrants are shaded. In each section the positive and negative quadrants cancel out.

It is useful to know also that any area which has the same moment of inertia about its two principal axes has the same moment about *any other* axis through the centroid, i.e., the *product of inertia* is zero about all axes. This applies to the circle, square, octagon, and similar doubly symmetrical shapes. *For these shapes any axis through the centroid may be used to compute the bending stresses.*

For sections which are not doubly symmetrical the moments of inertia about the two principal axes will be the maximum and minimum for the cross section. This applies to several of the shapes shown in Fig. 14·3.

If no axis of symmetry exists for the section, it is not possible to determine the direction of the principal axis by observation, but the procedure is to compute I_x, I_y, and I_{xy} for two convenient axes. From these quantities it is possible to find the location of the principal axes and the moments of inertia about them. (The computation of I_{xy} can easily be included in the table used for I_x and I_y.)

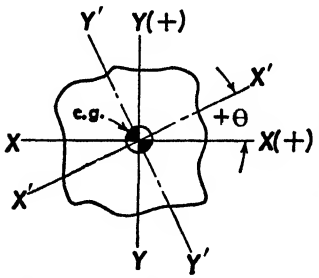


FIG. 14·4. Rotation of axes.

The following equations may be used to determine the moments of inertia about any set of axes (X' and Y') which have been rotated through an angle θ, if the values of I_x, I_y, and I_{xy} are known for the original axes (see Fig. 14·4).

$$I_{x'} = I_x \cos^2 \theta + I_y \sin^2 \theta - I_{xy} \sin 2\theta$$
$$I_{y'} = I_y \cos^2 \theta + I_x \sin^2 \theta + I_{xy} \sin 2\theta \qquad [14\cdot5]$$

The angle θ at which $I_{x'}$ and $I_{y'}$ reach their maximum or minimum values determines the location of the *principal axes* and may be found from the equation

$$\tan 2\theta = \frac{2I_{xy}}{I_y - I_x} \qquad [14\cdot6]$$

If this value for θ is substituted in Eqs. 14·5, the principal moments of inertia will be found; or the following equation may be used directly.

$$I = \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2} \qquad [14\cdot7]$$

Derivation of the foregoing equations has been omitted as it is seldom necessary to make use of them in routine analysis work. Problems of unsymmetrical bending may be solved without finding either the principal axes or the principal moments of inertia, by methods which will be described later. It is important, however, to be able to recognize unsymmetrical bending when it occurs, and in this connection a clear understanding of the significance of the principal axes is essential.

14·2. Unsymmetrical Bending. It will be assumed that the principal axes for a given cross section are known and that the applied bending moment axis does not coincide with one of these. The procedure to be used is as follows.

a. Resolve the applied bending moment into components acting about the two principal axes.

b. For any given point compute the stresses separately about each axis.

c. Add the stresses algebraically to obtain the net stress.

Figure 14·5, which shows the box beam of an airplane wing, is a typical example of unsymmetrical bending. If the resultant moment acts

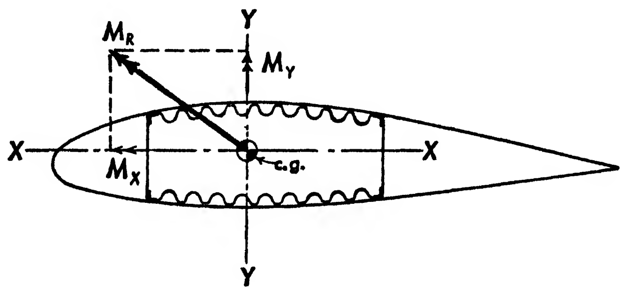


Fig. 14·5. Bending of airplane wing.

about the axis shown, it must be revolved into two components along the principal axes X–X and Y–Y. The stresses at any point are now determined with respect to each component and added algebraically. It is obviously necessary to be careful about the *signs* of the stresses in this process.

14·3. Effective Bending Moment. In the example used to introduce the subject of unsymmetrical bending (Sec. 14·1) it was found that the use of the moment axis as an axis of rotation induced an extra component of bending about an axis 90° away. This suggests a method of analysis in which the *induced* moment is actually accounted for, thereby permitting the axes of the applied moments to be used directly. Such methods have been developed, but their formulas are usually somewhat difficult to understand. For that reason the following development of the method is presented, and simplified formulas are obtained.

Consider Fig. 14·6, in which the axes X–X and Y–Y are not the principal axes. The bending moments M_x and M_y are assumed to be applied to the cross section shown. Positive values of moment are indicated by the conventions, using the right-hand rule. Hence positive values of M_x will produce *tension* (+) in the positive quadrant (positive values of x and y), but positive values of M_y will cause *compression* (−)

in this quadrant. (This must be considered in the final step in which the stresses caused by these two moments are combined.)

If, first, only the moment M_x is considered, it will be assumed that this causes a bending rotation about axis X–X (although it is known that

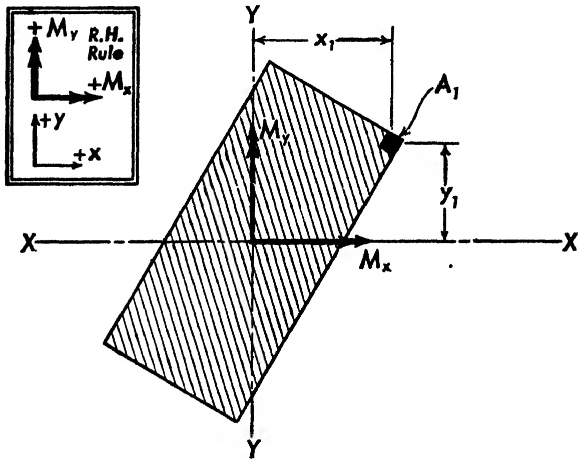


FIG. 14·6. General case of unsymmetrical bending.

this assumption will be incorrect.) The stress and force in element 1 will then be given by

$$f_1 = \frac{M_x y_1}{I_x} \qquad [a]$$

$$P_1 = f_1 A_1 = \frac{M_x y_1 A_1}{I_x} \qquad [b]$$

The force P_1 represents tension acting *on* the cross section, and the summation of the moment of all these forces about axis X–X will equal M_x. (P is *not* a reaction.)

These forces also have moment arms about axis Y–Y. The moment of P_1 about this axis is negative with respect to the convention adopted for positive values of M_y (see Fig. 14·6).

$$\Delta M_{y1} = -P_1 x_1$$

Substituting from Eq. *b* gives

$$\Delta M_{y1} = -\frac{M_x A_1 x_1 y_1}{I_x} \qquad [c]$$

If the values of ΔM_{y1}, etc., are now summed up over the entire cross section, their summation will represent an unbalanced moment ΔM_y, about axis Y–Y. This moment does not actually exist externally; consequently it will equalize itself within the beam by tending to cause rotation about the Y–Y axis and by causing bending stresses in accordance with the beam theory. The net value of the *induced* moment will be given by

$$\Delta M_y = -\frac{M_x}{I_x}\Sigma Axy = -M_x\frac{I_{xy}}{I_x} \qquad [d]$$

This is the moment induced by assuming rotation about an axis other than a *principal* axis. (Compare Sec. 13·9 in which an axial force was induced by assuming rotation about an axis other than one through the centroid of the cross section.)

To eliminate this unbalanced moment it is necessary to apply a hypothetical counteracting moment equal to ΔM_y and of opposite sign. The net result is that the section will be analyzed for a moment M_x and also for a hypothetical moment ΔM_y, the value of which is given by

$$\Delta M_y = M_x\frac{I_{xy}}{I_x} \qquad [e]$$

This is not the final answer, however, because the application of ΔM_y will in turn induce a moment about axis X–X. Although it would be possible to approach the answer by successive approximations, it can be seen that the final answer can be obtained directly by the solution of two equations. In the following development it will be assumed that external moments M_x and M_y are applied. The problem is to find effective values for these such that they can be used directly in the flexure formula, regardless of whether the moments act about the principal axes.

Let M = applied bending moment.
M' = effective bending moment.
ΔM = correction moment about one axis induced by bending about the other.
I_{xy} = product of inertia.

(Subscripts x and y refer to axes when used with M or I.)

From Eq. *e*

$$\Delta M_y = M_x'\frac{I_{xy}}{I_x}$$

Similarly,

$$\Delta M_x = M_y'\frac{I_{xy}}{I_y}$$

Let

$$U_x = \frac{I_{xy}}{I_x}$$

and

$$U_y = \frac{I_{xy}}{I_y}$$

Then

$$M_y = M_x' U_x$$
$$M_x = M_y' U_y$$

By definition

$$M_x' = M_x + \Delta M_x$$

and

$$M_y' = M_y + \Delta M_y$$

By substituting for ΔM_x,

$$\begin{aligned} M_x' &= M_x + M_y' U_y \\ &= M_x + (M_y + \Delta M_y) U_y \\ &= M_x + (M_y + M_x' U_x) U_y \\ &= M_x + M_y U_y + M_x' U_x U_y \end{aligned}$$

$$M_x' - M_x' U_x U_y = M_x + M_y U_y$$

The following equations for *effective bending moment* are obtained after the process is repeated for the Y–Y axis.

Effective bending moments

$$M_x' = \frac{M_x + M_y U_y}{1 - U_x U_y}$$
$$M_y' = \frac{M_y + M_x U_x}{1 - U_x U_y} \qquad [14{\cdot}8]$$

where

$$U_x = \frac{I_{xy}}{I_x}$$
$$U_y = \frac{I_{xy}}{I_y} \qquad [14{\cdot}9]$$

Since these factors determine the extent to which the bending moments are influenced by lack of symmetry, they will be called the *unsymmetrical factors* and have been designated here by the symbol U. If the product of inertia is zero, U becomes zero and Eqs. 14·8 show that the *effective* moments then equal the *applied* moments. This is in agreement with the previous findings concerning principal axes, as the bending moment axes would be the principal axes in such a case.

Note also that unsymmetrical bending about only *one* axis induces an effective bending moment about the other axis, in addition to changing the effective bending moment about the axis of application.

14·4. Stresses in Unsymmetrical Bending. The effective bending moments determined from Eqs. 14·8 may be used directly in the regular flexure formula, using the moments of inertia about the axes of the applied moments, i.e.,

$$f_x = \frac{M_x' y}{I_x}$$

$$f_y = \frac{M_y' x}{I_y}$$

The net stress at any point must, of course, be obtained by algebraic addition of the stresses caused by the two bending moments, giving:

Unsymmetrical bending stress

$$\boxed{f = \frac{M_x' y}{I_x} - \frac{M_y' x}{I_y}} \qquad [14 \cdot 10]$$

As previously noted the conventions used here are such that positive moments applied about two normal axes cause stresses of opposite sign in the positive quadrant. This accounts for the minus sign in Eq.14·10. If positive moments had been defined as those causing the *same* sign of stress in the positive quadrant, the opposite would be true. The conventions used here are consistent with the computation of the product of inertia, I_{xy}.

This method of attack is particularly useful in dealing with sections such as shown in Fig. 14·5 which have no axes of symmetry. It is possible to perform all computations with reference to the same set of axes, which greatly reduces the number of computations involved. This is a valuable feature in tabular computations such as used for routine stress analysis work.

14·5. Axis of Zero Stress (Neutral Axis). The relationship between x and y which determines the axis of zero stress is obtained by letting $f = 0$ in Eq. 14·10, giving

$$\frac{M_x' y}{I_x} = \frac{M_y' x}{I_y}$$

From which

Location of neutral axis

$$\frac{y}{x} = \frac{M_y'}{M_x'} \frac{I_x}{I_y} \qquad [14 \cdot 11]$$

Note. **When bending moments are measured about *principal axes*, the primes may be omitted.**

This equation determines the axis (through the centroid of the cross section) about which the relative rotation of the cross sections actually takes place. It is the *neutral axis for unsymmetrical bending* (see Sec. 13·8).

A knowledge of the position of this axis is useful in predicting the point of maximum stress on the cross section. Since linearity of stress distribution has been assumed for bending about both reference axes, it follows that the distribution from the axis of zero stress will also be linear, and the point *farthest away* from this axis will be most highly stressed.

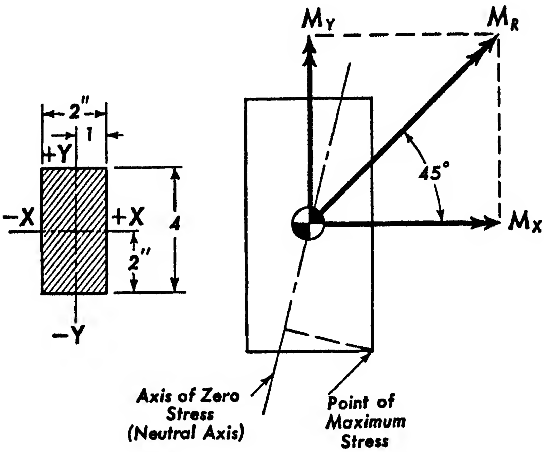


FIG. 14·7. Neutral axis in unsymmetrical bending.

Beams which have much less bending resistance about one axis than about the other (normal to it) are very sensitive to the location of the axis of the applied moment. (This is indicated by the term I_x/I_y in Eq. 14·11.) Figure 14·7 illustrates a simple example. Assume that an ordinary two-by-four is subjected to a bending moment about a 45° axis. The quantities involved are

$$\frac{M_y}{M_x} = 1.0$$

$$I_x = \frac{bh^3}{12} = \frac{2 \times 4^3}{12} = 10.67 \text{ in.}^4$$

$$I_y = \frac{bh^3}{12} = \frac{4 \times 2^3}{12} = 2.67 \text{ in.}^4$$

$$\frac{I_x}{I_y} = \frac{10.67}{2.67} = 4$$

Substituting these values in Eq. 14.11

$$\frac{y}{x} = 4$$

The line which corresponds to this relationship is shown in Fig. 14.7. It obviously does not coincide with the axis of the applied moment. The point of maximum stress is also shown.

14·6. Summary of General Method. In developing the various formulas used in bending analysis it has been convenient to start with the simplest ones and then to modify them one step at a time. Thus it

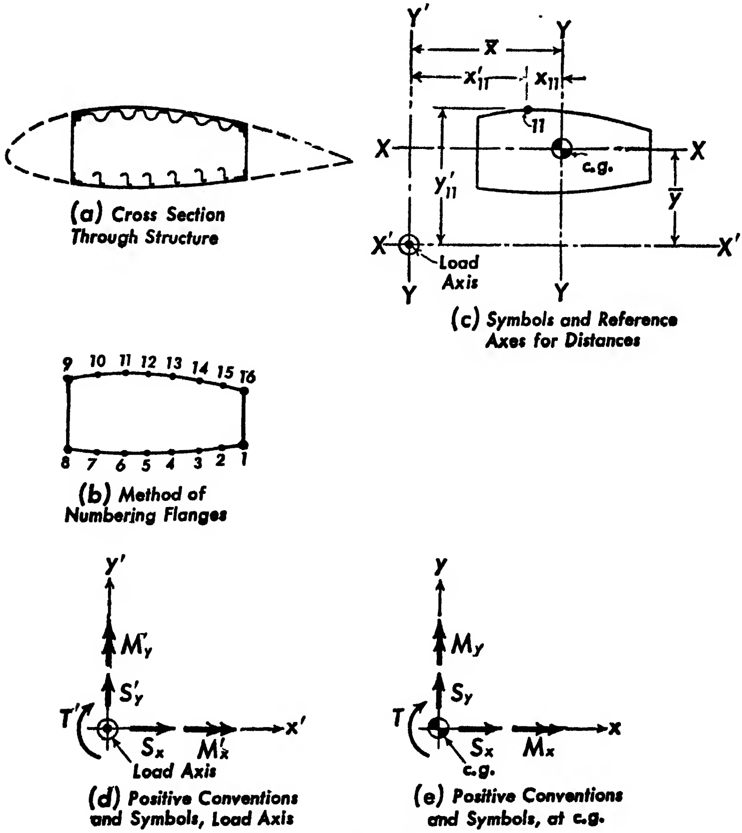


FIG. 14·8. Bending analysis of box beam.

was shown first that the moment of inertia must be computed about the axis through the *center of gravity*. Then it was shown how this computation could be performed about some other parallel axis, using the transfer formula to correct for the *induced axial force*. The next step was to

TABLE 14·2

BENDING COMPUTATIONS

①	②	③	④	⑤	⑥	⑦	⑧	⑨	⑩	⑪	⑫	⑬	⑭	⑮	⑯
Flange Number	Area of Flange	Distance from Ref. Axis		First Moment about Ref. Axis		Distance from c.g.		First Moment about c.g.		Second Moments		Product Moment	Partial Bending Stresses		Bending Stress
	A	x'	y'	Ax'	Ay'	x	y	Ax	Ay	Ax^2	Ay^2	Axy			f_b
Operation				② ③	② ④	③ $-\bar{x}$	④ $-\bar{y}$	② ⑦	② ⑧	⑦ ⑨	⑧ ⑩	⑧ ⑨	K_x ⑧	K_y ⑦	⑭ − ⑮
1 2 3 Etc.															
Σ															

Constants:

$M_x =$ ______

$M_y =$ ______

$\bar{x} = \frac{\Sigma ⑤}{\Sigma ②} =$ ______

$\bar{y} = \frac{\Sigma ⑥}{\Sigma ②} =$ ______

$I_x = \Sigma$ ⑫ $=$ ______

$I_y = \Sigma$ ⑪ $=$ ______

$I_{xy} = \Sigma$ ⑬ $=$ ______

$U_x = \frac{I_{xy}}{I_x} =$ ______

$U_y = \frac{I_{xy}}{I_y} =$ ______

$K_x = \frac{M_x + M_y U_y}{I_x(1 - U_x U_y)} =$ ______

$K_y = \frac{M_y + M_x U_x}{I_y(1 - U_x U_y)} =$ ______

show that the theory held only for the *principal axes*, but the equations were further generalized to cover unsymmetrical bending by the use of *effective* bending moments derived from the application of *unsymmetrical factors*. This procedure accounted for the *induced bending moments*.

Since the general method takes care of all of these problems, it is the one most frequently used in routine analysis work when the cross section does not have an axis of symmetry; also when the structure must be investigated for several different design conditions. A sample form will therefore be included to show how the computations are carried out. The example chosen is similar to the method developed in Ref. 27 and applies to a cellular type of structure. A cross section through such a structure is shown in Fig. 14·8, which also shows the conventions used and the manner in which the structure is represented by flange elements. Table 14·2 represents a typical tabular form which, in actual use, is printed on transparent paper and filled in by the computer, after which it can be blueprinted for direct inclusion in the stress analysis report.

14·7. Plane of Maximum Stress. In determining the stresses due to bending it is assumed that the beam is *cut* at the station in question, i.e., the section properties are computed with reference to some particular plane through the beam. It is generally obvious that the section should be taken as the *minimum* cross section, which is usually normal to the centerline of the beam. For instance, it would be entirely wrong to use Sec. A–A in Fig. 14·9. The proper section is shown as B–B. (It would be correct, however, to compute the bending moment either as Px or P_sd.)

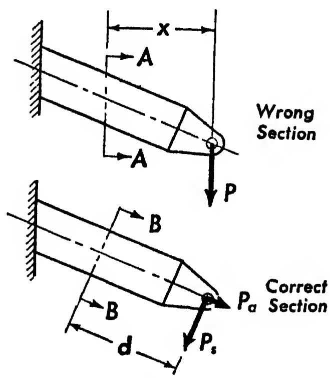


FIG. 14·9. Plane of cross section for bending computations.

Although the rule of *minimum cross section* usually gives the correct answer, a more general definition for the cross section of pure bending would be that cross section for which the application of a bending moment would produce no *induced* forces or torsional moments (moment vector normal to plane in question). This is simply an extension of the previous steps in the development of the conditions for pure bending, which may be summarized as follows.

a. The assumption of the wrong axis (not through the c.g.) for rotation of the cross sections causes an *induced axial force* for which a correction is made by using the transfer formula for moment of inertia.

b. The assumption that the axis of rotation coincides with the axis of the applied bending moment may cause an *induced bending moment* about an axis at 90° unless the axis in question is a *principal axis*. A correction may be made by using an *effective bending moment.*

c. The assumption of the wrong cross section through the c.g. may cause induced forces or a torsional moment in the plane of the cross section. It is usually correct to use the *minimum cross section.* For convenience it is sometimes desirable to use cross sections which are at a small angle to the plane of minimum cross section. Approximate corrections may then be made by methods such as suggested in Ref. 28.

14·8. Limitations of Flexure Formula. The following points should be noted in connection with the material presented on bending up to this point.

a. The beam or girder is assumed to be *straight,* not curved, before bending.

b. The flanges of the beam, or the top and bottom elements, are assumed to be *parallel.*

c. The stress distribution is not affected by *local* conditions, such as abrupt changes of cross section and end effects.

d. The strain variation across a section through the beam is assumed to be linear, i.e., plane sections remain plane.

e. Stress is proportional to strain, i.e., conditions are within the elastic range.

f. As a result of assumptions (*d*) and (*e*), the *stress distribution is assumed to be linear.*

PROBLEMS

14·1. Using Problem 13·7 as a basis, calculate the product of inertia and locate the principal axes. Calculate the principal moments of inertia by setting up a table. Check results by means of the equation for principal moments of inertia.

14·2. Assume that a 2 in. by 4 in. cantilever beam, 4 ft long, carries a concentrated load of 100 lb, applied in a plane through diagonally opposite edges. Calculate the bending stress at the root (supported end) of the beam, for each corner of the cross section. (*Note.* Resolve bending moment into values about principal axes.) Calculate, and show by means of a sketch, the location of the neutral axis, also the point of maximum stress. (Neglect shear stresses.)

14·3. Solve Problem 14·2 by computing the moments of inertia and the product of inertia for the axes connected with the loading plane. (Use Eqs. 14·5 and 14·6.) Find the effective bending moments and compute the bending stresses for each corner of the cross section, to check results of Problem 14.·2. Check the location of neutral plane by using Eq. 14·11.

14·4. Using Eq. 14·7 show that the sum of the moments of inertia about any two mutually perpendicular axes is equal to the sum of the principal moments of inertia, hence equal to the sum of the moments of inertia about any other similar pair of axes. (Compare Eq. 12·18.)

14·5. Using Fig. 14·8 as a basis, draw a cross section to a convenient scale (make the beam approx. 40 in. wide) and assign values to the flange unit areas. (Values to range from 0.25 to 0.75 sq in., corner members to be between 1.5 and 2.0 sq in.) Measure the distances from the reference axes used and set up Table 14·2 to find the centroid of area, moments of inertia, product of inertia, and the unsymmetrical factors, U_x and U_y.

14·6. In Problem 14·5 assume that a bending moment of 300,000 in.-lb acts so as to put the top surface in compression. Calculate the bending stress in each flange unit. Draw a line representing the neutral plane. (*Note.* The maximum bending stress should occur at a point farthest away from this line.)

14·7. In problem 14·5 assume that there exists also an additional bending moment of 10,000 in.-lb which tends to put the forward (left) edge of the box in compression. Calculate the bending stresses in each flange unit and locate the neutral plane.

14·8. Assume that in problem 14·6 an additional *axial* load of 20,000 lb acts at the centroid of area, normal to the plane of the cross section. Find the new stresses in the flange units. (Assume the axial load to be *compression.*)

14·9. In Fig. 14·9 assume that the beam cross section is rectangular and assign values in the following ranges.

Beam depth = 8 to 10 in.
Beam width = 2.6 to 2.9 in.
P = 7000 to 9000 lb
x = 30 in.
Angle with the horizontal = 30°
Find the axial stresses in the top and bottom fibers.

14·10. Prove that any beam of rectangular cross section, when loaded across one diagonal, will bend about the other diagonal. (*Note.* This can be done by proving that the stress in one corner will be zero. Write a general equation for the stress in a corner of a rectangle having dimensions a and b, using the moments about the principal axes.)

CHAPTER 15

BENDING (SPECIAL CASES)

15·1. Non-Linear Stress Distribution. Frequently the assumption of linear stress distribution in bending is considerably in error. Some typical examples are:

a. Composite beams (different materials).
b. Partially buckled shells.
c. Bending beyond the proportional limit (plastic range).
d. Curved beams (non-linear strain).
e. Warped cross sections (non-linear strain).

Since the bending theory has been so highly developed on the basis of linear stress distribution, it is very desirable to find some way to use this assumption for special cases. This can usually be done by the simple expedient of *effective areas* or *transformed sections.**

15·2. Effective Area. The method of effective areas has already been described under *axial* loads (Sec. 7·7) in connection with composite members which include different materials. The difference in the resistance to strain (measured by E) was taken care of by modifying or "weighting" the units of cross-sectional area by a factor based on relative values of E. The same method may be employed in bending.

Let n denote an **effectiveness factor,** defined by

$$n = \frac{\text{Stiffness of any unit of cross section}}{\text{Stiffness of a standard unit}}$$

Stiffness here means resistance to deformation and *standard* refers to any unit or material which may be selected as a basis for the calculation. (For example, in a composite beam made up of steel and aluminum alloy, the latter might be selected as standard.)

All the properties of the cross section may now be calculated in the usual manner, except that *each unit of area of the cross section is first multiplied by n.* This gives a fictitious section which is referred to as

* An excellent discussion of this subject will be found in *The Column Analogy* by Hardy Cross.[29]

the **transformed section.** The general types of equations for the transformed section are *

$$A_t = \Sigma nA = n_1A_1 + n_2A_2 + \text{etc.} \qquad [15\cdot1]$$

$$I_{xt} = \Sigma nAy^2 = n_1A_1y_1^2 + n_2A_2y_2^2 + \text{etc.} \qquad [15\cdot2]$$

The terms n_1, A_1, etc., may be thought of as the *effective areas.*

If the properties of the transformed section are used in the regular equations for axial stress and bending, the results must again be multiplied by the value of n for each unit to obtain the actual stress. Thus,

Axial

$$f_t = \frac{P}{A_t} \qquad f = nf_t \qquad [15\cdot3]$$

Bending

$$f_t = \frac{My}{I_t} \qquad f = nf_t \qquad [15\cdot4]$$

15·3. Composite Sections. If steel members were added to an aluminum-alloy beam as in Fig. 15·1, and if the aluminum alloy were used as standard, n would equal approximately 3 (ratio of E for the two materials). In computing the area, c.g., and moment of inertia, the actual area of the steel members would be multiplied by 3. The stresses thus determined would be correct for the aluminum-alloy parts but would have to be multiplied by 3 for the steel parts.

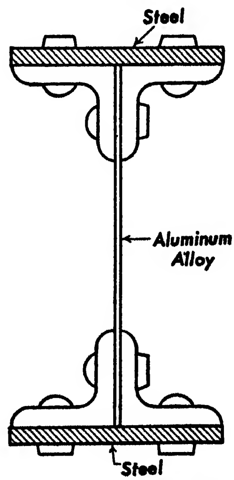


FIG. 15·1. Composite beam.

Obviously the factor 3 would cease to apply in the *plastic* range as it is based on the relative moduli of elasticity, and other methods of attack would be required.

The transformed section should not be considered as actually being different in shape, as this would lead to errors in computing moments of inertia. The mental picture should be that of a number of resisting *units* represented by points on the cross section and having relative resistances in proportion to their value of nA.

The method of transformed sections is used in the analysis of reinforced concrete beams. Although the modulus of elasticity of concrete is not

* Subscript t is here used to indicate "transformed."

constant, it is customary to assume that it is about one-fifteenth that of steel. If the concrete is used as standard, the value of n therefore becomes 15. In calculating ultimate strength the tension stresses are usually assumed to be resisted entirely by the steel reinforcing members, as the concrete will probably crack before the steel fails. For deflection calculations, however, it may be found more accurate to include the concrete in tension.

Since the proper design of reinforced concrete beams involves many special considerations it will not be taken up in further detail. The subject is well covered in most textbooks on strength of materials and building construction.

15·4. Buckling Effects. In dealing with thin shell structures such as are found in airplanes, the phenomenon of buckling is frequently encountered. If the shell is unstiffened by longitudinal members, it will fail when it reaches the buckling stress on the compression side. It is of course satisfactory to use the entire cross-sectional area in calculating bending stresses up to the point of buckling.

For a *stiffened* shell, however, it is possible for the thin sheet between stiffeners to buckle prematurely, after which the stiffeners may continue to resist more load. One method of handling this is to ignore the buckling and to assume that all the sheet is effective. When this is done, the allowable stress used must be an *average* stress obtained for the particular sheet-stringer combination employed. Even then some error is involved, as the true effective properties of the cross section are not obtained. (This error is usually small, however.)

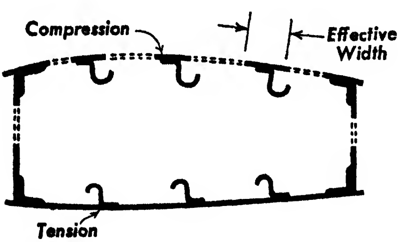


Fig. 15·2. Effects of buckling.

Another more accurate method is to leave out part of the buckled sheet in computing section properties. (This is analogous to letting n = zero for concrete in tension.) The remaining sheet is assumed to be fully effective, i.e., it has the same stress as the stiffener. The width of sheet thus assumed to be working with the stiffener is called the **effective width.** Figure 15·2 indicates how the transformed section appears (effective material shaded). Note that on the tension side of the beam all the sheet is assumed to be effective.

The theory of effective width is quite involved, but for approximate results it may be assumed that it depends on the sheet thickness alone. Thus a value of $30t$ for aluminum alloy is often used. For instance, if the sheet were 0.050 in. thick, the effective width would be 0.050×30

= 1.5 in. If the stiffeners were spaced at 6 in., this would represent an effectiveness of 1.5/6 = 0.25, which could be thought of as the value for n for the sheet.

For more refined methods see any modern textbook on airplane structures, such as Refs. 2, 3, 16, and 40.

15·5. Plastic Bending. In Chapt. 13 it was shown that the assumption of a linear stress distribution in bending depended on two other assumptions:

a. Linear strain distribution.
b. Proportionality of stress to strain.

The first of these two assumptions is satisfied for a straight beam by assuming that *plane cross sections remain plane.* This is the most basic

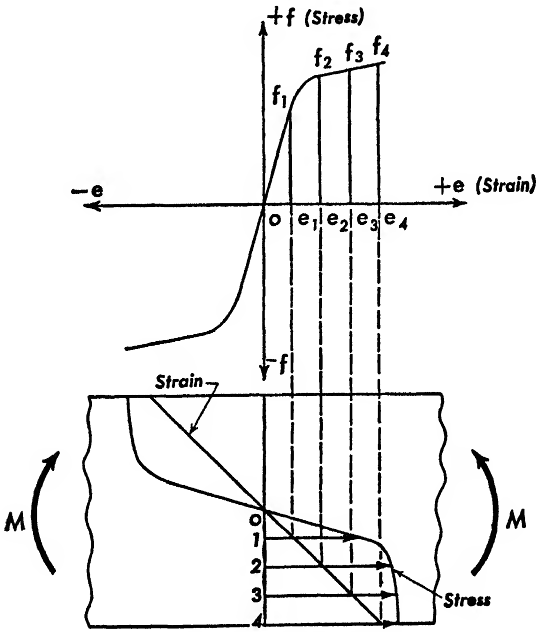


FIG. 15·3. Plastic bending.

of all the bending assumptions and is seldom violated to any appreciable extent.

In the plastic range it may be assumed that the first assumption is retained and that cross sections still remain plane during bending. But obviously the second assumption must be discarded, and the true variation of stress with strain as given by the stress-strain diagram must be considered.

Figure 15·3 shows an assumed case in which the strain variation for the beam is plotted to the same scale as the stress-strain diagram and directly under it. The stresses corresponding to different points on this line are obtained from the diagram and plotted on the beam. The stress distribution is no longer linear but takes on the general appearance of the stress-strain diagram itself (here rotated 90°).

It can be seen that as the degree of bending increases the strain progresses farther into the plastic range, and more of the stress diagram (for the beam) becomes curved. Different progressive stages of bending are indicated in Fig. 15·4. Note that the triangular shape of the stress

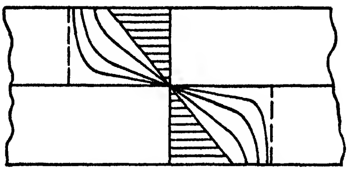


FIG. 15·4. Progressive plastic bending.

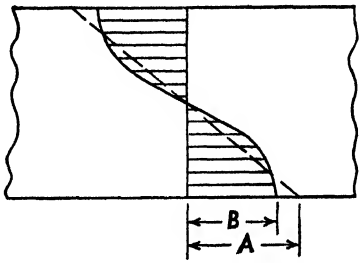


FIG. 15·5. Modulus of rupture relationship.

curve approaches a rectangle (dotted lines) as the plastic bending progresses.

If the strain at the outer fibers were known (or assumed), the stress distribution could be plotted as shown. The computation of the resisting moment could then be carried out by multiplying each element of area by its corresponding stress and again by its distance from the bending axis. The application of the direct method here would be simpler than any attempt to use effective areas or transformed sections.

15·6. Modulus of Rupture. Since the stress distribution curve changes shape for each degree of loading after entering the plastic range, it is impossible to represent it by a simple mathematical relationship, as was done for *elastic* bending, and the theory of plastic bending is therefore not utilized in structural analysis to any great extent. Instead, the error involved is usually taken care of in strength calculations by using a fictitious *allowable bending stress*, or *modulus of rupture*. This is obtained by testing a specimen in bending and substituting the allowable bending moment in the elastic formula, $f = My/I$, to determine the allowable stress. The modulus of rupture so obtained is, in general, considerably higher than the true maximum stress within the beam, as indicated by Fig. 15·5. If the true stress distribution were as shown by the solid line, the ratio involved would be A/B.

For solid bulky sections (not subject to buckling) it is almost always safe to assume that the allowable bending stress is at least as high as the ultimate tension stress.[48] For thin sections, tubes, and shells this may not be true because of buckling effects.

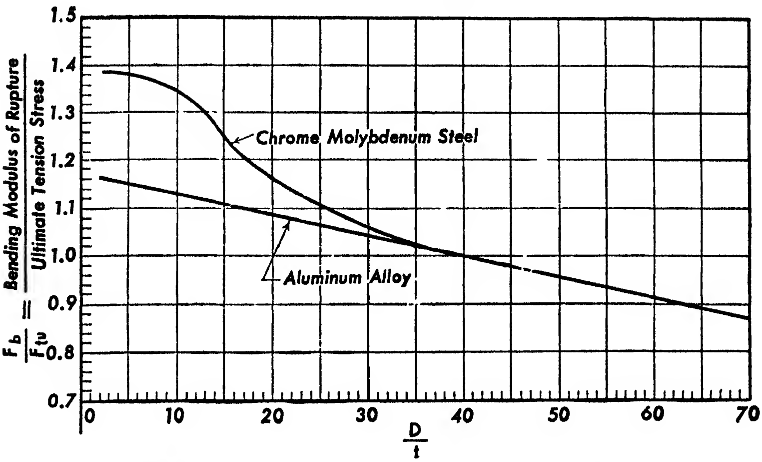


FIG. 15·6. Bending modulus of rupture for tubes. (D = outside diameter; t = thickness.)

Many factors influence the modulus of rupture in bending, so that a complete discussion is beyond the scope of this text. Besides, the allowable bending stresses are quite well established in different branches of structural engineering and can be obtained from design handbooks or government publications. One such chart, for tubes, is included as an example (Fig. 15·6). This was derived from standard charts used for aircraft work.

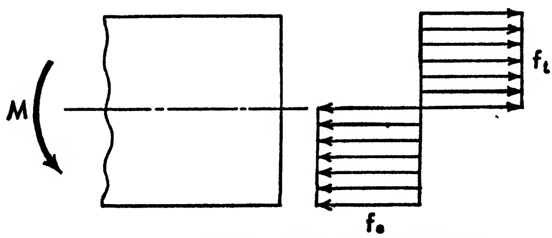


FIG. 15·7. Limiting case of plastic bending.

As a matter of interest, it might be noted that the limiting case of plastic bending would be a constant stress distribution over each half of the beam, as shown in Fig. 15·7. For the same stress in the outer fiber, this distribution would give a resisting moment 50 per cent greater than that given by the elastic (triangular) distribution, assuming a rectangular cross section. This means that the modulus of rupture in bending cannot exceed the ultimate tension stress by more than 50 per cent for a rectangular cross section. (The ratio may be greater for sections of the type shown in Fig. 13·15c.)

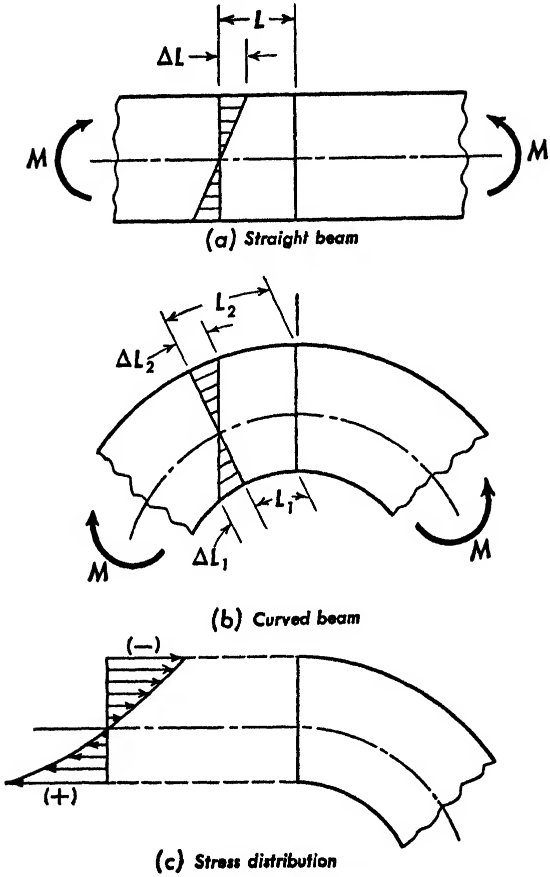


FIG. 15·8. Curved beam.

15·7. The Curved Beam. All the beams considered up to this point have been initially straight. In deriving the beam theory an element of unit or constant length, L, was assumed, as in Fig. 15·8a. The *strain* ($\Delta L/L$) was, therefore, directly proportional to ΔL, which was assumed to have a linear variation, based in turn on the assumption that the plane cross section remained plane during bending.

In the curved beam the assumption of plane cross sections is again used as a starting point. However, in dividing the beam into unit segments by means of plane cross sections, it is impossible to use segments cut by parallel lines; these lines must now radiate from the center of curvature as shown in Fig. 15·8*b*. This causes the length of the segment to be greater on the outside of the beam than on the inside.

In calculating the strain distribution over the cross section, the change in length (ΔL) at each point must be divided by the unit length at that point; for instance,

$$e_1 = \frac{\Delta L_1}{L_1}$$
$$e_2 = \frac{\Delta L_2}{L_2} \qquad [15\cdot5]$$

It can now be seen that when the cross sections remain plane during bending, giving a linear distribution of ΔL, the strain distribution will *not* be linear. It will vary not only with the distance from the neutral axis (as in the straight beam), but also with the change in length of the unit segment. Since this length is directly proportional to the radius from the center of curvature, the strain will tend to vary inversely as this radius. This gives a *hyperbolic* type of strain distribution, as indicated in Fig. 15·8*c*. If the stress is proportional to strain, it will have the same type of distribution. Hence, all the beam theories based on linear stress distribution are in error when applied to curved beams.

Although it is possible to develop a modified beam theory, using the hyperbolic stress distribution,* it is simpler to apply the method of *effective areas*, i.e., to use a *transformed* section. Since the departure from a linear stress distribution is due to the variation in length of the unit section, it should be possible to take care of this by modifying the unit *area* instead.

The axial resistance of a unit area of cross section is given by

$$P_1 = f_1 A_1 = E e_1 A_1$$

Substituting from Eq. 15·5,

$$P_1 = E A_1 \frac{\Delta L_1}{L_1}$$

which could also be written

$$P_1 = E\left(\frac{A_1}{L_1}\right)\Delta L_1$$

* The exact condition may be obtained from the theory of elasticity.[18] A close approximation is given by the Bach-Winkler formula, found in most textbooks. Reference 22 contains working tables for various types of curved beams.

The term (A_1/L_1) can obviously be considered as the *effective area,* making the equation identical with that used as a basis for the straight beam.

In computing the section properties, it is therefore necessary only to divide each element of area by its corresponding unit length. From Fig. 15·9 it can be seen that the following relationship holds.

$$\frac{1}{R_0} = \frac{L_2}{R_2}$$

Therefore

$$\frac{1}{L_2} = \frac{R_0}{R_2}$$

Since the term $1/L$ corresponds to the *effectiveness factor,* n (Sec. 15·2) the latter may be defined as

$$n = \frac{1}{L} = \frac{R_0}{R} \qquad [15\cdot6]$$

where R_0 = radius of curvature to element used as standard.
R = radius to any other element.

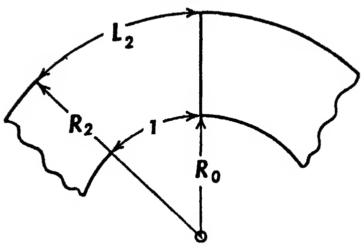


FIG. 15·9. Relationship between radius and unit length.

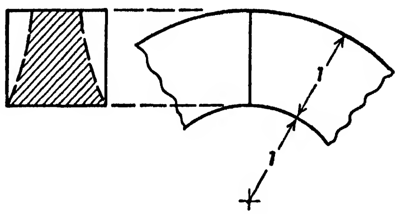


FIG. 15·10. Transformed section for a curved beam.

If the section properties are being computed with reference to some arbitrary axis, it is convenient to use this for the basic radius of curvature. Hardy Cross[29] recommends dividing each differential area by R, which is equivalent to assuming that the basic (or standard) radius of curvature is unity. Since the computed stresses must be multiplied by n to obtain actual stresses, it makes no difference what radius is assumed as standard.

Figure 15·10 shows how the transformed section would look if the inner radius of the beam were used as standard and the effectiveness factor applied to the *width* of the beam.

The transformed section derived in this manner converts the beam into an equivalent *straight* beam, both for bending and axial loading. It is, therefore, correct to use the centroid of the transformed section in dealing with pure axial loading, provided, of course, that the resulting axial stresses are finally multiplied by the *effectiveness factor n*. If pure axial loading is to cause a change in length without change in curvature, it is obvious from Fig. 15·8*b* that plane cross sections cannot remain parallel to their original positions during axial loading, as was assumed for the straight member. This accounts for the inconsistency encountered when an attempt is made to use straight beam theories without modifying the section properties.

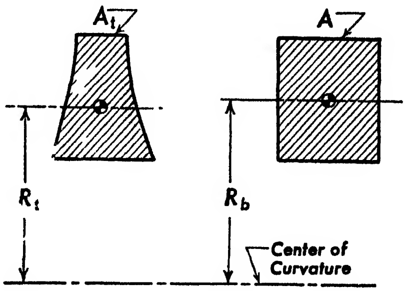


FIG. 15·11.

If the section properties of the actual section are known, or can be easily computed, it is convenient to follow a procedure suggested in Ref. 29 in which it is shown that it is necessary only to compute the *area* of the transformed section. The other properties may be obtained from the following equations, in which symbols are illustrated by Fig. 15·11 and R_0 is defined as for Eq. 15·6.

$$R_t = R_0 \frac{A}{A_t} \qquad [15\cdot 7]$$

$$I_t = A_t R_t (R_b - R_t)$$
$$= A R_0 (R_b - R_t) \qquad [15\cdot 8]$$

It is interesting to note that in the actual design of curved beams the most efficient cross section will have a shape similar to that of the transformed section. Thus for equal tension and compression stresses in the inner and outer fibers, a symmetrical cross section would be used for the straight beam. For a curved beam this would be transformed as indicated in Fig. 15·10, giving a wider section on the inside edge of the beam.

Regardless of the method used, it is convenient to express the results in terms of a *correction factor*, by which the stresses obtained from the straight beam formula may be multiplied. The maximum axial stress due to bending may then be obtained from the formula

$$f = K \frac{Mc}{I} \qquad [15\cdot 9]$$

where K = the curved beam correction factor.
M = the bending moment about a principal axis through the c.g.
c = the maximum distance from the principal axis.
I = the moment of inertia of the actual cross section (*not* the transformed section).

Values of K for typical sections are given in Ref. 22, based on the work of Wilson and Quereau.[30] Figure 15·12 is reproduced from Ref. 22, to show how the correction factor varies with the degree of curvature.

Dolan and Levine[39] suggest that this factor also be applied to the *axial* stresses for combined axial load and bending in curved beams. This is to take care of an apparent stress-concentration effect.

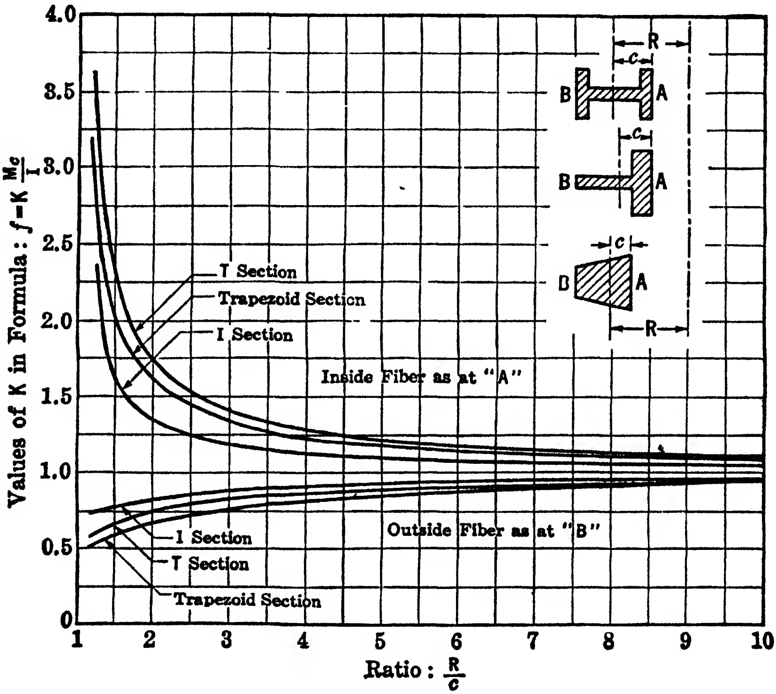


FIG. 15·12. Correction factors for curved beams. (Reproduced from Seely, *Advanced Mechanics of Materials*.)

15·8. Radial Loads in Curved Beams. From the discussion of *hoop tension* (Chapt. 7) and curved shear webs (Chapt. 11) it will be evident that radial forces or pressures must accompany any transmission of axial force through a curved path. For if w represents the *normal* or *radial* load per inch of length along the path (Fig. 15·13), the hoop-tension force required to balance it will be

$$P = wR \qquad [15\cdot10]$$

Conversely, if an axial force is transmitted through a curved path, it will cause radial forces equal to P/R lb per in. along the path. These must be resisted in some manner.

In the curved beam the actions of the tension and compression flanges balance each other through the web, as indicated in Fig. 15·13. For a

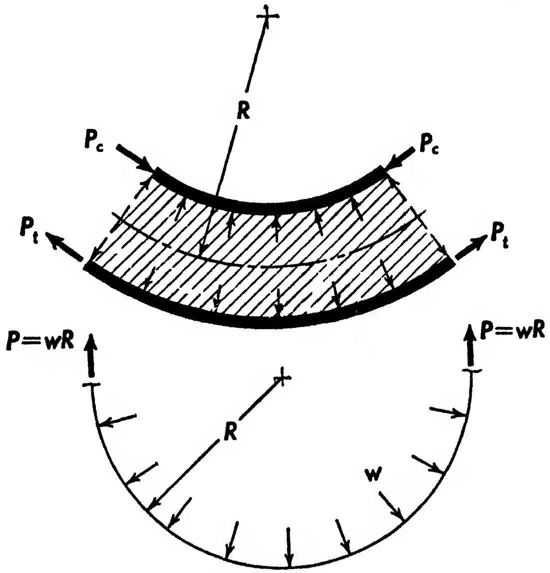


FIG. 15·13. Radial forces.

solid beam the web area is so large that these radial stresses may usually be neglected. But for a thin web or box type of beam special provisions may have to be made in the form of stiffeners or knee braces. The latter type of fitting is particularly useful in preventing the collapse of thin outstanding flanges on I-beams and similar structures (Fig. 15·14).

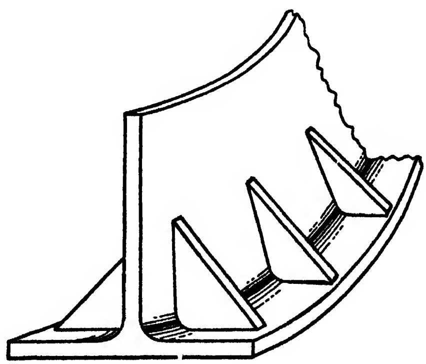


FIG. 15·14. Knee braces for curved beams.

The total radial load per stiffener (or brace) may be determined by multiplying the running load w by the distance between stiffeners (see Fig. 15·15).

$$P_r = wL = \frac{P_a L}{R} \quad [15 \cdot 11]$$

where P_a = the axial force.
L = the length supported by each member.
R = the radius of curvature of the axially loaded member.

Equation 15·11 may be used to calculate the *induced* radial loads caused by bending deflections, but the value of R must be calculated first (see Chapt. 16). In the analysis of shallow box beams such as used in airplane wings these loads may become critical for the ribs or other

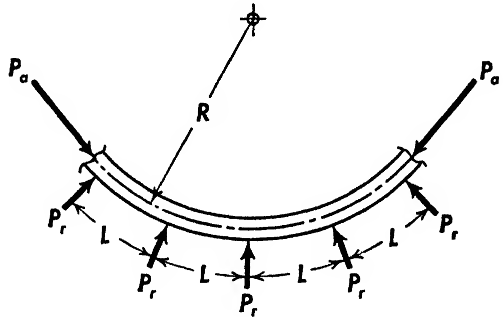


FIG. 15·15. Computation of radial loads.

internal supporting structure. They are sometimes called **crushing loads.**

15·9. Abrupt Curvature. The crushing loads may become very large if there is an abrupt change of angle of the beam, as shown in Fig. 15·16. The magnitude of the normal load may easily be computed for each

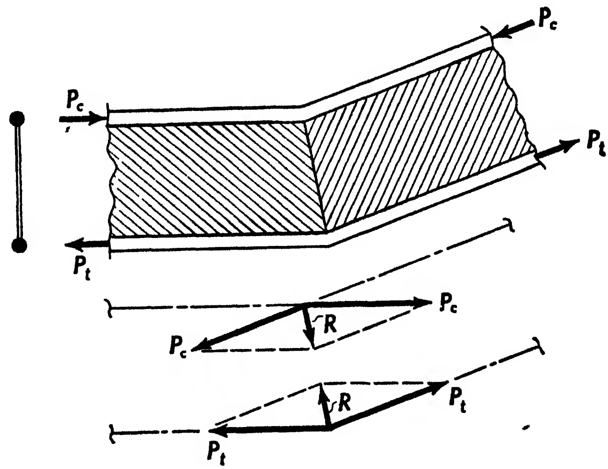


FIG. 15·16. Effects of abrupt curvature.

flange of a beam by finding the resultant of the two opposing axial loads at the point of curvature. Note that the direction of the crushing load will reverse if the direction of the bending moment is reversed, thus requiring a *tension* member across the beam at this point.

In solid beams it may be unnecessary to make this provision, but if the change of angle is quite large, it is well to determine the normal forces. This can be done by extending the process used for the flange type of beam, following the general methods used in developing the beam theory.

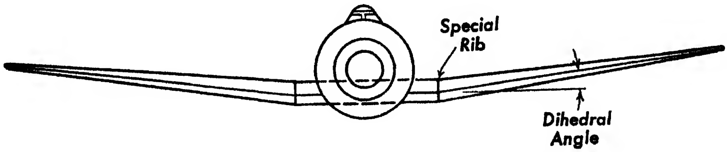


FIG. 15·17. Abrupt curvature of airplane wing beam.

A typical example of abrupt curvature is found in an airplane wing in which a dihedral angle exists at one or more joints (Fig. 15·17). A specially reinforced rib is required at such points.

15·10. Local Conditions. The assumption of a linear stress distribution in bending may be considerably in error near the points where high

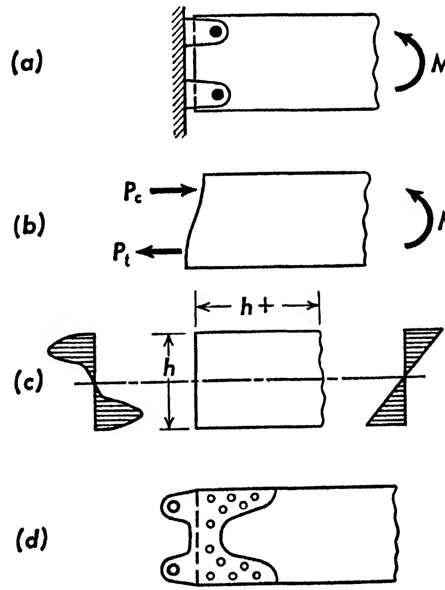


FIG. 15·18. Local conditions in bending.

local loads are introduced, or at points where the cross section changes abruptly. A typical example is found at the end of a beam, as shown in Fig. 15·18. If the attachment is made by only two bolts as in (*a*), the cross section will be distorted as in (*b*) (exaggerated); and the local stress distribution will appear roughly as in (*c*).

Because of the continuity of the structure, however, this departure from the beam theory does not affect a very large portion of the beam. According to *Saint-Venant's principle* (Sec. 7·9), the effect becomes negligible at distances greater than the depth of the beam. The exact conditions can be predicted analytically by the powerful methods of the theory of elasticity [18] or experimentally by the use of photoelasticity.[25]

From a practical point of view it is important that such high local stresses be avoided as much as possible. This may be done *by introducing the forces in the manner in which they will eventually be resisted.* Figure

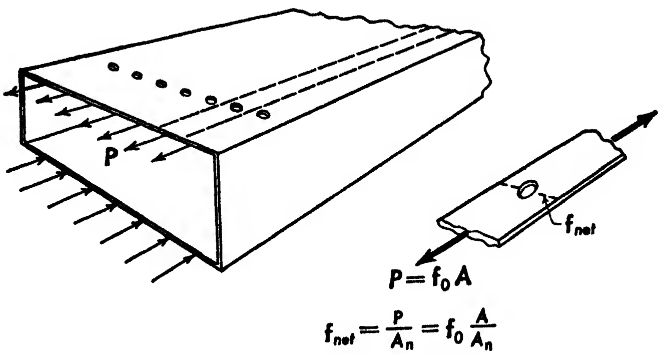


FIG. 15·19. Effect of holes.

15·18*d* shows how this might be accomplished for the case previously mentioned by employing a fitting which distributes the loads roughly in the manner predicted by the beam theory.

It is equally important to avoid abrupt changes in the distribution of the material along the beam. This applies particularly to large structures in which joints must be made, or in which the strength is varied by using sheets or members of graduated size. The entire theory of bending, up to this point, assumes that conditions are *continuous* through the cross section in question, i.e., that no appreciable changes in area, c.g., or moment of inertia occur for some distance on either side of the cross section. Obviously a reasonable degree of such variation can be introduced without any serious effects. If abrupt variations cannot be avoided, it is necessary to consider them as *stress concentrations* and to increase the computed stresses in some suitable manner.

Cut-outs are common in airplane construction and are typical examples of local discontinuity. Special reinforcements are usually required to avoid a considerable local decrease in strength. The methods of analyzing large cut-outs are beyond the scope of this volume, but the effect of bolt or rivet holes deserves attention.

If a beam is pierced by one or more holes at some particular point, it might be thought that the proper method of analysis would be to use a cross section through the holes in determining section properties. This is erroneous (unless the holes also extend spanwise and are closely spaced in the spanwise direction). The proper procedure is to obtain the section properties for the basic structure *without* the holes and to use these properties in determining the stress distribution over the cross section. The local effect of the holes is then obtained by increasing the the axial stress in the ratio of the basic area to the net area for suitable flange units or strips. The procedure is illustrated in Fig. 15·19.

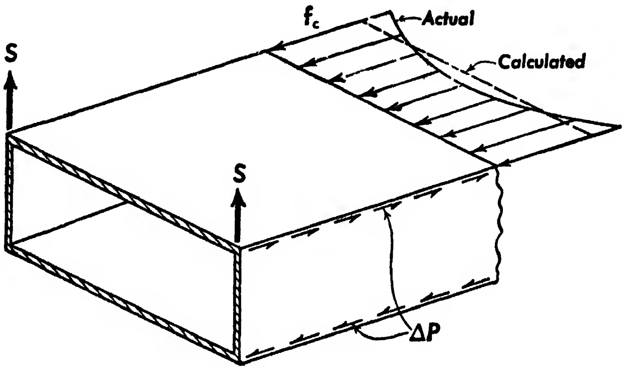


FIG. 15·20. Nature of shear lag.

15·11. Shear Lag. In thin shell structures, such as used in airplane wings and fuselages, the introduction of the shear loads along the edges tends to cause the cross sections to warp slightly. The effect on the bending stresses is to cause them to increase near the points of shear loading and to decrease at points farthest away from the application of shear loads, as indicated in Fig. 15·20. This is what happens when the loads are not introduced in accordance with the beam theory. It can be seen that the increments of axial force ΔP are not applied uniformly over the top and bottom portions of the beam; hence there is a lag between the actual and calculated stress distributions (*actual* shown by heavy line in Fig. 15·20, *calculated* shown by dotted line).

It is important to note that this phenomenon is due entirely to the flange load increments ΔP. It would not exist in pure bending, for instance, if the moments were applied through rigid end plates. It would also be eliminated if the increasing axial loads in the top and bottom of the beam were picked up by increasing corner flange areas at such a rate that no distribution over the shell was needed. (Taper of the beam in width has this effect.)

The methods of computing shear lag really belong under the heading of combined shear and bending and, in any case, are beyond the scope of this volume. The phenomenon can often be avoided or reduced to negligible proportions by proper design. (See Ref. 40 and NACA * reports for methods of calculation.)

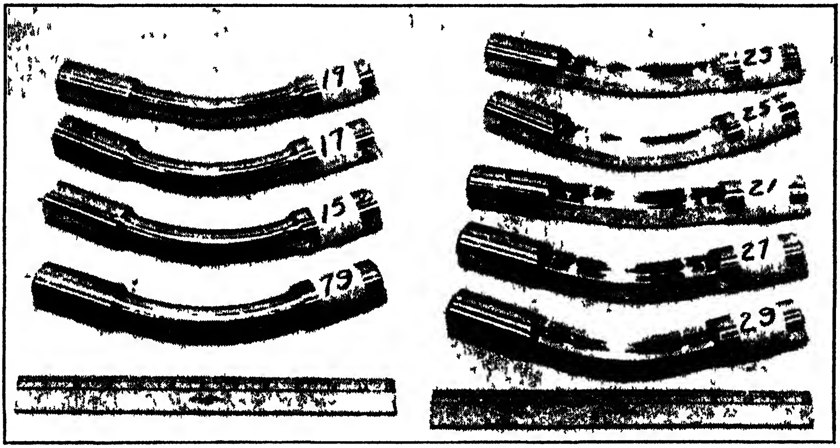
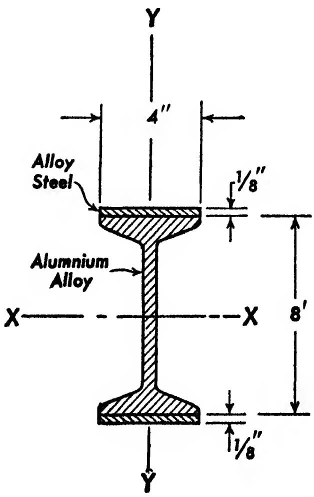


FIG. 15·21. Specimens which have been tested in bending. Note plastic deformation.

PROBLEMS



PROBLEM 15·1.

15·1. The aluminum-alloy I-beam shown in the figure has the following properties:

$$A = 5.40 \text{ sq in.}$$
$$I_x = 57.55 \text{ in.}^4$$
$$I_y = 3.73 \text{ in.}^4$$

(*a*) Find its ultimate bending strength (about each axis) before reinforcement, assuming an allowable stress of 50,000 psi.

(*b*) Using aluminum alloy as standard, calculate the transformed section properties of the beam when reinforced by steel plates as shown. (*Note.* Moment of inertia of steel plates about their own *X–X* axes may be neglected, but the value about the *Y–Y* axis must of course be used.)

* National Advisory Committee for Aeronautics.

15·2. (*a*) In Problem 15·1 find the maximum stress in the steel plate when the reinforced beam is subjected to a bending moment of 800,000 in.-lb (about X–X axis).

(*b*) What is the maximum stress in the aluminum alloy for this loading condition?

15·3. (*a*) Assuming that the allowable bending stress for the steel plate is 75,000 psi (compression) under bending about the Y–Y axis, calculate the strength of the reinforced beam under such bending.

(*b*) Compare with strength of plain aluminum alloy beam and explain results.

15·4. Work out Problem 15·1 assuming that the I-beam is alloy steel and the plates are aluminum alloy. Use an allowable stress of 90,000 psi for the steel and 50,000 psi for the aluminum alloy.

15·5. Work out Problem 15·2 for the assumptions used in Problem 15·4.

15·6. A reinforced concrete beam of rectangular cross section is 12 in. wide and 30 in. deep. Four steel rods, ¾ in. in diameter, are located 2 in. from one of the short sides. Assuming that the modulus of elasticity of the steel is 15 times that of the concrete find the transformed area, centroid, and moment of inertia (maximum), assuming all the concrete to be effective. Compare with the properties of the unreinforced beam.

15·7. In Problem 15·6 assume that half of the concrete is not effective in tension and find the bending moment which will produce a stress of 20,000 psi (tension) in the steel rods. (*Note.* Assume that the compression stress varies linearly from zero at middle of cross section.) Find the maximum compression stress in the concrete.

15·8. Draw an arbitrary stress-strain diagram for any desired material and compute the bending modulus of rupture on the assumption that failure occurs at the maximum elongation. Assume also that the diagram is identical for tension and compression. (*Note.* Use a rectangular cross section, which permits the areas under bending stress diagrams to be used directly. See Fig. 15·3.)

15·9. Assume that a tube has been designed on the basis that its ultimate bending stress equals the ultimate tension stress. What percentage of overstrength or understrength would this introduce, for the following examples? (Use Fig. 15·6.)

(*a*) 1.5 in. × 0.028 in. alloy steel
(*b*) 1.5 in. × 0.028 in. aluminum alloy
(*c*) 1.0 in. × 0.035 in. alloy steel
(*d*) ⅞ in. × 0.049 in. alloy steel
(*e*) ⅞ in. × 0.049 in. aluminum alloy

(*Note.* Above dimensions are outside diameter and wall thickness.)

15·10. Assuming a beam of rectangular cross section 1 in. wide, select a depth between 2.1 and 2.9 in. and a radius of curvature between 3.1 and 3.9 in., measured to the inside edge of beam. Divide the cross section into strips of approximately 0.25 in. and compute the area, centroid, and moment of inertia for the transformed section, using the inside radius as standard. Find the bending stress in the inside and outside fibers for a bending moment of 1000 in.-lb.

Compare with values for the *straight* beam and check accuracy of results by means of Fig. 15·12.

15·11. In a flange type of beam the radius of curvature under load is computed as 1200 in. The centroids of the flanges are 10 in. apart and the flange loads are 40,000 lb. Find the axial forces in "vertical" stiffeners placed 20 in. apart. (*Note.* Assume that both flanges have the same radius of curvature.)

15·12. Using the beam of Problem 15·11 find the transverse force caused by an abrupt change of angle, such as shown in Fig. 15·16. Use values of 5°, 10°, 15°, and 20°, and plot results to show effect of increasing angle. (*Note.* Neglect curvature of beam itself, in this problem.)

CHAPTER 16

BENDING DEFLECTIONS

16·1. Deflections in Pure Bending. In Chapt. 13, Sec. 13·3, it was shown that a truss subjected to a couple (pure bending) acquired a *curved* shape, which is characteristic of bending deflections. It was pointed out that the reason for the curvature was the change of angle caused by the distortion of each frame.

Consider several adjacent frames under the action of a couple, as shown in Fig. 16·1, in which the deflections are greatly exaggerated.

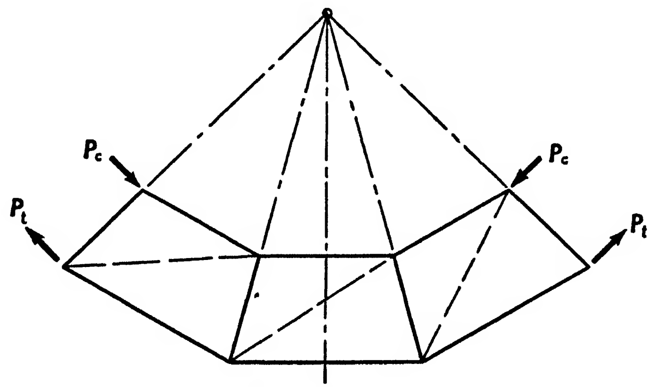


FIG. 16·1. Deflection of truss-type beam.

(Diagonals are present but remain unloaded.) If the flange members are the same size and length they will deflect equally in each frame and the verticals will acquire relative angles such that their projections pass through the same point. Under such conditions the beam will therefore have the general shape of a *circular arc* with a *center of curvature* and a *radius of curvature.*

If the individual frames are assumed to be very short the curve becomes smoother, finally becoming a perfect circular arc when the solid beam is considered. Conditions for such a beam under bending action are shown in Fig. 16·2 (deflections greatly exaggerated).

A unit length, AB, is measured off along the neutral axis, and parallel lines are drawn through A and B, normal to the axis of the beam. These represent the undeflected positions of two adjacent planes. As the bending progresses the line through B will remain normal to the axis of the beam and will rotate about the neutral axis, through the shaded area. If plane cross sections remain plane, as assumed by the beam theory, the line will remain straight and the shaded areas will be triangles. The maximum deflections are indicated by e_c (compression) and e_t (tension) and, of course, occur at the outer edges of the beam. Since a unit length is used as a basis these deflections are actually *strains* (strain = deflection per unit length).

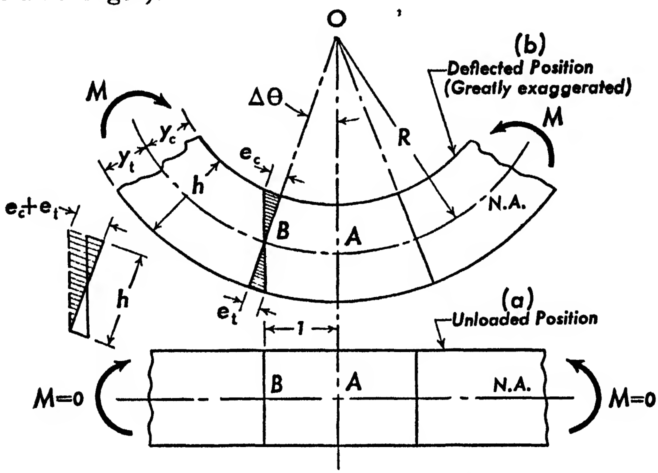


FIG. 16·2. Deflection of solid beam.

16·2. Radius of Curvature. If the line through B, normal to the beam axis, is now extended it will intersect a similar line through A at point O, which is the **center of curvature.** The radius of curvature OA may now be found from the relationship between the two shaded triangles and the large triangle OAB, all three of which are *similar*, giving

$$\frac{AB}{OA} = \frac{1}{R} = \frac{e_c}{y_c} = \frac{e_t}{y_t} \qquad [16\cdot1]$$

Another relationship may be found from the triangle indicated separately, giving three equivalent expressions for R.

$$\frac{1}{R} = \frac{e_c}{y_c} \qquad [16\cdot2a]$$

$$\frac{1}{R} = \frac{e_t}{y_t} \qquad [16\cdot2b]$$

$$\frac{1}{R} = \frac{e_c + e_t}{h} \qquad [16\cdot2c]$$

(See Fig. 16·2 for meaning of symbols.)

These equations are special cases of the general relationship.

Radius of curvature

$$\boxed{\frac{1}{R} = \frac{e}{y}} \qquad [16\cdot3]$$

where R = radius of curvature, measured to neutral axis.
e = strain at a point y.
y = distance to neutral axis.

This equation is based on the assumption that the *strain* distribution is linear (plane cross sections remain plane). It does not require that the *stress* distribution be linear, hence it may be used in the plastic range.

Equations 16·2 and 16·3 are expressed as reciprocals of the radius of curvature, as the degree of bending deflection is inversely proportional to the radius. Thus an infinite value of R corresponds to zero curvature and zero bending; hence it is appropriate that the term $1/R$ be used, as its value is zero for this condition.

16·3. Metal-Forming Strains. Equation 16·2 may be used to compute the strains corresponding to a given radius of curvature, as in metal-forming calculations. For example, assume that a round bar ½-in. in diameter is to be bent to an inside radius of 4 in. It can be assumed that the neutral axis will go through the center of the bar, hence the value for y_c or y_t is $0.5/2 = 0.25$ in. The radius of curvature at the neutral axis is $4 + 0.25 = 4.25$ in. By substituting in Eq. 16·2*a*,

$$\frac{e_t}{y_t} = \frac{1}{R}$$

$$e_t = \frac{y_t}{R} = \frac{0.25}{4.25} = 0.059$$

The same value of strain would of course be obtained for the compression fiber. By referring to the *stress-strain diagram* for the material it could be seen whether this strain would cause a permanent set, or whether it might exceed the maximum breaking strain. The important production problem of springback may also be analyzed by an extension of these simple relationships.*

The bending of a flat *spring* to a given radius can also be analyzed in this manner to see if the spring will recover its shape. The strain would have to remain in the elastic range. This could be readily determined by multiplying the strain by E to obtain the stress and noting whether this exceeded the proportional (elastic) limit.

* See Reference 31 for a discussion of springback and related problems.

16·4. Minimum Bend Radius. For sheets or bars of thickness t (assuming midplane to be neutral plane) Eq. 16·3 may be written

$$e = \frac{t}{2R} \qquad [16\cdot4]$$

Conversely, this equation can be simply expressed to determine the minimum R/t corresponding to any permissible elongation e_{max}:

$$\left(\frac{R}{t}\right)_{min} = \frac{1}{2e_{max}} \qquad [16\cdot4a]$$

where R is measured to the mid-plane.
e_{max} is the permissible elongation or strain.

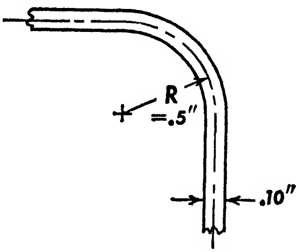


FIG. 16·3. Bend radius.

Example: Find the minimum R/t for a sheet of material having an elongation of 10 per cent.

$$\frac{R}{t} = \frac{1}{2 \times 0.10} = 5$$

For a sheet 0.10 in. thick, R would be 0.5 in., as shown in Fig. 16·3. On the drawing the inside radius would be shown as 0.5 − 0.05 = 0.45 in. (approximately 7/16).

Equation 16·4*a* indicates the importance of elongation in a material which is to be formed by bending. It is obviously impossible to make sharp bends in sheet material having elongations much less than the 10 per cent used in the preceding example.

16·5. Radius of Curvature—Flange Type Beam. In Sec. 13·4 a type of beam was described in which the bending moment is resisted entirely by two flanges, separated by a distance, h. (See Figs. 13·5 and 13·6.) For a given bending moment, M, the flange *forces* are given by Eq. 13·3 which may be written

$$P_t = P_c = \frac{M}{h}$$

where P_t = the force in the tension flange.
P_c = the force in the compression flange.

The flange *stresses* then become

$$f_t = \frac{P_t}{A_t}$$

$$f_c = \frac{P_c}{A_c}$$

In the elastic range the *strains* are obtained by dividing stress by E.

$$e_t = \frac{f_t}{E} = \frac{P_t}{A_t E} = \frac{M}{A_t E h}$$

$$e_c = \frac{f_c}{E} = \frac{P_c}{A_c E} = \frac{M}{A_c E h}$$

From Eq. 16·2c,

$$\frac{1}{R} = \frac{e_c + e_t}{h}$$

$$= \frac{M}{Eh^2}\left(\frac{1}{A_t} + \frac{1}{A_c}\right) \qquad [16\cdot5]$$

Equation 16·5 is not really necessary, as it is just as simple to compute the flange strains directly and substitute them in the equation. But it does indicate that the degree of curvature ($1/R$) is:

a. Directly proportional to M.
b. Inversely proportional to E.
c. Inversely proportional to h^2.

Thus the *stiffness* of a flange type of beam is proportional to the *square* of its depth. The term involving areas is also interesting as it shows that if either flange has a low cross-sectional area the deflection will be high, regardless of the area of the other flange. For high bending stiffness or rigidity it is therefore desirable to use a material having a high modulus of elasticity E, to separate the flanges as far as practicable, and to make the flange areas large and approximately equal.

16·6. Radius of Curvature Formula. Although the radius of curvature can be determined directly from the calculated bending strains (Eq. 16·2), it is usually more convenient to convert the formula into terms of bending moment and moment of inertia. Equation 16·3 states that

$$\frac{1}{R} = \frac{e}{y}$$

The flexure formula gives

$$f = \frac{My}{I}$$

Dividing both sides of this equation by E gives the strain (since $E = f/e$).

$$e = \frac{f}{E} = \frac{My}{EI}$$

Substituting this in the equation for $\frac{1}{R}$,

Radius of curvature $$\boxed{\frac{1}{R} = \frac{M}{EI}}$$ [16·6]

This is the companion formula of the *flexure formula* ($f = My/I$). It is the basis for all calculations of deflection due to bending. The term EI is of particular significance. It is often called the **flexural rigidity,** as it is a direct measure of the resistance to bending. It is comparable to the factor EA in the equation for deflection under *axial* force. Note that R is measured to the *neutral axis,* as it was derived in this manner in Sec. 16·2.

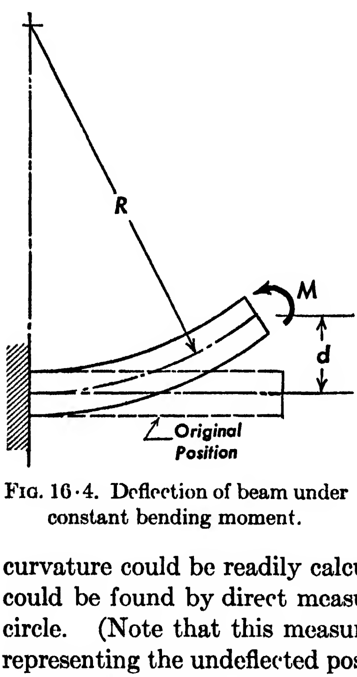


FIG. 16·4. Deflection of beam under constant bending moment.

16·7. Bending Deflection. One of the most common structural problems is the computation of deflections due to bending. Such deflections are in general considerably greater than those caused by axial or shear forces and are therefore of much practical significance in the design of buildings, bridges, airplanes, etc.

If a beam of constant cross section (or constant EI) were rigidly supported at one end and subjected to a constant bending moment over its entire length, it would appear as a circular arc, as shown in Fig. 16·4. The radius of curvature could be readily calculated from Eq. 16·6, and the deflection could be found by direct measurement or by using the equation of the circle. (Note that this measurement is made with reference to a line representing the undeflected position of the beam).

Quite frequently, however, the bending moment and value of EI are not constant. Besides, it is inconvenient to work with the equation of the circle even when it applies. For this reason the radius of curvature equation has been further modified so that it can be used more directly in computing deflections.

16·8. Slope of a Line. The modification used is a purely geometric one, based on the fact that the quantity $1/R$ actually measures the *rate of change of angle* (or slope) along a line. In Sec. 13·3 it was noted that this change of angle was the main source of truss deflections. Figure

13·4 showed the change of angle (greatly exaggerated) by the position of the verticals.

This is indicated more clearly by Fig. 16·5, in which the *slope* of the beam centerline or axis at point b is shown by a tangent drawn normal to the radius at that point. The angle $\Delta\theta$ is obviously the difference in slope between points a and b, which are separated by a distance Δx.

For small deflections (large R) it is quite accurate to assume that distances measured along the curved line are equal to those measured along

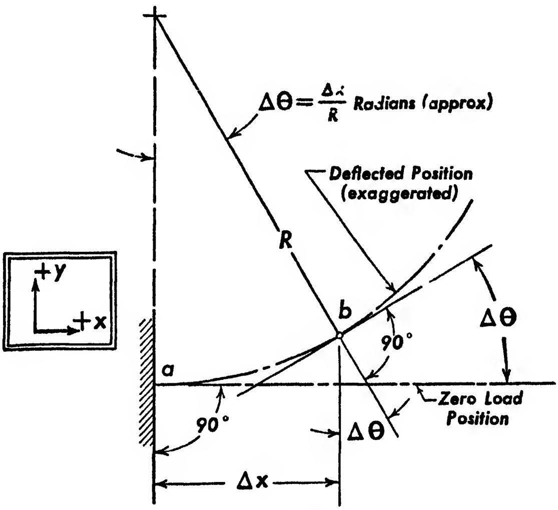


FIG. 16·5. Slope of beam.

the reference axis; hence it may be stated that the angle between the two radii equals $\Delta x/R$, in radians. Then,

$$\Delta\theta = \frac{\Delta x}{R} \qquad [16\cdot7]$$

This is the change in angle over a length Δx. The change over a *unit* length is obtained by dividing by Δx, giving

$$\frac{\Delta\theta}{\Delta x} = \frac{1}{R} \qquad [16\cdot8]$$

The term $\Delta\theta/\Delta x$ may be thought of as the *rate of change of slope* (θ) *with respect to distance along the axis* (x). The symbol Δ indicates that small increments or units are used. This expression is a purely geometric one which does not depend in any way on the assumptions of the beam theory. Its accuracy depends on the smallness of the increments selected for Δx. As these approach zero the formula becomes more accurate. For the hypothetical case of zero increments the formula would be written

$$\frac{d\theta}{dx} = \frac{1}{R} \qquad [16\cdot9]$$

in which the first term would be read "derivative of θ with respect to x." *

16·9. Calculation of Slope. For a circle having a constant value of R, the rate of change of slope will be given as constant by Eq. 16·8, so that the total slope over a distance x would be x/R.

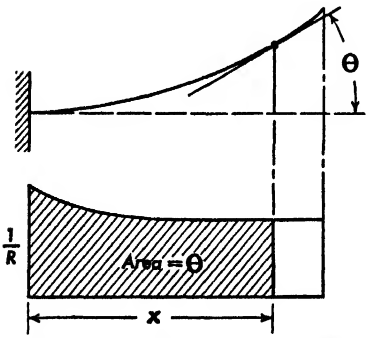


FIG. 16·6. Calculation of slope.

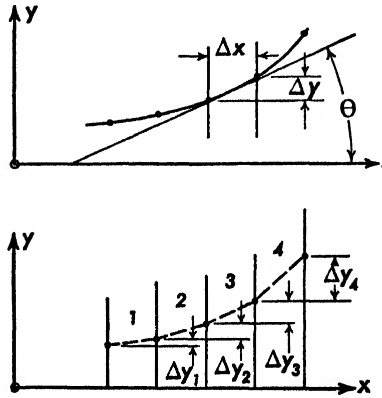


FIG. 16·7. Deflection of beam.

For a line having a variable radius of curvature the slope may be calculated by using methods of summation similar to those employed in Secs. 2·18 to 2·20. Since $1/R$ represents rate of change of θ with respect to x, the actual change over the increment Δx is obtained by multiplying by Δx (see Eq. 16·8). For a series of successive increments these values may be calculated and summed up by using the average values of $1/R$ over each increment.

The variation of $(1/R)$ over the length x may also be plotted to a suitable scale. The area under this curve, up to any point x, gives the value of θ at that point, as indicated in Fig. 16·6.

By selecting a number of "stations" along the span it is possible to derive a series of values for θ, which may then be plotted as a curve giving the slope at any point x. (This is *not* the deflection curve, however.)

16·10. Deflections in Terms of Radius of Curvature. If the *slope* of the beam at a series of points is known, or if a curve which indicates its

* As previously noted, the terminology of calculus is often employed in structural equations, even when it is not actually possible to apply the methods of calculus.

value at every point is given, the next step is to convert these values into deflections normal to the reference axis. The slope has the physical significance shown in Fig. 16·7. The value of θ at a certain point on a curved line represents the *rate of change of deflection* (y) *with respect to distance along the axis* (x). This could also be stated as the rate at which the curve departs from the axis. This may be expressed by the equation

$$\frac{\Delta y}{\Delta x} = \theta \qquad [16\cdot 10]$$

Actually this amounts to assuming that the curve is a series of straight lines over the lengths Δx, as indicated by the dotted lines in Fig. 16·7.

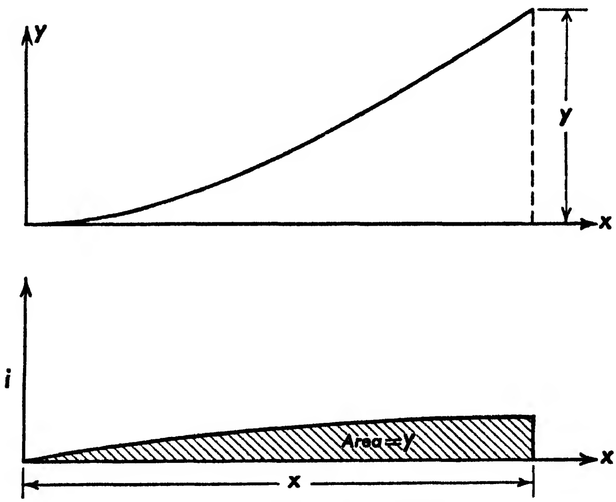


FIG. 16·8. Calculation of deflection.

As for the slope, the equation becomes more accurate as the increment Δx is decreased in size, and the ultimate form of the equation becomes

$$\frac{dy}{dx} = \theta \quad \text{(Compare Eq. 16·9.)} \qquad [16\cdot 11]$$

To obtain the total deflection at any point it is necessary to compute the values for all the increments and then add them, as was done in determining the slope. Here again the process may be performed numerically, graphically, or by integration. The picture is given by Fig. 16·8.

If the value for θ given by Eq. 16·11 is substituted in Eq. 16·9 the latter becomes:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{1}{R}$$

which may be written symbolically as

Curvature equation

$$\boxed{\frac{d^2y}{dx^2} = \frac{1}{R}} \qquad [16 \cdot 12]$$

The first term is read as "the second derivative of y with respect to x." The equation indicates that the *rate of change of the rate of change* of y with respect to x is equal to $1/R$. It may be considered as a simplified form of the exact equation which expresses the relationship between the radius of curvature of a line and its rectangular coordinates.

$$\frac{\dfrac{d^2y}{dx^2}}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}} = \frac{1}{R} \qquad [16 \cdot 12a]$$

The exact equation is not usually needed in structural work, as the only inaccuracy introduced by the simpler Eq. 16·12 is due to the assumption that distances measured along the line itself are equal to distances measured along the reference axis (see Fig. 16·5). The error becomes appreciable only when the ratio of x to R begins to exceed a value of about $1/10$.

16·11. The Elastic Curve. Up to this point the equations for the slope and deflected position of any line have been given in terms of $1/R$. This was done primarily to emphasize the fact that their relationships are geometric and that they do not depend in any way on the beam theory itself. Thus it is possible to use the foregoing equations in determining the deflection of a beam bent *beyond the elastic limit,* provided that the correct values for R are somehow obtained.

In Sec. 16·6 it was shown that the assumption of a *linear stress distribution* resulted in Eq. 16·6.

$$\frac{1}{R} = \frac{M}{EI}$$

This value may be substituted for $1/R$ in Eqs. 16·9 and 16·12, giving the following important formulas.

Slope equation

$$\frac{d\theta}{dx} = \frac{M}{EI} \qquad [16 \cdot 13]$$

Deflection equation

$$\boxed{\frac{d^2y}{dx^2} = \frac{M}{EI}} \qquad [16 \cdot 14]$$

Equation 16·14 is often called the equation of the *elastic curve,* as it applies only in the elastic range (because stress was assumed to be proportional to strain in deriving Eq. 16·6). It forms the basis for most of the mathematical processes used in handling more advanced problems in bending. It is seldom used directly in structural computations, so it is only important, at this point, to recognize the physical significance of the equation.

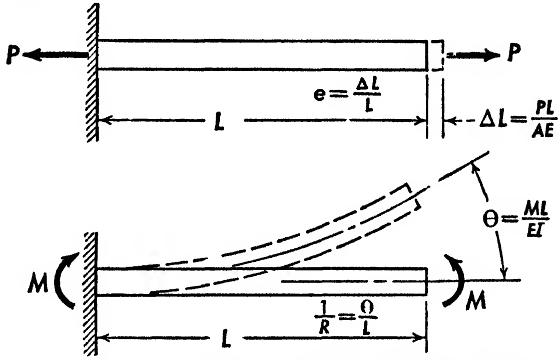


FIG 16·9. Comparison of axial loading and bending.

16·12. Length Effect. The **flexural rigidity** EI has been shown to be a measure of a beam's resistance to bending, just as the factor EA measures resistance to axial deformation. The primary effect of bending, however, is not deflection, but *rotation* or *curvature.* This can also be thought of as a *change of slope.* For axial loads the term P/EA gives the elongation per unit length, or the strain e. The total change in length is therefore given by the term PL/EA (Eq. 6·4c).

Figure 16·9 shows a comparison between axial loading and bending. The quantity which is comparable to axial strain is $1/R$, which might be thought of as the *angular strain.** This is equal to M/EI (Eq. 16·6). Therefore the total change of slope over the distance L equals ML/EI, for a constant bending moment, M.

Table 16·1 shows how the quantities compare.

TABLE 16·1

AXIAL	BENDING
P	M
A	I
e	$\frac{1}{R}$
ΔL	$\Delta\theta$

* This term is not commonly used in structural literature. It is used here mainly to assist in the comparison.

The comparable equations are given in Table 16·2.

TABLE 16.2

Axial	Bending
$f = \frac{P}{A}$	$\frac{f}{y} = \frac{M}{I}$
$e = \frac{f}{E} = \frac{P}{EA}$	$\frac{1}{R} = \frac{M}{EI}$
$\Delta L = eL = \frac{PL}{EA}$	$\Delta\theta = \left(\frac{1}{R}\right) L = \frac{ML}{EI}$

Although the change of slope is not often of direct interest, it is useful to know how this value is found for a given bending moment acting over a certain length. The following equation is therefore relatively important.

$$\Delta\theta = \frac{ML}{EI} \qquad [16\cdot15]$$

where $\Delta\theta$ = the change of slope over the length L.
M = the bending moment, assumed to be constant over length L.

It should be noted that the symbol Δ is retained in Eq. 16·15 because the *actual* slope at a given point will not necessarily be given by this equation. If θ is taken at zero at one end of the beam, however, Eq. 16·15 will give the actual slope at the other. Under such conditions the slope will vary linearly from zero at one end to a maximum at the other, as shown in Fig. 16·10.

Equation 16·15 can be transposed to indicate the *resistance to angular rotation.* If this quantity is defined as the bending moment required to cause a unit change of angle ($\Delta\theta = 1$), Eq. 16·15 becomes

$$\frac{M}{\Delta\theta} = \frac{EI}{L} \qquad [16\cdot16]$$

The term EI/L is of special significance in beam problems involving the *relative stiffness* of joined members. Since E is usually constant for a given structure, the term I/L gives the relative resistance to rotation. It is often referred to as the **stiffness factor.**

From Sec. 16·10 and Fig. 16·10 it is evident that the total deflection y over the length L is equal to the summation (or integral) of θ over this length, or the area under the curve in Fig. 16·10. Since this area is equal to $\frac{1}{2}\theta_{max}$, the total deflection for a constant bending moment is given by

Deflection for constant moment
$$y = \frac{1}{2}\frac{ML^2}{EI} \qquad [16 \cdot 17]$$

This equation implies that y is measured normal to a reference line for which the slope is zero, i.e., a tangent to the elastic curve. This is

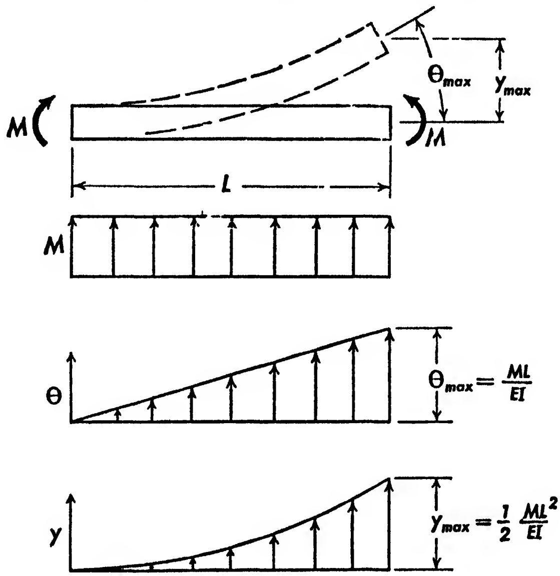


Fig. 16·10. Relationships for constant bending moment.

illustrated in Fig. 16·10. Figure 16·11 is more general. It is important to keep this in mind in dealing with beams which do not have zero slope at either end.

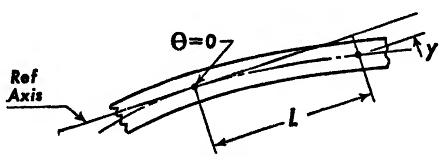


Fig. 16·11. Measurement of deflection.

Equation 16·17 reveals a very important point, which is that bending deflections (for constant moment) are proportional to the *square* of the length. This is obviously because the length enters twice into the summation process (once for slope and once for deflection).

Since I/L (or EI/L) was shown to be a measure of relative resistance to rotation, the term I/L^2 (or EI/L^2) is a similar measure of the relative resistance to deflection under a constant bending moment.

16·13. End Conditions. Most of the examples used up to this point have been cantilever beams, for which one end was completely constrained. For such a beam both the slope and the deflection may be considered to be zero at the constrained end, and the original (undeflected) axis of the beam may be used as a reference axis in computing deflections.

In Sec. 16·8 it was noted that the slope equation gave only the *change* of slope between two points due to the bending moment. If neither end of the beam is fully constrained the slope at the ends will not remain zero during loading, and the axis for which the deflections are computed will rotate.

The most common example of this is the simple beam which is supported at each end, but not constrained against rotation at the supports. The mid-portion of a beam loaded as in Fig. 16·12 represents such a beam subjected to a constant bending moment. The deflection may be computed by following the steps indicated in the figure.

The calculations are started as if the left end were constrained against turning; this enables the curves for slope and deflection to be started from zero, as shown. The summation is carried out and the deflection curve is found, as shown in (*e*). But the conditions of support are such that y must equal zero at the right end. This end of the beam is therefore figuratively pulled down to its point of support by rotating the entire beam about the left end. The deflection curve then appears as in (*f*).

This rotation process can be done by drawing the axis x–x through the two ends of the deflection curve and measuring the vertical distances between this axis and the beam axis. The angle through which the beam is figuratively rotated is indicated on (*e*) as θ_0. Since this applies to the entire beam, it represents a constant change of angle from end to end and would therefore appear as a horizontal line on Fig. 16·12*d*. The rotation is obviously opposite in sense to the original curvature of the beam, and therefore θ_0 would be laid off below the base line and the net (final) values obtained by taking the difference between the two curves. Figure 16·12*g* shows the final curve of θ for the simple beam. In practice it is simpler to draw the line for θ_0 *above* the base line and regard it as the new axis for measuring slope. It can be seen that this gives a negative value for θ_L and a positive value for θ_R. Since the beam is assumed to have a constant value of M, E, and I, the deflection is symmetrical about the midpoint; hence θ_L, θ_R, and θ_0 are equal in magnitude.

This general process is applicable to any beam simply supported at two points, regardless of the type of loading. It should be noted that

θ_0 may be taken as equal to y/x, in radians (it cannot be measured directly from the chart unless y and x are plotted to the same scale, which is impractical).

For the example shown, the rest of the deflection curve could be found by treating the two overhanging ends as cantilevers. Here again the

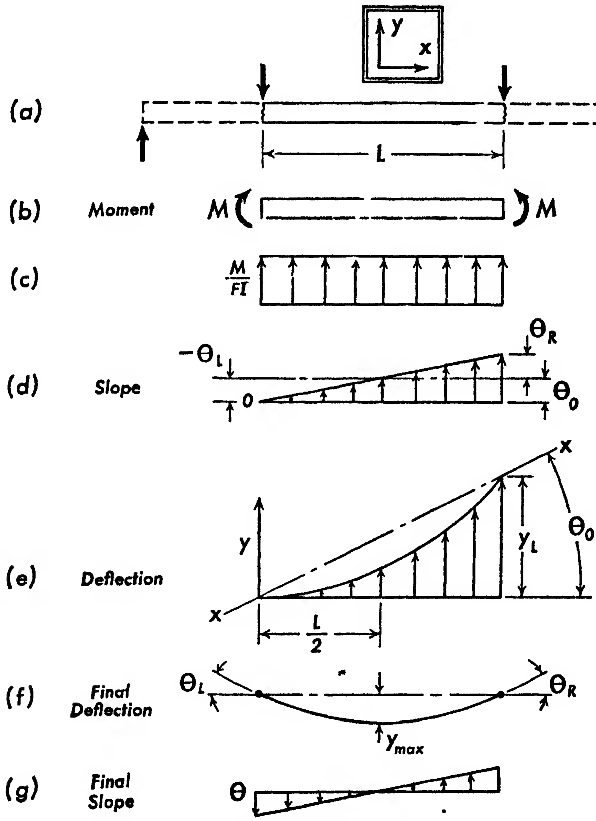


FIG. 16·12. Calculation of deflection of a simple beam.

computations could be carried out as if the slope at the supports were equal to zero. After the deflections were found, the curves would be rotated through the angles θ_L and θ_R to agree with actual conditions.

Note that rotation of any deflection curve through an angle θ (in radians) can be done numerically by adding (or subtracting) values of deflection equal to $x\theta$, where x is the distance along the axis measured from the center of rotation.

16·14. Maximum Deflection. In Fig. 16·12*f* it is obvious that the maximum deflection occurs at the mid-span of the beam. The value of this deflection may be found from Eq. 16·17 and Fig. 16·12*e*. Let y denote the hypothetical deflections of the beam (before it is rotated) and let y' be the deflection of the line x-x. The conditions at mid-span ($x = L/2$) are then as follows.

$$y' = \frac{1}{2}\frac{(L/2)^2M}{EI} = \frac{1}{8}\frac{L^2M}{EI}$$

$$y_0 = \frac{1}{2}y_L' = \frac{1}{2}\cdot\frac{1}{2}\frac{L^2M}{EI} = \frac{1}{4}\frac{L^2M}{EI}$$

$$y = y' - y_0 = \frac{1}{8}\frac{L^2M}{EI} - \frac{1}{4}\frac{L^2M}{EI}$$

$$y_{max} = \frac{1}{8}\frac{L^2M}{EI} \qquad [16\cdot18]$$

16·15. Conventions and Signs. In dealing with simple cases it is possible to dispense with a rigorous system of conventions, as a sketch showing the applied forces or bending moments will usually indicate the nature of the deflection curve. In general, however, it is desirable to establish a definite set of conventions and adhere to them throughout the solution of a deflection problem.

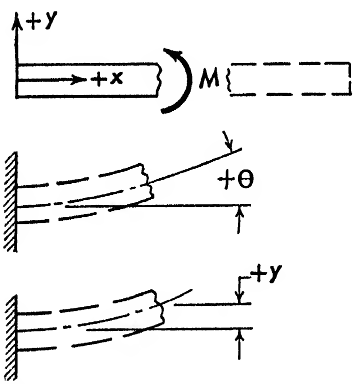


FIG. 16·13. Deflection conventions.

The important point in this connection is that there must be consistency between the positive conventions for moments, slope, and deflection. In defining a positive bending moment either the *space* or *structural* type of convention may be used (see Chapt. 3). If the space type of convention is adopted it must include a rule as to which side of the beam is being considered. Thus in Fig. 16·13 a positive bending moment is shown as a *counterclockwise* moment acting on a section *to the left* of the point at which the moment is measured. This convention must be adhered to even though the section to the left approaches zero.

In using the summation process to determine deflections it is necessary to start at some point for which the slope and deflection are known (or assumed) to be zero. This was also found to be true in computing bending moments, as it was necessary to start the process from a point of zero moment. These two starting points sometimes will not coincide on the beam. For instance, in a cantilever beam the bending moment is zero at one end, but the slope and deflection are zero at the other end.

If the summation processes are carried out numerically, using diagrams and working with areas, it is possible to make this transfer of reference points without any difficulty. But if mathematical methods are employed, the same reference point must be used throughout.

In the cantilever beam this could be done by selecting the fixed end as a base. The expression for bending moment would then become

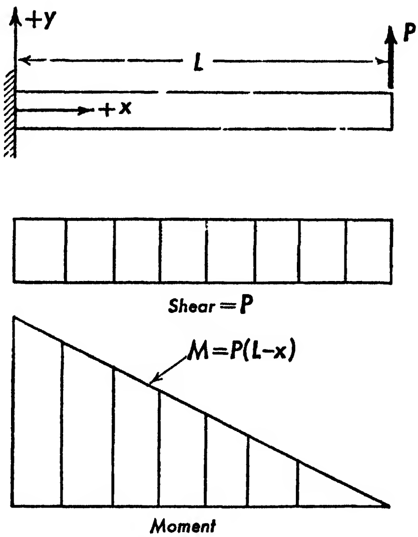


FIG 16·14 Concentrated loading.

$M = P(L - x)$, and this value would have to be used in deriving the formulas for slope and deflection (see Fig. 16·14).

16·16. Non-Uniform Bending. Although the theory of bending has been worked out for a *constant* bending moment, it is more common to find that the moment varies with distance along the beam. An obvious example is that of a concentrated load acting on a cantilever beam. Here the variation of the bending moment is linear, as shown in Fig. 16·14.

Although the general methods of summation previously described will take care of any variation of bending moment, very often the answers can be easily obtained mathematically. Appendix 3 gives the essential data on various types of beams and loading conditions in *non-dimensional* form. The curves may therefore be used for any specific case by multiplying their values by the factors peculiar to the problem in hand. As indicated in the figures, the numerical values for the curves have been obtained by assuming values of unity for the dimensional factors involved (M, P, E, I, L, etc.).

16·17. Fixed-End Beams. Although a detailed treatment of redundant structures is beyond the scope of this text, the fixed-end beam is so common that it will be discussed briefly. Figure 16·15*a* shows a beam that is fixed at one end and has simple support at the other. This gives *two* possible methods of force transmission, as shown by (*b*) and (*c*), making the beam *statically indeterminate*, or *redundant* (see Chapt. 5).

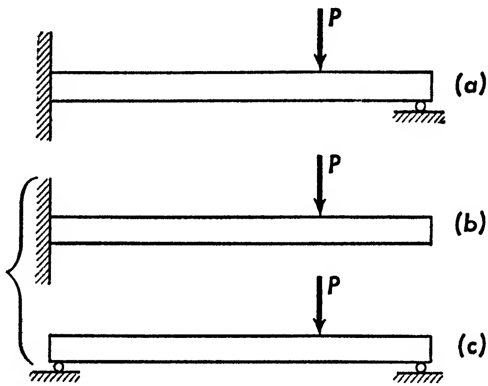


FIG. 16·15. Redundant beam.

In Sec. 16·13 it was shown how the computation of the bending deflection for a simple beam required a figurative rotation about one point of support in order to satisfy the requirement of zero deflection at the other. In Fig. 16·15 there are *two* end conditions that must be satisfied: the deflection must be zero at the right end, and the slope must be zero at the left end. This problem can be solved mathematically by using simultaneous equations. It could also be solved by a trial-and-error process in which the trial would consist in guessing how much of the force P is resisted by the right-hand support. If the guess happened to result in zero deflection at the right end and zero slope at the left end, it would be correct. If not, the results would indicate in which direction the error had been made, and a new trial would be carried out.

This method is laborious, but the underlying principle should be thoroughly understood, as it represents a process of self-adjustment which actually takes place under loading. The trial-and-error process is also foundation for many other powerful analytical methods which have become widely used.

It is obvious, of course, that the beam could be analyzed for two separate conditions such as represented by (*b*) and (*c*) in Fig. 16·15.

In general these conditions will bracket all intermediate conditions, but such assumptions are usually too severe to be used for practical purposes.

Figure 16·16 shows a beam that is fixed at both ends. This gives three statically determinate types of force transmission, as shown by (*b*), (*c*), and (*d*). The end conditions now become:

a. Zero relative deflection at ends.
b. Zero slope at left end.
c. Zero slope at right end.

If the loading is *symmetrical*, it is easy to solve the problem by applying to the slope (θ) curve the same principle that was applied to the deflection curve in solving the simple beam problem. First the slope curve for the simple beam is determined, as in Fig. 16·17*c*. Then the slope curve is rotated until its ends are at zero. This is equivalent to applying a straight-line slope curve which has end values exactly equal and

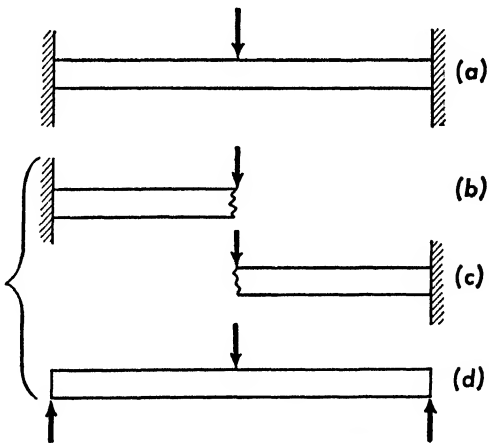


FIG. 16·16. Fixed-end beam.

opposite to those of the simple beam (dot-dash line). The final slope curve (*d*) is obtained by subtraction (using dash line).

It has previously been shown that a constant bending moment (more accurately, a constant value of M/EI) causes a straight-line slope curve. The total difference in slope between the two ends of a beam subjected to a constant bending moment is equal to ML/EI (Eq. 16·15). In Fig. 16·17*c* this corresponds to $2\theta_0$. The fixed-end moments may be obtained from this relationship. For the illustration $\theta_0 = PL/16$. Therefore the fixed-end moment is $PL/8$. This constant value is subtracted from that for the simple beam (Fig. 16·17*a*), giving the final bending moment curve shown in (*e*). The final deflection curve (*f*) is obtained by integrating the slope curve (*d*).

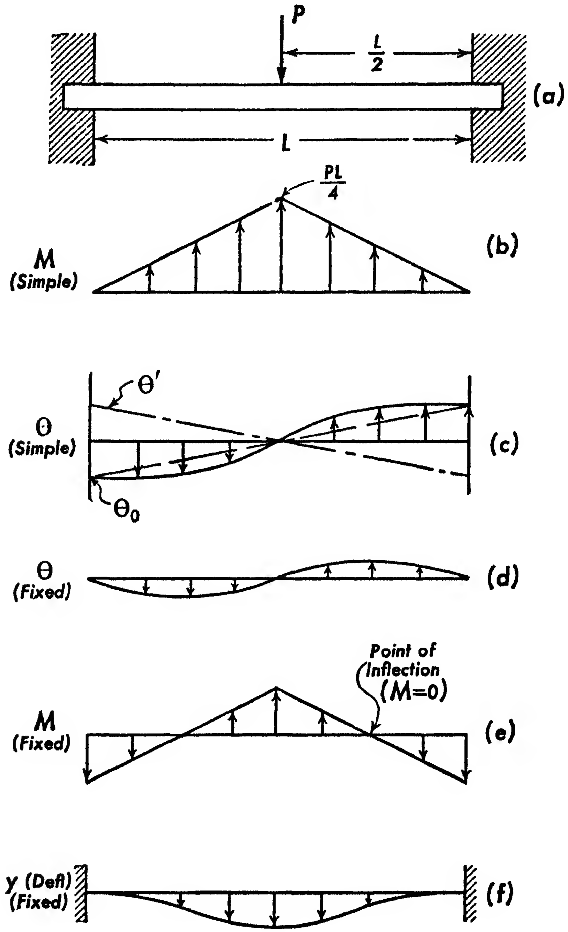


Fig. 16·17. Solution of fixed-end beam.

16·18. General Principles. It will be apparent from the examples already given that there are several general principles which apply to the relationships between the various curves used in computing deflections. The more important of these are as follows.

a. For zero relative deflection at each end the net area under the slope curve must be zero, i.e., the areas above the zero line must equal the areas below.

b. For complete fixity at each end the slope curve must start and end at zero.

c. Moving the slope curve vertically corresponds to rotation of the entire beam through the corresponding angle.

d. The slope curve may be rotated by applying a constant bending moment over the beam.

FIG. 16·18. Static test of *Lightning* (P-38) tail surface. This shows the deflection under very high loading, produced by means of shot-filled bags.

PROBLEMS

16·1. Assume that the beam of Fig. 16·1 is 8 in. deep, the top flanges are 1 sq in. in cross-sectional area, and the bottom flanges 1.5 sq in. The forces comprising the couple are each 50,000 lb. Using Eq. 16·2 find the value of $1/R$ for the beam.

(*a*) Both flanges aluminum alloy.
(*b*) Both flanges alloy steel.
(*c*) Top flange steel, lower flange aluminum alloy.

(*Note.* Verticals may be assumed to be close together, so as to cause a smooth curve.) Check results by using Eq. 16·5.

16·2. A square steel bar ½ in. thick is to be formed into a circular hoop.

(*a*) If the ultimate elongation is assumed to be 15 per cent, what is the minimum inside radius that may be used?
(*b*) Compute the approximate bending moment that must be applied, on the basis that the stress distribution is constant at a yield stress of 40,000 psi (as in Fig. 15·7).

16·3. In Problem 16·2, what is the minimum inside radius to which the bar may be bent without causing permanent set? (Assume that the yield stress and proportional limit have the same value.)

16·4. Assume that the bar of Problem 16·2 is to be used as a spring and is heat treated to raise the proportional limit to 150,000 psi.

(*a*) What mean radius of curvature (at neutral axis) may now be developed without causing permanent set?
(*b*) What maximum elongation is involved in bending to this radius?

16·5. A beam has two flanges composed of aluminum alloy bars, each of 0.5 sq in. cross section. The centroids of the bars are 12 in. apart. A bending moment of 200,000 in. lb is applied. Neglect the individual moments of inertia of the bars and compute the radius of curvature from Eq. 16·6. Check by means of Eq. 16·2.

16·6. In Problem 16·5 assume for the area of one flange a value between 0.35 and 0.45 sq in. and for the other a value between 0.55 and 0.6 sq in. Find the radius of curvature and check as in Problem 16·5.

16·7. Assume that the beam of Problem 16·5 is 120 in. long. What is the change of slope caused by the bending moment?

16·8. Find the change of slope for the beam used in Problem 16·6, assuming it to be 200 in. long.

16·9. What is the rate of change of slope, in degrees, of a line drawn with a radius of 250 in.? How far does this line depart from another line drawn tangent to it, in a distance of 20 in.?

16·10. A cantilever beam 40 in. long is subjected to a constant bending moment such that its slope at the free end is 3° with respect to that at the fixed end.

(*a*) What is the radius of curvature?
(*b*) What is the deflection?

16·11. Two cantilever beams of the same length are subjected to the same value of constant bending moment. One beam is aluminum alloy and the other is heat-treated alloy steel. The moment of inertia of the aluminum alloy beam is twice that of the steel beam. What are the relative deflections at the free ends, i.e., which beam is stiffer and how much?

16·12. In Problem 16·11 assume that the steel beam is 20 per cent longer than the aluminum-alloy beam. Find the relative slopes and deflections at the free ends.

16·13. A wooden cantilever beam 10 ft long is made from a 2 by 4. If the maximum safe bending stress at the root is assumed to be 8000 psi, find the maximum deflection permissible; also the maximum load. (Appendix 2 may be used. Assume $E = 1{,}200{,}000$ psi. Neglect weight of beam.) Work this problem for two conditions:

(*a*) Load in plane of major axis.
(*b*) Load in plane of minor axis.

16·14. Work out Problem 16·13, assuming the beam to be simply supported at each end and loaded in the middle by a concentrated load.

16·15. A bar of alloy steel 1 in. in diameter and 20 ft long is simply supported at each end. How much does it sag under its own weight? (See Appendix 2 for properties.)

16·16. Assume that the bar of Problem 16·15 is made from aluminum alloy which has a yield stress of 50,000 psi. What is the maximum length of bar that can be supported at each end without producing a permanent set? (Assume that yield stress and proportional limit are equal.)

16·17. Find the moment of inertia required for a carbon-steel beam which must support a uniformly distributed loading of 20 lb per in. (including its own weight), without exceeding a deflection of ¼ in., for the following cases.

(*a*) Cantilever beam 10 ft long.
(*b*) Simply supported beam 20 ft long.
(*c*) Continuous beam with supports every 20 ft. (Assume fixed ends at each support.)

16·18. In Problem 14·2 (page 230) calculate the displacement of the free end of the beam along the principal axes of the cross section. Show by means of a sketch that this displacement is *not* in the plane of the applied load. Assume E = 1,000,000 psi (approx. correct for soft wood). Appendix 3 may be used.

(*Note.* This problem shows that the principles of unsymmetrical bending developed in Chapt. 14 apply also to deflections. *Effective bending moments* (Sec. 14·3) may be used directly in the deflection equations to determine the components of the deflection along reference axes which are not the principal axes.)

16·19. Show that the *effective* resistance to bending in the plane of an applied moment equals $I(1 - U_xU_y)$. (Refer to Eqs. 16·6 and 14·10.)

CHAPTER 17

COMBINED BENDING AND SHEAR

17·1. Flange Type of Beam (Unit Method). In beginning the study of bending in Chapt. 13, it was noted that bending almost always resulted from the transmission of shear (transverse) forces. In Sec. 10·16 it was shown that a constant shear flow in a flange type of beam caused

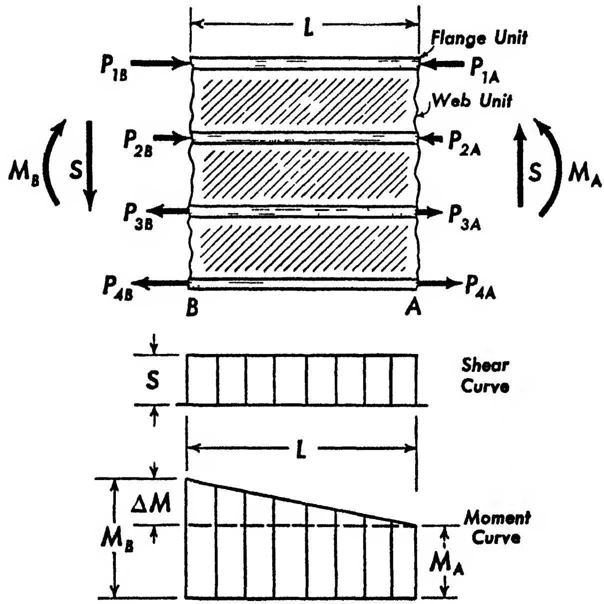


Fig. 17·1. Shear in multiflange beam.

an increasing axial force in the horizontal flanges. For such a beam the distribution of shear flow was assumed to be constant over the depth of the beam.

In Sec. 13·6 the theory of bending was developed by considering a *multiple-flange* type of beam, as shown in Fig. 13·7. This particular beam has three separate shear webs, and it cannot be assumed that the shear flow is the same across each of them. The distribution actually depends on the *axial* loads in the flange members, and hence on the bending theory used to determine these flange loads.

This can be seen by considering a small length of such a beam under combined shear and bending, as shown in Fig. 17·1. Also shown are portions of the shear and moment diagrams corresponding to this section of the beam.

The axial forces in the flanges are assumed to be known, or to have been calculated, at sections A and B. Since the bending moment at B is higher than at A the axial forces will be correspondingly higher. Taking flange 1, for example, the force P_{1B} will be greater than P_{1A}. Since these two forces are not in equilibrium, flange 1 would start moving toward the right, unless restrained by the shear web. It is therefore the *difference* between the axial forces at each end of the unit flange length which determines the shear loading for the web.

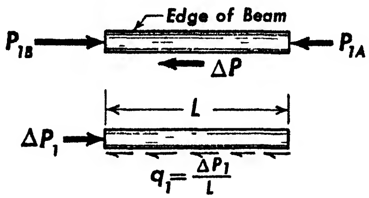


FIG. 17·2. Equilibrium of edge flange unit.

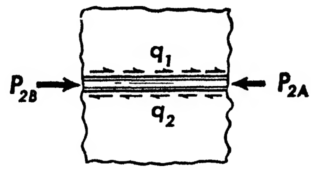


FIG. 17·3. Equilibrium of internal flange unit.

This is further illustrated by Fig. 17·2, in which the conditions are shown for flange 1. The web force obviously equals the difference in axial forces $(P_B - P_A) = \Delta P$. If this force is distributed uniformly along the flange by the shear web, it can be expressed as a *shear flow*, q, determined by dividing the force by the length. (Actually the flange is usually connected to the web by rivets or bolts; then the load per rivet would equal $\Delta P/n$, where n is the number of rivets in the length L.) Note that since flange 1 is on the free edge of the beam, all the unbalanced load must be resisted by only one shear web.

From Fig. 17·3 it can be seen that the shear load in the second web is equal to that in the first web, plus the difference in axial force over the unit length of flange 2. Therefore,

$$q_2 = q_1 + \frac{P_{2B} - P_{2A}}{L}$$

$$q_2 = q_1 + \frac{\Delta P_2}{L} \qquad [17\cdot1]$$

where ΔP_2 is the change in axial force in flange 2 over the length L. This equation can be put in a general form as follows.

Bending shear flow

$$q_n = q_{n-1} + \frac{\Delta P_n}{L} \tag{17·2}$$

where n = the number of the flange unit in question.
q_n = the shear flow on one side of the flange unit.
q_{n-1} = the shear flow on the other side.
ΔP_n = the change in axial force in flange unit n over length L.

Equation 17·2 is elementary in nature, but because it represents the most general form of the shear flow equation it is very important. The following points should be noted.

a. Since the shear flow in any web depends on the flow in the preceding one, it is necessary to know the flow in at least one web in order to get started on the computation.

b. If the beam has a "free" edge, the shear flow along that edge must be zero; hence the computation may be started from that edge.

c. If the beam has no free edges (as in a closed shell), equilibrium conditions may have to be applied to determine the shear flow at some arbitrary starting point. (This will be explained later.)

d. No new assumptions are introduced in deriving Eq. 17·2, i.e., the shear flow is indirectly based on the assumptions made in computing the axial flange loads due to the bending moment.

17·2. Calculation of Shear Flow. Equation 17·2 may be called the **bending shear flow formula** to distinguish it from the *torsional* shear flow formula (Eq. 12·2). It is most useful in analyzing shell structures in which the cross section is assumed to be represented by axial flange units connected by shear webs. In such analyses *it is arbitrarily assumed that the flange units resist only axial forces, and that the shear webs resist only shear flow.* (This assumption is very nearly correct in thin shell structures such as used in airplanes.)

Equation 10·10 (Sec. 10·7) is, of course, used to calculate the shear *stress,* after the shear flow is determined. This operation is nothing more than that of dividing the shear flow by the web thickness. It will be evident that the determination of the shear flow in a horizontal direction also determines it in the vertical direction, since the flow around the edges of a rectangular web in pure shear is constant (see Sec. 10·16).

Figure 17·4 illustrates a simple example. The computations follow.

$$q_1 = \frac{\Delta P_1}{L} = \frac{3600 - 3000}{10} = 60 \text{ lb/in.}$$

$$q_2 = q_1 + \frac{\Delta P_2}{L} = 60 + \frac{200}{10} = 80$$

$$q_3 = q_2 + \frac{\Delta P_3}{L} = 80 - \frac{200}{10} = 60$$

$$q_4 = q_3 + \frac{\Delta P_4}{L} = 60 - \frac{600}{10} = 0 \quad \text{(check)}$$

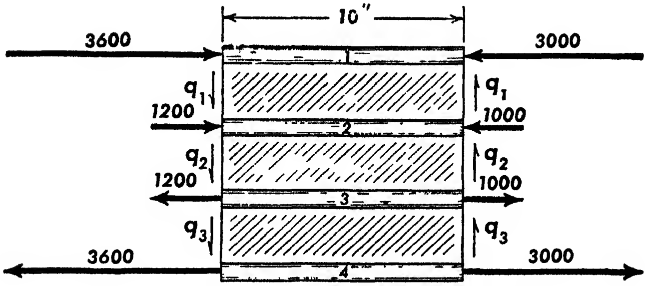


FIG. 17·4. Calculation of shear flow in webs.

Note that the computation was started from the free edge and that the sign of ΔP changed in going from flange 2 to flange 3. The computation for web 4 (which, of course, does not actually exist) is a useful check, as the shear flow must be zero on the free edge. If all the webs were 0.050 in. thick, the shear stresses would be $60/0.05 = 1200$ psi, and $80/0.05 = 1600$ psi, respectively.

Equation 17·2 does not contain any term directly representing the total shear load over the cross section of the beam. Likewise, in the previous example no value was given for such a load, yet the total shear flow obtained must obviously be equal to the applied shear load. The explanation for this is that Eq. 17·2 is based on the *change* of axial loads over a unit length. This corresponds to a change of bending moment, which in turn is caused by a shear load. Hence the value of the total shear load does actually enter into the picture.

(*Note.* An abrupt change in bending moment caused by the application of a *couple* acting in the flange direction is a different case. If the couple is applied in accordance with the distribution of axial loads given by the beam theory, no shear loads will be introduced in the webs.

If not so applied, the webs will act to redistribute the couple, in accordance with the principle of Saint-Venant. This is a local problem which is beyond the scope of this volume.)

In Fig. 17·1, for instance, the difference in bending moment between stations A and B can be shown to be equal to the shear load multiplied by the unit length L, as indicated by the shear and moment diagrams. This reveals one approximation that is made in using Eq. 17·2, which is that the *shear load is assumed to be constant over the unit length used.* This is equivalent to using a straight line between the two corresponding points on the bending moment curve.

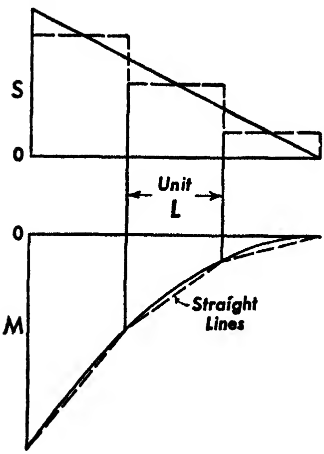


FIG. 17·5. Approximation involved in using unit lengths.

For constant shear loading this is, of course, correct, but for variable shear loading an error will be introduced, depending on the magnitude of the unit length used. The nature of this approximation is revealed by the dotted lines in Fig. 17·5, which illustrates uniformly varying shear (uniformly distributed load). This also indicates that under such conditions the shear flow thus calculated will be correct for the *midpoint* of the unit length. For rapidly varying shear loads this simple refinement may be used.

In the analysis of a shell type of structure the computations for shear flow by the *unit method* are easily included in the tabulation, as it is necessary only to multiply each flange unit area by its corresponding stress to obtain the axial force. The difference in this force, from one station to the next, is then divided by the length between stations to obtain the shear flow. As previously noted, however, it is not usually possible to find a starting point of zero shear flow unless the beam has a free edge or is symmetrically loaded. Hence the general computation methods will be deferred until this point can be further clarified.

17·3. The Solid Beam. As in the bending theory it is desirable to have a formula which can be used for the solid type of beam. The cross-sectional properties of such beams can often be determined by mathematical methods, but the flange method is not then used to determine the axial stresses and therefore cannot readily be employed to determine shear flow.

The classical equation for shear stress in a beam is actually a modification of the more general *unit method.* It can be derived from it by the following steps.

First, the length of the unit is chosen as unity (1 in., for instance), as shown in Fig. 17·6. The *change* in bending moment ΔM over this unit length is obtained by multiplying it (the unit length) by the shear, giving

$$\Delta M = S$$

Now if the cross section of the beam is assumed to be uniform over the unit length, the change in axial stress in the flanges may be computed from the flexure formula.

$$\Delta f = \frac{\Delta M y}{I} = \frac{Sy}{I}$$

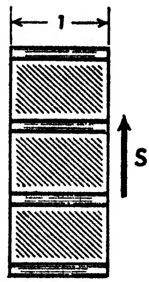


FIG. 17·6. Unit used in deriving shear flow equation. (Not shown in equilibrium.)

The change in axial force, for any flange, is then found by multiplying by its area, for instance,

$$\Delta P_1 = \frac{SA_1y_1}{I}$$

This equation gives the change in axial force over the unit length (1 in.); hence it gives the *shear flow* contributed by the flange in question. Thus,

$$\Delta q_1 = \frac{S}{I} A_1y_1$$

$$\Delta q_2 = \frac{S}{I} A_2y_2, \text{ etc.}$$

Since the shear flow between any two flanges is the *sum* of the shear flows from the free edge up to that point, it may be expressed as

$$q = \frac{S}{I} \Sigma Ay$$

The term ΣAy represents the first moment of area, or *static moment* (see Sec. 13·8). The shear flow equation may therefore be written:

Shear flow equation $$\boxed{q = \frac{SQ}{I}} \qquad [17\cdot3]$$

where q = the shear flow (lb/in.) in a horizontal or vertical plane.
S = the shear at the station in question.
I = the moment of inertia about the neutral axis (second moment of area).
Q = the static moment (first moment of area) of the cross-sectional area between a free edge and the point in question, the moments being taken about the neutral axis.

Figure 17·7 shows how this formula is applied to a solid beam. If it is desired to know the shear stress at section X–X the static moment of the shaded area must be computed. In making this computation the moment arm of each element of area must be taken to the neutral axis, as indicated by y_1 in Fig. 17·7.

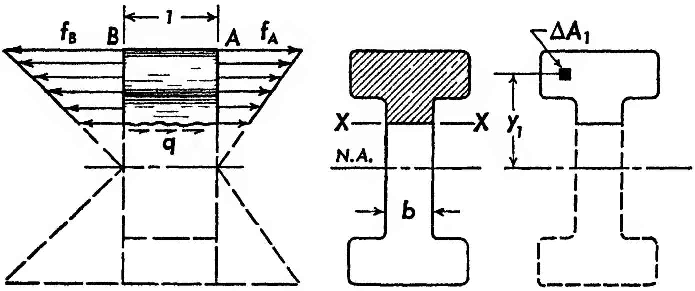


Fig. 17·7. Shear flow in solid beam.

The shear stress is found by dividing the shear flow by the minimum width or thickness through which it must be transmitted. This may be expressed by the equation

Shear stress equation

$$f_s = \frac{SQ}{bI} \qquad [17\cdot 4]$$

where b is the minimum width or thickness at the point in question.

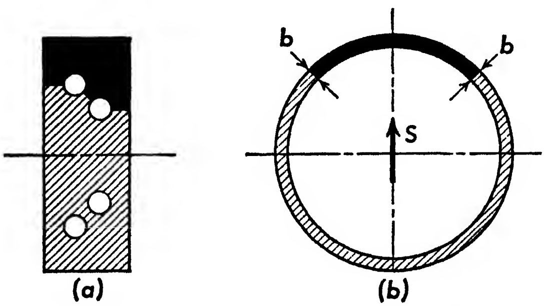


Fig 17·8. Choice of cross section for shear flow computation.

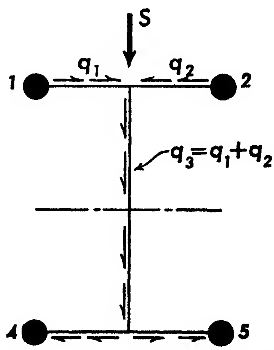


Fig. 17·9. Beam having more than two free edges.

Figure 17·7 shows how b is measured ordinarily and Fig. 17·8 shows two cross sections in which the minimum shear area does not occur straight across the beam. If the true significance of the shear formula is understood there should be no difficulty in deciding how to measure b.

Certain types of beams may have more than one free edge, as shown in Fig. 17·9, in which four flanges are connected by shear webs. Note that the shear flow for flanges 1 and 2 may be computed separately and that these two values are combined to obtain the shear flow in web 3.

17·4. Variation of Shear Stress. For a simple rectangular beam such as shown in Fig. 17·10, the shear stress variation over the depth can be obtained by direct summation or mathematical integration of the bending stress curve corresponding to the change in bending moment *over a unit length*. (Since the width b is constant, it is unnecessary to multiply axial stresses by the width and then divide later to get shear stress.)

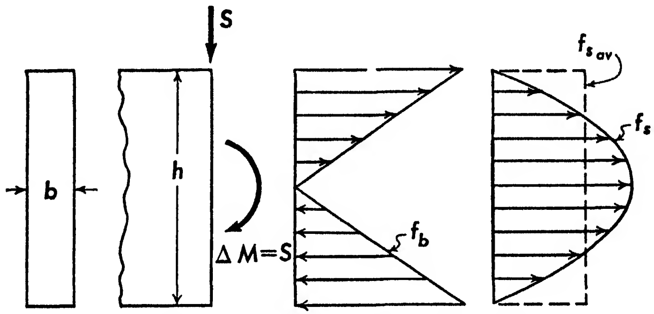


FIG. 17·10. Shear stress in a rectangular beam under transverse loading.

The resulting curve will have the shape of a parabola, as indicated, and the maximum value will occur at the neutral axis. This value will be numerically equal to the area under one-half of the bending stress curve (multiplied by proper scales, of course).

From the flexure formula

$$f_{b\,\max} = \frac{My}{I} = \frac{S(h/2)}{bh^3/12}$$

$$= \frac{6S}{bh^2}$$

The area under one of the triangles in Fig. 17·10 will then be equal to

$$f_{s\,\max} = \frac{1}{2} f_{b\,\max} \frac{h}{2}$$

$$= \frac{1}{2} \frac{6S}{bh^2} \frac{h}{2}$$

$$f_{s\,\max} = \frac{3}{2} \frac{S}{bh} \qquad [17\cdot5]$$

But

$$f_{s\,av} = \frac{S}{A} = \frac{S}{bh}$$

Hence

$$f_{s\,max} = \frac{3}{2} f_{s\,av} \qquad [17 \cdot 5a]$$

This example shows that the maximum shear stress (or flow) for the rectangular beam is half again as great as the shear stress for the idealized two-flange beam (Sec. 10·7).

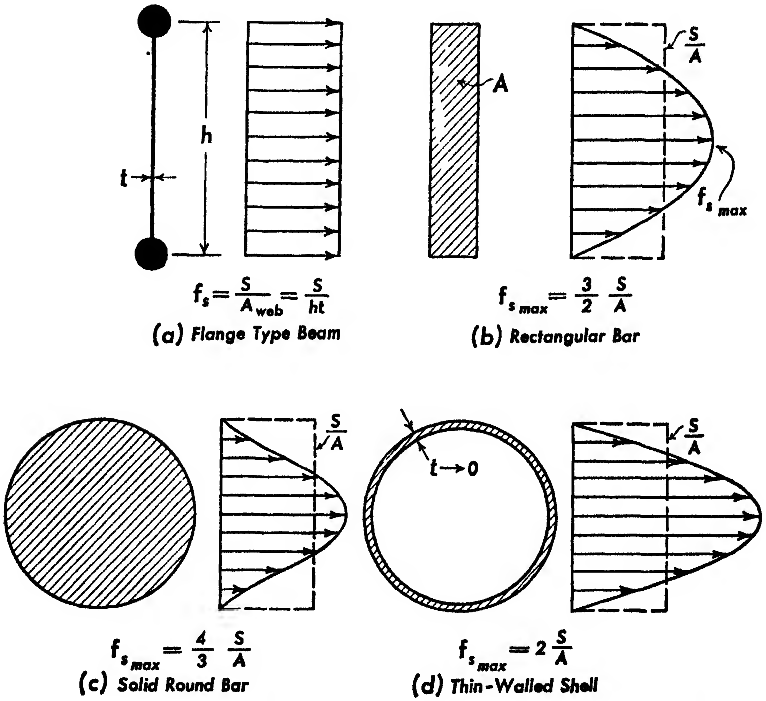


FIG. 17·11. Shear stress in beams of various cross sections (for transverse loading in vertical direction).

By similar methods it can be shown that the maximum shear stress of a circular cross section is *four-thirds* of the average shear stress, whereas that for a thin-walled tube is *twice* the average value. The comparisons are shown by Fig. 17·11.

The relationships shown in Fig. 17·11 are useful in estimating the maximum shear stress. Almost any normal cross section will correspond roughly to one of the shapes shown. It is important to note, however, that if the flange type of beam is used as a basis, the area of the flanges must not be included in the calculation of average shear stress. (It was not included in deriving the shear flow formula for the flange type of beam, in Chapt. 10.)

Since the shear flow for any beam always reaches its maximum value at the neutral axis, it is important to check carefully any cross section which shows a marked decrease in width in that region (Fig. 17·12*a*, for example). This same reasoning applies to *built-up* beams, which are joined at or near the neutral axis by means of bolts, nails, or other

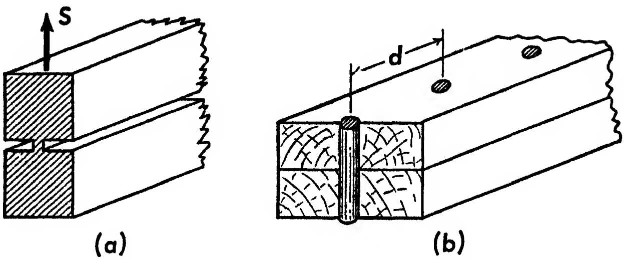


FIG. 17·12. Beams likely to be critical in shear.

means (Fig. 17·12*b*). The shear force on each connector may be computed by determining the shear *flow* and multiplying it by the spanwise distance between connectors (*d* in Fig. 17·12*b*).

The maximum *shear flow* at the neutral axis of a rectangular beam is (from Eq. 17·5*a*)

$$q = \frac{3}{2} q_{av} = \frac{3}{2}\frac{S}{h}$$

The simplicity of such calculations is illustrated by the following example. Assume that two two-by-fours are joined (as in Fig. 17·12*b*) with nails spaced 4 in. apart. If a vertical shear load of 100 lb is applied, find the horizontal shear force per nail.

$$P_s = dq_{max} = 4\frac{3}{2}\frac{S}{h} = 4\frac{3}{2}\frac{1000}{4} = 1500 \text{ lb}$$

17·5. Non-Connected Beams. It is of interest to know how the shear stresses are computed when two or more beams are joined in such a way that they deflect together but have no shear connection. (This would be true of the two beams in Fig. 17·12*b*, if the nails were omitted.) A common example of such a structure is the ordinary *leaf spring* in which the distribution of load depends on the relative deflection of the beams. Once this distribution has been determined, each beam is treated separately to find the shear stresses. Thus if a shear load were to be transmitted by three identical beams free to slide on each other (Fig. 17·13), each beam would be treated as a separate beam carrying one-third of the load. (Note the evidence of sliding action shown by the deflected position of the beams.) The bending stresses and deflections would also be computed by treating the beams individually.

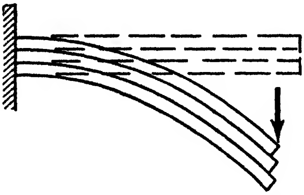


Fig. 17·13. Non-connected beams acting together. (Deflection exaggerated.)

17·6. Variable Depth (Tapered) Beams. The discussion up to this point has been based on the *square* frame and on beams of *constant depth.* Figure 17·14*a* shows an elementary frame in which the depth varies. Such beams are often called tapered beams. Figure 17·14*b* shows a model constructed from a wooden frame and a sheet-rubber web.

The tapered beam differs from the rectangular beam in two major respects.

a. The assumption of infinitely rigid frame members does not result in a condition of pure shear over the entire web.

b. Axial forces in the flange members have components normal to the centerline of the beam.

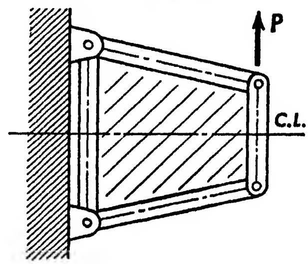


Fig. 17·14*a*. Tapered beam (frame).

The first point is not very important for beams that are not too sharply tapered. It is therefore customary to consider the web as having a constant shear flow over any cross section normal to the centerline.

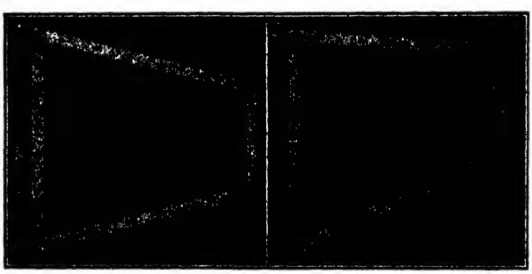


Fig. 17·14*b*. Tapered frame before and after loading. Web is made from sheet rubber. Note wrinkles in end having greater shear flow.

The second point must be evaluated, as its effect on the shear flow may be considerable. Assume that the flanges of the beam in Fig. 17·15 are carrying equal axial loads P at the cut section. (Forces shown are *from* the flanges.) The resultant of these forces acts at point O on the centerline of the beam and has a value equal to P_V, in the vertical direction. Point O is the point of intersection of the centerlines of the two flanges.

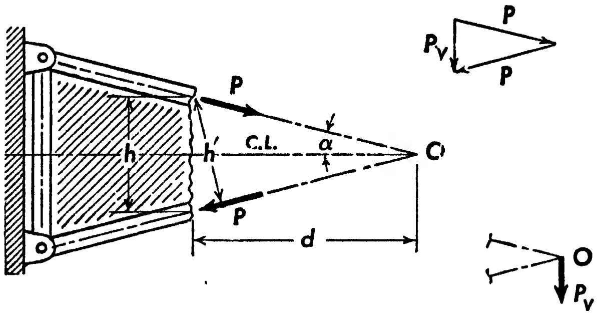


FIG. 17·15. Resistance of tapered depth beam.

The value of P for a given bending moment is equal to

$$P = \frac{M}{h'} \qquad [17\cdot6]$$

where M = the bending moment.
h' = the moment arm of one flange load, as indicated in Fig. 17·15.

By inspection it can be seen that

$$h' = h \cos \alpha$$

Hence

$$P = \frac{M}{h \cos \alpha} \qquad [17\cdot7]$$

If α is small (a few degrees or less), $\cos \alpha$ may be taken as 1.0, and the equation becomes the same as that for the rectangular beam (Eq. 13·3).

Figure 17·15 shows that the value of P_V is given by

$$P_V = 2P \sin \alpha$$

$$= 2\frac{M \sin \alpha}{h \cos \alpha}$$

Therefore

$$P_V = 2\frac{M}{h} \tan \alpha \qquad [17\cdot8]$$

Equation 17·8 indicates that if the bending moment is fairly large, the flange forces required to resist it may cause an induced shear load P_V of considerable magnitude. The *net* shear load to be carried by the web will be the algebraic sum of the external shear load and the induced shear load.

$$S_{\text{web}} = S - P_V \qquad [17\cdot 9]$$

or

$$S_{\text{web}} = S - 2\frac{M}{h}\tan\alpha \qquad [17\cdot 10]$$

(The negative sign is used because the force represented by the second term is usually opposite to the applied shear force.)

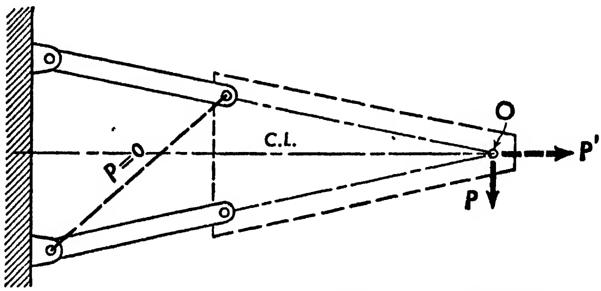


FIG. 17·16. Condition for zero load in diagonal.

Although it may not be obvious, any combination of shear and bending moment which has a resultant at point O will cause no shear force in the web. Thus a stress analysis of the truss shown in Fig. 17·16 would reveal zero load in the diagonal member (dotted line). A member of some sort would, of course, be needed, for stability, but from a purely theoretical point of view the open frame would carry the load without a diagonal or shear web.*

It will also be noted that an *axial* load applied at point O along the centerline of the truss will require no shear diagonal (or web). Hence the unbraced tapered truss will theoretically carry a load applied in any direction at point O, provided that it is in the plane of the truss.

When a single transverse load is involved, it is convenient to determine the shear in the web by the simple formula

$$S_{\text{web}} = P\frac{a}{b} \qquad [17\cdot 11]$$

where a and b are measured as shown in Fig. 17·17.

* This situation is often approached in airplane engine mount structures. Figure 17·16 might be considered as representing such a mount in either side or plan view, with the center of gravity of the engine located at point O.

The reference line used does not always represent the centerline of the beam, as indicated in Fig. 17·18. If the angles are not large (not more than several degrees), it is satisfactory to used the formula

$$S_{\text{web}} = S - \frac{M}{h}(\tan \alpha_1 + \tan \alpha_2) \qquad [17 \cdot 12]$$

where α_1 and α_2 are both considered as positive.

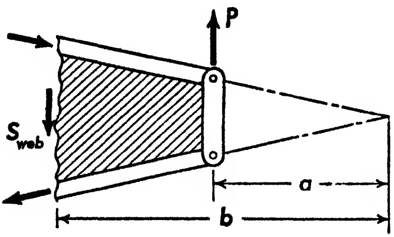


Fig. 17·17. Method of computing web shear for single force.

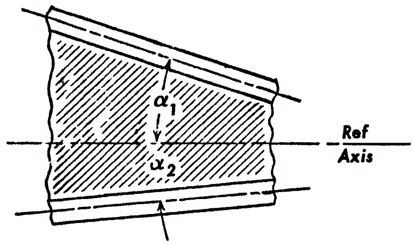


Fig. 17·18. Non-symmetrical case.

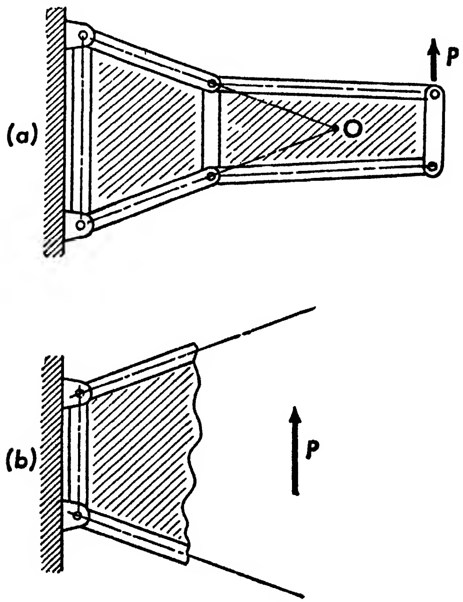


Fig. 17·19. Special cases of tapered beams.

As previously noted the use of the negative sign in Eqs. 17·10 and 17·12 is based on the idea that the bending moment caused by a certain shear load will always produce an induced shear that is opposite in sign to the external shear load. This in turn is based on the assumption that the depth of the beam is increasing in the direction toward which the shear load is being transmitted. If the resultant shear should act outside of the point of intersection, as shown in Fig. 17·19*a*, the induced shear would exceed the external shear and the shear in the web would be in a direction *opposite* to that of the applied load.

If conditions are as in Fig. 17·19*b*, the induced shear will *add* to the external shear (negative signs in Eqs. 17·10 and 17·12 would be changed to positive).

17·7. Tapered Shell. Equation 17·8 may also be used for the tapered circular shell (truncated cone) by replacing the quantity $h/2$ by the radius r, giving

$$P_V = \frac{M}{r} \tan \alpha \qquad [17\cdot 13]$$

where r = radius of median line of cross section.
α = angle of taper (see Fig. 17·20).

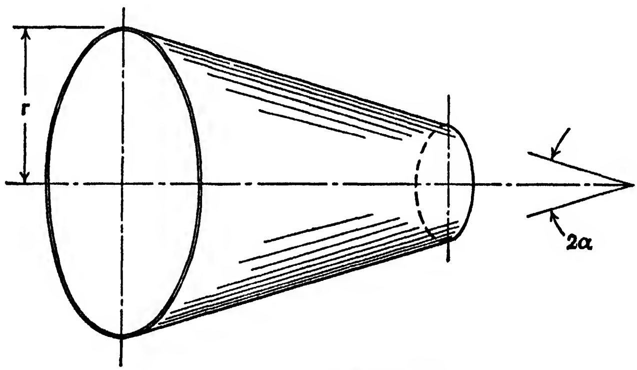


FIG. 17·20. Tapered shell.

The net transverse shear is then given by Eq. 17·9. The maximum shear stress may be computed by the methods of Sec. 17·4.

Equations 17·8 and 17·13 may be written in a more general form which may be used for linearly tapered beams of any cross section.

$$P_V = \frac{M}{b} \qquad [17\cdot 14]$$

where P_V = induced transverse shear force.
M = bending moment at the section.
b = distance to apex (see Fig. 17·17).

PROBLEMS

17·1. Using Fig. 17·4 as a basis, assume that the distance between centroids of the flanges is 4 in., that the bending moment at the left section is 320,000 in.-lb and at the right section 290,000 in.-lb. (Sections are 10 in. apart.) Assume that flanges 2 and 3 are each 1 sq in. in cross-sectional area and assign to flanges 1 and 4 some value between 1.1 and 1.9 sq in.

(*a*) What is the total transverse shear load?
(*b*) What is the shear flow in each web unit?
(*c*) What is the transverse shear load in each web unit?

(*Note.* Sum of shear loads should check total shear. Neglect web area in connection with flange loads.)

17·2. Draw a cross section similar to that of Fig. 17·7, using rectangular elements. Assign dimensions and compute *area*, *moment of inertia*, and *static moment* (latter to be computed for area above neutral axis). For a shear load of 1000 lb compute the maximum shear stress. Compare with average shear stress over cross section. (Actual should be somewhat greater than average.)

17·3. Using Problem 17·2 as a basis compute the shear stress for a section at the juncture of flange and web, also at two other intermediate points. Plot a curve showing shear stress variation over depth of beam. Show average shear stress by means of a dotted line.

17·4. A solid round bar 2 in. in diameter is carrying a shear load of 30,000 lb. What is the maximum shear stress? (Compute from Eq. 17·4 and check by Fig. 17·11.) See Appendix 1·1 for static moment.

17·5. Assume that the three beams shown in Fig. 17·13 are made from planks 12 in. wide and 2 in. thick and that the total applied load equals 3000 lb.

(*a*) Using Fig. 17·11 find the maximum shear stress in the wood.
(*b*) If the planks are glued together before the load is applied, what is the maximum shear stress?
(*c*) What is the shear stress in the planes of the glued joints?

17·6. A 3-in. diameter aluminum-alloy tube must carry a transverse shear force of 15,000 lb before failure. Consider the tube as a thin-walled tube and find the required wall thickness. (*Note.* Use Figs. 12·14, 17·11, and Appendix 2. A trial-and-error method must be used. Assume a value of allowable shear stress to start with and check against resulting D/t.)

17·7. Assign arbitrary centerline dimensions to the frame shown in Fig. 17·14*a* and assume that $P = 5000$ lb. Find the shear flow in the web at each end of the frame.

17·8. In Problem 17·7 locate the point at which the force must be applied to reduce the shear flow at the left end of the frame to half its original value.

17·9. A tapered thin-walled shell is 50 in. long, 40 in. in diameter at one end and 20 in. in diameter at the other. A transverse shear load of 1000 lb is applied at the small end and resisted at the other end. Find the maximum shear flow at each end and in the mid-section.

17·10. Compute the static moment (about neutral axis) of a thin-walled circular shell by dividing the cross section into small units of equal area. Use this value to show that the maximum shear stress is twice the average. (*Note.* The problem may be worked out by calculus, but it is desired to show the relative accuracy of using a number of finite units. Radius and thickness may be retained in symbol form. Only one quarter of the circle need be used. It should be divided into at least 10 parts.)

17·11. By taking moments about the (hypothetical) apex show that Eq. 17·14 should apply to *any* assumed distribution of axial stresses due to bending, provided that the axial forces on the cut section have lines of action which pass through the apex.

CHAPTER 18

COMBINED SHEAR AND TORSION

18·1. Symmetrical Members. A cross section loaded as indicated in Fig. 18·1 will be subjected to combined transverse shear and torsion (in addition to any bending moment which might be acting at that point). The transverse shear stresses due to the force P are determined by the methods outlined in Chapt. 17. The torsional moment is equal to Px, and the resulting torsional shear stresses are found by the methods outlined in Chapt. 12. The net shear stress at any point around the cross section will then be found by the algebraic addition of the shear stresses caused by the two loading conditions. The *maximum shear stress* will occur on the side nearest the applied force.

By using the thin-walled shell theory, the maximum shear stress will be found for the conditions shown in Fig. 18·1, under the following assumptions.

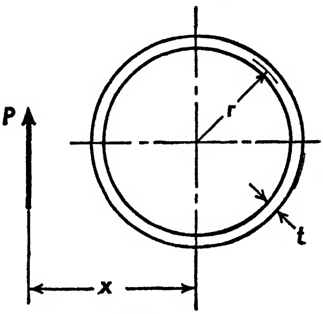


FIG. 18·1. Section through tube under combined shear and torsion.

$$P = 1000 \text{ lb}$$

$$x = 40 \text{ in.}$$

$$r = 10 \text{ in.}$$

$$t = 0.050 \text{ in.}$$

$$A = \pi r^2 = 314 \text{ sq in.}$$

The shear stress due to torsion will be found from Eq. 12·3.

$$f_{st} = \frac{M}{2At} = \frac{1000 \times 40}{2 \times 314 \times 0.05} = 1270 \text{ psi}$$

The maximum transverse shear stress is determined from Eq. 17·4, or from Fig. 17·11:

$$f_{ss} = \frac{2S}{A_s} = \frac{2 \times 1000}{2\pi r t}$$

$$= \frac{1000}{\pi \times 10 \times 0.05} = 637 \text{ psi}$$

$$f_s = f_{st} + f_{ss} = 1270 + 637 = 1907 \text{ psi}$$

On the opposite side the stress would be

$$f_s = 1270 - 637 = 633 \text{ psi}$$

On the top and bottom of the shell the stress due to direct shear would be zero, hence the total stress would be that due to torsion alone.

The stress at any intermediate point would be found by using the general formula for transverse shear stress (Eq. 17·4).

A *solid* member is treated in the same manner, except that the appropriate formulas must be used. Algebraic addition of the shear stresses is

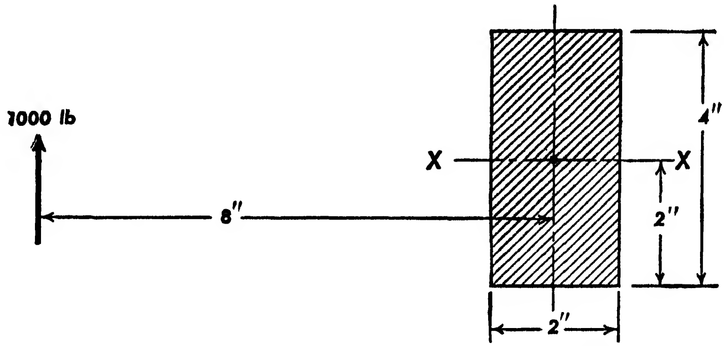


FIG. 18·2. Combined shear and torsion of solid bar.

not always correct, but the error involved will usually be on the safe side; i.e., computed stresses will tend to be too high.

For example, Fig. 18·2 shows a load of 1000 lb, acting 8 in. off the centerline of a two-by-four. The transverse shear stress will be greatest across the centerline X–X, and will be found from Eq. 17·5.

$$f_{ss} = \frac{3}{2}\frac{S}{bh} = \frac{3}{2}\frac{1000}{2 \times 4} = 188 \text{ psi}$$

The torsional shear stress will be found from Table 12·1 (page 192).

$$f_{st} = \frac{kM}{ab^2} = \frac{4.07 \times 1000 \times 8}{4 \times 2^2} = 2030 \text{ psi}$$

$$f_{s\,\max} = f_{ss} + f_{st} = 188 + 2030 = 2218 \text{ psi}$$

This direct addition of the maximum shear stresses is correct, as they act at the same point and in the same direction.

Note that these processes take care of the shear stresses only and would therefore give the complete answer only when the cross section has no axial bending stresses acting on it. If appreciable bending stresses exist, the maximum shear stresses may sometimes be greater than those found by considering only direct shear and torsion. Methods of combining axial and shear stresses are covered in Chapt. 20.

18·2. The Unsymmetrical Shell. In dealing with the symmetrical cross section, it was obvious that the loading condition should be converted into a direct shear load acting in the plane of symmetry and a torsional moment about the axis of the member. The true reason for the use of the plane of symmetry may not be immediately apparent, however. In a symmetrical cross section this plane represents an axis along which a transverse shear force may be applied *without inducing a torsional moment.* (Compare use of principal axes in unsymmetrical bending.)

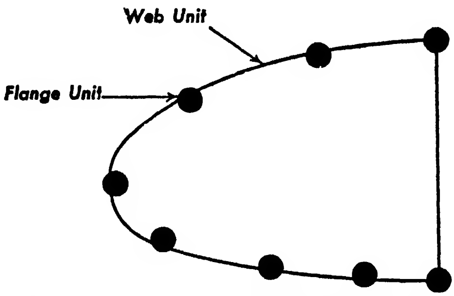


FIG. 18·3. Unsymmetrical closed section (idealized).

For an unsymmetrical cross section the location of such an axis cannot be determined by inspection alone. Methods of determining it for the *single* curved shear web are given in Chapt. 11, in which it was assumed that the beam was incapable of resisting torsion; hence the transverse shear force had to be applied at the so-called shear center.

A closed unsymmetrical section is capable of resisting both transverse shear and torsion. Such a section is shown in Fig. 18·3, which represents a D-tube construction often used in airplanes. The section is assumed to be made up of flange elements which carry only axial stresses, and web elements which resist only shear flow. (This condition is approached in thin sheet-metal structures stiffened by individual stringers. The assumptions may also be used for other types of structures in which the material is more evenly distributed.) It might appear that the solution could be obtained directly from Eq. 17·3 ($q = SQ/I$) by evaluating the static moment Q. Reference to Chapt. 17 will show, however, that Q is computed by starting with a point of zero shear flow (such as the outer edge of a beam). Since the location of this point cannot be predicted for an unsymmetrical cross section, it is not possible to use Eq. 17·3 directly.

This problem may be solved in a number of ways, but all are based on the fundamental shear flow Eq. 17·2 (repeated below):

$$q_n = q_{n-1} + \frac{\Delta P_n}{L}$$

Assuming that the analysis of the bending stresses in the flange units has been completed for two adjacent sections, the values for the flange increments ΔP are readily obtained.

It is now apparent that the shear flow in all the webs could be evaluated from Eq. 17·2, if the shear flow in any one web were known. The problem then resolves itself into one of finding the shear flow in any one web unit for the loading conditions imposed.

18·3. Cutting the Web. One method of attack is to make an arbitrary assumption that the shear flow in a particular web is *zero*, thereby hypothetically converting the closed section into an open section, i.e., cutting the web in question. The shear flow in all the other webs may then be evaluated from Eq. 17·2. The moment of these shear flows about a suitable reference point (such as the c.g.) can then be obtained by applying the principles of Sec. 11·2. If the shear flows in the webs have been taken as the internal *resisting* forces at the cut section, their total moment may be algebraically added to that of the *external* forces at the same section, acting about the same reference point. The net

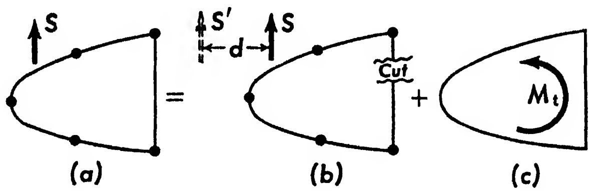


FIG. 18·4. Cutting the shell.

moment so obtained represents the torsion which must be resisted by the *closed* section, giving a constant shear flow in all the webs equal to $M/2A$ (see Eq. 12·2). This value is also the unknown shear flow in the cut web. The values for the other webs are obtained by algebraic addition of this torsional shear flow to the values previously obtained for the cut section.

The process involved may be visualized by considering the shell to be composed of two separate structures, as indicated in Fig. 18·4. The cut shell (*b*) can resist direct shear at the shear center only (indicated by dotted arrow, S') whereas the closed shell resists torsion only. The value of M would be equal to Sd. It is unnecessary, however, to determine the location of the shear center, as the moment of the shear flow may be used directly, as previously described.

18·4. Direct Method. Although the method of cutting the shell is useful in clarifying the physical action which takes place, it is not essential in solving the problem. Instead of assuming the shear flow in a given web unit to be zero it can be assigned a symbol, the value of which is unknown. The shear flow in each web will then consist of this unknown constant plus another value determined by the axial load increments in the flanges. The moment of the shear flows in all flanges may then be equated to the external torsional moment in order to find the unknown shear flow.

The general procedure will be shown for Fig. 18·5. Instead of equating the *resisting* shear flow to the external torsional moment it will be found more convenient (in stress analysis work) to determine the *equivalent* shear flow. This will give the loads acting *on* the cross section. For this reason Fig. 18·5 has been shown in two parts: (*a*) shows the external loading condition at the cut section, and (*b*) shows the assumed *equivalent* shear flow. A positive moment convention is established as shown in Fig. 18·5*a*.

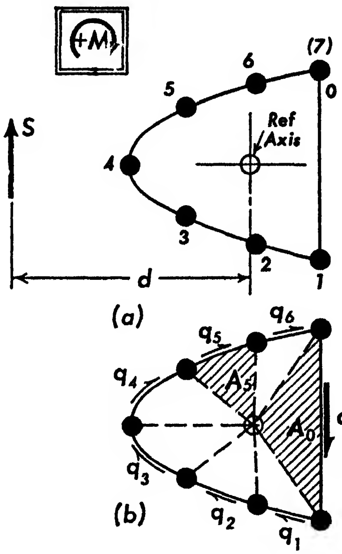


Fig. 18·5. Computation of shear flow.

Flange and web units are assigned suitable numbers. (It is convenient to assign the same number to the flange and to the web which follows it in the positive moment direction.)

If q_0 is now taken as the unknown shear flow, the values for other webs will be as follows.

$$q_0 = q_0$$

$$q_1 = q_0 + \frac{\Delta P_1}{L}$$

$$q_2 = q_1 + \frac{\Delta P_2}{L} = q_0 + \frac{\Delta P_1}{L} + \frac{\Delta P_2}{L}$$

$$q_3 = q_2 + \frac{\Delta P_3}{L} = q_0 + \frac{\Delta P_1}{L} + \frac{\Delta P_2}{L} + \frac{\Delta P_3}{L}, \text{ etc.}$$

It can be seen that the shear flow in any web is composed of two parts, one of which is q_0, the unknown shear flow in web No. 0. The other part represents the summation of differential flange forces (per unit length) up to the flange in question. For convenience this summation will be designated by the symbol H and will be called the **shear flow increment.*** It can be expressed by the equation

$$H_n = \frac{1}{L}(\Delta P_1 + \Delta P_2 + \cdots + \Delta P_n) \qquad [18 \cdot 1]$$

where H_n represents the shear flow increment for web n.
L is the length between cross sections.
ΔP is the change in flange force between sections.

The shear flow at any point can now be expressed by

$$q_n = q_0 + H_n \qquad [18 \cdot 2]$$

The *shear flow increment* H is a quantity which replaces and is analogous to the *static moment* Q. It is readily computed by progressive summation of the differential flange forces. Q may be derived from the equation for H by letting $q_0 = 0$ (at free edge) and assuming that the axial stress due to bending equals the distance from the bending axis.

The torsional moment of the shear flow in any web (about the reference point) is given by the following equation (from Eq. 12·1).

$$\begin{aligned} M_n &= 2q_nA_n \\ &= 2(q_0 + H_n)A_n \end{aligned} \qquad [18 \cdot 3]$$

where A_n is the segment of enclosed area corresponding to web n (shaded areas in Fig. 18·5*b*).

The values for several webs are written below.

$$\begin{aligned} \Delta M_0 &= 2q_0A_0 \\ \Delta M_1 &= 2q_1A_1 = 2q_0A_1 + 2H_1A_1 \\ \Delta M_2 &= 2q_2A_2 = 2q_0A_2 + 2H_2A_2 \\ \Delta M_3 &= 2q_3A_3 = 2q_0A_3 + 2H_3A_3, \text{ etc.} \end{aligned}$$

Adding

$$\Sigma\Delta M = 2\Sigma q_nA_n = 2q_0\Sigma A_n + 2\Sigma H_nA_n$$

The term ΣA_n obviously represents the entire enclosed area of the cross section. Hence the quantity $2q_0\Sigma A_n$ represents the moment of q_0 acting as a constant shear flow around the section.

* No standard term is yet in common use for this quantity. The symbol H has no special significance; it was chosen largely because most of the other letters had been used for other purposes.

An equation can now be written for the moment of the shear flow. as follows.

$$M_s = 2(q_0A + \Sigma H_nA_n) \qquad [18\cdot4]$$

in which M_s = total moment of shear flow in web elements.
A = total enclosed area = ΣA_n.
H_n = shear flow increment for a given web (Eq. 18·1).
A_n = segment of enclosed area corresponding to web n.

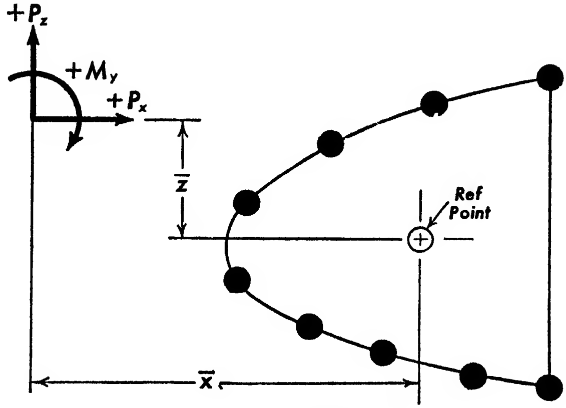


FIG. 18·6. General case.

In this equation all the terms on the right-hand side can be evaluated except the term q_0. (It is assumed that the bending stresses and flange forces have already been determined.) In order to find q_0 the moment M_s is equated to the moment of the external forces and moments acting at the section in question. (Obviously the moments must be taken about the same point.) If M_t represents the external torsional moment,

$$M_t = M_s = 2(q_0A + \Sigma H_nA_n)$$

from which

$$q_0 = \frac{1}{A}\left(\frac{M_t}{2} - \Sigma H_nA_n\right) \qquad [18\cdot5]$$

The shear flow in any given web is then found from Eq. 18·2 by adding (algebraically) the value of q_0 to the value of H_n already determined for that web. Equation 18·2 is repeated below, to emphasize its importance as the companion formula to Eq. 18·5.

$$q_n = q_0 + H_n$$

In Fig. 18·5, M_t would be equal to Sd. More generally the external forces would consist of two components and a torsional moment, as indicated in Fig. 18·6 where the value of M_t would be given by

$$M_t = P_z\bar{x} + P_x\bar{z} + M_y \qquad [18\cdot6]$$

It should be noted that the reference point selected does not have to be the c.g. of the cross section. However, this point will usually have been found in connection with the determination of bending stresses, and the calculations are simplified by using it for shear flow determination.

18·5. Sign Conventions. It is necessary to establish the proper relationship of signs between the axial load increments and shear flow in the webs. Figure 18·7 indicates how this can be done, using the customary conventions for axial forces. A downward-acting external (shear) force is assumed, which would cause tension in the top flange. The flange load increments are shown in Fig. 18·7, acting on the web unit. The direction of q required to maintain moment equilibrium of the flange unit is also shown. If this direction of the shear flow were in the direction assumed to be positive for torsional moment it would, of course, be given a positive sign.

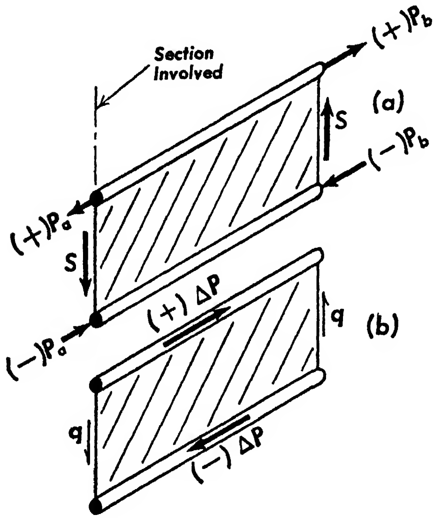


Fig. 18·7. Positive conventions.

Equation 18·1 shows that in determining q for a given web unit the value of ΔP used is that for the flange *just preceding* the web. Hence, if the positive direction of progression were downward, in Fig. 18·7 the top (tension) flange would be used. Since this flange increment has a positive value, it is satisfactory to permit the sign of the flange load to indicate the sign of the shear flow. If the positive direction of progression around the shell happened to be upward, the lower flange would be used, giving a negative value for q. This evidently agrees with the conventions, as the actual downward-acting direction of q is negative with respect to the torsion conventions.

It may be concluded without further proof that the use of the axial stress conventions (positive for tension, negative for compression) gives the correct signs for the increments of shear flow. It is important to remember, however, that the equivalent loading *on* the cross section, not the reaction *from* the cross section, is being considered.

18·6. Constant vs. Variable Cross Section. For beams of constant cross section it is not necessary to compute the bending stresses at two adjacent sections. The change in bending moment per unit length is equal to the shear; hence if a unit spanwise length is chosen, the change in axial flange stresses over this length may be found by substituting the shear for the moment in the flexure formula ($f = My/I$). The values so obtained must be multiplied by the flange areas to obtain the change in axial force. Thus, for any given flange,

$$\Delta f_1 = \frac{Sy_1}{I}$$

$$\Delta P_1 = \Delta f_1 A_1 = \frac{SA_1 y_1}{I}$$

If this quantity is evaluated for each flange unit, and the results are summed up from some arbitrary starting point, we obtain an alternative equation for the *shear flow increment*

$$H = \Sigma \Delta P = \frac{S \Sigma Ay}{I} = \frac{SQ}{I} \qquad [18\cdot 7]$$

This equation is identical with Eq. 17·3 for the shear flow in a symmetrical section. It will not give the complete answer for the unsymmetrical section, however, as it does not eliminate the problem of determining the point of zero shear flow or, conversely, the shear flow at some specific point. This requires the computation of the moment of the shear flow around the cross section, as previously described. It may be found convenient, however, to use the *static moment* Q and the transverse shear S in place of the unit method, which involves the computation of axial forces in all the flanges.

It should be clearly understood that the shear flow formula (SQ/I) is correct only when the beam has a uniform cross section. It may be used without appreciable error for beams having a gradual variation in cross section. If the beam varies only in depth, a correction may be made to account for the *induced* shear due to axial forces (see Sec. 17·6). *The use of the unit method makes such a correction unnecessary,* as the effects of variable depth are taken care of in the computation of the bending stresses. (See Ref. 27 for further discussion of the unit method as applied to aircraft structures.)

18·7. Abrupt Changes of Cross Section. If the distribution of flange material changes considerably between two adjacent cross sections there will be correspondingly large variations in axial loads in the flanges. These variations in axial load will cause high shear flows in the webs, if the unit method is employed, but will not affect the result if the shear flow formula (SQ/I) is used. The length of the unit which is chosen (distance between adjacent cross sections) will have a large effect on the computed shear flow. This is indicated by Fig. 18·8 which shows a flange having an abrupt change of cross section. As computed by the flexure formula the axial forces at sections A and B will be quite different, i.e., ΔP will be large. If the unit length chosen is small (as indicated by L_1) the shear flow increment $\Delta P/L$ will be very high. If a larger unit length (L_2) is used the shear flow will appear to be much lower.

The true conditions are quite complex, as the axial force in the flange will not actually change as abruptly as the cross-sectional area. In fact, the high local shear flow produced at such a point will tend to

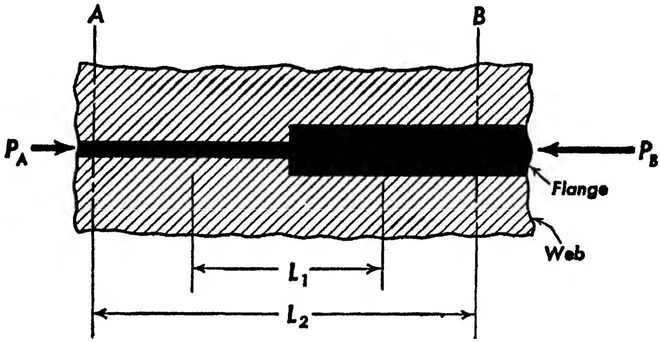


Fig. 18·8. Effect of abrupt change in flange area.

distort the cross section in a manner which will cause a more gradual variation in axial force and hence a lower shear flow. The exact solution is beyond the scope of this text, but several empirical rules may be used as a guide.

The first rule is that *abrupt changes of cross section should be avoided in design,* with the following exception. If the *entire cross section* is changed proportionately there will be no abrupt change in axial *force,* although the *stress* will vary. This condition is often present at joints. (The practice of staggering sheets of different thickness in a shell structure is therefore questionable, from this point of view.)

The second point to observe is that the unit length chosen should not be too small. No specific rules can be given on this, but in general it is unnecessary to use cross sections any closer than those usually selected for the computation of bending stresses. As the size of the structure increases, the unit length may be increased accordingly.

Finally, in extreme cases which cannot be avoided (such as cut-outs) some method of tapering the effective flange areas may be employed in the analysis. For example, a rough rule sometimes used involves the assumption that the effectiveness of any flange unit drops off linearly as it approaches a cut-out or discontinuity. This is illustrated in Fig. 18·9. For sheet-metal construction such as used in aircraft an average slope of b/a between 3 and 4 has been found to give reasonable results.[49] (The distance a is measured to the stringer involved.)

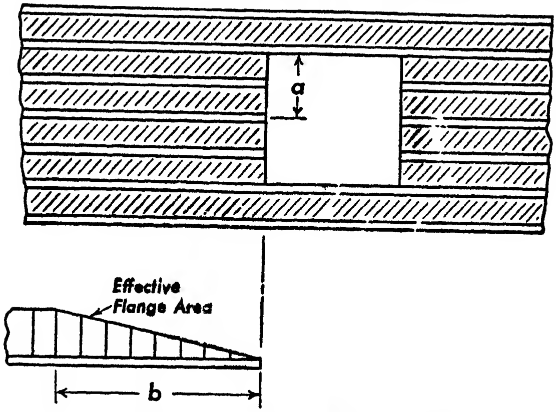


FIG. 18·9. Approximation of flange load variation at cutout.

18·8. Shear Center (Multi-Flange Beam). In Sec. 11·3 it was shown that the resultant of a *constant* shear flow in a curved web is located outside of the area enclosed by the web and a line joining the two flange units (see Fig. 11·8). The shear center of an open section having a *variable* shear flow may be found from Eq. 18·4, which gives the moment of the shear flow. In this case the summation is started from a free edge and the value of q_0 is therefore zero, giving the equation

$$M_s = 2\Sigma H_n A_n \qquad [18\cdot8]$$

or, for a uniform (non-tapered) section, the equation

$$M_s = 2\frac{S}{I}\Sigma Q_n A_n \qquad [18\cdot9]$$

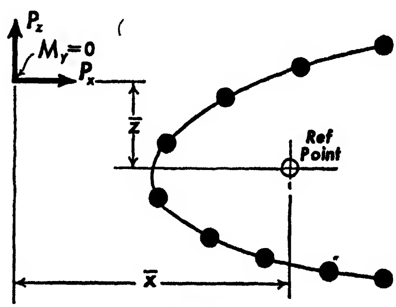


FIG. 18·10. Open section with variable shear flow.

Since the *shear center* is a point at which a shear (transverse) force may be applied without inducing torsion, its location must be such that M_s in Eq. 18·8 or 18·9 equals zero.

The general type of loading is given by Eq. 18·6 and Fig. 18·6. This figure is reproduced in Fig. 18·10, as an open section. The shear center will be located by values of $\bar{x}$ and $\bar{z}$ such that the moment of the external forces is exactly equal to the moment of the shear flows which they produce in the webs.

The location of $\bar{x}$ and $\bar{z}$ must be determined separately for two perpendicular axes (usually only one value is required). Considering only $\bar{x}$, for example, the value to be used in Eq. 18·6 would be P_z, and the moment of this force about the reference point is $P_z\bar{x}$. Substituting these quantities in Eq. 18·9 gives

$$\bar{x} = \frac{2}{I}\,\Sigma Q_n A_n \qquad [18\cdot 10]$$

This equation represents the general method of determining the shear center (or, more accurately, a line *through* the shear center) for any

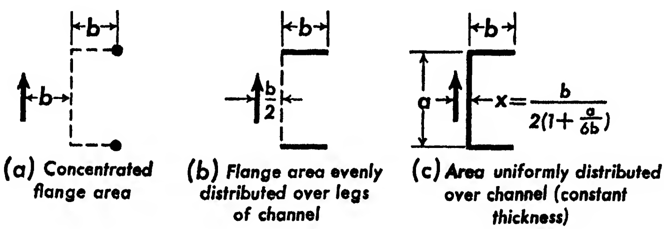


FIG. 18·11. Shear centers of channels.

open section of uniform cross section. The term $Q_n A_n$ represents one-half the moment of the shear flow in any web; A_n is determined as shown in Fig. 18·5.

Simple cross sections may be solved mathematically, by expres ing the shear flow as a function of certain dimensions and integrating to find the moment of the shear flow.* The results obtained for several sections are shown in Fig. 18·11.

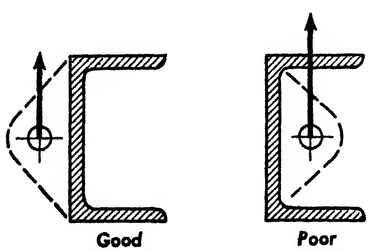


FIG. 18·12. Method of loading channel.

18·9. Significance of Shear Center. Although it is usually unnecessary to use the shear center directly in stress analysis work, the conception is very useful in design. For instance, if it is desired to apply a transverse load to a channel, the load should be introduced *outside* of the channel, at the shear center, as indicated in Fig. 18·12. Applying the load along the vertical edge or through the c.g. will induce torsional stresses which are unde-

* Sibert [33] gives a semi-graphical method for computing the moment of a variable shear flow.

sirable. At first sight this may seem irrational, but the principle is in actual use (for instance in automobile frames).

Another example, from aircraft work, will indicate the importance of the principle. Assume that a fuselage is composed of two vertical flange-type beams, without horizontal shear webs to complete the torsion box (Fig. 18·13). Torsion must therefore be resisted by a couple composed of opposite shear loads in the two beams. By visualizing the

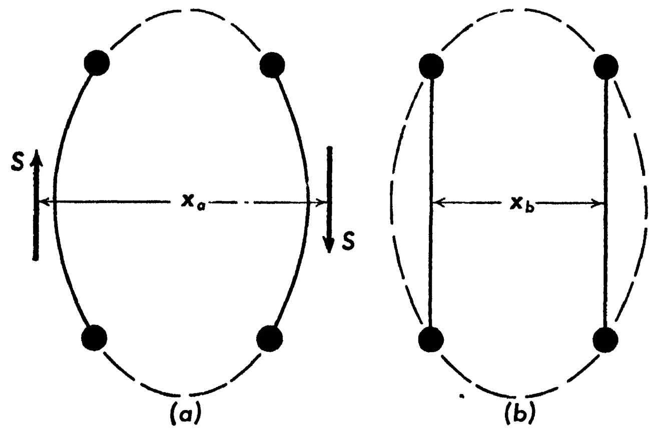


FIG. 18·13. Fuselage construction.

locations of the shear centers, it can be seen that the use of curved shear webs gives an effective moment arm much greater than that obtained by using vertical webs and therefore provides greater torsional resistance.

PROBLEMS

18·1. Assign to Fig. 18·1 any desired dimensions and loading. Find the maximum and minimum values of shear flow and shear stress (use thin-wall formulas).

18·2. In Fig. 18·2 change the width of the beam to 1 in. and find the allowable load, applied as shown, for a shear stress of 750 psi.

18·3. Draw a cross section similar to Fig. 18·6 and assign values to flange areas and dimensions. Assume that the shell is not tapered. Find the shear flow in each web unit for the following loading condition:

$$P_z = +10{,}000 \text{ lb}$$

$$P_x = -2000 \text{ lb}$$

$$M_y = +20{,}000 \text{ in.-lb}$$

18·4. Using the cross section of Problem 18·3, assume that the rear (vertical) web is removed. At what point must a vertical load be applied in order to be resisted by the open section?

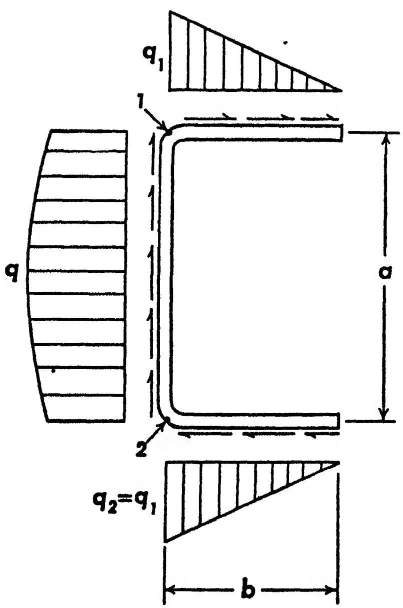
18·5. Draw a cross section of a wing such as shown in Fig. 14·8 and assign values to flange areas and dimensions. Assume that this represents a cross section of a tapered wing and draw another cross section representing a section 20 in. "inboard." This section is to be made about 10 per cent deeper, using same flange areas. Assume that a bending moment M_x = 800,000 in.-lb and an upward shear load of 8000 lb act at the c.g. of the smaller section. Find (by tabular methods):

(*a*) Axial stresses and forces in each flange unit, for each section.
(*b*) Shear flow in each web unit, between cross sections.

(Check approximately by comparing shear flow in front and rear webs with that obtained by assuming shear to be equally divided between them.)

18·6. Draw a cross section such as shown in Fig. 18·11*c*, using any desired dimensions. Find the shear center and sketch a fitting for proper application of the shear load.

18·7. Derive the formula given for the shear center of a channel of constant thickness, Fig. 18·11*c*. *Note.* The derivation may be simplified by assuming that the channel thickness is unity and by working with the median lines of the cross section. The sketch indicates the nature of the shear flow.



PROBLEM 18·7.

Procedure:

(*a*) Express q_1 in terms of *S*, *a*, and *b*, using Eq. 17·3.
(*b*) Write an expression for the moment of the shear flow about a point along the vertical web.
(*c*) Divide this moment by the shear (*S*) to find *x*.

18·8. In Sec. 13·4 it was shown how the effectiveness of a rectangular web, in carrying axial stresses in bending, could be expressed as fictitious flange areas. Prove this relationship by finding the areas of two flanges, separated by a distance *h*, such that their bending resistance equals that of a rectangular cross section of depth *h*. (*Note.* Assume that flanges have same axial stress as outer fibers of web. Express flange areas in terms of web area.)

CHAPTER 19

COMBINED BENDING AND AXIAL LOADING

19·1. Eccentric Loading. The determination of the net stress for combined bending and axial loading is a simple matter of algebraic addition. It is necessary only to be sure that the loading condition has been resolved into an equivalent axial force acting through the centroid of the cross section and a bending moment acting about an axis through the centroid.

Thus in Fig. 19·1 it is necessary to transfer the force P from its point of application to the axis through the centroid of the beam. The corre-

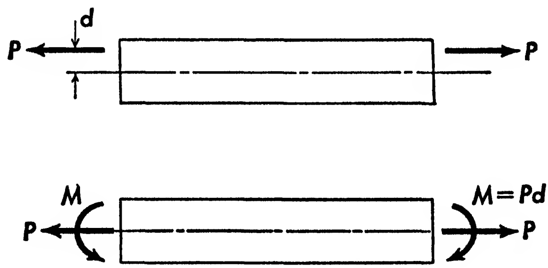


FIG. 19·1. Eccentric axial loading.

sponding bending moment Pd must then be used to determine the bending stresses in the beam.

19·2. Net Stresses. Figure 19·2 shows the stresses due to the bending and axial loads acting separately and in combination. The equation for net stress for bending about one axis is

Combined bending and axial loading

$$f = \frac{P}{A} \pm \frac{My}{I} \tag{19·1}$$

If there is bending about two principal axes (X and Y) the equation becomes

General case

$$f = \frac{P}{A} \pm \frac{M_x y}{I_x} \pm \frac{M_y x}{I_y} \tag{19·2}$$

For *unsymmetrical bending* the above equation may also be used by substituting the *effective bending moments* as derived in Sec. 14·3.

For *composite* and *curved* beams the properties of the *transformed section* should be used, and the resulting stresses must be multiplied by

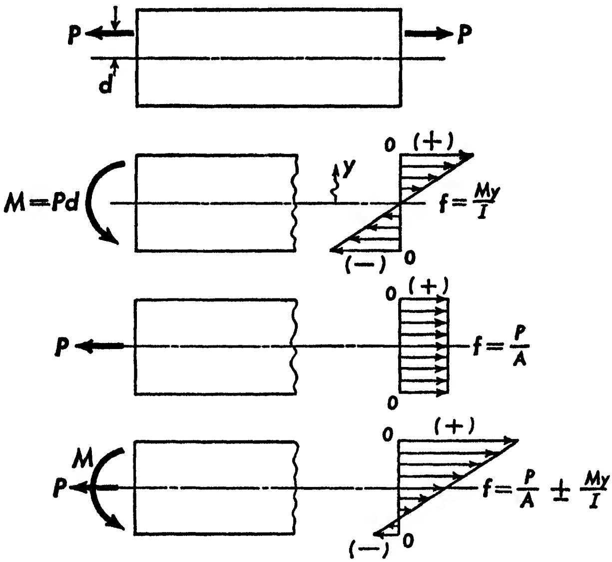


FIG. 19·2. Net stresses in eccentric loading.

the *effectiveness factor*, as described in Sec. 15·2. Note particularly that this applies to axial as well as to bending stresses.

19·3. Axis of Zero Stress. In combined bending and axial loading the axis through the centroid of the cross section, i.e., the *neutral axis* for pure bending, is no longer the axis of zero axial stress. Figure 19·2 shows that this point has moved toward the edge of the member. If the axial loading predominates there may be no axis of zero stress, as indicated in Fig. 19·3. Hence the neutral axis (or plane) should *not* be thought of generally as the axis (or plane) of zero stress, but should be defined as in Sec. 13·8, for *pure* bending.

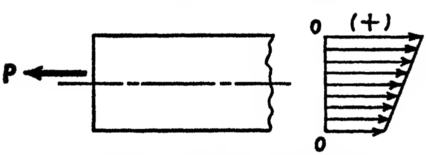


FIG. 19·3. Variable axial stress.

19·4. Reversal of Stress. Figures 19·2 and 19·3 illustrate an axial force applied eccentrically or off center with respect to the axis through the centroid. It is important to note that if the eccentricity is large enough it will cause an axial stress of opposite sign, as shown in Fig.

19·2, where an applied tension force caused a compression stress in part of the member.

This phenomenon becomes of some importance in members which are loaded in compression and are incapable of resisting tension stress. (A column made up of brick is a typical example.) It is always possible to compute the maximum stresses for a specific problem as a check. It is also possible to solve Eq. 19·2 to determine the area on the cross section within which the axial compression load may be applied without causing a tension stress. This area is called the *kern*. For a rectangular section this area is defined by the third points on the principal axes, as shown in Fig. 19·4. The shape and size of the *kern* may be derived for any cross section.

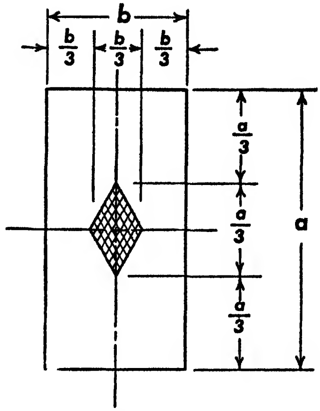


FIG. 19·4. The kern.

19·5. Curved Members. In Sec. 9·1 it was shown that initial curvature of a member could be neglected in truss analysis, by replacing the member by a hypothetical straight one (phantom member) through the two end points. If the axial force P is found by this method, it may then be applied as shown in Fig. 19·5. The conditions at any section of the beam are determined by transferring the axial force to the centroid and applying the resulting eccentric bending moment (Pd in Fig. 19·5).

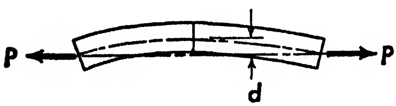


FIG. 19·5. Curved member under axial load.

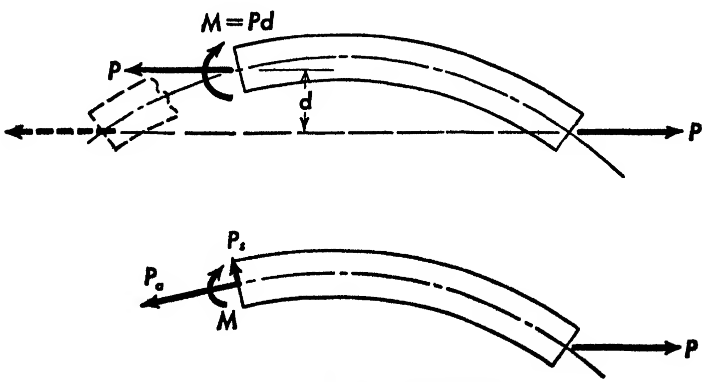


FIG. 19·6. General case of curved member.

If the axis of the member is not parallel with that of the force, the force will have to be further resolved into two components, one along the axis of the beam and the other normal to it, as shown in Fig. 19·6. See also Sec. 2·21 for method of rotating the forces to the proper angle. The axial and shear stresses are then found by methods already described.

19·6. Joint Eccentricities. Since a member usually begins to fail when its allowable stress is exceeded at any point on its cross section, for maximum load-carrying capacity the stress should be uniform, if possible. In axially loaded members (as in trusses) it is important that the

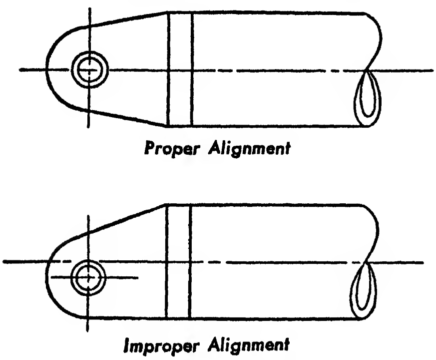


FIG. 19·7. Location of fittings for axial loading.

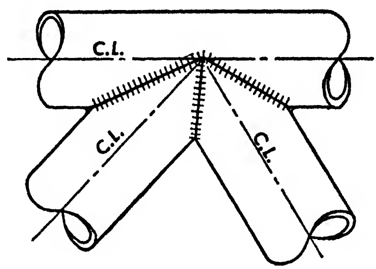


FIG. 19·8. Proper location of tubes in welded construction.

method of attachment be designed so that the resultant of the axial load acts through, or very near, the centroid of the cross section. This is accomplished by careful design of the fittings, or by proper location of the centerlines of the members in a welded assembly.

Figure 19·7 shows the right and wrong ways to design an efficient end fitting for a tension member. The eccentric fitting obviously causes additional bending stresses which reduce the efficiency of the member.

Figure 19·8 shows the proper way to join members of a truss. Note that the centerlines of all members intersect at a common point.

Eccentric fittings cannot always be avoided. The member in question may occasionally have an excess of strength which would take care of the additional bending stresses. The strength calculations should then include the effects of the eccentric loading. In general, however, eccentricities of this type should be avoided, particularly for *compression* members, as will be explained later.

19·7. Induced Bending. Since bending causes a lateral deflection of the member it will cause a variation in the eccentricity, as shown in Fig. 19·9. In tension the eccentricity will tend to decrease, thereby decreasing the bending moment and causing a more nearly uniform stress distribution (*b*). The more flexible the member is, the more nearly the stresses approach a uniform distribution. As noted in Chapt. 7, this is the reason why tension members are inherently the most efficient means of transmitting forces. Because flexible cables, chains, and tie rods have negligible bending resistance, they automatically line up with the load.

19·8. Secondary Bending (Beam-Columns). In compression loading the conditions are reversed, as shown in Fig. 19·9*c*. Such members are often called **beam-columns.** As the axial load is applied, the bending moment *Pd* increases. This causes an increase in the lateral deflection, which in turn further increases the bending moment. The additional

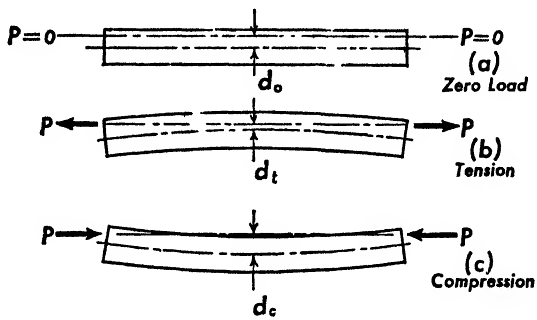


FIG. 19·9. Induced bending.

bending moment caused by the deflection is called *secondary bending* (this term is sometimes applied to the *total* bending moment). From Fig. 19·9*c* it can be seen that this extra bending moment will vary from end to end of the member and will have its maximum value near the middle.

The derivation of equations for the secondary bending moment is an advanced problem, as simple mathematical methods cannot be applied directly. It can also be seen that the principle of superposition will not hold, as the effects are not independent. The physical action can be easily understood, however, and is very important as a basis for the *column* theory. Assume that the *primary* bending moment curve for a beam is known, as shown in Fig. 19·10. (For eccentric loading this will be equal to the axial force multiplied by the eccentricity.) The deflection curve under primary bending may then be found by methods described in Chapt. 16. The additional (secondary) bending moments along the length of the beam will be equal to Py, where y is the deflection. The *total* bending moment curve is now found by adding the secondary and primary moments. If the deflection curve is again determined, it will show an increase over the first determination, causing still another increase in total bending moment. Obviously this process might be repeated *ad infinitum*, but either of two things would be found to happen:

a. The increase in secondary bending moment would become smaller and smaller with each succeeding operation, finally becoming negligible.

b. The increase in secondary bending moment would become larger and larger, finally causing failure, or excessive deflection.

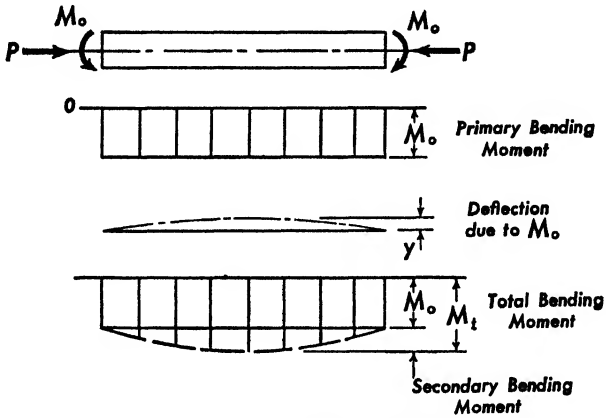


FIG. 19·10. Beam column.

Case (*a*) is called **stable,** as the conditions within the beam become stabilized at some definite value of deflection. Case (*b*) is called **unstable,** indicating that it is impossible to carry the loading which was assumed to be applied.

It is of course possible, even in the stable case, to reach the maximum allowable *stresses* in the beam before stability of deflections is reached. Hence calculations for secondary bending must usually be made for all beam-columns. It can be seen that the *yield stress* represents a sort of upper limit for allowable compression stresses in such cases, as any yielding represents a decrease in E, which in turn would cause further deflections, higher stresses, lower E, and so on until instability or failure occurs.

19·9. Columns.* The ideal column may be regarded as a hypothetical case of the eccentrically loaded compression member in which the eccentricity is zero. Actually it is impossible to attain zero eccentricity in practical structures, although it may be closely approached under laboratory conditions.

* The subjects of *instability* and *failure of materials* are beyond the scope of this volume, but an elementary explanation of columns is presented to emphasize the interrelation between buckling and bending and to serve as an introduction to column theory.

A more realistic conception of a column is that of a compression member for which the eccentricity is *practically*, but not quite, zero. This agrees more nearly with actual conditions and gives a better starting point for the column theory. It has been stated (Sec. 19·8) that under certain conditions the secondary deflections will be *unstable* in nature, i.e., they will continue to become larger automatically until the column fails or bends excessively.

If an unstable condition of this type exists, it will require only an infinitestimal eccentricity to start the process, after which the column will proceed to fail through instability. The theory of columns, and buckling in general, is based on the determination of the conditions under which this will happen.

19·10. Column Theory. Although the exact theory for columns requires more powerful mathematical methods than those used in this text, it is possible to derive a fairly close approximation by utilizing the beam theory already developed. This will aid in understanding the physical significance of the column formula.

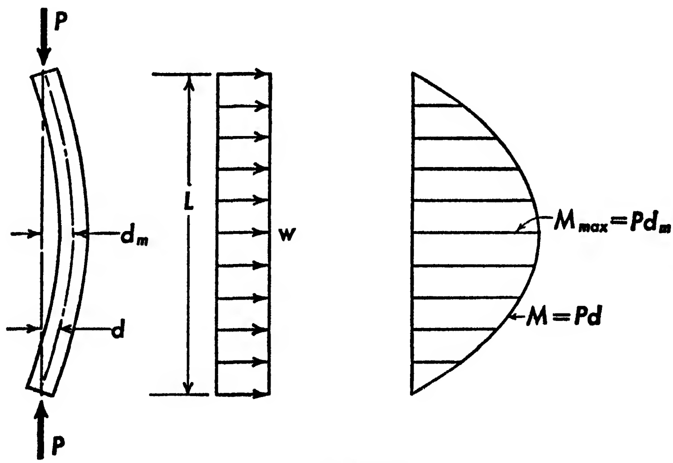


FIG. 19·11. Deflected column.

In Fig. 19·11 assume that the column has a very small lateral deflection, the maximum value of which is d_m. The bending moment will be equal to the end load P, times the eccentricity d. If the shape of the moment curve were known, it would be possible to integrate it and determine the maximum deflection in terms of the maximum moment. If this shape is not known to start with, an assumption of some sort must be made. (The theoretical curve is, of course, obtained from the exact theory and is that of a *sine* wave.)

The moment curve for a uniform lateral load has a shape something like that of a deflected column. It might therefore be used as a first approximation. In terms of M_{max}, the maximum deflection for such loading will be found (from Appendix 3·6) to be equal to

$$d_m = \frac{M_m L^2}{9.6EI} \qquad [19\cdot3]$$

If the maximum bending moment Pd_m is now substituted for M_m the solution gives

$$P = \frac{9.6EI}{L^2} \qquad [19\cdot4]$$

The value of end load thus found is such that the column will theoretically stay in any deflected position in which it is placed, i.e., it will have no bending stiffness whatever.

The deflected position assumed in deriving the above equation is, of course, not exactly correct. The true answer was found by Euler, who derived the basic column equation

Euler column equation

$$\boxed{P = \frac{\pi^2 EI}{L^2}} \qquad [19\cdot5]$$

Since the value of π^2 is approximately 9.86, it will be seen that the first approximation gave a result less than 3 per cent under the exact value. The mathematical difficulties of solving this problem are evidently centered around the determination of the shape of the deflection curve.

Equation 19·5 gives the critical load for a straight member. At any load *below* this value the column will return to its initial position after a slight displacement. At any load *above* this value the slightest displacement will cause a continually increasing deflection, indicating instability. Hence the critical load P is the *maximum* load that can be withstood in compression, regardless of the strength of the material. The only material property that affects the load is the modulus of elasticity E (see Sec. 19·13 for *plastic* range).

19·11. Column Strength vs. Bending Stiffness. In Sec. 16·12 it was shown that the resistance to deflection under the action of a bending moment is proportional to EI/L^2. Since this term is the basis for the Euler equation, it is evident that the buckling load for a column is directly proportional to its lateral stiffness. It must be kept in mind that instability failures are not really stress failures, but simply indicate that the effective lateral stiffness disappears. When that happens, the column shortens by *bending*, rather than by direct compression.

This principle can actually be utilized in experimental work to predict the end load at which instability will occur. In Fig. 19·12 the lateral bending stiffness is determined by applying a side load P_s and measuring the resulting lateral deflection d. The resistance to lateral deflection is given by the term P_s/d. If this operation is conducted at various values of end load P_c, it becomes possible to plot a curve such as shown in Fig. 19·12*b*. This curve will reach a value of zero at the end load which causes buckling and may be *extrapolated* as indicated by the dotted

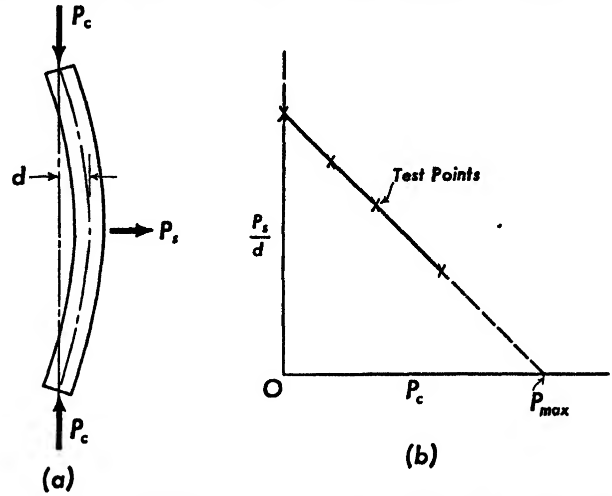


FIG. 19·12. Experimental method for predicting critical load.

line to predict the buckling load. (Actually the extrapolation may not be quite straight, as shown.) This procedure is a variation of Southwell's method, in which no side load is applied.[33]

19·12. Column Stress. It is sometimes desirable to express the critical condition for a column in terms of stress instead of force. This is done by dividing the Euler force by the cross-sectional area, giving

$$f = \frac{\pi^2 EI}{AL^2} \qquad [19\cdot 6]$$

Since I and A are both functions of the size and shape of the cross section, they can be combined into one term, I/A. This term measures the bending resistance per unit length, per unit area. It is more commonly denoted by its equivalent ρ^2, where ρ (rho) is the radius of gyration of the cross section (sometimes denoted by k). It is found from the equation

Radius of gyration $$\rho = \sqrt{\frac{I}{A}} \qquad [19\cdot 7]$$

Substituting this value in the Euler equation and combining it with L, the equation becomes

Euler stress equation

$$f = \frac{\pi^2 E}{(L/\rho)^2} \qquad [19 \cdot 8]$$

The term L/ρ is called the **slenderness ratio.** Equation 19·8 now contains only two variables, E and L/ρ. Hence a *column curve* can be

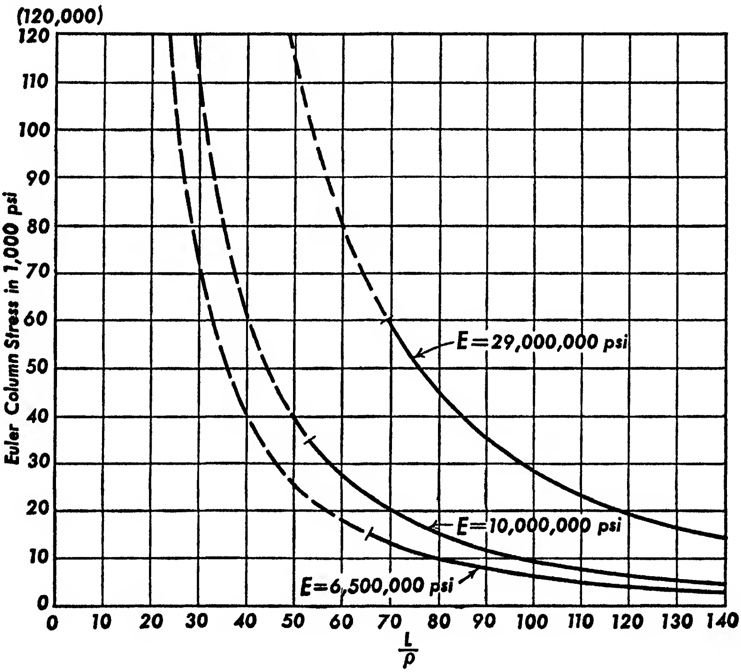


FIG. 19·13. Euler column curves.

constructed for all columns of a given material. Such curves are shown in Fig. 19·13, for steel, aluminum alloy, and magnesium alloy.

19·13. Short Columns. Euler's original equation was based on a constant value of E and was applicable only in the elastic range. In Chapt. 6 it was shown that the local value of E, given by the *tangent modulus,* drops off as the stress exceeds the proportional limit. It has been found [34] that a close approximation to actual conditions is obtained by substituting the tangent modulus in the Euler equation.*

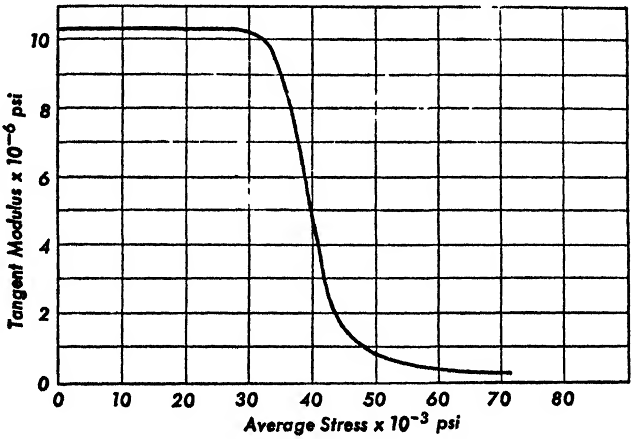
* The exact solution of this problem as developed by von Kármán [35] and others gives slightly different values based on the use of a double modulus. For practical work the direct use of the tangent modulus is satisfactory.

The equation may then be written

General column equation

$$f = \frac{\pi^2 E_t}{(L/\rho)^2} \qquad [19\cdot9]$$

The curve of column stress may be constructed for any given material by assuming different values of f, determining the tangent modulus E_t, and solving Eq. 19·9 for L/ρ, as shown in Fig. 19·14.



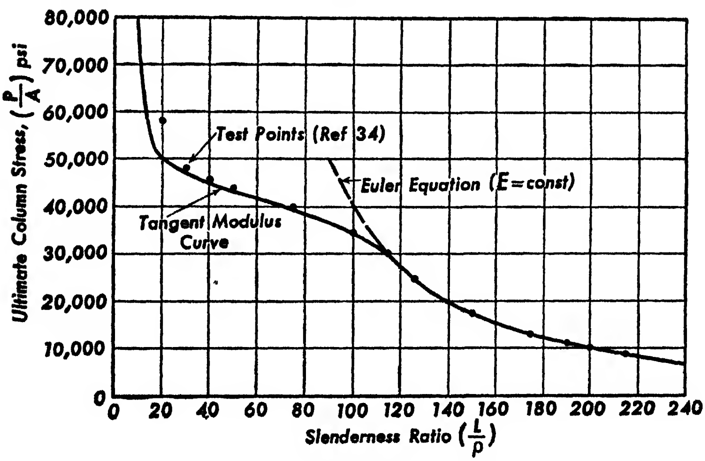


FIG. 19·14. Construction of short column curve.

Before the development of the tangent modulus theory it was necessary to represent the short-column range by empirical formulas. Many of these are in common use and are more convenient than the basic equation. The type of formula used depends on the way E_t varies for the

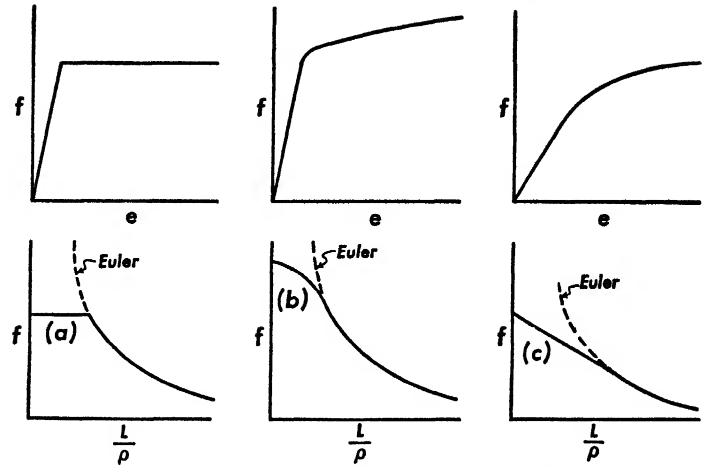


FIG. 19·15. Various short column curves.

material in question, i.e., on the shape of the stress-strain diagram. Figure 19·15 shows three types of short-column curves, together with the type of stress-strain diagram they represent.

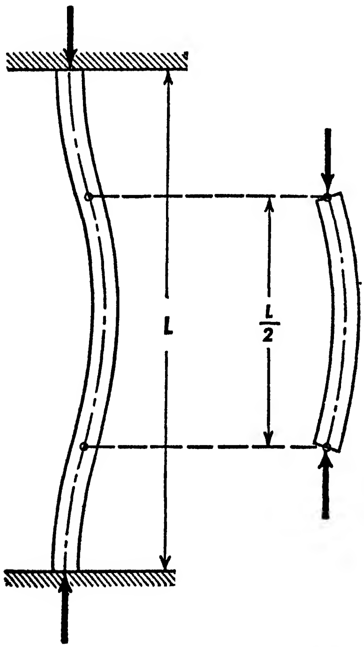


FIG. 19·16. Effects of end fixity (complete fixity shown, $c = 4$).

Curve (a) is a straight cut-off, which would apply if the stress-strain diagram had a constant value of stress in plastic range, or if an artificial reduction of E were caused by *local* buckling. Curve (b) is a *parabolic* type of curve, commonly used for steel. Curve (c) shows a *straight-line* formula which applies quite well to structural aluminum alloys.

19·14. End Constraint. The column formulas discussed up to this point are for *pin ends* (zero constraining moment). Either or both ends may, however, be *fixed* or may have an intermediate amount of constraint. The theory of columns shows that complete end fixity produces points of zero bending (points of inflection) at one-quarter of the length from the ends, as indicated in Fig. 19·16. This means that the middle half of the column behaves as a pin-ended column having a length equal to $L/2$. For a lesser degree of end fixity the *effective* column length will lie somewhere between $L/2$ and L

There are two alternative ways of taking care of this in the column formula. The most common is to modify the pin-ended column values

FIG 19·17 Failure of fuselage test specimen Note buckling of stringers acting as columns between frames

by a **fixity coefficient** c, which accounts for the effects of end constraint. For complete fixity Eq. 19·9 would become

$$f = \frac{\pi^2 E_t}{(\frac{1}{2}L/\rho)^2} = \frac{4\pi^2 E_t}{(L/\rho)^2} \qquad [19\cdot 10]$$

The factor 4 is here considered to be the coefficient of fixity Its value can obviously vary between 1 0 and 4 Denoting the coefficient by c generalizes the formulas.

General column formulas (coefficient type)

$$P = \frac{c\pi^2 E_t I}{L^2} \qquad [19\cdot 11]$$

or

$$f = \frac{c\pi^2 E_t}{(L/\rho)^2} \qquad [19\cdot 12]$$

The other method is to use an *effective length,* but the term c will then not be used in the equation. Denoting effective length by L_e, the general equations become

General column formulas (effective length)

$$P = \frac{\pi^2 E_t I}{L_e^2} \qquad [19\cdot13]$$

or

$$f = \frac{\pi^2 E}{(L_e/\rho)^2} \qquad [19\cdot14]$$

It can be seen that the relationship between the fixity coefficient and the effective length is given by

$$L_e = \frac{L}{\sqrt{c}}$$

or $\qquad [19\cdot15]$

$$c = \left(\frac{L}{L_e}\right)^2$$

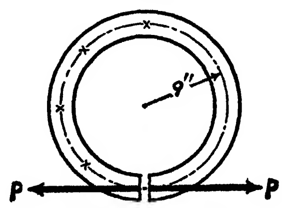
PROBLEMS

19·1. A solid round bar, 2 in. in diameter, is loaded by an axial tension force applied 3 in. from its center. Assume any value for the force (between 10,000 and 20,000 lb) and find the maximum compression and tension stresses in the bar (neglect possible secondary effects). Draw a sketch indicating the stress distribution across the section and locate the point of zero stress.

19·2. Using Eq. 19·1, find the location of the *kern* for a solid round cross section.

19·3. Work out Problem 19·2 for a round thin-walled tube. (Use mean radius and neglect effect of wall thickness.)

19·4. The split ring illustrated has a solid circular cross section 3 in. in diameter.



PROBLEM 19·4.

(*a*) Find the loading conditions on cross sections marked *X*.

(*b*) Find the maximum axial and shear stresses at each section (neglect curved beam effect).

19·5. A column 10 ft long, pinned at each end, must carry a load of 500 lb. Find the moment of inertia required for:

(*a*) Wood (E = 1,200,000 psi).
(*b*) Steel (E = 29,000,000 psi).
(*c*) Aluminum alloy (E = 10,000,000 psi).

19·6. Find the diameters and cross-sectional areas required for the three cases of problem 19·5, assuming, for each case

(*a*) Solid round bar.
(*b*) Thin-walled tube, having a D/t value of 30. (Find mean diameter and thickness.)

19·7. In Problem 19·6 determine the axial stress and note whether the long-column (unmodified Euler) formula will apply. Check results by using column *stress* formula to compute critical load. (Requires computation of radius of gyration.)

19·8. Draw (on graph paper) an arbitrary stress-strain diagram, to some suitable scale. Find the complete column curve as shown in Fig. 19·14.

19·9. Work out Problem 19·5, assuming that a fixity coefficient of 3.0 is developed at each end.

CHAPTER 20

COMBINED STRESSES

20·1. Division of Subject. The important subject of combined stresses may be divided into two parts:

a. Determination of the internal stress conditions under combined loading.

b. Failure of materials under combined loading.

Part (*a*) may be classified under the heading of *internal loads,* and will be taken up briefly in this chapter. Part (*b*) may be classified as *allowable loads* and will therefore be omitted from this volume.

20·2. Addition of Stresses. In Chapt. 2 the addition of forces by graphical and analytical methods was taken up. In dealing with stresses it is essential to remember that cross-sectional *area* is also involved. If stresses were to be added as if they were forces, the results would be in error except where the stresses act along the same axis.

The basic principle involved is simple. *Convert stresses into forces before addition.* As an obvious corollary, the resultant force so obtained must be converted back into stress. These conversions involve multiplying or dividing by the proper cross-sectional area. Analytical methods and equations are usually employed, but in order to clarify the process a general case will be worked out by vector methods, using the principles developed in Chapt. 2.

20·3. Vector Method. Consider any two uniform axial stress fields acting in the same plane at an angle θ (Fig. 20·1). The intensity of stress may be indicated by vectors, as shown, provided that the lines of the network are equally spaced. If it is further assumed that the stressed sheet or plate has a uniform thickness of unity (1 in., for instance) and that the lines of the network are spaced a unit distance apart, the forces represented by the vectors will be exactly equal to the stress intensity (since each vector acts on a unit cross-sectional area).

It will be assumed that these two stress fields act simultaneously, even though they might be caused by two different types of loading. Since the stresses are now represented by *force* vectors, the latter may be added vectorially. An element (shaded area in Fig. 20·1) may be isolated as a free body, as shown in Fig. 20·2*a*. It can then be split in two ways, as indicated in (*b*) and (*c*). Forces P_1 and P_2 may be added vectorially to obtain the reaction P_R. This force must be resolved into two components, P_N and P_S, which are the *normal* and *shear* forces acting on the cut section. Finally these forces must be divided by the

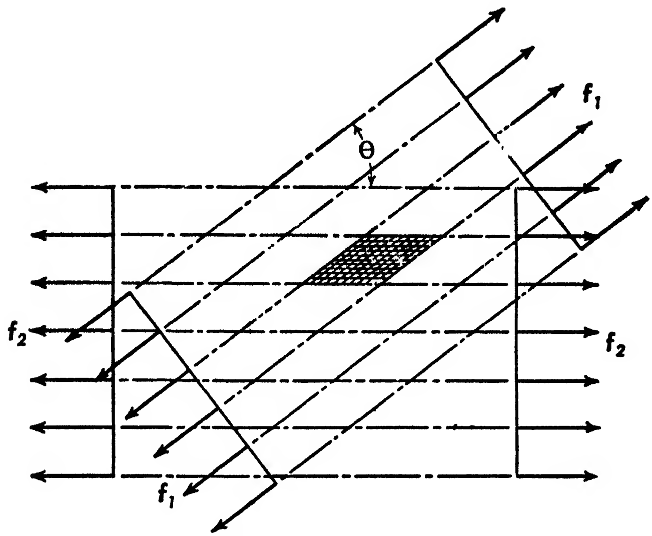


FIG. 20·1. Addition of stress fields.

area of the cut section, which is equal to the length of the cut side of the element.

It should be noted that the forces P_1 and P_2 may also be resolved into normal and shear components on the sides of the elements. This indi-

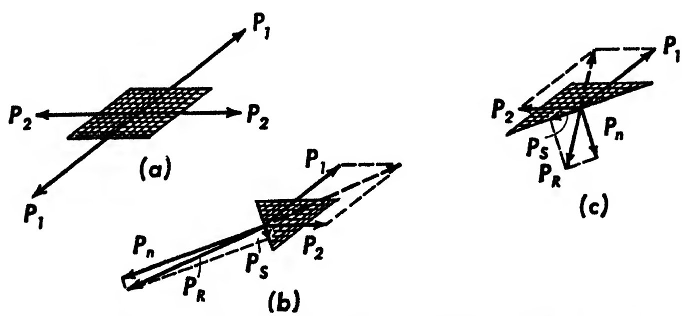


FIG. 20·2. Conditions for an element under combined stress.

cates that the effect of the first stress field on the second is to add a *shear* stress and a *normal* stress at right angles to the direction of the second stress field.

Two other conclusions may be drawn from this simple case.

a. It is obvious, from Fig. 20·2, that the length of the cut side does not remain unity, thus demonstrating that vector addition alone is not correct.

b. In general the resultant stress on a given cross section will be composed of a *normal* stress and a *shear* stress.

20·4. Stresses on Any Cross Section. The element selected in Fig. 20·2 permitted the calculation of the stresses along two lines only (the diagonals of the element). The stress condition along any line may be obtained by using an element of which one side is normal to the line, while the other two sides are parallel to the lines of axial stress. This is

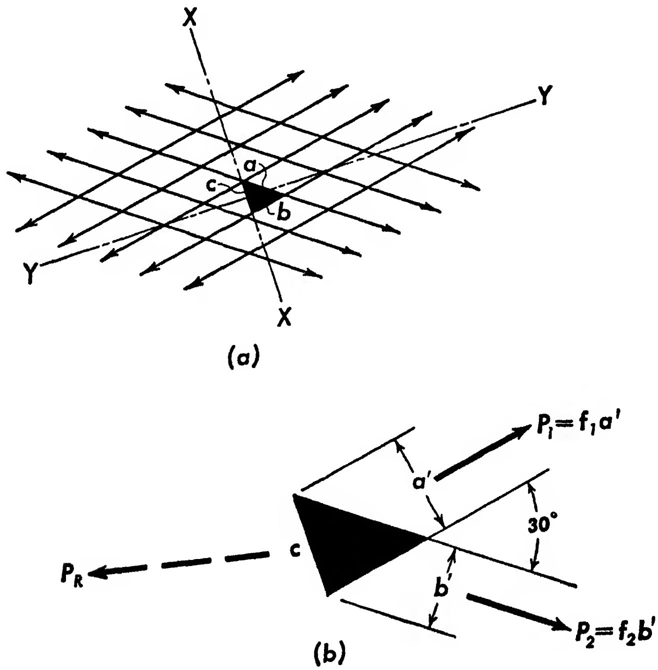


FIG. 20·3. Stress on any cross section.

illustrated in Fig. 20·3 in which the element is drawn so that side c is of unit length and is normal to the line Y–Y, along which the resultant stress condition is to be determined. Note that the sides a and b are no longer of unit length. Before adding the stresses vectorially they must therefore be multiplied by the projected area (length) of the respective sides.

Although this method is not generally used in determining combined stresses, an example will be worked out to demonstrate the principles involved.

Assume that uniform axial stress fields of 5000 and 10,000 psi, respectively, are superimposed at an angle of 30° as shown in Fig. 20·4, and that it is desired to know the stress condition at point O with respect to a section X–X (which may be drawn at any angle). A unit length, AB, is measured off along X–X, with its center at point O.

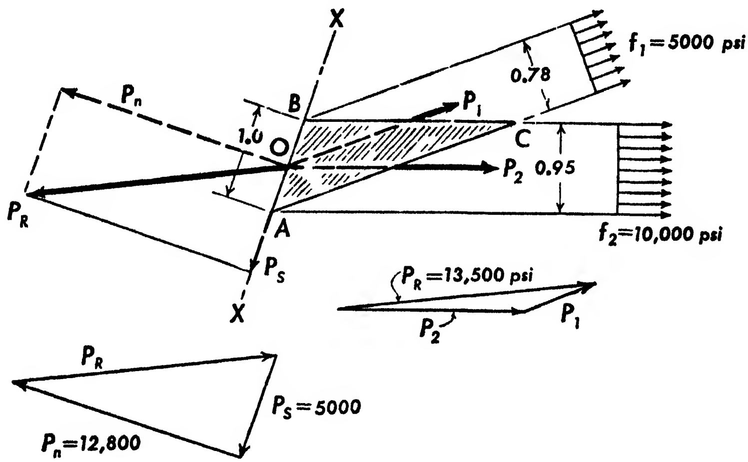


FIG. 20·4. Vector addition of axial stress fields.

Lines are then drawn through points A and B, parallel respectively to the lines of action of stresses f_1 and f_2. This establishes point C and the triangular element ABC.

The projected lengths of BC and AC normal to the stress fields acting on them are found to be 0.78 and 0.95. The *forces* acting on these sides of the triangle are, therefore,

$$P_1 = 5000 \times 0.78 = 3900 \text{ lb}$$

$$P_2 = 10,000 \times 0.95 = 9500 \text{ lb}$$

These are drawn to scale on Fig. 20·4. Their lines of action will pass through the midpoints of the sides of the element and will intersect at point O (this follows from the construction). The resultant force at point O may then be determined by vector addition of the two forces and is found to be equal to 13,500 lb. For equilibrium, the reaction must be in a direction opposite to the resultant force so P_R is drawn as shown.

Since P_R does not act normal to the side AB it must be resolved into two components P_N and P_S, normal to and parallel with this side. These values are found from the vector diagram to be equal to 12,800 and 5000 lb, respectively. Since the side AB has a length of unity, these values also represent the normal and shear stresses, i.e.,

$$f_n = 12{,}800 \text{ psi}$$

$$f_s = 5000 \text{ psi}$$

20·5. Analytical Methods (Pure Tension). Although it is possible to combine axial stresses vectorially by the method just described, it is

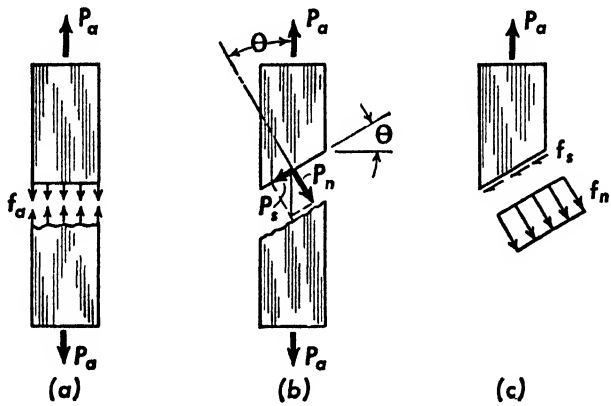


FIG. 20·5. Simple tension.

usually more convenient to use analytical methods. Such methods become rather cumbersome for the general case, and it is therefore best to consider first the simpler conditions of stress. Simple axial loading (tension or compression) is, of course, the most elementary condition.

Figure 20·5 shows the effects of cutting an axially loaded member at different angles. (The member shown may be assumed to have unit thickness and to have no axial stresses normal to the plane of the paper.) Figure 20·5*a* shows the usual cutting plane, through the minimum cross section. This plane, of course, represents the plane of maximum stress for a member of uniform cross section. The axial stress obtained in this manner will be represented by the symbol f_a. (Actually it will be either tension, f_t, or compression, f_c.)

Figure 20·5*b* shows the conditions existing along a different cutting plane located with respect to the normal plane by the angle θ. Note from (*b*) that θ also measures the angle between the line of action of the applied tension and the new line of action along which the normal stress is to be found. Following the basic rule that *forces* (not stresses) must be used in vectorial addition or resolution, the total axial force P is resolved into two components as follows:

P_n, a component *normal* to the cutting plane.

P_s, a component along (or parallel with) the cutting plane.

These two components are called the *normal* and *shear* components, respectively. The corresponding normal and shear stresses, f_n and f_s, are indicated in (c). Their values may be obtained by dividing the forces by the cross-sectional area. But the area along the cutting plane is obviously greater than the area used in calculating the tension stress (Fig. 20.5a). If A represents the minimum area, the area cut by a plane at an angle θ will be given by

$$A_\theta = \frac{A}{\cos\theta} \qquad [20\cdot 1]$$

The values of the normal and shear components are (from Fig. 20·5b)

$$P_n = P_a \cos\theta$$

$$P_s = P_a \sin\theta$$

By dividing each of these forces by the area A over which they are distributed,

$$f_n = \frac{P_n}{A_\theta} = \frac{P_a \cos^2\theta}{A}$$

$$f_s = \frac{P_s}{A_\theta} = \frac{P_a \sin\theta\cos\theta}{A}$$

These equations may be expressed in terms of the axial stress f_a (f_t or f_c), which is equal to P_a/A.

Components of axial stress

$$f_n = f_a \cos^2\theta \qquad [20\cdot 2a]$$

$$f_s = f_a \sin\theta\cos\theta \qquad [20\cdot 2b]$$

These are the basic equations for the components of axial stress.

Equations 20·2 may be divided by f_a, giving the *ratio* between the stresses on the diagonal plane and the maximum tension stress.

$$\frac{f_n}{f_a} = \cos^2\theta \qquad [20\cdot 3a]$$

$$\frac{f_s}{f_a} = \sin\theta\cos\theta \qquad [20\cdot 3b]$$

These ratios are plotted in Fig. 20·6 against the angle θ. It can be seen that the shear stress reaches its maximum value at an angle of 45° and

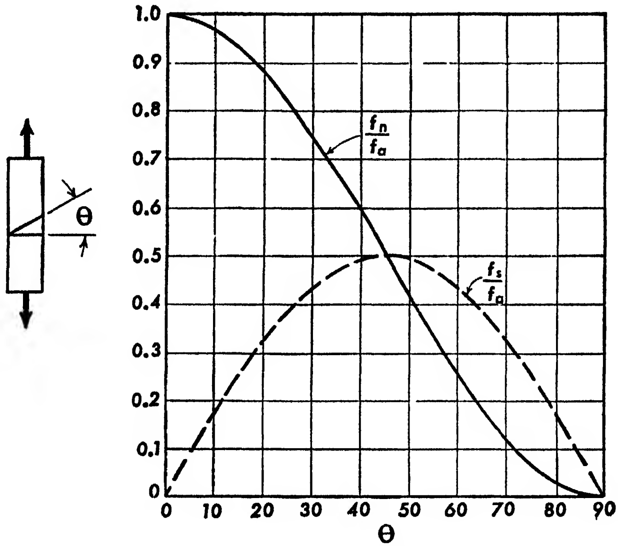


FIG. 20·6. Variation of normal and shear stress.

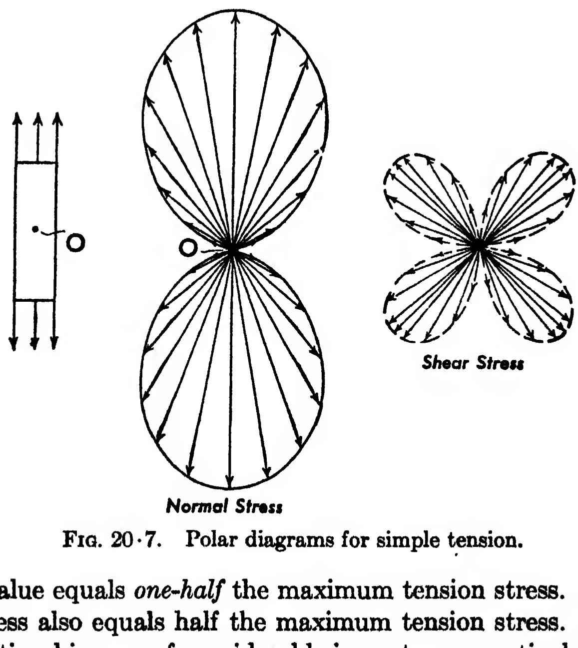


FIG. 20·7. Polar diagrams for simple tension.

that this value equals *one-half* the maximum tension stress. At 45° the normal stress also equals half the maximum tension stress. These two simple relationships are of considerable importance, particularly in dealing with the behavior of materials under stress. They may be summarized briefly as follows.

In simple tension:

a. The maximum tension stress occurs on a plane at right angles to the direction of stress.

b. The maximum shear stress occurs on a plane at 45° to the direction of stress and is equal to *half* the maximum tension stress.

c. The normal stress on the 45° plane is also equal to half the maximum tension stress.

The above relationships hold also for simple *compression* stress.

The relationships shown by Fig. 20·6 may also be plotted on a polar diagram, as in Fig. 20·7. In this figure the intensity of stress (in terms of the maximum tension stress) is given by the length of a vector. The direction of the stress is given by the direction of the vector. It should be realized that in simple tension all the stresses shown in Fig. 20·7 exist at every point.

20·6. Mohr's Circle. Equations 20·3 can be expressed graphically by a simple construction known as Mohr's circle. From trigonometry it is known that the following relationships are true.

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2} = \frac{1}{2} + \frac{1}{2}\cos 2\theta$$

$$\sin\theta\cos\theta = \frac{\sin 2\theta}{2}$$

Equations 20·3 may therefore be written

$$\frac{f_n}{f_a} = \frac{1}{2} + \frac{1}{2}\cos 2\theta \qquad [20\cdot 4a]$$

$$\frac{f_s}{f_a} = \frac{1}{2}\sin 2\theta \qquad [20\cdot 4b]$$

In Fig. 20·8 these values are plotted normal to each other. Equation 20·4*a* is satisfied by laying off a value of ½ and drawing a line at an angle 2θ, the length of this line being held constant at a value of ½. This construction also satisfies Eq. 20·4*b*. Now if the angle θ is varied, point *A* will describe a circle, giving the diagram shown in Fig. 20·9. This is a *non-dimensional* form of Mohr's circle, as ratios are used instead of actual stresses.

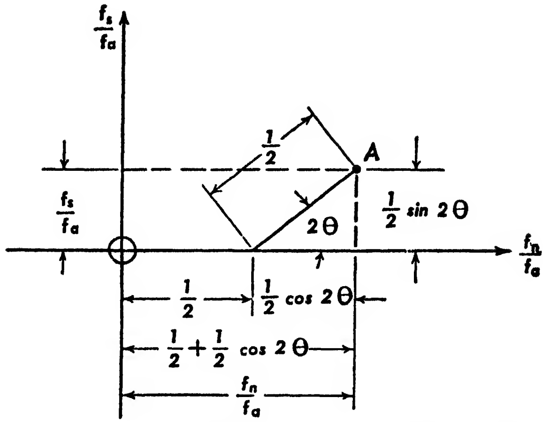


FIG. 20·8. Basis for Mohr's circle (non-dimensional).

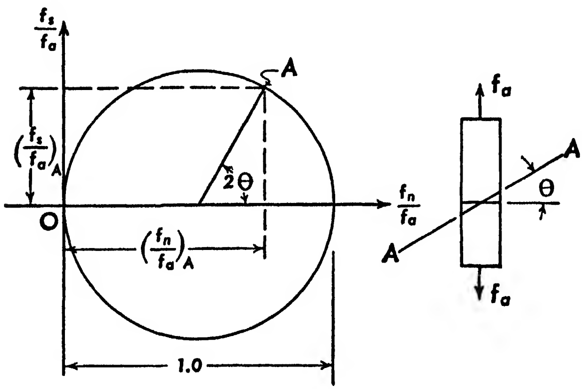


FIG. 20·9. Mohr's circle for simple tension (non-dimensional).

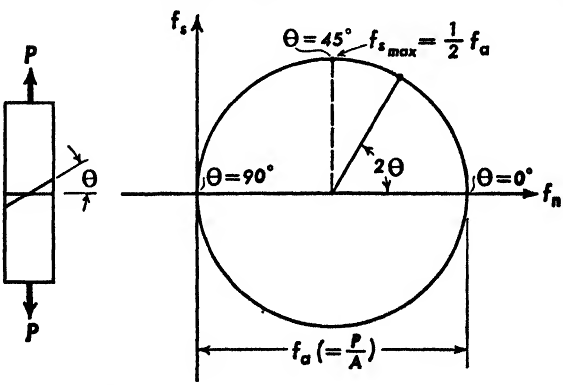


FIG. 20·10. Mohr's circle for simple tension.

If all values are multiplied by f_a the usual form of Mohr's circle is obtained as shown in Fig. 20·10. This simple diagram shows clearly

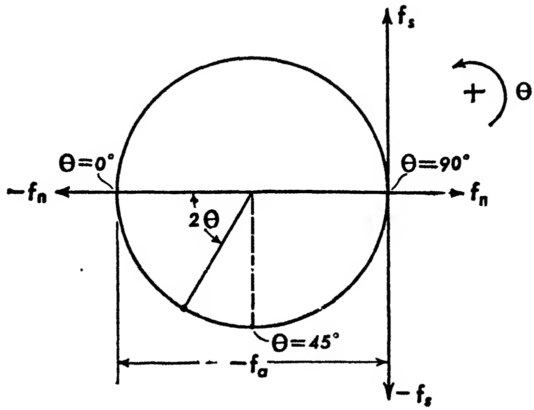


FIG. 20·11. Mohr's circle for compression.

the relationships between shear, normal, and maximum tension stress. For *compression* the value of f_a will be negative, when Eqs. 20·2 would be written

$$f_n = -\tfrac{1}{2}f_a - \tfrac{1}{2}f_a \cos 2\theta \qquad [20\cdot 5a]$$

$$f_s = -\tfrac{1}{2}f_a \sin 2\theta \qquad [20\cdot 5b]$$

The corresponding Mohr's circle is shown in Fig. 20·11. Note that the starting point ($\theta = 0$) is now at the point $f_n = -f_a$. The positive direction of rotation is maintained in measuring θ. The sign of the shear stress at 45° is negative, i.e., in a direction opposite to that for pure tension.

20·7. Sign of Shear Stress. In using analytical methods it is desirable to have a consistent system of conventions, particularly for shear stresses. The equations developed so far are based on the conditions in Fig. 20·5, which, since the shear stress was assumed to be positive, can be used as a basis. In order to avoid confusion, *two* cuts will be made, isolating an element or strip, as shown in Fig. 20·12. The shear stress will be con-

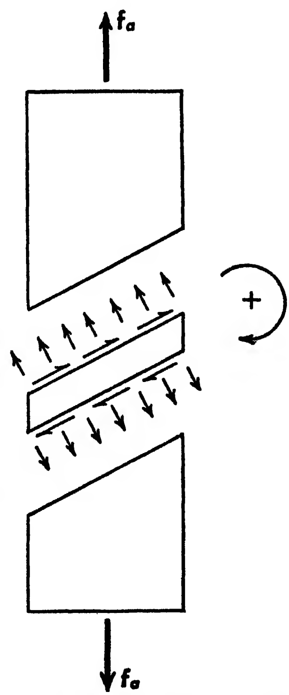


FIG. 20·12. Positive shear stress for Mohr's circle.

sidered as positive if it tends to rotate the element in a positive direction (shown here as clockwise).

The sign of the normal stress on the element will be governed by the conventions for axial stress, i.e., positive for tension, negative for compression.

20·8. Axial Stresses at Right Angles. The general method of Sec. 20·3 will now be converted into analytical form for stresses at right angles. Following the procedure suggested, an element will be selected

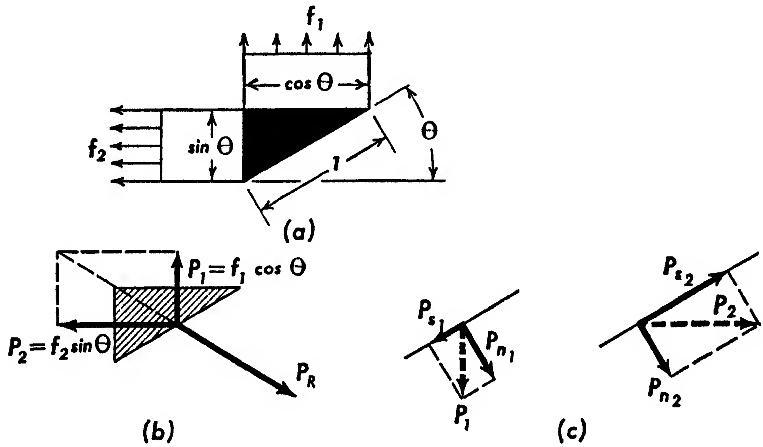


FIG. 20·13. Analytical addition of axial stresses at right angles.

so that the side along the desired cutting plane has a length of unity as in Fig. 20·13*a*. The other two sides will then have values of sin θ and cos θ, respectively. The corresponding forces are indicated in Fig. 20·13*b*. The normal and shear components of the resultant are shown separately in (*c*).

From these figures it can be seen that

$$P_1 = f_1 \cos\theta$$

$$P_2 = f_2 \cos\theta$$

$$P_{n1} = P_1 \cos\theta = f_1 \cos^2\theta$$

$$P_{n2} = P_2 \sin\theta = f_2 \sin^2\theta$$

$$P_{s1} = P_1 \sin\theta = f_1 \sin\theta\cos\theta$$

$$P_{s2} = -P_2 \cos\theta = -f_2 \sin\theta\cos\theta$$

Since the area over which these forces act is unity, they represent stresses directly, and since the stress components along the same line may be added directly,

Combined axial stresses at right angles

$$f_n = f_1 \cos^2 \theta + f_2 \sin^2 \theta$$
$$f_s = (f_1 - f_2) \sin \theta \cos \theta \qquad [20\cdot 6]$$

It must be remembered that in deriving these equations the stresses f_1 and f_2 were assumed to be pure axial stresses at right angles; that is, no shear stresses were acting along the lines of axial stress. In the equation for shear stress the term $\sin \theta \cos \theta$ reaches a maximum value of ½ (see Fig. 20·6); hence the *maximum* shear stress is given by the equation

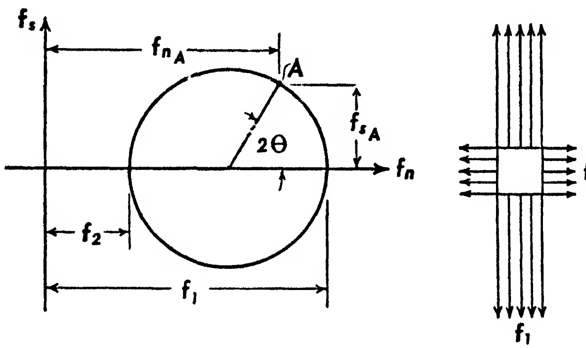


FIG. 20·14. Mohr's circle for combined axial stresses at right angles.

$$f_{s\,max} = \tfrac{1}{2}(f_1 - f_2) \qquad [20\cdot 7]$$

If f_1 and f_2 are equal and of the same sign (for example, both tension) the shear stress at any angle will obviously be zero. This is the only stress condition in which zero shear stress on all cutting planes will result. There will still be shear stresses on third-dimensional planes, however. To eliminate these it would be necessary to have uniform tension (or compression) in all *three* directions.

In constructing Mohr's circle the values of f_1 and f_2 are laid off along the same line, as the value of 2θ becomes 180° when the forces are acting at 90°. Figure 20·14 is typical. The circle is located by the values of f_1 and f_2.

From Fig. 20·14 it can be seen that if f_1 were to approach f_2 in value, the circle would become smaller, finally becoming a *point* when the tension stresses at 90° are equal. This, of course, agrees with the previous statement that the shear stress is zero on all planes in such cases.

Figure 20·15 shows polar diagrams for the normal and shear stresses under a condition of tension in two normal directions, one of the tension stresses having one-half the magnitude of the other. Compare Fig.

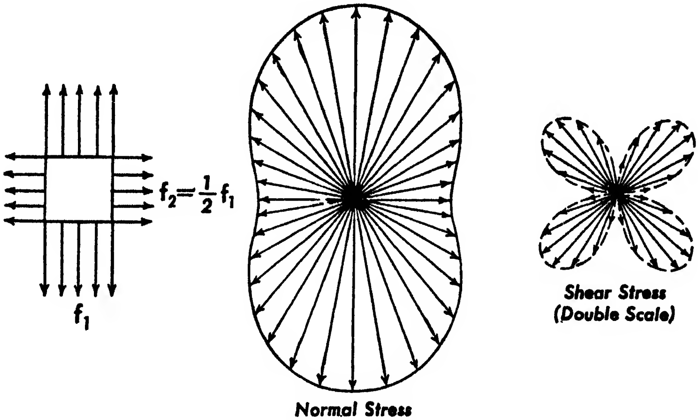


FIG. 20·15. Polar diagram for combined axial stress.

20·7, for simple tension, and note that the shear stresses are reduced by the presence of the second tension field.

For equal tension fields at right angles the polar diagram for normal stresses will become a circle and the shear diagram will vanish entirely.

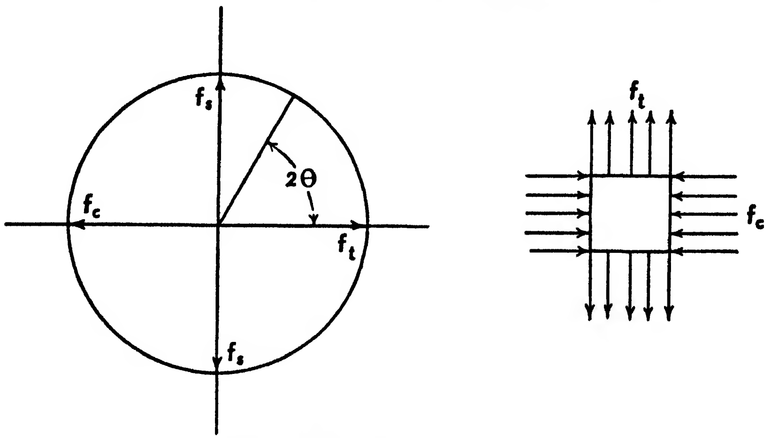


FIG. 20·16. Mohr's circle for pure shear.

Mohr's circle for equal tension and compression stresses at 90° is shown in Fig. 20·16. As described in Chapt. 10 this is a condition of *pure shear*, in which the maximum tension, compression, and shear stresses are all equal.

20·9. Principal Stresses. In adding axial stresses acting at right angles it was assumed that no shear stresses existed along the directions of the axial stresses. From Figs. 20·14 and 20·15 it can be seen that the two axial stress fields represent the maximum and minimum values of normal (axial) stress. These values occur along the line of zero shear stress. This leads to the important principle that in any combined stress condition there will be two mutually perpendicular *principal directions* along which the normal stresses are respectively a maximum and a minimum and for which the shear stresses are zero. The normal stresses along these directions are called the **principal stresses.** The term *principal planes* is also used to denote the planes of maximum and minimum stress.

It follows that Eqs. 20·6 and the construction of Mohr's circle, as described up to this point, hold only when the axial stresses are *principal stresses* and when the axis from which θ is measured is one of the *principal axes.*

20·10. Combined Axial and Shear Stresses. It is common, in stress analysis work, to find combinations of axial and shear stresses for which

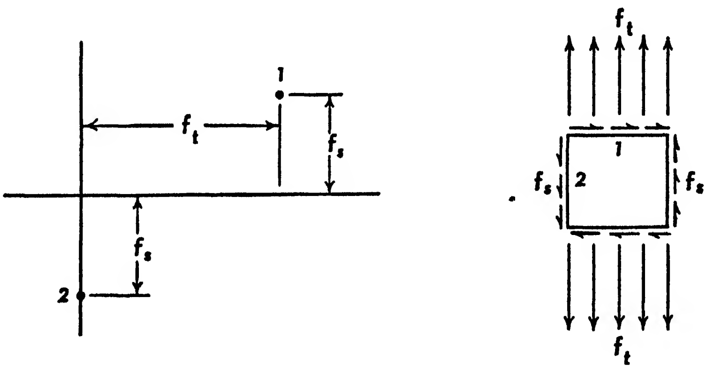


FIG. 20·17. Combined axial and shear stresses.

the resultant maximum normal and shear stresses must be obtained. (Combined axial loading and torsion is an example.) Combined tension and shear are shown in Fig. 20·17. Mohr's circle is found by locating points 1 and 2 corresponding to sides 1 and 2 of the element. Note that the shear stress on side 2 is always equal and opposite to that on side 1. Points 1 and 2 give the stress condition for the other two sides also. Since sides 1 and 2 are 90° apart, $2\theta = 180°$ and points 1 and 2 therefore lie on a diameter of the circle. This diameter is found by connecting the points with a straight line and the circle is drawn as shown in Fig.

20·18*a*. From Fig. 20·18*b* it can be seen that the radius of the circle is equal to

$$\sqrt{\left(\frac{f_t}{2}\right)^2 + (f_s)^2}$$

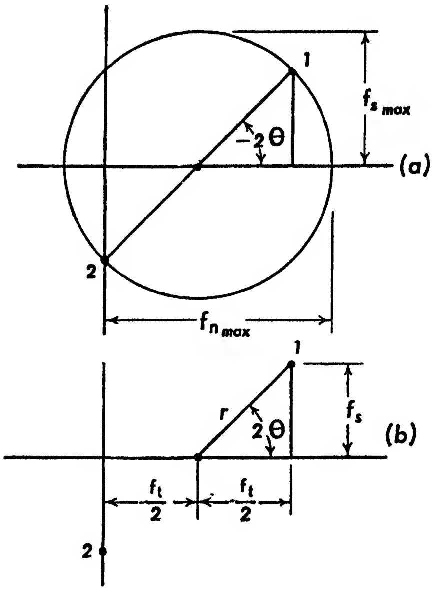


FIG. 20·18. Construction of Mohr's circle, combined axial and shear stresses.

This value therefore represents the maximum shear stress. Also,

$$f_{n\ \max} = \tfrac{1}{2}f_t + r$$

$$= \frac{f_t}{2} + f_{s\ \max}$$

$$= \frac{f_t}{2} + \sqrt{\left(\frac{f_t}{2}\right)^2 + (f_s)^2}$$

The equations for maximum normal and shear stresses may therefore be written

Maximum and minimum stresses in combined tension and shear

$$f_{s\ \max} = \sqrt{\left(\frac{f_t}{2}\right)^2 + (f_s)^2} \qquad [20\cdot8a]$$

$$f_{n\ \text{max-min}} = \frac{f_t}{2} \pm f_{s\ \max} \qquad [20\cdot8b]$$

The use of the ± sign in Eq. 20·8 indicates that the maximum stress will be obtained by using whichever sign gives the higher result.

Figure 20·18*b* shows that

$$\tan 2\theta = -\frac{f_s}{\frac{1}{2}f_t} = -\frac{2f_s}{f_t} \qquad [20\cdot 9]$$

From this equation the location of a *principal plane* may be found. A convenient rule to use in this connection is that the principal plane may be located by rotating a side of the element through the angle θ. The direction of rotation is found from Mohr's circle as that through which the point (representing the side in question) must be rotated to bring it back to the normal stress axis. Figure 20·19 shows how this would be done for a case such as previously described. Figure 20·18*a* shows that point 1 must be moved in a clockwise direction to reach the axis of normal stress. Therefore, side 1 in Fig. 20·19 must be rotated through the angle θ, in the same direction. The value of θ may be measured from the figure (which, of course, gives 2θ) or may be calculated from Eq. 20·9. Since positive values of θ were established as counterclockwise, Eq. 20·9 must have a negative sign.

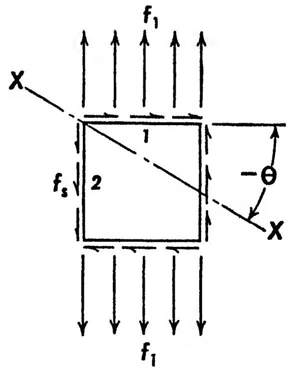


FIG. 20·19. Determination of principal axis.

The methods and equations developed for combined tension and shear are, of course, applicable in combined *compression* and shear.

20·11. General Case (Plane Stress). It will be assumed that known axial stresses act at right angles and, in addition, a shear stress acts on the planes normal to the axial stress lines. Such a condition is shown in Fig. 20·20*a*. The procedure in constructing Mohr's circle and finding the maximum normal and shearing stresses is essentially the same as for combined simple tension

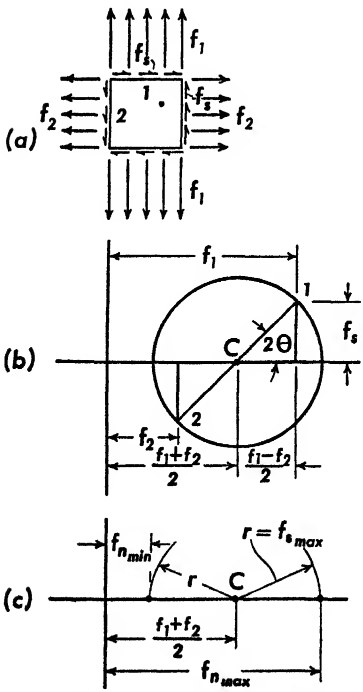


FIG. 20·20. General case of plane stress.

and shear. Points 1 and 2 are located as shown in Fig. 20·20*b* and the diameter of the circle is thus determined. Note that since f_2 is shown as tension it is laid off in the same direction as f_1.

From the construction of Fig. 20·20*c* the following equations may be derived.

General plane stress

$$f_{s\ \max} = \sqrt{\left(\frac{f_1 - f_2}{2}\right)^2 + (f_s)^2} \qquad [20\cdot 10a]$$

$$f_{n\ \text{max-min}} = \frac{f_1 + f_2}{2} \pm f_{s\ \max} \qquad [20\cdot 10b]$$

Figure 20·20*b* shows that

$$\tan 2\theta = -\frac{f_s}{\frac{1}{2}(f_1 - f_2)} = -\frac{2f_s}{f_1 - f_2} \qquad [20\cdot 11]$$

(Compare with Eq. 20·9)

20·12. Examples of Combined Stresses (Joints). A simple application of the foregoing principles is found in the *scarf* joint, commonly used

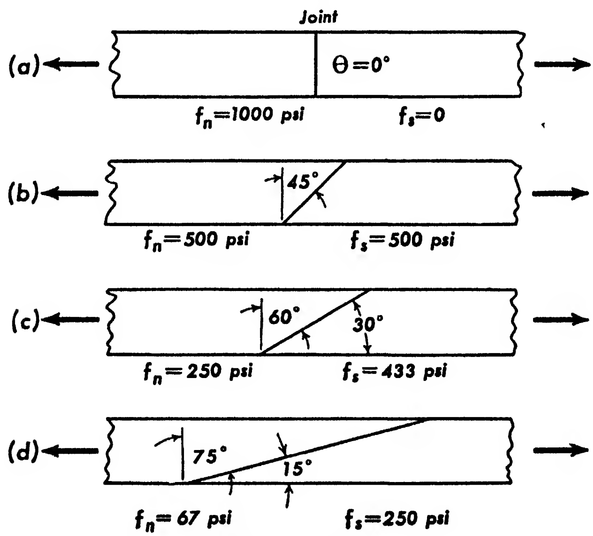


FIG. 20·21. Scarf joints.

in joining wood members and in welding. Figure 20·21 shows joints made at different angles. Figure 20·21*a* represents a butt joint in which the joint is subjected to the maximum tension stress (taken as 1000 psi for illustration). Figure 20·21*b* shows a 45° joint and indicates that the normal stress across the joint (tending to pull it apart) has been reduced to half the value for (*a*). A *shear* stress has been introduced, however,

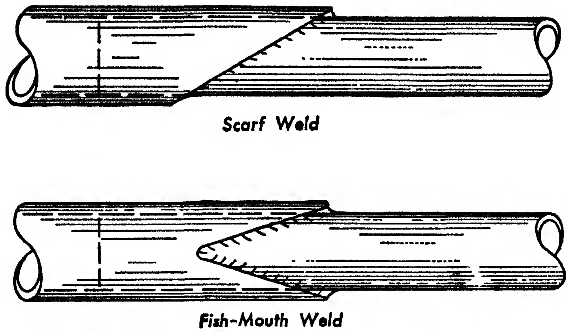


Fig. 20·22. Welds in tubing.

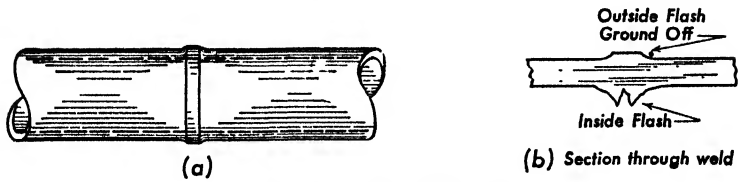


Fig. 20·23. Butt-flash weld.

equal to the normal stress. In (*c*) and (*d*) both the normal and shear stresses have been further reduced by using a flatter angle for the scarf.

If the joining material (such as glue or solder) is weaker than the basic material being joined, it is possible to obtain efficient joints by utilizing the scarf principle. In welded joints the same idea is usually applied, as the material in or near the weld may be weaker than the base material. Figure 20·22 shows typical welds used for aircraft tubing.

Electric butt-flash welds will develop the full strength of the part if properly made; hence scarfing is not used in such cases. Figure 20·23 shows a typical butt-flash weld, including an enlarged view of a section through the joint.

Bolts may be subjected to combined tension and shear as shown in Fig. 20·24. Equations 20·8 may be used to calculate the maximum normal and shear stresses, assuming a uniform stress distribution over the cross section. The breaking strength of a bolt under such conditions may be roughly predicted by assuming that failure will occur when either the maximum normal or shear stress reaches its breaking

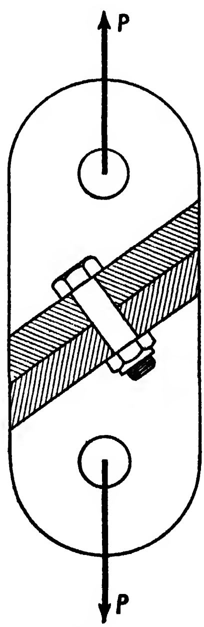


Fig. 20·24. Combined tension and shear on bolt.

value for simple tension or shear, respectively. This is slightly on the unsafe side, however, due to interaction effects. (Some notes on allowable combined stresses will be found at the end of this chapter.)

20·13. Combined Tension and Torsion. For members subjected to combined tension and torsion the maximum normal and shear stresses may be computed from Eqs. 20·8, as the directions of the applied stresses

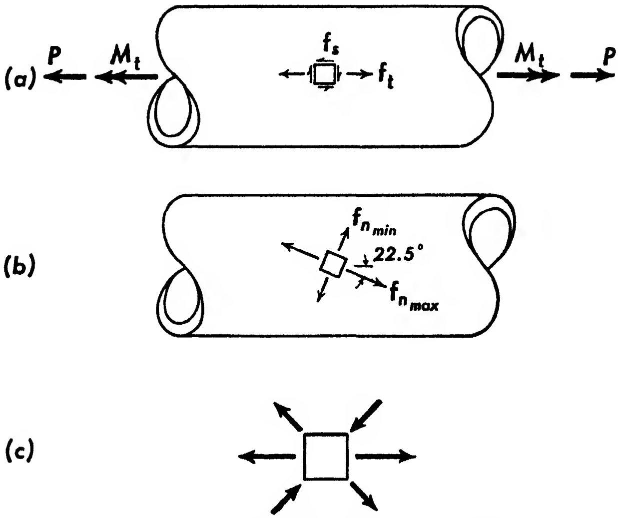


Fig. 20·25. Combined axial load and torsion.

are at right angles. Figure 20·25 shows such a loading condition. Assume that the values are as follows.

$$f_t = 20{,}000 \text{ psi}$$

$$f_s = 10{,}000 \text{ psi}$$

From Eqs. 20·8,

$$f_{s\,\max} = \sqrt{\left(\frac{20{,}000}{2}\right)^2 + (10{,}000)^2}$$

$$= 14{,}160 \text{ psi}$$

$$f_{n\,\max} = \frac{20{,}000}{2} + 14{,}160$$

$$= 24{,}160 \text{ psi}$$

From Eqs. 20·9,

$$\tan 2\theta = \frac{2 \times 10{,}000}{20{,}000} = 1.0$$

$$2\theta = 45°$$

$$\theta = 22.5°$$

These conditions are indicated in Fig. 20·25*b*. The general direction of the maximum normal stress can be predicted by noting the direction of the diagonal tension stress corresponding to the applied shear stress, as shown in (*c*), instead of using the sign conventions previously described. Obviously the maximum tension stress will occur on the side where the two tension stresses combine.

20·14. Combined Compression and Torsion. The maximum normal stress under this condition will be a compression stress. In thin-walled structures it is this stress which tends to cause buckling. It is therefore useful to have a means of calculating the maximum compression stress and the angle at which it acts. The methods used for combined tension and torsion may be employed. Since in a flat panel the sheet will tend to buckle or wrinkle along lines *normal* to the direction of maximum compression stress, the lines of wrinkling will tend to coincide with the *minimum* principal stress lines, which are always perpendicular to those for maximum normal stress. (This phenomenon often creates the erroneous impression that wrinkling is caused by the diagonal *tension* field.)

20·15. Combined Bending and Shear. The formulas for simple bending, torsion, and transverse shear permit the separate calculation of the axial and shear stresses at any point on a cross section. Hence it is possible to find the *maximum* normal and shear stresses by combining the independent values in accordance with Eqs. 20·8. If hoop tension were also present, due to internal pressure, Eqs. 20·10 would be employed.

A beam of rectangular cross section has a variable distribution of both axial and shear stress (see Chapts. 13 and 17). Therefore the value and direction of the principal stresses will change across the depth of the beam. Since the shear stress is zero at the outer edges the axial stress will be unaffected at this point. Likewise, the shear stress at the neutral axis will be unaffected by the bending stresses, which are zero in the neutral plane.

Figure 20·26 shows the variation of the maximum normal stresses over a cross section of a rectangular beam subjected to combined shear and bending. Note that the maximum axial stress still occurs at the outer fibers.

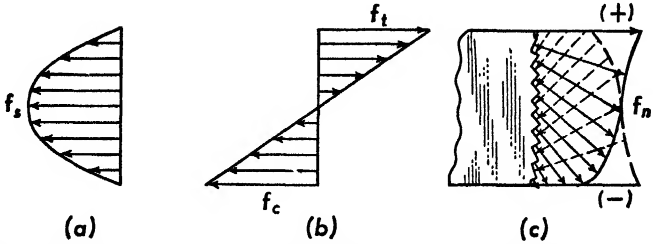


FIG. 20·26. Normal stresses in rectangular beam under combined shear and bending.

Figure 20·27, representing an I-beam, shows that the high shear stresses near the flange cause local normal stresses which exceed those computed for bending alone. Hence it is necessary to check the maximum normal stresses by using Eqs. 20·8. The most critical region will be near the juncture of flange and web.

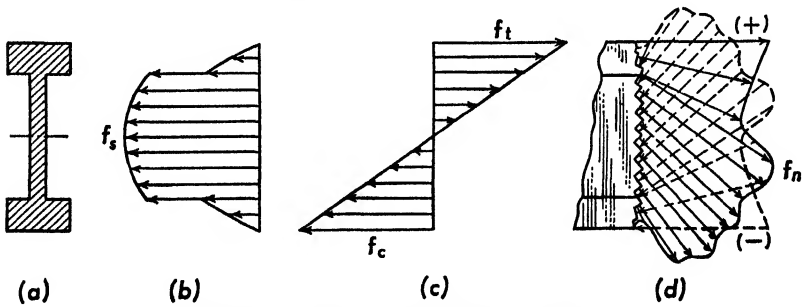


FIG. 20·27. Combined stresses in an I-beam.

20·16. Interaction Curves. Although it is not intended to cover, in this volume, the failure of materials under combined stresses, a brief description of the interaction curve method will serve as an introduction to this important subject. (The method was originally described in Ref. 36 as the *stress-ratio* method, but the term interaction curve appears to be more appropriate.)

The basic principle is that the strength under one loading condition is affected by the presence of another superimposed loading condition. This principle is made more general by dealing with *ratios*, instead of actual loads or stresses. For instance, assume that a tube is subjected to a bending moment equal to half the *ultimate* bending moment for the tube. It is then possible to determine, by tests or other means, how much *torsion* can be applied before failure occurs. This applied torsion can be expressed as a fraction of the ultimate moment for simple torsion. The *ratio* involved is, therefore,

$$R = \frac{\text{Applied load or stress}}{\text{Ultimate load or stress for simple loading}}$$

In terms of stress, the two values for combined bending and torsion would be

$$R_b = \frac{f_b}{F_b} \qquad [20 \cdot 12a]$$

$$R_s = \frac{f_s}{F_s} \qquad [20 \cdot 12b]$$

where R_b is the *stress-ratio* for bending.
R_s is the *stress-ratio* for torsion.
f_b is the applied bending stress.
F_b is the allowable bending stress for simple bending (modulus of rupture).
f_s and F_s correspond for torsion.

If a number of tests are made at different values of R_b, corresponding values of R_s at failure will be obtained. These values may then be plotted against each other to give the **interaction curve,** as indicated in Fig. 20·28.[37] The area under this curve represents safe combinations of loading; the curve itself represents these combinations which will produce failure. The *margin of safety* for a given loading condition is indicated by the location of the point A, representing the condition with reference to the *interaction curve.* (If both loadings increase simultaneously this measurement should be made along a diagonal line through the origin, as shown.)

This simple example almost completely explains the method. Further developments are concerned with the shape of the interaction curve, methods of expressing it mathematically, and equations for the margin of safety. One of the advantages of the method is that almost any theory of failure under combined loading can be expressed in the form of an interaction curve, usually with a considerable gain in simplicity.

Another feature is that the interaction curve makes full use of the large amount of data available on failure under *simple* loading conditions, as the end points of the curve are established by these conditions. (Some theories of failure cannot be made to check the simple conditions as special cases.) As a result of this, the interaction curve is always quite accurate when one loading condition predominates, even though few data may be available on combined conditions.

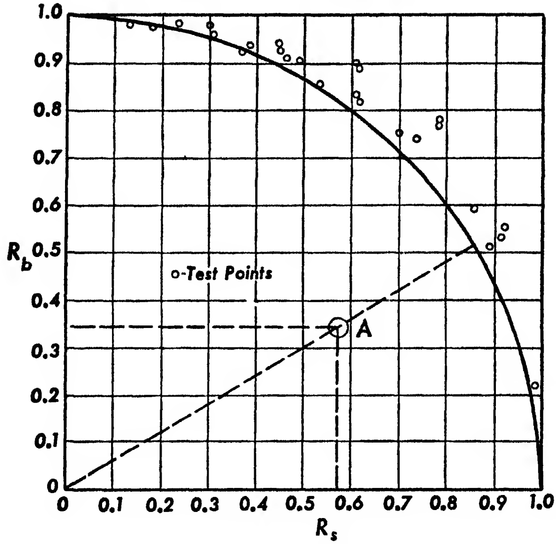


FIG. 20·28. Interaction curve for tubing under combined bending and torsion.

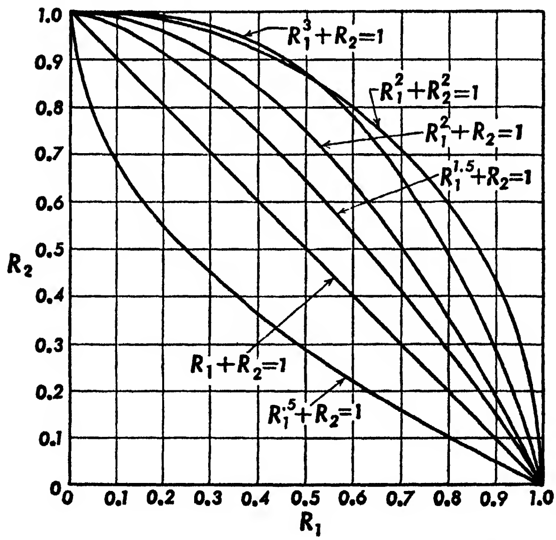


FIG. 20·29. Mathematical interaction curves.

20·17. Equations for Interaction Curves. The most convenient form for the equation of the interaction curve is

$$R_1{}^a + R_2{}^b = 1.0 \qquad [20\cdot13]$$

where 1 and 2 represent two simple types of loading, a and b are exponents which govern the shape of the curve.

Various curves are shown in Fig. 20·29. When a and b each equal 1.0, the curve is a straight line. When a and b each equal 2.0, the curve becomes circular. Other combinations give parabolas of various degrees. As a and b become larger, the interaction curve approaches the two limiting lines $R_1 = 1$ and $R_2 = 1$, indicating complete lack of interaction between the loading conditions. When a and b are less then 1.0, the curve approaches the other limits $R_1 = 0$ and $R_2 = 0$, indicating a high degree of interaction (usually involving secondary effects, as in combined compression and bending).

20·18. Specific Cases. A few special illustrations will show the application to actual conditions. Figure 20·28 gave a curve for combined bending and torsion of tubing (see Figs. 15·6 and 12·14 for the strength in simple loading).

Figure 20·30 shows a series of curves for steel bolts in combined tension and transverse shear. Note that these curves are given in terms of actual load, instead of stress. Such curves are convenient for routine calculations. The loading condition imposed is illustrated by Fig. 20·24.

Figure 20·31 shows a chart for the strength of tubing under combined axial compression and bending.[38] In this set of curves the effect of secondary bending (Sec. 19·8) was not included in calculating the value of f_b, hence the curves show a tendency to be concave downward as a result of secondary effects. (The bending moment was produced by transverse loading at the third points.) The factor B is a non-dimensional form of L/ρ, obtained by dividing the actual L/ρ by a standard value, $(L/\rho)_0$, which is the value at which the Euler curve intersects the column yield stress. (Roughly equal to the yield stress in tension.)[16]

In Fig. 20·31 the interaction curves do not all go through 1.0 for zero value of R_b. This is because the allowable compression stress was taken as the column *yield* stress, instead of the actual column stress obtained from the column formula (Eq. 19·12). If F_c were taken as the buckling stress for the column and if secondary bending were included in computing f_b, the family of curves would tend to converge to a single straight line curve, given by the equation

$$\frac{f_c}{F_c} + \frac{f_b'}{F_b} = 1.0 \qquad [20\cdot14]$$

where f_c = applied compression stress = P/A.
f_b' = maximum bending stress, including secondary effects.
F_c = buckling stress for the column.
F_b = modulus of rupture in simple bending.

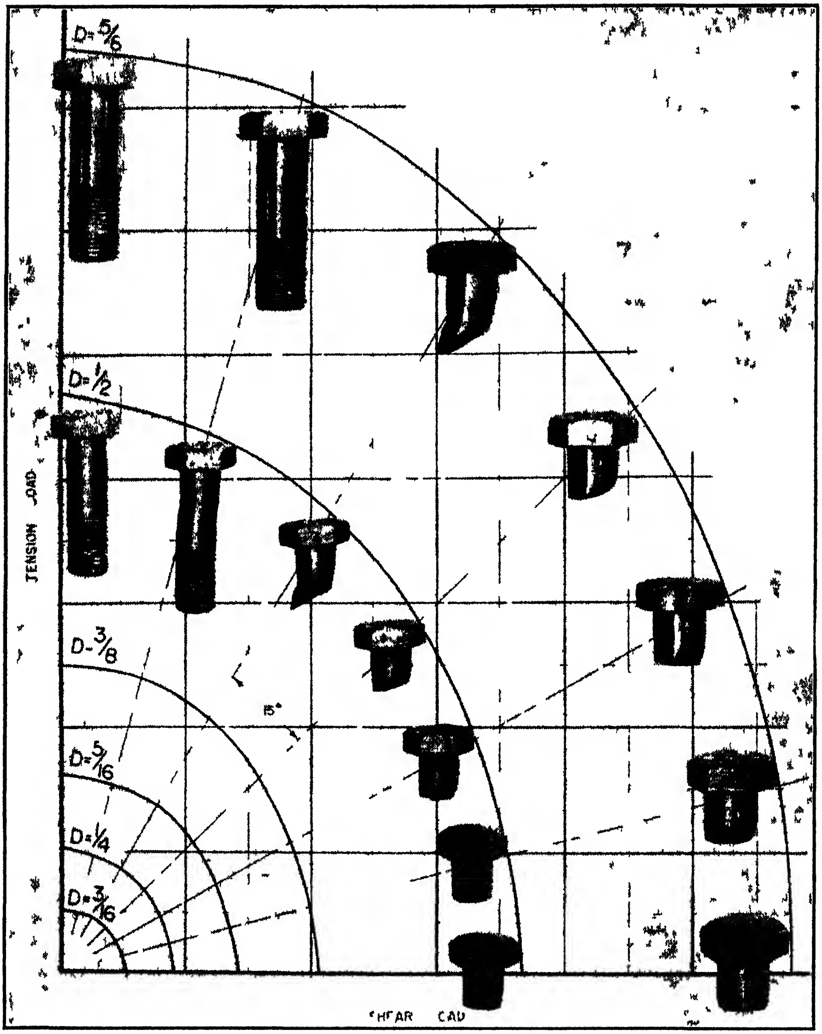


FIG 20 30 Interaction curves for steel bolts loaded in shear and tension.

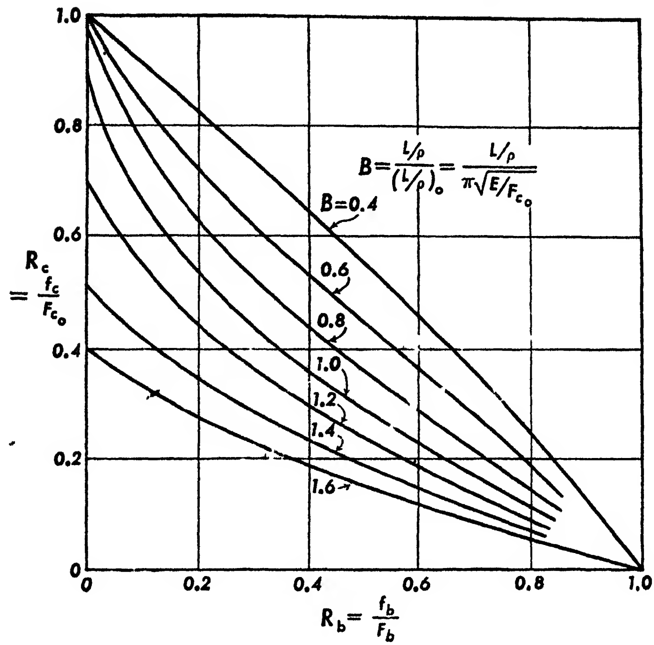


FIG. 20·31. Combined axial compression and bending of steel tubes (from Ref. 38).

20·19. Interaction Surfaces. Equation 20·13 may be expanded to include a third loading condition, giving the equation of a *surface.*

$$R_1^a + R_2^b + R_3^c = 1.0 \qquad [20\cdot 15]$$

For the case of combined *bending, compression,* and *torsion* the following equation may be used.

$$R_c + R_b' + R_s^2 = 1.0 \qquad [20\cdot 16]$$

where $R_c = \frac{f_c}{F_c}$ as in Eq. 20·14

$R_b' = \frac{f_b'}{F_b}$ as in Eq. 20·14

$R_s = \frac{f_s}{F_s} = \frac{\text{Applied torsion stress}}{\text{Allowable torsion stress}}$

Further discussion of this subject is beyond the scope of this volume. As noted in Ref. 2, the possibilities of the method have not been studied exhaustively. Recent developments indicate that it may be extended to include theories of failure of materials, as well as the failure of structural parts.

Reference 45 contains valuable information on the failure of materials under combined stresses.

PROBLEMS

20·1. A ¾-in. diameter bolt is carrying an axial load of 15,000 lb. Find the normal and shear stresses on planes which make the following angles with the transverse plane: 10°, 25°, 35°, 50°, 65°. Plot these stresses against angle. (Use formulas to work this problem.)

20·2. Assume that a certain structural material has an allowable shear stress of 20,000 psi but that its ultimate tension stress is unknown. What is the *maximum* ultimate tension stress that can possibly be expected?

20·3. In a sheetmetal forming operation the sheet is subjected to a maximum tension stress of 20,000 psi in one direction and a compression stress of 10,000 psi at right angles. What is the maximum shear stress thus produced? (Calculate by formula and check by Mohr's circle.)

20·4. A ¾-in. diameter bolt is subjected to a direct tension load of 9000 lb. At the same time it is loaded by a transverse shear force of 3000 lb. Assuming a uniform shear stress distribution, find the maximum normal and shear stresses. Check by means of Mohr's circle.

20·5. A thin-walled shell is subjected to combined bending and torsion. The maximum axial stress due to bending alone is 20,000 psi and the shear stress due to torsion alone is 5000 psi. What are the maximum normal and shear stresses? What is the maximum shear stress at a point halfway from the neutral axis to the outside of the shell? (Measured normal to N.A.)

20·6. In Problem 20·5 assume that a transverse shear force is added, such that the *average* shear stress due to transverse force alone is 3000 psi. What are the maximum normal and shear stresses at the neutral axis?

20·7. Draw, on a single chart, a series of Mohr's circles representing a constant tension stress of 10,000 psi along one principal axis and a series of stresses applied along the other axis, as follows.

Tension (psi)	*Compression (psi)*
0	0
2,000	−2,000
4,000	−4,000
6,000	−6,000
8,000	−8,000
10,000	−10,000

Calculate the maximum shear stress for each condition and check on the chart.

20·8. Draw a polar diagram of normal stress for the case in which the tension stress along one principal axis is one-third that along the other axis.

20·9. Select any material from Appendix 2 and draw, on the same chart, interaction curves for combined tension and shear loading, for the following assumptions:

(*a*) Failure occurs when maximum normal stress equals ultimate tension stress.
(*b*) Failure occurs when maximum shear stress equals ultimate shear stress.
(*c*) Failure occurs in accordance with Eq. 20·13, using a value of 2.0 for exponents *a* and *b*.

Note. Plot applied tension stress vertically and applied shear stress horizontally. Do not use ratios. Curves will show that Eq. 20·13 gives correct results for both simple loading conditions.)

20·10. In Fig. 20·24 assign values to the bolt diameter and to the angle of the test jig (latter to be between 31° and 39°). Find the normal and shear stresses with respect to the plane of sliding, for a load of 10,000 lb.

CHAPTER 21

JOINTS *

21·1. General. The design of joints is one of the most important structural problems, as the majority of failures occur at such points. In aircraft work many joints and fittings must be employed, not only to facilitate production, but also to provide changes of sheet thickness for greater weight economy. The importance of joint design has long been realized in the other branches of structural engineering.

The primary objective in joint design is to utilize as fully as possible the basic strength of the parts being joined. The degree to which this is attained is often referred to in terms of *joint efficiency.* Thus a joint which is 90 per cent efficient will develop 90 per cent of the strength of the (weaker) member which it joins. Efficiencies of 100 per cent may sometimes be attained by careful design. Joints in sheet metal cannot be made 100 per cent efficient by any of the conventional methods. A detailed study of joint strength would require an intimate knowledge of stress concentration effects and theories of failures; hence the following notes will be confined largely to the simpler aspects of the problem.

21·2. Types of Joint Failure. The common types of joint failure are shown in Fig. 21·1. Methods of calculating the strength for *shear, bearing,* and *tension* are covered by material already presented, as summarized below. The equations give the *allowable* loads.

Shear.

$$\text{Load per rivet or bolt} = A_s F_s \qquad [21\cdot 1]$$

where A_s = cross-sectional area of bolt.

F_s = allowable shear stress of bolt material (Sec. 6·15).

Becring.

$$\text{Load per rivet or bolt} = A_b F_b \qquad [21\cdot 2]$$

where A_b = projected area of thinner sheet = Dt.

F_b = allowable bearing stress for sheet or bolt (use lower) (Sec. 6·14).

* A complete discussion of this important subject is beyond the scope of this volume; this brief chapter has been included only to show the basic principles involved.

Tension.

$$\text{Load per rivet or bolt} = A_t F_t \qquad [21\cdot3]$$

where A_t = net area of cross section through bolt hole.
F_t = allowable stress in tension (Sec. 6·2).

(*Note.* For some materials a reduction factor should be used to account for stress concentration. An approximate reduction of 10 per

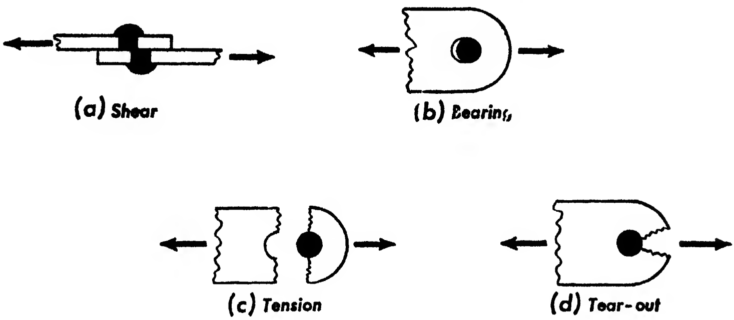


Fig. 21·1. Types of joint failure.

cent is in order for aluminum alloys. No reduction is usually made for steels.)

Tear-out. This is a complex type of failure, for which the following semi-empirical rules may be used. (From Lockheed data.)

$$\text{Load per rivet or bolt} = 2atF_s \qquad [21\cdot4]$$

where a = tear-out distance, measured as shown in Fig. 21·2. (*Note.* Use a_0, a_e, or a_s, depending on type of joint).
t = thickness of lug.
F_s = allowable shear stress for lug material.

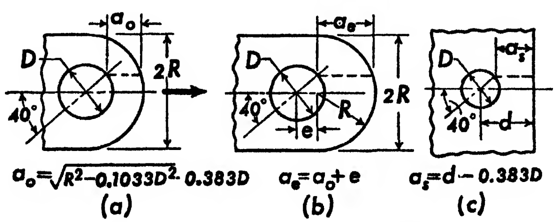


Fig. 21·2. Tear-out computations (see Eq. 21·4).

An edge distance, d (Fig. 21·2c), equal to at least twice the rivet diameter is usually required to prevent tear-out failure and avoid bulging of the sheet edge in riveting. Countersunk rivets should have even greater edge distances ($2\frac{1}{2}$ to 3 times rivet diameter). As noted in Fig. 21·2*c*, edge distance is measured to the center of the rivet, not to the edge of the hole.

21·3. Multiple Thickness Joints. In Fig. 21·3, (*b*) and (*c*) show common types of joints in which one rivet or bolt serves for several sheets or lugs. The total shear strength of the bolt is thereby increased

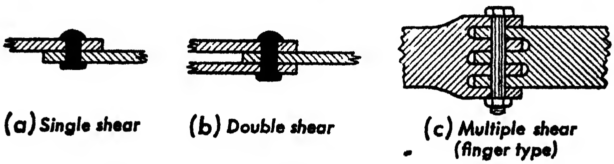


Fig 21·3. Types of joints.

by the number of different cross sections that must be cut when the bolt fails. Thus in Fig. 21·3*b* the shear strength of the rivet is doubled. In 21·3*c* the bolt would have to shear off on six planes, unless the fitting failed in some other manner; therefore Eq. 21·1 would be multiplied by 6. This type of fitting permits the use of small diameter bolts.

In the multiple thickness type of joint it is generally satisfactory to compute the bearing and tear-out strengths as the *sum* of the strengths of each sheet (in one direction of loading, of course). If any one sheet should be considerably weaker than the others, however, it would be safer to divide the load in proportion to the cross-sectional area of the sheets and then to treat each one separately. *Eccentric* joints must be treated in a different manner.

21·4. Continuous Joints. In sheetmetal structures continuous joints are common, as indicated in Fig. 21·4. The analysis may be made by

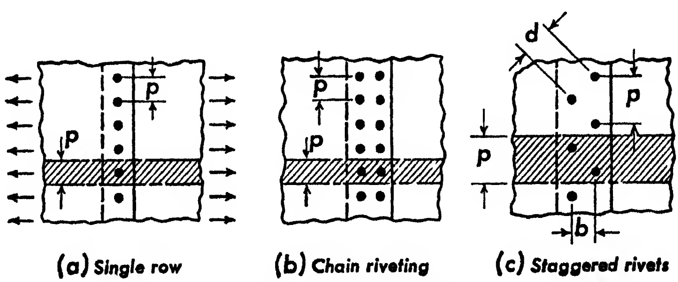


Fig. 21·4. Continuous joints.

considering a strip having a width equal to the rivet pitch p. Alternatively, an arbitrary width may be assumed (such as 1 in.) and the strength per inch of joint determined. (This may involve a fractional number of rivets per inch.)

Figure 21·4*c* shows a *staggered* type of rivet pattern, often used when seam tightness is a primary consideration. In this type of joint the rows of rivets should be far enough apart to eliminate failure of the sheet along a diagonal line between rivets. In general, the distance between rows (*b* in Fig. 21·4*c*) should be at least twice the rivet diameter. Even then the stagger will have an adverse effect on the flow of stress, causing some reduction in joint efficiency. (Because of this, the *chain* type of riveting in Fig. 21·4*b* appears to be more efficient than the staggered type, particularly for materials with relatively low elongation.)

21·5. Joint Efficiency. If the minimum strength for a strip through a joint is found (as shown in Fig. 21·4), the efficiency may easily be computed by calculating the strength of a strip of the same width (using the thickness of the thinner sheet). If the failure should be by tension between rivets the efficiency would be given by

$$\text{Efficiency (tension)} = K\frac{(p - D)}{p} \qquad [21\cdot5]$$

where p = the pitch.
D = the rivet diameter.
K = a stress-concentration factor, usually assumed to be 1.0, but which sometimes may be lower (0.9 is approximately correct for aluminum alloy.)

The general idea in designing for high efficiency is to increase the cross-sectional area over the region in which it must be reduced by bolt or

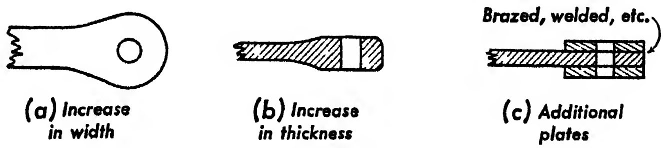


FIG. 21·5. Methods of improving joint efficiency.

rivet holes. This is possible in single joints or fittings, as indicated in Fig. 21·5.

Cover plates or doublers are frequently used in sheetmetal work, as shown in Fig. 21·6, which illustrates a *flush* type of joint (often required in aircraft). The doubler is used to provide greater strength in tear-out, bearing, and tension. The rivets attaching the doubler should be relatively small and not too closely spaced in order to avoid reducing the strength of the base sheet. The doubler may be scalloped to save weight as shown in Fig. 21·6. This type of joint is obviously eccentric and cannot be made as efficient as a symmetrical joint such as shown in Fig. 21·7.

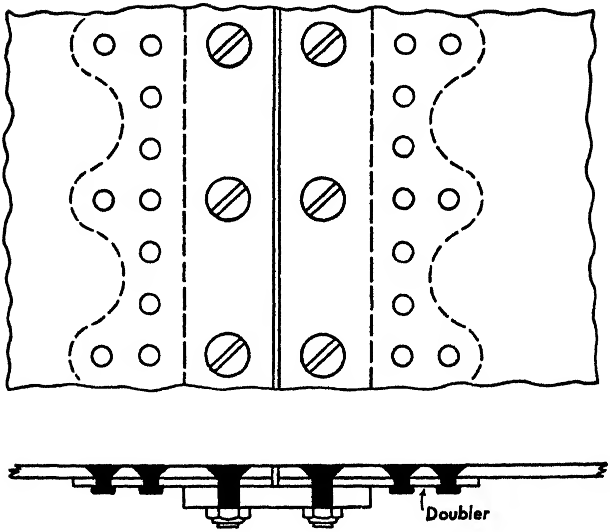


FIG. 21·6. Flush type of joint.

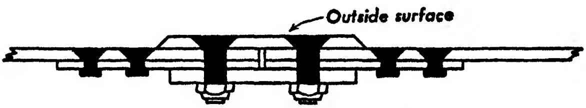


FIG. 21·7. Joint with reinforcing doubler.

FIG. 21·8. Typical failure of sheetmetal tension joint, showing tension, bearing, and tear-out failures. Note electric (resistance-type) strain gages attached to sheet.

21·6. Rivet Groups (Concentric).* If several rivets are used in a group it is necessary to determine the distribution of load between them. This distribution will depend on the deflection characteristics of the members and the rivets. For a long row of rivets, such as shown in Fig. 21·9*a*, it is customary to assume that the load is divided equally between rivets. Actually the load tends to be greater for the two end rivets, but a small amount of yielding will offset this. To obtain a more even distribution of load between rivets the joint could be made as in Fig. 21·9*b*. The principle involved here is that the cross-sectional area of the sheet should be reduced roughly in proportion to the rate at which the load is taken out of it. This results in more uniform stresses and avoids

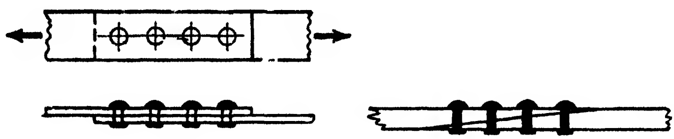


FIG. 21·9. Rivet rows.

large differences in the stress in adjacent sheets. Such stress differences cause different amounts of strain; these high relative strains are resisted by the adjacent rivets and produce high loads in them. (Figure 21.9*b* would be impractical for most sheetmetal joints, but may be used for fittings.)

In the foregoing discussion it was assumed that the load passes through the centroid of the rivet group, giving a symmetrical loading. If it is further assumed that each rivet resists deformation in proportion to its minimum strength (in shear or bearing), it is satisfactory to *add* the strength of all the rivets, even if they are not of the same size. (This assumption may not be valid if large stress differences between the joined parts cause one or more rivets to be greatly overloaded.)

21·7. Eccentric Rivet Groups. Fittings and joints must frequently transmit turning moments. Any eccentric loading may be resolved into an axial load and a turning moment, as shown in Fig. 21·10. The problem is to find the center of resistance for the rivet group. This center may be defined as the point about which the two joined parts could be rotated (relatively to each other) without inducing an axial resultant force, i.e., the resistance must be a couple, or its equivalent. The problem is therefore similar to that of torsion or bending; in fact, the same basic methods may be applied by considering each rivet to represent a

* The following discussion also applies to bolts, pins, etc.

resisting element, just as elements of cross-sectional area were assumed to resist bending or torsion.

To simplify the problem it is customary to assume that the resistance to deformation is proportional to the *cross-sectional area* of the rivet, or to its *diameter*, or to its *minimum strength* in shear or bearing. None of these assumptions is strictly true, but the last one probably gives the best all-round results. Usually, however, failure will occur by shearing a rivet, so it is common to use cross-sectional *areas* in computing the resistance. (If all the rivets are the same size the use of diameters or areas will give identical results.)

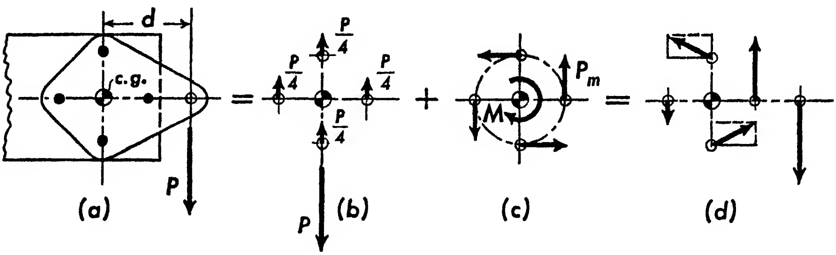


FIG 21·10. Eccentric rivet group.

Figure 21·10*a* shows a rivet group under eccentric loading. The rivets (or bolts) are assumed to be of the same size. The centroid of the group can be located by inspection, because of symmetry. The eccentric moment obviously equals *Pd*. The axial load is equally divided between rivets, as shown in (*b*). In (*c*) are shown the individual rivet loads resisting the turning moment. These will act normal to a radius line joining the rivet and the centroid. The moment resisted by each rivet is equal to the rivet load P_m times its moment arm. In this particular case the loads and moment arms (r) are the same for all rivets; hence the value of P_m may be found from the equation

$$4P_m r = M = Pd$$

$$P_m = \frac{Pd}{4r}$$

The resultant load from each rivet is found by vector addition, as shown in (*d*). Note that in this example the rivet nearest the applied force carries the largest load, while the load on one rivet reverses in direction.

In the more general case the center of resistance must be found as for a cross-sectional area (Sec. 7·6). If it is assumed that the resistance to deformation is equal to the cross-sectional area of the rivet the center of resistance will be located at the centroid of the rivet areas. This is found by the usual method of multiplying each area by its distance from a base line, adding these products, and dividing by the total area. The distance, d_0, in Fig. 21·11 would be found from the equation

$$d_0 = \frac{A_1x_1 + A_2x_2 + \cdots + A_7x_7}{A_1 + A_2 + \cdots + A_7} = \frac{\Sigma Ax}{\Sigma A} \qquad [21\cdot6]$$

(If the pattern is unsymmetrical about both axes another computation must be made for the Y-Y axis.)

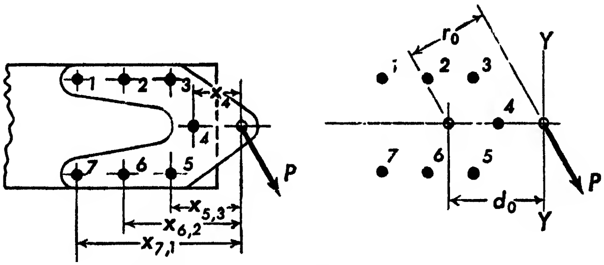


FIG. 21·11. Calculation of rivet loads.

The direct load resisted by each rivet will be given by

$$P_{d1} = \frac{A_1}{\Sigma A} P$$

$$P_{d2} = \frac{A_2}{\Sigma A} P, \text{ etc.} \qquad [21\cdot7]$$

For rotation about the c.g. the deformation of each rivet is proportional to its distance (r) from the c.g. Hence the *moment* resisted by each rivet will be

$$M_1 = \frac{A_1{r_1}^2}{\Sigma Ar^2} Pr_0$$

$$M_2 = \frac{A_2{r_2}^2}{\Sigma Ar^2} Pr_0, \text{ etc.} \qquad [21\cdot8]$$

where r is the radius from rivet center to c.g.

The loads due to moment will therefore be

$$P_{m1} = \frac{M_1}{r_1} = \frac{A_1r_1}{\Sigma Ar^2} Pr_0$$

$$P_{m2} = \frac{M_2}{r_2} = \frac{A_2r_2}{\Sigma Ar^2} Pr_0, \text{ etc.} \qquad [21\cdot9]$$

The final loads are found by vector addition of P_d and P_m.

If all the rivets are of the same size, i.e., if they have the same resistance to deformation, the area terms cancel out in Eq. 21·6, giving a simpler equation

$$d_0 = \frac{x_1 + x_2 + \cdots + x_7}{n} = \frac{\Sigma x}{n} \qquad [21\cdot10]$$

where n = number of rivets.

The loads are then given by

$$P_d = \frac{P}{n} \quad \text{(for each rivet)} \qquad [21\cdot11]$$

$$P_{m1} = \frac{r_1}{\Sigma r^2} Pr_0 \quad \text{(etc.)} \qquad [21\cdot12]$$

21·8. Location of Centroid (Graphical Method). A convenient method of locating the centroid of a rivet group is shown in Fig. 21·12. If all the rivets (or bolts) are of the same size, it is possible to find the centroid progressively by adding one rivet at a time. First, any two rivets are joined by a line and this is *bisected* to determine the centroid

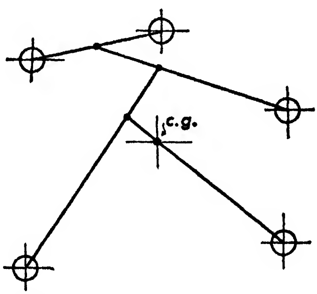


FIG. 21·12. Graphical location of centroid of rivet group.

of these rivets. This centroid is now connected to a third rivet and a point is located *one-third* of the distance from the centroid. This point is now connected with a fourth rivet and a point located *one-fourth* of the distance along the line, and so on until all the rivets have been included.

This method is useful in preliminary work and in checking, particularly if the rivet pattern is irregular. Relatively accurate results may be obtained by estimating the divisions of the lines by eye. The methods could, of course, be extended to cases in which all the rivets are not of the same size, by taking into account the relative areas (or diameters, or strengths).

PROBLEMS

21·1. Draw a sketch of a single row lap joint between two aluminum-alloy sheets, 0.040 in. thick, using $\frac{1}{8}$-in. diameter rivets and assigning any desired spacing and edge distance. Use the following data:

Ultimate tension stress for sheet	= 60,000 psi
Ultimate shear stress for sheet	= 40,000 psi
Ultimate bearing stress for sheet and rivets	= 90,000 psi
Ultimate shear stress for rivets	= 35,000 psi

Allow 10 per cent for stress concentration between rivet holes.

(*a*) Find strength per inch for the four possible types of failure.
(*b*) Calculate joint efficiency.

21·2. Assign any desired dimensions to the lug shown in Fig. 21·2*b* and calculate its minimum strength in tension, bearing, and tear-out. Use any material listed in Appendix 2.

21·3. Design a lug for a $\frac{1}{2}$-in. diameter bolt which will just develop the shear strength of the bolt. Assume bolt to be heat-treated alloy steel and lug to be aluminum alloy. Use properties given in Appendix 2. Lug to have shape of Fig. 21·2*b*.

21·4. Assign arbitrary dimensions to the fitting shown in Fig. 21·10 and assume that $P = 1000$ lb. Find the direction and magnitude of the load on each rivet.

21·5. Modify the fitting of Problem 21·4 by adding another bolt at the centroid of the group. Find load on each rivet. What percentage reduction resulted, for most highly loaded bolt?

21·6. Draw a fitting such as shown in Fig. 21·11 and assign values to dimensions. Assume that bolts 1, 2, 6, and 7 are $\frac{1}{4}$-in. diameter and bolts 3, 4, and 5 are $\frac{5}{16}$ in. diameter. Load $P = 50{,}000$ lb, acting at 30° from vertical. Find the load on each bolt (based on relative areas).

21·7. Work out Problem 21·6 using bolt diameters as basis instead of bolt areas.

21·8. Check centroid location in Problems 21·6 and 21·7 by method of Fig. 21·12, modifying it to account for difference in bolt sizes.

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TABLE 1·1

A = area
I_X = moment of inertia about principal axis X–X
I_Y = moment of inertia about principal axis Y–Y
I_N = moment of inertia about any axis N–N
Q_N = static moment about axis N–N

$$\rho = \text{radius of gyration} = \sqrt{\frac{I}{A}}$$

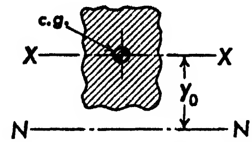
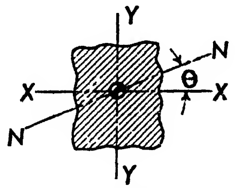
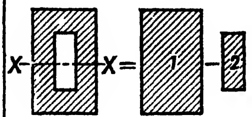
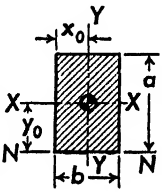
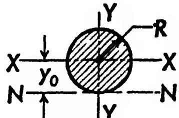
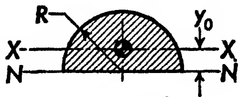
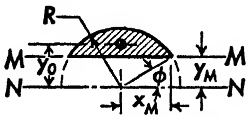
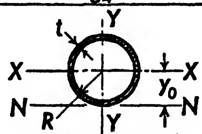
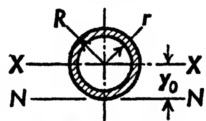
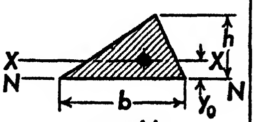
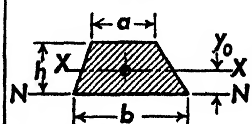
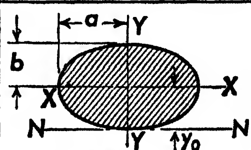
 $I_N = I_x + Ay_0^2$ $Q_N = Ay_0$	 $I_N = I_X \cos^2\Theta +$ $I_Y \sin^2\Theta - I_{XY}\sin 2\Theta$	 $A = A_1 - A_2$ $I = I_1 - I_2$ (Computed about same axis)
 $A = ab$ $x_0 = \frac{b}{2}$ $y_0 = \frac{a}{2}$ $I_X = \frac{ba^3}{12}$ $I_Y = \frac{ab^3}{12}$ $I_N = \frac{ba^3}{3}$	 $A = \pi R^2$ $y_0 = R$ $D = 2R$ $I_X = \frac{\pi D^4}{64} = \frac{\pi R^4}{4}$ $I_N = \frac{5\pi D^4}{64}$	 $A = 0.5\pi R^2$ $y_0 = \frac{4R}{3\pi}$ $I_X = 0.1098 R^4$ $I_N = \frac{\pi R^4}{8}$ $Q_N = \frac{2}{3} R^3$
 $A = \frac{\pi R^2}{2} - y_M x_M - R^2\phi$ $y_0 = \frac{Q_N}{A}$ $Q_N = \frac{2}{3}(R^2 - y_M^2)^{\frac{3}{2}}$	 (For thin walls only) $\bar{R} = R - 0.5t$ $y_0 = R$ $A = 2\pi \bar{R} t$ $I_X = \pi \bar{R}^3 t$	 $A = \pi(R^2 - r^2)$ $y_0 = R$ $I_X = \frac{\pi}{4}(R^4 - r^4)$ $Q_X = \frac{2}{3}(R^3 - r^3)$ (for half tube)
 $A = 0.5\,bh$ $y_0 = \frac{h}{3}$ $I_X = \frac{bh^3}{36}$ $I_N = \frac{bh^3}{12}$	 $A = h\left(\frac{a+b}{2}\right)$ $y_0 = \frac{h}{3}\left(\frac{b+2a}{b+a}\right)$ $I_X = \frac{h^3}{36}\left(\frac{b^2+4ab+a^2}{b+a}\right)$	 $A = \pi ab$ $y_0 = b$ $I_X = \frac{\pi ab^3}{4}$ Perimeter $= \pi(a+b)$ very approx.

TABLE 1·2

Properties of Round Tubing

This table gives the most commonly used section properties of round tubing. O.D. refers to outside diameter, t is wall thickness, Z is section modulus. A_s is equivalent shear area (divide transverse shear load by A_s to obtain maximum shear stress at neutral axis).

O.D.	t	A	ρ	I	Z	D/t	A_s
3/16	.028	.01403	.05726	.000046	.000491	6.70	.007169
	.035	.01677	.05531	.000051	.000547	5.36	.008680
1/4	.022	.01576	.08100	.000103	.000825	11.38	.007932
	.028	.01953	.07911	.000122	.000978	8.93	.009869
	.035	.02364	.07701	.000140	.001122	7.14	.01202
	.049	.03094	.07314	.000166	.001324	5.10	.01609
	.058	.03498	.07091	.000176	.001407	4.31	.01855
	.065	.03778	.06933	.000182	.001453	3.85	.02040
5/16	.022	.02008	.1030	.000213	.001363	14.20	.01008
	.028	.02503	.1011	.000256	.001636	11.16	.01260
	.035	.03051	.09889	.000298	.001910	8.93	.01542
	.049	.04056	.09476	.000364	.002331	6.38	.02078
	.058	.04637	.09228	.000395	.002527	5.39	.02399
	.065	.05054	.09048	.000414	.002648	4.81	.02643
	.095	.06491	.08392	.000457	.002925	3.29	.03632
3/8	.028	.03053	.1231	.000462	.002466	13.39	.01533
	.035	.03738	.1208	.000546	.002912	10.72	.01882
	.049	.05018	.1166	.000682	.003636	7.65	.02551
	.058	.05776	.1139	.000750	.003999	6.47	.02956
	.065	.06330	.1120	.000794	.004234	5.77	.03260
	.083	.07614	.1073	.000877	.004678	4.52	.04014
	.095	.08357	.1045	.000913	.004870	3.95	.04481
7/16	.028	.03602	.1451	.000759	.003468	15.63	.01807
	.035	.04426	.1428	.000903	.004128	12.50	.02224
	.049	.05981	.1384	.001146	.005240	8.93	.03022
	.058	.06915	.1357	.001274	.005824	7.54	.03519
	.065	.07607	.1337	.001360	.006215	6.73	.03887
	.083	.09244	.1287	.001532	.007002	5.27	.04790
	.095	.1022	.1257	.001614	.007379	4.61	.05373
1/2	.022	.03304	.1692	.000946	.003782	22.73	.01654
	.028	.04152	.1672	.001160	.004641	17.86	.02081
	.035	.05113	.1649	.001390	.005559	14.28	.02566
1/2	.049	.06943	.1576	.001786	.007144	10.20	.03499
	.058	.08054	.1576	.002001	.008002	8.62	.04082
	.065	.08883	.1555	.002148	.008592	7.69	.04516
	.083	.1087	.1503	.002457	.009828	6.02	.05577
	.095	.1209	.1471	.002615	.01046	5.26	.06264
	.120	.1433	.1409	.002844	.01137	4.17	.07635
9/16	.028	.04702	.1892	.001684	.005986	20.09	.02356
	.035	.05800	.1869	.002026	.007205	16.07	.02909
	.049	.07905	.1824	.002629	.009348	11.48	.03976
	.058	.09193	.1795	.002963	.01054	9.70	.04638
	.120	.1668	.1621	.004383	.01558	4.69	.08747
5/8	.028	.05252	.2113	.002345	.007503	22.32	.02630
	.035	.06487	.2090	.002833	.009065	17.85	.03252
	.049	.08867	.2044	.003704	.01185	12.77	.04456
	.058	.1033	.2016	.004195	.01342	10.79	.05201
	.065	.1144	.1993	.004543	.01454	9.62	.05787
	.083	.1413	.1939	.005311	.01700	7.53	.07202
	.095	.1582	.1904	.005732	.01834	6.58	.08092
	.120	.1904	.1835	.006412	.02052	5.21	.09876
11/16	.028	.05801	.2334	.003160	.009192	24.55	.02904
	.035	.07175	.2310	.003829	.01114	19.64	.03595
	.049	.09829	.2264	.005038	.01466	14.03	.04934
	.058	.1147	.2235	.005730	.01667	11.85	.05265
3/4	.028	.06351	.2555	.004145	.01105	26.80	.03179
	.035	.07862	.2531	.005036	.01343	21.42	.03938
	.049	.1079	.2485	.006661	.01776	15.30	.05413
	.058	.1261	.2455	.007601	.02027	12.94	.06334
	.065	.1399	.2433	.008278	.02208	11.53	.07037
	.083	.1739	.2376	.009820	.02619	9.04	.08805
	.095	.1955	.2340	.01070	.02854	7.89	.09934
	.120	.2375	.2267	.01221	.03256	6.25	.12179
	.156	.2911	.2171	.01372	.03660	4.81	.15225

TABLE 1·2 (*Continued*)

O.D.	t	A	ρ	I	Z	D/t	A_s
7/8	.028	.07451	.2996	.006689	.01529	31.23	.03728
	.035	.09236	.2973	.008161	.01865	25.00	.04624
	.049	.1272	.2925	.01088	.02487	17.85	.06376
	.058	.1489	.2896	.01248	.02853	15.10	.07471
	.065	.1654	.2865	.01365	.03121	13.47	.08306
	.083	.2065	.2815	.01637	.03742	10.54	.10403
	.095	.2328	.2778	.01797	.04107	9.21	.11781
	.120	.2846	.2703	.02079	.04753	7.29	.14491
	.156	.3524	.2601	.02384	.05450	5.61	.18184
1	.028	.08550	.3438	.01011	.02021	35.71	.04277
	.035	.1061	.3414	.01237	.02474	28.56	.05310
	.049	.1464	.3367	.01659	.03319	20.40	.07335
	.058	.1716	.3337	.01911	.03822	17.25	.08604
	.065	.1909	.3314	.02097	.04193	15.38	.09576
	.083	.2391	.3255	.02534	.05068	12.05	.12021
	.095	.2701	.3217	.02796	.05591	10.53	.13607
	.120	.3318	.3140	.03271	.06542	8.33	.16826
	.156	.4142	.3034	.03809	.07618	6.41	.21208
	.187	.4776	.2949	.04155	.08310	5.35	.24720
1 1/8	.028	.09640	.3880	.01453	.02582	40.18	.04822
	.035	.1198	.3856	.01782	.03168	32.10	.05994
	.049	.1656	.3808	.02402	.04270	22.95	.08292
	.058	.1944	.3780	.02775	.04933	19.40	.09742
	.065	.2165	.3755	.03052	.05425	17.30	.10855
	.083	.2717	.3696	.03711	.06597	13.55	.13644
	.095	.3074	.3657	.04111	.07309	11.84	.15460
	.120	.3789	.3578	.04852	.08625	9.38	.19165
1 1/4	.028	.1075	.4321	.02007	.03212	44.64	.05377
	.035	.1336	.4297	.02467	.03948	35.70	.06683
	.049	.1849	.4250	.03339	.05342	25.50	.09255
	.058	.2172	.4219	.03867	.06187	21.55	.10879
	.065	.2420	.4196	.04260	.06816	19.22	.12128
	.083	.3043	.4136	.05206	.08330	15.06	.15267
	.095	.3447	.4097	.05787	.09259	13.16	.17314
	.120	.4260	.4018	.06876	.1100	10.42	.21466
	.156	.5362	.3907	.08184	.1310	8.01	.27177

O.D.	t	A	ρ	I	Z	D/t	A_s
1 3/8	.028	.1185	.4764	.02689	.03911	49.11	.05927
	.035	.1473	.4739	.03309	.04814	39.25	.07368
	.049	.2041	.4691	.04492	.06534	28.05	.10215
	.058	.2400	.4661	.05213	.07583	23.70	.12018
	.065	.2675	.4638	.05753	.08367	21.15	.13400
	.083	.3369	.4577	.07059	.1027	16.57	.16896
	.095	.3820	.4538	.07867	.1144	14.47	.19172
	.120	.4731	.4457	.09400	.1367	11.46	.23804
	.156	.5974	.4345	.1129	.1642	8.81	.3025
	.187	.6979	.4252	.1262	.1835	7.35	.3553
1 1/2	.028	.1295	.5205	.03508	.04678	53.57	.06477
	.035	.1611	.5181	.04324	.05765	42.80	.08058
	.049	.2234	.5132	.05885	.07847	30.60	.1118
	.058	.2628	.5102	.06841	.09121	25.85	.1315
	.065	.2930	.5079	.07558	.1008	23.05	.1467
	.083	.3695	.5018	.09305	.1241	18.08	.1852
	.095	.4193	.4979	.1039	.1386	15.79	.2103
	.120	.5202	.4897	.1248	.1664	12.50	.2614
	.156	.6587	.4784	.1507	.2010	9.62	.3332
	.187	.7714	.4689	.1696	.2261	8.02	.3910
1 5/8	.028	.1405	.5647	.04480	.05514	58.04	.07026
	.035	.1748	.5622	.05528	.06803	46.40	.08743
	.049	.2426	.5575	.07540	.09279	33.15	.1214
	.058	.2855	.5544	.08776	.1080	28.00	.1429
	.065	.3186	.5520	.09707	.1195	25.00	.1595
	.083	.4021	.5459	.1198	.1475	19.58	.2015
	.095	.4566	.5420	.1341	.1651	17.11	.2289
	.120	.5674	.5338	.1617	.1990	13.54	.2849
1 3/4	.028	.1515	.6089	.05616	.06418	62.50	.07577
	.035	.1885	.6065	.06936	.07927	50.00	.09428
	.049	.2618	.6017	.09478	.1083	35.70	.1310
	.058	.3083	.5986	.1105	.1262	30.20	.1543
	.065	.3441	.5962	.1223	.1398	26.90	.1722
	.083	.4347	.5901	.1514	.1730	21.10	.2178
	.095	.4939	.5861	.1697	.1939	18.42	.2476
	.120	.6145	.5778	.2052	.2345	14.58	.3084

TABLE 1·2 (*Continued*)

O.D.	t	A	ρ	I	Z	D/t	A_s
1⅞	.028	.1625	.6531	.06930	.07392	66.96	.08126
	.035	.2023	.6507	.08565	.09136	53.60	.1012
	.049	.2811	.6458	.1172	.1250	38.25	.1406
	.058	.3311	.6427	.1368	.1459	32.30	.1657
	.065	.3696	.6404	.1516	.1617	28.80	.1850
	.083	.4673	.6342	.1880	.2005	22.60	.2340
	.095	.5312	.6302	.2110	.2251	19.74	.2662
	.120	.6616	.6219	.2559	.2730	15.63	.3319
2	.028	.1735	.6973	.08434	.08434	71.43	.08676
	.035	.2161	.6948	.1043	.1043	57.14	.1081
	.049	.3003	.6900	.1430	.1430	40.80	.1502
	.058	.3539	.6869	.1670	.1670	34.45	.1770
	.065	.3951	.6845	.1851	.1851	30.75	.1977
	.083	.4999	.6783	.2300	.2301	24.10	.2503
	.095	.5685	.6744	.2586	.2586	21.05	.2848
	.109	.6476	.6697	.2904	.2904	18.35	.3246
	.120	.7087	.6660	.3144	.3144	16.67	.3554
	.156	.9037	.6543	.3869	.3869	12.82	.4541
	.188	1.070	.6441	.4440	.4440	10.64	.5390
	.250	1.374	.6250	.5369	.5369	8.00	.6963
2⅛	.035	.2298	.7390	.1255	.1181	60.71	.1149
	.049	.3196	.7342	.1723	.1621	43.37	.1599
	.058	.3766	.7311	.2013	.1895	36.64	.1884
	.065	.4207	.7287	.2234	.2102	32.69	.2105
	.083	.5325	.7226	.2780	.2616	25.60	.2665
	.095	.6059	.7185	.3128	.2944	22.37	.3034
	.120	.7559	.7101	.3812	.3588	17.71	.3789
	.156	.9650	.6983	.4706	.4429	13.62	.4846
2¼	.035	.2436	.7832	.1494	.1328	64.29	.1218
	.049	.3388	.7783	.2052	.1824	45.90	.1695
	.058	.3994	.7753	.2401	.2134	38.80	.1998
	.065	.4462	.7728	.2665	.2369	34.60	.2232
	.083	.5651	.7667	.3322	.2953	27.15	.2828
	.095	.6432	.7626	.3741	.3325	23.70	.3221
	.120	.8030	.7543	.4568	.4061	18.75	.4025
	.156	1.026	.7424	.5656	.5028	14.42	.5149
	.188	1.218	.7320	.6526	.5801	11.97	.6124
	.250	1.571	.7126	.7977	.7090	9.00	.7936
2$\frac{5}{16}$	.250	1.620	.7345	.8740	.7559	9.25	.8103

O.D.	t	A	ρ	I	Z	D/t	A_s
2⅜	.049	.3581	.8225	.2423	.2040	48.47	.1791
	.058	.4222	.8194	.2835	.2387	40.95	.2112
	.065	.4717	.8170	.3149	.2651	36.54	.2360
	.083	.5976	.8109	.3930	.3309	28.61	.2991
	.095	.6805	.8068	.4429	.3730	25.00	.3406
	.120	.8501	.7984	.5419	.4563	19.79	.4259
	.156	1.088	.7865	.6727	.5664	15.22	.5459
	.188	1.292	.7761	.7780	.6551	12.63	.6492
	.250	1.669	.7565	.9551	.8043	9.50	.8429
	.375	2.356	.7194	1.220	1.027	6.33	1.2070
2½	.049	.3773	.8667	.2834	.2267	51.00	.1887
	.058	.4450	.8635	.3318	.2655	43.10	.2226
	.065	.4972	.8613	.3688	.2950	38.45	.2487
	.083	.6302	.8550	.4607	.3686	30.10	.3154
	.095	.7178	.8509	.5197	.4158	26.30	.3593
	.120	.8972	.8425	.6369	.5095	20.83	.4495
	.156	1.149	.8306	.7925	.6340	16.03	.5762
	.188	1.366	.8201	.9184	.7347	13.30	.6860
	.250	1.767	.8004	1.132	.9057	10.00	.8908
2¾	.049	.4158	.9551	.3793	.2759	56.10	.2080
	.058	.4905	.9521	.4446	.3233	47.40	.2453
	.065	.5483	.9496	.4944	.3596	42.30	.2743
	.083	.6954	.9434	.6189	.4501	33.15	.3479
	.095	.7924	.9393	.6991	.5084	28.95	.3966
	.120	.9915	.9308	.8590	.6247	22.92	.4965
	.156	1.271	.9188	1.073	.7805	17.63	.6372
	.188	1.513	.9082	1.248	.9078	14.63	.7592
	.250	1.964	.8883	1.549	1.127	11.00	.9885
3	.049	.4543	1.044	.4946	.3298	61.22	.2272
	.058	.5361	1.040	.5802	.3868	51.70	.2681
	.065	.5993	1.038	.6457	.4305	46.20	.2998
	.083	.7606	1.032	.8097	.5398	36.15	.3805
	.095	.8670	1.028	.9156	.6104	31.58	.4338
	.120	1.086	1.019	1.128	.7518	25.00	.5436
	.156	1.394	1.007	1.413	.9423	19.23	.6986
	.188	1.661	.9964	1.649	1.099	15.96	.8330
	.250	2.160	.9763	2.059	1.372	12.00	1.0859
	.375	3.093	.9375	2.718	1.812	8.00	1.5674

TABLE 1·2 (*Continued*)

O.D.	t	A	ρ	I	Z	D/t	A_s
3¼	.049	.4928	1.132	.6313	.3885	66.33	.2464
	.058	.5816	1.129	.7410	.4560	56.10	.2909
	.065	.6504	1.126	.8251	.5077	50.00	.3253
	.083	.8258	1.120	1.036	.6376	39.15	.4131
	.095	.9416	1.116	1.173	.7217	34.20	.4711
	.120	1.180	1.107	1.447	.8906	27.10	.5906
	.156	1.516	1.095	1.819	1.119	20.83	.7594
	.188	1.808	1.085	2.127	1.309	17.29	.9065
	.250	2.356	1.064	2.669	1.643	13.00	1.1834
3½	.065	.7014	1.215	1.035	.5914	53.80	.3508
	.083	.8910	1.208	1.301	.7435	42.20	.4457
	.095	1.016	1.204	1.474	.8422	36.85	.5083
	.120	1.274	1.196	1.822	1.041	29.15	.6376
	.156	1.639	1.184	2.296	1.312	22.44	.8208
	.188	1.956	1.173	2.691	1.538	18.62	.9804
	.250	2.553	1.152	3.390	1.937	14.00	1.2815
3¾	.065	.7525	1.303	1.278	.6814	57.60	.3763
	.083	.9562	1.297	1.608	.8576	45.20	.4783
	.095	1.091	1.293	1.823	.9722	39.50	.5457
	.120	1.368	1.284	2.256	1.204	31.25	.6845
	.156	1.761	1.272	2.849	1.520	24.04	.8817
	.188	2.104	1.261	3.346	1.784	19.95	1.0541
	.219	2.429	1.251	3.801	2.027	17.12	1.2180
	.250	2.749	1.241	4.231	2.256	15.00	1.3792
4	.065	.8035	1.392	1.556	.7779	61.50	.4018
	.083	1.021	1.385	1.960	.9799	48.20	.5107
	.095	1.166	1.381	2.223	1.111	42.10	.5833
	.120	1.463	1.372	2.755	1.378	33.33	.7320
	.156	1.884	1.360	3.485	1.743	25.64	.9430
	.188	2.251	1.349	4.099	2.050	21.28	1.1276
	.219	2.601	1.339	4.664	2.332	18.26	1.3038
	.250	2.945	1.329	5.200	2.600	16.00	1.4770
	.313	3.626	1.308	6.205	3.102	6.39	1.8576
4¼	.065	.8546	1.480	1.871	.8806	65.38	.4274
	.083	1.087	1.474	2.359	1.110	51.20	.5437
	.095	1.240	1.469	2.677	1.260	44.74	.6202

O.D.	t	A	ρ	I	Z	D/t	A_s
4¼	.120	1.557	1.461	3.322	1.564	35.42	.7789
	.134	1.733	1.456	3.673	1.741	31.75	.8672
	.156	2.006	1.448	4.210	1.981	27.24	1.0040
	.188	2.399	1.438	4.959	2.334	22.60	1.2014
	.219	2.773	1.427	5.650	2.659	19.40	1.3896
	.250	3.142	1.417	6.308	2.968	17.00	1.5753
	.313	3.871	1.396	7.548	3.552	13.58	1.9439
4½	.065	.9056	1.568	2.227	.9898	69.23	.4529
	.083	1.152	1.562	2.810	1.249	54.22	.5761
	.095	1.315	1.558	3.190	1.418	47.37	.6577
	.120	1.651	1.549	3.963	1.761	37.50	.8259
	.156	2.129	1.537	5.028	2.235	28.80	1.0654
	.188	2.547	1.526	5.930	2.636	23.94	1.2753
	.219	2.945	1.516	6.765	3.007	20.55	1.4755
	.250	3.338	1.505	7.562	3.361	18.00	1.6732
	.313	4.117	1.484	9.073	4.032	14.38	2.0662
	.375	4.860	1.464	10.42	4.632	12.00	2.4433
4¾	.065	.9567	1.657	2.625	1.105	73.08	.4784
	.083	1.217	1.650	3.314	1.396	57.23	.6087
	.095	1.389	1.646	3.765	1.585	50.00	.6947
	.120	1.745	1.638	4.680	1.971	39.58	.8729
	.156	2.251	1.625	5.946	2.504	30.45	1.1264
	.188	2.694	1.614	7.021	2.956	25.25	1.3486
	.219	3.117	1.604	8.019	3.376	21.69	1.5612
	.250	3.534	1.593	8.974	3.778	19.00	1.7711
	.313	4.363	1.573	10.79	4.543	15.18	2.1889
	.375	5.154	1.552	12.42	5.230	12.66	2.5897
5	.083	1.282	1.739	3.876	1.550	60.24	.6411
	.095	1.464	1.734	4.404	1.762	52.63	.7322
	.120	1.840	1.726	5.480	2.192	41.67	.9204
	.156	2.374	1.714	6.970	2.788	32.05	1.1879
	.188	2.842	1.703	8.239	3.295	26.60	1.4226
	.219	3.289	1.692	9.418	3.767	22.83	1.6470
	.250	3.731	1.682	10.55	4.220	20.00	1.8692
	.313	4.609	1.661	12.71	5.085	15.97	2.3114
	.375	5.449	1.640	14.66	5.866	13.33	2.7364

TABLE 1·3

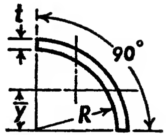
PROPERTIES OF ROUND BAR

This table gives the most commonly used section properties of round bar. O.D. refers to outside diameter, Z is section modulus. A_s is equivalent shear area (divide transverse shear load by A_s to obtain maximum shear stress at neutral axis).

O.D.	A	ρ	I	Z	A_s
1/8	.01227	.03125	.000012	.000192	.00920
1/4	.04909	.06250	.000192	.001534	.03682
3/8	.1104	.09375	.000971	.005177	.08280
1/2	.1963	.1250	.003068	.01227	.1472
5/8	.3068	.1562	.007490	.02397	.2301
3/4	.4418	.1875	.01553	.04142	.3314
7/8	.6013	.2188	.02877	.06577	.4510
1	.7854	.2500	.04909	.09817	.5891
1 1/4	1.227	.3125	.1198	.1917	.9203
1 1/2	1.767	.3750	.2485	.3313	1.3253
1 3/4	2.405	.4375	.4604	.5262	1.8038
2	3.142	.5000	.7854	.7854	2.3565
2 1/4	3.976	.5625	1.258	1.118	2.982
2 1/2	4.909	.6250	1.917	1.534	3.6818
2 3/4	5.940	.6875	2.807	2.042	4.4550
3	7.069	.7500	3.976	2.651	5.3018
3 1/2	9.621	.8750	7.366	4.209	7.2158
4	12.57	1.000	12.57	6.283	9.4275
4 1/2	15.90	1.125	20.13	8.946	11.9250
5	19.63	1.250	30.68	12.27	14.7225

TABLE 1·4

Areas of 90° Bends

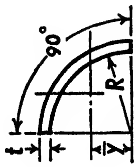


$\bar{y}$ = centroid
t = sheet thickness
R = radius of bend

t	Radius of Bend													t
	$\frac{1}{8}$	$\frac{5}{32}$	$\frac{3}{16}$	$\frac{7}{32}$	$\frac{1}{4}$	$\frac{9}{32}$	$\frac{5}{16}$	$\frac{11}{32}$	$\frac{3}{8}$	$\frac{13}{32}$	$\frac{7}{16}$	$\frac{15}{32}$	$\frac{1}{2}$	
.020	.004	.005	.006	.007	.008	.009	.010	.011	.012	.013	.014	.015	.016	.020
.025	.005	.007	.008	.009	.010	.012	.013	.014	.015	.016	.017	.019	.020	.025
.032	.007	.009	.010	.012	.013	.015	.017	.018	.019	.021	.022	.024	.026	.032
.040	.009	.011	.013	.015	.017	.019	.021	.023	.025	.027	.029	.031	.032	.040
.051	.012	.015	.017	.020	.022	.025	.027	.029	.032	.034	.037	.039	.042	.051
.064	.016	.019	.022	.025	.028	.032	.035	.038	.041	.044	.047	.050	.053	.064
.072	.018	.022	.025	.029	.032	.036	.040	.043	.047	.050	.054	.057	.061	.072
.081	.021	.025	.029	.033	.037	.041	.045	.049	.053	.057	.061	.065	.069	.081
.091	.024	.029	.033	.038	.042	.046	.051	.056	060	.065	.069	.073	.078	.091
.102	.028	.033	.038	.043	.048	.053	.058	.064	.068	.073	.078	.083	.088	.102
.125	.037	.043	.048	.056	.062	.069	.076	.082	.087	.094	.100	.106	.113	.125
.156	.050	.057	.064	.072	.080	.088	.095	.102	.111	.118	.126	.133	.141	.156
.1875	.064	.073	.082	.091	.100	.110	.119	.129	.137	.146	.155	.164	.174	.1875
.250	.100	.112	.123	.136	.148	.161	.173	.185	.196	.208	.221	.232	.245	.250

TABLE 1·5

Centroids of 90° Bends



$\bar{y}$ = centroid

t = sheet thickness

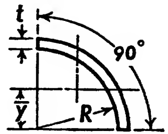
R = radius of bend

t	Radius of Bend 1/8	5/32	3/16	7/32	1/4	9/32	5/16	11/32	3/8	13/32	7/16	15/32	1/2	t
.020	.086	.106	.126	.146	.166	.185	.205	.225	.245	.265	.285	.305	.325	.020
.025	.088	.108	.128	.147	.167	.187	.207	.227	.247	.267	.287	.306	.326	.025
.032	.090	.110	.130	.150	.170	.190	.209	.229	.249	.269	.289	.308	.329	.032
.040	.093	.113	.133	.153	.172	.192	.212	.232	.252	.271	.291	.311	.331	.040
.051	.096	.116	.136	.156	.176	.196	.216	.236	.255	.275	.295	.315	.335	.051
.064	.101	.121	.141	.161	.180	.200	.220	.240	.260	.280	.299	.319	.339	.064
.072	.104	.124	.144	.163	.183	.203	.223	.242	.263	.282	.302	.322	.342	.072
.081	.108	.127	.147	.167	.186	.206	.226	.245	.266	.285	.305	.324	.345	.081
.091	.111	.130	.150	.170	.189	.209	.229	.249	.269	.288	.308	.328	.348	.091
.102	.115	.135	.154	.173	.194	.212	.234	.253	.272	.293	.312	.331	.352	.102
.125	.124	.143	.163	.181	.201	.221	.241	.260	.280	.301	.320	.339	.360	.125
.156		.154	.173	.193	.212	.232	.251	.271	.290	.310	.331	.349	.370	.156
.1875			.186	.204	.225	.244	.263	.281	.302	.323	.341	.362	.383	.1875
.250					.248	.265	.295	.305	.325	.345	.363	.385	.403	.250

TABLE 1·6

MOMENTS OF INERTIA OF 90° BENDS

Multiply all values in this table by 0.00001



$\bar{y}$ = centroid
t = sheet thickness
R = radius of bend

t	Radius of Bend											t
	$\frac{1}{8}$	$\frac{5}{32}$	$\frac{3}{16}$	$\frac{7}{32}$	$\frac{1}{4}$	$\frac{9}{32}$	$\frac{5}{16}$	$\frac{11}{32}$	$\frac{3}{8}$	$\frac{13}{32}$	$\frac{7}{16}$	
.020	.578	1.13	1.95	3.10	4.63	6.59	9.03	12.0	15.6	19.8	24.8	.020
.025	.722	1.41	2.44	3.88	5.78	8.23	11.3	15.0	19.5	24.8	31.0	.025
.032	.925	1.81	3.12	4.96	7.40	10.5	14.4	19.2	25.0	31.8	39.6	.032
.040	1.16	2.26	3.90	6.20	9.25	13.2	18.1	24.0	31.2	39.7	49.6	.040
.051	1.47	2.88	4.98	7.90	11.8	16.8	23.0	30.7	39.8	50.6	63.2	.051
.064	1.85	3.61	6.24	9.92	14.8	21.1	28.9	38.5	50.0	63.5	79.3	.064
.072	2.08	4.06	7.02	11.2	16.7	23.7	32.5	43.3	56.2	71.4	89.2	.072
.081	2.34	4.57	7.90	12.6	18.7	26.7	36.6	48.7	63.2	80.4	100.	.081
.091	2.63	5.14	8.88	14.1	21.0	30.0	41.1	54.7	71.0	90.3	113.	.091
.102	2.95	5.76	9.95	15.8	23.6	33.6	46.1	61.3	79.6	101.	126.	.102
.125	3.61	7.06	12.2	19.4	28.9	41.2	56.5	75.2	97.6	124.	155.	.125
.156	4.51	8.81	15.2	24.2	36.1	51.4	70.5	93.8	122.	155.	193.	.156
.188	5.42	10.6	18.3	29.1	43.4	61.7	84.7	113.	146.	186.	232.	.188
.250	7.22	14.1	24.4	38.8	57.8	82.3	113.	150.	195.	248.	310.	.250

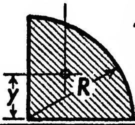
TABLE 1·6 (*Continued*)
Multiply all values by 0.00001

t	$\frac{15}{32}$	$\frac{1}{2}$	$\frac{17}{32}$	$\frac{9}{16}$	$\frac{19}{32}$	$\frac{5}{8}$	$\frac{21}{32}$	$\frac{11}{16}$	$\frac{23}{32}$	$\frac{3}{4}$	$\frac{25}{32}$	t
.020	30.5	37.0	44.4	52.7	62.0	72.3	83.6	96.2	110.	125.	141.	.020
.025	38.1	46.2	55.5	65.8	77.4	90.3	105.	120.	137.	156.	176.	.025
.032	48.8	59.2	71.0	84.3	99.1	116.	134.	154.	176.	200.	226.	.032
.040	61.0	74.0	88.8	105.	124.	144.	167.	192.	220.	250.	282.	.040
.051	77.7	94.4	113.	134.	158.	184.	213.	245.	280.	318.	360.	.051
.064	97.5	118.	142.	169.	198.	231.	268.	308.	352.	400.	452.	.064
.072	110.	133.	160.	190.	223.	260.	301.	346.	396.	450.	508.	.072
.081	123.	150.	180.	213.	251.	293.	339.	390.	445.	506.	572.	.081
.091	139.	168.	202.	240.	282.	329.	381.	438.	500.	568.	642.	.091
.102	155.	189.	226.	269.	316.	368.	427.	491.	560.	637.	720.	.102
.125	190.	231.	277.	329.	387.	452.	523.	601.	687.	780.	882.	.125
.156	238.	289.	346.	411.	483.	564.	652.	750.	857.	974.	1.100.	.156
.188	286.	347.	416.	494.	581.	677.	784.	902.	1,030.	1,170.	1.320.	.188
.250	381.	462.	555.	658.	774.	903.	1,050.	1,200.	1,370.	1,560.	1,760.	.250

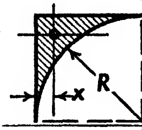
t	$\frac{13}{16}$	$\frac{27}{32}$	$\frac{7}{8}$	$\frac{29}{32}$	$\frac{15}{16}$	$\frac{31}{32}$	1	$1\frac{1}{4}$	$1\frac{1}{2}$	$1\frac{3}{4}$	2	t
.020	159.	178.	198.	220.	244.	269.	296.	578.	999.	1,590.	2,370.	.020
.025	198.	222.	248.	276.	305.	336.	370.	722.	1,250.	1,980.	2,960.	.025
.032	254.	284.	317.	353.	390.	430.	474.	925.	1,600.	2,540.	3,790.	.032
.040	318.	356.	397.	441.	488.	538.	592.	1,160.	2,000.	3,170.	4,740.	.040
.051	405.	453.	506.	562.	622.	686.	755.	1,470.	2,550.	4,040.	6,040.	.051
.064	508.	569.	635.	705.	781.	861.	947.	1,850.	3,200.	5.080.	7,580.	.064
.072	572.	640.	714.	793.	878.	968.	1,070.	2,080.	3.600.	5,710.	8,520.	.072
.081	643.	720.	803.	893.	988.	1,090.	1,200.	2,340.	4,050.	6,420.	9,590.	.081
.091	722.	809.	902.	1,000.	1,110.	1,220.	1,350.	2,630.	4,540.	7,220.	10,800.	.091
.102	810.	907.	1,010.	1,120.	1,240.	1,370.	1,510.	2,950.	5,100.	8,090.	12,100.	.102
.125	992.	1,110.	1,240.	1,380.	1,520.	1.680.	1,850.	3,610.	6,240.	9,910.	14,800.	.125
.156	1,240.	1,390.	1.550.	1,720.	1,900.	2,100.	2.310.	4.510.	7,790.	12,400.	18,500.	.156
.188	1,490.	1,670.	1,860.	2,070.	2,290.	2,520.	2,780.	5,420.	9,370.	14,900.	22,200.	.188
.250	1,980.	2,220.	2,480.	2,760.	3,050.	3,360.	3,700.	7,220.	12,500.	19,800.	29,600.	.250

TABLE 1·7

Areas and Centroids of Rounded and Filleted Corners



$A = 0.7854R^2$
$y = 0.4244R$
$I = 0.0549R^4$



$A = 0.2146R^2$
$x = 0.2234R$
$I = 0.0075R^4$

Radius: Decimal Equivalents	Radius: Fractions of inch: 8	16	32	64	Rounded corner: Area	Rounded corner: *y*	Filleted corner: Area	Filleted corner: *x*
.016				1	.00019	.00663	.00005	.00349
.031			1		.00077	.01326	.00021	.00698
.047				3	.00173	.01989	.00047	.01047
.063		1			.00307	.02653	.00084	.01396
.078				5	.00479	.03316	.00131	.01745
.094			3		.00690	.03979	.00189	.02094
.109				7	.00940	.04642	.00257	.02443
.125	1				.01227	.05305	.00335	.02793
.141				9	.01553	.05969	.00424	.03149
.156			5		.01918	.06631	.00524	.03491
.172				11	.02320	.07295	.00634	.03840
.188		3			.02761	.07958	.00754	.04189
.203				13	.03241	.08621	.00885	.04538
.219			7		.03758	.09284	.01027	.04887
.234				15	.04314	.09947	.01179	.05236
.250	2				.04909	.10610	.01341	.05585
.266				17	.05542	.11274	.01514	.05934
.281			9		.06213	.11937	.01698	.06283
.297				19	.06922	.12600	.01891	.06632
.313		5			.07670	.13263	.02096	.06981
.328				21	.08456	.13926	.02310	.07330
.344			11		.09281	.14589	.02536	.07679
.359				23	.10143	.15252	.02772	.08028
.375	3				.11045	.15915	.03018	.08378
.391				25	.11984	.16579	.03275	.08727
.406			13		.12962	.17242	.03542	.09076
.422				27	.13978	.17905	.03819	.09425
.438		7			.15033	.18568	.04108	.09774
.453				29	.16126	.19231	.04406	.10123
.469			15		.17257	.19894	.04715	.10472
.484				31	.18427	.20558	.05035	.10821
.500	4				.19635	.21220	.05365	.11170
.563		9			.24850	.23870	.06790	.12565
.625	5				.30680	.26530	.08383	.13961
.688		11			.37120	.29180	.10143	.15357
.750	6				.44180	.31830	.12071	.16753
.813		13			.51850	.34480	.14167	.18149
.875	7				.60130	.37140	.16430	.19545
.938		15			.69030	.39790	.18862	.20940
1.000	8				.78540	.42440	.21460	.22340

TABLE 2·1

Material Properties[1]—Metals

	Low-Carbon Steel	Alloy Steel (Aircraft Type)	Alloy Steel Heat Treated to 150,000 psi U.T.S.[5]	Aluminum Alloy (High-Strength Sheet)	Magnesium Alloy (High-Strength Sheet)
Modulus of elasticity, psi	28,000,000	29,000,000	29,000,000	10,000,000	6,500,000
Ultimate tension stress, psi	55,000	90,000	150,000	68,000	40,000
Tension yield stress,[2] psi	36,000	70,000	135,000	50,000	32,000
Ultimate shear stress, psi	35,000	55,000	90,000	42,000	20,000
Ultimate bearing stress,[3] psi	90,000	140,000	190,000	98,000	60,000
Endurance limit,[4] psi	25,000	45,000	78,000	18,000	12,000
Specific weight, lb/cu in.	0.2833	0.2833	0.2833	0.100	0.065

[1] These values are representative of several classes of materials, but do not necessarily agree exactly with government specifications which are subject to change. In actual design work the proper working stresses should be obtained for the particular material being used. Stresses listed here are minimum values likely to be obtained for the type of material noted, but have not been further reduced to provide any additional factor of safety.

[2] Stress at 0.002 permanent set.

[3] Nominal value, based on small deformation of hole or pin.

[4] For complete stress reversal. Based on bending tests of 300,000,000 cycles or more.

[5] Typical heat treatment for high-strength steel parts. Lower and higher values also used.

TABLE 2·2

Material Properties[1]—Wood

	Hardwoods (White Ash, Beech, Birch, Hickory, Maple, Oak)	Softwoods (Pine, Cypress, Fir)	Softwoods (Spruce, Hemlock, Cedar)
Modulus of elasticity, psi	1,600,000	1,500,000	1,200,000
Modulus of rupture (bending), psi	15,000	11,000	9,000
Ultimate shear stress, psi (parallel to grain)	1,400	800	700
Ultimate compression stress, psi (parallel to grain)	7,000	6,000	5,000
Ultimate compression stress, psi (perpendicular to grain)	1,800	1,100	900
Specific weight, lb/cu in.	0.026	0.019	0.015

[1] Mechanical properties of woods vary widely and are affected by moisture content, duration of loading, and other factors. The values listed are representative of several types of wood and may be used for approximate calculations. More complete data may be found in engineering handbooks and structural texts, for instance, *Modern Timber Design*, by Howard J. Hansen (Wiley, 1943).

Non-Dimensional Beam Curves

The following tables contain useful information on different types of beams under various loading conditions. The data have been converted into non-dimensional form by using unit values for distances and loadings. To convert back into dimensional form the values given in the curves or table should be multiplied by certain quantities, indicated in the group of equations at the top of each table. The center group of equations simply indicates the basic form of the non-dimensional curves.

In working with a beam of a certain span it will be found convenient to divide this length into 10 equal divisions and to use the values given in the lower right-hand table.

The particular conventions shown have been adopted in order to obtain consistency in the equations and charts.

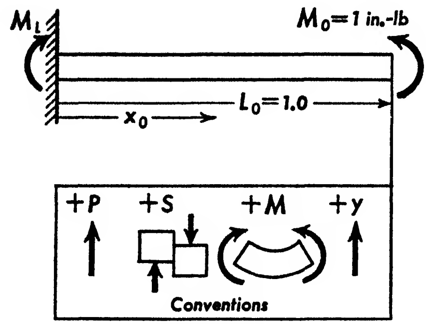
For example, if it is desired to find the deflection of a 20-ft beam at a distance 8 ft from the left end the value of x_0 would be $x/L = 8/20 = 0.40$. For a simply supported beam with uniform loading Table 3·6 would be used. The value y_0 would be found, from the lower graph or table, to be equal to 0.0125. This would be substituted in the equation for y, in the upper right-hand set of equations, together with the known values of loading (w), length (240 in.), modulus of elasticity (E), and moment of inertia (I).

TABLE 3·1

Beam Curves

(Non-dimensional)

Cantilever Beam—Constant Moment



$x = x_0 L$

$R_L = 0$

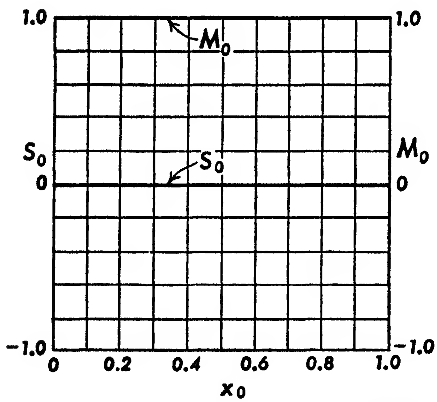
$M_L = M_0$

$S = S_0 M$

$M = M_0 M$

$\theta = \theta_0 \frac{ML}{EI}$

$y = y_0 \frac{ML^2}{EI}$

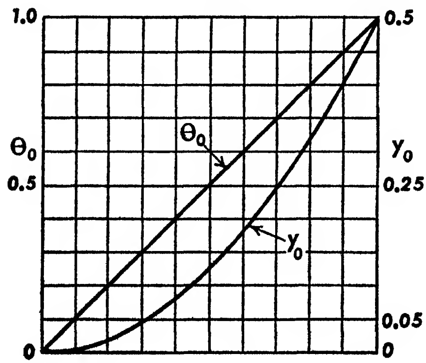


$S_0 = 0$

$M_0 = 1$

$\theta_0 = x_0$

$y_0 = \frac{x_0^2}{2}$



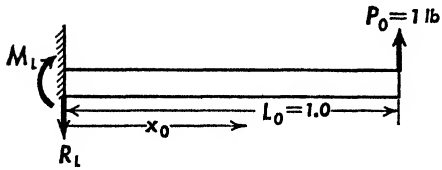
x_0	S_0	M_0	θ_0	y_0
0	0	1.0	0	0
0.1	↑	↑	0.1	0.005
0.2			0.2	0.020
0.3			0.3	0.045
0.4			0.4	0.080
0.5			0.5	0.125
0.6			0.6	0.180
0.7			0.7	0.245
0.8			0.8	0.320
0.9	↓	↓	0.9	0.405
1.0	0	1.0	1.0	0.500

TABLE 3·2

BEAM CURVES

(Non-dimensional)

CANTILEVER BEAM—CONCENTRATED LOAD



$x = x_0 L$

$R_L = -P$

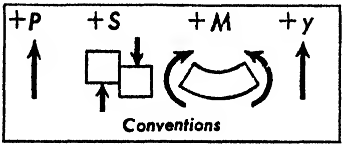
$M_L = PL$

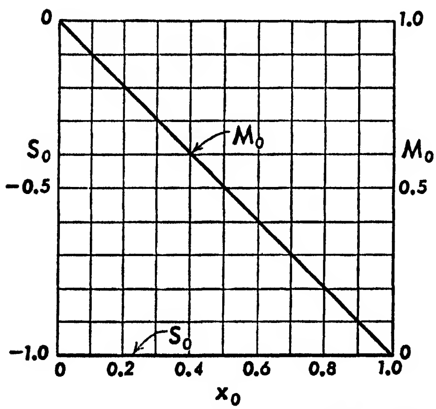
$S = S_0 P$

$M = M_0 PL$

$\Theta = \Theta_0 \frac{PL^2}{EI}$

$y = y_0 \frac{PL^3}{EI}$



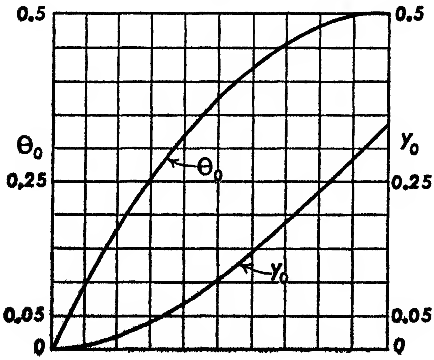


$S_0 = -1$

$M_0 = 1 - x_0$

$\Theta_0 = x_0 - \frac{x_0^2}{2}$

$y_0 = \frac{x_0^2}{2} - \frac{x_0^3}{6}$



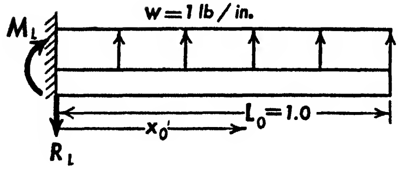
x_0	S_0	M_0	Θ_0	y_0
0	−1.0	1.0	0	0
0.1	↑	0.9	0.095	0.0048
0.2		0.8	0.180	0.0187
0.3		0.7	0.255	0.0405
0.4		0.6	0.320	0.0693
0.5		0.5	0.375	0.1044
0.6		0.4	0.420	0.1440
0.7		0.3	0.455	0.1878
0.8		0.2	0.480	0.2345
0.9	↓	0.1	0.495	0.2835
1.0	−1.0	0	0.500	0.3333

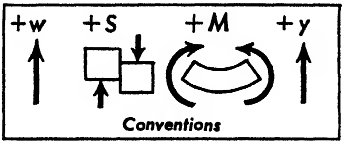
TABLE 3·3

Beam Curves

(Non-dimensional)

Cantilever Beam—Uniform Running Load





$$x = x_0 L$$

$$R_L = -wL$$

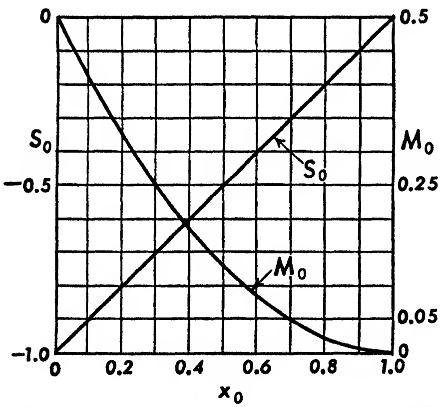
$$M_L = \frac{wL^2}{2}$$

$$S = S_0 wL$$

$$M = M_0 wL^2$$

$$\Theta = \Theta_0 \frac{wL^3}{EI}$$

$$y = y_0 \frac{wL^4}{EI}$$

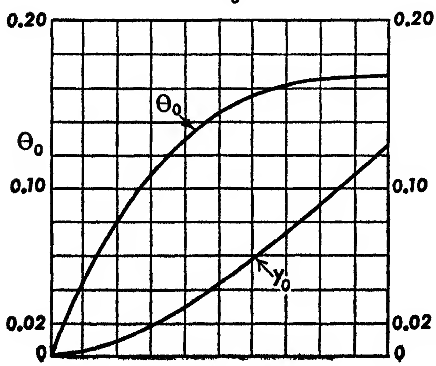


$$S_0 = -1 + x_0$$

$$M_0 = \frac{1}{2} - x_0 + \frac{x_0^2}{2}$$

$$\Theta_0 = \frac{x_0}{2} - \frac{x_0^2}{2} + \frac{x_0^3}{6}$$

$$y_0 = \frac{x_0^2}{4} - \frac{x_0^3}{6} + \frac{x_0^4}{24}$$

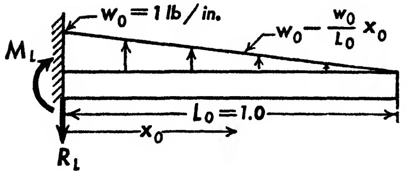


x_0	S_0	M_0	Θ_0	y_0
0	−1.0	0.500	0	0
0.1	−0.9	0.405	0.0451	0.0023
0.2	−0.8	0.320	0.0813	0.0088
0.3	−0.7	0.245	0.1095	0.0183
0.4	−0.6	0.180	0.1306	0.0304
0.5	−0.5	0.125	0.1458	0.0443
0.6	−0.4	0.080	0.1560	0.0594
0.7	−0.3	0.045	0.1623	0.0753
0.8	−0.2	0.020	0.1653	0.0918
0.9	−0.1	0.005	0.1665	0.1083
1.0	0	0	0.1667	0.1250

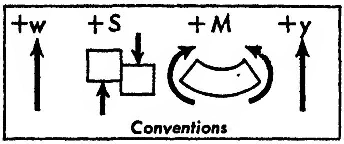
TABLE 3·4

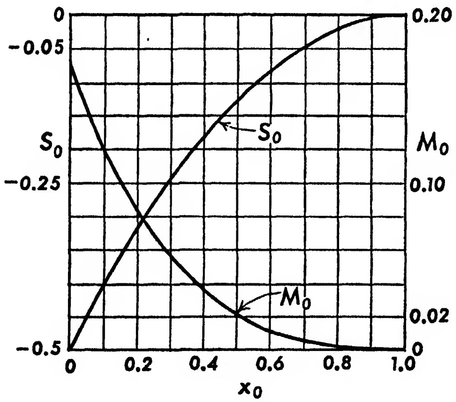
Beam Curves
(Non-dimensional)

Cantilever Beam—Uniformly Decreasing Load

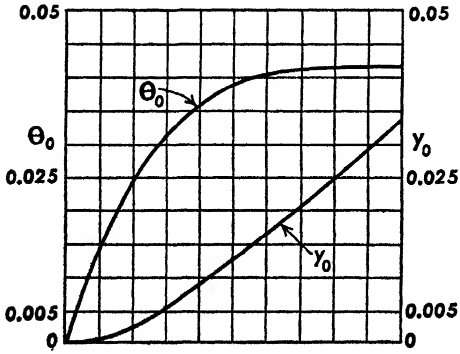


$$x = x_0 L$$
$$R_L = -\frac{wL}{2}$$
$$M_L = \frac{wL^2}{6}$$
$$S = S_0 wL$$
$$M = M_0 wL^2$$
$$\Theta = \Theta_0 \frac{wL^3}{EI}$$
$$y = y_0 \frac{wL^4}{EI}$$





$$S_0 = -\frac{1}{2} + x_0 - \frac{x_0^2}{2}$$
$$M_0 = \frac{1}{6} - \frac{x_0}{2} + \frac{x_0^2}{2} - \frac{x_0^3}{6}$$
$$\Theta_0 = \frac{x_0}{6} - \frac{x_0^2}{4} + \frac{x_0^3}{6} - \frac{x_0^4}{24}$$
$$y_0 = \frac{x_0^2}{12} - \frac{x_0^3}{12} + \frac{x_0^4}{24} - \frac{x_0^5}{120}$$

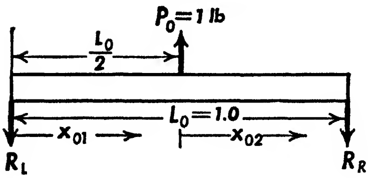


x_0	S_0	M_0	Θ_0	y_0
0	−0.500	0.1667	0	0
0.1	−0.405	0.1215	0.0144	0.0007
0.2	−0.320	0.0854	0.0245	0.0027
0.3	−0.245	0.0572	0.0317	0.0056
0.4	−0.180	0.0360	0.0363	0.0090
0.5	−0.125	0.0209	0.0390	0.0127
0.6	−0.080	0.0107	0.0406	0.0168
0.7	−0.045	0.0045	0.0414	0.0208
0.8	−0.020	0.0014	0.0415	0.0250
0.9	−0.005	0.0002	0.0417	0.0292
1.0	0	0	0.0417	0.0334

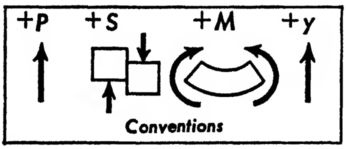
TABLE 3·5

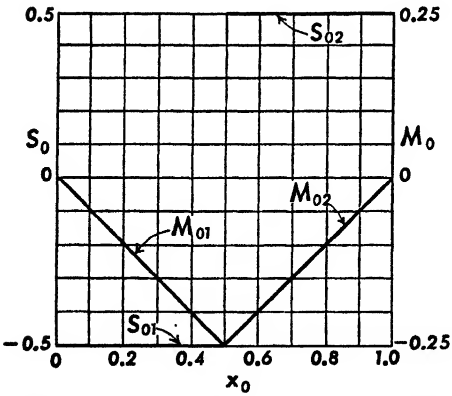
Beam Curves
(Non-dimensional)

Simple Beam—Concentrated Load at Mid-Point

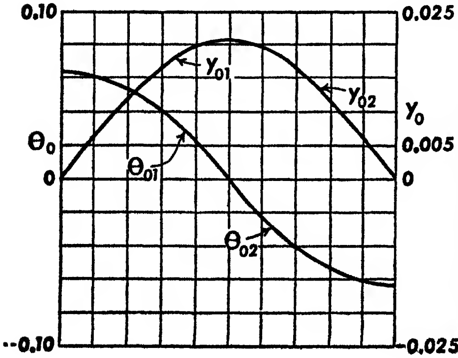


$$x = x_0 L$$
$$R_L = -\frac{P}{2}$$
$$R_R = -\frac{P}{2}$$
$$S = S_0 P$$
$$M = M_0 PL$$
$$\Theta = \Theta_0 \frac{PL^2}{EI}$$
$$y = y_0 \frac{PL^3}{EI}$$





$$S_{01} = -\frac{1}{2}$$
$$S_{02} = +\frac{1}{2}$$
$$M_{01} = -\frac{x_{01}}{2}$$
$$M_{02} = -\frac{1}{4} + \frac{x_{02}}{2}$$
$$\Theta_{01} = \frac{1}{16} - \frac{x_{01}^2}{4}$$
$$\Theta_{02} = -\frac{x_{02}}{4} + \frac{x_{02}^2}{4}$$
$$y_{01} = \frac{x_{01}}{16} - \frac{x_{01}^3}{12}$$
$$y_{02} = \frac{1}{48} - \frac{x_{02}^2}{8} + \frac{x_{02}^3}{12}$$

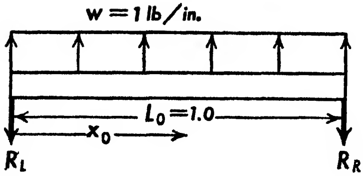


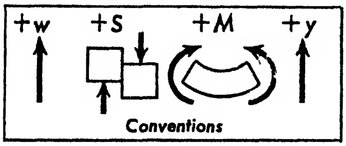
x_0	S_0	M_0	Θ_0	y_0
0	−0.5	0	0.0625	0
0.1		−0.05	0.0600	0.0062
0.2		−0.10	0.0525	0.0118
0.3		−0.15	0.0400	0.0166
0.4		−0.20	0.0225	0.0198
0.5	−0.5 +0.5	−0.25	0	0.0209
0.6		−0.20	−0.0225	0.0198
0.7		−0.15	−0.0400	0.0166
0.8		−0.10	−0.0525	0.0118
0.9		−0.05	−0.0600	0.0062
1.0	+0.5	0	−0.0625	0

TABLE 3·6

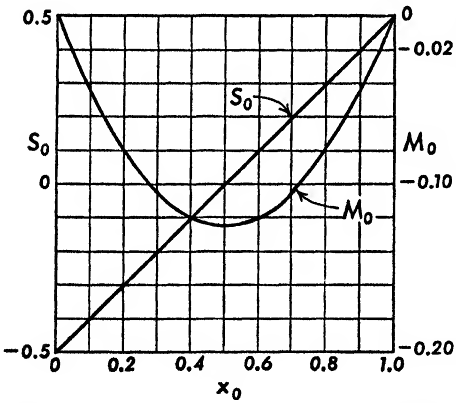
Beam Curves
(Non-dimensional)

Simple Beam—Uniform Running Load

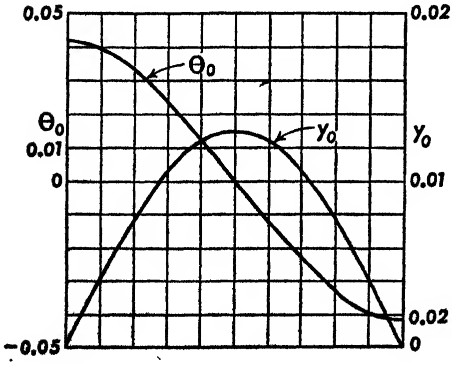




$$x = x_0 L$$
$$R_L = -\frac{wL}{2}$$
$$R_R = -\frac{wL}{2}$$
$$S = S_0 wL$$
$$M = M_0 wL^2$$
$$\Theta = \Theta_0 \frac{wL^3}{EI}$$
$$y = y_0 \frac{wL^4}{EI}$$



$$S_0 = -\frac{1}{2} + x_0$$
$$M_0 = -\frac{x_0}{2} + \frac{x_0^2}{2}$$
$$\Theta_0 = \frac{1}{24} - \frac{x_0^2}{4} + \frac{x_0^3}{6}$$
$$y_0 = \frac{x_0}{24} - \frac{x_0^3}{12} + \frac{x_0^4}{24}$$



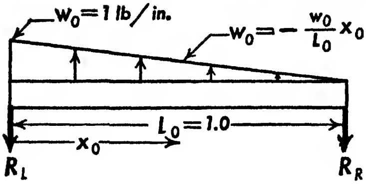
x_0	S_0	M_0	Θ_0	y_0
0	−0.5	0	0.0416	0
0.1	−0.4	−0.045	0.0394	0.0041
0.2	−0.3	−0.080	0.0330	0.0077
0.3	−0.2	−0.105	0.0236	0.0106
0.4	−0.1	−0.120	0.0124	0.0125
0.5	0	−0.125	0	0.0130
0.6	+0.1	−0.120	−0.0124	0.0125
0.7	0.2	−0.105	−0.0236	0.0106
0.8	0.3	−0.080	−0.0330	0.0077
0.9	0.4	−0.045	−0.0394	0.0041
1.0	0.5	0	−0.0416	0

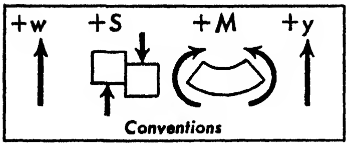
TABLE 3·7

Beam Curves

(Non-dimensional)

Simple Beam—Uniformly Decreasing Load





$$x = x_0 L$$

$$R_L = -\frac{wL}{3}$$

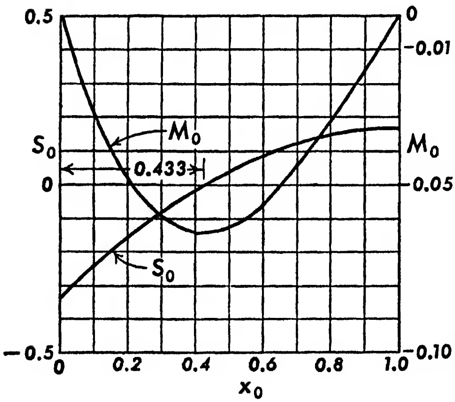
$$R_R = -\frac{wL}{6}$$

$$S = S_0 wL$$

$$M = M_0 wL^2$$

$$\Theta = \Theta_0 \frac{wL^3}{EI}$$

$$y = y_0 \frac{wL^4}{EI}$$

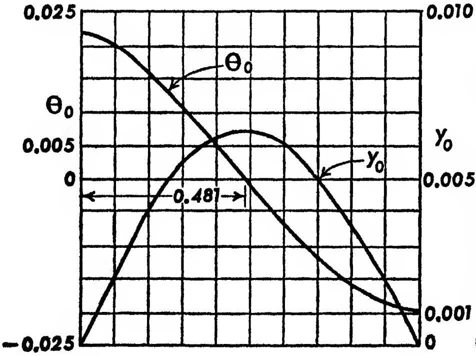


$$S_0 = -\frac{1}{3} + x_0 - \frac{x_0^2}{2}$$

$$M_0 = -\frac{x_0}{3} + \frac{x_0^2}{2} - \frac{x_0^3}{6}$$

$$\Theta_0 = \frac{1}{45} - \frac{x_0^2}{6} + \frac{x_0^3}{6} - \frac{x_0^4}{24}$$

$$y_0 = \frac{x_0}{45} - \frac{x_0^3}{18} + \frac{x_0^4}{24} - \frac{x_0^5}{120}$$

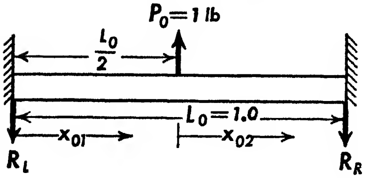


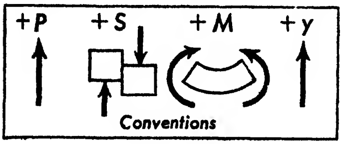
x_0	S_0	M_0	Θ_0	y_0
0	−0.3333	0	0.0222	0
0.1	−0.2383	−0.0285	0.0207	0.0021
0.2	−0.1533	−0.0480	0.0167	0.0041
0.3	−0.0783	−0.0595	0.0114	0.0055
0.4	−0.0133	−0.0640	+0.0051	0.0063
0.5	+0.0417	−0.0624	−0.0013	0.0064
0.6	0.0867	−0.0560	−0.0072	0.0061
0.7	0.1217	−0.0455	−0.0123	0.0050
0.8	0.1467	−0.0319	−0.0163	0.0037
0.9	0.1617	−0.0165	−0.0186	0.0019
1.0	0.1667	0	−0.0195	0

TABLE 3·8

Beam Curves
(Non-dimensional)

Fixed-End Beam—Concentrated Load at Mid-Point





$x = x_0 wL$

$R_L = -\frac{P}{2}$

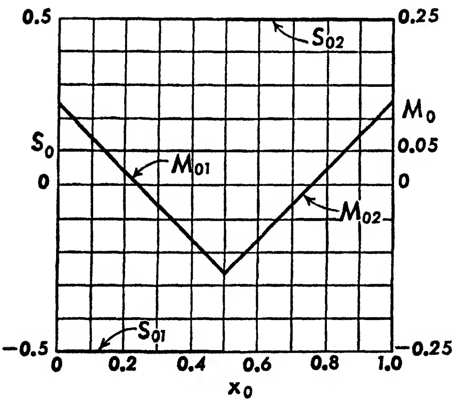
$R_R = -\frac{P}{2}$

$S = S_0 P$

$M = M_0 PL$

$\theta = \theta_0 \frac{PL^2}{EI}$

$y = y_0 \frac{PL^3}{EI}$



$S_{01} = -\frac{P}{2}$

$S_{02} = +\frac{P}{2}$

$M_{01} = \frac{1}{8} - \frac{x_{01}}{2}$

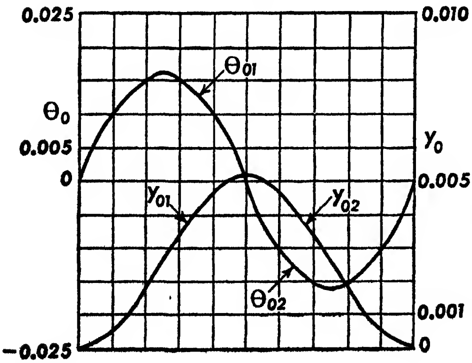
$M_{02} = -\frac{1}{8} + \frac{x_{02}}{2}$

$\theta_{01} = \frac{x_{01}}{8} - \frac{x_{01}^2}{4}$

$\theta_{02} = -\frac{x_{02}}{8} + \frac{x_{02}^2}{4}$

$y_{01} = \frac{x_{01}^2}{16} - \frac{x_{01}^3}{12}$

$y_{02} = \frac{1}{192} - \frac{x_{02}^2}{16} + \frac{x_{02}^3}{12}$



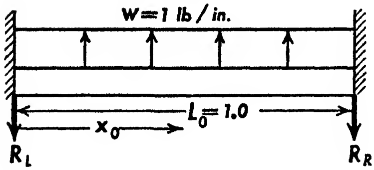
x_0	S_0	M_0	θ_0	y_0
0	−0.5	0.125	0	0
0.1	↑	0.075	0.0100	0.0005
0.2		+0.025	0.0150	0.0018
0.3		−0.025	0.0150	0.0034
0.4	↓	−0.075	0.0100	0.0047
0.5	−0.5 +0.5	−0.125	0	0.0052
0.6	↑	−0.075	−0.0100	0.0047
0.7		−0.025	−0.0150	0.0034
0.8		+0.025	−0.0150	0.0018
0.9	↓	0.075	−0.0100	0.0005
1.0	+0.5	0.125	0	0

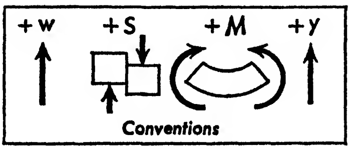
Table 3·9

Beam Curves

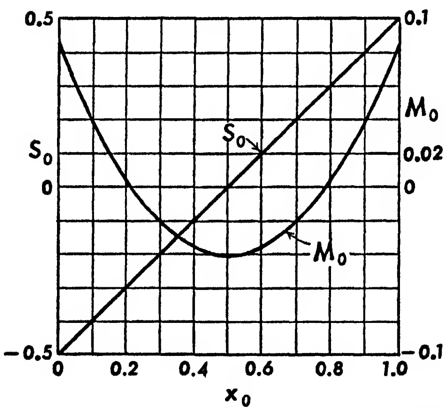
(Non-dimensional)

Fixed-End Beam—Uniform Running Load

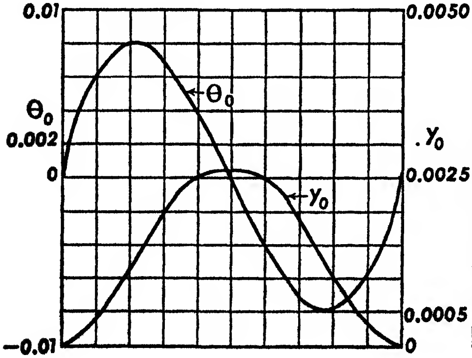




$$x = x_0 L$$
$$R_L = -\frac{wL}{2}$$
$$R_R = -\frac{wL}{2}$$
$$S = S_0 wL$$
$$M = M_0 wL^2$$
$$\Theta = \Theta_0 \frac{wL^3}{EI}$$
$$y = y_0 \frac{wL^4}{EI}$$



$$S_0 = -\frac{1}{2} + x_0$$
$$M_0 = \frac{1}{12} - \frac{x_0}{2} + \frac{x_0^2}{2}$$
$$\Theta_0 = \frac{x_0}{12} - \frac{x_0^2}{4} + \frac{x_0^3}{6}$$
$$y_0 = \frac{x_0^2}{24} - \frac{x_0^3}{12} + \frac{x_0^4}{24}$$



x_0	S_0	M_0	Θ_0	y_0
0	−0.5	0.0833	0	0
0.1	−0.4	0.0383	0.006	0.0003
0.2	−0.3	+0.0033	0.008	0.0011
0.3	−0.2	−0.0217	0.007	0.0019
0.4	−0.1	−0.0367	0.004	0.0025
0.5	0	−0.0417	0	0.0026
0.6	+0.1	−0.0367	−0.004	0.0025
0.7	+0.2	−0.0217	−0.007	0.0019
0.8	+0.3	+0.0033	−0.008	0.0011
0.9	+0.4	0.0383	−0.006	0.0003
1.0	+0.5	0.0833	0	0

ANSWERS TO PROBLEMS

CHAPTER 2

2·1. $P_x = 2266$ lb, $P_y = 1056$ lb.
2·2. $P_x = 6428$ lb, $P_z = 7660$ lb.
2·4. $P_r = 9785$ lb.
2·5. $R = 125$ lb, $\bar{z} = 1.2$ from 75-lb load.
2·6. $P_r = 18{,}300$ lb, 7.87 in. to right of A, 43° 17′ from X–X axis.
2·7. $P_x = +600$ lb, $P_y = +1970$ lb, $P_z = -3180$ lb.
2·10. 50%.
2·13. $S_{50} = 630$ lb, $S_{65} = 130$ lb, $S_{60} = 230$ lb, $S_{70} = 35.7$ lb.
2·15. $P_z = +23{,}890$ lb, $M_x = 425{,}000$ in.-lb, $P_y = -5940$ lb.
2·16. 1131 lb.
2·17. 849 lb vertical. 100 lb lateral.
2·18. $P_z = 10{,}195$ lb, $P_x = 234$ lb.

CHAPTER 3

3·1. Axial load at $A = 1378$ lb.
3·2. $P_x = 0$, $P_y = -12{,}500$ lb, $P_z = +21{,}650$ lb; $M_x = 246{,}500$ in.-lb, $M_y = 86{,}600$ in.-lb, $M_z = 50{,}000$ in.-lb.
3·3. At A, $T = 2516$ in.-lb, $M = 10{,}000$ in.-lb. At B, $T = 3752$ in.-lb, $M = 10{,}000$ in.-lb. At C $T = 3360$ in.-lb, $M = 10{,}000$ in.-lb.
3·4. Point A: $P = 0$, $S = +1000$ lb, $M = 0$.
Point B: $P = 707$ lb, $S = 707$ lb, $M = 14{,}140$ in.-lb.
Point C: $P = 1000$ lb, $S = 0$, $M = 20{,}000$ in.-lb.
Point D: $P = 707$ lb, $S = -707$ lb, $M = 14{,}140$ in.-lb.
Point E: $P = 0$, $S = 1000$, $M = 0$.
3·5. Point A: $P = 1000$ lb, $S = 0$, $M = 0$.
Point B: $P = 707$ lb, $S = -707$ lb, $M = -5860$ in.-lb.
Point C: $P = 0$, $S = -1000$ lb, $M = -20{,}000$ lb.
Point D: $P = -707$ lb, $S = -707$ lb, $M = -34{,}140$ in.-lb.
Point E: $P = -1000$, $S = 0$, $M = -40{,}000$ in.-lb.
3·6. Point A: $P = 900$ lb, $S = 500$ lb, $M = 0$.
Point B: $P = 990$ lb, $S = -283$ lb, $M = +1790$ in.-lb.
Point C: $P = 500$ lb, $S = -900$ lb, $M = -8000$ in.-lb.
Point D: $P = -283$ lb, $S = -990$ lb, $M = -23{,}630$ in.-lb.
Point E: $P = -900$ lb, $S = -500$ lb, $M = -36{,}000$ in.-lb.
3·7. Point A: $M = 0$, $T = 0$.
Point B: $M = 14{,}140$ in.-lb, $T = 5860$ in.-lb.
Point C: $M = 20{,}000$ in.-lb, $T = 20{,}000$ in.-lb.
Point D: $M = 14{,}140$ in.-lb, $T = 34{,}140$ in.-lb.
Point E: $M = 0$, $T = 40{,}000$ in.-lb.
3·13. $M_c = 0.645M$, $M_B = 0.430M$.
3·14. $M_c = 1.290M$, $M_B = 1.376M$.

CHAPTER 4

4·1. Shear = −148.4 lb, M = +539 in.-lb.
4·2. Nose wheel = 13,750 lb; main wheel = 15,000 lb per wheel.
4·3. A = 945 lb aft, B = 1215 lb forward.
4·4. $S_B = -55.7$ lb, $M_B = -427.1$ lb.
4·5. T = 1900 in. lb.

CHAPTER 5

5·1. (*a*) Members *OD*, *DB*, *DF*, *FB* are determinate; others indeterminate. (*b*) No. (*c*) Yes.
5·2. (*a*) Yes. (*b*) Yes. (*c*) No.
5·3. (*a*) Determinate; one-half. (*b*) Determinate; zero. (*c*) Determinate; three-fourths. (*d*) Mechanism; load cannot be carried.

CHAPTER 6

6·1. 0.0417 in.
6·2. Aluminum alloy.
6·3. (*a*) 20.033 in. (*b*) 47,075 psi.
6·4. x = 6.92 in.
6·13. 7200 psi tension.

CHAPTER 7

7·1. (*a*) Yes. (*b*) No.
7·2. Diameter = 0.521 in.
7·3. (*a*) $\bar{x} = 1.970$, $\bar{y} = 1.095$. (*b*) Steel. (*c*) 299,000 lb.
7·4. (*a*) 12,000 psi. (*b*) 6000 psi.
7·8. (*a*) 1922 psi. (*b*) 4700 psi.
7·9. 382 lb/ft.
7·10. 56,400 lb.

CHAPTER 8

8·2. 58.9%.

CHAPTER 10

10·1. ΔL = 0.0184 in.
10·3. (*a*) =0.000866 in. (*b*) =0.000613 in.
10·4. t = 0.0303 in. (min).
10·5. (*a*) 0.3150 degrees. (*b*) 0.264 in.
10·6. e_s = 0.004 radians. (*a*) f_s = 44,000 psi. (*b*) f_s = 10,000 psi.
10·9. q = 480 lb/in., t = 0.040 (min).
10·10. Same. Truss twice as heavy as web.

CHAPTER 11

11·1. f_s = 10,000 psi.
11·3. f_s = 5000 psi.
11·5. q = 34.5 lb/in.
11·8. $S = 0$, $M = 2\pi R^2 q$.
11·9. (*a*) $\pi r/2$. (*b*) $2a$. (*c*) a. (*d*) $\frac{\pi a}{2}$.

CHAPTER 12

12·2. (*a*) Same. (*b*) Weight of square tube = $\sqrt{2}$ times weight of round tube.
12·4. q = 104, t = 0.0208 (min).
12·5. 151,000 in.-lb.
12·6. 1.05 degrees.
12·7. 6.61 degrees.

12·8. (*a*) For 0.020, f_s = 9950 psi; For 0.040, f_s = 4975 psi; For 0.080, f_s = 2488 psi. (*b*) 0.0450 in.

12·9. (*a*) Low-carbon steel, 1.427 in.; alloy steel (normalized), 1.227 in.; alloy steel (H.T. 150,000), 1.043 in.; aluminum alloy, 1.342 in.; magnesium alloy, 1.720 in.

12·12. f_s = 22,200 psi. (*a*) 41 degrees. (*b*) 119 degrees. (*c*) 183 degrees.

12·13. Low-carbon steel, 12,900 in.-lb; alloy steel, 21,100 in.-lb; H.T. steel, 35,150 in.-lb; aluminum alloy, 15,940 in.-lb; magnesium alloy, 9380 in.-lb.

12·14. 1⅝ × 0.035 tube.

CHAPTER 13

13·1. 3000 lb each.

13·2. (*a*) .1034. (*b*) .329.

13·3. (*a*) .931 in. (*b*) 0.987 degree.

13·4. Tension area 0.1042 in.^{2}. Compression area 0.1667 in.^{2}.

13·5. (*a*) 1800 lb each. (*b*) 0.0397 tension area; 0.0675 compression area.

13·11. $f_b = \pm 86{,}920$ psi. A = 0.4999; I = 0.2300; Z = 0.2301.

CHAPTER 14

14·2. 0 psi, ±1610 psi.

CHAPTER 15

15·1. (*a*) M_x = 719,000 in.-lb, M_y = 93,300 in.-lb. (*b*) I_x = 105.3 in.^{4}, I_y = 7.59 in.^{4}, A = 8.30 in.^{2}.

15·2. (*a*) f_b = 90,900 psi. (*b*) f_b = 30,400 psi.

15·3. 98,300 in.-lb.

15·4. (*a*) M_x = 1,296,000 in.-lb, M_y = 168,000 in.-lb. (*b*) I_x = 183.2 in.^{4}, I_y = 12.15 in.^{4}, A = 16.67 in.^{2}.

15·5. (*a*) f_b = 50,600 psi. (*b*) f_b = 18,000 psi.

15·6. With rods, A = 386.5 in.^{2}, $\bar{y}$ = 14.10 in., I_x = 31,200 in.^{4}. Without rods, A = 360 in.^{2}, $\bar{y}$ = 15 in., I_x = 27,000 in.^{4}.

15·7. M = 812,500 in.-lb, f_c = 392.5 psi.

15·9. (*a*) 5% under. (*b*) 5% under. (*c*) 8% over. (*d*) 19% over. (*e*) 9% over.

15·11. 667 lb compression.

15·12.

θ	P
5	3,485
10	6,975
15	10,440
20	13,900

CHAPTER 16

16·1. (*a*) .001042. (*b*) .000359. (*c*) .000632.

16·2. (*a*) 1.42 in. (*b*) 1250 in.-lb.

16·3. 181 in.

16·4. (*a*) 48.3. (*b*) 0.517%.

16·5. 1800 in.

16·7. 3.82 degrees.

16·9. 0.2292 deg/in., 0.80 in.

16·10. (*a*) 765 in. (*b*) 1.046 in.

16·11. Steel beam is 1.45 times as stiff.
16·12. $y_{\text{Alum}} = 1.008 y_{\text{Steel}}$. $\theta_{\text{Alum}} = 1.207 \theta_{\text{Steel}}$.
16·13. (*a*) $y = 16.0$ in., $P = 356$ lb. (*b*) $y = 32.0$ in., $P = 178$ lb.
16·14. (*a*) $y = 4.00$ in., $P = 1422$ lb. (*b*) $y = 8.00$ in., $P = 711$ lb.
16·15. 6.84 in.
16·16. 58.9 ft.
16·17. (*a*) 74. (*b*) 123.2. (*c*) 24.6.
16·18. 0.309 in.

CHAPTER 17

17·4. 12,720 psi.
17·5. (*a*) 62.5 psi. (*b*) 62.5 psi. (*c*) 55.6 psi.
17·6. 0.120 in.
17·9. $q = 31.8$ lb/in. at loaded end, $q = 14.1$ lb/in. at mid-section, $q = 7.9$ lb/in. at supported end.

CHAPTER 18

18·2. 103 lb.

CHAPTER 19

19·2. Circle of radius $= R/4$.
19·3. Circle of radius $= R/2$.

19·4.

θ	*Axial*	*Shear*	*Moment*	f_n	f_s
45°	−.707*P*	.707*P*	2.64*P*	−1.093*P*	.1333*P*
90°	0	*P*	9.00*P*	3.392*P*	.1887*P*
135°	.707*P*	.707*P*	15.36*P*	5.89*P*	.1333*P*
180°	*P*	0	18*P*	6.93*P*	0

19·5. (*a*) 0.608 in.^{4}. (*b*) 0.0252 in.^{4}. (*c*) 0.073 in.^{4}.
19·6. (*a*) 1.88 in., 2.77 in.^{2}; 0.847 in., 0.563 in.^{2}; 1.104 in., 0.958 in.^{2}. (*b*) 2.614 in., 0.087 in.; 1.178 in., 0.0393 in.; 1.538 in., 0.0512 in.
19·7 (*a*) $f = 180$ psi, long column; $f = 880$ psi, long column; $f = 522$ psi, long column. (*b*) $f = 698$ psi, long column; $f = 3440$ psi, long column; $f = 2020$, psi long column.
19·9. (*a*) 0.2026 in.^{4}; (*b*) 0.0084 in.^{4}; (*c*) 0.0243 in.^{4} (One-third values of Prob. 19.5.)

CHAPTER 20

20·1.

Angle	*Normal stress*	*Shear stress*
10°	32,900	5,810
25°	27,850	13,000
35°	22,750	15,960
50°	14,020	16,750
65°	6,060	13,000

20·2. $F_t = 40{,}000$ psi.
20·3. $f_s = 15{,}000$ psi.
20·4. $f_n = 22{,}430$ psi, $f_s = 12{,}250$ psi.
20·5. $f_n = 21{,}180$ psi, $f_s = 11{,}180$ psi, $f_s = 7{,}070$ psi.
20·6. $f_n = 11{,}000$ psi, $f_s = 11{,}000$ psi.

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